Group Actions on Bicategories and Topological Quantum Field Theories

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Summary

Bicategories play an important rôle in the classification of fully-extended two-dimensional topological field theories. In order to describe topological field theories with more geometric structure, one needs more structure on the algebraic side, which is given by homotopy fixed points of a certain group action on a bicategory.

In the first chapter of this thesis, we develop the mathematical theory of group actions on bicategories. By categorifying the notion of a group action on a set, we arrive at a suitable definition of an action of a topological group on a bicategory. Given such an action, we provide an explicit definition of the bicategory of homotopy fixed points. This allows us to explicitly compute the bicategory of homotopy fixed points of certain group actions. Two fundamental examples show that even homotopy fixed points of trivial group actions give rise to additional structure: we show that a certain bigroupoid of semisimple symmetric Frobenius algebras is equivalent to the bicategory of homotopy fixed points of the trivial $SO(2)$-action on the core of fully-dualizable objects of the bicategory of algebras, bimodules and intertwiners. Furthermore, we show that homotopy fixed points of the trivial $SO(2)$-action on the bicategory of finite, linear categories are given by Calabi-Yau categories.

The next chapter deals with an additional equivariant structure on a functor between bicategories equipped with a group action. We show that such an equivariant functor induces a functor on homotopy fixed points. As an application, we consider the 2-functor which sends a finite-dimensional, semisimple algebra to its category of representations. This functor has got a natural $SO(2)$-equivariant structure, and thus induces a functor on homotopy fixed points. We show that this induced functor is pseudo-naturally isomorphic to an equivalence between Frobenius algebras and Calabi-Yau categories which we have constructed previously.

In the last two chapters, we classify fully-extended, 2-dimensional oriented topological field theories. We begin by constructing a non-trivial $SO(2)$-action on the framed bordism bicategory. The cobordism hypothesis for framed manifolds allows us to transport this action to the core of fully-dualizable objects of the target bicategory. We show that this action is given by the Serre automorphism and compute the bicategory of homotopy fixed points of this action. Finally, we identify this bigroupoid of homotopy fixed points with the bicategory of fully-extended oriented topological quantum field theory with values in an arbitrary symmetric monoidal bicategory. This proves the cobordism hypothesis for two-dimensional oriented cobordisms.

Zusammenfassung

In der Klassifizierung von vollständig erweiterten zweidimensionalen topologischen Feldtheorien spielen Bikategorien eine wichtige Rolle. Um topologische Feldtheorien mit zusätzlicher geometrischer Struktur zu beschreiben, benötigt man zusätzliche algebraische Struktur, die durch Homotopiefixpunkte einer Gruppenwirkung auf einer bestimmten Bikategorie gegeben ist.

Im ersten Kapitel entwickeln wir einen mathematischen Formalismus zur Beschreibung von Gruppenwirkungen auf Bikategorien. Gegeben die Wirkung einer topologischen Gruppe auf einer Bikategorie, konstruieren wir explizit eine Bikategorie von Homotopiefixpunkten dieser Wirkung. Dieser Formalismus ermöglicht uns, Homotopiefixpunkte von bestimmten Gruppenwirkungen explizit zu berechnen. Zwei fundamentale Beispiele zeigen nun, dass sogar Homotopiefixpunkte von trivialen Gruppenwirkungen zusätzliche Struktur sind: so ist die Bikategorie von endlichdimensionalen, halbeinfachen Frobeniusalgebren äquivalent zu der Bikategorie von Homotopiefixpunkten der trivialen $SO(2)$-Wirkung auf der Bikategorie von vollständig dualisierbaren Algebren und Bimoduln. Weiterhin zeigen wir, dass Homotopiefixpunkte der trivialen $SO(2)$-Wirkung auf der Bikategorie von endlichen, linearen Kategorien äquivalent zur Bikategorie von Calabi-Yau Kategorien sind.

Im nächsten Kapitel beschäftigen wir uns mit einer zusätzlichen äquivarianten Struktur auf einem Funktor zwischen Bikategorien mit einer Gruppenwirkung. Wir zeigen, dass solch ein äquivarianter Funktor zwischen zwei Bikategorien einen Funktor auf Homotopiefixpunkten induziert. Als Anwendung betrachten wir den 2-Funktor, der einer halbeinfachen Algebra ihre Darstellungskategorie zuweist. Dieser 2-Funktor hat eine natürliche $SO(2)$-äquivariante Struktur, und induziert daher einen Funktor auf Homotopiefixpunkten. Sodann identifizieren wir diesen induzierten Funktor mit einer bereits zuvor konstruierten Äquivalenz zwischen Frobeniusalgebren und Calabi-Yau Kategorien.

In den letzten beiden Kapiteln wenden wir uns der Klassifizierung von zweidimensionalen, vollständig erweiterten, orientierten topologischen Quantenfeldtheorien zu: wir konstruieren zunächst eine nicht-triviale $SO(2)$-Wirkung auf einem Skelett der Bikategorie von gerahmten Bordismen. Die Kobordismushypothese für gerahmte Mannigfaltigkeiten erlaubt uns, diese Wirkung auf den maximalen Untergruppoiden von vollständig dualisierbaren Objekten der Zielkategorie zu transportieren. Wir zeigen, dass diese $SO(2)$-Wirkung durch den Serre Automorphismus gegeben ist, und berechnen die Bikategorie von Homotopiefixpunkten. Schlussendlich identifizieren wir diese Bikategorie von Homotopiefixpunkten mit der Bikategorie von vollständig erweiterten, orientierten, zweidimensionalen topologischen Feldtheorien mit Werten in einer symmetrisch monoidalen Bikategorie. Dies beweist die Kobordismushypothese für zweidimensionale orientierte Kobordismen.

Contents

Introduction

This thesis investigates structures in 2-dimensional topological quantum field theories and is based at the interface of algebra and topology.

Giving a precise definition of quantum field theory is a notoriously difficult problem in mathematical physics. Historically, quantum field theory grew out of attempts to reconcile relativistic field theory, like classical electrodynamics, with quantum mechanics. Nowadays, there are essentially two main approaches for formalising quantum field theories. One approach is given by the framework of *algebraic* quantum field theory: here, axiomatic systems for quantum field theory were first developed by Wightman in [SW64] and then further abstracted by Haag and Kastler in [HK64]. More recent work of Brunetti, Fredenhagen and Verch in [BFV03] extends this approach to locally covariant field theories. These axiomatic systems formalise the assignment of an algebra of observables to certain patches of spacetime.

In this thesis, we use the second main approach for formalising quantum field theory, which is given by *functorial* quantum field theory. This approach formalises the assignment of a state space to patches of physical space and tries to axiomatize the output of the Feynman path integral. One of the main examples of functorial quantum field theories are given by *topological* quantum field theories as already considered in Witten's seminal paper [Wit89], which in physical terms should be imagined as quantum field theories in which the correlation functions only depend on the topological features of spacetime.

In order to turn this physical idea into mathematics, one employs the language of category theory. A category is a two-layered mathematical structure, which has a class of objects as one layer, and as a second layer a set of morphisms which act as "relations" between the objects. A "map" between two categories is called a functor: it maps objects and morphisms of the source category to objects and morphisms of the target category in a structure preserving manner.

As defined by Atiyah in [Ati88] and Segal in [Seg04], a topological quantum field theory is a functor between two categories, which is furthermore compatible with "cutting and gluing": the source category can be thought of as the category of spacetime, together with all possible ways to "cut and glue" particular patches together. Objects of this category are possible choices of "space", while morphisms are given by choices of "spacetime". A topological quantum field theory then assigns a vector space of states to each object of this category, and a linear operator to each morphism.

One drawback of Atiyah's definition is that it is not fully local: the setup of spacetime as an ordinary category only allows to "cut and glue" patches of space of codimension one. In order to allow gluing for spaces of arbitrary codimensions, one uses higher categories. A higher category is a multi-layered structure with additional layers of morphisms: these can be thought of relations between relations between relations, and so on. One can then try to define a higher categorical version of the category of spacetime, where cutting and gluing is also possible for spaces of lower dimension. This higher categorical approach allows us to define "fully local" topological field theories, which assign further algebraic data to spaces of lower dimensions. Fully local field theories are actually easier to describe than one might originally think: one expects from work of Lurie in [Lur09b] that framed, fully local field theories are determined by a single datum, namely their value on a point.

In order to describe field theories on spacetimes with more geometric structure, one needs more structure on the algebraic side. This structure is frequently given by homotopy fixed points of a certain group acting on a higher category. While a fixed point of a group G acting on a set X is just a point x of X, satisfying the equation $g.x = x$ for all group elements g in G, a homotopy fixed point of a group acting on a category consists of more data. In the categorical setting, it is unreasonable to demand an equality of objects on the nose. Instead, one demands the existence of an additional isomorphism $g.x \to x$, which then has to satisfy appropriate coherence equations.

In section 2.2 of this thesis, we develop the theory of a group acting on a bicategory, which is a higher category with three levels of information: objects, 1-morphisms, and 2-morphisms. We define a suitable generalization of a homotopy fixed point in the bicategorical setting, and compute these homotopy fixed points for the action of the group of rotations $SO(2)$ on a bicategory in theorem 2.34. Finally, theorem 5.5 and 5.8 show that these homotopy fixed points classify oriented, 2-dimensional topological quantum field theories with an arbitrary symmetric monoidal target bicategory $\mathcal{C}$. This extends work of Schommer-Pries in [SP09], who classified fully-extended 2-dimensional oriented field theories with values in the Morita bicategory in terms of separable, symmetric Frobenius algebras.

Topological quantum field theories

As originally defined by Atiyah in [Ati88], an oriented n-dimensional topological quantum field theory is a symmetric monoidal functor $Z : \mathrm{Cob}_n^{\mathrm{or}} \to \mathrm{Vect}$. Here, $\mathrm{Cob}_n^{\mathrm{or}}$ is the category of oriented cobordisms: objects of this category – previously referred to as space – are given by closed, oriented $(n-1)$-dimensional manifolds. Morphisms between two $(n-1)$-manifolds M and N are given by diffeomorphism classes of bordisms B, which are n-dimensional oriented manifolds with parametrized boundary, satisfying $\partial B \cong \overline{M} \sqcup N$. For instance, the bordism in figure 1 can be interpreted as a morphism $S^1 \sqcup S^1 \to S^1$ in the 2-dimensional oriented bordism category. Composition of bordisms is given by

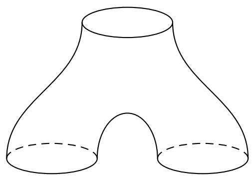

Figure 1.: The "pair of pants" bordism

"gluing" bordisms along a common boundary, which may require the choice of collars.

One reason explaining the original interest of mathematicians in topological field theories is that they provide invariants of manifolds, which are well-behaved under cutting and gluing. If M is an n-dimensional closed manifold, we may regard it as a bordism from the empty $(n-1)$-manifold to itself. Applying the functor Z then gives a linear map $Z(M) : Z(\emptyset) \to Z(\emptyset)$. Since Z is a monoidal functor, the vector space $Z(\emptyset)$ is canonically isomorphic to the ground field $\mathbb{K}$. Thus, we obtain a linear endomorphism of the ground field, which is nothing else than a scalar. In physical terms, the scalar $Z(M)$ should be interpreted as the output of the Feynman path integral. This can be made precise in topological field theories with finite gauge group as in Dijkgraaf-Witten theories, where the path integral reduces to a finite sum, cf. [FQ93].

Furthermore, the monoidal functor Z implements two crucial properties the path integral of a quantum field theory should have: *locality* and *gluing* properties. These two properties can be understood as follows: suppose that M is a closed n-manifold with a decomposition $M \cong M_0 \sqcup_N M_1$ of two other manifolds M_0 and M_1 along a closed submanifold N of codimension one. Since our topological field theory Z is a symmetric monoidal functor, the invariant $Z(M)$ can be computed in terms of $Z(M_0)$, $Z(M_1)$ and $Z(N)$. Thus, we might imagine "cutting up" our manifold M into simple pieces and computing $Z(M)$ in terms of this decomposition.

Unfortunately, this method becomes more difficult as the dimension of M grows. If M is of large dimension, it is generally not possible to simplify M much by cutting along submanifolds of codimension one. Thus, describing bordism categories of high-dimensional manifolds in terms of generators and relations becomes rather complicated, cf. [Juh14]. What we would really like to do is to iteratively cut up M into submanifolds of arbitrary codimension, and to recover the invariant $Z(M)$ in terms of this more general decomposition. This brings us to *fully-extended* or *fully-local* theories.

In order to deal with these theories one employs higher categories: one might try to define an n-dimensional fully-extended topological field theory to be a symmetric monoidal n-functor Z between an n-category of cobordisms and a suitable algebraic target n-category. If this was possible, then Z would be determined by a very small amount of data.

For instance, it is easy to see that a 1-dimensional topological field theory with values in the category of vector spaces is fully determined by a finite-dimensional vector space, which is given by evaluating the functor Z on a point. One might hope that this is true in general, and that a fully-extended n-dimensional topological field theory is determined by its value on a single point. Equivalently, one might be tempted to conjecture that the n-category of bordisms is "freely generated by a point". However, there are two more crucial ingredients in the conjecture that we have neglected so far:

1. As 1-dimensional field theories with values in vector spaces are completely determined by a single *finite-dimensional* vector space, we should expect that fully-extended field theories of higher dimensions with values in a higher category $\mathcal{C}$ are classified by objects in $\mathcal{C}$ which have to obey suitable finiteness conditions. An object satisfying these conditions is called *fully-dualizable*. A precise definition in

the bicategorical setting appears in section 4.1.

2. Experience with three dimensional topological field theories, as for instance considered in [RT91], has shown that it is beneficial to work with *framed* bordisms, instead of oriented ones. These framed bordisms possess the additional structure of a trivialization of the tangent bundle.

This brings us to the cobordism hypothesis as originally formulated by Baez and Dolan in [BD95]. They conjecture that the n-dimensional framed bordism category is equivalent to “the free stable weak n-category with duals on one object”, which implies that there is a bijection between framed fully-extended topological quantum field theories with values in a symmetric monoidal n-category $\mathcal{C}$, and dualizable objects of $\mathcal{C}$.

In order to prove this conjecture, one would have to define algebraic models of weak n-categories. However even for $n = 3$, the definition of a tricategory as in [GPS95] is rather unwieldy. In the 2-dimensional setting, the cobordism hypothesis for framed manifolds is proven in [Pst14]: framed, fully-extended 2-dimensional topological field theories with values in a symmetric monoidal bicategory $\mathcal{C}$ are classified by the core of fully-dualizable objects of the target.

A more homotopical approach to higher categories is the language of (∞, n)-categories: these are supposed to have a layer of k-morphism for every natural number k, which are invertible in an appropriate sense if $k > n$. Using (∞, n)-categories, Lurie gives an extensive sketch of a proof of the cobordism hypothesis in [Lur09b]. In this language, the cobordism hypothesis for framed manifolds is formulated as an equivalence of $(\infty, 0)$-categories

$$\mathrm{Fun}_{\otimes}(\mathrm{Cob}_n^{\mathrm{fr}}, \mathcal{C}) \cong \mathscr{K}(\mathcal{C}^{\mathrm{fd}}) \tag{0.1}$$

between fully-extended n-dimensional $\mathcal{C}$-valued framed topological field theories and the core of fully-dualizable objects of the symmetric monoidal (∞, n)-category $\mathcal{C}$.

In this thesis, we will avoid the language of (∞, n)-categories, and work with bi- and tricategories, which are more algebraic in nature. This allows us to exhibit equivalences as in equation (0.1) in two dimensions very explicitly. At the same time, we gain a new perspective on classical algebraic structures such as symmetric Frobenius algebras. These arise as homotopy fixed points of group actions on bicategories, which we discuss next.

Group actions on higher categories

In order to extend the cobordism hypothesis for framed manifolds as in equation (0.1) to manifolds with more geometric structure – say an orientation – we come to the second main player of this thesis: group actions on higher categories. We begin with the following observation: by definition, a framing on an n-manifold M is a trivialization $TM \to \mathbb{R}^n$ of its tangent bundle. By “rotating the framings”, we obtain an $O(n)$-action on the set of all framings. Now, the action on the set of all framing should induce an action on the higher category of framed bordisms. By precomposition, one obtains an action on $\mathrm{Fun}_{\otimes}(\mathrm{Cob}_n^{\mathrm{fr}}, \mathcal{C})$. Using the cobordism hypothesis for framed manifolds as in equation (0.1) yields an $O(n)$-action on the groupoid of fully-dualizable objects. Since the cobordism hypothesis for framed manifolds is not very explicitly available in the language of (∞, n)-categories, the $O(n)$-action on the core of fully-dualizable objects is also not directly given. In this thesis, we remedy the situation in the 2-dimensional case by giving a very explicit definition of the action in question in the language of symmetric monoidal bicategories.

Now, a relatively formal argument in the language of (∞, n)-categories proves the cobordism hypothesis for oriented manifolds. According to [Lur09b], "reducing the structure group" along the inclusion $SO(n) \hookrightarrow O(n)$ shows that there is an equivalence of $(\infty, 0)$-categories

$$\mathrm{Fun}_{\otimes}(\mathrm{Cob}_n^{\mathrm{or}}, \mathcal{C}) \cong \mathscr{K}(\mathcal{C}^{\mathrm{fd}})^{SO(n)} \tag{0.2}$$

between $\mathcal{C}$-valued oriented topological field theories, and homotopy fixed points of the $SO(n)$-action on the core of fully-dualizable objects of the symmetric monoidal (∞, n)-category $\mathcal{C}$. Again, as this equivalence arises due to a formal argument, it is not very explicitly available.

This thesis is concerned with making these statements explicit in the 2-dimensional case, using the language of symmetric monoidal bicategories. First of all, we set the stage by giving an explicit definition an action of a topological group on a bicategory in section 2.2. Then, we turn to homotopy fixed points of such actions. We show as a first main result in corollary 2.36 that homotopy fixed points of the *trivial* $SO(2)$-action on the core of fully-dualizable objects of the Morita bicategory Alg_2 are equivalent to semisimple symmetric Frobenius algebras.

As a second main result, we define an $SO(2)$-action on an algebraic skeleton of the framed bordism bicategory introduced by [Pst14] in chapter 4, and give an explicit description of the bicategory of homotopy fixed points of the induced action on the core of fully-dualizable objects of a symmetric monoidal bicategory $\mathcal{C}$. Finally, we show in section 5 that these homotopy fixed points classify $\mathcal{C}$-valued 2-dimensional fully-extended oriented topological field theories. In the following, we give a more detailed overview of this thesis.

Frobenius algebras as homotopy fixed points

While fixed points of a group action on a set form an ordinary subset, homotopy fixed points of a group action on a category as considered in [Kir02, EGNO15] provide additional structure.

In this thesis, we take one more step on the categorical ladder by considering group actions on bicategories. Furthermore, we also consider topological groups: given a topological group G, the fundamental 2-groupoid of G is a 3-group. We provide a detailed definition of an action of this 3-group on an arbitrary bicategory $\mathcal{C}$, and construct the bicategory of homotopy fixed points $\mathcal{C}^G$ of the action. Contrarily from the case of ordinary fixed points of group actions on sets, the bicategory of homotopy fixed points $\mathcal{C}^G$ is strictly "larger" than the bicategory $\mathcal{C}$. Hence, the usual fixed-point condition is promoted from a property to a structure.

We begin by studying homotopy fixed points of the *trivial* $SO(2)$-action on the core of fully-dualizable objects of Alg_2, which is the bicategory of algebras, bimodules and intertwiners. The reason for this is as follows: according to [Dav11, Proposition 3.2.8], the induced action on the core of fully-dualizable objects of Alg_2 which comes from "rotating the framing" on the framed bordism bicategory is actually trivializable. Hence, instead of considering the action coming from the framing, we may equivalently study the *trivial* $SO(2)$-action on $\mathrm{Alg}_2^{\mathrm{fd}}$. It is claimed in [FHLT10, Example 2.13] that the additional structure of a homotopy fixed point of the action on $\mathscr{K}(\mathrm{Alg}_2^{\mathrm{fd}})$ should be given by the structure of a symmetric Frobenius algebra.

In order to prove that symmetric Frobenius algebras are homotopy fixed points of the trivial $SO(2)$-action, we prove a more general result: the rather technical theorem 2.34 computes the bicategory of homotopy fixed points of an arbitrary $SO(2)$-action on an arbitrary bicategory $\mathcal{C}$. As a consequence, we obtain an explicit description of the bicategory of homotopy fixed points of the trivial $SO(2)$-action in theorem 2.35. This leads to the following theorem:

Theorem 1 (Corollary 2.36)**.** *Consider the trivial $SO(2)$-action on the core of fully-dualizable objects of* Alg_2*. Then, the bicategory of homotopy fixed points of this action is equivalent to the bigroupoid* Frob *of semisimple symmetric Frobenius algebras:*

$$\mathscr{K}(\mathrm{Alg}_2^{\mathrm{fd}})^{SO(2)} \cong \mathrm{Frob}\,. \tag{0.3}$$

Thus, unlike fixed points of the trivial action on a set, homotopy fixed points of the trivial $SO(2)$-action on $\mathscr{K}(\mathrm{Alg}_2^{\mathrm{fd}})$ are actually interesting, and come equipped with the additional structure of a symmetric Frobenius algebra. This theorem is a step towards the cobordism hypothesis for oriented manifolds, since the bigroupoid Frob classifies fully-extended, oriented 2-dimensional topological field theories with target Alg_2 by work of Schommer-Pries in [SP09].

Calabi-Yau categories and equivariant functors

In the second chapter of this thesis, we introduce Calabi-Yau categories as originally considered in [MS06]. These are finite, linear categories which have a Frobenius algebra structure on Hom-spaces. If Vect_2 is the bicategory of linear abelian categories, linear functors and natural transformations, we show the following algebraic result:

Theorem 2 (Corollary 3.12)**.** *Consider the trivial $SO(2)$-action on the core of fully-dualizable objects of* Vect_2*. Then, the bicategory of homotopy fixed points of this action is equivalent to the bigroupoid* CY *of Calabi-Yau categories:*

$$\mathscr{K}(\mathrm{Vect}_2^{\mathrm{fd}})^{SO(2)} \cong \mathrm{CY}\,. \tag{0.4}$$

The next part of this chapter deals with relating the two theorems above to each other. By [BDSPV15, Appendix A], the weak 2-functor $\mathrm{Rep} : \mathrm{Alg}_2^{\mathrm{fd}} \to \mathrm{Vect}_2^{\mathrm{fd}}$ which sends a separable algebra to its category of finitely-generated modules is an equivalence between the fully-dualizable objects of the bicategories Alg_2 and Vect_2. We extend this result to

Frobenius algebras and Calabi-Yau categories by showing that the category of finitely-generated representations of a separable, symmetric Frobenius algebra carries a canonical structure of a finite, semisimple Calabi-Yau category. The Calabi-Yau structure on the representation category is given by the composite of the Hattori-Stallings trace with the Frobenius form. This allows us to construct a 2-functor

$$\mathrm{Rep}^{\mathrm{fg}} : \mathrm{Frob} \to \mathrm{CY} \tag{0.5}$$

between the bigroupoid of Frobenius algebras Frob, and the bigroupoid of Calabi-Yau categories CY. We then show:

Theorem 3 (Theorem 3.37)**.** *The weak 2-functor* $\mathrm{Rep}^{\mathrm{fg}} : \mathrm{Frob} \to \mathrm{CY}$ *is an equivalence of bigroupoids.*

We are now in the following situation: theorem 1, theorem 2 and theorem 3 give three equivalences of bicategories in the following diagram:

$$\begin{tikzcd} \mathscr{K}(\mathrm{Alg}_2^{\mathrm{fd}})^{SO(2)} \arrow[r, dashed, "?"] \arrow[d, "\text{Theorem 1}"'] & \mathscr{K}(\mathrm{Vect}_2^{\mathrm{fd}})^{SO(2)} \arrow[d, "\text{Theorem 2}"] \\ \mathrm{Frob} \arrow[r, "\text{Theorem 3}"'] & \mathrm{CY} \end{tikzcd} \tag{0.6}$$

The question is now if there is a canonical arrow $\mathscr{K}(\mathrm{Alg}_2^{\mathrm{fd}})^{SO(2)} \to \mathscr{K}(\mathrm{Vect}_2^{\mathrm{fd}})^{SO(2)}$ which makes the diagram commute. In order to answer this question, we introduce the new concept of a "G-equivariant structure" on a weak 2-functor $F : \mathcal{C} \to \mathcal{D}$ between two bicategories endowed with the action of a topological group. We show that a 2-functor with such a G-equivariant structure induces a 2-functor $F^G : \mathcal{C}^G \to \mathcal{D}^G$ on homotopy fixed point bicategories. As an application, we consider the trivial action of the topological group $SO(2)$ on the core of fully-dualizable objects of Alg_2 and Vect_2. Since the action is trivial, the 2-functor sending an algebra to its category of representations has a canonical $SO(2)$-equivariant structure. Thus, it induces a 2-functor

$$\mathrm{Rep}^{SO(2)} : \mathscr{K}(\mathrm{Alg}_2^{\mathrm{fd}})^{SO(2)} \to \mathscr{K}(\mathrm{Vect}_2^{\mathrm{fd}})^{SO(2)} \tag{0.7}$$

on homotopy fixed points. Our second result in this section shows that this induced functor $\mathrm{Rep}^{SO(2)}$ fits into the diagram in equation (0.6). More precisely, we show the following:

Theorem 4 (Theorem 3.41)**.** *Let* $\mathrm{Rep}^{\mathrm{fg}} : \mathrm{Frob} \to \mathrm{CY}$ *be the equivalence of bigroupoids constructed by hand in theorem 3.37, and let* $\mathrm{Rep}^{SO(2)}$ *be the weak 2-functor in equation* (0.7)*. Then, the diagram of weak 2-functors*

$$\begin{tikzcd} (\mathscr{K}(\mathrm{Alg}_2^{\mathrm{fd}}))^{SO(2)} \arrow[r, "\mathrm{Rep}^{SO(2)}"] \arrow[d, "\wr"'] & (\mathscr{K}(\mathrm{Vect}_2^{\mathrm{fd}}))^{SO(2)} \arrow[d, "\wr"] \\ \mathrm{Frob} \arrow[r, "\mathrm{Rep}^{\mathrm{fg}}"'] & \mathrm{CY} \end{tikzcd} \tag{0.8}$$

commutes up to a pseudo-natural isomorphism.

These results are related to topological quantum field theories as follows: the 2-dimensional cobordism hypothesis for framed manifolds, which has been proven in the language of symmetric monoidal bicategories in [Pst14], asserts that a framed, fully-extended, 2-dimensional topological quantum field theory is classified by its value on the positively framed point. However, one needs more data to classify *oriented* theories, which is given by the datum of an homotopy fixed point of a certain $SO(2)$-action on the target bicategory. We will prove this statement in chapter 5.

Hence, in the language of topological quantum field theory, the equivalence between fully-dualizable objects of Alg_2 and Vect_2 as proven in [BDSPV15, Appendix A] shows that framed 2-dimensional topological quantum field theories with target space Alg_2 are equivalent to field theories with target Vect_2. Our results now imply that this equivalence of *framed* theories with target spaces Alg_2 and Vect_2 extends to an equivalence of *oriented* theories with target bicategories Alg_2 and Vect_2.

Calabi-Yau objects and the cobordism hypothesis for oriented manifolds

In chapter 4, we gather the ingredients for the proof of the cobordism hypothesis for oriented manifolds in two dimensions, which will be proven in chapter 5.

We first clarify the situation on the algebraic side by giving a detailed description of the $SO(2)$-action on the core of fully-dualizable objects of an arbitrary symmetric monoidal bicategory $\mathcal{C}$. This action is essentially given by the Serre automorphism: for each fully-dualizable object X of $\mathcal{C}$, the Serre-automorphism is a 1-morphism $S_X : X \to X$, which corresponds in the setting of topological field theories to the generator of $\pi_1(SO(2))$. We show that the collection of these 1-morphisms are natural with respect to 1-equivalences of $\mathcal{C}$, and thus constitute a pseudo-natural endotransformation of the identity functor on $\mathscr{K}(\mathcal{C}^{\mathrm{fd}})$. Furthermore, this pseudo-natural transformation turns out to be monoidal. This allows us to construct a *monoidal* $SO(2)$-action on the core of fully-dualizable objects.

Next, we explicitly construct an $SO(2)$-action on a skeletal version of the framed bordism bicategory. We use the description of this bicategory in terms of generators and relations as given in [Pst14].

By the cobordism hypothesis for framed manifolds, which has been proven in the setting of bicategories in [Pst14], there is an equivalence of bicategories

$$\mathrm{Fun}_{\otimes}(\mathrm{Cob}^{\mathrm{fr}}_{2,1,0}, \mathcal{C}) \cong \mathscr{K}(\mathcal{C}^{\mathrm{fd}}). \tag{0.9}$$

This equivalence allows us to transport the $SO(2)$-action on the framed bordism bicategory to the core of fully-dualizable objects of $\mathcal{C}$. We then prove in proposition 4.48 that this induced $SO(2)$-action on $\mathscr{K}(\mathcal{C}^{\mathrm{fd}})$ is given precisely by the Serre automorphism. This shows that the Serre automorphism has indeed a geometric origin, as expected from [Lur09b].

In chapter 5, we prove the cobordism hypothesis for 2-dimensional, oriented manifolds. In fact, we prove a slightly more general result: first of all, we define Calabi-Yau objects in an arbitrary symmetric monoidal bicategory $\mathcal{C}$. These generalise both symmetric Frobenius algebras and Calabi-Yau categories. In general, Calabi-Yau objects do not need to be fully-dualizable. As a first result, we show:

Theorem 5 (Theorem 5.5). *Let $\mathcal{C}$ be a symmetric monoidal bicategory. Consider the $SO(2)$-action by the Serre automorphism on $\mathscr{K}(\mathcal{C}^{\mathrm{fd}})$. Then, there is an equivalence of bicategories*

$$\mathrm{CY}(\mathcal{C}^{\mathrm{fd}}) \cong \mathscr{K}(\mathcal{C}^{\mathrm{fd}})^{SO(2)} \tag{0.10}$$

between the bicategory of fully-dualizable Calabi-Yau objects in $\mathcal{C}$ and the bicategory of homotopy fixed points of the $SO(2)$-action on $\mathscr{K}(\mathcal{C}^{\mathrm{fd}})$.

Conjecturally, non-fully-dualizable Calabi-Yau objects classify non-compact field theories, cf. [Lur09b]. However, we will not pursue non-compact theories in this thesis.

The second result in this chapter relates fully-dualizable Calabi-Yau objects to oriented 2-dimensional topological field theories. In [SP09], Schommer-Pries gives generators and relations of the oriented bordism bicategory, and proves that oriented field theories with target Alg_2 are classified by the bigroupoid Frob of semisimple symmetric Frobenius algebras. Using this presentation of the oriented bordism bicategory, we prove a stronger theorem which classifies 2-dimensional topological field theories with arbitrary target:

Theorem 6 (Theorem 5.8). *Let $\mathcal{C}$ be a symmetric monoidal bicategory. Then, there is an equivalence*

$$\mathrm{Fun}_{\otimes}(\mathrm{Cob}^{\mathrm{or}}_{2,1,0}, \mathcal{C}) \cong \mathrm{CY}(\mathcal{C}^{\mathrm{fd}}) \tag{0.11}$$

between fully-extended 2-dimensional $\mathcal{C}$-valued oriented topological quantum field theories and fully-dualizable Calabi-Yau objects in $\mathcal{C}$.

Combining these two theorems with the results of the previous chapter yields the cobordism hypothesis for oriented manifolds in dimension two:

Corollary (Corollary 5.9). *Let $\mathcal{C}$ be a symmetric monoidal bicategory, and consider the $SO(2)$-action on $\mathscr{K}(\mathcal{C}^{\mathrm{fd}})$ by the Serre automorphism. Then, there is an equivalence of bigroupoids*

$$\mathrm{Fun}_{\otimes}(\mathrm{Cob}^{\mathrm{or}}_{2,1,0}, \mathcal{C}) \cong (\mathscr{K}(\mathcal{C}^{\mathrm{fd}}))^{SO(2)}. \tag{0.12}$$

Summarizing, the main results of this thesis are theorem 2.34 which allows us to identify homotopy fixed points of the trivial $SO(2)$-action with semisimple symmetric Frobenius algebras, together with theorems 5.5 and 5.8 which allow us to prove the cobordism hypothesis for oriented, 2-dimensional manifolds.

Publications

The chapters of this thesis are based on the following publications and preprints:

Chapter 2: J. Hesse, C. Schweigert, and A. Valentino, *Frobenius algebras and homotopy fixed points of group actions on bicategories*, Theory Appl. Categ. **32** (2017), no. 18, 652–681. arXiv:1607.05148

Chapter 3: J. Hesse, *An equivalence between Frobenius algebras and Calabi-Yau categories*, Accepted for publication in J. Homotopy Relat. Struct. (2017). doi:10.1007/s40062-017-0181-3. arXiv:1609.06475

Chapter 4: J. Hesse and A. Valentino, *The Serre automorphism via homotopy actions and the Cobordism Hypothesis for oriented manifolds*, ArXiv e-prints (2017). arXiv:1701.03895

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1. Preliminaries

In this chapter, we introduce the main players of the thesis. We begin by highlighting the main features of symmetric monoidal bicategories. We then comment on fully-dualizable objects in symmetric monoidal bicategories, and introduce the ideas underlying the cobordism hypothesis for 2-dimensional manifolds. This chapter does not contain original material and is meant to serve as a gentle introduction to the more technical parts of the thesis. References for symmetric monoidal bicategories and fully-dualizable objects in symmetric monoidal bicategories include [SP09], [Pst14]. The theory of group action on categories already appeared in [Kir02] and is expanded in [EGNO15]. The section concerning the cobordism hypothesis is based on [BD95] and [Lur09b].

1.1. Symmetric monoidal bicategories

We begin by introducing the necessary background of symmetric monoidal bicategories. We assume familiarity with the definitions of a category, as well as functors and natural transformations.

Recall that a braided monoidal category consist a category $\mathcal{C}$, a functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, together with a unit object $1_{\mathcal{C}}$ and natural isomorphisms

$$\begin{aligned} \gamma_{X,Y} &: X \otimes Y \xrightarrow{\sim} Y \otimes X && \text{called braiding,} \\ a_{X,Y,Z} &: X \otimes (Y \otimes Z) \xrightarrow{\sim} (X \otimes Y) \otimes Z && \text{called associator,} \\ r_X &: X \otimes 1_{\mathcal{C}} \xrightarrow{\sim} X && \text{called right unitor,} \\ l_X &: X \xrightarrow{\sim} 1_{\mathcal{C}} \otimes X && \text{called left unitor} \end{aligned} \tag{1.1}$$

for each object X, Y and Z of $\mathcal{C}$. The isomorphisms in equation (1.1) are then required to fulfil certain coherence conditions. Schematically, these coherence conditions are given by

- a pentagon axiom for the associators,
- two triangle diagrams for the unitors,
- another triangle axiom relating the braiding and the unitors,
- and two hexagon diagrams relating the braiding and the associators.

A braided monoidal category $\mathcal{C}$ is called symmetric if the braiding $\gamma_{X,Y}$ additionally fulfils the inverse law $\gamma_{Y,X} \circ \gamma_{X,Y} = \text{id}$. For complete definitions, see either [BK01] or [Lan98]. The prime example of a symmetric monoidal category is the category of vector spaces with the usual tensor product. Other examples include the category of sets where natural choices of the monoidal product are either given by the disjoint union or by the cartesian product. Furthermore, the category of cobordisms is symmetric monoidal with respect to disjoint union.

One central result concerning monoidal categories is Mac Lanes's coherence theorem [Lan98], which states that every diagram in a monoidal category which is made up of associators and unitors commutes automatically. Thus, every monoidal category is equivalent to a strict monoidal category, where the associator and the unitors can be taken to be identities. However, coherence for higher categories is more complicated: we will see that although every bicategory is equivalent to a strict 2-category, not every tricategory is equivalent to a strict 3-category.

We now step up the categorical ladder and give the data underlying a bicategory. Informally speaking, a bicategory has an additional layer of morphisms. Instead of only considering morphisms between objects (which will be called 1-morphisms), we introduce additional morphisms between 1-morphisms, which will be called 2-morphisms. More formally, a bicategory $\mathcal{C}$ consists of the following collection of data:

- a class of objects $\mathrm{Ob}(\mathcal{C})$,
- for each pair of objects X and Y of $\mathcal{C}$, a category $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ with identity object id_X. The objects of $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ are called 1-morphisms of $\mathcal{C}$, while the morphisms of $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ are called 2-morphisms of $\mathcal{C}$.
- For each triple of objects X, Y, Z, a functor
$$c_{X,Y,Z} : \mathrm{Hom}_{\mathcal{C}}(X,Y) \times \mathrm{Hom}_{\mathcal{C}}(Y,Z) \to \mathrm{Hom}_{\mathcal{C}}(X,Z)$$
called horizontal composition,
- a family of natural isomorphisms called associators, making composition associative up to natural isomorphism,
- a family of natural isomorphisms called unitors, making the identity objects id_X neutral elements of the composition up to natural isomorphism.

The associators are then required to fulfil the pentagon identity, while there is the usual triangle axiom for the unitors. Similarly to ordinary categories, one introduces to notion of "functors" between bicategories. If $\mathcal{C}$ and $\mathcal{D}$ are bicategories, a weak 2-functor $F : \mathcal{C} \to \mathcal{D}$ consists of the following data:

- a map $F : \mathrm{Ob}(\mathcal{C}) \to \mathrm{Ob}(\mathcal{D})$,
- a family of functors $F_{X,Y} : \mathrm{Hom}_{\mathcal{C}}(X,Y) \to \mathrm{Hom}_{\mathcal{D}}(FX,FY)$ for each pair of objects X and Y of $\mathcal{C}$,
- a family of natural isomorphisms making the functor $F_{X,Y}$ compatible with the (weak) associative law,

- a family of natural isomorphisms making the functor $F_{X,X}$ compatible with the identities id_X in a weak sense.

Furthermore, the existence of certain modifications is required, which act as higher coherence cells between the natural transformation. Again, we will not spell out the coherence conditions and refer to the fourth chapter of [Bén67] for a detailed definition.

We now raise up to the challenge of defining *monoidal* bicategories. By comparing definitions, one realizes that a bicategory with one object is exactly a monoidal category. Thus, we could have defined a monoidal category to be a bicategory with one object. This also works in higher categorical dimensions: Schommer-Pries [SP09] defines a monoidal bicategory to be a tricategory with one object, while monoidal tricategories are defined to be quadcategories with one object in [Hof11]. Now, we need one more piece of data, which is the braiding. Braided monoidal bicategories have already appeared in [McC00].

Let us spell out these definitions a bit more detail: as one would expect, a symmetric monoidal bicategory consists of a bicategory $\mathcal{C}$, together with a weak 2-functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, a unit object $1_{\mathcal{C}}$ and natural transformations similar to the isomorphisms in equation (1.1). At this point, we observe an important feature which often turns up in categorification: we will turn a *property* into a *structure*. Instead of requiring conditions at the level of 1-morphisms as in the definition of a monoidal category, we have to provide additional 2-cells which sit in the diagrams of 1-cells of the data in equation (1.1). This additional data consists of seven modifications:

- a modification π inside the pentagon diagram for the coherence of associators,
- a modification λ in the triangle diagram for the left unitors,
- a modification ρ in the triangle diagram for the right unitors,
- a modification μ, making the left unitors compatible with the right unitors,
- two modifications R and S in the hexagon diagrams concerning the braiding,
- and an additional modification σ which weakens the inverse law of the symmetric braiding.

For a detailed definition, we refer to [SP09, Definition 2.1]. Now, one has to come up with the right coherence axioms for this data. A priori, it is not clear which axioms the data of a symmetric monoidal bicategory, or equivalently the data of a tricategory with one object, should satisfy. This problem gets more and more extreme as the categorical dimension grows. While trying to work out a suitable definition of an ∞-groupoid, Grothendieck wrote in a letter to Daniel Quillen:

> Here one seems caught at first sight in an infinite chain of ever "higher", and presumably, messier structures, where one is going to get hopelessly lost, unless one discovers some simple guiding principle for shedding some clarity in the mess. ([Gro83])

At least for low categorical dimensions, the “simple guiding principle” Grothendieck speaks about seems to be given by the polyhedra of Stasheff [Sta63]: these are a series of polytopes whose vertices enumerate the way to parenthesize the tensor product of n objects. Thus, they can be used as a guiding principle to come up with coherence axioms related to associativity. For instance, in the case of tricategories, the associativity axiom (HTA1) of [GPS95] is recognizable as algebraic incarnation of the Stasheff polytope K_5 (picture taken from [Dev12]) as in figure 1.1. For a beautiful exposition including many pictures, see [Sta16]. Work of Trimble [Tri95] which is expanded in [Hof11] explains how the tricategorical conditions governing unitality can also be deduced from the associahedra of Stasheff. This gives an ansatz for finding the appropriate coherence conditions for tricategories, and thus for symmetric monoidal bicategories.

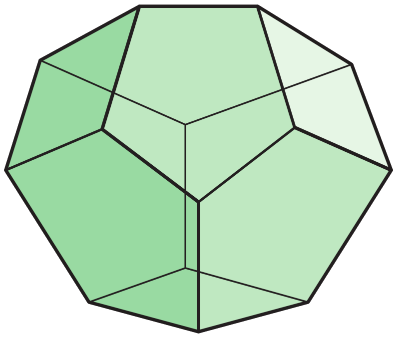


Figure 1.1.: The K_5 associahedron of Stasheff

We now come to coherence for tricategories: unlike in the case for monoidal categories, *not* every tricategory is equivalent to a strict 3-category. The coherence statement for tricategories of [GPS95] states instead that every tricategory is triequivalent to a Gray-category, which is a certain kind or semi-strict 3-category in which composition is strictly associative and unital, but the interchange law only holds up to isomorphism. Thus, it is unreasonable to expect that symmetric monoidal bicategories, which are by definition tricategories with one object, can be completely strictified. However, due to work of Schommer-Pries in [SP09], every symmetric monoidal bicategory can be strictified to a “quasistrict” symmetric monoidal bicategory, in which most, but not all, coherence data is trivial. This theorem allows us to introduce a graphical calculus for symmetric monoidal bicategories as explained in [Bar14]. We will use this graphical calculus in chapters 4 and 5, where we work with the framed cobordism bicategory.

1.2. Fully-dualizable objects

In order to motivate the notion of "fully-dualizable objects" which we will use in chapter 4, we begin by giving an exposition to duals in monoidal categories. This material is well-known and rather standard, cf. [BK01] and [Pst14].

In chapter 4, we will generalize the discussion to symmetric monoidal bicategories. Although the case of symmetric monoidal bicategories is more laborious due to the more complicated coherence axioms, one of the main ideas already appears in the setting for symmetric monoidal categories. Formulated as a slogan, one might say that "the space of duality data is contractible". By this, we mean that the choice of duality data for a dualizable object in a symmetric monoidal category is unique up to unique isomorphism. The bicategorical version of these statements will play an important role in the sequel. Let us begin with the more basic version for duals in monoidal categories:

Definition 1.1. Let $\mathcal{C}$ be a symmetric monoidal category. A dual pair consists of an object X, an object X^* which we call the (left) dual of X, and two morphisms

$$\begin{aligned} \mathrm{ev}_X &: X \otimes X^* \to 1 \\ \mathrm{coev}_X &: 1 \to X^* \otimes X \end{aligned} \tag{1.2}$$

so that two triangle equations are satisfied. We call an object X of $\mathcal{C}$ which admits a choice of dual pair dualizable.

One might wonder whether dualizability is a property or a structure. In order to settle this question, we proceed as follows: it is not difficult to define the notion of a morphism between dual pairs, and thus one arrives at the notion of a category of dual pairs in a symmetric monoidal category $\mathcal{C}$, which we shall denote by $\mathrm{DualPair}(\mathcal{C})$. By [Pst14, Theorem 1.6], this category of duals pairs is actually a groupoid, and thus every morphism between dual pairs is invertible. Furthermore, this groupoid is contractible: the forgetful functor to the maximal subgroupoid $\mathscr{K}(\mathcal{C}^d)$ of dualizable objects of $\mathcal{C}$

$$\begin{aligned} \mathrm{DualPair}(\mathcal{C}) &\to \mathscr{K}(\mathcal{C}^d) \\ (X, X^*, \mathrm{ev}_X, \mathrm{coev}_X) &\mapsto X \end{aligned} \tag{1.3}$$

is an equivalence of categories. Thus, the additional structure of a dual pair is "property-like": if an object is dualizable, every choice of additional duality data is equivalent.

In order to generalize this statement to bicategories, we need a suitable definition of a dual pair in a symmetric monoidal bicategory. If $\mathcal{C}$ is a symmetric monoidal bicategory, one defines a dual pair to consist of an object X of $\mathcal{C}$, its dual X^*, two 1-morphisms as in equation (1.2), and two additional 2-cells living in the usual triangle diagrams. However, this naive definition of a dual pair in a symmetric monoidal bicategory does not satisfy an analogous version of the above theorem. Therefore, one has to restrict to a suitable class of dual pairs which satisfies additional coherence equations, cf. [Pst14, Section 2]. These dual pairs are called *coherent*. One then sets up a bicategory of coherent dual pairs, and proves that this bicategory is actually a 2-groupoid, which is furthermore contractible.

We now come to the stronger condition of fully-dualizability: the idea is to put additional conditions on the evaluation and coevaluation maps. It turns out that the correct notion of fully-dualizability is to require the existence of all adjoints, meaning that both evaluation and coevaluation have left- and right adjoints which in turn have both adjoints of their on, and so on. In the bicategorical setting however, it is sufficient to require the existence of a left- and right adjoint of the evaluation and coevaluation: if these exist, the left- and right adjoints will have adjoints themselves automatically by [Pst14, Theorem 3.9]. In this case, we call the collection of duality data, together with the adjoints of the evaluation and the coevaluation a fully-dual pair. If an object X in a symmetric monoidal bicategory can be completed into such a fully-dual pair, it is called fully-dualizable. Again, one needs to restrict to coherent fully-dual pairs by requiring an additional coherence equation to show that the groupoid of *coherent* fully-dual pairs is contractible. These coherent fully-dual pair will feature as the bicategory $\mathbb{F}_{cfd}$ in chapter 4.

1.3. Group actions on bicategories

Next, we come to the concept of group actions on (bi)-categories. Recall that if X is a set, and G is a group, a G-action on X is a group homomorphism $\rho : G \to \mathrm{Aut}(X)$. A fixed point of this G-action is an element x of X, satisfying $g.x = x$ for all group elements g of G.

Generalizing the definition of a G-action on a set appropriately leads to the following definition: if $\mathcal{C}$ is a category, we denote by $\mathrm{Aut}(\mathcal{C})$ the category of auto-equivalences of $\mathcal{C}$. If $\mathcal{C}$ is a monoidal category, we write $\mathrm{Aut}_{\otimes}(\mathcal{C})$ for the category of monoidal auto-equivalences. Furthermore, let $\underline{G}$ be the discrete monoidal category with G as objects, and only identity morphisms. We then define:

Definition 1.2. An action of a group G on a category $\mathcal{C}$ is a monoidal functor $\underline{G} \to \mathrm{Aut}(\mathcal{C})$. A monoidal group action on a monoidal category $\mathcal{C}$ is a monoidal functor $\underline{G} \to \mathrm{Aut}_{\otimes}(\mathcal{C})$.

Unpacking this definition shows that a G-action on a category $\mathcal{C}$ consists of an equivalence of categories $F_g := \rho(g) : \mathcal{C} \to \mathcal{C}$ for each $g \in G$, as well as natural isomorphisms $\gamma_{g,h} : F_g \circ F_h \to F_{gh}$, satisfying the usual axioms of a monoidal functor.

In order to give an appropriate notion of a fixed point of such an action, we have to be a bit careful: requiring equations at the level of objects like the fixed-point condition is considered to be "evil", since it breaks the principle of equivalence. Thus, we only consider objects to be isomorphic instead of equal, and remember the choice of isomorphism. This leads to the following as considered in [Kir01, Kir02, Kir04]. Here, we follow a more modern exposition.

Definition 1.3 ([EGNO15, Definition 2.7.2])**.** A G-equivariant object for a G-action on a category $\mathcal{C}$ consists of an object X of $\mathcal{C}$, together with an isomorphisms $u_g : F_g(X) \to X$ for every $g \in G$, so that the following diagram commutes:

$$\begin{tikzcd} F_g(F_h(X)) \arrow[r, "F_g(u_h)"] \arrow[d, "\gamma_{g,h}"'] & F_g(X) \arrow[d, "u_g"] \\ F_{gh}(X) \arrow[r, "u_{gh}"'] & X \end{tikzcd} \tag{1.4}$$

It is now possible to define a whole category $\mathcal{C}^G$ of equivariant objects of a G-action on a category, where the objects are given as in definition 1.3, and the morphisms are given by morphisms in $\mathcal{C}$ which commute with the trivializing isomorphisms u_g. This category is called the “equivariantization” in [EGNO15].

By unpacking definitions, one shows:

Example 1.4. Consider the *trivial* G-action on the category of finite-dimensional vector spaces. Then, there is an equivalence between the category of G-equivariant objects of Vect^G and $\mathrm{Rep}(G)$, the category of representations of G.

Indeed, a homotopy fixed point of the trivial G-action on Vect consists of a vector space V, together with a family of linear maps $u_g : V \to V$ for any $g \in G$. The commutative diagram in equation (1.4) then demands that the map $g \to u_g$ is a group homomorphism.

As another example, let us mention the following:

Example 1.5. Let G be a finite group. Then, G acts on Vect_G, the category of G-graded vector spaces, by conjugation. The category of homotopy fixed points Vect_G^G is equivalent to the Drinfeld center of Vect_G.

Indeed, a homotopy fixed point of this action consists of a family of isomorphisms

$$u_h : \bigoplus_{g\in G} V_{hgh^{-1}} \to \bigoplus_{g\in G} V_g \tag{1.5}$$

which is nothing else than a choice of isomorphism $V_{hg} \cong V_{gh}$. This reproduces the definition of the Drinfeld center of Vect_G.

Next, we generalize these definitions to bicategories. In order to do this, we have to reformulate the definition of G-equivariant object in a more categorical manner and introduce a bit more notation: for a group G, we denote with BG the category with one object and G as morphisms. Similarly, if $\mathcal{C}$ is a monoidal category, $B\mathcal{C}$ will denote the bicategory with one object and $\mathcal{C}$ as the endomorphism category of this object.

Now note that in equivalent, but more categorical terms, a G-action on a set X can be defined to be a functor $\rho : BG \to \mathrm{Set}$ which sends the one object of the category BG to the set X. Let $\Delta : BG \to \mathrm{Set}$ be the constant functor sending the one object of BG to the set with one element. Then, we claim that the set of fixed points X^G of the action ρ stands in bijection to the set of natural transformations from the constant functor Δ to ρ, which is exactly the limit of the functor ρ:

Lemma 1.6. *Let $\rho : BG \to \mathrm{Set}$ be a functor with $\rho(*) = X$, and let $\Delta : BG \to \mathrm{Set}$ be the constant functor which sends the one object of BG to the 1-element set and every morphism of BG to the identity. Then there is a bijection of sets*

$$X^G \cong \lim_{BG} \rho \cong \mathrm{Nat}(\Delta, \rho). \tag{1.6}$$

Proof. Indeed, let us show that X^G, together with the inclusions $X^G \to X$ is the universal cone of the diagram ρ: first it is easy to see that the fixed point set X^G, together with the inclusion $\iota : X^G \to X$ is indeed a cone over ρ. Now, if (N, φ) is another cone over ρ, (hence $\varphi : N \to X$) note that we must have that $\varphi(x) \in X^G$, since $\rho(g)(\varphi(x)) = \varphi(x)$ as (N, φ) was supposed to be a cone. Therefore, we can define the universal map $u : N \to X^G$ to be $u(n) := \varphi(n)$.

$$\begin{tikzcd}
& N \arrow[d, dashed, "u"] \arrow[ddl, bend right, "\varphi"'] \arrow[ddr, bend left, "\varphi"] & \\
& X^G \arrow[dl, "\iota"'] \arrow[dr, "\iota"] & \\
X \arrow[rr, "\rho(g)"'] & & X
\end{tikzcd} \tag{1.7}$$

This shows that the limit of ρ stands in bijection to the fixed point set. Finally, one shows that the set $\mathrm{Nat}(\Delta, \rho)$ is a universal cone as well, and thus is isomorphic to the limit. $\square$

Categorifying this notion of a G-action on a set reproduces the definition of a discrete group acting on a category as introduced in definition 1.2, as one can show by unpacking definition.

Remark 1.7. Let G be a discrete group and let $\mathcal{C}$ be a category. Let $B\underline{G}$ be the 2-category with one object and $\underline{G}$ as the category of endomorphisms of the single object $*$. Furthermore, let Cat be the 2-category of categories, functors and natural transformations. A G-action on $\mathcal{C}$ as in definition 1.2 is equivalent to a weak 2-functor $\rho : B\underline{G} \to \mathrm{Cat}$ with $\rho(*) = \mathcal{C}$.

Next, we would like to define the homotopy fixed point category of this action to be a suitable limit of the action, just as in equation (1.6). The appropriate notion of a limit of a weak 2-functor with values in a bicategory appears in the literature as a *pseudo-limit* or *indexed limit*, which we will simply denote by lim. We will only consider limits indexed by the constant functor. For background, we refer the reader to [Lac10], [Kel89], [Str80] or [Fio06]. We are now in the position to introduce the following definition:

Definition 1.8. Let G be a discrete group, let $\mathcal{C}$ be a category, and let $\rho : B\underline{G} \to \mathrm{Cat}$ be a G-action on $\mathcal{C}$. Then, the category of homotopy fixed points $\mathcal{C}^G$ is defined to be the pseudo-limit of ρ.

Just as in the 1-categorical case as in equation (1.6), it is shown in [Kel89] that the limit of any weak 2-functor with values in Cat is equivalent to the category of pseudo-natural transformations and modifications $\mathrm{Nat}(\Delta, \rho)$. Hence, we have an equivalence of categories

$$\mathcal{C}^G \cong \lim \rho \cong \mathrm{Nat}(\Delta, \rho). \tag{1.8}$$

Here, $\Delta : B\underline{G} \to \mathrm{Cat}$ is the constant functor sending the one object of $B\underline{G}$ to the terminal category with one object and only the identity morphism. By spelling out definitions, one shows:

Proposition 1.9. *Let $\rho : B\underline{G} \to \mathrm{Cat}$ be a G-action on a category $\mathcal{C}$, and suppose that $\rho(e) = \mathrm{id}_{\mathcal{C}}$, i.e. the action respects the unit strictly. Then, the homotopy fixed point category $\mathcal{C}^G$ in definition 1.8 is equivalent to category of G-equivariant objects of [EGNO15], which has been introduced in definition 1.3.*

Proof. By the equivalence of categories in equation (1.8), we may assume that objects of $\mathcal{C}^G$ are pseudo-natural transformations $\Delta \to \rho$. Such a pseudo-natural transformation consists of

- a functor $\Theta(*) : \Delta(*) \to \rho(*) = \mathcal{C}$ which is uniquely specified by its image of the one object of $\Delta(*)$ and thus is equivalently given by just an object $x = \Theta(*)(*)$ of $\mathcal{C}$,
- for every $g \in G$, a natural isomorphism

$$\begin{tikzcd} \Delta(*) \arrow[r, "\Theta(*)"] \arrow[d, "\Delta(g)"'] & \rho(*) \arrow[d, "\rho(g)"] \arrow[dl, Rightarrow, "\Theta(g)"] \\ \Delta(*) \arrow[r, "\Theta(*)"'] & \rho(*) \end{tikzcd} \tag{1.9}$$

  which is uniquely specified on its one component $\Theta(g)_*$ and thus is equivalent to an isomorphism

$$\Theta(g)_* : F_g(x) \to x. \tag{1.10}$$

This data has to satisfy three conditions: first of all, it has to be natural with respect to 2-morphisms in BG. Since there are only identity 2-morphisms, this condition is automatically satisfied. Second, there is a compatibility condition with respect to units which has to be fulfilled due to the requirement that the action respects the units strictly. Finally, there is a compatibility condition with respect to composition of 1-morphisms in BG, which reproduces the diagram in equation (1.4). Hence, we exactly reproduced the definition of [EGNO15]. $\square$

In chapter 2, we will generalize this discussion to homotopy fixed points of group actions on bicategories. We will define the bicategory of homotopy fixed points as a suitable trilimit in the tricategory of bicategories. We will be able to explicitly describe this trilimit as a bicategory of tritransformations, trimodifications and perturbations. This explicit description allows us to actually compute homotopy fixed points.

1.4. The cobordism hypothesis

As we will prove the cobordism hypothesis for oriented, 2-dimensional manifolds in chapter 5, let us explain the basic idea behind the proof. We begin with an exposition of the 1-dimensional case. First, we recall the definition of a framing:

Definition 1.10. Let M be an n-dimensional manifold, and let $k \geq n$ be a natural number. A k-framing of M is a trivialization of the vector bundle $T^kM := TM \oplus \mathbb{R}^{k-n}$. In detail, this is a choice of k sections $s_1, \ldots, s_k$ of the stabilized tangent bundle T^kM, so that the vectors $s_1(x), \ldots, s_k(x)$ form a basis of the tangent space T^kM at every point x.

We then define the framed bordism category $\mathrm{Cob}^{\mathrm{fr}}_{1,0}$ as the symmetric monoidal category having 1-framed points as objects and (isotopy classes of) 1-framed bordisms as morphisms.

Given a symmetric monoidal category $\mathcal{C}$, the cobordism hypothesis for framed, 1-dimensional manifolds states that evaluating a symmetric monoidal functor on the positively framed point induces an equivalence of groupoids

$$\begin{aligned} \mathrm{Fun}_{\otimes}(\mathrm{Cob}^{\mathrm{fr}}_{1,0}, \mathcal{C}) &\to \mathscr{K}(\mathcal{C}^d) \\ Z &\mapsto Z(+) \end{aligned} \tag{1.11}$$

between the category of symmetric monoidal functors, and the groupoid of dualizable objects in $\mathcal{C}$. This statement consists of two essentially different parts:

1. First of all, the cobordism hypothesis states that the category of symmetric monoidal functors $\mathrm{Fun}_{\otimes}(\mathrm{Cob}^{\mathrm{fr}}_{1,0}, \mathcal{C})$ is actually a groupoid, and thus every monoidal natural transformation is invertible. In proposition 1.11 which will be proven in a relatively formal manner below, we will see that a more general statement holds.

2. The second part of the statement tells us that every symmetric monoidal functor $Z : \mathrm{Cob}^{\mathrm{fr}}_{1,0} \to \mathcal{C}$ is already determined by its value on the positively framed point, which furthermore has to be dualizable. This statement is shown in [Pst14] and relies on the fact that a dualizable object gives us enough data to define a 1-dimensional framed topological field theory. One way to show this is to notice that the 1-dimensional framed bordism category is freely generated by the dual pair given by the positively and negatively framed points, as well as the left and right elbows in figure 1.2. As the elbows can be though of "evaluation" and "coevaluation", the data of a dualizable object in $\mathcal{C}$ and a symmetric monoidal functor $\mathrm{Cob}^{\mathrm{fr}}_{1,0} \to \mathcal{C}$ actually agree.

Figure 1.2.: The right- and left elbows

To show the mathematics behind the first statement, we give a formal proof of a more general statement. A generalization to bicategories can be found in [Pst14] and [FSW11].

Proposition 1.11. *Let $\mathcal{C}$ and $\mathcal{D}$ be symmetric monoidal categories with duals, and let F, $G : \mathcal{C} \to \mathcal{D}$ be symmetric monoidal functors. Then, every monoidal natural transformation $\nu : F \Rightarrow G$ is invertible.*

Proof. As $\mathcal{C}$ is symmetric, there is a canonical isomorphism $\delta_X : X \to X^{**}$. To simplify notation, we write $d_X := \mathrm{coev}_X$ and $b_X := \mathrm{ev}_X$. Now, we claim that an inverse to $\nu_X : F(X) \to G(X)$ is given by

$$\nu_X{}^{-1} := F(\delta_X{}^{-1}) \circ G(d_{X^*}) \otimes \mathrm{id} \circ \mathrm{id} \otimes \nu_{X^*} \otimes \mathrm{id} \circ \mathrm{id} \otimes F(b_{X^*}) \circ G(\delta_X). \tag{1.12}$$

In order to see that this map is indeed an inverse, consider the diagram in figure 1.3 on page 12.

Here, the middle row is given by the map $\nu_X{}^{-1}$. By applying the functors F and G to the S-relation of duality, the compositions along the first and last row are identities. By composing the first horizontal arrow ν_X with the middle row, we see that $\nu_X{}^{-1} \circ \nu_X = \mathrm{id}_X$. Composing the middle row with the last horizontal arrow shows that $\nu_X \circ \nu_X{}^{-1} = \mathrm{id}_X$.

The diagram is commutative for the following reason: the first and last squares in the top and bottom row commute due to the naturality of ν applied to δ_X and δ_X^{-1}. The fourth square in the top row commutes due to the naturality of ν applied to the evaluation $d_{X^*} : X^{**} \otimes X^* \to 1_{\mathcal{C}}$. Similarly, the second square in the bottom row commutes by applying the naturality of ν to the coevaluation $b_{X^*} : 1_{\mathcal{C}} \to X^* \otimes X^{**}$. All other squares commute trivially.

Naturality and monoidality of ν^{-1} now follow from the naturality and monoidality of ν. Indeed, if $f : X \to Y$ is a morphism in $\mathcal{C}$, then $G(f) \circ \nu_X = \nu_Y \circ F(f)$ since ν is natural. This is equivalent to $F(f) \circ \nu_X{}^{-1} = \nu_Y{}^{-1} \circ G(f)$, which shows naturality of ν^{-1}. By a similar algebraic manipulation, one shows that ν^{-1} is indeed monoidal. □

We now come to the framed cobordism hypothesis in two dimensions: here one needs to take more care to define the framed bordism bicategory. Roughly speaking, objects of the symmetric monoidal bicategory $\mathrm{Cob}^{\mathrm{fr}}_{2,1,0}$ are given by 2-framed points, 1-morphisms are given by 2-framed 1-dimensional bordisms, and 2-morphisms are given by (isotopy-classes of) 2-framed 2-bordisms.

In [Pst14], the cobordism hypothesis for framed, 2-dimensional manifolds is proven by giving a description of the framed bordism bicategory in terms of generators and

$$
\begin{tikzcd}
F(X) \arrow[r, "F(\delta_X)"] \arrow[d, "\nu_X"] & F(X^{**}) \arrow[r, "\mathrm{id}\otimes F(b_{X^*})"] \arrow[d, "\nu_{X^{**}}"] & F(X^{**})\otimes F(X^*)\otimes F(X^{**}) \arrow[r, equal] \arrow[d, "\nu_{X^{**}}\otimes\mathrm{id}\otimes\mathrm{id}"] & F(X^{**})\otimes F(X^*)\otimes F(X^{**}) \arrow[r, "F(d_{X^*})\otimes\mathrm{id}"] \arrow[d, "\nu_{X^{**}}\otimes\nu_{X^*}\otimes\mathrm{id}"] & F(X^{**}) \arrow[r, "F(\delta_x{}^{-1})"] \arrow[d, equal] & F(X) \arrow[d, equal] \\
G(X) \arrow[r, "G(\delta_X)"] \arrow[d, equal] & G(X^{**}) \arrow[r, "\mathrm{id}\otimes F(b_{X^*})"] \arrow[d, equal] & G(X^{**})\otimes F(X^*)\otimes F(X^{**}) \arrow[r, "\mathrm{id}\otimes\nu_{X^*}\otimes\mathrm{id}"] \arrow[d, "\mathrm{id}\otimes\nu_{X^*}\otimes\nu_{X^{**}}"] & G(X^{**})\otimes G(X^*)\otimes F(X^{**}) \arrow[r, "G(d_{X^*})\otimes\mathrm{id}"] \arrow[d, "\mathrm{id}\otimes\mathrm{id}\otimes\nu_{X^{**}}"] & F(X^{**}) \arrow[r, "F(\delta_X{}^{-1})"] \arrow[d, "\nu_{X^{**}}"] & F(X) \arrow[d, "\nu_X"] \\
G(X) \arrow[r, "G(\delta_X)"] & G(X^{**}) \arrow[r, "\mathrm{id}\otimes G(b_{X^*})"] & G(X^{**})\otimes G(X^*)\otimes G(X^{**}) \arrow[r, equal] & G(X^{**})\otimes G(X^*)\otimes G(X^{**}) \arrow[r, "G(d_{X^*})\otimes\mathrm{id}"] & G(X^{**}) \arrow[r, "G(\delta_X{}^{-1})"] & G(X)
\end{tikzcd}
$$

Figure 1.3.: Every monoidal natural transformation is invertible

relations. In two dimensions, the cobordism hypothesis then states that there is an equivalence of bigroupoids

$$\mathrm{Fun}_{\otimes}(\mathrm{Cob}^{\mathrm{fr}}_{2,1,0}, \mathcal{C}) \to \mathscr{K}(\mathcal{C}^{\mathrm{fd}}) \tag{1.13}$$

between the bigroupoid of symmetric monoidal 2-functors and the core of fully-dualizable objects of the target bicategory $\mathcal{C}$. Note that unlike in the 1-dimensional case where we only had to require dualizability, 2-dimensional framed field theories are classified by the groupoid of *fully-dualizable* objects. This is due to the fact that in the bordism bicategory, there are additional 2-cells which serve as units and counits of adjunctions between the 1-morphisms.

Now, we come to the cobordism hypothesis for oriented manifolds: as an orientation is an additional piece of structure, one should expect an additional piece of information on the algebraic side. We will see that this additional structure is given by the datum of a homotopy fixed point of a certain $SO(2)$-action on the core of fully-dualizable objects.

In chapter 4, we will define a non-trivial monoidal $SO(2)$-action on an algebraic skeleton of the framed bordism bicategory, and show that it induces an $SO(2)$-action on the bicategory of symmetric monoidal functors $\mathrm{Fun}_{\otimes}(\mathrm{Cob}^{\mathrm{fr}}_{2,1,0}, \mathcal{C})$, where $\mathcal{C}$ is an arbitrary symmetric monoidal bicategory. Using the cobordism hypothesis for framed manifolds as in equation (1.13) allows us to transport the $SO(2)$-action to the core of fully-dualizable objects. This action only depends on a pseudo-natural equivalence of the identity functor and is called the Serre automorphism. Geometrically, this automorphism corresponds to the non-trivial generator of $\pi_1(SO(2)) \cong \mathbb{Z}$.

The cobordism hypothesis for oriented manifolds then states that there is an equivalence of bigroupoids

$$\mathrm{Fun}_{\otimes}(\mathrm{Cob}^{\mathrm{or}}_{2,1,0}, \mathcal{C}) \to \mathscr{K}(\mathcal{C}^{\mathrm{fd}})^{SO(2)} \tag{1.14}$$

between "representations" of the oriented bordism bicategory and homotopy fixed points of this $SO(2)$-action on the core of fully-dualizable objects of the target bicategory. This statement will be proven in chapter 5.

2. Frobenius algebras and homotopy fixed points of group actions on bicategories

In this chapter, which is based on results in [HSV17], we explicitly show that the bigroupoid of finite-dimensional, semisimple, symmetric Frobenius algebras is equivalent to the bigroupoid of homotopy fixed points of the trivial $SO(2)$-action on the core of the bicategory of finite-dimensional, semisimple algebras, bimodules and intertwiners. This result is motivated by the two-dimensional cobordism hypothesis for oriented manifolds which will be proven in chapter 5, and can hence be interpreted in the realm of topological quantum field theory.

We begin by recalling the definition of the bigroupoid Frob of symmetric Frobenius algebras in section 2.1. This involves the concept of compatible Morita contexts between symmetric Frobenius algebras. Although most of the material has already appeared in [SP09], we give full definitions to mainly fix the notation. We give a very explicit description of compatible Morita contexts between semisimple symmetric Frobenius algebras not present in [SP09], which will be needed to relate the bicategory of symmetric Frobenius algebras and compatible Morita contexts to the bicategory of homotopy fixed points of the trivial $SO(2)$-action.

In section 2.2, we recall the notion of a group action on a category and of its homotopy fixed points. By categorifying this notion, we arrive at the definition of a group action on a bicategory and its homotopy fixed points. This definition is formulated in the language of tricategories. Since we prefer to work with bicategories, we explicitly spell out the definition in remark 2.20. Although quite technical, these definitions are parts of the main results of this chapter since they allow us to give an explicit description of the bicategory of homotopy fixed points later on.

Given a weak 2-functor $F : \mathcal{C} \to \mathcal{D}$ between two bicategories endowed with the action of a topological group, we introduce the concept of an "equivariantization" of this 2-functor in section 2.3. As in the case of homotopy fixed points, this will be additional structure on the functor. We show that a 2-functor with such a G-equivariant structure induces a 2-functor $F^G : \mathcal{C}^G \to \mathcal{D}^G$ on homotopy fixed point bicategories. An application of this formalism related to Frobenius algebras and Calabi-Yau categories will be given in chapter 3.

In section 2.4, we compute the bicategory of homotopy fixed points of a certain $SO(2)$-action on an arbitrary bicategory. As a consequence, we obtain an explicit description of the bicategory of homotopy fixed points of the *trivial* $SO(2)$-action on an arbitrary bicategory. Corollary 2.36 then shows an equivalence of bicategories between the bicategory of homotopy fixed points of the trivial $SO(2)$-action on the core of fully-dualizable objects on the bicategory Alg_2, and the bigroupoid Frob introduced earlier:

$$(\mathscr{K}(\mathrm{Alg}_2^{\mathrm{fd}}))^{SO(2)} \cong \mathrm{Frob}\,. \tag{2.1}$$

We note that the bicategory Frob has also appeared in [Dav11, Proposition 3.3.2] as a certain bicategory of functors. We clarify the relationship between this functor bicategory and the bicategory of homotopy fixed points $(\mathscr{K}(\mathrm{Alg}_2^{\mathrm{fd}}))^{SO(2)}$ in remark 2.38.

Throughout the chapter, we use the following conventions: all algebras considered will be over an algebraically closed field $\mathbb{K}$. All Frobenius algebras appearing will be symmetric.

2.1. Frobenius algebras and Morita contexts

In this section we recall some basic notions regarding Morita contexts, mostly with the aim of setting up notations. We will mainly follow [SP09] and [Bas76], though we point the reader to remark 2.7 for a slight difference in the statement of the compatibility condition between the Morita context and the Frobenius forms.

Definition 2.1. A Frobenius algebra (A, λ) consists of an associative, unital $\mathbb{K}$-algebra A, together with a linear map $\lambda : A \to \mathbb{K}$, so that the pairing

$$\begin{aligned} A \otimes_{\mathbb{K}} A &\to \mathbb{K} \\ a \otimes b &\mapsto \lambda(ab) \end{aligned} \tag{2.2}$$

is non-degenerate. A Frobenius algebra is called symmetric if $\lambda(ab) = \lambda(ba)$ for all a and b in A.

Definition 2.2. Let A and B be two algebras. A Morita context $\mathcal{M}$ consists of a quadruple $\mathcal{M} := ({}_BM_A, {}_AN_B, \varepsilon, \eta)$, where ${}_BM_A$ is a (B, A)-bimodule, ${}_AN_B$ is an (A, B)-bimodule, and

$$\begin{aligned} \varepsilon &: {}_AN \otimes_B M_A \to {}_AA_A \\ \eta &: {}_BB_B \to {}_BM \otimes_A N_B \end{aligned} \tag{2.3}$$

are isomorphisms of bimodules, so that the two diagrams

$$\begin{tikzcd} {}_BM \otimes_A N_B \otimes_B M_A \arrow[r, "\mathrm{id}_M \otimes \varepsilon"] & {}_BM \otimes_A A_A \arrow[d] \\ {}_BB \otimes_B M_A \arrow[u, "\eta \otimes \mathrm{id}_M"] \arrow[r] & {}_BM_A \end{tikzcd} \tag{2.4}$$

$$\begin{tikzcd} {}_AN \otimes_B M \otimes_A N_B \arrow[d, "\varepsilon \otimes \mathrm{id}_N"] & {}_AN \otimes_B B_B \arrow[l, "\mathrm{id}_N \otimes \eta"'] \arrow[d] \\ {}_AA \otimes_A N_B \arrow[r] & {}_AN_B \end{tikzcd} \tag{2.5}$$

commute.

These two conditions are not independent from each other, as the next lemma shows.

Lemma 2.3 ([Bas68, Lemma 3.3])**.** *In the situation of definition 2.2, diagram* (2.4) *commutes if and only if diagram* (2.5) *commutes.*

Proof. First suppose that diagram (2.5) commutes. Then,

$$\begin{aligned}
\eta^{-1}((\eta^{-1}(m\otimes n).m'\otimes n') &= \eta^{-1}(m\otimes n).\eta^{-1}(m'\otimes n') && (\eta^{-1}\text{ is left }B\text{-linear})\\
&= \eta^{-1}(m\otimes n.\eta^{-1}(m'\otimes n')) && (\eta^{-1}\text{ is right }B\text{-linear})\\
&= \eta^{-1}(m\otimes \varepsilon(n\otimes m').n') && (\text{since (2.5) commutes})\\
&= \eta^{-1}(m.\varepsilon(n\otimes m')\otimes n').
\end{aligned} \tag{2.6}$$

Since η^{-1} is an isomorphism, applying η to both sides of the above equations shows that diagram (2.4) commutes.

Now suppose that diagram (2.4) commutes. Then,

$$\begin{aligned}
\varepsilon(n\otimes m.\varepsilon(n'\otimes m')) &= \varepsilon(n\otimes m).\varepsilon(n'\otimes m') && (\varepsilon\text{ is right }A\text{-linear})\\
&= \varepsilon(\varepsilon(n\otimes m).n'\otimes m') && (\varepsilon\text{ is left }A\text{-linear})\\
&= \varepsilon(n.\eta^{-1}(m\otimes n')\otimes m') && (\text{since (2.4) commutes})\\
&= \varepsilon(n\otimes \eta^{-1}(m\otimes n').m')
\end{aligned} \tag{2.7}$$

Applying ε^{-1} to both sides of this equations shows that diagram (2.5) commutes. □

Note that Morita contexts are the adjoint 1-equivalences in the bicategory Alg_2 of algebras, bimodules and intertwiners. The 2-morphisms in this bicategory are given by morphisms between Morita contexts, which we define next.

Definition 2.4. Let $\mathcal{M} := ({}_BM_A, {}_AN_B, \varepsilon, \eta)$ and $\mathcal{M}' := ({}_BM'_A, {}_AN'_B, \varepsilon', \eta')$ be two Morita contexts between two algebras A and B. A morphism of Morita contexts consists of a morphism of (B, A)-bimodules $f : M \to M'$ and a morphism of (A, B)-bimodules $g : N \to N'$, so that the two diagrams

$$\begin{tikzcd}
{}_BM\otimes_A N_B \arrow[r, "f\otimes g"] & {}_BM'\otimes_A N'_B \\
B \arrow[u, "\eta"] \arrow[ur, "\eta'"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
{}_AN\otimes_B M_A \arrow[r, "g\otimes f"] \arrow[d, "\varepsilon"'] & {}_AN'\otimes_B M'_A \arrow[dl, "\varepsilon'"] \\
A &
\end{tikzcd} \tag{2.8}$$

commute.

If the algebras in question have the additional structure of a symmetric Frobenius form $\lambda : A \to \mathbb{K}$, we formulate a compatibility condition between the Morita context and the Frobenius forms. We begin with the following two observations: if A is an algebra, the map

$$\begin{aligned}
A/[A,A] &\to A\otimes_{A\otimes A^{\mathrm{op}}} A\\
[a] &\mapsto a\otimes 1
\end{aligned} \tag{2.9}$$

is an isomorphism of vector spaces, with inverse given by $a \otimes b \mapsto [ab]$. Furthermore, if B is another algebra, and ${}_BM_A$ and ${}_AN_B$ are (A,B)-bimodules, there is a canonical isomorphism of vector spaces

$$\begin{aligned} \tau : (N \otimes_B M) \otimes_{A\otimes A^{\mathrm{op}}} (N \otimes_B M) &\to (M \otimes_A N) \otimes_{B\otimes B^{\mathrm{op}}} (M \otimes_A N) \\ n \otimes m \otimes n' \otimes m' &\mapsto m \otimes n' \otimes m' \otimes n. \end{aligned} \tag{2.10}$$

Using these results, together with the next lemma, we formulate a compatibility condition between Morita context and Frobenius forms.

Lemma 2.5. *Let A and B be two algebras, and let $({}_BM_A, {}_AN_B, \varepsilon, \eta)$ be a Morita context between A and B. Then, there is a canonical isomorphism of vector spaces*

$$\begin{aligned} f : A/[A,A] &\to B/[B,B] \\ [a] &\mapsto \sum_{i,j} \left[\eta^{-1}(m_j.a \otimes n_i)\right] \end{aligned} \tag{2.11}$$

where n_i and m_j are defined by

$$\varepsilon^{-1}(1_A) = \sum_{i,j} n_i \otimes m_j \in N \otimes_B M. \tag{2.12}$$

Proof. Consider the following chain of isomorphisms:

$$\begin{aligned} f : A/[A,A] &\cong A \otimes_{A\otimes A^{\mathrm{op}}} A && \text{(by equation 2.9)} \\ &\cong (N \otimes_B M) \otimes_{A\otimes A^{\mathrm{op}}} (N \otimes_B M) && \text{(using } \varepsilon\otimes\varepsilon) \\ &\cong (M \otimes_A N) \otimes_{B\otimes B^{\mathrm{op}}} (M \otimes_A N) && \text{(by equation 2.10)} \\ &\cong B \otimes_{B\otimes B^{\mathrm{op}}} B && \text{(using } \eta\otimes\eta) \\ &\cong B/[B,B] && \text{(by equation 2.9)} \end{aligned} \tag{2.13}$$

Chasing through those isomorphisms, we see that the map f is given by

$$\begin{aligned} f([a]) &= \sum_{i,j,k,l} \left[\eta^{-1}(m_j \otimes n_k) \cdot \eta^{-1}(m_l \otimes a.n_i)\right] \\ &= \sum_{i,j,k,l} \left[\eta^{-1}(m_j \otimes n_k.\eta^{-1}(m_l \otimes a.n_i))\right] \\ &= \sum_{i,j,k,l} \left[\eta^{-1}(m_j \otimes \varepsilon(n_k \otimes m_l).a.n_i)\right] && \text{(since (2.5) commutes)} \\ &= \sum_{i,j} \left[\eta^{-1}(m_j \otimes a.n_i)\right] && \text{(by definition of } \textstyle\sum_{k,l} n_k \otimes m_l) \\ &= \sum_{i,j} \left[\eta^{-1}(m_j.a \otimes n_i)\right] \end{aligned} \tag{2.14}$$

as claimed. ∎

The isomorphism f described in lemma 2.5 allows us to introduce the following relevant definition.

Definition 2.6. Let (A, λ^A) and (B, λ^B) be two symmetric Frobenius algebras, and let $({}_BM_A, {}_AN_B, \varepsilon, \eta)$ be a Morita context between the algebras A and B. Since the Frobenius algebras are symmetric, the Frobenius forms necessarily factor through $A/[A, A]$ and $B/[B, B]$. We call the Morita context *compatible* with the Frobenius forms, if the diagram

$$\begin{tikzcd} A/[A,A] \arrow[rr, "f"] \arrow[dr, "\lambda^A"'] & & B/[B,B] \arrow[dl, "\lambda^B"] \\ & \mathbb{K} & \end{tikzcd} \tag{2.15}$$

commutes. Using the notation from lemma 2.5, this means that

$$\lambda^A([a]) = \sum_i \lambda^B\left(\left[\eta^{-1}(m_i.a \otimes n_i)\right]\right) \tag{2.16}$$

for all $a \in A$.

Remark 2.7. The definition of a compatible Morita context of [SP09, Definition 3.72] requires another compatibility condition on the coproduct of the unit of the Frobenius algebras. However, a calculation using proposition 2.10 shows that the condition of [SP09] is already implied by our condition on Frobenius form of definition 2.6; thus the two definitions of compatible Morita context do coincide.

For later use, we give a very explicit way of expressing the compatibility condition between Morita context and Frobenius forms: if (A, λ^A) and (B, λ^B) are two semisimple symmetric Frobenius algebras, and $({}_BM_A, {}_BN_A, \varepsilon, \eta)$ is a Morita context between them, the algebras A and B are isomorphic to direct sums of matrix algebras by Artin-Wedderburn:

$$A \cong \bigoplus_{i=1}^{r} M_{d_i}(\mathbb{K}), \qquad \text{and} \qquad B \cong \bigoplus_{j=1}^{r} M_{n_j}(\mathbb{K}). \tag{2.17}$$

By theorem 3.3.1 of [EGH$^+$11], the simple modules $(S_1, \ldots, S_r)$ of A and the simple modules $(T_1, \ldots, T_r)$ of B are given by $S_i := \mathbb{K}^{d_i}$ and $T_i := \mathbb{K}^{n_i}$, and every module is a direct sum of copies of those. Since simple finite-dimensional representations of $A \otimes_{\mathbb{K}} B^{\mathrm{op}}$ are given by tensor products of simple representations of A and B^{op} by theorem 3.10.2 of [EGH$^+$11], the most general form of ${}_BM_A$ and ${}_AN_B$ is given by

$$\begin{aligned} {}_BM_A &:= \bigoplus_{i,j=1}^{r} \alpha_{ij}\, T_i \otimes_{\mathbb{K}} S_j \\ {}_AN_B &:= \bigoplus_{k,l=1}^{r} \beta_{kl}\, S_k \otimes_{\mathbb{K}} T_l \end{aligned} \tag{2.18}$$

where α_{ij} and β_{kl} are multiplicities. First, we show that these multiplicities must be trivial. This works for Morita contexts between finite-dimensional, semisimple algebras and does not require a Frobenius structure.

Lemma 2.8. *Let $({}_BM_A, {}_AN_B, \varepsilon, \eta)$ be a Morita context between two finite-dimensional, semisimple algebras A and B. Then, the multiplicities in equation* (2.18) *are trivial after a possible reordering of the simple modules: $\alpha_{ij} = \delta_{ij} = \beta_{ij}$ and the two bimodules M and N are actually given by*

$$\begin{aligned} {}_BM_A &= \bigoplus_{i=1}^{r} T_i \otimes_{\mathbb{K}} S_i \\ {}_AN_B &= \bigoplus_{j=1}^{r} S_j \otimes_{\mathbb{K}} T_j. \end{aligned} \tag{2.19}$$

Proof. Suppose for a contradiction that there is a term of the form $(T_i \oplus T_j) \otimes S_k$ in the direct sum decomposition of M. Let $f : T_i \to T_j$ be a non-trivial linear map, and define $\varphi \in \mathrm{End}_A((T_i \oplus T_j) \otimes S_k)$ by setting $\varphi((t_i + t_j) \otimes s_k) := f(t_i) \otimes s_k$. The A-module map φ induces an A-module endomorphism on all of ${}_AM_B$ by extending φ with zero on the rest of the direct summands. Since $\mathrm{End}_A({}_BM_A) \cong B$ as algebras by theorem 3.5 of [Bas68], the endomorphism φ must come from left multiplication, which cannot be true for an arbitrary linear map f. This shows that the bimodule M is given as claimed in equation (2.19). The statement for the other bimodule N follows analogously. □

Lemma 2.8 shows how the bimodules underlying a Morita context of semisimple algebras look like. Next, we consider the Frobenius structure, using the following lemma.

Lemma 2.9 ([Koc03, Lemma 2.2.11])**.** *Let (A, λ) be a symmetric Frobenius algebra. Then, every other symmetric Frobenius form on A is given by multiplying the Frobenius form with a central invertible element of A.*

By lemma 2.9, we conclude that the Frobenius forms on the two semisimple algebras A and B are given by

$$\lambda^A = \bigoplus_{i=1}^{r} \lambda_i^A \, \mathrm{tr}_{M_{d_i}(\mathbb{K})} \qquad \text{and} \qquad \lambda^B = \bigoplus_{i=1}^{r} \lambda_i^B \mathrm{tr}_{M_{n_i}(\mathbb{K})} \tag{2.20}$$

where λ_i^A and λ_i^B are non-zero scalars and tr is the usual trace of matrices. We can now state the following proposition, which will be used in the proof of corollary 2.36.

Proposition 2.10. *Let (A, λ^A) and (B, λ^B) be two semisimple symmetric Frobenius algebras and suppose that $\mathcal{M} := (M, N, \varepsilon, \eta)$ is a Morita context between them. Let λ_i^A and λ_j^B be scalars as in equation* (2.20)*, and define two invertible central elements*

$$\begin{aligned} a &:= (\lambda_1^A, \ldots, \lambda_r^A) \in \mathbb{K}^r \cong Z(A) \\ b &:= (\lambda_1^B, \ldots, \lambda_r^B) \in \mathbb{K}^r \cong Z(B). \end{aligned} \tag{2.21}$$

Then, the following are equivalent:

1. *The Morita context $\mathcal{M}$ is compatible with the Frobenius forms in the sense of definition 2.6.*

2. *We have $m.a = b.m$ for all $m \in {}_BM_A$ and $n.b^{-1} = a^{-1}.n$ for all $n \in {}_AN_B$.*

3. *For every $i = 1, \ldots, r$, we have that $\lambda_i^A = \lambda_i^B$.*

Proof. With the form of M and N determined by equation (2.19), we see that the only isomorphisms of bimodules $\varepsilon : N \otimes_B M \to A$ and $\eta : B \to M \otimes_A N$ must be given by multiples of the identity matrix on each direct summand:

$$\begin{aligned} \varepsilon : N \otimes_A M \cong \bigoplus_{i=1}^r M_i(d_i \times d_i, \mathbb{K}) &\to \bigoplus_{i=1}^r M_i(d_i \times d_i, \mathbb{K}) = A \\ \sum_{i=1}^r M_i &\mapsto \sum_{i=1}^r \varepsilon_i M_i. \end{aligned} \tag{2.22}$$

Similarly, η is given by

$$\begin{aligned} \eta : B = \bigoplus_{i=1}^r M_i(n_i \times n_i, \mathbb{K}) &\mapsto M \otimes_A B \cong \bigoplus_{i=1}^r M_i(n_i \times n_i, \mathbb{K}) \\ \sum_{i=1}^r M_i &\mapsto \sum_{i=1}^r \eta_i M_i. \end{aligned} \tag{2.23}$$

Here, ε_i and η_i are non-zero scalars. Using the definition of Morita context and lemma 2.3 shows that the quadruple $(M, N, \varepsilon, \eta)$ is a Morita context if and only if $\varepsilon_i = \eta_i^{-1}$. By calculating the action of the elements a and b defined above in a basis, we see that conditions (2) and (3) of the above proposition are equivalent.

Next, we show that (1) and (3) are equivalent. In order to see when the Morita context is compatible with the Frobenius forms, we calculate the map $f : A/[A, A] \to B/[B, B]$ from equation (2.15). One way to do this is to notice that $[A, A]$ consists precisely of trace-zero matrices (cf. [AM57]); thus the map

$$\begin{aligned} A/[A, A] &\to \mathbb{K}^r \\ [A_1 \oplus A_2 \oplus \cdots \oplus A_r] &\mapsto (\mathrm{tr}(A_1), \cdots, \mathrm{tr}(A_r)) \end{aligned} \tag{2.24}$$

is an isomorphism of vector spaces. Using this identification, we see that the map f is given by

$$\begin{aligned} f : A/[A, A] &\to B/[B, B] \\ [A_1 \oplus A_2 \oplus \cdots \oplus A_r] &\mapsto \bigoplus_{i=1}^r \mathrm{tr}_{M_{d_i}}(A_i) \left[E_{11}^{(n_i \times n_i)}\right]. \end{aligned} \tag{2.25}$$

Note that this map is independent of the scalars ε_i and η_i coming from the Morita context. Now, the two Frobenius algebras A and B are Morita equivalent via a compatible Morita context if and only if the diagram in equation (2.15) commutes. This is the case if and only if $\lambda_i^A = \lambda_i^B$ for all i, as a straightforward calculation in a basis shows. □

Having established how compatible Morita contexts between semisimple algebras over an algebraic closed field look like, we arrive at following definition.

Definition 2.11. Let $\mathbb{K}$ be an algebraically closed field. Let Frob be the bicategory where

- objects are given by semisimple, symmetric Frobenius algebras,
- 1-morphisms are given by compatible Morita contexts, as in definition 2.6,
- 2-morphisms are given by isomorphisms of Morita contexts.

Note that Frob has got the structure of a symmetric monoidal bigroupoid, where the monoidal product is given by the tensor product over the ground field, which is the monoidal unit.

The bicategory Frob classifies oriented, fully-extended 2-dimensional topological field theories with values in Alg_2 by a theorem of Chris Schommer-Pries:

Theorem 2.12 ([SP09])**.** *The weak 2-functor*

$$\begin{aligned} \mathrm{Fun}_{\otimes}(\mathrm{Cob}^{\mathrm{or}}_{2,1,0}, \mathrm{Alg}_2) &\to \mathrm{Frob} \\ Z &\mapsto Z(+) \end{aligned} \tag{2.26}$$

is an equivalence of bicategories.

In the next sections of this chapter, we will identify the bigroupoid Frob with the bicategory of homotopy fixed points of the trivial $SO(2)$-action on the maximal subgroupoid of fully-dualizable objects of the Morita bicategory Alg_2.

2.2. Group actions on bicategories and their homotopy fixed points

In this section, we give an explicit definition of group actions of topological groups on bicategories and their homotopy fixed points. We recall the following notation: for a group G, we denote with BG the category with one object and G as morphisms. Similarly, if $\mathcal{C}$ is a monoidal category, $B\mathcal{C}$ will denote the bicategory with one object and $\mathcal{C}$ as endomorphism category of this object. Furthermore, we denote by $\underline{G}$ the discrete monoidal category associated to G, i.e. the category with the elements of G as objects, only identity morphisms, and monoidal product given by group multiplication.

Recall from section 1.3 that a G-action on a set X can be defined as a functor $\rho : BG \to \mathrm{Set}$ with $\rho(*) = X$, and that the set of fixed points is given by the limit of this functor. As also discussed in the same section, a G-action on a category $\mathcal{C}$ can be defined as a weak 2-functor $\rho : B\underline{G} \to \mathrm{Cat}$ with $\rho(*) = \mathcal{C}$. The category of homotopy fixed points is then given as the 2-limit of this functor.

Next, we step up the categorical ladder once more, and define an action of a group G on a bicategory. Moreover, we would also like to account for the case where our group is equipped with a topology. This will be done by considering the fundamental 2-groupoid of G, referring the reader to [HKK01] for additional details.

Definition 2.13. Let G be a topological group. The fundamental 2-groupoid of G is the monoidal bicategory $\Pi_2(G)$ where

- objects are given by points of G,
- 1-morphisms are given by paths between points,
- 2-morphisms are given by homotopy classes of homotopies between paths, called *2-tracks*.

The monoidal product of $\Pi_2(G)$ is given by the group multiplication on objects, by pointwise multiplication of paths on 1-morphisms, and by pointwise multiplication of 2-tracks on 2-morphisms. Notice that this monoidal product is associative on the nose, and all other monoidal structure like associators and unitors can be chosen to be trivial.

We are now ready to give a definition of a G-action on a bicategory. Although the definition we give uses the language of tricategories as defined in [GPS95] or [Gur07], we provide a bicategorical description in remark 2.16.

Definition 2.14. Let G be a topological group, and let $\mathcal{C}$ be a bicategory. A G-action on $\mathcal{C}$ is defined to be a trifunctor

$$\rho : B\Pi_2(G) \to \text{Bicat} \tag{2.27}$$

with $\rho(*) = \mathcal{C}$. Here, $B\Pi_2(G)$ is the tricategory with one object and with $\Pi_2(G)$ as endomorphism-bicategory, and Bicat is the tricategory of bicategories.

Remark 2.15. If $\mathcal{C}$ is a bicategory, let $\text{Aut}(\mathcal{C})$ be the bicategory consisting of auto-equivalences of of $\mathcal{C}$, pseudo-natural isomorphisms and invertible modifications. Observe that $\text{Aut}(\mathcal{C})$ has the structure of a monoidal bicategory, where the monoidal product is given by composition. Since there are two ways to define the horizontal composition of pseudo-natural transformation, which are *not* equal to each other, there are actually two monoidal structures on $\text{Aut}(\mathcal{C})$. It turns out that these two monoidal structures are equivalent; see [GPS95, Section 5] for a discussion in the language of tricategories, or appendix A for more details.

With either monoidal structure of $\text{Aut}(\mathcal{C})$ chosen, note that as in definition 1.2 we could equivalently have defined a G-action on a bicategory $\mathcal{C}$ to be a weak monoidal 2-functor $\rho : \Pi_2(G) \to \text{Aut}(\mathcal{C})$.

The next remark explicitly unpacks this definition. The notation introduced here will also be used in our explicit description of homotopy fixed points in remark 2.20.

Remark 2.16 (Unpacking Definition 2.14)**.** Unpacking the definition of a weak monoidal 2-functor $\rho : \Pi_2(G) \to \text{Aut}(\mathcal{C})$, as for instance in [SP09, Definition 2.5], or equivalently of a trifunctor $\rho : B\Pi_2(G) \to \text{Bicat}$, as in [GPS95, Definition 3.1], shows that a G-action on a bicategory $\mathcal{C}$ consists of the following data:

- for each group element $g \in G$, an equivalence of bicategories $F_g := \rho(g) : \mathcal{C} \to \mathcal{C}$,

- for each path $\gamma : g \to h$ between two group elements, the action assigns a pseudo-natural isomorphism $\rho(\gamma) : F_g \to F_h$,
- for each 2-track $m : \gamma \to \gamma'$, an invertible modification $\rho(m) : \rho(\gamma) \to \rho(\gamma')$.
- There is additional data making ρ into a weak 2-functor, namely: if $\gamma_1 : g \to h$ and $\gamma_2 : h \to k$ are paths in G, we obtain invertible modifications
$$\phi_{\gamma_2\gamma_1} : \rho(\gamma_2) \circ \rho(\gamma_1) \to \rho(\gamma_2 \circ \gamma_1). \tag{2.28}$$
- Furthermore, for every $g \in G$ there is an invertible modification $\phi_g : \mathrm{id}_{F_g} \to \rho(\mathrm{id}_g)$ between the identity endotransformation on F_g and the value of ρ on the constant path id_g.

  There are three compatibility conditions for this data: one condition making ϕ_{γ_2,γ_1} compatible with the associators of $\Pi_2(G)$ and $\mathrm{Aut}(\mathcal{C})$, and two conditions with respect to the left and right unitors of $\Pi_2(G)$ and $\mathrm{Aut}(\mathcal{C})$.
- Finally, there is data for the monoidal structure, which is given by:
  - A pseudo-natural isomorphism
$$\chi : \rho(g) \otimes \rho(h) \to \rho(g \otimes h), \tag{2.29}$$
  - a pseudo-natural isomorphism
$$\iota : \mathrm{id}_{\mathcal{C}} \to F_e, \tag{2.30}$$
  - for each triple (g, h, k) of group elements, an invertible modification ω in the diagram
$$\begin{tikzcd} F_g \otimes F_h \otimes F_k \arrow[r, "\chi_{g,h}\otimes\mathrm{id}"] \arrow[d, "\mathrm{id}\otimes\chi_{h,k}"'] & F_{gh} \otimes F_k \arrow[d, "\chi_{gh,k}"] \arrow[dl, Rightarrow, "\omega_{g,h,k}"'] \\ F_g \otimes F_{hk} \arrow[r, "\chi_{g,hk}"] & F_{ghk} \end{tikzcd} \tag{2.31}$$
  - an invertible modification γ in the triangle below
$$\begin{tikzcd} & F_e \otimes F_g \arrow[dr, "\chi_{e,g}"] \arrow[d, phantom, "\Downarrow\gamma"] & \\ \mathrm{id}_{\mathcal{C}} \otimes F_g \arrow[ur, "\iota\otimes\mathrm{id}"] \arrow[rr, "\mathrm{id}_{F_g}"'] & {} & F_g \end{tikzcd} \tag{2.32}$$
  - another invertible modification δ in the triangle
$$\begin{tikzcd} & F_g \otimes F_e \arrow[dr, "\chi_{g,e}"] \arrow[d, phantom, "\Uparrow\delta"] & \\ F_g \otimes \mathrm{id}_{\mathcal{C}} \arrow[ur, "\mathrm{id}\otimes\iota"] \arrow[rr, "\mathrm{id}_{F_g}"'] & {} & F_g \end{tikzcd} \tag{2.33}$$

The data $(\rho, \chi, \iota, \omega, \gamma, \delta)$ then has to obey equations (HTA1) and (HTA2) in [GPS95, p. 17], which are given in figure 2.1 and figure 2.2. In these figures, we use the following notation: the tensor product in the diagrams is suppressed, for instance $F_g F_h$ means $F_g \otimes F_h = F_g \circ F_h$. Furthermore, the identity natural transformation of F_g is denoted by 1_g.

$$
\begin{tikzcd}
 & F_{xy}F_zF_w \arrow[rr, "\chi_{xy,z}\,1_w"] & {} \arrow[d, Rightarrow, "\omega_{x,y,z}\,1_w"] & F_{xyz}F_w \arrow[dr, bend left, "\chi_{xyz,w}"] \arrow[dd, Rightarrow, "\omega_{x,yz,w}"] & \\
F_xF_yF_zF_w \arrow[ur, bend left, "(\chi_{x,y}\,1_z)\,1_w"] \arrow[rr, bend left, "(1_x\,\chi_{y,z})\,1_w"] \arrow[rr, bend right, "1_x\,(\chi_{y,z}\,1_w)"'] \arrow[rr, phantom, "\cong"] \arrow[dr, bend right, "1_x\,(1_y\,\chi_{z,w})"'] & & F_xF_{yz}F_w \arrow[ur, "\chi_{x,yz}1_w"'] \arrow[dr, "1_x\,\chi_{yz,w}"] \arrow[d, Rightarrow, "1_x\,\omega_{y,z,w}"] & & F_{xyzw} \\
 & F_xF_yF_{zw} \arrow[rr, "1_x\,\chi_{y,zw}"'] & {} & F_xF_{yzw} \arrow[ur, bend right, "\chi_{x,yzw}"'] &
\end{tikzcd}
$$

$$\Vert$$

$$
\begin{tikzcd}
 & F_{xy}F_zF_w \arrow[rr, "\chi_{xy,z}\,1_w"] \arrow[dr, "1_{xy}\,\chi_{z,w}"] & & F_{xyz}F_w \arrow[dr, bend left, "\chi_{xyz,w}"] \arrow[dl, Rightarrow, "\omega_{xy,z,w}"] & \\
F_xF_yF_zF_w \arrow[ur, bend left, "(\chi_{x,y}\,1_z)\,1_w"] \arrow[rr, phantom, "\cong"] \arrow[dr, bend right, "1_x\,(1_y\,\chi_{z,w})"'] & & F_{xy}F_{zw} \arrow[rr, "\chi_{xy,zw}"] \arrow[dr, Rightarrow, "\omega_{x,y,zw}"] & & F_{xyzw} \\
 & F_xF_yF_{zw} \arrow[rr, "1_x\,\chi_{y,zw}"'] \arrow[ur, "\chi_{x,y}\,1_{zw}"] & & F_xF_{yzw} \arrow[ur, bend right, "\chi_{x,yzw}"'] &
\end{tikzcd}
$$

Figure 2.1.: Equation (HTA1) for G-actions

Just as in the case of a group action on a set and a group action on a category, we would like to define the bicategory of homotopy fixed points of a group action on a bicategory as a suitable limit. However, the theory of trilimits is not very well established. Therefore we will take the description of homotopy fixed points as natural transformations as in equation (1.6) as a definition, and define homotopy fixed points of a group action on a bicategory as the bicategory of pseudo-natural transformations between the constant functor and the action.

$$\begin{tikzcd}
& & F_{xy} \arrow[d, phantom, "\overset{\omega_{x,e,y}}{\Longrightarrow}"] & & \\
& & F_xF_eF_y \arrow[ddll, "\chi_{x,e}\,1_y"'] \arrow[ddrr, "1_x\,\chi_{e,y}"] & & \\
& & F_x\,\mathrm{id}_{\mathcal{C}}\,F_y \arrow[u, bend left, "(1_x\iota)1_y"] \arrow[u, bend right, "1_x(\iota 1_y)"'] \arrow[u, phantom, "\cong"] \arrow[drr, "1_x\,1_y"] \arrow[l, phantom, "\Uparrow \delta\,1_y"] \arrow[r, phantom, "\Downarrow 1_x\,\gamma"] \arrow[d, phantom, "\cong"] & & \\
F_xF_y \arrow[uuurr, bend left, "\chi_{x,y}"] \arrow[urr, "1_x\,1_y"'] \arrow[rrrr, "1_x\,1_y"'] & & & & F_xF_y \arrow[uuull, bend right, "\chi_{x,y}"']
\end{tikzcd}$$

$$\Vert$$

$$\begin{tikzcd}
& F_{xy} \arrow[d, phantom, "\cong"] & \\
F_xF_y \arrow[ur, bend left, "\chi_{x,y}"] \arrow[rr, "1_x\,1_y"'] & & F_xF_y \arrow[ul, bend right, "\chi_{x,y}"']
\end{tikzcd}$$

Figure 2.2.: Equation (HTA2) for G-actions

Definition 2.17. Let G be a topological group and $\mathcal{C}$ a bicategory. Let

$$\rho : B\Pi_2(G) \to \mathrm{Bicat} \tag{2.34}$$

be a G-action on $\mathcal{C}$. The bicategory of homotopy fixed points $\mathcal{C}^G$ is defined to be

$$\mathcal{C}^G := \mathrm{Nat}(\Delta, \rho). \tag{2.35}$$

Here, Δ is the constant functor which sends the one object of $B\Pi_2(G)$ to the terminal bicategory with one object, only the identity 1-morphism and only identity 2-morphism. The bicategory $\mathrm{Nat}(\Delta, \rho)$ then has objects given by tritransformations $\Delta \to \rho$, 1-morphisms are given by modifications, and 2-morphisms are given by perturbations.

Remark 2.18. The notion of the "equivariantization" of a strict 2-monad on a 2-category has already appeared in [MN14, Section 6.1]. Note that definition 2.17 is more general than the definition of [MN14], in which some modifications have been assumed to be trivial.

Remark 2.19. In principle, even higher-categorical definitions are possible: for instance in [FV15] a homotopy fixed point of a higher character ρ of an ∞-group is defined to be a (lax) morphism of ∞-functors $\Delta \to \rho$.

Remark 2.20 (Unpacking objects of $\mathcal{C}^G$)**.** Since unpacking the definition of homotopy fixed points is not entirely trivial, we spell it out explicitly in the subsequent remarks, following [GPS95, Definition 3.3]. In the language of bicategories, a homotopy fixed point consists of:

- an object c of $\mathcal{C}$,
- a pseudo-natural equivalence

$$\begin{tikzcd} \Pi_2(G) \arrow[r, bend left=50, "\Delta_c"] \arrow[r, bend right=50, "\mathrm{ev}_c \circ \rho"'] \arrow[r, phantom, "\Downarrow \Theta"] & \mathcal{C} \end{tikzcd} \tag{2.36}$$

  where Δ_c is the constant functor which sends every object to $c \in \mathcal{C}$, and ev_c is the evaluation at the object c. In components, the pseudo-natural transformation Θ consists of the following:

  – for every group element $g \in G$, a 1-equivalence in $\mathcal{C}$

$$\Theta_g : c \to F_g(c), \tag{2.37}$$

  – and for each path $\gamma : g \to h$, an invertible 2-morphism Θ_γ in the diagram

$$\begin{tikzcd} c \arrow[r, "\Theta_g"] \arrow[d, "\mathrm{id}_c"'] & F_g(c) \arrow[d, "\rho(\gamma)_c"] \arrow[dl, phantom, "\Swarrow \Theta_\gamma"] \\ c \arrow[r, "\Theta_h"'] & F_h(c) \end{tikzcd} \tag{2.38}$$

which is natural with respect to 2-tracks.

- Furthermore, we have an invertible modification Π in the diagram

$$
\begin{tikzcd}
\Pi_2(G)\times\Pi_2(G) \arrow[rr, bend left, "\Delta_c"] \arrow[dr, "\rho\times\Delta_c"] \arrow[ddr, bend right, "\rho\times\rho"'] \arrow[ddd, bend right=60, "\otimes"'] & \arrow[d, phantom, "\Downarrow\Theta\times 1"] & \mathcal{C} \\
\arrow[r, phantom, "\swarrow 1\times\Theta"] & \mathrm{Aut}(\mathcal{C})\times\mathcal{C} \arrow[ur, "\mathrm{ev}"] & \\
 & \mathrm{Aut}(\mathcal{C})\times\mathrm{Aut}(\mathcal{C}) \arrow[u, "\mathrm{id}\times\mathrm{ev}_c"] \arrow[dr, "\otimes"] \arrow[dl, phantom, "\swarrow\chi"] & \cong \\
\Pi_2(G) \arrow[rr, bend right, "\rho"'] & & \mathrm{Aut}(\mathcal{C}) \arrow[uuu, "\mathrm{ev}_c"']
\end{tikzcd}
\tag{2.39}
$$

$$\Downarrow\Pi$$

$$
\begin{tikzcd}
\Pi_2(G)\times\Pi_2(G) \arrow[r, "\Delta_c"] \arrow[d, "\otimes"'] \arrow[dr, phantom, "\cong", near start] & \mathcal{C} \\
\Pi_2(G) \arrow[r, "\rho"'] \arrow[ur, "\Delta_c"] \arrow[ur, phantom, "\Theta", near end] & \mathrm{Aut}(\mathcal{C}) \arrow[u, "\mathrm{ev}_c"']
\end{tikzcd}
$$

which in components means that for every tuple of group elements (g, h) we have an invertible 2-morphism $\Pi_{g,h}$ in the diagram below.

$$
\begin{tikzcd}
c \arrow[r, "\Theta_g"] \arrow[rrr, bend right=50, "\Theta_{gh}"'] & F_g(c) \arrow[r, "F_g(\Theta_h)"] & F_g(F_h(c)) \arrow[r, "\chi^c_{g,h}"] \arrow[d, phantom, "\Downarrow\Pi_{g,h}"] & F_{gh}(c) \\
 & & \, &
\end{tikzcd}
\tag{2.40}
$$

- For the unital structure, there is another invertible modification M, which only has the component given in the diagram shown below,

$$
\begin{tikzcd}
c \arrow[r, bend left=50, "\Theta_e"] \arrow[r, bend right=50, "\iota_c"'] \arrow[r, phantom, "\Downarrow M"] & F_e(c)
\end{tikzcd}
\tag{2.41}
$$

  with the 1-morphism ι given as in equation (2.30).

This ends the description of the data of a homotopy fixed point. We now come to the axioms the data defined above have to satisfy. Since we defined a homotopy fixed point to be a certain tritransformation, we shall require the three axioms of a tritransformation to hold for the data defied above. Using the equation in [GPS95, p.21-22] we find the first condition for the data (c, Θ, Π, M) of a homotopy fixed point:

$$
\begin{tikzcd}
 & F_xF_y\,c \arrow[rr, "F_xF_y(\Theta_z)"] \arrow[dr, "\chi^c_{x,y}"] & & F_xF_yF_z\,c \arrow[dr, "\chi^{F_z(c)}_{x,y}"] \arrow[d, phantom, "\cong"] & \\
F_x\,c \arrow[ur, "F_x(\Theta_y)"] \arrow[rr, phantom, "\Searrow \Pi_{x,y}"] & & F_{xy}\,c \arrow[rr, "F_{xy}(\Theta_z)"] \arrow[dr, phantom, "\Swarrow \Pi_{xy,z}"] & & F_{xy}F_z\,c \arrow[dl, "\chi_{xy,z}"] \\
 & c \arrow[ul, "\Theta_x"] \arrow[ur, "\Theta_{xy}"'] \arrow[rr, "\Theta_{xyz}"'] & & F_{xyz}\,c &
\end{tikzcd}
$$

$$
\|
$$

$$
\begin{tikzcd}
 & F_xF_y\,c \arrow[rr, "F_xF_y(\Theta_z)"] \arrow[d, phantom, "\Swarrow F_x(\Pi_{y,z})"] & & F_xF_yF_z\,c \arrow[dl, "F_x(\chi^c_{y,z})"'] \arrow[dr, "\chi^{F_z(c)}_{x,y}"] & \\
F_x\,c \arrow[ur, "F_x(\Theta_y)"] \arrow[rr, "F_x(\Theta_{yz})"] & {} \arrow[d, phantom, "\Searrow \Pi_{x,yz}"] & F_xF_{yz}\,c \arrow[dr, "\chi^c_{x,yz}"] & \Leftarrow\omega_{x,y,z} & F_{xy}F_z\,c \arrow[dl, "\chi_{xy,z}"] \\
 & c \arrow[ul, "\Theta_x"] \arrow[rr, "\Theta_{xyz}"'] & & F_{xyz}\,c &
\end{tikzcd}
\tag{2.42}
$$

The second axiom we require is due to the equation on p.23 of [GPS95] and demands that we have

$$
\begin{tikzcd}
& F_e\,c \arrow[rr, "F_e(\Theta_x)"] \arrow[rr, phantom, shift right=4ex, "\Downarrow \Pi_{e,x}"] & & F_eF_x\,c \arrow[ddr, "\chi_{e,x}"] & \\
& & \arrow[d, phantom, "\cong"] & & \\
c \arrow[uur, "\Theta_e"] \arrow[rrrr, bend left=40, "\Theta_x" description] \arrow[rr, "\Theta_x"'] & & F_x\,c \arrow[rr, "\mathrm{id}_{F_x(c)}"'] & & F_x\,c
\end{tikzcd}
$$

$$
\Vert \tag{2.43}
$$

$$
\begin{tikzcd}
& F_e\,c \arrow[rr, "F_e(\Theta_x)"] & & F_eF_x\,c \arrow[ddr, "\chi_{e,x}"] \arrow[dd, phantom, "\Downarrow \gamma"] & \\
& \arrow[r, phantom, "\cong"] & & & \\
c \arrow[uur, bend left=40, "\Theta_e"] \arrow[uur, bend right=20, "\iota_c"'] \arrow[uur, phantom, "\Rightarrow M"] \arrow[rr, "\Theta_x"'] & & F_x\,c \arrow[uur, "\iota_{F_x(c)}"] \arrow[rr, "\mathrm{id}_{F_x(c)}"'] & & F_x\,c
\end{tikzcd}
$$

Finally, the equation on p.25 of [GPS95] demands that

$$
\begin{tikzcd}
F_x\,c \arrow[r, "F_x(\Theta_e)"] \arrow[dr, phantom, "\Downarrow \Pi_{x,e}"] & F_xF_e\,c \arrow[d, "\chi_{x,e}"] \\
c \arrow[u, "\Theta_x"] \arrow[r, "\Theta_x"'] & F_x\,c
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
F_x\,c \arrow[rr, "F_x(\iota_c)"] \arrow[rr, bend left=50, "F_x(\Theta_e)"] \arrow[rr, phantom, bend left=25, "\Downarrow F_x(M)"] \arrow[ddrr, "\mathrm{id}_{F_x(c)}" description] & & F_xF_e\,c \arrow[dd, "\chi_{x,e}"] \arrow[dl, phantom, "\Swarrow \delta^{-1}"] \\
& \arrow[dl, phantom, "\cong"] & \\
c \arrow[uu, "\Theta_x"] \arrow[rr, "\Theta_x"'] & & F_x\,c
\end{tikzcd}
\tag{2.44}
$$

Remark 2.21. Suppose that (c, Θ, Π, M) and (c', Θ', Π', M') are homotopy fixed points. We now spell out what a 1-morphism between these fixed points is. Since we have defined a homotopy fixed point to be a tritransformation, a 1-morphism between these homotopy fixed points will be a trimodification. In detail, this trimodification consists of

- a 1-morphism $f : c \to c'$ in $\mathcal{C}$,
- an invertible modification m in the diagram below.

$$\begin{tikzcd}[column sep=huge]
\Pi_2(G) \arrow[r, bend left=60, "\Delta_c"{name=T}] \arrow[r, "\mathrm{ev}_c \circ \rho" description, ""{name=M}] \arrow[r, bend right=60, "\mathrm{ev}_{c'} \circ \rho"'{name=B}] \arrow[Rightarrow, from=T, to=M, "\Theta"'] \arrow[Rightarrow, from=M, to=B, "\mathrm{ev}_f * \mathrm{id}"'] & \mathcal{C}
\end{tikzcd}
\overset{m}{\Rrightarrow}
\begin{tikzcd}[column sep=huge]
\Pi_2(G) \arrow[r, bend left=60, "\Delta_c"{name=T}] \arrow[r, "\Delta_{c'}" description, ""{name=M}] \arrow[r, bend right=60, "\mathrm{ev}_{c'} \circ \rho"'{name=B}] \arrow[Rightarrow, from=T, to=M, "\Delta_f"'] \arrow[Rightarrow, from=M, to=B, "\Theta'"'] & \mathcal{C}
\end{tikzcd}
\tag{2.45}$$

In components, m_g is given by the diagram

$$\begin{tikzcd}
c \arrow[r, "\Theta_g"] \arrow[d, "f"'] & F_g(c) \arrow[d, "F_g(f)"] \arrow[dl, Rightarrow, "m_g"] \\
c' \arrow[r, "\Theta'_g"'] & F_g(c')
\end{tikzcd}
\tag{2.46}$$

The data (f, m) of a 1-morphism between homotopy fixed points has to satisfy the two axioms of a trimodification. Following the two equations as on p.25 and p. 26 of [GPS95], we find the first condition to be:

$$\begin{tikzcd}
c \arrow[r, bend left=60, "\Theta_e"{name=T}] \arrow[r, "\iota_c"'{name=M}] \arrow[d, "f"'] \arrow[Rightarrow, from=T, to=M, "M"] & F_e(c) \arrow[d, "F_e(f)"] \arrow[dl, Rightarrow, "\iota_f"] \\
c' \arrow[r, "\iota_{c'}"'] & F_e(c')
\end{tikzcd}
=
\begin{tikzcd}
c \arrow[r, "\Theta_e"] \arrow[d, "f"'] & F_e(c) \arrow[d, "F_e(f)"] \arrow[dl, Rightarrow, "m_e"] \\
c' \arrow[r, "\Theta'_e"{name=M}] \arrow[r, bend right=60, "\iota_{c'}"'{name=B}] \arrow[Rightarrow, from=M, to=B, "M'"] & F_e(c')
\end{tikzcd}
\tag{2.47}$$

The second axiom the data of a 1-morphism between homotopy fixed points must satisfy is given below in equation (2.48).

$$
\begin{tikzcd}
c \arrow[r, "\Theta_g"] \arrow[dd, "f"'] \arrow[rrr, bend right, "\Theta_{gh}"'] & F_g(c) \arrow[r, "F_g(\Theta_h)"] & F_g(F_h(c)) \arrow[r, "\chi^c_{g,h}"] \arrow[d, phantom, "\Downarrow \Pi_{g,h}"] & F_{gh}(c) \arrow[dd, "F_{gh}(f)"] \\
 & & {} \arrow[d, phantom, "\Downarrow m_{gh}"] & \\
c' \arrow[rrr, "\Theta'_{gh}"'] & & {} & F_{gh}(c')
\end{tikzcd}
$$

$$
\| \tag{2.48}
$$

$$
\begin{tikzcd}
c \arrow[r, "\Theta_g"] \arrow[dd, "f"'] & F_g(c) \arrow[r, "F_g(\Theta_h)"] \arrow[d, "F_g(f)"'] & F_g(F_h(c)) \arrow[r, "\chi^c_{g,h}"] \arrow[d, "F_g(F_h(f))"] \arrow[dl, phantom, "\swarrow F_g(m_h)"] & F_{gh}(c) \arrow[dd, "F_{gh}(f)"] \\
{} & F_g(c') \arrow[r, "F_g(\Theta'_h)"'] \arrow[l, phantom, "\Leftarrow m_g"'] \arrow[d, phantom, "\Downarrow \Pi'_{g,h}"] & F_g(F_h(c')) \arrow[dr, "\chi^{c'}_{g,h}"'] \arrow[r, phantom, "\cong"] & {} \\
c' \arrow[ur, "\Theta'_g"'] \arrow[rrr, "\Theta'_{gh}"'] & {} & & F_{gh}(c')
\end{tikzcd}
$$

Remark 2.22. The condition saying that m, as introduced in equation (2.45), is a modification will be vital for the proof of theorem 2.35 and states that for every path $\gamma : g \to h$ in G, we must have the following equality of 2-morphisms in the two diagrams:

$$
\begin{tikzcd}
c \arrow[r, "\Theta_g"] \arrow[dr, "f"'] \arrow[ddr, bend right=60, "\mathrm{id}_c"'] & F_g(c) \arrow[r, "F_g(f)"] \arrow[d, phantom, "\Downarrow m_g"] & F_g(c') \arrow[r, "\rho(\gamma)^{c'}"] \arrow[d, phantom, "\Downarrow \Theta'_\gamma"] & F_h(c') \\
& c' \arrow[ur, "\Theta'_g"] \arrow[r, "\mathrm{id}_{c'}"'] \arrow[dr, phantom, "\cong"] & c' \arrow[ur, "\Theta'_h"'] & \\
& c \arrow[r, "f"'] & c' \arrow[uur, bend right=40, "\Theta'_h"'] &
\end{tikzcd}
$$

$$
\| \tag{2.49}
$$

$$
\begin{tikzcd}
c \arrow[r, "\Theta_g"] \arrow[ddr, bend right=60, "\mathrm{id}_c"'] & F_g(c) \arrow[r, "F_g(f)"] \arrow[d, "\rho(\gamma)^c"] & F_g(c') \arrow[r, "\rho(\gamma)^{c'}"] \arrow[d, phantom, "\Downarrow \rho(\gamma)_f^{-1}"] & F_h(c') \\
\arrow[r, phantom, "\Leftarrow \Theta_\gamma"'] & F_h(c) \arrow[urr, bend right=20, "F_h(f)"'] \arrow[r, phantom, ""] & \arrow[d, phantom, "\Downarrow m_h"] & \\
& c \arrow[u, "\Theta_h"] \arrow[r, "f"'] & c' \arrow[uur, bend right=40, "\Theta'_h"'] &
\end{tikzcd}
$$

Next, we come to 2-morphisms of the bicategory $\mathcal{C}^G$ of homotopy fixed points:

Remark 2.23. Let $(f, m), (\xi, n) : (c, \Theta, \Pi, M) \to (c', \Theta', \Pi', M')$ be two 1-morphisms of homotopy fixed points. A 2-morphism of homotopy fixed points consists of a perturbation between those trimodifications. In detail, a 2-morphism of homotopy fixed points consists of a 2-morphism $\alpha : f \to \xi$ in $\mathcal{C}$, so that the following equation is satisfied:

$$
\begin{tikzcd}
c \arrow[r, "\Theta_g"] \arrow[d, "f"] \arrow[d, bend right=80, "\xi"', ""{name=X}] \arrow[dr, phantom, "\swarrow m_g"] & F_g(c) \arrow[d, "F_g(f)"] \\
c' \arrow[r, "\Theta'_g"'] & F_g(c')
\end{tikzcd}
\quad \overset{\Leftarrow \alpha}{} \quad
=
\quad
\begin{tikzcd}
c \arrow[r, "\Theta_g"] \arrow[d, "\xi"'] \arrow[dr, phantom, "\swarrow n_g"] & F_g(c) \arrow[d, "F_g(\xi)"'] \arrow[d, bend left=80, "F_g(f)"] \\
c' \arrow[r, "\Theta'_g"'] & F_g(c')
\end{tikzcd}
\quad \overset{\Leftarrow F_g(\alpha)}{}
\tag{2.50}
$$

Let us give an example of a group action on bicategories and its homotopy fixed points:

Example 2.24. Let G be a discrete group, and let $\mathcal{C}$ be any bicategory. Suppose $\rho : \Pi_2(G) \to \mathrm{Aut}(\mathcal{C})$ is the trivial G-action. Then, by remark 2.20 a homotopy fixed point, i.e. an object of $\mathcal{C}^G$ consists of

- an object c of $\mathcal{C}$,
- a 1-equivalence $\Theta_g : c \to c$ for every $g \in G$,
- a 2-isomorphism $\Pi_{g,h} : \Theta_h \circ \Theta_g \to \Theta_{gh}$,
- a 2-isomorphism $M : \Theta_e \to \mathrm{id}_c$.

This is exactly the same data as a functor $B\underline{G} \to \mathcal{C}$, where $B\underline{G}$ is the bicategory with one object, G as morphisms, and only identity 2-morphisms. Extending this analysis to 1- and 2-morphisms of homotopy fixed points shows that we have an equivalence of bicategories

$$\mathcal{C}^G \cong \mathrm{Fun}(B\underline{G}, \mathcal{C}). \tag{2.51}$$

When one specializes to $\mathcal{C} = \mathrm{Vect}_2$, the functor bicategory $\mathrm{Fun}(B\underline{G}, \mathcal{C})$ is also known as $\mathrm{Rep}_2(G)$, the bicategory of 2-representations of G. Thus, we have an equivalence of bicategories $\mathrm{Vect}_2^G \cong \mathrm{Rep}_2(G)$. This result generalizes the 1-categorical statement that the homotopy fixed point 1-category of the trivial G-action on Vect is equivalent to $\mathrm{Rep}(G)$, cf. [EGNO15, Example 4.15.2].

Next, we give an example coming from tensor categories:

Example 2.25 ([BGM17, Theorem 5.4])**.** Let G be a finite group. A G-graded extension of a finite tensor category $\mathcal{C}$ is a decomposition $\mathcal{D} = \oplus_{g \in G} \mathcal{C}_g$ with $\mathcal{C}_1 = \mathcal{C}$. If the tensor category $\mathcal{D}$ is strict monoidal, there is a G-action on the 2-category ${}_{\mathcal{C}}\mathrm{Mod}^{\mathrm{op}}$ of left $\mathcal{C}$-module categories. Furthermore, there is an equivalence of 2-categories

$$({}_{\mathcal{C}}\mathrm{Mod}^{\mathrm{op}})^G \cong {}_{D}\mathrm{Mod}. \tag{2.52}$$

2.3. Induced functors on homotopy fixed points

In this section, we introduce the notion of a G-equivariant structure on a weak 2-functor between two bicategories equipped with a G-action. We show that if such a structure exists, we obtain an induced functor on homotopy fixed points. We use this theory in section 3.3.1, where we show that the 2-functor sending an algebra to its category of representation is $SO(2)$-equivariant, and induces an equivalence between the bigroupoid of Frobenius algebras and the bigroupoid of Calabi-Yau categories.

Definition 2.26. Let $\rho : B\Pi_2(G) \to \mathrm{Bicat}$ be a G-action on the bicategory $\rho(*) = \mathcal{C}$, and let $\rho' : B\Pi_2(G) \to \mathrm{Bicat}$ be a G-action on another bicategory $\rho'(*) = \mathcal{D}$. Let $H : \mathcal{C} \to \mathcal{D}$ be a weak 2-functor. A G-equivariant structure for the weak 2-functor H consists of:

- a pseudo-natural transformation T in the diagram

$$\begin{tikzcd} \Pi_2(G) \arrow[r, "\rho"] \arrow[d, "\rho'"'] & \mathrm{Aut}(\mathcal{C}) \arrow[d, "H\circ\_"] \arrow[dl, Rightarrow, "T"] \\ \mathrm{Aut}(\mathcal{D}) \arrow[r, "\_\circ H"'] & \mathrm{Fun}(\mathcal{C},\mathcal{D}) \end{tikzcd} \tag{2.53}$$

  which very explicitly consists of the data:
  - a pseudo-natural transformation

$$T_g : H \circ F_g \to F'_g \circ H \tag{2.54}$$

    for every $g \in G$, explaining the name G-equivariant structure,
  - For every path $\gamma : g \to h$, an invertible modification T_γ in the diagram

$$\begin{tikzcd} H\circ F_g \arrow[r, "T_g"] \arrow[d, "\mathrm{id}_H * \rho(\gamma)"'] & F'_g \circ H \arrow[d, "\rho'(\gamma)*\mathrm{id}_H"] \arrow[dl, Rightarrow, "T_\gamma"] \\ H \circ F_h \arrow[r, "T_h"'] & F'_h \circ H \end{tikzcd} \tag{2.55}$$

- for every tuple of group elements (g, h), an invertible modifications $P_{g,h}$ in the diagram

$$\begin{tikzcd} H\circ F_g \circ F_h \arrow[r, "T_g * \mathrm{id}_{F_h}"] \arrow[d, "\mathrm{id}_H * \chi_{g,h}"'] & F'_g \circ H \circ F_h \arrow[r, "\mathrm{id}_{F'_g} * T_h"] \arrow[d, phantom, "\Downarrow P_{g,h}"] & F'_g \circ F'_h \circ H \arrow[r, "\chi'_{g,h} * \mathrm{id}_H"] & F'_{gh} \circ H \\ H \circ F_{gh} \arrow[urrr, bend right, "T_{gh}"'] & & & \end{tikzcd} \tag{2.56}$$

- a modification N

$$\begin{tikzcd} H \arrow[r, equal] \arrow[d, equal] & H \circ \mathrm{id}_{\mathcal{C}} \arrow[r, "\mathrm{id}_H * \iota"] \arrow[d, phantom, "\Downarrow N"] & H \circ F_e \arrow[r, "T_e"] & F'_e \circ H \\ \mathrm{id}_{\mathcal{D}} \circ H \arrow[urrr, bend right, "\iota' * \mathrm{id}_H"'] & & & \end{tikzcd} \tag{2.57}$$

so that the three equations of a tritransformation in definition 3.3 of [GPS95] are fulfilled.

Remark 2.27. We have defined a G-equivariant structure on a weak 2-functor H in such a way that it induces a tritransformation $\rho \to \rho'$ between the two actions. It is crucial to remark that the G-equivariant structure induces a weak 2-functor H^G on homotopy fixed point bicategories:

$$H^G : \mathcal{C}^G = \mathrm{Nat}(\Delta, \rho) \to \mathrm{Nat}(\Delta, \rho') = \mathcal{D}^G. \tag{2.58}$$

Explicitly, the induced functor on homotopy fixed points is given as follows:

Definition 2.28. Suppose that $H : \mathcal{C} \to \mathcal{D}$ is a weak 2-functor between bicategories endowed with G-actions ρ and ρ', and suppose that H possesses a G-equivariant structure as in definition 2.26. Then, the induced functor $H^G : \mathcal{C}^G \to \mathcal{D}^G$ is given as follows: On objects (c, Θ, Π, M) as defined in remark 2.20 of the homotopy fixed point bicategory $\mathcal{C}^G$ we define:

- on the object c of $\mathcal{C}$, we have $H^G(c) := H(c)$,
- on the pseudo-natural equivalence Θ, we define the functor on the 1-cell $\Theta_g : c \to F_g(c)$ by
$$H^G(\Theta_g) := \left(H(c) \xrightarrow{H(\Theta_g)} H(F_g(c)) \xrightarrow{T_g(c)} F'_g(H(c)) \right), \tag{2.59}$$
where F_g and F'_g are data given by the action as defined in remark 2.16, whereas on the 2-dimensional component Θ_γ in the diagram
$$\begin{tikzcd} c \arrow[r, "\Theta_g"] \arrow[d, "\mathrm{id}_c"'] & F_g(c) \arrow[d, "\rho(\gamma)_c"] \arrow[dl, phantom, "\Theta_\gamma"] \\ c \arrow[r, "\Theta_h"'] & F_h(c) \end{tikzcd} \tag{2.60}$$
we assign the 2-morphism
$$H^G(\Theta_\gamma) := \begin{tikzcd} H(c) \arrow[r, "H(\Theta_g)"] \arrow[d, "\mathrm{id}_{H(c)}"'] & H(F_g(c)) \arrow[r, "T_g(c)"] \arrow[d, "H(\rho(\gamma)_c)"] \arrow[dl, phantom, "H(\Theta_\gamma)"] & F'_g(H(c)) \arrow[d, "\rho'(\gamma)_{H(c)}"] \arrow[dl, phantom, "T_\gamma(c)"] \\ H(c) \arrow[r, "H(\Theta_h)"'] & H(F_h(c)) \arrow[r, "T_h(c)"'] & F'_h(H(c)) \end{tikzcd} \tag{2.61}$$
- For the modification Π, we assign the 2-morphism

$$\begin{tikzcd} Hc \arrow[r, "H(\Theta_g)"] \arrow[dd, "H(\Theta_{gh})"'] & HF_gc \arrow[r, "T_g(c)"] \arrow[dr, "H(F_g(\Theta_h))"] \arrow[d, phantom, "\Leftarrow H(\Pi_{g,h})"] & F'_gHc \arrow[r, "F_g(H(\Theta_h))"] & F'_gHF_hc \arrow[r, "F'_g(T_h(c))"] \arrow[d, phantom, "\Downarrow P_{g,h}"] & F'_gF'_hHc \arrow[dd, "\chi'_{g,h}(H(c))"] \\ & & HF_gF_hc \arrow[ur, "T_g(F_h(c))"] \arrow[dll, "H(\chi_{g,h}(c))"] & & \\ HF_{gh}c \arrow[rrrr, "T_{gh}(c)"'] & & & & F'_{gh}Hc \end{tikzcd} \tag{2.62}$$

- The induced functor sends the modification M to the modification in the diagram

$$\begin{tikzcd} Hc \arrow[r, "H(\iota_c)"] \arrow[r, bend left=60, "H(\Theta_e)"] \arrow[dr, "\iota'_{H(c)}"'] & HF_e c \arrow[d, "T_e(c)"] \arrow[dl, Rightarrow, "N_c"] \\ & F'_e Hc \end{tikzcd} \qquad H(M) \tag{2.63}$$

It is a straightforward, but tedious to see that $(H^G(c), H^G(\Theta), H^G(\Pi), H^G(M))$ is a homotopy fixed point in $\mathcal{D}^G$.

Definition 2.29. If $(f, m) : (c, \Theta, \Pi, M) \to (\tilde{c}, \tilde{\Theta}, \tilde{\Pi}, \tilde{M})$ is a morphism of homotopy fixed points in $\mathcal{C}^G$ as in remark 2.21, the induced functor H^G is given on 1-morphisms of homotopy fixed points by $H^G(f) := H(f)$, and by

$$H^G(m_g) := \quad \begin{tikzcd} Hc \arrow[r, "H(\Theta_g)"] \arrow[d, "H(f)"'] & HF_g c \arrow[r, "T_g(c)"] \arrow[d, "HF_g f"] \arrow[dl, Rightarrow, "H(m_g)"] & F'_g Hc \arrow[d, "F'_g Hf"] \arrow[dl, Rightarrow, "T_g(f)"] \\ H\tilde{c} \arrow[r, "H(\tilde{\Theta}_g)"'] & HF_g \tilde{c} \arrow[r, "T_g(\tilde{c})"'] & F'_g H\tilde{c} \end{tikzcd} \tag{2.64}$$

Definition 2.30. If (f, m) and (ξ, n) are two 1-morphisms of homotopy fixed points in $\mathcal{C}^G$, and $\sigma : ((f, m) \to (\xi, n))$ is a 2-morphism of homotopy fixed points as in remark 2.23, the induced functor on 2-morphisms is given by $H^G(\sigma) := H(\sigma)$.

Having this language available allows one to prove the following:

Remark 2.31 ([BGM17, Theorem 3.1])**.** Let G be a finite group acting on a strict 2-category $\mathcal{C}$. Then, $\mathcal{C}$ is G-equivalent to a 2-category with a *strict* G-action.

However, in the remainder of this thesis we will consider actions by topological groups, which cannot be easily strictified in this manner.

2.4. Computing homotopy fixed points

In this section, we compute the bicategory of homotopy fixed points of a certain $SO(2)$-action on an arbitrary bicategory. Applying the description of homotopy fixed points in remark 2.20 to the trivial action of the topological group $SO(2)$ on an arbitrary bicategory yields theorem 2.35. Specifying the bicategory in question to be the maximal subgroupoid $\mathscr{K}(\mathrm{Alg}_2)$ of the fully-dualizable objects of the Morita-bicategory then shows in corollary 2.36 that homotopy fixed points of the trivial $SO(2)$-action on $\mathscr{K}(\mathrm{Alg}_2^{\mathrm{fd}})$ are given by semisimple symmetric Frobenius algebras.

We begin by defining an $SO(2)$-action on an arbitrary bicategory, starting from a pseudo-natural transformation of the identity functor on $\mathcal{C}$.

Definition 2.32. Let $\mathcal{C}$ be a bicategory, and let $\alpha : \mathrm{id}_{\mathcal{C}} \to \mathrm{id}_{\mathcal{C}}$ be a pseudo-natural equivalence of the identity functor on $\mathcal{C}$. Since $\Pi_2(SO(2))$ is equivalent to the bicategory with one object, $\mathbb{Z}$ worth of morphisms, and only identity 2-morphisms, we may define an $SO(2)$-action $\rho : \Pi_2(SO(2)) \to \mathrm{Aut}(\mathcal{C})$ by the following data:

- for every group element $g \in SO(2)$, we assign the identity functor of $\mathcal{C}$,
- for the generator $1 \in \mathbb{Z}$, we assign the pseudo-natural transformation of the identity functor given by α.
- Since there are only identity 2-morphisms in $\mathbb{Z}$, we have to assign these to identity 2-morphisms in $\mathcal{C}$.
- For composition of 1-morphisms, we assign the identity modifications $\rho(a+b) := \rho(a) \circ \rho(b)$.
- In order to make ρ into a monoidal 2-functor, we have to assign additional data which we can choose to be trivial. In detail, we set $\rho(g \otimes h) := \rho(g) \otimes \rho(h)$, and $\rho(e) := \mathrm{id}_{\mathcal{C}}$. Finally, we choose ω, γ and δ as in equations (2.31), (2.32) and (2.33) to be identities.

Our main example is the action of the Serre automorphism on the core of fully-dualizable objects:

Example 2.33. If $\mathcal{C}$ is a symmetric monoidal bicategory, consider the maximal sub-groupoid $\mathscr{K}(\mathcal{C}^{\mathrm{fd}})$ of fully-dualizable objects of $\mathcal{C}$. By proposition 4.9, the Serre automorphism defines a pseudo-natural equivalence of the identity functor on $\mathscr{K}(\mathcal{C}^{\mathrm{fd}})$. By definition 2.32, we obtain an $SO(2)$-action on $\mathscr{K}(\mathcal{C}^{\mathrm{fd}})$, which we denote by ρ^S. We will show that the Serre automorphism is actually a *monoidal* natural transformation, so that we obtain a *monoidal* $SO(2)$-action. The explicit description of the bicategory of homotopy fixed points in the next theorem also shows that $\mathcal{C}^{SO(2)}$ is a symmetric monoidal bicategory if the original bicategory $\mathcal{C}$ was assumed to be symmetric monoidal.

The next theorem computes the bicategory of homotopy fixed points $\mathcal{C}^{SO(2)}$ of the action in definition 2.32. This theorem generalizes [HSV17, Theorem 4.1], which only computes the bicategory of homotopy fixed points of the *trivial* $SO(2)$-action.

Theorem 2.34. *Let $\mathcal{C}$ be a bicategory, and let $\alpha : \mathrm{id}_{\mathcal{C}} \to \mathrm{id}_{\mathcal{C}}$ be a pseudo-natural equivalence of the identity functor on $\mathcal{C}$. Let ρ be the $SO(2)$-action on $\mathcal{C}$ as in definition 2.32. Then, the bicategory of homotopy fixed points $\mathcal{C}^{SO(2)}$ is equivalent to the bicategory with*

- *objects: (c, λ) where c is an object of $\mathcal{C}$ and $\lambda : \alpha_c \to \mathrm{id}_c$ is a 2-isomorphism,*

- *1-morphisms* $(c,\lambda) \to (c',\lambda')$ *in* $\mathcal{C}^G$ *are given by 1-morphisms* $f : c \to c'$ *in* $\mathcal{C}$*, so that the diagram*

$$\begin{tikzcd} \alpha_{c'} \circ f \arrow[d, "\lambda' * \mathrm{id}_f"'] & f \circ \alpha_c \arrow[l, "\alpha_f"'] \arrow[r, "\mathrm{id}_f * \lambda"] & f \circ \mathrm{id}_c \arrow[d] \\ \mathrm{id}_c \circ f \arrow[rr] & & f \end{tikzcd} \tag{2.65}$$

commutes,

- *2-morphisms of* $\mathcal{C}^{SO(2)}$ *are given by 2-morphisms in* $\mathcal{C}$*.*

Proof. First, notice that we do not require any conditions on the 2-morphisms of $\mathcal{C}^{SO(2)}$. This is due to the fact that $\pi_2(SO(2))$ is trivial. Hence, all naturality conditions with respect to 2-morphisms in $\Pi_2(SO(2))$ are automatically fulfilled.

In order to prove the theorem, we need to explicitly unpack the definition of the bicategory of homotopy fixed points $\mathcal{C}^G$. This is done in remark 2.20. The idea of the proof is to show that the forgetful functor which on objects of $\mathcal{C}^G$ forgets the data Θ, Π and M is an equivalence of bicategories. In order to show this, we need to analyze the bicategory of homotopy fixed points. We start with the objects of $\mathcal{C}^G$.

By definition, a homotopy fixed point of this action consists of

- an object c of $\mathcal{C}$,
- a 1-equivalence $\Theta : c \to c$,
- for every $n \in \mathbb{Z}$, an invertible 2-morphism $\Theta_n : \alpha_c^n \circ \Theta \to \Theta \circ \mathrm{id}_c$ so that (Θ, Θ_n) fulfill the axioms of a pseudo-natural transformation,
- a 2-isomorphism $\Pi : \Theta \circ \Theta \to \Theta$ which obeys the modification square,
- another 2-isomorphism $M : \Theta \to \mathrm{id}_c$,

so that the following equations hold: equation (2.42) demands that

$$\Pi \circ (\mathrm{id}_\Theta * \Pi) = \Pi \circ (\Pi * \mathrm{id}_\Theta), \tag{2.66}$$

whereas equation (2.43) demands that Π is equal to the composition of 2-morphisms

$$\Theta \circ \Theta \xrightarrow{\mathrm{id}_\Theta * M} \Theta \circ \mathrm{id}_c \cong \Theta, \tag{2.67}$$

and finally equation (2.44) tells us that Π must also be equal to the composition

$$\Theta \circ \Theta \xrightarrow{M * \mathrm{id}_\Theta} \mathrm{id}_c \circ \Theta \cong \Theta. \tag{2.68}$$

Hence Π is fully specified by M. An explicit calculation using the two equations above then confirms that equation (2.66) is automatically fulfilled. Indeed, by composing with

Π^{-1} from the right, it suffices to show that $\mathrm{id}_\Theta * \Pi = \Pi * \mathrm{id}_\Theta$. Suppose for simplicity that $\mathcal{C}$ is a strict 2-category. Then,

$$\begin{aligned} \mathrm{id}_\Theta * \Pi &= \mathrm{id}_\Theta * (M * \mathrm{id}_\Theta) && \text{by equation (2.68)} \\ &= (\mathrm{id}_\Theta * M) * \mathrm{id}_\Theta && \\ &= \Pi * \mathrm{id}_\Theta && \text{by equation (2.67).} \end{aligned} \tag{2.69}$$

Adding appropriate associators shows that this is true in a general bicategory.

Note that by using the modification M, the 2-morphism $\Theta_n : \alpha_n^c \to \Theta \circ \mathrm{id}_c$ is equivalent to a 2-morphism $\lambda_n : \alpha_c \to \mathrm{id}_c$. Here, α_c^n is the n-times composition of 1-morphism α_c. Indeed, define λ_n by setting

$$\lambda_n := \left(\alpha_c \cong \alpha_c \circ \mathrm{id}_c \xrightarrow{\mathrm{id}_{\alpha_c} * M^{-1}} \alpha_c \circ \Theta \xrightarrow{\Theta_n} \Theta \circ \mathrm{id}_c \cong \Theta \xrightarrow{M} \mathrm{id}_c \right). \tag{2.70}$$

In a strict 2-category, the fact that Θ is a pseudo-natural transformation requires that $\lambda_0 = \mathrm{id}_c$ and that $\lambda_n = \lambda_1 * \cdots * \lambda_1$. In a bicategory, similar equations hold by adding coherence morphisms. Thus, λ_n is fully determined by λ_1. In order to simplify notation, we set $\lambda := \lambda_1 : \alpha_c \to \mathrm{id}_c$.

A 1-morphism of homotopy fixed points $(c, \Theta, \Theta_n, \Pi, M) \to (c', \Theta', \Theta'_n, \Pi', M')$ consists of:

- a 1-morphism $f : c \to c'$,
- an invertible 2-morphism $m : f \circ \Theta \to \Theta' \circ f$ which fulfills the modification square. Note that m is equivalent to a 2-isomorphism $m : f \to f'$ which can be seen by using the 2-morphism M.

The condition due to equation (2.47) demands that the following 2-isomorphism

$$f \circ \Theta \xrightarrow{\mathrm{id}_f * M} f \circ \mathrm{id}_c \cong f \tag{2.71}$$

is equal to the 2-isomorphism

$$f \circ \Theta \xrightarrow{m} \Theta' \circ f \xrightarrow{M' * \mathrm{id}_f} \mathrm{id}_{c'} \circ f \cong f \tag{2.72}$$

and thus is equivalent to the equation

$$m = \left(f \circ \Theta \xrightarrow{\mathrm{id}_f * M} f \circ \mathrm{id}_c \cong f \cong \mathrm{id}_{c'} \circ f \xrightarrow{M'^{-1} * \mathrm{id}_f} \Theta' \circ f \right). \tag{2.73}$$

Thus, m is fully determined by M and M'. The condition due to equation (2.48) reads

$$m \circ (\mathrm{id}_f * \Pi) = (\Pi' * \mathrm{id}_f) \circ (\mathrm{id}_{\Theta'} * m) \circ (m * \mathrm{id}_\Theta) \tag{2.74}$$

and is automatically satisfied, as an explicit calculation confirms: indeed, if $\mathcal{C}$ is a strict 2-category we have that

$$
\begin{aligned}
&(\Pi' * \mathrm{id}_f) \circ (\mathrm{id}_{\Theta'} * m) \circ (m * \mathrm{id}_\Theta) \\
&\quad = (\Pi' * \mathrm{id}_f) \circ \left[\mathrm{id}_{\Theta'} * (M'^{-1} * \mathrm{id}_f \circ \mathrm{id}_f * M)\right] \circ \left[(M'^{-1} * \mathrm{id}_f \circ \mathrm{id}_f * M) * \mathrm{id}_\Theta\right] \\
&\quad = (\Pi' * \mathrm{id}_f) \circ (\mathrm{id}_{\Theta'} * M'^{-1} * \mathrm{id}_f) \circ (\mathrm{id}_{\Theta'} * \mathrm{id}_f * M) \\
&\qquad\qquad \circ (M'^{-1} * \mathrm{id}_f * \mathrm{id}_\Theta) \circ (\mathrm{id}_f * M * \mathrm{id}_\Theta) \\
&\quad = (\Pi' * \mathrm{id}_f) \circ (\Pi'^{-1} * \mathrm{id}_f) \circ (\mathrm{id}_{\Theta'} * \mathrm{id}_f * M) \circ (M'^{-1} * \mathrm{id}_f * \mathrm{id}_\Theta) \circ (\mathrm{id}_f * \Pi) \\
&\quad = (\mathrm{id}_{\Theta'} * \mathrm{id}_f * M) \circ (M'^{-1} * \mathrm{id}_f * \mathrm{id}_\Theta) \circ (\mathrm{id}_f * \Pi) \\
&\quad = (M^{-1} * \mathrm{id}_f) \circ (\mathrm{id}_f * M) \circ (\mathrm{id}_f * \Pi) \\
&\quad = m \circ (\mathrm{id}_f * \Pi)
\end{aligned}
$$

as desired. Here, we have used equation (2.73) in the first and last line, and equations (2.67) and (2.68) in the third line. Adding associators shows that this is true for an arbitrary bicategory.

Now, it suffices to look at the modification square of m in equation (2.49). This condition is equivalent to the commutativity of the diagram

$$
\begin{tikzcd}
\alpha_{c'} \circ f \circ \Theta \arrow[d, "\mathrm{id}_{\alpha_{c'}} * m"'] & f \circ \alpha_c \circ \Theta \arrow[l, "\alpha_f * \mathrm{id}_\Theta"'] \arrow[r, "\mathrm{id}_f * \Theta_1"] & f \circ \Theta \arrow[d, "m"] \\
\alpha_{c'} \circ \Theta' \circ f \arrow[rr, "\Theta'_1 * \mathrm{id}_f"'] & & \Theta' \circ f
\end{tikzcd}
\tag{2.75}
$$

Substituting m as in equation (2.73) and Θ_1 for $\lambda := \lambda_1$ as defined in equation (2.70), one confirms that this diagram commutes if and only if the diagram in equation (2.65) commutes.

If (f, m) and (g, n) are 1-morphisms of homotopy fixed points, a 2-morphism of homotopy fixed points consists of a 2-isomorphism $\beta : f \to g$ in $\mathcal{C}$. The condition coming from equation (2.50) then demands that the diagram

$$
\begin{tikzcd}
f \circ \Theta \arrow[r, "m"] \arrow[d, "\beta * \mathrm{id}_\Theta"'] & \Theta' \circ f \arrow[d, "\mathrm{id}_{\Theta'} * \beta"] \\
g \circ \Theta \arrow[r, "n"'] & \Theta' \circ g
\end{tikzcd}
\tag{2.76}
$$

commutes. Using the fact that both m and n are uniquely specified by M and M', one quickly confirms that this diagram commutes automatically. Indeed, suppose for simplicity that $\mathcal{C}$ is a strict 2-category. Composing diagram (2.76) with $M' * \mathrm{id}_g$ from the left and with $\mathrm{id}_f * M^{-1}$ yields after simplifying and using equation (2.73) to replace m and n

$$
(M' * M^{-1}) \circ (\mathrm{id}_{\Theta'} * \alpha) \circ (M'^{-1} * M) = \alpha * \mathrm{id}_\Theta. \tag{2.77}
$$

This equation is always fulfilled since α is a 2-morphism between f and g.

Our detailed analysis of the bicategory $\mathcal{C}^G$ shows that the forgetful functor U which forgets the data Θ, M, and Π on objects and assigns Θ_1 to λ, which forgets the data m on 1-morphisms, and which is the identity on 2-morphisms is an equivalence of bicategories. Indeed, let (c, λ) be an object in the strictified homotopy fixed point bicategory. Choose $\Theta := \mathrm{id}_c$, $M := \mathrm{id}_\Theta$ and Π as in equation (2.67). Then, $U(c, \Theta, M, \Pi, \lambda) = (c, \lambda)$. This shows that the forgetful functor is essentially surjective on objects. Since m is fully determined by M and M', it is clear that the forgetful functor is essentially surjective on 1-morphisms. Since (2.76) commutes automatically, the forgetful functor is bijective on 2-morphisms and thus an equivalence of bicategories. □

Let us first apply this theorem to the trivial action: by choosing the pseudo-natural transformation α in theorem 2.34 to be the identity transformation, we obtain the following theorem about homotopy fixed points of the trivial $SO(2)$-action on an arbitrary bicategory.

Theorem 2.35. *Let $\mathcal{C}$ be a bicategory, and let $\rho : \Pi_2(SO(2)) \to \mathrm{Aut}(\mathcal{C})$ be the trivial $SO(2)$-action on $\mathcal{C}$. Then, the bicategory of homotopy fixed points $\mathcal{C}^{SO(2)}$ is equivalent to the bicategory where*

- *objects are given by pairs (c, λ) where c is an object of $\mathcal{C}$, and $\lambda : \mathrm{id}_c \to \mathrm{id}_c$ is a 2-isomorphism,*
- *1-morphisms $(c, \lambda) \to (c', \lambda')$ are given by 1-morphisms $f : c \to c'$ in $\mathcal{C}$, so that the diagram of 2-morphisms*

$$\begin{tikzcd}
f \arrow[r, "\sim"] \arrow[d, "\wr"'] & f \circ \mathrm{id}_c \arrow[r, "\mathrm{id}_f * \lambda"] & f \circ \mathrm{id}_c \arrow[d, "\wr"] \\
\mathrm{id}_{c'} \circ f \arrow[r, "\lambda' * \mathrm{id}_f"'] & \mathrm{id}_{c'} \circ f \arrow[r, "\sim"'] & f
\end{tikzcd} \tag{2.78}$$

  commutes, where $*$ denotes horizontal composition of 2-morphisms. The unlabeled arrows are induced by the canonical coherence isomorphisms of $\mathcal{C}$.
- *2-morphisms of $\mathcal{C}^{SO(2)}$ are given by 2-morphisms $\alpha : f \to f'$ in $\mathcal{C}$.*

□

In the following, we specialise theorem 2.35 to the case of symmetric Frobenius algebras and Calabi-Yau categories.

2.4.1. Symmetric Frobenius algebras as homotopy fixed points

In order to state the next corollary, recall that the fully-dualizable objects of the Morita bicategory Alg_2 consisting of algebras, bimodules and intertwiners are precisely given by the finite-dimensional, semisimple algebras, cf. [SP09]. Furthermore, recall that the core $\mathscr{K}(\mathcal{C})$ of a bicategory $\mathcal{C}$ consists of all objects of $\mathcal{C}$, the 1-morphisms are given by 1-equivalences of $\mathcal{C}$, and the 2-morphisms are restricted to be isomorphisms.

Corollary 2.36. *Let $\mathcal{C} = \mathscr{K}(\mathrm{Alg}_2^{\mathrm{fd}})$, and consider the trivial $SO(2)$-action on $\mathcal{C}$. Then $\mathcal{C}^{SO(2)}$ is equivalent to the bicategory of finite-dimensional, semisimple symmetric Frobenius algebras* Frob*, as defined in definition 2.11. This implies a bijection of isomorphism-classes of semisimple symmetric Frobenius algebras and homotopy fixed points of the trivial $SO(2)$-action on $\mathscr{K}(\mathrm{Alg}_2^{\mathrm{fd}})$.*

Proof. Indeed, by theorem 2.35, an object of $\mathcal{C}^{SO(2)}$ is given by a finite-dimensional semisimple algebra A, together with an isomorphism of Morita contexts $\mathrm{id}_A \to \mathrm{id}_A$. By definition, a morphism of Morita contexts consists of two intertwiners of (A, A)-bimodules $\lambda_1, \lambda_2 : A \to A$. The diagrams in definition 2.4 then require that $\lambda_1 = \lambda_2^{-1}$. Thus, λ_2 is fully determined by λ_1. Let $\lambda := \lambda_1$. Since λ is an automorphism of (A, A)-bimodules, it is fully determined by $\lambda(1_A) \in Z(A)$. This gives A, by lemma 2.9, the structure of a symmetric Frobenius algebra.

We analyze the 1-morphisms of $\mathcal{C}^{SO(2)}$ in a similar way: if (A, λ) and (A', λ') are finite-dimensional semisimple symmetric Frobenius algebras, a 1-morphism in $\mathcal{C}^{SO(2)}$ consists of a Morita context $\mathcal{M} : A \to A'$ so that (2.78) commutes.

Suppose that $\mathcal{M} = ({}_{A'}M_A, {}_AN_{A'}, \varepsilon, \eta)$ is a Morita context, and let $a := \lambda(1_A)$ and $a' := \lambda'(1_{A'})$. Then, the condition that (2.78) commutes demands that

$$\begin{aligned} m.a &= a'.m \\ a^{-1}.n &= n.a'^{-1} \end{aligned} \tag{2.79}$$

for every $m \in M$ and every $n \in N$. By proposition 2.10 this condition is equivalent to the fact that the Morita context is compatible with the Frobenius forms as in definition 2.6.

One now shows by hand that the 2-morphisms of $\mathcal{C}^{SO(2)}$ and Frob stand in bijection which proves the statement of the theorem. $\square$

Remark 2.37. We have just shown that homotopy fixed points of the trivial $SO(2)$-action on $\mathscr{K}(\mathrm{Alg}_2^{\mathrm{fd}})$ are symmetric Frobenius algebras. However, this equivalence is not canonical, as it relies on lemma 2.9 by choosing a Frobenius form in the first place. The reason the equivalence of corollary 2.36 is not canonical comes from choosing the trivial $SO(2)$-action. We will construct an $SO(2)$-action on the core of fully-dualizable objects of an arbitrary bicategory in chapter 4 given by the Serre automorphism. In the example of Alg_2, the action of the Serre-automorphism will be trivializable and thus agree with the trivial action. However, there is a *canonical* equivalence between homotopy fixed points of the Serre-automorphism on $\mathscr{K}(\mathrm{Alg}_2^{\mathrm{fd}})$ and symmetric Frobenius algebras. Thus, the choice of Frobenius form we had to make in corollary 2.36 comes from considering the trivial action instead of the action of the Serre automorphism.

Remark 2.38. In [Dav11, Proposition 3.3.2], the bigroupoid Frob of corollary 2.36 is shown to be equivalent to the bicategory of 2-functors $\mathrm{Fun}(B^2\mathbb{Z}, \mathscr{K}(\mathrm{Alg}_2^{\mathrm{fd}}))$. Assuming a homotopy hypothesis for bigroupoids, as well as an equivariant homotopy hypothesis in a bicategorical framework, this bicategory of functors should agree with the bicategory of homotopy fixed points of the trivial $SO(2)$-action on $\mathscr{K}(\mathrm{Alg}_2^{\mathrm{fd}})$ in corollary 2.36. Concretely, one might envision the following strategy for an alternative proof of corollary 2.36, which should roughly go as follows:

1. By [Dav11, Proposition 3.3.2], there is an equivalence of bigroupoids $\mathrm{Frob} \cong \mathrm{Fun}(B^2\mathbb{Z}, \mathscr{K}(\mathrm{Alg}_2^{\mathrm{fd}}))$.

2. Then, use the homotopy hypothesis for bigroupoids. By this, we mean that the fundamental 2-groupoid should induce an equivalence of tricategories
$$\Pi_2 : \mathrm{Top}_{\leq 2} \to \mathrm{BiGrp} \tag{2.80}$$
Here, the right hand-side is the tricategory of bigroupoids, whereas the left hand side is a suitable tricategory of 2-types. Such an equivalence of tricategories induces an equivalence of bicategories
$$\mathrm{Fun}(B^2\mathbb{Z}, \mathscr{K}(\mathrm{Alg}_2^{\mathrm{fd}})) \cong \Pi_2(\mathrm{Hom}(BSO(2), X)), \tag{2.81}$$
where X is a 2-type representing the bigroupoid $\mathscr{K}(\mathrm{Alg}_2^{\mathrm{fd}})$.

3. Now, consider the trivial homotopy $SO(2)$-action on the 2-type X. Using the fact that we work with the trivial $SO(2)$-action, we obtain a homotopy equivalence $\mathrm{Hom}(BSO(2), X) \cong X^{hSO(2)}$, cf. [Dav11, Page 50].

4. In order to identify the 2-type $X^{hSO(2)}$ with our definition of homotopy fixed points, we additionally need an equivariant homotopy hypothesis: namely, we need to use that a homotopy action of a topological group G on a 2-type Y is equivalent to a G-action on the bicategory $\Pi_2(Y)$ as in definition 2.14. Furthermore, we also need to assume that the fundamental 2-groupoid is G-equivariant, namely that there is an equivalence of bicategories $\Pi_2(Y^{hG}) \cong \Pi_2(Y)^G$. Using this equivariant homotopy hypothesis for the trivial $SO(2)$-action on the 2-type X then should give an equivalence of bicategories
$$\Pi_2(X^{hSO(2)}) \cong \Pi_2(X)^{SO(2)} \cong (\mathscr{K}(\mathrm{Alg}_2^{\mathrm{fd}}))^{SO(2)}. \tag{2.82}$$

Combining all four steps gives an equivalence of bicategories between the bigroupoid of Frobenius algebras and homotopy fixed points:
$$\begin{aligned} \mathrm{Frob} &\cong \mathrm{Fun}(B^2\mathbb{Z}, \mathscr{K}(\mathrm{Alg}_2^{\mathrm{fd}})) && \text{by (1)} \\ &\cong \Pi_2(\mathrm{Hom}(BSO(2), X)) && \text{by (2)} \\ &\cong \Pi_2(X^{hSO(2)}) && \text{by (3)} \\ &\cong (\mathscr{K}(\mathrm{Alg}_2^{\mathrm{fd}}))^{SO(2)} && \text{by (4).} \end{aligned} \tag{2.83}$$

In order to turn this argument into a full proof, we would need to provide a proof of the homotopy hypothesis for bigroupoids in equation (2.80), as well as a proof for the equivariant homotopy hypothesis in equation (2.82). While the homotopy hypothesis as formulated in equation (2.80) is widely believed to be true, we are not aware of a proof of this statement in the literature. A step in this direction is [MS93], which proves that the homotopy categories of 2-types and 2-groupoids are equivalent. We however really need the full tricategorical version of this statement as in equation (2.80), since we need to identify the (higher) morphisms in BiGrp with (higher) homotopies. In [Gur11], Gurski sets up a tricategory of topological spaces and shows that the fundamental 2-groupoid is indeed a trifunctor. Notice that statements of this type are rather subtle, see [KA91, Sim98].

While certainly interesting and conceptually illuminating, a proof of the equivariant homotopy hypothesis in a bicategorical language in equation (2.82) is beyond the scope of this thesis, which aims to give an *algebraic* description of homotopy fixed points on bicategories. Although an equivariant homotopy hypothesis for ∞-groupoids follows from [Lur09a, Theorem 4.2.4.1], we are not aware of a proof of the bicategorical statement in equation (2.82). We also note that an equivariant homotopy hypothesis for strict group actions of discrete groups on ordinary categories has been proven in [BMO$^+$15] in the language of model categories.

3. An equivalence between Frobenius algebras and Calabi-Yau categories

In this chapter, which is based on results of [Hes16], we show that the bigroupoid Frob of separable, symmetric Frobenius algebras over an algebraically closed field and the bigroupoid CY of Calabi-Yau categories are equivalent. To this end, we construct a trace on the category of finitely-generated representations of a separable symmetric Frobenius algebra, given by the composite of the Frobenius form with the Hattori-Stallings trace.

The second part of this chapter deals with relating the equivalence of bicategories Frob $\cong$ CY of theorem 3.37 with homotopy fixed points. Recall that in chapter 2, we identified the bigroupoid Frob of Frobenius algebras with homotopy fixed points of the trivial $SO(2)$-action on the bigroupoid of fully-dualizable objects of the Morita bicategory Alg_2. Here, we consider the trivial $SO(2)$-action on the fully-dualizable objects of the bigroupoid of 2-vector spaces Vect_2, and show in corollary 3.12 that homotopy fixed points of this action are given by Calabi-Yau categories.

Now recall that in section 2.3, we defined the concept of "equivariantization" of a 2-functor between bicategories endowed with a G-action, and gave an explicit description of the induced 2-functor on homotopy fixed points. As an example, we endow both Alg_2 and Vect_2 with the trivial $SO(2)$-action and compute the induced functor $\mathrm{Rep}^{SO(2)}$ on homotopy fixed points. Theorem 3.41 then shows that this induced 2-functor is naturally isomorphic to the 2-functor $\mathrm{Rep}^{\mathrm{fg}}$ constructed in section 3.2, and thus the diagram

$$\begin{tikzcd}
(\mathscr{K}(\mathrm{Alg}_2^{\mathrm{fd}}))^{SO(2)} \arrow[r, "\mathrm{Rep}^{SO(2)}"] \arrow[d, "\wr"'] & (\mathscr{K}(\mathrm{Vect}_2^{\mathrm{fd}}))^{SO(2)} \arrow[d, "\wr"] \\
\mathrm{Frob} \arrow[r, "\mathrm{Rep}^{\mathrm{fg}}"'] & \mathrm{CY}
\end{tikzcd} \tag{3.1}$$

commutes up to a pseudo-natural isomorphism. Here, the unlabeled equivalences are due to corollary 2.36 and corollary 3.12.

The chapter is organized as follows: in section 3.1, we recall the definition of the bicategory CY of finitely semisimple Calabi-Yau categories, as originally considered in [MS06]. In corollary 3.12 we show that the bicategory of Calabi-Yau categories is equivalent to the bicategory of homotopy fixed points of the trivial $SO(2)$-action on the core of fully-dualizable objects of Vect_2, which is the bicategory of finite, linear categories.

In section 3.2, we construct a weak 2-functor $\mathrm{Rep}^{\mathrm{fg}} : \mathrm{Frob} \to \mathrm{CY}$ which sends a separable symmetric Frobenius algebra to its category of finitely-generated modules. We endow this category of representations with the Calabi-Yau structure given by the composite of the Frobenius form with the Hattori-Stallings trace in definition 3.21, and show in theorem 3.37 that this functor is an equivalence of bicategories. Section 3.3 is devoted to the proof of theorem 3.37.

Throughout the chapter we use the following conventions: all algebras considered will be over an algebraically closed field $\mathbb{K}$. All Frobenius algebras appearing will be symmetric.

3.1. Calabi-Yau categories

The main player of this section are Calabi-Yau categories, which we define below. Let $\mathbb{K}$ be a field, and let Vect be the category of $\mathbb{K}$-vector spaces. Recall the following terminology: a linear category is an abelian category with a compatible enrichment over Vect. A linear functor is an additive functor which is also a functor of Vect-enriched categories.

Definition 3.1. Following [BDSPV15, Appendix A], we call a linear category $\mathcal{C}$ finite, if

1. there are only finitely many isomorphism classes of simple objects of $\mathcal{C}$,
2. the category $\mathcal{C}$ has enough projectives,
3. every object of $\mathcal{C}$ has finite length, and
4. the Hom-spaces of $\mathcal{C}$ are finite-dimensional.

Definition 3.2. Let $\mathbb{K}$ be a field. A Calabi-Yau category $(\mathcal{C}, \mathrm{tr}^{\mathcal{C}})$ is a $\mathbb{K}$-linear, finite, semisimple category $\mathcal{C}$, together with a family of $\mathbb{K}$-linear maps

$$\mathrm{tr}^{\mathcal{C}}_c : \mathrm{End}_{\mathcal{C}}(c) \to \mathbb{K} \tag{3.2}$$

for each object c of $\mathcal{C}$, so that:

1. for each $f \in \mathrm{Hom}_{\mathcal{C}}(c, d)$ and for each $g \in \mathrm{Hom}_{\mathcal{C}}(d, c)$, we have that
$$\mathrm{tr}^{\mathcal{C}}_c(g \circ f) = \mathrm{tr}^{\mathcal{C}}_d(f \circ g), \tag{3.3}$$
2. for all objects c and d of $\mathcal{C}$, the induced pairing
$$\begin{aligned} \langle - , - \rangle_{\mathcal{C}} : \mathrm{Hom}_{\mathcal{C}}(c, d) \otimes_{\mathbb{K}} \mathrm{Hom}_{\mathcal{C}}(d, c) &\to \mathbb{K} \\ f \otimes g &\mapsto \mathrm{tr}^{\mathcal{C}}_c(g \circ f) \end{aligned} \tag{3.4}$$
is a non-degenerate pairing of $\mathbb{K}$-vector spaces.

We will call the collection of morphisms $\mathrm{tr}^{\mathcal{C}}_c$ a trace on $\mathcal{C}$. Note that an equivalent way of defining a Calabi-Yau structure on a linear category $\mathcal{C}$ is given by specifying a natural isomorphism

$$\mathrm{Hom}_{\mathcal{C}}(c, d) \to \mathrm{Hom}_{\mathcal{C}}(d, c)^*, \tag{3.5}$$

cf. [Sch13, Proposition 4.1].

Remark 3.3. The space of all endomorphisms of an object of a Calabi-Yau category has got the structure of a semisimple symmetric Frobenius algebra.

Remark 3.4. The definition of a Calabi-Yau category generalizes the notion of a *trace* in a symmetric monoidal category. A nice exposition of traces in symmetric monoidal categories can be found in [PS14]. A generalization of traces to bicategories using the theory of shadows also has appeared in [PS13, PS16].

We begin with two preparatory lemmas:

Lemma 3.5. *Let $(\mathcal{C}, \mathrm{tr}^{\mathcal{C}})$ be a Calabi-Yau category. Then, the trace is automatically additive: for each $f \in \mathrm{End}_{\mathcal{C}}(x)$ and each $g \in \mathrm{End}_{\mathcal{C}}(y)$, we have that*

$$\mathrm{tr}^{\mathcal{C}}_{x\oplus y}(f \oplus g) = \mathrm{tr}^{\mathcal{C}}_{x}(f) + \mathrm{tr}^{\mathcal{C}}_{y}(g). \tag{3.6}$$

Proof. Denote by $p_x : x \oplus y \leftrightarrow x : \iota_x$ and by $p_y : x \oplus y \leftrightarrow y : \iota_y$ the canonical projections and inclusions. Then,

$$\mathrm{id}_{x\oplus y} = \iota_x \circ p_x + \iota_y \circ p_y. \tag{3.7}$$

We calculate using the linearity and cyclicity of the trace:

$$\begin{aligned}
\mathrm{tr}^{\mathcal{C}}_{x\oplus y}(f+g) &= \mathrm{tr}^{\mathcal{C}}_{x\oplus y}((f+g)\circ(\iota_x \circ p_x + \iota_y \circ p_y)) \\
&= \mathrm{tr}^{\mathcal{C}}_{x\oplus y}((f+g)\circ \iota_x \circ p_x) + \mathrm{tr}^{\mathcal{C}}_{x\oplus y}((f+g)\circ \iota_y \circ p_y) \\
&= \mathrm{tr}^{\mathcal{C}}_{x}(p_x \circ (f+g)\circ \iota_x) + \mathrm{tr}^{\mathcal{C}}_{y}(p_y \circ (f+g)\circ \iota_y) \\
&= \mathrm{tr}^{\mathcal{C}}_{x}(f) + \mathrm{tr}^{\mathcal{C}}_{y}(g).
\end{aligned} \tag{3.8}$$

□

Adapting the proof of [Sta65, Section 1] to the setting of linear categories proves the following lemma:

Lemma 3.6. *Let $(\mathcal{C}, \mathrm{tr}^{\mathcal{C}})$ be a Calabi-Yau category, and let $x_1, \ldots, x_n$ be objects of $\mathcal{C}$. Let $x := \oplus_{i=1}^n x_i$, and let $f \in \mathrm{End}_{\mathcal{C}}(x)$. Since $\mathcal{C}$ is an additive category, we may write the morphism f in matrix form as*

$$f = \begin{pmatrix} f_{11} & \cdots & f_{1n} \\ \vdots & & \vdots \\ f_{n1} & \cdots & f_{nn} \end{pmatrix} \tag{3.9}$$

where the entries f_{ij} are morphisms $f_{ij} \in \mathrm{Hom}_{\mathcal{C}}(x_j, x_i)$. Then,

$$\mathrm{tr}^{\mathcal{C}}_{x}(f) = \sum_{i=1}^{n} \mathrm{tr}^{\mathcal{C}}_{x_i}(f_{ii}). \tag{3.10}$$

Proof. First suppose that f is a block matrix of the form

$$f = \begin{pmatrix} 0_p & X \\ 0 & 0_q \end{pmatrix}. \tag{3.11}$$

Define an endomorphism J of x by the square matrix

$$J := \begin{pmatrix} I_p & 0 \\ 0 & 0_q \end{pmatrix} \tag{3.12}$$

with I_p the diagonal $p \times p$-matrix with diagonal entries $(\mathrm{id}_{x_1}, \dots, \mathrm{id}_{x_p})$. Then,

$$f = J \circ f = J \circ f - f \circ J, \tag{3.13}$$

and by the symmetry and the linearity the trace we then have

$$\mathrm{tr}_x^{\mathcal{C}}(f) = \mathrm{tr}_x^{\mathcal{C}}(J \circ f - f \circ J) = \mathrm{tr}_x^{\mathcal{C}}(J \circ f) - \mathrm{tr}_x^{\mathcal{C}}(J \circ f) = 0. \tag{3.14}$$

Similarly, one shows that if

$$f = \begin{pmatrix} 0_p & 0 \\ X & 0_q \end{pmatrix}, \tag{3.15}$$

then $\mathrm{tr}_x^{\mathcal{C}}(f) = 0$. Combining these two results and using the linearity of the trace again shows that

$$\mathrm{tr}_x^{\mathcal{C}}\left(\begin{pmatrix} 0_p & X \\ Y & 0_q \end{pmatrix}\right) = 0. \tag{3.16}$$

Now, suppose that f is an arbitrary block matrix of the from

$$f = \begin{pmatrix} A_p & X \\ Y & B_q \end{pmatrix}, \tag{3.17}$$

where $A_p \in \mathrm{End}_{\mathcal{C}}(x_p)$ and $B_q \in \mathrm{End}_{\mathcal{C}}(x_q)$, where $x_p = \oplus_{i=1}^n x_i$ and $x_q = \oplus_{j=p+1}^n x_j$. Then,

$$\begin{aligned} \mathrm{tr}_x^{\mathcal{C}}(f) &= \mathrm{tr}_x^{\mathcal{C}}\left(\begin{pmatrix} A_p & 0 \\ 0 & 0_q \end{pmatrix}\right) + \mathrm{tr}_x^{\mathcal{C}}\left(\begin{pmatrix} 0_p & 0 \\ 0 & B_q \end{pmatrix}\right) && \text{(by equation (3.16))} \\ &= \mathrm{tr}_{x_p}^{\mathcal{C}}(A_p) + \mathrm{tr}_{x_q}^{\mathcal{C}}(B_q) && \text{(by additivity).} \end{aligned} \tag{3.18}$$

Now, the original claim follows inductively from the last equation. ☐

We now define functors between Calabi-Yau categories:

Definition 3.7. Let $(\mathcal{C}, \mathrm{tr}^{\mathcal{C}})$ and $(\mathcal{D}, \mathrm{tr}^{\mathcal{D}})$ be two Calabi-Yau categories. A linear functor $F : \mathcal{C} \to \mathcal{D}$ is called a Calabi-Yau functor, if

$$\mathrm{tr}_c^{\mathcal{C}}(f) = \mathrm{tr}_{F(c)}^{\mathcal{D}}(F(f)) \tag{3.19}$$

for each object $c \in \mathcal{C}$ and each $f \in \mathrm{End}_{\mathcal{C}}(c)$. Equivalently, one may require that

$$\langle Ff, Fg \rangle_{\mathcal{D}} = \langle f, g \rangle_{\mathcal{C}} \tag{3.20}$$

for every pair of morphisms $f : c \to c'$ and $g : c' \to c$ in $\mathcal{C}$.

If F, $G : \mathcal{C} \to \mathcal{D}$ are two Calabi-Yau functors between Calabi-Yau categories, a natural transformation of Calabi-Yau functors is just an ordinary natural transformation.

Definition 3.8. Let CY be the bigroupoid consisting of

objects: Calabi-Yau categories, which are by definition finite and semisimple,

1-morphisms: equivalences of Calabi-Yau categories as in definition 3.7,

2-morphisms: natural isomorphisms.

Note that CY has got a symmetric monoidal structure, given by the Deligne tensor product of abelian categories, cf. [Del90]. We begin by showing some basic properties of Calabi-Yau categories.

Lemma 3.9. *Let F, $G : \mathcal{C} \to \mathcal{D}$ be two functors between two Calabi Yau categories $(\mathcal{C}, \mathrm{tr}^{\mathcal{C}})$ and $(\mathcal{D}, \mathrm{tr}^{\mathcal{D}})$. If F and G are naturally isomorphic, then F is a Calabi-Yau functor if and only if G is a Calabi-Yau functor.*

Proof. Let x be an object of $\mathcal{C}$, and let $f \in \mathrm{End}_{\mathcal{C}}(x)$. Let $\eta : F \Rightarrow G$ be a natural isomorphism. Since η is natural, the following diagram commutes:

$$\begin{tikzcd} F(x) \arrow[r, "F(f)"] \arrow[d, "\eta_x"] & F(x) \arrow[d, "\eta_x"] \\ G(x) \arrow[r, "G(f)"] & G(x) \end{tikzcd} \tag{3.21}$$

If F is a Calabi-Yau functor, then

$$\begin{aligned} \mathrm{tr}^{\mathcal{C}}_x(f) &= \mathrm{tr}^{\mathcal{D}}_{F(x)}(F(f)) && \text{(since } F \text{ is Calabi-Yau)} \\ &= \mathrm{tr}^{\mathcal{D}}_{F(x)}(\eta_x^{-1} \circ G(f) \circ \eta_x) && \text{(since } \eta \text{ is natural)} \\ &= \mathrm{tr}^{\mathcal{D}}_{G(x)}(G(f)) && \text{(since the trace is symmetric).} \end{aligned} \tag{3.22}$$

This shows that G is a Calabi-Yau functor.

If G is a Calabi-Yau functor, then

$$\begin{aligned} \mathrm{tr}^{\mathcal{C}}_x(f) &= \mathrm{tr}^{\mathcal{D}}_{G(x)}(G(f)) && \text{(since } G \text{ is Calabi-Yau)} \\ &= \mathrm{tr}^{\mathcal{D}}_{G(x)}(\eta_x \circ F(f) \circ \eta_x^{-1}) && \text{(since } \eta \text{ is natural)} \\ &= \mathrm{tr}^{\mathcal{D}}_{F(x)}(F(f)) && \text{(since the trace is symmetric).} \end{aligned} \tag{3.23}$$

This shows that F is a Calabi-Yau functor. □

Lemma 3.10. *Let $(\mathcal{C}, \mathrm{tr}^{\mathcal{C}})$ and $(\mathcal{D}, \mathrm{tr}^{\mathcal{D}})$ be two Calabi-Yau categories, and let*

$$F : \mathcal{C} \rightleftarrows \mathcal{D} : G \tag{3.24}$$

be an equivalence of linear categories. Then F is a Calabi-Yau functor if and only if G is a Calabi-Yau functor.

Proof. Let $\eta : FG \Rightarrow \mathrm{id}_{\mathcal{D}}$ and $\nu : GF \Rightarrow \mathrm{id}_{\mathcal{D}}$ be natural isomorphisms.
If F is a Calabi-Yau functor and $f \in \mathrm{End}_{\mathcal{D}}(x)$ for some $x \in \mathcal{D}$, then

$$\begin{aligned} \mathrm{tr}_x^{\mathcal{D}}(f) &= \mathrm{tr}_x^{\mathcal{D}}(\eta_x \circ FG(f) \circ \eta_x^{-1}) && \text{(since } \eta \text{ is natural)} \\ &= \mathrm{tr}_{FG(x)}^{\mathcal{D}}(FG(f)) && \text{(since the trace is symmetric)} \\ &= \mathrm{tr}_{G(x)}^{\mathcal{C}}(G(f)) && \text{(since } F \text{ is Calabi-Yau).} \end{aligned} \tag{3.25}$$

Hence, G is Calabi-Yau.
If on the other hand G is already Calabi-Yau and $f \in \mathrm{End}_{\mathcal{C}}(x)$ for some $x \in \mathcal{C}$, then

$$\begin{aligned} \mathrm{tr}_x^{\mathcal{C}}(f) &= \mathrm{tr}_x^{\mathcal{C}}(\nu \circ GF(f) \circ \nu^{-1}) && \text{(since } \nu \text{ is natural)} \\ &= \mathrm{tr}_{GF(x)}^{\mathcal{C}}(GF(f)) && \text{(since the trace is symmetric)} \\ &= \mathrm{tr}_{F(x)}^{\mathcal{D}}(F(f)) && \text{(since } G \text{ is Calabi-Yau).} \end{aligned} \tag{3.26}$$

This shows that F is Calabi-Yau. $\square$

Next, we show that *every* finite, linear, semisimple category admits the structure of a Calabi-Yau category. This is analogous to the fact that every finite-dimensional, semisimple algebra admits the structure of a Frobenius algebra.

Lemma 3.11. *Let $\mathcal{C}$ be a finite semisimple linear category over an algebraically closed field $\mathbb{K}$ with n simple objects. Then, $\mathcal{C}$ has got a structure of a Calabi-Yau category. Furthermore, the set of Calabi-Yau structures on $\mathcal{C}$ stands in bijection to $(\mathbb{K}^*)^n$.*

Proof. If $\mathcal{C}$ has got the structure of a Calabi-Yau category, the trace $\mathrm{tr}^{\mathcal{C}}$ will be additive by lemma 3.6. Hence, the trace $\mathrm{tr}^{\mathcal{C}}$ is uniquely determined by the endomorphism algebras of the simple objects. If X is a simple object of $\mathcal{C}$, Schur's lemma shows that $\mathrm{End}_{\mathcal{C}}(X) \cong \mathbb{K}$ as vector spaces, since the ground field $\mathbb{K}$ is algebraically closed and $\mathcal{C}$ is finite. One now checks that choosing

$$\mathrm{tr}_X^{\mathcal{C}} : \mathrm{End}_{\mathcal{C}}(X) \cong \mathbb{K} \to \mathbb{K} \tag{3.27}$$

to be the identity for every simple object X indeed defines the structure of a Calabi-Yau category on $\mathcal{C}$. This shows the first claim.

Now note that for a simple object X, due to its symmetry the trace $\mathrm{tr}_X^{\mathcal{C}}$ is unique up to multiplication with an invertible central element in $Z(\mathrm{End}_{\mathcal{C}}(X)) \cong \mathbb{K}$. Thus, the trace $\mathrm{tr}_X^{\mathcal{C}}$ on $\mathrm{End}_{\mathcal{C}}(X)$ is unique up to a non-zero element in $\mathbb{K}$. Taking direct sums now shows the second claim. $\square$

3.1.1. Calabi-Yau categories as homotopy fixed points

We now apply theorem 2.35 which computes homotopy fixed points of the trivial $SO(2)$-action on an arbitrary bicategory to Calabi-Yau categories. Let Vect_2 be the bicategory consisting of linear, abelian categories, linear functors, and natural transformations. The fully-dualizable objects of Vect_2 are then precisely the finite, semisimple linear categories as in definition 3.1, cf. [BDSPV15, Appendix A]. As a corollary to theorem 2.35 we obtain:

Corollary 3.12. *Let $\mathbb{K}$ be an algebraically closed field. Let $\mathcal{C} = \mathscr{K}(\mathrm{Vect}_2^{\mathrm{fd}})$, the core of fully-dualizable objects of* Vect_2*, and consider the trivial $SO(2)$-action on $\mathcal{C}$. Then $\mathcal{C}^{SO(2)}$ is equivalent to* CY*, the bicategory of Calabi-Yau categories.*

Proof. Indeed, by theorem 2.35 a homotopy fixed point of the trivial $SO(2)$-action on $\mathscr{K}(\mathrm{Vect}_2^{\mathrm{fd}})$ consists of a finite, linear, semisimple category $\mathcal{C}$, together with a natural isomorphism $\lambda : \mathrm{id}_{\mathcal{C}} \to \mathrm{id}_{\mathcal{C}}$. Let $X_1, \dots, X_n$ be the simple objects of $\mathcal{C}$. Then, the natural transformation $\lambda : \mathrm{id}_{\mathcal{C}} \to \mathrm{id}_{\mathcal{C}}$ is fully determined by giving an endomorphism $\lambda_X : X \to X$ for every simple object X. Since λ is an invertible natural transformation, the λ_X must be central invertible elements in $\mathrm{End}_{\mathcal{C}}(X)$. As we work over an algebraically closed field, Schur's Lemma shows that $\mathrm{End}_{\mathcal{C}}(X) \cong \mathbb{K}$ as vector spaces. Hence, the structure of a natural transformation of the identity functor of $\mathcal{C}$ boils down to choosing a non-zero scalar for each simple object of $\mathcal{C}$. This structure is equivalent to giving $\mathcal{C}$ the structure of a Calabi-Yau category by lemma 3.11.

Now note that by equation (2.78) in theorem 2.35, 1-morphisms of homotopy fixed points consist of equivalences of categories $F : \mathcal{C} \to \mathcal{C}'$ so that $F(\lambda_X) = \lambda'_{F(X)}$ for every object X of $\mathcal{C}$. This is exactly the condition saying that F must a Calabi-Yau functor.

Finally, one sees that 2-morphisms of homotopy fixed points are given by natural isomorphisms of Calabi-Yau functors. □

3.2. Constructing an equivalence between Frobenius algebras and Calabi-Yau categories

The purpose of this section is to construct a weak 2-functor $\mathrm{Rep}^{\mathrm{fg}} : \mathrm{Frob} \to \mathrm{CY}$, which sends a separable, symmetric Frobenius algebra to its category of finitely generated modules, equipped with the additional structure of a Calabi-Yau category, which comes from the Frobenius form. This weak 2-functor will turn out to be an equivalence of bigroupoids. The construction uses standard material about separable algebras and projective modules, which we recall below. For further background, we refer to [Pie82], [SY11], [AW92], or [Lam12].

3.2.1. Separable algebras and projective modules

We begin by recalling some standard material about separable algebras and finitely-generated modules. In the following, R will always be a commutative, unital ring. All algebras considered will be associative and unital. If A is an R-algebra, the enveloping algebra $A^e := A^{\mathrm{op}} \otimes A$ is an A^e-module in a natural way. One way of defining separability of an algebra is as follows:

Definition 3.13. An R-algebra A is said to be separable, if A is a projective A^e module.

An equivalent way of encoding separability goes via the multiplication $\mu : A \otimes A^{\mathrm{op}} \to A$:

Proposition 3.14 ([Pie82])**.** *Let A be an R-algebra. Then, the following are equivalent:*

1. *The algebra A is separable.*

2. *The sequence*
$$0 \to \ker \mu \to A^e \xrightarrow{\mu} A \to 0 \tag{3.28}$$
is split exact.

3. *There is an* $e \in A^e$ *with* $\mu(e) = 1_A$ *and* $xe = ex$ *for all* $x \in A$.

We will call the element $e \in A^e$ of the last proposition a separability idempotent. One reason we are interested in separable algebras is due to their nice categorical properties. Following the proof of proposition 1.5 in [Jan66], we show the following lemma.

Lemma 3.15. *Let* A *be separable* R*-algebra. If* P *is an* A*-module which is projective as an* R*-module, then* P *is projective as an* A*-module.*

Proof. Let P be a projective R-module, and let
$$0 \to M \xrightarrow{\alpha} N \to P \to 0 \tag{3.29}$$
be an exact sequence of A-modules. Since P is a projective R-module, there is an $f \in \mathrm{Hom}_R(N, M)$, so that $f \circ \alpha = \mathrm{id}_M$. Let $e = \sum_{i=1}^n x_i \otimes y_i \in A \otimes_R A$ be a separability idempotent for A. Recall that this means that
$$\sum_{i=1}^n x_i y_i = 1_A, \tag{3.30}$$
and
$$(a \otimes 1_A)e = (1_A \otimes a)e \tag{3.31}$$
for all $a \in A$.

Now, define a map $f' : N \to M$ as
$$f'(n) := \sum_{i=1}^n x_i.f(y_i.n). \tag{3.32}$$

We claim that f' is a morphism of A-modules which splits the exact sequence in equation (3.29). Indeed, since
$$\sum_{i=1}^n (ax_i) \otimes y_i = \sum_{i=1}^n x_i \otimes (y_i a) \tag{3.33}$$
by equation (3.31), applying $f(-.n)$ to the second tensorand yields
$$\sum_{i=1}^n ax_i \otimes f(y_i.n) = \sum_{i=1}^n x_i \otimes f(y_i a.n). \tag{3.34}$$

Therefore,
$$a.f'(n) = \sum_{i=1}^n ax_i.f(y_i.n) = \sum_{i=1}^n x_i.f(y_i a.n) = f'(a.n). \tag{3.35}$$

This shows that f' is a morphism of A-modules.

Furthermore,

$$(f' \circ \alpha)(m) = \sum_{i=1}^{n} x_i.f(y_i.\alpha(m)) = \sum_{i=1}^{n} x_i.f(\alpha(y_i.m)) = \sum_{i=1}^{n} x_i y_i.m = m. \tag{3.36}$$

This shows that the exact sequence splits as A-modules. □

Corollary 3.16. *If A is a separable algebra over a field $\mathbb{K}$, then every module over A is projective. Hence, A is semisimple.*

Proof. Since $\mathbb{K}$ is a field, every $\mathbb{K}$-module M is projective. By lemma 3.15, M is also projective as an A-module. □

Another nice property of separable algebras is that they are finitely generated:

Proposition 3.17 ([Pie82])**.** *Let A be a separable R-algebra that is projective as an R-module. Then A is a finitely-generated R-module.*

Corollary 3.18. *Any separable algebra over a field is finitely-generated and thus finite-dimensional.*

Proof. This follows from the above proposition since every $\mathbb{K}$-algebra is free as a $\mathbb{K}$-module and thus projective. □

Next, we review the so-called dual basis lemma for projective modules. References for the following are [Lam12, AW92]. If M is a left A-module, the dual module

$$M^* := \mathrm{Hom}_A(M, A) \tag{3.37}$$

is a right A-module with right action given by $(f.a)(m) := f(m).a$. These modules admit a "basis" in the following sense:

Lemma 3.19 (Dual basis lemma)**.** *Let R be a commutative ring, let A be a R-algebra, and let P be a left A-module. The following are equivalent:*

1. *The module P is finitely generated and projective.*
2. *There are $f_1, \ldots, f_n \in P^*$ and $p_1, \ldots, p_n \in P$ (sometimes called dual- or projective basis of P) so that*
$$x = \sum_{i=1}^{n} f_i(x).p_i \tag{3.38}$$
for all $x \in P$.
3. *The map*
$$\begin{aligned} \Psi_{P,P} : P^* \otimes_A P &\to \mathrm{End}_A(P) \\ f \otimes p &\mapsto (x \mapsto f(x).p) \end{aligned} \tag{3.39}$$
is an isomorphism of R-modules.

4. For any other left A-module M, the map

$$\begin{aligned} \Psi_{P,M} : P^* \otimes_A M &\to \mathrm{Hom}_A(P, M) \\ f \otimes m &\mapsto (x \mapsto f(x).m) \end{aligned} \tag{3.40}$$

is an isomorphism of R-modules.

Proof. (1) $\Rightarrow$ (2): Suppose that P is projective and finitely generated. Let $p_1, \ldots, p_n$ be a set of generators for P, and let F be a free module on n generators $x_1, \ldots, x_n$. Define an map $g : F \to P$ by setting $g(x_i) := p_i$ for $1 \leq i \leq n$. Since g is defined on generators, the map g is a morphism of R-modules which is even surjective, since the p_i generate P.

Since P is projective, every epimorphism splits. Thus, there is a morphism of A-modules $h : P \to F$, so that $g \circ h = \mathrm{id}_P$. Since F is a free finitely-generated A-module on n generators, it is isomorphic to A^n. Let $\pi_i : F \cong A^n \to A$ be the projections, and define $f_i := \pi_i \circ h$. Then,

$$h(a) = \sum_{i=1}^{n} f_i(a).x_i \tag{3.41}$$

since F is free, and by applying g to equation (3.41), we see that

$$a = (g \circ h)(a) = \sum_{i=1}^{n} g(f_i(a).x_i) = \sum_{i=1}^{n} f_i(a).g(x_i) = \sum_{i=1}^{n} f_i(a).p_i \tag{3.42}$$

for every $a \in P$. This shows (2).

(2) $\Rightarrow$ (1): Suppose that $f_1, \ldots, f_n$ and $p_1, \ldots, p_n$ is a dual basis. Then, the p_i generate P by equation (3.38). Thus, P is finitely generated. As in the proof of (1) $\Rightarrow$ (2), let F be a free module on generators $x_1, \ldots, x_n$ and define a surjective A-linear map $g : F \to P$ by $g(x_i) := p_i$. Now, define a map

$$\begin{aligned} h : P &\to F \\ x &\mapsto \sum_{i=1}^{n} f_i(x).x_i. \end{aligned} \tag{3.43}$$

Then, h splits g because

$$(g \circ h)(x) = g\left(\sum_{i=1}^{n} f_i(x).x_i\right) = \sum_{i=1}^{n} f_i(x).p_i = x. \tag{3.44}$$

Therefore, P is a direct summand of the free module F and hence projective.

(2) $\Rightarrow$ (4): Let $f_1, \ldots, f_n \in P^*$ and $p_1, \ldots, p_n \in P$ be a projective basis of P as in (2). We claim: the map

$$\begin{aligned} \varphi : \mathrm{Hom}_A(P, M) &\to P^* \otimes_A M \\ g &\mapsto \sum_{i=1}^{n} f_i \otimes g(p_i) \end{aligned} \tag{3.45}$$

is an inverse to $\Psi_{P,M}$. Indeed,

$$(\Psi_{P,M} \circ \varphi)(g) = \Psi_{P,M}\left(\sum_{i=1}^{n} f_i \otimes g(p_i)\right) = x \mapsto \sum_{i=1}^{n} f_i(x).g(p_i) = g. \tag{3.46}$$

The last equality follows by applying g to equation (3.38). On the other hand,

$$\begin{aligned}(\varphi \circ \Psi_{P,M})(p^* \otimes m) &= \varphi(x \mapsto p^*(x).m) = \sum_{i=1}^{n} f_i \otimes p^*(p_i).m = \sum_{i=1}^{n} f_i.p^*(p_i) \otimes m \\ &= \sum_{i=1}^{n}(x \mapsto f_i(x).p^*(p_i)) \otimes m = p^* \otimes m.\end{aligned} \tag{3.47}$$

In the last equality, we have used that P^* is a right A-module. The last equality follows again by applying p^* to equation (3.38). This shows that $\Psi_{P,M}$ is an isomorphism.

(4) $\Rightarrow$ (3) is trivial, since we may choose $M := P$.

(3) $\Rightarrow$ (2): Suppose that $\Psi_{P,P} : P^* \otimes_A P \to \mathrm{End}_A(P)$ is an isomorphism. Then,

$$\Psi_{P,P}^{-1}(\mathrm{id}_P) = \sum_{i,j}^{n} f_i \otimes p_j \tag{3.48}$$

is a dual basis. Indeed,

$$\sum_{i=1}^{n} f_i(x).p_i = \Psi_{P,P}\left(\sum_{i=1}^{n} f_i \otimes p\right)(x) = \mathrm{id}_P(x) = x. \tag{3.49}$$

$\square$

Corollary 3.20. *Let A be a separable algebra over a field $\mathbb{K}$, and let M be a finitely generated A-module. Then, the map $\Psi_{M,M} : M^* \otimes_A M \to \mathrm{End}_A(M)$ of lemma 3.19 is an isomorphism of A-modules.*

Proof. This follows from the fact that *every* module over a separable $\mathbb{K}$-algebra is projective, which is proven in corollary 3.16. Hence, by the first part of lemma 3.19, the map $\Psi_{M,M} : M^* \otimes_A M \to \mathrm{Hom}_A(M, M)$ is an isomorphism. $\square$

This corollary enables us to define a trace for finitely-generated modules over a separable symmetric Frobenius algebra, as the next subsection shows.

3.2.2. A Calabi-Yau structure on the representation category of a Frobenius algebra

Here, we will show that the category of finitely generated modules over a separable symmetric Frobenius algebra over an algebraically closed field $\mathbb{K}$ has the structure of a Calabi-Yau category in the sense of definition 3.2, and thus construct the 2-functor $\mathrm{Rep}^{\mathrm{fg}} : \mathrm{Frob} \to \mathrm{CY}$ on objects.

Definition 3.21. Let (A, λ) be a separable symmetric Frobenius algebra over a field $\mathbb{K}$ with Frobenius form $\lambda : A \to \mathbb{K}$. Let M be a finitely-generated left A-module. Denote by

$$\begin{aligned} \mathrm{ev} : M^* \otimes_A M &\to A \\ f \otimes m &\mapsto f(m) \end{aligned} \tag{3.50}$$

the evaluation.

Since M is finitely generated, the map $\Psi_{M,M} : \mathrm{End}_A(M) \to M^* \otimes_A M$ is an isomorphism by corollary 3.20. We define a trace $\mathrm{tr}^\lambda_M : \mathrm{End}_A(M) \to \mathbb{K}$ by the composition

$$\mathrm{tr}^\lambda_M : \mathrm{End}_A(M) \xrightarrow{\Psi^{-1}_{M,M}} M^* \otimes_A M \xrightarrow{\mathrm{ev}} A \xrightarrow{\lambda} \mathbb{K}. \tag{3.51}$$

Remark 3.22. As defined here, the trace tr^λ_M is the composition of the Hattori-Stallings trace with the Frobenius form λ. For more on the Hattori-Stallings trace, see [Hat65], [Sta65] and [Bas76].

Example 3.23. Let (A, λ) be a separable symmetric Frobenius algebra over a field $\mathbb{K}$. Suppose that F is a free A-module with basis $e_1, \ldots, e_n$. Then,

$$\mathrm{tr}^\lambda_F(\mathrm{id}_F) = n\lambda(1_A). \tag{3.52}$$

Example 3.24. As a second example, let $A := M_n(\mathbb{K})$ be the algebra of $n \times n$-matrices over $\mathbb{K}$ with Frobenius form λ given by the usual trace of matrices. Then, $M := \mathbb{K}^n$ is a projective (but not free), simple A-module. We claim:

$$\mathrm{tr}^\lambda_M(\mathrm{id}_M) = 1. \tag{3.53}$$

Indeed, let $e_1, \ldots, e_n$ be a vector space basis of $\mathbb{K}^n$. This basis also generates $\mathbb{K}^n$ as an A-module. Define for each $1 \leq i \leq n$ a $\mathbb{K}$-linear map $f_i^* : \mathbb{K}^n \to M_n(\mathbb{K}) = A$ by setting

$$f_i^*(e_k) := \delta_{i,1} E_{k,1}, \tag{3.54}$$

where $E_{k,1}$ is the square matrix with $(k,1)$-entry given by one and zero otherwise. A short calculation confirms that the f_i^* are even morphisms of A-modules. Indeed, if $M \in A$, then

$$(M.f_i^*(e_k))_{p,q} = \delta_{1,i}\delta_{1,q} M_{p.k} = (f_i^*(M.e_k))_{p,q}. \tag{3.55}$$

Next, we claim that

$$\Psi^{-1}_{M,M} = \sum_{i=1}^n f_i^* \otimes e_i \in M^* \otimes_A M. \tag{3.56}$$

Indeed,

$$\begin{aligned} \Psi_{M,M}\left(\sum_{i=1}^n f_i^* \otimes e_i\right)(e_k) &= \sum_{i=1}^n f_i^*(e_k).e_i \\ &= \sum_{i=1}^n \delta_{i,1} E_{k,1} e_i \\ &= E_{k,1} e_1 = e_k. \end{aligned} \tag{3.57}$$

Thus,

$$\mathrm{tr}^{\lambda}_{M}(\mathrm{id}_M) = \lambda\left(\sum_{i=1}^{n} f_i^*(e_i)\right) = \lambda\left(\sum_{i=1}^{n} E_{i,1}\delta_{1,i}\right) = \lambda(E_{1,1}) = 1. \tag{3.58}$$

Next, we show that $\mathrm{tr}^{\lambda}_{M}$ has indeed the properties of a trace. In order to show that the trace is symmetric, we need an additional lemma first.

Lemma 3.25. *Let A be an $\mathbb{K}$-algebra, and let M and N be left A-modules. Define a linear map*

$$\begin{aligned} \xi : (M^* \otimes_A N) \times (N^* \otimes_A M) &\to M^* \otimes_A M \\ (f \otimes n, g \otimes m) &\mapsto f \otimes g(n).m. \end{aligned} \tag{3.59}$$

Then, the following diagram commutes:

$$\begin{tikzcd} (M^* \otimes_A N) \times (N^* \otimes_A M) \arrow[r, "\xi"] \arrow[d, "\Psi_{M,N}\times\Psi_{N,M}"] & M^* \otimes_A M \arrow[d, "\Psi_{M,M}"] \\ \mathrm{Hom}_A(M,N) \times \mathrm{Hom}_A(N,M) \arrow[r, "\circ"] & \mathrm{Hom}_A(M,M) \end{tikzcd} \tag{3.60}$$

Here, the horizontal map at the bottom is given by composition of morphisms of A-modules and $\Psi_{M,M}$ is defined as in equation (3.40)*.*

Proof. We calculate:

$$\begin{aligned} (\Psi_{M,M} \circ \xi)(f \otimes n, g \otimes m) &= \Psi(f \otimes g(n).m) \\ &= x \mapsto f(x)g(n).m. \end{aligned} \tag{3.61}$$

On the other hand,

$$\begin{aligned} \Psi_{M,N}(g \otimes m) \circ \Psi_{N,M}(f \otimes n) &= (x \mapsto g(x).m) \circ (x \mapsto f(x).n) = x \mapsto g(f(x).n).m \\ &= x \mapsto f(x)g(n).m. \end{aligned} \tag{3.62}$$

Comparing the right hand-side of equation (3.61) with the right hand-side of equation (3.62) shows that the diagram commutes. □

We are now ready to show that the trace is symmetric:

Lemma 3.26. *Let (A, λ) be a separable, symmetric Frobenius algebra over a field $\mathbb{K}$. Let M and N be finitely-generated A-modules, and let $f : M \to N$ and $g : N \to M$ be morphisms of A-modules. Then, the trace is symmetric:*

$$\mathrm{tr}^{\lambda}_{M}(g \circ f) = \mathrm{tr}^{\lambda}_{N}(f \circ g). \tag{3.63}$$

Proof. Write

$$\begin{aligned} \Psi^{-1}_{M,N}(f) &= \sum_{i,j} m_i^* \otimes n_j \in M^* \otimes_A N \qquad \text{and} \\ \Psi^{-1}_{N,M}(g) &= \sum_{k,l} x_k^* \otimes y_l \in N^* \otimes_A M. \end{aligned} \tag{3.64}$$

We calculate:

$$\begin{aligned}
\mathrm{tr}^{\lambda}_{M}(g \circ f) &= (\lambda \circ \mathrm{ev} \circ \Psi^{-1}_{M,M})(g \circ f) \\
&= (\lambda \circ \mathrm{ev}) \left(\sum_{i,j,k,l} m_i^* \otimes x_k^*(n_j).y_l \right) \qquad \text{(by lemma 3.25)} \\
&= \lambda \left(\sum_{i,j,k,l} m_i^*(x_k^*(n_j).y_l) \right) \\
&= \sum_{i,j,k,l} \lambda(x_k^*(n_j) \cdot m_i^*(y_l)).
\end{aligned} \tag{3.65}$$

On the other hand,

$$\begin{aligned}
\mathrm{tr}^{\lambda}_{N}(f \circ g) &= (\lambda \circ \mathrm{ev} \circ \Psi^{-1}_{N,N})(f \circ g) \\
&= (\lambda \circ \mathrm{ev}) \left(\sum_{i,j,k,l} x_k^* \otimes m_i^*(y_l).n_j \right) \qquad \text{(by lemma 3.25)} \\
&= \lambda \left(\sum_{i,j,k,l} x_k^*(m_i^*(y_l).n_j) \right) \\
&= \sum_{i,j,k,l} \lambda(m_i^*(y_l) \cdot x_k^*(n_j)).
\end{aligned} \tag{3.66}$$

Since λ is symmetric, the right hand-sides of equations (3.65) and (3.66) agree. This shows that the trace is symmetric. ☐

Historically, the Hattori-Stallings trace has been defined by using bases. This is also possible for the trace in definition 3.21, as the next remark shows.

Remark 3.27. Let (A, λ) be a separable, symmetric Frobenius algebra over a field $\mathbb{K}$, and let M be a finitely-generated A-module.

If M is a free A-module, we may express the trace in a basis of M: if $f \in \mathrm{End}_A(M)$, choose a basis $e_1, \dots, e_n$ of M, and let $e_1^*, \dots, e_n^*$ be the dual basis. Let $(A_f)_{ij} := e_i^*(f(e_j)) \in A$. Then,

$$\mathrm{tr}^{\lambda}_{M}(f) = (\lambda \circ \mathrm{ev}) \left(\sum_{i=1}^{n} e_i^* \otimes f(e_i) \right) = \sum_{i=1}^{n} \lambda(e_i^*(f(e_i))) = \sum_{i=1}^{n} \lambda((A_f)_{ii}) \in \mathbb{K}. \tag{3.67}$$

Since the trace is symmetric by lemma 3.26, we know that $\mathrm{tr}^{\lambda}_{M}(g \circ f \circ g^{-1}) = \mathrm{tr}^{\lambda}_{M}(f)$ for any isomorphism g, and therefore the trace is independent of the basis. If M is only projective and not necessarily free, there is an A-module Q so that $F := M \oplus Q$ is free. Let $e_1 = m_1 \oplus q_1, \dots, e_n = m_n \oplus q_n$ be a basis of F, and let $e_1^* = m_1^* \oplus q_1^*, \dots, e_n^* = m_1^* \oplus q_n^*$ be the dual basis. Using the additivity of the trace as in lemma 3.5, we have

$$\mathrm{tr}^{\lambda}_{M}(f) = \mathrm{tr}^{\lambda}_{M \oplus Q}(f \oplus 0) = \sum_{i=1}^{n} \lambda(e_i^*((f \oplus 0)(e_i))) = \sum_{i=1}^{n} \lambda(m_i^*(f(m_i))). \tag{3.68}$$

Since the trace is symmetric, this expression is independent of the basis of F. Now, a standard argument (cf. [Sta65, 1.7]) shows that equation (3.68) is also independent of the complement Q. For the sake of completeness, we recall this argument here.

Suppose there is another Q' so that $M \oplus Q' := F'$ is free. Let α be the isomorphism

$$\alpha : Q \oplus F' = Q \oplus (M \oplus Q') \cong Q' \oplus (M \oplus Q) = Q' \oplus F. \tag{3.69}$$

Using that the trace is additive as in lemma 3.5 shows that

$$\begin{aligned} \mathrm{tr}^{\lambda}_{M\oplus Q}(f \oplus 0_Q) &= \mathrm{tr}^{\lambda}_{M\oplus Q\oplus F'}(f \oplus 0_Q \oplus 0_{F'}) \qquad \text{and} \\ \mathrm{tr}^{\lambda}_{M\oplus Q'}(f \oplus 0_{Q'}) &= \mathrm{tr}^{\lambda}_{M\oplus Q'\oplus F}(f \oplus 0_{Q'} \oplus 0_F). \end{aligned} \tag{3.70}$$

Since

$$(\mathrm{id}_M \oplus \alpha)^{-1} \circ (f \oplus 0_{Q'} \oplus 0_F) \circ (\mathrm{id}_M \oplus \alpha) = f \oplus 0_{Q'} \oplus 0_{F'}, \tag{3.71}$$

using the cyclic invariance of the traces shows that

$$\mathrm{tr}^{\lambda}_{M\oplus Q}(f \oplus 0_Q) = \mathrm{tr}^{\lambda}_{M\oplus Q'}(f \oplus 0_{Q'}), \tag{3.72}$$

as required. Thus, equation (3.68) is independent of the complement Q.

Next, we show that the trace is non-degenerate.

Lemma 3.28. *Let (A, λ) be a separable, symmetric Frobenius algebra over an algebraically closed field $\mathbb{K}$, and let M and N be finitely-generated A-modules. Then, the bilinear pairing of vector spaces induced by the trace in definition 3.21*

$$\begin{aligned} \langle -, - \rangle : \mathrm{Hom}_A(M, N) \times \mathrm{Hom}_A(N, M) &\to \mathbb{K} \\ (f, g) &\mapsto \mathrm{tr}^{\lambda}_M(g \circ f) \end{aligned} \tag{3.73}$$

is non-degenerate.

Proof. By Artin-Wedderburn's theorem, the algebra A is isomorphic to a direct product of matrix algebras over $\mathbb{K}$:

$$A \cong \prod_{i=1}^{r} M_{n_i}(\mathbb{K}). \tag{3.74}$$

Since the sum of the usual trace of matrices gives each A the structure of a symmetric Frobenius algebra, lemma 2.9 shows that the Frobenius form λ of A is given by

$$\lambda = \sum_{i=1}^{r} \lambda_i \mathrm{tr}_i, \tag{3.75}$$

where $\mathrm{tr}_i : M_{n_i}(\mathbb{K}) \to \mathbb{K}$ is the usual trace of matrices and $\lambda_i \in \mathbb{K}^*$ are non-zero scalars.

Recall that a module over a finite-dimensional algebra is finite-dimensional (as a vector space) if and only if it is finitely generated as a module, cf. [SY11, Proposition 2.5]. A classical theorem in representation theory (cf. theorem 3.3.1 in [EGH+11]) asserts that the only finite-dimensional simple modules of A are given by $V_1 := \mathbb{K}^{n_1}, \dots V_r := \mathbb{K}^{n_r}$. Since the category $(A\text{-Mod})^{\mathrm{fg}}$ is semisimple, we may decompose the finitely-generated A-modules M and N as the direct sum of simple modules:

$$\begin{aligned} M &\cong \left(\bigoplus_{i_1=1}^{l_1} \mathbb{K}^{n_1}\right) \oplus \left(\bigoplus_{i_2=1}^{l_2} \mathbb{K}^{n_2}\right) \oplus \dots \oplus \left(\bigoplus_{i_r=1}^{l_r} \mathbb{K}^{n_r}\right) \\ N &\cong \left(\bigoplus_{i_1=1}^{l'_1} \mathbb{K}^{n_1}\right) \oplus \left(\bigoplus_{i_2=1}^{l'_2} \mathbb{K}^{n_2}\right) \oplus \dots \oplus \left(\bigoplus_{i_r=1}^{l'_r} \mathbb{K}^{n_r}\right). \end{aligned} \tag{3.76}$$

By Schur's lemma, any $f \in \mathrm{Hom}_A(M, N)$ is given by $f = f_1 \oplus f_2 \oplus \ldots \oplus f_r$ where f_i is a $l'_i \times l_i$-matrix. Similarly, any $g \in \mathrm{Hom}_A(N, M)$ is given by $g = g_1 \oplus g_2 \oplus \ldots \oplus g_r$ where each g_i is a $l_i \times l'_i$ matrix. Thus,

$$\begin{aligned} \mathrm{tr}^\lambda_M(g \circ f) &= \mathrm{tr}^\lambda_M((g_1 f_1) \oplus (g_2 f_2) \oplus \ldots \oplus g_r f_r) \\ &= \sum_{i=1}^{r} \mathrm{tr}^\lambda_{(\mathbb{K}^{n_i})^{l_i}}(g_i f_i) && \text{(by additivity)} \\ &= \sum_{i=1}^{r} \sum_{j=1}^{l_i} \mathrm{tr}^\lambda_{\mathbb{K}^{n_i}}((g_i f_i)_{j,j}) && \text{(by lemma 3.6)} \\ &= \sum_{i=1}^{r} \sum_{j=1}^{l_i} \sum_{k=1}^{l'_i} \mathrm{tr}^\lambda_{\mathbb{K}^{n_i}}\left((g_i)_{j,k} \circ (f_i)_{k,j}\right). \end{aligned} \tag{3.77}$$

Since f was assumed to be non-zero, at least one $(f_i)_{j,k}$ is non-zero. Suppose that $(f_{\tilde{i}})_{\tilde{j},\tilde{k}} \in \mathrm{End}_A(\mathbb{K}^{n_i})$ is not the zero morphism. By Schur's lemma, $(f_{\tilde{i}})_{\tilde{j},\tilde{k}}$ is an isomorphism. Now define $g \in \mathrm{Hom}_A(N, M)$ as

$$(g_i)_{j,k} := (\lambda_{\tilde{i}})^{-1} \delta_{i,\tilde{i}} \delta_{j,\tilde{j}} \delta_{k,\tilde{k}} (f_{\tilde{i}})^{-1}_{\tilde{k},\tilde{j}}. \tag{3.78}$$

Then, by example 3.24,

$$\mathrm{tr}^\lambda_M(g \circ f) = (\lambda_{\tilde{i}})^{-1} \mathrm{tr}^\lambda_{\mathbb{K}^{n_{\tilde{i}}}}(\mathrm{id}_{\mathbb{K}^{n_{\tilde{i}}}}) = 1_{\mathbb{K}} \neq 0. \tag{3.79}$$

□

We summarize the situation with the following proposition:

Proposition 3.29. *Let (A, λ) be a separable symmetric Frobenius algebra over an algebraically closed field $\mathbb{K}$. Then, the category of finitely-generated A-modules $(A\text{-Mod})^{\mathrm{fg}}$ has got the structure of a Calabi-Yau category with trace $\mathrm{tr}^\lambda_M : \mathrm{End}_A(M) \to \mathbb{K}$ as defined in equation (3.51).*

Proof. Since A a separable $\mathbb{K}$-algebra, A is finite-dimensional by corollary 3.18. By lemma B.5, all finitely generated A-modules are necessarily finite-dimensional. It is well-known that the category of finite-dimensional modules over a finite-dimensional algebra is a finite, linear category, cf. [DSPS14]. Since A is a separable $\mathbb{K}$-algebra, all A-modules are projective by corollary 3.16. Hence, $(A\text{-Mod})^{\mathrm{fg}}$ is semisimple.

If M is a finitely-generated A-module, the trace $\mathrm{tr}^{\lambda}(M) : \mathrm{End}(M) \to \mathbb{K}$ as defined in equation (3.51) is symmetric by lemma 3.26, while the induced bilinear form is non-degenerate by lemma 3.28. This shows that $(A\text{-Mod})^{\mathrm{fg}}$ is a Calabi-Yau category. □

The following example shows that the assumption that the algebra A is separable is a necessary condition.

Example 3.30 (Counter-example)**.** Let $\mathbb{K}$ be a field of characteristic two, and consider the group algebra $A := \mathbb{K}[\mathbb{Z}_2]$. Then, $A \cong \mathbb{K}[x]/(x^2-1) \cong \mathbb{K}[x]/(x-1)^2$. This is in fact a Frobenius algebra with Frobenius form $\lambda(g) = \delta_{g,e}$, which is not separable. Let S be the trivial representation, and consider a projective two-dimensional representation of A which we shall call P. Here, the non-trivial generator g of A acts on P by the matrix

$$g = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}. \tag{3.80}$$

One easily computes that

$$\mathrm{Hom}(P,S) \cong \left\{ \begin{pmatrix} 0 & b \end{pmatrix} \mid b \in \mathbb{K} \right\}, \qquad \text{and} \qquad \mathrm{Hom}(S,P) \cong \left\{ \begin{pmatrix} a \\ 0 \end{pmatrix} \mid a \in \mathbb{K} \right\}. \tag{3.81}$$

We claim that there is no trace on the representation category of A. Indeed, let $\mathrm{tr}_S : \mathrm{End}(S) \to \mathbb{K}$ be any linear map. Then, the pairing

$$\mathrm{Hom}(S,P) \otimes \mathrm{Hom}(P,S) \to \mathbb{K} \tag{3.82}$$

$$\begin{pmatrix} a \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & b \end{pmatrix} \mapsto \mathrm{tr}_S \left(\begin{pmatrix} 0 & b \end{pmatrix} \begin{pmatrix} a \\ 0 \end{pmatrix} \right) = 0 \tag{3.83}$$

is always degenerate. Therefore, a non-degenerate pairing does not exist.

3.2.3. Constructing the 2-functor $\mathrm{Rep}^{\mathrm{fg}}$ on 1-morphisms

The next step of the construction will be the value of $\mathrm{Rep}^{\mathrm{fg}}$ on 1-morphisms of Frob, which are compatible Morita contexts. To these, we will have to assign equivalences of Calabi-Yau categories. Let us recall a classical theorem from Morita theory:

Theorem 3.31 ([Bas68, Theorem 3.4 and 3.5])**.** *Let A and B be R-algebras, and let $({}_BM_A, {}_AN_B, \varepsilon, \eta)$ be a Morita context between A and B. Then,*

1. *M and N are both finitely-generated and projective as B-modules.*
2. *An A-module X is finitely generated over A if and only if $M \otimes_A X$ is finitely generated over B.*

3. The functor

$$M \otimes_A - : A\text{-Mod} \to B\text{-Mod} \tag{3.84}$$

is an equivalence of linear categories.

This theorem suggests that we should define $\mathrm{Rep}^{\mathrm{fg}}$ on Morita contexts by the functor $M \otimes_A -$. In order for this to be well-defined, this functor should be a Calabi-Yau functor as in definition 3.7 if the Morita context is compatible with the Frobenius forms as in definition 2.6. In order to show this, we need an additional lemma:

Lemma 3.32. *Let A and B be two separable $\mathbb{K}$-algebras. Let $\mathcal{M} = (M, N, \varepsilon, \eta)$ be a Morita context between A and B. Write*

$$\varepsilon^{-1}(1_A) = \sum_{i,j} n_i \otimes m_j \in N \otimes_B M. \tag{3.85}$$

For another finitely-generated left A-module T, define a linear map

$$\begin{aligned} \xi : T^* \otimes_A T &\to (M \otimes_A T)^* \otimes_B (M \otimes_A T) \\ t^* \otimes t &\mapsto \left(\left(x \otimes y \mapsto \sum_i \eta^{-1}\left(x.t^*(y) \otimes n_i\right)\right) \otimes \sum_j m_j \otimes t \right). \end{aligned} \tag{3.86}$$

Then, the following diagram commutes.

$$\begin{tikzcd} T^* \otimes_A T \arrow[r, "\xi"] \arrow[d, "\Psi_{T,T}"'] & (M \otimes_A T)^* \otimes_B (M \otimes_A T) \arrow[d, "\Psi_{M\otimes_A T, M\otimes_A T}"] \\ \mathrm{End}_A(T) \arrow[r, "\mathrm{id}_M \otimes -"'] & \mathrm{End}_B(M \otimes_A T) \end{tikzcd} \tag{3.87}$$

Proof. First note that

$$\sum_{i,j} \eta^{-1}(x \otimes n_i).m_j = x \tag{3.88}$$

for every x in M, since ε and η are part of a Morita context. Now, we calculate:

$$(\mathrm{id}_M \otimes - \circ \Psi_{T,T})(t^* \otimes t)(x \otimes y) = (\mathrm{id}_M \otimes -)(y \mapsto t^*(y).t)(x \otimes y) = x \otimes t^*(y).t. \tag{3.89}$$

On the other hand,

$$\begin{aligned} &(\Psi_{M\otimes_A T, M\otimes_A T} \circ \xi)(t^* \otimes t)(x \otimes y) = \\ &= (\Psi_{M\otimes_A T, M\otimes_A T}) \left(\left(x \otimes y \mapsto \sum_i \eta^{-1}\left(x.t^*(y) \otimes n_i\right)\right) \otimes \sum_j m_j \otimes t \right) (x \otimes y) \\ &= \sum_{i,j} \eta^{-1}(x.t^*(y) \otimes n_i).(m_j \otimes t) \\ &= \sum_{i,j} \eta^{-1}(x.t^*(y) \otimes n_i).m_j \otimes t \\ &= x.t^*(y) \otimes t, \end{aligned} \tag{3.90}$$

where in the last line, we have used equation (3.88). This shows that the diagram commutes. □

The next proposition shows that the compatibility condition on the Morita context between two Frobenius algebras in definition 2.6 is equivalent to the fact that tensoring with the bimodule M of the Morita context is a Calabi-Yau functor:

Proposition 3.33. *Let (A, λ^A) and (B, λ^B) be two separable symmetric Frobenius algebras over an algebraically closed field $\mathbb{K}$, and let $(M, N, \varepsilon, \eta)$ be a Morita context between A and B. Endow $\mathrm{Rep}^{\mathrm{fg}}(A)$ and $\mathrm{Rep}^{\mathrm{fg}}(B)$ with the Calabi-Yau structure as in definition 3.21. Then, the Morita context is compatible with the Frobenius forms λ^A and λ^B as in definition 2.6 if and only if*

$$(M \otimes_A -) : \mathrm{Rep}(A)^{\mathrm{fg}} \to \mathrm{Rep}(B)^{\mathrm{fg}} \tag{3.91}$$

is a Calabi-Yau functor as in definition 3.7.

Proof. Let ${}_AT$ be a finitely-generated left A-module. By definition, the functor $M \otimes_A -$ is a Calabi-Yau functor if and only if

$$\mathrm{tr}^{\lambda^B}_{M\otimes_A T}(\mathrm{id}_M \otimes f) = \mathrm{tr}^{\lambda^A}_T(f) \tag{3.92}$$

for all $f \in \mathrm{End}_A(T)$. We have to calculate the left hand-side: Let $f \in \mathrm{End}_A(T)$ and write

$$\Psi^{-1}_{T,T}(f) = \sum_{i,j} t_i^* \otimes t_j \in T^* \otimes_A T. \tag{3.93}$$

Using n_i and m_j as introduced in formula (3.85), lemma 3.32 shows that

$$\begin{aligned} \Psi^{-1}_{M\otimes_A T, M\otimes_A T}(\mathrm{id}_M \otimes f) &= \xi \circ \Psi^{-1}_{T,T}(f) \\ &= \sum_{i,j} \xi(t_i^* \otimes t_j) \\ &= \left(x \otimes y \mapsto \sum_{k,i} \eta^{-1}\left(x.t_i^*(y) \otimes n_k\right) \right) \otimes \sum_{l,j} m_l \otimes t_j. \end{aligned} \tag{3.94}$$

Hence,

$$\begin{aligned} \mathrm{tr}^{\lambda^B}_{M\otimes_A T}(\mathrm{id}_M \otimes f) &= (\lambda^B \circ \mathrm{ev} \circ \Psi^{-1}_{M\otimes_A T, M\otimes_A T})(\mathrm{id}_M \otimes f) \\ &= \sum_{i,j,k,l} \lambda^B(\eta^{-1}(m_l.t_i^*(t_j) \otimes n_k)). \end{aligned} \tag{3.95}$$

Since

$$\mathrm{tr}^{\lambda^A}_T(f) = \sum_{i,j} \lambda^A(t_i^*(t_j)), \tag{3.96}$$

the functor $M \otimes_A -$ is a Calabi-Yau functor if and only if the right hand sides of equations (3.95) and (3.96) agree for every $t_i \in T^*$ and $t_j \in T$. Using the fact that the Frobenius forms are symmetric, and thus factor through $A/[A, A]$, this is the case if and only if the Morita context is compatible with the Frobenius forms as in equation (2.16). □

Definition 3.34. Using proposition 3.33 enables us to define the 2-functor $\mathrm{Rep}^{\mathrm{fg}}$ on 1-morphisms of the bicategory Frob: we assign to a compatible Morita context $\mathcal{M} := (M, N, \varepsilon, \eta)$ between two separable symmetric Frobenius algebras A and B the equivalence of Calabi-Yau categories $\mathrm{Rep}^{\mathrm{fg}}(\mathcal{M})$ given by

$$\mathrm{Rep}^{\mathrm{fg}}(\mathcal{M}) := (M \otimes_A -) : \mathrm{Rep}^{\mathrm{fg}}(A) \to \mathrm{Rep}^{\mathrm{fg}}(B). \tag{3.97}$$

3.2.4. Constructing the 2-functor $\mathrm{Rep}^{\mathrm{fg}}$ on 2-morphisms

Let $(M, N, \varepsilon, \eta)$ and $(M', N', \varepsilon', \eta')$ be two compatible Morita contexts between two separable symmetric Frobenius algebras A and B, and let $\alpha : M \to M'$ and $\beta : N \to N'$ be a morphism of Morita contexts. We define a natural transformation $\mathrm{Rep}^{\mathrm{fg}}((\alpha, \beta)) : (M \otimes_A -) \to (M' \otimes_A -)$ as follows: for every left A-module ${}_AX$, we define the component of the natural transformation as

$$\mathrm{Rep}^{\mathrm{fg}}((\alpha, \beta))_X := (\alpha \otimes \mathrm{id}_X) : M \otimes_A X \to M' \otimes_A X. \tag{3.98}$$

This is indeed a natural transformation because for every intertwiner $f : {}_AX \to {}_AY$ of left A-modules, the following diagram

$$\begin{tikzcd} M \otimes_A X \arrow[r, "\alpha\otimes \mathrm{id}_X"] \arrow[d, "\mathrm{id}_M \otimes f"'] & M' \otimes_A X \arrow[d, "\mathrm{id}_{M'} \otimes f"] \\ M \otimes_A Y \arrow[r, "\alpha\otimes \mathrm{id}_Y"'] & M' \otimes_A Y \end{tikzcd} \tag{3.99}$$

commutes.

Thus, we have obtained the following weak 2-functor $\mathrm{Rep}^{\mathrm{fg}} : \mathrm{Frob} \to \mathrm{CY}$, which sends

- a symmetric Frobenius algebra (A, λ) to the Calabi-Yau category $\left(\mathrm{Rep}^{\mathrm{fg}}(A), \mathrm{tr}^{\lambda}\right)$,
- a compatible Morita context $({}_BM_A, {}_BN_A, \varepsilon, \eta)$ to the Calabi-Yau functor

$$\left(M \otimes_A - : (\mathrm{Rep}^{\mathrm{fg}}(A), \mathrm{tr}^{\lambda^A}) \to (\mathrm{Rep}^{\mathrm{fg}}(B), \mathrm{tr}^{\lambda^B})\right), \tag{3.100}$$

- and a morphism of Morita contexts

$$((\alpha, \beta) : ({}_BM_A, {}_BN_A, \varepsilon, \eta) \to ({}_BM'_A, {}_BN'_A, \varepsilon', \eta')) \tag{3.101}$$

  to the natural transformation $(\alpha \otimes \mathrm{id}_- : (M \otimes_A -) \to (M' \otimes_A -))$.

Observe that by the definition of the Deligne tensor product, this weak 2-functor is compatible with the symmetric monoidal structures of Frob and CY, and thus can be equipped with the additional structure of a symmetric monoidal 2-functor. This follows from the fact that a Calabi-Yau structure on $(A \otimes B)$-Mod $\cong$ A-Mod $\boxtimes$ B-Mod is equivalent to a Calabi-Yau structure on A-Mod and on B-Mod.

3.3. Proving the equivalence

The aim of this section is to prove that the weak 2-functor $\mathrm{Rep}^{\mathrm{fg}} : \mathrm{Frob} \to \mathrm{CY}$ of section 3.2 is an equivalence of bicategories. This will be done in several steps. First, we show that $\mathrm{Rep}^{\mathrm{fg}}$ is essentially surjective. Let $\mathbb{K}$ be an algebraically closed field, and let $(\mathcal{C}, \mathrm{tr}^{\mathcal{C}})$ be a Calabi-Yau category over $\mathbb{K}$. Let $X_1, \ldots, X_n$ be representatives of the isomorphism classes of simple objects of $\mathcal{C}$, and define an object P of $\mathcal{C}$ as $P := \oplus_{i=1}^n X_i$. Then, $A := \mathrm{End}_{\mathcal{C}}(P)$ is a separable, symmetric Frobenius algebra over $\mathbb{K}$ with Frobenius form λ given by $\lambda := \mathrm{tr}_P^{\mathcal{C}}$. By proposition 3.29, the category $(A\text{-Mod})^{\mathrm{fg}}$ has the structure of a Calabi-Yau category. We claim:

Proposition 3.35. *The functor*

$$\mathrm{Hom}_{\mathcal{C}}(P, -) : \mathcal{C} \to (A\text{-Mod})^{\mathrm{fg}} \tag{3.102}$$

is an equivalence of Calabi-Yau categories.

Proof. It is clear that $\mathrm{tr}_P^{\mathcal{C}}$ endows $A := \mathrm{End}_{\mathcal{C}}(P)$ with the structure of a symmetric Frobenius algebra. Since $\mathcal{C}$ is semisimple, the object P is the sum of the finitely many simple objects P_i. By Schur's lemma, the endomorphism algebra of a simple object P_i is isomorphic to a matrix algebra over a division algebra. This is equivalent to the separability of $\mathrm{End}_{\mathcal{C}}(P)$, cf. [SY11, Theorem 11.11].

We now show that the functor $\mathrm{Hom}_{\mathcal{C}}(P, -)$ is an equivalence of Calabi-Yau categories. An exercise of [EGH$^+$11] which is proven in the appendix in proposition B.4 asserts that the functor $\mathrm{Hom}_{\mathcal{C}}(P, -)$ is an equivalence of linear categories. Thus, our claim amounts to showing that this functor is compatible with the traces as required in definition 3.7.

Write an object Y of $\mathcal{C}$ as an arbitrary sum of simple objects, so that $Y = \oplus_{j=1}^m Y_j$, and let $f \in \mathrm{End}_{\mathcal{C}}(Y)$. Since $\mathcal{C}$ is an additive category, we can represent f as an $m \times m$ matrix

$$f = \begin{pmatrix} f_{1,1} & \cdots & f_{1,m} \\ \vdots & & \vdots \\ f_{m,1} & \cdots & f_{m,m} \end{pmatrix} \tag{3.103}$$

where $f_{k,l} \in \mathrm{Hom}_{\mathcal{C}}(Y_l, Y_k)$.

Similarly, any $g \in \mathrm{Hom}_{\mathcal{C}}(P, Y)$ is naturally a $m \times n$ matrix with entries

$$g = \begin{pmatrix} g_{1,1} & \cdots & g_{1,m} \\ \vdots & & \vdots \\ g_{n,1} & \cdots & g_{n,m} \end{pmatrix} \tag{3.104}$$

where $g_{i,k} \in \mathrm{Hom}_{\mathcal{C}}(Y_k, Y_i)$.

Under this identification, $A = \mathrm{End}_{\mathcal{C}}(P)$ acts on $\mathrm{Hom}_{\mathcal{C}}(P, Y)$ as $a.f := f \cdot a$ where $f \cdot a$ is the matrix product of f and a.

Then, the morphism $\mathrm{Hom}_{\mathcal{C}}(P, f)$ is given by

$$\begin{aligned} \mathrm{Hom}_{\mathcal{C}}(P, f) : \mathrm{Hom}_{\mathcal{C}}(P, Y) &\to \mathrm{Hom}_{\mathcal{C}}(P, Y) \\ g &\mapsto f \cdot g \end{aligned} \tag{3.105}$$

where $f \cdot g$ is the matrix product of f and g. As a first step to calculate the trace in $(A\text{-Mod})^{\text{fg}}$, we claim that

$$\Psi^{-1}_{\mathrm{Hom}_{\mathcal{C}}(P,Y),\mathrm{Hom}_{\mathcal{C}}(P,Y)}(\mathrm{Hom}_{\mathcal{C}}(P,f)) = \delta^* \otimes \tilde{f}, \tag{3.106}$$

as an element of $\mathrm{Hom}_{\mathcal{C}}(P,Y)^* \otimes_A \mathrm{Hom}_{\mathcal{C}}(P,Y)$, where $\delta^* \in \mathrm{Hom}_{\mathcal{C}}(P,Y)^*$ and $\tilde{f} \in \mathrm{Hom}_{\mathcal{C}}(P,Y)$ are defined as follows. First, define the $m \times n$-matrix

$$\tilde{f}_{k,r} := \begin{cases} f_{k,r} & \text{if } r \le m, \\ 0 & \text{else.} \end{cases} \tag{3.107}$$

Now, given a $m \times n$ matrix $g \in \mathrm{Hom}_{\mathcal{C}}(P,Y)$, the element $\delta^*(g)$ of A is defined to be an $n \times n$ matrix with entries

$$\delta^*(g)_{r,l} := \begin{cases} g_{r,l} & \text{if } r \le m, \\ 0 & \text{else.} \end{cases} \tag{3.108}$$

Then, if $g \in \mathrm{Hom}_{\mathcal{C}}(P,Y)$,

$$\begin{aligned} (\Psi((\delta)^* \otimes \tilde{f})(g))_{k,l} &= (\delta^*(g).\tilde{f})_{k,l} \\ &= (\tilde{f} \cdot \delta^*(g))_{k,l} \\ &= \sum_{r=1}^{n} \tilde{f}_{k,r} \cdot \delta^*(g)_{r,l} \\ &= \sum_{r=1}^{m} f_{k,r} \circ g_{r,l} \\ &= (\mathrm{Hom}_{\mathcal{C}}(P,f)(g))_{k,l}. \end{aligned} \tag{3.109}$$

This shows equation (3.106).

We may now calculate the trace of the morphism $\mathrm{Hom}_{\mathcal{C}}(P,f)$ in $(A\text{-Mod})^{\text{fg}}$. By definition of the trace in $(A\text{-Mod})^{\text{fg}}$, we have

$$\mathrm{tr}^{\lambda}_{\mathrm{Hom}_{\mathcal{C}}(P,Y)}(\mathrm{Hom}_{\mathcal{C}}(P,f)) = (\mathrm{tr}_P \circ \mathrm{ev} \circ \Psi^{-1}_{\mathrm{Hom}_{\mathcal{C}}(P,Y),\mathrm{Hom}_{\mathcal{C}}(P,Y)})(\mathrm{Hom}_{\mathcal{C}}(P,f)). \tag{3.110}$$

Now,

$$\begin{aligned} \mathrm{tr}^{\lambda}_{\mathrm{Hom}_{\mathcal{C}}(P,Y)}(\mathrm{Hom}_{\mathcal{C}}(P,f)) &= (\mathrm{tr}_P \circ \mathrm{ev})(\delta^* \otimes \tilde{f}) && \text{(by equation (3.106))} \\ &= \mathrm{tr}_P(\delta^*(\tilde{f})) \\ &= \sum_{i=1}^{n} \delta^*(\tilde{f})_{ii} && \text{(by lemma 3.6)} \\ &= \sum_{i=1}^{m} f_{i,i} \\ &= \mathrm{tr}^{\mathcal{C}}_Y(f) && \text{(by lemma 3.6).} \end{aligned} \tag{3.111}$$

This shows that $\mathrm{Hom}_{\mathcal{C}}(P,-)$ is a Calabi-Yau functor. □

Next, we follow the exposition in [Bas68, Proposition 3.1] and show that the functor $\mathrm{Rep}^{\mathrm{fg}}$ is essentially surjective on 1-morphisms. In detail:

Proposition 3.36. *Let $\mathbb{K}$ an algebraically closed field, and let (A, λ^A) and (B, λ^B) be two separable, symmetric Frobenius algebras. Endow $(A\text{-Mod})^{\mathrm{fg}}$ and $(B\text{-Mod})^{\mathrm{fg}}$ with the Calabi-Yau structure described in proposition 3.29, and let*

$$F : (A\text{-Mod})^{\mathrm{fg}} \rightleftarrows (B\text{-Mod})^{\mathrm{fg}} : G \tag{3.112}$$

be an equivalence of Calabi-Yau categories.

Then, there is a compatible Morita context $\mathcal{M}$ between A and B, so that $\mathrm{Rep}^{\mathrm{fg}}(\mathcal{M})$ is naturally isomorphic to the functor F.

Proof. The proof essentially works by an application of Eilenberg-Watts and by checking that everything is compatible with the traces. Define a (B, A)-bimodule M as $M := F(A)$ which is naturally a left B-module, and a right A-module by using the map

$$A \cong \mathrm{End}_A(A) \xrightarrow{F} \mathrm{End}_B(M). \tag{3.113}$$

The Eilenberg-Watts theorem then shows that the functor F is naturally isomorphic to $M \otimes_A -$ (cf. theorem 1 in [Wat60]). Thus, by lemma 3.9, F is a Calabi-Yau functor if and only if $M \otimes_A -$ is a Calabi-Yau functor. Similarly, there is an (A, B)-bimodule N given by $N := G(B)$, so that the functor G is naturally isomorphic to $N \otimes_B -$.

Furthermore, there are isomorphisms of bimodules

$$\begin{aligned} \varepsilon &: N \otimes_B M \cong G(M) \cong G(M \otimes_A A) \cong G(F(A)) \cong A \\ \eta &: B \cong F(G(B)) \cong F(N \otimes_B B) \cong F(N) \cong M \otimes_A N \end{aligned} \tag{3.114}$$

since F and G is an equivalence of categories. We claim that we can choose these isomorphisms in such a way that $(M, N, \varepsilon, \eta)$ becomes a Morita context.

Indeed, by lemma 2.3 it suffices to show that diagram (2.5) commutes. Let $r_B : N \otimes_B B \to N$ be right-multiplication, and let $l_A : A \otimes_B N \to N$ be left-multiplication.

Since ε, η, r_B and l_A are isomorphisms of bimodules, there is a $u \in \mathrm{Aut}_{(A,B)}(N)$, so that

$$r_B \circ \mathrm{id}_N \otimes \eta^{-1} = u \circ l_A \circ \varepsilon \otimes \mathrm{id}_N. \tag{3.115}$$

In particular,

$$u \in \mathrm{Hom}_A(N, N) \cong \mathrm{Hom}_A(G(B), G(B)) \cong \mathrm{Hom}_B(B, B). \tag{3.116}$$

Since every morphism of left B-modules $u \in \mathrm{Hom}_B(B, B)$ is given by right multiplication with an element of B, we may identify u with this element. Since u is also a morphism of right B-modules, the element u is in the center of B.

Now define an isomorphism of (B, B)-bimodules

$$\begin{aligned} \tilde{\eta}^{-1} : M \otimes_A N &\to B \\ m \otimes n &\mapsto u.\eta^{-1}(n \otimes m). \end{aligned} \tag{3.117}$$

Thus, if we replace η by $\tilde{\eta}$ we have made diagram (2.4) commute. Hence, $(M, N, \varepsilon, \tilde{\eta})$ is a Morita context, which is compatible with the Frobenius forms by proposition 3.33. □

The following theorem summarizes the discussion:

Theorem 3.37. *The weak 2-functor* $\mathrm{Rep}^{\mathrm{fg}} : \mathrm{Frob} \to \mathrm{CY}$ *is an equivalence of bigroupoids.*

Proof. In a first step, proposition 3.29 shows the representation category of a separable symmetric Frobenius algebra has indeed got the structure of a finite, semisimple Calabi-Yau category. Furthermore, this assignment is essentially surjective on objects: given a finite, semisimple Calabi-Yau category $\mathcal{C}$, proposition 3.35 shows how to construct a Frobenius algebra A so that $\mathrm{Rep}^{\mathrm{fg}}(A)$ and $\mathcal{C}$ are equivalent as Calabi-Yau categories.

Now, given a compatible Morita context between two Frobenius algebras A and B, proposition 3.33 shows how to construct a Calabi-Yau functor between the representation categories. Furthermore, this assignment is essentially surjective by proposition 3.36.

Finally, one shows by hand that $\mathrm{Rep}^{\mathrm{fg}}$ induces a bijection on 2-morphisms of Frob and CY, which carry no additional structures or properties. $\square$

Remark 3.38. Note that the bigroupoid Frob, as well as the bigroupoid CY have additional symmetric monoidal structure. It is not difficult to see that the weak 2-functor $\mathrm{Rep}^{\mathrm{fg}}$ is compatible with this symmetric monoidal structure, and is thus an equivalence of symmetric monoidal bigroupoids.

3.3.1. The functor of representations as equivarinatization

So far, we have constructed an equivalence of bicategories $\mathrm{Rep}^{\mathrm{fg}} : \mathrm{Frob} \to \mathrm{CY}$. In this section, we show that this equivalence is actually the "equivariantization" of the 2-functor sending an algebra to its category of modules.

In order to do so, we use the notion of a "equivariantization" of a weak 2-functor between bicategories equipped with a G-action from section 2.3, where G is a topological group. Let us briefly recall the relevant definitions: for a group G, we denote with BG the category with one object and G as morphisms. Similarly, if $\mathcal{C}$ is a monoidal (bi-)category, $B\mathcal{C}$ will denote the (tri-)bicategory with one object and $\mathcal{C}$ as endomorphism (bi-)category of this object.

For a topological group G, let $\Pi_2(G)$ be its fundamental 2-groupoid, and $B\Pi_2(G)$ the tricategory with one object called $*$ and $\Pi_2(G)$ as endomorphism bicategory. A G-action on a bicategory $\mathcal{C}$ was defined to be a trifunctor $\rho : B\Pi_2(G) \to \mathrm{Bicat}$ with $\rho(*) = \mathcal{C}$, where Bicat is the tricategory of bicategories. Furthermore, given a G-action ρ on a bicategory, we define the bicategory of homotopy fixed points $\mathcal{C}^G$ to be the bicategory $\mathrm{Nat}(\Delta, \rho)$ where objects are given by tritransformations between the constant functor Δ and ρ, 1-morphisms are modifications, and 2-morphisms are perturbations. In the following, we will use the notation of section 2.2 concerning homotopy fixed points. Using this notation, we were able to give a concrete definition of a G-equivariant structure on a weak 2-functor between two bicategories equipped with a G-action in section 2.3. This concrete definition allows us to show the following lemma.

Lemma 3.39. *Let* $F : \mathscr{K}(\mathrm{Alg}_2^{\mathrm{fd}}) \to \mathscr{K}(\mathrm{Vect}_2^{\mathrm{fd}})$ *be any functor, and let* $\rho : \Pi_2(SO(2)) \to \mathrm{Aut}(\mathscr{K}(\mathrm{Alg}_2^{\mathrm{fd}}))$ *and* $\rho' : \Pi_2(SO(2)) \to \mathrm{Aut}(\mathscr{K}(\mathrm{Vect}_2^{\mathrm{fd}}))$ *be the trivial* $SO(2)$*-actions.*

Then, F has got a canonical $SO(2)$-equivariant structure given by taking identities everywhere.

Proof. We need to provide the data in definition 2.26: Since both actions are trivial, we may choose $T_g : \mathrm{Rep} \to \mathrm{Rep}$ to be the identity pseudo-natural transformation for every $g \in G$, and $T_\gamma : \mathrm{id}_{\mathrm{Rep}} \circ T_g \to T_h \circ \mathrm{id}_{\mathrm{Rep}}$ to be the identity modification for every path $\gamma : g \to h$. Furthermore, we may also choose P_{gh} and N to be the identity modifications. □

Corollary 3.40. *Choosing F to be the weak 2-functor* Rep *which sends an algebra to its category of finitely-generated modules shows that* Rep *has a canonical $SO(2)$-equivariant structure.*

Since the representation functor is $SO(2)$-equivariant, it induces a functor on homotopy fixed point bicategories by definitions 2.28, 2.29 and 2.30. We claim:

Theorem 3.41. *The diagram*

$$\begin{tikzcd} (\mathscr{K}(\mathrm{Alg}_2^{\mathrm{fd}}))^{SO(2)} \arrow[r, "\mathrm{Rep}^{SO(2)}"] \arrow[d, "\wr"'] & (\mathscr{K}(\mathrm{Vect}_2^{\mathrm{fd}}))^{SO(2)} \arrow[d, "\wr"] \\ \mathrm{Frob} \arrow[r, "\mathrm{Rep}^{\mathrm{fg}}"'] & \mathrm{CY} \end{tikzcd} \tag{3.118}$$

commutes up to a pseudo-natural isomorphism. Here, the unlabeled equivalences are induced by corollary 2.36 and corollary 3.12, while the functor $\mathrm{Rep}^{\mathrm{fg}}$ *is constructed in section 3.2.*

Proof. Let

$$\begin{aligned} F &: (\mathscr{K}(\mathrm{Alg}_2^{\mathrm{fd}}))^{SO(2)} \cong \mathrm{Frob} \\ G &: (\mathscr{K}(\mathrm{Vect}_2^{\mathrm{fd}}))^{SO(2)} \cong \mathrm{CY} \end{aligned} \tag{3.119}$$

be the equivalences of bicategories in corollaries 2.36 and 3.12. By theorem 2.35, the bicategory $(\mathscr{K}(\mathrm{Alg}_2^{\mathrm{fd}}))^{SO(2)}$ is equivalent to a bicategory where objects are given by separable algebras A, together with an isomorphism of Morita contexts $\lambda : \mathrm{id}_A \to \mathrm{id}_A$, where id_A is the identity Morita context, consisting of the algebra A considered as an (A, A)-bimodule. If (A, λ) is an object of $(\mathscr{K}(\mathrm{Alg}_2^{\mathrm{fd}}))^{SO(2)}$, we need to construct an equivalence of Calabi-Yau categories

$$\eta_{(A,\lambda)} : (G \circ \mathrm{Rep}^{SO(2)})(A, \lambda) \to (\mathrm{Rep}^{\mathrm{fg}} \circ F)(A, \lambda). \tag{3.120}$$

By definition 2.28, the value of $\mathrm{Rep}^{SO(2)}$ on A is given by $\mathrm{Rep}^{\mathrm{fg}}(A)$, the category of finitely-generated modules of A. The value of $\mathrm{Rep}^{SO(2)}$ on λ is given by the natural isomorphism defined as follows: if ${}_AM$ is an A-module, the natural transformation $\mathrm{Rep}^{SO(2)}(\lambda)$ of the identity functor on $\mathrm{Rep}^{\mathrm{fg}}(A)$ is given in components by

$$\mathrm{Rep}^{SO(2)}(\lambda)_M := \left(M \cong A \otimes_A M \xrightarrow{\lambda \otimes \mathrm{id}_M} A \otimes_A M \cong M \right). \tag{3.121}$$

We know that A is isomorphic to a direct sum of matrix algebras:

$$A \cong \bigoplus_{i=1}^{r} M_{d_i}(\mathbb{K}). \tag{3.122}$$

Let

$$(\lambda_1, \ldots, \lambda_r) \in \mathbb{K}^r \cong Z(A) \cong \mathrm{End}_{(A,A)}(A) \tag{3.123}$$

be the scalars corresponding to the isomorphism of Morita contexts λ. Then, the Calabi-Yau structure on $(G \circ \mathrm{Rep}^{SO(2)})(A, \lambda)$ is given as follows: it suffices to write down a trace for the simple modules, because $\mathrm{Rep}^{\mathrm{fg}}(A)$ is semisimple. If X_i is a simple A-module, chasing through the equivalence G shows that the trace is given by identifying the division algebra $\mathrm{End}_{\mathrm{Rep}^{\mathrm{fg}}(A)}(X_i)$ with the algebraically closed ground field $\mathbb{K}$ by Schur's lemma, and then (up to a permutation of the simple modules) multiplying with the scalar λ_i.

On the other hand, chasing through the equivalence of bicategories F in corollary 2.36, we see that the Frobenius algebra $F(A, \lambda)$ is given by the separable algebra A as in equation (3.122), together with the Frobenius form given by taking direct sums of matrix traces, multiplied with the scalars λ_i in equation (3.123). Using the construction of the functor $\mathrm{Rep}^{\mathrm{fg}}$ in section 3.2 shows that the Calabi-Yau category $(\mathrm{Rep}^{\mathrm{fg}} \circ F)(A, \lambda)$ is given by the linear category $\mathrm{Rep}^{\mathrm{fg}}(A)$, together with the Calabi-Yau structure given by the composite of the Frobenius form with the Hattori-Stallings trace. For a simple module X_i, this Calabi-Yau structure is given by multiplying with the scalar λ_i under the identification $\mathrm{End}_{\mathrm{Rep}^{\mathrm{fg}}(A)}(X_i) \cong \mathbb{K}$. Thus, we have succeeded in finding an equivalence η as required in equation (3.120). Going through the equivalences F and G, we check that η is even pseudo-natural. This shows that the diagram (3.118) commutes up to a pseudo-natural isomorphism. $\square$

4. The Serre automorphism as a homotopy action

In this chapter, which is based on results of [HV17], we explicitly construct a non-trivial $SO(2)$-action on a skeletal version of the 2-dimensional framed bordism bicategory. By the 2-dimensional cobordism hypothesis for framed manifolds, we obtain an $SO(2)$-action on the core of fully-dualizable objects of the target bicategory. This action is shown to coincide with the one given by the Serre automorphism. We give an explicit description of the bicategory of homotopy fixed points of this action, and prove that this bicategory classifies oriented 2-dimensional topological quantum field theories.

The chapter is organized as follows: in section 4.1 we recall the notion of a fully-dualizable object in a symmetric monoidal bicategory $\mathcal{C}$. For each such an object X, we define the Serre automorphism as a certain 1-endomorphism of X. We show that the Serre automorphism is a pseudo-natural transformation of the identity functor on $\mathscr{K}(\mathcal{C}^{\mathrm{fd}})$, which is moreover monoidal. This suffices to define an $SO(2)$-action on $\mathscr{K}(\mathcal{C}^{\mathrm{fd}})$. As a corollary of theorem 2.34 of section 2, we then obtain an explicit description of the bicategory of homotopy fixed points of this action.

Section 4.2 investigates *monoidal* group actions on bicategories. We obtain a general criterion for when such an action is trivializable, and comment on the $SO(2)$-action on invertible field theories.

In section 4.3, we introduce a skeletal version of the framed bordism bicategory by generators and relations, and define a non-trivial $SO(2)$-action on this bicategory. By the framed cobordism hypothesis, we obtain an $SO(2)$-action on $\mathscr{K}(\mathcal{C}^{\mathrm{fd}})$, which we prove to be given by the Serre automorphism.

4.1. Fully-dualizable objects and the Serre automorphism

The aim of this section is to introduce the main players of the present chapter. On the algebraic side, these are fully-dualizable objects in a symmetric monoidal bicategory $\mathcal{C}$, and the Serre automorphism. Although parts of the following material has already appeared in the literature, we recall the relevant definitions in order to fix notation. For background on symmetric monoidal bicategories, we refer the reader to [Pst14] and [SP09]. We begin by recalling dual pairs in a monoidal bicategory, following [Pst14, Definition 2.3].

Definition 4.1. A dual pair in a symmetric monoidal bicategory $\mathcal{C}$ consists of an object X, an object X^*, two 1-morphisms

$$\begin{aligned} \mathrm{ev}_X &: X \otimes X^* \to 1 \\ \mathrm{coev}_X &: 1 \to X^* \otimes X \end{aligned} \tag{4.1}$$

and two invertible 2-morphisms α and β in the diagrams below.

$$\begin{tikzcd}
 & & X \otimes (X^* \otimes X) \arrow[r, "a"] & (X \otimes X^*) \otimes X \arrow[dr, "\mathrm{ev}_X \otimes \mathrm{id}_X"] & & \\
 & X \otimes 1 \arrow[ur, "\mathrm{id}_X \otimes \mathrm{coev}_X"] \arrow[rrr, phantom, "\Downarrow\alpha"] & & & 1 \otimes X \arrow[dr, "l"] & \\
X \arrow[ur, "r"] \arrow[rrrrr, "\mathrm{id}_X"'] & & & & & X
\end{tikzcd} \tag{4.2}$$

$$\begin{tikzcd}
 & & (X^* \otimes X) \otimes X^* \arrow[r, "a"] & X^* \otimes (X \otimes X^*) \arrow[dr, "\mathrm{id}_{X^*} \otimes \mathrm{ev}_X"] & & \\
 & 1 \otimes X^* \arrow[ur, "\mathrm{coev}_X \otimes \mathrm{id}_{X^*}"] \arrow[rrr, phantom, "\Downarrow\beta"] & & & X^* \otimes 1 \arrow[dr, "r"] & \\
X^* \arrow[ur, "l"] \arrow[rrrrr, "\mathrm{id}_{X^*}"'] & & & & & X^*
\end{tikzcd} \tag{4.3}$$

We call an object X of $\mathcal{C}$ dualizable if it can be completed to a dual pair. A dual pair is said to be coherent if the "swallowtail" equations are satisfied, as in [Pst14, Def. 2.6].

Remark 4.2. Given a dual pair, it is always possible to modify the 2-cell β in such a way that the swallowtail are fulfilled, cf. [Pst14, Theorem 2.7].

Dual pairs can be organized into a bicategory by defining appropriate 1- and 2-morphisms between them. The bicategory of dual pairs turns out to be a 2-groupoid. Moreover, the bicategory of coherent dual pairs is equivalent to the core of dualizable objects in $\mathcal{C}$. In particular, this shows that any two coherent dual pairs over the same dualizable object are equivalent.

We now come to the stronger concept of fully-dualizability.

Definition 4.3. An object X in a symmetric monoidal bicategory is called fully-dualizable if it can be completed into a dual pair and the evaluation and coevaluation maps admit both left- and right adjoints.

Note that if left- and right adjoints exists, the adjoint maps will have adjoints themselves, since we work in a bicategorical setting [Pst14, Theorem 3.9]. Thus, definition 4.3 agrees with the definition of [Lur09b] in the special case of bicategories.

4.1.1. The Serre automorphism

Recall that by definition, the evaluation morphism for a fully dualizable object X admits both a right-adjoint ev_X^R and a left adjoint ev_X^L. We use these adjoints to define the Serre automorphism of X:

Definition 4.4. Let X be a fully-dualizable object in a symmetric monoidal bicategory. The Serre automorphism of X is defined to be the following composition of 1-morphisms:

$$S_X : X \cong X \otimes 1 \xrightarrow{\mathrm{id}_X \otimes \mathrm{ev}_X^R} X \otimes X \otimes X^* \xrightarrow{\tau_{X,X} \otimes \mathrm{id}_{X^*}} X \otimes X \otimes X^* \xrightarrow{\mathrm{id}_X \otimes \mathrm{ev}_X} X \otimes 1 \cong X. \tag{4.4}$$

Notice that the Serre automorphism is actually a 1-equivalence of X, since an inverse is given by the 1-morphism

$$S_X^{-1} = (\mathrm{id}_X \circ \mathrm{ev}_X) \circ (\tau_{X,X} \otimes \mathrm{id}_{X^*}) \circ (\mathrm{id}_X \otimes \mathrm{ev}_X^L), \tag{4.5}$$

cf. [DSPS13, Proposition 2.3.3]. The next lemma is well-known, cf. [Lur09b, Proposition 4.2.3] or [Pst14, Proposition 3.8] and can be shown easily graphically.

Lemma 4.5. *Let X be fully-dualizable in $\mathcal{C}$. Then, there are 2-isomorphisms*

$$\begin{aligned} \mathrm{ev}_X^R &\cong \tau_{X^*,X} \circ (\mathrm{id}_{X^*} \otimes S_X) \circ \mathrm{coev}_X \\ \mathrm{ev}_X^L &\cong \tau_{X^*,X} \circ (\mathrm{id}_{X^*} \otimes S_X^{-1}) \circ \mathrm{coev}_X \,. \end{aligned} \tag{4.6}$$

Next, we show that the Serre automorphism is actually a pseudo-natural transformation of the identity functor on the maximal subgroupoid of fully-dualizable objects of $\mathcal{C}$, as suggested in [SP14]. To the best of our knowledge, a proof of this statement has not appeared in the literature so far, hence we illustrate the details in the following. We begin by showing that the evaluation 1-morphism is "dinatural". Recall the definition of the dual morphism in a monoidal bicategory:

Definition 4.6. Let $\mathcal{C}$ be a symmetric monoidal bicategory, and let X and Y be dualizable objects of $\mathcal{C}$. Let $f : X \to Y$ a 1-morphism. We define the dual morphism $f^* : Y^* \to X^*$ by the composition

$$Y^* \cong 1 \otimes Y^* \xrightarrow{\mathrm{coev}_X \otimes \mathrm{id}_{Y^*}} X^* \otimes X \otimes Y^* \xrightarrow{\mathrm{id}_{X^*} \otimes f \otimes \mathrm{id}_{Y^*}} X^* \otimes Y \otimes Y^* \xrightarrow{\mathrm{id}_{X^*} \otimes \mathrm{ev}_Y} X^* \otimes 1 \cong X^*. \tag{4.7}$$

Lemma 4.7. *Let X be dualizable in $\mathcal{C}$. The evaluation 1-morphism ev_X is "dinatural": for every 1-morphism $f : X \to Y$ between dualizable objects, there is a natural 2-isomorphism ev_f in the diagram below.*

$$\begin{tikzcd} X \otimes Y^* \arrow[r, "\mathrm{id} \otimes f^*"] \arrow[d, "f \otimes \mathrm{id}"'] & X \otimes X^* \arrow[d, "\mathrm{ev}_X"] \arrow[dl, Rightarrow, "\mathrm{ev}_f"] \\ Y \otimes Y^* \arrow[r, "\mathrm{ev}_Y"'] & 1 \end{tikzcd} \tag{4.8}$$

Proof. We explicitly write out the definition of f^* and define ev_f to be the composition of the 2-morphisms in the diagram below.

$$
\mathrm{ev}_f := \begin{tikzcd}
X\,1\,Y^* \arrow[r, "\mathrm{id}\,\mathrm{coev}_X\,\mathrm{id}"] \arrow[dr, phantom, "\Swarrow \alpha\,\mathrm{id}_{\mathrm{id}}"] & X\,X^*\,X\,Y^* \arrow[r, "\mathrm{id}\,\mathrm{id}\,f\,\mathrm{id}"] \arrow[d, "\mathrm{ev}_X\ \mathrm{id}\,\mathrm{id}"] \arrow[dr, phantom, "\cong"] & X\,X^*\,Y\,Y^* \arrow[r, "\mathrm{id}\,\mathrm{id}\,\mathrm{ev}_Y"] \arrow[d, "\mathrm{ev}_X\ \mathrm{id}\,\mathrm{id}"] \arrow[dr, phantom, "\cong"] & X\,X^*\,1 \arrow[r, "\mathrm{id}\,r"] \arrow[d, "\mathrm{ev}_X\ \mathrm{id}"] & X\,X^* \arrow[dd, bend left, "\mathrm{ev}_X"] \arrow[dl, phantom, "\Swarrow r_{\mathrm{ev}_X}"] \\
 & 1\,X\,Y^* \arrow[r, "\mathrm{id}\,f\,\mathrm{id}"] \arrow[d, "l\,\mathrm{id}\,\mathrm{id}"] \arrow[dr, phantom, "\Swarrow l_f\,\mathrm{id}_{\mathrm{id}}"] & 1\,Y\,Y^* \arrow[r, "\mathrm{id}\,\mathrm{ev}_Y"] \arrow[d, "l\,\mathrm{id}"] \arrow[dr, phantom, "\Swarrow l_{\mathrm{ev}_Y}"] & 1\,1 \arrow[dr, bend left, "r"] \arrow[dr, bend right, "l"'] \arrow[dr, phantom, "\cong"] & \\
X\,Y^* \arrow[uu, bend left, "\mathrm{id}\,l"] \arrow[uu, bend right, "r\,\mathrm{id}"'] \arrow[uu, phantom, "\cong"] \arrow[r, "\mathrm{id}\,\mathrm{id}"] \arrow[rr, bend right, "f\,\mathrm{id}"', "\cong"] & X\,Y^* \arrow[r, "f\,\mathrm{id}"] & Y\,Y^* \arrow[rr, "\mathrm{ev}_Y"] & & 1
\end{tikzcd}
\tag{4.9}
$$

∎

In order to show that the Serre automorphism is pseudo-natural, we also need to show the dinaturality of the right adjoint of the evaluation.

Lemma 4.8. *For a fully-dualizable object X of $\mathcal{C}$, the right adjoint ev^R of the evaluation is "dinatural" with respect to 1-equivalences: for every 1-equivalence $f : X \to Y$ between fully-dualizable objects, there is a natural 2-isomorphism ev_f^R in the diagram below.*

$$
\begin{tikzcd}
1 \arrow[r, "\mathrm{ev}_X^R"] \arrow[d, "\mathrm{ev}_Y^R"'] & X \otimes X^* \arrow[d, "f\otimes\mathrm{id}"] \arrow[dl, phantom, "\Swarrow \mathrm{ev}_f^R"] \\
Y \otimes Y^* \arrow[r, "\mathrm{id}\otimes f^*"'] & Y \otimes X^*
\end{tikzcd}
\tag{4.10}
$$

Proof. In a first step, we show that $f \otimes (f^*)^{-1} \circ \mathrm{ev}_X^R$ is a right-adjoint to $\mathrm{ev}_X \circ (f^{-1} \otimes f^*)$. In formula:

$$(\mathrm{ev}_X \circ f^{-1} \otimes f^*)^R = f \otimes (f^*)^{-1} \circ \mathrm{ev}_X^R \,. \tag{4.11}$$

Indeed, let

$$
\begin{aligned}
\eta_X &: \mathrm{id}_{X\otimes X^*} \to \mathrm{ev}_X^R \circ \mathrm{ev}_X \\
\varepsilon_X &: \mathrm{ev}_X \circ \mathrm{ev}_X^R \to \mathrm{id}_1
\end{aligned}
\tag{4.12}
$$

be the unit and counit of the right-adjunction of ev_X and its right adjoint ev_X^R. We construct unit and counit for the adjunction in equation (4.11). Let

$$
\begin{aligned}
&\tilde{\varepsilon} : \mathrm{ev}_X \circ (f^{-1} \otimes f^*) \circ (f \otimes (f^*)^{-1}) \circ \mathrm{ev}_X^R \cong \mathrm{ev}_X \circ \mathrm{ev}_X^R \xrightarrow{\varepsilon_X} \mathrm{id}_1 \\
&\tilde{\eta} : \mathrm{id}_{Y\otimes Y^*} \cong (f \otimes (f^*)^{-1}) \circ (f^{-1} \otimes f^*) \xrightarrow{\mathrm{id}*\eta_X*\mathrm{id}} (f \otimes (f^*)^{-1}) \circ \mathrm{ev}_X^R \circ \mathrm{ev}_X \circ (f^{-1} \otimes f^*).
\end{aligned}
\tag{4.13}
$$

Now, one checks that the quadruple

$$(\mathrm{ev}_X \circ (f^{-1} \otimes f^*),\ (f \otimes (f^*)^{-1}) \circ \mathrm{ev}_X^R,\ \tilde{\varepsilon}, \tilde{\eta}) \tag{4.14}$$

fulfills indeed the axioms of an adjunction. This follows from the fact that the quadruple $(\mathrm{ev}_X, \mathrm{ev}_X^R, \varepsilon_X, \eta_X)$ is an adjunction. This shows equation (4.11).

Now, notice that due to the dinaturality of the evaluation in lemma 4.7, we have a natural 2-isomorphism

$$\mathrm{ev}_Y \cong \mathrm{ev}_X \circ (f^{-1} \otimes f^*). \tag{4.15}$$

Combining this 2-isomorphism with equation (4.11) shows that the right adjoint of ev_Y is given by $f \otimes (f^*)^{-1} \circ \mathrm{ev}_X^R$. Since all right-adjoints are isomorphic, the 1-morphism $f \otimes (f^*)^{-1} \circ \mathrm{ev}_X^R$ is isomorphic to ev_Y^R, as desired. □

We can now prove the following proposition:

Proposition 4.9. *Let $\mathcal{C}$ be a symmetric monoidal bicategory. Denote by $\mathscr{K}(\mathcal{C}^{\mathrm{fd}})$ the maximal sub-bigroupoid of fully-dualizable objects of $\mathcal{C}$. Then, the Serre automorphism S is a pseudo-natural isomorphism of the identity functor on $\mathscr{K}(\mathcal{C}^{\mathrm{fd}})$.*

Proof. Let $f : X \to Y$ be a 1-morphism in $\mathscr{K}(\mathcal{C}^{\mathrm{fd}})$. We need to provide a natural 2-isomorphism in the diagram

$$\begin{tikzcd} X \arrow[r, "S_X"] \arrow[d, "f"'] \arrow[dr, phantom, "\swarrow S_f"] & X \arrow[d, "f"] \\ Y \arrow[r, "S_Y"'] & Y \end{tikzcd} \tag{4.16}$$

By spelling out the definition of the Serre automorphism, we see that this is equivalent to filling the following diagram with natural 2-cells:

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"'] & X\,1 \arrow[r, "\mathrm{id}_X\ \mathrm{ev}_X^R"] \arrow[d, "f\,\mathrm{id}"] & X\,X\,X^* \arrow[r, "\tau_{X,X}\ \mathrm{id}_{X^*}"] \arrow[d, "f\,f\,(f^*)^{-1}"] & X\,X\,X^* \arrow[r, "\mathrm{id}_X\ \mathrm{ev}_X"] \arrow[d, "f\,f\,(f^*)^{-1}"] & X\,1 \arrow[r] \arrow[d, "f\,\mathrm{id}"] & X \arrow[d, "f"] \\ Y \arrow[r] & Y\,1 \arrow[r, "\mathrm{id}_Y\ \mathrm{ev}_Y^R"'] & Y\,Y\,Y^* \arrow[r, "\tau_{Y,Y}\ \mathrm{id}_{Y^*}"'] & Y\,Y\,Y^* \arrow[r, "\mathrm{id}_Y\ \mathrm{ev}_Y"'] & Y\,1 \arrow[r] & Y \end{tikzcd} \tag{4.17}$$

The first, the last and the middle square can be filled with a natural 2-cell due to the fact that $\mathcal{C}$ is a symmetric monoidal bicategory. The square involving the evaluation commutes up to a 2-cell using the mate of the 2-cell of lemma 4.7, while the square involving the right adjoint of the evaluation commutes up a 2-cell using the mate of the 2-cell of lemma 4.8. The so-constructed 2-morphism S_f is pseudo-natural since it is constructed as a composition of pseudo-natural isomorphisms: the 2-cell of lemma 4.7 in diagram (4.9) is itself defined by composing various natural 2-isomorphisms. □

We now come to a main result of this thesis: using that the Serre automorphism is a pseudo-natural transformation defines an $SO(2)$-action on the core of fully-dualizable objects of an arbitrary symmetric monoidal bicategory by definition 2.32. As a corollary to theorem 2.34 we then obtain an explicit description of the bicategory of homotopy fixed points of this action.

Corollary 4.10. *Let $\mathcal{C}$ be a symmetric monoidal bicategory, and consider the $SO(2)$-action of the Serre automorphism on $\mathscr{K}(\mathcal{C}^{\mathrm{fd}})$ as in example 2.33. Then, the bicategory of homotopy fixed points $\mathscr{K}(\mathcal{C}^{\mathrm{fd}})^{SO(2)}$ is equivalent to a bicategory where*

- *objects are given by pairs (X, λ_X) with X a fully-dualizable object of $\mathcal{C}$ and $\lambda_X : S_X \to \mathrm{id}_X$ is a 2-isomorphism which trivializes the Serre automorphism,*
- *1-morphisms are given by 1-equivalences $f : X \to Y$ in $\mathcal{C}$, so that the diagram*

$$\begin{tikzcd} S_Y \circ f \arrow[d, "\lambda_Y * \mathrm{id}_f"'] & f \circ S_X \arrow[l, "S_f"'] \arrow[r, "\mathrm{id}_f * \lambda_X"] & f \circ \mathrm{id}_X \arrow[d] \\ \mathrm{id}_X \circ f \arrow[rr] & & f \end{tikzcd} \tag{4.18}$$

  commutes, and
- *2-morphisms are given by 2-isomorphisms in $\mathcal{C}$.*

Proof. This follows directly from theorem 2.34. □

Remark 4.11. Recall that we have defined the bicategory of homotopy fixed points $\mathcal{C}^G$ as the bicategory of tritransformations $\mathrm{Nat}(\Delta, \rho)$. This bicategory should coincide with the tri-limit of the action considered as a trifunctor $\rho : B\Pi_2(G) \to \mathrm{Bicat}$. Since we only consider symmetric monoidal bicategories and the action of the Serre automorphism is monoidal by proposition 4.23, we actually obtain an action with values in SymMonBicat, the tricategory of symmetric monoidal bicategories. It would be interesting to compute the limit of the action in the tricategory of *symmmetric monoidal* bicategories. We expect that this trilimit computed in SymMonBicat is given by $\mathcal{C}^G$ as a bicategory, with the symmetric monoidal structure induced by the symmetric monoidal structure of $\mathcal{C}$.

Remark 4.12. By either using a result of Davidovich in [Dav11] or using the results in section 4.1.3, the action via the Serre automorphism on $\mathscr{K}(\mathrm{Alg}_2^{\mathrm{fd}})$ is trivializable. The category of homotopy fixed points $\mathscr{K}(\mathrm{Alg}_2^{\mathrm{fd}})^{SO(2)}$ is then equivalent to the bigroupoid of semisimple symmetric Frobenius algebras by corollary 2.36.

Similarly, the action of the Serre automorphism on $\mathscr{K}(\mathrm{Vect}_2^{\mathrm{fd}})$ is trivializable by section 4.1.2. The bicategory of homotopy fixed points of this action is equivalent to the bicategory of Calabi-Yau categories by corollary 3.12.

We now come to two examples: we explicitly compute the Serre automorphism in the bicategories Alg_2 and of Vect_2, and show that it is trivializable.

4.1.2. The Serre automorphism in 2-vector spaces

In this section, we calculate the Serre automorphism in $\mathscr{K}(\mathrm{Vect}_2^{\mathrm{fd}})$, the bigroupoid of fully-dualizable 2-vector spaces over an algebraically closed field $\mathbb{K}$. Recall that the fully-dualizable objects of Vect_2 are given by the finite, linear, semisimple categories as in definition 3.1. Let $\mathcal{C}$ be a finite, semisimple, linear category, with simple objects

$X_1, \dots, X_n$. Then, we can choose the dual of $\mathcal{C}$ to be the functor category $\mathcal{C}^* := \mathrm{Fun}_\oplus(\mathcal{C}, \mathrm{Vect})$ with objects consisting of additive functors $\mathcal{C} \to \mathrm{Vect}$. Note that $\mathcal{C}^*$ is finitely semisimple as well, with simple objects given by the functors $F_1, \dots, F_n$, which are defined on simple objects by $F_i(X_j) = \mathbb{K}\, \delta_{ij}$. The evaluation 1-morphism is given by:

$$\begin{aligned} \mathrm{ev} : \mathcal{C} \boxtimes \mathcal{C}^* &\to \mathrm{Vect} \\ X \boxtimes F &\mapsto F(X), \end{aligned} \tag{4.19}$$

where $\boxtimes$ is the Deligne tensor product of abelian categories, cf. [Del90]. Since all functors considered here are additive, it suffices to give the value of the coevaluation on the ground field $\mathbb{K}$, the only simple object of Vect. We define the coevaluation as

$$\begin{aligned} \mathrm{coev} : \mathrm{Vect} &\to \mathcal{C}^* \boxtimes \mathcal{C} \\ \mathbb{K} &\mapsto \bigoplus_{i=1}^{n} F_i \boxtimes X_i. \end{aligned} \tag{4.20}$$

In order to compute the Serre automorphism, we have to compute the right adjoint of the evaluation functor. We claim:

Lemma 4.13. *The right adjoint functor* ev^R *of the evaluation* ev *exists, and is fully determined by setting*

$$\begin{aligned} \mathrm{ev}^R : \mathrm{Vect} &\to \mathcal{C} \boxtimes \mathcal{C}^* \\ \mathbb{K} &\mapsto \bigoplus_{i=1}^{n} X_i \boxtimes F_i. \end{aligned} \tag{4.21}$$

Proof. Let X_k be a simple object of $\mathcal{C}$, and F_l be a simple object of $\mathcal{C}^*$. Then, we have a chain of natural isomorphisms

$$\begin{aligned} \mathrm{Hom}_{\mathcal{C}\boxtimes\mathcal{C}^*}(X_k \boxtimes F_l, \mathrm{ev}^R(\mathbb{K})) &\cong \bigoplus_{i=1}^{n} \mathrm{Hom}_{\mathcal{C}}(X_k, X_i) \otimes_{\mathbb{K}} \mathrm{Hom}_{\mathcal{C}^*}(F_l, F_i) \\ &\cong \mathbb{K}\, \delta_{kl} \cong \mathrm{Hom}_{\mathrm{Vect}}(F_l(X_k), \mathbb{K}) \\ &= \mathrm{Hom}_{\mathrm{Vect}}(\mathrm{ev}(X_k \boxtimes F_l), \mathbb{K}). \end{aligned} \tag{4.22}$$

This shows the adjunction formula for simple objects. The general case follows by taking direct sums. □

Now, we are ready to calculate the Serre automorphism.

Lemma 4.14. *Let* $\mathcal{C}$ *be a fully-dualizable object of* Vect_2 *(i.e. a finite semisimple linear category). Then, the Serre automorphism* $S : \mathcal{C} \to \mathcal{C}$ *is pseudo-naturally isomorphic to the identity functor on* $\mathcal{C}$*.*

Proof. By definition of the Serre automorphism in definition 4.4, the value of the Serre automorphism of a simple object X_k in $\mathscr{K}(\mathrm{Vect}_2^{\mathrm{fd}})$ is given by

$$S(X_k) = \bigoplus_{i=1}^{n} X_i \boxtimes F_i(X_k) \cong \bigoplus_{i=1}^{n} X_i \boxtimes \delta_{ik}\mathbb{K} \cong X_k \boxtimes \mathbb{K} \cong X_k. \tag{4.23}$$

Thus, the Serre automorphism is naturally isomorphic to the identity. □

Lemma 4.15. *Let $\mathcal{C}$ be a fully dualizable object in* Vect_2. *Then, there is a 1:1 correspondence between trivializations of the Serre automorphism S and Calabi-Yau structures on $\mathcal{C}$.*

Proof. The previous lemma 4.14 shows that the Serre automorphism itself is trivial. Thus, a trivialization of S corresponds bijectively to an invertible endotransformation of the identity functor on $\mathcal{C}$. Let $\eta : \mathrm{id}_{\mathcal{C}} \Rightarrow \mathrm{id}_{\mathcal{C}}$ be an invertible natural transformation, and let $X \in \mathcal{C}$ be simple. Since the natural transformation η is additive, it is fully determined by its components on simple objects. Since η_X is natural, it must be an element in the center of $\mathrm{End}_{\mathcal{C}}(X)$. Thus,

$$\eta_X \in Z(\mathrm{End}_{\mathcal{C}}(X)) \cong \mathbb{K}. \tag{4.24}$$

Since η is an isomorphism, η_X cannot be zero. Hence, η_X is uniquely determined by a non-zero scalar. Taking direct sums shows that a natural transformation is uniquely determined by a non-zero scalar for each simple object of $\mathcal{C}$. Now, the statement follows from lemma 3.11. □

Remark 4.16. Here, we have shown by hand that trivializations of the Serre automorphism in Vect_2 stand in bijection to Calabi-Yau categories. However, theorem 2.34 gives a more general result: this theorem shows that the bicategory of homotopy fixed points of the action of the Serre automorphism on $\mathscr{K}(\mathrm{Vect}_2^{\mathrm{fd}})$ is equivalent to the bicategory of Calabi-Yau categories.

4.1.3. The Serre automorphism in the Morita bicategory

Here, we follow the exposition in [FHLT10], but give more details as needed. In Alg_2, there is a natural notion of duals: if A is an algebra, denote by A^{op} the opposite algebra, and let $A^e := A \otimes_{\mathbb{K}} A^{\mathrm{op}}$ be the enveloping algebra. Then, evaluation and coevaluation are given by the algebra A as $(A^e, \mathbb{K})$ and $(\mathbb{K}, A^e)$-modules respectively. An object is called fully dualizable, if the evaluation map admits both a left and a right adjoint. In Alg_2 over a field, this is the case if and only if the algebra in question is separable, cf. [SP09, Definition 2.70]. If $\mathbb{K}$ is an algebraically closed field, the separable algebras are finite-dimensional and semisimple.

Lemma 4.17. *Let A be a fully dualizable algebra. Then, the right-adjoint of the evaluation* $\mathrm{ev}_A^R : \mathbb{K} \to A \otimes A^{\mathrm{op}}$ *is given by* $A^* := \mathrm{Hom}_{\mathbb{K}}(A, \mathbb{K})$*, regarded as an* $(A^e, \mathbb{K})$*-module.*

Proof. By definition, if the right-adjoint of the evaluation exists, we have a natural isomorphism

$$\mathrm{Hom}_{\mathrm{Vect}}(A \otimes_{A^e} N, V) \cong \mathrm{Hom}_{A^e\text{-Mod}}(N, \mathrm{ev}^R \otimes_{\mathbb{K}} V) \tag{4.25}$$

for any N $(A^e, \mathbb{K})$-module and any vector space V. Now, choose $V := \mathbb{K}$ and $N := A^e$. Then we obtain

$$A^* = \mathrm{Hom}_{\mathbb{K}}(A, \mathbb{K}) \cong \mathrm{Hom}_{\mathbb{K}}(A \otimes_{A^e} A^e, \mathbb{K}) \cong \mathrm{Hom}_{A^e}(A^e, \mathrm{ev}^R) \cong \mathrm{ev}^R, \tag{4.26}$$

as claimed. □

Lemma 4.18. *Let A be a fully-dualizable algebra. Then, the Serre automorphism of A is given by $S_A = A^* := \mathrm{Hom}_{\mathbb{K}}(A, \mathbb{K})$, which we regard as an (A, A)-bimodule.*

Proof. This follows from the definition of the Serre automorphism in definition 4.4. Indeed, using that the evaluation is given by the algebra A as an $(A^e, \mathbb{K})$-module together with lemma 4.17 shows that $S_A \cong A^*$ as (A, A)-bimodules. □

It now follows that the Serre automorphism is trivializable in Alg_2: a trivialization of the Serre automorphisms consists of an isomorphism $A \to A^*$ of (A, A)-bimodules. This is nothing else than the structure of a Frobenius algebra. Since every finite-dimensional, semisimple algebra admits the structure of a Frobenius algebra, the Serre automorphism in Alg_2 is trivializable.

Remark 4.19. Here, we have shown by hand that trivializations of the Serre automorphism in Alg_2 stand in bijection to Frobenius algebras. However, we obtain a more general result using theorem 2.34: using the fact that the Serre automorphism is trivializable in Alg_2, this theorem shows that the bicategory of homotopy fixed points of the action of the Serre automorphism in Alg_2 is equivalent to the bicategory of Frobenius algebras.

4.1.4. Monoidality of the Serre automorphism

We now return to the abstract definition of the Serre automorphism in an arbitrary symmetric monoidal bicategory as in definition 4.4. We show that the Serre automorphism respects the monoidal structure, and is actually a *monoidal* pseudo-natural transformation. This fact will be used to show that the action of the Serre automorphism on the core of fully-dualizable objects is monoidal. We begin with the following two lemmas:

Lemma 4.20. *Let $\mathcal{C}$ be a monoidal bicategory, and let X and Y be dualizable objects of $\mathcal{C}$. Then, there is a 1-equivalence $\xi : (X \otimes Y)^* \cong Y^* \otimes X^*$.*

Proof. Consider the 1-morphism $(X \otimes Y)^* \to Y^* \otimes X^*$ in $\mathcal{C}$ defined by

$$(\mathrm{id}_{Y^*} \otimes \mathrm{id}_{X^*} \otimes \mathrm{ev}_{X \otimes Y}) \circ (\mathrm{id}_{Y^*} \otimes \mathrm{coev}_X \otimes \mathrm{id}_Y \otimes \mathrm{id}_{(X \otimes Y)^*}) \circ (\mathrm{coev}_Y \otimes \mathrm{id}_{(X \otimes Y)^*}), \quad (4.27)$$

and consider the 1-morphism $Y^* \otimes X^* \to (X \otimes Y)^*$ in $\mathcal{C}$ defined by

$$(\mathrm{id}_{(X \otimes Y)^*} \otimes \mathrm{ev}_X) \circ (\mathrm{id}_{(X \otimes Y)^*} \otimes \mathrm{id}_X \otimes \mathrm{ev}_Y \otimes \mathrm{id}_{X^*}) \circ (\mathrm{coev}_{X \otimes Y} \otimes \mathrm{id}_Y^* \otimes \mathrm{id}_{X^*}). \quad (4.28)$$

These two 1-morphisms are (up to invertible 2-cells) inverse to each other. This shows the claim. □

Now, we show that the evaluation 1-morphism respects the monoidal structure:

Lemma 4.21. *For a dualizable object X of a symmetric monoidal bicategory $\mathcal{C}$, the evaluation 1-morphism is monoidal. More precisely: the following diagram commutes up* *to a natural 2-cell.*

$$\begin{tikzcd}
(X\otimes Y)\otimes(X\otimes Y)^* \arrow[rr, "\mathrm{ev}_{X\otimes Y}"] \arrow[d, "\mathrm{id}_{X\otimes Y}\otimes\xi"'] & & 1 \arrow[d] \\
(X\otimes Y)\otimes Y^*\otimes X^* \arrow[r, "\mathrm{id}_X\otimes\tau_{Y\otimes Y^*,X^*}"'] & X\otimes X^*\otimes Y\otimes Y^* \arrow[r, "\mathrm{ev}_X\otimes\mathrm{ev}_Y"'] & 1\otimes 1
\end{tikzcd} \tag{4.29}$$

Here, the 1-equivalence ξ is due to lemma 4.20.

Proof. Consider the diagram in figure 4.1 on page 83: here, the composition of the horizontal arrows at the top, together with the two arrows on the vertical right are exactly the 1-morphism in equation (4.29). The other arrow is given by $\mathrm{ev}_{X\otimes Y}$. We have not written down the tensor product, and left out isomorphisms of the form $1\otimes X\cong X\cong X\otimes 1$ for readability. $\square$

We can now establish the monoidality of the right adjoint of the evaluation via the following lemma:

Lemma 4.22. *Let $\mathcal{C}$ be a symmetric monoidal bicategory, and let X and Y be fully-dualizable objects. Then, the right adjoint of the evaluation is monoidal. More precisely: if $\xi : (X\otimes Y)^* \to Y^*\otimes X^*$ is the 1-equivalence of lemma 4.20, the following diagram commutes up to a natural 2-cell.*

$$\begin{tikzcd}
1 \arrow[r, "\mathrm{ev}^R_{X\otimes Y}"] \arrow[d, "\mathrm{ev}^R_X\otimes\mathrm{ev}^R_Y"'] & X\otimes Y\otimes(X\otimes Y)^* \arrow[d, "\mathrm{id}_{X\otimes Y}\otimes\xi"] \\
X\otimes X^*\otimes Y\otimes Y^* \arrow[r, "\mathrm{id}_X\otimes\tau_{X^*,Y\otimes Y^*}"'] & X\otimes Y\otimes Y^*\otimes X^*
\end{tikzcd} \tag{4.30}$$

Proof. In a first step, we show that the right adjoint of the 1-morphism

$$(\mathrm{ev}_X\otimes\mathrm{ev}_Y)\circ(\mathrm{id}_X\otimes\tau_{Y\otimes Y^*,X^*})\circ(\mathrm{id}_{X\otimes Y}\otimes\xi) \tag{4.31}$$

is given by the 1-morphism

$$(\mathrm{id}_{X\otimes Y}\otimes\xi^{-1})\circ(\mathrm{id}_X\circ\tau_{X^*,Y\otimes Y^*})\circ(\mathrm{ev}^R_X\otimes\mathrm{ev}^R_Y). \tag{4.32}$$

Indeed, if

$$\begin{aligned}\eta_X &: \mathrm{id}_{X\otimes X^*}\to\mathrm{ev}^R_X\circ\mathrm{ev}_X\\ \varepsilon_X &: \mathrm{ev}_X\circ\mathrm{ev}^R_X\to\mathrm{id}_1\end{aligned} \tag{4.33}$$

are the unit and counit of the right-adjunction of ev_X and its right adjoint ev^R_X, we construct adjunction data for the adjunction in equations (4.31) and (4.32) as follows. Let $\tilde\varepsilon$ and $\tilde\eta$ be the following 2-morphisms:

$$\begin{aligned}\tilde\varepsilon :\ &(\mathrm{ev}_X\otimes\mathrm{ev}_Y)\circ(\mathrm{id}_X\otimes\tau_{Y\otimes Y^*,X^*})\circ(\mathrm{id}_{X\otimes Y}\otimes\xi)\circ(\mathrm{id}_{X\otimes Y}\otimes\xi^{-1})\\ &\qquad\qquad\circ(\mathrm{id}_X\otimes\tau_{X^*,Y\otimes Y^*})\circ(\mathrm{ev}^R_X\otimes\mathrm{ev}^R_Y)\\ \cong\ &(\mathrm{ev}_X\otimes\mathrm{ev}_Y)\circ(\mathrm{id}_X\otimes\tau_{Y\otimes Y^*,X^*})\circ(\mathrm{id}_X\otimes\tau_{X^*,Y\otimes Y^*})\circ(\mathrm{ev}^R_X\otimes\mathrm{ev}^R_Y)\\ \cong\ &(\mathrm{ev}_X\otimes\mathrm{ev}_Y)\circ(\mathrm{ev}^R_X\otimes\mathrm{ev}^R_Y)\xrightarrow{\varepsilon_X\otimes\varepsilon_Y}\mathrm{id}_1\end{aligned} \tag{4.34}$$

$$
\begin{tikzcd}
XY(XY)^* \arrow[r, "\mathrm{id}_{XY}\ \mathrm{coev}_Y\ \mathrm{id}_{(XY)^*}"] \arrow[d, equal] \arrow[dr, phantom, "\mathrm{id}_{\mathrm{id}_X}\ \alpha_Y\ \mathrm{id}_{\mathrm{id}_{(XY)^*}}"] & XYY^*Y(XY)^* \arrow[r, "\mathrm{id}_{XYY^*}\ \mathrm{coev}_X\ \mathrm{id}_{Y(XY)^*}"] \arrow[d, "\mathrm{id}_X\ \mathrm{ev}_Y\ \mathrm{id}_{Y(XY)^*}"] \arrow[dr, phantom, "\cong"] & XYY^*X^*XY(XY)^* \arrow[r, "\mathrm{id}_{XYY^*X^*}\ \mathrm{ev}_{XY}"] \arrow[d, "\mathrm{id}_X\ \mathrm{ev}_Y\ \mathrm{id}_{X^*XY(XY)^*}"] \arrow[dr, phantom, "\cong"] & XYY^*X^* \arrow[r, "\mathrm{id}_X\ \tau_{YY^*,X^*}"] \arrow[d, "\mathrm{id}_X\ \mathrm{ev}_Y\ \mathrm{id}_{X^*}"] \arrow[dr, phantom, "\cong"] & XX^*YY^* \arrow[d, "\mathrm{ev}_X\ \mathrm{ev}_Y"] \\
XY(XY)^* \arrow[r, "\mathrm{id}_{XY(XY)^*}"'] \arrow[drr, bend right, equal] \arrow[drr, phantom, "\alpha_X\ \mathrm{id}_{\mathrm{id}_{(XY)^*}}"] & XY(XY)^* \arrow[r, "\mathrm{id}_X\ \mathrm{coev}_X\ \mathrm{id}_{Y(XY)^*}"'] & XX^*XY(XY)^* \arrow[r, "\mathrm{id}_{XX^*}\ \mathrm{ev}_{XY}"'] \arrow[d, "\mathrm{ev}_X\ \mathrm{id}_{XY(XY)^*}"] \arrow[drr, phantom, "\cong"] & XX^* \arrow[r, "\mathrm{ev}_X"'] & 1 \arrow[d, equal] \\
& & XY(XY)^* \arrow[rr, "\mathrm{ev}_{XY}"] & & 1
\end{tikzcd}
$$

Figure 4.1.: Diagram for the proof of lemma 4.21

and

$$
\begin{aligned}
\tilde{\eta} : \mathrm{id}_{X\otimes Y\otimes(X\otimes Y)^*} &\cong (\mathrm{id}_{X\otimes Y} \otimes \xi^{-1}) \circ (\mathrm{id}_{X\otimes Y} \otimes \xi) \\
&\cong (\mathrm{id}_{X\otimes Y} \otimes \xi^{-1}) \circ (\mathrm{id}_X \otimes \tau_{X^*,Y\otimes Y^*}) \\
&\qquad \circ (\mathrm{id}_X \otimes \tau_{Y\otimes Y^*,X^*}) \circ (\mathrm{id}_{X\otimes Y} \otimes \xi) \\
&\xrightarrow{\mathrm{id}\otimes\eta_X\otimes\eta_Y\otimes\mathrm{id}} (\mathrm{id}_{X\otimes Y} \otimes \xi^{-1}) \circ (\mathrm{id}_X \otimes \tau_{X^*,Y\otimes Y^*}) \circ (\mathrm{ev}_X^R \otimes \mathrm{ev}_Y^R) \\
&\qquad\qquad \circ (\mathrm{ev}_X \otimes \mathrm{ev}_Y) \circ (\mathrm{id}_X \otimes \tau_{Y\otimes Y^*,X^*}) \circ (\mathrm{id}_{X\otimes Y} \otimes \xi).
\end{aligned}
\tag{4.35}
$$

One now shows that the two 1-morphisms in equations (4.31) and (4.32), together with the two 2-morphisms $\tilde{\varepsilon}$ and $\tilde{\eta}$ form an adjunction. This shows that the two 1-morphisms in equations (4.31) and (4.32) are adjoint.

Next, notice that the 1-morphism in equation (4.31) is 2-isomorphic to the 1-morphism $\mathrm{ev}_{X\otimes Y}$ by lemma 4.21. Thus, the right adjoint of $\mathrm{ev}_{X\otimes Y}$ is given by the right adjoint of the 1-morphism in equation (4.31), which is the 1-morphism in equation (4.32) by the argument above. Since all adjoints are equivalent, this shows the lemma. □

We are now ready to prove that the Serre automorphism is a monoidal pseudo-natural transformation.

Proposition 4.23. *Let $\mathcal{C}$ be a symmetric monoidal bicategory. Then, the Serre automorphism is a* monoidal *pseudo-natural transformation of* $\mathrm{Id}_{\mathscr{K}(\mathcal{C}^{\mathrm{fd}})}$.

Proof. We have to provide invertible 2-cells

$$
\begin{aligned}
\Pi &: S_{X\otimes Y} \to S_X \otimes S_Y \\
M &: \mathrm{id}_1 \to S_1.
\end{aligned}
\tag{4.36}
$$

By definition of the Serre automorphism in definition 4.4, it suffices to show that the evaluation and its right adjoint are monoidal, since the braiding τ will be monoidal by definition. The monoidality of the evaluation is proven in lemma 4.21, while the monoidality of its right adjoint follows from lemma 4.22. These two lemmas thus provide an invertible 2-cell $S_{X\otimes Y} \cong S_X \otimes S_Y$. The second 2-cell $\mathrm{id}_1 \to S_1$ can be constructed in a similar way, by noticing that $1 \cong 1^*$. □

Remark 4.24. If $\mathcal{C}$ is a symmetric monoidal bicategory, we have constructed an action of the Serre automorphism on $\mathscr{K}(\mathcal{C}^{\mathrm{fd}})$ in definition 2.32. The last proposition shows that this action is a *monoidal* action.

4.2. Monoidal homotopy actions

In this section, we investigate monoidal group actions on symmetric monoidal bicategories. Recall that by a (homotopy) action of a topological group G on a bicategory $\mathcal{C}$, we mean a weak monoidal 2-functor $\rho : \Pi_2(G) \to \mathrm{Aut}(\mathcal{C})$, where $\Pi_2(G)$ is the fundamental 2-groupoid of G, and $\mathrm{Aut}(\mathcal{C})$ is the bicategory of auto-equivalences of $\mathcal{C}$. For details on homotopy actions of groups on bicategories, we refer the reader to section 2.2. In order to simplify the exposition, we introduce the following definition:

Definition 4.25. Let G be a topological group. We will say that G is 2-truncated if $\pi_2(G,x)$ is trivial for every base point $x \in G$.

Moreover, we will need also the following definition:

Definition 4.26. Let $\mathcal{C}$ be a symmetric monoidal bicategory. We will say that $\mathcal{C}$ is 1-connected if it is monoidally equivalent to B^2H, for some abelian group H.

In the following, we denote with $\mathrm{Aut}_{\otimes}(\mathcal{C})$ the bicategory of *monoidal* auto-equivalences of $\mathcal{C}$: these are *monoidal* 2-functors, *monoidal* pseudo-natural transformations, and *monoidal* modifications. For detailed definitions, we refer to [SP09].

Definition 4.27. Let $\mathcal{C}$ be a symmetric monoidal category and G be a topological group. A *monoidal* homotopy action of G on $\mathcal{C}$ is a weak monoidal 2-functor

$$\rho : \Pi_2(G) \to \mathrm{Aut}_{\otimes}(\mathcal{C}). \tag{4.37}$$

We now prove a general criterion for when monoidal homotopy actions are trivializable.

Proposition 4.28. *Let $\mathcal{C}$ be a symmetric monoidal bicategory, and let G be a path connected topological group. Assume that G is 2-truncated, and that $\mathrm{Aut}_{\otimes}(\mathcal{C})$ is 1-connected, for some abelian group H. If $H^2_{grp}(\pi_1(G,e),H) \simeq 0$, then any monoidal homotopy action of G on $\mathcal{C}$ is pseudo-naturally isomorphic to the trivial action.*

Proof. Let $\rho : \Pi_2(G) \to \mathrm{Aut}_{\otimes}(\mathcal{C})$ be a weak monoidal 2-functor. Since $\mathrm{Aut}_{\otimes}(\mathcal{C})$ was assumed to be monoidally equivalent to B^2H for some abelian group H, the group action ρ is equivalent to a 2-functor $\rho : \Pi_2(G) \to B^2H$. Due to the fact that G is path connected and 2-truncated, there is an equivalence of monoidal bicategories $\Pi_2(G) \simeq B\pi_1(G,e)$, where $\pi_1(G,e)$ is regarded as a discrete monoidal category. Thus, the homotopy action ρ is equivalent to a weak monoidal 2-functor $B\pi_1(G,e) \to B^2H$.

We claim that such functors are classified by $H^2_{grp}(\pi_1(G,e),H)$ up to pseudo-natural isomorphism. Indeed, let $F : B\pi_1(G,e) \to B^2H$ be a weak monoidal 2-functor. It is easy to see that F is trivial as a weak 2-functor, since we must have $F(*) = *$ on objects, $F(\gamma) = \mathrm{id}_*$ on 1-morphisms, and $B\pi_1(G)$ only has identity 2-morphisms. Thus, the only non-trivial data of F can come from the monoidal structure on F. The 1-dimensional components of the pseudo-natural transformations $\chi_{a,b} : F(a) \otimes F(b) \to F(a \otimes b)$ must be trival since there are only identity 1-morphisms in B^2H. The 2-dimensional components of this pseudo-natural transformation consists of a 2-morphism $\chi_{\gamma,\gamma'}$ in B^2H for every pair of 1-morphisms $\gamma : a \to b$ and $\gamma' : a' \to b'$ in $B\pi_1(G)$ in the diagram in equation (4.38) below.

$$\begin{tikzcd} F(a) \otimes F(a') \arrow[r, "\chi_{a,a'}"] \arrow[d, "F(\gamma)\otimes F(\gamma')"'] & F(a \otimes a') \arrow[d, "F(\gamma\otimes\gamma')"] \arrow[dl, Rightarrow, "\chi_{\gamma,\gamma'}"] \\ F(b) \otimes F(b') \arrow[r, "\chi_{b,b'}"'] & F(b \otimes b') \end{tikzcd} \tag{4.38}$$

Hence, we obtain a 2-cochain $\pi_1(G) \times \pi_1(G) \to H$, which obeys the cocycle condition due to the coherence equations of a monoidal 2-functor, cf. [SP09, Definition 2.5].

One now checks that a monoidal pseudo-natural transformation between two such functors is exactly a 2-coboundary, which shows the claim. Since we assumed that $H^2_{grp}(\pi_1(G,e),H) \simeq 0$, the original action ρ must be trivializable. $\square$

In order to give an example for the last proposition, we show that the bicategory $\mathrm{Alg}_2^{\mathrm{fd}}$ of finite-dimensional, semisimple algebras, bimodules and intertwiners, equipped with the monoidal structure given by the *direct sum* fulfills the conditions of proposition 4.28.

Lemma 4.29. *Let $\mathbb{K}$ be an algebraically closed field. Let $\mathcal{C} = \mathrm{Alg}_2^{\mathrm{fd}}$ be the bicategory where objects are given by finite-dimensional, semisimple algebras, equipped with the monoidal structure given by the direct sum. Then, $\mathrm{Aut}_\otimes(\mathcal{C})$ is equivalent to $B^2\mathbb{K}^*$.*

Proof. Let $F : \mathrm{Alg}_2^{\mathrm{fd}} \to \mathrm{Alg}_2^{\mathrm{fd}}$ be a weak monoidal 2-equivalence, and let A be a finite-dimensional, semisimple algebra. Then, A is isomorphic to a direct sum of matrix algebras. Calculating up to Morita equivalence and using that F has to preserve the single simple object $\mathbb{K}$ of Alg_2, we have

$$F(A) \cong F\left(\bigoplus_i M_{n_i}(\mathbb{K})\right) \cong \bigoplus_i F\left(M_{n_i}(\mathbb{K})\right) \cong \bigoplus_i F(\mathbb{K}) \cong \bigoplus_i \mathbb{K} \cong \bigoplus_i M_{n_i}(\mathbb{K}) \cong A. \tag{4.39}$$

Thus, the functor F is pseudo-naturally isomorphic to the identity functor on $\mathrm{Alg}_2^{\mathrm{fd}}$.

Now, let $\eta : F \to G$ be a monoidal pseudo-natural isomorphism between two endofunctors of Alg_2. Since both F and G are pseudo-naturally isomorphic to the identity, we may consider instead a pseudo-natural isomorphism $\eta : \mathrm{id}_{\mathrm{Alg}_2^{\mathrm{fd}}} \to \mathrm{id}_{\mathrm{Alg}_2^{\mathrm{fd}}}$. We claim that up to an invertible modification, the 1-equivalence $\eta_A : A \to A$ must be given by the bimodule ${}_AA_A$, which is the identity 1-morphism on A in Alg_2. Indeed, since η_A is assumed to be linear, it suffices to consider the case of $A = M_n(\mathbb{K})$ and to take direct sums. It is well-known that the only simple modules of A are given by $\mathbb{K}^n$. Thus,

$$\eta_A = (\mathbb{K}^n)^\alpha \otimes_{\mathbb{K}} (\mathbb{K}^n)^\beta, \tag{4.40}$$

where α and β are multiplicities. Now, lemma 2.8 ensures that these multiplicities are trivial, and thus we have $\eta_A = {}_AA_A$ up to an invertible intertwiner. This shows that up to invertible modifications, all 1-morphisms in $\mathrm{Aut}_\otimes(\mathrm{Alg}_2^{\mathrm{fd}})$ are identities.

Next, let m be an invertible endo-modification of the pseudo-natural transformation $\mathrm{id}_{\mathrm{id}_{\mathrm{Alg}_2^{\mathrm{fd}}}}$. Then, the component $m_A : {}_AA_A \to {}_AA_A$ is an element of $\mathrm{End}_{(A,A)}(A) \cong \mathbb{K}$. This shows that the 2-morphisms of $\mathrm{Aut}^\otimes(\mathrm{Alg}_2^{\mathrm{fd}})$ stand in bijection to $\mathbb{K}^*$. $\square$

Remark 4.30. Notice that the symmetric monoidal structure on $\mathrm{Alg}_2^{\mathrm{fd}}$ considered above is *not* the standard one, which is instead the one induced by the tensor product of algebras, and which is the monoidal structure relevant for the remainder of the paper.

The last lemmas imply the following:

Lemma 4.31. *Any monoidal $SO(2)$-action on $\mathrm{Alg}_2^{\mathrm{fd}}$ equipped with the monoidal structure given by the direct sum is trivial.*

Proof. Since $\pi_1(SO(2), e) \cong \mathbb{Z}$, and $H^2_{grp}(\mathbb{Z}, \mathbb{K}^*) \cong H^2(S^1, \mathbb{K}^*) \cong 0$, proposition 4.28 and lemma 4.29 ensure that any monoidal $SO(2)$-action on $\mathrm{Alg}_2^{\mathrm{fd}}$ is trivializable. $\square$

Corollary 4.32. *Since* $\mathrm{Alg}_2^{\mathrm{fd}}$ *and* $\mathrm{Vect}_2^{\mathrm{fd}}$ *are equivalent as additive categories, any $SO(2)$-action on* $\mathrm{Vect}_2^{\mathrm{fd}}$ *via linear morphisms is trivializable.*

Remark 4.33. The last two results rely on the fact that $\mathrm{Aut}_{\otimes}(\mathrm{Alg}_2^{\mathrm{fd}})$ and $\mathrm{Aut}_{\otimes}(\mathrm{Vect}_2^{\mathrm{fd}})$ are 1-connected as additive categories. This is due to the fact that fully-dualizable part of either Alg_2 or Vect_2 is semisimple. An example in which the conditions in proposition 4.28 do *not* hold is provided by the bicategory of Landau-Ginzburg models.

4.2.1. Invertible Field Theories

In the subsection, we consider the case of 2-dimensional oriented *invertible* topological field theories: such theories are in many ways easier to describe than arbitrary topological field theories, and play an important role in condensed matter physics and homotopy theory, as suggested in [Fre14a, Fre14b, FH16]. Using the cobordism hypothesis for oriented manifolds which we will prove in chapter 5, we classify oriented invertible theories in terms of homotopy fixed points. Furthermore, we provide a general criterion when the $SO(2)$-action on the bigroupoid of invertible theories is trivializable.

Denote with $\mathrm{Pic}(\mathcal{C})$ the *Picard groupoid* of a symmetric monoidal bicategory $\mathcal{C}$: it is defined as the maximal subgroupoid of $\mathcal{C}$ where the objects are invertible with respect to the monoidal structure of $\mathcal{C}$. Also recall that if $\mathrm{Cob}_{2,1,0}^{\mathrm{fr}}$ denotes the framed bordism bicategory, the bicategory of symmetric monoidal functors $\mathrm{Fun}_{\otimes}(\mathrm{Cob}_{2,1,0}^{\mathrm{fr}}, \mathcal{C})$ is equipped with a monoidal structure which is defined pointwise.

Definition 4.34. An invertible, framed, 2-dimensional, fully-extended topological quantum field theory with values in a symmetric monoidal bicategory $\mathcal{C}$ is an invertible object in the monoidal bicategory of functors $\mathrm{Fun}_{\otimes}(\mathrm{Cob}_{2,1,0}^{\mathrm{fr}}, \mathcal{C})$. The bigroupoid of invertible framed TQFTs with values in $\mathcal{C}$ is given by $\mathrm{Pic}(\mathrm{Fun}_{\otimes}(\mathrm{Cob}_{2,1,0}^{\mathrm{fr}}, \mathcal{C}))$.

Remark 4.35. Equivalently, an invertible field theory assigns to the point in $\mathrm{Cob}_{2,1,0}^{\mathrm{fr}}$ an invertible object in $\mathcal{C}$, and to any 1- and 2-dimensional manifold it assigns invertible 1- and 2-morphisms.

Since the cobordism hypothesis for framed manifolds provides a *monoidal* equivalence between the bicategory of symmetric monoidal functors $\mathrm{Fun}_{\otimes}(\mathrm{Cob}_{2,1,0}^{\mathrm{fr}}, \mathcal{C})$ and $\mathscr{K}(\mathcal{C}^{\mathrm{fd}})$, the space of invertible, framed topological field theories is given by $\mathrm{Pic}(\mathscr{K}(\mathcal{C}^{\mathrm{fd}}))$, since taking the Picard groupoid behaves well with respect to monoidal equivalences.

We begin by proving the following lemma.

Lemma 4.36. *Let $\mathcal{C}$ be a symmetric monoidal bicategory. Then, there is an equivalence of symmetric monoidal bicategories*

$$\mathrm{Pic}(\mathscr{K}(\mathcal{C}^{\mathrm{fd}})) \cong \mathrm{Pic}(\mathcal{C}). \tag{4.41}$$

Proof. First note that $\mathscr{K}(\mathcal{C}^{\mathrm{fd}})$ is a monoidal 2-groupoid, so there is an equivalence of bicategories $\mathrm{Pic}(\mathscr{K}(\mathcal{C}^{\mathrm{fd}})) \cong \mathrm{Pic}(\mathcal{C}^{\mathrm{fd}})$. Now, it suffices to show that every object X in $\mathrm{Pic}(\mathcal{C})$ is already fully-dualizable. Indeed, denote the tensor-inverse of X by X^{-1}. By definition, we have 1-equivalences $X \otimes X^{-1} \cong 1$ and $1 \cong X^{-1} \otimes X$, which serve as evaluation and coevaluation. These maps may be promoted to adjoint 1-equivalences by [SP09, Proposition A.27]. Thus, the evaluation and coevaluation also admit adjoints, which suffices for fully-dualizability in the bicategorical setting. □

Notice that given a monoidal bicategory $\mathcal{C}$, any monoidal autoequivalence of $\mathcal{C}$ preserves the Picard groupoid of $\mathcal{C}$, since it preserves invertibility of objects and (higher) morphisms. In particular, we have a monoidal 2-functor

$$\mathrm{Aut}_{\otimes}(\mathcal{C}) \to \mathrm{Aut}_{\otimes}(\mathrm{Pic}(\mathcal{C})) \tag{4.42}$$

obtained by restriction. Since the $SO(2)$-action on $\mathrm{Aut}_{\otimes}(\mathcal{C})$ induced by the action on $\mathrm{Cob}^{\mathrm{fr}}_{2,1,0}$ in definition 4.42 is monoidal, it induces an action on $\mathrm{Pic}(\mathcal{C})$. To proceed, we need the following lemma:

Lemma 4.37. *Let $\mathcal{C}$ be a symmetric monoidal bicategory, and assume that* $\mathrm{Pic}(\mathcal{C})$ *is monoidally equivalent to $B^2\mathbb{K}^*$. Then, there is an equivalence of bicategories*

$$\mathrm{Aut}_{\otimes}(\mathrm{Pic}(\mathcal{C})) \simeq \mathrm{Iso}(\mathbb{K}^*), \tag{4.43}$$

where the bicategory on the right hand side is regarded as a discrete bicategory.

Proof. Since $\mathrm{Pic}(\mathcal{C}) \simeq B^2\mathbb{K}^*$ by assumption, we have to describe the Picard groupoid of the category of monoidal functors from $B^2\mathbb{K}^*$ to $B^2\mathbb{K}^*$.

First, notice that the monoidal bicategory $B^2\mathbb{K}^*$ is computadic in the sense of [SP09]: it admits a presentation with only one object, only the identity 1-morphism, $\mathbb{K}^*$ as the set of 1-morphisms, and no relations between the 2-morphisms. The cofibrancy theorem in [SP09, Theorem 2.78] ensures that every monoidal 2-functor out of a computadic monoidal bicategory is equivalent to a *strict* monoidal functor. It is clear that *strict* monoidal auto-equivalences of $B^2\mathbb{K}$ are classified by $\mathrm{Iso}(\mathbb{K}^*)$ up to natural isomorphism. In order to see that the 1- and 2- morphisms of $\mathrm{Aut}_{\otimes}(B^2\mathbb{K}^*)$ are trivial, we use the cofibrancy theorem again to strictify monoidal pseudo-natural transformations. In detail, if $F, F' : B^2\mathbb{K}^* \to B^2\mathbb{K}^*$ are two weak monoidal 2-functors, and $\eta : F \to F'$ is a monoidal pseudo-natural equivalence, the confibrancy theorem ensures that η is equivalent to a *strict* monoidal pseudo-natural transformation, which means we may choose the data Π and M in [SP09, Figure 2.7] to be identity 2-morphisms. Thus, η is fully determined by a 1-morphism $\eta_* : F(*) \to F(*)$ in $B^2\mathbb{K}^*$ which has to be the identity, and by a 2-morphism η_{id_*} in $B^2\mathbb{K}^*$ filling the naturality square. This 2-morphism however is also fixed to be trivial by the unitality conditions of a monoidal pseudo-natural transformation in [SP09, Axiom MBTA2 and Axiom MBTA3].

We now come to the last layer of information: any monoidal modification between two monoidal pseudo-natural transformations is fixed to be the identity modification by the unitality requirement in [SP09, Axiom BMBM2]. □

Examples of symmetric monoidal bicategories satisfying the assumption of lemma 4.37 are $\mathrm{Alg}_2^{\mathrm{fd}}$ and $\mathrm{Vect}_2^{\mathrm{fd}}$. In the general case, we have the following lemma:

Lemma 4.38. *Let $\mathcal{C}$ be a symmetric monoidal bicategory such that $\mathrm{Pic}(\mathcal{C})$ is monoidally equivalent to $B^2\mathbb{K}^*$. Then, any monoidal $SO(2)$-action on $\mathrm{Pic}(\mathcal{C})$ is trivializable.*

Proof. Since we have monoidal equivalences $\Pi_2(SO(2)) \simeq B\mathbb{Z}$ and $\mathrm{Aut}_\otimes(\mathrm{Pic}(\mathcal{C})) \simeq \mathrm{Iso}(\mathbb{K}^*)$, monoidal actions correspond to monoidal 2-functors $B\mathbb{Z} \to \mathrm{Iso}(\mathbb{K}^*)$. Monoidality implies that the single object of $B\mathbb{Z}$ is sent to the identity isomorphism of $\mathbb{K}^*$, which correspond to the identity functor on $\mathrm{Pic}(\mathcal{C})$. This forces the functor to be trivial on objects. It is clear that the action is also trivial on 1- and 2-morphisms. Since there are no non-trivial morphisms in $\mathrm{Iso}(\mathbb{K}^*)$, the monoidal structure on the action ρ must also be trivial. □

Finally, we need the following lemma:

Lemma 4.39. *Let $\mathcal{C}$ be a symmetric monoidal bicategory, and let ρ_S be the $SO(2)$-action on $\mathscr{K}(\mathcal{C}^{\mathrm{fd}})$ by the Serre automorphism as in example 2.33. Since this action is monoidal, it induces an action on $\mathrm{Pic}(\mathscr{K}(\mathcal{C}^{\mathrm{fd}})) \cong \mathrm{Pic}(\mathcal{C})$ by lemma 4.36. We then have an equivalence of monoidal bicategories*

$$\mathrm{Pic}\left((\mathscr{K}(\mathcal{C}^{\mathrm{fd}}))^{SO(2)}\right) \cong \mathrm{Pic}(\mathcal{C})^{SO(2)}. \tag{4.44}$$

Proof. Theorem 2.34 allows us to compute the two bicategories of homotopy fixed points explicitly: we see that objects of both bicategories are given by invertible objects X of $\mathcal{C}$, together with the choice of a trivialization of the Serre automorphism. The 1-morphisms of both bicategories are given by 1-equivalences between invertible objects of $\mathcal{C}$, so that the diagram in equation (4.18) commutes, while 2-morphisms are given by 2-isomorphisms in $\mathcal{C}$. □

The implication of the above lemmas is the following: when $\mathcal{C}$ is a symmetric monoidal bicategory with $\mathrm{Pic}(\mathcal{C}) \cong B^2\mathbb{K}^*$, the action of the Serre automorphism on framed, invertible field theories with values in $\mathcal{C}$ is trivializable. Thus *all* framed invertible 2-dimensional field theories wih values in $\mathcal{C}$ can be turned into orientable ones.

Remark 4.40. Let us also compare these results with proposition 4.28, which showed that if $\mathcal{C}$ is a symmetric monoidal bicategory with $\mathrm{Aut}_\otimes(\mathcal{C}) \cong B^2H$ for some abelian group H, and G is a 2-truncated topological group so that $H^2_{grp}(\pi_1(G,e), H)$ is trivial, than any monoidal G-action on $\mathcal{C}$ is trivializable.

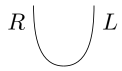
However, the assumptions of the above lemmas are slightly different: here we make an assumption about the homotopy type of the bicategory we want to act on, instead of the homotopy type of its automorphism category: instead of making an assumption on $\mathrm{Aut}_\otimes(\mathrm{Pic}(\mathcal{C}))$ (which is proven to be equivalent to $\mathrm{Iso}(\mathbb{K}^*)$ in lemma 4.37), we assume that $\mathrm{Pic}(\mathcal{C})$ itself has got the homotopy type of a $B^2\mathbb{K}^*$.

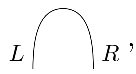
4.3. The 2-dimensional framed bordism bicategory

In this section, we introduce a skeleton of the framed bordism bicategory $\mathrm{Cob}_{2,1,0}^{\mathrm{fr}}$: this symmetric monoidal bicategory $\mathbb{F}_{cfd}$ is the free bicategory of a coherent fully-dual pair as introduced in [Pst14, Definition 3.13].

Using this presentation, we define a non-trivial $SO(2)$-action on the skeleton of the framed bordism bicategory. If $\mathcal{C}$ is an arbitrary symmetric monoidal bicategory, the action on $\mathbb{F}_{cfd}$ will induce an action on the functor bicategory $\mathrm{Fun}_{\otimes}(\mathbb{F}_{cfd}, \mathcal{C})$ of symmetric monoidal functors. Using the cobordism hypothesis for framed manifolds, which has been proven in the bicategorical framework in [Pst14], we obtain an $SO(2)$-action on $\mathscr{K}(\mathcal{C}^{\mathrm{fd}})$. We show that this induced action coming from the framed bordism bicategory is exactly the action given by the Serre automorphism. We begin by recalling the definition of the symmetric monoidal bicategory $\mathbb{F}_{cfd}$. Instead of writing down diagrams as in [Pst14], we use a "wire diagram" graphical calculus as introduced in [Bar14].

Definition 4.41. The symmetric monoidal bicategory $\mathbb{F}_{cfd}$ is generated by

- 2 generating objects L and R,
- 4 generating 1-morphisms, given by
  - a 1-morphism $\mathrm{coev} : 1 \to R \otimes L$, which we write as 

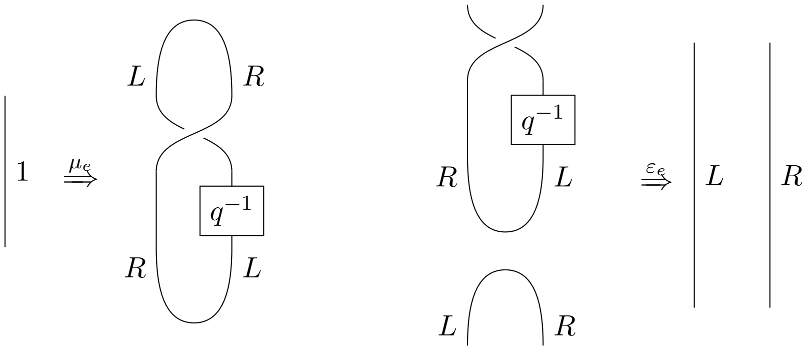
,
  - $\mathrm{ev} : L \otimes R \to 1$ which we write as 

,
  - a 1-morphism $q : L \to L$,
  - another 1-morphism $q^{-1} : L \to L$,
- 12 generating 2-cells given by
  - isomorphisms α, β, α^{-1} and β^{-1} as in definition 4.1, which in pictorial form are given as follows:

(4.45)

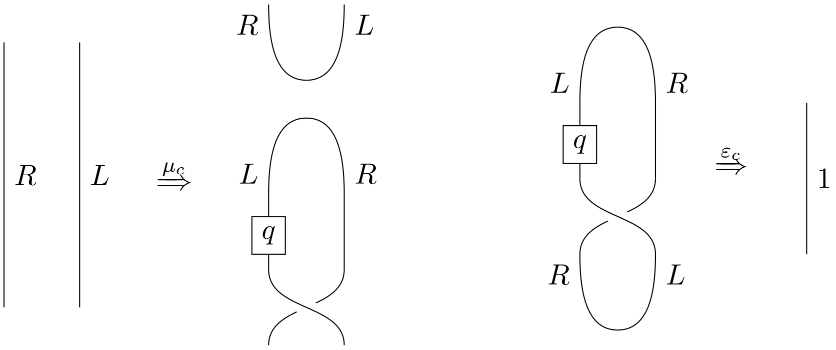
  - isomorphisms $\psi : qq^{-1} \cong \mathrm{id}_L : \psi^{-1}$ and $\phi : q^{-1}q \cong \mathrm{id}_L : \phi^{-1}$,
  - 2-cells $\mu_e : \mathrm{id}_1 \to \mathrm{ev} \circ \mathrm{ev}^L$ and $\varepsilon_e : \mathrm{ev}^L \circ \mathrm{ev} \to \mathrm{id}_{L \otimes R}$, where ev^L is defined as $\mathrm{ev}^L := \tau \circ (\mathrm{id}_R \otimes q^{-1}) \circ \mathrm{coev}$. In pictorial form, these 2-morphisms are given

as follows:



(4.46)

– 2-cells $\mu_c : \mathrm{id}_{R\otimes L} \to \mathrm{coev} \circ \mathrm{coev}^L$ and $\varepsilon_c : \mathrm{coev}^L \circ \mathrm{coev} \to \mathrm{id}_1$, where coev^L is defined by $\mathrm{coev}^L := \mathrm{ev} \circ (q \otimes \mathrm{id}_R) \circ \tau$. In pictorial form, these 2-morphisms are given as follows:

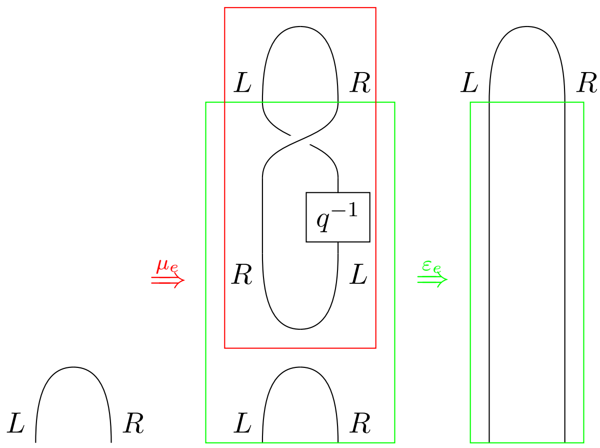


(4.47)

This ends the description of the data of $\mathbb{F}_{cfd}$. We now come to the relations. We demand that:

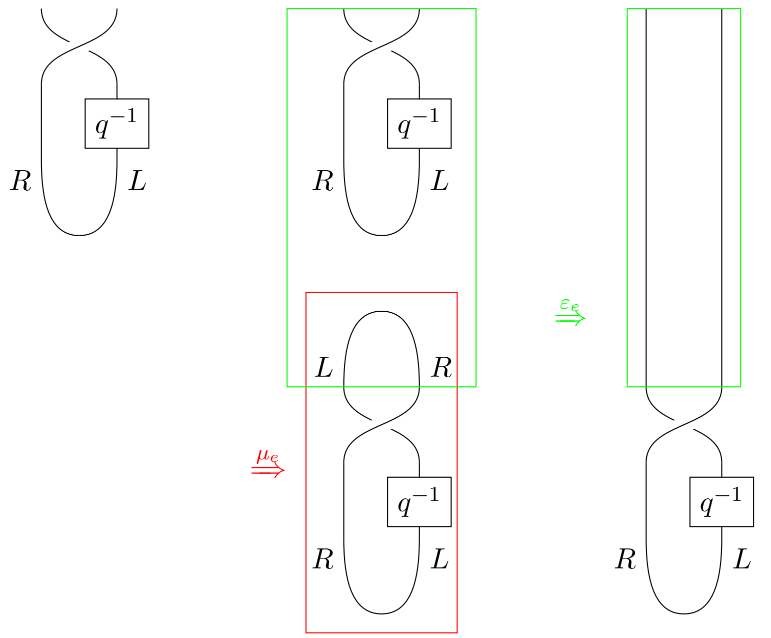
- the 2-cells α and α^{-1}, as well as β and β^{-1}, as well as ϕ and ϕ^{-1}, as well as ψ and ψ^{-1} are inverses to each other,

- the 2-cells μ_e and ε_e satisfy the two Zorro equations, which in pictorial form demand that the following composition of 2-morphisms



(4.48)

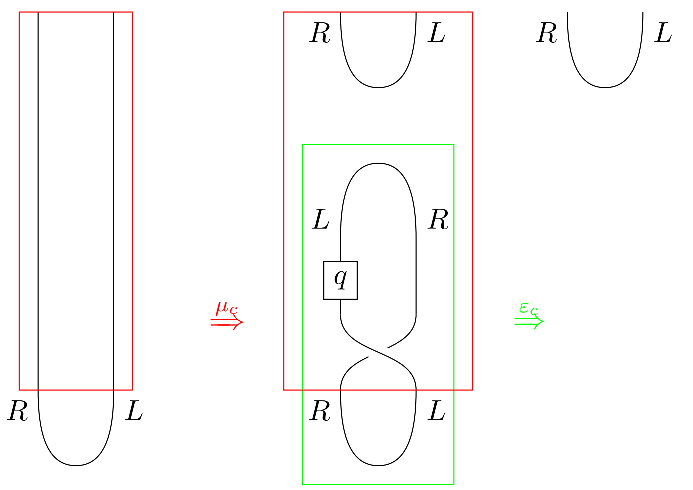
is equal to $\mathrm{id}_{\mathrm{ev}}$, and that the following composition of 2-morphisms



(4.49)

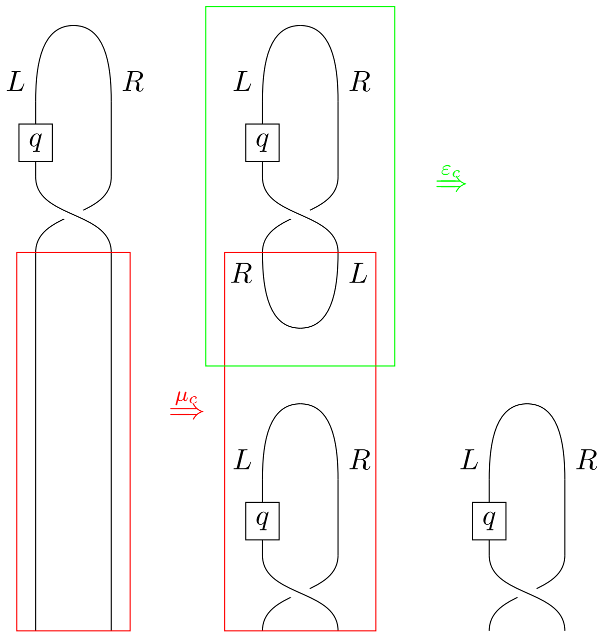
is equal to $\mathrm{id}_{\mathrm{ev}^L}$.

- Furthermore, we demand that μ_c and ε_c satisfy two Zorro equations, which in pictorial form demand that the composition



(4.50)

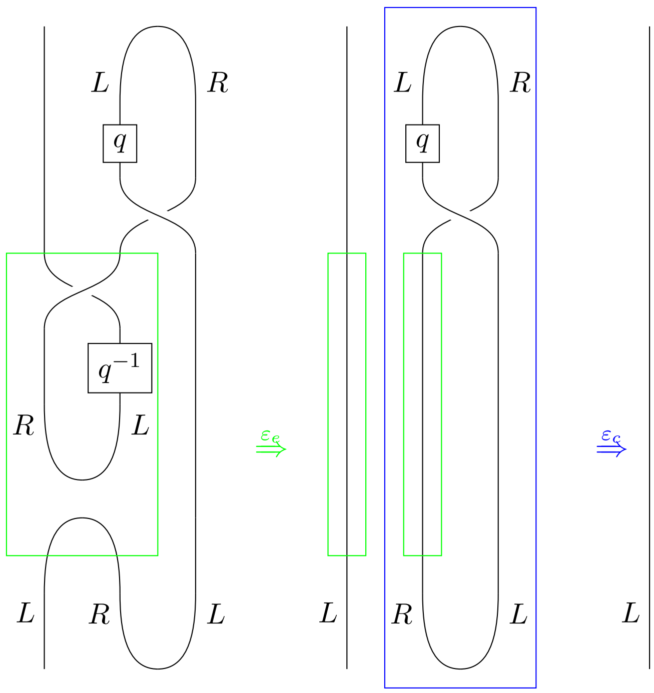
is equal to $\mathrm{id}_{\mathrm{coev}}$, and the composition of the following 2-morphisms



(4.51)

is equal to $\mathrm{id}_{\mathrm{coev}^L}$.

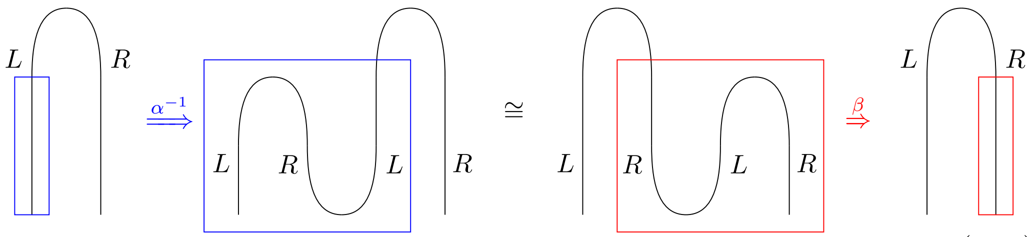
- The 2-cells ϕ and ψ satisfy triangle identities,
- and finally the cusp-counit equations in figure 5 and 6 on p.33 of [Pst14] are satisfied, which in graphical form demands that the composition of the following 2-morphisms



(4.52)

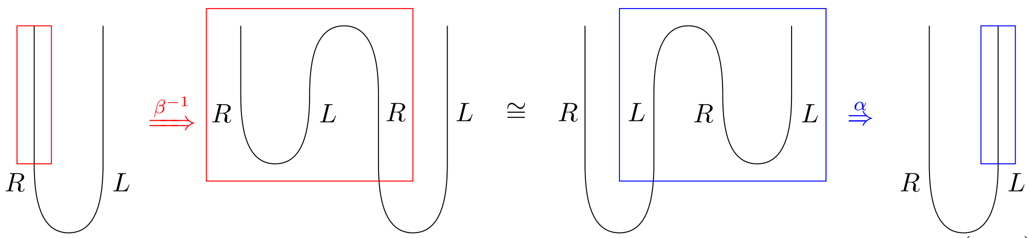
is equal to the composition of 2-morphisms of figure 4.2 on page 95,

- the swallowtail equations in figure 3 and 4 on p.15 of [Pst14] are satisfied, which in graphical form demand that the composition of the 2-morphisms



(4.53)

is equal to $\mathrm{id}_{\mathrm{ev}}$, and that the composition of the following 2-morphisms



(4.54)

is equal to $\mathrm{id}_{\mathrm{coev}}$.

This ends the description of $\mathbb{F}_{cfd}$, the bicategory freely generated by a coherent fully-dualizable object.

4.3.1. An action on the framed bordism bicategory

We now proceed to construct a non-trivial $SO(2)$-action on $\mathbb{F}_{cfd}$. This action will be vital for the remainder of this thesis.

By definition 2.32 it suffices to construct a pseudo-natural equivalence of the identity functor on $\mathbb{F}_{cfd}$ in order to construct an $SO(2)$-action. This pseudo-natural transformation is given as follows:

Definition 4.42. Let $\mathbb{F}_{cfd}$ be the free symmetric monoidal bicategory on a coherent fully-dual object as in definition 4.41. We construct a pseudo-natural equivalence $\alpha : \mathrm{id}_{\mathbb{F}_{cfd}} \to \mathrm{id}_{\mathbb{F}_{cfd}}$ of the identity functor on $\mathbb{F}_{cfd}$ as follows:

- for every object c of $\mathbb{F}_{cfd}$, we need to provide a 1-equivalence $\alpha_c : c \to c$.
  - For the object L of $\mathbb{F}_{cfd}$, we define $\alpha_L := q : L \to L$,
  - for the object R of $\mathbb{F}_{cfd}$, we set $\alpha_R := (q^{-1})^*$, which in pictorial form is given

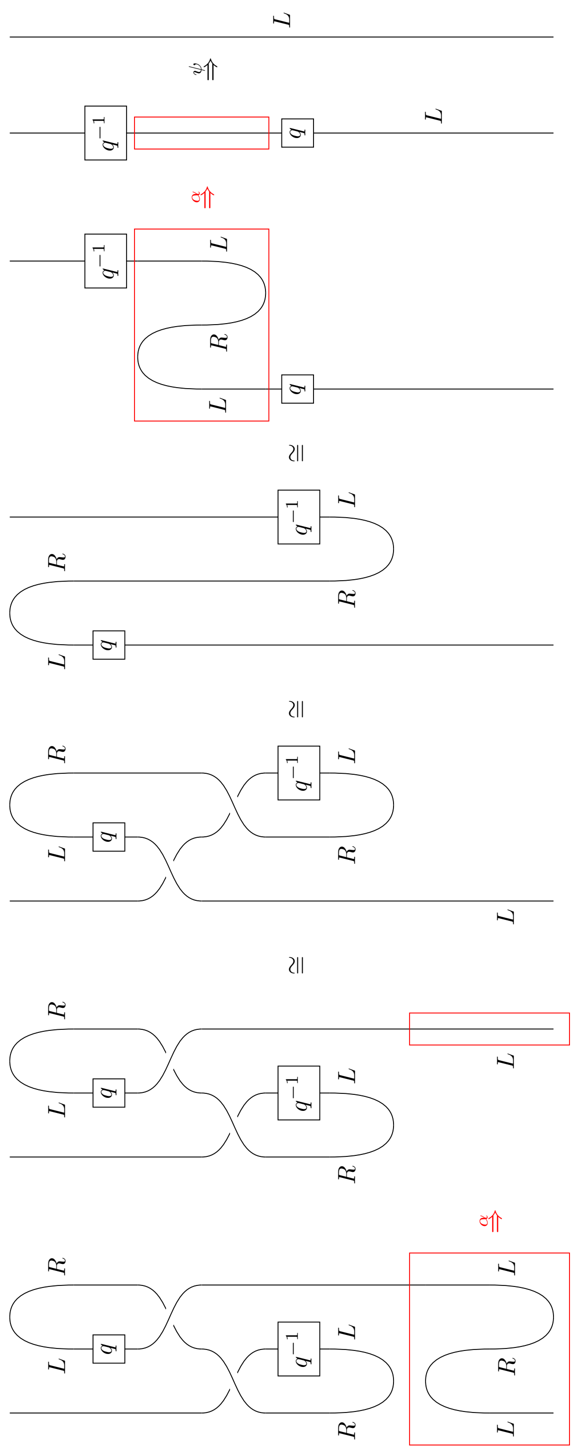
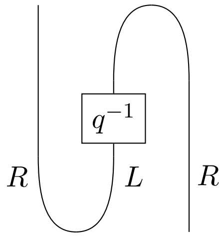


Figure 4.2.: The cusp-counit equations

by

$(q^{-1})^* :=$ 

(4.55)

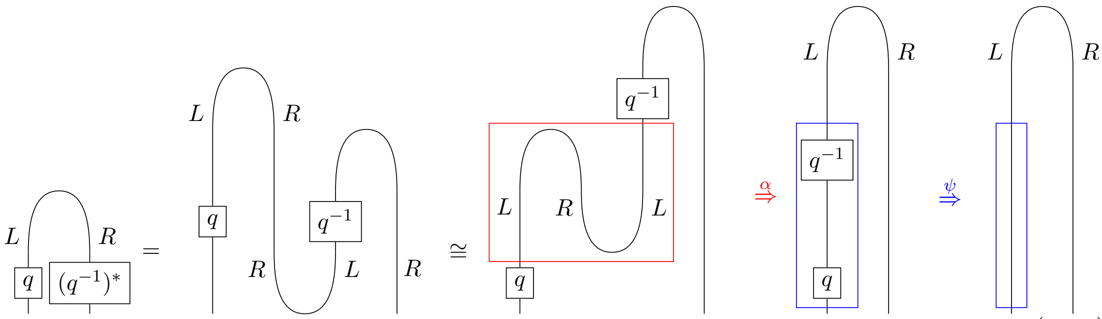
- for every 1-morphism $f : c \to d$ in $\mathbb{F}_{cfd}$, we need to provide a 2-isomorphism

$$\alpha_f : f \circ \alpha_c \to \alpha_d \circ f. \tag{4.56}$$

  - For the 1-morphism $q : L \to L$ of $\mathbb{F}_{cfd}$ we define the 2-isomorphism $\alpha_q := \mathrm{id}_{q \circ q}$.
  - For the 1-morphism $q^{-1} : L \to L$ we define the 2-isomorphism

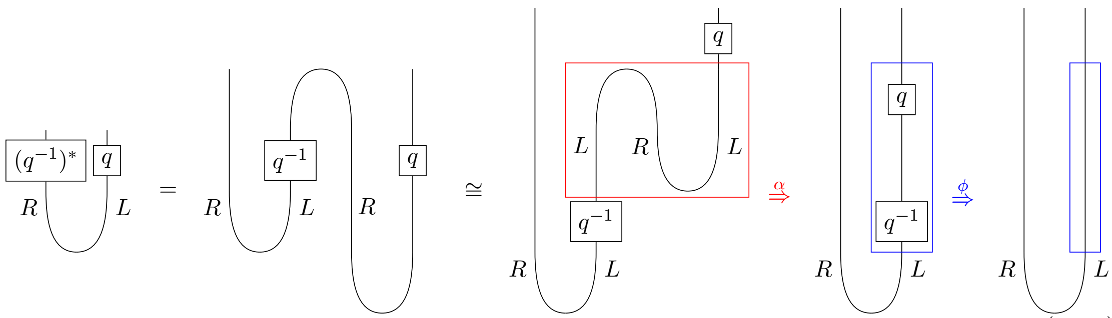
$$\alpha_{q^{-1}} := \left(q^{-1} \circ q \xrightarrow{\phi} \mathrm{id}_L \xrightarrow{\psi^{-1}} q \circ q^{-1} \right). \tag{4.57}$$

  - For the evaluation $\mathrm{ev} : L \otimes R \to 1$, we define the 2-isomorphism α_{ev} to be the following composition:



(4.58)

  - For the coevaluation $\mathrm{coev} : 1 \to R \otimes L$, we define the 2-isomorphism α_{coev} to be the composition



(4.59)

One now checks that this defines a pseudo-natural transformation of $\mathrm{id}_{\mathbb{F}_{cfd}}$. Using definition 2.32 gives us a non-trivial $SO(2)$-action on $\mathbb{F}_{cfd}$.

Remark 4.43. Note that the $SO(2)$-action on $\mathbb{F}_{cfd}$ does *not* send generators to generators: for instance, the 1-morphism $(q^{-1})^*$ in equation (4.55) is not part of the generating data of $\mathbb{F}_{cfd}$.

Remark 4.44. Notice that the pseudo-natural equivalence $\alpha : \mathrm{id}_{\mathbb{F}_{cfd}} \to \mathrm{id}_{\mathbb{F}_{cfd}}$ constructed in definition 4.42 is a *symmetric monoidal* pseudo-natural transformation. This follows from the fact that $\mathbb{F}_{cfd}$ is the *free* symmetric monoidal bicategory generated by a coherent fully-dual pair. Thus, we obtain an $SO(2)$-action on $\mathbb{F}_{cfd}$ via symmetric monoidal morphisms.

4.3.2. Induced action on functor categories

Starting from the action defined on $\mathbb{F}_{cfd}$, we induce an action on the bicategory of functors $\mathrm{Fun}(\mathbb{F}_{cfd}, \mathcal{C})$ for an arbitrary bicategory $\mathcal{C}$. The construction of the induced action on the bicategory of functors is a rather general. We provide details in the following.

Definition 4.45. Let $\rho : \Pi_2(G) \to \mathrm{Aut}(\mathcal{C})$ be a G-action on a bicategory $\mathcal{C}$, and let $\mathcal{D}$ be another bicategory. We define a G-action $\tilde{\rho} : \Pi_2(G) \to \mathrm{Aut}(\mathrm{Fun}(\mathcal{C}, \mathcal{D}))$ which is induced by ρ as follows:

- on objects $g \in \Pi_2(G)$, we define an endofunctor $\tilde{\rho}(g)$ of $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ on objects F on $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ by $\tilde{\rho}(g)(F) := F \circ \rho(g^{-1})$. If $\alpha : F \to G$ is a 1-morphism in $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$, we define

$$\tilde{\rho}(g)(\alpha) := \begin{tikzcd} F\rho(g^{-1})c \arrow[r, "\alpha_{\rho(g^{-1})(c)}"] \arrow[d, "F\rho(g^{-1})(f)"'] & G\rho(g^{-1})c \arrow[d, "G\rho(g^{-1})(f)"] \arrow[dl, Rightarrow, "\alpha_{\rho(g^{-1})(f)}"] \\ F\rho(g^{-1})d \arrow[r, "\alpha_{\rho(g^{-1})(d)}"'] & G\rho(g^{-1})d \end{tikzcd} \tag{4.60}$$

  If $m : \alpha \to \beta$ is a 2-morphism in $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$, the value of $\tilde{\rho}(\gamma)$ is given by

$$\tilde{\rho}(\gamma)(m)_x := m_{\rho(g^{-1})(x)}. \tag{4.61}$$

- On 1-morphisms $\gamma : g \to h$ of $\Pi_2(G)$, which are paths, we define a 1-morphism $\tilde{\rho}(\gamma)$ in $\mathrm{Aut}(\mathrm{Fun}(\mathcal{C}, \mathcal{D}))$ between the two endofunctors $F \mapsto F \circ \rho(g^{-1})$ and $F \mapsto F \circ \rho(h^{-1})$ of $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ as follows:
  - for each 2-functor $F : \mathcal{C} \to \mathcal{D}$, we need to provide a pseudo-natural transformation $\tilde{\rho}(\gamma)_F : F \circ \rho(g^{-1}) \to F \circ \rho(h^{-1})$ which we define via the diagram

$$\begin{tikzcd} F\rho(g^{-1})x \arrow[r, "F(\rho(\gamma^{-1})_x)"] \arrow[d, "F\rho(g^{-1})(f)"'] & F\rho(h^{-1})x \arrow[d, "F\rho(h^{-1})(f)"] \arrow[dl, Rightarrow, "F(\rho(\gamma^{-1})_f)"] \\ F\rho(g^{-1})y \arrow[r, "F(\rho(\gamma^{-1})_y)"'] & F\rho(h^{-1})y \end{tikzcd} \tag{4.62}$$

Here, γ^{-1} is the "inverse" path of γ given by $t \mapsto \gamma(t)^{-1}$, and $f : x \to y$ is a 1-morphism in $\mathcal{C}$.

– For every pseudo-natural transformation $\alpha : F \to G$, we need to provide a modification $\tilde{\rho}(\gamma)_\alpha$ in the diagram

$$\begin{tikzcd} \tilde{\rho}(g)(F) \arrow[r, "\tilde{\rho}(\gamma)_F"] \arrow[d, "\tilde{\rho}(g)(\alpha)"'] & \tilde{\rho}(h)(F) \arrow[d, "\tilde{\rho}(h)(\alpha)"] \arrow[dl, phantom, "\swarrow \tilde{\rho}(\gamma)_\alpha"] \\ \tilde{\rho}(g)(G) \arrow[r, "\tilde{\rho}(\gamma)_G"'] & \tilde{\rho}(h)(G) \end{tikzcd} \tag{4.63}$$

which we define by

$$\tilde{\rho}(\gamma)_\alpha := \alpha^{-1}_{\rho(\gamma^{-1})_x}. \tag{4.64}$$

- For the 2-morphisms in $\mathrm{Aut}(\mathrm{Fun}(\mathcal{C}, \mathcal{D}))$ we proceed in a similar fashion: if $m : \gamma \to \gamma'$ is a 2-track, we have to provide a 2-morphism $\tilde{\rho}(m) : \tilde{\rho}(\gamma) \to \tilde{\rho}(\gamma')$ which can be done by explicitly writing down diagrams as above.

The rest of the data of a monoidal functor $\tilde{\rho}$ is induced from the data of the monoidal functor ρ.

For $\mathcal{C}$ and $\mathcal{D}$ symmetric monoidal bicategories, the bicategory of symmetric monoidal functors $\mathrm{Fun}_\otimes(\mathcal{C}, \mathcal{D})$ acquires a monoidal structure by "pointwise evaluation" of functors. Such a monoidal structure is also symmetric. The following result is straightforward, and follows from "symmetric monoidal whiskering" of [SP09].

Lemma 4.46. *Let $\mathcal{C}$ and $\mathcal{D}$ be symmetric monoidal bicategories, and let ρ be a monoidal action of a group G on $\mathcal{C}$. Then ρ induces a monoidal action*

$$\tilde{\rho} : \Pi_2(G) \to \mathrm{Aut}_\otimes(\mathrm{Fun}_\otimes(\mathcal{C}, \mathcal{D})). \tag{4.65}$$

Example 4.47. Our main example for induced actions is the $SO(2)$-action on $\mathbb{F}_{cfd}$ as in definition 4.42. This action only depends on a pseudo-natural equivalence α of the identity functor on $\mathrm{id}_{\mathbb{F}_{cfd}}$. Consequently, the induced action on $\mathrm{Fun}(\mathbb{F}_{cfd}, \mathcal{C})$ also only depends on a pseudo-natural equivalence of the identity functor on $\mathrm{Fun}(\mathbb{F}_{cfd}, \mathcal{C})$. Using the definition above, we construct this induced pseudo-natural equivalence $\tilde{\alpha}$ as follows.

- For every 2-functor $F : \mathcal{C} \to \mathcal{D}$, we need to provide a pseudo-natural equivalence $\tilde{\alpha}_F : F \to F$, which is given by the diagram

$$\tilde{\alpha}_F := \begin{tikzcd} Fx \arrow[r, "F(\alpha_x^{-1})"] \arrow[d, "F(f)"'] & Fx \arrow[d, "F(f)"] \arrow[dl, phantom, "\swarrow F(\alpha_f^{-1})"] \\ Fy \arrow[r, "F(\alpha^{-1})_y"'] & Fy \end{tikzcd} \tag{4.66}$$

- for every pseudo-natural transformation $\beta : F \to G$, we need to give a modification $\tilde{\alpha}_\beta$, which we define by the diagram

$$\begin{tikzcd} Fx \arrow[r, "F(\alpha_x^{-1})"] \arrow[d, "\beta_x"'] & Fx \arrow[d, "\beta_x"] \arrow[dl, phantom, "\swArrow \beta^{-1}_{(\alpha_x^{-1})}"] \\ Gx \arrow[r, "G(\alpha^{-1})_x"'] & Gx \end{tikzcd} \tag{4.67}$$

This defines a pseudo-natural equivalence of the identity functor on $\mathrm{Fun}(\mathbb{F}_{cfd}, \mathcal{C})$. By definition 2.32, we obtain an $SO(2)$-action on $\mathrm{Fun}(\mathbb{F}_{cfd}, \mathcal{C})$. Note that $\mathbb{F}_{cfd}$ is even a *symmetric monoidal* bicategory. The $SO(2)$-action on $\mathbb{F}_{cfd}$ of definition 4.42 is via symmetric monoidal homomorphisms by remark 4.44. Hence, if $\mathcal{C}$ is also symmetric monoidal, then lemma 4.46 provides a monoidal action on $\mathrm{Fun}_\otimes(\mathbb{F}_{cfd}, \mathcal{C})$.

4.3.3. Induced action on the core of fully-dualizable objects

In this subsection, we compute the $SO(2)$-action on the core of fully-dualizable objects coming from the $SO(2)$-action on $\mathbb{F}_{cfd}$. Starting from the $SO(2)$-action on $\mathbb{F}_{cfd}$ as by definition 4.42, we have shown in the previous subsection how to induce an $SO(2)$-action on the bicategory of symmetric monoidal functors $\mathrm{Fun}_\otimes(\mathbb{F}_{cfd}, \mathcal{C})$ for $\mathcal{C}$ some symmetric monoidal bicategory. By the cobordism hypothesis for framed manifolds, we obtain an induced $SO(2)$-action on $\mathscr{K}(\mathcal{C}^{\mathrm{fd}})$. More precisely, denote by

$$\begin{aligned} E_L : \mathrm{Fun}_\otimes(\mathbb{F}_{cfd}, \mathcal{C}) &\to \mathscr{K}(\mathcal{C}^{\mathrm{fd}}) \\ Z &\mapsto Z(L) \end{aligned} \tag{4.68}$$

the evaluation functor. The cobordism hypothesis for framed manifolds in two dimensions [Pst14, Lur09b] states that E_L is an equivalence of symmetric monoidal bicategories. Hence, the composition of the $SO(2)$-action on $\mathrm{Fun}_\otimes(\mathbb{F}_{cfd}, \mathcal{C})$ and (the inverse of) E_L provides an $SO(2)$-action on $\mathscr{K}(\mathcal{C}^{\mathrm{fd}})$. The next proposition shows that this action is equivalent to the action ρ^S induced by the Serre automorphism which is illustrated in example 2.33.

Proposition 4.48. *Let ρ be the $SO(2)$-action on $\mathbb{F}_{cfd}$ given in definition 4.42, and let $\mathcal{C}$ be a symmetric monoidal bicategory. By definition 4.45, we obtain a monoidal $SO(2)$-action on* $\mathrm{Fun}_\otimes(\mathbb{F}_{cfd}, \mathcal{C})$*. Then, the monoidal $SO(2)$-action induced by the evaluation in equation* (4.68) *on* $\mathscr{K}(\mathcal{C}^{\mathrm{fd}})$ *is equivalent to the action by the Serre automorphism ρ^S.*

Proof. Let

$$\rho : \Pi_2(SO(2)) \to \mathrm{Aut}(\mathrm{Fun}_\otimes(\mathbb{F}_{cfd}, \mathcal{C})) \tag{4.69}$$

be the $SO(2)$-action on the bicategory of symmetric monoidal functors $\mathrm{Fun}_\otimes(\mathbb{F}_{cfd}, \mathcal{C})$ as in example 4.47. This action only depends on a pseudo-natural transformation α on the identity functor on $\mathrm{Fun}_\otimes(\mathbb{F}_{cfd}, \mathcal{C})$. By [Pst14], the 2-functor in equation (4.68) which evaluates a framed field theory on the object L is an equivalence of bicategories. Thus, we obtain an $SO(2)$-action ρ' on $\mathscr{K}(\mathcal{C}^{\text{fd}})$. This action is given as follows. By definition 2.32, we only need to provide a pseudo-natural transformation of the identity functor of $\mathscr{K}(\mathcal{C}^{\text{fd}})$. In order to write down this pseudo-natural transformation, note that the functor

$$\begin{aligned} \text{Aut}(\text{Fun}_{\otimes}(\mathbb{F}_{cfd}, \mathcal{C})) &\to \text{Aut}(\mathscr{K}(\mathcal{C}^{\text{fd}})) \\ F &\mapsto E_L \circ F \circ E_L^{-1} \end{aligned} \tag{4.70}$$

is an equivalence. Hence, the induced pseudo-natural transformation of $\text{id}_{\mathscr{K}(\mathcal{C}^{\text{fd}})}$ is given as follows:

- for each fully-dualizable object c of $\mathcal{C}$, we assign the 1-morphism $\alpha'_c : c \to c$ defined by
$$\alpha'_c := E_L\left(\alpha_{\left(E_L^{-1}(c)\right)}\right). \tag{4.71}$$
- For each 1-equivalence $f : c \to d$ between fully-dualizable objects of $\mathcal{C}$, we define a 2-isomorphism $\alpha'_f : f \circ \alpha'_c \to \alpha'_d \circ f$ by the formula
$$\alpha'_f := E_L\left(\alpha_{\left(E_L^{-1}(f)\right)}\right). \tag{4.72}$$

Here, α is the pseudo-natural transformation as in example 4.47. In order to see that α'_c is given by the Serre automorphism of the fully-dualizable object c, note that the 1-morphism $q : L \to L$ of $\mathbb{F}_{cfd}$ is mapped to the Serre automorphism $S_{Z(L)}$ by the equivalence in equation (4.68). This follows from lemma 4.5 which characterizes the Serre-automorphism in terms of the left- and right adjunction of the evaluation. Now notice that 2-cells μ_e, ε_e, μ_c and μ_e in the framed bordism bicategory define adjunctions using the 1-morphism q. This shows that q is mapped to the Sere-automorphism. □

Corollary 4.49. *Let ρ be the $SO(2)$-action on $\mathbb{F}_{cfd}$ given in definition 4.42, and let $\mathcal{C}$ be a symmetric monoidal bicategory. Consider the $SO(2)$-action ρ^S on $\mathscr{K}(\mathcal{C}^{\text{fd}})$ induced by the Serre automorphism. Then the evaluation morphism ev_L induces an equivalence of bicategories*

$$\text{Fun}_{\otimes}(\mathbb{F}_{cfd}, \mathcal{C})^{SO(2)} \to \mathscr{K}(\mathcal{C}^{\text{fd}})^{SO(2)}. \tag{4.73}$$

Proof. By proposition 4.48, the equivalence of equation (4.68) is $SO(2)$-equivariant. Thus, it induces an equivalence on homotopy fixed points, see. section 2.3 for an explicit description. It is also possible to construct this equivalence directly: by theorem 2.34, the bicategory of homotopy fixed points $\text{Fun}_{\otimes}(\mathbb{F}_{cfd}, \mathcal{C})^{SO(2)}$ is equivalent to the bicategory where

- objects are given by symmetric monoidal functors $Z : \mathbb{F}_{cfd} \to \mathcal{C}$, together with a modification $\lambda_Z : \tilde{\alpha}_Z \to \text{id}_Z$. Explicitly, this means: if α is the endotransformation of the identity functor of $\mathbb{F}_{cfd}$ as in definition 4.42, we obtain two 2-isomorphisms in $\mathcal{C}$:
$$\begin{aligned} \lambda_L &: Z(q^{-1}) \to \text{id}_{Z(L)} \\ \lambda_R &: Z(((q^{-1})^*)^{-1}) \to \text{id}_{Z(R)} \end{aligned} \tag{4.74}$$
which are compatible with evaluation and coevaluation,

- 1-morphisms are given by symmetric monoidal pseudo-natural transformations $\mu : Z \to Z'$, so that the analogue of the diagram in equation (2.65) commutes,
- 2-morphisms are given by symmetric monoidal modifications.

Now notice that $Z(q)$ is precisely the Serre automorphism of the object $Z(L)$. Thus, λ_L provides a trivialization of (the inverse of) the Serre automorphism. Applying theorem 2.34 again to the action of the Serre automorphism on the core of fully-dualizable objects shows that the functor $Z \mapsto (Z(L), \lambda_L)$ is an equivalence of homotopy fixed point bicategories. $\square$

5. Calabi-Yau objects and the cobordism hypothesis for oriented manifolds

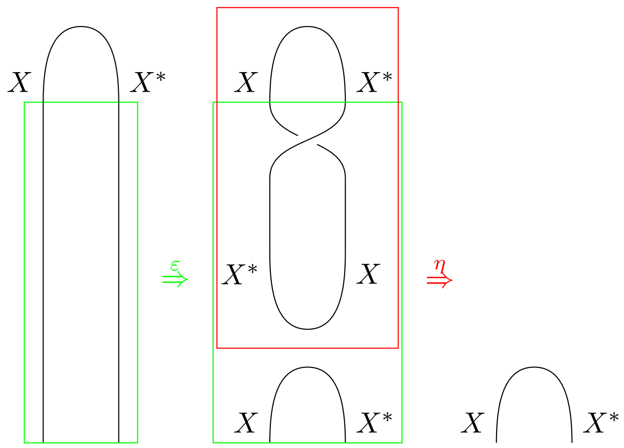
In this chapter, we define Calabi-Yau objects in an arbitrary symmetric monoidal bicategory $\mathcal{C}$. First, we show that fully-dualizable Calabi-Yau objects are equivalent to homotopy fixed points of the $SO(2)$-action by the Serre automorphism on $\mathscr{K}(\mathcal{C}^{\mathrm{fd}})$. Conjecturally, Calabi-Yau objects which are not fully-dualizable are related to non-compact field theories. We then prove that the 2-groupoid of 2-dimensional oriented topological quantum field theories with values in a symmetric monoidal bicategory $\mathcal{C}$ is equivalent to the bigroupoid of Calabi-Yau objects in $\mathcal{C}$. This proves the cobordism hypothesis for 2-dimensional, oriented manifolds. We begin with a central definition:

Definition 5.1. Let $\mathcal{C}$ be a symmetric monoidal bicategory. A Calabi-Yau object in $\mathcal{C}$ consists of the following data:

- a dualizable object X of $\mathcal{C}$,
- and a 2-morphism

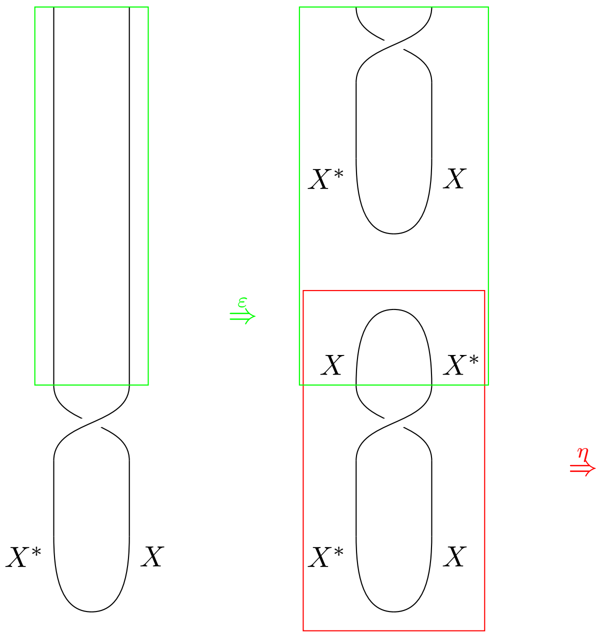
$$\eta : \mathrm{ev}_X \circ \tau_{X^*,X} \circ \mathrm{coev}_X \to \mathrm{id}_1. \tag{5.1}$$

The data (X, η) is then required to fulfill the following condition: we demand the existence of a 2-morphism $\varepsilon : \mathrm{id}_{X\otimes X^*} \to \tau_{X^*,X} \circ \mathrm{coev}_X \circ \mathrm{ev}_X$, so that ε and η are unit and counit of an adjunction between ev_X and $\tau_{X^*,X} \circ \mathrm{coev}_X$. Explicitly, this means that the composition of the following 2-morphisms



(5.2)

must be equal to $\mathrm{id}_{\mathrm{ev}_x}$, and the composition of the following 2-morphisms



(5.3)

must be also equal to the identity. We call the 1-morphism

$$\dim(X) := \mathrm{ev}_X \circ \tau_{X^*,X} \circ \mathrm{coev}_X \in \mathrm{End}_{\mathcal{C}}(1) \tag{5.4}$$

the dimension of X.

Example 5.2. If $\mathcal{C} = \mathrm{Alg}_2$ is the bicategory of algebras, bimodules and intertwiners, every object is dualizable: the dual of an algebra A is given by the opposite algebra A^{op}, and evaluation and coevaluation are the bimodules $A_{A\otimes A^{\mathrm{op}}}$ and ${}_{A\otimes A^{\mathrm{op}}}A$. The additional structure of a Calabi-Yau object is nothing else than a symmetric Frobenius form on A.

Unsurprisingly, there is a whole bicategory of Calabi-Yau objects. The 1-morphisms of this bicategory are defined as follows:

Definition 5.3. Let (X, η_X) and (Y, η_Y) be two Calabi-Yau objects in a symmetric monoidal bicategory $\mathcal{C}$. A 1-morphism of Calabi-Yau objects is a 1-equivalence $f : X \to Y$ in $\mathcal{C}$, so that the following diagram of 2-morphisms in $\mathcal{C}$ commutes.

$$\begin{tikzcd}
\dim(X) \arrow[dd, "\dim(f)"'] \arrow[dr, "\eta_X"] & \\
& \mathrm{id}_1 \\
\dim(Y) \arrow[ur, "\eta_Y"'] &
\end{tikzcd} \tag{5.5}$$

Here, $\dim(f) : \dim(X) \to \dim(Y)$ is the 2-isomorphism in $\mathcal{C}$ which is induced from the 1-morphism $f : X \to Y$. An explicit description of this 2-morphism follows from the "dinaturality" of the evaluation and the coevaluation in lemma 4.7.

The 2-morphisms of the bicategory of Calabi-Yau objects are given as follows:

Definition 5.4. A 2-morphism of Calabi-Yau objects in $\mathcal{C}$ is a 2-isomorphism in $\mathcal{C}$. Given a symmetric monoidal bicategory $\mathcal{C}$, this defines the bigroupoid $\mathrm{CY}(\mathcal{C})$ of Calabi-Yau objects in $\mathcal{C}$.

Next, we show that fully-dualizable Calabi-Yau objects are precisely homotopy fixed points of the action given by the Serre automorphism:

Theorem 5.5. *Let $\mathcal{C}$ be a symmetric monoidal bicategory. Then, there is an equivalence of bigroupoids*

$$\mathrm{CY}(\mathcal{C}^{\mathrm{fd}}) \cong (\mathscr{K}(\mathcal{C}^{\mathrm{fd}}))^{SO(2)}, \tag{5.6}$$

where the $SO(2)$-action on $\mathscr{K}(\mathcal{C}^{\mathrm{fd}})$ is given by the Serre automorphism as in example 2.33.

Proof. First note that corollary 4.10 provides an explicit description of $(\mathscr{K}(\mathcal{C}^{\mathrm{fd}}))^{SO(2)}$: objects are given by pairs (X, λ_X), where X is a fully-dualizable object of $\mathcal{C}$, and $\lambda_X : S_X \to \mathrm{id}_X$ is a 2-isomorphism which trivializes the Serre automorphism of X. Given such a homotopy fixed point, we define a Calabi-Yau object in $\mathscr{K}(\mathcal{C}^{\mathrm{fd}})$ by taking the trace of the Serre automorphism: indeed, by using the pseudo-naturality of the braiding, one sees that the trace of the Serre automorphism is given by

$$\mathrm{tr}(S_X) \cong \mathrm{ev}_X \circ \mathrm{ev}_X^R . \tag{5.7}$$

Since the Serre automorphism is trivializable in $(\mathscr{K}(\mathcal{C}^{\mathrm{fd}}))^{SO(2)}$, the right adjoint of the evaluation is equivalent to the coevaluation composed with the braiding, see lemma 4.5. Now, we use that X is fully-dualizable by observing that there is a counit $\eta_X : \mathrm{ev}_X \circ \mathrm{ev}_X^R \to \mathrm{id}_1$ of the right-adjunction of the evaluation. Thus, the counit of the right-adjunction gives X the structure of a Calabi-Yau object.

For the other direction, we proceed as follows: If X is a fully-dualizable Calabi-Yau object in $\mathcal{C}$, we need to construct a trivialization of the Serre automorphism. For this, it suffices to show that

$$\mathrm{ev}_X^R = \tau_{X^*,X} \circ \mathrm{coev}_X, \tag{5.8}$$

up to a 2-isomorphism. Then, the structure of a Calabi-Yau object provides a trivialization of the Serre automorphism by lemma 4.5.

In order to show equation (5.8), it suffices to show that the 1-morphism $\tau_{X^*,X} \circ \mathrm{coev}_X$ is a right-adjoint of the evaluation, since the category of all adjoints is contractible. Now, the unit and the counit of the Calabi-Yau object are precisely the unit and the counit for the right adjunction of $\tau_{X^*,X} \circ \mathrm{coev}_X$ and the evaluation. Thus, starting from a Calabi-Yau object, we have constructed a trivialization of the Serre automorphism. Since X was assumed to be fully-dualizable, this data is exactly an object in $\mathscr{K}(\mathcal{C}^{\mathrm{fd}})^{SO(2)}$.

Furthermore, one checks that a 1-morphism in $\mathscr{K}(\mathcal{C}^{\mathrm{fd}})^{SO(2)}$ gives rise to a 1-morphism in $\mathrm{CY}(\mathcal{C})$ and vice versa. It is trivial to see that the 2-morphisms of the two bigroupoids stand in bijection. □

Remark 5.6. By equation (5.8), the right adjoint of the evaluation of a fully-dualizable Calabi-Yau object is given by the $\tau_{X^*,X} \circ \mathrm{coev}_X$. We claim that the left-adjoint of the evaluation is given by the same formula

$$\mathrm{ev}^L = \tau_{X^*,X} \circ \mathrm{coev}_X, \tag{5.9}$$

and thus left- and right-adjoint of the evaluation agree. Indeed, by theorem 5.5, a fully-dualizable Calabi-Yau object is equivalent to a trivialization $\lambda_X : S_X \to \mathrm{id}_X$ of the Serre automorphism. The trivialization of S_X also provides a trivialization of S_X^{-1}. By lemma 4.5, the left adjoint of the evaluation will then be as claimed.

We now come to the second main result of this section: the classification of fully-extended oriented 2-dimensional topological quantum field theories in terms of Calabi-Yau objects. Recall that Schommer-Pries has given a presentation of the 2-dimensional oriented bordism bicategory via generators and relations. For the benefit of the reader, we recall this theorem here.

Theorem 5.7 ([SP09, Theorem 3.50])**.** *The oriented 2-dimensional bordism bicategory, as a symmetric monoidal bicategory, has the generators and relations as in figure 5.1 on page 107.*

Using the presentation for the oriented bordisms bicategory, Schommer-Pries was able to classify oriented 2-dimensional field theories with values in Alg_2 by showing that there is an equivalence of bicategories

$$\mathrm{Fun}_\otimes(\mathrm{Cob}^{\mathrm{or}}_{2,1,0}, \mathrm{Alg}_2) \cong \mathrm{Frob} \tag{5.10}$$

where Frob is the bigroupoid of Frobenius algebras introduced in definition 2.11. We now prove a stronger theorem where the target space is allowed to be an arbitrary symmetric monoidal bicategory.

Theorem 5.8. *Let $\mathcal{C}$ be a symmetric monoidal bicategory. Then, the following 2-functor is an equivalence of bicategories:*

$$\begin{aligned} \mathrm{Fun}_\otimes(\mathrm{Cob}^{\mathrm{or}}_{2,1,0}, \mathcal{C}) &\to \mathrm{CY}(\mathcal{C}^{\mathrm{fd}}) \\ Z &\mapsto (Z(+),\, Z(\ominus)) \\ (\nu : Z \to \bar{Z}) &\mapsto (\nu_+ : Z(+) \to \bar{Z}(+)) \\ (m : \nu \to \nu') &\mapsto (m_+ : \nu_+ \Rightarrow \nu'_+). \end{aligned} \tag{5.11}$$

Proof. In order to simplify notation, set $X := Z(+)$ and $X^* := Z(-)$. To see that X is a fully-dualizable object in $\mathcal{C}$, note that the value of Z on the elbows provides evaluation and coevaluation. The value of Z on the cusps provides the 2-morphisms α and β in definition 4.1. Thus, X is dualizable. Now, the cup, the cap and the saddle give units and counits for left- and right-adjunction between evaluation and coevaluation. Thus, X is fully-dualizable by [Pst14, Theorem 3.9].

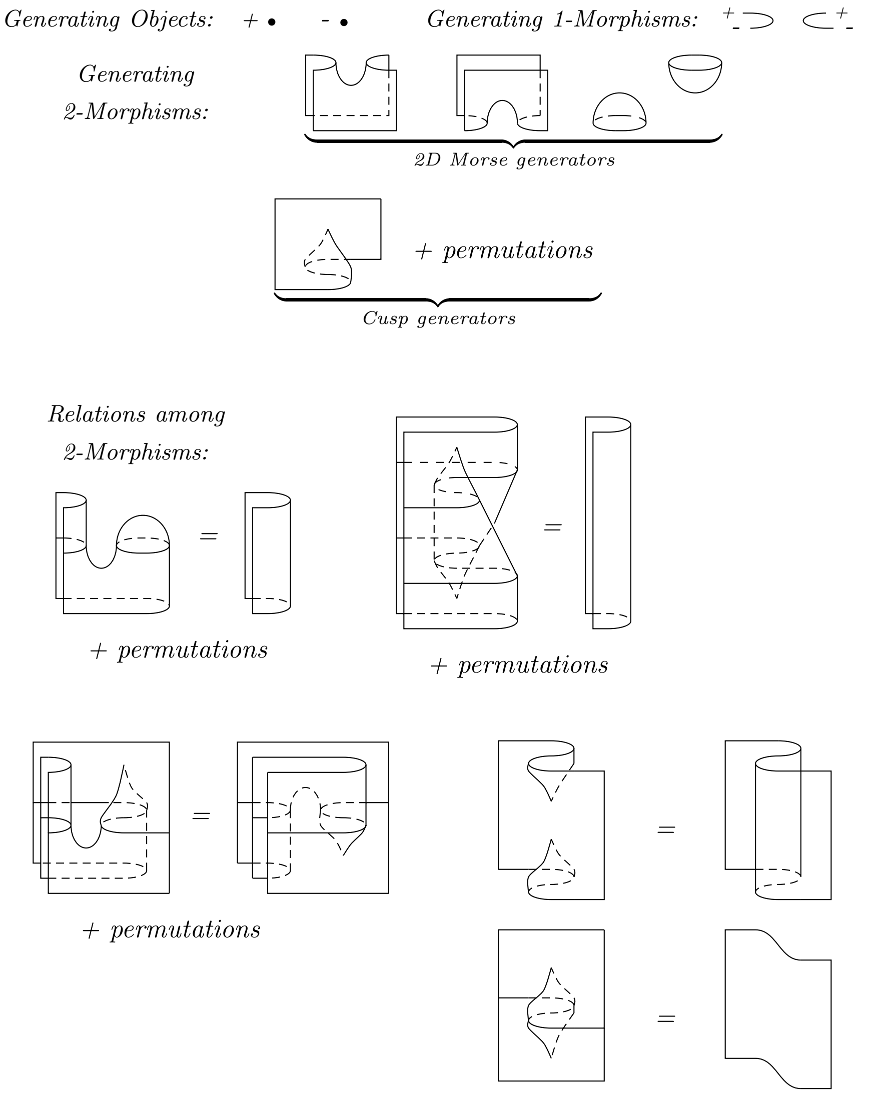


Figure 5.1.: Generators and relations for the oriented bordism bicategory

Next, note that X, together with $\eta := Z(\ ) : \dim(X) \to \mathrm{id}_1$ is a Calabi-Yau object in $\mathcal{C}^{\mathrm{fd}}$; the corresponding unit $\varepsilon : \mathrm{id}_{X\otimes X^*} \to \tau_{X^*,X} \circ \mathrm{coev}_X \circ \mathrm{ev}_X$ is given by the value of Z on the saddle. The relations among the 2-morphisms in the oriented bordism bicategory then ensure that (X, η) really is a Calabi-Yau object.

In order to see that the functor in equation (5.11) is well-defined on 1-morphisms, let $\nu : Z \to \bar{Z}$ be pseudo-natural transformation. Note that naturality with respect to the cup is equivalent to the equation

$$Z(\ ) \circ \nu_{S^1} = \bar{Z}(\ ), \tag{5.12}$$

where for better readability we have left out associators and unitors. Here $\nu_{S^1} : \bar{Z}(S^1) \to Z(S^1)$ is the 2-isomorphism which fills the naturality square. This is exactly the condition requiring that $\nu_+ : Z(+) \to \bar{Z}(+)$ is a 1-morphism of Calabi-Yau objects. This shows that the 2-functor in equation (5.11) indeed takes vales in $\mathrm{CY}(\mathcal{C}^{\mathrm{fd}})$.

We now show that the 2-functor in equation (5.11) is an equivalence of bicategories. In order to see essential surjectivity, it suffices to give the values of Z on the generators of the bordism bicategory. Suppose that (X, η) is a fully-dualizable Calabi-Yau object. Let ε be the unit of left adjunction of evaluation and coevaluation coming from the definition of Calabi-Yau object. Note that in the oriented bordism bicategory, the elbows also form a right-adjunction, with unit given by the cap and counit given by the opposite saddle.

Define a fully-extended 2-dimensional topological quantum field theory Z by setting

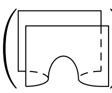
$$Z(+) := X, \qquad Z(-) := X^*. \tag{5.13}$$

On the 1-morphisms of the bordism bicategory, set

$$\begin{aligned} Z(\ {}^{+}_{-}\ ) &:= \tau_{X,X^*} \circ \mathrm{coev}_X , \\ Z(\ {}^{+}_{-}\ ) &:= \mathrm{ev}_X . \end{aligned} \tag{5.14}$$

Now define on the 2-morphisms of the oriented bordism bicategory as follows:

$$Z(\ ) := \eta,$$

Z  $:= \varepsilon. \quad (5.15)$

In order to define the value of Z on the cap and on the other saddle, let

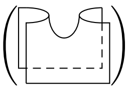
$$\begin{aligned} \eta^L &: \mathrm{ev}^L \circ \mathrm{ev} \to \mathrm{id}_{X\otimes X^*} \\ \varepsilon^L &: \mathrm{id}_1 \to \mathrm{ev} \circ \mathrm{ev}^L \end{aligned} \tag{5.16}$$

be the unit and counit of the left-adjunction between the evaluation and its left adjoint ev^L. By remark 5.6, the left-adjoint ev^L is also given by $\tau_{X^*,X} \circ \mathrm{coev}$. Thus, η^L and ε^L induce 2-morphisms

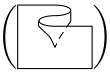
$$\begin{aligned} \eta^L &: \tau_{X^*,X} \circ \mathrm{coev}_X \circ \mathrm{ev}_X \to \mathrm{id}_{X\otimes X^*} \\ \varepsilon^L &: \mathrm{id}_1 \to \mathrm{ev}_X \circ \tau_{X^*,X} \circ \mathrm{coev}_X . \end{aligned} \tag{5.17}$$

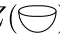
Now, define

$Z($$) : = \varepsilon^L$

$Z\Bigg($$\Bigg) : = \eta^L.$ (5.18)

Finally, assign the value of Z on the cusps to be the 2-morphisms α and β (and their inverses) in definition 4.1. We set

$Z\Bigg($$\Bigg) := \alpha$ and $Z\Bigg($$\Bigg) := \alpha^{-1}.$ (5.19)

With these definitions, Z becomes is a symmetric monoidal 2-functor with $Z($$) = \eta$. Thus, the functor in (5.11) is essentially surjective on objects.

In order to see that evaluating at the positive point is essentially surjective on 1-morphisms, let $f : X \to Y$ be a 1-morphism in $\mathrm{CY}(\mathcal{C}^{\mathrm{fd}})$. Let Z and $\bar{Z}$ be the topological quantum field theories constructed above with $Z(+) = X$ and $\bar{Z}(+) = Y$. We now need to construct a pseudo-natural transformation $\nu : Z \to \bar{Z}$. Define on objects of the oriented bordism bicategory

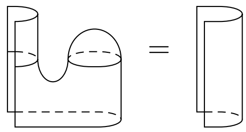
$$\begin{aligned} \nu_+ &: = f : Z(+) \to \bar{Z}(+) \\ \nu_- &: = (f^*)^{-1} : Z(-) \to \bar{Z}(-). \end{aligned} \tag{5.20}$$

On the 1-morphisms of the bordism bicategory, which are just the elbows, we define 2-cells which fill the naturality square. For the left elbow, we have to provide a 2-morphism $\nu_{(\subset^{+}_{-})}$ in the diagram

$$\begin{tikzcd} Z(+\sqcup -) \arrow[r, "\nu_+\otimes\nu_-"] \arrow[d, "Z(\subset^{+}_{-})"'] & \bar{Z}(+\sqcup -) \arrow[d, "\bar{Z}(\subset^{+}_{-})"] \arrow[dl, phantom, "\swarrow \nu_{(\subset^{+}_{-})}"] \\ Z(\emptyset) \arrow[r, "\nu_\emptyset"'] & \bar{Z}(\emptyset) \end{tikzcd} \tag{5.21}$$

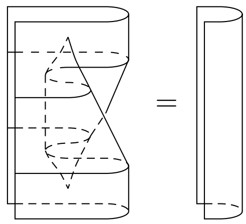
which follows from the "dinaturality" of the evaluation and is explicitly constructed in lemma 4.7. The 2-cell for the other elbow is constructed similarly, by observing that the coevaluation is also dinatural. Now, we need to check that ν is indeed a pseudo-natural transformation and is natural with respect to 2-morphisms of the bordism bicategory. Naturality with respect to the cup is equivalent to equation (5.12), which just says that our original 1-morphism f is a morphism of Calabi-Yau objects. As argued in [SP09, Section 3.7] naturality with respect to the two cusps is equivalent to the fact that there is an ambidextrous adjunction between $Z(+)$ and $\bar{Z}(+)$, which even is an adjoint equivalence by naturality with respect to the saddles. Thus, ν really is a pseudo-natural transformation, and the 2-functor in equation (5.11) is essentially surjective on 1-morphisms.

In order to check that the functor induces a bijection on 2-morphisms, we have to check the relations among the 2-morphism of both bicategories. First of all, the equation



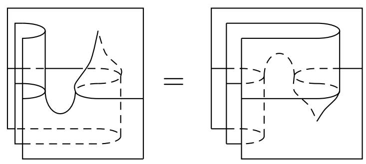
(5.22)

in the oriented bordism bicategory, together with its permutations, demands that the two elbows, together with the saddles and the cup and cap form an ambidextrous adjunction. In $\mathscr{K}(\mathcal{C}^{\mathrm{fd}})^{SO(2)}$, this equation is satisfied since $Z(+)$ is fully-dualizable. Next, the swallowtail equation



(5.23)

are satisfied in $\mathscr{K}(\mathcal{C}^{\mathrm{fd}})^{SO(2)}$ because every fully-dualizable object may be completed into a coherent dual pair, which satisfies these equations, cf. [Pst14, Theorem 2.7]. This leaves the cusp-flip equations,



(5.24)

which are satisfied since $Z(+)$ can be made into a *coherent* fully-dual pair, which demands that these equations are satisfied, cf. [Pst14, Theorem 3.16]. The last two relations among the 2-morphisms in the bordism bicategory demand that the two cusps are inverses to each other, and are satisfied since the two 2-morphisms α and β are isomorphisms. Thus, the 2-functor in equation (5.11) induces a bijection on 2-morphisms and is an equivalence of bicategories. □

Combining theorem 5.8 and theorem 5.5 now shows the 2-dimensional cobordism hypothesis for oriented manifolds:

Corollary 5.9. *Let $\mathcal{C}$ be a symmetric monoidal bicategory, and consider the $SO(2)$-action on $\mathscr{K}(\mathcal{C}^{\mathrm{fd}})$ by the Serre automorphism. Then, there is an equivalence of bigroupoids*

$$\mathrm{Fun}_{\otimes}(\mathrm{Cob}^{\mathrm{or}}_{2,1,0}, \mathcal{C}) \cong (\mathscr{K}(\mathcal{C}^{\mathrm{fd}}))^{SO(2)}. \tag{5.25}$$

□

6. Outlook

Here, we give an outlook towards several generalizations and extensions of this thesis. As we have proven the 2-dimensional cobordism hypothesis for oriented manifolds in corollary 5.9, one obvious aim would be to give a detailed proof of the cobordism hypothesis for higher dimensions. Our proof relies on the description of the framed- and oriented bordism bicategories in terms of generators and relations, as well as an explicit description of fully-dualizable objects. In three dimensions, [DSPS13] gives a description of the fully-dualizable objects in a 3-category of tensor categories, while [BDSPV14] gives an explicit description of the $(3+2+1)$-dimensional bordism bicategory and proceeds to classify extended $(3+2+1)$-theories in terms of modular tensor categories in [BDSPV15]. In order to understand the action of $SO(3)$ and to compute homotopy fixed points, a 4-categorical setup is needed. While an algebraic model of a fully weak 4-category has appeared in [Hof11] under the name quadcategory, it remains to be seen if one can effectively work with this model. In contrast to the algebraic approach, a detailed description of the (∞, n)-category of bordisms has appeared in [Sch14].

Another immediate application of theorem 5.8 would be to classify topological field theories with vales in symmetric monoidal bicategories other than Vect_2 or Alg_2. One candidate is the bicategory of spans of 2-vector spaces as considered in [Mor11, Mor15]. However, a formal argument in [Hau14] in the language of ∞-categories suggests that the $SO(2)$-action on iterated spans should be trivializable.

More interesting targets would be the bicategory of Landau-Ginzburg models, or the bicategory of differential graded algebras. By [CM16], the whole bicategory of Landau-Ginzburg models is fully-dualizable. In order to use theorem 5.8 of the present thesis, one would have to explicitly compute the Serre automorphism of a fully-dualizable object in the bicategory of Landau-Ginzburg models.

The situation for dg-algebras is a bit more involved. Here, fully-dualizable objects are given by smooth and proper dg-algebras as defined in [KS09]. Computing the Serre automorphism in this bicategory is likely to be more difficult than in Alg_2 or in Vect_2, since a trivialization of the Serre automorphism already consists of an infinite amount of data. Work in this directions using the language of A_∞-algebras includes for instance [Cos07].

A geometric extension of this thesis would be to work directly with the framed bordism bicategory, instead of the algebraic skeleton of [Pst14]. This would allow to give a precise meaning to "rotating the framings" on the framed bordism bicategory, and to define the $SO(2)$-action geometrically.

Another extension of this thesis could be towards non-compact, or open-closed theories: theorem 5.8 classifies 2-dimensional, oriented, fully-extended topological quantum field theories by fully-dualizable Calabi-Yau objects. According to [Lur09b], oriented, non-compact field theories are classified by the whole bicategory of Calabi-Yau objects, which do not need to be fully-dualizable. In order to prove this in the language of bicategories, one should give generators and relations of the non-compact version of the oriented bordism bicategory, and then consider symmetric monoidal functors out of it.

6.1. The homotopy hypothesis

Next, we given an outlook on the homotopy hypothesis. While this thesis deals with homotopy actions in an essentially algebraic way using the language of symmetric monoidal bicategories, one might hope to make the guiding principle of the homotopy hypothesis sufficiently precise, so that one can do computations on the topological side, and then transfer the results to the algebraic world. This approach is for instance taken in [Dav11]. As we work with group actions on symmetric monoidal bicategories, there are three essentially different types of structures that one has to take care of:

1. First of all, one has to take the purely bicategorical aspect into account. This means that one expects that there is a suitable tricategory of 2-types, which is equivalent to the tricategory of bicategories via the fundamental 2-groupoid. In [Gur11], Gurski sets up an appropriate tricategory of topological spaces and gives a detailed description of the tricategorical structure of the fundamental 2-groupoid. Restricting this tricategory of topological spaces to the tricategory of 2-types should show that the fundamental 2-groupoid induces an equivalence of tricategories.

   Further work in [CCG11] considers the geometric realization of a bicategory, which should act as an inverse to the fundamental 2-groupoid. A model-categorical result going in this direction is proven in [MS93]: there is a Quillen-equivalence between the category of 2-types and the category of strict 2-categories.

2. As we deal with *symmetric monoidal* bicategories, which is extra structure, we should expect additional structure on the topological side. As originally proven by Segal using the theory of Γ-spaces in [Seg74], the classifying space of a symmetric monoidal category is an E_∞-space. Moreover, one should also expect a result in the other direction: namely that the fundamental groupoid of an E_n-space has a symmetric monoidal structure. Results in both directions appear in [GO13]: it is proven that the classifying space of a symmetric monoidal bicategory is an E_∞-space. In the other direction, the authors show that for $n \geq 4$, the fundamental 2-groupoid of an E_n-space carries the additional structure of a symmetric monoidal bicategory.

3. Finally, we have to take the additional data of a group action into account. Here, the following results are available: in [BMO$^+$15], it is proven that there is a model structure on the category of categories equipped with a strict G-action of a finite group. This model category is then shown to be Quillen equivalent to the model category of G-spaces. It would be interesting to extend this result to non-strict G-actions of topological groups on bicategories and thus to prove an equivariant homotopy hypothesis for weak G-actions.

6.2. Homotopy orbits

Finally, we indicate another way of proving the cobordism hypothesis for oriented manifolds by using homotopy orbits. Recall that in chapter 4, we have constructed an $SO(2)$-action on the bicategory $\mathbb{F}_{cfd}$, which is a skeleton of the framed bordism bicategory. We have shown how the action on $\mathbb{F}_{cfd}$ induces an action on the bicategory of symmetric monoidal functors $\mathrm{Fun}_{\otimes}(\mathbb{F}_{cfd}, \mathcal{C})$, and that via the framed cobordism hypothesis the induced action on $\mathscr{K}(\mathcal{C}^{\mathrm{fd}})$ agrees with the action of the Serre automorphism. As a consequence, we are able to provide an equivalence of bicategories

$$\mathrm{Fun}_{\otimes}(\mathbb{F}_{cfd}, \mathcal{C})^{SO(2)} \cong \mathscr{K}(\mathcal{C}^{\mathrm{fd}})^{SO(2)} \tag{6.1}$$

in corollary 4.49. We could then in principle deduce the cobordism hypothesis for oriented manifolds from equation (6.1), once we provide an equivalence of bicategories

$$\mathrm{Fun}_{\otimes}(\mathbb{F}_{cfd}, \mathcal{C})^{SO(2)} \cong \mathrm{Fun}(\mathrm{Cob}^{\mathrm{or}}_{2,1,0}, \mathcal{C}). \tag{6.2}$$

The above equivalence can be proven directly by using a presentation of the oriented bordism bicategory via generators and relations, given in [SP09]. In fact, this equivalence of bicategories follows directly from corollary 4.49 and from corollary 5.9.

Here, we want to comment on an alternative approach. Namely, in order to provide an equivalence as in equation (6.2), it suffices to identify the oriented bordism bicategory with the *colimit* of the $SO(2)$-action on $\mathbb{F}_{cfd}$. Indeed, recall that one may define a G-action on a bicategory $\mathcal{C}$ to be a trifunctor $\rho : B\Pi_2(G) \to \mathrm{Bicat}$ with $\rho(*) = \mathcal{C}$. We then define the bicategory of homotopy orbits or co-invariants $\mathcal{C}_G$ to be the tricategorical colimit of the action.

Furthermore, recall that we work with *symmetric monoidal* bicategories, and that the action of the Serre automorphism is monoidal. Thus, we obtain a diagram $\rho : \Pi_2(SO(2)) \to \mathrm{SymMonBicat}$, with values in the tricategory of symmetric monoidal bicategories.

It follows from theorem 2.34 that the bicategory of homotopy fixed points of this action has a monoidal structure, which is induced from the monoidal structure on $\mathcal{C}$. This observation allows us to make the following conjecture:

Conjecture 6.1. Let $\rho : \Pi_2(SO(2)) \to \mathrm{SymMonBicat}$ be the action of the Serre automorphism on the core of fully-dualizable objects of a symmetric monoidal bicategory $\mathcal{C}$. Then, the trilimit of this diagram exists in the tricategory of symmetric monoidal bicategories and is given as a bicategory by $\mathscr{K}(\mathcal{C}^{\mathrm{fd}})^{SO(2)}$ as in theorem 2.34. Furthermore, the monoidal structure on $\mathscr{K}(\mathcal{C}^{\mathrm{fd}})^{SO(2)}$ coming from the monoidal structure on $\mathcal{C}$ agrees with the monoidal structure of the tricategorical limit in SymMonBicat.

Now, consider a monoidal G-action on a symmetric monoidal bicategory $\mathcal{C}$, and suppose that the tricategorical colimit of the action in SymMonBicat exists. Then, we obtain an equivalence of bicategories

$$\mathrm{Fun}_{\otimes}(\mathcal{C}_G, \mathcal{D}) \cong \mathrm{Fun}_{\otimes}(\mathcal{C}, \mathcal{D})^G \tag{6.3}$$

for an arbitrary symmetric monoidal bicategory $\mathcal{D}$. The following conjecture is then natural:

Conjecture 6.2. Consider the $SO(2)$-action on the skeletal version of the framed bordism bicategory $\mathbb{F}_{cfd}$ as in defintion 4.42. The tricategorical colimit of this action with values in SymMonBicat exists and is monoidally equivalent to the oriented bordism bicategory:

$$(\mathbb{F}_{cfd})_{SO(2)} \cong \mathrm{Cob}^{\mathrm{or}}_{2,1,0}\,. \tag{6.4}$$

Remark 6.3. We believe that this is not an isolated phenomenon, in the sense that any higher bordism category equipped with additional tangential structure should be obtained by taking an appropriate colimit of a G-action on the framed bordism category.

Given conjecture 6.2 and equation 6.3, we obtain the following sequence of monoidal equivalences of bicategories:

$$\begin{aligned} \mathrm{Fun}_{\otimes}(\mathrm{Cob}^{\mathrm{or}}_{2,1,0}, \mathcal{C}) &\cong \mathrm{Fun}_{\otimes}((\mathbb{F}_{cfd})_{SO(2)}, \mathcal{C}) && \text{(by conjecture 6.2)} \\ &\cong \mathrm{Fun}_{\otimes}(\mathbb{F}_{cfd}, \mathcal{C})^{SO(2)} && \text{(by equation 6.3)} \\ &\cong \mathscr{K}(\mathcal{C}^{\mathrm{fd}})^{SO(2)} && \text{(by corollary 4.49).} \end{aligned} \tag{6.5}$$

Hence conjecture 6.2 implies the cobordism hypothesis for oriented 2-manifolds. Notice that the chain of equivalences in equation (6.5) is natural in $\mathcal{C}$.

On the other hand, the cobordism hypothesis for oriented manifolds in 2-dimensions implies conjecture 6.2, provided that the colimit exists. Indeed, by using a tricategorical version of the Yoneda lemma, as developed for instance in [Buh15], the chain of equivalences

$$\begin{aligned} \mathrm{Fun}_{\otimes}(\mathrm{Cob}^{\mathrm{or}}_{2,1,0}, \mathcal{C}) &\cong \mathscr{K}(\mathcal{C}^{\mathrm{fd}})^{SO(2)} \\ &\cong \mathrm{Fun}_{\otimes}(\mathrm{Cob}^{\mathrm{fr}}_{2,1,0}, \mathcal{C})^{SO(2)} \\ &\cong \mathrm{Fun}_{\otimes}((\mathbb{F}_{cfd})_{SO(2)}, \mathcal{C}) \end{aligned} \tag{6.6}$$

implies that $\mathrm{Cob}^{\mathrm{or}}_{2,1,0}$ is equivalent to $(\mathbb{F}_{cfd})_{SO(2)}$, due to the uniqueness of representable objects.

It would then be of great interest to develop concrete constructions of homotopy coinvariants of group actions on (symmetric monoidal) bicategories in order to directly verify the equivalence in conjecture 6.2, and to extend the above arguments to general tangential G-structures.

A. Weak endofunctors as a monoidal bicategory

Given a bicategory $\mathcal{C}$, the bicategory of endofunctors $\mathrm{End}(\mathcal{C})$ has got the structure of a monoidal bicategory, which we explain below. Most of the material can be deduced from the theory of "symmetric monoidal whiskering" of [SP09]. As we only consider the case of group actions via auto-equivalences, we restrict to the bigroupoid $\mathrm{Aut}(\mathcal{C})$.

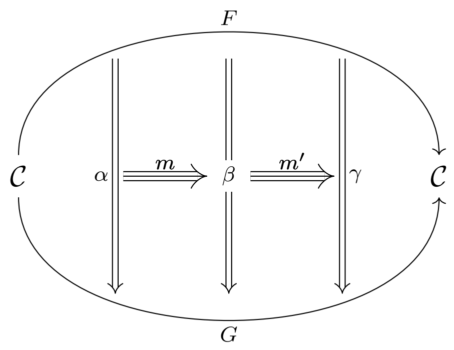
Definition A.1. Let $\mathcal{C}$ be a bicategory. Then, $\mathrm{Aut}(\mathcal{C})$ is the monoidal bicategory where

- objects are given by equivalences of bicategories $F : \mathcal{C} \to \mathcal{C}$,
- 1-morphisms are given by invertible pseudo-natural transformations,
- 2-morphisms are given by invertible modifications.

For the definitions of a bicategory, functors between bicategories, pseudo-natural transformations, and modifications we refer to [Bén67].

First, we set up the Hom-categories $\mathrm{Aut}(\mathcal{C})(F, G)$. The vertical composition of 2-morphisms in $\mathrm{Aut}(\mathcal{C})$ is given by vertical composition of modifications, as explained below.

Definition A.2. Let $\mathcal{C}$ be a bicategory, let F, $G : \mathcal{C} \to \mathcal{C}$ be weak 2-functors, let α, β, $\gamma : F \to G$ be pseudo-natural transformations, and let $m : \alpha \to \beta$ and $m' : \beta \to \gamma$ be modifications as in the diagram below.



(A.1)

Define the vertical composition $m' \circ m : \alpha \to \gamma$ of m and m' to be the modification given in components by the 2-morphism $(m' \circ m)_c := m'_c \circ m_c$ between the 1-morphisms

$$\alpha_c \xrightarrow{m_c} \beta_c \xrightarrow{m'_c} \gamma_c \tag{A.2}$$

for every object $c \in \mathcal{C}$.

Next, we define the horizontal composition in $\mathrm{Aut}(\mathcal{C})$: First, let us specify the composition functor

$$\mathrm{Aut}(\mathcal{C})(G,H)\times\mathrm{Aut}(\mathcal{C})(F,G)\to\mathrm{Aut}(\mathcal{C})(F,H) \tag{A.3}$$

on objects:

Definition A.3. Let $\mathcal{C}$ be a bicategory, and let F, G, $H:\mathcal{C}\to\mathcal{C}$ be weak 2-functors. Let $\alpha: F\to G$ and $\beta: G\to H$ be two pseudo-natural transformations as in the diagram below.

$$\begin{tikzcd} \mathcal{C} \arrow[rr, bend left=60, "F"{name=F}] \arrow[rr, "G"{description, name=G}] \arrow[rr, bend right=60, "H"'{name=H}] & & \mathcal{C} \arrow[from=F, to=G, Rightarrow, "\alpha"'] \arrow[from=G, to=H, Rightarrow, "\beta"'] \end{tikzcd} \tag{A.4}$$

We define their composition $\beta\circ\alpha: F\to H$ to be the pseudo-natural transformations given by the following data:

- for every object $c\in\mathcal{C}$, the 1-morphism

$$F(c)\xrightarrow{\alpha_c}G(c)\xrightarrow{\beta_c}H(c), \tag{A.5}$$

- for every 1-morphism $f: c\to d$ in $\mathcal{C}$, the invertible 2-morphism $(\beta\circ\alpha)_f$ in the 2-cell

$$\begin{tikzcd} F(c) \arrow[r, "\beta_c\circ\alpha_c"] \arrow[d, "F(f)"'] & H(c) \arrow[d, "H(f)"] \arrow[dl, Rightarrow, "(\beta\circ\alpha)_f"] \\ F(d) \arrow[r, "\beta_d\circ\alpha_d"'] & H(d) \end{tikzcd} \tag{A.6}$$

defined by the following composition:

$$\begin{tikzcd} H(f)\circ(\beta_c\circ\alpha_c) \arrow[r] & (H(f)\circ\beta_c)\circ\alpha_c \arrow[r, "\beta_f*\mathrm{id}"] & (\beta_d\circ G(f))\circ\alpha_c \arrow[dll] \\ \beta_d\circ(G(f)\circ\alpha_c) \arrow[r, "\mathrm{id}*\alpha_f"'] & \beta_d\circ(\alpha_d\circ F(f)) \arrow[r] & (\beta_d\circ\alpha_d)\circ F(f). \end{tikzcd} \tag{A.7}$$

Here, the unlabeled arrows are induced by the associators of $\mathcal{C}$.

Now, we shall define horizontal composition on 2-morphisms of $\mathrm{Aut}(\mathcal{C})$.

Definition A.4. Let $\mathcal{C}$ be a bicategory, let F, G, $H:\mathcal{C}\to\mathcal{C}$ be weak 2-functors, let α, $\beta: F\to G$ and α', $\beta': G\to H$ be pseudo-natural transformations, and suppose

$m : \alpha \to \beta$ and $m' : \alpha' \to \beta'$ are modifications as below.

$$\begin{tikzcd}[column sep=huge]
\mathcal{C} \arrow[rr, bend left=80, "F"] \arrow[rr, "G" description] \arrow[rr, bend right=80, "H"'] \arrow[rr, phantom, bend left=40, "\alpha\Downarrow \overset{m}{\Rrightarrow} \Downarrow\beta"] \arrow[rr, phantom, bend right=40, "\alpha'\Downarrow \overset{m'}{\Rrightarrow} \Downarrow\beta'"'] & & \mathcal{C}
\end{tikzcd} \tag{A.8}$$

Define the horizontal composition $m' * m : \alpha' \circ \alpha \to \beta' \circ \beta$ in components by the 2-morphism

$$(m' * m)_c := m_c * m'_c : \alpha_c \circ \alpha'_c \to \beta_c \circ \beta'_c \tag{A.9}$$

for all $c \in \mathcal{C}$.

Furthermore, the associator and unitors of $\mathrm{Aut}(\mathcal{C})$ are induced by the associator and unitors of $\mathcal{C}$.

Next, we explain the monoidal structure of $\mathrm{Aut}(\mathcal{C})$. We will see that there are two canonical monoidal structures on $\mathrm{Aut}(\mathcal{C})$, since there are two ways of defining horizontal composition of pseudo-natural transformations, which are not equal to each other, but only related by an invertible modification. On objects, the monoidal product is given by the composition of weak 2-functors, which we define below.

Definition A.5. Let $\mathcal{C}$ be a bicategory, and let (F, ϕ), $(G, \psi) : \mathcal{C} \to \mathcal{C}$ be two weak 2-functors. We define their composite $F \otimes G := F \circ G$ as follows:

- on objects of $\mathcal{C}$ we have the function $F \circ G : \mathrm{Ob}(\mathcal{C}) \to \mathrm{Ob}(\mathcal{C})$,
- on Hom-categories, we have the composition of ordinary functors

$$F_{Ga,Gb} \circ G_{ab} : \mathcal{C}(a, b) \to \mathcal{C}(FG(a), FG(b)). \tag{A.10}$$

- For the identity 1-morphisms, we set the composite to be

$$\mathrm{id}_{FG(a)} \xrightarrow{\psi_{Ga}} G(\mathrm{id}_{Ga}) \xrightarrow{F(\phi_a)} FG(\mathrm{id}_a). \tag{A.11}$$

- For composition of 1-morphisms, we set the composite to be

$$FG(g) \circ FG(f) \xrightarrow{\psi_{G(g),G(f)}} F(G(g) \circ G(f)) \xrightarrow{F(\phi_{gf})} FG(g \circ f). \tag{A.12}$$

One can show that this composition defines another weak 2-functor $F \circ G : \mathcal{C} \to \mathcal{C}$. Note that composition of functors is strictly associative. This follows from the fact that composition of 2-morphisms is strictly associative and that a weak 2-functor is strictly associative on 2-morphisms. For example, if (H, λ), (G, ψ) and $(F, \phi) : \mathcal{C} \to \mathcal{C}$ are three weak 2-functors, the compositions for the units of $(H \circ G) \circ F$ is given by

$$HG(\phi_a) \circ H(\psi_{Fa}) \circ \lambda_{GF(a)} : \mathrm{id}_{HGFa} \to HGF(\mathrm{id}_a) \tag{A.13}$$

whereas the compositions for the units of $H \circ (G \circ F)$ is given by

$$H(G(\phi_a) \circ \psi_{Fa}) \circ \lambda_{GF(a)} : \mathrm{id}_{HGFa} \to HGF(\mathrm{id}_a). \tag{A.14}$$

Since the functor H is strictly associative on 2-morphisms, both composites are actually equal. A similar formula holds for the other natural transformation involving ϕ_{gf}.

The monoidal product of 1-morphisms of $\mathrm{Aut}(\mathcal{C})$ can be defined in two ways:

Definition A.6. Let $\mathcal{C}$ be a bicategory, let F, F' G, $G' : \mathcal{C} \to \mathcal{C}$ be weak 2-functors, and let $\alpha : F \to F'$ and $\beta : G \to G'$ be two pseudo-natural transformations as in the diagram below.

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F", ""{name=U, below}] \arrow[r, bend right, "F'"', ""{name=D}] \arrow[Rightarrow, from=U, to=D, "\alpha"'] & \mathcal{C} \arrow[r, bend left, "G", ""{name=U2, below}] \arrow[r, bend right, "G'"', ""{name=D2}] \arrow[Rightarrow, from=U2, to=D2, "\beta"'] & \mathcal{C} \end{tikzcd} \tag{A.15}$$

Then, there are two different ways of defining the horizontal composition of α and β:

1. The first way of composing α and β is given by
   - for every objects a of $\mathcal{C}$, we define a 1-morphism by the composition

$$(\beta \otimes_1 \alpha)_a : (G \circ F)(a) \xrightarrow{G(\alpha_a)} (G \circ F')(a) \xrightarrow{\beta_{F'(a)}} (G' \circ F')(a), \tag{A.16}$$

   - for every $f : a \to b$ in $\mathcal{C}$, we define the 2-morphism $(\beta \otimes_1 \alpha)_f$ to be the composition

$$\begin{tikzcd} G'(F'(f)) \circ (\beta_{F'(a)} \circ G(\alpha_a)) \arrow[r] & (G'(F'(f)) \circ \beta_{F'(a)}) \circ G(\alpha_a) \arrow[dl, "\beta_{F'(f)}*\mathrm{id}"] \\ (\beta_{F'(b)} \circ G(F'(f))) \circ G(\alpha_a) \arrow[r] & \beta_{F'(b)} \circ (G(F'(f)) \circ G(\alpha_a)) \arrow[dl, "\mathrm{id}*G(\alpha_f)"] \\ \beta_{F'(b)} \circ (G(\alpha_b) \circ G(F(f))) \arrow[r] & (\beta_{F'(b)} \circ G(\alpha_b)) \circ G(F(f)). \end{tikzcd} \tag{A.17}$$

2. The second way of composing α and β is given by:

- for every object a of $\mathcal{C}$, we define a 1-morphism $(\beta \otimes_2 \alpha)_a$ by the composition

$$(G \circ F)(a) \xrightarrow{\beta_{F(a)}} (G' \circ F)(a) \xrightarrow{G'(\alpha_a)} (G' \circ F')(a), \tag{A.18}$$

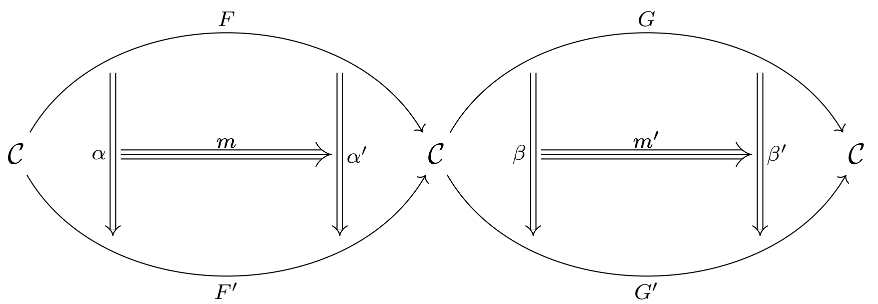
- for every $f : a \to b$ in $\mathcal{C}$, we define a 2-morphism $(\beta \otimes_2 \alpha)_f$ to be the composition

$$\begin{tikzcd}
(G' \circ F')(f) \circ (G'(\alpha_a) \circ \beta_{F(a)}) \arrow[r] & (G'(F'(f)) \circ G'(\alpha_a)) \circ \beta_{F(a)} \arrow[dl, "G'(\alpha_f)*\mathrm{id}"] \\
(G'(\alpha_b) \circ G'(F(f))) \circ \beta_{F(a)} \arrow[r] & G'(\alpha_b) \circ (G'(F(f)) \circ \beta_{F(a)}) \arrow[dl, "\mathrm{id}*\beta_{F(f)}"] \\
G'(\alpha_b) \circ (\beta_{F(b)} \circ G(F(f))) \arrow[r] & (G'(\alpha_b) \circ \beta_{F(b)}) \circ G(F(f)).
\end{tikzcd} \tag{A.19}$$

These two ways of defining the horizontal composition are *not* equal; however there is an invertible modification between the two choices of horizontal composition, cf. [GPS95, Section 5.6] for a proof in the language of tricategories.

Next, we need to define a monoidal product of 2-morphisms in $\mathrm{Aut}(\mathcal{C})$.

Definition A.7. Let $\mathcal{C}$ be a bicategory, let F, F', G, $G' : \mathcal{C} \to \mathcal{C}$ be weak 2-functors, α, $\alpha' : F \to G$ and β, $\beta' : G \to G'$ pseudo-natural transformations, and let $m : \alpha \to \alpha'$ and $m' : \beta \to \beta'$ be modifications as in the diagram below.



(A.20)

Depending on which tensor product we have chosen for the pseudo-natural transformations, we define the tensor product of modifications as follows:

1. If we have chosen the tensor product $\otimes_1$, we define the modification $m' \otimes_1 m : \beta \otimes_1 \alpha \to \beta' \otimes_1 \alpha'$ for an object a of $\mathcal{C}$ to have the components

$$(m' \otimes_1 m)_a := m'_{F(a)} * G(m_a) : \beta_{F'(a)} \circ G(\alpha_a) \to \beta'_{F'(a)} \circ G(\alpha'_a) \tag{A.21}$$

as in the diagram below.

$$\begin{tikzcd}[column sep=huge]
(G\circ F)(a) \arrow[r, bend left=50, "G(\alpha_a)"{name=U}] \arrow[r, bend right=50, "G(\alpha'_a)"'{name=D}] \arrow[Rightarrow, from=U, to=D, "G(m_a)"] & (G\circ F')(a) \arrow[r, bend left=50, "\beta_{F'(a)}"{name=U2}] \arrow[r, bend right=50, "\beta'_{F'(a)}"'{name=D2}] \arrow[Rightarrow, from=U2, to=D2, "m'_{F(a)}"] & (G'\circ F')(a)
\end{tikzcd} \tag{A.22}$$

2. For the second tensor product $\otimes_2$, we define the tensor product $m' \otimes_2 m$ at the component $a \in \mathcal{C}$ by

$$(m \otimes_2 m')_a := G'(\alpha'_a) * m'_{F(a)} : G'(\alpha_a) \circ \beta_{F(a)} \to G'(\alpha_a) \circ \beta'_{F(a)} \tag{A.23}$$

as in the diagram below.

$$\begin{tikzcd}[column sep=huge]
(G\circ F)(a) \arrow[r, bend left=50, "\beta_{F(a)}"{name=U}] \arrow[r, bend right=50, "\beta'_{F(a)}"'{name=D}] \arrow[Rightarrow, from=U, to=D, "m'_{F(a)}"] & (G'\circ F)(a) \arrow[r, bend left=50, "G'(\alpha_a)"{name=U2}] \arrow[r, bend right=50, "G'(\alpha'_a)"'{name=D2}] \arrow[Rightarrow, from=U2, to=D2, "G'(m_a)"] & (G'\circ F')(a)
\end{tikzcd} \tag{A.24}$$

In the following, we will have to choose a monoidal structure on $\mathrm{Aut}(\mathcal{C})$ which we will simply call $\otimes$; just to make a choice, define $\otimes := \otimes_1$.

As a last piece of monoidal structure on $\mathrm{Aut}(\mathcal{C})$, there are invertible modifications

$$\begin{gathered}
\phi^{\otimes}_{F,G} : \mathrm{id}_{F\otimes G} \to \mathrm{id}_F \otimes \mathrm{id}_G \\
\phi^{\otimes}_{(\beta,\beta'),(\alpha,\alpha')} : \beta \otimes \beta' \circ \alpha \otimes \alpha' \to (\beta' \circ \alpha') \otimes (\beta \circ \alpha)
\end{gathered} \tag{A.25}$$

which are induced from the unitors and associators of $\mathcal{C}$. Explicitly, the last equation should be read as follows: $\alpha : F \to G$, $\alpha' : F' \to G'$, $\beta : G \to H$ and $\beta' : G' \to H'$ are pseudo-natural transformations between weak 2-functors. Then ϕ is a modification in the diagram below.

$$\left(\begin{tikzcd}[column sep=huge]
\mathcal{C} \arrow[r, bend left=60, "F\otimes F'"{name=U}] \arrow[r, "G\otimes G'" description, ""{name=M}] \arrow[r, bend right=60, "H\otimes H'"'{name=D}] \arrow[Rightarrow, from=U, to=M, "\alpha\otimes\alpha'"'] \arrow[Rightarrow, from=M, to=D, "\beta\otimes\beta'"'] & \mathcal{C}
\end{tikzcd} \right) \xrightarrow{\phi^{\otimes}_{(\beta,\beta'),(\alpha,\alpha')}} \left(\begin{tikzcd}
\mathcal{C} \arrow[r, bend left=50, "F'"{name=U}] \arrow[r, bend right=50, "H'"'{name=D}] \arrow[Rightarrow, from=U, to=D, "\beta'\circ\alpha'"'] & \mathcal{C} \arrow[r, bend left=50, "F"{name=U2}] \arrow[r, bend right=50, "H"'{name=D2}] \arrow[Rightarrow, from=U2, to=D2, "\beta\circ\alpha"] & \mathcal{C}
\end{tikzcd} \right) \tag{A.26}$$

The rest of the data making $\mathrm{Aut}(\mathcal{C})$ into a monoidal bicategory is relatively trivial: we have to choose associators, which are pseudo-natural transformation between the functors $(F \otimes G) \otimes H \to F \otimes (G \otimes H)$. On objects of $\mathrm{Aut}(\mathcal{C})$, we may choose these associators to be trivial, since composition of weak 2-functors is associative on the nose. If $\alpha : F \to F'$, $\beta : G \to G'$ and $\gamma : H \to H'$ are pseudo-natural transformations, we need to choose a modification $m_{\alpha,\beta,\gamma}$ in the diagram below.

$$\begin{tikzcd}
(F \otimes G) \otimes H \arrow[r, "\mathrm{id}"] \arrow[d, "(\alpha\otimes\beta)\otimes\gamma"'] & F \otimes (G \otimes H) \arrow[d, "\alpha\otimes(\beta\otimes\gamma)"] \arrow[dl, Rightarrow, "m_{\alpha,\beta,\gamma}"'] \\
(F' \otimes G') \otimes H' \arrow[r, "\mathrm{id}"] & F' \otimes (G' \otimes H')
\end{tikzcd} \tag{A.27}$$

This modification is induced from the associator and unitors of $\mathcal{C}$.

The last piece of structure making $\mathrm{Aut}(\mathcal{C})$ into a monoidal bicategory are modifications between quadruple applications of the monoidal associator defined above. These can be chosen to be identities, since the associator is the identity on objects.

B. Finite linear categories as categories of modules

Here, we will show that finite, linear categories are equivalent to the representation category of a finite-dimensional algebra. We begin by giving abelian categories the structure of a Vect-module, cf. [Kir02].

B.1. Abelian categories as Vect-modules

Let $\mathcal{C}$ be a linear category. Then, the functor category $\mathrm{End}(\mathcal{C})$ becomes a linear category by adding natural transformations pointwise. Furthermore, $\mathrm{End}(\mathcal{C})$ has got the structure of a strict monoidal category where the monoidal product is given by the composition of functors.

Definition B.1. A Vect-module structure on a linear category $\mathcal{C}$ is a linear, monoidal functor $F : \mathrm{Vect} \to \mathrm{End}(\mathcal{C})$.

We claim that there is only one (up to unique isomorphism) such functor. Indeed, since F is monoidal there is an isomorphism $F(\mathbb{K}) \cong \mathrm{id}_{\mathcal{C}}$. Since F is additive, we have that

$$F(\mathbb{K}^n) \cong \bigoplus_{i=1}^{n} \mathrm{id}_{\mathcal{C}}. \tag{B.1}$$

Therefore, F is uniquely specified on objects. If $A : \mathbb{K}^n \to \mathbb{K}^m$ is a matrix, then $F(A)$ is a natural transformation in

$$\mathrm{Hom}\left(\bigoplus_{i=1}^{n} \mathrm{id}_{\mathcal{C}}, \bigoplus_{j=1}^{m} \mathrm{id}_{\mathcal{C}}\right) \cong \bigoplus_{i=1}^{n}\bigoplus_{j=1}^{m} \mathrm{Hom}(\mathrm{id}_{\mathcal{C}}, \mathrm{id}_{\mathcal{C}}) \tag{B.2}$$

which is given by $F(A)_{ij} = A_{ij}\, \mathrm{id}_{\mathrm{id}_{\mathcal{C}}}$: the ij-th component of $F(A)$ is given by a multiplying the number A_{ij} with the identity natural transformation between the identity functor and the identity functor on $\mathcal{C}$. Hence, the functor F is uniquely specified on morphisms as well.

We regard $\mathcal{C}$ as a right Vect-module category and thus define a functor $\boxtimes : \mathcal{C} \times \mathrm{Vect} \to \mathrm{Vect}$ by setting $X \boxtimes V := F(V)(X)$. The fact that F is monoidal and additive translates into the coherent isomorphisms

$$X \boxtimes (V \otimes_{\mathbb{K}} W) \cong (X \boxtimes V) \boxtimes W \tag{B.3}$$

$$X \boxtimes (V \oplus W) \cong (X \boxtimes V) \oplus (X \boxtimes W) \tag{B.4}$$

for all $X \in \mathcal{C}$ and vector spaces V and W.

Lemma B.2. *Let $V \in \mathrm{Vect}$ and let X and Y be objects of $\mathcal{C}$. Then, there are the following isomorphisms which are natural in X and in Y:*

1. $\mathrm{Hom}_{\mathcal{C}}(X, Y \boxtimes V) \cong \mathrm{Hom}_{\mathcal{C}}(X, Y) \otimes_{\mathbb{K}} V$,
2. $\mathrm{Hom}_{\mathcal{C}}(X \boxtimes V, Y) \cong V^* \otimes_{\mathbb{K}} \mathrm{Hom}_{\mathcal{C}}(X, Y)$,
3. $\mathrm{Hom}_{\mathcal{C}}(X \boxtimes V, Y) \cong \mathrm{Hom}_{\mathrm{Vect}}(V, \mathrm{Hom}_{\mathcal{C}}(X, Y))$.

Proof. Since V is a finite-dimensional vector space, it is isomorphic to $\mathbb{K}^n$. For the first isomorphism, we calculate using equation (B.4):

$$\begin{aligned} \mathrm{Hom}_{\mathcal{C}}(X, Y \boxtimes V) &\cong \mathrm{Hom}(X, Y \boxtimes \mathbb{K}^n) \cong \mathrm{Hom}_{\mathcal{C}}(X, \bigoplus_n Y \boxtimes \mathbb{K}) \\ &\cong \bigoplus_n \mathrm{Hom}_{\mathcal{C}}(X, Y) \cong \mathrm{Hom}_{\mathcal{C}}(X, Y) \otimes_{\mathbb{K}} V. \end{aligned} \tag{B.5}$$

The second isomorphism is similar. The third isomorphism follows from the second one by dimensional reasoning. □

Remark B.3. The second part of the lemma states that $X \boxtimes V$ is an object representing the contravariant functor $F_{X,V}(Y) = V^* \otimes_{\mathbb{K}} \mathrm{Hom}_{\mathcal{C}}(X, Y)$. Note that due to the Yoneda lemma, representable objects are unique up to unique isomorphism. Hence, we could have alternatively defined the object $X \boxtimes V$ as the object representing the functor $F_{X,V}$. For instance, this perspective is taken in [Kir02].

B.2. Finite linear categories as categories of modules

In the sequel, we will show that a finite linear category is equivalent to a category of modules over a finite-dimensional algebra. We will mostly follow the exposition in [EGNO15]; other references for this fact include [DSPS14] and [EGH$^+$11].

Proposition B.4. *Let $\mathcal{C}$ be a finite linear category. Let X_i be representatives of the (finitely many) isomorphism classes of simple objects of $\mathcal{C}$. Since $\mathcal{C}$ has enough projectives, there are projective objects P_i, and epimorphisms $P_i \to X_i$. Let $P := \oplus_i P_i$ be a projective object of $\mathcal{C}$, and define a finite-dimensional associative $\mathbb{K}$-algebra*

$$A := \mathrm{Hom}_{\mathcal{C}}(P, P) \tag{B.6}$$

with multiplication $f \cdot g := g \circ f$.

Let $(A\text{-Mod})^{\mathrm{fg}}$ be the category of finitely generated left A-modules. Then, the functor

$$\mathrm{Hom}_{\mathcal{C}}(P, -) : \mathcal{C} \to (A\text{-Mod})^{\mathrm{fg}} \tag{B.7}$$

is an equivalence of linear categories.

In order to prove this, we need a few auxiliary lemmas together with a well-known fact from representation theory.

Lemma B.5. *Let A be a finite-dimensional algebra over a field $\mathbb{K}$, and let M be an A-module. Then, M is finite-dimensional as a $\mathbb{K}$-vector space if and only if M is finitely-generated as an A-module.*

Proof. If M is a finite-dimensional vector space over $\mathbb{K}$, there exists a vector space basis $\mathcal{B} := \{x_1, \ldots, x_n\}$ of M. We claim that this vector space basis generates M as an A-module. Indeed, if $m \in M$, there are $\lambda_1, \ldots, \lambda_n \in \mathbb{K}$, so that

$$m = \sum_{i=1}^{n} \lambda_i x_i, \tag{B.8}$$

since $\mathcal{B}$ is a vector space basis of M. Hence,

$$m = \sum_{i=1}^{n} (1_A \cdot \lambda_i).x_i. \tag{B.9}$$

This shows that $\mathcal{B}$ generates M as an A-module.

If on the other hand M is finitely generated as an A-module, let $x_1, \ldots, x_n$ be a set of generators of M. Thus, for an $m \in M$ there are $a_1, \ldots, a_n \in A$ so that

$$m = \sum_{i=1}^{n} a_i.x_i. \tag{B.10}$$

Since A is a finite-dimensional vector space over $\mathbb{K}$ by assumption, we may choose a vector space basis $y_1, \ldots, y_k$ of A. Hence, for each $a_i \in A$, there are $\lambda_{ij} \in \mathbb{K}$, so that

$$a_i = \sum_{j=1}^{k} \lambda_{ij} y_j. \tag{B.11}$$

Thus,

$$m = \sum_{i=1}^{n} a_i.x_i = \sum_{i=1}^{n} \sum_{j=1}^{k} \lambda_{ij} (y_j.x_i). \tag{B.12}$$

This shows that the $y_j.x_i$ form a vector-space basis of M. Thus, M is a finite-dimensional $\mathbb{K}$-vector space. $\square$

Lemma B.6. *Let A be a finite-dimensional algebra. Then, the induction functor is left-adjoint to the restriction functor:*

$$\mathrm{Ind}_A : \mathbb{K}\text{-Mod} \leftrightarrows A\text{-Mod} : \mathrm{Res}^A_{\mathbb{K}}. \tag{B.13}$$

Hence, we have an isomorphism

$$\mathrm{Hom}_{A\text{-Mod}}(A \otimes_{\mathbb{K}} V, M) \cong \mathrm{Hom}_{\mathrm{Vect}}(V, M) \tag{B.14}$$

which is natural in V and M.

Proof. This is a special case of Frobenius reciprocity. Indeed, if $f : A \otimes_{\mathbb{K}} V \to M$ is an intertwiner, then $v \mapsto f(1 \otimes v)$ is a linear map $V \to M$. On the other hand, if $g : V \to M$ is a linear map, then $a \otimes v \mapsto a.g(v)$ is a morphism of A-modules. It is easy to see that these two constructions are inverse to each other. □

Lemma B.7. *Let $P = \oplus_i P_i$ be the object of $\mathcal{C}$ as defined in proposition B.4. We claim: for each $X \in \mathcal{C}$, there exists a finite-dimensional vector space V and an epimorphism $P \boxtimes V \to X$.*

Proof. We argue by induction on the length of X. If X is simple, we may choose V to be $\mathbb{K}$, since by construction of P there is an epimorphism $P \boxtimes \mathbb{K} \cong P \to X$.

If X is not simple, there exists a short exact sequence

$$0 \to X' \to X \to X'' \to 0 \tag{B.15}$$

where the length of both X' and X'' are strictly less than the length of X. By the inductive hypothesis, there are vector spaces V' and V'', together with epimorphisms $f' : P \boxtimes V' \to X'$ and $f'' : P \boxtimes V'' \to X''$. Now let $V := V' \oplus V''$, and define a epimorphism $P \boxtimes V \to X$ as follows: by equation (B.4), we have an isomorphism

$$P \boxtimes (V' \oplus V'') \cong (P \boxtimes V') \oplus (P \boxtimes V''). \tag{B.16}$$

Since in abelian categories, finite products agree with finite coproducts, giving a morphism $P \boxtimes V \to X$ is equivalent to specifying morphisms $P \boxtimes V' \to X$ and $P \boxtimes V'' \to X$. Hence, there is a map $f' \oplus f'' : P \boxtimes V \to X$. This map is even an epimorphism, since f' and f'' are epimorphisms, and finite products of epimorphisms are epimorphisms. The situation is depicted in the diagram below. □

$$\begin{tikzcd}
0 \arrow[r] & P \boxtimes V' \arrow[r] \arrow[d, two heads, "f'"] & P \boxtimes (V' \oplus V'') \arrow[r] \arrow[d, two heads, "f' \oplus f''"] & P \boxtimes V'' \arrow[r] \arrow[d, two heads, "f''"] & 0 \\
0 \arrow[r] & X' \arrow[r] & X \arrow[r, two heads] & X'' \arrow[r] & 0
\end{tikzcd}$$

Figure B.1.: Diagram for lemma B.7

Following the proof of theorem 7.10.1 in [EGNO15], we are now ready to prove proposition B.4.

Proof of proposition B.4. We will show first that the functor $\mathrm{Hom}_{\mathcal{C}}(P, -)$ is fully faithful. Let X and Y be objects of $\mathcal{C}$, and suppose that X is of the form $X = P \boxtimes V$ for some vector space V. By lemma B.2, there is a natural isomorphism $\mathrm{Hom}_{\mathcal{C}}(P, X) \cong A \otimes_{\mathbb{K}} V$ of vector spaces. When regarding $\mathrm{Hom}_{\mathcal{C}}(P, X)$ and $A \otimes_{\mathbb{K}} V$ as A-modules, this becomes even an isomorphism of A-modules. Thus we calculate using lemma B.2 and lemma B.6:

$$\begin{aligned} \mathrm{Hom}_{A\text{-Mod}}(\mathrm{Hom}_{\mathcal{C}}(P,X),\mathrm{Hom}_{\mathcal{C}}(P,Y)) &\cong \mathrm{Hom}_{A\text{-Mod}}(A\otimes_{\mathbb{K}} V,\mathrm{Hom}_{\mathcal{C}}(P,Y)) \\ &\cong \mathrm{Hom}_{\mathrm{Vect}}(V,\mathrm{Hom}_{\mathcal{C}}(P,Y)) \\ &\cong \mathrm{Hom}_{\mathcal{C}}(P\boxtimes V,Y) \\ &\cong \mathrm{Hom}_{\mathcal{C}}(X,Y). \end{aligned} \tag{B.17}$$

This shows that the functor $\mathrm{Hom}_{\mathcal{C}}(P,-)$ is fully faithful on objects of the form $X = P\boxtimes V$.

Now let $X \in \mathcal{C}$ be arbitrary. By lemma B.7, there is a vector space V together with an epimorphism $f : P\boxtimes V \to X$. Applying lemma B.7 again gives another vector space W with an epimorphism $g : P\boxtimes W \to \ker f$. Let $k : \ker f \to P\boxtimes V$ be the canonical morphism. By construction, the horizontal sequence

$$\begin{tikzcd} P\boxtimes W \arrow[rr, "k\circ g"] \arrow[dr, "g"] & & P\boxtimes V \arrow[r, "f"] & X \arrow[r] & 0 \\ & \ker f \arrow[ur, "k"] \arrow[dr] & & & \\ 0 \arrow[ur] & & 0 & & \end{tikzcd} \tag{B.18}$$

is exact, since both diagonal sequences are exact.

Since the functor $\mathrm{Hom}_{\mathcal{C}}(P,-)$ is exact, the sequence

$$\mathrm{Hom}_{\mathcal{C}}(P,P\boxtimes W) \to \mathrm{Hom}_{\mathcal{C}}(P,P\boxtimes V) \to \mathrm{Hom}_{\mathcal{C}}(P,X) \to 0 \tag{B.19}$$

is exact.

For the sake of simplifying notation, let $F := \mathrm{Hom}_{\mathcal{C}}(P,-)$. Since Hom is always left exact, the first row in the diagram is exact. Since P is projective and thus $\mathrm{Hom}_{\mathcal{C}}(P,-)$ is exact, the second row is also exact.

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Hom}_{\mathcal{C}}(X,Y) \arrow[r] \arrow[d, "F"] & \mathrm{Hom}_{\mathcal{C}}(P\boxtimes V,Y) \arrow[r] \arrow[d, "F"] & \mathrm{Hom}_{\mathcal{C}}(P\boxtimes W,Y) \arrow[d, "F"] \\ 0 \arrow[r] & \underset{A\text{-Mod}}{\mathrm{Hom}}(F(X),F(Y)) \arrow[r] & \underset{A\text{-Mod}}{\mathrm{Hom}}(F(P\boxtimes V),F(Y)) \arrow[r] & \underset{A\text{-Mod}}{\mathrm{Hom}}(F(P\boxtimes W),F(Y)) \end{tikzcd}$$

By the first step, the second and third vertical arrows are isomorphisms. Using the 5-lemma shows that the first vertical arrow is an isomorphism as well. Therefore, the functor $F = \mathrm{Hom}_{\mathcal{C}}(P,-)$ is fully faithful.

We now show that the functor $\mathrm{Hom}_{\mathcal{C}}(P,-)$ is essentially surjective. Let M be a finitely-generated A-module, which is finite-dimensional by lemma B.5. First, define two morphisms $P\boxtimes (A\otimes_{\mathbb{K}} M) \rightrightarrows P\boxtimes M$ as follows: since $\mathrm{Hom}_{\mathcal{C}}(X,P)$ is a right A-module, and M is a left A-module, there are two natural transformations between the two functors

$$F_{P,A\otimes_{\mathbb{K}}M}(X) = \mathrm{Hom}_{\mathcal{C}}(X,P)\otimes_{\mathbb{K}} A\otimes_{\mathbb{K}} M \qquad \text{and} \tag{B.20}$$

$$F_{P,M}(X) = \mathrm{Hom}_{\mathcal{C}}(X,P)\otimes_{\mathbb{K}} M \tag{B.21}$$

given by the right action of A on $\mathrm{Hom}_{\mathcal{C}}(X,P)$ and by the left action of A on M. Recall that by remark B.3, the object $P \boxtimes (A \otimes_{\mathbb{K}} M)$ represents the functor $F_{P,A\otimes_{\mathbb{K}}M}$, and the object $P \boxtimes M$ represents the functor $F_{P,M}$.

Therefore, we can define two maps $P \boxtimes (A \otimes_{\mathbb{K}} M) \rightrightarrows P \boxtimes M$ as the images of the two natural transformations under the isomorphism induced by the Yoneda lemma

$$\mathrm{Nat}(F_{P,A\otimes_{\mathbb{K}}M}, F_{P,M}) \cong \mathrm{Hom}_{\mathcal{C}}(P \boxtimes (A \otimes_{\mathbb{K}} M), P \boxtimes M). \tag{B.22}$$

Finally, define an object X_M of $\mathcal{C}$ as the coequalizer of these two morphisms.

We claim that there is a natural isomorphism of left A-modules $\mathrm{Hom}_{\mathcal{C}}(P, X_M) \cong M$. Since P is projective, the functor $\mathrm{Hom}(P,-)$ is exact and thus preserves finite colimits. Hence, we calculate

$$\begin{aligned}
\mathrm{Hom}_{\mathcal{C}}(P, X_M) &\cong \mathrm{Hom}_{\mathcal{C}}(P, \operatorname*{colim}_{\mathcal{C}} P \boxtimes (A \otimes_{\mathbb{K}} M) \rightrightarrows P \boxtimes M) \\
&\cong \operatorname*{colim}_{A\text{-Mod}} (\mathrm{Hom}_{\mathcal{C}}(P, P \boxtimes (A \otimes_{\mathbb{K}} M)) \rightrightarrows \mathrm{Hom}_{\mathcal{C}}(P, P \boxtimes M)) \\
&\cong \operatorname*{colim}_{A\text{-Mod}} (\mathrm{End}_{\mathcal{C}}(P) \otimes_{\mathbb{K}} A \otimes_{\mathbb{K}} M \rightrightarrows \mathrm{End}_{\mathcal{C}}(P) \otimes_{\mathbb{K}} M) \\
&\cong \mathrm{End}_{\mathcal{C}}(P) \otimes_A M \\
&\cong A \otimes_A M \\
&\cong M.
\end{aligned} \tag{B.23}$$

In the third line, we have used that the isomorphism of vector spaces $\mathrm{Hom}_{\mathcal{C}}(P, P \boxtimes M) \cong \mathrm{End}_{\mathcal{C}}(P) \otimes_{\mathbb{K}} M$ from lemma B.2 is even an isomorphism of A-modules.

This calculation shows that $\mathrm{Hom}_{\mathcal{C}}(P,-)$ is essentially surjective and thus an equivalence of linear categories. □

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Eidesstattliche Erklärung

Hiermit erkläre ich an Eides statt, dass ich die vorliegende Dissertationsschrift selbst verfasst und keine anderen als die angegebenen Quellen und Hilfsmittel benutzt habe.

Ort, Datum

Unterschrift