The affine special Kähler/projective special Kähler correspondence and related constructions

Dissertation

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Dedicated to my parents, and to my lovely wife.

Abstract

In this thesis we study various geometric correspondences that are motivated by constructions in string theory.

The first part of this thesis considers the Kähler/Kähler correspondence and its curvature properties. We show that the Kähler/Kähler correspondence can be recovered from the more general twist construction, which is due to A. Swann. We present results on the behavior of the Ricci curvature under this correspondence using a formula by A. Futaki.

In the second part we formulate a correspondence between affine and projective special Kähler manifolds of the same dimension. We show as an application that under this correspondence the affine special Kähler manifolds in the image of the rigid r-map are mapped to one-parameter deformations of projective special Kähler manifolds in the image of the supergravity r-map. The above one-parameter deformations are interpreted as perturbative α'-corrections in heterotic and type-II string compactifications with $\mathcal{N} = 2$ supersymmetry. Moreover, we prove that the completeness of the deformed supergravity r-map metric depends only on the already well-understood completeness of the undeformed metric and the sign of the deformation parameter. We remark on the striking similarity of this situation to the HK/QK correspondence and its application to the c-map.

In the last chapter we provide a detailed review of algebraic completely integrable systems and prove a theorem of D. Freed stating that the base of such an integrable system is affine special Kähler. We formulate our statement of this result slightly more precisely than it appeared in its original paper. Finally, we show that the semi-flat metric appearing in a certain integrable system is in fact equivalent to the natural hyper-Kähler structure on the cotangent bundle of the associated special Kähler manifold.

Zusammenfassung

In dieser Dissertation studieren wir verschiedene geometrische Korrespondenzen die ihren Ursprung in aus der String Theorie stammenden Konstruktionen haben.

Im ersten Teil dieser Arbeit untersuchen wir die Kähler/Kähler Korrespondenz und ihre Krümmungseigenschaften. Wir zeigen, dass die Kähler/Kähler Korrespondenz als Spezialfall der allgemeineren Twist Konstruktion von A. Swann auftritt. Außerdem stellen wir Resultate über Verhalten der Ricci Krümmung unter dieser Korrespondenz vor.

Im zweiten Teil formulieren wir eine Korrespondenz zwischen affin und projektiv speziellen Kählermannigfaltigkeiten der selben Dimension. Wir zeigen, dass unter dieser Korrespondenz die affin speziellen Kählermannigfaltigkeiten im Bild der rigiden r-Abbildung auf eine Einparameterfamilie von projektiv speziellen Kählermannigfaltigkeiten im Bild der lokalen r-Abbildung abgebildet werden. Die obigen Einparameterdeformationen werden als perturbative α'-Korrekturen in heterotischen und Typ-II String Kompaktifizierungen mit $\mathcal{N} = 2$ Supersymmetrie interpretiert. Außerdem zeigen wir, dass die Vollständigkeit der deformierten lokalen r-Abbildungsmetrik nur von der bereits gut untersuchten Vollständigkeit der undeformierten Metrik und dem Vorzeichen des Deformationsparameters abhängt. Wir betonen die starke Ähnlichkeit dieser Situation zum Fall der HK/QK Korrespondenz und dessen Anwendung auf die c-Abbildung.

Im letzen Teil geben wir einen detaillierten Überblick über algebraisch vollständig integrable Systeme und beweisen ein Theorem von D. Freed, das besagt, dass die Basis eines solchen integrablen Systems affin speziell Kähler ist. Wir formulieren unsere Behauptung ein wenig präziser als im Originalpaper von Freed. Abschließend zeigen wir, dass die halbflache Metrik die in einem bestimmten integrablen System auftaucht tatsächlich äquivalent zur natürlichen hyper-Kählerstruktur des Kotangentialbündels der zugehörigen affin speziellen Kählermannigfaltigkeit ist.

Contents

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Chapter 1

Introduction

1.1 Motivation

The notion of special geometry is one that first appeared in the physics literature as the scalar target geometries of $\mathcal{N} = 2$ supersymmetric theories in four spacetime dimensions [dWVP84]. On the side of mathematics, the respective geometries occuring in rigid supersymmetry and supergravity correspond to what is called *affine special* (ASK) and *projective special Kähler* (PSK) [Fre99, ACD02]. Dimensional reduction from the four-dimensional vector multiplets to three-dimensional hypermultiplets leads to interesting geometric constructions called the *rigid c-map* [CFG89, Cor98, Fre99, Hit99, ACD02] and the *supergravity c-map* [FS90]. The rigid c-map associates a hyper-Kähler manifold of dimension $4n$ to any affine special Kähler manifold of dimension $2n$. The local c-map associates a quaternionic Kähler manifold of dimension $4n$ to any projective special Kähler manifold of dimension $2n - 2$. The constructed quaternionic Kähler metric is explicit but rather complicated in contrast to the hyper-Kähler metric of the rigid c-map. It was shown in [ACDM15] that the supergravity c-map can be understood as a special case of a much more general construction, the hyper-Kähler/quaternionic Kähler (HK/QK) correspondence [Hay08], as is summarized in the following diagram:

$$\begin{tikzcd}
M^{2n}_{\mathrm{CASK}} \arrow[r, mapsto, "\text{rigid c-map}"] \arrow[d, "\mathbb{C}^*\text{ bundle}"'] & N^{4n}_{\mathrm{HK}} \arrow[r, mapsto, dashed, "\text{conification}"] \arrow[dr, mapsto, "\text{HK/QK corr.}"] & \hat{N}^{4n+4}_{\mathrm{HK}} \arrow[d, "\text{Swann bundle}"] \\
\overline{M}^{2n-2}_{\mathrm{PSK}} \arrow[rr, mapsto, "\text{supergravity c-map}"] & & \overline{N}^{4n}_{\mathrm{QK}} .
\end{tikzcd} \tag{1.1.1}$$

In this diagram, M and $\hat{N}$ are the respective $\mathbb{C}^*$- and Swann-bundles of the projective special and quaternionic Kähler manifolds $\overline{M}$ and $\overline{N}$. The manifolds N and $\overline{N}$ are obtained via the respective c-maps from M and $\overline{M}$. In order to understand the supergravity

c-map in terms of the rigid c-map, it is necessary to give a conification procedure to construct $\hat{N}$ from N. This is achieved by the conification method developed in [ACM13]. The resulting relation between the hyper-Kähler manifold N and the quaternionic Kähler manifold $\overline{N}$ is obtained from the HK/QK correspondence [Hay08, ACM13, ACDM15]. The HK/QK correspondence can essentially be applied to any hyper-Kähler manifold with a Hamiltonian Killing vector field and depends on the choice of a Hamiltonian function which is unique up to a constant. Consequently, one recovers not only the supergravity c-map but a one-parameter deformation thereof. This deformation was identified as the one-loop deformed supergravity c-map metric [RLSV06].

The conification procedure of the HK/QK correspondence can also be applied to (pseudo)-Kähler[1] manifolds carrying an isometric Hamiltonian flow, thus giving a Kähler/Kähler (K/K) correspondence [ACM13, ACDM15]. Our interest in this correspondence was twofold:

For one, unlike in the hyper-Kähler and quaternionic Kähler case, Kähler manifolds are not automatically Einstein. It is thus an interesting question to ask under which conditions the K/K correspondence preserves and/or generates Einstein metrics.

Secondly, the K/K correspondence seemed to be the correct candidate for the analogous situation in the case of the supergravity r-map, introduced in [dWVP92], which is the map induced by dimensional reduction of five-dimensional to four dimensional vector multiplets. The situation is portrayed in the following diagram:

$$\begin{tikzcd}
\mathcal{U}^n_{\mathrm{CASR}} \arrow[r, mapsto, "\text{rigid r-map}"'] \arrow[d] & M^{2n}_{\mathrm{ASK}} \arrow[r, dashed, mapsto, "?"] \arrow[dr, dashed, mapsto, "?"] & \hat{N}^{2n+2}_{\mathrm{CASK}} \arrow[d, "\mathbb{C}^* \text{ bundle}"] \\
\mathcal{H}^{n-1}_{\mathrm{PSR}} \arrow[rr, mapsto, "\text{supergravity r-map}"] & & \overline{M}^{2n}_{\mathrm{PSK}} .
\end{tikzcd} \tag{1.1.2}$$

Here, $\mathcal{U}$ is a *conical affine special real* (CASR) domain containing the *projective special real* (PSR) manifold $\mathcal{H}$. It seemed likely to expect that the K/K correspondence would provide the link between the affine special Kähler manifold M in the image of the rigid r-map and the projective special Kähler manifold $\overline{M}$ in the image of the supergravity r-map. However, M does not carry a distinguished holomorphic Hamiltonian vector field to which the K/K correspondence could be applied.

It was therefore natural to ask whether the K/K correspondence could be modified in order to provide a link between affine special Kähler and projective special Kähler geometry such that in the special case of the r-map we would recover Diagram (1.1.2).

[1] In our conventions, metrics are of indefinite signature if not specified otherwise.

1.2 Main results

In [MS15], Macia and Swann showed that the HK/QK correspondence can be recovered as a combination of the twist construction with the concept of a so-called *elementary deformation*. In particular, they proved that there is essentially only a one parameter degree of freedom in constructing a quaternionic Kähler manifold of the same dimension using this method.

As is the case with the HK/QK correspondence, the twist method can also be applied to Kähler manifolds. In Theorem 2.1.18 we give necessary and sufficient conditions for the twist of an elementary deformation to be Kähler. We present an alternative proof of the K/K correspondence (Theorem 2.3.3) using the twist method, establishing, in particular, that the K/K correspondence can be recovered from a combination of a twist and an elementary deformation. We also show that in the Kähler case there are more degrees of freedom in the construction of Kähler manifolds, see Proposition 2.1.21 and Example 2.1.22.

We study the curvature properties of the K/K correspondence and derive the following result in the case of a conical Kähler manifold M of dimension $2n$: If ξ is the Euler vector field of the conical structure and f is Hamiltonian function with respect to the Hamiltonian Killing vector field $Z = J\xi$ we show in Theorem 2.3.9 that the K/K correspondence yields an Einstein metric with Einstein constant $\lambda = \sigma(2n+2)$ only if M is Ricci flat, where σ is the signature of f.

As the main result of this thesis, we establish the ASK/PSK correspondence that relates affine special Kähler manifolds to projective special Kähler manifolds of the same dimension, providing the missing link of Diagram (1.1.2) as a special case. This is done by giving a new conification procedure that maps affine special Kähler manifolds of dimension $2n$ to conical affine special Kähler manifolds of dimension $2n+2$. The conification does not, unlike in the case of the K/K and HK/QK correspondence, require the existence of a Hamiltonian Killing vector field. Instead it relies on the fact that affine special Kähler manifolds of dimension $2n$ can locally be realized as a Lagrangian submanifold in $\mathbb{C}^{2n}$ with induced geometric data, whereas projective special Kähler manifolds of complex dimension n are locally realized as the projectivization of a Lagrangian cone in $\mathbb{C}^{2n+2}$, c.f. [ACD02]. Thus in order to relate an affine special Kähler manifold M to a projective special Kähler manifold of the same complex dimension, we essentially have to map a Lagrangian submanifold $\mathcal{L} \subset \mathbb{C}^{2n}$ to a Lagrangian cone $\hat{\mathcal{L}} \subset \mathbb{C}^{2n+2}$. This is done by embedding $\mathcal{L}$ into the affine hyperplane $\{z^0 = 1\} \subset \mathbb{C} \times \mathbb{C}^{2n}$, where z^0 is the coordinate on the first factor. Then we take $\hat{\mathcal{L}}$ to be the graph over $\{1\} \times \mathcal{L}$ with respect to a certain

holomorphic function $f : \mathcal{L} \cong \{1\} \times \mathcal{L} \to \mathbb{C}$. The function f is what we call a *Lagrangian potential*, c.f. Definition 3.2.3, and is unique up to a complex constant C. As it turns out the real part of the constant C does not influence the resulting geometry. However, changing the imaginary part $c := \mathrm{Im}(C)$ leads to a one-parameter family of projective special Kähler manifolds $(\overline{M}_c, \overline{g}_c)$.

We discuss global properties of the construction by introducing a flat principal bundle with structure group $G_{\mathrm{SK}} := \mathrm{Sp}(\mathbb{R}^{2n}) \ltimes \mathrm{Heis}_{2n+1}(\mathbb{C})$. The group G_{SK} acts on pairs $(\mathcal{L}, f)$ of Lagrangian submanifolds that are local realizations of the affine special Kähler manifold and Lagrangian potentials f of $\mathcal{L}$. Moreover, it acts simply transitively on the set of special Kähler pairs $\mathcal{F}(U) = \{(\phi, F)\}$ of an open subset $U \subset M$ of holomorphic Kählerian Lagrangian immersions ϕ that locally realize U as a Lagrangian submanifold and corresponding holomorphic prepotentials F, cf. Definition 3.1.7. The relation between Lagrangian prepotentials and holomorphic prepotentials is shown in Lemma 3.2.9. In terms of a prepotential F and special coordinates $z := (z^1, \dots, z^n)$ on U, the conification construction can be understood as a homogenization of $F(z)$ to a holomorphic function

$$\hat{F}(Z^0, Z) = (Z^0)^2 F(Z/Z^0), \tag{1.2.1}$$

homogeneous of degree two in the coordinates $(Z^0, Z) := (Z^0, Z^1, \dots, Z^n) = (Z^0, Z^0 z)$ of $\mathbb{C}^* \times U$, cf. Remark 3.4.6.

The group G_{SK} is a central extension of the group $\mathrm{Aff}_{\mathrm{Sp}(\mathbb{R}^{2n})}(\mathbb{C}^{2n})$, which acts simply transitively on the set of Kählerian Lagrangian immersions of U. The central extension to G_{SK} is necessary to encompass the correct transformation behavior of holomorphic prepotentials F. Although the group action G_{SK} is equivariant with respect to the conification $\mathcal{L} \mapsto \hat{\mathcal{L}}$ (and $F \mapsto \hat{F}$), it does not leave the induced Kähler metrics on $\hat{\mathcal{L}}$ invariant, in contrast to the real subgroup $G := \mathrm{Sp}(\mathbb{R}^{2n}) \ltimes \mathrm{Heis}_{2n+1}(\mathbb{R})$. In Theorem 3.4.11 we prove that the conification is globally well defined if the holonomy of the flat connection of the principal G_{SK}-bundle is contained in the real group G and a certain notion of non-degeneracy is satisfied.

Our main application of the ASK/PSK correspondence is a one-parameter deformation of the supergravity r-map metric. It is obtained by applying the conification to the affine special Kähler manifold M obtained from the conical affine special real manifold $\mathcal{U}$ via the rigid r-map, as displayed in the following diagram:

$$\begin{tikzcd}
\mathcal{U}^n_{\mathrm{CASR}} \arrow[r, mapsto, "\text{rigid r-map}"] \arrow[d] & M^{2n}_{\mathrm{ASK}} \arrow[r, mapsto, dashed, "\text{conification}"] & \hat{N}^{2n+2}_{\mathrm{CASK}} \arrow[d, "\mathbb{C}^* \text{ bundle}"] \\
\mathcal{H}^{n-1}_{\mathrm{PSR}} \arrow[rr, mapsto, "\text{supergravity r-map}"] & & \overline{M}^{2n}_{\mathrm{PSK}} \,. \arrow[ul, mapsto, "\text{ASK/PSK corr.}"']
\end{tikzcd} \tag{1.2.2}$$

In Theorem 3.7.2 we give a global description of the resulting one-parameter family of projective special Kähler manifolds $(\overline{M}_c, \bar{g}_c)$, where $(\overline{M}_0, \bar{g}_0) = (\overline{M}, \bar{g})$ recovers the undeformed projective special Kähler manifold obtained from the supergravity r-map. We analyze completeness of the resulting one-parameter family. First of all, the underformed Riemannian manifold $(\overline{M}, \bar{g})$ is complete if and only if the projective special real manifold $\mathcal{H} \subset \mathbb{R}^n$ is a connected component of a global level set $\{x \in \mathbb{R}^n \mid h(x) = 1\}$ of a homogeneous cubic polynomial h [CHM12, CNS16]. Recall that the level set is required to be locally strictly convex for $\mathcal{H}$ to be projective special real and Riemannian. Assuming the undeformed metric $(\overline{M}, \bar{g})$ to be complete, we show that $(\overline{M}_c, \bar{g}_c)$ is Riemannian and complete if and only if $c < 0$. These results should be contrasted with the more involved completeness theorems for one-loop deformed c-map spaces [CDS16]. In the case of projective special Kähler manifolds with cubic prepotential the completeness of the supergravity c-map metric was shown to be preserved precisely under one-loop deformations with positive deformation parameter. However, for general c-map spaces this result has been established only under the additional assumption of regular boundary behavior for the initial projective special Kähler manifold, which is satisfied, for instance, for quadratic prepotentials. As in the case of the one-loop deformed c-map, the isometry type of the deformed r-map space $(\overline{M}_c, \bar{g}_c)$ depends only on the sign of c (positive, negative, or zero). Note that the completeness of $\overline{M}_0$ implies that $\overline{M}_1$ is neither isometric to $\overline{M}_0$ nor to $\overline{M}_{-1}$, since the latter two manifolds are then complete whereas $\overline{M}_1$ is incomplete. Computing the scalar curvature in examples, see Examples 3.7.4 and 3.7.5, we complete this analysis by showing that $\overline{M}_0$ and $\overline{M}_{-1}$ are in general not isometric. Incidentally, most, but not all, of the above results extend from cubic polynomials to general homogeneous functions, say of degree $k > 1$, see Remark 3.7.3. For instance, it is not known whether the above necessary and sufficient completeness criterion for projective special real manifolds [CNS16, Theorem 2.5] holds for polynomials of quartic and higher degree.

We note that the above one-parameter deformation can be interpreted as perturbative α'-corrections in heterotic and type-II string compactifications with $N = 2$ supersymmetry.

We study further properties of the principal G_{SK} bundle of affine special Kähler manifolds. In Theorem 3.5.4 we show that a complex manifold M of complex dimension n is affine special Kähler if and only if it admits a flat affine bundle $A \to M$ modelled over the complexification of a flat symplectic vector bundle together with a global holomorphic section Φ satisfying certain properties. We identify this bundle as the associated bundle to the principal G_{SK}-bundle with respect to the affine representation $\bar{\rho} : G_{\mathrm{SK}} \to \mathrm{Aff}_{\mathrm{Sp}(\mathbb{R}^{2n})}(\mathbb{C}^{2n})$. Over a local trivialization, the global section Φ gives

a Kählerian Lagrangian immersion ϕ. This result is a generalization of the statement that the affine special Kähler structure on M is locally induced by Kählerian Lagrangian immersions ϕ [ACD02].

We identify the data used to construct the algebraic completely integrable system $\mathcal{M}_0 \to M$ of [GMN10] as a special application of our theorem, see Proposition 3.5.1 and Section 4.2. In this case, the global holomorphic section Φ takes values in a vector bundle, implying that the holonomy of the principal G_{SK}-bundle is contained in $\mathrm{Sp}(\mathbb{R}^{2n}) \times \mathbb{C} \subset G_{\mathrm{SK}}$, c.f. Proposition 3.5.2. This provides a surprising potential application of the ASK/PSK correspondence to this class of integrable systems.

Finally, in Theorem 4.2.8 we show that the hyper-Kähler structure given on $\mathcal{M}_0$ is equivalent, up to rescaling and reordering of complex structures, to the c-map hyper-Kähler structure of T^*M.

1.3 Outline

This thesis is structured as follows. In Chapter 2 we give an introduction to Swann's twist method and develop a formula due to Futaki [Fut87], relating the Ricci curvatures of Kähler quotients. We give necessary and sufficient conditions for the twist of an elementary deformation to be Kähler, use the twist method to give an alternative proof of the K/K correspondence, and derive our curvature results for the K/K correspondence applied to conical Kähler manifolds.

In Chapter 3 we introduce the notion of special Kähler geometry and establish our conification construction and the ASK/PSK correspondence.

In Section 3.6 we derive our completeness results in terms of elementary deformations before we give our main application to of the ASK/PSK correspondence the r-map in Section 3.7.

In Chapter 4 we begin by giving a detailed introduction to algebraic completely integrable systems from a differential geometric viewpoint following [Fre99, GS90, Cor15]. We reproduce Freed's result that the base of an algebraic integrable system is affine special Kähler [Fre99]. Our statement of the theorem is slightly more precise than Freed's, cf. Remark 4.1.16. In Section 4.2 we show that the semi-flat hyper-Kähler structure of a certain integrable system [GMN,N] is equivalent to the natural hyper-Kähler structure on the cotangent bundle of the associated affine special Kähler manifold [CFG89, Cor98, Fre99, Hit99].

Chapter 2

Twisting Kähler geometries

The central theme of this chapter is the twisting of Kähler geometries, by which we understand constructions that produce new Kähler manifolds from a given Kähler manifold (M, g, J) with some additional data.

In Section 2.1 we introduce Swann's twist method in the context of circle actions. It can be used to construct Kähler manifolds from a Kähler manifold endowed with an isometric Hamiltonian S^1-action. In particular, we give necessary and sufficient conditions for the twist of an elementary deformation of a Kähler metric to be Kähler.

In Section 2.2 we recall the notion of a Kähler quotient and reproduce a formula by Futaki [Fut87] that relates the Ricci curvatures of such quotients.

In Section 2.3 we give an alternative proof of the K/K correspondence using the twist method. We close this chapter by applying our results from Section 2.2 to conical Kähler manifolds.

2.1 The Swann-Twist

Swann's twist construction [Swa10] is a method of equivariantly lifting the action of a k-torus T on a manifold M to a torus action on a principal $(S^1)^k$-bundle $P \to M$ that commutes with the principal action and preserves a principal connection. This allows to construct the quotient space $W = P/T$ and relate tensor fields on M with tensor fields on W.

In Sections 2.1.1 and 2.1.2 we give a description of the twist construction and elementary deformations for circle actions, following [Swa10].

In Section 2.1.3 we will present a method similar to [MS14] to produce Kähler metrics using the twist method and show that we can recover the Kähler/Kähler correspondence

in this way.

2.1.1 Lifting of actions

Let M be an n-dimensional manifold carrying the action of a group G, where G is either $\mathbb{R}$ or S^1. We denote by $Z \in \mathfrak{X}(M)$ a vector field that generates the G-action.

Definition 2.1.1. Let $F \in \Omega^2(M)$ be a closed two-form. We say that Z (or the action generated by Z) is *F-Hamiltonian* if there is a function $a \in C^\infty(M)$ such that

$$da = -Z \lrcorner F, \tag{2.1.1}$$

i.e., $[Z \lrcorner F] = 0 \in H^1(M)$. The function a is called a *moment map* of Z with respect to F.

Let $\pi : P \to M$ be a principal S^1-bundle with connection θ such that its curvature $d\theta = \pi^* F$ is given by a closed two-form F representing an element of $H^2(M, \mathbb{Z})$.

Proposition 2.1.2 ([Swa10, Proposition 2.1])**.** *The action induced by Z lifts to an action preserving the connection form θ and commuting with the principal action if and only if Z is F-Hamiltonian.*

Proof. We make the Ansatz

$$\mathring{Z} = \tilde{Z} + \mathring{a} X_P, \tag{2.1.2}$$

for the lifted infinitesimal action on P, where $\tilde{Z}$ is the horizontal lift of Z with respect to θ, $\mathring{a}$ is a function on P, and X_P is the fundamental vector field of the principal S^1-action of P. We compute

$$L_{\mathring{Z}}\theta = d(\mathring{Z} \lrcorner \theta) + \mathring{Z} \lrcorner d\theta = d\mathring{a} + \mathring{Z} \lrcorner \pi^* F = d\mathring{a} + \pi^*(Z \lrcorner F). \tag{2.1.3}$$

By evaluating $X_P \lrcorner L_{\mathring{Z}}\theta$, we see that $L_{\mathring{Z}}\theta = 0$ implies that $\mathring{a}$ is necessarily the pullback of a function on M, say $\mathring{a} = \pi^* a$. But then $L_{\mathring{Z}}\theta = 0$ if and only if $\pi^*(da + Z \lrcorner F) = 0$ if and only if $Z \lrcorner F = -\,da$ if and only if Z is F-Hamiltonian with moment map a. Computing

$$[\mathring{Z}, X_P] = [\tilde{Z}, X_P] + [\mathring{a} X_P, X_P] = -\,d\mathring{a}(X_P) X_P, \tag{2.1.4}$$

we see that $L_{\mathring{Z}}\theta = 0$ already implies $[\mathring{Z}, X_P] = 0$. □

Definition 2.1.3. If Z is F-Hamiltonian with moment map a we call the tuple (Z, F, a) *twist data*. We call the vector field $\mathring{Z} \in \mathfrak{X}(P)$ defined as in (2.1.2) the *lift* of Z with respect to the twist data (Z, F, a).

Remark 2.1.4. Note that moment maps are unique up to a constant. Each moment map a determines a lift $\mathring{Z} = \tilde{Z} + \mathring{a}X_P$ for $\mathring{a} := \pi^* a$.

Remark 2.1.5. It is shown in [Swa10, Proposition 2.3] that for a given closed integral two-form F and a vector field Z coming from an arbitrary S^1-action on M there is a choice of moment map $a \in C^\infty(M)$ and a principal bundle $\pi : P \to M$ with connection θ and curvature $\pi^* F$ such that the lift $\mathring{Z} = \tilde{Z} + \mathring{a}X_P$ in fact generates an S^1-action covering the S^1-action generated by Z on M. Here, a is unique up to an integral constant. If we allow the constant to be rational, then the corresponding lift covers the action of a finite covering of the S^1-action on M.

Lemma 2.1.6. *Let $\mathring{Z}$ be a lift with respect to the twist data (Z, F, a). Then $L_{\mathring{Z}}\tilde{X} = \widetilde{L_Z X}$ for any vector field X on M.*

Proof. Recall that since the horizontal distribution $\mathcal{H} = \ker\theta$ is invariant under the principal action, we have $[X_P, \tilde{X}] = 0$. Also, if X, Y are vector fields on M, then $[\tilde{X}, \tilde{Y}] = \widetilde{[X, Y]} + \theta([\tilde{X}, \tilde{Y}])X_P$ and $\theta([\tilde{X}, \tilde{Y}]) = -\pi^* F(\tilde{X}, \tilde{Y})$. Now we compute

$$\begin{aligned} L_{\mathring{Z}}\tilde{X} &= [\mathring{Z}, \tilde{X}] = [\tilde{Z}, \tilde{X}] + [\mathring{a}X_P, \tilde{X}] \\ &= \widetilde{[Z, X]} + \theta([\tilde{Z}, \tilde{X}])X_P - d\mathring{a}(\tilde{X})X_P + \mathring{a}[X_P, \tilde{X}] \\ &= \widetilde{L_Z X} + \theta([\tilde{Z}, \tilde{X}])X_P + \pi^* F(\tilde{Z}, \tilde{X})X_P = \widetilde{L_Z X}. \end{aligned} \tag{2.1.5}$$

□

2.1.2 The twist construction

Let M be a manifold with an F-Hamiltonian vector field Z with respect to a closed integral two-form F. We assume that Z is nowhere vanishing, i.e., the $\mathbb{R}$- or S^1-action generated by Z is locally free. Let $\pi : P \to M$ be a principal S^1-bundle with connection θ and curvature $d\theta = \pi^* F$ and let $\mathring{Z} = \tilde{Z} + \mathring{a}X_P$ be a lift of Z to P with respect to a moment map $a \in C^\infty(M)$. We assume that $\mathring{Z}$ is transverse to the horizontal distribution $\mathcal{H} = \ker\theta$ or, equivalently, that the function $a \in C^\infty(M)$ has no zeroes on M.

Definition 2.1.7. If the quotient space $W := P/\langle\mathring{Z}\rangle$ is smooth, we call W the *twist* of M with respect to the *twist-data* (F, Z, a) as above.

Let W be a twist of M with respect to the twist-data (F, Z, a) and with projection maps

$$\begin{tikzcd} & P \arrow[dl, "\pi"'] \arrow[dr, "\pi_W"] & \\ M & & W. \end{tikzcd} \tag{2.1.6}$$

By assumption, both maps π and π_W are transversal to the horizontal distribution $\mathcal{H}$. Note first that since X_P commutes with $\mathring{Z}$, X_P is invariant along fibers of π_W and, hence, descends to a non-zero vector field Z_W which is π_W-related to X_P on W, giving an S^1-action that is covered by the principal action on P.

Due to the transversality, $\theta(\mathring{Z}) = \mathring{a} \neq 0$ and $\mathring{\theta} := \mathring{a}^{-1}\theta$ defines a connection 1-form on P with horizontal distribution $\ker \mathring{\theta} = \ker \theta = \mathcal{H}$. Thus π and π_W induce isomorphisms

$$T_{\pi(p)}M \cong \mathcal{H}_p \cong T_{\pi_W(p)}W, \tag{2.1.7}$$

for $p \in P$.

This makes it possible to define pull-backs of tensor fields along π and π_W by setting

$$\pi^*(X \otimes \alpha)_p := (d\pi|_{\mathcal{H}_p})^{-1}(X_{\pi(p)}) \otimes (\pi^*\alpha)_p, \quad p \in P, \tag{2.1.8}$$

for a vector field X and a one-form α. If X is a vector field on M, the pull-back coincides with the horizontal lift $\tilde{X}$ of X. We write $\hat{Y} := (\pi_W)^*Y$ for the horizontal lift of a vector field Y on W. By definition and by the invariance of the horizontal distribution, pull-backs of tensor fields are invariant with respect to the corresponding principal action.

Definition 2.1.8. Let α be a tensor field on M and α_W a tensor field on W. We say α_W is $\mathcal{H}$-related to α, written as $\alpha_W \sim_{\mathcal{H}} \alpha$, if $\pi^*\alpha = \pi_W^*\alpha_W$ on $\mathcal{H}$.

Lemma 2.1.9. *If $\alpha \sim_{\mathcal{H}} \alpha_W$ then α is Z-invariant.*

Proof. Suppose $\alpha \sim_{\mathcal{H}} \alpha_W$ for tensor fields α on M and α_W on W of type (p,q). Denote by $\mathring{\varphi}_t$ the flow of $\mathring{Z}$. Let $x \in P$ and $Y_1, \dots, Y_p \in \mathcal{H}_p$. Then, since $\pi_W^*\alpha_W$ and θ are $\mathring{Z}$-invariant,

$$\begin{aligned}
(\pi^*\alpha)_x(Y_1, \dots, Y_p) &= (\pi_W^*\alpha_W)_x(Y_1, \dots, Y_p) = (\mathring{\varphi}_t^*)_x(\pi_W^*\alpha_W)(Y_1, \dots, Y_p) \\
&= (\pi_W^*\alpha_W)_{\mathring{\varphi}_t(x)}(d\mathring{\varphi}_t(Y_1), \dots, d\mathring{\varphi}_t(Y_p)) \\
&= (\pi^*\alpha)_{\mathring{\varphi}_t(x)}(d\mathring{\varphi}_t(Y_1), \dots, d\mathring{\varphi}_t(Y_p)) \\
&= (\mathring{\varphi}_t^*)_x(\pi^*\alpha)(Y_1, \dots, Y_p),
\end{aligned} \tag{2.1.9}$$

which shows that $\pi^*\alpha$ is invariant under $\mathring{Z}$. By Lemma 2.1.6 it follows that α is invariant under Z. □

Conversely, if α is Z-invariant, then $\pi^*\alpha$ is $\mathring{Z}$-invariant and therefore projects down along π_W to give a well-defined tensor field α_W that is $\mathcal{H}$-related to α. The following lemma shows the uniqueness of α_W.

Lemma 2.1.10 ([Swa10, Lemma 3.4])**.** *For each Z-invariant q-form α on M there is a unique q-form α_W on W, $\mathcal{H}$-related to α given by*

$$\pi_W^* \alpha_W = \pi^* \alpha - \theta \wedge \pi^* \left(a^{-1} Z \lrcorner \alpha \right). \tag{2.1.10}$$

Proof. Denote by $\Omega^q_{hor}(P) = \{\alpha \in \Omega^q(P) \mid X_P \lrcorner \alpha = 0\}$ the space of q-forms on P that are horizontal with respect to π. We claim that

$$\Omega^q(P) = \Omega^q_{hor}(P) \oplus \theta \wedge \Omega^{q-1}_{hor}(P). \tag{2.1.11}$$

Let $\mathrm{pr}_v(\alpha) := \theta \wedge (X_P \lrcorner \alpha)$ and $\mathrm{pr}_h(\alpha) := \alpha - \mathrm{pr}_v(\alpha)$ for $\alpha \in \Omega^q(P)$. Then it is straightforward to check that pr_v and pr_h are projections onto $\theta \wedge \Omega^{q-1}_{hor}(P)$ and $\Omega^q_{hor}(P)$, respectively. Moreover, $\alpha = \mathrm{pr}_h(\alpha) + \mathrm{pr}_v(\alpha)$ and $\mathrm{pr}_v \circ \mathrm{pr}_h = \mathrm{pr}_h \circ \mathrm{pr}_v = 0$, proving Eq. (2.1.11).

Now let $\alpha_W \sim_{\mathcal{H}} \alpha$. Then we can write $\pi_W^* \alpha_W = \pi^* \alpha + \theta \wedge \beta$ for a unique $\beta \in \Omega^{q-1}_{hor}$. We compute

$$\begin{aligned} 0 = \mathring{Z} \lrcorner \pi_W^* \alpha_W &= \mathring{Z} \lrcorner \pi^* \alpha + \mathring{Z} \lrcorner (\theta \wedge \beta) \\ &= \pi^*(Z \lrcorner \alpha) + \mathring{a}\beta - \theta \wedge (\tilde{Z} \lrcorner \beta). \end{aligned} \tag{2.1.12}$$

Evaluating on $\mathcal{H}$ yields $\beta = -\pi^* \left(a^{-1} Z \lrcorner \alpha \right)$. Thus α_W is uniquely determined by α and Eq. (2.1.10) holds. □

Corollary 2.1.11 ([Swa10, Corollary 3.6])**.** *Let $\alpha_W \sim_{\mathcal{H}} \alpha$. Then*

$$d\alpha_W \sim_{\mathcal{H}} d\alpha - a^{-1} F \wedge Z \lrcorner \alpha. \tag{2.1.13}$$

Proof. This follows from a straightforward computation by differentiating Eq. (2.1.10) and using $L_Z \alpha = 0$. □

Remark 2.1.12 (Duality). The curvature of the connection $\mathring{\theta} = \mathring{a}^{-1}\theta$ is given by

$$\pi_W^* F_W = \pi^*(a^{-1} F) - \theta \wedge \pi^*(a^{-2} Z \lrcorner F), \tag{2.1.14}$$

where F_W is the two-form $\mathcal{H}$-related to $a^{-1}F$. Moreover,

$$\begin{aligned} \pi_W^*(Z_W \lrcorner F_W) = X_P \lrcorner \pi_W^* F_W &= X_P \lrcorner (\pi^*(a^{-1}F) - \theta \wedge \pi^*(a^{-2} Z \lrcorner F)) \\ &= -\pi^*(a^{-2} Z \lrcorner F) = -\pi^* \, d(a^{-1}), \end{aligned} \tag{2.1.15}$$

so the action of Z_W is F_W-Hamiltonian and the function a_W that is $\mathcal{H}$-related to a^{-1} is a moment map. This shows that M is the twist of W with respect to the twist data (F_W, Z_W, a_W).

Lemma 2.1.13 ([Swa10, Lemma 3.7]). *Let $X_W \sim_{\mathcal{H}} X$ and $Y_W \sim_{\mathcal{H}} Y$ be vector fields. Then*

$$[X_W, Y_W] \sim_{\mathcal{H}} [X, Y] + a^{-1}F(X, Y)Z. \tag{2.1.16}$$

Proof. Since $X_W \sim_{\mathcal{H}} X$ and $Y_W \sim_{\mathcal{H}} Y$ we have $\tilde{X} = \hat{X}_W$ and $\tilde{Y} = \hat{Y}_W$. We know that $[\tilde{X}, \tilde{Y}] = \widetilde{[X, Y]} - \pi^* F(\tilde{X}, \tilde{Y}))X_P$ and $[\widehat{X}_W, \widehat{Y}_W] = \widehat{[X_W, Y_W]} - \pi_W^* F_W(\hat{X}_W, \hat{Y}_W)\mathring{Z}$. Then

$$\begin{aligned}
\widehat{[X_W, Y_W]} &= [\hat{X}_W, \hat{Y}_W] + \pi_W^* F_W(\hat{X}_W, \hat{Y}_W))\mathring{Z} \\
&= [\tilde{X}, \tilde{Y}] + \pi_W^* F_W(\tilde{X}, \tilde{Y})\mathring{Z} \\
&= [\tilde{X}, \tilde{Y}] + \left(\pi^*(a^{-1}F) - \theta \wedge \pi^*(a^{-2}Z \lrcorner F)\right)(\tilde{X}, \tilde{Y})\mathring{Z} \\
&= [\tilde{X}, \tilde{Y}] + \pi^*(a^{-1}F)(\tilde{X}, \tilde{Y})(\tilde{Z} + \mathring{a}X_P) \\
&= [\tilde{X}, \tilde{Y}] + \pi^* F(\tilde{X}, \tilde{Y}))X_P + \pi^*(a^{-1}F(X, Y))\tilde{Z} \\
&= \widetilde{[X, Y]} + \pi^*(a^{-1}F(X, Y))\tilde{Z},
\end{aligned} \tag{2.1.17}$$

proving the lemma. □

If J is an almost complex structure on M that is invariant under the action of Z, then we can define an almost complex structure J_W on W that is $\mathcal{H}$-related to J.

Lemma 2.1.14 ([Swa10, Lemma 3.9]). *If J is a complex structure on M, then J_W is a complex structure on W if and only if $F \in \Omega^{(1,1)}(M)$.*

Proof. Applying Lemma 2.1.13 to the Nijenhuis tensor, defined as

$$N_J(X, Y) = [JX, JY] - J[JX, Y] - J[X, JY] - [X, Y] \tag{2.1.18}$$

we find that

$$N_{J_W} \sim_{\mathcal{H}} N_J + Z \otimes (F(J\cdot, J\cdot) - F(\cdot, \cdot)) - JZ \otimes (F(J\cdot, \cdot) + F(\cdot, J\cdot)). \tag{2.1.19}$$

Since J is complex, $N_{J_W} = 0$ if and only if F is of type $(1, 1)$. □

Remark 2.1.15 (Local twist). A smooth twist does not exist in general. However, we can define a local version of the twist construction as follows. Let $\pi : P \to M$ be an S^1-principal bundle as above with lift $\mathring{Z}$ with respect to twist data (Z, F, a). Choose a submanifold $W \subset P$ that is transverse to the foliation induced by $\mathring{Z}$ and a tubular neighborhood U of W. Then we can identify W with the leaf space of the local foliation on U and we let $\pi_W : U \to W$ be the quotient map. By the preceding discussion, a Z-invariant tensor field α on M then defines a well-defined tensor-field on W.

2.1.3 Twisting Kähler structures

Let (M, g, J) be a $2n$ dimensional Kähler manifold with Kähler form $\omega = g(J\cdot, \cdot)$. Let F be a closed integral two-form of type $(1,1)$ and suppose that the vector field Z is F-Hamiltonian, preserves the Kähler structure, and has non-vanishing norm. Let $a \in C^\infty(M)$ such that $da = -Z \lrcorner F$. With respect to the twist data (Z, F, a) we let W either be a smooth twist, in case it exists, or, as explained in Remark 2.1.15, a transverse submanifold.

Our goal is to use the twist to construct a Kähler structure on W using the Kähler structure on M. By Lemma 2.1.14 we know that J_W is a complex structure on W. However, the unique two-form ω_W that is $\mathcal{H}$-related to ω is in general not closed, as is apparent from Eq. (2.1.13).

To remedy this, we will consider the twist of a deformation of the Kähler metric on the distribution spanned by Z and JZ. Let $\alpha = g(JZ, \cdot)$ and $\beta = g(Z, \cdot)$.

Definition 2.1.16. An *elementary deformation* g^N of g (respectively $\omega^N = g^N(J\cdot, \cdot)$ of ω) with respect to Z is given by

$$g^N = h_1 g + h_2(\alpha^2 + \beta^2), \tag{2.1.20}$$

$$\omega^N = h_1 \omega + h_2 \beta \wedge \alpha, \tag{2.1.21}$$

for Z-invariant functions $h_1, h_2 \in C^\infty(M)$ such that $h_1 + \beta(Z)h_2 \neq 0$.

Remark 2.1.17. The condition $h_1 + \beta(Z)h_2 \neq 0$ is equivalent to g^N being non-degenerate.

Theorem 2.1.18. *Let ω_W be a twist of ω^N with respect to the twist data (Z, F, a). Then (W, ω_W, J_W) is Kähler if and only if*

$$d\omega^N = a^{-1}(h_1 + h_2\beta(Z))F \wedge \alpha. \tag{2.1.22}$$

In particular, the Kähler form ω_W and metric g_W are given by

$$\pi_W^* \omega_W = \pi^* \omega^N - \theta \wedge \pi^* \left(\frac{h_1 + h_2\beta(Z)}{a} \alpha \right), \tag{2.1.23}$$

$$\pi_W^* g_W = \pi^* g^N - 2\theta \cdot \pi^* \left(\frac{h_1 + h_2\beta(Z)}{a} \beta \right) + \pi^* \left(\frac{h_1 + h_2\beta(Z)}{a^2} \beta(Z) \right) \theta^2. \tag{2.1.24}$$

Remark 2.1.19. Splitting

$$g = \check{g} + \frac{1}{\beta(Z)}(\alpha^2 + \beta^2) \quad \text{and} \quad \omega = \check{\omega} + \frac{1}{\beta(Z)} \beta \wedge \alpha,$$

into the distribution spanned by Z and JZ and its orthogonal complement, Eqs. (2.1.23) and (2.1.24) yield

$$\pi_W^*\omega_W = h_1\check{\omega} + \frac{h_1 + h_2\beta(Z)}{\beta(Z)}\left(\beta - \frac{\beta(Z)}{a}\theta\right)\wedge\alpha, \tag{2.1.25}$$

$$\pi_W^* g_W = h_1\check{g} + \frac{h_1 + h_2\beta(Z)}{\beta(Z)}\left(\alpha^2 + \left(\beta - \frac{\beta(Z)}{a}\theta\right)^2\right), \tag{2.1.26}$$

where tensor fields on M are understood as being pulled back by π.

Proof of Theorem 2.1.18. Applying Corollary 2.1.11 to $d\omega_W$, we see that ω_W is closed if and only if ω^N is closed with respect to $d_W = d - a^{-1}F\wedge Z\lrcorner$, which is equivalent to Eq. (2.1.22). Let $X_W \sim_{\mathcal{H}} X$ and $Y_W \sim_{\mathcal{H}} Y$. Then $\widetilde{JX} = \widehat{J_W X_W}$ by the definition of J_W. Since $\pi^*\omega^N$ and $\pi_W^*\omega_W$ agree on $\mathcal{H}$, it follows that ω_W is of type $(1,1)$ with respect to the complex structure J_W.

To verify the formula for g_W, we write $\pi_W^* g_W = \pi^* g^N + \gamma\cdot\theta + f^2\theta^2$ for γ a horizontal one-form and f a function on P. Then from

$$0 = \pi_W^* g_W(\mathring{Z},\cdot) = \pi^*(g^N(Z,\cdot)) + \frac{1}{2}\gamma + \theta\cdot\left(af + \frac{1}{2}\gamma(\tilde{Z})\right), \tag{2.1.27}$$

we find comparing the horizontal and vertical parts that $\gamma = -2\pi^*\left(\frac{h_1+h_2\beta(Z)}{a}\beta\right)$ and $f = \pi^*\left(\frac{h_1+h_2\beta(Z)}{a^2}\beta(Z)\right)$, yielding Eq. (2.1.24). □

In the following we will give two examples of Kähler twists. The first one yields a metric that is identical to the metric obtained through the Kähler/Kähler correspondence, as will be shown in Section 2.3. The second example is an adaptation of a twist of a hyper-Kähler structure to a hyper-Kähler structure in [Swa14].

Proposition 2.1.20 (K/K correspondence)**.** *Let f be a moment map of Z with respect to ω. Then $a = f_1 := f - \frac{1}{2}\beta(Z)$ is a moment map of Z with respect to $F := \omega - \frac{1}{2}\,d\beta$. Set $h_1 = \frac{\sigma}{2f}$, $h_2 = -\frac{\sigma}{4f^2}$, where $\sigma = \operatorname{sign} f$, and assume that f and f_1 have no zeroes on M. Then the twist of $g^N = h_1 g + h_2(\alpha^2+\beta^2)$ yields the Kähler metric g_W given by*

$$\pi_W^* g_W = \pi^* g^N - 2\theta\cdot\pi^*\left(\frac{\sigma}{2f^2}\beta\right) + \pi^*\left(\frac{\sigma\beta(Z)}{2f_1 f^2}\right)\theta^2 \tag{2.1.28}$$

$$= \frac{1}{2|f|}\left(\check{g} + \frac{f_1}{f\beta(Z)}\left(\alpha^2 + \left(\beta - \frac{\beta(Z)}{f_1}\theta\right)^2\right)\right), \tag{2.1.29}$$

and Kähler form ω_W given by

$$\pi_W^*\omega_W = \pi^*\omega^N - \theta \wedge \pi^*\left(\frac{\sigma}{2f^2}\alpha\right) \tag{2.1.30}$$

$$= \frac{1}{2|f|}\left(\breve{\omega} + \frac{f_1}{f\beta(Z)}\left(\beta - \frac{\beta(Z)}{f_1}\theta\right) \wedge \alpha\right). \tag{2.1.31}$$

Proof. The function $h_1 + h_2\beta(Z) = \frac{\sigma f_1}{2f^2}$ has no zeroes by assumption. Thus g^N and its twist g_W are non-degenerate. The two-form $d\beta$ is of type $(1,1)$ since Z is holomorphic and, hence, F is of type $(1,1)$. We compute

$$Z \lrcorner F = -\,df + \frac{1}{2}Z \lrcorner d\beta = -\,df - \frac{1}{2}\,d(\beta(Z)) = -\,df_1, \tag{2.1.32}$$

thus (Z, F, a) are twist data. To see that g_W is Kähler, we verify Eq. (2.1.22). First, note that $a^{-1}(h_1 + h_2\beta(Z)) = \frac{h_1}{f}$ and then compute

$$\begin{aligned} d\omega^N &= dh_1 \wedge \omega + \underbrace{dh_2 \wedge \beta \wedge \alpha}_{=0} + h_2\, d\beta \wedge \alpha \\ &= \frac{h_1}{f}\omega \wedge \alpha - \frac{h_1}{2f}\, d\beta \wedge \alpha \\ &= \frac{h_1}{f}(\omega - \frac{1}{2}\, d\beta) \wedge \alpha \\ &= a^{-1}(h_1 + h_2\beta(Z))F \wedge \alpha. \end{aligned} \tag{2.1.33}$$

Hence, g_W defines a Kähler metric on W by Theorem 2.1.18. □

Proposition 2.1.21. *Let f be a moment map of Z with respect to ω. Choose $h_1 \equiv 1$, h_2 a polynomial in f, $F = d(h_2\beta)$, and $a = 1 + h_2\beta(Z)$, assuming $a \neq 0$ on M. Then the twist ω_W of ω^N is Kähler with Kähler form ω_W given by*

$$\pi_W^*\omega_W = \pi^*\omega^N - \theta \wedge \pi^*\alpha \tag{2.1.34}$$

$$= \breve{\omega} + \frac{a}{\beta(Z)}\left(\beta - \frac{\beta(Z)}{a}\theta\right) \wedge \alpha. \tag{2.1.35}$$

The Kähler metric g_W is given by

$$\pi_W^* g_W = \pi^* g^N - 2\theta \cdot \pi^*\beta + \frac{\beta(Z)}{1 + h_2\beta(Z)}\theta^2 \tag{2.1.36}$$

$$= \breve{g} + \frac{a}{\beta(Z)}\left(\alpha^2 + \left(\beta - \frac{\beta(Z)}{a}\theta\right)^2\right). \tag{2.1.37}$$

Proof. First note that since h_2 is a polynomial in f, h_2 is Z-invariant and F is of type $(1,1)$. Also, we easily verify $Z \lrcorner F = -\,da$. Finally we compute

$$d\omega^N = dh_2 \wedge \beta \wedge \alpha + h_2\, d\beta \wedge \alpha = d(h_2\beta) \wedge \alpha = F \wedge \alpha, \tag{2.1.38}$$

and it follows from Theorem 2.1.18 that the twist is Kähler. □

Example 2.1.22. Let (Z, F, a) and h_1, h_2 as in Proposition 2.1.21. We work in a local trivialization of the S^1-principal bundle P with local connection form $\theta = ds + h_2\beta$, where s is the coordinate on S^1. In this trivialization, the lifted action with respect to the twist data (Z, F, a) is given by $\mathring{Z} = \tilde{Z} + (1 + h_2\beta(Z))\partial_s = Z + (1 + h_2\beta(Z) - \theta(Z))\partial_s = Z + \partial_s$. We choose $W = \{s = 0\}$ as the transverse submanifold, and we find the following local formulas for the twists g_W and ω_W of g^N and ω^N:

$$\omega_W = \omega + h_2\beta \wedge \alpha - \theta \wedge \alpha = \omega, \tag{2.1.39}$$

$$g_W = g + h_2 \left(\alpha^2 - \frac{1}{a}\beta^2 \right) \tag{2.1.40}$$

$$= \check{g} + \frac{a}{\beta(Z)} \left(\alpha^2 + \frac{1}{a^2}\beta^2 \right). \tag{2.1.41}$$

Hence, the twist leaves the Kähler form invariant but deforms the complex structure locally along the distribution spanned by Z and JZ.

2.2 Ricci curvature of Kähler quotients

Let (M, g, J) be a $2n$ dimensional (pseudo)-Kähler manifold with Kähler form $\omega = g(J\cdot, \cdot)$ and suppose there is a non-vanishing time- or spacelike Hamiltonian Killing vector field Z, inducing an S^1-action, with moment map $\mu \in C^\infty(M)$, $d\mu = -\omega(Z, \cdot)$. Assume that $m \in \mathbb{R}$ is a regular value of μ. Then $N := \mu^{-1}(m) \subset M$ is a smooth submanifold that is invariant under Z. We assume that the induced S^1-action on N is free. Let $M' := N/S^1$ be the quotient and denote by $\iota : N \hookrightarrow M$ and $\pi : N \to M'$ the inclusion and the quotient map, respectively.

The map π is a principal S^1-bundle with vertical distribution $\mathcal{Z} := \ker d\pi$ spanned by the vector field Z. The metric defines a J- and S^1-invariant complementary distribution $E := \mathcal{Z}^\perp$, giving the orthogonal decompositions

$$TN = E \oplus \mathcal{Z}, \text{ and} \tag{2.2.1}$$

$$\iota^* TM = TN \oplus J\mathcal{Z}. \tag{2.2.2}$$

In fact, E is the horizontal distribution given by the connection one-form

$$g(Z,\cdot)/g(Z,Z) \in \Omega^1(N, \mathfrak{s}^1 \cong \mathbb{R}). \tag{2.2.3}$$

For any vector field X on N we denote by vX and hX the vertical and horizontal components of X, respectively. We define a (pseudo)-Riemannian metric on M' via

$$g'_{\pi(p)}(d\pi(U), d\pi(V)) := g_p(U,V), \quad U,V \in E_p, p \in M. \tag{2.2.4}$$

This turns $\pi : (N, \iota^* g) \to (M', g')$ into a (pseudo)-Riemannian submersion. We call a vector field X on N *basic* if it is the horizontal lift of a vector field X' on M', i.e., $X = \widetilde{X'}$.

We denote by ∇, $\nabla^N = \iota^*\nabla$, and ∇' the Levi-Civita connections of M, N, and M', respectively.

Proposition 2.2.1 ([FPI04, Proposition 1.1])**.** *If X, Y are horizontal lifts of vector fields X', Y' on M', then $h(\nabla^N_X Y)$ is the horizontal lift of $\nabla'_{X'} Y'$.*

Proof. Note first that $g(X,Y) = g'(X',Y') \circ \pi$ from Eq. (2.2.4). Let Z be the horizontal lift of an arbitrary vector field Z' on M'. We find $X(g(Y,Z)) = X(g'(Y',Z') \circ \pi) = X'(g'(Y',Z')) \circ \pi$ and $g([X,Y],Z) = g(h[X,Y],Z) = g'([X',Y'],Z') \circ \pi$, where we have used that $h[X,Y]$ is the horizontal lift of $[X',Y']$. Now, using the Koszul-formula,

$$\begin{aligned} 2g(h\nabla^N_X Y, Z) &= 2g(\nabla^N_X Y, Z) \\ &= X(g(Y,Z)) + Y(g(X,Z)) - Z(g(X,Y)) \\ &\quad + g([X,Y],Z) + g([Z,X],Y) - g([Y,Z],X) \\ &= g'(\nabla'_{X'} Y', Z') \circ \pi. \end{aligned} \tag{2.2.5}$$

Since Z' was arbitrary and π is surjective, the claim follows. □

It is well known [HKLR87, Fut87] that g' is a Kähler metric with Kähler form ω' given by $\pi^*\omega' = \iota^*\omega$.

Definition 2.2.2. The Kähler manifold M', constructed above, is called the *Kähler-quotient* of M with respect to the S^1-action, the moment map μ and the regular value k, and we write $M' = M /\!\!/ S^1$.

We define the $(1,2)$-tensor field $A : E \times E \to \mathcal{Z}$ by

$$A(X,Y) = A_X Y = v(\nabla^N_X Y).$$

The tensor field A is one of the fundamental tensor fields of the (pseudo)-Riemannian submersion π as defined in [O'N66], satisfies

$$A(X,Y) = \frac{1}{2}v[X,Y], \tag{2.2.6}$$

and is related to the second fundamental form $I\!I : TN \times TN \to TN^{\perp}$ of $N \subset M$ as follows.

Proposition 2.2.3 ([Kob87])**.** *Let $X, Y \in E$. Then*

$$\begin{aligned} A(X, JY) &= J(I\!I(X,Y)) \\ I\!I(X, JY) &= J(A(X,Y)). \end{aligned} \tag{2.2.7}$$

In particular, $A(JX, JY) = A(X,Y)$ and $I\!I(JX, JY) = I\!I(X,Y)$.

Proof. Let X, Y be horizontal vector fields, then

$$\begin{aligned} \nabla_X Y &= \nabla^N_X Y + I\!I(X,Y) \\ &= h(\nabla^N_X Y) + v(\nabla^N_X Y) + I\!I(X,Y) \\ &= h(\nabla^N_X Y) + A(X,Y) + I\!I(X,Y). \end{aligned} \tag{2.2.8}$$

The claim follows by comparing $\nabla_X(JY)$ and $J\nabla_X Y$ and using the directness of the sum $\iota^* TM = T^h N \oplus T^v N \oplus TN^{\perp}$. □

Proposition 2.2.4. *Let $\pi^* F$ be the curvature of the connection Eq.* (2.2.3) *of the S^1-bundle $\pi : N \to M'$, given by a closed two-form $F \in \Omega^1(M')$. Then*

$$A(X,Y) = -\frac{1}{2}\pi^* F(X,Y)Z, \tag{2.2.9}$$

and F is of type $(1,1)$.

Proof. Let X, Y be horizontal vector fields on N. The first claim follows from Eq. (2.2.6) and the standard formula $v[X,Y] = -\pi^* F(X,Y)Z$. The second claim is a direct consequence of Eq. (2.2.9) and Proposition 2.2.3. □

Let $F = \mathcal{Z} \oplus J\mathcal{Z}$ such that $\iota^* TM = E \oplus F$. By J-invariance of E and F we have orthogonal decompositions into $\pm i$-eigenspaces

$$E \otimes \mathbb{C} = E^{1,0} \oplus E^{0,1}, \tag{2.2.10}$$

$$F \otimes \mathbb{C} = F^{1,0} \oplus F^{0,1}, \tag{2.2.11}$$

$$\iota^* T^{(1,0)} M = E^{1,0} \oplus F^{1,0}. \tag{2.2.12}$$

Note that $E^{1,0}$ is integrable as we can write $E^{1,0} = \iota^* T^{1,0}M \cap (TN \otimes \mathbb{C})$.

Since Z is a (real) holomorphic vector field the section $Z_{\mathbb{C}} := \frac{1}{2}(Z - iJZ)$ of $F^{1,0}$ is holomorphic. Its (possibly negative) norm $\|Z\|^2 := g(Z, \overline{Z})$ is invariant under Z and therefore descends to a function $\|\check{Z}\|^2$ on M'.

We will now derive a formula connecting the Ricci tensors of M and M' following the treatment of [Fut87].

The connection $\nabla^N = \iota^*\nabla$ of $\iota^* T^{(1,0)}M = E^{(1,0)} \oplus F^{(1,0)}$ induces connections on $E^{(1,0)}$ and $F^{(1,0)}$ which we will denote by ∇^h and ∇^v, respectively. By the same symbols we will also denote the induced connections on the line bundles $\det T^{(1,0)}M$, $\det E^{(1,0)}$, and $\det F^{(1,0)}$. Let θ, θ^h and θ^v be the corresponding connection forms of ∇^N, ∇^h and ∇^v with respect to the frames $X_1 \wedge \ldots \wedge X_{n-1} \wedge Z_{\mathbb{C}}$, $X_1 \wedge \ldots \wedge X_{n-1}$ and $Z_{\mathbb{C}}$, respectively, where $\{X_i\}_{i=1}^{n-1}$ is a pseudo-orthonormal basis of basic vector fields of $E^{(1,0)}$ such that $\|X_i\|^2 = \epsilon_i \in \{\pm 1\}$. Then, by virtue of the wedge product, $\theta = \theta^h + \theta^v$.

Let X be a basic vector field. Then, using Proposition 2.2.1, we find

$$\begin{aligned}
&\theta'(d\pi(X))\, d\pi(X_1) \wedge \ldots \wedge d\pi(X_{n-1}) \\
&\quad = \nabla'_{d\pi(X)}\, d\pi(X_1) \wedge \ldots \wedge d\pi(X_{n-1}) \\
&\quad = \sum_{i=1}^{n-1} d\pi(X_1) \wedge \ldots \wedge \nabla'_{d\pi(X)}\, d\pi(X_i) \wedge \ldots \wedge d\pi(X_{n-1}) \\
&\quad = \sum_{i=1}^{n-1} d\pi(X_1) \wedge \ldots \wedge d\pi(\nabla^h_X X_i) \wedge \ldots \wedge d\pi(X_{n-1}) \\
&\quad = \sum_{i=1}^{n-1} \pi_*(X_1 \wedge \ldots \wedge \nabla^h_X X_i \wedge \ldots \wedge X_{n-1}) \\
&\quad = \pi_*\left(\nabla^h_X (X_1 \wedge \ldots \wedge X_{n-1})\right) \\
&\quad = \theta^h(X)\, d\pi(X_1) \wedge \ldots \wedge d\pi(X_{n-1}),
\end{aligned} \tag{2.2.13}$$

hence, $\pi^*\theta' = \theta^h \circ \mathrm{pr}_h =: \theta^h_h$. Set $\theta^h_v := \theta^h \circ \mathrm{pr}_v$ and $\theta^v_i := \theta^v \circ \mathrm{pr}_i$ for $i \in \{h, v\}$.

Denote by ρ' and ρ the Ricci form of M' and M. Then

$$\begin{aligned}
\pi^*\rho' &= i\,\pi^*\, d\theta' = i\, d(\pi^*\theta') = i\, d\theta^h_h \\
&= i\, d(\theta - \theta^h_v - \theta^v) \\
&= (\iota^*\rho) - i(d\theta^h_v + d\theta^v),
\end{aligned} \tag{2.2.14}$$

where we have used $\theta = \theta^h_h + \theta^h_v + \theta^v$ and that the curvature form of the canonical line bundle over a Kähler manifold is given by i times its Ricci form.

Remark 2.2.5. Note that the curvature form of the canonical line bundle $\Lambda^{(n,0)}M$ is given by $-\,d\theta$ as $\Lambda^{(n,0)}M = (\det T^{(1,0)}M)^*$.

Lemma 2.2.6. *Let $Y \in E^{(1,0)}$ and $\sigma = \operatorname{sign}(\|Z\|^2)$. Then*

(i) $d\theta_h^v = \pi^*(\bar\partial\partial \log(\sigma\|\check{Z}\|^2))$,

(ii) $d\theta_v^v(Y,\overline{Y}) = i(JA(Y,\overline{Y})\log(\sigma\|Z\|^2))$, *and*

(iii) $d\theta_v^h(Y,\overline{Y}) = 2\sum_{i=1}^{n-1}\epsilon_i\, g(A(Y,\overline{Y}), A(X_i,\overline{X}_i))$.

Proof. (i) Since $Z_{\mathbb{C}}$ is holomorphic $\nabla^v Z_{\mathbb{C}}$ and θ_h^v are 1-forms of type $(1,0)$. Furthermore, we have

$$\begin{aligned}\theta_h^v(Y)Z_{\mathbb{C}} = \nabla_Y^v Z_{\mathbb{C}} &= \frac{g(\nabla_Y Z_{\mathbb{C}}, \overline{Z}_{\mathbb{C}})}{\|Z_{\mathbb{C}}\|^2} Z_{\mathbb{C}} \\ &= (Y\log(\sigma\|Z_{\mathbb{C}}\|^2))Z_{\mathbb{C}} = (\partial\log(\sigma\|Z\|^2))(Y)Z_{\mathbb{C}}.\end{aligned} \tag{2.2.15}$$

Thus, $d\theta_h^v = d\pi^*(\partial\log(\sigma\|\check{Z}\|^2)) = \pi^*(\bar\partial\partial\log(\sigma\|\check{Z}\|^2)) = \pi^*(\bar\partial\partial\log(\sigma\|\check{Z}\|^2))$.

(ii) We first note that

$$d\theta_v^\bullet(Y,\overline{Y}) = -\theta^\bullet(v[Y,\overline{Y}]), \quad \text{for } \bullet \in \{h,v\}, \tag{2.2.16}$$

since $\theta_v^\bullet = \theta^\bullet \circ v$ vanishes on E by definition. Let $X = \frac{1}{2}(v[Y,\overline{Y}] - iJv[Y,\overline{Y}]) = A(Y,\overline{Y}) - iJA(Y,\overline{Y})$. Then, since $Z_{\mathbb{C}}$ is holomorphic,

$$\begin{aligned}\theta^v(v[Y,\overline{Y}])Z_{\mathbb{C}} = \nabla^v_{v[Y,\overline{Y}]}Z_{\mathbb{C}} = \nabla^v_{X-\overline{X}}Z_{\mathbb{C}} &= \nabla^v_X Z_{\mathbb{C}} \\ &= (X\log(\sigma\|Z\|^2))Z_{\mathbb{C}} \\ &= -i(JA(Y,\overline{Y})\log(\sigma\|Z\|^2))Z_{\mathbb{C}}.\end{aligned} \tag{2.2.17}$$

The last equation holds, since $\|Z\|^2$ is constant along fibers.

(iii) This follows from Eq. (2.2.16),

$$\begin{aligned}&\theta^h(v[Y,\overline{Y}])X_1\wedge\ldots\wedge X_{n-1} \\ &\quad= \nabla^h_{v[Y,\overline{Y}]}(X_1\wedge\ldots\wedge X_{n-1}) \\ &\quad= \sum_{i=1}^{n-1} X_1\wedge\ldots\wedge\nabla^h_{v[Y,\overline{Y}]}X_i\wedge\ldots\wedge X_{n-1} \\ &\quad= \sum_{i=1}^{n-1} X_1\wedge\ldots\wedge\epsilon_i\, g(\nabla_{v[Y,\overline{Y}]}X_i,\overline{X}_i)X_i\wedge\ldots\wedge X_{n-1} \\ &\quad= \sum_{i=1}^{n-1}\epsilon_i\, g(\nabla_{v[Y,\overline{Y}]}X_i,\overline{X}_i)X_1\wedge\ldots\wedge X_{n-1},\end{aligned} \tag{2.2.18}$$

and

$$\begin{aligned} g(\nabla_{v[Y,\overline{Y}]}X_i, \overline{X}_i) &= g(\nabla_{X_i} v[Y,\overline{Y}], \overline{X}_i) - g([v[Y,\overline{Y}], X_i], \overline{X}_i) \\ &= g(\nabla_{X_i} v[Y,\overline{Y}], \overline{X}_i) \\ &= X_i(g(v[Y,\overline{Y}], \overline{X}_i)) - g(v[Y,\overline{Y}], \nabla_{X_i}\overline{X}_i) \\ &= -g(v[Y,\overline{Y}], \nabla_{X_i}\overline{X}_i) \\ &= -g(v[Y,\overline{Y}], v(\nabla_{X_i}\overline{X}_i)) \\ &= -2g(A(Y,\overline{Y}), A(X_i,\overline{X}_i)), \end{aligned} \tag{2.2.19}$$

where we have used that $[v[Y,\overline{Y}], X_i]$ is vertical since X_i is basic. □

Proposition 2.2.7 ([Fut87, Proposition 3.12])**.** *Let* Ric' *and* Ric *be the Ricci curvature of* M' *and* M*, respectively. Then for* $Y \in E^{1,0}$ *we have*

$$\begin{aligned} \pi^*\mathrm{Ric}'(Y,\overline{Y}) = &\,\mathrm{Ric}(Y,\overline{Y}) - \pi^*(\bar{\partial}\partial \log(\sigma\|Z\|^2))(Y,\overline{Y}) \\ &- iJA(Y,\overline{Y})\log(\sigma\|Z\|^2) \\ &- 2\sum_{i=1}^{n-1} \epsilon_i\, g(A(Y,\overline{Y}), A(X_i,\overline{X}_i)), \end{aligned} \tag{2.2.20}$$

where $\{X_i\}$ *is a basic orthonormal basis of* $E^{(1,0)}$*.*

Proof. The Ricci form is defined as $\rho(X,Y) = \mathrm{Ric}(JX,Y)$ for $X,Y \in \mathfrak{X}(M)$. Hence, $\rho(Y,\overline{Y}) = i\,\mathrm{Ric}(Y,\overline{Y})$. Using Eq. (2.2.14) and Lemma 2.2.6 the claim follows. □

Corollary 2.2.8. *Let* M *be Kähler-Einstein. If* $\|Z\|^2$ *is constant on level sets of* μ *and* $A \equiv 0$ *then* M' *is Kähler-Einstein.*

Proof. By our assumptions all terms but the first vanish on the right hand side of Eq. (2.2.20). Let $Y \in E^{1,0}$. Then

$$\pi^*\,\mathrm{Ric}'(Y,\overline{Y}) = \mathrm{Ric}(Y,\overline{Y}) = \lambda g(Y,\overline{Y}) = \lambda\pi^* g'(Y,\overline{Y}), \tag{2.2.21}$$

where λ is the Einstein constant of M. Thus M' is Einstein. □

Example 2.2.9. Let $(S^{2n+1}, g_S, Z, \eta, \Phi)$ be a regular pseudo Sasaki-Einstein manifold with contact form η, Reeb vector field Z, and $(1,1)$-tensor field Φ. We set $\sigma = \mathrm{sign}(g_S(Z,Z))$. Pseudo Sasakian manifolds that are Einstein have Einstein constant $2n\sigma$. Its Riemannian cone $(\hat{M} = S \times \mathbb{R}_{>0}, \hat{g} = \sigma\, dr^2 + r^2 g_S, \hat{\omega} = \frac{\sigma}{2}\, d(r^2\eta))$ is Kähler-Einstein and Ricci-flat and the Euler vector field $r\partial_r$ satisfies $\hat{J}r\partial_r = Z$ and $\hat{g}(Z,Z) = \sigma r^2$. By regularity of Z, S is a principal S^1 bundle $\pi : S \to M'$ over a Kähler-Einstein manifold

(M', g', ω') where $\pi^*\omega' = \frac{1}{2}\sigma\, d\eta$ and $g_S = \pi^* g' + \sigma\eta^2$. M' corresponds to the Kähler quotient of $\hat{M}$ with respect to the holomorphic Killing vector field Z on the level-set $\{r = 1\} = S \times \{1\} \cong S$.

We have

$$A(Y, \overline{Y}) = \frac{1}{2} v[Y, \overline{Y}] = -\frac{1}{2}\, d\eta(Y, \overline{Y})Z = -\sigma\, \pi^*\omega'(Y, \overline{Y})Z = -i\sigma\, \pi^* g'(Y, \overline{Y})Z. \quad (2.2.22)$$

Using $\nabla_{r\partial r} Z = Z$ we find

$$\hat{J}Z(\log(\sigma\|Z_{\mathbb{C}}\|^2)) = -r\, dr(\log(\sigma\|Z_{\mathbb{C}}\|^2)) = -2. \quad (2.2.23)$$

Thus

$$i\hat{J}A(Y, \overline{Y}) \log(\sigma\|Z_{\mathbb{C}}\|^2) = -2\sigma\, \pi^* g'(Y, \overline{Y}), \quad (2.2.24)$$

and

$$\begin{aligned} 2\sum_{i=1}^{n-1} \epsilon_i\, \hat{g}(A(Y, \overline{Y}), A(X_i, \overline{X}_i)) &= -2\pi^* g'(Y, \overline{Y})\hat{g}(Z, Z) \sum_{i=1}^{n} \epsilon_i\, \underbrace{\pi^* g'(X_i, \overline{X}_i)}_{=\epsilon_i} \\ &= -2n\sigma\pi^* g'(Y, \overline{Y}). \end{aligned} \quad (2.2.25)$$

Finally, using Eq. (2.2.20),

$$\pi^* \operatorname{Ric}'(Y, \overline{Y}) = 2\sigma(n+1)\pi^* g'(Y, \overline{Y}). \quad (2.2.26)$$

So M' is Kähler-Einstein with Einstein constant $\sigma(2n+2)$. This result is well-known, see, for instance, [BG08, Theorem 11.1.3] for the Riemannian case.

2.3 The Kähler/Kähler correspondence

Let (M, g, J) be a $2n$ dimensional (pseudo)-Kähler manifold with Kähler form $\omega = g(J\cdot, \cdot)$ and a non-vanishing Hamiltonian Killing vector field Z such that $g(Z, Z)$ is nowhere zero. Let f be a Hamiltonian function of Z, i.e., $df = -Z \lrcorner\, \omega$ and assume that f and $f_1 := f - \frac{1}{2}g(Z, Z)$ are nowhere vanishing. Set $\beta := g(Z, \cdot)$ and $\alpha := g(JZ, \cdot) = -\, df$.

Assume there is a principal S^1-bundle $\pi : P \to M$ with connection θ and curvature $d\theta = \pi^*(\omega - \frac{1}{2}\beta(Z))$, and endow it with the metric

$$g_P := \pi^* g + \frac{2}{f_1}\theta^2, \quad (2.3.1)$$

and the tensor field

$$\eta := \theta + \frac{1}{2}\beta, \quad (2.3.2)$$

where we have identified tensor fields on M with their pullbacks to tensor fields on P. Denote by X_P the fundamental vector field of P. Let $\hat{M} := P \times \mathbb{R}$ with the coordinate t on the $\mathbb{R}_{>0}$-factor. On $\hat{M}$ we introduce the following tensor fields.

$$\begin{aligned} \hat{\xi} &:= \partial_t, \\ \hat{g} &:= e^{2t}(g_P + 2f\,dt^2 + 2\,df\,dt), \\ \hat{\theta} &:= e^{2t}\left(\theta + \frac{1}{2}\beta\right), \text{ and} \\ \hat{\omega} &:= d\hat{\theta}, \end{aligned} \tag{2.3.3}$$

where we have again identified tensor fields on M and P with their canonical pullback to $\hat{M}$. We will also denote the canonical lift of X_P to $\hat{M}$ by the same symbol.

Definition 2.3.1. A *conical Kähler manifold* $(\hat{M}, \hat{g}, \hat{J}, \hat{\xi})$ is a pseudo-Kähler manifold $(\hat{M}, \hat{g}, \hat{J})$ and a vector field $\hat{\xi}$ such that $\hat{g}(\hat{\xi}, \hat{\xi})$ has no zeroes on $\hat{M}$ and $\nabla\hat{\xi} = \mathrm{id}$, where ∇ is the Levi-Civita connection of $\hat{M}$.

Theorem 2.3.2 ([ACM13, Theorem 1])**.** *Given (M, g, J) and Z, then the manifold $(\hat{M}, \hat{g}, \hat{J} = \hat{g}^{-1}\hat{\omega}, \hat{\xi})$, constructed as above, is a conical Kähler manifold.*

On $\hat{M}$ the vector fields $\mathring{Z} := \hat{J}\hat{\xi} = \tilde{Z} + f_1 X_P$ and X_P are holomorphic Killing vector fields and commute. We have $X_P \lrcorner\, \hat{\omega} = -\,d(e^{2t})$, so $P \cong P \times \{1\} = \{e^{2t} = 1\}$ and we recover M as the Kähler quotient of $\hat{M}$ with respect to the moment map e^{2t} and the regular value $m = 1$.

Moreover, the manifold $(\hat{M}, \hat{g})$ is a metric cone over a pseudo-Sasakian manifold (S, g_S). The norm of the Euler vector field $\hat{\xi}$ defines the radial coordinate $r^2 = |\hat{g}(\hat{\xi}, \hat{\xi})| = 2|f|e^{2t}$ and $S = \{r = 1\}$ is diffeomorphic to P. The metric $\hat{g}$ takes the form $\hat{g} = \sigma\, dr^2 + r^2 g_S$, where $\sigma = \mathrm{sign}(\hat{g}(\hat{\xi}, \hat{\xi}))$.

Theorem 2.3.3 ([ACDM15, Theorem 3])**.** *The tensor field*

$$\tilde{g}_P := g_P - \frac{1}{2f}\alpha^2 - \frac{2}{f}\left(\theta + \frac{1}{2}\beta\right)^2 \tag{2.3.4}$$

on P is invariant under Z_P and has a one-dimensional kernel $\mathbb{R}\mathring{Z}$. Let W be a submanifold of P which is transversal to the vector field $\mathring{Z}$. Then

$$g' := \frac{1}{2|f|}\tilde{g}_P|_W \tag{2.3.5}$$

is a possibly indefinite Kähler metric on W.

Remark 2.3.4. (1) The above relation between the Kähler manifold (M, g, J) together with its Hamiltonian function f and the Kähler manifold W is what is called the K/K correspondence.

(2) In the original proof in [ACDM15], the authors show that the metric g' corresponds under the identification $S \cong P$ to the transverse Kähler metric of the Sasakian structure of S. We give an alternative proof using the twist construction.

Proof of Theorem 2.3.3. We note that $\hat{J}\hat{\xi} = \mathring{Z} = \tilde{Z} + f_1 X_P$ is a lift of Z with respect to the twist data (Z, F, a), where we have set $a := f_1$ and $F = \omega - \frac{1}{2}\beta$. So let $W \subset P$ be transverse to $\mathring{Z}$. We choose $h_1 = \frac{1}{2|f|}$ and $h_2 = -\frac{h_1}{2f}$. We have already shown in Proposition 2.1.20 that the twist g_W of $g^N = h_1 g + h_2(\alpha^2 + \beta^2)$ with respect to the above data is Kähler. We compute (identifying tensors on M with their pullback to P)

$$\begin{aligned}
\pi_W^* g_W &= \pi^* g^N - 2\theta \cdot \pi^* \left(\frac{\sigma}{2f^2}\beta\right) + \pi^* \left(\frac{\sigma\beta(Z)}{2f_1 f^2}\right)\theta^2 \\
&= \left(\frac{1}{2|f}g - \frac{1}{4|f|f}(\alpha^2 + \beta^2)\right) - 2\theta \cdot \left(\frac{\sigma}{2f^2}\beta\right) + \left(\frac{\sigma\beta(Z)}{2f_1 f^2}\right)\theta^2 \\
&= \frac{1}{2|f|}\left(g - \frac{1}{2f}(\alpha^2 + \beta^2) - \frac{2}{f}\beta\theta + \frac{\beta(Z)}{f_1 f}\theta^2\right) \\
&= \frac{1}{2|f|}\left(g_P - \underbrace{\left(\frac{2}{f_1} - \frac{\beta(Z)}{f_1 f}\right)}_{=\frac{2}{f}}\theta^2 - \frac{2}{f}\theta \cdot \beta - \frac{1}{2f}\beta^2 - \frac{1}{2f}\alpha^2\right) \\
&= \frac{1}{2|f|}\left(g_P - \frac{1}{2f}\alpha^2 - \frac{2}{f}\left(\theta + \frac{1}{2}\beta\right)^2\right).
\end{aligned} \tag{2.3.6}$$

Hence $g' = \pi_W^* g_W|_W = g_W$ is Kähler. □

2.3.1 Curvature properties of the Kähler/Kähler correspondence for conical Kähler manifolds

Let (M, g, J) be a Kähler manifold with Kähler form $\omega = g(J\cdot, \cdot)$, non-vanishing Hamiltonian Killing vector field Z, $\beta = g(Z, \cdot)$, $-df = -Z \lrcorner \omega$, $f_1 = f - \frac{1}{2}\beta(Z)$, and f, f_1 both non-zero. Set $\sigma := \mathrm{sign}(f)$.

Lemma 2.3.5 ([ACM13, Lemma 1])**.** *f_1 is constant if and only if $\nabla_Z Z = JZ$.*

Proof. Let $h = \frac{1}{2}\beta(Z)$. We have

$$dh = g(\nabla Z, Z) = -g(\nabla_Z Z, \cdot) = \omega(\nabla_Z JZ, \cdot). \tag{2.3.7}$$

Thus $df_1 = d(f - h) = -\omega(Z + J\nabla_Z Z, \cdot)$ which is identically zero if and only if $\nabla_Z Z = JZ$. □

Lemma 2.3.6. *We have $\omega - \frac{1}{2}\, d\beta = 0$ if and only if $\nabla Z = J$.*

Proof. We compute

$$\begin{aligned} d\beta(X,Y) &= X(g(Z,Y)) - Y(g(Z,X)) - g(Z,[X,Y]) \\ &= -g(X,\nabla_Y Z) + g(Y,\nabla_X Z) \\ &= -2g(X,\nabla_Y Z) = -2\omega(X,\nabla_Y JZ). \end{aligned} \tag{2.3.8}$$

Thus

$$\left(\omega - \frac{1}{2}\, d\beta\right)(X,Y) = \omega(X, Y + \nabla_Y JZ), \tag{2.3.9}$$

which is identically zero if and only if $\nabla Z = J$. □

Following the treatment of [Dyc15, Section 4.1.1] in the case of the HK/QK correspondence applied to conical hyper-Kähler manifolds, we will apply the K/K correspondence to a conical Kähler manifold (M, g, J, ξ) with $Z = J\xi$.

We set $g(\xi,\xi) = \lambda r^2$ where $\lambda = \operatorname{sign}(g(\xi,\xi))$. Then a moment map of Z is given by $f = \frac{\lambda}{2}(r^2 + c)$ and $f_1 = \frac{\lambda}{2}c$. We denote the sign of f by σ. By Lemma 2.3.6 we can choose $P = M \times S^1$ and $\mathring{Z} = Z + \lambda\frac{c}{2}\partial_s$, denoting by s the coordinate in S^1. The submanifold $W := \{s = 0\} \cong M$ is transverse to $\mathring{Z}$. The metric obtained from the K/K correspondence Eq. (2.3.5) on W is then given by

$$\begin{aligned} g' &= \frac{1}{2|f|}\left(g_P - \frac{1}{2f}\alpha^2 - \frac{2}{f}\left(\theta + \frac{1}{2}\beta\right)^2\right)\Bigg|_W \\ &= \frac{\sigma}{2f}(g - \frac{1}{2f}(\alpha^2 + \beta^2)) \\ &= \frac{\sigma}{r^2 + c}g - \frac{\sigma}{(r^2+c)^2}(\alpha^2 + \beta^2). \end{aligned} \tag{2.3.10}$$

Example 2.3.7. Consider $M = \mathbb{C}^n \setminus \{0\}$ with standard coordinates given by $(z^1, \dots, z^n)$ and standard metric $g = dz^i\, d\bar{z}^i$. Its conical vector field is given by $\xi = r\partial r = 2\operatorname{Re}(z^i \partial_{z^i})$, we set $Z = J\xi$, and we find $\alpha = -\frac{1}{2}(z^i\, d\bar{z}^i + \bar{z}^i\, dz^i)$ and $\beta = \frac{i}{2}(z^i\, d\bar{z}^i - \bar{z}^i\, dz^i)$. Set $f = \frac{1}{2}(\|z\|^2 + c)$ and $\sigma = \operatorname{sign}(f)$. If $c > 0$, then $f > 0$ on M. For $c < 0$ we restrict M to $M_c = \{c + \|z\|^2 < 0\} = \{f < 0\}$. Then Eq. (2.3.10) reads

$$g' = \sigma\frac{dz^i\, d\bar{z}^i}{c + \|z\|^2} - \sigma\frac{\bar{z}^i z^j\, dz^i\, d\bar{z}^j}{(c + \|z\|^2)^2}. \tag{2.3.11}$$

This shows that (M_c, g') for $c < 0$ is isometric to the complex hyperbolic space $\mathbb{C}H^n$. Note that for $c > 0$ the metric g' on $\mathbb{C}^n \setminus \{0\}$ extends to all of $\mathbb{C}^n$. Hence $(\mathbb{C}^n, g')$ is isometric to $\{[z^0 : \ldots : z^n] \mid z^0 \neq 0\} \subset \mathbb{C}P^n$.

Proposition 2.3.8. *Let (M, g, J, ξ) be conical Kähler and set $Z = J\xi$. If the conical Kähler manifold $\hat{M}$ obtained from Theorem 2.3.2 is Einstein (or, equivalently, Ricci-flat) then M is necessarily Ricci-flat as well.*

Proof. Since M is conical and $Z = J\xi$, the principal S^1-bundle $P \to M$ is flat by Lemma 2.3.6, hence, the tensor A from Section 2.2 is zero. We have $\hat{g}(X_P, X_P) = \frac{2}{f_1} e^{2t}$ and the moment map of X_P is simply e^{2t}. So X_P has constant norm on level sets if and only if f_1 is constant on M. The statement then follows from Lemma 2.3.5 and Corollary 2.2.8. □

Theorem 2.3.9. *Let (M, ξ) be a $2n$-dimensional conical Kähler manifold with Hamiltonian function f. Set $\sigma = \operatorname{sign}(f)$. If every manifold W that is obtained from M and f via the K/K correspondence is Einstein with Einstein constant $\sigma(2n + 2)$, then M is necessarily Ricci-flat.*

Proof. Note from Eq. (2.3.3) that the signature of $\hat{g}(\xi, \xi)$ is determined by $\sigma = \operatorname{sign}(f)$. Let $(S \subset \hat{M}, g_S)$ be the Sasaki submanifold over which $\hat{M}$ is the metric cone. We choose g_S such that the norm of the Reeb flow is given by σ. Assume that every W obtained from M and f is Einstein with Einstein constant $\lambda := \sigma(2n + 2)$. Since any such W is isometric to a submanifold of S transverse to the Reeb foliation this is equivalent to the transverse metric of the Sasakian structure to be Einstein with Einstein constant λ. From this we conclude that $\hat{M}$ is Ricci flat, see, for instance, Example 2.2.9, or [BG08, Theorem 11.1.3, Lemma 11.1.5] for the Riemannian case. By Proposition 2.3.8 it follows that M is necessarily Ricci-flat. □

Example 2.3.10. Another interesting case arises when applying the K/K correspondence to a conical Kähler manifold (M, g, J, ξ) with $Z = 2J\xi$. We will assume that M is a cone over a regular Sasaki manifold that fibers over a Kähler manifold $(\check{M}, \check{g})$. If M is only locally a cone or S is not regular, we can instead choose $\check{M}$ as a submanifold of S transverse to the local Reeb flow of S. Let $\lambda = \operatorname{sign}(g(\xi, \xi))$ and $r^2 := |g(\xi, \xi)|$. It was shown in [ACDM15, Theorem 4] that when applying the K/K correspondence to M with $f = \lambda r^2$, one obtains a product manifold

$$(W, 2g') = (\mathbb{R}^{>0} \times S^1 \times \check{M}, -\lambda g_{\mathbb{C}H^1} + \check{g}), \tag{2.3.12}$$

where $g_{\mathbb{C}H^1} = \frac{1}{4\rho^2}(d\rho^2 + d\tilde{\phi}^2)$, ρ is a coordinate on $\mathbb{R}^{>0}$, and $\tilde{\phi} = -4s$ is related to the coordinate s on the S^1-factor. The metric $g_{\mathbb{C}H^1}$ is Einstein with Einstein constant $\Lambda_{\mathbb{C}H^1} = -4$. Recall that a product metric is Einstein if and only if the factors have the same Einstein constant. Hence, in this case, the metric g' is Einstein if and only if $\check{g}$ is Einstein with Einstein constant $\Lambda' = -\lambda 4$.

Chapter 3

The ASK/PSK correspondence

In this chapter we will establish the ASK/PSK correspondence relating affine special Kähler manifolds to projective special Kähler manifolds.

We begin in Section 3.1 with an introduction to special Kähler geometry, mostly following [ACD02].

In Sections 3.2 to 3.4 we introduce the necessary technical tools needed to formulate the conification construction and the ASK/PSK correspondence.

Section 3.5 contains the generalization of the statement that the affine special Kähler structure of an affine special Kähler manifold is locally induced by Kählerian Lagrangian immersions [ACD02].

In Section 3.6 we prove a completeness result for a one-parameter deformation of a positive definite Hessian, which will be specialized in Section 3.7 to the case of the r-map.

Finally, Section 3.7 will contain our results of the application of the ASK/PSK correspondence to the case of the r-map, cf. Diagram Eq. (1.2.2).

3.1 Special Kähler geometry

Definition 3.1.1. An *affine special Kähler manifold* (M, J, g, ∇) is a pseudo-Kähler manifold (M, J, g) with symplectic form $\omega := g(J\cdot, \cdot)$ endowed with a flat torsion-free connection ∇ such that $\nabla\omega = 0$ and $d^{\nabla} J = 0$.

An affine special Kähler manifold has the property that a ∇-parallel one-form α is holomorphic as a section of the holomorphic cotangent bundle (T^*M, J). This follows from the next proposition.

Proposition 3.1.2 ([ACD02, Proposition 1])**.** *Let ∇ be a flat torsion-free connection on a complex manifold (M, J). Then $\mathrm{d}^\nabla J = 0$ if and only if $d(\alpha \circ J) = 0$ for all local ∇-parallel one-forms α on M.*

Proof. Let α be a local ∇-parallel 1-form, and X, Y be local vector fields such that $\nabla JX = \nabla JY = 0$. We compute

$$\begin{aligned}(d^\nabla J)(X,Y) &= (\nabla_X J)Y - (\nabla_Y J)X \\ &= \nabla_X JY - J\nabla_X Y - \nabla_Y JX + J\nabla_Y X \\ &= -J[X,Y],\end{aligned} \tag{3.1.1}$$

and

$$\begin{aligned}d(\alpha \circ J)(X,Y) &= X(\underbrace{\alpha(JY)}_{\text{const.}}) - Y(\underbrace{\alpha(JX)}_{\text{const.}}) - \alpha(J[X,Y]) \\ &= -\alpha \circ J([X,Y]).\end{aligned} \tag{3.1.2}$$

This shows that $d^\nabla J = 0$ if and only if $d(\alpha \circ J) = 0$ for all local ∇-parallel one-forms α on M. □

Definition 3.1.3. Let M be a complex manifold of complex dimension n and consider the complex vector space $T^*\mathbb{C}^n = \mathbb{C}^{2n}$ endowed with the canonical coordinates $(z^1, \dots, z^n, w_1, \dots, w_n)$, standard complex symplectic form $\Omega = \sum_{i=1}^n dz^i \wedge dw_i$, standard real structure $\tau : \mathbb{C}^{2n} \to \mathbb{C}^{2n}$ and Hermitian form $\gamma = \frac{\sqrt{-1}}{2}\Omega(\cdot, \tau\cdot)$. A holomorphic immersion $\phi : M \to \mathbb{C}^{2n}$ is called *Lagrangian* (respectively, *Kählerian*) if $\phi^*\Omega = 0$ (respectively, if $\phi^*\gamma$ is non-degenerate). ϕ is called *totally complex* if $d\phi(T_pM) \cap \tau\, d\phi(T_pM) = 0$ for all $p \in M$.

Remark 3.1.4. Our conventions differ slightly from [ACD02, CDM17] in that we have set $\omega = g(J\cdot, \cdot)$ in contrast to $\omega = g(\cdot, J\cdot)$. One consequence of this is that a Hermitian form γ (which, in both conventions, is $\mathbb{C}$-linear in its first argument) and a Kähler structure (g, ω) are related via $\gamma = g - i\omega$. Also, our Hermitian structure on $\mathbb{C}^{2n}$ differs from the Hermitian structure of [ACD02] by a factor of $\frac{1}{2}$.

Proposition 3.1.5 ([ACD02])**.** *Let $\phi : M \to \mathbb{C}^{2n}$ be a holomorphic immersion.*

(1) ϕ is totally complex if and only if its real part $\operatorname{Re}\phi : M \to \mathbb{R}^{2n}$ is an immersion.

(2) If ϕ is Lagrangian, then ϕ is Kählerian if and only if it is totally complex.

A Kählerian Lagrangian immersion $\phi : M \to \mathbb{C}^{2n}$ induces on M the structure of an affine special Kähler manifold. Locally, an affine special Kähler manifold can always be realized as a Kählerian Lagrangian immersion. This is reflected in the following proposition.

Proposition 3.1.6 ([ACD02])**.** *Let (M, J, g, ∇) be a simply connected affine special Kähler manifold of complex dimension n. Then there exists a Kählerian Lagrangian immersion $\phi : M \to \mathbb{C}^{2n}$ inducing the affine special Kähler structure (J, g, ∇) on M. Moreover, ϕ is unique up to a transformation of $\mathbb{C}^{2n}$ by an element in $\mathrm{Aff}_{\mathrm{Sp}(\mathbb{R}^{2n})}(\mathbb{C}^{2n})$.*

More precisely, the action of the group $\mathrm{Aff}_{\mathrm{Sp}(\mathbb{R}^{2n})}(\mathbb{C}^{2n})$ on the set of Kählerian Lagrangian immersions $\phi : M \to \mathbb{C}^{2n}$ is simply transitive, as can be proven along the lines of the proof of simple transitivity in Proposition 3.2.10.

Definition 3.1.7. Let $\phi : M \to \mathbb{C}^{2n}$ be a Kählerian Lagrangian immersion of an affine special Kähler manifold M. Denote by $\lambda = w^t\, dz = \sum_{i=1}^n w_i\, dz^i$ the Liouville form of $\mathbb{C}^{2n}$. A function $F : M \to \mathbb{C}$ is called a *prepotential* of ϕ if $dF = \phi^*\lambda$.

Remark 3.1.8. (1) The function $K := \gamma(\phi, \phi)$ is a Kähler potential of the Kähler form ω, i.e., $\omega = \frac{i}{2}\partial\bar{\partial}K$.

(2) Let M be a local affine special Kähler manifold given as a Kählerian Lagrangian immersion $\phi : M \to \mathbb{C}^{2n}$. Then the pullback of the canonical coordinates of $T^*\mathbb{C}^n = \mathbb{C}^{2n}$ gives functions $z^1, \dots, z^n, w_1, \dots, w_n : M \to \mathbb{C}$ such that $\phi = (z, w) := (z^1, \dots, z^n, w_1, \dots, w_n)$. It can always be achieved that $z, w : M \to \mathbb{C}^n$ are holomorphic coordinate systems by replacing ϕ with $x \circ \phi$ for some $x \in \mathrm{Sp}(\mathbb{R}^{2n})$ and restricting M if necessary, c.f. [ACD02, Section 1.2]. In this case, we call (z, w) a *conjugate pair of special holomorphic coordinates.*

(3) Let $\phi = (z, w) : M \to \mathbb{C}^{2n}$ be a Kählerian Lagrangian immersion of an affine special Kähler manifold given by a conjugate pair of special holomorphic coordinates (z, w) and let $F : M \to \mathbb{C}$ be a prepotential of ϕ. Then we can identify $M \cong z(M) \subset \mathbb{C}^n$ and ϕ with $dF : M \to T^*M = \mathbb{C}^{2n}$. In particular, $\phi(M) = \left\{(z, w) \in \mathbb{C}^{2n} \mid w_i = \frac{\partial F}{\partial z^i}\right\}$ is the graph of dF over M. In this case, $M \subset \mathbb{C}^n$ is called an *affine special Kähler domain* and $K(p) = \sum_{i=1}^n \mathrm{Im}(\bar{z}^i F_i)$ where $F_i := \frac{\partial F}{\partial z^i}$.

Definition 3.1.9. A *conical affine special Kähler manifold* $(\hat{M}, \hat{J}, \hat{g}, \hat{\nabla}, \xi)$ is an affine special Kähler manifold $(\hat{M}, \hat{J}, \hat{g}, \hat{\nabla})$ and a vector field ξ such that $\hat{g}(\xi, \xi) \neq 0$ and $\hat{\nabla}\xi = \hat{D}\xi = \mathrm{Id}$, where $\hat{D}$ is the Levi-Civita connection of $\hat{g}$.

Note that contrary to [CHM12, Definition 3] here we are not making any assumptions on the signature of the metric $\hat{g}$.

A conical affine special Kähler manifold $\hat{M}$ of complex dimension $n+1$ locally admits Kählerian Lagrangian immersions $\Phi : U \to \mathbb{C}^{2n+2}$ that are equivariant with respect to the local $\mathbb{C}^*$-action defined by $Z = \xi - iJ\xi$ and scalar multiplication on $\mathbb{C}^{2n}$ [ACD02]. As a consequence, the function $\hat{K} := \frac{1}{2}\hat{g}(Z, \overline{Z}) = \hat{g}(\xi, \xi)$ is a globally defined Kähler potential of $\hat{M}$. Indeed, if $p \in U$, then $\hat{K}(p) = \hat{g}_p(\xi, \xi) = \hat{\gamma}(\Phi(p), \Phi(p))$, where $\hat{\gamma} = \frac{i}{2}\hat{\Omega}(\cdot, \bar{\cdot})$ for the standard symplectic form $\hat{\Omega}$ of $\mathbb{C}^{2n+2}$.

If the vector field Z generates a principal $\mathbb{C}^*$-action then the symmetric tensor field

$$g' := -\frac{\hat{g}}{\hat{K}} + \frac{(\partial\hat{K})(\bar{\partial}\hat{K})}{\hat{K}^2} \tag{3.1.3}$$

induces a Kähler metric $\bar{g}$ on the quotient manifold $\overline{M} := \hat{M}/\mathbb{C}^*$, compare [CDS16, Proposition 2]. It follows that $\pi^*\bar{g} = g'$ and $\pi^*\bar{\omega} = -\frac{i}{2}\partial\bar{\partial}\log|\hat{K}|$, where $\bar{\omega} = \bar{g}(J\cdot, \cdot)$ is the Kähler form of $\overline{M}$ and $\pi : \hat{M} \to \overline{M}$ is the canonical projection. Set $\mathcal{D} := \operatorname{span}\{\xi, J\xi\}$. Note that if $\hat{K} > 0$, then the signature of $\bar{g}$ is minus the signature of $\hat{g}|_{\mathcal{D}^\perp}$, whereas if $\hat{K} < 0$, then the signature of $\bar{g}$ agrees with the signature of $\hat{g}|_{\mathcal{D}^\perp}$.

Definition 3.1.10. The quotient $(\overline{M}, \bar{g})$ is called a *projective special Kähler manifold.*

Remark 3.1.11. Let $\Phi = (Z, W) : M \to \mathbb{C}^{2n+2}$ be an equivariant Kählerian Langrangian immersion such that (Z, W) is a conjugate pair of special holomorphic coordinates. Identify $M \cong Z(M) \subset \mathbb{C}^{n+1}$. Then the prepotential $F : M \to \mathbb{C}$ can be chosen to be homogeneous of degree 2 such that $\Phi = dF$.

3.2 Symplectic group actions

3.2.1 Linear representation of the central extension of the affine symplectic group

Let $G = \mathrm{Sp}(\mathbb{R}^{2n}) \ltimes \mathrm{Heis}_{2n+1}(\mathbb{R})$ be the extension of the real Heisenberg group by the group of automorphisms $\mathrm{Sp}(\mathbb{R}^{2n})$. The complexification of G is the group $G_\mathbb{C} = \mathrm{Sp}(\mathbb{C}^{2n}) \ltimes \mathrm{Heis}_{2n+1}(\mathbb{C})$ which contains G as a real subgroup. Given two elements $x = (X, s, v)$ and $x' = (X', s', v') \in G_\mathbb{C}$, where $X, X' \in \mathrm{Sp}(\mathbb{C}^{2n})$, $s, s' \in \mathbb{C} = Z(G)$, and $v, v' \in \mathbb{C}^{2n}$, their product in $G_\mathbb{C}$ is given by

$$x \cdot x' = \left(XX', s + s' + \frac{1}{2}\Omega(v, Xv'), Xv' + v\right), \tag{3.2.1}$$

where Ω is the symplectic form on $\mathbb{C}^{2n}$.

The group $G_{\mathbb{C}}$ is a central extension of the group $\mathrm{Aff}_{\mathrm{Sp}(\mathbb{C}^{2n})}(\mathbb{C}^{2n})$ of affine transformations of $\mathbb{C}^{2n}$ with linear part in $\mathrm{Sp}(\mathbb{C}^{2n})$. The two groups are related by the quotient homomorphism

$$G_{\mathbb{C}} \to \mathrm{Aff}_{\mathrm{Sp}(\mathbb{C}^{2n})}(\mathbb{C}^{2n}) = G_{\mathbb{C}}/Z(G_{\mathbb{C}}), \quad (X, s, v) \mapsto (X, v). \tag{3.2.2}$$

This induces an affine representation $\bar{\rho}$ of $G_{\mathbb{C}}$ on $\mathbb{C}^{2n}$ with image $\mathrm{Aff}_{\mathrm{Sp}(\mathbb{C}^{2n})}(\mathbb{C}^{2n})$ whose restriction to the real group G has the image $\bar{\rho}(G) = \mathrm{Aff}_{\mathrm{Sp}(\mathbb{R}^{2n})}(\mathbb{R}^{2n})$. In the complex symplectic vector space $\mathbb{C}^{2n}$ we use standard coordinates $(z^1, \dots, z^n, w_1, \dots, w_n)$ in which the complex symplectic form is $\Omega = \sum dz^i \wedge dw_i$.

We will now show that $\bar{\rho}$ can be extended to a linear symplectic representation

$$\rho : G_{\mathbb{C}} \to \mathrm{Sp}(\mathbb{C}^{2n+2}) \tag{3.2.3}$$

in the sense that the group $\rho(G_{\mathbb{C}})$ preserves the affine hyperplane $\{z^0 = 1\} \subset \mathbb{C}^{2n+2}$ with respect to standard coordinates $(z^0, w_0, z^1 \dots, z^n, w_1, \dots w_n)$ on $\mathbb{C}^{2n+2} = \mathbb{C}^2 \oplus \mathbb{C}^{2n}$ and the distribution spanned by ∂_{w_0} inducing on the symplectic affine space $\{z^0 = 1\}/\langle \partial_{w_0} \rangle \cong \mathbb{C}^{2n}$ the symplectic affine representation $\bar{\rho}$.

Remark 3.2.1. Notice that $\{z^0 = 1\}/\langle \partial_{w_0} \rangle$ is precisely the symplectic reduction of $\mathbb{C}^{2n+2}$ with respect to the holomorphic Hamiltonian group action generated by the vector field ∂_{w_0}. The group $\rho(G_{\mathbb{C}}) \subset \mathrm{Sp}(\mathbb{C}^{2n+2})$ preserves the Hamiltonian z^0 of that action and, hence, ρ induces a symplectic affine representation on the reduced space. Similarly, we will consider the initial real symplectic affine space $\mathbb{R}^{2n}$ as the symplectic reduction of the real symplectic vector space $\mathbb{R}^{2n+2}$ in the context of the real group G.

Proposition 3.2.2. *(i) The map*

$$x = (X, s, v) \mapsto \rho(x) = \begin{pmatrix} 1 & 0 & 0 \\ -2s & 1 & \hat{v}^t \\ v & 0 & X \end{pmatrix}, \quad \hat{v} := X^t \Omega_0 v = \Omega_0 X^{-1} v, \tag{3.2.4}$$

where $\Omega_0 = \begin{pmatrix} 0 & \mathrm{id} \\ -\mathrm{id} & 0 \end{pmatrix}$ *is the matrix representing the symplectic form on* $\mathbb{C}^{2n}$*, defines a faithful linear symplectic representation* $\rho : G_{\mathbb{C}} \to \mathrm{Sp}(\mathbb{C}^{2n+2})$*, which induces the affine symplectic representation* $\bar{\rho} : G_{\mathbb{C}} \to \mathrm{Aff}_{\mathrm{Sp}(\mathbb{C}^{2n})}(\mathbb{C}^{2n})$ *in the sense explained above.*

(ii) The image $\rho(G_{\mathbb{C}}) \subset \mathrm{Sp}(\mathbb{C}^{2n+2})$ *consists of the transformations in* $\mathrm{Sp}(\mathbb{C}^{2n+2})$ *which preserve the hyperplane* $\{z^0 = 1\} \subset \mathbb{C}^{2n+2}$ *and the complex rank one distribution* $\langle \partial_{w_0} \rangle$*. The image* $\rho(G) \subset \mathrm{Sp}(\mathbb{R}^{2n+2}) \subset \mathrm{Sp}(\mathbb{C}^{2n+2})$ *is the group that additionally preserves the real structure of* $\mathbb{C}^{2n+2}$*.*

Proof. We first observe that, for $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, an element of $\mathrm{GL}(2n+2, \mathbb{K})$ preserves $\{z^0 = 1\}$ and $\langle \partial_{w_0} \rangle$ if and only if it is of the form

$$\begin{pmatrix} 1 & 0 & 0 \\ -2s & c & w^t \\ v & 0 & X \end{pmatrix}, \tag{3.2.5}$$

where $s \in \mathbb{K}$, $0 \neq c \in \mathbb{K}$, $v, w \in \mathbb{K}^{2n}$, and $X \in \mathrm{GL}(2n, \mathbb{K})$. One then checks that such a transformation is symplectic if and only if $X \in \mathrm{Sp}(\mathbb{K}^{2n})$, $c = 1$, and $w = \hat{v}$. Clearly an element in $\mathrm{GL}(2n, \mathbb{K})$ preserves the real structure of $\mathbb{C}^{2n}$ if and only if $\mathbb{K} = \mathbb{R}$. This proves (ii) and shows that the linear transformation $\rho(x)$ induces the affine transformation $\bar{\rho}(x) \in \mathrm{Aff}_{\mathrm{Sp}(\mathbb{C}^{2n})}(\mathbb{C}^{2n})$ for all $x \in G_{\mathbb{C}}$.

To check that ρ is a representation we put $\mu(x) := -2s$, $\gamma(x) := \hat{v} = X^t \Omega_0 v$. Then we compute

$$\mu(xx') = \mu(x) + \mu(x') - \omega(v, Xv') = \mu(x) + \mu(x') + \hat{v}^t v', \tag{3.2.6}$$

which coincides with the matrix element of $\rho(x)\rho(x')$ in the second row and first column. Next we compute the column vector

$$\gamma(xx') = (XX')^t \Omega_0 (v + Xv') = (X')^t (\gamma(x) + \Omega_0 v') = (X')^t \gamma(x) + \gamma(x'), \tag{3.2.7}$$

the entries of which coincide with the last $2n$ entries of the second row of $\rho(x)\rho(x')$. From these properties one sees immediately that ρ is a representation. It is obviously faithful, since X, s, and v appear in the matrix $\rho(x)$. □

We define the subgroup $G_{\mathrm{SK}} = \mathrm{Sp}(\mathbb{R}^{2n}) \ltimes \mathrm{Heis}_{2n+1}(\mathbb{C}) \subset G_{\mathbb{C}}$ to be the extension of the complex Heisenberg group by $\mathrm{Sp}(\mathbb{R}^{2n})$. It contains the real group G as a subgroup and is a central extension of the affine group $\bar{\rho}(G_{\mathrm{SK}}) = \mathrm{Aff}_{\mathrm{Sp}(\mathbb{R}^{2n})}(\mathbb{C}^{2n})$. We will show that G_{SK} acts on pairs (ϕ, F) of Kählerian Lagrangian immersions and prepotentials. This gives a transformation formula, see Eq. (3.2.16), of prepotentials of affine special Kähler manifolds which generalizes de Wit's formula (9) in [dW96a] from the case of linear to affine symplectic transformations.

3.2.2 Representation of $G_{\mathbb{C}}$ on Lagrangian pairs

Let $\mathcal{L} \subset \mathbb{C}^{2n}$ be a Lagrangian submanifold and denote by η be the canonical $\mathrm{Sp}(\mathbb{R}^{2n})$-invariant 1-form given by $\eta_q := \Omega(q, \cdot)$, for $q \in \mathbb{C}^{2n}$. In Darboux coordinates $(z^1, \ldots, z^n, w_1, \ldots, w_n)$ we can write η as $\eta = \sum z^i \, dw_i - w_i \, dz^i$. Since $d\eta = 2\Omega$, this form is closed when restricted to $\mathcal{L}$.

Definition 3.2.3. We call a function $f : \mathcal{L} \to \mathbb{C}$ a *Lagrangian potential* of $\mathcal{L}$ if $df = -\eta|_{\mathcal{L}}$ and a pair $(\mathcal{L}, f)$ a *Lagrangian pair* if $\mathcal{L} \subset \mathbb{C}^{2n}$ is a Lagrangian submanifold and f is a Lagrangian potential of $\mathcal{L}$.

Proposition 3.2.4. *The group $G_{\mathbb{C}}$ acts on the set of pairs $(\mathcal{L}, f)$, where $\mathcal{L} \subset \mathbb{C}^{2n}$ is a Lagrangian submanifold and f is a holomorphic function on $\mathcal{L}$. The action is defined as follows. Given $x = (X, s, v) \in G_{\mathbb{C}}$ and a pair $(\mathcal{L}, f)$ as above, we define*

$$x \cdot (\mathcal{L}, f) := (x\mathcal{L}, x \cdot f), \tag{3.2.8}$$

where $x\mathcal{L} := \bar{\rho}(x)\mathcal{L}$ and $x \cdot f$ is function on $x\mathcal{L}$ defined as

$$x \cdot f := f \circ x^{-1} + \Omega(\cdot, v) - 2s. \tag{3.2.9}$$

Moreover, if f is a Lagrangian potential of $\mathcal{L}$, then $x \cdot f$ is a Lagrangian potential of $x\mathcal{L}$.

Proof. For the neutral element $e \in G_{\mathbb{C}}$, clearly $e \cdot (\mathcal{L}, f) = (\mathcal{L}, f)$. Let $q \in \mathcal{L}$ and $x, x' \in G_{\mathbb{C}}$ with $x = (X, s, v)$ and $x' = (X', s', v')$. Then

$$\begin{aligned} x \cdot (x' \cdot f)(xx'q) &= (x' \cdot f)(x'q) + \Omega(xx'q, v) - 2s \\ &= f(q) + \Omega(x'q, v') + \Omega(xx'q, v) - 2s - 2s' \\ &= f(q) + \Omega(xx'q, v + Xv') - 2\left(s + s' + \frac{1}{2}\Omega(v, Xv')\right) \\ &= (xx') \cdot f(xx'q), \end{aligned} \tag{3.2.10}$$

where we have used the second-to-last equation that

$$\begin{aligned} \Omega(x'q, v') &= \Omega(Xx'q, Xv') \\ &= \Omega(xx'q - v, Xv') \\ &= \Omega(xx'q, Xv') - \Omega(v, Xv'). \end{aligned} \tag{3.2.11}$$

This shows that Eq. (3.2.8) defines an action of $G_{\mathbb{C}}$. Now let f be a Lagrangian potential of $\mathcal{L}$ and set $\tilde{q} = xq$. Then

$$\begin{aligned} d_{\tilde{q}}(x \cdot f) &= d_q f \circ d(x^{-1}) + d_{\tilde{q}}(\Omega(\cdot, v)) \\ &= -\eta_q \circ X^{-1} + \Omega(\cdot, v) \\ &= -\Omega(q, X^{-1}\cdot) + \Omega(\cdot, v) \\ &= -\Omega(Xq + v, \cdot) = -\eta_{\tilde{q}}, \end{aligned} \tag{3.2.12}$$

hence, $x \cdot f$ is a Lagrangian potential of $x \cdot \mathcal{L}$. □

Definition 3.2.5. We call a Lagrangian submanifold $\mathcal{L} \subset \mathbb{C}^{2n}$ *Kählerian* if the Hermitian form $\gamma = \sqrt{-1}\Omega(\cdot, \tau\cdot)$ is non-degenerate when restricted to $\mathcal{L}$. Similarly, a Lagrangian pair $(\mathcal{L}, f)$ is called *Kählerian* if $\mathcal{L}$ is Kählerian.

Lemma 3.2.6. *A Lagrangian submanifold $\mathcal{L} \subset \mathbb{C}^{2n}$ is Kählerian if and only if $\mathcal{L}$ is totally complex, i.e., $T_q\mathcal{L} \cap \tau T_q\mathcal{L} = \{0\}$ for all $q \in \mathcal{L}$.*

Proof. Since the inclusion $\iota : \mathcal{L} \to \mathbb{C}^{2n}$ is a holomorphic Lagrangian immersion, the statement follows from Prop. 3.1.5. □

Corollary 3.2.7. *The group $G_{\mathrm{SK}} \subset G_{\mathbb{C}}$ acts on the set of Kählerian Lagrangian pairs.*

Proof. The group G_{SK} acts on $\mathbb{C}^{2n}$ as the group $\bar{\rho}(G_{\mathrm{SK}}) = \mathrm{Aff}_{\mathrm{Sp}(\mathbb{R}^{2n})}(\mathbb{C}^{2n})$ which is the affine linear group that leaves invariant the complex symplectic form Ω and the real structure τ and, hence, also the Hermitian form $\gamma = \sqrt{-1}\Omega(\cdot, \tau\cdot)$. This shows that if $(\mathcal{L}, f)$ is a Kählerian Lagrangian pair, then $x \cdot (\mathcal{L}, F) = (\bar{\rho}(x)\mathcal{L}, x \cdot f)$ is again a Kählerian Lagrangian pair for all $x \in G_{\mathrm{SK}}$. □

3.2.3 Representation of G_{SK} on special Kähler pairs

Definition 3.2.8. Let (M, J, g, ∇) be a connected affine special Kähler manifold of complex dimension n and let $U \subset M$ be an open subset of M. We call a pair (ϕ, F) a *special Kähler pair* of U if $\phi : U \to \mathbb{C}^{2n}$ is a Kählerian Lagrangian immersion inducing on U the restriction of the special Kähler structure (J, g, ∇) and F is a prepotential of ϕ. We denote the set of special Kähler pairs of U by $\mathcal{F}(U)$.

The following Lemma shows how the notions of prepotentials and Lagrangian potentials are related.

Lemma 3.2.9. *Let M be a special Kähler manifold together with a Kählerian Lagrangian embedding $\phi : M \to \phi(M) \subset \mathbb{C}^{2n}$ inducing the special Kähler structure of M. Set $\mathcal{L} := \phi(M)$ and $(z, w) := \phi$. Then a Lagrangian potential f of $\mathcal{L}$ defines a prepotential F of ϕ via*

$$F = \frac{1}{2}(\phi^* f + z^t w), \tag{3.2.13}$$

and vice versa.

Proof. Let f be a Lagrangian potential of $\mathcal{L}$. We compute

$$\begin{aligned} dF &= \frac{1}{2}(\phi^* df + d(z^t w)) \\ &= \frac{1}{2}(-\phi^*\eta + w^t\, dz + z^t\, dw) \\ &= \frac{1}{2}(w^t\, dz - z^t\, dw + w^t\, dz + z^t\, dw) \\ &= w^t\, dz. \end{aligned} \tag{3.2.14}$$

Since ϕ is a biholomorphism onto its image, the converse follows easily. □

Proposition 3.2.10. *Let M be a connected affine special Kähler manifold of complex dimension n and $U \subset M$ an open subset such that $\mathcal{F}(U) \neq \emptyset$. Then the group G_{SK} acts simply transitively on $\mathcal{F}(U)$. The action is defined as follows. Given $x = (X, s, v) \in G_{\mathrm{SK}}$ and a special Kähler pair (ϕ, F) of U, we define*

$$x \cdot (\phi, F) := (x\phi, x \cdot F), \tag{3.2.15}$$

where $x\phi := \bar{\rho}(x) \circ \phi$ and

$$x \cdot F := F - \frac{1}{2} z^t w + \frac{1}{2} z'^t w' + \frac{1}{2}(x\phi)^* \Omega\,(\cdot, v) - s, \tag{3.2.16}$$

where $(z, w) := \phi$ and $(z', w') := x\phi$ are the components of ϕ and $x\phi$, respectively.

Proof. We begin by showing that Eq. (3.2.15) defines a G_{SK}-action on $\mathcal{F}(U)$. Clearly, the neutral element of G_{SK} acts trivially. We can locally rewrite Eq. (3.2.16) as

$$\begin{aligned} 2x \cdot F - z'^t w' &= 2F - z^t w + (x\phi)^* \Omega(\cdot, v) - 2s \\ &= (x\phi)^*(f \circ x^{-1} + \Omega(\cdot, v) - 2s) \\ &= (x\phi)^*(x \cdot f) \end{aligned} \tag{3.2.17}$$

where f is the Lagrangian potential locally corresponding to F according to Lemma 3.2.9, i.e., $\phi^* f = 2F - z^t w$. This shows that $x \cdot F$ is a prepotential, namely the prepotential locally corresponding to the Lagrangian potential $x \cdot f$ via $x\phi$. The remaining group action axioms now follow easily from Proposition 3.2.4.

It remains to show that the action is simply transitive. Let (ϕ, F), (ϕ', F') be two special Kähler pairs of U. Since ϕ and ϕ' are both Kählerian Lagrangian immersions inducing same special Kähler structure, we know from Prop. 3.1.6 that there is an element $(X, v) \in \mathrm{Aff}_{\mathrm{Sp}(\mathbb{R}^{2n})}(\mathbb{C}^{2n})$ such that $\phi' = (X, v) \circ \phi$. Since prepotentials are unique up to a constant, there is an $s \in \mathbb{C}$ such that $x \cdot F = F'$ for $x = (X, s, v) \in G_{\mathrm{SK}}$. This shows

$x \cdot (\phi, F) = (\phi', F')$ and, hence, the transitivity. To see that the action is free, assume that $x \cdot (\phi, F) = (\phi, F)$ for some $x = (X, s, v) \in G_{\mathrm{SK}}$. Then $X \circ \phi + v = \phi$. Differentiating and taking the real part gives $(X - \mathrm{id}_{2n}) \circ \mathrm{Re}\, d\phi = 0$. Since ϕ is Kählerian, $\mathrm{Re}\,\phi$ is an immersion and this implies $X = \mathrm{id}_{2n}$. But then from $X \circ \phi + v = \phi$ it also follows that $v = 0$. Finally, $x \cdot (\phi, F) = (\phi, F - s)$ implies $s = 0$ and, hence, x is the identity of G_{SK}. □

Corollary 3.2.11. *Under the assumptions of Prop. 3.2.10, the subgroup* $\mathrm{Sp}(\mathbb{R}^{2n}) \subset G_{\mathrm{SK}}$ *acts by*

$$x \cdot (\phi, F) = \left(\phi' = x\phi, F' = x \cdot F = F - \frac{1}{2} z^t w + \frac{1}{2} z'^t w'\right) \tag{3.2.18}$$

on the set of special Kähler pairs (ϕ, F)*. In particular, in the case of conical affine special Kähler manifolds,* $\mathrm{Sp}(\mathbb{R}^{2n})$ *acts on the set of homogeneous prepotentials of degree 2.*

Remark 3.2.12. By Corollary 3.2.11, the function $F - \frac{1}{2} z^t w$ is invariant under the above action of $\mathrm{Sp}(\mathbb{R}^{2n})$ in the sense that

$$F' - \frac{1}{2} z'^t w' = F - \frac{1}{2} z^t w. \tag{3.2.19}$$

This is precisely the statement of de Wit, see eq. (10) in [dW96a], that $F - \frac{1}{2} z^t w$ transforms as a symplectic function under linear symplectic transformations.

In terms of the Lagrangian potentials f and f' corresponding to F and F', eq. (3.2.19) is equivalent to

$$f \circ \phi = f' \circ \phi'. \tag{3.2.20}$$

3.3 Conification of Lagrangian submanifolds

The aim is to associate (under some assumptions) a Lagrangian cone $\hat{\mathcal{L}} \subset \mathbb{C}^{2n+2}$ with a Lagrangian submanifold $\mathcal{L} \subset \mathbb{C}^{2n}$, and vice versa.

Fix a linear symplectic splitting $\mathbb{C}^{2n+2} = \mathbb{C}^2 \times \mathbb{C}^{2n}$ of the symplectic vector space $\mathbb{C}^{2n+2}$ with its standard symplectic form $\hat{\Omega}$ and linear Darboux coordinates z^0, w_0 in $\mathbb{C}^2$ such that the symplectic form on $\mathbb{C}^2$ is given by $dz^0 \wedge dw_0$. Then the symplectic vector space $\mathbb{C}^{2n}$ with its standard symplectic form Ω is recovered as the symplectic reduction with respect to the Hamiltonian flow of the function z^0 as explained in Rem. 3.2.1. Let $\pi : \{z^0 = 1\} \to \{z^0 = 1\}/\langle \partial_{w_0} \rangle = \mathbb{C}^{2n}$ be the quotient map and $\iota : \{z^0 = 1\} \hookrightarrow \mathbb{C}^{2n+2}$ the inclusion.

In one direction, let $\mathcal{L}$ be a Lagrangian submanifold of $\mathbb{C}^{2n}$. A submanifold $\hat{\mathcal{L}}_1 \subset \{z^0 = 1\} \subset \mathbb{C}^{2n+2}$ is called a *lift* of $\mathcal{L}$ if the projection

$$\pi|_{\hat{\mathcal{L}}_1} : \hat{\mathcal{L}}_1 \to \mathcal{L} \tag{3.3.1}$$

is a diffeomorphism. Equivalently, a lift is a section over $\mathcal{L}$ of the trivial $\mathbb{C}$-bundle $\pi : \{z^0 = 1\} \to \mathbb{C}^{2n}$. Hence, a lift $\hat{\mathcal{L}}_1$ is of the form $\hat{\mathcal{L}}_1 = \{(1, f(q), q) \mid q \in \mathcal{L}\}$ for a function $f : \mathcal{L} \to \mathbb{C}$.

Proposition 3.3.1. *Let $\hat{\mathcal{L}}_1$ be a lift of a Lagrangian submanifold $\mathcal{L} \subset \mathbb{C}^{2n}$ with respect to the function $f : \mathcal{L} \to \mathbb{C}$. Then the cone $\hat{\mathcal{L}} := \mathbb{C}^* \cdot \hat{\mathcal{L}}_1$ is Lagrangian if and only if f is a Lagrangian potential.*

Proof. By the above $\hat{\mathcal{L}}_1 = \{(1, f(q), q) \mid q \in \mathcal{L}\}$. To show that $\hat{\mathcal{L}} := \mathbb{C}^* \cdot \hat{\mathcal{L}}_1$ is Lagrangian it is sufficient to show that $\hat{\Omega}(p, \hat{X}_p) = 0$ for all $p \in \hat{\mathcal{L}}_1$ and $\hat{X}_p \in T_p\hat{\mathcal{L}}_1$. A tangent vector $\hat{X}_p \in T_p\hat{\mathcal{L}}_1$ is of the form $\hat{X}_p = df(X)\partial_{w_0} + X$ for $X \in T_q\mathcal{L}$ with $q = \pi(p)$. Then

$$\begin{aligned}\hat{\Omega}(p, \hat{X}_p) &= \hat{\Omega}(\partial_{z^0} + f(q)\partial_{w_0} + q, \hat{X}_p) \\ &= dz^0 \wedge dw_0\,(\partial_{z^0} + f(q)\partial_{w_0}, df(X)\partial_{w_0}) + \Omega(q, X) \\ &= df(X) + \eta_q(X).\end{aligned} \tag{3.3.2}$$

Hence, $\hat{\mathcal{L}}$ is Lagrangian if and only if $df = -\eta|_{\mathcal{L}}$. □

Definition 3.3.2. Let $\hat{\mathcal{L}}_1$ be the lift of the Lagrangian pair $(\mathcal{L}, f)$. We call the Lagrangian cone $\mathrm{con}(\mathcal{L}, f) := \mathbb{C}^* \cdot \hat{\mathcal{L}}_1$ the *conification* of $(\mathcal{L}, f)$.

Conversely, let $\hat{\mathcal{L}} \subset \mathbb{C}^{2n+2}$ be a Lagrangian cone such that the submanifold $\hat{\mathcal{L}}_1 := \hat{\mathcal{L}} \cap \{z^0 = 1\}$ is transverse to the Hamiltonian vector field ∂_{w_0} and each integral curve intersects $\hat{\mathcal{L}}_1$ at most once. We will call Lagrangian cones with this property *regular*. Then we define $\mathcal{L} \subset \mathbb{C}^{2n}$ as the image of $\hat{\mathcal{L}}_1$ under the quotient map $\pi : \{z^0 = 1\} \to \{z^0 = 1\}/\langle\partial_{w_0}\rangle = \mathbb{C}^{2n}$. Since the pullback $\pi^*\Omega$ of the symplectic form Ω on $\mathbb{C}^{2n}$ is given by $\pi^*\Omega = \iota^*\hat{\Omega}$, it follows that $\mathcal{L}$ is Lagrangian. By the regularity, the function $f := w_0 \circ (\pi|_{\hat{\mathcal{L}}_1})^{-1}$ is a well-defined function on $\mathcal{L}$ and $\hat{\mathcal{L}}_1$ is of the form $\hat{\mathcal{L}}_1 = \{(1, f(q), q) \mid q \in \mathcal{L}\}$. In particular, $\hat{\mathcal{L}}_1$ is the lift of $\mathcal{L}$ with respect to the function f.

Definition 3.3.3. We call the pair $\mathrm{red}(\hat{\mathcal{L}}) := (\mathcal{L}, f)$ the *reduction* of the regular Lagrangian cone $\hat{\mathcal{L}} \subset \mathbb{C}^{2n+2}$.

Proposition 3.3.4. *With respect to a splitting $\mathbb{C}^{2n+2} = \mathbb{C}^2 \times \mathbb{C}^{2n}$ and linear Darboux coordinates z^0, w_0 of $\mathbb{C}^2$, we obtain a one-to-one correspondence*

$$\begin{array}{c}\{\textit{Regular Lagrangian cones in } \mathbb{C}^{2n+2}\} \\ \Big\updownarrow \textit{1:1} \\ \{\textit{Lagrangian pairs } (\mathcal{L}, f) \textit{ in } \mathbb{C}^{2n}\},\end{array}$$

given by conification and reduction.

Moreover, conification and reduction are equivariant with respect to the action of the group $G_{\mathbb{C}}$, i.e., $\operatorname{con}(x \cdot (\mathcal{L}, f)) = \rho(x) \operatorname{con}(\mathcal{L}, f)$ *and* $\operatorname{red}(\rho(x)\hat{\mathcal{L}}) = x \cdot \operatorname{red}(\hat{\mathcal{L}})$ *for* $x \in G_{\mathbb{C}}$.

Proof. Let $\hat{\mathcal{L}} \subset \mathbb{C}^{2n+2}$ be a regular Lagrangian cone. We have already seen that $\hat{\mathcal{L}}_1 = \hat{\mathcal{L}} \cap \{z^0 = 1\}$ is the same as the lift of the pair $(\mathcal{L}, f) := \operatorname{red}(\hat{\mathcal{L}})$. Since the cone $\hat{\mathcal{L}} = \mathbb{C}^* \cdot \hat{\mathcal{L}}_1$ is Lagrangian, it follows from Prop. 3.3.1 that f is a Lagrangian potential and, hence, $\operatorname{con}(\operatorname{red}(\hat{\mathcal{L}})) = \hat{\mathcal{L}}$. Conversely, if $(\mathcal{L}, f)$ is a Lagrangian pair and $\hat{\mathcal{L}}_1 \subset \{z^0 = 1\}$ is the lift of $\mathcal{L}$ with respect to f, then $\operatorname{con}(\mathcal{L}, f) = \mathbb{C}^* \cdot \hat{\mathcal{L}}_1$ is a regular Lagrangian cone by Prop. 3.3.1. Since $\operatorname{con}(\mathcal{L}, f) \cap \{z^0 = 1\} = \hat{\mathcal{L}}_1$, it follows that $\operatorname{red}(\operatorname{con}(\mathcal{L}, f)) = (\mathcal{L}, f)$. This shows $\operatorname{red} = \operatorname{con}^{-1}$.

Now let $x = (X, s, v) \in G_{\mathbb{C}}$ and $\hat{\mathcal{L}}_1$ be the lift of a Lagrangian pair $(\mathcal{L}, f)$. Then

$$\begin{aligned}
\rho(x)\hat{\mathcal{L}}_1 &= \rho(x)\{(1, f(q), q) \in \mathbb{C}^{2n+2} \mid q \in \mathcal{L}\} \\
&= \{(1, f(q) + \hat{v}^t q - 2s, xq) \in \mathbb{C}^{2n+2} \mid q \in \mathcal{L}\} \\
&= \{(1, f(q) + \Omega(xq, v) - 2s, xq) \in \mathbb{C}^{2n+2} \mid q \in \mathcal{L}\} \\
&= \{(1, f(x^{-1}q') + \Omega(q', v) - 2s, q') \in \mathbb{C}^{2n+2} \mid q' \in x\mathcal{L}\} \\
&= \{(1, x \cdot f(q'), q') \in \mathbb{C}^{2n+2} \mid q' \in x\mathcal{L}\}.
\end{aligned} \tag{3.3.3}$$

This shows that $\rho(x)\hat{\mathcal{L}}_1$ is the lift of the Lagrangian pair $x \cdot (\mathcal{L}, f) = (x\mathcal{L}, x \cdot f)$. Since the action of $G_{\mathbb{C}}$ on $\mathbb{C}^{2n+2}$ leaves level-sets of z^0 and the distribution spanned by ∂_{w_0} invariant, it follows that

$$\operatorname{con}(x \cdot (\mathcal{L}, f)) = \mathbb{C}^* \cdot (\rho(x)\hat{\mathcal{L}}_1) = \rho(x)(\mathbb{C}^* \cdot \hat{\mathcal{L}}_1) = \rho(x) \operatorname{con}(\mathcal{L}, \hat{w}_0). \tag{3.3.4}$$

The equivariance of red follows immediately from $\operatorname{red} = \operatorname{con}^{-1}$. □

Proposition 3.3.5. *Let $(\mathcal{L}, f)$ be a Lagrangian pair such that $\mathcal{L}$ is Kählerian. If there is a point $q \in \mathcal{L}$ such that q is real and $f(q) \notin \mathbb{R}$, then there is an open neighborhood $U \subset \mathcal{L}$ of q such that the Lagrangian cone $\hat{U} := \operatorname{con}(U, f) \subset \hat{\mathcal{L}} := \operatorname{con}(\mathcal{L}, f)$ is Kählerian.*

Proof. Let $q \in \mathcal{L}$ be real such that $f(q) \notin \mathbb{R}$ and choose an arbitrary $\hat{v} \in T_p\hat{\mathcal{L}} \cap \tau T_p\hat{\mathcal{L}}$ for $p = (1, f(q), q) \in \hat{\mathcal{L}}$. Since $T_p\hat{\mathcal{L}} = \operatorname{span}_{\mathbb{C}}(p) \oplus T_q\mathcal{L}$, we have $\hat{v} = \lambda(1, f(q), q) + (0, df(v), v)$ for $\lambda \in \mathbb{C}$ and $v \in T_q\mathcal{L}$. The condition $\hat{v} - \tau\hat{v} = 0$ gives three equations

$$0 = \lambda - \overline{\lambda}, \tag{3.3.5}$$

$$0 = \lambda f(q) - \overline{\lambda f(q)} + df(v) - \overline{df(v)}, \tag{3.3.6}$$

$$0 = \lambda q - \overline{\lambda q} + v - \overline{v}. \tag{3.3.7}$$

From the first, we immediately see that $\lambda \in \mathbb{R}$. From the third we find $v - \overline{v} = \lambda(\overline{q} - q) = 0$ since q is a real point. But $v - \overline{v} = 0$ is only possible if $v = 0$ as $\mathcal{L}$ is Kählerian. The second equation then implies $\lambda(f(q) - \overline{f(q)}) = 0$ which, as $f(q) \notin \mathbb{R}$, is only possible if $\lambda = 0$. Hence, $\hat{v} = 0$ and this shows $T_p\hat{\mathcal{L}} \cap \tau T_p\hat{\mathcal{L}} = 0$. Since $\hat{\mathcal{L}}$ is Lagrangian, this is equivalent to the Hermitian form $\hat{\gamma} = \hat{\Omega}(\cdot, \tau\cdot)$ being non-degenerate when restricted to $\hat{\mathcal{L}}$ at the point p. By continuity, it is then also non-degenerate on a neighborhood $\hat{U}_1 \subset \hat{\mathcal{L}}_1 = \hat{\mathcal{L}} \cap \{z^0 = 1\}$ of p. Non-degeneracy is invariant under multiplication by $z^0 \in \mathbb{C}^*$, which acts by homothety on the Hermitian form $\hat{\gamma}$. Therefore, $\hat{\gamma}|_{\hat{\mathcal{L}}}$ is non-degenerate on $\hat{U} := \mathbb{C}^* \cdot \hat{U}_1$ which is the conification of the Lagrangian pair (U, f) for $U = \pi(\hat{U}_1)$. □

Proposition 3.3.6. *If $(\mathcal{L}, f)$ is a Lagrangian pair and $\mathcal{L}$ is Kählerian, then there is an open subset $U \subset \mathcal{L}$ and an element $x \in G_{\mathrm{SK}}$ such that the cone* $\mathrm{con}(x \cdot (U, f))$ *is Kählerian.*

Proof. Let $(\mathcal{L}, f)$ be a Lagrangian pair such that $\mathcal{L}$ is Kählerian. If $\mathcal{L}$ does not have real points, set $\mathcal{L}' = \mathcal{L} - q$ for an arbitray $q \in \mathcal{L}$. Then $0 \in \mathcal{L}'$ is a real point and we can choose a Lagrangian potential f' such that $f'(0) \notin \mathbb{R}$. This determines an element $x \in G_{\mathrm{SK}}$ such that $(\mathcal{L}', f') = x \cdot (\mathcal{L}, f)$. The statement now follows from Prop. 3.3.5. □

3.4 Conification of affine special Kähler manifolds

3.4.1 Conification of special Kähler pairs

Since special Kähler pairs locally correspond to Lagrangian pairs we can use the results from the previous chapter to give a conification procedure for special Kähler pairs.

Proposition 3.4.1. *Let (ϕ, F) be a special Kähler pair of an affine special Kähler manifold M and denote by $(z, w) := \phi$ the components of ϕ as before. Set $\hat{M} := \mathbb{C}^* \times M = \{(z^0, p) \in \mathbb{C}^* \times M\}$ with $\mathbb{C}^*$-action defined by $\lambda \cdot (z^0, p) := (\lambda z^0, p)$. Then the map*

$$\begin{aligned} \Phi : \hat{M} &\to \mathbb{C}^{2n+2} \\ (z^0, p) &\mapsto z^0(1, (2F - z^t w)(p), \phi(p)) \end{aligned} \tag{3.4.1}$$

is a $\mathbb{C}^*$-equivariant Lagrangian immersion of $\hat{M}$.

Proof. Consider open subsets $\hat{U}$ of $\hat{M}$ of the form $\hat{U} = \mathbb{C}^* \times U$ where $U \subset M$ is open such that $\phi|_U$ is an embedding. Let $(\mathcal{L}, f)$ be the Lagrangian pair corresponding to $(\phi, F)|_U$ by Lemma 3.2.9. Then $\Phi(z^0, p) = z^0(1, f(\phi(p)), \phi(p))$ for all $(z^0, p) \in \hat{U}$, i.e.,

$\Phi(\hat{U}) = \mathrm{con}(\mathcal{L}, f)$. This shows that Φ is a Lagrangian immersion. The equivariance is obvious. □

Definition 3.4.2. Let (ϕ, F) be a special Kähler pair of an affine special Kähler manifold M. We call the complex manifold $\hat{M} = \mathbb{C}^* \times M$ together with the map $\Phi : \hat{M} \to \mathbb{C}^{2n+2}$ the *conification* of the special Kähler pair (ϕ, F) and we write $\Phi = \mathrm{con}(\phi, F)$. We say that the special Kähler pair (ϕ, F) is *non-degenerate* if the immersion Φ is Kählerian and $\hat{\gamma}(\Phi, \Phi) \neq 0$.

Proposition 3.4.3. *Let (ϕ, F) be a special Kähler pair of an affine special Kähler manifold M. Then conification is equivariant with respect to the action of G_{SK} in the sense that $\mathrm{con}(x \cdot (\phi, F)) = \rho(x) \circ \mathrm{con}(\phi, F)$ for $x \in G_{\mathrm{SK}}$.*

Proof. This follows since conification locally corresponds to the conification of Lagrangian pairs. □

Theorem 3.4.4. *Let (ϕ, F) be a non-degenerate special Kähler pair of an affine special Kähler manifold M. Then $\Phi = \mathrm{con}(\phi, F)$ induces on $\hat{M}$ the structure of a conical affine special Kähler manifold. This structure is independent of the representative of the equivalence class of (ϕ, F) in $\mathcal{F}(M)/G$.*

Proof. Let Φ be the conification of a non-degenerate special Kähler pair (ϕ, F). Then Φ is by definition a Kählerian Lagrangian immersion of $\hat{M}$ inducing the special Kähler metric $\hat{g} = \mathrm{Re}\,\Phi^*(\hat{\gamma})$. Since Φ is also equivariant with respect to the $\mathbb{C}^*$-action, it follows that the real part $\xi := \mathrm{Re}(Z)$ of the vector field Z generating the $\mathbb{C}^*$ action satisfies $\nabla\xi = D\xi = \mathrm{Id}$. Its length is given by

$$\hat{g}(\xi, \xi) = \hat{\gamma}(\Phi, \Phi) = |Z^0|^2 (\mathrm{Im}\, f + K) \neq 0, \tag{3.4.2}$$

where $f = 2F - z^t w$ for $(z, w) := \phi$ and $K = \gamma(\phi, \phi)$. This shows that Φ induces on $\hat{M}$ a conical affine special Kähler structure.

Let $(\phi', F') \in \mathcal{F}(M)$ with $\Phi' = \mathrm{con}(\phi', F')$. Then $(\phi', F') = x \cdot (\phi, F)$ for a unique $x \in G_{\mathbb{C}}$ and by Proposition 3.4.3 $\Phi' = \rho(x) \circ \Phi$. Now Φ and Φ' induce the same conical affine Kähler structure on $\hat{M}$ if and only if $\rho(x) \in \mathrm{Sp}(\mathbb{R}^{2n+2})$ which is the case if and only if $x \in G$. □

Proposition 3.4.5. *Let (ϕ, F) be a special Kähler pair defined on $U \subset M$ and set $f = 2F - z^t w$ for $(z, w) := \phi$ and $K = \gamma(\phi, \phi)$. Then (ϕ, F) is non-degenerate if and only if $\mathrm{Im}\, f + K \neq 0$ and $\overline{\omega} := -\frac{i}{2}\partial\overline{\partial} \log |\mathrm{Im}\, f + K|$ is non-degenerate.*

Proof. This follows easily from Eqs. (3.1.3) and (3.4.2). □

Remark 3.4.6. A special Kähler domain $M \subset \mathbb{C}^n$ with coordinates $z^1, \dots, z^n$ of $\mathbb{C}^n$ and prepotential $F : M \to \mathbb{C}$ determines a special Kähler pair (ϕ, F) by setting $\phi = dF : M \to T^*\mathbb{C}^n = \mathbb{C}^{2n}$. Then the conification

$$\begin{aligned} \hat{M} &= \{(Z^0, Z^1, \dots, Z^n) \in \mathbb{C}^* \times \mathbb{C}^n \mid Z^i/Z^0 \in M, i = 1, \dots, n\}, \\ \Phi &= \operatorname{con}(dF, F) : \hat{M} \to \mathbb{C}^{2n+2} \end{aligned} \tag{3.4.3}$$

is the graph of $d\hat{F}$, where $\hat{F}$ is a holomorphic homogeneous function of degree 2 given by

$$\hat{F}(Z^0, \dots, Z^n) = (Z^0)^2 F\left(\frac{Z^1}{Z^0}, \dots, \frac{Z^n}{Z^0}\right). \tag{3.4.4}$$

The special Kähler pair (ϕ, F) is non-degenerate if and only if the matrix given by $\operatorname{Im}\left(\frac{\partial^2 \hat{F}}{\partial Z^I \partial Z^J}\right)$ for $I, J = 0, \dots, n$ is invertible and

$$\begin{aligned} \hat{K}(Z^0, \dots, Z^n) &= \sum_{I=0}^{n} \operatorname{Im}\left(\overline{Z}^I \frac{\partial F}{\partial Z^I}\right) \\ &= \left|Z^0\right|^2 \left(K(z^1, \dots, z^n) + \operatorname{Im}(f(z^1, \dots, z^n))\right) \end{aligned} \tag{3.4.5}$$

is non-zero, where $z^i = Z^i/Z^0$, $f = 2F - \sum_{i=1}^n z^i \frac{\partial F}{\partial z^i}$, and $K = \sum_{i=1}^n \operatorname{Im}(\overline{z}^i \frac{\partial F}{\partial z^i})$. Note that in this case, $\hat{K} = \hat{\gamma}(\Phi, \Phi)$ is the Kähler potential, $\operatorname{Im}\left(\frac{\partial^2 \hat{F}}{\partial Z^I \partial Z^J}\right) = \frac{\partial^2 \hat{K}}{\partial Z^I \partial \overline{Z}^J}$ are the components of the metric, and

$$\begin{aligned} K'(z^1, \dots, z^n) &:= -\log|K(z^1, \dots, z^n) + \operatorname{Im}(f(z^1, \dots, z^n))| \\ &= -\log|\hat{K}(1, z^1, \dots, z^n)| \end{aligned} \tag{3.4.6}$$

gives a Kähler potential of the projective special Kähler metric $\overline{g}$ defined on $\hat{M}/\mathbb{C}^* \cong M$.

Example 3.4.7. Let $M \subset \mathbb{C}^n$ with standard coordinates $(z^1, \dots, z^n)$ be an affine special Kähler domain with a holomorphic prepotential given by $F = \sum_{i,j=1}^n a_{ij} z^i z^j + \frac{1}{2}C$ for $a_{ij} = a_{ji}, C \in \mathbb{C}$. Note how the parameter C does not affect the affine special Kähler geometry of M. We have $K = 2\sum_{i,j=1}^n z^i \overline{z}^j \operatorname{Im}(a_{ij})$ and $f = 2F - \sum_{i=1}^n z^i \frac{\partial F}{\partial z^i} = C$. Consider the conification of the special Kähler pair (dF, F). We denote by $(Z^0, \dots, Z^n)$ the homogeneous coordinates on $\mathbb{C}^* \times M$. The holomorphic prepotential $\hat{F}$ of the conification is then given by $\hat{F}(Z^0, Z) = \sum_{i,j=1}^n a_{ij} Z^i Z^j + C(Z^0)^2$. The matrix

$$\left(\operatorname{Im} \frac{\partial^2 \hat{F}}{\partial Z^I \partial Z^J}\right)_{I,J=0,\dots,n} = \begin{pmatrix} \operatorname{Im} C & 0 \\ 0 & 2(\operatorname{Im} a_{ij})_{i,j=1,\dots,n} \end{pmatrix} \tag{3.4.7}$$

is non-degenerate if and only if $c := \operatorname{Im} C \neq 0$. Thus (dF, F) is non-degenerate if and only if $c \neq 0$ and $K + \operatorname{Im} f = K + c \neq 0$ on M.

Assuming (dF, F) is non-degenerate, then the projective special Kähler metric $\bar{g}$ on M is given by

$$\begin{aligned}\bar{g} &= -\sum_{i,j=1}^{n} \frac{\partial^2}{\partial z^i \partial \bar{z}^j} \log|K+c|\, dz^i\, d\bar{z}^j \\ &= -\frac{1}{K+c} g + \frac{1}{(K+c)^2}(\partial K)(\bar{\partial} K),\end{aligned} \tag{3.4.8}$$

where g is the affine special Kähler metric of M.

3.4.2 The ASK/PSK correspondence

In this section we will give a global description of the conification procedure of the previous section and establish the ASK/PSK correspondence which will assign a projective special Kähler manifold to any affine special Kähler manifold given a non-degenerate special Kähler pair. For this, we will prove that every affine special Kähler manifold admits a flat principal G_{SK}-bundle. Using this bundle, we show that if the holonomy of the flat connection is contained in the group $G \subset G_{\mathrm{SK}}$, then the local conification of a non-degenerate special Kähler pair (ϕ, F) can be extended to the largest domain on which analytic continuation of (ϕ, F) is non-degenerate.

Lemma 3.4.8. *Let G be a Lie group and $\mathcal{F}$ be a presheaf on a manifold M with values in the category of principal homogeneous G-spaces. Then the disjoint union of stalks $P := \dot{\cup}_{p\in M}\mathcal{F}_p$ carries the structure of a principal G-bundle $\pi : P \to M$ with a flat connection 1-form θ such that the horizontal sections of P over U are given by $\mathcal{F}(U)$.*

Proof. Fix a point $p \in M$ and a neighborhood U of p such that $\mathcal{F}(U) \neq \emptyset$. We claim that evaluation of sections, i.e., the map taking a section $s \in \mathcal{F}(U)$ to its germ $[s]_p \in \mathcal{F}_p$, is a bijection. Let $[s_V]_p \in \mathcal{F}_p$, where $s_V \in \mathcal{F}(V)$ for some open neighborhood V of p. Without loss of generality, we can assume $V \subset U$. If $s \in F(U)$ is a section, then there is a unique $x \in G$ such that $x \cdot s|_V = s_V$. Hence, $x \cdot s$ and s_V define the same germ at p. This shows the surjectivity. Now let $s, \tilde{s} = x \cdot s \in \mathcal{F}(U)$ such that $[s]_p = [\tilde{s}]_p$. Then there is a neighborhood $V \subset U$ of P such that $s|_V = \tilde{s}|_V$. Since $s = x \cdot \tilde{s}$ for a unique $x \in G$ this implies $x = e$, where $e \in G$ is the neutral element, showing the injectivity. It follows that the stalks of $\mathcal{F}$ are also principal homogeneous G-spaces with G-action defined as $x \cdot [s]_p = [x \cdot s]_p$.

Set $P = \dot{\cup}_{p\in M}\mathcal{F}_p$ and $\pi : P \to M$, $[s]_p \mapsto p$. We can now consider a section $s \in \mathcal{F}(U)$ as a section of P over U by setting $s(p) := [s]_p$. Choose an open covering $\mathcal{U} = (U_\alpha)_{\alpha\in I}$ such that $\mathcal{F}(U_\alpha) \neq \emptyset$ and for each U_α pick a section $s_\alpha \in \mathcal{F}(U_\alpha)$. Define G-equivariant

maps $\Psi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times G$ such that $\Psi_\alpha(s_\alpha(p)) = (p, e)$. These maps are bijective by the first part of the proof. Let $U_{\alpha\beta} = U_\alpha \cap U_\beta$ be a non-empty overlap. Then $\mathcal{F}(U_{\alpha\beta}) \neq \emptyset$ and by the simply transitive action of G on $\mathcal{F}(U_{\alpha\beta})$ there is a unique $x_{\alpha\beta} \in G$ such that $s_\alpha = x_{\alpha\beta} s_\beta$, showing that the transition maps

$$\begin{aligned} \Psi_{\alpha\beta}(p, g) &:= (\Psi_\beta \circ {\Psi_\alpha}^{-1})(p, g) \\ &= \Psi_\beta(g \cdot s_\alpha(p)) = \Psi_\beta(g x_{\alpha\beta} \cdot s_\beta(p)) = (p, g x_{\alpha\beta}) \end{aligned} \tag{3.4.9}$$

are smooth and the transition functions $g_{\alpha\beta} : U_{\alpha\beta} \to G_{\mathrm{SK}}$, $g_{\alpha\beta}(p) = x_{\alpha\beta}$ are constant. On a non-empty overlap $U_{\alpha\beta\gamma} = U_\alpha \cap U_\beta \cap U_\gamma$ we have $s_\beta = x_{\beta\gamma} \cdot s_\gamma$ and $s_\alpha = x_{\alpha\beta} \cdot s_\beta = x_{\alpha\beta} x_{\beta_\gamma} \cdot s_\gamma$. Hence, the transition functions satisfy $g_{\alpha\gamma} = g_{\alpha\beta} g_{\beta\gamma}$. This shows that $\pi : P \to M$ is a principal G_{SK} bundle, see, e.g., [KN63, Chapter 1, Proposition 5.2]).

The transformation rule for local connection 1-forms $\theta_\alpha \in \Omega^1(U_\alpha, \mathrm{Lie}(G_{\mathrm{SK}}))$ is

$$\theta_\beta = \mathrm{Ad}({g_{\alpha\beta}}^{-1})\theta_\alpha + {g_{\alpha\beta}}^{-1}\, dg_{\alpha\beta} \tag{3.4.10}$$

for transition functions $g_{\alpha\beta} : U_{\alpha\beta} \to G$. In our case, the transition functions $g_{\alpha\beta}(p) = x_{\alpha\beta}$ are constant. Thus we see that setting $\theta_\alpha = 0$ defines a flat connection 1-form θ on P.

In the above we have seen that a section $s \in \mathcal{F}(U)$ gives a local trivialization $\Psi : \pi^{-1}(U) \to U \times G$. A section $\tilde{s}$ of $\pi^{-1}(U)$ is horizontal with respect to θ if and only if it is constant in this trivialization. Thus it is of the form $\tilde{s}(p) = [x \cdot s]_p$ for some $x \in G$. Under the identification $\mathcal{F}_p \cong \mathcal{F}(U)$, $\tilde{s}$ thus corresponds to $x \cdot s \in \mathcal{F}(U)$, completing the proof. □

Now let (M, J, g, ∇) be an affine special Kähler manifold of complex dimension n. Consider the map $\mathcal{F}$ assigning to each open subset U of M the set $\mathcal{F}(U)$ of special Kähler pairs of U. The map $\mathcal{F}$ is a sheaf with values in the category of G_{SK}-principal homogeneous spaces. The restriction map is given by $(\phi, F)|_V = (\phi|_V, F|_V)$. By Lemma 3.4.8 the sheaf $\mathcal{F}$ thus defines a flat principal G_{SK}-bundle $\pi : P \to M$ with flat connection 1-form θ where $P = \dot{\bigcup}_{p \in M} \mathcal{F}_p$.

Definition 3.4.9. We call the flat principal G_{SK}-bundle of germs of special Kähler pairs $\pi : P \to M$ the *bundle of special Kähler pairs.*

Definition 3.4.10. (1) We call a germ u in the fiber P_p *non-degenerate* if there is a non-degenerate special Kähler pair (ϕ, F) of an open neighborhood of p such that $[(\phi, F)]_p = u$. Note that every fiber contains at least one non-degenerate germ by Proposition 3.3.6.

(2) Let $u = [(\phi, F)]_p$ be a non-degenerate germ in the fiber P_p and (ϕ, F) be a non-degenerate special Kähler pair. Define $dom(u) \subset M$ to be the set of points in M that are connected to p via a path γ along which the analytic continuation of (ϕ, F) is non-degenerate. We call $dom(u)$ the *domain of non-degeneracy of* u.

Note that analytic continuation of a special Kähler pair (ϕ, F) defined on a neighborhood of a point p along a path γ corresponds to parallel transport of the germ $u = [(\phi, F)]_p \in P_p$ along γ. Therefore, if u is non-degenerate, then a point $p' \in M$ is in $dom(u)$ if and only if there is a horizontal path from u to the fiber over p' such that all points of γ are non-degenerate.

Theorem 3.4.11. *Let M be a connected affine special Kähler manifold of complex dimension n and $\pi : P \to M$ be the bundle of special Kähler germs of M with its flat connection 1-form θ. Assume that* $\mathrm{Hol}(\theta) \subset G$. *Let $u \in P$ be a non-degenerate point. Then the manifold $\hat{M}_u := \mathbb{C}^* \times dom(u)$ carries a conical affine special Kähler structure.*

Proof. Due to the condition on the holonomy, we can reduce the bundle $\pi : P \to M$ and the connection 1-form θ to a $\mathrm{Hol}(\theta)$-bundle

$$P(u) := \{u' \in P \mid \text{there is a } \theta\text{-horizontal path connecting } u \text{ and } u'\} \subset P.$$

First note that if $u' \in P(u)_{p'}$ is a non-degenerate germ in the fiber over p', then all germs in the fiber are non-degenerate. Indeed, if $u'' \in P(u)_{p'}$, then $u'' = x \cdot u'$ for some $x \in \mathrm{Hol}(\theta) \subset G$. Thus if (ϕ', F') is the non-degenerate special Kähler pair corresponding to u' then $\mathrm{con}(x \cdot (\phi', F')) = \rho(x)\,\mathrm{con}(\phi', F')$ is Kählerian since $\rho(x) \in \mathrm{Sp}(\mathbb{R}^{2n})$ for all $x \in G$.

By the definition of $dom(u)$ the fibers of $P(u)|_{dom(u)}$ are all non-degenerate. Hence, we can find an open covering $\mathcal{U} = (U_\alpha)_{\alpha \in I}$ of $dom(u)$ and non-degenerate special Kähler pairs $(\phi_\alpha, F_\alpha) \in \mathcal{F}(U_\alpha)$ such that $[(\phi_\alpha, F_\alpha)]_p \in P(u)_p$ for all $p \in dom(u)$. This gives a covering $\hat{\mathcal{U}} = (\hat{U}_\alpha) := (\mathbb{C}^* \times U_\alpha)_{\alpha \in I}$ and conic Kählerian Lagrangian immersions $\Phi_\alpha = \mathrm{con}(\phi_\alpha, F_\alpha) : \hat{U}_\alpha \to \mathbb{C}^{2n+2}$. The induced conical affine special Kähler structure on $\hat{U}_\alpha$ is independent of the choice of special Kähler pairs (ϕ_α, F_α) for each $\alpha \in I$ by Theorem 3.4.4 and agrees on overlaps, since the transistion functions take values in $\mathrm{Sp}(\mathbb{R}^{2n+2})$. This shows that the Φ_α induce a well-defined conical affine special Kähler structure on $\hat{M}_u = \mathbb{C}^* \times dom(u)$. □

The $\mathbb{C}^*$-action on $\hat{M}_u$ is principal. Hence, the quotient $\overline{M}_u = \hat{M}_u/\mathbb{C}^*$ is projective special Kähler with metric $\bar{g}_u$ given by Eq. (3.1.3). In particular, a Kähler potential of $\bar{g}_u$ is given by $K'_u(p) := -\log|\hat{K}_u(1, p)|$ for $p \in dom(u)$.

Definition 3.4.12. We call the map taking the affine special Kähler manifold (M, g) and a special Kähler germ u of M to the projective special Kähler manifold $(\overline{M}_u, \bar{g}_u)$ the *ASK/PSK correspondence.*

3.5 Affine bundles and affine special Kähler structures

Let (V, Ω, ∇) be a flat real symplectic vector bundle of rank $2n$ over a complex manifold M of complex dimension n such that $\nabla\Omega = 0$. Since V is flat, we can choose trivializations such that the components of Ω and the transition functions $g_{ij} : U_{ij} \to \mathrm{Sp}(\mathbb{R}^{2k}) \subset \mathrm{GL}(\mathbb{C}^{2k})$ are constant and, hence, holomorphic. This shows that the complexification $V_{\mathbb{C}} := V \otimes \mathbb{C}$ is a holomorphic bundle and the complex-linear extension of Ω (also denoted by Ω) defines a holomorphic symplectic structure on $V_{\mathbb{C}}$. The connection ∇ extends to a complex connection on $V_{\mathbb{C}}$ which we also denote by ∇. Moreover, $\gamma := \frac{i}{2}\Omega(\cdot, \bar{\cdot})$ defines a Hermitian metric on $V_{\mathbb{C}}$. Note that if $\Phi : M \to V_{\mathbb{C}}$ is a holomorphic section, then we can interpret the covariant derivative $\nabla\Phi : TM \to V_{\mathbb{C}}$ as a morphism of holomorphic vector bundles.

Proposition 3.5.1. *Let (M, J) be a complex manifold of complex dimension n and (V, Ω, ∇) a flat real symplectic vector bundle of rank $2n$ such that $\nabla\Omega = 0$. If there is a global holomorphic section $\Phi : M \to V_{\mathbb{C}}$ such that*

(i) $(\nabla\Phi)^*\Omega = 0$ *and*

(ii) $(\nabla\Phi)^*\gamma$ *is non-degenerate,*

then M carries the structure of an affine special Kähler manifold. Moreover, $V_{\mathbb{C}}$ is associated to the principal G_{SK}-bundle $P \to M$ of special Kähler pairs with the linear representation $G_{\mathrm{SK}} \to \mathrm{Sp}(\mathbb{R}^{2n})$ acting on $\mathbb{C}^{2n}$.

Proof. Write $\Phi = \rho + i\xi$ for sections $\rho, \xi : M \to V$. Since Φ is holomorphic and ∇ is complex, we have $\nabla\rho \circ J = -\nabla\xi$. Condition (ii) implies that γ pulls back to a Hermitian metric on TM and, hence, $g - i\omega := (\nabla\Phi)^*\gamma$ defines a (pseudo)-Kähler structure on (M, J). Computing ω in terms of ρ and ξ, we find

$$\begin{aligned}\omega &= -\operatorname{Im}(\nabla\Phi)^*\gamma = -\frac{1}{2}\operatorname{Re}\Omega(\nabla\Phi, \overline{\nabla\Phi}) \\ &= -\frac{1}{2}\left(\Omega(\nabla\rho, \nabla\rho) - \Omega(\nabla\xi, \nabla\xi)\right) \\ &= -(\nabla\rho)^*\Omega(\cdot, \cdot),\end{aligned} \tag{3.5.1}$$

where in the last equation we have made use of condition (i) which implies $\Omega(\nabla\rho, \nabla\rho) = \Omega(\nabla\xi, \nabla\xi)$. Since ω is non-degenerate, $\nabla\rho$ and $\nabla\xi$ are isomorphisms of vector bundles. Therefore, the connection ∇ pulls back via $\nabla\rho$ to a flat connection ∇' on TM, i.e., a vector field X is parallel if and only if $\nabla_X\rho$ is a parallel section of V. This also shows that $\nabla'\omega = 0$, since $\Omega(s, s')$ is constant for parallel sections s, s' of V.

To prove that M is affine special Kähler, it remains to show that ∇' is torsion-free and satisfies $\mathrm{d}^{\nabla'} J = 0$. For arbitrary vector fields X, Y on M, we compute

$$\begin{aligned} \mathrm{d}^{\nabla}\rho(\mathrm{d}^{\nabla'}\mathrm{id}(X,Y)) &= \mathrm{d}^{\nabla}\rho(\nabla'_X Y - \nabla'_Y X - [X,Y]) \\ &= \nabla_X \mathrm{d}^{\nabla}\rho(Y) - \nabla_Y \mathrm{d}^{\nabla}\rho(X) - \mathrm{d}^{\nabla}\rho([X,Y]) \\ &= \mathrm{d}^{\nabla}(\mathrm{d}^{\nabla}\rho)(X,Y) = ((\mathrm{d}^{\nabla})^2\rho)(X,Y) = 0, \end{aligned} \tag{3.5.2}$$

and

$$\begin{aligned} \mathrm{d}^{\nabla}\rho(\mathrm{d}^{\nabla'}J(X,Y)) &= \mathrm{d}^{\nabla}\rho(\nabla'_X(JY) - \nabla'_Y(JX) - J[X,Y]) \\ &= \nabla_X \mathrm{d}^{\nabla}\rho(JY) - \nabla_Y \mathrm{d}^{\nabla}\rho(JX) - \mathrm{d}^{\nabla}\rho(J[X,Y]) \\ &= -\nabla_X \mathrm{d}^{\nabla}\xi(Y) + \nabla_Y \mathrm{d}^{\nabla}\xi(X) + \mathrm{d}^{\nabla}\xi([X,Y]) \\ &= -\mathrm{d}^{\nabla}(\mathrm{d}^{\nabla}\xi)(X,Y) = -(\mathrm{d}^{\nabla})^2\xi(X,Y) = 0. \end{aligned} \tag{3.5.3}$$

Since $\mathrm{d}^{\nabla}\rho = \nabla\rho$ is an isomorphism, this shows $\mathrm{d}^{\nabla'}\mathrm{id} = \mathrm{d}^{\nabla'}J = 0$. It follows that M is an affine special Kähler manifold.

It remains to show that $V_{\mathbb{C}}$ is associated to the bundle of special Kähler pairs G_{SK} defined by the affine special Kähler structure on M. Let $(U_\alpha, \Psi_\alpha)_{\alpha\in I}$ be a local trivialization of $V_{\mathbb{C}}$, i.e., $\Psi_\alpha : V_{\mathbb{C}} \to U_\alpha \times \mathbb{C}^{2n}$ corresponding to a choice of a ∇-parallel Darboux frame of V on each U_α. The (constant) transition functions $g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{Sp}(\mathbb{R}^{2n})$ are defined via

$$\Psi_\beta \circ {\Psi_\alpha}^{-1}(x, v) = (x, {g_{\alpha\beta}}^{-1} v). \tag{3.5.4}$$

Note that in our conventions, transition functions act from the right whereas actions on principal bundles are left actions. Each α defines a map $\phi_\alpha : U_\alpha \to \mathbb{C}^{2n}$ via

$$\Psi_\alpha \circ \Phi|_{U_\alpha}(x) = (x, \phi_\alpha(x)), \tag{3.5.5}$$

which is a Kählerian Lagrangian immersion by conditions (i) and (ii). Hence, there is an element $x_{\alpha\beta} \in G_{\mathrm{SK}}$ such that $\bar{\rho}(x_{\alpha\beta}) \circ \phi_\beta = \phi_\alpha$. This element $x_{\alpha\beta}$ is unique for an arbitrary choice of prepotential F_α such that (ϕ_α, F_α) is a special Kähler pair for each α. However, the element $\bar{\rho}(x_{\alpha\beta}) \in \mathrm{Sp}(\mathbb{R}^{2n})$ is independent of this choice.

Let $x \in U_\alpha \cap U_\beta$. We compute

$$\begin{aligned}(x, \phi_\beta(x)) &= \Psi_\beta \circ \Phi(x) \\ &= \Psi_\beta \circ {\Psi_\alpha}^{-1} \circ \Psi_\alpha \circ \Phi(x) \\ &= \Psi_\beta \circ {\Psi_\alpha}^{-1}(x, \phi_\alpha(x)) \\ &= \Psi_\beta \circ {\Psi_\alpha}^{-1}(x, \bar{\rho}(x_{\alpha\beta}) \circ \phi_\beta(x)) \\ &= (x, {g_{\alpha\beta}}^{-1} \circ \bar{\rho}(x_{\alpha\beta}) \circ \phi_\beta(x)).\end{aligned} \tag{3.5.6}$$

It follows by Proposition 3.2.10 that $g_{\alpha\beta} = \bar{\rho}(x_{\alpha\beta})$. Hence, the transition functions of $V_{\mathbb{C}}$ are related to the transition functions of the G_{SK}-bundle $P \to M$ of special Kähler pairs via the linear representation $G_{\mathrm{SK}} \to \mathrm{Sp}(\mathbb{R}^{2n})$. □

The existence of a global section $\Phi : M \to V_{\mathbb{C}}$ in the sense of Proposition 3.5.1 is closely linked to the holonomy of the flat connection θ of the principal G_{SK}-bundle $P \to M$.

Proposition 3.5.2. *Let M be an affine special Kähler manifold of complex dimension n and P its bundle of special Kähler pairs with flat connection 1-form θ. Let $V_{\mathbb{C}}$ be the associated bundle of P with respect the linear representation $G_{\mathrm{SK}} \to \mathrm{Sp}(\mathbb{R}^{2n})$. Then $V_{\mathbb{C}}$ has a global holomorphic section in the sense of Proposition 3.5.1 if and only if* $\mathrm{Hol}(\theta) \subset \mathrm{Sp}(\mathbb{R}^{2n}) \times \mathbb{C} \subset G_{\mathrm{SK}}$.

Proof. A global holomorphic section Φ in the sense of Proposition 3.5.1 gives a covering $\mathcal{U} = (U_\alpha)_{\alpha \in I}$ and Kählerian Lagrangian immersions $\phi_\alpha : U_\alpha \to \mathbb{C}^{2n}$ for each α such that on overlaps $U_{\alpha\beta} := U_\alpha \cap U_\beta \neq \emptyset$, $\phi_\alpha = X_{\alpha\beta} \circ \phi_\beta$ for $X_{\alpha\beta} \in \mathrm{Sp}(\mathbb{R}^{2n})$. For each α, choose a prepotential F_α of ϕ_α. Then on $U_{\alpha\beta}$: $(\phi_\alpha, F_\alpha) = x_{\alpha\beta} \cdot (\phi_\beta, F_\beta)$ for $x_{\alpha\beta} = (X_{\alpha\beta}, 0, s_{\alpha\beta}) \in G_{\mathrm{SK}}$ where $s_{\alpha\beta} \in \mathbb{C}$ is determined uniquely from our choice of prepotentials. The family $(x_{\alpha\beta})$ is a cocycle with values in the subgroup $\mathrm{Sp}(\mathbb{R}^{2n}) \times \mathbb{C} \subset G_{\mathrm{SK}}$. This shows that P reduces to a principal $\mathrm{Sp}(\mathbb{R}^{2n}) \times \mathbb{C}$-bundle. It follows that $\mathrm{Hol}(\theta) \subset \mathrm{Sp}(\mathbb{R}^{2n}) \times \mathbb{C}$.

Conversely, if $\mathrm{Hol}(\theta) \subset \mathrm{Sp}(\mathbb{R}^{2n}) \times \mathbb{C}$, then we can cover M with Kählerian Lagrangian immersions ϕ_α, differing only by a linear transformation in $\mathrm{Sp}(\mathbb{R}^{2n})$. They determine a well-defined global holomorphic section $\Phi : M \to V_{\mathbb{C}}$ satisfying conditions (i) and (ii) of Proposition 3.5.1. □

Definition 3.5.3. Let $V \to M$ be a vector bundle. An *affine bundle* modelled on the vector bundle $V \to M$ is a fiber bundle $A \to M$ such that the following conditions hold.

(1) Each fiber A_p over $p \in M$ is an affine vector space over the vector space V_p.

(2) There is a bundle atlas of A such that the transition functions are given by affine isomorphisms whose linear parts give transition functions of V.

Let $A \to M$ be a flat complex affine bundle modelled on the complex vector bundle $V_{\mathbb{C}}$ from above. Since A is flat, there is a covering $\mathcal{U} = (U_\alpha)_{\alpha \in I}$ of M and local horizontal sections s_α of A giving local identifications of $A|_{U_\alpha} \cong V_{\mathbb{C}}|_{U_\alpha}$. Let $\Phi : M \to A$ be a holomorphic section of A. Thus we can locally identify Φ on U_α with a local holomorphic section Φ_α of $V_{\mathbb{C}}$ over U_α. We set $\nabla\Phi : TM \to V_{\mathbb{C}}$ by defining $\nabla\Phi|_{U_\alpha} = \nabla\Phi_\alpha$. This is well-defined, for if s'_α is a different choice of local horizontal sections of A, then $s'_\alpha - s_\alpha$ is a ∇-parallel section of $V_{\mathbb{C}}$ over U_α and $\Phi'_\alpha = \Phi_\alpha - (s'_\alpha - s_\alpha)$.

The following theorem is a generalization of Proposition 3.1.6.

Theorem 3.5.4. *Let $A \to M$ be a flat complex affine bundle modelled on the complex vector bundle $V_{\mathbb{C}} = V \otimes \mathbb{C}$ where (V, Ω, ∇) is a flat real symplectic vector bundle such that $\nabla\Omega = 0$. If there is a global holomorphic section $\Phi : M \to A$ such that*

(i) $(\nabla\Phi)^*\Omega = 0$ *and*

(ii) $(\nabla\Phi)^*\gamma$ *is non-degenerate,*

then M carries the structure of an affine special Kähler manifold. Moreover, A is associated to the principal G_{SK}-bundle of special Kähler pairs with the affine representation $\bar{\rho} : G_{\mathrm{SK}} \to \mathrm{Aff}_{\mathrm{Sp}(\mathbb{R}^{2n})}(\mathbb{C}^{2n})$ acting on $\mathbb{C}^{2n}$.

Conversely, if M is affine special Kähler, then the associated complex affine bundle $A \to M$ corresponding to the affine representation $\bar{\rho} : G_{\mathrm{SK}} \to \mathrm{Aff}_{\mathrm{Sp}(\mathbb{R}^{2n})}(\mathbb{C}^{2n})$ acting on $\mathbb{C}^{2n}$ has a global section $\Phi : M \to A$ satisfying conditions (i) and (ii).

Proof. Note that in general we can only locally write $\Phi_\alpha = \rho_\alpha + i\xi_\alpha$ on an open subset U_α for functions $\rho_\alpha, \xi_\alpha : U_\alpha \to V|_{U_\alpha}$. However, the bundle morphisms $\nabla\rho$ and $\nabla\xi$ are still well-defined. Hence, it follows from the proof of Proposition 3.5.1 that M is affine special Kähler. A similar argument also shows that A is associated to G_{SK} via the representation $\bar{\rho}$.

For the converse, we simply note that a coverinng of M with Kählerian Lagrangian immersions ϕ_α defines a well-defined global section Φ in a similar way as in the proof of Proposition 3.5.2. □

Remark 3.5.5. With respect to a ∇-parallel frame $(\lambda_1, \ldots, \lambda_{2n})$ of $V|_{U_\alpha}$, we can write the local function $\rho_\alpha = \mathrm{Re}\,\Phi_\alpha : U_\alpha \to V|_{U_\alpha}$ as

$$\rho = \sum_{i=1}^{2n} \rho^i \lambda_i. \tag{3.5.7}$$

The functions $(\rho^1, \dots, \rho^{2n})$ then define an affine coordinate system around any point $p \in U_\alpha$. Indeed, $\nabla \rho_\alpha = \sum_{i=1}^{2n} d\rho^i \lambda_i$ is an isomorphism of vector bundles, which shows that the differentials $d\rho^i$ are linearly independent. Since $(\mathrm{d}^\nabla)^2 = 0$ it follows that $\nabla\, d\rho^i = 0$. The symplectic form ω then takes the form $\omega = -\omega_{ij}\, d\rho^i \wedge d\rho^j$ for $\omega_{ij} := \Omega(\lambda_i, \lambda_j) \in \mathbb{R}$.

3.6 Completeness of Hessian metrics associated with a hyperbolic centroaffine hypersurface

In this section we will prove a completeness result for a one-parameter deformation of a positive definite Hessian metric with Hesse potential of the form $-\log h$ where h is a homogeneous function on a domain in $\mathbb{R}^n$. The latter metric is isometric to a product of the form $dr^2 + g_{\mathcal{H}}$, where $g_{\mathcal{H}}$ is proportional to the canonical metric on a centroaffine hypersurface $\mathcal{H} \subset \mathbb{R}^n$. This will be specialized in Section 3.7 to the case of a cubic polynomial h and related to the r-map.

Let $\mathcal{U} \subset \mathbb{R}^n$ be a domain such that $\mathbb{R}^{>0} \cdot \mathcal{U} \subset \mathcal{U}$ and let $h : \mathcal{U} \to \mathbb{R}$ be a smooth positive homogeneous function of degree $k > 1$. Then $\mathcal{H} := \{h = 1\} \subset \mathcal{U}$ is a smooth hypersurface and $\mathcal{U} = \mathbb{R}^{>0} \cdot \mathcal{H}$. We assume that for $g_{\mathcal{U}} := -\partial^2 h$ the metric $g_{\mathcal{H}} := \iota^* g_{\mathcal{U}}$ is positive definite, where $\iota : \mathcal{H} \hookrightarrow \mathcal{U}$ is the inclusion. The manifold $\left(\mathcal{H}, \frac{1}{k} g_{\mathcal{H}}\right)$ is a hyperbolic centroaffine hypersurface in the sense of [CNS16].

Definition 3.6.1. If h is a cubic homogeneous polynomial, then the manifold $(\mathcal{H}, g_{\mathcal{H}})$, defined as above, is called a *projective special real manifold.*

Let $g' := -\partial^2 \log h = \frac{1}{h} g_{\mathcal{U}} + \frac{1}{h^2}(dh)^2$. Denote by $\xi := x^i \partial_{x^i}$ the position vector field on $\mathcal{U}$ and by $E \subset T\mathcal{U}$ the distribution of tangent spaces tangent to the level sets of h. Then $T\mathcal{U}$ decomposes into

$$T\mathcal{U} = E \oplus \langle \xi \rangle\,. \tag{3.6.1}$$

Proposition 3.6.2. *The bilinear form* $\check{g} := g_{\mathcal{U}} - \frac{g_{\mathcal{U}}(\xi,\cdot)^2}{g_{\mathcal{U}}(\xi,\xi)}$ *is positive semidefinite with kernel* $\mathbb{R}\xi$*, and we can write*

$$g_{\mathcal{U}} = \check{g} - \frac{k-1}{kh}(dh)^2, \tag{3.6.2}$$

$$g' = \frac{1}{h}\check{g} + \frac{1}{kh^2}(dh)^2. \tag{3.6.3}$$

In particular, $g_{\mathcal{U}}$ *is a Lorentzian metric,* g' *is a Riemannian metric on* $\mathcal{U}$*, and the decomposition (3.6.1) is orthogonal with respect to* $g_{\mathcal{U}}$ *and* g'*.*

Proof. By homogeneity of h, we have $dh(\xi) = kh$, $g_{\mathcal{U}}(\xi, \cdot) = -(k-1)\,dh$ and $g_{\mathcal{U}}(\xi, \xi) = -k(k-1)h$. This implies $\check{g}|_{E\times E} = g_{\mathcal{U}}|_{E\times E} > 0$ and, hence, $\ker \check{g} = \mathbb{R}\xi$. Observing that $\frac{g_{\mathcal{U}}(\xi,\cdot)^2}{g_{\mathcal{U}}(\xi,\xi)} = -\frac{(k-1)}{kh}(dh)^2$ we obtain the formulas for $g_{\mathcal{U}}$ and g'. The distributions E and $\mathbb{R}\xi$ are obviously orthogonal with respect to $\check{g}$ and $(dh)^2$ and, therefore, also with respect to $g_{\mathcal{U}}$ and g' which are linear combinations (with functions as coefficients) of these two tensors. □

Definition 3.6.3. For $c \in \mathbb{R}$ we define the bilinear symmetric form

$$g'_c := -\partial^2 \log(h + c) = \frac{1}{h+c} g_{\mathcal{U}} + \frac{1}{(h+c)^2}(dh)^2 \tag{3.6.4}$$

on the set

$$\mathcal{U}_c := \begin{cases} \{x \in \mathcal{U} \mid h(x) + c > 0\} & \text{for } c \leq 0, \\ \{x \in \mathcal{U} \mid h(x) - c(k-1) > 0\} & \text{for } c > 0. \end{cases} \tag{3.6.5}$$

Proposition 3.6.4. *(1) As in Proposition 3.6.2 we can write*

$$g'_c = \frac{1}{h+c}\check{g} + \frac{h - c(k-1)}{kh}\frac{1}{(h+c)^2}(dh)^2. \tag{3.6.6}$$

(2) The metric g'_c is Riemannian on $\mathcal{U}_c$.

(3) If $cc' > 0$, then $(\mathcal{U}_c, g'_c)$ is isometric to $(\mathcal{U}_{c'}, g'_{c'})$.

Proof. (1) Equation (3.6.6) follows by inserting (3.6.2) into (3.6.4).

(2) The positive definiteness of g'_c follows directly from Eq. (3.6.6) since the coefficients of the two terms are positive.

(3) Scalar multiplication by $\lambda > 0$ is a diffeomorphism on $\mathcal{U}$. Let $\phi_\lambda : \mathcal{U}_c \to \mathcal{U}$ be the restriction. Using the homogeneity of h it easily follows that $\phi_\lambda(\mathcal{U}_c) = \mathcal{U}_{\lambda^k c}$.

Computing

$$\begin{aligned} \phi_\lambda^* g'_c &= \phi_\lambda^* \left(\frac{1}{h+c} g_{\mathcal{U}} + \frac{1}{(h+c)^2}(dh)^2 \right) \\ &= \frac{1}{\lambda^k h + c}\lambda^k g_{\mathcal{U}} + \frac{1}{(\lambda^k h + c)^2}\lambda^{2k}(dh)^2 \\ &= \frac{1}{h + \lambda^{-k}c} g_{\mathcal{U}} + \frac{1}{(h + \lambda^{-k}c)^2}(dh)^2 \\ &= g'_{\lambda^{-k}c} \end{aligned} \tag{3.6.7}$$

we see that for $\lambda = (c'/c)^{1/k}$ we have $\phi_\lambda^*(g'_{c'}) = g'_c$. Hence, ϕ_λ gives the required isometry. □

Theorem 3.6.5. *Assume that g' is a complete metric on $\mathcal{U}$ and $c < 0$. Then g'_c is a complete metric on $\mathcal{U}_c$.*

Remark 3.6.6. The metric g' on $\mathcal{U}$ is complete if and only if $g_{\mathcal{H}}$ is complete, since $(\mathcal{U}, g')$ is isometric to $(\mathbb{R} \times \mathcal{H}, dr^2 + g_{\mathcal{H}})$.

Proof. Denote by $L(\gamma)$ and $L'_c(\gamma)$ the Riemannian length of a curve γ in $\mathcal{U}_c$ with respect to g' and g'_c, respectively. Note first that

$$\begin{aligned} g'_c - g' &= \left(\frac{1}{h+c} - \frac{1}{h}\right)\check{g} + \frac{1}{k}\Bigg(\underbrace{\frac{h - c(k-1)}{h}}_{>1}\frac{1}{(h+c)^2} - \frac{1}{h^2}\Bigg)(dh)^2 \\ &\geq \frac{1}{k}\left(\frac{1}{(h+c)^2} - \frac{1}{h^2}\right)(dh)^2 \geq 0 \end{aligned} \tag{3.6.8}$$

on $\mathcal{U}'$. Hence, $L'_c(\gamma) \geq L(\gamma)$ for any curve γ in $\mathcal{U}_c$.

Now, for some $T > 0$ let $\gamma : [0, T) \to \mathcal{U}_c$ be a curve that is not contained in any compact set in $\mathcal{U}_c$. If γ already has infinite length with respect to g' then it also has infinite length with respect to g'_c by Eq. (3.6.8) and we are done.

Assume that $L(\gamma) < \infty$. Since g' is complete, there exists a compact set $K \subset \mathcal{U}$ such that $\gamma \subset K$. Then $\{\gamma(t)\}$ has a limit point $p \in \mathcal{U}$ that is not in $\mathcal{U}_c$ because otherwise $\overline{\{\gamma(t)\}} \subset \mathcal{U}_c$ is a compact subset of $\mathcal{U}_c$ containing γ which is a contradiction. By continuity of h, this limit point lies in $\{h + c = 0\}$. Hence, we can find a sequence $t_i \in [0, T)$, $t_i \to T$, such that $h(\gamma(t_i)) \to -c$.

Using the estimate

$$\begin{aligned} g'_c &= \frac{1}{h+c}\check{g} + \frac{h - c(k-1)}{kh}(d\log(h+c))^2 \\ &\geq \frac{1}{k}(d\log(h+c))^2 \end{aligned} \tag{3.6.9}$$

we find

$$\begin{aligned} L'_c(\gamma) &\geq \frac{1}{\sqrt{k}}\int_0^{t_i}\left|\frac{\partial}{\partial t}\log(h(\gamma(t)) + c)\right| dt \\ &\geq \frac{1}{\sqrt{k}}\left|\log(h(\gamma(t_i)) + c) - \log(h(\gamma(0)) + c)\right| \xrightarrow{t_i \to T} \infty \end{aligned} \tag{3.6.10}$$

Hence, any curve that is not contained in any compact set in $\mathcal{U}_c$ has infinite length with respect to g'_c. This is equivalent to the completeness of g'_c. □

Remark 3.6.7. In the case of $c > 0$ the metric g'_c is not complete. One can find a curve with limit point in $\{h - c(k-1) = 0\}$ that has finite length.

The following lemma will be used in the proof of Theorem 3.7.2 in the next section.

Lemma 3.6.8. *Let (M_1^n, g_1) be a complete Riemannian manifold. Then the metric*

$$g := \begin{pmatrix} g_1 & 0 \\ 0 & g_1 \end{pmatrix} \tag{3.6.11}$$

defined on the product $M = M_1 \times \mathbb{R}^n$ is complete.

Proof. This is a special case of [CHM12, Theorem 2]. □

3.7 Application to the r-map

Let us first recall the definition of the supergravity r-map, following [CHM12].

Let $(\mathcal{H}, g_{\mathcal{H}})$ be a projective special real manifold defined by a homogeneous cubic polynomial h such that $\mathcal{H} \subset \{h = 1\}$. Set $\mathcal{U} := \mathbb{R}^{>0} \cdot \mathcal{H}$ and define $g_{\mathcal{U}} := -\partial^2 h$.

Define $\overline{M} = \mathbb{R}^n + i\mathcal{U} \subset \mathbb{C}^n$ with coordinates $(z^i = y^i + \sqrt{-1}x^i)_{i=1,\dots,n} \in \mathbb{R}^n + i\mathcal{U}$. We endow $\overline{M}$ with a Kähler metric $\bar{g}$ defined by the Kähler potential $K(z) = -\log h(x)$. As a matrix, this metric is given by

$$\bar{g} = \frac{1}{4}\begin{pmatrix} -\partial^2 \log h(x) & 0 \\ 0 & -\partial^2 \log h(x) \end{pmatrix}. \tag{3.7.1}$$

Take note that $\bar{g}$ is positive definite and is the quotient metric of the conical affine special Kähler manifold $\mathbb{C}^* \times \overline{M}$ defined by the prepotential

$$\hat{F}(Z^0, \dots, Z^n) = -h(Z^1, \dots, Z^n)/Z^0, \tag{3.7.2}$$

where Z^0 is the coordinate in the $\mathbb{C}^*$-factor and $Z^i := Z^0 z^i$ for $i = 1, \dots, n$.

Definition 3.7.1. The correspondence $(\mathcal{H}, g_{\mathcal{H}}) \mapsto (\overline{M}, \bar{g})$ is called the *supergravity r-map.*

Related to the projective special real manifold $(\mathcal{H}, g_{\mathcal{H}})$ is the so-called conical affine special real manifold $(\mathcal{U}, g_{\mathcal{U}})$. The rigid r-map assigns it to the affine special Kähler manifold $(M := \overline{M}, g)$ with metric g induced by the holomorphic prepotential $F(z) = -h(z)$. As a matrix with respect to the real coordinates (y^i, x^i), this metric is given by

$$g = \begin{pmatrix} -\partial^2 h(x) & 0 \\ 0 & -\partial^2 h(x) \end{pmatrix}. \tag{3.7.3}$$

Let $\mathcal{U}_c$ be defined as in Eq. (3.6.5) and set $M_c = \mathbb{R}^n + i\mathcal{U}_c \subset M$. Note that $M_0 = M$.

Theorem 3.7.2. *Applying the ASK/PSK correspondence to the special Kähler pair*

$$(\phi_c, F_c) := (dF, F - 2\sqrt{-1}c), \tag{3.7.4}$$

defined on M_c with $F(z) = -h(z)$ and $c \in \mathbb{R}$ gives a projective special Kähler manifold $(\overline{M}_c, \bar{g}_c)$. For $c = 0$ we recover the supergravity r-map metric $\bar{g} = \bar{g}_0$. For any pair $c, c' \in \mathbb{R}$ such that $cc' > 0$, the obtained manifolds $(\overline{M}_c, \bar{g}_c)$ and $(\overline{M}_{c'}, \bar{g}_{c'})$ are isometric. Moreover, if $c < 0$ and $(\mathcal{H}, g_{\mathcal{H}})$ is complete, then $(\overline{M}_c, \bar{g}_c)$ is complete.

Proof. We will use Proposition 3.4.5 to show that $(dF, F - 2\sqrt{-1}c)$ is a non-degenerate special Kähler pair on M_c. Set $f(z) = 2(F - 2\sqrt{-1}c) - \sum_{i=1}^n z^i \frac{\partial F}{\partial z^i} = h(z) - 4\sqrt{-1}c$ and $K(z) = \sum_{i=1}^n \operatorname{Im}\left(\bar{z}^i \frac{\partial F}{\partial z^i}\right)$. Using the identity

$$\operatorname{Im} h(z) = \sum_{i=1}^n \operatorname{Im}\left(\bar{z}^i \frac{\partial}{\partial z^i} h(z)\right) - 4h(\operatorname{Im} z), \tag{3.7.5}$$

we compute $\operatorname{Im} f(z) + K(z) = -4(h(\operatorname{Im} z) + c)$, which is nonzero on M_c. The function $K' := -\log|\operatorname{Im} f + K| = -\log(4|h(\operatorname{Im} z) + c|)$ defines a symmetric bilinear tensorfield $\bar{g}_c = \sum_{i,j=1}^n \frac{\partial^2 K'}{\partial z^i \partial \bar{z}^j}\, dz^i\, d\bar{z}^j$ which, as a matrix, is of the form

$$\bar{g}_c = \frac{1}{4}\begin{pmatrix} -\partial^2 \log(h(x) + c) & 0 \\ 0 & -\partial^2 \log(h(x) + c) \end{pmatrix} = \frac{1}{4}\begin{pmatrix} g_c'(x) & 0 \\ 0 & g_c'(x) \end{pmatrix}, \tag{3.7.6}$$

where ∂^2 is the real Hessian operator with respect to the real coordinates x and g_c' is the deformed metric of the previous section. Hence, we see that $\bar{g}_c$ is positive definite by Proposition 3.6.4. This proves that $(dF, F - 2\sqrt{-1}c)$ is a non-degenerate special Kähler pair on M_c. In particular, $\bar{g}_c$ is the projective special Kähler metric that is obtained via Eq. (3.1.3) from the conical affine special Kähler metric $\hat{g}$ on the cone $\mathbb{C}^* \times M_c$ with structure induced by $\operatorname{con}\big(dF, F - 2\sqrt{-1}c\big)$. The supergravity r-map metric is recovered for $c = 0$. If $g_{\mathcal{H}}$ is complete and $c < 0$, then $\bar{g}_c$ is complete by Theorem 3.6.5 and Lemma 3.6.8. It was proven in Proposition 3.6.4.(3) that scalar multiplication on $\mathcal{U}$ by $\lambda > 0$ induces a family of isometries $\phi_\lambda : (\mathcal{U}_c, g_c') \to (\mathcal{U}_{\lambda^3 c}, g_{\lambda^3 c}')$. The differential defines a corresponding family of isometries $d\phi_\lambda : (\overline{M}_c = T\mathcal{U}_c, \bar{g}_c) \to (\overline{M}_{\lambda^3 c} = T\mathcal{U}_{\lambda^3 c}, \bar{g}_c)$. □

Remark 3.7.3. The above proof shows that the family of Kähler manifolds $(\overline{M}_c, \bar{g}_c)$ with $\bar{g}_c$ given by Eq. (3.7.6) is still defined if the projective special real manifold is replaced by a general hyperbolic centroaffine hypersurface associated with a homogeneous function $\tilde{h}$. The statements about completeness and isometries relating members of the family $(\overline{M}_c, \bar{g}_c)$ remain true under the assumption that the centroaffine hypersurface is complete. However, the metrics $\bar{g}_c$ are in general no longer projective special Kähler. In fact, the

ASK/PSK correspondence can not be applied, as the Kähler metric g obtained by the generalized r-map is in general no longer affine special Kähler. However, it turns out that the metrics g and $\bar{g}_c$ are related by an elementary deformation, c.f., Definition 2.1.16 or [MS14, Definition 1], with the symmetry replaced by the vector field $X = \operatorname{grad} \tilde{h}(x)$ and $g_\alpha := g(X,\cdot)^2 + g(JX,\cdot)^2 = (d\tilde{h})^2 + (d\tilde{h} \circ J)^2$. Indeed, the metric $\bar{g}_c$ is of the form

$$\begin{aligned} \bar{g}_c &= f_1 g + f_2 g_\alpha \\ &= \frac{1}{4}\left(\frac{1}{\tilde{h}+c}\, g + \frac{1}{(\tilde{h}+c)^2}\left((d\tilde{h})^2 + (d\tilde{h}\circ J)^2\right)\right), \end{aligned} \tag{3.7.7}$$

for $f_1 = \frac{1}{4(\tilde{h}+c)}$ and $f_2 = \frac{1}{4(\tilde{h}+c)^2}$. Its Kähler potential is $-\log(\tilde{h}(\operatorname{Im} z) + c)$.

Example 3.7.4. Consider the complete projective special real manifold

$$\mathcal{H} = \{(x,y,z) \in \mathbb{R}^3 \mid x(xy - z^2) = 1, x > 0\} \tag{3.7.8}$$

and set $\mathcal{U} = \mathbb{R}^{>0} \cdot \mathcal{H}$. Computing the scalar curvature of the metric $g'_c := -\partial^2 \log(h+c)$ for $h = x(xy - z^2)$ and $c \in \mathbb{R}$, for example with Mathematica [Wol] using the RGTC package [Bon03], gives

$$\operatorname{scal}_{g'_c} = -\frac{3(h^2 - 11ch + 6c^2)}{4(h-2c)^2}. \tag{3.7.9}$$

For $c = 0$ we find that $\operatorname{scal}_{g'_c} = -\frac{3}{4}$ is constant. For $c \neq 0$ we can further substitute $u := h/c$ and find

$$\operatorname{scal}_{g'_c} = -\frac{3(u^2 - 11u + 6)}{4(u-2)^2} \tag{3.7.10}$$

which is constant only on the level sets of h. This shows that the deformed metrics are in general not isometric to the undeformed metric. Since the manifold $(\mathcal{U}_c, g'_c)$ is contained in $(\overline{M}_c, \bar{g}_c)$ as a totally geodesic submanifold, this shows that the deformed metrics are in general not isometric to the undeformed metric.

Example 3.7.5. Consider the complete projective special real manifold

$$\mathcal{H} = \{(x,y,z) \in \mathbb{R}^3 \mid xyz = 1, x > 0, y > 0\} \tag{3.7.11}$$

and set $\mathcal{U} = \mathbb{R}^{>0} \cdot \mathcal{H}$. Computing the scalar curvature of the metric $g'_c := -\partial^2 \log(h+c)$ for $h = xyz$ and $c \in \mathbb{R}$, gives

$$\operatorname{scal}_{g'_c} = \frac{3c(4h^2 - 3ch + 2c^2)}{2h(h-2c)^2}. \tag{3.7.12}$$

For $c = 0$ we find that $\operatorname{scal}_{g'_c} = 0$ is constant. For $c \neq 0$ we can substitute $u := h/c$ and find

$$\operatorname{scal}_{g'_c} = \frac{3(4u^2 - u + 2)}{2u(u-2)^2} \tag{3.7.13}$$

which is constant only on the level sets of h.

Chapter 4

Special Kähler geometry of integrable systems

The base of an algebraic completely integrable system carries an affine special Kähler structure. This fact was first asserted by Donagi and Witten [DW96b] and proved by Freed [Fre99]. In Section 4.1 we give an introduction to algebraic integrable systems and give a detailed proof of Freed's result, following a combination of [GS90, Fre99]. Our statement of Freed's theorem is slightly more precise, see Remark 4.1.16

In Section 4.2 we analyze the integrable system of [GMN10] and show its relation to the natural hyper-Kähler structure on the cotangent bundle.

4.1 Integrable systems and Freed's theorem

Definition 4.1.1. An *algebraic completely integrable system* $(\pi : X \to M, \eta, \{\rho_b\})$ is a holomorphic submersion $\pi : X \to M$ such that

(1) (X, η) is a complex symplectic manifold with holomorphic symplectic form $\eta \in \Omega^{(2,0)}(X)$,

(2) the fibers $X_b := \pi^{-1}(b), b \in M$ are compact connected Lagrangian submanifolds of X, and

(3) there is a continuous family $\{\rho_b\}_{b\in M}$ where $\rho_b \in \Omega^2(X_b)$ is a Hodge form on X_b, i.e., a closed, positive form of type $(1, 1)$ representing an integral cohomology class.

Definition 4.1.2. Let $V \cong C^n$ be a complex vector space and $\Gamma \subset V$ be a lattice of rank $2n$. We call the quotient $M = V/\Gamma$ a *complex torus*. A complex torus M is called an

Abelian variety if M is a projective algebraic variety, i.e., if it admits an embedding into some projective space. By an *affine torus* or an *affine Abelian variety* we understand a principal homogeneous space of a complex torus or an Abelian variety, respectively.

We recall Kodaira's embedding theorem, see, e.g., [GH78].

Theorem 4.1.3 (Kodaira)**.** *A compact comlex manifold M admits a holomorphic embedding into complex projective space if and only if M admits a Hodge form.*

For the most part of this section we follow [Fre99] and have adapted some proofs of [GS90] to the holomorphic setting.

Lemma 4.1.4 ([GS90, Theorem 44.14])**.** *Let X be a holomorphic symplectic manifold with holomorphic submersion $\pi : X \to M$ such that the fibers are compact connected Lagrangian submanifolds. Then the following holds.*

(i) There is a holomorphic fiberwise transitive action of T^*M on X.

(ii) The fibers are affine tori.

(iii) Each (local) holomorphic 1-form α on M defines a (local) automorphism κ_α of the fibration such that

$$\kappa_\alpha^* \eta = \eta + d\pi^* \alpha \tag{4.1.1}$$

holds.

Proof. Let $b \in M$. Then for every $x \in X_b = \pi^{-1}(b)$ the dual of the map $d\pi_x$ gives a rise to a short exact sequence

$$0 \longrightarrow T_b^* M \xrightarrow{(d\pi_x)^*} T_x^* X \longrightarrow T_x^* X_b \longrightarrow 0, \tag{4.1.2}$$

from which we deduce that $(d\pi_x)^*$ gives an identification between $T_b^* M$ and its image $(T_x X_b)^\circ \subset T_x^* X$. Here $(T_x X_b)^\circ$ is the *annihilator* of $T_x X_b$, which is the space of covectors vanishing on $T_x X_b$. Since X_x is a Lagrangian submanifold η_x identifies $(T_x X_b)^\circ$ with $T_x X_b$:

$$T_b^* M \xrightarrow[(d\pi_x)^*]{\sim} (T_x X_b)^\circ \xrightarrow[\eta_x^{-1}]{\sim} T_x X_b. \tag{4.1.3}$$

In particular, for every $b \in B$ we have

$$T X_b \cong \pi^*(T_b^* M). \tag{4.1.4}$$

So every $\xi \in T_b^* M$ gives rise to a holomorphic vector field $\hat{\xi}$ on X_b which is tangent to X_b by setting $\hat{\xi}_x := \eta_x^{-1} \circ (d\pi)_x^*(\xi)$ for $x \in X_b$.

Suppose $\xi = -\,df_b$ for some holomorphic function f. Then for $x \in X_b$

$$-\,d(\pi^* f)_x = -\pi_x^*(df_b) = (d\pi_x)^*(\xi) = (\hat{\xi} \lrcorner\, \eta)_x. \tag{4.1.5}$$

So $\hat{\xi}$ is just the restriction to X_b of the Hamiltonian vector field $Z_{\pi^* f}$ of the function $\pi^* f$ on X.

For $\xi_i = -(df_i)_b \in T_b^* M$, $i \in \{1, 2\}$, we compute on X_b

$$[\hat{\xi}_1, \hat{\xi}_2] = [Z_{\pi^* f_1}, Z_{\pi^* f_2}] = Z_{\{\pi^* f_1, \pi^* f_2\}} = Z_{\eta(\hat{\xi}_1, \hat{\xi}_2)} = 0. \tag{4.1.6}$$

This shows that the map $\hat{\cdot} : T_b^* M \to \mathfrak{X}(X_b)$, $\xi \mapsto \hat{\xi}$ is a complex Lie algebra homomorphism.

Since X_b is compact and connected, and $T^* M$ is connected, this exponentiates to a holomorphic action $\Psi : T_b^* M \times X \to X$. If $\kappa_{\xi,t} := \exp(\hat{\xi} t)$ is the complex holomorphic flow of $\hat{\xi}$ for some $\xi \in T_b^* M$, then the action is simply given by $\Psi(\xi, x) = \kappa_{\xi,1}(x) =: \kappa_\xi(x)$. Denote by $\Psi_x : T_b^* M \to X_b$, $\Psi_x(\xi) = \kappa_\xi(x)$ the orbit map of $x \in X_b$. Then $d(\Psi_x)_0 : T_b^* M \to T_x X_b$ is the isomorphism of Eq. (4.1.3). This implies that the action is locally transitive and, since the fibers are compact and connected, also transitive. It follows that the isotropy subgroups of any two points are conjugate, and must be the same due to the fact that $T_b^* M$ is an Abelian group. Denote this subgroup by $\Lambda_b \subset T_b^* M$. Λ_b is necessarily discrete and therefore a lattice, giving the fibers the structure of complex affine tori. This shows (i) and (ii).

Now let α be a local holomorphic 1-form on M considered as a local section of $T^* M$. Then using the fiberwise action we get a vertical holomorphic vector field $\hat{\alpha} = \eta^{-1}(\pi^* \alpha)$ locally over M on X.

Using the closedness of η and the definition of $\hat{\alpha}$, we find

$$L_{\hat{\alpha}} \eta = d(\hat{\alpha} \lrcorner\, \eta) + \hat{\alpha} \lrcorner\, d\eta = d(\pi^* \alpha). \tag{4.1.7}$$

Let $\kappa_{\alpha,t} = \exp(t\hat{\alpha})$ be the complex holomorphic flow of $\hat{\alpha}$. Note that the flow is vertical, i.e., $\pi \circ \kappa_{\alpha,t} = \pi$. Using

$$\frac{d}{dt} \kappa_{\alpha,t}^* = \kappa_{\alpha,t}^* L_{\hat{\alpha}}, \tag{4.1.8}$$

and

$$\kappa_\alpha^* = \mathrm{id} + \int_0^1 \frac{d}{dt} \kappa_{\alpha,t}^* \, dt, \tag{4.1.9}$$

we find

$$\begin{aligned}\kappa_\alpha^*\eta &= \eta + \int_0^1 \frac{d}{dt}\kappa_{\alpha,t}^*\eta\,dt = \eta + \int_0^1 \kappa_{\alpha,t}^* L_{\hat\alpha}\eta\,dt \\ &= \eta + \int_0^1 \kappa_{\alpha,t}^*\,d(\pi^*\alpha)\,dt = \eta + \int_0^1 d((\pi\circ\kappa_{\alpha,t})^*\alpha)\,dt \\ &= \eta + \int_0^1 d(\pi^*\alpha)\,dt = \eta + d\pi^*\alpha.\end{aligned} \tag{4.1.10}$$

This shows (iii). □

Corollary 4.1.5. *The fibers of an algebraic completely integrable system are affine Abelian varieties.*

Proof. This follows from (ii) of the previous lemma and Kodaira's embedding theorem, as ρ_b is a Hodge form on the fiber X_b for all $b \in M$. □

Let $\hat\alpha$ be a vector field on the fiber X_b that is invariant under the action of T_b^*M, i.e., $\kappa_\beta^*\hat\alpha = \hat\alpha$ for all $\beta \in T_b^*M$. Then $\hat\alpha$ is determined by its value $\hat\alpha_x \in T_xX_b \cong T_b^*M$ at an arbitrary point $x \in X_b$. It follows that we can identify invariant vertical vector fields with sections of the bundle $V := T^*M$.

Denote by $\Lambda \subset V$ the subbundle of those elements acting trivially on X. Then each fiber X_b is a principal homogeneous space for the complex Lie group $G_b := V_b/\Lambda_b$, and by specifying a point x_0 in X_b we can identify (X_b, x_0) with the complex Lie group $V_b/\Lambda_b = T_b^*M/\Lambda_b$.

Denote by $T^*M/\Lambda = \bigcup_{b\in M} T_b^*M/\Lambda_b$ the bundle of fiberwise quotients. In the next steps, we want to show that one can even identify T^*M/Λ with X at least over open neighborhoods of M by giving a local Lagrangian section of π.

Lemma 4.1.6. *For each $b \in M$ and $x \in X_b$ there exists a neighborhood U of b and a holomorphic section $s : U \to X$ of $\pi : X \to M$ such that $s(b) = x$, and $s^*\eta = 0$, i.e., s is a local* Lagrangian section.

Proof. Since π is a holomorphic submersion, there exist local holomorphic sections by the implicit function theorem. Choose a contractible open neighborhood U of a point $b \in M$ such that there is a local holomorphic section $\tilde s : U \to X|_U$. Then $[\tilde s^*\eta] = 0$[1], so $\tilde s^*\eta = d\alpha$ for a local holomorphic 1-form α. Setting $s = \kappa_{-\alpha}\circ\tilde s$ we find

$$s^*\eta = \tilde s^*(\kappa_{-\alpha}^*\eta) = \tilde s^*(\eta - \pi^*\,d\alpha) = d\alpha - d\alpha = 0, \tag{4.1.11}$$

and hence s is Lagrangian. □

[1] because U is contractible, $H^2(U) = 0$

For a local Lagrangian section s we define an equivariant map $\chi : T^*M|_U \to X|_U$ via

$$\chi(\alpha) := \alpha \cdot s(b) = \kappa_\alpha \circ s(b), \tag{4.1.12}$$

for $\alpha \in T_b^*M$, $b \in U$.

Lemma 4.1.7 ([GS90, Theorem 44.2])**.** *The map χ is a local holomorphic symplectic bundle morphism, i.e., $\chi^*\eta = \eta_0$, where η_0 is the canonical holomorphic symplectic form of T^*M.*

Proof. Since the statement is local we will assume $M = U$.

By the equivariance of χ it is sufficient to show that χ is a holomorphic symplectic map along the zero section $M_0 \subset T^*M$. Note that by definition $\chi(M_0) = s(M)$. Let $(b, 0) \in M_0$. We identify $T_{(b,0)}T^*M \cong T_bM \oplus T_b^*M$ and $T_{s(b)}X \cong T_{s(b)}s(B) \oplus T_{s(b)}X_b$. The tangent space T_bM and the tangent space to the fiber T_b^*M are Lagrangian. Their images under $d\chi$ are given by $d\chi(T_bM) = T_{s(b)}s(B)$ and $d\chi(T_b^*M) = T_{s(b)}X_b$. Now $T_{s(b)}X_b$ is Lagrangian since X_b is a Lagrangian submanifold and $T_{s(b)}s(B)$ is Lagrangian because $s^*\eta = 0$. So the equation $\chi^*\eta = \eta_0$ holds for tangent vectors lying in the same Lagrangian factor. Thus in order to show $\chi^*\eta = \eta_0$ it suffices to show

$$\chi^*\eta(\xi, X) = \eta_0(\xi, X) = \xi(X), \tag{4.1.13}$$

for $X \in T_bM$ and $Y \in T_b^*M$. By definition of χ we have $d\chi(\xi) = \hat{\xi}_{s(b)}$ and $d\chi(X) = ds_b(X)$. We compute

$$\begin{aligned}
(\chi^*\eta)_{(b,0)}(\xi, X) &= \eta_{s(b)}(d\chi(\xi), d\chi(X)) \\
&= \eta_{s(b)}(\hat{\xi}, ds_b(X)) \\
&= \hat{\xi} \lrcorner \eta_{s(b)}(ds_b(X)) \\
&= ((d\pi)^*_{s(b)}\xi)(ds_b(X)) \\
&= \xi(d\pi \circ ds_b(X)) = \xi(X),
\end{aligned} \tag{4.1.14}$$

which proves the lemma. □

Lemma 4.1.8 ([GS90, Theorem 44.3])**.** *The bundle $\Lambda \subset V$ of elements acting trivial on X is a complex Lagrangian submanifold of $V = T^*M$.*

Proof. The statement follows from Lemma 4.1.7 and the fact that $\Lambda = \chi^{-1}(s(B))$. For if $\xi \in T_b^*M$ such that $\chi(\xi) = s(b)$ we can apply the holomorphic inverse function theorem to construct a local inverse $\chi^{-1} : \chi(U) \to U$ on an open neighborhood U of ξ such that $\chi^{-1}(s(B) \cap \chi(U)) \subset \Lambda$. Therefore Λ is a complex Lagrangian submanifold of T^*M which is locally biholomorphic to the Lagrangian submanifold $s(B) \subset X$ via the holomorphic symplectic map χ. □

Lemma 4.1.9. *The continuous family $\{\rho_b\}_{b\in M}$ of Hodge forms defines a smooth section $\rho : M \to \bigwedge^2 TM$ such that $\rho(b)$ can be identified with an invariant Hodge form on the fiber X_b.*

Proof. We can assume that ρ_b is invariant under the left multiplication L_g of $G_b = V_b/\Lambda_b$. Indeed, we can instead take the form

$$\rho_b' := \int_{g\in G_b} L_g^* \rho_b \, d\mu, \tag{4.1.15}$$

where we interpret ρ_b as a smooth function on X_b with values in $\bigwedge^2 TX_b$ and $d\mu$ is a volume form on X_b with volume 1 which is induced by a translational invariant volume form on V_b. The form ρ_b' will still represent the same integral cohomology class as ρ_b since left multiplication induces the identity on $H_2(M, \mathbb{Z})$. For details, we refer to [Cor15]. The continuous family of Hodge-forms thus gives a continuous section $\rho : M \to \bigwedge^2 V^* = \bigwedge^2 TM$, where $\rho(b)$ corresponds to the invariant form $\rho_b' \in \Omega^2(X_b)$ via the identification $TX_b \cong \pi^*(T^*M)$. By identifying $\Lambda_b \cong H_1(X_b, \mathbb{Z})$, $\rho(\lambda, \lambda') \in \mathbb{Z}$ for local sections λ, λ' of Λ. It follows that the section ρ is smooth and defines an invariant Hodge form ρ_b' on each fiber X_b, proving the lemma. □

Corollary 4.1.10. *Each algebraic integrable system $(\pi : X \to M, \eta, \{\rho_b\})$ has a canonically associated algebraic integrable system $(A \to M, \hat{\eta}_0, [\hat{\rho}_b])$ whose fibers are Abelian varieties.*

Proof. For a local Lagrangian section s the map χ descends to the quotient $A = T^*M/\Lambda$ and gives a local fiberwise identification of $(X_b, s(b))$ with the complex Lie group $A_b = T_b^*M/\Lambda_b$. The invariant Hodge form ρ_b' defined in Eq. (4.1.15) pulls back to a Hodge form on A_b that is, by its invariance, independent of the choice of the local Lagrangian section s. Hence, the continuous family of Hodge forms $\{\rho_b\}_{b\in M}$ on X defines a continuous family of polarizations $\{\hat{\rho}_b\}$ on A.

Locally we can identify Λ with the $\mathbb{Z}$-span of a local system of holomorphic sections $\alpha_1, \ldots, \alpha_{2n}$ such that at each $b \in M$ the forms comprise a (real) basis of Λ_b. Since Λ is Lagrangian, the forms are closed. Therefore the action of Λ on T^*M is holomorphic symplectic by Eq. (4.1.1) and, hence, the canonical holomorphic symplectic form η_0 of T^*M induces a well-defined holomorphic symplectic form $\hat{\eta}_0$ on the quotient. This proves the claim. □

Lemma 4.1.11. *Let ω be a non-degenerate skew-symmetric bilinear form on a vector space $V \cong \mathbb{R}^{2n}$ that is integer-valued on a lattice $\Gamma \subset V$ of full rank. Then there is a*

basis $(\lambda^1, \dots, \lambda^n, \gamma_1, \dots, \gamma_n)$ *of* Γ *such that*

$$\omega = \sum_{i=1}^{n} \delta_i \lambda^{i^*} \wedge \gamma_i^*, \quad \delta_i \in \mathbb{N} \tag{4.1.16}$$

and $\delta_i \mid \delta_{i+1}$ *for* $i = 1, \dots, n-1$.

We refer to [GH78, Chapter 2.6] for a proof.

Remark 4.1.12. The components of $\delta := (\delta_1, \dots, \delta_n) \in \mathbb{N}^n$ are called the *elementary divisors* of the non-degenerate form ω. We conclude that the family $\{\rho_b\}$ of Hodge forms determines a set $\delta = (\delta_1, \dots, \delta_n) \in \mathbb{N}^n$ of elementary divisors associated to the algebraic completely integrable system.

Definition 4.1.13. Let $(e_1, \dots, e_{2n})$ be a standard basis of $\mathbb{R}^{2n}$, $\delta \in \mathbb{Z}^n$ a set of elementary divisors, and set $\omega_\delta = \sum_{i=1}^n \delta_i e_i^* \wedge e_{i+n}^*$. We define

$$\mathrm{Sp}(\delta, \mathbb{R}) = \{A \in \mathrm{GL}(\mathbb{R}^{2n}) \mid A^* \omega_\delta = \omega_\delta\} \tag{4.1.17}$$

and $\mathrm{Sp}(\delta, \mathbb{Z}) = \mathrm{Sp}(\delta, \mathbb{R}) \cap \mathrm{GL}(\mathbb{Z}^{2n})$. Note that $\mathrm{Sp}(\delta, \mathbb{R}) \cong \mathrm{Sp}(\mathbb{R}^{2n})$.

Lemma 4.1.14. *Let* λ *be a local section of* Λ. *Then* $d(\lambda \circ J) = 0$.

Proof. Since Λ is a complex submanifold of T^*M that is locally biholomorphic to a local Lagrangian section of X, λ is a (real) holomorphic section of T^*M. Since a real holomorphic form λ is closed if and only if its $(1,0)$-part $\lambda - iJ^*\lambda$ is closed, it follows that $d(\lambda \circ J) = 0$. □

Remark 4.1.15. A non-degenerate bilinear form ω on a vector space V defines an isomorphism $\omega : V \to V^*$ via $v \mapsto \omega(v, \cdot)$. Likewise, the inverse $\omega^{-1} : V^* \to V$ defines a non-degenerate bilinear form on V^* via $\omega^{-1}(\xi, \eta) := \langle \omega^{-1}(\xi), \eta \rangle = \eta(\omega^{-1}(\xi))$, also called the *dual* of ω.

Remark 4.1.16. The following theorem is due to Freed [Fre99]. We remark that our formulation of part (2) is slightly more precise than the original statement. In particular, we do not need to assume that the ∇-parallel lattice $\Lambda \subset T^*M$ is complex and Lagrangian. Instead, we show that this follows directly from the special Kähler condition. It was also necessary to add the condition that the dual ω^{-1} of the special Kähler form is integral on Λ in order to get a continuous family of Hodge-forms on the quotient $A = T^*M/\Lambda$.

Theorem 4.1.17. *(1) Let* $(X \to M, \eta, \{\rho_b\})$ *be an algebraic completely integrable system. Then the base* M *has a canonically induced special Kähler structure* (J, ω, ∇). *The holonomy of* ∇ *is contained in the subgroup* $\mathrm{Sp}(\delta, \mathbb{Z})$.

(2) Let (M, J, ω, ∇) be a special Kähler manifold such that there exists a ∇-parallel lattice $\Lambda \subset TM^*$. Assume that ω^{-1} is integral when restricted to Λ. Then the quotient $A = T^*M/\Lambda$ admits a canonical holomorphic form η and a continuous family of Hodge-forms $\{\rho_b\}$ such that $(A \to M, \eta, \{\rho_b\})$ is an algebraic completely integrable system.

Proof. (1) Let $(\lambda_1, \ldots, \lambda_{2n})$ be a local frame of Λ. Then

$$\nabla(f\lambda_i) := df \otimes \lambda_i \tag{4.1.18}$$

defines a flat connection ∇. Since Λ is a complex Lagrangian submanifold by Lemma 4.1.8, the one-forms λ_i are closed and holomorphic. Hence, any ∇-parallel one-form is closed and holomorphic. Let T^∇ be the torsion of ∇ and let α be a ∇-parallel one-form. Then

$$0 = d\alpha(X, Y) = \alpha(T^\nabla(X, Y)), \tag{4.1.19}$$

shows that ∇ is torsion-free. Since α is holomorphic, we also have that $d\alpha \circ J = 0$ by Lemma 4.1.14. This implies $d^\nabla J = 0$ by Proposition 3.1.2.

Let ρ be the smooth section of Lemma 4.1.9. Since $\rho(b)$ is positive, of type $(1, 1)$, and integral when restricted to Λ, it follows that $\omega := \rho^{-1} \in \Omega^2(M)$ is a Kähler form that is parallel with respect to ∇. This shows that M carries an affine special Kähler structure.

Since the connection ∇ preserves both ρ and Λ and ρ is an integral non-degenerate skew-symmetric bilinear form on Λ, the holonomy of ∇ must be contained in the group $\mathrm{Sp}(\delta, \mathbb{Z})$ defined in Definition 4.1.13 by Lemma 4.1.11.

(2) As Λ is ∇-parallel, any local section λ of Λ is closed since ∇ is torsion-free and holomorphic by Proposition 3.1.2. This shows that Λ is a complex Lagrangian submanifold of T^*M. Thus the canonical holomorphic symplectic form on T^*M descends to a holomorphic symplectic form η on the quotient $A := T^*M/\Lambda$.

The dual $\rho := \omega^{-1}$ of ω defines an invariant 2-form ρ_b on each fiber $A_b := T_b^*M/\Lambda_b$ which is closed, positive definite and of type $(1, 1)$ because ω is a Kähler form. By assumption, ρ_b is integral on Λ_b and hence $[\rho_b] \in H^{1,1}(A_b) \cap H^2(A_b, \mathbb{Z})$. Thus, $\{\rho_b\}$ is a continuous family of Hodge-forms, and $(A \to M, \eta, \{\rho_b\})$ is an algebraic integrable system. $\square$

4.2 The semi-flat metric

Recall the initial data for the construction [GMN10, Nei14]:

- (M, J) a complex manifold of real dimension $2n$.
- $\Gamma \to M$ a bundle of lattices of rank $2n$. Also, the bundles $V = \Gamma \otimes \mathbb{R}$ and $V_{\mathbb{C}} = \Gamma \otimes \mathbb{C}$. The bundle Γ induces a flat connection on V and $V_{\mathbb{C}}$ which we will denote by $\hat{\nabla}$.
- A symplectic, skew-symmetric, and integer-valued bilinear form $\langle \cdot, \cdot \rangle$ on Γ. We will also denote the inverse pairing on Γ^* by $\langle \cdot, \cdot \rangle$. Also, we define $\langle \cdot \wedge \cdot \rangle$ a bilinear form on $T^*M \otimes V^*_{\mathbb{C}}$ with values in $\Lambda^2 T^*M$ by

$$\langle \alpha \otimes \xi \wedge \beta \otimes \eta \rangle := \alpha \wedge \beta \langle \xi, \eta \rangle.$$

  for $\alpha, \beta \in T^*_u M$ and $\xi, \eta \in V^*_{\mathbb{C}}|_u$, $u \in M$.
- $\Phi : M \to V^*_{\mathbb{C}}$ a holomorphic section such that

$$\langle \mathrm{d}^{\hat{\nabla}} \Phi \wedge \mathrm{d}^{\hat{\nabla}} \Phi \rangle = 0 \qquad \text{(Lagrangian)}, \tag{4.2.1}$$

$$\langle \mathrm{d}^{\hat{\nabla}} \Phi \wedge \mathrm{d}^{\hat{\nabla}} \bar{\Phi} \rangle > 0 \qquad \text{(Positivity)}. \tag{4.2.2}$$

Remark 4.2.1. One can check $\langle \varphi \wedge \varphi \rangle = 2\langle \varphi, \varphi \rangle$ and $\langle \varphi \wedge \bar{\varphi} \rangle = 2\,\mathrm{Re}\langle \varphi, \bar{\varphi} \rangle$ for $\varphi \in T^*M \otimes \Gamma^*_{\mathbb{C}}$, where $\langle \varphi, \varphi \rangle(X, Y) = \langle \varphi(X), \varphi(Y) \rangle$.

The bundle $(V^*, \Omega, \hat{\nabla})$ is a flat real symplectic vector bundle of rank $2n$ with a $\hat{\nabla}$-parallel symplectic form $\Omega = \langle \cdot, \cdot \rangle$, Hermitian form $\gamma = \frac{i}{2}\Omega(\cdot, \bar{\cdot})$, and a global holomorphic section $\Phi : M \to V^*_{\mathbb{C}}$ such that $(\hat{\nabla}\Phi)^*\Omega = 0$ and $(\hat{\nabla}\Phi)^*\gamma > 0$. Hence, it immediately follows from Proposition 3.5.1 that Φ induces an affine special Kähler structure on M, which we denote by (g, J, ∇). We let $\omega = g(J\cdot, \cdot)$ be its Kähler form.

4.2.1 Hyper-Kähler structure on the cotangent bundle

We follow [ACD02] to construct a hyper-Kähler structure on T^*M.

At each $p \in T^*M$ the flat connection ∇ induces a decomposition

$$T_p(T^*M) = \mathcal{H}^{\nabla}_p \oplus T^v_p(T^*M) \cong T_u M \oplus T^*_u M, \quad \pi(p) = u, \tag{4.2.3}$$

of the tangent space to T^*M, where $T^v_p(T^*M) = \ker d\pi_p$ is the vertical subspace and $\mathcal{H}^{\nabla}$ is the integrable horizontal distribution defined by the connection ∇.

We define the following structures on T^*M with respect to the splitting (4.2.3),

$$g_{T^*M} = \begin{pmatrix} g & 0 \\ 0 & g^{-1} \end{pmatrix}, \quad J_1 = \begin{pmatrix} J & 0 \\ 0 & J^* \end{pmatrix}, \quad J_2 = \begin{pmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{pmatrix}, \quad J_3 = J_1 J_2, \tag{4.2.4}$$

where we have identified ω as an isomorphism $TM \xrightarrow{\sim} T^*M$ via $\omega(v)(w) := \omega(v, w)$. It is easy to verify that J_1, J_2, and J_3 are almost complex structures.

Theorem 4.2.2 ([ACD02, Theorem 11])**.** *$(T^*M, g_{T^*M}, J_1, J_2, J_3)$ is a hyper-Kähler manifold.*

Proof. In affine coordinates J_2 has constant coefficients and, hence, is a complex structure. To see $J_1 J_2 = -J_2 J_1$ one uses $J^* \circ \omega = -\omega \circ J$. We compute $\omega_\alpha = g_{T^*M}(J_\alpha \cdot, \cdot) = g_{T^*M} \circ J$:

$$\omega_1 = \begin{pmatrix} \omega & 0 \\ 0 & \omega^{-1} \end{pmatrix}, \quad \omega_2 = \begin{pmatrix} 0 & -J^* \\ J & 0 \end{pmatrix}, \quad \omega_3 = \begin{pmatrix} 0 & -\mathrm{Id}_n \\ \mathrm{Id}_n & 0 \end{pmatrix}. \tag{4.2.5}$$

So let $\rho^1, \dots, \rho^{2n}$ be affine coordinates on M and $p_1, \dots, p_{2n}$ be the conjugate momenta corresponding to the ρ^i such that $\omega = \frac{1}{2}\omega_{ij}\, d\rho^i \wedge d\rho^j$. Then the local forms are given by

$$\omega_1 = \frac{1}{2}\omega_{ij}\, d\rho^i \wedge d\rho^j + \frac{1}{2}\omega^{ij}\, dp_i \wedge dp_j \tag{4.2.6}$$

$$\omega_2 = (J^*\, d\rho^i) \wedge dp_i \tag{4.2.7}$$

$$\omega_3 = d\rho^i \wedge dp_i. \tag{4.2.8}$$

From the local description we see that they are closed. Indeed, the coefficients of ω are constant in affine coordinates and $J^*\, d\rho^i$ is closed, as $\mathrm{d}^\nabla J = 0$. By Hitchin's Lemma [Hit87, Lemma 6.8], the closedness of ω_α implies that J_1, J_2, and J_3 are complex structures. Therefore $(T^*M, g_{T^*M}, J_1, J_2, J_3)$ is hyper-Kähler. □

Remark 4.2.3. The map assigning the hyper-Kähler manifold $(T^*M, g_{T^*M}, J_1, J_2, J_3)$ to the affine special Kähler manifold (M, g, J, ∇) is called the rigid c-map. We call the natural hyper-Kähler structure on T^*M the rigid c-map hyper-Kähler structure. The metric g_{T^*M} is called the rigid c-map metric.

We are interested in the associated hyper-Kähler structure on the tangent bundle TM. The symplectic form ω gives a vector bundle isomorphism $\varphi : TM \to TM^*, (u, X) \mapsto (u, \omega(X))$ with differential $d\varphi = (\mathrm{Id}_{TM}, \omega)$. We define $g_{TM} = \varphi^* g_{T^*M}$, $\tilde{\omega}_\alpha = \varphi^* \omega_\alpha$, and $\tilde{J}_\alpha = \varphi^* J_\alpha$.

As matrices with respect to the splitting (4.2.3) and

$$T(TM) := \mathcal{H}^\nabla \oplus T^v(TM) \cong TM \oplus TM \tag{4.2.9}$$

the differential $d\varphi$ corresponds to the matrix

$$d\varphi = \begin{pmatrix} \mathrm{Id}_{TM} & 0 \\ 0 & \omega \end{pmatrix},$$

and we can compute the matrix forms of the hyper-Kähler structure as follows. For the metric, we find

$$g_{TM} = \varphi^* g_{T^*M} = \begin{pmatrix} \mathrm{Id}_{TM} & 0 \\ 0 & -\omega \end{pmatrix} \begin{pmatrix} g & 0 \\ 0 & g^{-1} \end{pmatrix} \begin{pmatrix} \mathrm{Id}_{TM} & 0 \\ 0 & \omega \end{pmatrix} = \begin{pmatrix} g & 0 \\ 0 & g \end{pmatrix}, \tag{4.2.10}$$

which is the canonical metric on the tangent bundle.

The forms $\tilde{\omega}_\alpha = \varphi^*\omega_\alpha = (d\varphi)^t \circ \omega_\alpha \circ d\varphi$ and complex structures $\tilde{J}_\alpha = (d\varphi)^{-1} J_\alpha (d\varphi)$ are given by

$$\tilde{\omega}_1 = \begin{pmatrix} \omega & 0 \\ 0 & -\omega \end{pmatrix}, \quad \tilde{\omega}_2 = \begin{pmatrix} 0 & g \\ -g & 0 \end{pmatrix}, \quad \tilde{\omega}_3 = -\begin{pmatrix} 0 & \omega \\ \omega & 0 \end{pmatrix}, \tag{4.2.11}$$

$$\tilde{J}_1 = \begin{pmatrix} J & 0 \\ 0 & -J \end{pmatrix}, \quad \tilde{J}_2 = \begin{pmatrix} 0 & -\mathrm{Id} \\ \mathrm{Id} & 0 \end{pmatrix}, \quad \tilde{J}_3 = -\begin{pmatrix} 0 & J \\ J & 0 \end{pmatrix}. \tag{4.2.12}$$

Let $\rho^1, \dots, \rho^{2n}$ be affine coordinates on M and $\epsilon^1, \dots, \epsilon^{2n}$ be the adapted fiber coordinates corresponding to the ρ^i. Then the local forms are given by

$$\tilde{\omega}_1 = \frac{1}{2}\omega_{ij}\, d\rho^i \wedge d\rho^j - \frac{1}{2}\omega_{ij}\, d\epsilon^i \wedge d\epsilon^j, \tag{4.2.13}$$

$$\tilde{\omega}_2 = -(g^*\, d\rho^i) \wedge d\epsilon^i = -(J^*\, d\rho^i \circ \omega) \wedge d\epsilon^i = -\omega_{ij}(J^*\, d\rho^i) \wedge d\epsilon^j, \tag{4.2.14}$$

$$\tilde{\omega}_3 = -\omega_{ij}\, d\rho^i \wedge d\epsilon^j \tag{4.2.15}$$

Corollary 4.2.4. *$(TM, g_{TM}, \tilde{J}_1, \tilde{J}_2, \tilde{J}_3)$ is a hyper-Kähler manifold.*

Proof. This follows immediately from the proof of Theorem 4.2.2 since the forms $\tilde{\omega}_\alpha$ are closed. □

Ultimately, we will also need to make a different choice of basis for the complex structures to be able to compare the local formulas later. Set $\hat{J}_3 = \tilde{J}_1$ and $\hat{J}_1 = -\tilde{J}_3$. Then

$$\hat{\omega}_1 = \begin{pmatrix} 0 & \omega \\ \omega & 0 \end{pmatrix}, \quad \hat{\omega}_2 = \begin{pmatrix} 0 & g \\ -g & 0 \end{pmatrix}, \quad \hat{\omega}_3 = \begin{pmatrix} \omega & 0 \\ 0 & -\omega \end{pmatrix}. \tag{4.2.16}$$

4.2.2 Structure of the bundle of lattices

We now describe the integrable system defined by our data.

Definition 4.2.5. We set

$$\mathcal{M}_0 := \mathrm{Hom}(\Gamma, \mathbb{R}/\mathbb{Z}) = \left\{\theta : \Gamma \to \mathbb{R}/\mathbb{Z} \mid \theta(\gamma+\gamma') = \theta(\gamma) + \theta(\gamma')\right\}, \tag{4.2.17}$$

$$\mathcal{M} := \mathrm{Hom}_{tw}(\Gamma, \mathbb{R}/\mathbb{Z}) = \left\{\theta : \Gamma \to \mathbb{R}/\mathbb{Z} \mid \theta(\gamma+\gamma') = \theta(\gamma) + \theta(\gamma') + \frac{1}{2}\langle\gamma,\gamma'\rangle\right\}, \tag{4.2.18}$$

and we call $\mathcal{M}$ the set of twisted characters of Γ.

Remark 4.2.6. (1) We can identify $\mathrm{Hom}(\Gamma, \mathbb{R}/\mathbb{Z}) \cong \Gamma^* \otimes_{\mathbb{Z}} \mathbb{R}/\mathbb{Z} \cong V^*/\Gamma^*$, hence, $\mathcal{M}_0$ gives an algebraic integrable system over M whose fibers are Abelian varieties.

(2) $\mathcal{M}_u$ is affine over $\mathcal{M}_0|_u$. Indeed, if $\theta, \theta' \in \mathcal{M}_u$ then $\theta - \theta' \in \mathcal{M}_0|_u$. In particular, if $\theta_0 \in \mathcal{M}_u$, then $\mathcal{M}_u = \theta_0 + \mathcal{M}_0|_u$. Thus $\mathcal{M}$ is an algebraic integrable system over M which can locally be identified with $\mathcal{M}_0$.

Proposition 4.2.7. $\mathcal{M}_u \neq \emptyset$

Proof. By Lemma 4.1.11 we choose a local frame $(\lambda^1, \ldots, \lambda^n, \gamma_1, \ldots, \gamma_n)$ of V such that $\langle\cdot,\cdot\rangle = \sum_{i=1}^n \delta_i \lambda^{i*} \wedge \gamma_i^*$, for elementary divisors $\delta_i \in \mathbb{N}$. For $\gamma \in \Gamma$ set

$$\theta_0(\gamma) := \frac{1}{2}\sum_{i=0}^n d_i \lambda^{i*}(\gamma)\gamma_i^*(\gamma) + \mathbb{Z}. \tag{4.2.19}$$

Then

$$\begin{aligned}
\theta_0(\gamma+\gamma') &= \frac{1}{2}\sum_{i=0}^n d_i \lambda^{i*}(\gamma+\gamma')\gamma_i^*(\gamma+\gamma') + \mathbb{Z} \\
&= \theta_0(\gamma) + \theta_0(\gamma') + \frac{1}{2}\sum_{i=0}^n d_i \left(\lambda^{i*}(\gamma)\gamma_i^*(\gamma') + \lambda^{i*}(\gamma')\gamma_i^*(\gamma)\right) + \mathbb{Z} \\
&= \theta_0(\gamma) + \theta_0(\gamma') + \frac{1}{2}\sum_{i=0}^n d_i \left(\lambda^{i*}(\gamma)\gamma_i^*(\gamma') - \lambda^{i*}(\gamma')\gamma_i^*(\gamma)\right) + \mathbb{Z} \\
&= \theta_0(\gamma) + \theta_0(\gamma') + \frac{1}{2}\langle\gamma,\gamma'\rangle + \mathbb{Z},
\end{aligned} \tag{4.2.20}$$

and, thus, $\theta_0 \in \mathcal{M}_u$. □

Let $(\gamma_1, \ldots, \gamma_{2n})$ be a local frame of Γ and $(\gamma_1^*, \ldots, \gamma_{2n}^*)$ its dual. We have already seen in Remark 3.5.5 that this defines an affine coordinate system $(\rho^1, \ldots, \rho^{2n})$ on M.

The evaluation maps $\epsilon^i : V^* \to \mathbb{R}$ given by $\epsilon^i(\gamma^*) := \gamma^*(\gamma_i)$ are coordinate functions of the fibers.

Consider the decomposition

$$TV^* = \mathcal{H}^{\hat{\nabla}} \oplus T^v V^* \cong TM \oplus V^* \tag{4.2.21}$$

induced by $\hat{\nabla}$. Let $\Theta \in \Omega^1(V^*, V^*)$ be the connection 1-form of $\hat{\nabla}$. That is, Θ vanishes on the horizontal part and gives the identification $T^v_\gamma V^* \cong V^*|_{\pi(\gamma)}$ on the vertical space. In the coordinates (ρ^i, ϵ^j), Θ can be written as

$$\Theta = d\epsilon^j \otimes d\rho^j. \tag{4.2.22}$$

Recall that $\nabla\rho := \operatorname{Re}\hat{\nabla}\Phi : TM \to V^*$ is an isomorphism of vector bundles.

Theorem 4.2.8. *The forms*

$$\omega_3 = \frac{R}{4}\pi^*\langle\hat{\nabla}\Phi \wedge \hat{\nabla}\bar{\Phi}\rangle - \frac{1}{2R}\langle\Theta \wedge \Theta\rangle \tag{4.2.23}$$

$$\omega_+ = \omega_1 + i\omega_2 = \langle\pi^*\hat{\nabla}\Phi \wedge \Theta\rangle \tag{4.2.24}$$

define a hyper-Kähler structure on $V^* \cong TM$ for all $R \in \mathbb{R}^*$ which agrees up to the multiplicative factor R and a permutation of the complex structure with the hyper-Kähler structure of Eq. (4.2.16).

Proof. Let $\Phi = \rho + i\xi$. Then $\rho = \rho^i \otimes \gamma_i^*$ and $(\rho^1, \ldots, \rho^{2n}, \epsilon^1, \ldots, \epsilon^{2n})$ form a local coordinate system of V^*. Using $\langle\hat{\nabla}\Phi \wedge \hat{\nabla}\overline{\Phi}\rangle = 2\langle\hat{\nabla}\rho \wedge \hat{\nabla}\rho\rangle$, $\hat{\nabla}\rho = d\rho^j \otimes \gamma_j^*$, $\Theta = d\epsilon^j \otimes \gamma_j^*$, and $\omega_{ij} = \langle\gamma_i^*, \gamma_j^*\rangle$ we find

$$\begin{aligned}
\omega_3 &= \frac{R}{4}\pi^*\langle\hat{\nabla}\Phi \wedge \hat{\nabla}\overline{\Phi}\rangle - \frac{1}{2R}\langle\Theta \wedge \Theta\rangle \\
&= \frac{R}{2}\pi^*\langle\hat{\nabla}\rho \wedge \hat{\nabla}\rho\rangle - \frac{1}{2R}\langle\Theta \wedge \Theta\rangle \\
&= \frac{1}{2}\left(R\, d\rho^i \wedge d\rho^j \langle\gamma_i^*, \gamma_j^*\rangle - \frac{1}{R}\, d\epsilon^i \wedge d\epsilon^j \langle\gamma_i^*, \gamma_j^*\rangle\right) \\
&= \frac{1}{2}\left(R\omega_{ij}\, d\rho^i \wedge d\rho^j - \frac{1}{R}\omega_{ij} d\epsilon^i \wedge d\epsilon^j\right),
\end{aligned} \tag{4.2.25}$$

and

$$\begin{aligned}
\omega_+ &= \langle\pi^*\hat{\nabla}\rho \wedge \Theta\rangle + i\langle\pi^*\hat{\nabla}\xi \wedge \Theta\rangle \\
&= \langle\pi^*\hat{\nabla}\rho \wedge \Theta\rangle - i\langle\pi^*\hat{\nabla}\rho \circ J \wedge \Theta\rangle \\
&= d\rho^i \wedge d\epsilon^j \langle\gamma_i^*, \gamma_j^*\rangle - i(J^* d\rho^i) \wedge d\epsilon^j \langle\gamma_i^*, \gamma_j^*\rangle \\
&= \omega_{ij}\, d\rho^i \wedge d\epsilon^j - i\omega_{ij}(J^* d\rho^i) \wedge d\epsilon^j \\
&= (\omega^*\, d\rho^i) \wedge d\epsilon^i - i(\omega \circ J)^*\, d\rho^i \wedge d\epsilon^i.
\end{aligned} \tag{4.2.26}$$

Thus

$$\omega_1 = \begin{pmatrix} 0 & \omega \\ \omega & 0 \end{pmatrix}, \quad \omega_2 = \begin{pmatrix} 0 & \omega \circ J \\ -\omega \circ J & 0 \end{pmatrix}, \quad \omega_3 = \begin{pmatrix} R\omega & 0 \\ 0 & -\frac{1}{R}\omega \end{pmatrix}, \tag{4.2.27}$$

and

$$\omega_1 = \omega_{ij} d\rho^i \wedge d\epsilon^j \tag{4.2.28}$$

$$\omega_2 = -\omega_{ij} J^* d\rho^i \wedge d\epsilon^j \tag{4.2.29}$$

$$\omega_3 = \frac{1}{2}\left(R\omega_{ij} d\rho^i \wedge d\rho^j - \frac{1}{R}\omega_{ij} d\epsilon^i \wedge \epsilon^j \right). \tag{4.2.30}$$

The differential of the fibre bundle isomorphism $\varphi_R := R\,\hat{\nabla}\rho : TM \to V^*$, $(x, X) \mapsto (x, R\,\hat{\nabla}\rho(X))$ with respect to the bases $\left(\frac{\partial}{\partial\rho^1}, \ldots, \frac{\partial}{\partial\rho^{2n}}\right)$ of TM and $(\gamma_1^*, \ldots, \gamma_{2n}^*)$ of V^* is

$$d\varphi_R = \begin{pmatrix} \mathrm{Id} & 0 \\ 0 & R\,\mathrm{Id} \end{pmatrix}, \tag{4.2.31}$$

so using $\varphi_R^*\omega_\alpha = (d\varphi_R)^t \omega_\alpha (d\varphi_R)$ we obtain

$$\varphi^*\omega_1 = R\,\hat{\omega}_1, \quad \varphi^*\omega_2 = R\,\hat{\omega}_2, \quad \varphi^*\omega_3 = R\,\hat{\omega}_3. \tag{4.2.32}$$

This is up to a factor of R the hyper-Kähler structure on TM induced by the c-map hyper-Kähler structure on T^*M with respect to the choice of complex structures as in Eq. (4.2.16).

Hence, (ω_3, ω_+) is a hyper-Kähler structure on V^*. □

Corollary 4.2.9. *The hyper-Kähler on V^* induces a hyper-Kähler structure on $\mathcal{M}_0$.*

Proof. Since the structure is invariant under translation along the fibers of the bundle V^*, it induces a well-defined hyper-Kähler structure on $\mathcal{M}_0 \cong V^*/\Gamma^*$. □

Outlook

In Theorem 2.1.18 we give necessary and sufficient conditions for the twist of an elementary deformation with respect to some twist data to be Kähler. Together with an example (Proposition 2.1.20) of such a Kähler twist, this proves that, unlike in the hyper-Kähler/quaternionic Kähler case, the twist construction provides more degrees of freedom to produce Kähler manifolds. In this context it would be worthwile to research further examples of twists of elementary deformations.

In Section 2.3.1 we have analyzed the behavior of the Ricci curvature and the Einstein condition for the special class of conical Kähler manifolds where the Hamiltonian Killing vector field is related to the Euler vector field. In this respect it would be desirable to find more general formulas relating the curvatures of the Kähler manifolds in the hope of using the K/K correspondence to construct new Einstein metrics.

For the global ASK/PSK correspondence it would be worthwile to try to get a better understanding of the holonomy condition for the flat connection of the principal G_{SK}-bundle. In Sections 3.5 and 4.2 we have already seen special cases where the holonomy reduces to $\mathrm{Sp}(\mathbb{R}^{2n}) \times \mathbb{C}$. The factor $\mathbb{C}$ can be interpreted as the (constant) difference of a prepotential and its analytic continuation along closed loops. It is a natural question to ask, under which conditions this difference is real, independent of the chosen path.

In Section 4.2 we show that the integrable system $\mathcal{M}_0$ carries a hyper-Kähler structure that is induced from the natural hyper-Kähler structure on the cotangent bundle T^*M of the affine special Kähler manifold M. It is, however, still unclear whether this also induces a hyper-Kähler structure on the bundle $\mathcal{M}$ of twisted characters. Recall that the bundle $\mathcal{M}$ is affine over $\mathcal{M}_0$, thus the difference of two local sections s, s' of $\mathcal{M}$ gives a section of $\mathcal{M}_0$. If these sections can be chosen in such a way that $s - s'$ is in fact a $\hat{\nabla}$-parallel section of $\mathcal{M}_0$, then this would give a well-defined hyper-Kähler structure on $\mathcal{M}$. We can formulate the following question: Does the bundle of twisted characters $\mathcal{M}$ carry a flat connection? We close this chapter by pointing out the relevance of the quadratic form Eq. (4.2.19) to this question.

Bibliography

[ACD02] D. Alekseevsky, V. Cortés, and C. Devchand, *Special complex manifolds*, J. Geom. Phys. **42** (2002), no. 1, 85–105.

[ACDM15] D. V. Alekseevsky, V. Cortés, M. Dyckmanns, and T. Mohaupt, *Quaternionic Kähler metrics associated with special Kähler manifolds*, J. Geom. Phys. **92** (2015), 271–287.

[ACM13] D. V. Alekseevsky, V. Cortés, and T. Mohaupt, *Conification of Kähler and hyper-Kähler manifolds*, Commun. Math. Phys. **324** (2013), no. 2, 637–655.

[BG08] C. Boyer and K. Galicki, *Sasakian geometry*, Oxford university press, 2008.

[Bon03] S. Bonanos, *Capabilities of the Mathematica package "Riemannian Geometry and Tensor Calculus"*, Proceedings of the 10th Hellenic Relativity Conference on Recent Developments in Gravity: Kalithea/Chalkidiki, Greece, 30 May-2 June 2002, World Scientific Publishing Company Incorporated, 2003, p. 174.

[CDM17] V. Cortés, P.-S. Dieterich, and T. Mohaupt, *ASK/PSK-correspondence and the r-map*, Preprint, `arXiv:1702.02400 [math.DG]` (2017).

[CDS16] V. Cortés, M. Dyckmanns, and S. Suhr, *Completeness of projective special Kähler and quaternionic Kähler manifolds*, to appear (accepted 26-12-2016) in "New perspectives in differential geometry: special metrics and quaternionic geometry" in honour of Simon Salamon (Rome, 16-20 November 2015), `arXiv:1607.07232 [math.DG]` (2016).

[CFG89] S. Cecotti, S. Ferrara, and L. Girardello, *Geometry of type II superstrings and the moduli of superconformal field theories*, Int. J. Mod. Phys. A **4** (1989), no. 10, 2475–2529.

[CHM12] V. Cortés, X. Han, and T. Mohaupt, *Completeness in Supergravity Constructions*, Commun. Math. Phys. **311** (2012), no. 1, 191–213.

[CNS16] V. Cortés, M. Nardmann, and S. Suhr, *Completeness of hyperbolic centroaffine hypersurfaces*, Comm. Anal. Geom. **24** (2016), no. 1, 59–92.

[Cor98] V. Cortés, *On hyper Kähler manifolds associated to Lagrangian Kähler submanifolds of $T^*\mathbb{C}^n$*, Trans. Amer. Math. Soc. **350** (1998), no. 8, 3193–3205.

[Cor15] ———, *Integrable systems*, Lecture notes, 2015.

[dW96a] B. de Wit, *N=2 electric-magnetic duality in a chiral background*, Nucl. Phys. B **49** (1996), no. 1-3, 191–200.

[DW96b] R. Donagi and E. Witten, *Supersymmetric Yang-Mills theory and integrable systems*, Nucl. Phys. B **460** (1996), no. 2, 299–334.

[dWVP84] B. de Wit and A. Van Proeyen, *Potentials and symmetries of general gauged N=2 supergravity-Yang-Mills models*, Nucl. Phys. B **245** (1984), 89–117.

[dWVP92] ———, *Special geometry, cubic polynomials and homogeneous quaternionic spaces*, Commun. Math. Phys. **149** (1992), no. 2, 307–333.

[Dyc15] M. Dyckmanns, *The hyper-Kähler/quaternionic Kähler correspondence and the geometry of the c-map*, Ph.D. thesis, Universität Hamburg, 2015.

[FPI04] M. Falcitelli, A. M. Pastore, and S. Ianus, *Riemannian submersions and related topics*, World Scientific, 2004.

[Fre99] D. S. Freed, *Special Kähler manifolds*, Commun. Math. Phys. **203** (1999), no. 1, 31–52.

[FS90] S. Ferrara and S. Sabharwal, *Quaternionic manifolds for type II superstring vacua of Calabi-Yau spaces*, Nucl. Phys. B **332** (1990), no. 2, 317–332.

[Fut87] A. Futaki, *The Ricci curvature of symplectic quotients of Fano manifolds*, Tohoku Math. J. (2) **39** (1987), no. 3, 329–339.

[GH78] P. Griffiths and J. Harris, *Principles of algebraic geometry*, Wiley-Interscience, 1978.

[GMN10] D. Gaiotto, G. W. Moore, and A. Neitzke, *Four-dimensional wall-crossing via three-dimensional field theory*, Commun. Math. Phys. **299** (2010), no. 1, 163–224.

[GS90] V. Guillemin and S. Sternberg, *Symplectic techniques in physics*, Cambridge University Press, 1990.

[Hay08] A. Haydys, *HyperKähler and quaternionic Kähler manifolds with S^1-symmetries*, J. Geom. Phys. **58** (2008), no. 3, 293–306.

[Hit87] N. J. Hitchin, *The self-duality equations on a Riemann surface*, Proc. of the London Math. Soc. **3** (1987), no. 1, 59–126.

[Hit99] ———, *The moduli space of complex Lagrangian submanifolds*, Asian J. Math **3** (1999), no. 1, 77–91.

[HKLR87] N. J. Hitchin, A. Karlhede, U. Lindström, and M. Roček, *Hyperkähler metrics and supersymmetry*, Commun. Math. Phys. **108** (1987), no. 4, 535–589.

[KN63] S. Kobayashi and K. Nomizu, *Foundations of differential geometry*, vol. 1, Wiley, 1963.

[Kob87] S. Kobayashi, *Submersions of CR submanifolds*, Tohoku Math. J. (2) **39** (1987), no. 1, 95–100. MR 876456 (88d:32032)

[MS14] O. Macia and A. Swann, *Elementary deformations and the hyperKähler-quaternionic Kähler correspondence*, in "Real and complex submanifolds", Daejoen, Korea, August 2014, Springer Proc. Math. Stat., vol. 106, Springer, Tokyo, 2014.

[MS15] ———, *Twist geometry of the c-map*, Commun. Math. Phys. **336** (2015), no. 3, 1329–1357.

[Nei14] A. Neitzke, *Notes on a new construction of hyperkähler metrics*, Homological Mirror Symmetry and Tropical Geometry, Springer, 2014, pp. 351–375.

[O'N66] B. O'Neill, *The fundamental equations of a submersion.*, Michigan Math. J. **13** (1966), no. 4, 459–469.

[RLSV06] D. Robles-Llana, F. Saueressig, and S. Vandoren, *String loop corrected hypermultiplet moduli spaces*, J. High Energy Phys. **2006** (2006), no. 03, 081.

[Swa10] A. Swann, *Twisting Hermitian and hypercomplex geometries*, Duke Math. J. **155** (2010), no. 2, 403–431.

[Swa14] ———, *Twist geometry of elementary deformations*, Talk presented at a conference on "Lorentzian and conformal Geometry" in Greifswald, Germany. Available at `http://home.imf.au.dk/swann/talks/Greifswald-2014.pdf`, March 2014.

[Wol] Wolfram Research, Inc., *Mathematica 10.0*.

Publications

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This collaboration was part of my doctoral project. Chapter 3 is based on this publication. Key ideas, such as the determination of the correct group extension, the lift of the group action as in Section 3.2.1, and the local conification (Section 3.3) were developed in joint discussions with my co-authors. The physical interpretation of the deformation of the supergravity r-map metric as an α'-correction is due to T. Mohaupt. The completeness result of Section 3.6, the explicit transformation behavior of Lagrangian potentials and prepotentials (Sections 3.2.2 and 3.2.3), and the description of the principal G_{SK}-bundle as well as the global description of the ASK/PSK correspondence (Section 3.4.2) are results of my own work.