Universität Hamburg
DER FORSCHUNG | DER LEHRE | DER BILDUNG

Department of Mathematics

Dissertation with the aim of achieving a doctoral degree at the
Faculty of Mathematics, Informatics and Natural Sciences

The symplectic fermion ribbon quasi-Hopf algebra and the $SL(2, \mathbb{Z})$-action on its centre

submitted by Vanda Farsad

Hamburg, 2017

Chairman of the Examination Commission:
Prof. Dr. Sören Christensen

Reviewers:
Prof. Dr. Ingo Runkel
Prof. Dr. Yorck Sommerhäuser

Day of oral defense: 14.06.2017

To my dad

Declaration

I hereby declare, on oath, that I have written the present dissertation by my own and have not used other than the acknowledged resources and aids.

Hamburg, July 29, 2017

Acknowledgements

First I like to thank my supervisor Ingo Runkel for his support, his patience, his guidance, and for making my time as a PhD student a great experience in my life. I am also indebted to him for proof-reading this thesis.

No less my thanks are due to Azat Gainutdinov for the pleasant collaboration and many helpful discussion.

I also like to thank Lorenz Hilfiker, Ehud Meir, Tobias Ohrmann, Jan Priel and Louis-Hadrien Robert for helpful discussions, and for being more than just colleagues. In addition, my thanks are due to Lorenz Hilfiker for proof-reading this thesis.

Finally, thanks to my family for supporting me throughout the last decade, and thanks to my girlfriend for being both supportive and patient.

Contents

Introduction

The investigations in this thesis are motivated by conformal field theories, that is, quantum field theories which are invariant under conformal transformations. Besides its applications to string theory, statistical mechanics, and condensed matter physics, conformal field theories in two dimensions are also interesting from a mathematical point of view. There is an infinite-dimensional algebra of infinitesimal conformal transformations, which together with some finiteness conditions provide rare examples of exactly solvable quantum field theories.

Two-dimensional conformal field theories and three-dimensional topological field theories are closely connected. This connection is well-understood for so-called rational conformal field theories and for topological field theories constructed from modular tensor categories. It is due to the fact that modular tensor categories also arise as representation categories of vertex operator algebras associated to rational conformal field theories. Modular tensor categories are, in particular, finitely semisimple (see below). The theory becomes more involved if one drops the semisimplicity requirement while keeping the finiteness condition. This leads to logarithmic conformal field theories and to non-semisimple finite ribbon categories.

A modular tensor category is a finitely semisimple linear and abelian category, which is in addition a ribbon category (a braided tensor category with ribbon twist, see Section 2.1), which has a simple tensor unit, and whose braiding satisfies a certain non-degeneracy condition (Definition 2.13). The importance of modular tensor categories is due to the fact that they contain the necessary data to define a 3-2-1 extended topological field theory [**RT2, Tu, BDSPV**].

Modular tensor categories can be obtained from finite-dimensional ribbon Hopf algebras H which are semisimple as algebras and which are *factorisable* [**RS, Ta**]. Factorisability means that the monodromy matrix $M = R_{21}R \in H \otimes H$ determined by the R-matrix of H is non-degenerate as a copairing.

Three-dimensional topological field theories generate representations of mapping class groups of surfaces (possibly with marked points in the extended case). It turns out that if one drops the semisimplicity requirement representations of mapping class groups can still be obtained [**Ly2, KL**], now without an underlying 3-2-1 topological field theory in the sense of [**BDSPV**].

The relevant algebraic structure is now a finite abelian ribbon category with simple tensor unit, whose braiding satisfies a (more complicated) non-degeneracy condition [**Ly1, Ly2**].

We refer to such categories as factorisable finite ribbon tensor categories. Again, finite-dimensional factorisable ribbon Hopf algebras provide examples, now without the semisimplicity requirement.

In the first part of this thesis we apply the general formalism of [**Ly1, Ly2, KL**] to finite-dimensional ribbon *quasi*-Hopf algebras A. We express the relevant non-degeneracy condition on the braiding in terms of the defining data of A (see Section 5.2.3) and compute the action of $SL(2,\mathbb{Z})$ – the mapping class group of the torus – on the centre $Z(A)$ of A (Theorem 5.19).

Our motivation for this is two fold: firstly, it provides us the explicit expressions we need in the second part for the symplectic fermion calculation; secondly, it is easy to detect when a finite tensor category originates from a quasi-Hopf algebra, as we explain next.

Let $\mathcal{C}$ be a finite tensor category over a field $\Bbbk$. If there exists a fiber functor $F : \mathcal{C} \to \mathbf{vect}_\Bbbk$, by reconstruction one can find a Hopf algebra H such that $\mathcal{C} \cong \mathbf{Rep}\, H$ as linear tensor categories [**Ul**], see also [**Ma2**, Sec. 9.4]. If we only require F to be *multiplicative*, i.e. that there are isomorphisms $F(U \otimes V) \cong F(U) \otimes F(V)$ natural in U, V but not subject to coherence conditions, then reconstruction results in a *quasi*-Hopf algebra [**Ma2**, Sec. 9.4]. While it may be difficult to determine whether there is a fiber functor $\mathcal{C} \to \mathbf{vect}_\Bbbk$, there is a very simple criterion for the existence of multiplicative functors:

Theorem ([**EGNO**, Prop. 6.1.14])**.** *A finite tensor category over an algebraically closed field is equivalent as a linear tensor category to the representation category of a finite-dimensional quasi-Hopf algebra iff the Perron-Frobenius dimensions of its simple objects are integers.*

The Perron-Frobenius dimensions of a simple object $X \in \mathcal{C}$ is the positive real number given by the maximal non-negative eigenvalue of the linear map $[X \otimes -]$ on the $\mathbb{C}$-linearised Grothendieck ring $\mathbb{C} \otimes_\mathbb{Z} \mathrm{Gr}(\mathcal{C})$, see e.g. [**EGNO**]. For $\mathbf{Rep}\, A$, the Perron-Frobenius dimension of an object is simply the dimension of the underlying vector space.

We now describe in more detail the construction of [**Ly1, Ly2, KL**], see also [**FS**, Sec. 4] for a review. In this thesis we will only be interested in the action of the mapping class group of the torus, i.e. of $SL(2,\mathbb{Z})$.

Let $\mathcal{C}$ be a factorisable finite ribbon tensor category over a field $\Bbbk$. Let $\mathcal{L} \in \mathcal{C}$ be the coend for the functor $\mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{C}$: $(U, V) \mapsto U^* \otimes V$. As we review in Section 4.2, using the universal property of the coend, one can endow $\mathcal{L}$ with the structure of a Hopf algebra in the braided category $\mathcal{C}$ (Definition 2.5), together with a Hopf pairing $\omega_\mathcal{L} : \mathcal{L} \otimes \mathcal{L} \to \mathbf{1}$. The category $\mathcal{C}$ is called *factorisable* if $\omega_\mathcal{L}$ is non-degenerate. Using once more the universal property one defines endomorphisms $\mathcal{S}$, $\mathcal{T}$ of $\mathcal{L}$ which induce a projective action of $SL(2,\mathbb{Z})$ on $\mathcal{C}(\mathbf{1}, \mathcal{L})$ (see Section 5.1.1). One finds that $\mathcal{C}(\mathbf{1}, \mathcal{L}) \cong \mathrm{End}(\mathrm{id}_\mathcal{C})$, and so one obtains a projective action of $SL(2,\mathbb{Z})$ on $\mathrm{End}(\mathrm{id}_\mathcal{C})$.

Let A be a finite-dimensional ribbon quasi-Hopf algebra (see Section 1 for conventions and details). We show that, as for Hopf algebras [**Ly2, Ke2**], the coend $\mathcal{L}$ in $\mathbf{Rep}\, A$ is given by the coadjoint representation on A^* (Proposition 5.8). We compute the structure morphisms of the Hopf algebra $\mathcal{L}$, as well as the pairing $\omega_{\mathcal{L}}$, in terms of the data of A (Theorem 5.10). Note that $\mathrm{End}(\mathrm{id}_{\mathbf{Rep}\,A}) \cong Z(A)$, the centre of A. Using our explicit expressions, we give the action of the S- and T-generators of $SL(2,\mathbb{Z})$ on $Z(A)$ in terms of the defining data of A and an integral, see Proposition 5.16 and Theorem 5.19. This generalises results for Hopf algebras in [**LM**] to quasi-Hopf algebras.

In the second part of this thesis we present our main result. The conjecture in [**GR2**] gives rise to an isomorphism of projective representations between two $SL(2,\mathbb{Z})$-actions associated to a C_2-cofinite, simple, self-dual and non-negatively graded vertex operator algebra $\mathcal{V}$. The first action is obtained by modular transformations on the space of so-called pseudo-trace functions of $\mathcal{V}$ [**Mi, AN**]. For the second action one uses that $\mathbf{Rep}\,\mathcal{V}$ is conjecturally a factorisable finite ribbon tensor category and thus carries a projective $SL(2,\mathbb{Z})$-action on $\mathrm{End}(\mathrm{id}_{\mathbf{Rep}\,\mathcal{V}})$ as described above (see e.g. [**GR2**, Sec. 5] for details).

We investigate a family of examples which are the so-called symplectic fermions, and which are parametrised by $\mathrm{N} \in \mathbb{Z}_{>0}$, the “number of pairs of symplectic fermions”. Denote the even part of the symplectic fermion vertex operator super-algebra $\mathcal{V}$ defined in [**Ab**] by $\mathcal{V}_{\mathrm{ev}}$. Following [**DR1, Ru**] $\mathbf{Rep}\,\mathcal{V}_{\mathrm{ev}}$ is conjecturally ribbon equivalent to the category of N pairs of symplectic fermions $\mathcal{SF}$, see Section 6 and Section 10.1 or [**DR3**, Conj. 7.4] for a precise formulation. We prove that $\mathcal{SF}$ in turn is ribbon equivalent to the representation category of a factorisable quasi-Hopf algebra $\mathsf{Q} = \mathsf{Q}(\mathrm{N},\beta)$, see Theorem 7.6 and Corollary 7.8.

Q is an associative unital algebra over $\mathbb{C}$ with generators K and $\mathsf{f}_i^{\pm}, i = 1,\dots,\mathrm{N}$, and defining relations, for $i,j = 1,\dots,\mathrm{N}$,

$$\{\mathsf{f}_i^{\pm},\mathsf{K}\} = 0 \ , \quad \{\mathsf{f}_i^{+},\mathsf{f}_j^{-}\} = \delta_{i,j}\,\tfrac{1}{2}(\mathbf{1}-\mathsf{K}^2) \ , \quad \{\mathsf{f}_i^{\pm},\mathsf{f}_j^{\pm}\} = 0 \ , \quad \mathsf{K}^4 = \mathbf{1} \ ,$$

where $\{x,y\} = xy+yx$ is the anticommutator. It has some interesting properties. For even N and $\beta = \pm 1$, Q is actually a Hopf algebra. In these cases the Hopf algebra Q is isomorphic to the Drinfeld double of a generalization of Sweedlers Hopf algebra, see Section 7.3. Moreover, if $\beta = 1$ this isomorphism is an isomorphism of quasi-triangular Hopf algebras.

Using the results of the first part the $SL(2,\mathbb{Z})$-action on $\mathrm{End}(\mathrm{id}_{\mathbf{Rep}\,\mathsf{Q}})$ is computed (see also [**GR1**] for the case $\mathrm{N} = 1$). The $SL(2,\mathbb{Z})$-action on the space of pseudo-trace functions follows from [**GR2**]. The conclusion of this thesis is that the $SL(2,\mathbb{Z})$-actions do indeed agree projectively, see Section 10.2 – this provides the first example of such a comparison for non-semisimple theories in the literature.

Let us briefly outline the proof of Theorem 7.6. The proof is a bit extensive which is why we split it in two steps. In step 1, we introduce a quasi-bialgebra $\mathsf{S} = \mathsf{S}(\mathrm{N},\beta)$ in **Svect** which has half the dimension of Q. We give an element $r \in \mathsf{S} \otimes \mathsf{S}$ which, as we prove, defines a braiding on its category $\mathbf{Rep}\,\mathsf{S}$ of finite-dimensional representations in **Svect**. Though r is not a universal R-matrix as the braiding involves the parity involution natural automorphism of **Svect**, see Section B.5. Then we show that $\mathbf{Rep}\,\mathsf{S}$ is equivalent to $\mathcal{SF}(\mathrm{N},\beta)$ as a braided monoidal category.

In step 2, we present a quasi-bialgebra $\hat{\mathsf{Q}}(\mathrm{N},\beta)$ in **vect** and show a braided monoidal equivalence between $\mathbf{Rep}\,\mathsf{S}$ and $\mathbf{Rep}\,\hat{\mathsf{Q}}$. We present a twisting of $\hat{\mathsf{Q}}$ into Q, and therefore proof that $\mathbf{Rep}\,\mathsf{Q}$ is braided monoidally equivalent to $\mathbf{Rep}\,\mathsf{S}$ and thus to $\mathcal{SF}$. Finally, using the equivalence $\mathcal{SF} \to \mathbf{Rep}\,\mathsf{Q}$ we transport the ribbon element from $\mathcal{SF}$ to $\mathbf{Rep}\,\mathsf{Q}$.

This thesis is organised as follows.

Part 1 : In Section 1 we give our conventions for factorisable ribbon quasi-Hopf algebras. In Section 2 we recall standard notions in category theory, in particular, a precise definition of ribbon categories. We also give our conventions for Hopf algebras in braided tensor categories. A brief review of conformal field theory, and vertex operator algebras and their modules is presented in Section 3. In Section 4 we review the reconstruction theory for Hopf algebras of [**Ma1**] in the special case of the identity functor, leading to the universal Hopf algebra, and an equivalent description of the universal Hopf algebra in terms of coends. The latter is the formalism used in [**Ly1**] and in the rest of this thesis. In Section 5 the $SL(2,\mathbb{Z})$-action of [**Ly1**] and the theory of internal characters of [**FS, Sh1**] are recalled. It contains the explicit computation of the Hopf algebra structure maps of the universal Hopf algebra $\mathcal{L}$ in $\mathbf{Rep}\,A$. We state the factorisability condition on $\mathbf{Rep}\,A$ in terms of the defining data of A and show that it is equivalent to the definition in [**BT**]. Moreover, we present our main results of the first part, namely the explicit computation of the Hopf algebra structure maps of the universal Hopf algebra $\mathcal{L}$ in $\mathbf{Rep}\,A$, and the explicit expressions for the S- and T-action on the centre $Z(A)$ of a ribbon quasi-Hopf algebra A.

Part 2 : Section 6 starts with a review of the finite ribbon category of N pairs of symplectic fermions $\mathcal{SF}$ introduced in [**DR1, Ru**], which is conjecturally equivalent to $\mathbf{Rep}\,\mathcal{V}_{\mathrm{ev}}$ [**DR3**, Conj. 7.4]. The ribbon quasi-Hopf algebra Q is introduced in Section 7, and we state our first main result of the second part, namely a ribbon equivalence between $\mathcal{SF}$ and $\mathbf{Rep}\,\mathsf{Q}$. Using the results from the first part we calculate in Section 8 the Hopf algebra structure on the coend $\mathcal{L}$ in $\mathbf{Rep}\,\mathsf{Q}$ and an integral. Finally, we present in Section 9 our last main result: the $SL(2,\mathbb{Z})$-action on symplectic fermion pseudo-trace functions agrees projectively with the $SL(2,\mathbb{Z})$-action on the centre of Q.

Basis of the thesis

This thesis is based primarily on the following papers:

Paper I. V. Farsad, A. M. Gainutdinov and I. Runkel,
$SL(2,\mathbb{Z})$ action for ribbon quasi-Hopf algebras, 1702.01086 [math.QA].

Paper II. V. Farsad, A.M. Gainutdinov, I. Runkel,
The symplectic fermion ribbon quasi-Hopf algebra and the $SL(2,\mathbb{Z})$-action on its centre, 1706.08164 [math.QA].

Contribution to the papers by the author of the thesis

Paper I. Large parts of the paper were developed in discussions with the coauthors. The Hopf structure on the coend $\mathcal{L}$ in the representation category of a quasi-Hopf A, as well as the $SL(2,\mathbb{Z})$-action on the centre of A was developed by myself and independently calculated by the coauthors.

Paper II. Large parts of the paper were developed in collaboration with the coauthors. I worked out the details of the proof of the ribbon equivalence between $\mathbf{Rep}\,\mathsf{Q}$ and $\mathcal{SF}$. The transport of the coassociator from $\mathcal{SF}$ to $\mathbf{Rep}\,\mathsf{Q}$ was calculated by myself. In discussion with the coauthors I developed the isomorphism of the Drinfeld double of $H(\mathrm{N})$ and the Hopf algebra $\mathsf{Q}(\mathrm{N},\beta)$ for even N and $\beta = \pm 1$, as well as the quasi-triangular variant.

Part 1

$SL(2,\mathbb{Z})$-action for ribbon quasi-Hopf algebras

1. Ribbon quasi-Hopf algebras

In this chapter we introduce our conventions for ribbon quasi-Hopf algebras A. In Section 2.2 we will show that these objects have enough structure to build a finite ribbon category (with an associator which can be different from the canonical associator in $\mathbf{vect}_{\Bbbk}$).

1.1. Conventions

We begin with the definition of a quasi-Hopf algebra A [**Dr2**] and we mainly follow the conventions in [**CP**, Sec. 16.1].

We will use Sweedler's sum notation with primes $'$ for the coproduct $\Delta(a) \in A \otimes A$ of an element $a \in A$, and with subscript numbers $_{1,2,\dots}$ for elements of tensor products of A. For example,

$$\Delta(a) = \sum_{(a)} a' \otimes a'' \quad , \qquad X = \sum_{(X)} X_1 \otimes X_2 \otimes X_3 \quad \text{for } X \in A^{\otimes 3} \ . \tag{1.1}$$

Definition 1.1. *A quasi-Hopf algebra* over a field $\Bbbk$ is a unital associative algebra A over $\Bbbk$ together with

- an algebra homomorphism $\epsilon : A \to \mathbb{C}$ (the *counit*),
- an algebra homomorphism $\Delta : A \to A \otimes A$ (the *coproduct*),
- an algebra anti-homomorphism $S : A \to A$ (the *antipode*),
- a multiplicatively invertible element $\Phi \in A \otimes A \otimes A$ (the *coassociator*),
- elements $\boldsymbol{\alpha}$, $\boldsymbol{\beta} \in A$ (the *evaluation* and *coevaluation element*, respectively).

These data are subject to the conditions:

- counitality and coassociativity:

$$(\epsilon \otimes \mathrm{id}) \circ \Delta \ = \ \mathrm{id} \ = \ (\mathrm{id} \otimes \epsilon) \circ \Delta \ , \tag{1.2}$$

$$\big((\Delta \otimes \mathrm{id})(\Delta(a))\big) \cdot \Phi \ = \ \Phi \cdot \big((\mathrm{id} \otimes \Delta)(\Delta(a))\big) \quad \text{for all } a \in A \ , \tag{1.3}$$

- the coassociator Φ is counital and a 3-cocycle:

$$(\mathrm{id} \otimes \epsilon \otimes \mathrm{id})(\Phi) = \mathbf{1} \otimes \mathbf{1} \ , \tag{1.4}$$

$$(\Delta \otimes \mathrm{id} \otimes \mathrm{id})(\Phi) \cdot (\mathrm{id} \otimes \mathrm{id} \otimes \Delta)(\Phi) = (\Phi \otimes \mathbf{1}) \cdot (\mathrm{id} \otimes \Delta \otimes \mathrm{id})(\Phi) \cdot (\mathbf{1} \otimes \Phi) \ , \tag{1.5}$$

- the antipode conditions:

$$\sum_{(a)} S(a')\,\boldsymbol{\alpha}\,a'' = \epsilon(a)\,\boldsymbol{\alpha}\ , \qquad \sum_{(a)} a'\,\boldsymbol{\beta}\,S(a'') = \epsilon(a)\,\boldsymbol{\beta} \quad \text{for all } a \in A\ , \tag{1.6}$$

$$\sum_{(\Phi)} S(\Phi_1)\,\boldsymbol{\alpha}\,\Phi_2\,\boldsymbol{\beta}\,S(\Phi_3) = \mathbf{1}\ , \qquad \sum_{(\Phi^{-1})} (\Phi^{-1})_1\,\boldsymbol{\beta}\,S((\Phi^{-1})_2)\,\boldsymbol{\alpha}\,(\Phi^{-1})_3 = \mathbf{1}\ , \tag{1.7}$$

for an expansion $\Phi = \sum_{(\Phi)} \Phi_1 \otimes \Phi_2 \otimes \Phi_3 \in A \otimes A \otimes A$ and similarly for Φ^{-1}, cf. (1.1).

Remark 1.2.

(1) We note that the antipode S, as well as $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are uniquely determined up to the conjugation by a unique element U: if the triple $\tilde{S}$, $\tilde{\boldsymbol{\alpha}}$, $\tilde{\boldsymbol{\beta}}$ gives another antipode structure in A then there exists a unique element $U \in A$ such that

$$\tilde{S}(a) = US(a)U^{-1}\ , \qquad \tilde{\boldsymbol{\alpha}} = U\boldsymbol{\alpha}\ , \qquad \tilde{\boldsymbol{\beta}} = \boldsymbol{\beta}U^{-1}\ , \tag{1.8}$$

see [**Dr2**, Prop. 1.1] for details.

(2) Every Hopf algebra is also a quasi-Hopf algebra for which $\Phi = \mathbf{1}\otimes\mathbf{1}\otimes\mathbf{1}$ and $\boldsymbol{\alpha} = \boldsymbol{\beta} = \mathbf{1}$.

Let us denote by τ the symmetric braiding in vector spaces, i.e. for vector spaces U, V and $u \in U$, $v \in V$ we set

$$\tau_{U,V}(u \otimes v) \ = \ v \otimes u\ . \tag{1.9}$$

Definition 1.3. A quasi-Hopf A is *quasi-triangular* if it is equipped with an invertible element $R \in A \otimes A$, called *the universal R-matrix*, such that

- R relates the coproduct with the opposite coproduct $\Delta^{\mathrm{op}} := \tau \circ \Delta$ as

$$R\,\Delta(a) = \Delta^{\mathrm{op}}(a)\,R \quad \text{for all } a \in A\ , \tag{1.10}$$

- the quasi-triangularity conditions hold:

$$\begin{aligned} (\Delta \otimes \mathrm{id})(R) &= (\Phi^{-1})_{231}\,R_{13}\,\Phi_{132}\,R_{23}\,\Phi^{-1}\ , \\ (\mathrm{id} \otimes \Delta)(R) &= \Phi_{312}\,R_{13}\,\Phi^{-1}_{213}\,R_{12}\,\Phi\ . \end{aligned} \tag{1.11}$$

Here we set $\Phi_{231} = \sum_{(\Phi)} \Phi_2 \otimes \Phi_3 \otimes \Phi_1$ and $R_{13} = \sum_{(R)} R_1 \otimes \mathbf{1} \otimes R_2$, *etc.*

We will often use the *monodromy element*

$$M \ := \ R_{21}\,R \ \in \ A \otimes A\ . \tag{1.12}$$

Definition 1.4. A finite-dimensional quasi-triangular quasi-Hopf algebra A is called *factorisable* if $Q \in A \otimes A$ is a non-degenerate copairing,

$$Q \ = \sum_{(X),(W)} S(W_3 X_2')W_4 X_2'' \otimes S(W_1 X_1')W_2 X_1''\ , \tag{1.13}$$

where $X \in A^{\otimes 2}$, $W \in A^{\otimes 4}$ are defined as

$$(1.14) \qquad X = \sum_{(\Phi)} \Phi_1 \otimes \Phi_2 \boldsymbol{\beta} S(\Phi_3) \ ,$$

$$W = (\mathbf{1} \otimes \boldsymbol{\alpha} \otimes \mathbf{1} \otimes \boldsymbol{\alpha}) \cdot (\mathbf{1} \otimes \Phi^{-1}) \cdot (\mathbf{1} \otimes M \otimes \mathbf{1}) \cdot (\mathbf{1} \otimes \Phi) \cdot (\mathrm{id} \otimes \mathrm{id} \otimes \Delta)(\Phi^{-1}) \ ,$$

M was given in (1.12).

Thus, Definition 1.4 states that A is factorisable if and only if $Q = \sum_{i \in I} a_i \otimes b_i$ for two bases $\{a_i \mid i \in I\}$ and $\{b_i \mid i \in I\}$ of A. This definition give rise to a canonical non-degeneracy definition on the braiding in categorical terms which we discuss below in Section 5.2.3.

Remark 1.5. Let us specialise the factorisability condition to the case that A is a Hopf algebra. Then we have the trivial coassociator $\Phi = \mathbf{1}^{\otimes 3}$ and $\boldsymbol{\alpha} = \boldsymbol{\beta} = \mathbf{1}$. Equation (1.13) reduces to $Q = \sum_{(M)} S(M_2) \otimes M_1$. Applying the isomorphism $A \otimes A \cong \mathrm{Hom}(A^*, A)$ of vector spaces $(a \otimes b \mapsto (\varphi \mapsto \varphi(b)a))$ on Q we get a linear map $\phi \mapsto S \circ \big((\phi \otimes \mathrm{id})(M)\big)$. This is equal to the well-know Drinfeld mapping [**Dr1**] composed with the antipode. We conclude that $Q \in A \otimes A$ is a non-degenerate copairing if and only if the Drinfeld mapping is invertible. The latter condition is the usual definition of a factorisable Hopf algebra [**RS**].

1.2. Some special elements

1.2.1. The Drinfeld twist. By definition, the antipode of a quasi-Hopf algebra is an algebra anti-homomorphism. However, in contrast to Hopf algebras it is in general not a coalgebra anti-homomorphism, i.e. the equality $\Delta\big(S(a)\big) = (S \otimes S)(\Delta^{\mathrm{op}}(a))$ may not hold. Instead, the right hand side is conjugated by the Drinfeld twist [**Dr2**]. The Drinfeld twist is the invertible element $\boldsymbol{f} \in A \otimes A$ given by

$$(1.15) \qquad \boldsymbol{f} = \sum_{(\Phi)} (S \otimes S)(\Delta^{\mathrm{op}}(\Phi_1)) \cdot \gamma \cdot \Delta\big(\Phi_2 \boldsymbol{\beta} S(\Phi_3)\big)$$

with

$$(1.16) \qquad \gamma = \sum_{(X)} (S(X_2)\boldsymbol{\alpha} X_3) \otimes (S(X_1)\boldsymbol{\alpha} X_4) \quad \text{where} \quad X = (\mathbf{1} \otimes \Phi) \cdot (\mathrm{id} \otimes \mathrm{id} \otimes \Delta)(\Phi^{-1}) \ .$$

In terms of $\boldsymbol{f}$, Δ and Δ^{op} are related by (see [**Dr2**])

$$(1.17) \qquad \boldsymbol{f} \Delta\big(S(a)\big) \boldsymbol{f}^{-1} = (S \otimes S)(\Delta^{\mathrm{op}}(a)) \ , \qquad a \in A \ .$$

1.2.2. Drinfeld element. The *Drinfeld element* is defined as

$$(1.18) \qquad \boldsymbol{u} = \sum_{(\Phi),(R)} S(\Phi_2 \boldsymbol{\beta} S(\Phi_3))\, S(R_2)\, \boldsymbol{\alpha}\, R_1\, \Phi_1 \ .$$

It satisfies

$$(1.19) \qquad S^2(a) = \boldsymbol{u} a \boldsymbol{u}^{-1} \ ,$$

for any $a \in A$, see [**AC**, Sect. 3].

1.2.3. Ribbon element. A quasi-triangular quasi-Hopf algebra A is called *ribbon* if it contains a ribbon element $\boldsymbol{v}$ defined in the same way as for ordinary Hopf algebras [**So**]:

Definition 1.6. A non-zero central element $\boldsymbol{v} \in A$ is called a *ribbon element* if it satisfies

$$M \cdot \Delta(\boldsymbol{v}) \;=\; \boldsymbol{v} \otimes \boldsymbol{v}, \qquad S(\boldsymbol{v}) = \boldsymbol{v}. \tag{1.20}$$

In a ribbon quasi-Hopf algebra A we have the identities [**AC, So**]

$$\boldsymbol{v}^2 = \boldsymbol{u} S(\boldsymbol{u}) \quad , \quad \epsilon(\boldsymbol{v}) = 1 \ , \tag{1.21}$$

where $\boldsymbol{u}$ is the canonical Drinfeld element defined in (1.18).

2. Category theory

In this chapter we recall some standard categorical notions which will be needed in the rest of the thesis. In particular, we will show how to build a ribbon category from the data of a ribbon quasi-Hopf algebra. We assume the reader is familiar with basics notions like functor, natural transformation, etc.. We will sometimes abbreviate the notation for the Hom-set $\mathrm{Hom}_{\mathcal{C}}(U, V)$ of a category $\mathcal{C}$ and objects $U, V \in \mathcal{C}$ by $\mathcal{C}(U, V)$.

2.1. Ribbon categories

2.1.1. Monoidal categories. A *monoidal category* $\mathcal{C}$ is a category with the following additional data:

- a bifunctor $\otimes \colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ called *tensor product*,
- on object $\mathbf{1}$ called *tensor unit*,
- three natural isomorphisms
  - the *associator* $\alpha_{U,V,W} \colon U \otimes (V \otimes W) \to (U \otimes V) \otimes W$, where α is natural in each argument,
  - the *left unitor* $\lambda_U \colon \mathbf{1} \otimes U \to U$ and the *right unitor* $\rho_U \colon U \otimes \mathbf{1} \to U$,

These data need to satisfy two coherence conditions:

- the pentagon diagram,

$$\begin{tikzcd}
U \otimes (V \otimes (W \otimes Z)) \arrow[rr, "\alpha_{U,V,W\otimes Z}"] \arrow[d, "\mathrm{id}_U \otimes \alpha_{V,W,Z}"'] & & (U \otimes V) \otimes (W \otimes Z) \arrow[d, "\alpha_{U\otimes V,W,Z}"] \\
U \otimes ((V \otimes W) \otimes Z) \arrow[r, "\alpha_{U,V\otimes W,Z}"] & (U \otimes (V \otimes W)) \otimes Z \arrow[r, "\alpha_{U,V,W}\otimes \mathrm{id}_Z"] & ((U \otimes V) \otimes W) \otimes Z
\end{tikzcd}$$

- the triangle diagram,

$$\begin{tikzcd}
U \otimes (\mathbf{1} \otimes V) \arrow[rr, "\alpha_{U,\mathbf{1},V}"] \arrow[dr, "\mathrm{id}_U \otimes \lambda_V"'] & & (U \otimes \mathbf{1}) \otimes V \arrow[dl, "\rho_U \otimes \mathrm{id}_V"] \\
& U \otimes V &
\end{tikzcd}$$

A monoidal category where the natural isomorphisms α, λ and ρ are identities is called *strict*. By Mac Lane's coherence theorem ([**McL**, Chapter VII.2]) every monoidal category is monoidally equivalent to a strict one.

2.1.2. Rigid monoidal categories. A monoidal category $\mathcal{C}$ is said to have *left duals* if for each $U \in \mathcal{C}$ there is an object $U^* \in \mathcal{C}$ together with morphisms

$$\mathrm{ev}_U : U^* \otimes U \to \mathbf{1} \quad , \quad \mathrm{coev}_U : \mathbf{1} \to U \otimes U^* , \tag{2.1}$$

called *evaluation* and *coevaluation* map, which satisfy the two zig-zag identities:[1]

$$\begin{aligned}&\Big[U \xrightarrow{\sim} \mathbf{1}U \xrightarrow{\mathrm{coev}_U \otimes \mathrm{id}_U} (UU^*)U \xrightarrow{\sim} U(U^*U) \xrightarrow{\mathrm{id}_U \otimes \mathrm{ev}_U} U\mathbf{1} \xrightarrow{\sim} U\Big] = \mathrm{id}_U , \\ &\Big[U^* \xrightarrow{\sim} U^*\mathbf{1} \xrightarrow{\mathrm{id}_U \otimes \mathrm{coev}_U} U^*(UU^*) \xrightarrow{\sim} (U^*U)U^* \xrightarrow{\mathrm{ev}_U \otimes \mathrm{id}_U} \mathbf{1}U^* \xrightarrow{\sim} U^*\Big] = \mathrm{id}_{U^*} .\end{aligned} \tag{2.2}$$

Similarly, we say $\mathcal{C}$ has *right duals* if for each $U \in \mathcal{C}$ there is an object ${}^*U \in \mathcal{C}$ together with morphisms

$$\widetilde{\mathrm{ev}}_U : U \otimes {}^*U \to \mathbf{1} \quad , \quad \widetilde{\mathrm{coev}}_U : \mathbf{1} \to {}^*U \otimes U . \tag{2.3}$$

subject to the zig-zag identities.

For a monoidal category $\mathcal{C}$ with left duals one obtains a contravariant functor $(-)^* : \mathcal{C} \to \mathcal{C}$. For a morphism $f : U \to V$, the image $f^* : V^* \to U^*$ under $(-)^*$ is

$$\begin{aligned}f^* = \Big[V^* &\xrightarrow{\sim} V^*\mathbf{1} \xrightarrow{\mathrm{id}_{V^*} \otimes \mathrm{coev}_U} V^*(UU^*) \xrightarrow{\mathrm{id}_{V^*} \otimes f \otimes \mathrm{id}_{U^*}} V^*(VU^*) \\ &\xrightarrow{\alpha_{V^*,V,U^*}} (V^*V)U^* \xrightarrow{\mathrm{ev}_V \otimes \mathrm{id}_{U^*}} \mathbf{1}U^* \xrightarrow{\sim} U^*\Big] .\end{aligned} \tag{2.4}$$

An analogous remark applies to right duals and ${}^*(-) : \mathcal{C} \to \mathcal{C}$.

A *rigid monoidal category* is a monoidal category which has both right and left duals.

One can show that if U has a left (right) dual it is unique up to unique isomorphisms, see e.g. [**EGNO**, Prop. 2.10.5]. This justifies the notation U^* (${}^*U$). Moreover, if U has a right and a left dual it is clear that

$${}^*(U^*) \cong U \cong ({}^*U)^* , \tag{2.5}$$

but it is not guaranteed that ${}^*U \cong U^*$, or equivalently, $U \cong U^{**}$.

Remark 2.1. If U, V are objects with left duals, then $U^* \otimes V^*$ is a left dual of $V \otimes U$ with evaluation map given by

$$\begin{aligned}\tilde{\gamma}_{V,U} = \Big[(U^*V^*)(VU) &\xrightarrow{\alpha^{-1}_{U^*,V^*,VU}} U^*(V^*(VU)) \xrightarrow{\mathrm{id} \otimes \alpha_{V^*,V,U}} U^*((V^*V)U) \\ &\xrightarrow{\mathrm{id} \otimes \mathrm{ev}_V \otimes \mathrm{id}} U^*(\mathbf{1}U) \xrightarrow{\mathrm{id} \otimes \lambda_U} U^*U \xrightarrow{\mathrm{ev}_U} \mathbf{1}\Big] .\end{aligned} \tag{2.6}$$

[1] When giving morphisms involving associator and unit isomorphism, we often write them as sequences of arrows, where for better readability we omit the tensor product symbol between objects and only write "∼" for a composition of coherence isomorphisms of the monoidal category.

Similarly, one defines the coevaluation morphism. Recall that if a (left or right) dual exists it is unique up to unique isomorphism. In a monoidal category $\mathcal{C}$ with left duals, the unique isomorphism from $U^* \otimes V^*$ to any left dual $(V \otimes U)^*$ is given by

$$\begin{aligned}\gamma_{V,U} = \Big[U^*V^* &\xrightarrow{\rho^{-1}_{U^*V^*}} (U^*V^*)\mathbf{1} \xrightarrow{\mathrm{id}\otimes\mathrm{coev}_{VU}} (U^*V^*)((VU)(VU)^*) \\ &\xrightarrow{\alpha_{U^*V^*,VU,(VU)^*}} ((U^*V^*)(VU))(VU)^* \xrightarrow{\tilde{\gamma}_{V,U}\otimes\mathrm{id}} \mathbf{1}(VU)^* \xrightarrow{\lambda_{(VU)^*}} (VU)^*\Big]\end{aligned} \tag{2.7}$$

which is natural in U, V. We will also need the isomorphism from $U^{**} \otimes V^{**}$ to $(U \otimes V)^{**}$ given by

$$\hat{\gamma}_{V,U} := (\gamma^{-1}_{U,V})^* \circ \gamma_{V^*,U^*} \ , \tag{2.8}$$

which is natural in U and V, too. It defines a monoidal structure on the (covariant) functor $(-)^{**} : \mathcal{C} \to \mathcal{C}$ (see the definition in [**EGNO**, Def. 2.4.1]).

If a natural isomorphism $\delta\colon \mathrm{id} \Rightarrow (-)^{**}$ exists, which is monoidal, i.e.

$$\delta_{U\otimes V} = \hat{\gamma}_{V,U} \circ (\delta_U \otimes \delta_V) \ , \tag{2.9}$$

$\mathcal{C}$ is called *pivotal*. We refer to [**EGNO**, Def. 2.4.8] for the definition of monoidal natural transformations between two general functors.

In a pivotal category one can define the *left* and *right trace* of an endomorphism $f \in \mathrm{End}(U)$,

$$\begin{aligned}\mathrm{Tr}^L_U(f) &= \Big[\ \mathbf{1} \xrightarrow{\mathrm{coev}_U} U \otimes U^* \xrightarrow{f\otimes\mathrm{id}} U \otimes U^* \xrightarrow{\delta_U\otimes\mathrm{id}} U^{**} \otimes U^* \xrightarrow{\mathrm{ev}_{U^*}} \mathbf{1}\ \Big] \ , \\ \mathrm{Tr}^R_U(f) &= \Big[\ \mathbf{1} \xrightarrow{\mathrm{coev}_{U^*}} U^* \otimes U^{**} \xrightarrow{\mathrm{id}\otimes\delta_U^{-1}} U^* \otimes U \xrightarrow{\mathrm{id}\otimes f} U^* \otimes U \xrightarrow{\mathrm{ev}_U} \mathbf{1}\ \Big] \ .\end{aligned} \tag{2.10}$$

A rigid category where all left and right trace functions coincide is called *spherical*. For these categories we define the *dimension* of an object U as $\mathrm{Tr}_U(\mathrm{id}) = \mathrm{Tr}^R_U(\mathrm{id})$.

Finally, we want to note that if a monoidal category has left duals, there is a canonical isomorphism $\mathbf{1}^* \to \mathbf{1}$ given by

$$\mathbf{1}^* \xrightarrow{\rho^{-1}_{\mathbf{1}^*}} \mathbf{1}^*\mathbf{1} \xrightarrow{\mathrm{ev}_{\mathbf{1}}} \mathbf{1} \ . \tag{2.11}$$

When writing $\mathbf{1}^* \xrightarrow{\sim} \mathbf{1}$ below, we refer to this isomorphism.

2.1.3. Braided monoidal categories. A *braided monoidal category* $\mathcal{C}$ is a monoidal category with a natural isomorphism $c_{U,V}\colon U \otimes V \to V \otimes U$ called *braiding* such that the hexagon identities are satisfied:

$$
\begin{tikzcd}
& (U\otimes V)\otimes W \arrow[r, "c_{U\otimes V,W}"] & W\otimes(U\otimes V) \arrow[dr, "\alpha_{W,U,V}"] & \\
U\otimes(V\otimes W) \arrow[ur, "\alpha_{U,V,W}"] \arrow[dr, "\mathrm{id}_U\otimes c_{V,W}"'] & & & (W\otimes U)\otimes V \ , \\
& U\otimes(W\otimes V) \arrow[r, "\alpha_{U,W,V}"] & (U\otimes W)\otimes V \arrow[ur, "c_{U,W}\otimes \mathrm{id}_V"'] &
\end{tikzcd}
$$

$$
\begin{tikzcd}
& U\otimes(V\otimes W) \arrow[r, "c_{U,V\otimes W}"] & (V\otimes W)\otimes U \arrow[dr, "\alpha^{-1}_{V,W,U}"] & \\
(U\otimes V)\otimes W \arrow[ur, "\alpha^{-1}_{U,V,W}"] \arrow[dr, "c_{U,V}\otimes \mathrm{id}_W"'] & & & V\otimes(W\otimes U) \ . \\
& (V\otimes U)\otimes W \arrow[r, "\alpha^{-1}_{V,U,W}"] & V\otimes(U\otimes W) \arrow[ur, "\mathrm{id}_V\otimes c_{U,W}"'] &
\end{tikzcd}
$$

Proposition 2.2 ([**Ka**, XIII.1.2]). *For any object U in a braided monoidal category we have*

$$\lambda_U \circ c_{U,\mathbf{1}} = \rho_U \ , \quad \rho_U \circ c_{\mathbf{1},U} = \lambda_U \ , \quad c_{U,\mathbf{1}} = c^{-1}_{\mathbf{1},U} \ . \tag{2.12}$$

If $\mathcal{C}$ has left duals one gets right duals for free by setting ${}^*U = U^*$ and defining

$$\widetilde{\mathrm{ev}}_U \ = \ \Big[\, UU^* \xrightarrow{c_{U,U^*}} U^*U \xrightarrow{\mathrm{ev}_U} \mathbf{1} \,\Big] \quad , \quad \widetilde{\mathrm{coev}}_U \ = \ \Big[\, \mathbf{1} \xrightarrow{\mathrm{coev}_U} UU^* \xrightarrow{c^{-1}_{U,U^*}} U^*U \,\Big] \ . \tag{2.13}$$

Hence, in braided monoidal categories left and right duals are isomorphic. Moreover, one can define Drinfeld's canonical isomorphism $u_U : U \to U^{**}$ between U and its double dual U^{**}, as well as a variant of it which we call $\tilde{u}_U$:

$$\begin{aligned} u_U \ = \ \Big[\, U &\xrightarrow{\rho_U^{-1}} U\mathbf{1} \xrightarrow{\mathrm{id}_U \otimes \mathrm{coev}_{U^*}} U(U^*U^{**}) \xrightarrow{\alpha_{U,U^*,U^{**}}} (UU^*)U^{**} \\ &\xrightarrow{\widetilde{\mathrm{ev}}_U \otimes \mathrm{id}} \mathbf{1}U^{**} \xrightarrow{\lambda_{U^{**}}} U^{**} \,\Big] \ , \end{aligned} \tag{2.14}$$

$$\begin{aligned} \tilde{u}_U \ = \ \Big[\, U &\xrightarrow{\lambda_U^{-1}} \mathbf{1}U \xrightarrow{\widetilde{\mathrm{coev}}_{U^*} \otimes \mathrm{id}_U} (U^{**}U^*)U \xrightarrow{\alpha^{-1}_{U^{**},U^*,U}} U^{**}(U^*U) \\ &\xrightarrow{\mathrm{id} \otimes \mathrm{ev}_U} U^{**}\mathbf{1} \xrightarrow{\rho_{U^{**}}} U^{**} \,\Big] \ . \end{aligned} \tag{2.15}$$

We have

$$u_{U\otimes V} = \hat{\gamma}_{V,U} \circ (u_U \otimes u_V) \circ c^{-1}_{U,V} \circ c^{-1}_{V,U} \quad , \quad \tilde{u}_{U\otimes V} = \hat{\gamma}_{V,U} \circ (\tilde{u}_U \otimes \tilde{u}_V) \circ c_{V,U} \circ c_{U,V} \ , \tag{2.16}$$

see [**EGNO**, Prop. 8.9.3]. That is, u and $\tilde{u}$ are monoidal if and only if $\mathcal{C}$ is symmetric, i.e. $c_{V,U} \circ c_{U,V} = \mathrm{id}_{U\otimes V}$. If that is the case, $\mathcal{C}$ is pivotal.

Summarising, a braided monoidal category with left duals is automatically rigid, but not pivotal in general.

2.1.4. Ribbon categories. A braided monoidal category with left duals is *ribbon* if it is equipped with a natural isomorphism $\theta\colon \mathrm{id} \Rightarrow \mathrm{id}$ (the *ribbon twist*), which satisfies, for all $U, V \in \mathcal{C}$,

$$\theta_{U\otimes V} = (\theta_U \otimes \theta_V) \circ c_{V,U} \circ c_{U,V} \qquad \text{and} \qquad \theta_{U^*} = (\theta_U)^* \ . \tag{2.17}$$

In a ribbon category, there is an alternative way to get right duals from left duals, by defining right duality morphisms as

$$
\begin{aligned}
\widetilde{\mathrm{ev}}_U &= \left[\, UU^* \xrightarrow{\theta_U \otimes \mathrm{id}_{U^*}} UU^* \xrightarrow{c_{U,U^*}} U^*U \xrightarrow{\mathrm{ev}_U} \mathbf{1} \,\right] , \\
\widetilde{\mathrm{coev}}_U &= \left[\, \mathbf{1} \xrightarrow{\mathrm{coev}_U} UU^* \xrightarrow{c^{-1}_{U^*,U}} U^*U \xrightarrow{\mathrm{id}_{U^*} \otimes \theta_U^{-1}} U^*U \,\right] .
\end{aligned} \tag{2.18}
$$

Replacing in (2.14)–(2.15) the maps (2.13) with those above gives a natural isomorphism $\delta\colon \mathrm{id} \Rightarrow (-)^{**}$. Equivalently, one can define

$$
\delta_U : U \to U^{**} \quad , \quad \delta_U = u_U \circ \theta_U = \tilde{u}_U \circ \theta_U^{-1} , \tag{2.19}
$$

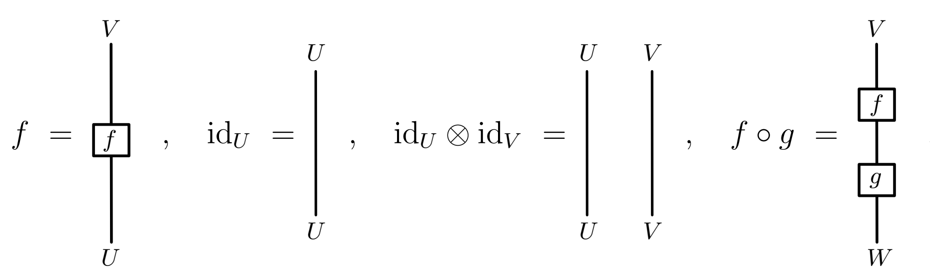
with u_U and $\tilde{u}_U$ from (2.14)–(2.15). To understand the second identity note that the axioms in (2.17) and the naturality of θ imply $\mathrm{ev}_V \circ c_{V,V^*} \circ (\theta_V \otimes \mathrm{id}_{V^*}) = \mathrm{ev}_V \circ c^{-1}_{V^*,V} \circ (\theta_V^{-1} \otimes \mathrm{id}_{V^*})$. By recalling (2.16) and (2.17) one sees that δ is monoidal, i.e. a pivotal structure on $\mathcal{C}$. In fact, $\mathcal{C}$ is spherical [**EGNO**, 8.10.12]. Moreover, δ provides an equivalent way to define the right duality morphisms in (2.18),

$$
\widetilde{\mathrm{ev}}_U = \left[\, UU^* \xrightarrow{\delta_U \otimes \mathrm{id}} U^{**}U^* \xrightarrow{\mathrm{ev}_{U^*}} \mathbf{1} \,\right] , \quad \widetilde{\mathrm{coev}}_U = \left[\, \mathbf{1} \xrightarrow{\mathrm{coev}_{U^*}} U^*U^{**} \xrightarrow{\mathrm{id} \otimes \delta_U^{-1}} U^*U \,\right] . \tag{2.20}
$$

Remark 2.3. The statements above which we did not prove or for which we did not provide a reference can be easily checked with the graphical calculus introduced in the next section.

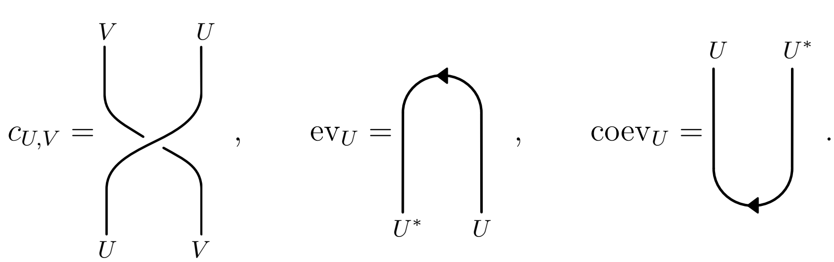
2.1.5. Graphical calculus. Below, we will use string diagram notation for morphisms. Our diagrams are read from bottom to top. A morphisms $f\colon U \to V$ will be denoted by a string with a box while the identity is just a string without a box. The tensor product will be represented by two strings, and the composition $f \circ g$ of two morphisms $f\colon U \to V$ and $g\colon W \to U$ by putting f on the top of g:

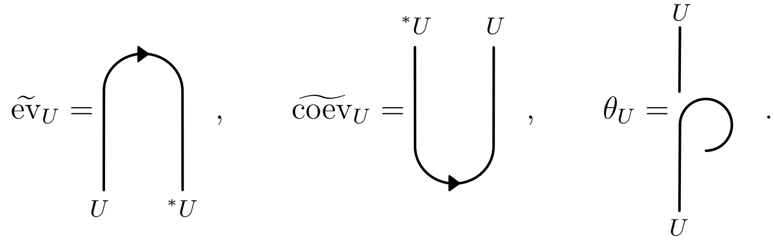
(2.21)



The diagrams we will use for the braiding, ribbon twist, and the right/left duality maps are

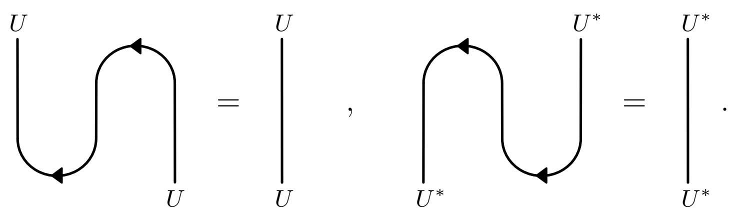
(2.22)



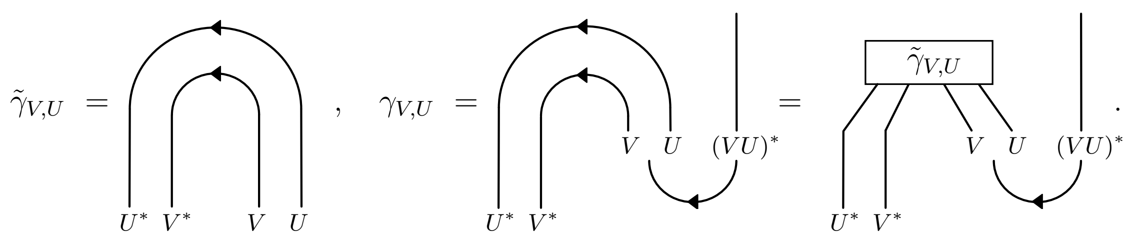
(2.23)


It is easy to see that in general two isotopic diagrams do not correspond to the same morphism. For example, the picture of the ribbon twist is clearly isotopic to a straight line, but $\theta = \mathrm{id}$ holds only in symmetric categories. It turns out that this problem only arise for morphisms which include the ribbon twist. Reshetikhin and Turaev [**RT1**] were able to solve this problem by, roughly speaking, using ribbons instead of lines. We avoid doing this by remembering that we must never apply any string manipulations on the twist.

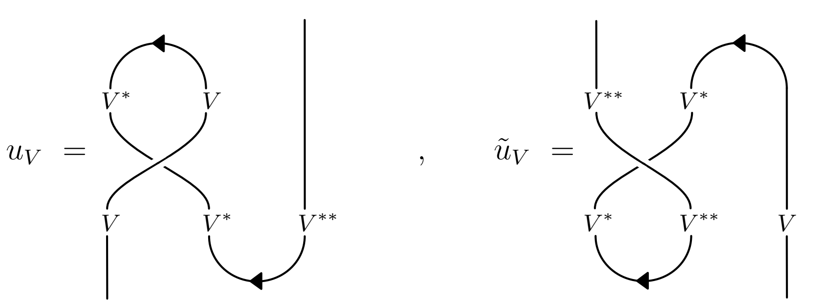
Let us use the graphical calculus in order to simplify identities and definitions we introduced above. For example, the zig-zag conditions in (2.2) can be graphically expressed as

(2.24)


In string diagram notation, the definitions (2.6)–(2.7) look much simpler:

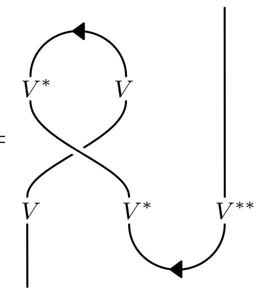
(2.25)


The structure morphisms in (2.14)–(2.15) read

(2.26)


By using string manipulation it is easy to see that

(2.27) $\tilde{u}_V \;=$



.

2.2. The category **Rep** A

Let A be a quasi-triangular quasi-Hopf algebra. In this section we introduce our conventions for the structure maps in the category **Rep** A of finite-dimensional representations of A.

The $\Bbbk$-linear category of finite-dimensional left A-modules will be denoted by

$$\mathbf{Rep}\,A \ . \tag{2.28}$$

Let us explain what we mean by that: Objects (called *modules*) in **Rep** A are pairs (M, ρ_M) where $M \in \mathbf{vect}_\Bbbk$ and $\rho_M : A \otimes M \to M$ (called *action*) is linear in $A \otimes M$ and $\rho_M(ab, m) = \rho_M(a, \rho_M(b, m))$ for every $a, b \in A$ and $m \in M$. For brevity, we will most of the time refer to (M, ρ_M) as M and denote the action by a dot. In this notation the last identity reads $(ab).m = a.(b.m)$. $\mathrm{Hom}_A(M, N)$ is the subspace of all $\Bbbk$-linear maps $f : M \to N$ satisfying $f(a.m) = a.f(m)$ for every $a \in A$. We call such maps *intertwiners*.

In this thesis, we will only consider quasi-Hopf algebras A which satisfy the following

> *Assumption:* The unit isomorphisms λ_U and ρ_U in **Rep** A coincide with those in $\mathbf{vect}_\Bbbk$.

The data of a quasi-triangular quasi-Hopf algebra A from Section 1 allows one to turn **Rep** A into a $\Bbbk$-linear braided category with left duals as follows.

- The associativity isomorphism $\alpha_{U,V,W} : U \otimes (V \otimes W) \to (U \otimes V) \otimes W$ for the tensor product (over $\Bbbk$) of A-modules U, V, W is given by

$$\alpha_{U,V,W}(u \otimes v \otimes w) = \Phi.(u \otimes v \otimes w) \ , \tag{2.29}$$

  for any elements $u \in U$, $v \in V$, $w \in W$. The 3-cocycle condition (1.5) on Φ is equivalent to the commutativity of the pentagon diagram in Section 2.1.1 for α.
- The antipode structure on A gives rise to left duals for **Rep** A. Namely, the left dual U^* of U in **Rep** A is the vector space dual to U together with the A-action

$$(a \cdot f)(u) := f(S(a)u) \ , \qquad u \in U, \quad f \in U^*, \quad a \in A \ . \tag{2.30}$$

The elements $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ enter the definition of the evaluation and coevaluation morphisms as

$$\mathrm{ev}_U : \ \phi \otimes u \mapsto \phi(\boldsymbol{\alpha}.u) \ , \qquad \mathrm{coev}_U : \ 1 \mapsto \sum_i (\boldsymbol{\beta}.u_i) \otimes u_i^* \ , \tag{2.31}$$

where $\phi \in U^*$, $u \in U$, and $\{u_i\}$ is a basis of U with $\{u_i^*\}$ its corresponding dual basis. The zig-zag identities in (2.2) are equivalent to (1.7) and the intertwiner property of ev_U and coev_U are equivalent (1.6).

- The braiding isomorphisms $\sigma_{U,V}$ in $\mathbf{Rep}\,A$ are defined in terms of the universal R-matrix as

$$\sigma_{U,V}(u \otimes v) = \tau_{U,V}(R.(u \otimes v)) \ . \tag{2.32}$$

The isomorphisms $\sigma_{U,V}$ satisfy the hexagon axioms of a braided monoidal category (see Section 2.1.3) iff quasi-triangularity (1.11) holds. Applying the linear map $\mathrm{id} \otimes \epsilon \otimes \mathrm{id}$ to both equations in (1.11) and using the counital condition (1.4), we obtain the following result for a quasi-Hopf algebra under our *Assumption* [**Dr2**, Sec. 3]:

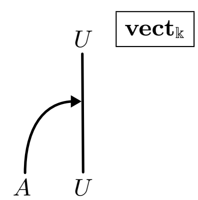
$$(\epsilon \otimes \mathrm{id})(R) = \mathbf{1} = (\mathrm{id} \otimes \epsilon)(R) \ . \tag{2.33}$$

These equalities correspond to the commutativity of the diagram involving the left and right unit isomorphisms and the braiding in (2.12).[2]

Recall the monodromy element M from (1.12). It describes the double braiding in $\mathbf{Rep}\,A$: $\sigma_{V,U} \circ \sigma_{U,V}(u \otimes v) = M.(u \otimes v)$.

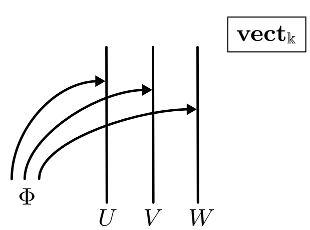
Let us give string diagrams for the structure maps above. The action $A \otimes U \to U$ of A on a left A-module U will be written as

(2.34)



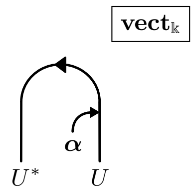
Here, the $\mathbf{vect}_{\Bbbk}$ in a box indicates that the corresponding string diagram is to be taken in $\mathbf{vect}_{\Bbbk}$. For the structure morphisms (2.29), (2.31) and (2.32) we get

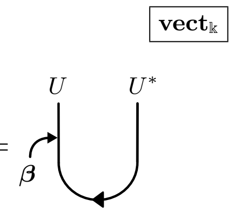
(2.35) $\alpha_{U,V,W}(u \otimes v \otimes w) \ =$



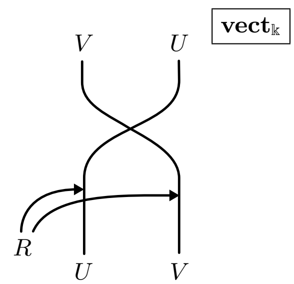
,

[2] One can also demand the identities (2.33) as part of the definition of quasi-triangularity in Definition 1.3. In this case it is not necessary to require that R is invertible, see [**BN2**].

(2.36) $\mathrm{ev}_U(u) \;=$ 

, $\mathrm{coev}_U(u) \;=$ 

,

(2.37) $\sigma_{U,V}(u \otimes v) \;=$ 

,

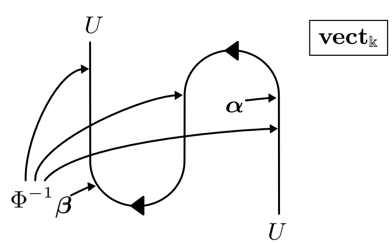
for any $u \in U$, $v \in V$ and $w \in W$. Note that since the string diagrams are in $\mathbf{vect}_{\Bbbk}$, the diagrams for duality maps and symmetric braiding represent the duality maps and braiding of $\mathbf{vect}_{\Bbbk}$, not those of $\mathbf{Rep}\,A$.

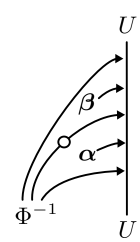
We will denote the standard pivotal structure of $\mathbf{vect}_{\Bbbk}$ by

$$\delta^{\mathbf{vect}} : (-) \to (-)^{**} \quad , \quad \delta^{\mathbf{vect}}_V(v) = \langle -, v\rangle \ , \tag{2.38}$$

where $V \in \mathbf{vect}_{\Bbbk}$, $v \in V$ and $\langle -, -\rangle$ denotes the pairing between V^* and V.

Let us explain how some of the axioms in Definition 1.1 arise from a categorical point of view. For instance, since $\alpha_{U,V,W}$ needs to be an intertwiner we have to make sure that $\Phi \cdot (\mathrm{id}_A \otimes \Delta) \circ \Delta(a) \,.\, u \otimes v \otimes w = (\Delta \otimes \mathrm{id}_A) \circ \Delta(a) \cdot \Phi \,.\, u \otimes v \otimes w$ holds for every $u \in U$, $v \in V$, $w \in W$ and $U, V, W \in \mathbf{Rep}\,A$. But this is equivalent to (1.3). Next consider the zig-zag condition. As a picture in $\mathbf{vect}_{\Bbbk}$ the left hand side of the first equality in (2.24) is given by

(2.39) 

$=$ 

,

where we made use of the zig-zag identity in $\mathbf{vect}_{\Bbbk}$. This picture is equal to the identity map if and only if the second equality in (1.7) holds. Similarly, the other quasi-Hopf algebra axioms in Definition 1.1 can be motivated.

2.2.1. The Drinfeld twist. In this section we want to explain how the formulas for $\boldsymbol{f}$ in (1.15) and equation (1.17) arise.

Lemma 2.4 ([**Dr2**]). *As morphisms in* $\mathbf{Rep}\, A$*,* $\gamma_{N,M}$ *and* $\tilde{\gamma}_{N,M}$ *from* (2.7) *and* (2.6)*, respectively, are given by*

$$\big(\gamma_{N,M}(\varphi \otimes \psi)\big)(n \otimes m) = (\psi \otimes \varphi)(\boldsymbol{f} .\, n \otimes m) \ , \tag{2.40}$$

$$\tilde{\gamma}_{N,M}(\varphi \otimes \psi \otimes n \otimes m) = (\psi \otimes \varphi)(\gamma .\, n \otimes m) \ , \tag{2.41}$$

where $m \in M$*,* $n \in N$*,* $\varphi \in M^*$*,* $\psi \in N^*$*, and* $\boldsymbol{f}$ *and* γ *are defined in* (1.15) *and* (1.16)*, respectively.*

Proof. We begin with $\tilde{\gamma}$. Let $X = (\mathbf{1} \otimes \Phi) \cdot (\mathrm{id} \otimes \mathrm{id} \otimes \Delta)(\Phi^{-1})$. Then (2.6) gives

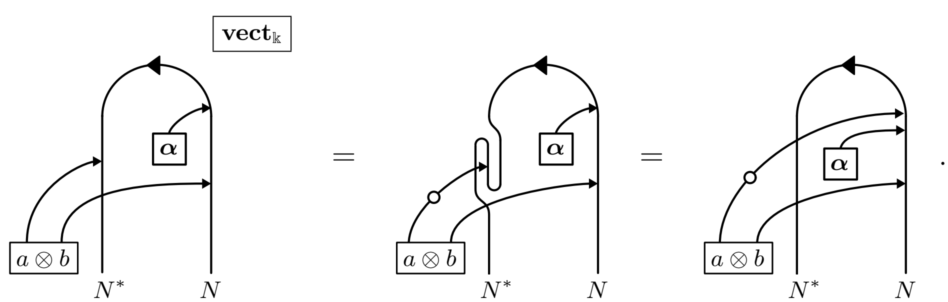
$$\tilde{\gamma}_{N,M}(\varphi \otimes \psi \otimes n \otimes m) = \mathrm{ev}_M \circ (\mathrm{id} \otimes \mathrm{ev}_N \otimes \mathrm{id})(X \,.\, \varphi \otimes \psi \otimes n \otimes m) \ . \tag{2.42}$$

Note that

$$\mathrm{ev}_N(a \otimes b \,.\, \psi \otimes n) = \psi(S(a)\boldsymbol{\alpha} b \,.\, n) \ . \tag{2.43}$$

Indeed, we have

(2.44)

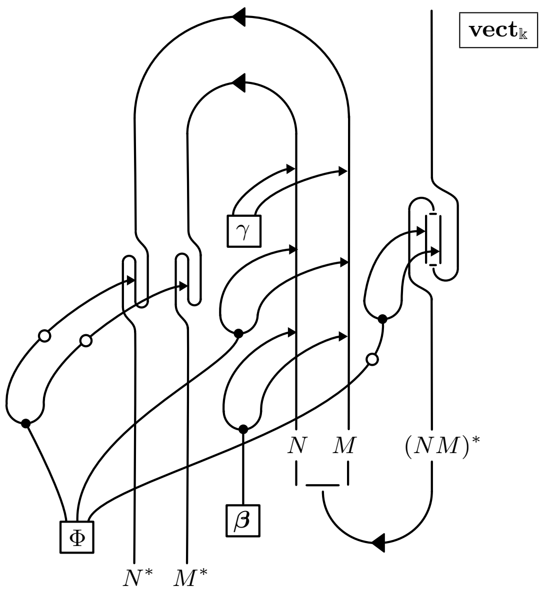


It follows then from the above equalities that

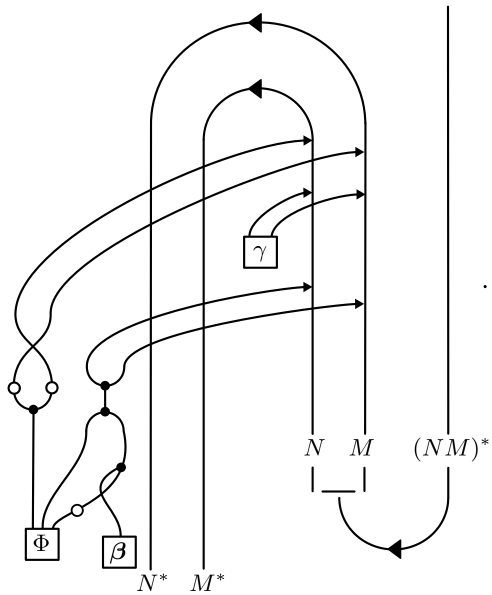
$$\begin{aligned}
\tilde{\gamma}_{N,M}(\varphi \otimes \psi \otimes n \otimes m) &= \sum_{(X)} \mathrm{ev}_M(X_1 \otimes X_4 \,.\, \varphi \otimes m)\, \psi(S(X_2)\boldsymbol{\alpha} X_3 \,.\, n) \\
&= \sum_{(X)} \varphi(S(X_1)\boldsymbol{\alpha} X_4 \,.\, m)\, \psi(S(X_2)\boldsymbol{\alpha} X_3 \,.\, n) \\
&= \sum_{(X)} (\psi \otimes \varphi)(S(X_2)\boldsymbol{\alpha} X_3 \otimes S(X_1)\boldsymbol{\alpha} X_4 \,.\, n \otimes m) \\
&= (\psi \otimes \varphi)(\gamma \,.\, n \otimes m) \ .
\end{aligned} \tag{2.45}$$

Equation (2.40) follows by recalling (2.7) and using the identity

(2.46)



$=$



.

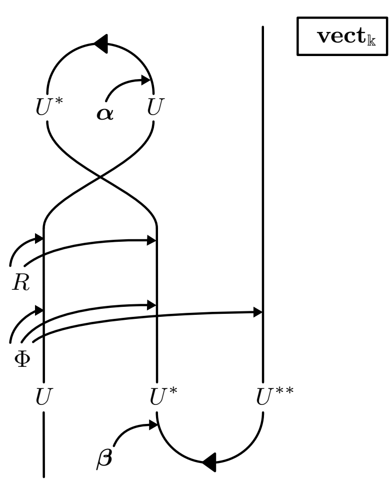
□

Note that the product of the elements acting on $N \otimes M$ on the RHS of (2.46) is precisely the Drinfeld twist from (1.15).

An easy check now shows that (1.17) follows from the intertwiner property of γ.

2.2.2. Drinfeld element. Recall Drinfeld's canonical natural isomorphism $u : \mathrm{id}_{\mathcal{C}} \Rightarrow (-)^{**}$ from (2.14). In terms of the data of the quasi-triangular quasi-Hopf algebra A, the morphism u_U, for $U \in \mathbf{Rep}\, A$, can be written as

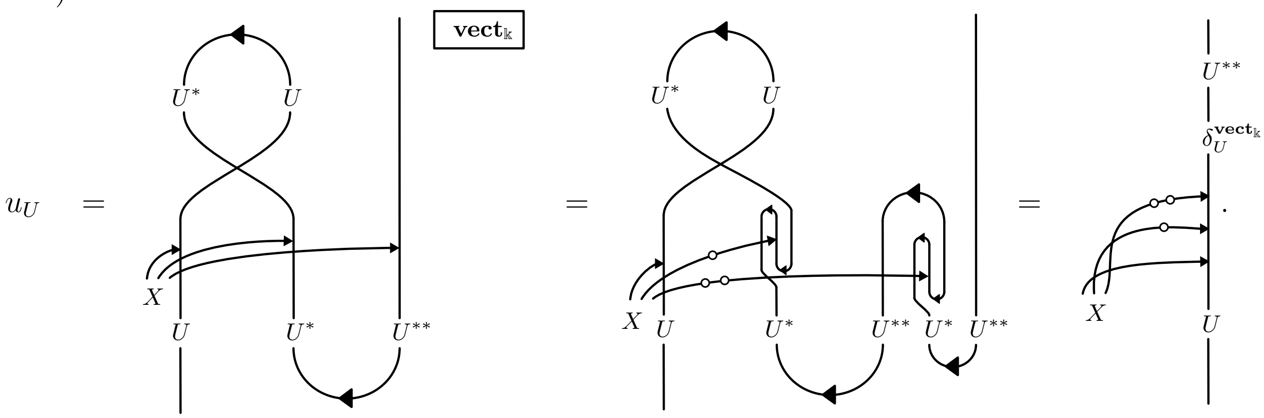
(2.47) $u_U \quad =$



.

Abbreviate $X = (\boldsymbol{\alpha} \otimes \mathbf{1} \otimes \mathbf{1}) \cdot (R \otimes \mathbf{1}) \cdot \Phi \cdot (\mathbf{1} \otimes \boldsymbol{\beta} \otimes \mathbf{1})$ and recall the standard pivotal structure of $\mathbf{vect}_{\Bbbk}$ in (2.38). Then we have

(2.48)



We conclude that

$$u_U = \delta_U^{\mathbf{vect}_{\Bbbk}} \circ (\boldsymbol{u}\,.\,(-)) \ , \tag{2.49}$$

where $\boldsymbol{u} \in A$ is the Drinfeld element in (1.18).

The corresponding calculation for the variant $\tilde{u}$ in (2.15) or (2.27) gives the same expression with R replaced by R^{-1}. For later reference, we state it explicitly:

$$\tilde{u}_U = \delta_U^{\mathbf{vect}_{\Bbbk}} \circ (\tilde{\boldsymbol{u}}\,.\,(-)) \quad , \quad \tilde{\boldsymbol{u}} = \sum_{(\Phi),(R^{-1})} S(\Phi_2 \boldsymbol{\beta} S(\Phi_3))\, S((R^{-1})_2)\, \boldsymbol{\alpha}\, (R^{-1})_1\, \Phi_1 \ . \tag{2.50}$$

2.2.3. Ribbon twist. Let now A be a finite-dimensional ribbon quasi-Hopf algebra. By convention, the ribbon twist θ_U on an object U is given by acting with the inverse ribbon element $\boldsymbol{v}^{-1}$:

$$\theta_U \;=\; \boldsymbol{v}^{-1}.(-) \ . \tag{2.51}$$

Since $\mathbf{Rep}\,A$ is a braided monoidal category with left duals and a twist, it is rigid as explained in Section 2.1.4. In fact, it is a finite braided tensor category over $\Bbbk$.

2.2.4. Pivotal structure. Recall from Section 2.1.4 that for a ribbon category $\mathcal{C}$ with ribbon twist θ, one can define a pivotal structure $\delta_X : X \to X^{**}$ by $\delta_X = u_X \circ \theta_X = \tilde{u}_X \circ \theta_X^{-1}$.

Combining (2.49) and (2.51), we see that in the category $\mathbf{Rep}\,A$ the pivotal structure takes the form

$$\delta_U \;=\; \delta_U^{\mathbf{vect}} \circ (\boldsymbol{v}^{-1}\boldsymbol{u}\,.\,(-)) \;:\; U \to U^{**} \ . \tag{2.52}$$

The right evaluation and coevaluation morphisms are (recall (2.20)),

$$\widetilde{\mathrm{ev}}_U :\; w \otimes \phi \mapsto \phi(S(\boldsymbol{\alpha})\boldsymbol{v}^{-1}\boldsymbol{u}.w) \quad , \quad \widetilde{\mathrm{coev}}_U :\; 1 \mapsto \sum_i w_i^* \otimes (\boldsymbol{u}^{-1}\boldsymbol{v}S(\boldsymbol{\beta}).w_i) \ , \tag{2.53}$$

where $\phi \in U^*$, $w \in U$, and $\{w_i\}$ is a basis of U with dual basis $\{w_i^*\}$.

Using the ribbon structure, we can give a relation between the two variants $\boldsymbol{u}$ and $\tilde{\boldsymbol{u}}$ of the Drinfeld element. Namely, from $u_X \circ \theta_X = \tilde{u}_X \circ \theta_X^{-1}$ we get $\boldsymbol{u}\boldsymbol{v}^{-1} = \tilde{\boldsymbol{u}}\boldsymbol{v}$ and combining this with (1.21) gives

$$\tilde{\boldsymbol{u}} = S(\boldsymbol{u}^{-1}) \ . \tag{2.54}$$

Applying S to both sides and using (1.19) and (2.50) gives an explicit formula for $\boldsymbol{u}^{-1}$. An alternative expression is given in [**BN2**, Thm. 2.6].

2.3. Hopf algebras in braided categories

The definition of a Hopf algebra over a field has a natural generalisation to braided monoidal categories, see e.g. [**Ma3**].

Definition 2.5. Let $\mathcal{C}$ be a braided monoidal category. *A Hopf algebra H in $\mathcal{C}$* is an object H together with morphisms

$$\begin{aligned} &\text{(product)} \quad \mu_H : H \otimes H \to H \ , && \text{(coproduct)} \quad \Delta_H : H \to H \otimes H \ , \\ &\text{(unit)} \quad \eta_H : \mathbf{1} \to H \ , && \text{(counit)} \quad \varepsilon_H : H \to \mathbf{1} \ , \\ &\text{(antipode)} \quad S_H : H \to H \ . \end{aligned} \tag{2.55}$$

These data are subject to the conditions

- associativity and unitality:

$$\begin{aligned} \big[\, H(HH) \xrightarrow{\mathrm{id}\otimes\mu_H} HH \xrightarrow{\mu_H} H \,\big] \;&=\; \big[\, H(HH) \xrightarrow{\sim} (HH)H \xrightarrow{\mu_H\otimes\mathrm{id}} HH \xrightarrow{\mu_H} H \,\big] \ , \\ \big[\, H \xrightarrow{\sim} \mathbf{1}H \xrightarrow{\eta_H\otimes\mathrm{id}} HH \xrightarrow{\mu_H} H \,\big] \;=\; \mathrm{id}_H \;&=\; \big[\, H \xrightarrow{\sim} H\mathbf{1} \xrightarrow{\mathrm{id}\otimes\eta_H} HH \xrightarrow{\mu_H} H \,\big] \ . \end{aligned} \tag{2.56}$$

- coassociativity and counitality: same as above but with all arrows reversed, μ_H replaced by Δ_H and η_H by ε_H.
- Δ_H, ε_H are algebra homomorphisms:

$$\begin{aligned} &\Delta_H \circ \eta_H = (\eta_H \otimes \eta_H) \circ \lambda_{\mathbf{1}}^{-1} \ , \\ &\big[\, HH \xrightarrow{\mu_H} H \xrightarrow{\Delta_H} HH \,\big] \;=\; \big[\, HH \xrightarrow{\Delta_H\otimes\Delta_H} (HH)(HH) \xrightarrow{\sim} H((HH)H) \\ &\qquad\qquad \xrightarrow{\mathrm{id}\otimes c_{H,H}\otimes\mathrm{id}} H((HH)H) \xrightarrow{\sim} (HH)(HH) \xrightarrow{\mu_H\otimes\mu_H} HH \,\big] \ , \end{aligned} \tag{2.57}$$

  and

$$\varepsilon_H \circ \mu_H = \lambda_{\mathbf{1}} \circ (\varepsilon_H \otimes \varepsilon_H) \quad , \quad \varepsilon_H \circ \eta_H = \mathrm{id}_{\mathbf{1}} \ . \tag{2.58}$$

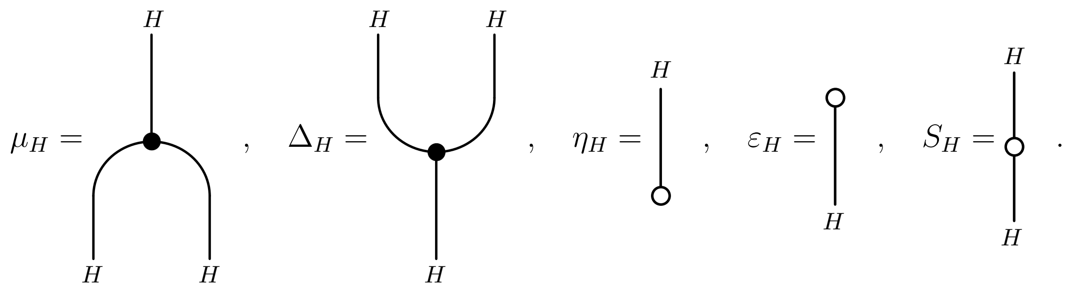
- antipode condition:

$$\begin{aligned} \big[\, H \xrightarrow{\varepsilon_H} \mathbf{1} \xrightarrow{\eta_H} H \,\big] \;&=\; \big[\, H \xrightarrow{\Delta_H} HH \xrightarrow{S_H\otimes\mathrm{id}} HH \xrightarrow{\mu_H} H \,\big] \\ &=\; \big[\, H \xrightarrow{\Delta_H} HH \xrightarrow{\mathrm{id}\otimes S_H} HH \xrightarrow{\mu_H} H \,\big] \ . \end{aligned} \tag{2.59}$$

As a consequence of the Hopf-algebra axioms, we get that S_H is an algebra and a coalgebra anti-homomorphism, in particular we have $S_H \circ \mu_H = \mu_H \circ c_{H,H} \circ (S_H \otimes S_H)$, see [**Ma3**, Lem. 2.3].

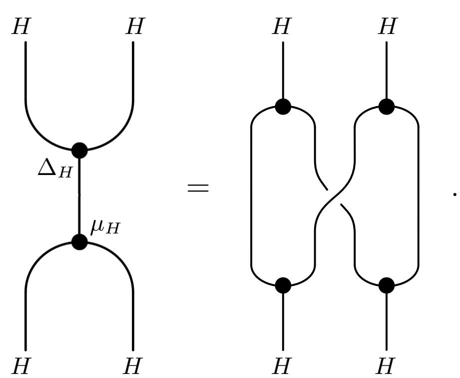
When using string diagram notation to depict morphisms involving Hopf algebras, we use the following notation for its structure morphisms:

(2.60)



For example, the second condition in (2.57), i.e. compatibility of Δ_H with μ_H, reads

(2.61)

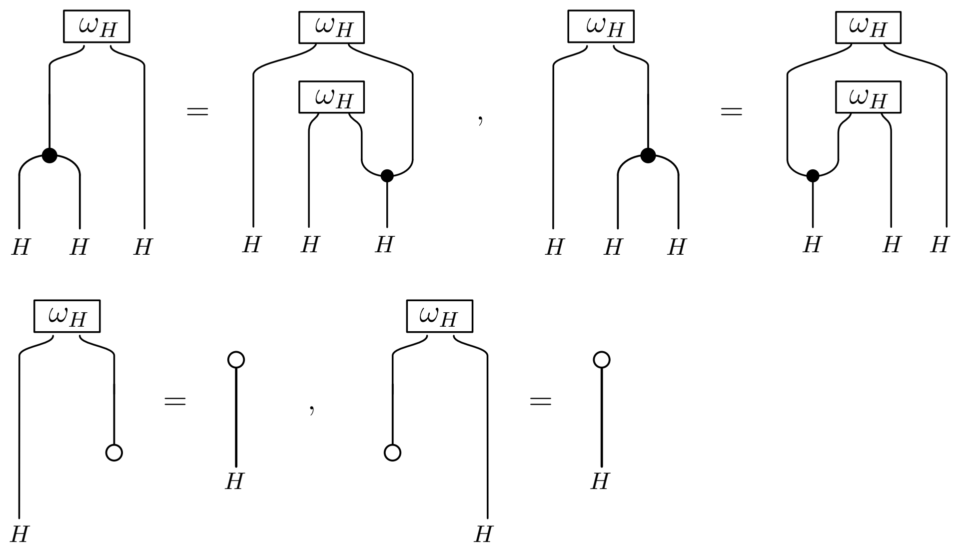


The appearance of the braiding as opposed to the inverse braiding in the above condition is a choice, related to the choice made in defining the tensor product of algebras: for (associative, unital) algebras (A, μ_A, η_A) and (B, μ_B, η_B) we define the algebra $A \otimes B$ to have structure morphisms

$$\begin{aligned}
\mu_{A\otimes B} &= \big[\,(AB)(AB) \xrightarrow{\sim} (A((BA)B) \xrightarrow{\mathrm{id}\otimes c_{B,A}\otimes \mathrm{id}} (A((AB)B) \\
&\qquad \xrightarrow{\sim} (AA)(BB) \xrightarrow{\mu_A\otimes\mu_B} AB\,\big]\,, \\
\eta_{A\otimes B} &= \big[\,\mathbf{1} \xrightarrow{\sim} \mathbf{11} \xrightarrow{\eta_A\otimes\eta_B} AB\,\big]\,.
\end{aligned} \tag{2.62}$$

By definition, a *Hopf pairing* for a Hopf algebra H in $\mathcal{C}$ is a morphism $\omega_H : H \otimes H \to \mathbf{1}$ which makes the multiplication μ_H and the coproduct Δ_H, as well as the unit η_H and the counit ε_H, each others adjoints. In terms of string diagrams, this means

(2.63)



Translating back into formulas, for example the first of the above identities becomes

$$\text{(2.64)} \qquad \Big[(HH)H \xrightarrow{\mu_H \otimes \mathrm{id}} HH \xrightarrow{\omega_H} \mathbf{1}\Big] \\ = \Big[(HH)H \xrightarrow{\mathrm{id}\otimes\mathrm{id}\otimes\Delta_H} (HH)(HH) \xrightarrow{\sim} H((HH)H) \\ \xrightarrow{\mathrm{id}\otimes\omega_H\otimes\mathrm{id}} H(\mathbf{1}H) \xrightarrow{\sim} HH \xrightarrow{\omega_H} \mathbf{1}\Big] .$$

If the braided monoidal category $\mathcal{C}$ is equipped with left duals, for each Hopf algebra H in $\mathcal{C}$ we obtain the *(left) dual Hopf algebra* H^*. Its structure maps are

$$\text{(2.65)} \qquad \begin{aligned} \mu_{H^*} &= \Big[H^*H^* \xrightarrow{\gamma_{H,H}} (HH)^* \xrightarrow{(\Delta_H)^*} H^*\Big] , \\ \Delta_{H^*} &= \Big[H^* \xrightarrow{(\mu_H)^*} (HH)^* \xrightarrow{\gamma_{H,H}^{-1}} H^*H^*\Big] , \\ \eta_{H^*} &= \Big[\mathbf{1} \xrightarrow{\sim} \mathbf{1}^* \xrightarrow{(\varepsilon_H)^*} H^*\Big] , \\ \varepsilon_{H^*} &= \Big[H^* \xrightarrow{(\eta_H)^*} \mathbf{1}^* \xrightarrow{\sim} \mathbf{1}\Big] , \end{aligned}$$

where we used the isomorphism $\gamma_{H,H}$ from (2.7) and the isomorphism (2.11). The antipode is given by $S_{H^*} = (S_H)^*$.

Given a Hopf pairing ω_H on H, we can define the map

$$\text{(2.66)} \qquad \mathcal{D}_H := \big[\, H \xrightarrow{\sim} H\mathbf{1} \xrightarrow{\mathrm{id}\otimes\mathrm{coev}_H} H(HH^*) \xrightarrow{\sim} (HH)H^* \xrightarrow{\omega_H\otimes\mathrm{id}} \mathbf{1}H^* \xrightarrow{\sim} H^* \,\big]$$

The definitions above are set up such that $\mathcal{D}_H$ is a homomorphism of Hopf algebras.

Definition 2.6. Let H be a Hopf algebra in a braided monoidal category with left duals. A Hopf pairing ω_H for H is called *non-degenerate* if the morphism $\mathcal{D}_H$ in (2.66) is an isomorphism.

In Sections 4.3, 5.3 we will also need integrals and cointegrals for a Hopf algebra H in $\mathcal{C}$. These are morphisms $\Lambda_H : \mathrm{Int}H \to H$ and $\Lambda^{\mathrm{co}}_H : H \to \mathrm{Int}H$ with an invertible object $\mathrm{Int}H$ which are subject to the corresponding condition in the following list:

$$
\begin{array}{llrcl}
(2.67) & \text{(left integral)} & \mu_H \circ (\mathrm{id}_H \otimes \Lambda_H) & = & \Lambda_H \circ \lambda_{\mathrm{Int}H} \circ (\varepsilon_H \otimes \mathrm{id}_{\mathrm{Int}H}) \ , \\
 & \text{(right integral)} & \mu_H \circ (\Lambda_H \otimes \mathrm{id}_H) & = & \Lambda_H \circ \rho_{\mathrm{Int}H} \circ (\mathrm{id}_{\mathrm{Int}H} \otimes \varepsilon_H) \ , \\
 & \text{(left cointegral)} & (\mathrm{id}_H \otimes \Lambda^{\mathrm{co}}_H) \circ \Delta_H & = & (\eta_H \otimes \mathrm{id}_{\mathrm{Int}H}) \circ \lambda^{-1}_{\mathrm{Int}H} \circ \Lambda^{\mathrm{co}}_H \ , \\
 & \text{(right cointegral)} & (\Lambda^{\mathrm{co}}_H \otimes \mathrm{id}_H) \circ \Delta_H & = & (\mathrm{id}_{\mathrm{Int}H} \otimes \eta_H) \circ \rho^{-1}_{\mathrm{Int}H} \circ \Lambda^{\mathrm{co}}_H \ .
\end{array}
$$

If $\mathrm{Int}H \simeq \mathbf{1}$ one can use the integrals and cointegrals to test non-degeneracy of a Hopf pairing ω_H:

Lemma 2.7 ([**Ke2**])**.** *Assume that $H \in \mathcal{C}$ is a Hopf algebra with a right cointegral $\Lambda^{\mathrm{co}}_H : H \to \mathbf{1}$ and a Hopf pairing ω_H. The Hopf pairing ω_H is non-degenerate iff there exists a morphism $\Lambda_H : \mathbf{1} \to H$ such that the cointegral Λ^{co}_H factors through ω_H:*

$$\Lambda^{\mathrm{co}}_H = \Big[H \xrightarrow{\sim} \mathbf{1}H \xrightarrow{\Lambda_H \otimes \mathrm{id}} HH \xrightarrow{\omega_H} \mathbf{1} \Big] \ .$$

If such a Λ_H exists, it is automatically a left integral for H. A similar statement can be made for left cointegrals.

2.4. Coends and ends

We introduce here the concept of dinatural transformations between two functors. Let $\mathcal{C}$ and $\mathcal{D}$ be any categories and let $F\colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{D}$ and $G\colon \mathcal{C} \times \mathcal{C}^{\mathrm{op}} \to \mathcal{D}$ be two functors. The next definition is a slight modification of the one in [**McL**, Ch. IX.4] – the order of categories for the second functor is different.

Definition 2.8. A *dinatural transformation* from the functor F to the functor G is a family of morphisms $\phi \equiv (\phi_U\colon F(U,U) \to G(U,U))_{U\in\mathcal{C}}$ in $\mathcal{D}$, written $\phi\colon F \overset{..}{\longrightarrow} G$, that makes the diagram

(2.68)
$$
\begin{tikzcd}
 & F(V,V) \arrow[r, "\phi_V"] & G(V,V) \arrow[dr, "G(\mathrm{id},f)"] & \\
F(V,U) \arrow[ur, "F(\mathrm{id},f)"] \arrow[dr, "F(f,\mathrm{id})"'] & & & G(V,U) \\
 & F(U,U) \arrow[r, "\phi_U"] & G(U,U) \arrow[ur, "G(f,\mathrm{id})"'] &
\end{tikzcd}
$$

commute for all $U, V \in \mathcal{C}$ and $f \in \mathcal{C}(U,V)$.

For the definition of coends and ends, we consider only the case where one of the functors, F or G, is a “constant” functor: e.g. $G\colon U \times V \mapsto B$ for all $U, V \in \mathcal{C}$ and an object $B \in \mathcal{D}$, and morphisms get mapped to id_B. Definition 2.8 then reduces to the following one.

Definition 2.9.

(1) A *dinatural transformation* from the functor F to an object $B \in \mathcal{D}$ is a family of morphisms $\phi \equiv (\phi_U \colon F(U,U) \to B)_{U\in\mathcal{C}}$ in $\mathcal{D}$, written $\phi \colon F \xrightarrow{\cdot\cdot} B$, which makes the diagram

(2.69)
$$\begin{tikzcd} F(V,U) \arrow[r, "F(\mathrm{id},f)"] \arrow[d, "F(f,\mathrm{id})"'] & F(V,V) \arrow[d, "\phi_V"] \\ F(U,U) \arrow[r, "\phi_U"] & B \end{tikzcd}$$

commutative for all $U, V \in \mathcal{C}$ and $f \in \mathcal{C}(U,V)$.

(2) A *dinatural transformation* from an object $B \in \mathcal{D}$ to the functor G is a family of morphisms $\phi \equiv (\phi_U \colon B \to G(U,U))_{U\in\mathcal{C}}$ in $\mathcal{D}$ which makes the diagram

(2.70)
$$\begin{tikzcd} B \arrow[r, "\phi_V"] \arrow[d, "\phi_U"'] & G(V,V) \arrow[d, "G(\mathrm{id},f)"] \\ G(U,U) \arrow[r, "G(f,\mathrm{id})"] & G(V,U) \end{tikzcd}$$

commutative for all $U, V \in \mathcal{C}$ and $f \in \mathcal{C}(U,V)$.

Definition 2.10.

(1) A *coend* (C,ι) of a functor $F \colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{D}$ is an object $C \in \mathcal{D}$ endowed with a dinatural transformation $\iota \colon F \xrightarrow{\cdot\cdot} C$, see Definition 2.9 (1), satisfying the following universal property: for any dinatural transformation $\phi \colon F \xrightarrow{\cdot\cdot} B$ there is a unique morphism $g \in \mathcal{D}(C,B)$ such that the following diagram commutes for all $U \in \mathcal{C}$:

(2.71)
$$\begin{tikzcd} & F(U,U) \arrow[dl, "\iota_U"'] \arrow[dr, "\phi_U"] & \\ C \arrow[rr, dashed, "\exists!\, g"] & & B \end{tikzcd}$$

In other words, any dinatural transformation $\phi \colon F \xrightarrow{\cdot\cdot} B$ factors through the coend of F in a unique way: $\phi_U = g \circ \iota_U$ for all $U \in \mathcal{C}$.

(2) An *end* (E,j) of a functor $G \colon \mathcal{D} \to \mathcal{C} \times \mathcal{C}^{\mathrm{op}}$ is an object $E \in \mathcal{D}$ endowed with a dinatural transformation $j \colon E \xrightarrow{\cdot\cdot} G$, see Definition 2.9 (2), satisfying the following universal property: for any dinatural transformation $\phi \colon B \xrightarrow{\cdot\cdot} G$ there is a unique morphism $g \in \mathcal{D}(B,E)$ such that the following diagram commutes for all $U \in \mathcal{C}$:

(2.72)
$$\begin{tikzcd} & G(U,U) & \\ E \arrow[ur, "j_U"] & & B \arrow[ul, "\phi_U"'] \arrow[ll, dashed, "\exists!\, g"'] \end{tikzcd}$$

or equivalently: $\phi_U = j_U \circ g$ for all $U \in \mathcal{C}$.

Coends and ends are unique up to unique isomorphism, so that we may refer to 'the coend' and 'the end'. A common notation for the coend is $\int^{U\in\mathcal{C}} F(U,U)$, and $\int_{U\in\mathcal{C}} G(U,U)$ for the end. For brevity, we will often just denote the coend by C instead of (C,ι), and the by E instead of (E,j).

We will also need multiple or iterated coends $\int^{U\in\mathcal{B}}\int^{V\in\mathcal{C}} F(U,U,V,V)$ of a functor

$$F\colon \mathcal{B}^{\mathrm{op}}\times\mathcal{B}\times\mathcal{C}^{\mathrm{op}}\times\mathcal{C}\to\mathcal{D}\ . \tag{2.73}$$

These are defined by considering first the functor F as the functor $\tilde{F}\colon \mathcal{C}^{\mathrm{op}}\times\mathcal{C}\to \mathrm{Fun}(\mathcal{B}^{\mathrm{op}}\times\mathcal{B},\mathcal{D})$ to the category of functors from $\mathcal{B}^{\mathrm{op}}\times\mathcal{B}$ to $\mathcal{D}$, and assuming that the coend of $\tilde{F}$ exists as an object in this category of functors. We can then consider the coend of this coend-object $\int^{V\in\mathcal{C}} F(-,-,V,V)$, which is by definition the iterated coend of F from above. Alternatively, we could first take the coend (or "integration") over objects in $\mathcal{B}$ as the object $\int^{U\in\mathcal{B}} F(U,U,-,-)$ in the category of functors from $\mathcal{C}^{\mathrm{op}}\times\mathcal{C}$ to $\mathcal{D}$ and then take the corresponding coend (or "integration") over objects in $\mathcal{C}$. This gives another iterated coend $\int^{V\in\mathcal{C}}\int^{U\in\mathcal{B}} F(U,U,V,V)$. Finally, one can consider the "double coend", that is, the coend for the functor

$$(\mathcal{B}\times\mathcal{C})^{\mathrm{op}}\times(\mathcal{B}\times\mathcal{C}) \xrightarrow{\sim} \mathcal{B}^{\mathrm{op}}\times\mathcal{B}\times\mathcal{C}^{\mathrm{op}}\times\mathcal{C} \xrightarrow{F} \mathcal{D}\ , \tag{2.74}$$

which we write as $\int^{(U,V)\in\mathcal{B}\times\mathcal{C}} F(U,U,V,V)$. The iterated coends and the double coend can be compared by a "Fubini theorem" (see e.g. [**McL**, IX.8]).

Proposition 2.11. *Let $\mathcal{B},\mathcal{C}$ be categories and let $F\colon \mathcal{B}^{\mathrm{op}}\times\mathcal{B}\times\mathcal{C}^{\mathrm{op}}\times\mathcal{C}\to\mathcal{D}$ be a functor. Consider the three coends*

$$\int^{U\in\mathcal{B}}\int^{V\in\mathcal{C}} F(U,U,V,V) \quad , \quad \int^{(U,V)\in\mathcal{B}\times\mathcal{C}} F(U,U,V,V) \quad , \quad \int^{V\in\mathcal{C}}\int^{U\in\mathcal{B}} F(U,U,V,V)\ .$$

If any one of them exists, then so do the other two, and all three are canonically isomorphic.

We can similarly define higher iterated and multiple coends, up to a unique isomorphism.

Remark 2.12.

(1) We can define the category $DIN(F)$ of dinatural transformations for F: objects are pairs (B,ι), for dinatural transformations $\iota\colon F \overset{..}{\longrightarrow} B$, and morphisms are defined as $DIN(F)\big((B,\iota),(B',\phi)\big) := \{f\in\mathcal{D}(B,B') : \phi = f\circ\iota\}$. Coends are then the initial objects in $DIN(F)$. (And *ends* are the terminal objects, but we only use these in Section 4.3.3 below.)

(2) We will later use the following important property of a coend C: to define a morphism $C\to B$ (e.g. for $B = C\otimes C$ below) it is enough to fix a morphism from $F(U,U)$ to B for all $U\in\mathcal{C}$ such that it is dinatural. This is due to the universal property of C: there is a one-to-one correspondence between the set $\mathrm{Din}(F,B)$ of dinatural transformations $\iota\colon F \overset{..}{\longrightarrow} B$ and the set $\mathcal{D}(C,B)$. Similarly, by Proposition 2.11 the iterated coend $C = \int^{U\in\mathcal{B}}\int^{V\in\mathcal{C}} F(U,U,V,V)$ of a functor $F\colon \mathcal{B}^{\mathrm{op}}\times\mathcal{B}\times\mathcal{C}^{\mathrm{op}}\times\mathcal{C}\to\mathcal{D}$ has the following

universal property: any transformation of F dinatural in both arguments (for $\mathcal{B}$ and $\mathcal{C}$) to an object in $\mathcal{D}$ factors uniquely through the iterated coend.

2.5. Further relevant notions

In this section we collect more categorical notions which we need later. Let $\Bbbk$ be a algebraically closed field with characteristic 0.

Definition 2.13 ([**Tu**]). A *modular tensor category* is a finitely semisimple $\Bbbk$-linear and abelian category, which is in addition a ribbon category, which has a simple tensor unit, and a non-degenerate s-matrix.

Let us roughly explain what this means.

A $\Bbbk$-*linear abelian* category $\mathcal{C}$ is, in particular, a category for which all Hom-sets are $\Bbbk$-vector spaces, and which has a notion of direct sum. An object in $\mathcal{C}$ is *simple* if it has no proper subobject. A subobject of an object V is an object U together with an monomorphism $i\colon U \to V$. $\mathcal{C}$ is *finitely semisimple* if there are up to isomorphisms finitely many simple objects $\{U_i \mid i \in I\}$ in $\mathcal{C}$, and every object is a finite direct sum of simple objects. For a finitely semisimple $\Bbbk$-linear and abelian ribbon category the *s-matrix* is defined by $s_{i,j} = \mathrm{Tr}_{U_i\otimes U_j}(c_{U_j,U_i} \circ c_{U_i,U_j}) \in \Bbbk$. Here, we used that ribbon categories are spherical (see Sec. 2.1.4). The s-matrix together with the t-matrix $t_{i,j} = \delta_{i,j}\theta_{U_i}$ provide a projective $SL(2,\mathbb{Z})$-action on the Hom-space $\mathcal{C}(\mathbf{1}, \bigoplus_{i\in I} U_i^* \otimes U_i)$ [**Tu**, Sec. II.3.9].

Definition 2.14. Let $\mathcal{F}\colon \mathcal{C} \to \mathcal{D}$ and $\mathcal{G}\colon \mathcal{D} \to \mathcal{C}$ be functors. If the functors from $\mathcal{D}^{\mathrm{op}} \times \mathcal{C}$ to the category of sets,

$$(U,V) \mapsto \mathcal{C}(\mathcal{G}(U),V) \quad , \quad (U,V) \mapsto \mathcal{D}(U,\mathcal{F}(V)) \tag{2.75}$$

are natural isomorphic, we call $\mathcal{F}$ the *right adjoint* of $\mathcal{G}$ and $\mathcal{G}$ the *left adjoint* of $\mathcal{F}$.

Definition 2.15. Let $\mathcal{C}$ and $\mathcal{D}$ be abelian categories, $\mathcal{F}\colon \mathcal{C} \to \mathcal{D}$ an additive functor (i.e., in particular, $\mathcal{F}(0) = 0$).

- $\mathcal{F}$ is called *left exact* if exactness of $0 \to U \to V \to W$ implies exactness of $0 \to \mathcal{F}(U) \to \mathcal{F}(V) \to \mathcal{F}(W)$, for all $U, V, W \in \mathcal{C}$.
- $\mathcal{F}$ is called *right exact* if exactness of $U \to V \to W \to 0$ implies exactness of $\mathcal{F}(U) \to \mathcal{F}(V) \to \mathcal{F}(W) \to 0$, for all $U, V, W \in \mathcal{C}$.

We say a morphism $f\colon U \to W$ *factors through* $g\colon V \to W$ if there is a morphism $h\colon U \to V$ such that $f = g \circ h$.

Definition 2.16. Let $\mathcal{C}$ be a category.

(1) An object $P \in \mathcal{C}$ is called *projective* if every morphism $f\colon P \to U$ factors through every epimorphism $e\colon V \to U$.

(2) A *projective cover* is a pair (P, π) which consists of a projective object P and an epimorphism $\pi\colon P \to U$, such that every epimorphism $P' \to U$ with projective source object P', factors through π.

Projective objects generalise the notion of projective modules in the categorical setting.

Next, we give a categorical product between abelian categories introduced in [**De**, Sec. 5.1] (see also [**GRW**, Def. 3.3]). Let $\mathcal{C}, \mathcal{D}$ and $\mathcal{B}$ be $\Bbbk$-linear abelian categories. Denote by $\mathrm{Fun}_{\mathrm{r.ex.}}(\mathcal{C} \times \mathcal{D}, \mathcal{B})$ the category of functors from $\mathcal{C} \times \mathcal{D}$ to $\mathcal{B}$ which are $\Bbbk$-linear and right exact in each argument.

Definition 2.17. Let $\mathcal{C}$ and $\mathcal{D}$ be $\Bbbk$-linear abelian categories. The *Deligne product* $\mathcal{C} \boxtimes \mathcal{D}$ is an abelian category together with a bifunctor $\boxtimes\colon \mathcal{C} \times \mathcal{D} \to \mathcal{C} \boxtimes \mathcal{D}$ which is right exact and $\Bbbk$-linear in both arguments and satisfies the following condition: For all $\Bbbk$-linear abelian categories $\mathcal{B}$,

$$\mathrm{Fun}_{\mathrm{r.ex.}}(\mathcal{C} \boxtimes \mathcal{D}, \mathcal{B}) \to \mathrm{Fun}_{\mathrm{r.ex.}}(\mathcal{C} \times \mathcal{D}, \mathcal{B}) \quad , \quad \mathcal{F} \mapsto \mathcal{F} \circ \boxtimes \tag{2.76}$$

is an equivalence of categories.

Finally, we explain what we mean when we refer to a finite tensor category.

Definition 2.18 ([**EO**])**.** Let $\Bbbk$ be a field. A *finite tensor category* $\mathcal{C}$ is a finite abelian $\Bbbk$-linear rigid monoidal category with simple tensor unit $\mathbf{1}$, such that the tensor product is a $\Bbbk$-linear functor in each argument.

Here *finite* means that $\mathcal{C}$ is equivalent, as a $\Bbbk$-linear category, to the category of finite-dimensional representations of a finite-dimensional $\Bbbk$-algebra. In particular, $\mathcal{C}$ is essentially small. As $\mathcal{C}$ is rigid, the tensor product is automatically exact in each argument, see e.g. [**BK**, Prop. 2.1.8]. We note that in a finite tensor category, the dual of a projective object is again projective (see e.g. [**EGNO**, Sec. 6.1]), and so each projective object is also injective.

3. Vertex operator algebras

In this section we give a brief introduction to the theory of vertex operator algebras. The reader should not expect a comprehensive treatment into this vast theory. Rather, we aim to provide some context for the original work of this thesis.

3.1. CFTs in a nutshell

This section is a short review of [**Sc**, Part 1].

3.1.1. Conformal group. We start by considering general semi-Riemann manifolds. Later we will focus on the Euclidean case.

Definition 3.1. Let (M, g) be a semi-Riemann manifold and $U \subset M$ open. A smooth map $f\colon U \to M$ with full rank (i.e. an immersion) is called a *conformal transformation* if there is a smooth function $\lambda\colon U \to \mathbb{R}_{>0}$ such that for every $p \in U$ and every $X, Y \in T_pM$

$$g_p(d_pfX, d_pfY) = \lambda(p)^2\, g_p(X, Y) \ . \tag{3.1}$$

λ is called the *conformal factor of f*.

Recall that the angle $\angle(X, Y)$ of two non-zero tangent vectors $X, Y \in T_pM$ with non-vanishing norm is defined as $\cos\angle(X, Y)\|X\|_p\|Y\|_p = g_p(X, Y)$ where $\|X\|_p := \sqrt{|g_p(X, X)|}$. Thus, conformal transformation are precisely the immersions which preserve angles.

Definition 3.2. Let M be a connected semi-Riemann manifold. A *conformal compactification* of M is a compact semi-Riemann manifold K together with an embedding $\iota\colon M \to K$ such that

- $\iota(M)$ is dense in K.
- For every conformal transformation $f\colon U \subset M \to M$ there exists a conformal transformation $\hat{f}\colon K \to K$ called *conformal extension* such that the following diagram commutes:

$$\begin{tikzcd} U \arrow[r, "f"] \arrow[d, "\iota"'] & M \arrow[d, "\iota"] \\ K \arrow[r, "\hat{f}"] & K \end{tikzcd}$$

Note that a conformal compactification (if it exists) is unique up to isomorphisms.

For the Euclidean space $(\mathbb{R}^d, \langle\cdot,\cdot\rangle)$ the formula in (3.1) reduces to

$$\langle DfX, DfY\rangle = \lambda(p)^2 \langle X, Y\rangle \ ,$$

where Df denotes the Jacobi matrix of f. We will now give a classification of conformal transformations for the Euclidean space which depends on the dimension.

Theorem 3.3 ([**Sc**, Thm. 1.11 & Thm. 1.9])**.** *Let $f\colon U \to \mathbb{R}^d$ be a map and $U \subset \mathbb{R}^d$ open.*

- $d = 2$: *f is a conformal transformation iff it is a locally invertible holomorphic or antiholomorphic function. In this case the conformal factor is $\sqrt{|\det Df|}$.*
- $d > 2$: *If additionally U is connected then f is a conformal transformation iff it is equal to any composition of*
  - *a translation*
  - *a rotation*
  - *a dilatation $p \mapsto ap$ for any $a \in \mathbb{R}_{>0}$*
  - *a special conformal transformation (an inversion $p \mapsto \frac{p}{\langle p,p\rangle}$ followed by a translation and again an inversion)*

Let $\gamma\colon \mathbb{R}^{d+2} \to P^{d+1}\mathbb{R}$ be the canonical map to the $(d+1)$-dimensional projective space $P^{d+1}\mathbb{R}$. We define

$$N^d = \{\gamma(x) \mid x \in \mathbb{R}^{d+2}, -x_0^2 + x_1^2 + \ldots + x_{d+1}^2 = 0\} \subset P^{d+1}\mathbb{R} \ . \tag{3.2}$$

Theorem 3.4 ([**Sc**, Thm. 2.9 & Thm. 2.11])**.** *Let $\mathbb{R}^d$ be the Euclidean space. If $d > 2$ every conformal transformation in $\mathbb{R}^d$ has a conformal extension on N^d. In the case $d = 2$ every injective conformal transformation on all of $\mathbb{C}$ with at most one singular point has a conformal extension on N^2.*

Remark 3.5. In contrast to the $d > 2$ case not every conformal transformation in $\mathbb{R}^2 \cong \mathbb{C}$ has a conformal extension. For example, the injective holomorphic function

$$z \mapsto \sqrt{z} \quad , \quad \mathrm{Re(z)} > 0 \ ,$$

can even not be extended to all of $\mathbb{C}$.

Hence, in the case $d = 2$ a conformal compactification does not exist. A way to bypass this issue is to consider only such conformal transformations as described in Theorem 3.4 (which are often called *global*, see [**Sc**, Def. 2.10]). With this restriction the conformal compatification of $\mathbb{R}^2$ exists and is the Riemann sphere $P^1\mathbb{C}$, see [**Sc**, Sec. 2.3].

Definition 3.6. The *conformal group* of $\mathbb{R}^d$ is the connected component of the identity in the Lie group of conformal diffeomorphism on N^d the compactification of $\mathbb{R}^d$.

Theorem 3.7 ([**Sc**, Sec. 2.3])**.** *Under the restriction explained above the (global) conformal group of* $\mathbb{R}^2$ *is the group of Möbius transformations. These are all the maps* $P^1\mathbb{C} \to P^1\mathbb{C}$ *of the form*

$$z \mapsto \frac{az+b}{cz+d} \quad , \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{C}) \ . \tag{3.3}$$

Remark 3.8. It is straightforward to check that a Möbius transformation is a composition of translations, rotations, dilations and special conformal transformations ($z \mapsto \frac{z}{cz+1}$). Recall the second statement in Theorem 3.3. Hence, the conformal group of $\mathbb{R}^d, d \geq 2$, consists of compositions of translations, rotations, dilations and special conformal transformations.

3.1.2. Virasoro algebra. In two-dimensional Euclidean space a complex function $f(z)$ transforms under an infinitesimal holomorphic conformal transformation $z \mapsto z + \varepsilon\omega(z)$, $\varepsilon \ll 1$, as

$$f(z) \mapsto f(z) - \varepsilon\omega(z)\partial f(z) + \mathcal{O}(\varepsilon^2) = f(z) - \varepsilon \sum_{n\in\mathbb{Z}} \omega_n z^{n+1} \partial f(z) + \mathcal{O}(\varepsilon^2) \ , \tag{3.4}$$

where $\omega(z)$ is a general local holomorphic function. The complex Lie algebra of all vector fields which can be associated to infinitesimal conformal transformations for which ω is a Laurent polynomial in z is spanned by the vector fields

$$L_m := -z^{m+1}\frac{\partial}{\partial z} \quad , \quad m \in \mathbb{Z} \ , \tag{3.5}$$

with commutation relations,

$$[L_m, L_n] = (n-m)L_{m+n} \ . \tag{3.6}$$

This algebra is known as the *Witt algebra* and we denoted it by W. The finite subalgebra with basis $\{L_{-1}, L_0, L_1\}$ and its antiholomorphic counterpart can be associated to the conformal group: Indeed, we have

$$\begin{aligned} \text{Translation} &: \quad L_{-1} + \bar{L}_{-1} \quad , \quad \mathrm{i}(L_{-1} - \bar{L}_{-1}) \\ \text{Scale trans.} &: \quad L_0 + \bar{L}_0 \\ \text{Rotation} &: \quad \mathrm{i}(L_0 - \bar{L}_0) \\ \text{Special conf. trans.} &: \quad (L_1 + \bar{L}_1) \quad , \quad \mathrm{i}(L_1 - \bar{L}_1) \end{aligned} \tag{3.7}$$

Conformal field theories are invariant under conformal transformations, i.e. the quantum field theory has conformal symmetry. The quantization of classical systems with symmetries yields representations of the classical symmetry group in the projective space of a Hilbert space, the so-called projective representations. In order to get representations on the Hilbert space one has to consider, in general, a central extension of the symmetry group. It turns out (see e.g. [**Sc**, Sec. 3]) that in two-dimensional conformal field theories the relevant algebra has a non-trivial central extension: The *Virasoro algebra Vir* is a central extension of the Witt algebra. That is,

$$
\begin{aligned}
Vir &= W \oplus \mathbb{C}\, C\ , \\
[L_m, L_n] &= (n-m)L_{m+n} - \frac{1}{12}(m^3 - m)\delta_{m+n,0} C\ , \\
[L_m, C] &= 0\ .
\end{aligned} \tag{3.8}
$$

Hence, if one want to build a two-dimensional conformal field theory one would expect that the Hilbert space carries an action of $Vir \oplus Vir$.

3.2. Vertex operator algebras

Let V be a vector space and z a formal variable. We define the following vector spaces (see [**Sc**, Sec. 10.1]),

$$V[[z^{\pm}]] = \Big\{ \sum_{n\in\mathbb{Z}} a_n z^n \mid a_n \in V \Big\} \quad \textit{(formal distribution)}, \tag{3.9}$$

$$V[[z]] = \Big\{ \sum_{n\in\mathbb{Z}_{\geq 0}} a_n z^n \mid a_n \in V \Big\} \quad \textit{(formal power series)}, \tag{3.10}$$

$$V((z)) = \Big\{ \sum_{n\in\mathbb{Z}_{\geq k}} a_n z^n \mid a_n \in V,\ k \in \mathbb{Z} \Big\} \quad \textit{(formal Laurent series)}, \tag{3.11}$$

and we shall also use analogous notation for several variables. A formal distribution

$$a(z) = \sum_{n\in\mathbb{Z}} a_{(n)} z^n \in \mathrm{End}(V)[[z^{\pm}]] \tag{3.12}$$

is called a *field* if for any $v \in V$ we have $a_{(n)}v = 0$ for n large enough.

In this section we consider only vector spaces over $\mathbb{C}$.

Definition 3.9 ([**Sc**, Def. 10.18])**.** A *vertex algebra* (VA) consists of the following data:

- a vector space $\mathcal{V}$ (*space of states*),
- a distinguished vector $\mathbf{1} \in \mathcal{V}$ (*vacuum vector*),
- an endomorphism $T \in \mathrm{End}(\mathcal{V})$ (*translation operator*),
- a linear map $Y \colon \mathcal{V} \to \mathrm{End}(\mathcal{V})[[z^{\pm}]]$ (*vertex operator*) taking any $a \in \mathcal{V}$ to a field

$$Y \colon a \mapsto Y(a,z) = \sum_{n\in\mathbb{Z}} a_{(n)} z^{-n-1}\ , \quad a_{(n)} \in \mathrm{End}(\mathcal{V})\ , \tag{3.13}$$

such the following axioms are satisfied:

- $Y(\mathbf{1}, z) = \mathrm{id}_{\mathcal{V}}$, $Y(a,z)\mathbf{1} \in \mathcal{V}[[z]]$ and $a_{(-1)}\mathbf{1} = a$ (*vacuum axioms*),
- $[T, Y(a,z)] = \partial_z Y(a,z)$ and $T\mathbf{1} = 0$ (*translation axioms*),
- For any $a, b \in \mathcal{V}$ there exists a $N \geq 0$ such that $(z-w)^N\, [Y(a,z), Y(b,w)] = 0$ (*locality axiom*).

Remark 3.10.

- Equivalently, one can define the vertex operator as a map $Y\colon \mathcal{V}\otimes\mathcal{V}\to\mathcal{V}((z)), a\otimes b\mapsto Y(a,z)b$.
- The vacuum axioms imply that $Y(a,z)\mathbf{1}=a+\sum_{n>0}(a_{(-n-1)}\mathbf{1})z^n$.
- T is completely fixed by the above axioms: $T(a)=a_{(-2)}\mathbf{1}$, see [**Sc**, Rem. 10.21].

The locality axiom is equivalent to the following (*Borcherds*) identity ([**Kac**, Prop. 4.8]): For all vectors $a,b,c\in\mathcal{V}$ and all integers $p,q,r\in\mathbb{Z}$, we have

$$\sum_{k=0}^{\infty}\binom{p}{k}(a_{(r+k)}b)_{(p+q-k)}=\sum_{k=0}^{\infty}(-1)^k\binom{r}{k}\left(a_{(p+r-k)}b_{(q+k)}-(-1)^r b_{(q+r-k)}a_{(p+k)}\right)\ . \tag{3.14}$$

Setting $r=0$ and $p=0$ in (3.14), we get

$$\left(a_{(p)}b_{(q)}-b_{(q)}a_{(p)}\right)=\sum_{k=0}^{\infty}\binom{p}{k}(a_{(k)}b)_{(p+q-k)}\ , \tag{3.15}$$

$$(a_{(r)}b)_{(q)}=\sum_{k=0}^{\infty}(-1)^k\binom{r}{k}\left(a_{(r-k)}b_{(q+k)}-(-1)^r b_{(q+r-k)}a_{(k)}\right)\ , \tag{3.16}$$

which are called the *commutator* and *associator* formula, respectively.

Example 3.11. Commutative vertex algebras, i.e. VAs with $[Y(a,z),Y(b,w)]=0$ for all $a,b\in\mathcal{V}$, are in one-to-one correspondence with commutative, associative, unital algebras with derivation. To see that, the unit of the algebra has to be associated with the vacuum vector and the derivation with the translation operator. For details see [**Sc**, Prop. 10.27 & 10.28].

Theorem 3.12 ([**FB**, Thm. 3.1.1])**.** *Let $\mathcal{V}$ be a vertex algebra and $a(z)$ a field acting on $\mathcal{V}$. Assume that there is a vector $b\in\mathcal{V}$ such that*

$$a(z)\mathbf{1}=Y(b,z)\mathbf{1} \tag{3.17}$$

and $a(z)$ is local with respect to $Y(c,z)$ for any c in $\mathcal{V}$. Then $a(z)=Y(b,z)$.

Theorem 3.13 ([**FB**, Thm. 4.4.1])**.** *Let $\mathcal{V}$ be a vector space with $0\neq\mathbf{1}\in\mathcal{V}$, $T\in\mathrm{End}(\mathcal{V})$ and $I\subseteq\mathcal{V}$ a linearly independent subset. Suppose that $\{Y_a\}_{a\in I}$ are linear operators*

$$Y_a\colon\mathcal{V}\to\mathrm{End}\,\mathcal{V}((z))\quad,\quad Y_a(z)\,b=\sum_{n\in\mathbb{Z}}a_{(n)}b\,z^{-n-1}\ , \tag{3.18}$$

such that for all $a\in I$, as well as for $\mathbf{1}$ and T the vacuum, translation and locality axioms in (3.9) *hold. Moreover, $\{Y_a\}_{a\in I}$ shall generate $\mathcal{V}$ in the sense that*

$$\mathcal{V}=\mathrm{span}\{a^1_{(-n_1)}\cdots a^k_{(-n_k)}\mathbf{1}\mid a^i\in I, n_i\geq 1, k\geq 0\}\ . \tag{3.19}$$

Then there is an unique vertex algebra $(\mathcal{V},Y,T,\mathbf{1})$ with

$$\begin{aligned} Y(a^1_{(-n_1-1)}\cdots a^k_{(-n_k-1)}\mathbf{1},z)&=((n_1)!\cdots(n_k)!)^{-1}:\partial_z^{n_1}Y_{a^1}(z)\cdots\partial_z^{n_k}Y_{a^k}(z):\quad,\\ Y(a,z)&=Y_a(z)\ . \end{aligned} \tag{3.20}$$

Here, we used the *normally ordered product* which is defined for two fields $a(z)$, $b(z)$ as

$$:a(z)b(z): \; = \sum_{n\in\mathbb{Z}} \left(\sum_{m<0} a_{(m)} b_{(n)} z^{-m-1} + \sum_{m>0} b_{(n)} a_{(m)} z^{-m-1} \right) \tag{3.21}$$

and extended recursively, for fields $a_1(z), \ldots, a_n(z)$, to

$$:a_1(z)\cdots a_n(z): \; = \; :a_1(z) :a_2(z)\cdots a_n(z):: \quad . \tag{3.22}$$

Definition 3.14. A *vertex operator algebra* (VOA) of charge $c \in \mathbb{C}$ is a vertex algebra $\mathcal{V}$ equipped with a distinguished vector $\omega \in \mathcal{V}$ (*conformal vector*) such that the linear operators L_n in

$$Y(\omega, z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2} \qquad \text{(note that } \omega_{(n)} = L_{n-1}) \tag{3.23}$$

satisfy the Virasoro relations in (3.8) with central charge $c \cdot \mathrm{id}_{\mathcal{V}}$. Moreover,

- L_0 acts semisimply on $\mathcal{V}$ with integral eigenvalues of L_0 which provides a grading of $\mathcal{V} = \bigoplus_{k=K}^{\infty} \mathcal{V}_k$, which is bounded below and $\dim(\mathcal{V}_k) < \infty$,
- the vertex operator is homogeneous (in terms of the L_0-grading of $\mathcal{V}$), i.e. for homogeneous $a, b \in \mathcal{V}$ we have $\deg(a_n b) = \deg(a) + \deg(b) - n - 1$,
- $\deg(\mathbf{1}) = 0$ and $\deg(\omega) = 2$,
- $T = L_{-1}$.

3.3. $\mathcal{V}$-modules

There are several definitions of modules over a vertex operator algebra. Some are not graded (which are basically modules for a vertex algebra), and some comprise a $\mathbb{N}$, $\mathbb{Q}$, $\mathbb{R}$ or $\mathbb{C}$-grading. While the $\mathbb{N}$-graded modules have a lower-truncation, the others are given by the eigenvalues of L_0, and may or may not have finite-dimensional graded pieces. We will describe three of these notions.

From now we will always assume that a VOA $\mathcal{V}$ is of *CFT type*, that is, $\mathcal{V} = \bigoplus_{k=0}^{\infty} \mathcal{V}_k$ and $\mathcal{V}_0 = \mathbb{C}\mathbf{1}$.

Definition 3.15 ([**ABD**, Def. 2.3])**.** Let $\mathcal{V}$ be a vertex operator algebra. A *weak $\mathcal{V}$-module* is a vector space M together with an action

$$Y^M \colon \mathcal{V} \otimes M \to M((z)) \quad , \quad a \mapsto Y^M(a, z) = \sum_{n\in\mathbb{Z}} a^M_{(n)} z^{-n-1} \tag{3.24}$$

such that

- $Y^M(\mathbf{1}, z) = \mathrm{id}_M$ (vacuum axiom),
- for all vectors $a, b, c \in \mathcal{V}$ and all integers $p, q, r \in \mathbb{Z}$ the (*Borcherds*) identity holds,

$$\sum_{k=0}^{\infty} \binom{p}{k} (a_{(r+k)} b)^M_{(p+q-k)} = \sum_{k=0}^{\infty} (-1)^k \binom{r}{k} \left(a^M_{(p+r-k)} b^M_{(q+k)} - (-1)^r b^M_{(q+r-k)} a^M_{(p+k)} \right) \; . \tag{3.25}$$

Weak modules admit a representation of the Virasoro algebra via $Y^M(\omega, z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}$, and satisfy $Y^M(Tv, z) = \partial_z Y^M(v, z)$ [**FHL**, Sec. 4.1].

Definition 3.16 ([**ABD**, Def. 2.4])**.** Let $\mathcal{V}$ be a vertex operator algebra. An *admissible* $\mathcal{V}$-module is an $\mathbb{N}$-gradable weak module $M = \bigoplus_{n=1}^{\infty} M_n$, such that for a homogeneous $a \in \mathcal{V}_r$

$$a^M_{(m)} \colon M_n \to M_{n+r-m-1} \ . \tag{3.26}$$

Definition 3.17 ([**ABD**, Def. 2.5])**.** An *ordinary* $\mathcal{V}$-module $M = \bigoplus_{\lambda\in\mathbb{C}} M_\lambda$ over a VOA $\mathcal{V}$ is a weak $\mathcal{V}$-module such that

- $\dim M_\lambda < \infty$,
- for all $\lambda \in \mathbb{C}$, $M_{\lambda+n} = 0$ if the integer n is small enough,
- L_0 acts semisimply on M and $L_0 m = \lambda m$ for $m \in M_\lambda$.

It may seem as if $\mathbb{N}$-gradeability of admissible modules were the strongest condition. It turns out that due to the finiteness condition on the graded pieces we actually have [**DLM1**, Lem.3.4]:

$$\{\text{ordinary modules}\} \subseteq \{\text{admissible modules}\} \subseteq \{\text{weak modules}\} \ . \tag{3.27}$$

An additional important finiteness condition is C_2-cofiniteness which was introduced by Zhu, [**Zh**]: A VOA $\mathcal{V}$ is called *C_2-cofinite* if $\mathcal{V}/C_2(\mathcal{V})$ is finite dimensional, where $C_2(\mathcal{V})$ is the subspace spanned by the vectors $a_{(-2)}b$ for any $a, b \in \mathcal{V}$. Under the assumption of C_2-cofiniteness [**ABD**] proved that every admissible module is a direct sum of simple admissible modules if and only if every weak module is a direct sum of simple ordinary modules. The first condition is known as *rationality* of $\mathcal{V}$ while the latter describes *regularity* of $\mathcal{V}$. If there is an admissible module which is not completely reducible we call $\mathcal{V}$ *logarithmic*. It is shown in [**DLM2**] that C_2-cofiniteness implies that $\mathcal{V}$ has only finitely many irreducible modules. A C_2-cofinite VOA is finitely generated and every weak module is $\mathbb{N}$-gradable such that the graded pieces are a direct sum of generalised L_0-eigenspaces, see e.g [**Mi**, Thm. 2.7].

We can build categories out of weak, admissible or ordinary $\mathcal{V}$-modules by taking such modules as objects and defining $\mathrm{Hom}_{\mathcal{V}}(M, N) = \{\psi \colon M \to N \text{ linear} \mid \psi(Y(a, z)b) = Y(a, z)\psi(b)\}$.

Clearly, $\mathcal{V}$ itself is a ordinary module by taking $Y^{\mathcal{V}} = Y$. We say that $\mathcal{V}$ is *simple* when $\mathcal{V}$ is irreducible as an ordinary $\mathcal{V}$-module. The next theorem was proved by Huang (see [**Hu**] and the references therein).

Theorem 3.18. *Let $\mathcal{V}$ be a rational C_2-cofinite VOA of CFT type, such that $\mathcal{V}$ is simple and isomorphic to its contragredient $\mathcal{V}' = \bigoplus_{k=0}^{\infty} \mathcal{V}_k^*$ as an ordinary $\mathcal{V}$-module. Then the category of ordinary $\mathcal{V}$-modules has a natural structure of a modular tensor category.*

The logarithmic case is less well understood. Define

$$\mathbf{Rep}\,\mathcal{V} \tag{3.28}$$

as the category of *generalised* ordinary $\mathcal{V}$-modules (the only difference to ordinary modules is that here the modules are graded by the generalised eigenspaces of L_0). It is shown in [**HLZ**] that $\mathbf{Rep}\,\mathcal{V}$ is a finite monoidal braided category; the proof of rigidity is still missing.

3.4. Modular invariance of 1-point functions

Let $\mathcal{V}$ be a C_2-cofinite VOA of CFT type and $\mathbb{H}$ the upper half plane in $\mathbb{C}$. The $\mathbb{C}$-linear space of *one-point functions* $C_1(\mathcal{V})$ consists of functions

$$\xi\colon \mathcal{V}\times\mathbb{H}\to\mathbb{C} \tag{3.29}$$

which are linear in $\mathcal{V}$, holomorphic in $\mathbb{H}$ and fulfil certain differential equations (see [**Zh**] or [**DLM2**] for details). The vector space $C_1(\mathcal{V})$ is finite-dimensional and invariant under the modular action given by ([**Zh**, Thm. 5.1.1])

$$\xi|_\gamma(v,\tau) := (c\tau-d)^{-h}\,\xi\Big(v,\frac{a\tau+b}{c\tau+d}\Big) \quad,\quad \gamma\in\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in SL(2,\mathbb{Z})\ , \tag{3.30}$$

where $v\in\mathcal{V}_{[h]}$, and $\mathcal{V}=\oplus_{n=0}^{\infty}\mathcal{V}_{[n]}$ is a second grading introduced in [**Zh**].

For a homogeneous element a in $\mathcal{V}$ we define the *zero mode*

$$o(a) = a_{(\deg a-1)}\in\mathrm{End}(\mathcal{V}) \tag{3.31}$$

and extend it by linearity to all of $\mathcal{V}$. Note that by (3.26) $o(a)$ acts grade-preservingly on admissible $\mathcal{V}$-modules and since it commutes with L_0 it also acts grade-preservingly on L_0-graded modules.

Let M be an admissible $\mathcal{V}$-module (with finite-dimensional graded pieces). Zhu ([**Zh**]) showed that the trace functions

$$\mathrm{Tr}_M\big(o(a)q^{L_0-c/24}\big) \quad,\quad q=e^{2\pi\mathrm{i}\tau}\text{ and }\tau\in\mathbb{H} \tag{3.32}$$

lie in $C_1(\mathcal{V})$. In other words, every admissible $\mathcal{V}$-module gives a vector in the space of 1-point functions. Moreover, he proved that when $\mathcal{V}$ is rational the trace functions of inequivalent irreducible modules form a basis in $C_1(\mathcal{V})$.

Miyamoto [**Mi**] generalized this to the non-semisimple case by introducing so-called pseudo-trace functions. Arike and Nagatomo presented in [**AN**] a simpler notion of pseudo-trace functions at the expense of the spanning set property. They considered modules $M=\bigoplus_{\lambda\in\mathbb{C}}M_\lambda$ over a C_2-cofinite, $\mathbb{Z}_{\geq 0}$-graded VOA $\mathcal{V}$, which are finitely-generated and graded by generalised L_0-eigenvalues with only countably many non-zero graded pieces. *Pseudo-trace functions* are then defined as follows (see also [**GR2**]): Let M be a $\mathcal{V}$-module which is projective as a module over the $\mathbb{C}$-algebra $E:=\mathrm{End}_{\mathcal{V}}(M)$. Moreover, define the *central forms* of E as

$$C(E) := \{\varphi\in E\to\mathbb{C}\mid \varphi(xy)=\varphi(yx)\text{ for all }x,y\in E\}\ . \tag{3.33}$$

The pseudo-trace function ξ^{φ}_{M} on $\mathcal{V} \times \mathbb{H}$ is defined as

$$\xi^{\varphi}_{M}(a,\tau) \coloneqq t^{\varphi}_{M}\big(o(a)q^{L_0-c/24}\big) = \sum_{\lambda\in\mathbb{C}} t^{\varphi}_{M_\lambda}\big(o(a)q^{L_0-c/24}\big)\ , \tag{3.34}$$

where $t^{\varphi}_{\mathcal{G}} : \mathcal{G} \to \mathbb{C}$ is a Hattori-Stallings trace, and $c = -2\mathrm{N}$ is the central charge of $\mathcal{V}_{\mathrm{ev}}$. We refer to [**AN**] and [**GR2**, Sec. 4] for details (see also [**GR2**, Prop. 5.2] on how the above definition relates to [**AN**]). We only stress that if M is simple then $E \cong \mathbb{C}$ and the pseudo-trace function is just the usual trace function on M. Let G be a projective generator of $\mathbf{Rep}\,\mathcal{V}$. [**AN**] proved that for every M the assignment $\varphi \to \xi^{\varphi}_{M}(a,\tau)$ is a map $C(E) \to C_1(\mathcal{V})$. In [**GR2**] it is conjectured that for $M = G$ this map is bijective.

4. Hopf-algebras and coends

4.1. Reconstruction of a Hopf algebra

Given a Hopf algebra $\mathcal{A}$ over a field $\mathbb{k}$, let $\mathbf{coRep}_{\mathbb{k}}\mathcal{A}$ denote the category of finite-dimensional right corepresentations of a $\mathcal{A}$ over $\mathbb{k}$. We have the following well-known reconstruction theorem [**Ul**] (see also [**CP**, Thm. 5.1.11]):

Theorem 4.1. *Let $\mathcal{C}$ be a $\mathbb{k}$-linear abelian monoidal category with left duals and let $\mathcal{F} : \mathcal{C} \to \mathbf{vect}_{\mathbb{k}}$ be a fiber functor (i.e. a $\mathbb{k}$-linear exact faithful monoidal functor into the category of finite-dimensional $\mathbb{k}$-vector spaces). Then there exists a Hopf algebra $\mathcal{A}$ over $\mathbb{k}$ such that $\mathcal{F}$ factors monoidally as $\mathcal{C} \to \mathbf{coRep}_{\mathbb{k}}\mathcal{A} \xrightarrow{\text{forget}} \mathbf{vect}_{\mathbb{k}}$, where $\mathcal{C} \to \mathbf{coRep}_{\mathbb{k}}\mathcal{A}$ is a $\mathbb{k}$-linear equivalence of monoidal categories.*

This theorem can be formulated in a more general context [**Ma1**, Thm. 2.2], which we now review. Let $\mathcal{C}$ be a monoidal category with left duals, let $\mathcal{V}$ be a braided monoidal category with left duals, and let $\mathcal{F} : \mathcal{C} \to \mathcal{V}$ be a monoidal functor (so $\mathcal{V}$ is a replacement for $\mathbf{vect}_{\mathbb{k}}$ while $\mathcal{F}$ is a replacement for the fiber functor). In this situation there is a universal property which characterises a Hopf algebra $\mathcal{A}$ internal to $\mathcal{V}$ such that $\mathcal{F}$ factors monoidally as $\mathcal{C} \to \mathbf{coRep}_{\mathcal{V}}\mathcal{A} \xrightarrow{\text{forget}} \mathcal{V}$. Here, $\mathbf{coRep}_{\mathcal{V}}\mathcal{A}$ is the category of $\mathcal{A}$-comodules internal to $\mathcal{V}$. Existence of $\mathcal{A}$ is guaranteed under certain completeness conditions [**Ma1**], and we describe such a situation in Section 4.3.1 below.

If $\mathcal{C}$ is in addition braided, one has the natural choice $\mathcal{V} = \mathcal{C}$, $\mathcal{F} = \mathrm{id}$, and we obtain a universal property characterising a Hopf algebra in $\mathcal{C}$ which only depends on $\mathcal{C}$. We will refer to this Hopf algebra as the *universal Hopf algebra for $\mathcal{C}$*.

4.1.1. The universal Hopf algebra $\mathcal{A}$. Let $\mathcal{C}$ be a braided monoidal category with left duals. We now review the construction in [**Ma1**] in the special case $\mathcal{V} = \mathcal{C}$ and $\mathcal{F} = \mathrm{id}$. As an object in $\mathcal{C}$, $\mathcal{A}$ is defined to represent the functor $\mathcal{N}\colon \mathcal{C} \to \mathbf{Set}$ which on objects is given by

$$\mathcal{N} :\ V \mapsto \mathrm{Nat}(\mathrm{id}, \mathrm{id} \otimes V)\ . \tag{4.1}$$

Here, $\mathrm{id} \otimes V : \mathcal{C} \to \mathcal{C}$ is the functor that sends an object to its tensor product with V. By the overall assumption that all our categories are essentially small, the functor $\mathcal{N}$ does indeed land in $\mathbf{Set}$.

If a representing object exists, by definition it is equipped with a family of natural isomorphisms

$$\varphi_V : \quad \mathcal{C}(\mathcal{A}, V) \xrightarrow{\sim} \mathcal{N}(V) = \mathrm{Nat}(\mathrm{id}, \mathrm{id} \otimes V) \ , \tag{4.2}$$

and the pair $(\mathcal{A}, \varphi_V)$ is uniquely defined up to a unique isomorphism.

In particular, for $V = \mathcal{A}$ we have the natural transformation

$$\tilde{\imath} := \varphi_{\mathcal{A}}(\mathrm{id}) \ . \tag{4.3}$$

In terms of the morphisms $\tilde{\imath}_X : X \mapsto X \otimes \mathcal{A}$, which are natural in X, we define the Hopf algebra structure on $\mathcal{A}$ as follows.

(1) The *comultiplication* $\Delta_{\mathcal{A}} : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ is defined by

$$\varphi_{\mathcal{A}\otimes\mathcal{A}}(\Delta_{\mathcal{A}})_X \ = \ \big[\, X \xrightarrow{\tilde{\imath}_X} X\mathcal{A} \xrightarrow{\tilde{\imath}_X \otimes \mathrm{id}_{\mathcal{A}}} (X\mathcal{A})\mathcal{A} \xrightarrow{\sim} X(\mathcal{A}\mathcal{A}) \,\big] \ . \tag{4.4}$$

(2) The *counit* $\varepsilon_{\mathcal{A}} : \mathcal{A} \to \mathbf{1}$ is defined via the right unit isomorphism of $\mathcal{C}$,

$$\varphi_{\mathbf{1}}(\varepsilon_{\mathcal{A}})_X \ = \ \big[\, X \xrightarrow{\sim} X\mathbf{1} \,\big] \ . \tag{4.5}$$

In order to define the multiplication, we need to consider the functor $\mathcal{N}^2 : \mathcal{C} \to \mathbf{Set}$ acting on objects as

$$\mathcal{N}^2 : \ V \mapsto \mathrm{Nat}(- \otimes -, (- \otimes -) \otimes V) \ , \tag{4.6}$$

where the natural transformations are between two functors $\mathcal{C} \times \mathcal{C} \to \mathcal{C}$. $\mathcal{N}^2$ is a kind of square version of $\mathcal{N}$. It is shown in [**Ma1**, Lem. 2.3] that this functor is representable by $\mathcal{A} \otimes \mathcal{A}$. The corresponding natural isomorphisms

$$\varphi^2_V : \quad \mathcal{C}(\mathcal{A} \otimes \mathcal{A}, V) \xrightarrow{\sim} \mathcal{N}^2(V) \tag{4.7}$$

are given by

$$\begin{aligned} \varphi^2_V(f)_{X,Y} \ = \ \big[\, & XY \xrightarrow{\tilde{\imath}_X \otimes \tilde{\imath}_Y} (X\mathcal{A})(Y\mathcal{A}) \xrightarrow{\sim} X((\mathcal{A}Y)\mathcal{A}) \\ & \xrightarrow{\mathrm{id}\otimes c_{A,Y}\otimes \mathrm{id}} X((Y\mathcal{A})\mathcal{A}) \xrightarrow{\sim} X(Y(\mathcal{A}\mathcal{A})) \xrightarrow{\mathrm{id}\otimes f} X(YV) \xrightarrow{\sim} (XY)V \,\big] \ . \end{aligned} \tag{4.8}$$

(3) *The multiplication* $\mu_{\mathcal{A}} \in \mathcal{C}(\mathcal{A} \otimes \mathcal{A}, \mathcal{A})$ is defined by the equality

$$\varphi^2_{\mathcal{A}}(\mu_{\mathcal{A}})_{X,Y} \ = \ \big[\, XY \xrightarrow{\tilde{\imath}_{X\otimes Y}} (XY)\mathcal{A} \,\big] \ . \tag{4.9}$$

(4) *The unit* is given by $\eta_{\mathcal{A}} := \lambda_{\mathcal{A}} \circ \tilde{\imath}_{\mathbf{1}} : \mathbf{1} \to \mathcal{A}$, where $\lambda_{\mathcal{A}} : \mathbf{1} \otimes \mathcal{A} \to \mathcal{A}$ is the left-unit isomorphism of $\mathcal{C}$.

(5) *The antipode* $S_{\mathcal{A}} : \mathcal{A} \to \mathcal{A}$ is defined by the equality

$$\begin{aligned} \varphi_{\mathcal{A}}(S)_X \ = \ \big[\, & X \xrightarrow{\sim} \mathbf{1}X \xrightarrow{\mathrm{coev}_X \otimes \mathrm{id}} (XX^*)X \xrightarrow{\mathrm{id}\otimes \tilde{\imath}_{X^*} \otimes \mathrm{id}} (X(X^*\mathcal{A}))X \\ & \xrightarrow{\mathrm{id}\otimes c^{-1}_{\mathcal{A},X^*}\otimes \mathrm{id}} (X(\mathcal{A}X^*))X \xrightarrow{\sim} X(\mathcal{A}(X^*X)) \xrightarrow{\mathrm{id}\otimes \mathrm{ev}_X} X(\mathcal{A}\mathbf{1}) \xrightarrow{\sim} X\mathcal{A} \,\big] \ . \end{aligned} \tag{4.10}$$

The following theorem is shown in [**Ma1**, Sec. 2] (in the more general case of monoidal functors $\mathcal{F} : \mathcal{C} \to \mathcal{V}$).

Theorem 4.2 (Part 1)**.** *Let $\mathcal{A}$ be an object representing $\mathcal{N}$ in* (4.1)*. Then (1)-(5) endow $\mathcal{A}$ with the structure of a Hopf algebra in $\mathcal{C}$.*

The above construction also provides us with a functor $\mathcal{R} : \mathcal{C} \to \mathbf{coRep}_{\mathcal{C}}\mathcal{A}$. Namely, for each $X \in \mathcal{C}$ consider the morphism $\tilde{\imath}_X : X \to X \otimes \mathcal{A}$. It is immediate from the definition of $\mathcal{A}$ that this defines a right comodule structure on X. Naturality of $\tilde{\imath}$ implies that morphisms in $\mathcal{C}$ become comodule morphisms in $\mathbf{coRep}_{\mathcal{C}}\mathcal{A}$. Furthermore, it is straightforward to check that $\mathcal{R}$ is strictly monoidal (i.e. via the identity morphisms $\mathcal{R}(X) \otimes \mathcal{R}(Y) = \mathcal{R}(X \otimes Y)$, $\mathcal{R}(\mathbf{1}) = \mathbf{1}$). Altogether we see that the identity functor on $\mathcal{C}$ factors monoidally as

$$\mathrm{id}_{\mathcal{C}} \;=\; \big[\,\mathcal{C} \xrightarrow{\mathcal{R}} \mathbf{coRep}_{\mathcal{C}}\mathcal{A} \xrightarrow{\text{forget}} \mathcal{C}\,\big] \;. \tag{4.11}$$

(And this is indeed an equality, not just an equivalence.)

Theorem 4.2 (Part 2)**.** *The Hopf algebra $\mathcal{A}$ is universal in the sense that if $\mathcal{A}'$ is another Hopf algebra in $\mathcal{C}$ such that the identity functor factors monoidally as $\mathcal{C} \to \mathbf{coRep}_{\mathcal{C}}\mathcal{A}' \to \mathcal{C}$ as in* (4.11)*, then there exists a unique Hopf algebra map $\mathcal{A} \to \mathcal{A}'$ and thus the corresponding functor $\mathbf{coRep}_{\mathcal{C}}\mathcal{A} \to \mathbf{coRep}_{\mathcal{C}}\mathcal{A}'$ such that the functor $\mathcal{C} \to \mathbf{coRep}_{\mathcal{C}}\mathcal{A}'$ equals the composition $\mathcal{C} \to \mathbf{coRep}_{\mathcal{C}}\mathcal{A} \to \mathbf{coRep}_{\mathcal{C}}\mathcal{A}'$.*

4.1.2. Hopf pairing for $\mathcal{A}$.

The "inverse monodromy" natural isomorphisms $c^{-1}_{X,Y} \circ c^{-1}_{Y,X} : X \otimes Y \to X \otimes Y$ of $\mathcal{C}$ can be used to define a pairing on the universal Hopf algebra $\mathcal{A}$. Namely, define the morphism $\omega_{\mathcal{A}} : \mathcal{A} \otimes \mathcal{A} \to \mathbf{1}$ via

$$\varphi^2_{\mathbf{1}}(\omega_{\mathcal{A}})_{X,Y} \;=\; \big[\,XY \xrightarrow{c^{-1}_{Y,X}} YX \xrightarrow{c^{-1}_{X,Y}} XY \xrightarrow{\sim} (XY)\mathbf{1}\,\big] \;, \tag{4.12}$$

where the family $\varphi^2_{\mathbf{1}}(f)_{X,Y}$ was defined in (4.8). The following proposition is a corollary to Theorem 4.5 and Proposition 4.8 below, where we give an alternative description of the universal Hopf algebra $\mathcal{A}$ and of the pairing $\omega_{\mathcal{A}}$ in terms of coends.

Proposition 4.3. *Suppose that the universal Hopf algebra $\mathcal{A} \in \mathcal{C}$ exists. Then the pairing $\omega_{\mathcal{A}} : \mathcal{A} \otimes \mathcal{A} \to \mathbf{1}$ is a Hopf pairing for $\mathcal{A}$.*

We remark that if one were to use the monodromy isomorphism $c_{Y,X} \circ c_{X,Y}$ instead of the inverse monodromy in (4.12) one would not obtain a Hopf pairing in the sense of (2.63).

In particular, if the universal Hopf algebra exists it is automatically equipped with a Hopf algebra map

$$\mathcal{D}_{\mathcal{A}} :\; \mathcal{A} \to \mathcal{A}^* \;, \tag{4.13}$$

as defined in (2.66). We note that in case $\mathcal{C}$ is the category of representations of a Hopf algebra over a field, $\mathcal{D}_{\mathcal{A}}$ specialises to the Drinfeld map composed with an isomorphism to the double dual (see Remark 1.5 (2) below).

Later in this paper we will be particularly interested in situations where $\mathcal{D}_{\mathcal{A}}$ is invertible, or, equivalently, where $\omega_{\mathcal{A}}$ is non-degenerate (recall Definition 2.6).

In this section, we recall standard facts about coends in braided finite tensor categories and write explicitly the Hopf-algebra structure on a particular coend $\int^{U\in\mathcal{C}} U^* \otimes U$ in terms of the (braided monoidal) structure morphisms of the category.

4.2. The universal Hopf-algebra via coends

4.2.1. The coend $\mathcal{L}$. Let $\mathcal{C}$ be a braided monoidal category with left duals. Recall the definition of a coend in Section 1. We will now focus on the coend of the functor

$$F \;:=\; \left[\mathcal{C}^{\mathrm{op}} \times \mathcal{C} \xrightarrow{(-)^* \times \mathrm{id}_{\mathcal{C}}} \mathcal{C} \times \mathcal{C} \xrightarrow{\;\otimes\;} \mathcal{C}\right] , \tag{4.14}$$

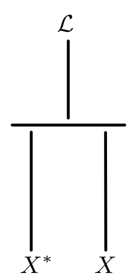
i.e. the functor which acts on objects and morphisms as $(U, V) \mapsto U^* \otimes V$ and $(f, g) \mapsto f^* \otimes g$. We denote this coend as

$$\mathcal{L} := \int^{U\in\mathcal{C}} U^* \otimes U \ , \tag{4.15}$$

and the corresponding family of dinatural transformations as

$$\iota_X : X^* \otimes X \longrightarrow \mathcal{L} \quad , \quad X \in \mathcal{C} \ . \tag{4.16}$$

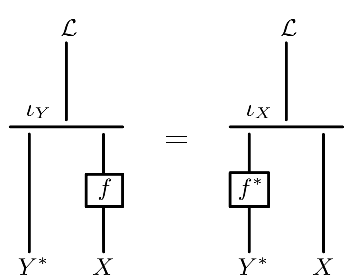
We will abbreviate the morphism ι_X in string diagrams as

(4.17) $\iota_X \;:=$ 

Dinaturality means here that for all $X, Y \in \mathcal{C}$ and $f \in \mathcal{C}(X, Y)$ we have

$$\iota_Y \circ (\mathrm{id}_{Y^*} \otimes f) = \iota_X \circ (f^* \otimes \mathrm{id}_X) \ . \tag{4.18}$$

In string diagram notation, this equality reads

(4.19) 

.

Remark 4.4. Consider the functor $F : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \times \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \longrightarrow \mathcal{C}$ which is defined on objects and morphisms as

$$F: \quad (U,V,X,Y) \mapsto U^* \otimes V \otimes X^* \otimes Y \ , \qquad (f,g,h,k) \mapsto f^* \otimes g \otimes h^* \otimes k \ . \tag{4.20}$$

It follows from the relation to iterated coends in Proposition 2.11 that the double coend is given by

$$\int^{(U,V)\in\mathcal{C}\times\mathcal{C}} F(U,U,V,V) \ = \ \mathcal{L}\otimes\mathcal{L} \ , \tag{4.21}$$

with dinatural family $\iota_{U,V} = \iota_U \otimes \iota_V$. As explained in Remark 2.12 (2), by the universal property of $\mathcal{L}\otimes\mathcal{L}$, any dinatural family $\phi_{U,V}$ from $U^*\otimes U\otimes V^*\otimes V$ (dinatural in U and V) to an object $B \in \mathcal{C}$ uniquely factors through a map $g : \mathcal{L}\otimes\mathcal{L} \to B$ as $\phi_{U,V} = g\circ(\iota_U\otimes\iota_V)$.

4.2.2. Hopf algebra structure and Hopf pairing on $\mathcal{L}$. Let again $\mathcal{C}$ be a braided monoidal category with left duals and assume that the coend $\mathcal{L}$ defined in (4.15) exists. Following [**Ly1**] we now use the universal properties of the coends $\mathcal{L}$ and $\mathcal{L}\otimes\mathcal{L}$ (as in Remark 2.12 (2) and Remark 4.4) to define the structure morphisms of a Hopf algebra on $\mathcal{L}$ and endow it with a Hopf pairing. For example, instead of giving the map $\Delta_{\mathcal{L}} : \mathcal{L} \to \mathcal{L}\otimes\mathcal{L}$ explicitly, we give the corresponding dinatural transformation $U^*\otimes U \to \mathcal{L}\otimes\mathcal{L}$, etc.

In giving the dinatural transformations defining the product, coproduct, etc. on $\mathcal{L}$, we will write out all coherence isomorphisms of $\mathcal{C}$ explicitly. This involves choices and we will use the precise form given below to derive explicit expressions for all structure morphisms in the case of representations of a quasi-triangular quasi-Hopf algebra in Section 5.2. Other combinations of associator and unit isomorphisms lead to the same structure morphisms, but the explicit formulas in terms of the data of the quasi-Hopf algebra would look differently.

Recall our Hopf-algebra conventions in Definition 2.5 and also the isomorphism $\gamma_{U,V}$ from (2.7). The structure morphisms on $\mathcal{L}$ are defined as follows.

(1) *(Multiplication)* $\mu_{\mathcal{L}} : \mathcal{L}\otimes\mathcal{L} \to \mathcal{L}$ is determined by the universal property of the double coend $\mathcal{L}\otimes\mathcal{L}$ via, for all $U,V \in \mathcal{C}$,

$$\begin{aligned} \mu_{\mathcal{L}}\circ(\iota_U\otimes\iota_V) \ = \ \Big[(U^*U)(V^*V) &\xrightarrow{\alpha^{-1}_{U^*,U,V^*V}} U^*(U(V^*V)) \\ &\xrightarrow{\mathrm{id}\otimes c_{U,V^*V}} U^*((V^*V)U) \xrightarrow{\mathrm{id}\otimes\alpha^{-1}_{V^*,V,U}} U^*(V^*(VU)) \\ &\xrightarrow{\alpha_{U^*,V^*,VU}} (U^*V^*)(VU) \xrightarrow{\gamma_{V,U}\otimes\mathrm{id}} (VU)^*(VU) \xrightarrow{\iota_{V\otimes U}} \mathcal{L} \Big] \ . \end{aligned} \tag{4.22}$$

(2) *(Unit)* $\eta_{\mathcal{L}} : \mathbf{1} \to \mathcal{L}$ is defined directly as

$$\begin{aligned} \eta_{\mathcal{L}} \ &= \ \Big[\mathbf{1} \xrightarrow{\lambda_{\mathbf{1}}^{-1}} \mathbf{1}\mathbf{1} \xrightarrow{(2.11)^{-1}\otimes\mathrm{id}} \mathbf{1}^*\mathbf{1} \xrightarrow{\iota_{\mathbf{1}}} \mathcal{L} \Big] \\ &\overset{(*)}{=} \ \Big[\mathbf{1} \xrightarrow{\mathrm{coev}_{\mathbf{1}}} \mathbf{1}\mathbf{1}^* \xrightarrow{\lambda_{\mathbf{1}^*}} \mathbf{1}^* \xrightarrow{\rho_{\mathbf{1}^*}^{-1}} \mathbf{1}^*\mathbf{1} \xrightarrow{\iota_{\mathbf{1}}} \mathcal{L} \Big] \ , \end{aligned} \tag{4.23}$$

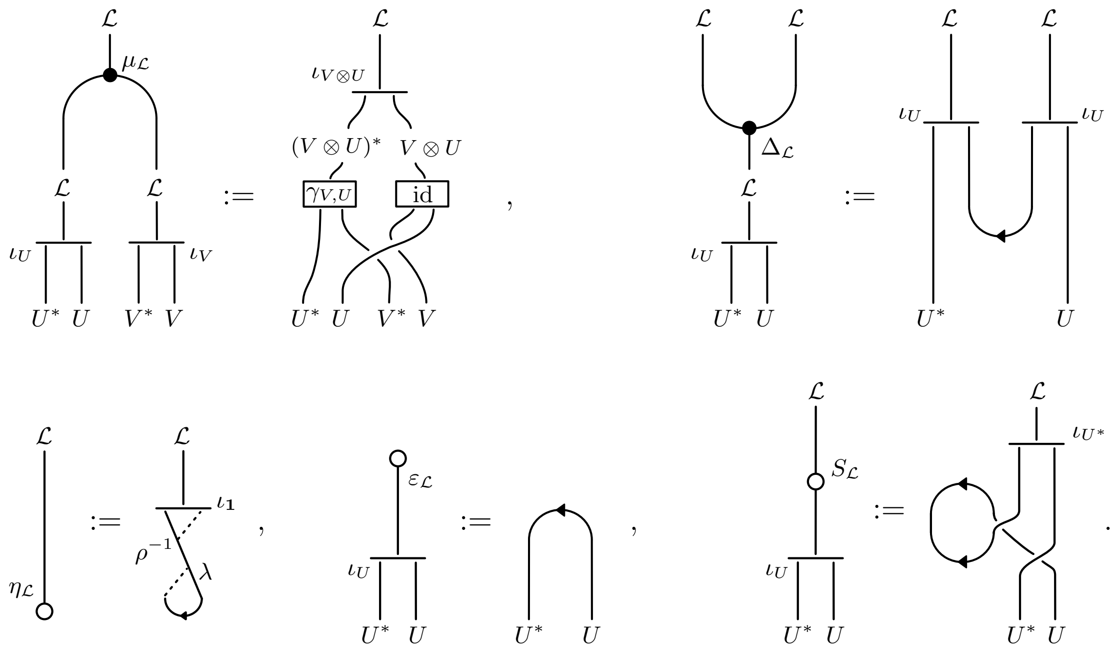


FIGURE 1. Hopf algebra structure on the coend $\mathcal{L} \in \mathcal{C}$. Here, $\gamma_{V,U}$ is the canonical isomorphism $U^* \otimes V^* \to (V \otimes U)^*$ defined by (2.7).

where (*) follows from the zig-zag identity for $\mathrm{ev}_{\mathbf{1}}$ and $\mathrm{coev}_{\mathbf{1}}$ and naturality of the unit-isomorphisms λ, ρ.

(3) *(Coproduct)* $\Delta_\mathcal{L} : \mathcal{L} \to \mathcal{L} \otimes \mathcal{L}$ is determined by the universal property of the coend $\mathcal{L}$ via, for all $U \in \mathcal{C}$,

$$\Delta_\mathcal{L} \circ \iota_U = \Big[U^*U \xrightarrow{\mathrm{id}\otimes\lambda_U^{-1}} U^*(\mathbf{1}U) \xrightarrow{\mathrm{id}\otimes\mathrm{coev}_U\otimes\mathrm{id}} U^*((UU^*)U) \xrightarrow{\mathrm{id}\otimes\alpha^{-1}_{U,U^*,U}} U^*(U(U^*U)) \xrightarrow{\alpha_{U^*,U,U^*U}} (U^*U)(U^*U) \xrightarrow{\iota_U\otimes\iota_U} \mathcal{L}\otimes\mathcal{L} \Big] . \tag{4.24}$$

(4) *(Counit)* $\varepsilon_\mathcal{L} : \mathcal{L} \to \mathbf{1}$ is determined by, for all $U \in \mathcal{C}$,

$$\varepsilon_\mathcal{L} \circ \iota_U \ , \ = \Big[U^*U \xrightarrow{\mathrm{ev}_U} \mathbf{1} \Big] . \tag{4.25}$$

(5) *(Antipode)* $S_\mathcal{L} : \mathcal{L} \to \mathcal{L}$ is determined by, for all $U \in \mathcal{C}$,

$$S_\mathcal{L} \circ \iota_U = \Big[U^*U \xrightarrow{c_{U^*,U}} UU^* \xrightarrow{\tilde{u}_U\otimes\mathrm{id}} U^{**}U^* \xrightarrow{\iota_{U^*}} \mathcal{L} \Big] . \tag{4.26}$$

These defining relations are given in string diagram notation in Figure 1.

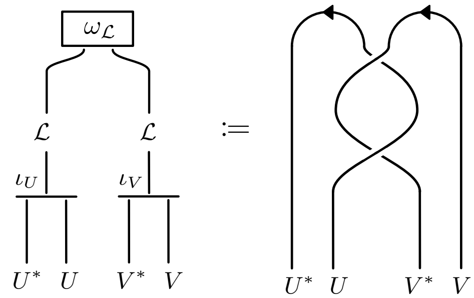
Finally, we define a pairing on $\mathcal{L}$ using the universal property of $\mathcal{L}\otimes\mathcal{L}$ via the condition that for all $U, V \in \mathcal{C}$,

$$\omega_\mathcal{L} \circ (\iota_U \otimes \iota_V) = \Big[(U^*U)(V^*V) \xrightarrow{\alpha^{-1}_{U^*,U,V^*V}} U^*(U(V^*V)) \xrightarrow{\mathrm{id}\otimes\alpha_{U,V^*,V}} U^*((UV^*)V) \tag{4.27}$$

$$\xrightarrow{\mathrm{id}\otimes(c_{V^*,U}\circ c_{U,V^*})\otimes\mathrm{id}} U^*((UV^*)V) \xrightarrow{\mathrm{id}\otimes\alpha^{-1}_{U,V^*,V}} U^*(U(V^*V))$$
$$\left.\xrightarrow{\mathrm{id}\otimes\mathrm{id}\otimes\mathrm{ev}_V} U^*(U\mathbf{1}) \xrightarrow{\mathrm{id}\otimes\rho_U} U^*U \xrightarrow{\mathrm{ev}_U} \mathbf{1}\right] .$$

In string diagram notation this reads

(4.28)



We gather all the structures so far defined on the coend $\mathcal{L}$ in the following theorem.

Theorem 4.5 ([**Ly1**])**.** *Let $\mathcal{C}$ be a braided monoidal category with left duals. If the coend $\mathcal{L}$ from* (4.15) *exists, the morphisms defined by* (4.22)–(4.26) *turn it into a Hopf algebra in $\mathcal{C}$. The pairing* (4.27) *is a Hopf pairing for $\mathcal{L}$.*

4.2.3. Relation between the universal Hopf algebra in $\mathcal{C}$ and the coend $\mathcal{L}$. Let $\mathcal{C}$ be a braided monoidal category with left duals. At this point we have introduced two Hopf algebras in $\mathcal{C}$ together with a Hopf pairing, subject to existence of solutions to certain universal properties: on the one hand, the universal Hopf algebra $\mathcal{A}$ from Theorem 4.2, and on the other hand the coend $\mathcal{L}$ from Theorem 4.5. In this section we show that $\mathcal{A}$ and $\mathcal{L}$ are canonically isomorphic as Hopf algebras with Hopf pairing.

To describe the isomorphism, we need to consider two functors $\mathcal{D}, \mathcal{N} \colon \mathcal{C} \to \mathbf{Set}$. Denote by $\mathrm{Din}(F, V)$ the set of dinatural transformations $j \colon F \xrightarrow{..} V$, where $F = (-)^* \otimes (-) : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{C}$ is the functor from (4.14). The first functor is $\mathcal{D} := \mathrm{Din}(F, -)$. The second functor is $\mathcal{N} = \mathrm{Nat}(\mathrm{id}, \mathrm{id} \otimes -)$ as already defined in (4.1). We have the following simple lemma.

Lemma 4.6. *The family of maps of sets $\zeta_V \colon \mathcal{D}(V) \to \mathcal{N}(V)$, $V \in \mathcal{C}$, given by, for $j \in \mathcal{D}(V)$,*

$$(\zeta_V(j))_X = \tilde{j}_X \;:=\; \left[\, X \xrightarrow{\sim} \mathbf{1}X \xrightarrow{\mathrm{coev}_X \otimes \mathrm{id}} (XX^*)X \xrightarrow{\sim} X(X^*X) \xrightarrow{\mathrm{id}\otimes j_X} XV \,\right] , \tag{4.29}$$

defines a natural isomorphism $\zeta : \mathcal{D} \to \mathcal{N}$.

Proof. Using the zig-zag property of the evaluation and coevaluation maps, one easily checks that the map

$$\zeta_V^{-1}\colon \quad \tilde{j}_X \;\mapsto\; \left[X^*X \xrightarrow{\mathrm{id}\otimes\tilde{j}_X} X^*(XV) \xrightarrow{\sim} (X^*X)V \xrightarrow{\mathrm{ev}_X \otimes \mathrm{id}} \mathbf{1}V \xrightarrow{\sim} V\right] \tag{4.30}$$

is the inverse to (4.29). The naturality of ζ follows from dinaturality of j. $\square$

Suppose the coend $(\mathcal{L}, (\iota_V)_{V\in\mathcal{C}})$ from (4.15) exists in $\mathcal{C}$. Recall from Remark 2.12 (2) that there is a natural isomorphism $\mathcal{C}(\mathcal{L}, V) \xrightarrow{\sim} \mathrm{Din}(F, V)$, given by $f \mapsto f \circ \iota$. In other words, the coend $\mathcal{L}$ represents the functor $\mathcal{D}$, while by definition the universal Hopf algebra $\mathcal{A}$ represents the functor $\mathcal{N}$, see Section 4.1.1. Therefore, we have following corollary to Lemma 4.6.

Corollary 4.7. *If $(\mathcal{L}, \iota)$ is a coend for the functor $F = (-)^* \otimes (-)$, then $(\mathcal{L}, \varphi)$ with*

$$\varphi_V : \mathcal{C}(\mathcal{L}, V) \to \mathrm{Nat}(\mathrm{id}, \mathrm{id} \otimes V) \quad , \quad (\varphi_V(f))_X = (\zeta_V(f \circ \iota))_X : X \to X \otimes V \tag{4.31}$$

represents $\mathcal{N}$, with ζ_V defined in (4.29).

Conversely, if $(\mathcal{A}, \varphi)$ represents $\mathcal{N}$, then $(\mathcal{A}, \iota_X = \zeta_{\mathcal{A}}^{-1}(\tilde{\iota}_X))$, with ζ^{-1} from (4.30) and $\tilde{\iota}$ from (4.3), is a coend for the functor F.

In particular, the coend $\mathcal{L}$ exists in $\mathcal{C}$ if and only if the representing object $\mathcal{A}$ exists.

Proposition 4.8. *Suppose the coend $(\mathcal{L}, \iota)$ exists, and denote by $(\mathcal{L}, \varphi)$ the corresponding representing object for $\mathcal{N}$ obtained from Corollary 4.7. The Hopf algebra structure morphisms and the Hopf pairing defined on $\mathcal{L}$ in Theorem 4.2 and Proposition 4.3 via the representing object property of $(\mathcal{L}, \varphi)$ are equal to those defined on $\mathcal{L}$ in Theorem 4.5 via the coend property of $(\mathcal{L}, \iota)$.*

The proof of this proposition is given in Appendix A.

In what follows we will mostly work with the description of the universal Hopf algebra as a coend, as this is the framework used in [**Ly1, Ly2**] to obtain mapping class group actions on certain Hom spaces, see Section 5.1.1 below.

4.3. Factorisable finite tensor categories

In this section we apply the construction of the universal Hopf algebra in the case of braided finite tensor categories (see Definition 2.18). It is known that for such categories, the universal Hopf algebra exists (see Section 4.3.1). The Hopf pairing appears as one of several equivalent ways of characterising factorisability of such a category (see Sections 4.3.2 and 4.3.3).

The finiteness condition implies the following useful representability property (see e.g. [**DSS**, Cor. 1.10]).

Lemma 4.9. *Let $\mathcal{A}$ be a finite $\Bbbk$-linear abelian category and let $\mathcal{F} : \mathcal{A} \to \mathbf{vect}_\Bbbk$ be a $\Bbbk$-linear left exact functor from $\mathcal{A}$ to finite-dimensional $\Bbbk$-vector spaces. Then $\mathcal{F}$ is representable, i.e. there is $A \in \mathcal{A}$ such that $\mathcal{A}(A, -)$ is naturally isomorphic to $\mathcal{F}$.*

This result follows from the more general observation that a $\Bbbk$-linear left exact functor between two finite $\Bbbk$-linear abelian categories admits a left adjoint (see e.g. [**DSS**, Cor. 1.9]). Indeed, in the case of the above lemma, if we denote the left adjoint of $\mathcal{F}$ by $\mathcal{G}$, we have $\mathcal{F}(X) \cong \mathrm{Hom}_\Bbbk(\Bbbk, \mathcal{F}(X)) \cong \mathcal{A}(\mathcal{G}(\Bbbk), X)$.

4.3.1. Existence of the universal Hopf algebra. Let $\Bbbk$ be a field and let $\mathcal{C}$ be a $\Bbbk$-linear braided finite tensor category. Using exactness of the tensor product, it is straightforward to verify that the functor $\mathcal{N} : \mathcal{C} \to \mathbf{vect}_\Bbbk$ from (4.1) is left exact: Indeed, let $f \in \mathcal{C}(V, W)$ be injective. Then

$$\mathcal{N}(f)\colon \mathrm{Nat}(\mathrm{id}, \mathrm{id} \otimes V) \to \mathrm{Nat}(\mathrm{id}, \mathrm{id} \otimes W) \quad , \quad \eta \mapsto ((\mathrm{id} \otimes f) \circ \eta)_X \tag{4.32}$$

must be injective, too.

Lemma 4.9 now implies [**Ma1, Ly1**]:

Proposition 4.10. *The universal Hopf algebra from Theorem 4.2 exists in* $\mathcal{C}$.

Since we will mostly use the coend perspective (Proposition 4.8), we will denote the universal Hopf algebra in $\mathcal{C}$ by $\mathcal{L}$ in what follows.

We can describe $\mathcal{L}$ more explicitly as a cokernel. Since $\mathcal{C}$ is finite, it contains a projective generator G. The coend can be written as a quotient of $G^* \otimes G$:

Proposition 4.11 ([**KL**, Cor. 5.1.8])**.** *We have the short exact sequence*

$$0 \longrightarrow K \longrightarrow G^* \otimes G \xrightarrow{\ \pi\ } \mathcal{L} \longrightarrow 0 \tag{4.33}$$

where K is the image of the map $\bigoplus_i (f_i^* \otimes \mathrm{id} - \mathrm{id} \otimes f_i)$: $\bigoplus_i G^* \otimes G \longrightarrow G^* \otimes G$ and the direct sum is taken over a basis $\{f_i \in \mathrm{End}_{\mathcal{C}}(G)\}$.

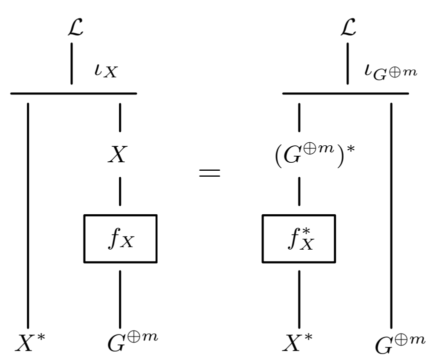
Note that K in the above proposition is independent of the choice of the basis in $\mathrm{End}_{\mathcal{C}}(G)$.

Remark 4.12. The coend $\mathcal{L}$ is equipped with the family of dinatural transformations ι_X : $X^* \otimes X \to \mathcal{L}$. The latter can be defined in terms of the surjective map π from (4.33) as follows: note first that the map π in (4.33) is by definition ι_G; we fix then a surjective map $f_X : G^{\oplus m} \to X$ (for some $m \geq 1$) and define ι_X by the equality of the (composition of the) maps

$$[X^* \otimes G^{\oplus m} \xrightarrow{\mathrm{id}_{X^*} \otimes f_X} X^* \otimes X \xrightarrow{\iota_X} \mathcal{L}] = [X^* \otimes G^{\oplus m} \xrightarrow{f_X^* \otimes \mathrm{id}} (G^{\oplus m})^* \otimes G^{\oplus m} \xrightarrow{\iota_{G^{\oplus m}}} \mathcal{L}] \tag{4.34}$$

or graphically

(4.35)



where $\iota_{G^{\oplus m}}$ is defined using (4.34) for $X = G$ and recall that $\iota_G = \pi$ is given to us.

We can describe $\mathcal{L}$ as a quotient of an even smaller projective object. Namely, the surjective map π in (4.33) factors through a 'diagonal' product of the projective covers in $\mathcal{C}$. Indeed, let

$$\mathrm{Irr}(\mathcal{C}) \tag{4.36}$$

be a choice of representatives of the isomorphism classes of simple objects in $\mathcal{C}$ and let us denote by P_U a choice of projective cover of $U \in \mathrm{Irr}(\mathcal{C})$.

Proposition 4.13. *We have the short exact sequence*

$$0 \longrightarrow W \longrightarrow \bigoplus_{U\in\mathrm{Irr}(\mathcal{C})} P_U^* \otimes P_U \xrightarrow{\tilde{\pi}} \mathcal{L} \longrightarrow 0 \tag{4.37}$$

where the object W (i.e. the kernel of $\tilde{\pi}$) is the subobject of $G^* \otimes G$ spanned by the images, for $U, V \in \mathrm{Irr}(\mathcal{C})$,

$$\mathrm{im}(f^* \otimes \mathrm{id}_{P_U}) \ , \ \mathrm{im}(\mathrm{id}_{P_V^*} \otimes f) \ , \quad \textit{for} \quad f : P_U \to P_V,\ U \neq V \ , \tag{4.38}$$

and

$$\mathrm{im}(f^* \otimes \mathrm{id}_{P_U} - \mathrm{id}_{P_{U^*}} \otimes f) \ , \quad \textit{for} \quad f : P_U \to P_U \ . \tag{4.39}$$

Proof. In the exact sequence (4.33), we can choose G as the minimal projective generator $G = \oplus_{U\in\mathrm{Irr}(\mathcal{C})} P_U$ in $\mathcal{C}$. Denote the (primitive) idempotents in $\mathrm{End}_{\mathcal{C}}(G)$ by $e_U : G \to P_U \to G$, for $U \in \mathrm{Irr}(\mathcal{C})$. The image of the map $e_U^* \otimes \mathrm{id} - \mathrm{id} \otimes e_U \in \mathrm{End}_{\mathcal{C}}(G^* \otimes G)$ equals $\bigoplus_{V\neq U} P_U^* \otimes P_V \oplus P_V^* \otimes P_U$. Therefore, the kernel K of π contains all $P_U^* \otimes P_V$ such that $V \ncong U$, and the map π factors through a map $\tilde{\pi}$ from the diagonal part of $G^* \otimes G$ to $\mathcal{L}$:

$$\pi : \quad G^* \otimes G \longrightarrow \bigoplus_{U\in\mathrm{Irr}(\mathcal{C})} P_U^* \otimes P_U \xrightarrow{\tilde{\pi}} \mathcal{L} \ . \tag{4.40}$$

We compute the kernel W of $\tilde{\pi}$ using the rest of the basis elements in $\mathrm{End}_{\mathcal{C}}(G)$, those in the radical, and it gives the span of (4.38) and (4.39). □

Remark 4.14. If $\mathcal{C}$ is semisimple, then by using (4.37) with $P_U = U$ we see that the coend $\mathcal{L}$ is the direct sum over (isomorphism classes of) simple objects:

$$\mathcal{L} = \bigoplus_{U\in\mathrm{Irr}(\mathcal{C})} U^* \otimes U \ . \tag{4.41}$$

The corresponding family of dinatural transformations can be easily described using the corresponding embeddings $i_U : U^* \otimes U \to \mathcal{L}$, for details see [**Ke2**, Lem. 2].

4.3.2. Factorisability of a braided finite tensor category. Let $\Bbbk$ be a field and let $\mathcal{C}$ be a $\Bbbk$-linear braided finite tensor category.

By Proposition 4.10, the universal Hopf algebra $\mathcal{L}$ exists in $\mathcal{C}$, and by Theorem 4.5 it is equipped with a Hopf pairing $\omega_{\mathcal{L}} : \mathcal{L} \otimes \mathcal{L} \to \mathbf{1}$. Recall from Definition 2.6 that $\omega_{\mathcal{L}}$ is called non-degenerate if the Hopf algebra homomorphism $\mathcal{D}_{\mathcal{L}} : \mathcal{L} \to \mathcal{L}^*$ in (2.66) is an isomorphism. Note that the kernel of $\mathcal{D}_{\mathcal{L}}$ is the left annihilator of $\omega_{\mathcal{L}}$.

Definition 4.15. $\mathcal{C}$ is called *factorisable* if the Hopf pairing $\omega_{\mathcal{L}}$ is non-degenerate.[1]

In particular, in a factorisable finite tensor category the universal Hopf algebra $\mathcal{L}$ is self-dual as a Hopf algebra.

Next we recall three more natural non-degeneracy conditions on the braiding of $\mathcal{C}$ and then quote a theorem from [**Sh2**] which states that for $\Bbbk$ algebraically closed, they are all equivalent to Definition 4.15. A fourth equivalent condition will be given later (Proposition 4.19). We will need the $\Bbbk$-linear map

$$\Omega : \mathcal{C}(\mathbf{1}, \mathcal{L}) \longrightarrow \mathcal{C}(\mathcal{L}, \mathbf{1}) \quad , \quad a \mapsto \omega_{\mathcal{L}} \circ (a \otimes \mathrm{id}) \circ \lambda_{\mathcal{L}}^{-1} \ . \tag{4.42}$$

The three conditions are:

(1) The linear map Ω from (4.42) is an isomorphism.
(2) Every transparent object in $\mathcal{C}$ is isomorphic to a direct sum of tensor units. ($T \in \mathcal{C}$ is *transparent* if for all $X \in \mathcal{C}$, $c_{X,T} \circ c_{T,X} = \mathrm{id}_{T \otimes X}$.)
(3) The canonical braided monoidal functor $\mathcal{C} \boxtimes \overline{\mathcal{C}} \to \mathcal{Z}(\mathcal{C})$ is an equivalence. (Here, $\boxtimes$ is the Deligne product, $\overline{\mathcal{C}}$ is the same tensor category as $\mathcal{C}$, but has inverse braiding, and $\mathcal{Z}(\mathcal{C})$ is the Drinfeld centre of $\mathcal{C}$.)

Theorem 4.16 ([**Sh2**])**.** *A braided finite tensor category over an algebraically closed field is factorisable if and only if any one of the conditions (1)–(3) is satisfied.*

Remark 4.17. In the case that $\mathcal{C}$ is semisimple, this equivalence was already known from [**Br, Mü**]. $\mathcal{C}$ is then a modular tensor category (minus the ribbon structure), and condition (1) above encodes the non-degeneracy of the S-matrix whose entries given by the quantum traces of the monodromy of pairs of simple objects (see [**Sh2**, Sec. 5.1] for details).

The following result is instrumental in the construction of the projective $SL(2,\mathbb{Z})$-action below.

[1]This definition is due to [**Ly1, KL**] where the name 'modular' is used. The usual definition of "modular tensor category" implies semisimplicity. But in view of fact that the qualifier "modular" is motivated by the projective action of the modular group, it would equally make good sense to speak of "modular fusion category" and of "modular finite tensor category". However, to avoid confusion we stick to the term "factorisable" in this paper, which is motivated from the application to Hopf algebras and quasi-Hopf algebras (see Section 1).

Proposition 4.18 ([**Ly1**] and [**KL**, Sect. 5.2.3])**.** *If $\mathcal{C}$ is factorisable, the coend $\mathcal{L}$ has a two-sided (that is, a simultaneous left and right) integral $\Lambda_{\mathcal{L}} : \mathbf{1} \to \mathcal{L}$ satisfying*

$$\omega_{\mathcal{L}} \circ (\Lambda_{\mathcal{L}} \otimes \Lambda_{\mathcal{L}}) \circ \lambda_{\mathbf{1}}^{-1} \;=\; k\,\mathrm{id}_{\mathbf{1}} \tag{4.43}$$

for some $k \in \Bbbk^{\times}$. If $\Lambda'_{\mathcal{L}}$ is another such integral, then $\Lambda'_{\mathcal{L}} = r\,\Lambda_{\mathcal{L}}$ for some $r \in \Bbbk^{\times}$.

If $\Bbbk$ has square roots we can normalise $\Lambda_{\mathcal{L}}$ in (4.43) such that $k = 1$. In this normalisation, $\Lambda_{\mathcal{L}}$ is unique up to a sign.

4.3.3. Factorisability as an isomorphism between end and coend. Let $\mathcal{C}$ be a finite braided tensor category over a field $\Bbbk$. Recall our functor $F = (-)^* \otimes (-) \colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{C}$ from (4.14) and that the coend $\mathcal{L}$ represents the functor $\mathcal{D} = \mathrm{Din}(F, -) \colon \mathcal{C} \to \mathbf{vect}_{\Bbbk}$. The coend $\mathcal{L}$ can be thought of as the dual notion to the *end* of the functor

$$G := (-) \otimes (-)^* : \quad \mathcal{C} \times \mathcal{C}^{\mathrm{op}} \to \mathcal{C} \tag{4.44}$$

(note the change of the order in the tensor product). The end of G is an object in $\mathcal{C}$ representing the functor $\mathrm{Din}(-, G) \colon \mathcal{C} \to \mathbf{vect}_{\Bbbk}$, see Definition 2.9 (2). We will denote such an object (if it exists) as Γ together with its family of dinatural transformations $j_X \colon \Gamma \to X \otimes X^*$. The dinaturality condition is now (compare with (4.18))

$$(\mathrm{id}_Y \otimes f^*) \circ j_Y = (f \otimes \mathrm{id}_{X^*}) \circ j_X \qquad \text{for all} \quad f \colon X \to Y \;. \tag{4.45}$$

An end Γ exists by similar arguments as for coends in Section 4.3.1. The existence also follows from Lemma 4.20 given below.

Consider a family of maps

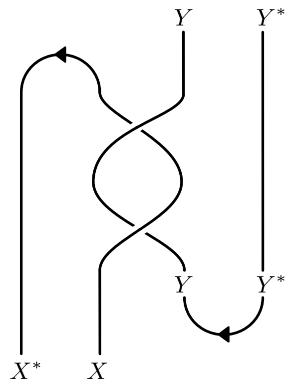
$$T_{X,Y} : \quad X^* \otimes X \to Y \otimes Y^* \qquad , \quad X, Y \in \mathcal{C} \;. \tag{4.46}$$

Suppose that for fixed Y the family $(t_X)_{X \in \mathcal{C}}$ with $t_X := T_{X,Y}$ is a dinatural transformation from F to $Y \otimes Y^*$, and that for fixed X the family $(t'_Y)_{Y \in \mathcal{C}}$ with $t'_Y := T_{X,Y}$ is a dinatural transformation from $X^* \otimes X$ to G (see Definition 2.9). Using the universal properties of ends and coends, one easily checks that there exists a unique $D \colon \mathcal{L} \to \Gamma$ such that $T_{X,Y}$ factors as $T_{X,Y} = j_Y \circ D \circ \iota_X$ for all $X, Y \in \mathcal{C}$.

Consider now a special ("Hopf tangle") dinatural transformation $\mathbb{T}_{X,Y}$ defined explicitly as

$$\begin{aligned} \mathbb{T}_{X,Y} \;:=\; \Big[X^*X \xrightarrow{\sim} X^*(X\mathbf{1}) &\xrightarrow{\mathrm{id}\otimes \mathrm{coev}_Y} X^*(X(YY^*)) \xrightarrow{\mathrm{id}\otimes\alpha_{X,Y,Y^*}} X^*((XY)Y^*) \\ &\xrightarrow{\mathrm{id}\otimes(c_{Y,X}\circ c_{X,Y})\otimes\mathrm{id}} X^*((XY)Y^*) \xrightarrow{\mathrm{id}\otimes\alpha^{-1}_{X,Y,Y^*}} X^*(X(YY^*)) \\ &\xrightarrow{\alpha_{X^*,X,YY^*}} (X^*X)(YY^*) \xrightarrow{\mathrm{ev}_X\otimes\mathrm{id}} \mathbf{1}(YY^*) \xrightarrow{\sim} YY^* \Big] \;. \end{aligned} \tag{4.47}$$

The corresponding diagram is

(4.48) $\mathbb{T}_{X,Y} =$  .

By the above discussion, this map factors through a unique map $\mathbb{D}_{\mathcal{L},\Gamma} : \mathcal{L} \to \Gamma$ such that

$$\mathbb{T}_{X,Y} = j_Y \circ \mathbb{D}_{\mathcal{L},\Gamma} \circ \iota_X \ . \tag{4.49}$$

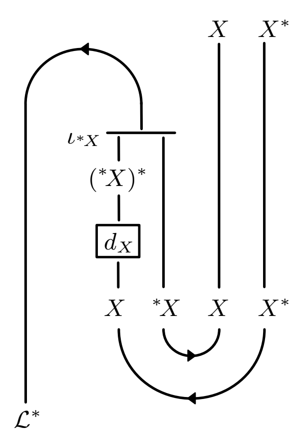
We call $\mathbb{D}_{\mathcal{L},\Gamma}$ the *Drinfeld map* for the category $\mathcal{C}$. The reason for this is two-fold. Firstly, in case $\mathcal{C}$ is the category of finite-dimensional representations of a finite-dimensional quasi-triangular Hopf algebra, and for an appropriate choice of end and coend, $\mathbb{D}_{\mathcal{L},\Gamma}$ is precisely the Drinfeld map, see Remark 5.15 below. Secondly, invertibility of the map $\mathbb{D}_{\mathcal{L},\Gamma}$ provides another equivalent formulation of factorisability of $\mathcal{C}$:

Proposition 4.19. *$\mathcal{C}$ is factorisable iff $\mathbb{D}_{\mathcal{L},\Gamma}$ is invertible.*

The proof requires the following lemma, which makes use of the canonical isomorphism $X \to ({}^*X)^*$ in a rigid category, which is given by

$$\begin{aligned} d_X \ = \ \big[\, X \xrightarrow{\sim} X\mathbf{1} \xrightarrow{\mathrm{id}\otimes \mathrm{coev}_{{}^*X}} X({}^*X({}^*X)^*) \\ \xrightarrow{\sim} (X\,{}^*X)({}^*X)^* \xrightarrow{\widetilde{\mathrm{ev}}_X\otimes \mathrm{id}} \mathbf{1}({}^*X)^* \xrightarrow{\sim} ({}^*X)^* \,\big] \ . \end{aligned} \tag{4.50}$$

Lemma 4.20. *The pair $(\mathcal{L}, \iota)$ is a coend of $F = (-)^* \otimes (-)$ iff the pair $(\mathcal{L}^*, \hat{\iota})$ with*

(4.51) $\hat{\iota}_X =$ 

is an end for the functor $G = (-) \otimes (-)^*$.

PROOF. The dinaturality condition (4.45) on $\hat{\iota}_X$ is easily verified from the diagram (4.51) using the dinaturality of ι and naturality of d_X. The universal property of $(\mathcal{L}^*, \hat{\iota})$ is proven using the universal property of $(\mathcal{L}, \iota)$. $\square$

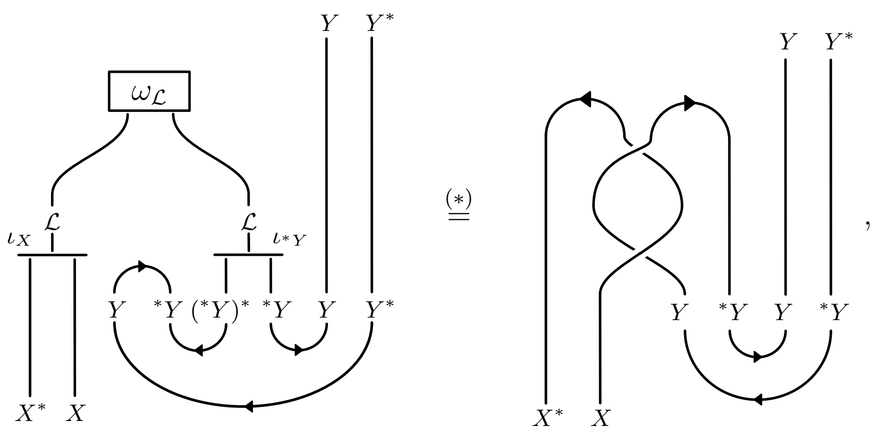
PROOF OF PROPOSITION 4.19. By definition, $\mathcal{C}$ is factorisable if the map $\mathcal{D}_{\mathcal{L}}$ defined in (2.66) is invertible (Definitions 4.15 and 2.6).

Consider the end $(\mathcal{L}^*, \hat{\iota})$ from Lemma 4.20. We will start by showing that $\mathbb{D}_{\mathcal{L},\mathcal{L}^*} = \mathcal{D}_{\mathcal{L}}$. By the unique factorisation property explained above, it is enough to show that for all $X, Y \in \mathcal{C}$ we have $\hat{\iota}_Y \circ \mathbb{D}_{\mathcal{L},\mathcal{L}^*} \circ \iota_X = \hat{\iota}_Y \circ \mathcal{D}_{\mathcal{L}} \circ \iota_X$. Using (4.49) we see that we need to show

$$\hat{\iota}_Y \circ \mathcal{D}_{\mathcal{L}} \circ \iota_X = \mathbb{T}_{X,Y} \tag{4.52}$$

with $\mathbb{T}_{X,Y}$ as in (4.47). Substituting (2.66), (4.28), (4.51) and (4.50) into the left hand side of (4.52) and using the zig-zag identity for the duality maps on $\mathcal{L}$ once gives

(4.53)



where in (*) the zig-zag identity for duality morphisms was used once again. Using the zig-zag identity once more, we find the right hand side of (4.52).

For an end (Γ, j), there is a unique isomorphism $\phi : (\mathcal{L}^*, \hat{\iota}) \xrightarrow{\sim} (\Gamma, j)$ such that the diagram

$$\begin{tikzcd} & X \otimes X^* & \\ \mathcal{L}^* \arrow[ur, "\hat{\iota}_X"] \arrow[rr, "\phi"] & & \Gamma \arrow[ul, "j_X"'] \end{tikzcd} \tag{4.54}$$

commutes. Then we have $\mathbb{D}_{\mathcal{L},\Gamma} = \phi \circ \mathbb{D}_{\mathcal{L},\mathcal{L}^*}$ because by definition we have

$$j_Y \circ \mathbb{D}_{\mathcal{L},\Gamma} \circ \iota_X = \mathbb{T}_{X,Y} = \hat{\iota}_Y \circ \mathbb{D}_{\mathcal{L},\mathcal{L}^*} \circ \iota_X = j_Y \circ \phi \circ \mathbb{D}_{\mathcal{L},\mathcal{L}^*} \circ \iota_X \ . \tag{4.55}$$

Therefore, $\mathbb{D}_{\mathcal{L},\Gamma}$ is invertible iff $\mathcal{D}_{\mathcal{L}}$ is and so iff the Hopf pairing $\omega_{\mathcal{L}}$ is non-degenerate. $\square$

5. $SL(2,\mathbb{Z})$-action for ribbon quasi-Hopf algebras

5.1. $SL(2,\mathbb{Z})$-action for factorisable finite tensor categories

For factorisable finite ribbon categories $\mathcal{C}$ with universal Hopf algebra $\mathcal{L}$, one can define a projective $SL(2,\mathbb{Z})$-action on $\mathcal{C}(\mathbf{1},\mathcal{L})$ (Section 5.1.1), and on the $\Bbbk$-vector space $\mathrm{End}(\mathrm{id}_\mathcal{C})$. The second action is independent of the choice of $\mathcal{L}$ (Proposition 5.3). In Section 5.1.2 we discuss different ways to transport internal characters from $\mathcal{C}(\mathbf{1},\mathcal{L})$ to $\mathrm{End}(\mathrm{id}_\mathcal{C})$.

5.1.1. Projective $SL(2,\mathbb{Z})$-action. Let $\Bbbk$ be a field and let $\mathcal{C}$ be a factorisable finite tensor category over $\Bbbk$. We assume that (4.43) has a solution with $k = 1$ (e.g. if $\Bbbk$ is algebraically closed). We furthermore assume that $\mathcal{C}$ is ribbon, and we will denote the ribbon twist on $V \in \mathcal{C}$ by $\theta_V \colon V \to V$.

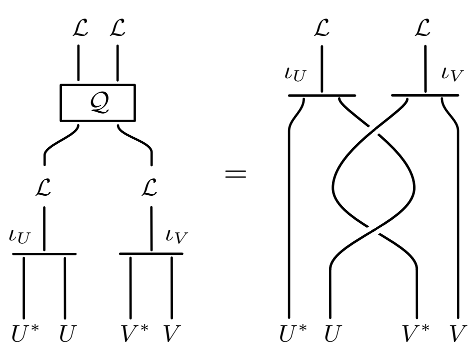
In this section we review from [**Ly1**] the projective $SL(2,\mathbb{Z})$-action on the Hom-space $\mathcal{C}(1,\mathcal{L})$ and on the vector space $\mathrm{End}(\mathrm{id}_\mathcal{C})$, the natural endomorphisms of the identity functor.

We start by introducing the monodromy morphism $\mathcal{Q}\colon \mathcal{L}\otimes\mathcal{L}\to\mathcal{L}\otimes\mathcal{L}$ as

$$
\begin{aligned}
(5.1)\qquad \mathcal{Q}\circ(\iota_U\otimes\iota_V) \;=\; \Big[&(U^*U)(V^*V) \xrightarrow{\alpha^{-1}_{U^*,U,V^*V}} U^*(U(V^*V)) \xrightarrow{\mathrm{id}\otimes\alpha_{U,V^*,V}} U^*((UV^*)V) \\
&\xrightarrow{\mathrm{id}\otimes(c_{V^*,U}\circ c_{U,V^*})\otimes\mathrm{id}} U^*((UV^*)V) \xrightarrow{\mathrm{id}\otimes\alpha^{-1}_{U,V^*,V}} U^*(U(V^*V)) \\
&\xrightarrow{\alpha_{U^*,U,V^*V}} (U^*U)(V^*V) \xrightarrow{\iota_U\otimes\iota_V} \mathcal{L}\otimes\mathcal{L}\Big]\,,
\end{aligned}
$$

or, in string diagram notation,

(5.2)



The Hopf pairing in (4.27) is related to $\mathcal{Q}$ as

$$
(5.3)\qquad \omega_\mathcal{L} \;=\; \big[\,\mathcal{L}\mathcal{L} \xrightarrow{\mathcal{Q}} \mathcal{L}\mathcal{L} \xrightarrow{\varepsilon_\mathcal{L}\otimes\varepsilon_\mathcal{L}} \mathbf{1}\mathbf{1} \xrightarrow{\sim} \mathbf{1}\,\big]\,.
$$

Following [**Ly1**], we introduce two endomorphisms $\mathcal{S}, \mathcal{T} : \mathcal{L} \to \mathcal{L}$, which will then be used to define the action of the S- and T-generator of $SL(2,\mathbb{Z})$. We take the integral given to us by Proposition 4.18 to be normalised such that

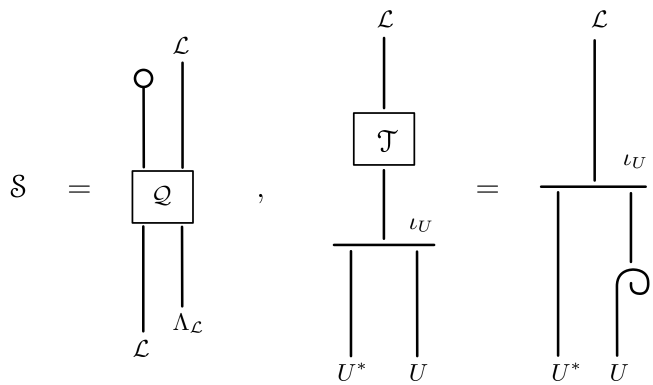
$$\omega_{\mathcal{L}} \circ (\Lambda_{\mathcal{L}} \otimes \Lambda_{\mathcal{L}}) \circ \lambda_{\mathbf{1}}^{-1} \;=\; \mathrm{id}_{\mathbf{1}} \tag{5.4}$$

(as is possible by our assumptions on $\Bbbk$). Recall that this fixes $\Lambda_{\mathcal{L}}$ up to a sign. We set

$$\mathcal{S} = \lambda_{\mathcal{L}} \circ (\varepsilon_{\mathcal{L}} \otimes \mathrm{id}) \circ \mathcal{Q} \circ (\mathrm{id} \otimes \Lambda_{\mathcal{L}}) \circ \rho_{\mathcal{L}}^{-1} \;, \quad \mathcal{T} \circ \iota_U = \iota_U \circ (\mathrm{id} \otimes \theta_U) \;\text{ for all }\; U \in \mathcal{C} \;. \tag{5.5}$$

In terms of string diagrams we have

(5.6)



Theorem 5.1 ([**Ly1**]). *The endomorphisms* $\mathcal{S}, \mathcal{T}$ *from* (5.5) *satisfy*

$$(\mathcal{S}\mathcal{T})^3 = \lambda\, \mathcal{S}^2 \;, \qquad \mathcal{S}^2 = S_{\mathcal{L}}^{-1} \;, \qquad , \tag{5.7}$$

for some constant $\lambda \in \Bbbk^*$.

It is not hard to verify directly from the definition in (4.26) that the antipode of $\mathcal{L}$ squares to the ribbon twist,

$$S_{\mathcal{L}} \circ S_{\mathcal{L}} \;=\; \theta_{\mathcal{L}} \;. \tag{5.8}$$

Thus, the relations in Theorem 5.1 also imply $\mathcal{S}^4 = \theta_{\mathcal{L}}^{-1}$.

By the S- and T-generators of $SL(2,\mathbb{Z})$ we mean the $2{\times}2$ matrices $\mathbf{S} = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ and $\mathbf{T} = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$, respectively. One can describe $SL(2,\mathbb{Z})$ as the group freely generated by $\mathbf{S}$ and $\mathbf{T}$ subject to the relations

$$(\mathbf{S}\mathbf{T})^3 = \mathbf{S}^2 \quad , \quad \mathbf{S}^4 = \mathrm{id} \;. \tag{5.9}$$

Therefore, as an immediate consequence of Theorem 5.1 we have:

Corollary 5.2. *The* $\Bbbk$*-vector space* $\mathcal{C}(\mathbf{1}, \mathcal{L})$ *carries a projective action of* $SL(2,\mathbb{Z})$ *where the action of* $\mathbf{S}$ *and* $\mathbf{T}$ *is given by, for* $f \in \mathcal{C}(\mathbf{1}, \mathcal{L})$,

$$\mathbf{S}.f := \mathcal{S} \circ f \quad , \quad \mathbf{T}.f := \mathcal{T} \circ f \;. \tag{5.10}$$

Proof. The first relation in (5.9) is just the first relation in (5.7) (up to the projectivity factor). The second relation in (5.9) follows from (5.7) and (5.8), together with naturality of the ribbon twist: $\mathbf{S}^4.f = \mathcal{S}^4 \circ f = \theta_{\mathcal{L}}^{-1} \circ f = f \circ \theta_{\mathbf{1}}^{-1} = f$. $\square$

The universal Hopf algebra $\mathcal{L}$ is only unique up to unique isomorphism, and in the above projective representation of $SL(2,\mathbb{Z})$ already the underlying vector space depends on $\mathcal{L}$. It can be helpful to have a variant of the action which is manifestly independent of the choice of $\mathcal{L}$ (up to the choice of the sign of the integral). This can be achieved by transporting the action to $\mathrm{End}(\mathrm{id}_\mathcal{C})$, as we explain next (see [**Ly1, Sh1**]).

We start by defining two $\Bbbk$-linear isomorphisms

$$\rho : \mathcal{C}(\mathcal{L},\mathbf{1}) \to \mathcal{C}(\mathbf{1},\mathcal{L}) \quad , \quad \psi : \mathrm{End}(\mathrm{id}_\mathcal{C}) \to \mathcal{C}(\mathcal{L},\mathbf{1}) \ . \tag{5.11}$$

Their values on $f \in \mathcal{C}(\mathcal{L},\mathbf{1})$ and $\alpha \in \mathrm{End}(\mathrm{id}_\mathcal{C})$ are determined by

$$\begin{aligned} \rho(f) \ &= \ \big[\,\mathbf{1} \xrightarrow{\Lambda_\mathcal{L}} \mathcal{L} \xrightarrow{\Delta_\mathcal{L}} \mathcal{L}\mathcal{L} \xrightarrow{f\otimes \mathrm{id}} \mathbf{1}\mathcal{L} \xrightarrow{\sim} \mathcal{L}\,\big] \ , \\ \psi(\alpha)\circ \iota_X \ &= \ \big[\,X^*X \xrightarrow{\mathrm{id}\otimes\alpha_X} X^*X \xrightarrow{\mathrm{ev}_X} \mathbf{1}\,\big] \quad \text{for all } X \in \mathcal{C} \ . \end{aligned} \tag{5.12}$$

The inverses of ρ and ψ can be given explicitly. For $a \in \mathcal{C}(\mathbf{1},\mathcal{L})$ and $f \in \mathcal{C}(\mathcal{L},\mathbf{1})$ we have

$$\begin{aligned} \rho^{-1}(a) \ &= \ \big[\,\mathcal{L} \xrightarrow{\sim} \mathbf{1}\mathcal{L} \xrightarrow{a\otimes\mathrm{id}} \mathcal{L}\mathcal{L} \xrightarrow{S_\mathcal{L}\otimes\mathrm{id}} \mathcal{L}\mathcal{L} \xrightarrow{\mu_\mathcal{L}} \mathcal{L} \xrightarrow{\Lambda^{\mathrm{co}}_\mathcal{L}} \mathbf{1}\,\big] \ , \\ \psi^{-1}(f)_X \ &= \ \big[\,X \xrightarrow{\sim} \mathbf{1}X \xrightarrow{\mathrm{coev}_X\otimes\mathrm{id}} (XX^*)X \xrightarrow{\sim} X(X^*X) \xrightarrow{\mathrm{id}\otimes\iota_X} X\mathcal{L} \xrightarrow{\mathrm{id}\otimes f} X\mathbf{1} \xrightarrow{\sim} X\,\big] \ , \end{aligned} \tag{5.13}$$

where the second line applies again for all $X \in \mathcal{C}$. In terms of the isomorphisms ψ and ρ, together with the isomorphism Ω from (4.42) and the ribbon twist θ, we define $\mathcal{S}_\mathcal{C}, \mathcal{T}_\mathcal{C} \in \mathrm{End}_k(\mathrm{End}(\mathrm{id}_\mathcal{C}))$ as

$$\begin{aligned} \mathcal{S}_\mathcal{C} \ &= \ \big[\ \mathrm{End}(\mathrm{id}_\mathcal{C}) \xrightarrow{\psi} \mathcal{C}(\mathcal{L},\mathbf{1}) \xrightarrow{\rho} \mathcal{C}(\mathbf{1},\mathcal{L}) \xrightarrow{\Omega} \mathcal{C}(\mathcal{L},\mathbf{1}) \xrightarrow{\psi^{-1}} \mathrm{End}(\mathrm{id}_\mathcal{C})\,\big] \ , \\ \mathcal{T}_\mathcal{C} \ &= \ \big[\ \mathrm{End}(\mathrm{id}_\mathcal{C}) \xrightarrow{\theta\circ(-)} \mathrm{End}(\mathrm{id}_\mathcal{C})\,\big] \ . \end{aligned} \tag{5.14}$$

We collect the results reviewed in this section in the following proposition.

Proposition 5.3. *The $\Bbbk$-vector space $\mathrm{End}(\mathrm{id}_\mathcal{C})$ carries a projective action of $SL(2,\mathbb{Z})$ where the actions of* $\mathbf{S}$ *and* $\mathbf{T}$ *are given by, for $\alpha \in \mathrm{End}(\mathrm{id}_\mathcal{C})$,*

$$\mathbf{S}.\alpha := \mathcal{S}_\mathcal{C}(\alpha) \quad , \quad \mathbf{T}.\alpha := \mathcal{T}_\mathcal{C}(\alpha) \ . \tag{5.15}$$

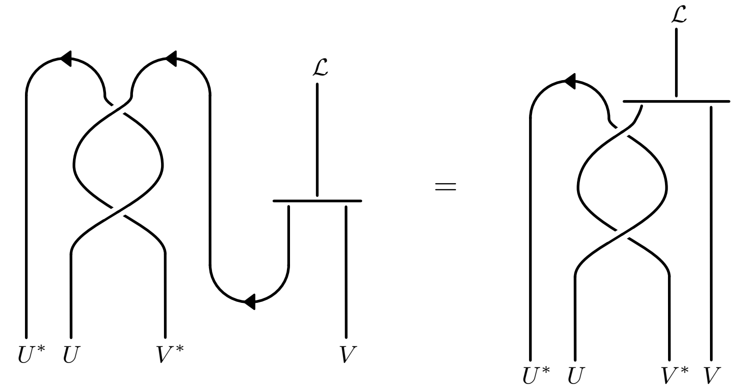
Moreover, $\mathcal{S}_\mathcal{C}$ and $\mathcal{T}_\mathcal{C}$ in (5.14) are independent of $(\mathcal{L},\Lambda_\mathcal{L})$ up to the choice of sign of $\Lambda_\mathcal{L}$.

Proof. We first show the identity

$$\big[\,\mathcal{L}\mathcal{L} \xrightarrow{\mathrm{id}\otimes\Delta_\mathcal{L}} \mathcal{L}(\mathcal{L}\mathcal{L}) \xrightarrow{\sim} (\mathcal{L}\mathcal{L})\mathcal{L} \xrightarrow{\omega_\mathcal{L}\otimes\mathrm{id}} \mathbf{1}\mathcal{L} \xrightarrow{\sim} \mathcal{L}\,\big] \ = \ \big[\,\mathcal{L}\mathcal{L} \xrightarrow{\mathcal{Q}} \mathcal{L}\mathcal{L} \xrightarrow{\varepsilon_\mathcal{L}\otimes\mathrm{id}} \mathbf{1}\mathcal{L} \xrightarrow{\sim} \mathcal{L}\,\big] \ . \tag{5.16}$$

To establish this equality we verify that it holds when precomposed with $\iota_U \otimes \iota_V$ for all $U,V \in \mathcal{C}$. Indeed, substituting the defining relations in (4.24), (4.27), (5.1) and (4.25) one finds that (5.16) is equivalent to the following identity, which we give in terms of string diagrams

(5.17)



This identity clearly holds by the zig-zag identity for duality morphisms. Thus also (5.16) holds.

Precomposing (5.16) with $\mathrm{id} \otimes \Lambda_{\mathcal{L}}$ and comparing to the definition of Ω and ρ in (4.42) and (5.12), as well as to the definition of $\mathcal{S}$ in (5.5), shows that for all $f \in \mathcal{C}(\mathbf{1}, \mathcal{L})$,

$$\rho(\Omega(f)) \;=\; \mathcal{S} \circ f\ . \tag{5.18}$$

It is now straightforward to check that conjugating the action of $\mathbf{S}$ and $\mathbf{T}$ in (5.10) with the $\Bbbk$-linear isomorphism $\rho \circ \psi$ gives the action in (5.15).

Next we show the independence of $\mathcal{S}_{\mathcal{C}}$ and $\mathcal{T}_{\mathcal{C}}$ from the choice of universal Hopf algebra. For $\mathcal{T}_{\mathcal{C}}$ there is nothing to do. For $\mathcal{S}_{\mathcal{C}}$, let $(\mathcal{L}', \iota')$ be another choice of coend, and let $h : \mathcal{L} \to \mathcal{L}'$ be the unique isomorphism satisfying $h \circ \iota_X = \iota'_X$ for all $X \in \mathcal{C}$. Then using the defining relations in Figure 1 for the Hopf algebra structure and the Hopf pairing (4.28) but written for the coend $\mathcal{L}'$ and equating maps corresponding to the same dinatural transformations, we find the relations

$$\begin{aligned} \mu_{\mathcal{L}} &= h^{-1} \circ \mu_{\mathcal{L}'} \circ (h \otimes h)\ , & \Delta_{\mathcal{L}} &= (h^{-1} \otimes h^{-1}) \circ \Delta_{\mathcal{L}'} \circ h\ , \\ \omega_{\mathcal{L}} &= \omega_{\mathcal{L}'} \circ (h \otimes h)\ , & \varepsilon_{\mathcal{L}} &= \varepsilon_{\mathcal{L}'} \circ h\ . \end{aligned} \tag{5.19}$$

From this and Proposition 4.18 we get the relation $\Lambda_{\mathcal{L}'} = \pm\, h \circ \Lambda_{\mathcal{L}}$ between the normalised integrals. Denote by Ω', ρ', ψ' the maps in (4.42) and (5.12), but computed for $\mathcal{L}'$. Using the relations between $\mathcal{L}$ and $\mathcal{L}'$ just stated, one easily verifies that, for $x \in \mathcal{C}(\mathbf{1}, \mathcal{L}')$, $f \in \mathcal{C}(\mathcal{L}', \mathbf{1})$, $\alpha \in \mathrm{End}(\mathrm{id}_{\mathcal{C}})$,

$$\begin{aligned} \Omega'(x) &= \Omega(h^{-1} \circ x) \circ h^{-1}\ , & \psi'(\alpha) &= \psi(\alpha) \circ h^{-1}\ , \\ \rho'(f) &= h \circ \rho(f \circ h)\ , & (\psi')^{-1}(f) &= \psi^{-1}(f \circ h)\ . \end{aligned} \tag{5.20}$$

Substituting this into the definition of $\mathcal{S}_{\mathcal{C}}$ in (5.14) one arrives at $\mathcal{S}'_{\mathcal{C}} = \mathcal{S}_{\mathcal{C}}$. □

5.1.2. Internal characters and corresponding natural endomorphisms. Let $\Bbbk$ be an algebraically closed field and let $\mathcal{C}$ be a $\Bbbk$-linear factorisable and pivotal finite tensor category.

For the comparison to the conformal field theory calculation of the $SL(2,\mathbb{Z})$-action from [**GR2**] in the companion paper [**FGR2**] and for the explicit form of the Verlinde formula in Section 5.2.5 below, we will need to recall the definition and some properties of internal characters.

The *internal character* of $M \in \mathcal{C}$ is the element $\chi_M \in \mathcal{C}(\mathbf{1},\mathcal{L})$ given by [**FSS, Sh1**]

$$\chi_M \ = \ \big[\,\mathbf{1} \xrightarrow{\widetilde{\mathrm{coev}}_M} M^* \otimes M \xrightarrow{\iota_M} \mathcal{L}\,\big] \ , \tag{5.21}$$

where we use the convention in [**GR2**].

Denote by $\mathrm{Gr}(\mathcal{C})$ the Grothendieck ring of $\mathcal{C}$, and recall the definition of $\mathrm{Irr}(\mathcal{C})$ from (4.36). As $\mathcal{C}$ is finite, $\mathrm{Gr}(\mathcal{C})$ is the free $\mathbb{Z}$-linear span of $[U]$, $U \in \mathrm{Irr}(\mathcal{C})$. We will abbreviate $\mathrm{Gr}_{\Bbbk}(\mathcal{C}) := \Bbbk \otimes_{\mathbb{Z}} \mathrm{Gr}(\mathcal{C})$ for the $\Bbbk$-linearised Grothendieck ring. The following theorem was proved in [**Sh1**, Cor. 4.2] under more general assumptions (in particular for non-braided $\mathcal{C}$).

Theorem 5.4. *The assignment $M \mapsto \chi_M$ induces a $\Bbbk$-linear map $\chi : \mathrm{Gr}_{\Bbbk}(\mathcal{C}) \to \mathcal{C}(\mathbf{1}, L)$. The map χ is injective.*

We remark that if in addition $\Bbbk$ is of characteristic zero, then also the composition $\mathrm{Gr}(\mathcal{C}) \to \mathrm{Gr}_{\Bbbk}(\mathcal{C}) \xrightarrow{\chi} \mathcal{C}(\mathbf{1}, L)$ is injective.

Next we use the maps ρ^{-1} and ψ^{-1} from (5.13) to transport χ_M to $\mathrm{End}(\mathrm{id}_{\mathcal{C}})$,

$$\phi_M \ := \ \psi^{-1}(\rho^{-1}(\chi_M)) \ . \tag{5.22}$$

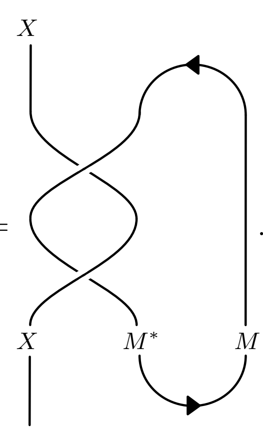
From the proof of Proposition 5.3 and from Theorem 5.4 we conclude:

Corollary 5.5. *The ϕ_M only depend on the class $[M]$ of M in $\mathrm{Gr}(\mathcal{C})$ and are independent of the choice of coend $\mathcal{L}$. The set $\{\phi_U \,|\, U \in \mathrm{Irr}(\mathcal{C})\} \subset \mathrm{End}(\mathrm{id}_{\mathcal{C}})$ is $\Bbbk$-linearly independent.*

Proof. Using the notation from the proof of Proposition 5.3, it remains to note that in addition to (5.20) we have $\chi'_M = h \circ \chi_M$ and $(\rho')^{-1}(x) = \rho^{-1}(h^{-1} \circ x) \circ h^{-1}$. $\square$

After applying $\mathcal{S}_{\mathcal{C}}$ to ϕ_M, the expression simplifies to a "Hopf link operator" as considered in [**CG**], see [**GR2**, Rem. 3.10],

(5.23) $\mathcal{S}_{\mathcal{C}}(\phi_M)_X \ =$



.

In terms of formulas, this reads

$$\begin{aligned}(5.24)\qquad \mathcal{S}_{\mathcal{C}}(\phi_M)_X \;=\; \big[X &\xrightarrow{\rho_X^{-1}} X\mathbf{1} \xrightarrow{\mathrm{id}\otimes\widetilde{\mathrm{coev}}_M} X(M^*M) \xrightarrow{\alpha_{X,M^*,M}} (XM^*)M \\ &\xrightarrow{(c_{M^*,X}\circ c_{X,M^*})\otimes\mathrm{id}} (XM^*)M \xrightarrow{\alpha^{-1}_{X,M^*,M}} X(M^*M) \xrightarrow{\mathrm{id}\otimes\mathrm{ev}_M} X\mathbf{1} \xrightarrow{\rho_X} X\big]\ .\end{aligned}$$

Combining Theorem 5.4 and [**Sh1**, Thm. 3.11 & Prop. 3.14] gives (see also [**CG**] and [**GR2**, Thm. 3.9]):

Theorem 5.6. *The assignment* $[M] \mapsto \mathcal{S}_{\mathcal{C}}(\phi_M)$ *is an injective* $\Bbbk$*-algebra homomorphism* $\mathrm{Gr}_{\Bbbk}(\mathcal{C}) \to \mathrm{End}(\mathrm{id}_{\mathcal{C}})$*.*

This theorem implies the categorical Verlinde formula stated in [**GR2**, Thm. 3.9] (see [**Tu**, Thm. 4.5.2] for the semisimple case, i.e. the case of modular tensor categories)

$$(5.25)\qquad \mathcal{S}_{\mathcal{C}}^{-1}\big(\mathcal{S}_{\mathcal{C}}(\phi_U)\circ\mathcal{S}_{\mathcal{C}}(\phi_V)\big) \;=\; \sum_W N_{UV}^{\;W}\,\phi_W\ .$$

If $\Bbbk$ has characteristic zero, it allows one to compute the structure constants $N_{UV}^{\;W}$ of $\mathrm{Gr}(\mathcal{C})$.

In case that $\mathcal{C}$ is ribbon, a short calculation with string diagrams shows that the antipode of $\mathcal{L}$ acts on internal characters as $S_{\mathcal{L}}\circ\chi_M = \chi_{M^*}$. Combining this with $\mathcal{S}^2 = S_{\mathcal{L}}^{-1}$ from Theorem 5.1 and transporting everything to $\mathrm{End}(\mathrm{id}_{\mathcal{C}})$ gives:

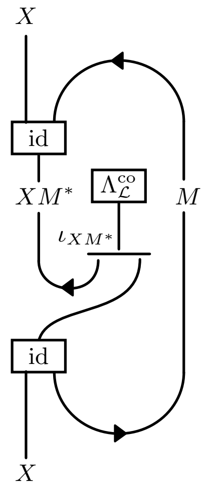
Lemma 5.7. *Let* $\mathcal{C}$ *be in addition ribbon. Then* $\mathcal{S}_{\mathcal{C}}(\mathcal{S}_{\mathcal{C}}(\phi_M)) = \phi_{M^*}$*.*

In this sense, we can think of $\mathcal{S}_{\mathcal{C}}^2$ as implementing "charge conjugation" on the internal characters (and on their images in $\mathrm{End}(\mathrm{id}_{\mathcal{C}})$).

For $\mathcal{C}$ pivotal (but not necessarily ribbon), after a short calculation with string diagrams one can find the following expression for the ϕ_M, which is a bit lengthy as we write out associators for later use:

$$\begin{aligned}(5.26)\qquad (\phi_M)_X \;=\; \big[X &\xrightarrow{\sim} \mathbf{1}(X\mathbf{1}) \xrightarrow{\mathrm{coev}_{XM^*}\otimes\mathrm{id}\otimes\widetilde{\mathrm{coev}}_M} \{(XM^*)(XM^*)^*\}\,\big(X(M^*M)\big) \\ &\xrightarrow{\mathrm{id}\otimes\alpha_{X,M^*,M}} \{(XM^*)(XM^*)^*\}\,\big((XM^*)M\big) \\ &\xrightarrow{\alpha^{-1}_{XM^*,(XM^*)^*,(XM^*)M}} (XM^*)\,\big\{(XM^*)^*\big((XM^*)M\big)\big\} \\ &\xrightarrow{\mathrm{id}\otimes\alpha_{(XM^*)^*,XM^*,M}} (XM^*)\,\big\{\big((XM^*)^*(XM^*)\big)M\big\} \\ &\xrightarrow{\mathrm{id}\otimes(\Lambda_{\mathcal{L}}^{\mathrm{co}}\circ\iota_{XM^*})\otimes\mathrm{id}} (XM^*)\,\{\mathbf{1}M\} \xrightarrow{\sim} (XM^*)M \\ &\xrightarrow{\alpha^{-1}_{X,M^*,M}} X(M^*M) \xrightarrow{\mathrm{id}\otimes\mathrm{ev}_M} X\mathbf{1} \xrightarrow{\sim} X\,\big]\ ,\end{aligned}$$

or graphically

(5.27) $(\phi_M)_X \;=$  .

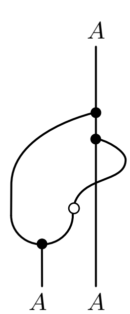
For a more detailed study of the properties of ϕ_M, see [**GR3**].

5.2. Coends for quasi-triangular quasi-Hopf algebras

We begin this section by describing explicitly the coend object $\mathcal{L}$ in **Rep** A for a quasi-triangular quasi-Hopf algebra A. Then we give explicit expressions for its Hopf-algebra structure morphisms and Hopf pairing in terms of the defining data of A.

All string diagrams in this section are taken in $\mathbf{vect}_{\Bbbk}$, and we will drop the label $\boxed{\mathbf{vect}_{\Bbbk}}$ from the diagrams. Recall that this means that duality maps in string diagrams refer to those of $\mathbf{vect}_{\Bbbk}$, not those of **Rep** A.

5.2.1. The coend $\mathcal{L}$ in Rep A. To describe the coend, we will need to discuss the coadjoint representation of A, as well as an equivalent way of writing it. The *adjoint representation* $\rho_A^{\mathrm{adj}} : A \otimes A \to A$ of A on itself is given by

(5.28) $\rho_A^{\mathrm{adj}} \;=$  .

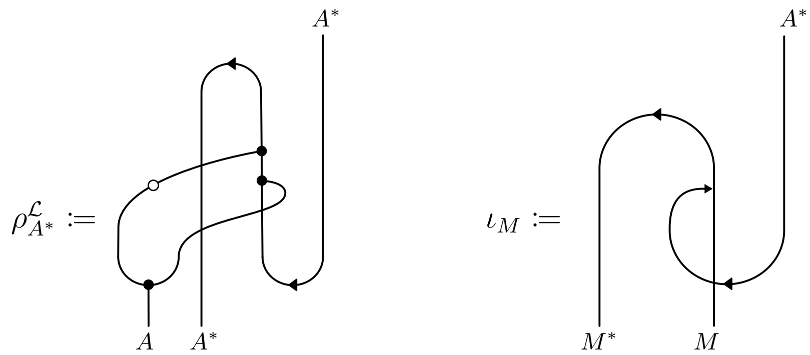
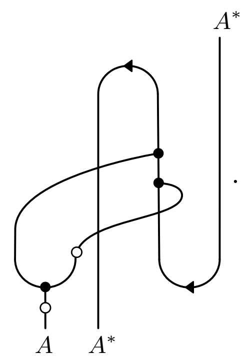


FIGURE 1. The coend object $\mathcal{L} = A^*$ for a quasi-Hopf algebra A. The diagrams here are in $\mathbf{vect}_{\Bbbk}$, i.e. the braidings are the usual flips of the vector spaces, etc.

By definition, the dual $\rho^{\mathrm{adj}^*}_{A^*} : A \otimes A^* \to A^*$ of the adjoint representation is given by

(5.29) $$\rho^{\mathrm{adj}^*}_{A^*} \;=$$  .

We will show below that, as for Hopf algebras (see [**Ly2, Ke2**]), the coend $\mathcal{L}$ in $\mathbf{Rep}\,A$ can be taken to be the dual of the adjoint representation. However, again as for Hopf algebras [**Ke2**], we find it convenient to work with the action $\rho^{\mathcal{L}}_{A^*}$ on A^* from Figure 1, which we refer to as the *coadjoint representation*.

The dual of the adjoint representation and the coadjoint representation are isomorphic. Indeed, define the map $E : A \to A$ as

(5.30) $$E(a) \;=\; \sum_{(\boldsymbol{f})} S^{-1}\big(\boldsymbol{f}'\, a\, S(\boldsymbol{f}'')\big)$$

where $\boldsymbol{f}$ is defined in (1.15). It is straightforward to verify from invertibility of $\boldsymbol{f}$ that E is invertible, and from (1.17) that

(5.31) $$E^* \;:\; (A^*, \rho^{\mathrm{adj}^*}_{A^*}) \longrightarrow (A^*, \rho^{\mathcal{L}}_{A^*})$$

is an isomorphism of A-modules.

We are now ready to show that $(A^*, \rho^{\mathcal{L}}_{A^*})$ can serve as the coend in $\mathbf{Rep}\,A$.

Proposition 5.8. *Let A be a finite-dimensional quasi-Hopf algebra over a field $\Bbbk$.*

(1) *The coend object* (4.15) *in* $\mathbf{Rep}\,A$ *can be chosen to be the coadjoint representation* $\mathcal{L} = (A^*, \rho^{\mathcal{L}}_{A^*})$*, together with the dinatural family* $\iota \equiv (\iota_M\colon M^* \otimes M \to A^*)$ *given by (see Figure 1)*

$$\iota_M : \quad \varphi \otimes m \ \mapsto \ \big(a \mapsto \varphi(a.m)\big) \ , \qquad m \in M,\ \varphi \in M^*,\ a \in A \ .$$

(2) *The unique morphism* $g : \mathcal{L} \to B$ *from* (2.71) *is, for a dinatural transformation* (B, ϕ),

$$g = \Big[\mathcal{L} = A^* \xrightarrow{\sim} A^* \otimes \mathbf{1} \xrightarrow{\mathrm{id}\otimes\eta_A} A^* \otimes A \xrightarrow{\phi_A} B\Big] \ ,$$

where η_A *is the unit morphism of* A *and* ϕ_A *is evaluated on the regular representation.*

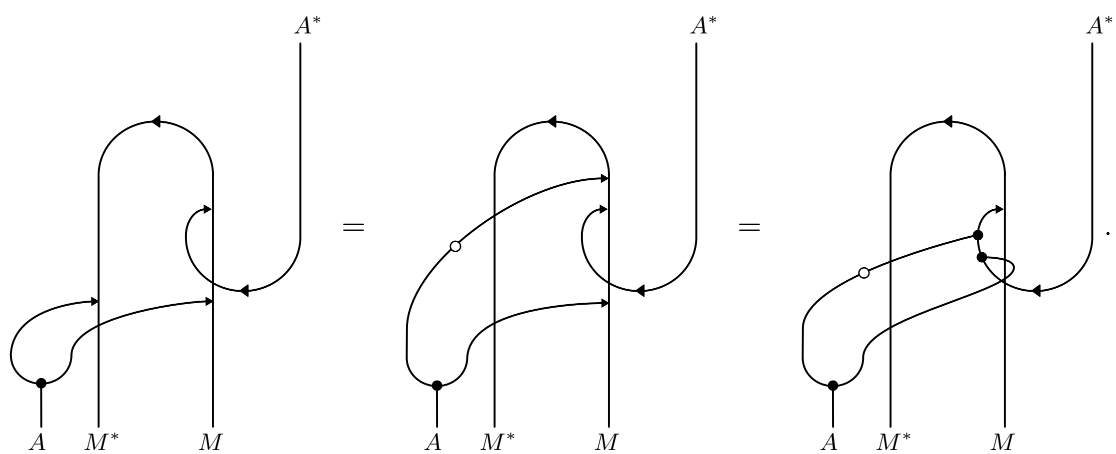
In the case that A is a Hopf algebra, this proposition was proven in [**Ly2**, Sec. 3.3] and [**Ke2**, Lem. 3]. The proof for the quasi-Hopf case is very similar and we reproduce it here for completeness.

Proof of Proposition 5.8. We show that $(\mathcal{L}, \iota)$ satisfies the universal property of the coend. As part of the argument, we show that g is as stated in part (2).

• ι_M *is an* A*-module intertwiner:* We use the identity

$$\iota_M(a\,.\,(\varphi \otimes m)) = \big(h \mapsto \sum_{(a)} \varphi\big(S(a')ha''.m\big)\big) \quad , \quad h \in A \ . \tag{5.32}$$

Indeed, we have

(5.33) 

Then the intertwiner property follows from $\sum_{(a)} \varphi(S(a')(-)a''.m) = \rho^{\mathcal{L}}_{A^*}(a \otimes \varphi(-\,.m)) = \rho^{\mathcal{L}}_{A^*}(a \otimes \iota_M(\varphi \otimes m))$. Pictorially, it can be seen easily by adding a zig-zag at the last string of the last diagram in (5.33) and moving the coadjoint action of a on the other end.

• ι_M *is dinatural:* We need to show that for all A-module maps $f : M \to N$ we have $\iota_N \circ (\mathrm{id} \otimes f) = \iota_M \circ (f^* \otimes \mathrm{id})$ as maps $N^* \otimes M \to A^*$. Evaluating on $\varphi \in N^*$, $m \in M$ and $a \in A$ gives

$$\begin{aligned}\big[\iota_N \circ (\mathrm{id} \otimes f)(\varphi \otimes m)\big](a) &= \varphi(a.f(m)) = \varphi(f(a.m)) = (f^*(\varphi))(a.m) \\ &= \big[\iota_M \circ (f^* \otimes \mathrm{id})(\varphi \otimes m)\big](a) \ .\end{aligned} \tag{5.34}$$

• *g from part (2) is an A-module map:* For $x \in A$ denote by $L_x, R_x : A \to A$ the left and right multiplication by x: $L_x(a) = xa$ and $R_x(a) = ax$. Note that R_x is an A-module intertwiner of A seen as a left module over itself. Since ϕ is dinatural, we have

$$\phi_A \circ (\mathrm{id} \otimes R_x) = \phi_A \circ ((R_x)^* \otimes \mathrm{id}) \ . \tag{5.35}$$

The coadjoint action of $a \in A$ is $\rho^{\mathcal{L}}_{A^*}(a \otimes -) = \big(\sum_{(a)} L_{S(a')} \circ R_{a''}\big)^* : A^* \to A^*$, see Figure 1. We compute, for $\varphi \in A^*$,

$$\begin{aligned} g(a.\varphi) &= \sum_{(a)} g(R^*_{a''} \circ L^*_{S(a')}(\varphi)) \overset{\text{def.}\, g}{=} \sum_{(a)} \phi_A\big((R^*_{a''} \circ L^*_{S(a')}(\varphi)) \otimes 1_A\big) \\ &\overset{(5.35)}{=} \sum_{(a)} \phi_A\big((L^*_{S(a')}(\varphi)) \otimes R_{a''}(1_A)\big) = \sum_{(a)} \phi_A\big((L^*_{S(a')} \otimes L_{a''})(\varphi \otimes 1_A)\big) \\ &\overset{(*)}{=} a.\phi_A\big(\varphi \otimes 1_A\big) \overset{\text{def.}\, g}{=} a.g(\varphi) \ . \end{aligned} \tag{5.36}$$

Here, (*) amounts to the observation that $\sum_{(a)} L^*_{S(a')} \otimes L_{a''}$ gives the action of a on $A^* \otimes A$, where A is the left regular module and A^* is the corresponding dual module, together with the fact that ϕ_A is an A-module intertwiner.

• *g makes* (2.71) *commute:* We need to show that for every A-module M, $g \circ \iota_M = \phi_M$. Consider the map $R_m : A \to M$, $R_m(a) = a.m$. This is an A-module intertwiner, for A the left regular module over itself. We compute, for $\varphi \in M^*$ and $m \in M$,

$$\begin{aligned} g \circ \iota_M(\varphi \otimes m) &= \phi_A\big(\iota_M(\varphi \otimes m) \otimes 1_A\big) = \phi_A\big(\varphi(-.m) \otimes 1_A\big) \\ &= \phi_A\big((R_m)^*(\varphi) \otimes 1_A\big) \overset{(*)}{=} \phi_M\big(\varphi \otimes (R_m(1_A))\big) = \phi_M\big(\varphi \otimes m\big) \ , \end{aligned} \tag{5.37}$$

where (*) is dinaturality of ϕ.

• *g is unique:* Let $h : \mathcal{L} \to B$ be an A-module map satisfying $h \circ \iota_M = \phi_M$ for all A-modules M. We need to show that $h = g$. But this is immediate if one chooses $M = A$. Indeed, for all $\varphi \in A^*$ we have $\iota_A(\varphi \otimes 1_A) = \varphi$ and so

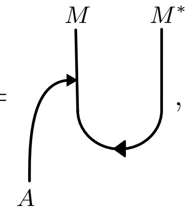
$$h(\varphi) = h \circ \iota_A(\varphi \otimes 1_A) = \phi_A(\varphi \otimes 1_A) = g(\varphi) \ , \tag{5.38}$$

by definition of g. □

We make a similar statement for an end Γ, recall the discussion in Section 4.3.3. This statement is proven in [**Sa**, Lem. 5.4] and can also be verified by arguments analogous to those in Proposition 5.8.

Proposition 5.9. *Let A be a finite-dimensional quasi-Hopf algebra over a field $\Bbbk$. The end object Γ in* **Rep** *A can be chosen to be the adjoint representation $\Gamma = (A, \rho^{\mathrm{adj}}_A)$, together with* *the dinatural family* $j \equiv (j_M \colon A \to M \otimes M^*)$ *given by*

$$j_M : \quad a \;\mapsto\; \sum_i (a.m_i) \otimes m_i^* \;= \tag{5.39}$$



where m_i *is a basis in* M *and* m_i^* *is the dual basis.*

5.2.2. Hopf structure and Hopf pairing on $\mathcal{L}$. In this section we present explicit expressions for the structure maps (4.22)–(4.26) and (4.27) which define a Hopf structure and a Hopf pairing on the universal Hopf algebra $\mathcal{L} = A^*$ in $\mathbf{Rep}\, A$.

To work with elements rather than with functionals, we dualise all structure morphisms. We will use the notation $\langle -,- \rangle$ for the contraction of an element in A^* with an element in A, and of an element in $A^* \otimes A^*$ with an element in $A \otimes A$:

$$\begin{aligned} &\langle -,- \rangle : A^* \otimes A \to \Bbbk \ , && \langle \varphi, a \rangle = \varphi(a) \ , \\ &\langle -,- \rangle : A^* \otimes A^* \otimes A \otimes A \to \Bbbk \ , && \langle \varphi \otimes \psi, a \otimes b \rangle = \varphi(b)\psi(a) \ . \end{aligned} \tag{5.40}$$

Note the order of arguments in the second contraction. The convention is such that $\langle -,- \rangle = \tilde\gamma^{\mathbf{vect}_\Bbbk}_{A,A}$, i.e. (2.25) applied to the category of vector spaces. In terms of this bracket notation, we define $\hat\mu_{\mathcal{L}} : A \to A \otimes A$, etc., as follows. For all $a, b \in A$ and $f, g \in A^*$,

$$\begin{aligned} \big\langle \mu_{\mathcal{L}}(f \otimes g)\,,\, a \big\rangle &= \big\langle f \otimes g\,,\, \hat\mu_{\mathcal{L}}(a) \big\rangle && , \quad \hat\mu_{\mathcal{L}} : A \to A \otimes A \ , \\ \big\langle \Delta_{\mathcal{L}}(f)\,,\, a \otimes b \big\rangle &= \big\langle f\,,\, \hat\Delta_{\mathcal{L}}(a \otimes b) \big\rangle && , \quad \hat\Delta_{\mathcal{L}} : A \otimes A \to A \ , \\ \eta_{\mathcal{L}}(1) &= \big(a \mapsto \hat\eta_{\mathcal{L}}(a) \big) && , \quad \hat\eta_{\mathcal{L}} : A \to \Bbbk \ , \\ \varepsilon_{\mathcal{L}}(f) &= f(\hat\varepsilon_{\mathcal{L}}) && , \quad \hat\varepsilon_{\mathcal{L}} \in A \ , \\ \big\langle S_{\mathcal{L}}(f)\,,\, a \big\rangle &= \big\langle f\,,\, \hat S_{\mathcal{L}}(a) \big\rangle && , \quad \hat S_{\mathcal{L}} : A \to A \ , \\ \omega_{\mathcal{L}}(f \otimes g) &= \big\langle f \otimes g\,,\, \hat\omega_{\mathcal{L}} \big\rangle && , \quad \hat\omega_{\mathcal{L}} \in A \otimes A \ . \end{aligned} \tag{5.41}$$

Theorem 5.10. *Let A be a finite-dimensional quasi-triangular quasi-Hopf algebra over a field $\Bbbk$. The Hopf algebra structure and Hopf pairing from Theorem 4.5 applied to the universal Hopf algebra in* $\mathbf{Rep}\, A$ *from Proposition 5.8 are given by the maps in* (5.41) *with*

$$\begin{aligned} \hat\mu_{\mathcal{L}}(a) &= \sum_{(\Phi),(\Psi),(\tilde\Psi),(R)} \Big[S(\Phi_2 \Psi_1 R_2' \tilde\Psi_3') \otimes S(\Phi_1 \tilde\Psi_1) \Big] \cdot \boldsymbol{f} \\ &\qquad\qquad \cdot \Delta(a \Phi_3) \cdot \Big[(\Psi_2 R_2'' \tilde\Psi_3'') \otimes (\Psi_3 R_1 \tilde\Psi_2) \Big] \ , \\ \hat\Delta_{\mathcal{L}}(a \otimes b) &= \sum_{(D)} S(D_1) b D_2 S(D_3) a D_4 \ , \\ \hat\eta_{\mathcal{L}}(a) &= \varepsilon(\boldsymbol{\beta}\, a) \ , \\ \hat\varepsilon_{\mathcal{L}} &= \boldsymbol{\alpha} \ , \end{aligned} \tag{5.42}$$

$$\begin{aligned}\hat{S}_{\mathcal{L}}(a) &= \sum_{(R)} S(aR_1)\tilde{\boldsymbol{u}}R_2 \ , \\ \hat{\omega}_{\mathcal{L}} &= \sum_{(W)} S(W_3)W_4 \otimes S(W_1)W_2 \ .\end{aligned}$$

In these expressions, $\Psi = \Phi^{-1}$, $\tilde{\Psi}$ *is another copy of* Φ^{-1}, $\boldsymbol{f}$ *is the Drinfeld twist from* (1.15), $\tilde{\boldsymbol{u}}$ *was given in* (2.50), *the elements* $D, W \in A^{\otimes 4}$ *are defined as*

$$\begin{aligned}(5.43)\qquad D &= (\mathrm{id}\otimes\mathrm{id}\otimes\Delta)(\Phi)\cdot(\mathbf{1}\otimes\Phi^{-1})\cdot(\mathbf{1}\otimes\boldsymbol{\beta}\otimes\mathbf{1}\otimes\mathbf{1}) \ , \\ W &= (\mathbf{1}\otimes\boldsymbol{\alpha}\otimes\mathbf{1}\otimes\boldsymbol{\alpha})\cdot(\mathbf{1}\otimes\Phi^{-1})\cdot(\mathbf{1}\otimes M\otimes\mathbf{1})\cdot(\mathbf{1}\otimes\Phi)\cdot(\mathrm{id}\otimes\mathrm{id}\otimes\Delta)(\Phi^{-1}) \ ,\end{aligned}$$

and M *was defined in* (1.12).

Proof. Let $M, N \in \mathbf{Rep}\,A$ and $m \in M$, $n \in N$, $\varphi \in M^*$, $\psi \in N^*$. By (4.22) the multiplication on $\mathcal{L}$ is determined by the equality $X = Y$ with

$$\begin{aligned}(5.44)\quad X &= \mu_{\mathcal{L}}\circ(\iota_M\otimes\iota_N)(\varphi\otimes m\otimes\psi\otimes n) \ , \\ Y &= \iota_{N\otimes M}\circ(\gamma_{N,M}\otimes\mathrm{id})\Big((\mathrm{id}\otimes\mathrm{id}\otimes\Delta)(\Phi)\cdot(\mathbf{1}\otimes\Phi^{-1}) \\ &\qquad .\,(\mathrm{id}\otimes\tau_{M,N^*N})\Big\{\big(\mathbf{1}\otimes(\mathrm{id}\otimes\Delta)(R)\big)\cdot(\mathrm{id}\otimes\mathrm{id}\otimes\Delta)(\Phi^{-1})\,.\,\varphi\otimes m\otimes\psi\otimes n\Big\}\Big) \ .\end{aligned}$$

By abbreviating $\Psi, \tilde{\Psi} = \Phi^{-1}$ we get

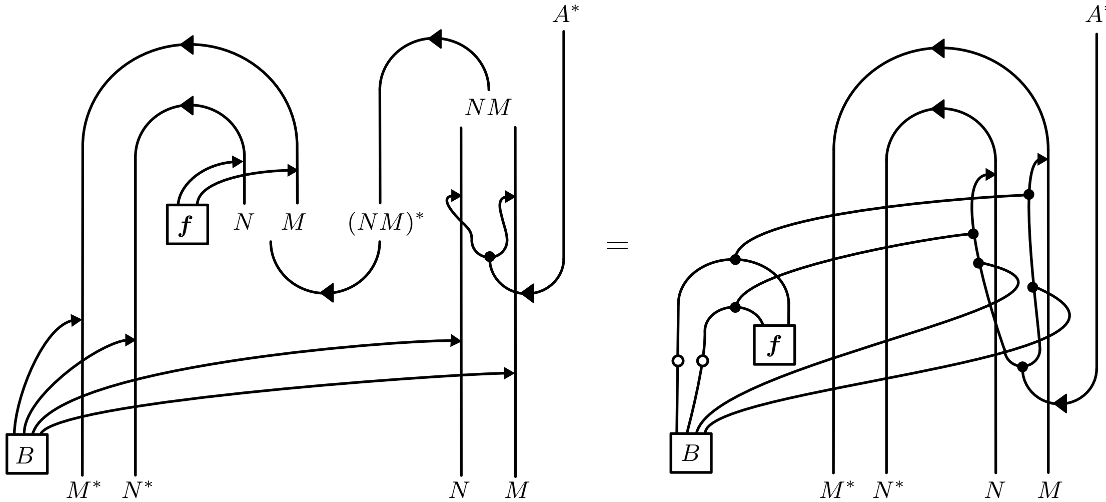
$$\begin{aligned}(5.45)\qquad Y = \sum_{(\Phi),(\Psi),(\tilde{\Psi}),(R)} \iota_{N\otimes M}\circ(\gamma_{N,M}\otimes\mathrm{id})\Big(&\Phi_1\tilde{\Psi}_1\otimes\Phi_2\Psi_1R_2'\tilde{\Psi}_3'\otimes\Phi_3'\Psi_2R_2''\tilde{\Psi}_3''\otimes\Phi_3''\Psi_3R_1\tilde{\Psi}_2 \\ &.\,\varphi\otimes\psi\otimes n\otimes m\Big) \ .\end{aligned}$$

Note that for $B \in A^{\otimes 4}$ we have

$$\begin{aligned}&\iota_{N\otimes M}\circ(\gamma_{N,M}\otimes\mathrm{id})(B\,.\,\varphi\otimes\psi\otimes n\otimes m) \\ (5.46)\qquad &=\Big(a\mapsto\sum_{(B)}(\psi\otimes\varphi)\big(\big(S(B_2)\otimes S(B_1)\big)\cdot\boldsymbol{f}\cdot\Delta(a)\cdot(B_3\otimes B_4)\,.\,n\otimes m\big)\Big) \ .\end{aligned}$$

Indeed, with (2.40) we get

(5.47)



Thus (5.45) becomes, for $a \in A$,

$$Y(a) = \sum_{(\Phi),(\Psi),(\tilde{\Psi}),(R)} (\psi \otimes \varphi)\Big\{ \big(S(\Phi_2 \Psi_1 R_2' \tilde{\Psi}_3') \otimes S(\Phi_1 \tilde{\Psi}_1) \big) \cdot \boldsymbol{f} \cdot \Delta(a\Phi_3) \cdot \big(\Psi_2 R_2'' \tilde{\Psi}_3'' \otimes \Psi_3 R_1 \tilde{\Psi}_2 \big) . n \otimes m \Big\} \Big) = \big\langle f \otimes g \,,\, \hat{\mu}_{\mathcal{L}}(a) \big\rangle \,, \tag{5.48}$$

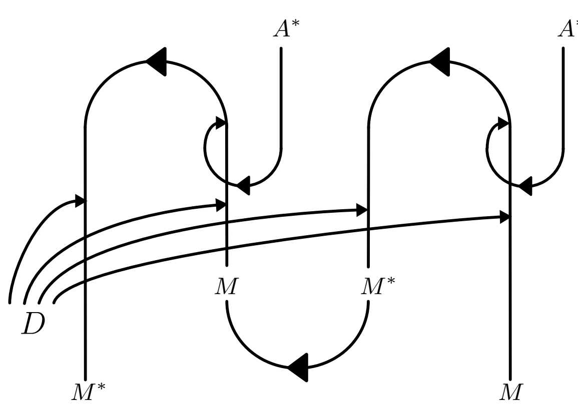
where $f, g \in A^*$ are given by $f = \varphi((-).m)$ and $g = \psi((-).n)$. Since, with the same notation, $(\iota_M \otimes \iota_N)(\varphi \otimes m \otimes \psi \otimes n) = f \otimes g$, from (5.44) we get $X(a) = \big\langle \mu_{\mathcal{L}}(f \otimes g) \,,\, a \big\rangle$. Altogether,

$$\big\langle \mu_{\mathcal{L}}(f \otimes g) \,,\, a \big\rangle = X(a) = Y(a) = \big\langle f \otimes g \,,\, \hat{\mu}_{\mathcal{L}}(a) \big\rangle \tag{5.49}$$

which proves the statement for $\mu_{\mathcal{L}}$.

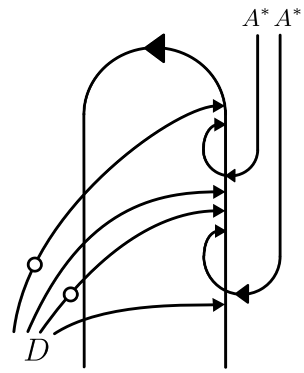
Next we calculate the coproduct. Again, let $M \in \mathbf{Rep}\, A$, $m \in M$ and $\varphi \in M^*$. Applying (4.24) it is easy to see that $\Delta_{\mathcal{L}}$ is determined by

(5.50) $\qquad \Delta_{\mathcal{L}} \circ \iota_M \; =$



with D from (5.43). Using string manipulation the RHS in (5.50) can by written as

(5.51)



.

Thus (5.50) becomes, for $a \otimes b \in A \otimes A$,

$$\Delta_{\mathcal{L}} \circ \iota_M(\varphi \otimes m) = \big(a \otimes b \mapsto \varphi(S(D_1)bD_2S(D_3)aD_4 \,.\, m)\big) \ . \tag{5.52}$$

Again, setting $f = \varphi((-)\,.\,m) \in A^*$ we get

$$\big\langle \Delta_{\mathcal{L}}(f)\,,\, a \otimes b \big\rangle = \sum_{(D)} f\big(S(D_1)bD_2S(D_3)aD_4\big) = \big\langle f\,,\, \hat{\Delta}_{\mathcal{L}}(a \otimes b)\big\rangle \ . \tag{5.53}$$

Applying (4.23) and (4.25) the equalities for $\hat{\eta}_{\mathcal{L}}$ and $\hat{\varepsilon}_{\mathcal{L}}$ in (5.42) are obvious.

Next we proof the formula for the antipode. By (4.26) and (2.50) we have

$$\begin{aligned} S_{\mathcal{L}} \circ \iota_M(\varphi \otimes m) &= \iota_{M^*} \circ (\delta_U^{\mathbf{vect}_\Bbbk} \otimes \mathrm{id})(\tilde{\boldsymbol{u}} \otimes \mathbf{1} \cdot R_{21} \,.\, m \otimes \varphi) \\ &= \Big(a \mapsto \sum_{(R)} \Big\langle (\tilde{\boldsymbol{u}} R_2 \,.\, m)^{**}\,,\, aR_1 \,.\, \varphi)\Big\rangle\Big) \\ &= \Big(a \mapsto \sum_{(R)} \varphi(S(aR_1)\tilde{\boldsymbol{u}} R_2 \,.\, m)\Big) \ . \end{aligned} \tag{5.54}$$

Now it is straightforward to prove the equality for $\hat{S}_{\mathcal{L}}$ in (5.42).

Finally, we calculate the Hopf pairing. By (4.27) $\omega_{\mathcal{L}}$ is determined by

$$\begin{aligned} \omega_{\mathcal{L}} \circ (\iota_M \otimes \iota_N)(\varphi \otimes m \otimes \psi \otimes n) &= (\mathrm{ev}_M^{\mathbf{vect}_\Bbbk} \otimes \mathrm{ev}_N^{\mathbf{vect}_\Bbbk})(W \,.\, \varphi \otimes m \otimes \psi \otimes n) \\ &= \sum_{(W)} (\varphi \otimes \psi)(S(W_1)W_2 \otimes S(W_3)W_4 \,.\, m \otimes n) \end{aligned} \tag{5.55}$$

with W from (5.43). Hence, for $f = \varphi((-).m)$ and $g = \psi((-).n)$ we get

$$\begin{aligned} \omega_{\mathcal{L}}(f \otimes g) &= (f \otimes g)(S(W_1)W_2 \otimes S(W_3)W_4) \\ &= \big\langle f \otimes g\,,\, S(W_3)W_4 \otimes S(W_1)W_2 \big\rangle = \big\langle f \otimes g\,,\, \hat{\omega}_{\mathcal{L}} \big\rangle \ . \end{aligned} \tag{5.56}$$

$\square$

Remark 5.11. We note that in the case A is a quasi-triangular Hopf algebra, the universal Hopf algebra $\mathcal{L}$ in $\mathbf{Rep}\,A$ has the following structural maps on $\mathcal{L} = A^*$, using (5.41) and applying Theorem 5.10:

$$(5.57)\qquad \hat{\mu}_{\mathcal{L}}(a) \;=\; \sum_{(a),(R)} S(R_2')a'R_2'' \otimes a''R_1 \;, \qquad \hat{\Delta}_{\mathcal{L}}(a\otimes b) \;=\; b\cdot a \;,$$
$$\hat{\eta}_{\mathcal{L}}(a) \;=\; \varepsilon(a) \;, \qquad \hat{\varepsilon}_{\mathcal{L}} \;=\; \mathbf{1} \;,$$
$$\hat{S}_{\mathcal{L}}(a) \;=\; \sum_{(R)} S(\boldsymbol{u}^{-1}\,a\,R_1)R_2 \;, \qquad \hat{\omega}_{\mathcal{L}} \;=\; \sum_{(M)} S(M_2)\otimes M_1 \;,$$

where $\boldsymbol{u} = \sum_{(R)} S(R_2)R_1$, see (1.18). The structure maps for the universal Hopf algebra have also been explicitly computed in [**LM**, Sec. 4], [**Ly2**, Sec. 3.4] (in different conventions) and in [**Vi**, Lem. 4.4]. The structure maps for H in [**Vi**, Lem. 4.4] are precisely those for A above with the opposite $\hat{\mu}_{\mathcal{L}}$ and $\hat{\Delta}_{\mathcal{L}}$.

5.2.3. Equivalent factorisability conditions. We give two equivalent ways to phrase factorisability of a quasi-triangular quasi-Hopf algebra as defined in Definition 1.4.

Proposition 5.12. *A finite-dimensional quasi-triangular quasi-Hopf algebra A is factorisable if and only if its representation category* $\mathbf{Rep}\,A$ *is factorisable in the sense of Definition 4.15.*

Proof. Recall from Definitions 4.15 that $\mathbf{Rep}\,A$ is called factorisable if $\omega_{\mathcal{L}}$ is non-degenerate. Non-degeneracy of $\omega_{\mathcal{L}}$ in turn by definition means that the morphism $D_{\mathcal{L}} : \mathcal{L} \to \mathcal{L}^*$ from (2.66) is invertible. In the quasi-Hopf case, $D_{\mathcal{L}}$ is an A-module map $A^* \to A^{**}$. We will describe $D_{\mathcal{L}}$ in terms of an element $\hat{\mathcal{D}}_{\mathcal{L}} \in A\otimes A$ as, for $\varphi \in A^*$,

$$(5.58)\qquad D_{\mathcal{L}}(\varphi) = \delta_A^{\mathbf{vect}} \circ \big((\mathrm{id}\otimes\varphi)(\hat{\mathcal{D}}_{\mathcal{L}})\big) \;.$$

A short computation shows that in terms of $\hat{\omega}_{\mathcal{L}}$ as given in Theorem 5.10 we have

$$(5.59)\qquad \hat{\mathcal{D}}_{\mathcal{L}} = \sum_{(X),(\hat{\omega}_{\mathcal{L}})} S(X_2')\,(\hat{\omega}_{\mathcal{L}})_1 X_2'' \,\otimes\, S(X_1')\,(\hat{\omega}_{\mathcal{L}})_2\, X_1''$$

with X as in (1.14). Substituting the explicit expression for $\hat{\omega}_{\mathcal{L}}$ from Theorem 5.10 and replacing $\hat{\mathcal{D}}_{\mathcal{L}}$ with Q gives the expression in Definition 1.4. □

The second equivalent definition was originally phrased in [**BT**]. The authors also introduce a notion of factorisability for quasi-triangular quasi-Hopf algebras. In this section we recall the definition in [**BT**] and show that it is equivalent to Definition 1.4.

Let A be a finite-dimensional quasi-triangular quasi-Hopf algebra and consider the linear map $\mathcal{Q}^{\mathrm{BT}} : A^* \to A$ defined by (see [**BT**, Prop. 2.2 (i)] but note that their Φ is our Φ^{-1})

$$(5.60)\qquad \mathcal{Q}^{\mathrm{BT}} :\; \phi \mapsto (\phi\otimes\mathrm{id})(\mathcal{M}^{\mathrm{BT}}) \;.$$

The element $\mathcal{M}^{\mathrm{BT}} \in A \otimes A$ is defined as

$$\mathcal{M}^{\mathrm{BT}} = \sum_{(R),(\tilde{R}),(\Phi),(\boldsymbol{p}),(\tilde{\boldsymbol{q}})} \tilde{\boldsymbol{q}}_1(\Phi^{-1})_1 R_2 \tilde{R}_1 \, \boldsymbol{p}_1 \otimes \tilde{\boldsymbol{q}}_2'(\Phi^{-1})_2 R_1 \tilde{R}_2 \, \boldsymbol{p}_2 S\big(\tilde{\boldsymbol{q}}_2''(\Phi^{-1})_3\big) \ , \tag{5.61}$$

where $\tilde{R}$ is a copy of R, and $\boldsymbol{p}$ and

$$\boldsymbol{p} = \sum_{(\Phi)} \Phi_1 \otimes \Phi_2 \boldsymbol{\beta} S(\Phi_3) \ , \qquad \tilde{\boldsymbol{q}} = \sum_{(\Phi)} S(\Phi_1)\boldsymbol{\alpha}\Phi_2 \otimes \Phi_3 \ . \tag{5.62}$$

Recall the end $\Gamma = (A, j)$ from Proposition 5.9 and the map $\mathbb{D}_{\mathcal{L},\Gamma}$ defined by (4.49).

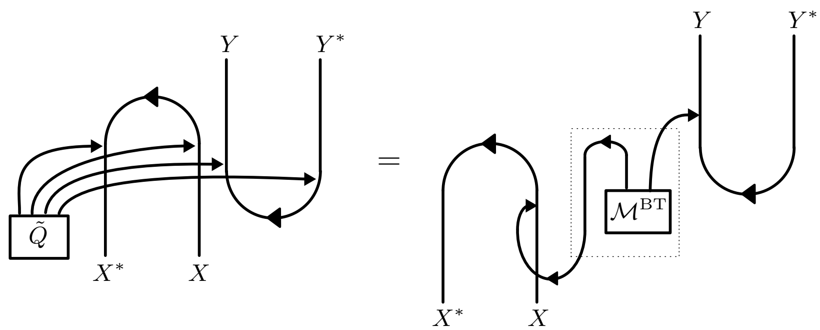
Lemma 5.13. $\mathcal{Q}^{\mathrm{BT}} = \mathbb{D}_{\mathcal{L},\Gamma}$.

Proof. It is enough to show that for all $X, Y \in \mathbf{Rep}\, A$ we have $j_Y \circ \mathcal{Q}^{\mathrm{BT}} \circ \iota_X = j_Y \circ \mathbb{D}_{\mathcal{L},\Gamma} \circ \iota_X$. By definition, $j_Y \circ \mathbb{D}_{\mathcal{L},\Gamma} \circ \iota_X = \mathbb{T}_{X,Y}$, so that it remains to show $j_Y \circ \mathcal{Q}^{\mathrm{BT}} \circ \iota_X = \mathbb{T}_{X,Y}$. We will verify this by computing $\mathbb{T}_{X,Y}$ in (4.47) explicitly. Let

$$\tilde{Q} = (\mathbf{1} \otimes \boldsymbol{\alpha} \otimes \mathbf{1} \otimes \mathbf{1}) \cdot [(\mathrm{id} \otimes \mathrm{id} \otimes \Delta)\Phi] \cdot \big(\mathrm{id} \otimes (\Phi^{-1} \cdot (M \otimes \mathbf{1}) \cdot \Phi)\big) \cdot (\mathbf{1} \otimes \mathbf{1} \otimes \boldsymbol{\beta} \otimes \mathbf{1}) \tag{5.63}$$

then we obtain

(5.64) $\qquad \mathbb{T}_{X,Y} \quad =$



,

where $\mathcal{M}^{\mathrm{BT}} = (\mu \otimes \mu) \circ [S \otimes \mathrm{id} \otimes \mathrm{id} \otimes S](\tilde{Q})$ and a direct calculation shows that it equals to (5.61), as the notation suggests. The morphism within the dotted frame in RHS of (5.64) is then obviously the map $\mathcal{Q}^{\mathrm{BT}}$ and therefore we finally have that $\mathbb{T}_{X,Y} = j_Y \circ \mathcal{Q}^{\mathrm{BT}} \circ \iota_X$. $\square$

In [**BT**], a quasi-triangular quasi-Hopf algebra A is called factorisable if the map $\mathcal{Q}^{\mathrm{BT}}$ defined in (5.60) is an isomorphism. As an immediate consequence of Lemma 5.13 and Proposition 4.19 we get:

Corollary 5.14. *For a finite-dimensional quasi-triangular quasi-Hopf algebra A, the map $\mathcal{Q}^{\mathrm{BT}}$ from* (5.60) *is an isomorphism if and only if A is factorisable in the sense of Definition 1.4.*

Remark 5.15. Consider the case that A is a quasi-triangular Hopf algebra and use the end $\Gamma = (A, j)$ from Proposition 5.9. In this case, the map $\mathbb{D}_{\mathcal{L},\Gamma}$ defined by (4.49) is precisely the Drinfeld map. Indeed, for $\Phi = \mathbf{1}^{\otimes 3}$ and $\boldsymbol{\alpha} = \boldsymbol{\beta} = \mathbf{1}$, the expression for $\mathcal{M}^{\mathrm{BT}}$ in (5.61) reduces to $M = R_{21}R$ and $\mathcal{Q}^{\mathrm{BT}}$ from (5.60) (which is equal to $\mathbb{D}_{\mathcal{L},\Gamma}$ by Lemma 5.13) becomes the Drinfeld map, cf. Remark 1.5 (2). We also note that we have defined so far two maps from

A^* to A – the map $(\delta_A^{\mathbf{vect}})^{-1} \circ D_{\mathcal{L}}$ in Remark 1.5 and the Drinfeld map $\mathbb{D}_{\mathcal{L},\Gamma}$. The difference is that the first intertwines the coadjoint and its dual actions, while the second intertwines the coadjoint and adjoint actions. Both can be used to test factorisability of A.

5.2.4. Integrals and cointegrals. Let A be a factorisable quasi-Hopf algebra over an algebraically closed field $\Bbbk$ (actually, we only need $\Bbbk$ to contain square roots).

From Proposition 4.18 we know that $\mathcal{L}$ has a unique-up-to-scalar two-sided integral $\Lambda_{\mathcal{L}} : \mathbf{1} \to \mathcal{L}$. We will impose the normalisation $\omega_{\mathcal{L}} \circ (\Lambda_{\mathcal{L}} \otimes \Lambda_{\mathcal{L}}) \circ \lambda_{\mathbf{1}}^{-1} = \mathrm{id}_{\mathbf{1}}$, which makes $\Lambda_{\mathcal{L}}$ unique up to sign. Given a two-sided integral, we obtain a two-sided cointegral $\Lambda_{\mathcal{L}}^{\mathrm{co}} : \mathcal{L} \to \mathbf{1}$ by

$$\Lambda_{\mathcal{L}}^{\mathrm{co}} := \Omega(\Lambda_{\mathcal{L}}) \ , \tag{5.65}$$

where Ω was given in (4.42) (cf. Lemma 2.7). By construction, we have

$$\Lambda_{\mathcal{L}}^{\mathrm{co}} \circ \Lambda_{\mathcal{L}} = \mathrm{id}_{\mathbf{1}} \ . \tag{5.66}$$

There are no closed formulas for the integral or the cointegral. Instead one needs to compute the spaces of solutions to the linear conditions (2.67). We will spell out these conditions for A in terms of the structure maps given in Theorem 5.10.

Let $\hat{\Lambda}_{\mathcal{L}} \in A^*$ and $\hat{\Lambda}_{\mathcal{L}}^{\mathrm{co}} \in A$ be defined as, for $a \in A$, $f \in A^*$,

$$\Lambda_{\mathcal{L}}(1) = \big(a \mapsto \hat{\Lambda}_{\mathcal{L}}(a)\big) \quad , \quad \Lambda_{\mathcal{L}}^{\mathrm{co}}(f) = f(\hat{\Lambda}_{\mathcal{L}}^{\mathrm{co}}) \ . \tag{5.67}$$

Conditions (2.67) turn into the following linear relations on $\hat{\Lambda}_{\mathcal{L}}$ and $\hat{\Lambda}_{\mathcal{L}}^{\mathrm{co}}$, for all $a \in A$,

$$\begin{aligned} (\mathrm{id} \otimes \hat{\Lambda}_{\mathcal{L}}) \circ \hat{\mu}_{\mathcal{L}}(a) &= \boldsymbol{\alpha} \cdot \hat{\Lambda}_{\mathcal{L}}(a) &&= (\hat{\Lambda}_{\mathcal{L}} \otimes \mathrm{id}) \circ \hat{\mu}_{\mathcal{L}}(a) \ , \\ \hat{\Delta}_{\mathcal{L}}(\hat{\Lambda}_{\mathcal{L}}^{\mathrm{co}} \otimes a) &= \hat{\Lambda}_{\mathcal{L}}^{\mathrm{co}} \cdot \varepsilon(\boldsymbol{\beta} a) &&= \hat{\Delta}_{\mathcal{L}}(a \otimes \hat{\Lambda}_{\mathcal{L}}^{\mathrm{co}}) \ , \end{aligned} \tag{5.68}$$

where $\hat{\mu}_{\mathcal{L}}$ and $\hat{\Delta}_{\mathcal{L}}$ are introduced in (5.42). The normalisation condition is quadratic and reads

$$(\hat{\Lambda}_{\mathcal{L}} \otimes \hat{\Lambda}_{\mathcal{L}})(\hat{\omega}_{\mathcal{L}}) = 1 \ , \tag{5.69}$$

and the relative normalisation of integral and cointegral is fixed by (5.66) to be $\hat{\Lambda}_{\mathcal{L}}(\hat{\Lambda}_{\mathcal{L}}^{\mathrm{co}}) = 1$.

A *left (resp. right) integral for a quasi-Hopf algebra A* is an element $\mathbf{c} \in A$ satisfying $a \cdot \mathbf{c} = \varepsilon(a)\mathbf{c}$ (resp. $\mathbf{c} \cdot a = \varepsilon(a)\mathbf{c}$) for all $a \in A$, see e.g. [**BC**]. Finite-dimensional quasi-Hopf algebras possess a one-dimensional space of left and of right integrals [**HN**] (see also [**PO**] for existence).

If one knows a non-zero left or right integral for A, then the following proposition provides a shortcut for computing $\Lambda_{\mathcal{L}}^{\mathrm{co}}$.

Proposition 5.16. *Let A be a finite-dimensional quasi-Hopf algebra over some field, and let $\mathbf{c} \in A$ be left (resp. right) integral for A. Then $\langle -, \mathbf{c} \rangle \in \mathcal{L}^* = A^{**}$ is a left (resp. right) cointegral for $\mathcal{L}$.*

PROOF. This is an immediate consequence of the explicit form of the structure maps of $\mathcal{L}$ given in Theorem 5.10. The two conditions in the second line of (5.68) become

$$\text{left:}\quad \hat{\Delta}_{\mathcal{L}}(\mathbf{c}\otimes a)=\mathbf{c}\cdot\varepsilon(\boldsymbol{\beta}a)\qquad,\qquad \text{right:}\quad \hat{\Delta}_{\mathcal{L}}(a\otimes\mathbf{c})=\mathbf{c}\cdot\varepsilon(\boldsymbol{\beta}a)\ . \tag{5.70}$$

Now substitute the expression for $\hat{\Delta}_{\mathcal{L}}$ in terms of D as given in (5.42). The left hand side in each case in (5.70) then becomes

$$\text{left:}\quad \hat{\Delta}_{\mathcal{L}}(\mathbf{c}\otimes a)=\sum_{(D)}\varepsilon(D_1)\varepsilon(a)\varepsilon(D_2)\varepsilon(D_3)\mathbf{c}D_4\ , \tag{5.71}$$

$$\text{right:}\quad \hat{\Delta}_{\mathcal{L}}(a\otimes\mathbf{c})=\sum_{(D)}S(D_1)\mathbf{c}\varepsilon(D_2)\varepsilon(D_3)\varepsilon(a)\varepsilon(D_4)\ , \tag{5.72}$$

where we used $\varepsilon\circ S=\varepsilon$, which holds for quasi-Hopf algebras by [**Dr2**, part 7 of remark on p. 1425]. Substituting the explicit form of D from (5.43) and using the counitality condition (1.4) on Φ^{-1} we get for RHS of (5.71)

$$\sum_{(\Phi)}\varepsilon(\boldsymbol{\beta}a)\mathbf{c}\,\varepsilon(\Phi_1)\varepsilon(\Phi_2)\varepsilon(\Phi_3')\Phi_3''=\varepsilon(\boldsymbol{\beta}a)\mathbf{c}\big\{(\varepsilon\otimes\mathrm{id})\circ\Delta\big\}\big((\varepsilon\otimes\varepsilon\otimes\mathrm{id})(\Phi)\big)=\varepsilon(\boldsymbol{\beta}a)\mathbf{c}\ , \tag{5.73}$$

where the counitality condition was used once more, and similarly for RHS of (5.72). This shows that both the identities in (5.70) hold. □

In particular, for factorisable A we know by Proposition 4.18 that the coend $\mathcal{L}$ has a one-dimensional space of two-sided integrals (and hence of cointegrals). The above proposition hence shows that A has two-sided integrals, and a non-zero such two-sided integral $\mathbf{c}$ spans the space of cointegrals of $\mathcal{L}$.

5.2.5. Internal characters. Let A be a factorisable ribbon quasi-Hopf algebra over a field $\Bbbk$. Then $\mathbf{Rep}\,A$ is in particular pivotal, and we can use the results of Section 5.1.2. In this section we will give explicit expressions for the natural endomorphisms ϕ_M and $\mathcal{S}_{\mathcal{C}}(\phi_M)$ in terms of elements of the centre of A. Recall from (5.22) that ϕ_M, and hence also $\mathcal{S}_{\mathcal{C}}(\phi_M)$, is an image of the internal characters χ_M as defined in (5.21).

Denote by $\xi\colon \mathcal{Z}(A)\to\mathrm{End}(\mathrm{id}_{\mathbf{Rep}\,A})$ the $\Bbbk$-algebra isomorphism between the centre of A and the natural endomorphisms of the identity functor on $\mathbf{Rep}\,A$. Explicitly, for all $V\in\mathbf{Rep}\,A$, $v\in V$ and $z\in\mathcal{Z}(A)$,

$$(\xi(z)_V)(v)=z.v\ . \tag{5.74}$$

We can use ξ to represent the S-transformation of ϕ_V in terms of a central element $\boldsymbol{\chi}_V\in\mathcal{Z}(A)$ as $\xi(\boldsymbol{\chi}_V)=S_{\mathbf{Rep}\,A}(\phi_V)$. A short calculation starting from (5.24) gives

$$\boldsymbol{\chi}_V\ =\sum_{(\Phi),(\Psi),(M)}\mathrm{tr}_V\Big(\boldsymbol{u}^{-1}\boldsymbol{v}\,S\big(\Psi_2M_2\Phi_2\boldsymbol{\beta}\big)\boldsymbol{\alpha}\,\Psi_3\Phi_3\Big)\,\Psi_1\,M_1\,\Phi_1\ , \tag{5.75}$$

where $\Psi=\Phi^{-1}$. We obtain the following corollary to Theorem 5.6.

Corollary 5.17. *Let the field* $\Bbbk$ *be algebraically closed and of characteristic zero. Then* $[V] \mapsto \boldsymbol{\chi}_V$ *is an injective ring-homomorphism* $\mathrm{Gr}(\mathbf{Rep}\,A) \to \mathcal{Z}(A)$.

The Verlinde-type formula (5.25) takes the form

$$\boldsymbol{\chi}_U\, \boldsymbol{\chi}_V = \sum_{W \in \mathrm{Irr}(\mathbf{Rep}\,A)} N_{UV}^{\;W}\, \boldsymbol{\chi}_W \tag{5.76}$$

By linear independence of the $\boldsymbol{\chi}_W$, this determines the structure constants $N_{UV}^{\;W}$ of $\mathrm{Gr}(\mathbf{Rep}\,A)$ uniquely (for $\Bbbk$ as in the corollary). We stress that it is not necessary to compute the centre of A to evaluate this formula, but one does need to know all simple A-modules U and be able to compute their characters $\mathrm{tr}_U(-)$.

For completeness we also give the central element $\boldsymbol{\phi}_V \in \mathcal{Z}(A)$ representing ϕ_V from (5.26) via $\xi(\boldsymbol{\phi}_V) = \phi_V$. We set

$$\tilde{F} = \sum_{(\Phi),(\Psi)} \Psi_1 \boldsymbol{\beta} S(\Phi_1\Psi_2)\hat{\Lambda}^{\mathrm{co}}_{\mathcal{L}} \Phi_2 \Psi_3' \otimes \Phi_3 \Psi_3'' \;,$$

$$F = (\mathrm{id}\otimes\mu)\circ(\mathrm{id}\otimes S\otimes \mathrm{id})\big((\mathbf{1}\otimes\mathbf{1}\otimes\boldsymbol{\alpha})\cdot\Phi^{-1}\cdot(\Delta\otimes\mathrm{id})(\tilde{F})\cdot\Phi\cdot(\mathbf{1}\otimes\boldsymbol{\beta}\otimes\mathbf{1})\big) \;. \tag{5.77}$$

Note that $F \in A\otimes A$. Then, we have

$$\boldsymbol{\phi}_V \;=\; \sum_{(F)} F_1\, \mathrm{tr}_V\big(\boldsymbol{u}^{-1}\, \boldsymbol{v}\, F_2\big) \;. \tag{5.78}$$

Remark 5.18. Let A be a finite-dimensional factorisable ribbon Hopf algebra over $\Bbbk$. Recall [**Dr1**] that the space qCh of q-characters of A is defined as the space of invariants in A^* under the coadjoint action, e.g. the *quantum traces* $\mathrm{qTr}_V := \mathrm{Tr}_V\big(\boldsymbol{u}^{-1}\,\boldsymbol{v}\,(-)\big)$, which are the internal characters $\chi_V(1) \in A^*$, are q-characters. The two families of elements, $\boldsymbol{\chi}_V$ and $\boldsymbol{\phi}_V$, simplify to

$$\boldsymbol{\chi}_V \;= \sum_{(M)} M_1\, \mathrm{tr}_V\big(\boldsymbol{u}^{-1}\boldsymbol{v}\, S(M_2)\big) \;\overset{(*)}{=}\; S^{-1}\circ\big(\mathrm{qTr}_V(-)\otimes\mathrm{id}\big)(M) \tag{5.79}$$

and (recall that we found $\hat{\Lambda}^{\mathrm{co}}_{\mathcal{L}} = \mathbf{c}$ and here $F = (\mathrm{id}\otimes S)\circ\Delta(\mathbf{c})$)

$$\boldsymbol{\phi}_V \;=\; \sum_{(c)} \mathbf{c}'\, \mathrm{tr}_V\big(\boldsymbol{u}^{-1}\,\boldsymbol{v}\, S(\mathbf{c}'')\big) \;\overset{(**)}{=}\; S^{-1}\circ\big(\mathrm{qTr}_V(-)\otimes\mathrm{id}\big)\circ\Delta(\pm\mathbf{c}) \;, \tag{5.80}$$

where for $(*)$ we used the identity[1] $(S\otimes S)(R) = R$, while for $(**)$ we used that $S(\mathbf{c}) = \lambda\mathbf{c}$ for $\lambda^2 = 1$ (due to $S^2|_{\mathcal{Z}(A)} = \mathrm{id}$), so λ is a sign, and thus $\Delta^{\mathrm{op}}(\lambda\mathbf{c}) = (S\otimes S)\circ\Delta(\mathbf{c})$. (Note that the integral $\mathbf{c}$ is normalised up to a sign only.)

The central elements $\boldsymbol{\chi}_V$ and $\boldsymbol{\phi}_V$ are related to q-characters in the following way. The (algebra) map $S\circ\xi^{-1}\circ\psi^{-1}\circ\Omega$ from the space of q-characters to $\mathcal{Z}(A)$ is the well-known *Drinfeld mapping* given by $\mathbb{D}_{A^*,A}\colon \phi(\cdot) \mapsto (\phi\otimes\mathrm{id})M$, see [**Dr1**], while the map $S\circ\xi^{-1}\circ\psi^{-1}\circ\rho^{-1}\colon \mathrm{qCh} \to \mathcal{Z}(A)$ is the *Radford mapping* given by $\phi(\cdot) \mapsto (\phi\otimes\mathrm{id})\Delta(\boldsymbol{c})$, see [**Ra**] for its definition and

[1] This results in $(\mathrm{id}\otimes S)(M) = \tau\circ(\mathrm{id}\otimes S^{-1})(M)$ and then $(\mathrm{id}\otimes\phi)\circ(\mathrm{id}\otimes S)(M) = (\phi\otimes\mathrm{id})\circ(\mathrm{id}\otimes S^{-1})(M) = S^{-1}\circ\big((\phi\otimes\mathrm{id})(M)\big)$, for $\phi\in A^*$.

properties. The central elements (5.79) and (5.80) are then images of the q-characters qTr_V under the Drinfeld and Radford mappings (composed with S^{-1}), correspondingly.

5.3. $SL(2,\mathbb{Z})$-action for ribbon quasi-Hopf algebras

In this section, we assume that A is a factorisable quasi-Hopf algebra and we express the S- and T-transformations from (5.5) in $\mathbf{Rep}\,A$, and compute the resulting action on the centre $\mathcal{Z}(A)$ of A. To start with, we evaluate the map $\mathcal{Q}$ from (5.1) in $\mathbf{Rep}\,A$. One finds, for $a,b\in A$, $f,g\in A^*$,

$$\big\langle \mathcal{Q}(f\otimes g)\,,\, a\otimes b\big\rangle \;=\; \big\langle f\otimes g\,,\, \hat{\mathcal{Q}}(a\otimes b)\big\rangle \tag{5.81}$$

with

$$\begin{aligned}&\hat{\mathcal{Q}}(a\otimes b) = \big(S(X_3)aX_4\big)\otimes\big(S(X_1)bX_2\big)\ ,\\ &X = (\mathrm{id}\otimes\mathrm{id}\otimes\Delta)(\Phi)\cdot(\mathbf{1}\otimes\Phi^{-1})\cdot(\mathbf{1}\otimes M\otimes\mathbf{1})\cdot(\mathbf{1}\otimes\Phi)\cdot(\mathrm{id}\otimes\mathrm{id}\otimes\Delta)(\Phi^{-1})\ .\end{aligned}\tag{5.82}$$

Then the S- and T-transformations from (5.5) take the form, for $a\in A$, $f\in A^*$,

$$\begin{aligned}\big\langle \mathcal{S}(f)\,,\, a\big\rangle \;&=\; \big\langle \mathcal{Q}(f\otimes\Lambda_{\mathcal{L}})\,,\, a\otimes\boldsymbol{\alpha}\big\rangle\ ,\\ \big\langle \mathcal{T}(f)\,,\, a\big\rangle \;&=\; \big\langle f\,,\, \boldsymbol{v}^{-1}a\big\rangle\ .\end{aligned}\tag{5.83}$$

Our next aim is to evaluate the action of the S- and T-generators on $\mathrm{End}(\mathrm{id}_{\mathcal{C}})$ as given in (5.14) in the case $\mathcal{C}=\mathbf{Rep}\,A$. To do so, we use the isomorphism ξ from (5.74) and will give the corresponding action on elements of $\mathcal{Z}(A)$ instead. The result is:

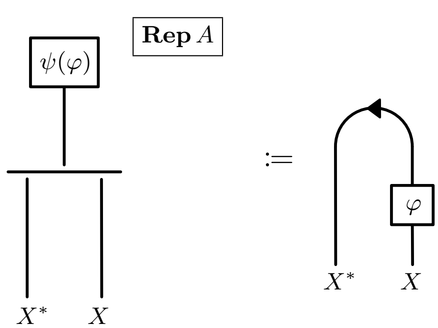
Theorem 5.19. *The S- and T-transformations on $\mathcal{Z}(A)$ are given by the following linear maps $\mathcal{Z}(A)\to\mathcal{Z}(A)$: for $z\in\mathcal{Z}(A)$,*

$$\begin{aligned}\mathcal{S}_{\mathcal{Z}}(z)\;&=\sum_{\Psi,(\hat\omega_{\mathcal{L}})}\Psi_1\,\boldsymbol{\beta}\,S(\Psi_2)\,(\hat\omega_{\mathcal{L}})_1\,\Psi_3\,\hat\Lambda_{\mathcal{L}}\Big(\hat\Delta_{\mathcal{L}}\big((\hat\omega_{\mathcal{L}})_2\otimes\boldsymbol{\alpha}z\big)\Big)\ ,\\ \mathcal{T}_{\mathcal{Z}}(z)\;&=\boldsymbol{v}^{-1}z\ ,\end{aligned}\tag{5.84}$$

where $\Psi=\Phi^{-1}$, and $\hat\Delta_{\mathcal{L}}$ and $\hat\omega_{\mathcal{L}}$ are defined in (5.42).

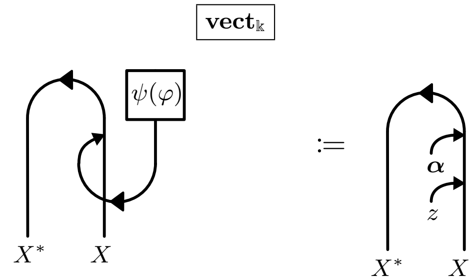
Proof. We compute the ingredients of (5.14). Let $\varphi\in\mathrm{End}(\mathrm{id}_{\mathcal{C}})$ be the natural transformation which acts by a central element $z\in\mathcal{Z}(A)$, i.e. $\xi(z)=\varphi$. Then

(5.85)



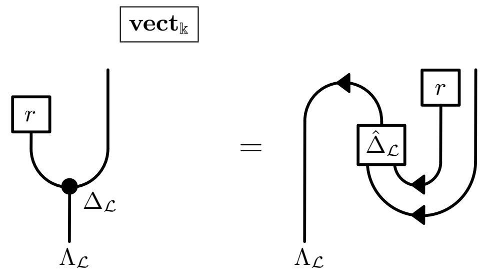
or equivalently as a diagram in $\mathbf{vect}_{\Bbbk}$

(5.86)



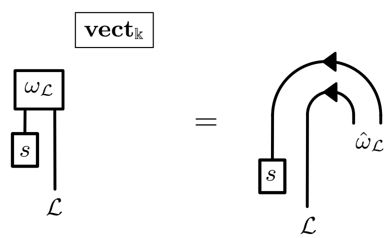
that is, $\psi(\varphi) = \langle -, \boldsymbol{\alpha} z\rangle \in A^{**}$. Let $r \in \mathrm{Hom}_A(\mathcal{L}, \mathbf{1})$ and $s \in \mathrm{Hom}_A(\mathbf{1}, \mathcal{L})$, and write $r = \langle -, \tilde{r}\rangle$ for some $\tilde{r} \in A$. The other maps in (5.14) are given by

(5.87) $\rho(r) =$



$= \hat{\Lambda}_{\mathcal{L}}\big(\hat{\Delta}_{\mathcal{L}}(- \otimes \tilde{r})\big)$,

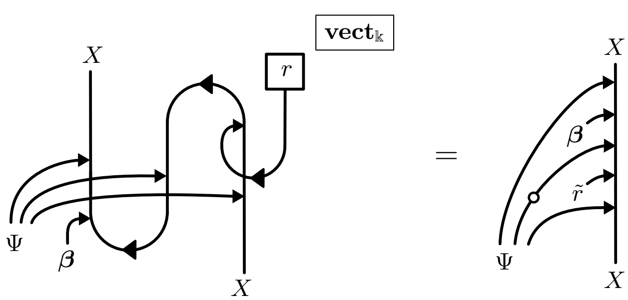
(5.88) $\Omega(s) =$



$= \langle -, (\mathrm{id} \otimes s)(\hat{\omega}_{\mathcal{L}})\rangle$,

(5.89)

$\psi^{-1}(r)_X =$



$= \sum_{(\Psi)} \Psi_1 \, \boldsymbol{\beta}\, S(\Psi_2)\, \tilde{r}\, \Psi_3 \,.\, (-)$,

where $\Psi = \Phi^{-1}$. Now it is easy to see that (5.14) reduces to (5.84). $\square$

Recall that in Section 5.2.5 we introduced the special central elements $\boldsymbol{\chi}_V$ and $\boldsymbol{\phi}_V$ that are related to the internal characters χ_V. As a corollary of Theorem 5.84 and by definition $\xi(\boldsymbol{\chi}_V) = S_{\mathbf{Rep}\,A}(\phi_V)$, we have the following S-transformation of these elements.

Corollary 5.20. *$\boldsymbol{\chi}_V = \mathcal{S}_{\mathcal{Z}}(\boldsymbol{\phi}_V)$ and $\boldsymbol{\phi}_{V^*} = \mathcal{S}^2_{\mathcal{Z}}(\boldsymbol{\phi}_V)$.*

Next we give the S-transformation on A as $\hat{\mathcal{S}} : A \to A$, using (5.83),

(5.90) $$\langle \mathcal{S}(f), a\rangle = \langle f, \hat{\mathcal{S}}(a)\rangle \ ,$$

for any $f \in A^*$ and $a \in A$. We easily get

$$\hat{\mathcal{S}}(a) = (\hat{\Lambda}_{\mathcal{L}} \otimes \mathrm{id})\Big[\hat{\mathcal{Q}}(a \otimes \boldsymbol{\alpha})\Big] = \sum_{(X)} \hat{\Lambda}_{\mathcal{L}}\Big(S(X_3)aX_4\Big)S(X_1)\boldsymbol{\alpha}X_2 \tag{5.91}$$

or equivalently, using the relation (5.16) we have

$$\hat{\mathcal{S}}(a) = \sum_{(\Phi),(\hat{\omega}_{\mathcal{L}})} \hat{\Lambda}_{\mathcal{L}}\Big(\hat{\Delta}_{\mathcal{L}}\big(S(\Phi'_3)\, a\, \Phi''_3 \otimes S(\Phi'_2)(\hat{\omega}_{\mathcal{L}})_1\Phi''_2\big)\Big)S(\Phi'_1)(\hat{\omega}_{\mathcal{L}})_2\Phi''_1\ . \tag{5.92}$$

For $\hat{\mathcal{T}}$, we obviously have $\hat{\mathcal{T}}(a) = \boldsymbol{v}^{-1}a$. Following (5.7) and the definition in (5.90), $\hat{\mathcal{S}}$ and $\hat{\mathcal{T}}$ satisfy

$$(\hat{\mathcal{S}}\hat{\mathcal{T}})^3 = \lambda\,\hat{\mathcal{S}}^2\ , \qquad \hat{\mathcal{S}}^2 = \hat{S}_{\mathcal{L}}^{-1}\ , \qquad \lambda \in \Bbbk^\times\ , \tag{5.93}$$

with the antipode $\hat{S}_{\mathcal{L}}$ as in Theorem 5.10. We note then the S-transformation (5.90) with (5.92) simplifies on linear forms $f \in \mathcal{C}(\mathbf{1},\mathcal{L})$, or on the invariants $\hat{f} := f(1)$ of the coadjoint action (recall that $\mathcal{S}\colon \mathcal{L} \to \mathcal{L}$ is an intertwiner of the coadjoint action of A on $\mathcal{L} = A^*$ and thus $\mathcal{S}$ acts on the space invariants):

$$\mathcal{S}(\hat{f})(a) = \hat{f}\big(\hat{\mathcal{S}}(a)\big) = \sum_{(\hat{\omega}_{\mathcal{L}})} \hat{\Lambda}_{\mathcal{L}}\Big(\hat{\Delta}_{\mathcal{L}}\big(a \otimes (\hat{\omega}_{\mathcal{L}})_1\big)\Big)\hat{f}\big((\hat{\omega}_{\mathcal{L}})_2\big)\ , \qquad a \in A\ , \tag{5.94}$$

where we used the counitality of Φ. The two projective $SL(2,\mathbb{Z})$ actions, on $\mathcal{Z}(A)$ in (5.84) and on $\mathcal{C}(\mathbf{1},\mathcal{L})$ in (5.94), are related by conjugation with the isomorphism $\rho\circ\psi\circ\xi$ between the centre $\mathcal{Z}(A)$ and the space of coadjoint invariants. (Note that the action (5.94) agrees with the one from the categorical formula (5.18)).

Similarly, we have a projective $SL(2,\mathbb{Z})$ representation on the space $\mathcal{C}(\mathcal{L},\mathbf{1})$ of *co*invariants of the coadjoint action that is identified by $\delta_A^{\mathbf{vect}}$ with the subspace $\boldsymbol{\alpha}\cdot\mathcal{Z}(A) \subset A$. Indeed, by (5.86) the image of the isomorphism $\psi\circ\xi : \mathcal{Z}(A) \to \mathcal{C}(\mathcal{L},\mathbf{1})$ is $\delta_A^{\mathbf{vect}}(\boldsymbol{\alpha}\cdot\mathcal{Z}(A))$. In general, $\boldsymbol{\alpha}\cdot\mathcal{Z}(A)$ is different from the centre $\mathcal{Z}(A)$ and from the space of invariants of the adjoint action, which is $\boldsymbol{\beta}\cdot\mathcal{Z}(A)$ (in the Hopf algebra case, the three spaces are identical). The S-transformation acts on $\boldsymbol{\alpha}\cdot\mathcal{Z}(A)$ by restricting $\hat{\mathcal{S}}$ to it. When evaluated on $\boldsymbol{\alpha}z$, the result (5.92) simplifies to

$$\hat{\mathcal{S}}(\boldsymbol{\alpha}z) = \sum_{(\hat{\omega}_{\mathcal{L}})} \hat{\Lambda}_{\mathcal{L}}\Big(\hat{\Delta}_{\mathcal{L}}\big(\boldsymbol{\alpha}z \otimes (\hat{\omega}_{\mathcal{L}})_1\big)\Big)(\hat{\omega}_{\mathcal{L}})_2\ , \qquad z \in \mathcal{Z}(A)\ , \tag{5.95}$$

where we again used the counitality of Φ. By construction, $\hat{\mathcal{S}}$ commutes with the A_{op}-action on A given by $\rho(b)\colon a \mapsto \sum_{(b)} S(b')ab''$ [2] and thus $\hat{\mathcal{S}}$ acts on the spaces of invariants of this action, which is indeed $\boldsymbol{\alpha}\cdot\mathcal{Z}(A)$.

[2] Note the different position of the antipode S here with respect to the adjoint action.

Remark 5.21. If A is a factorisable Hopf algebra, recalling Remark 5.11 we have the S-transformation (5.91) (or equivalently (5.92)) on A as

$$\hat{\mathcal{S}}(a) = \sum_{(M)} \hat{\Lambda}_{\mathcal{L}}\big(S(M_2)a\big)M_1 = \sum_{(M)} \hat{\Lambda}_{\mathcal{L}}\big(M_1 a\big)S^{-1}(M_2)\ , \quad a \in A\ , \tag{5.96}$$

where we used $(S\otimes S)(M) = \tau_{A,A}(M)$, while $\mathcal{S}_Z$ from (5.84) becomes

$$\mathcal{S}_{\mathcal{Z}}(z) = \sum_{(M)} \hat{\Lambda}_{\mathcal{L}}\big(zM_1\big)S(M_2)\ , \quad z \in \mathcal{Z}(A)\ . \tag{5.97}$$

When $\hat{\mathcal{S}}$ is restricted to $\mathcal{Z}(A)$ the two formulas become equal: $\mathcal{S}_{\mathcal{Z}}(z) = S^2\big(\hat{\mathcal{S}}(z)\big)$, but S^2 is identity on the centre as $S^2(a) = \boldsymbol{u}a\boldsymbol{u}^{-1}$. We note that (5.97) agrees with the S-transformation obtained in [**Ke1**, Sec. 2] [3], which is a slight rewriting of the "quantum Fourier" S-transformation originally obtained in [**LM**].

[3]Note that $\hat{\Lambda}_{\mathcal{L}}\colon A \to \Bbbk$ is an intertwiner for the A_{op}-action on A by $\rho(b)\colon a \mapsto \sum_{(b)} S(b')ab''$. Therefore, the right integral condition in RHS of (5.68) (the one in the context of the coends) simplifies to the standard right-cointegral condition for a Hopf algebra: $(\hat{\Lambda}_{\mathcal{L}}\otimes \mathrm{id})\circ\Delta(a) = \hat{\Lambda}_{\mathcal{L}}(a)\mathbf{1}$. (But the LHS of (5.68) does not simplify to the standard left-cointegral condition for a Hopf algebra.) The right (co)integral μ_D used in [**Ke1**, Section 2] thus coincides with our $\hat{\Lambda}_{\mathcal{L}}$ (possibly up to a sign), as the normalisation is also the same.

Part 2

The symplectic fermion ribbon quasi-Hopf algebra and the $SL(2,\mathbb{Z})$-action on its centre

6. The ribbon category $\mathcal{SF}$

In this section we review the definition of the categories $\mathcal{SF}(\mathrm{N},\beta)$ introduced in [**DR1, Ru**], see also [**DR2**] for a summary. These are finite ribbon categories (see Sections 2.1 and Definition 2.18 for notation and conventions) which depend on two parameters:

$$\mathrm{N} \in \mathbb{N} = \{1,2,\dots\} \qquad \text{and} \qquad \beta \in \mathbb{C} \quad \text{such that} \quad \beta^4 = (-1)^{\mathrm{N}} \ . \tag{6.1}$$

We will refer to $\mathcal{SF}(\mathrm{N},\beta)$ as the *category of N pairs of symplectic fermions*, and we will abbreviate $\mathcal{SF} := \mathcal{SF}(\mathrm{N},\beta)$.

6.1. $\mathcal{SF}$ as an abelian category

Let Λ be the $2^{2\mathrm{N}}$-dimensional Graßmann algebra over $\mathbb{C}$ generated by a_i, b_i with defining relations

$$\{a_i, a_j\} = \{b_i, b_j\} = \{a_i, b_j\} = 0 \ , \qquad i,j = 1, \dots, \mathrm{N} \ . \tag{6.2}$$

By giving $c \in \{a_1, b_1, \dots, a_\mathrm{N}, b_\mathrm{N}\}$ odd parity and defining $\Delta(c) = c \otimes \mathbf{1} + \mathbf{1} \otimes c$, $\varepsilon(c) = 0$ and $S(c) = -c$ we get a Hopf algebra in **Svect**. Let $\mathbf{Rep}_{\mathrm{s.v.}}\Lambda$ be the category of finite dimensional super-vector spaces with a Λ-action. The category $\mathcal{SF} = \mathcal{SF}(\mathrm{N},\beta)$ is defined as

$$\mathcal{SF} := \mathcal{SF}_0 \oplus \mathcal{SF}_1 \quad \text{where} \quad \mathcal{SF}_0 := \mathbf{Rep}_{\mathrm{s.v.}}\Lambda \ , \ \mathcal{SF}_1 := \mathbf{Svect}. \tag{6.3}$$

Note that $\mathcal{SF}_0$ is not semi-simple. A choice of representatives for the isomorphism classes of simple objects in $\mathcal{SF}$ is

$$\begin{aligned} \mathbf{1} &:= \mathbb{C}^{1|0} \in \mathcal{SF}_0 \ , & T &:= \mathbb{C}^{1|0} \in \mathcal{SF}_1 \ , \\ \Pi\mathbf{1} &:= \mathbb{C}^{0|1} \in \mathcal{SF}_0 \ , & \Pi T &:= \mathbb{C}^{0|1} \in \mathcal{SF}_1 \ , \end{aligned} \tag{6.4}$$

where Π denotes the parity exchange endofunctor on **Svect**, and the Λ-action on $\mathbf{1}$ and $\Pi\mathbf{1}$ is trivial. The projective covers of the simple objects are

$$\begin{aligned} P_{\mathbf{1}} &= \Lambda \ , & P_T &= T \ , \\ P_{\Pi\mathbf{1}} &= \Pi\Lambda \ , & P_{\Pi T} &= \Pi T \ . \end{aligned} \tag{6.5}$$

We will now endow $\mathcal{SF}$ step by step with the structure of a ribbon category.

6.2. Tensor product

Given two objects $X, Y \in \mathcal{SF}$ we define a tensor product functor $*\colon \mathcal{SF} \times \mathcal{SF} \to \mathcal{SF}$ as follows:

$$(6.6)\qquad X * Y \;=\; \begin{cases} \begin{array}{llll} X & Y & X * Y & \\ \mathcal{SF}_0 & \mathcal{SF}_0 & X \otimes_{\mathbf{Rep}_{\text{s.v.}}\Lambda} Y & \in \mathcal{SF}_0 \\ \mathcal{SF}_0 & \mathcal{SF}_1 & F(X) \otimes_{\mathbf{Svect}} Y & \in \mathcal{SF}_1 \\ \mathcal{SF}_1 & \mathcal{SF}_0 & X \otimes_{\mathbf{Svect}} F(Y) & \in \mathcal{SF}_1 \\ \mathcal{SF}_1 & \mathcal{SF}_1 & \Lambda \otimes_{\mathbf{Svect}} X \otimes_{\mathbf{Svect}} Y & \in \mathcal{SF}_0 \end{array} \end{cases}$$

Here, $F : \mathbf{Rep}_{\text{s.v.}}\Lambda \to \mathbf{Svect}$ stands for the forgetful functor. By $X \otimes_{\mathbf{Rep}_{\text{s.v.}}\Lambda} Y$ we mean the tensor product in **Svect** with Λ-action via the coproduct. In more detail, denote by

$$(6.7)\qquad \tau^{\text{s.v.}}_{X,Y} : X \otimes_{\mathbf{Svect}} Y \longrightarrow Y \otimes_{\mathbf{Svect}} X \quad , \quad \tau^{\text{s.v.}}_{X,Y}(x \otimes y) = (-1)^{|x||y|} y \otimes x \ ,$$

the symmetric braiding in **Svect**, where x, y are homogeneous elements. Then the action of an element $g \in \Lambda$ on $x \otimes y \in X \otimes_{\mathbf{Svect}} Y$ is then given by

$$(6.8)\qquad \begin{aligned} g.(x \otimes y) &= (\rho^X \otimes \rho^Y) \circ (\mathrm{id} \otimes \tau^{\text{s.v.}}_{\Lambda,X} \otimes \mathrm{id}) \circ (\Delta(g) \otimes x \otimes y) \\ &= \sum_{(g)} (-1)^{|g''||x|} (g'.x) \otimes (g''.y) \quad \text{where} \quad \Delta(g) = \sum_{(g)} g' \otimes g'' \ , \end{aligned}$$

where ρ^X and ρ^Y give the action of Λ on X and Y.

On morphism, we define the tensor product in all cases beside the last one to be $f * g = f \otimes g$. If $f, g \in \mathcal{SF}_1$ we set $f * g = \mathrm{id}_\Lambda \otimes f \otimes g$.

For the remainder of this section we will drop the subscripts from the tensor products $\otimes_{\mathbf{Svect}}$ and $\otimes_{\mathbf{Rep}_{\text{s.v.}}\Lambda}$ for brevity.

6.3. Associator

Both, the associator and the braiding depend on β from (6.1) and on a copairing C on Λ given by [**DR1**, Eqn. (5.5)]

$$(6.9)\qquad C := \sum_{i=1}^{\mathrm{N}} b_i \otimes a_i - a_i \otimes b_i \quad \in \ \Lambda \otimes \Lambda \ .$$

The associator is a natural family of isomorphisms

$$\alpha^{\mathcal{SF}}_{X,Y,Z} : \quad X * (Y * Z) \to (X * Y) * Z \ ,$$

which is defined sector by sector by the following eight expressions (see [**Ru**, Thm. 6.2] and [**DR1**, Sect. 5.2 & Thm. 2.5]):

$X\,Y\,Z$	$X * (Y * Z)$	$(X * Y) * Z$	$\alpha^{\mathcal{SF}}_{X,Y,Z} : X * (Y * Z) \to (X * Y) * Z$
0 0 0	$\underline{X} \otimes \underline{Y} \otimes \underline{Z}$	$\underline{X} \otimes \underline{Y} \otimes \underline{Z}$	$\mathrm{id}_{X \otimes Y \otimes Z}$

0 0 1	$X \otimes Y \otimes Z$	$X \otimes Y \otimes Z$	$\mathrm{id}_{X \otimes Y \otimes Z}$
0 1 0	$X \otimes Y \otimes Z$	$X \otimes Y \otimes Z$	$\exp\big(C^{(13)}\big)$
1 0 0	$X \otimes Y \otimes Z$	$X \otimes Y \otimes Z$	$\mathrm{id}_{X \otimes Y \otimes Z}$
0 1 1	$\underline{X} \otimes \underline{\mathbb{\Lambda}} \otimes Y \otimes Z$	$\underline{\mathbb{\Lambda}} \otimes X \otimes Y \otimes Z$	$\Big[\big\{\mathrm{id}_{\mathbb{\Lambda}} \otimes (\rho^X \circ (S \otimes \mathrm{id}_X))\big\} \circ \big\{\Delta \otimes \mathrm{id}_X\big\} \circ \tau^{\text{s.v.}}_{X,\mathbb{\Lambda}}\Big] \otimes \mathrm{id}_{Y \otimes Z}$
1 0 1	$\underline{\mathbb{\Lambda}} \otimes X \otimes Y \otimes Z$	$\underline{\mathbb{\Lambda}} \otimes X \otimes Y \otimes Z$	$\exp\big(C^{(13)}\big)$
1 1 0	$\underline{\mathbb{\Lambda}} \otimes X \otimes Y \otimes Z$	$\underline{\mathbb{\Lambda}} \otimes X \otimes Y \otimes \underline{Z}$	$\big\{\mathrm{id}_{\mathbb{\Lambda} \otimes X \otimes Y} \otimes \rho^Z\big\} \circ \big\{\mathrm{id}_{\mathbb{\Lambda}} \otimes \tau^{\text{s.v.}}_{\mathbb{\Lambda}, X \otimes Y} \otimes \mathrm{id}_Z\big\} \circ \big\{\Delta \otimes \mathrm{id}_{X \otimes Y \otimes Z}\big\}$
1 1 1	$X \otimes \mathbb{\Lambda} \otimes Y \otimes Z$	$\mathbb{\Lambda} \otimes X \otimes Y \otimes Z$	$\big\{\phi \otimes \mathrm{id}_{X \otimes Y \otimes Z}\big\} \circ \big\{\tau^{\text{s.v.}}_{X,\mathbb{\Lambda}} \otimes \mathrm{id}_{Y \otimes Z}\big\}$

The underlines mark on which tensor factors $\mathbb{\Lambda}$ acts (hence they appear only when the triple tensor product lies in $\mathcal{SF}_0$, i.e. when an even number of sectors '1' appear). With $C^{(13)}$ denote $C_1 \otimes \mathbf{1} \otimes C_2$ where $C = \sum_{(C)} C_1 \otimes C_2$. Hence, the action of $C^{(13)}$ is given by

$$C^{(13)}(x \otimes y \otimes z) = (-1)^{(|x|+|y|)} \sum_{i=1}^{\mathrm{N}} b_i.x \otimes y \otimes a_i.z - a_i.x \otimes y \otimes b_i.z \ , \tag{6.10}$$

for homogeneous x, y, z. The linear map $\phi : \mathbb{\Lambda} \to \mathbb{\Lambda}$ is given by

$$\phi = (\mathrm{id} \otimes (\Lambda^{\mathrm{co}}_{\mathbb{\Lambda}} \circ \mu_{\mathbb{\Lambda}})) \circ (\exp(-C) \otimes \mathrm{id}) \ \ , \tag{6.11}$$

where $\mu_{\mathbb{\Lambda}}$ is the multiplication in $\mathbb{\Lambda}$ and $\Lambda^{\mathrm{co}}_{\mathbb{\Lambda}} \in \mathbb{\Lambda}^*$ is a specific cointegral for $\mathbb{\Lambda}$ [**DR1**, Eqn. (5.16)]. Namely, $\Lambda^{\mathrm{co}}_{\mathbb{\Lambda}}$ is non-vanishing only in the top degree of $\mathbb{\Lambda}$, and there it takes the value

$$\Lambda^{\mathrm{co}}_{\mathbb{\Lambda}}(a_1 b_1 \cdots a_{\mathrm{N}} b_{\mathrm{N}}) = \beta^{-2} \ . \tag{6.12}$$

6.4. Braiding

For $X \in \mathbf{Svect}$ denote by

$$\omega_X : X \xrightarrow{\sim} X \quad , \quad \omega_X(x) = (-1)^{|x|}\, x \ , \tag{6.13}$$

the parity involution on X. The family $X \mapsto \omega_X$ is a natural monoidal isomorphism of the identity functor on **Svect**.

The braiding on $\mathcal{SF}$ is given – again sector by sector – by the following family of natural isomorphisms $c_{X,Y}$ (see [**Ru**, Thm. 6.4] and [**DR1**, Sect. 5.2 & Thm. 2.8]):

$$
\begin{array}{ccl}
X & Y & c_{X,Y} \;:\; X * Y \to Y * X \\
0 & 0 & \tau^{\text{s.v.}}_{X,Y} \circ \exp(-C) \\
0 & 1 & \tau^{\text{s.v.}}_{X,Y} \circ \big\{\kappa \otimes \mathrm{id}_Y\big\} \\
1 & 0 & \tau^{\text{s.v.}}_{X,Y} \circ \big\{\mathrm{id}_X \otimes \kappa\big\} \circ \{\mathrm{id}_X \otimes \omega_Y\} \\
1 & 1 & \beta \cdot \big(\mathrm{id}_{\mathbb{\Lambda}} \otimes \tau^{\text{s.v.}}_{X,Y}\big) \circ \big\{R_{\kappa^{-1}} \otimes \mathrm{id}_X \otimes \omega_Y\big\}
\end{array} \tag{6.14}
$$

Here, $\kappa := \exp\big(\frac{1}{2}\hat{C}\big)$ where $\hat{C} = \mu_{\mathbb{\Lambda}}(C) = \sum_{i=1}^{\mathrm{N}} -2a_i b_i$ and R_a is the right multiplication with $a \in \mathbb{\Lambda}$:

$$
R_a \colon \mathbb{\Lambda} \to \mathbb{\Lambda}, \qquad R_a = \mu^{\mathbb{\Lambda}} \circ (id_{\mathbb{\Lambda}} \otimes a) \ . \tag{6.15}
$$

If a is parity-even, this is indeed a morphism in **Svect**, and, since it commutes with the $\mathbb{\Lambda}$-action, it is also a morphism in $\mathcal{SF}$.

Recall the definition of a factorisable braided tensor category from Definition 4.15. A key property of $\mathcal{SF}$ is:

Proposition 6.1. *The finite braided tensor category $\mathcal{SF}$ is factorisable.*

Proof. In [**DR1**, Proposition 5.3] it is shown that the full subcategory of transparent objects in $\mathcal{SF}$ is **vect**. Thus $\mathcal{SF}$ fulfils one of equivalent factorisability conditions in [**Sh2**] (see Section 4.3.2 for a summary in our notation). □

6.5. Left duality

Our conventions for duality morphisms are given in Section 2.1.2. By convention, for $X \in$ **Svect** we choose the left-dual X^* and the associated duality maps $\mathrm{ev}^{\mathbf{Svect}}_X$, $\mathrm{coev}^{\mathbf{Svect}}_X$ to be the same as for the underlying vector space.

For $X \in \mathcal{SF}_i$, $i = 0, 1$, we define the left dual object $X^* \in \mathcal{SF}_i$ to be the dual in **Svect** as super-vector space. For $X \in \mathcal{SF}_0$, X^* is furthermore equipped with the $\mathbb{\Lambda}$-action

$$
\rho_{X^*} \colon \mathbb{\Lambda} \otimes X^* \to X^* \ , \quad g \otimes \varphi \mapsto \big(x \mapsto (-1)^{|g|(|\varphi|+1)} \varphi(g.x)\big) \ . \tag{6.16}
$$

The evaluation and coevaluation maps in $\mathcal{SF}$ are induced from those in **Svect** as follows [**DR1**, Sec. 3.6] (note that for $X \in \mathcal{SF}_1$ we have $X^* * X = \mathbb{\Lambda} \otimes X^* \otimes X$):

$$
\begin{array}{ccc}
X & \mathrm{ev}^{\mathcal{SF}}_X : X^* * X \to \mathbf{1} & \mathrm{coev}^{\mathcal{SF}}_X : \mathbf{1} \to X * X^* \\
0 & \mathrm{ev}^{\mathbf{Svect}}_X & \mathrm{coev}^{\mathbf{Svect}}_X \\
1 & \varepsilon_{\mathbb{\Lambda}} \otimes \mathrm{ev}^{\mathbf{Svect}}_X & \Lambda_{\mathbb{\Lambda}} \otimes \mathrm{coev}^{\mathbf{Svect}}_X
\end{array} \tag{6.17}
$$

Here, $\varepsilon_{\mathbb{\Lambda}}$ is the counit for $\mathbb{\Lambda}$, and $\Lambda_{\mathbb{\Lambda}} = \beta^2 a_1 b_1 \cdots a_{\mathrm{N}} b_{\mathrm{N}}$ is the integral for $\mathbb{\Lambda}$ normalised with respect to the cointegral in (6.12) such that $\Lambda^{\mathrm{co}}_{\mathbb{\Lambda}}(\Lambda_{\mathbb{\Lambda}}) = 1$.

6.6. Ribbon twist

The ribbon twist isomorphisms θ_X are given in [**DR1**, Prop. 4.17]:

$$\begin{array}{cc} X & \theta_X : X \to X \\ 0 & \exp(-\hat{C}) \\ 1 & \beta^{-1} \cdot \omega_X \end{array} \quad : \tag{6.18}$$

The twist isomorphisms satisfy

$$\theta_{X*Y} = (\theta_X \otimes \theta_Y) \circ c_{Y,X} \circ c_{X,Y} \quad , \quad \theta_{X^*} = \theta_X^* \ . \tag{6.19}$$

For later reference we note that on the four simple objects in (6.4), the twist is given by

$$\theta_{\mathbf{1}} = \mathrm{id}_{\mathbf{1}} \ , \quad \theta_{\Pi\mathbf{1}} = \mathrm{id}_{\Pi\mathbf{1}} \ , \quad \theta_T = \beta^{-1}\,\mathrm{id}_T \ , \quad \theta_{\Pi T} = -\beta^{-1}\,\mathrm{id}_{\Pi T} \ . \tag{6.20}$$

We summarise Proposition 6.1 and the ribbon structure on $\mathcal{SF}$ reviewed in this section by the following theorem.

Theorem 6.2. *$\mathcal{SF}(N,\beta)$ is a factorisable finite ribbon category for all choices of N and β as in* (6.1)*.*

7. The ribbon quasi-Hopf algebra Q

In this section we define the central object of this paper, the family of ribbon quasi-Hopf algebras $\mathsf{Q}(\mathrm{N},\beta)$, where the parameters N and β are constrained as in (6.1). The main result of this section is a ribbon equivalence between $\mathbf{Rep}\,\mathsf{Q}(\mathrm{N},\beta)$ and $\mathcal{SF}(\mathrm{N},\beta)$.

Our conventions on quasi-Hopf algebras, universal R-matrices, etc., are given in Section 1. In this section we will abbreviate $\mathsf{Q} := \mathsf{Q}(\mathrm{N},\beta)$.

7.1. Definition of Q

In defining the ribbon quasi-Hopf algebra Q, we will first list all of its data – starting with the product and ending with the ribbon element – and only afterwards we will prove that it satisfies the necessary properties.

We start by giving Q as an associative unital algebra over $\mathbb{C}$ via generators and relations. The generators are

$$(7.1) \qquad \mathsf{K} \quad \text{and} \quad \mathsf{f}_i^{\pm}, \qquad i = 1,\dots,\mathrm{N} \ .$$

Define the elements

$$(7.2) \qquad \boldsymbol{e}_0 := \tfrac{1}{2}(\mathbf{1}+\mathsf{K}^2) \quad , \quad \boldsymbol{e}_1 := \tfrac{1}{2}(\mathbf{1}-\mathsf{K}^2) \ .$$

Using these, the defining relations of Q can be written as, for $i,j = 1,\dots,\mathrm{N}$,

$$(7.3) \qquad \{\mathsf{f}_i^{\pm},\mathsf{K}\} = 0 \ , \quad \{\mathsf{f}_i^{+},\mathsf{f}_j^{-}\} = \delta_{i,j}\boldsymbol{e}_1 \ , \quad \{\mathsf{f}_i^{\pm},\mathsf{f}_j^{\pm}\} = 0 \ , \quad \mathsf{K}^4 = \mathbf{1} \ ,$$

where $\{x,y\} = xy + yx$ is the anticommutator. With these relations, $\boldsymbol{e}_0$ and $\boldsymbol{e}_1$ become central idempotents in Q. The corresponding decomposition of Q into ideals is

$$(7.4) \qquad \mathsf{Q} = \mathsf{Q}_0 \oplus \mathsf{Q}_1 \qquad \text{where} \qquad \mathsf{Q}_i := \boldsymbol{e}_i\mathsf{Q} \ .$$

It is easy to check that restricted to each ideal, the algebra structure of Q becomes

$$(7.5) \qquad \mathsf{Q}_0 = \mathsf{\Lambda}_{2\mathrm{N}} \rtimes \mathbb{C}\mathbb{Z}_2 \ , \qquad \mathsf{Q}_1 = \mathsf{Cl}_{2\mathrm{N}} \rtimes \mathbb{C}\mathbb{Z}_2 \ .$$

Here, $\mathsf{\Lambda}_{2\mathrm{N}}$ is the Graßmann algebra of the 2N generators $\mathsf{f}_i^{\pm}\boldsymbol{e}_0$ while $\mathbb{C}\mathbb{Z}_2$ stands for the group algebra of $\mathbb{Z}_2$ generated by $\mathsf{K}\boldsymbol{e}_0$. Similarly, $\mathsf{Cl}_{2\mathrm{N}}$ is the Clifford algebra generated by $\mathsf{f}_i^{\pm}\boldsymbol{e}_1$, while the generator of $\mathbb{Z}_2$ is $\mathsf{i}\mathsf{K}\boldsymbol{e}_1$. In particular, we see that

$$(7.6) \qquad \dim_{\mathbb{C}} \mathsf{Q} = 2^{2\mathrm{N}+2}$$

and that a basis of Q is

$$\mathsf{Q} = \mathrm{span}_{\mathbb{C}} \left\{ \mathsf{f}_{i_1}^{\varepsilon_1} \cdots \mathsf{f}_{i_m}^{\varepsilon_m} \mathsf{K}^n \,\middle|\, 0 \le m \le \mathrm{N}, 1 \le i_1 < i_2 < \ldots < i_m \le \mathrm{N}, \varepsilon_j = \pm, n \in \mathbb{Z}_4 \right\} . \tag{7.7}$$

Next we define the quasi-bialgebra structure of Q. The coproduct on generators is

$$\begin{aligned} \Delta(\mathsf{K}) &= \mathsf{K} \otimes \mathsf{K} - (1 + (-1)^{\mathrm{N}}) \boldsymbol{e}_1 \otimes \boldsymbol{e}_1 \cdot \mathsf{K} \otimes \mathsf{K} , \\ \Delta(\mathsf{f}_i^{\pm}) &= \mathsf{f}_i^{\pm} \otimes \mathbf{1} + \omega_{\pm} \otimes \mathsf{f}_i^{\pm} , \qquad\qquad \omega_{\pm} := (\boldsymbol{e}_0 \pm \mathrm{i} \boldsymbol{e}_1) \mathsf{K} . \end{aligned} \tag{7.8}$$

We will show in Lemma 7.3 below that Δ is well-defined and an algebra map. The counit is

$$\varepsilon(\mathsf{K}) = 1 , \qquad \varepsilon(\mathsf{f}_i^{\pm}) = 0 . \tag{7.9}$$

We remark that K itself is group-like only for odd N. However, one quickly verifies that

$$\mathsf{K}^2 , \ \mathsf{K}^{\mathrm{N}} \quad \text{are group-like for all N} . \tag{7.10}$$

The co-associator and its inverse are

$$\Phi^{\pm 1} = \mathbf{1} \otimes \mathbf{1} \otimes \mathbf{1} + \boldsymbol{e}_1 \otimes \boldsymbol{e}_1 \otimes \left\{ (\mathsf{K}^{\mathrm{N}} - \mathbf{1}) \boldsymbol{e}_0 + (\beta^2 (\pm \mathrm{i} \mathsf{K})^{\mathrm{N}} - \mathbf{1}) \boldsymbol{e}_1 \right\} . \tag{7.11}$$

For the quasi-Hopf algebra structure on Q we still need to specify the antipode S and the evaluation and coevaluation elements $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$. They are:

$$\begin{aligned} S(\mathsf{K}) &= \mathsf{K}^{(-1)^{\mathrm{N}}} = (\boldsymbol{e}_0 + (-1)^{\mathrm{N}} \boldsymbol{e}_1) \mathsf{K} , & \boldsymbol{\alpha} &= \mathbf{1} , \\ S(\mathsf{f}_k^{\pm}) &= \mathsf{f}_k^{\pm} (\boldsymbol{e}_0 \pm (-1)^{\mathrm{N}} \, \mathrm{i} \boldsymbol{e}_1) \mathsf{K} , & \boldsymbol{\beta} &= \boldsymbol{e}_0 + \beta^2 (\mathrm{i} \mathsf{K})^{\mathrm{N}} \boldsymbol{e}_1 . \end{aligned} \tag{7.12}$$

Remark 7.1. For even N and $\beta^2 = 1$ we have $\boldsymbol{\alpha} = \boldsymbol{\beta} = \mathbf{1}$ and $\Phi = \mathbf{1} \otimes \mathbf{1} \otimes \mathbf{1}$, so that (pending the verification that the above data verifies the axioms) in these cases Q is a Hopf algebra. For odd N on the other hand, the coproduct fails to be coassociative. For example,

$$\begin{aligned} (\Delta \otimes \mathrm{id}) \circ \Delta(\mathsf{f}_i^-) &= \mathsf{f}_i^- \otimes \mathbf{1} \otimes \mathbf{1} + \omega_- \otimes \mathsf{f}_i^- \otimes \mathbf{1} \\ &\quad + \mathsf{K} \otimes \mathsf{K} \cdot \big(\boldsymbol{e}_0 \otimes \boldsymbol{e}_0 - \mathrm{i} \boldsymbol{e}_0 \otimes \boldsymbol{e}_1 - \mathrm{i} \boldsymbol{e}_1 \otimes \boldsymbol{e}_0 - (-1)^{\mathrm{N}} \boldsymbol{e}_1 \otimes \boldsymbol{e}_1 \big) \otimes \mathsf{f}_i^- , \\ (\mathrm{id} \otimes \Delta) \circ \Delta(\mathsf{f}_i^-) &= \mathsf{f}_i^- \otimes \mathbf{1} \otimes \mathbf{1} + \omega_- \otimes \mathsf{f}_i^- \otimes \mathbf{1} \\ &\quad + \mathsf{K} \otimes \mathsf{K} \cdot (\boldsymbol{e}_0 \otimes \boldsymbol{e}_0 - \mathrm{i} \boldsymbol{e}_0 \otimes \boldsymbol{e}_1 - \mathrm{i} \boldsymbol{e}_1 \otimes \boldsymbol{e}_0 - \boldsymbol{e}_1 \otimes \boldsymbol{e}_1) \otimes \mathsf{f}_i^- . \end{aligned} \tag{7.13}$$

Next we introduce a quasi-triangular structure on Q. Define the Cartan factor $\rho_{n,m}$ as

$$\rho_{n,m} = \frac{1}{2} \sum_{i,j=0}^{1} (-1)^{ij} \mathrm{i}^{-in+jm} \mathsf{K}^i \otimes \mathsf{K}^j , \qquad n, m \in \{0, 1\} . \tag{7.14}$$

The universal R-matrix and its inverse are defined as

$$R = \Big(\sum_{n,m \in \{0,1\}} \beta^{nm} \rho_{n,m} \boldsymbol{e}_n \otimes \boldsymbol{e}_m \Big) \cdot \prod_{k=1}^{\mathrm{N}} (\mathbf{1} \otimes \mathbf{1} - 2 \mathsf{f}_k^- \omega_- \otimes \mathsf{f}_k^+) , \tag{7.15}$$

$$R^{-1} = \prod_{k=1}^{\mathrm{N}} (\mathbf{1} \otimes \mathbf{1} + 2\mathsf{f}_k^- \omega_- \otimes \mathsf{f}_k^+) \cdot \Big(\sum_{n,m \in \{0,1\}} \beta^{-nm} \rho_{n,m} \boldsymbol{e}_n \otimes \boldsymbol{e}_m \Big) \ .$$

Finally, the ribbon element of Q and its inverse are

$$\boldsymbol{v} = (\boldsymbol{e}_0 - \beta \mathrm{i} \mathsf{K} \boldsymbol{e}_1) \cdot \prod_{k=1}^{\mathrm{N}} (\mathbf{1} - 2\mathsf{f}_k^+ \mathsf{f}_k^-) \ , \tag{7.16}$$

$$\boldsymbol{v}^{-1} = (\boldsymbol{e}_0 - \beta^{-1} \mathrm{i} \mathsf{K} \boldsymbol{e}_1) \cdot \prod_{k=1}^{\mathrm{N}} (\mathbf{1} + 2\mathsf{f}_k^+ \mathsf{f}_k^- \mathsf{K}^2) \ . \tag{7.17}$$

Proposition 7.2. *The data* $(\mathsf{Q}, \cdot, \mathbf{1}, \Delta, \varepsilon, \Phi, S, \boldsymbol{\alpha}, \boldsymbol{\beta}, R, \boldsymbol{v})$ *defines a ribbon quasi-Hopf algebra.*

The proof of this proposition will be given after some preparation. The following lemma is straightforward check on the generators of Q.

Lemma 7.3. *The map* Δ *defined in* (7.8) *is an algebra map.*

Since Δ is an algebra map, we can use it to define a tensor product functor

$$\otimes\colon\ \mathbf{Rep}\,\mathsf{Q} \times \mathbf{Rep}\,\mathsf{Q} \to \mathbf{Rep}\,\mathsf{Q} \ . \tag{7.18}$$

The central idempotents $\boldsymbol{e}_0$ and $\boldsymbol{e}_1$ behave under the coproduct as

$$\Delta(\boldsymbol{e}_0) = \boldsymbol{e}_0 \otimes \boldsymbol{e}_0 + \boldsymbol{e}_1 \otimes \boldsymbol{e}_1 \quad , \quad \Delta(\boldsymbol{e}_1) = \boldsymbol{e}_0 \otimes \boldsymbol{e}_1 + \boldsymbol{e}_1 \otimes \boldsymbol{e}_0 \ , \tag{7.19}$$

so that the tensor product (7.18) respects the $\mathbb{Z}_2$-grading $\mathbf{Rep}\,\mathsf{Q} = \mathbf{Rep}\,\mathsf{Q}_0 \oplus \mathbf{Rep}\,\mathsf{Q}_1$.

Denote by

$$\tau_{X,Y}\colon\ X \otimes Y \longrightarrow Y \otimes X \quad , \quad \tau_{X,Y}(x \otimes y) = y \otimes x \ , \tag{7.20}$$

the symmetric braiding in **vect**. By acting with Φ, R nd $\boldsymbol{v}^{-1}$ we get families of isomorphisms in $\mathbf{Rep}\,\mathsf{Q}$

$$\alpha^{\mathbf{Rep}\,\mathsf{Q}}_{M,N,K}\colon\quad M \otimes (N \otimes K) \to (M \otimes N) \otimes K \ , \qquad m \otimes n \otimes k \mapsto \Phi.(m \otimes n \otimes k) \ , \tag{7.21}$$

$$c^{\mathbf{Rep}\,\mathsf{Q}}_{M,N}\colon\quad M \otimes N \to N \otimes M \ , \qquad m \otimes n \mapsto \tau_{M,N}(R.(m \otimes n)) \ , \tag{7.22}$$

$$\theta^{\mathbf{Rep}\,\mathsf{Q}}_{M}\colon\quad M \to M \ , \qquad m \mapsto \boldsymbol{v}^{-1}.m \ . \tag{7.23}$$

We will show in Lemma 7.5 below that the linear maps (7.21)–(7.23) are indeed morphisms in $\mathbf{Rep}\,\mathsf{Q}$, that they are natural and that $\alpha^{\mathbf{Rep}\,\mathsf{Q}}$ gives an associator, $c^{\mathbf{Rep}\,\mathsf{Q}}$ a braiding and $\theta^{\mathbf{Rep}\,\mathsf{Q}}$ a twist in $\mathbf{Rep}\,\mathsf{Q}$.

Definition 7.4. Let $\mathcal{C}$ be a monoidal category and $\mathcal{D}$ a category with a tensor product functor $\otimes_{\mathcal{D}}\colon \mathcal{D} \times \mathcal{D} \to \mathcal{D}$ and tensor unit $\mathbf{1}_{\mathcal{D}}$. A functor $\mathcal{F}\colon \mathcal{C} \to \mathcal{D}$ is called *multiplicative* if there exists a family of natural isomorphisms $\Theta_{U,V}\colon \mathcal{F}(U \otimes_{\mathcal{C}} V) \to \mathcal{F}(U) \otimes_{\mathcal{D}} \mathcal{F}(V)$ and an isomorphism $\Theta_{\mathbf{1}}\colon \mathcal{F}(\mathbf{1}) \to \mathbf{1}$.

If $\mathcal{D}$ in the above definition is a monoidal category as well (i.e. equipped with associator and unit isomorphisms), a multiplicative functor is also called a *quasi-tensor functor*, see e.g. [**EGNO**, Def. 4.2.5], i.e. we dropped the coherence conditions with the associators from the definition of the tensor functor.

Let $\mathcal{F}\colon \mathcal{C} \to \mathcal{D}$ be a multiplicative equivalence. We now describe how $\mathcal{F}$ can be used to transport structure from $\mathcal{C}$ to $\mathcal{D}$.

- By *transporting the associator from* $\mathcal{C}$ *to* $\mathcal{D}$ *along* $\mathcal{F}$ we mean seeking a natural family $\alpha^{\mathcal{D}}_{X,Y,Z}$, for $X, Y, Z \in \mathcal{D}$, such that for all $U, V, W \in \mathcal{C}$, the diagram

$$\begin{tikzcd}
\mathcal{F}\big(U \otimes_{\mathcal{C}} (V \otimes_{\mathcal{C}} W)\big) \arrow[rr, "\mathcal{F}(\alpha^{\mathcal{C}}_{U,V,W})"] \arrow[d, "\Theta_{U,V\otimes_{\mathcal{C}}W}"'] & & \mathcal{F}\big((U \otimes_{\mathcal{C}} V) \otimes_{\mathcal{C}} W\big) \arrow[d, "\Theta_{U\otimes_{\mathcal{C}}V,W}"] \\
\mathcal{F}(U) \otimes_{\mathcal{D}} \mathcal{F}(V \otimes_{\mathcal{C}} W) \arrow[d, "\mathrm{id}\otimes\Theta_{V,W}"] & & \mathcal{F}(U \otimes_{\mathcal{C}} V) \otimes_{\mathcal{D}} \mathcal{F}(W) \arrow[d, "\Theta_{U,V}\otimes\mathrm{id}"] \\
\mathcal{F}(U) \otimes_{\mathcal{D}} \big(\mathcal{F}(V) \otimes_{\mathcal{D}} \mathcal{F}(W)\big) \arrow[rr, "\alpha^{\mathcal{D}}_{\mathcal{F}(U),\mathcal{F}(V),\mathcal{F}(W)}"] & & \big(\mathcal{F}(U) \otimes_{\mathcal{D}} \mathcal{F}(V)\big) \otimes_{\mathcal{D}} \mathcal{F}(W)
\end{tikzcd} \tag{7.24}$$

  commutes. Since $\mathcal{F}$ is an equivalence such an $\alpha^{\mathcal{D}}$ exists. Moreover, because $\alpha^{\mathcal{D}}$ is natural and $\mathcal{F}$ is essentially surjective, $\alpha^{\mathcal{D}}$ is unique. By construction $\alpha^{\mathcal{D}}$ satisfies the pentagon condition. The tensor unit structure (i.e. left and right unit morphisms) can be transported in the same way which makes $\mathcal{D}$ a monoidal category.
- Suppose $\mathcal{C}$ is in addition braided. *Transporting the braiding* from $\mathcal{C}$ to $\mathcal{D}$ means determining a family $c^{\mathcal{D}}_{X,Y}$ such that the following diagram commutes for all $U, V \in \mathcal{C}$:

$$\begin{tikzcd}
\mathcal{F}(U \otimes_{\mathcal{C}} V) \arrow[r, "\mathcal{F}(c^{\mathcal{C}}_{U,V})"] \arrow[d, "\Theta_{U,V}"] & \mathcal{F}(V \otimes_{\mathcal{C}} U) \arrow[d, "\Theta_{V,U}"] \\
\mathcal{F}(U) \otimes_{\mathcal{D}} \mathcal{F}(V) \arrow[r, "c^{\mathcal{D}}_{\mathcal{F}(U),\mathcal{F}(V)}"] & \mathcal{F}(V) \otimes_{\mathcal{D}} \mathcal{F}(U)
\end{tikzcd} \tag{7.25}$$

  For the same reasons as above this diagram determines the $c^{\mathcal{D}}_{X,Y}$ uniquely and turns $\mathcal{D}$ into a braided category.
- Suppose $\mathcal{C}$ is in addition ribbon. By *transporting the ribbon twist* we mean giving a natural family $\theta^{\mathcal{D}}_X$ such that for all $U \in \mathcal{C}$, $\theta^{\mathcal{D}}_{\mathcal{F}(U)} = \mathcal{F}(\theta^{\mathcal{C}}_U)$. Again, the family $\theta^{\mathcal{D}}_X$ is unique and turns $\mathcal{D}$ into a ribbon category (with braided monoidal structure as above).

Lemma 7.5. *The linear maps* $\alpha^{\mathbf{Rep}\,\mathsf{Q}}$, $c^{\mathbf{Rep}\,\mathsf{Q}}$, $\theta^{\mathbf{Rep}\,\mathsf{Q}}$ *in* (7.21)–(7.23) *are natural isomorphisms in* $\mathbf{Rep}\,\mathsf{Q}$. *There exists a multiplicative equivalence* $\mathcal{F}\colon \mathcal{SF} \to \mathbf{Rep}\,\mathsf{Q}$ *which transports*

- *the associator* (6.3) *of* $\mathcal{SF}$ *to* $\alpha^{\mathbf{Rep}\,\mathsf{Q}}$, *and the unit isomorphisms of* $\mathcal{SF}$ *to those of the underlying vector spaces in* $\mathbf{Rep}\,\mathsf{Q}$,

- *the braiding* (6.4) *of $\mathcal{SF}$ to* $c^{\mathbf{Rep}\,\mathsf{Q}}$,
- *the ribbon twist* (6.6) *of $\mathcal{SF}$ to* $\theta^{\mathbf{Rep}\,\mathsf{Q}}$.

The proof of this lemma is lengthy and tedious (and fills half of this paper). It is spread across the Appendices B and C. In Appendix B we transport the structure morphisms of $\mathcal{SF}$ to an intermediate category of representations of a quasi-bialgebra in **Svect**. In Appendix C we transport the structure morphisms further to a quasi-bialgebra in **vect** which, finally, we exhibit to be a twisting of Q.

Proof of Proposition 7.2. By Lemma 7.5 $\alpha^{\mathbf{Rep}\,\mathsf{Q}}$ fulfils the pentagon identity (since $\alpha^{\mathcal{SF}}$ does and $\mathcal{F}$ is multiplicative) and $c^{\mathbf{Rep}\,\mathsf{Q}}$ the hexagon identities. Hence, $\mathbf{Rep}\,\mathsf{Q}$ is braided monoidal. We conclude that $(\mathsf{Q},\cdot,\mathbf{1},\Delta,\varepsilon,R)$ is a quasi-triangular quasi-bialgebra.

We will now show that $S,\boldsymbol{\alpha},\boldsymbol{\beta}$ define a quasi-Hopf structure on Q, see Section 1.1 for definitions. A straightforward calculation shows that S, as defined on generators in (7.12), is compatible with the relations on Q and hence provides an algebra anti-homomorphism on Q. It remains to show the identities

$$\sum_{(a)} S(a')\boldsymbol{\alpha}a'' = \varepsilon(a)\boldsymbol{\alpha}\ , \qquad \sum_{(a)} a'\boldsymbol{\beta}S(a'') = \varepsilon(a)\boldsymbol{\beta}\ , \tag{7.26}$$

for all $a \in \mathsf{S}$ and

$$\sum_{(\Phi)} S(\Phi_1)\boldsymbol{\alpha}\Phi_2\boldsymbol{\beta}S(\Phi_3) = \mathbf{1}\ , \qquad \sum_{(\Phi^{-1})} (\Phi^{-1})_1\boldsymbol{\beta}S((\Phi^{-1})_2)\boldsymbol{\alpha}(\Phi^{-1})_3 = \mathbf{1}\ . \tag{7.27}$$

For example, to see the first equality in (7.27) one computes

$$\begin{aligned}\sum_{(\Phi)} S(\Phi_1)\boldsymbol{\alpha}\Phi_2\boldsymbol{\beta}S(\Phi_3) &= \boldsymbol{e}_0 + \boldsymbol{e}_1\big(\beta^2(\mathrm{i}\mathsf{K})^{\mathrm{N}}\big)S\big(\beta^2(\mathrm{i}\mathsf{K})^{\mathrm{N}}\big)\\ &= \boldsymbol{e}_0 + \boldsymbol{e}_1\mathsf{K}^{\mathrm{N}}((-1)^{\mathrm{N}}\mathsf{K})^{\mathrm{N}} = \boldsymbol{e}_0 + \boldsymbol{e}_1(-1)^{\mathrm{N}^2-\mathrm{N}} = \mathbf{1}\ .\end{aligned} \tag{7.28}$$

To see the first identity in (7.26) we define the linear map $P := \mu\circ(S\otimes\mathrm{id})\colon \mathsf{Q}\otimes\mathsf{Q}\to\mathsf{Q}$ where μ denotes the multiplication in Q and check that

$$\begin{aligned}P\big(\Delta(\mathsf{f}_i^{\pm})\big) &= S(\mathsf{f}_i^{\pm}) + S\big((\boldsymbol{e}_0 \pm \mathrm{i}\boldsymbol{e}_1)\mathsf{K}\big)\mathsf{f}_i^{\pm}\\ &= \mathsf{f}_i^{\pm}\mathsf{K}(\boldsymbol{e}_0 \pm (-1)^{\mathrm{N}}\,\mathrm{i}\boldsymbol{e}_1) + (\boldsymbol{e}_0 \pm (-1)^{\mathrm{N}}\,\mathrm{i}\boldsymbol{e}_1)\mathsf{K}\mathsf{f}_i^{\pm} = 0\ ,\\ P\big(\Delta(\mathsf{K})\big) &= (\boldsymbol{e}_0 + (-1)^{\mathrm{N}}\boldsymbol{e}_1)\mathsf{K}^2 - (1+(-1)^{\mathrm{N}})\boldsymbol{e}_1\mathsf{K}^2 = \mathbf{1}\ .\end{aligned} \tag{7.29}$$

For general basis elements $\mathsf{f}_{i_1}^{\varepsilon_1}\cdots\mathsf{f}_{i_m}^{\varepsilon_m}\mathsf{K}^n$ from (7.7) and by defining $f_1 := \mathsf{f}_{i_1}^{\varepsilon_1}$ and $f := \mathsf{f}_{i_2}^{\varepsilon_1}\cdots\mathsf{f}_{i_m}^{\varepsilon_m}\mathsf{K}^n$ we get

$$\begin{aligned}P\big(\Delta(\mathsf{f}_{i_1}^{\varepsilon_1}\cdots\mathsf{f}_{i_m}^{\varepsilon_m}\mathsf{K}^n)\big) &= P\big(\Delta(f_1)\Delta(f)\big) = \sum_{(f_1),(f)} S(f_1'f')f_1''f''\\ &= \sum_{(f)} S(f')P\big(\Delta(f_1)\big)f'' \overset{(7.29)}{=} 0\ ,\end{aligned} \tag{7.30}$$

$$P\big(\Delta(\mathsf{K}^n)\big) = \sum_{(\mathsf{K}^{n-1})} S\big((\mathsf{K}^{n-1})'\big) P\big(\Delta(\mathsf{K})\big)(\mathsf{K}^{n-1})'' = P\big(\Delta(\mathsf{K}^{n-1})\big) = \mathbf{1} ,$$

where the last equality follows by induction in n. The second identity in (7.26) can be shown in a similar way.

Having a quasi-Hopf structure we can now conclude from Lemma 7.5 that $\boldsymbol{v}$ defines a ribbon element in Q. In particular, $S(\boldsymbol{v}) = \boldsymbol{v}$ follows from the duality property of the twist $\theta_{U^*} = (\theta_U)^*$ (one can verify that if this equality holds for one choice of left duality $(-)^*$ on the category in question it holds for all choices of left duality). □

Another important consequence of Lemma 7.5 is the following theorem.

Theorem 7.6. *$\mathcal{SF}(N,\beta)$ is ribbon equivalent to* $\mathbf{Rep}\,\mathsf{Q}(N,\beta)$ *for all choices of N and β as in* (6.1).

The precise definition of the equivalence functor $\mathcal{F}\colon \mathcal{SF} \to \mathbf{Rep}\,\mathsf{Q}$ is given in Section C.8.

Remark 7.7. For fixed N, the quasi-triangular quasi-Hopf algebras $\mathsf{Q}(\mathrm{N},\beta)$ for the four possible choices of β differ by abelian 3-cocycles for $\mathbb{Z}_2$. The 3rd abelian group cohomology of $\mathbb{Z}_2$ is $H^3_{ab}(\mathbb{Z}_2,\mathbb{C}^\times) = \mathbb{Z}_4$ and it describes possible braided monoidal structures on $\mathbf{Rep}\,\mathbb{Z}_2$, up to braided monoidal equivalence (see [**JS**] for details). A generator of $H^3_{ab}(\mathbb{Z}_2,\mathbb{C}^\times)$ is given by the class of (ω,σ), where ω is a 3-cocycle for group-cohomology with (writing $\mathbb{Z}_2 = \{0,1\}$ additively) $\omega(1,1,1) = -1$ and 1 else, and σ is a 2-cochain with only non-trivial value $\sigma(1,1) = \mathrm{i}$. Multiplying the coassociator Φ by $\sum_{a,b,c\in\mathbb{Z}_2}\omega(a,b,c)\cdot \boldsymbol{e}_a\otimes\boldsymbol{e}_b\otimes\boldsymbol{e}_c$ and the R-matrix by $\sum_{a,b\in\mathbb{Z}_2}\sigma(a,b)\cdot\boldsymbol{e}_a\otimes\boldsymbol{e}_b$ is equivalent to changing β to $\mathrm{i}\beta$. Repeating this makes β run through its four possibilities.

7.2. Factorisability of Q

A finite-dimensional quasi-triangular quasi-Hopf algebra is called *factorisable* if its representation category is factorisable in the sense of Definition 1.4. A direct definition in terms of the data of Q can be found in Remark 1.5. It is shown in Corollary 5.14 that this definition is equivalent to the one given in [**BT**, Def. 2.1]. We obtain the following corollary to Theorem 7.6:

Corollary 7.8. Q *is a factorisable ribbon quasi-Hopf algebra.*

Below in Lemma 8.2 we will give an alternative proof by verifying non-degeneracy of the Hopf pairing of the universal Hopf algebra in $\mathbf{Rep}\,\mathsf{Q}$ by direct calculation.

7.3. $\mathsf{Q}(2n,1)$ as a Drinfeld double

In Remark 7.1 we saw that for $\mathrm{N} = 2n$ and $\beta^2 = 1$, $\mathsf{Q}(\mathrm{N},\beta)$ is a Hopf algebra, not only a quasi-Hopf algebra. Combining this with Corollary 7.8 shows that $\mathsf{Q}(2n,\pm1)$ is a factorisable ribbon Hopf algebra. It turns out that as a quasi-triangular Hopf algebra, $\mathsf{Q}(2n,1)$ is isomorphic to a Drinfeld double, as we now explain.

Let $H = H(\mathrm{N})$ be the algebra generated by the elements k and f_i, $i = 1,\dots,\mathrm{N}$, subject to the relations

$$\{f_i, f_j\} = 0 \ , \qquad \{f_i, k\} = 0 \ , \qquad k^2 = \mathbf{1} \ . \tag{7.31}$$

We define the coproduct, counit and antipode on $H(\mathrm{N})$ as ($i = 1,\dots,\mathrm{N}$)

$$\begin{aligned} &\Delta(f_i) = f_i \otimes k + \mathbf{1} \otimes f_i \ , \qquad \Delta(k) = k \otimes k \ , \\ &\varepsilon(f_i) = 0 \ , \qquad \varepsilon(k) = 1 \ , \qquad S(f_i) = -f_i k \ , \qquad S(k) = k \ . \end{aligned} \tag{7.32}$$

One can easily verify that we get an injective Hopf-algebra homomorphism $H(\mathrm{N}) \to \mathsf{Q}(\mathrm{N},1)$ which is defined on generators by (see Appendix D for details)

$$k \longmapsto \omega_- = (\boldsymbol{e}_0 - i\boldsymbol{e}_1)\mathsf{K} \ , \qquad f_i \longmapsto \mathsf{f}_i^- \omega_- \ , \quad i = 1,\dots,\mathrm{N} \ . \tag{7.33}$$

This embedding also proves that $H(\mathrm{N})$ is indeed a Hopf algebra.

Remark 7.9. Above we defined $H(\mathrm{N})$ for even N, but the definition works just as well for odd N. In this case, the map (7.33) defines an embedding $H(\mathrm{N}) \to \mathsf{Q}(\mathrm{N}+1,1)$ (or into any $\mathsf{Q}(2n,1)$ with $2n > \mathrm{N}$) and therefore $H(\mathrm{N})$ is a Hopf algebra for all $\mathrm{N} \in \mathbb{N}$. Note that $H(1)$ is Sweedler's 4-dimensional Hopf algebra, so that for $\mathrm{N} > 1$ we have a generalisation of Sweedler's Hopf algebra which is different from the Taft Hopf algebra. In fact, $H(\mathrm{N})$ is the Hopf algebra associated to (or bosonisation of) the super-group algebra $\Lambda\mathbb{C}^{\mathrm{N}}$, see e.g. [**AEG**, Sec. 3.4].

Proposition 7.10. *For $N \geq 1$, the Drinfeld double of $H(N)$ is isomorphic to $\mathsf{Q}(N,\beta)$ as a $\mathbb{C}$-algebra. For even N and $\beta = \pm 1$ the Drinfeld double of $H(N)$ is isomorphic to $\mathsf{Q}(N,\beta)$ as a Hopf algebra. Moreover, if $\beta = 1$ this isomorphism is an isomorphism of quasi-triangular Hopf algebras.*

The proof of this proposition is given in Appendix D.

Remark 7.11.

(1) The double of $H(\mathrm{N})$ has also been constructed in [**BN1**]. It appears in [**GS**] in the classification of factorisable tensor categories which contain $\mathbf{Rep}\,H(\mathrm{N})$ as a Lagrangian subcategory (and of which $\mathcal{SF}(\mathrm{N},\beta)$ provide four of the 16 possible cases).

(2) Following Remark 7.7, the quasi-triangular quasi-Hopf algebras $\mathsf{Q}(2n,\beta)$ for the other three choices of β are simple modifications of the Drinfeld double of $H(2n)$ by the 3rd abelian cohomology classes of $\mathbb{Z}_2$.

(3) It is shown in [**DR3**, Thm. 6.11] that $\mathcal{SF}(\mathrm{N},\beta)$ contains a Lagrangian algebra L iff N is even and $\beta = 1$ (see [**DR3**] for definition and references). This implies that precisely in these cases $\mathcal{SF}(\mathrm{N},\beta)$ is equivalent as a braided category to the Drinfeld centre $\mathcal{Z}(\mathcal{D})$ of some other (non-unique) finite tensor category $\mathcal{D}$ [**DMNO**, Cor. 4.1], e.g. one may choose $\mathcal{D} = L$-mod. Proposition 7.10 shows that $\mathcal{D}$ can also be taken to be $\mathbf{Rep}\,H(\mathrm{N})$.

7.4. Some special elements of Q

The Drinfeld twist $\boldsymbol{f}$ of a quasi-Hopf algebra expresses the deviation of the antipode from being an anti-coalgebra map via

$$\boldsymbol{f}\Delta(S(a)) = (S\otimes S)(\Delta^{\mathrm{op}}(a))\boldsymbol{f} \;, \qquad a\in A \;. \tag{7.34}$$

Its expression in terms of quasi-Hopf algebra structure maps is given in (1.15) following [**Dr2**]. When evaluated for Q, one quickly checks that the general expression reduces to

$$\boldsymbol{f} = \boldsymbol{e}_0\otimes\mathbf{1} + \boldsymbol{e}_1\otimes\mathsf{K}^{\mathrm{N}}\boldsymbol{e}_0 + \beta^2(-\mathrm{i}\mathsf{K})^{\mathrm{N}}\boldsymbol{e}_1\otimes\boldsymbol{e}_1 \;. \tag{7.35}$$

As reviewed in Section 2.2.2, the canonical Drinfeld element $\boldsymbol{u}$ and the corresponding element $\tilde{\boldsymbol{u}}$ with inverse braiding defined as

$$\begin{aligned}\boldsymbol{u} &= \sum_{(\Phi),(R)} S\big(\Phi_2\boldsymbol{\beta}S(\Phi_3)\big)S(R_2)\boldsymbol{\alpha}R_1\Phi_1 \;,\\ \tilde{\boldsymbol{u}} &= \sum_{(\Phi),(R^{-1})} S\big(\Phi_2\boldsymbol{\beta}S(\Phi_3)\big)S\big((R^{-1})_2\big)\boldsymbol{\alpha}(R^{-1})_1\Phi_1 \;.\end{aligned} \tag{7.36}$$

Lemma 7.12. *In Q the elements $\boldsymbol{u}$, $\tilde{\boldsymbol{u}}$ and $\boldsymbol{u}^{-1}$ take the form*

$$\begin{aligned}\boldsymbol{u} &= \Big(\boldsymbol{e}_0\mathsf{K} + \boldsymbol{e}_1\beta(-\mathrm{i}\mathsf{K})^N\Big)\cdot\prod_{i=1}^{N}(\mathbf{1}-2\mathsf{f}_i^+\mathsf{f}_i^-) \;,\\ \tilde{\boldsymbol{u}} = \boldsymbol{u}^{-1} &= \big(\boldsymbol{e}_0\mathsf{K} + \boldsymbol{e}_1\beta^{-1}(-\mathrm{i}\mathsf{K})^N\big)\cdot\prod_{i=1}^{N}(\mathbf{1}+2\mathsf{f}_i^+\mathsf{f}_i^-\mathsf{K}^2) \;.\end{aligned} \tag{7.37}$$

Proof. We start with the expression for $\boldsymbol{u}$ in (7.37). We give the details for sector $\mathbf{0}$, the computation in sector $\mathbf{1}$ is similar.

In sector $\mathbf{0}$, the first equality in (7.36) reduces to $\boldsymbol{u}\cdot\boldsymbol{e}_0 = S(R_2)R_1\cdot\boldsymbol{e}_0$. Define the linear map $T\colon \mathsf{Q}\otimes\mathsf{Q}\to\mathsf{Q},\ a\otimes b\mapsto S(b)a$, so that $\boldsymbol{u}\cdot\boldsymbol{e}_0 = T(R)\cdot\boldsymbol{e}_0$. The last factor of R in (7.15) can be written as

$$X := \prod_{i=1}^{\mathrm{N}}(\mathbf{1}\otimes\mathbf{1} - 2\mathsf{f}_i^-\omega_-\otimes\mathsf{f}_i^+) = \mathbf{1}\otimes\mathbf{1} + \sum_{\substack{1\le m\le \mathrm{N}\\ 1\le i_1<\ldots<i_m\le\mathrm{N}}} (-2)^m\mathsf{f}_{i_1}^-\omega_-\cdots\mathsf{f}_{i_m}^-\omega_-\otimes\mathsf{f}_{i_1}^+\cdots\mathsf{f}_{i_m}^+ \;. \tag{7.38}$$

We need to compute

$$T(R)\boldsymbol{e}_0 = T\big(\tfrac12(\mathbf{1}\otimes\mathbf{1}+\mathbf{1}\otimes\mathsf{K}+\mathsf{K}\otimes\mathbf{1}-\mathsf{K}\otimes\mathsf{K})\cdot X\big)\cdot\boldsymbol{e}_0 \;. \tag{7.39}$$

To do so we use that for $a,b,x,y\in\mathsf{Q}$ such that $S(y)T(a\otimes b) = \pm T(a\otimes b)S(y)$ we have

$$T(a\otimes b\cdot x\otimes y) = \pm T(a\otimes b)\cdot T(x\otimes y) \;. \tag{7.40}$$

Then, together with $\omega_-\boldsymbol{e}_0 = \mathsf{K}\boldsymbol{e}_0$,

$$T\big(\tfrac12(\mathbf{1}\otimes\mathbf{1}-\mathsf{K}\otimes\mathsf{K})\cdot X\big)\cdot\boldsymbol{e}_0 = (1-\mathsf{K}^2)T(X)\boldsymbol{e}_0 = 0 \tag{7.41}$$

$$
\begin{aligned}
T\big(\tfrac{1}{2}(\mathbf{1}\otimes\mathsf{K}+\mathsf{K}\otimes\mathbf{1})\cdot X\big)\cdot \boldsymbol{e}_0 &= \mathsf{K}\Big(\mathbf{1}+\sum 2^m \mathsf{f}^+_{i_m}\mathsf{K}\cdots\mathsf{f}^+_{i_1}\mathsf{K}\cdot\mathsf{f}^-_{i_1}\mathsf{K}\cdots\mathsf{f}^-_{i_m}\mathsf{K}\Big)\cdot\boldsymbol{e}_0\\
&= \mathsf{K}\Big(\mathbf{1}+\sum(-2)^m \mathsf{f}^+_{i_1}\mathsf{f}^-_{i_1}\cdots\mathsf{f}^+_{i_m}\mathsf{f}^-_{i_m}\Big)\cdot\boldsymbol{e}_0\\
&= \mathsf{K}\prod_{i=1}^{\mathrm{N}}(\mathbf{1}-2\mathsf{f}^+_i\mathsf{f}^-_i)\cdot\boldsymbol{e}_0\ ,
\end{aligned}
$$

as required.

The expression for $\boldsymbol{u}^{-1}$ is an immediate consequence of the following identity, which is easily verified:

$$
\Big(\prod_{i=1}^{\mathrm{N}}(\mathbf{1}-2\mathsf{f}^+_i\mathsf{f}^-_i)\Big)^{-1} = \prod_{i=1}^{\mathrm{N}}(\mathbf{1}+2\mathsf{f}^+_i\mathsf{f}^-_i\mathsf{K}^2)\ . \tag{7.42}
$$

Finally, $\tilde{\boldsymbol{u}}$ can be calculated from the identity $\tilde{\boldsymbol{u}} = S(\boldsymbol{u}^{-1})$, see (2.54). □

In a ribbon quasi-Hopf algebra, we define the balancing element as [**AC**]

$$
\boldsymbol{g} = \boldsymbol{\beta}S(\boldsymbol{\alpha})\boldsymbol{v}^{-1}\boldsymbol{u}\ , \tag{7.43}
$$

with the canonical Drinfeld element $\boldsymbol{u}$ defined in (7.36). The balancing element $\boldsymbol{g}$ is group-like. We recall from Section 2.2.4 that the pivotal structure is given by the action with $\boldsymbol{v}^{-1}\boldsymbol{u}$ (as in the Hopf case). The balancing $\boldsymbol{g}$ appears in the expression for the categorical trace $\mathrm{tr}^{\mathcal{C}}$ of a morphism $f: M\to M$:

$$
\mathrm{tr}^{\mathcal{C}}(f) = \tilde{\mathrm{ev}}_M\circ(f\otimes\mathrm{id})\circ\mathrm{coev}_M = \mathrm{tr}_M(\boldsymbol{g}\circ f)\ , \tag{7.44}
$$

where we used the expressions for evaluation and coevaluation in (2.31) and (2.53). In particular, the quantum dimension of M is $\mathrm{tr}_M(\boldsymbol{g})$.

Combining (7.12), (7.16) and (7.37), in Q we find the balancing element to be

$$
\boldsymbol{g} = (\boldsymbol{e}_0 - (-1)^{\mathrm{N}}\,\mathrm{i}\beta^2\boldsymbol{e}_1)\mathsf{K}\ . \tag{7.45}
$$

Remark 7.13. For even N the element $\boldsymbol{g}$ equals $\omega_\pm$ for $\beta^2=\mp 1$, while for N odd $\boldsymbol{g}=\mathsf{K}^{\pm1}$ for $\beta^2=\mp\mathrm{i}$. Given a pivotal structure on a monoidal category with left duals, all other pivotal structures are obtained by composing the given one with natural monoidal automorphisms of the identity functor. In **Rep** Q, these are given by acting with group-like elements in the centre of Q. It follows from Proposition 7.15 below that these are precisely $\{\mathbf{1},\mathsf{K}^2\}$. Modifying the pivotal structure by K^2 has the effect of replacing β^2 by $-\beta^2$ in (7.45).

7.5. Integrals

A *two-sided integral* of a quasi-Hopf algebra A is an element $\boldsymbol{c}\in A$ such that [**HN**, Def. 4.1]

$$
\boldsymbol{c}a = \varepsilon(a)\boldsymbol{c} = a\boldsymbol{c}\quad,\quad\text{for all } a\in A\ . \tag{7.46}
$$

In [**BT**, Sec. 6] it is shown that factorisable quasi-Hopf algebras are unimodular. Together with [**HN**, Thm. 4.3] this shows that the space of two-sided integrals is one-dimensional. It is easy to see that every element in Q of the form

$$\boldsymbol{c} = \nu\, 2^{\mathrm{N}} \beta^2\, \mathsf{f}_1^+ \mathsf{f}_1^- \dots \mathsf{f}_{\mathrm{N}}^+ \mathsf{f}_{\mathrm{N}}^-\, \boldsymbol{e}_0 (1 + \mathsf{K}) \quad , \quad \nu \in \mathbb{C} \ , \tag{7.47}$$

satisfies (7.46). Indeed, $\boldsymbol{c}\,\mathsf{K} = \boldsymbol{c} = \mathsf{K}\boldsymbol{c}$ and $\boldsymbol{c}\,\mathsf{f}_i^\pm = 0 = \mathsf{f}_i^\pm \boldsymbol{c}$. The prefactor $\nu 2^{\mathrm{N}} \beta^2$ will be convenient later, when a normalisation condition will require the constant ν to be a sign (see Proposition 8.4 below).

7.6. The centre of Q

We define

$$\boldsymbol{e}_1^\pm := \frac{1}{2} \boldsymbol{e}_1 \big(\mathbf{1} \mp \mathrm{i}\mathsf{K} \prod_{i=1}^{\mathrm{N}} (\mathbf{1} - 2\mathsf{f}_i^+ \mathsf{f}_i^-) \big) \ . \tag{7.48}$$

Lemma 7.14. *The elements $\boldsymbol{e}_1^\pm$ are central orthogonal idempotents. Moreover, the ribbon twist acts on $\boldsymbol{e}_1^\pm$ by a scalar:*

$$\boldsymbol{v}^{-1} \cdot \boldsymbol{e}_1^\pm = \pm\, \beta^{-1} \boldsymbol{e}_1^\pm \ . \tag{7.49}$$

Proof. It is easy to see the commutativity property of $\boldsymbol{e}_1^\pm$. For example, since

$$\mathsf{K}(\mathbf{1} - 2\mathsf{f}_i^+ \mathsf{f}_i^-) \mathsf{f}_i^+ \boldsymbol{e}_1 = -\mathsf{K}\mathsf{f}_i^+ \boldsymbol{e}_1 = \mathsf{f}_i^+ \mathsf{K} (\mathbf{1} - 2\mathsf{f}_i^+ \mathsf{f}_i^-) \boldsymbol{e}_1 \tag{7.50}$$

$\boldsymbol{e}_1^\pm$ commutes with f_i^+.

The orthogonality and idempotent property follow immediately from $\boldsymbol{e}_1 (\mathbf{1} - 2\mathsf{f}_i^+ \mathsf{f}_i^-)^2 = \boldsymbol{e}_1$, see (7.42). In order to prove (7.49) we express $\boldsymbol{e}_1^\pm$ in terms of $\boldsymbol{v}$ from (7.16) as

$$\boldsymbol{e}_1^\pm = \frac{1}{2} \boldsymbol{e}_1 (\mathbf{1} \pm \beta^{-1} \boldsymbol{v}) \quad . \tag{7.51}$$

Together with $\beta^{-1} \boldsymbol{v} \boldsymbol{e}_1 = \beta \boldsymbol{v}^{-1} \boldsymbol{e}_1$, this gives

$$\boldsymbol{v}^{-1} \boldsymbol{e}_1^\pm = \tfrac{1}{2} \boldsymbol{e}_1 (\boldsymbol{v}^{-1} \pm \beta^{-1} \mathbf{1}) = \pm \beta^{-1} \tfrac{1}{2} \boldsymbol{e}_1 (\pm \beta \boldsymbol{v}^{-1} + \mathbf{1}) = \pm \beta^{-1} \boldsymbol{e}_1^\pm \quad . \tag{7.52}$$

□

Proposition 7.15. *The centre of Q is $\mathcal{Z}(\mathsf{Q}) = \mathcal{Z}_0 \oplus \mathcal{Z}_1$, where*

$$\begin{aligned} \mathcal{Z}_0 \ &:= \ \mathrm{span}_{\mathbb{C}} \Big\{ \boldsymbol{e}_0 \prod_{j=1}^{2k} \mathsf{f}_{i_j}^{\varepsilon_j} \,\Big|\, k \le N/2\,,\ i_j = 1, \dots, N,\ \varepsilon_j = \pm \Big\} \ \oplus \ \mathbb{C}\, \mathsf{K} \boldsymbol{e}_0 \prod_{i=1}^{N} \mathsf{f}_i^+ \mathsf{f}_i^- \ , \\ \mathcal{Z}_1 \ &:= \ \mathrm{span}_{\mathbb{C}} \big\{ \boldsymbol{e}_1^+, \boldsymbol{e}_1^- \big\} \ . \end{aligned} \tag{7.53}$$

It has dimension $3 + 2^{2N-1}$.

PROOF. From (7.5) we know that the ideal Q_1 is a direct sum of two matrix algebras, so its centre is two-dimensional and is therefore spanned by the central idempotents $\boldsymbol{e}_1^{\pm}$ from Lemma 7.14. It remains to compute the centre $\mathcal{Z}_0$ of Q_0. For an element of Q_0 to commute with K, it must be a sum of monomials in $\mathsf{f}_i^{\pm}\boldsymbol{e}_0$ of even degree, where each monomial can be multiplied by K, that is,

$$\mathcal{Z}_0 \subset \operatorname{span}_{\mathbb{C}}\Big\{\mathsf{K}^{\delta}\prod_{j=1}^{2k}\mathsf{f}_{i_j}^{\varepsilon_j}\boldsymbol{e}_0 \,\Big|\, k \leq \mathrm{N}/2\,,\; i_j = 1,\ldots,\mathrm{N}\,,\; \varepsilon_j = \pm\,,\; \delta \in \{0,1\}\Big\}\;. \tag{7.54}$$

Any monomial from RHS of (7.54) with $\delta = 0$ obviously commutes with $\mathsf{f}_i^{\pm}$, for $1 \leq i \leq \mathrm{N}$, while a monomial with $\delta = 1$ commutes with all $\mathsf{f}_i^{\pm}$ iff it is annihilated by all $\mathsf{f}_i^{\pm}$. This gives the expression for $\mathcal{Z}_0$ in (7.53). Since the $\mathbb{C}$-linear span of the even degree monomials (multiplied by $\boldsymbol{e}_0$) has the dimension $2^{2\mathrm{N}-1}$, the overall dimension of the centre is

$$\dim_{\mathbb{C}} \mathcal{Z}(\mathsf{Q}) = 3 + 2^{2\mathrm{N}-1}\;. \tag{7.55}$$

□

7.7. Simple and projective Q-modules

Recall the decomposition (7.4) of Q onto the direct sum of two algebras $\mathsf{Q}_0 \oplus \mathsf{Q}_1$ where the first is the Graßmann algebra times $\mathbb{Z}_2$, which is non-semisimple, while the second is the Clifford algebra times $\mathbb{Z}_2$, which is semisimple. Therefore, the algebra Q has up to an isomorphism only four simple modules that we will denote as $\mathsf{X}_s^{\pm}$, with $s = 0, 1$, and where $\mathsf{X}_0^{\pm} \in \mathbf{Rep}\,\mathsf{Q}_0$ while $\mathsf{X}_1^{\pm} \in \mathbf{Rep}\,\mathsf{Q}_1$. They are of highest-weight type: $\mathsf{X}_0^{\pm}$ are one-dimensional of weights ± 1 with respect to K and with zero action of $\mathsf{f}_i^{\pm}$, i.e. they are spanned by $v_0^{\pm}$ such that

$$\mathsf{K}.v_0^{\pm} = \pm v_0^{\pm}\;, \qquad \mathsf{f}_i^{\pm}.v_0^{\pm} = 0\;; \tag{7.56}$$

$\mathsf{X}_1^{\pm}$ are of the highest weights $\pm\mathrm{i}$, i.e. they are generated by $v_1^{\pm}$ such that

$$\mathsf{K}.v_1^{\pm} = \pm\mathrm{i}v_1^{\pm}\;, \qquad \mathsf{f}_i^{-}.v_1^{\pm} = 0\;. \tag{7.57}$$

A basis of $\mathsf{X}_1^{\pm}$ is given by the set

$$\Big\{\, v_i^{\pm} := \prod_{k=1}^{\mathrm{N}}(\mathsf{f}_k^{+})^{i_k}.v_1^{\pm} \,\Big|\, i = (i_1,\ldots,i_{\mathrm{N}})\,,\; i_k \in \{0,1\}\,\Big\}\;. \tag{7.58}$$

In particular, the dimension of $\mathsf{X}_1^{\pm}$ is 2^{N}. In terms of the primitive central idempotents $\boldsymbol{e}_1^{\pm}$ from (7.48), the modules $\mathsf{X}_1^{\pm}$ are $\mathsf{Q}\boldsymbol{e}_1^{\pm}$ and the generating vector $v_1^{\pm}$ can be obtained within Q as $\prod_{i=1}^{\mathrm{N}}\mathsf{f}_i^{-}\boldsymbol{e}_1^{\pm}$. The modules $\mathsf{X}_1^{\pm}$ are therefore projective.

We note that

$$\boldsymbol{e}_0^{\pm} = \frac{1}{2}(\mathbf{1} \pm \mathsf{K})\boldsymbol{e}_0 \tag{7.59}$$

are primitive (non-central) idempotents and $\mathsf{K}\boldsymbol{e}_0^{\pm} = \boldsymbol{e}_0^{\pm}\mathsf{K} = \pm\boldsymbol{e}_0^{\pm}$. The module $\mathsf{P}_0^{\pm} := \mathsf{Q}\boldsymbol{e}_0^{\pm}$ is therefore a projective cover for $\mathsf{X}_0^{\pm}$. It is indecomposable but reducible and has the basis

$$\mathsf{P}_0^{\pm} \;:\; \mathrm{span}\Big\{ \Big(\prod_{k=1}^{\mathrm{N}} (\mathsf{f}_k^+)^{i_k^+} (\mathsf{f}_k^-)^{i_k^-} \Big) \boldsymbol{e}_0^{\pm} \,\Big|\, i_k^+, i_k^- \in \{0,1\} \Big\} \; . \tag{7.60}$$

The dimension of $\mathsf{P}_0^{\pm}$ is thus $2^{2\mathrm{N}}$ and each of them has $2^{2\mathrm{N}-1}$ copies of $\mathsf{X}_0^{\pm}$ in its composition series. We can finally conclude that the *Cartan matrix*[1] $\mathsf{C}(\mathsf{Q})$ is

$$\mathsf{C}(\mathsf{Q}) \;=\; \begin{pmatrix} 2^{2\mathrm{N}-1} & 2^{2\mathrm{N}-1} & 0 & 0 \\ 2^{2\mathrm{N}-1} & 2^{2\mathrm{N}-1} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \; . \tag{7.61}$$

One can check now the dimension of Q by decomposing it as the left regular representation: $\mathsf{Q} = \mathsf{P}_0^+ \oplus \mathsf{P}_0^- \oplus 2^{\mathrm{N}}\mathsf{X}_1^+ \oplus 2^{\mathrm{N}}\mathsf{X}_1^-$ and this indeed gives $\dim \mathsf{Q} = 2^{2\mathrm{N}+2}$.

7.8. Basic algebra

The basic algebra of Q is $E := \mathrm{End}_{\mathsf{Q}}(G_{\mathsf{Q}})$ where G_{Q} is the minimal projective generator

$$G_{\mathsf{Q}} = \mathsf{P}_0^+ \oplus \mathsf{P}_0^- \oplus \mathsf{X}_1^+ \oplus \mathsf{X}_1^- \; . \tag{7.62}$$

In what follows, we will need a description of this algebra. Recall from (7.5) that $\mathbb{A} := \mathbb{A}_{2\mathrm{N}}$ is the subalgebra in Q_0 generated by $\mathsf{f}_i^{\pm}\boldsymbol{e}_0$.

Lemma 7.16. *$E^{\mathrm{op}} = (\mathbb{A} \oplus \mathbb{C}e_{\mathsf{X}}) \rtimes \mathbb{C}\mathbb{Z}_2$, where*

- *e_{X} denotes the idempotent corresponding to $\mathsf{X}_1^+ \oplus \mathsf{X}_1^-$, i.e. it acts as identity on $\mathsf{X}_1^+ \oplus \mathsf{X}_1^-$ and as zero otherwise,*
- *$a \in \mathbb{A}$ acts on $\mathsf{P}_0^+ \oplus \mathsf{P}_0^- = \mathsf{Q}_0$ by right multiplication,*
- *the generator κ of the group algebra $\mathbb{C}\mathbb{Z}_2$ acts on $\mathsf{P}_0^+ \oplus \mathsf{P}_0^-$ by right multiplication with K which is $\pm\mathrm{id}$ on $\mathsf{P}_0^{\pm}$; on $\mathsf{X}_1^{\pm}$ it equally acts by $\pm\mathrm{id}$,*
- *the algebra structure is that of $\mathbb{A} \oplus \mathbb{C}e_{\mathsf{X}}$ with $\kappa a = -a\kappa$ for all $a = \mathsf{f}_i^{\pm}\boldsymbol{e}_0$, $\kappa e_{\mathsf{X}} = e_{\mathsf{X}}\kappa$.*

Proof. We have the decomposition $\mathrm{End}(G_{\mathsf{Q}}) = E_0 \oplus E_1$ where $E_0 = \mathrm{End}_{\mathsf{Q}}(\mathsf{P}_0^+ \oplus \mathsf{P}_0^-)$ and $E_1 = \mathrm{End}_{\mathsf{Q}}(\mathsf{X}_1^+ \oplus \mathsf{X}_1^-)$. The latter algebra is a direct sum of two matrix algebras of dimension 1 each, and is isomorphic to the algebra $\mathbb{C}e_{\mathsf{X}} \rtimes \mathbb{C}\mathbb{Z}_2$ from the statement. We then note that $\mathsf{P}_0^+ \oplus \mathsf{P}_0^-$ is equal to the left regular representation of Q_0, with the action by left multiplication. The algebra centralising this action is given by $\mathsf{Q}_0^{\mathrm{op}}$ which acts by right multiplication. Therefore $E_0^{\mathrm{op}} = \mathsf{Q}_0$. We then recall from (7.5) that $\mathsf{Q}_0 = \mathbb{A} \rtimes \mathbb{C}\mathbb{Z}_2$, with the same $\mathbb{Z}_2$ action as in the statement. The fact that K acts by $\pm\mathrm{id}$ on the direct summands $\mathsf{P}_0^{\pm}$ follows from the identification $\mathsf{P}_0^{\pm} := \mathsf{Q}\boldsymbol{e}_0^{\pm}$, recall (7.59). This finally proves the lemma. $\square$

[1]Its matrix elements are the multiplicities of the simple Q-module V in the composition series of the projective cover P_U of U, i.e. $\mathsf{C}(\mathsf{Q})_{U,V} = \mathrm{Hom}_{\mathsf{Q}}(P_V, P_U)$.

Remark 7.17. In the above lemma we give E^{op} instead of E as we describe the endomorphism algebra via a right action. However, from the defining relations of Q it is easy to give an algebra isomorphism $E^{\mathrm{op}} \to E$, e.g. via the antipode S. We also note that $\mathbf{Rep}\,\mathsf{Q}$ is equivalent to $\mathbf{Rep}\,E$ as abelian categories, i.e. Q and E are Morita equivalent, and E is the minimal algebra with such a property.

We recall then the equivalence stated in Theorem 7.6 (we need only the equivalence of abelian $\mathbb{C}$-linear categories) and given by the functor $\mathcal{F}\colon \mathcal{SF} \to \mathbf{Rep}\,\mathsf{Q}$. Under this functor, the projective covers are mapped as $\mathcal{F}(P_{\mathbf{1}}) = \mathsf{P}_0^+$, $\mathcal{F}(P_{\Pi\mathbf{1}}) = \mathsf{P}_0^-$, $\mathcal{F}(T) = \mathsf{X}_0^+$, and $\mathcal{F}(\Pi T) = \mathsf{X}_0^-$, recall (6.5). In particular, the minimal projective generator in $\mathcal{SF}$

$$G_{\mathcal{SF}} := P_{\mathbf{1}} \oplus P_{\Pi\mathbf{1}} \oplus T \oplus \Pi T = (\mathbb{\Lambda} \oplus T) \otimes \mathbb{C}^{1|1} \tag{7.63}$$

goes to $\mathcal{F}(G_{\mathcal{SF}}) \cong G_{\mathsf{Q}}$.

We denote the functor inverse to $\mathcal{F}$, as a $\mathbb{C}$-linear functor, by $\mathcal{J} : \mathbf{Rep}\,\mathsf{Q} \to \mathcal{SF}$ and $\mathcal{J} = \mathcal{E} \circ \mathcal{H}$, where $\mathcal{E}$ is given in the proof of Proposition B.3 and $\mathcal{H}$ in the proof of Proposition C.3. To describe $\mathrm{End}_{\mathcal{SF}}(G_{\mathcal{SF}})$ explicitly, consider the isomorphism $\psi : \mathbb{\Lambda} \otimes \mathbb{C}^{1|1} \xrightarrow{\sim} \mathcal{J}(\mathsf{Q}_0)$ given by

$$\psi\colon\ f \otimes v \mapsto f \cdot \big(v_0 \boldsymbol{e}_0^+ + v_1 \boldsymbol{e}_0^-\big)\ , \qquad f \in \mathbb{\Lambda},\ v \in \mathbb{C}^{1|1}\ , \tag{7.64}$$

where $v_{0/1} \in \mathbb{C}$ is the even/odd component of v. Using this isomorphism, we have that the endomorphism R_a of Q_0 from Lemma 7.16 given by the right multiplication with $a \in \mathbb{\Lambda}$ goes under the functor $\mathcal{J}$ to

$$\begin{aligned} \psi^{-1} \circ \mathcal{J}(R_a) \circ \psi\colon\ \mathbb{\Lambda} \otimes \mathbb{C}^{1|1} &\to \mathbb{\Lambda} \otimes \mathbb{C}^{1|1}\ , \\ f \otimes v &\mapsto f \cdot a \otimes \Pi^{\deg a} v\ . \end{aligned} \tag{7.65}$$

Indeed, the shift of the degree part is due to

$$\psi^{-1}(R_a(f\boldsymbol{e}_0^{\pm})) = \psi^{-1}(f\boldsymbol{e}_0^{\pm} \cdot a) = \psi^{-1}(f \cdot a\boldsymbol{e}_0^{\mp}) = f \cdot a \otimes \Pi(-)\ , \tag{7.66}$$

for odd a. Then, as a corollary to Lemma 7.16 we get:

Corollary 7.18. *For the minimal projective generator $G_{\mathcal{SF}}$ the opposite algebra of* $\mathrm{End}_{\mathcal{SF}}(G_{\mathcal{SF}})$ *is* $(\mathbb{\Lambda} \oplus \mathbb{C}e_T) \rtimes \mathbb{C}\mathbb{Z}_2$ *where*

- e_T *is the idempotent corresponding to* $T \oplus \Pi T$,
- *the generator* κ *acts by* id *on* $P_{\mathbf{1}}$ *and* T *and by* $-$id *on* $P_{\Pi\mathbf{1}}$ *and* ΠT,
- *the element* $a \in \mathbb{\Lambda}$ *acts as in* (7.65),
- *the algebra structure is that of* $\mathbb{\Lambda} \oplus \mathbb{C}e_T$ *with* $\kappa a = -a\kappa$ *for odd* a, $\kappa e_T = e_T\kappa$.

8. Properties of the coend in $\mathbf{Rep}\,\mathsf{Q}$

In this section we investigate the universal Hopf algebra $\mathcal{L}$ of $\mathbf{Rep}\,\mathsf{Q}$, which can be expressed as a coend of a certain functor. We refer to Sections 4.1 and 4.2 for general background and references on the universal Hopf algebra of a braided monoidal category with duals, and to Section 5.2 for the particular case of categories of representations of quasi-triangular quasi-Hopf algebras.

We give explicitly the Hopf algebra structure and Hopf pairing on the universal Hopf algebra $\mathcal{L}$ of $\mathbf{Rep}\,\mathsf{Q}$, we verify by direct calculation that the Hopf pairing is non-degenerate, and we describe the integrals and cointegrals of $\mathcal{L}$. In section 8.4, we also calculate the central elements corresponding to internal characters of $\mathcal{L}$.

8.1. The universal Hopf algebra $\mathcal{L}$

By Proposition 5.8, universal Hopf algebra – described by a coend $\mathcal{L}$ in $\mathbf{Rep}\,\mathsf{Q}$ – can be chosen to be the object $\mathcal{L} = \mathsf{Q}^*$ equipped with the coadjoint action

$$(8.1) \qquad \mathsf{Q} \otimes \mathsf{Q}^* \to \mathsf{Q}^* \ , \qquad a \otimes \varphi \mapsto \sum_{(a)} \varphi\big(S(a')(-)a''\big) \ ,$$

and the dinatural transformation

$$(8.2) \qquad \iota_M \colon M^* \otimes M \to \mathsf{Q}^* \ , \qquad \varphi \otimes m \mapsto \big(a \mapsto \varphi(a.m)\big) \ , \quad M \in \mathbf{Rep}\,\mathsf{Q}$$

(see Figure 1 for a string diagram representation).

We define the contraction maps

$$(8.3) \qquad \begin{aligned} &\langle -,- \rangle : \mathsf{Q}^* \otimes \mathsf{Q} \to \mathbb{C} \ , && \langle \varphi, a \rangle = \varphi(a) \ , \\ &\langle -,- \rangle : \mathsf{Q}^* \otimes \mathsf{Q}^* \otimes \mathsf{Q} \otimes \mathsf{Q} \to \mathbb{C} \ , && \langle \varphi \otimes \psi, a \otimes b \rangle = \varphi(b)\psi(a) \ . \end{aligned}$$

As in Section 5.2.2, we will express the Hopf algebra structure maps of $\mathcal{L}$ in terms of their duals as follows ($f, g \in \mathsf{Q}^*$, $a, b \in \mathsf{Q}$):

$$(8.4) \qquad \begin{aligned} \big\langle \mu_{\mathcal{L}}(f \otimes g)\, ,\, a \big\rangle &= \big\langle f \otimes g\, ,\, \hat{\mu}_{\mathcal{L}}(a) \big\rangle && , \quad \hat{\mu}_{\mathcal{L}} : \mathsf{Q} \to \mathsf{Q} \otimes \mathsf{Q} \ , \\ \big\langle \Delta_{\mathcal{L}}(f)\, ,\, a \otimes b \big\rangle &= \big\langle f\, ,\, \hat{\Delta}_{\mathcal{L}}(a \otimes b) \big\rangle && , \quad \hat{\Delta}_{\mathcal{L}} : \mathsf{Q} \otimes \mathsf{Q} \to \mathsf{Q} \ , \\ \eta_{\mathcal{L}}(1) &= \big(\, a \mapsto \hat{\eta}_{\mathcal{L}}(a)\,\big) && , \quad \hat{\eta}_{\mathcal{L}} : \mathsf{Q} \to \mathbb{C} \ , \\ \varepsilon_{\mathcal{L}}(f) &= f(\hat{\varepsilon}_{\mathcal{L}}) && , \quad \hat{\varepsilon}_{\mathcal{L}} \in \mathsf{Q} \ , \end{aligned}$$

$$\begin{aligned}\big\langle S_{\mathcal{L}}(f)\,,\,a\big\rangle &= \big\langle f\,,\,\hat{S}_{\mathcal{L}}(a)\big\rangle \qquad , \quad \hat{S}_{\mathcal{L}} : \mathsf{Q}\to\mathsf{Q}\ ,\\ \omega_{\mathcal{L}}(f\otimes g) &= \big\langle f\otimes g\,,\,\hat{\omega}_{\mathcal{L}}\big\rangle \qquad , \quad \hat{\omega}_{\mathcal{L}}\in\mathsf{Q}\otimes\mathsf{Q}\ .\end{aligned}$$

Proposition 8.1. *Via the dualisation in* (8.4)*, the Hopf algebra structure maps and the Hopf pairing on the coend $\mathcal{L}=\mathsf{Q}^*$ are given by*

$$\begin{aligned}\hat{\mu}_{\mathcal{L}}(a) &= \sum_{\substack{(R),(a),\\ n,m\in\mathbb{Z}_2}} \big(\mathsf{K}^{Nnm}R_2\mathsf{K}^{Nm}\big)\triangleright a'\otimes(\mathsf{K}^{Nn(1-m)}\triangleright a'')R_1\cdot\boldsymbol{e}_n\otimes\boldsymbol{e}_m\ , \qquad (8.5)\\ \hat{\Delta}_{\mathcal{L}}(a\otimes b) &= ba\ ,\\ \hat{\eta}_{\mathcal{L}}(a) &= \varepsilon(a)\ ,\\ \hat{\varepsilon}_{\mathcal{L}} &= \mathbf{1}\ ,\\ \hat{S}_{\mathcal{L}}(a) &= \sum_{(R)} S(aR_1)\tilde{\boldsymbol{u}}R_2\ ,\\ \hat{\omega}_{\mathcal{L}} &= \sum_{(M)} S(M_2)\otimes M_1\boldsymbol{e}_0+S(\mathsf{K}^{-N}M_2\mathsf{K}^N)\otimes M_1\boldsymbol{e}_1\ ,\end{aligned}$$

where $h\triangleright a:=\sum_{(h)}S(h')ah''$ defines an action of Q^{op} on Q, and $\tilde{\boldsymbol{u}}$ was defined in (7.37)*.*

Proof. Applying Theorem 5.10 the only non-trivial equalities in (8.5) are the first two and last one. Note that for the calculation of $\hat{\eta}_{\mathcal{L}}$ we used $\varepsilon(\boldsymbol{\beta})=1$. By the same theorem we know that

$$\begin{aligned}\hat{\mu}_{\mathcal{L}}(a) = \sum_{(\Phi),(\Psi),(\tilde{\Psi}),(R)} &\Big[S(\Phi_2\Psi_1R_2'\tilde{\Psi}_3')\otimes S(\Phi_1\tilde{\Psi}_1)\Big]\cdot\boldsymbol{f}\\ &\cdot\Delta(a\Phi_3)\cdot\Big[(\Psi_2R_2''\tilde{\Psi}_3'')\otimes(\Psi_3R_1\tilde{\Psi}_2)\Big]\ ,\end{aligned} \qquad (8.6)$$

where $\Psi=\Phi^{-1}$, $\tilde{\Psi}$ is another copy of Φ^{-1}, and $\boldsymbol{f}$ is the Drinfeld twist from (7.35). We compute the result sector by sector. For this, it is useful to expand $\boldsymbol{f}$ and $\Phi^{\pm1}$ sector by sector first:

(8.7)

	$\boldsymbol{f}$
00	$\mathbf{1}\otimes\mathbf{1}$
01	$\mathbf{1}\otimes\mathbf{1}$
10	$\mathbf{1}\otimes\mathsf{K}^{\mathrm{N}}$
11	$\beta^2(-\mathrm{i})^{\mathrm{N}}\mathbf{1}\otimes\mathsf{K}^{\mathrm{N}}$

	$\Phi^{\pm1}$
000	$\mathbf{1}\otimes\mathbf{1}\otimes\mathbf{1}$
001	$\mathbf{1}\otimes\mathbf{1}\otimes\mathbf{1}$
010	$\mathbf{1}\otimes\mathbf{1}\otimes\mathbf{1}$
100	$\mathbf{1}\otimes\mathbf{1}\otimes\mathbf{1}$
011	$\mathbf{1}\otimes\mathbf{1}\otimes\mathbf{1}$
101	$\mathbf{1}\otimes\mathbf{1}\otimes\mathbf{1}$
110	$\mathbf{1}\otimes\mathbf{1}\otimes\mathsf{K}^{\mathrm{N}}$
111	$\beta^2(\pm\mathrm{i})^{\mathrm{N}}\mathbf{1}\otimes\mathbf{1}\otimes\mathsf{K}^{\mathrm{N}}$

The elements $\boldsymbol{f}$ and $\Phi^{\pm1}$ are recovered from these tables by summing over the idempotents $\boldsymbol{e}_a\otimes\boldsymbol{e}_b$ (resp. $\boldsymbol{e}_a\otimes\boldsymbol{e}_b\otimes\boldsymbol{e}_c$) multiplied with the corresponding entry of the table.

We now give the contribution of each sector to $\hat{\mu}_{\mathcal{L}}(a)$. In doing so, we indicate which sectors of $\boldsymbol{f}, \Phi, \Psi, \tilde{\Psi}, R$ contribute to the expression (the equalities in sectors **ij** are true up the multiplication with the corresponding idempotents $\boldsymbol{e}_i \otimes \boldsymbol{e}_j$, here we omit them for brevity):

00: $\boldsymbol{f} : 00\ ,\ \Phi : 000\ ,\ \Psi : 000\ ,\ \tilde{\Psi} : 000\ ,\ R : 00$

$$\sum_{(R)} [S(R_2') \otimes \mathbf{1}] \cdot \Delta(a) \cdot [R_2'' \otimes R_1] = \sum_{(R),(a)} R_2 \triangleright a' \otimes a'' R_1$$

01: $\boldsymbol{f} : 01\ ,\ \Phi : 101\ ,\ \Psi : 001\ ,\ \tilde{\Psi} : 110\ ,\ R : 10$

$$\sum_{(R)} \left[S(R_2'\mathsf{K}^{\mathrm{N}}) \otimes \mathbf{1}\right] \cdot \Delta(a) \cdot \left[R_2''\mathsf{K}^{\mathrm{N}} \otimes R_1\right] = \sum_{(R),(a)} (R_2\mathsf{K}^{\mathrm{N}}) \triangleright a' \otimes a'' R_1$$

10: $\boldsymbol{f} : 10\ ,\ \Phi : 011\ ,\ \Psi : 110\ ,\ \tilde{\Psi} : 000\ ,\ R : 00$

$$\begin{aligned} &\sum_{(R)} [S(R_2') \otimes \mathbf{1}] \cdot \mathbf{1} \otimes \mathsf{K}^{\mathrm{N}} \cdot \Delta(a) \cdot \left[R_2'' \otimes (\mathsf{K}^{\mathrm{N}} R_1)\right] \\ &= \sum_{(R),(a)} R_2 \triangleright a' \otimes \mathsf{K}^{\mathrm{N}} a'' \mathsf{K}^{\mathrm{N}} R_1 = \sum_{(R),(a)} R_2 \triangleright a' \otimes (\mathsf{K}^{\mathrm{N}} \triangleright a'') R_1 \end{aligned}$$

11: $\boldsymbol{f} : 11\ ,\ \Phi : 110\ ,\ \Psi : 111\ ,\ \tilde{\Psi} : 110\ ,\ R : 10$

$$\begin{aligned} &\left(\beta^2(-\mathrm{i})^{\mathrm{N}}\right)^2 \sum_{(R)} \left[S(R_2'\mathsf{K}^{\mathrm{N}}) \otimes \mathbf{1}\right] \cdot \mathsf{K}^{\mathrm{N}} \otimes \mathbf{1} \cdot \Delta(a\mathsf{K}^{\mathrm{N}}) \cdot \left[R_2''\mathsf{K}^{\mathrm{N}} \otimes \mathsf{K}^{\mathrm{N}} R_1\right] \\ &= \sum_{(R),(a)} S(R_2'\mathsf{K}^{\mathrm{N}})\mathsf{K}^{\mathrm{N}} a' \mathsf{K}^{\mathrm{N}} R_2'' \mathsf{K}^{\mathrm{N}} \otimes a'' \mathsf{K}^{2\mathrm{N}} R_1 \\ &= \sum_{(R),(a)} S(\mathsf{K}^{\mathrm{N}} R_2' \mathsf{K}^{\mathrm{N}}) a' \mathsf{K}^{\mathrm{N}} R_2'' \mathsf{K}^{\mathrm{N}} \otimes a'' R_1 \\ &= \sum_{(R),(a)} (\mathsf{K}^{\mathrm{N}} R_2 \mathsf{K}^{\mathrm{N}}) \triangleright a' \otimes a'' R_1\ . \end{aligned}$$

Combining the four sectors above results in the expression for $\hat{\mu}_{\mathcal{L}}(a)$ given in (8.5).

In order to determine $\hat{\Delta}_{\mathcal{L}}$ we have to calculate

$$\hat{\Delta}_{\mathcal{L}}(a \otimes b) = \sum_{(D)} S(D_1) b D_2 S(D_3) a D_4 \tag{8.8}$$

where $D = (\mathrm{id} \otimes \mathrm{id} \otimes \Delta)(\Phi) \cdot \mathbf{1} \otimes \Phi^{-1} \cdot \mathbf{1} \otimes \boldsymbol{\beta} \otimes \mathbf{1} \otimes \mathbf{1}$, see (5.42) and (5.43). Note that we need D only in sectors **0000** and **1111** which are $\mathbf{1}^{\otimes 4}$ and $\mathbf{1} \otimes \mathsf{K}^{\mathrm{N}} \otimes \mathsf{K}^{\mathrm{N}} \otimes \mathbf{1}$, respectively. It is then immediate that (8.8) reduces to $\hat{\Delta}_{\mathcal{L}}(a \otimes b) = ba$ as claimed in (8.5).

The Hopf pairing is given by (see (5.42) and (5.43))

$$\hat{\omega}_{\mathcal{L}} \ = \ \sum_{(W)} S(W_3) W_4 \otimes S(W_1) W_2 \tag{8.9}$$

with $W = (\mathbf{1} \otimes \boldsymbol{\alpha} \otimes \mathbf{1} \otimes \boldsymbol{\alpha}) \cdot (\mathbf{1} \otimes \Phi^{-1}) \cdot (\mathbf{1} \otimes M \otimes \mathbf{1}) \cdot (\mathbf{1} \otimes \Phi) \cdot (\mathrm{id} \otimes \mathrm{id} \otimes \Delta)(\Phi^{-1})$. Recall from (7.12) and (8.7), that $\boldsymbol{\alpha} = \mathbf{1}$ and that Φ is non-trivial only in the third tensor factor. Thus W simplifies to

$$W = (\mathbf{1} \otimes M \otimes \mathbf{1}) \cdot (\mathrm{id} \otimes \mathrm{id} \otimes \Delta)(\Phi^{-1}) \ . \tag{8.10}$$

To compute $\hat{\omega}_{\mathcal{L}}$ we only need the following four sectors of W:

(8.11)

	W
0000	$\mathbf{1} \otimes M \otimes \mathbf{1}$
0011	$\mathbf{1} \otimes M \otimes \mathbf{1}$
1100	$(\mathbf{1} \otimes M \otimes \mathbf{1}) \cdot (\mathbf{1} \otimes \mathbf{1} \otimes \mathsf{K}^{\mathrm{N}} \otimes \mathsf{K}^{\mathrm{N}})$
1111	$(\mathbf{1} \otimes M \otimes \mathbf{1}) \cdot (\mathbf{1} \otimes \mathbf{1} \otimes \mathsf{K}^{\mathrm{N}} \otimes \mathsf{K}^{\mathrm{N}})$

From this it is straightforward to read off the expression for $\hat{\omega}_{\mathcal{L}}$ in (8.5). □

8.2. Non-degeneracy of the monodromy matrix

We compute the monodromy matrix (or double braiding) $M = R_{21}R \in \mathsf{Q} \otimes \mathsf{Q}$. For this, we need the identities (for any $n, m \in \mathbb{Z}_2$)

$$\mathsf{f}^{\pm} \otimes \mathsf{f}^{\mp}\omega_- \cdot \rho_{n,m} \cdot \boldsymbol{e}_n \otimes \boldsymbol{e}_m = (-1)^{m+1}\rho_{n,m} \cdot \mathsf{f}^{\pm}\omega_- \otimes \mathsf{f}^{\mp} \cdot \boldsymbol{e}_n \otimes \boldsymbol{e}_m \tag{8.12}$$

(note that $\omega_- = (\boldsymbol{e}_0 - \mathrm{i}\boldsymbol{e}_1)\mathsf{K}$ appears in different tensor factors) and

$$(\rho_{m,n})_{21} \cdot \rho_{n,m} \cdot \boldsymbol{e}_n \otimes \boldsymbol{e}_m = (-1)^{nm}\mathsf{K}^m \otimes \mathsf{K}^n \cdot \boldsymbol{e}_n \otimes \boldsymbol{e}_m \ , \tag{8.13}$$

where $(\rho_{m,n})_{21}$ stands for the flip of $\rho_{m,n}$ from (7.14). Then, using the expression (7.15) for the R-matrix the computation is straightforward:

$$\begin{aligned} M = R_{21}R = \sum_{n,m=0}^{1} \left(-\beta^2\right)^{nm} \mathsf{K}^m\boldsymbol{e}_n \otimes \mathsf{K}^n\boldsymbol{e}_m \\ \times \prod_{j=1}^{\mathrm{N}} \Big\{ \big(\mathbf{1} \otimes \mathbf{1} + 2(-1)^m \mathsf{f}_j^+\omega_- \otimes \mathsf{f}_j^-\big)\big(\mathbf{1} \otimes \mathbf{1} - 2\mathsf{f}_j^-\omega_- \otimes \mathsf{f}_j^+\big) \Big\} \ . \end{aligned} \tag{8.14}$$

Lemma 8.2. *The M-matrix is non-degenerate, i.e., it can be written as $M = \sum_{I\in X} g_I \otimes f_I$, where $\{g_I\}_{I\in X}$ and $\{f_I\}_{I\in X}$ are two bases of Q, for some indexing set X.*

Proof. Using the expression (8.14), we can rewrite M as

$$\begin{aligned} M = \sum_{n,m=0}^{1} \sum_{s_1,t_1=0}^{1} \cdots \sum_{s_{\mathrm{N}},t_{\mathrm{N}}=0}^{1} \left(-\beta^2\right)^{nm} 2^{\sum_i (s_i+t_i)} (-1)^{\sum_i (mt_i+s_i)} \\ \times \mathsf{K}^m\boldsymbol{e}_n \prod_{j=1}^{\mathrm{N}} (\tilde{\mathsf{f}}_j^+)^{t_j} (\tilde{\mathsf{f}}_j^-)^{s_j} \otimes \mathsf{K}^n\boldsymbol{e}_m \prod_{j=1}^{\mathrm{N}} (\mathsf{f}_j^-)^{t_j} (\mathsf{f}_j^+)^{s_j} \end{aligned} \tag{8.15}$$

where $\sum_i$ stands for $\sum_{i=1}^{\mathrm{N}}$ and $\tilde{\mathsf{f}}_j^{\pm} = \mathsf{f}_j^{\pm}\omega_-$. This expression suggests to introduce two bases in Q,

$$\begin{aligned} f_I &= \mathsf{K}^n \boldsymbol{e}_m \prod_{j=1}^{\mathrm{N}} (\mathsf{f}_j^-)^{t_j} (\mathsf{f}_j^+)^{s_j} \ , \\ g_I &= \big(-\beta^2\big)^{nm} 2^{\sum_i (s_i+t_i)} (-1)^{\sum_i (m t_i + s_i)} \mathsf{K}^m \boldsymbol{e}_n \prod_{j=1}^{\mathrm{N}} (\tilde{\mathsf{f}}_j^+)^{t_j} (\tilde{\mathsf{f}}_j^-)^{s_j} \ , \end{aligned} \tag{8.16}$$

where the indices I run over the set

$$X = \big\{ (n, m, s_1, t_1, \dots, s_{\mathrm{N}}, t_{\mathrm{N}}) \,\big|\, n, m, s_j, t_j \in \mathbb{Z}_2,\ j = 1, \dots, \mathrm{N} \big\} \ . \tag{8.17}$$

We then have $M = \sum_{I \in X} g_I \otimes f_I$ and thus the statement of the lemma is proven. □

As a consequence of Lemma 8.2, we have that $\hat{\omega}_{\mathcal{L}}$ is also non-degenerate (the antipode S and the conjugation with K map a basis to another basis) and this is a direct proof of Corollary 7.8.

8.3. Integrals and cointegrals for the coend

Since $\mathbf{Rep}\,\mathsf{Q}$ is factorisable (Corollary 7.8), $\mathcal{L} = \mathsf{Q}^*$ has a one-dimensional space of two-sided integrals $\Lambda_{\mathcal{L}} : \mathbb{C} \to \mathcal{L}$ (see Proposition 4.18, due to [**Ly1**]). This space of integrals contains an element (unique up to a sign) normalised such that $\omega_{\mathcal{L}} \circ (\Lambda_{\mathcal{L}} \otimes \Lambda_{\mathcal{L}}) = \mathrm{id}_{\mathbb{C}}$. From Lemma 2.7 (due to [**Ke2**]) we furthermore know that there is a two-sided cointegral $\Lambda^{\mathrm{co}}_{\mathcal{L}} \colon \mathcal{L} \to \mathbf{1}$ such that $\Lambda^{\mathrm{co}}_{\mathcal{L}} = \omega_{\mathcal{L}} \circ (\Lambda_{\mathcal{L}} \otimes \mathrm{id}_{\mathcal{L}})$.

In this section we give $\Lambda_{\mathcal{L}}$ and $\Lambda^{\mathrm{co}}_{\mathcal{L}}$ for $\mathcal{L} = \mathsf{Q}^*$ in the above normalisation. We refer to Section 2.3 for our conventions for integrals and cointegrals in braided categories.

By Proposition 5.16 the integrals on Q from (7.47) define cointegrals on $\mathcal{L} = \mathsf{Q}^*$ by

$$\Lambda^{\mathrm{co}}_{\mathcal{L}} = \langle - \,, \boldsymbol{c} \rangle \in \mathsf{Q}^{**} \ . \tag{8.18}$$

To describe the integrals of $\mathcal{L}$, let $\tilde{B}_m = \{\mathsf{K}^m, \mathsf{f}_1^+\mathsf{K}^m, \mathsf{f}_1^-\mathsf{K}^m, \mathsf{f}_1^+\mathsf{f}_1^-\mathsf{K}^m, \ \dots\ , \mathsf{f}_1^+\mathsf{f}_1^- \cdots \mathsf{f}_{\mathrm{N}}^+\mathsf{f}_{\mathrm{N}}^-\mathsf{K}^m\}$, so that $B = \bigcup_{m=0}^{3} \tilde{B}_m$ is a basis in Q. We use the basis dual to B to define the element $\hat{\Lambda}_{\mathcal{L}} \in \mathsf{Q}^*$ as

$$\hat{\Lambda}_{\mathcal{L}} = (-1)^{\mathrm{N}} \nu \beta^2 2^{1-\mathrm{N}} \Big(\prod_{i=1}^{\mathrm{N}} \mathsf{f}_i^+ \mathsf{f}_i^- \Big)^* \ , \tag{8.19}$$

where $\nu \in \mathbb{C}$ is the same constant as in the definition of the integral for Q in (7.47). In what follows, we will use several times the following simple lemma (whose proof we omit).

Lemma 8.3. *Let $\{w_I\}_{I \in X}$, for some indexing set X, be a linearly independent subset of Q such that each w_I is a monomial in the generators $\mathsf{f}_i^{\pm}$ and K. Assume that a central element*

$z \in \mathcal{Z}(\mathsf{Q})$ can be written as

$$z = \sum_{I \in X} \alpha_I w_I \ , \qquad \alpha_I \in \mathbb{C} \ , \ \alpha_I \neq 0 \ . \tag{8.20}$$

Then each w_I commutes with K.

Proposition 8.4. *The linear map $\Lambda_{\mathcal{L}} \colon \mathbb{C} \to \mathcal{L}$ given by $\Lambda_{\mathcal{L}}(1) = \hat{\Lambda}_{\mathcal{L}}$ is a two-sided integral for $\mathcal{L}$. For $\nu \in \{\pm 1\}$ this integral satisfies*

$$\omega_{\mathcal{L}} \circ (\Lambda_{\mathcal{L}} \otimes \Lambda_{\mathcal{L}}) = \mathrm{id}_{\mathbb{C}} \quad , \quad \Lambda^{\mathrm{co}}_{\mathcal{L}} \circ \Lambda_{\mathcal{L}} = \mathrm{id}_{\mathbb{C}} \ . \tag{8.21}$$

Proof. We know that $\Lambda^{\mathrm{co}}_{\mathcal{L}}$ in (8.18) is a cointegral for $\mathcal{L}$. To show that $\Lambda_{\mathcal{L}}$ is a two-sided integral, by Lemma 2.7 it is enough to verify the identity $\Lambda^{\mathrm{co}}_{\mathcal{L}} = \omega_{\mathcal{L}} \circ (\Lambda_{\mathcal{L}} \otimes \mathrm{id}_{\mathcal{L}})$. In the present setting, this is equivalent to $\langle -, \boldsymbol{c} \rangle = \omega_{\mathcal{L}}(\hat{\Lambda}_{\mathcal{L}} \otimes -)$ or

$$\boldsymbol{c} = \sum_{(\omega_{\mathcal{L}})} (\hat{\omega}_{\mathcal{L}})_1 \, \hat{\Lambda}_{\mathcal{L}}\big((\hat{\omega}_{\mathcal{L}})_2\big) = \sum_{(M)} \hat{\Lambda}_{\mathcal{L}}(M_1) S(M_2) \ , \tag{8.22}$$

where for the last equality we used Lemma 8.3 (each non-zero term in the sum commutes with K).

RHS of (8.22) =

$$\begin{aligned}
&\sum_{m,n=0}^{1} (-\beta^2)^{mn} 2^{2\mathrm{N}} (-1)^{\mathrm{N}(m+1)} \overbrace{\hat{\Lambda}_{\mathcal{L}}(\mathsf{K}^m \boldsymbol{e}_n \mathsf{f}^+_1 \omega_- \mathsf{f}^-_1 \omega_- \dots \mathsf{f}^+_{\mathrm{N}} \omega_- \mathsf{f}^-_{\mathrm{N}} \omega_-)}^{=\delta_{m,0}\, \nu \beta^2 2^{-\mathrm{N}}} S(\mathsf{K}^n \boldsymbol{e}_m \mathsf{f}^-_1 \mathsf{f}^+_1 \dots \mathsf{f}^-_{\mathrm{N}} \mathsf{f}^+_{\mathrm{N}}) \\
&\qquad = (-2)^{\mathrm{N}} \nu \beta^2 \big(S(\boldsymbol{e}_0 \mathsf{f}^-_1 \mathsf{f}^+_1 \dots \mathsf{f}^-_{\mathrm{N}} \mathsf{f}^+_{\mathrm{N}}) + S(\boldsymbol{e}_0 \mathsf{f}^-_1 \mathsf{f}^+_1 \dots \mathsf{f}^-_{\mathrm{N}} \mathsf{f}^+_{\mathrm{N}} \mathsf{K}) \big) \\
&\qquad = 2^{\mathrm{N}} \nu \beta^2 \boldsymbol{e}_0 \mathsf{f}^+_1 \mathsf{f}^-_1 \dots \mathsf{f}^+_{\mathrm{N}} \mathsf{f}^-_{\mathrm{N}} (1 + \mathsf{K}) = \boldsymbol{c} \ .
\end{aligned}$$

It remains to show that the normalisation condition holds. Indeed, we have

$$\begin{aligned}
\omega_{\mathcal{L}}(\hat{\Lambda}_{\mathcal{L}} \otimes \hat{\Lambda}_{\mathcal{L}}) &= \langle \hat{\Lambda}_{\mathcal{L}} \otimes \hat{\Lambda}_{\mathcal{L}} \, , \, \hat{\omega}_{\mathcal{L}} \rangle \overset{(8.22)}{=} \hat{\Lambda}_{\mathcal{L}}(\boldsymbol{c}) \\
&= 2^{\mathrm{N}} \nu \beta^2 \hat{\Lambda}_{\mathcal{L}}(\boldsymbol{e}_0 \mathsf{f}^+_1 \mathsf{f}^-_1 \dots \mathsf{f}^+_{\mathrm{N}} \mathsf{f}^-_{\mathrm{N}}) = \nu^2 = 1 \ .
\end{aligned} \tag{8.23}$$

□

8.4. Internal characters and ϕ_M

We first recall that the *internal character* of $V \in \mathbf{Rep}\,\mathsf{Q}$ is the intertwiner from the trivial representation to the coend $\mathcal{L}$ given by (see [**FSS, Sh1**])

$$\chi_V \ = \ \big[\, \mathbf{1} \xrightarrow{\widetilde{\mathrm{coev}}_V} V^* \otimes V \xrightarrow{\iota_V} \mathcal{L} \,\big] \ , \tag{8.24}$$

where we follow conventions Section 5.1.2. Via the isomorphism $\mathrm{Hom}_{\mathsf{Q}}(\mathbf{1}, \mathcal{L}) \to \mathrm{End}(\mathrm{Id}_{\mathbf{Rep}\,\mathsf{Q}})$ given in (5.22), the internal characters χ_V correspond to natural endomorphisms ϕ_V of the identity functor. For $\mathcal{C} = \mathbf{Rep}\,\mathsf{Q}$, under the categorical modular S-transformation $\mathcal{S}_{\mathcal{C}} \colon \mathrm{End}(Id_{\mathcal{C}}) \to \mathrm{End}(Id_{\mathcal{C}})$ in (5.14) the images ϕ_V of the internal characters are mapped to natural endomorphisms $\mathcal{S}_\mathcal{C}(\phi_V)$ which are the "Hopf link operators" in (5.23). We also recall from Corollary 5.5 and Theorem 5.6 that the assignments $[V] \mapsto \phi_V$ and $[V] \mapsto \mathcal{S}_\mathcal{C}(\phi_V)$ are injective linear maps $\mathrm{Gr}_\mathbb{C}(\mathcal{C}) \to \mathrm{End}(\mathrm{id}_\mathcal{C})$ (the second map is actually an algebra map). Both, ϕ_V and $\mathcal{S}_\mathcal{C}(\phi_V)$ will be useful in the computation of the S-transformation on the centre $Z(\mathsf{Q})$ in Section 9 and when comparing $SL(2,\mathbb{Z})$-actions in Section 10.

Let us identify χ_V with their images $\chi_V(1) \in \mathsf{Q}^*$. These are then given by the q-characters

$$\chi_V(-) = \mathrm{Tr}_V(\varkappa \cdot -), \qquad \varkappa = \boldsymbol{u}^{-1}\boldsymbol{v}S(\boldsymbol{\beta}) \ , \tag{8.25}$$

where explicitly

$$\varkappa = (\boldsymbol{e}_0 - \mathrm{i}\beta^2\boldsymbol{e}_1)\mathsf{K} \ . \tag{8.26}$$

The $\phi_V \in \mathrm{End}(Id_{\mathbf{Rep}\,\mathsf{Q}})$ correspond to central elements $\boldsymbol{\phi}_V \in \mathcal{Z}(\mathsf{Q})$, see Section 5.2.5. We now compute the $\boldsymbol{\phi}_V$ for all simple modules V.

Lemma 8.5. *We have*

$$\boldsymbol{\phi}_V = \sum_{(\boldsymbol{c})} \boldsymbol{c}' \otimes \chi_V(S(\boldsymbol{c}'')) \tag{8.27}$$

and, in particular,

$$\boldsymbol{\phi}_{\mathsf{X}_0^\pm} = \nu\, 2^N \beta^2 (\mathsf{K} \pm \mathbf{1})\boldsymbol{e}_0 \prod_{i=1}^{N} \mathsf{f}_i^+\mathsf{f}_i^- \ , \qquad \boldsymbol{\phi}_{\mathsf{X}_1^\pm} = \pm\,\nu\, 2^{N+1}\boldsymbol{e}_1^\pm \ . \tag{8.28}$$

Proof. Recall from Section 5.2.5 that

$$\boldsymbol{\phi}_V \ = \ \sum_{(F)} F_1\, \mathrm{tr}_V\big(\boldsymbol{u}^{-1}\,\boldsymbol{v}\,F_2\big) \ , \tag{8.29}$$

where

$$F = \varepsilon(\boldsymbol{\beta}) \sum_{(\Psi),(\Phi),(\boldsymbol{c})} \Psi_1\boldsymbol{c}'\Phi_1 \otimes S(\Psi_2\boldsymbol{c}''\Phi_2\boldsymbol{\beta})\boldsymbol{\alpha}\Psi_3\Phi_3 \quad \text{and} \quad \Psi = \Phi^{-1} \ . \tag{8.30}$$

It is straightforward to see that for Q we have

$$F = \sum_{(\boldsymbol{c})} \boldsymbol{c}' \otimes S(\boldsymbol{c}''\boldsymbol{\beta}) \ , \tag{8.31}$$

which together with (8.29) proves (8.27).

Next we compute from (7.47) that

$$\begin{aligned} \Delta(\boldsymbol{c}) = a\big(\mathbf{1}\otimes\mathbf{1} + \mathsf{K}^2\otimes\mathsf{K}^2\big)\big(\mathbf{1}\otimes\mathbf{1} + \Delta(\mathsf{K})\big) \\ \times \prod_{i=1}^{\mathrm{N}} \big(\mathsf{f}_i^+\mathsf{f}_i^- \otimes \mathbf{1} + \mathsf{f}_i^+\omega_- \otimes \mathsf{f}_i^- - \mathsf{f}_i^-\omega_+ \otimes \mathsf{f}_i^+ + \mathsf{K}^2 \otimes \mathsf{f}_i^+\mathsf{f}_i^-\big) \ , \end{aligned} \tag{8.32}$$

where $a = \nu\, 2^{\mathrm{N}-1}\beta^2$.

We note that for any $x \in \mathsf{Q}$ we get

$$\mathrm{Tr}_{\mathsf{X}_i^{\pm}}(\boldsymbol{e}_j x) = \delta_{i,j}\mathrm{Tr}_{\mathsf{X}_i^{\pm}}(x) \ , \quad i,j \in \{0,1\} \ . \tag{8.33}$$

Recall in Section 7.7 that $\mathsf{X}_0^{\pm}$ have the trivial action of $\mathsf{f}_i^{\pm}$. Therefore, in the traces over $\mathsf{X}_0^{\pm}$ only the basis elements K^n, $0 \le n \le 3$, have non-zero contribution. Then for $V = \mathsf{X}_0^{\pm}$ and using (8.25) together with (8.26) we get

$$\boldsymbol{\phi}_{\mathsf{X}_0^{\pm}} = \sum_{(\boldsymbol{c})} \mathrm{Tr}_{\mathsf{X}_0^{\pm}}(\mathsf{K}\boldsymbol{c}'')\boldsymbol{c}' = 2a(\mathsf{K} \pm \mathbf{1})\boldsymbol{e}_0 \prod_{i=1}^{\mathrm{N}} \mathsf{f}_i^+\mathsf{f}_i^- \ , \tag{8.34}$$

where we used that only the components with $\boldsymbol{c}'' = \mathsf{K}^n$ contribute to the expression, and that $S(\mathsf{K}^n\boldsymbol{e}_0) = \mathsf{K}^n\boldsymbol{e}_0$. This gives the first expression in (8.28).

For the computation of $\boldsymbol{\phi}_{\mathsf{X}_1^{\pm}}$ we first recall that $\mathsf{X}_1^{\pm}$ are simple projective, as discussed in Section 7.7. It then follows form [**GR3**, Eq. (7.5)] that $\boldsymbol{\phi}_{\mathsf{X}_1^{\pm}} = b_{\pm}\boldsymbol{e}_1^{\pm}$ for some non-zero $b_{\pm} \in \mathbb{C}$. To determine $b_{\pm}$ it is enough to act on the highest-weight vector $v_0^{\pm} \in \mathsf{X}_1^{\pm}$ defined by (7.57). We then note that in the expression

$$\boldsymbol{\phi}_{\mathsf{X}_1^{\pm}}.v_0^{\pm} = -\mathrm{i}\beta^2 \sum_{(\boldsymbol{c})} \mathrm{Tr}_{\mathsf{X}_1^{\pm}}\big(\mathsf{K}S(\boldsymbol{c}'')\big)\boldsymbol{c}'.v_0^{\pm} \tag{8.35}$$

only the term $a(\mathbf{1}\otimes\mathbf{1} + \mathsf{K}^2\otimes\mathsf{K}^2)(\mathbf{1}\otimes\mathbf{1} + \Delta(\mathsf{K}))\big(\mathsf{K}^{2\mathrm{N}} \otimes \prod_{i=1}^{\mathrm{N}} \mathsf{f}_i^+\mathsf{f}_i^-\big)$ in $\Delta(\boldsymbol{c})$ gives a non-zero contribution: the other terms either have f_i^- in $\boldsymbol{c}'$, and therefore are zero on $v_0^{\pm}$, or have f_k^+ without f_k^- in $\boldsymbol{c}''$ and therefore zero in the trace. Within the trace $\mathrm{Tr}_{\mathsf{X}_1^{\pm}}(-)$, we replace $\boldsymbol{e}_1 S(\prod_{i=1}^{\mathrm{N}} \mathsf{f}_i^+\mathsf{f}_i^-) = \boldsymbol{e}_1 \prod_{i=1}^{\mathrm{N}} \mathsf{f}_i^-\mathsf{f}_i^+$. Therefore, we need to calculate the trace of the operator $\mathsf{K}^n \prod_{i=1}^{\mathrm{N}} \mathsf{f}_i^-\mathsf{f}_i^+$. For any n, this operator in the basis $v_i^{\pm}$ from (7.58) is given by a diagonal matrix with one-dimensional eigenspace of non-zero eigenvalue – spanned by $v_0^{\pm}$. We thus have $\mathrm{Tr}_{\mathsf{X}_1^{\pm}}\big(\mathsf{K}^n \prod_{i=1}^{\mathrm{N}} \mathsf{f}_i^-\mathsf{f}_i^+\big) = (\pm\mathrm{i})^n$. Using this, a simple calculation finally gives

$$\boldsymbol{\phi}_{\mathsf{X}_1^{\pm}} = \pm\nu\, 2^{\mathrm{N}+1}\boldsymbol{e}_1^{\pm} \ , \tag{8.36}$$

which agrees with the second expression in (8.28). $\square$

The Hopf link operators $\mathcal{S}_{\mathcal{C}}(\phi_V) \in \mathrm{End}(Id_{\mathbf{Rep}\,\mathsf{Q}})$ correspond to central elements $\boldsymbol{\chi}_V \in \mathcal{Z}(\mathsf{Q})$ given by

$$\boldsymbol{\chi}_V \;=\; \sum_{(\Phi),(\Psi),(M)} \mathrm{tr}_V\Big(\varkappa\, S\big(\Psi_2 M_2 \Phi_2\big)\boldsymbol{\alpha}\,\Psi_3\Phi_3\Big)\,\Psi_1\, M_1\, \Phi_1 \ , \tag{8.37}$$

see (5.75). We now compute the $\boldsymbol{\chi}_V$ for all simple Q-modules V.

Lemma 8.6. *We have*

$$\boldsymbol{\chi}_V \;= \sum_{(M)} \mathrm{tr}_V\big(\varkappa\, S(M_2)\big)M_1 \tag{8.38}$$

and on simple V*:*

$$\boldsymbol{\chi}_{\mathsf{X}_0^\pm} = \boldsymbol{e}_1 \pm \boldsymbol{e}_0 \ , \qquad \boldsymbol{\chi}_{\mathsf{X}_1^\pm} = \pm\beta^2 4^N \mathsf{K}\boldsymbol{e}_0 \prod_{j=1}^{N} \mathsf{f}_j^+\mathsf{f}_j^- + 2^N(\boldsymbol{e}_1^+ - \boldsymbol{e}_1^-) \ . \tag{8.39}$$

PROOF. It is straightforward to see that for Q the expression in (8.37) reduces to

$$\boldsymbol{\chi}_V \ = \sum_{(M)} \mathrm{tr}_V\big(\varkappa\, S(M_2)\big) M_1 \ . \tag{8.40}$$

We first compute

$$\boldsymbol{\chi}_{\mathsf{X}_0^\pm} \ = \sum_{(M)} \mathrm{tr}_{\mathsf{X}_0^\pm}\big(\mathsf{K}\, S(M_2)\big) M_1 = \boldsymbol{e}_1 \pm \boldsymbol{e}_0 \ , \tag{8.41}$$

where we used that only the Cartan part of M in (8.14) contributes into the trace over $\mathsf{X}_0^\pm$.

Next, to compute $\boldsymbol{\chi}_{\mathsf{X}_1^\pm}$ we study the action of these central elements on all the four projective covers, actually on generating vectors of the covers. Recall that they are $v_1^\pm$ for $\mathsf{X}_1^\pm$ and $\boldsymbol{e}_0^\pm$ for $\mathsf{P}_0^\pm$, recall (7.59). The point is that only very few terms in an expansion of $\boldsymbol{\chi}_{\mathsf{X}_1^\pm}$ will contribute to the action. To start we act on v_1^α, $\alpha = \pm$,

$$\boldsymbol{\chi}_{\mathsf{X}_1^\pm}.v_1^\alpha = -\mathrm{i}\beta^2 \sum_{(M)} \mathrm{tr}_{\mathsf{X}_1^\pm}\big(\mathsf{K}\, S(M_2)\big) M_1.v_1^\alpha = \alpha 2^N v_1^\alpha \ , \tag{8.42}$$

where again only the Cartan part of M (actually just the $n = m = 1$ term) contributed into the non-zero action. Next, we have

$$\boldsymbol{\chi}_{\mathsf{X}_1^\pm}.\boldsymbol{e}_0^\alpha = -\mathrm{i}\beta^2 \sum_{(M)} \mathrm{tr}_{\mathsf{X}_1^\pm}\big(\mathsf{K}\, S(M_2)\big) M_1.\boldsymbol{e}_0^\alpha = \pm\alpha\beta^2 4^N \prod_{j=1}^{\mathrm{N}} \mathsf{f}_j^+\mathsf{f}_j^- \boldsymbol{e}_0^\alpha \ , \tag{8.43}$$

where we also noticed that only one term in the expansion of M:

$$M = (-1)^N 4^N (\mathsf{K}\boldsymbol{e}_0 \otimes \boldsymbol{e}_1) \prod_{j=1}^{\mathrm{N}} \mathsf{f}_j^+\mathsf{f}_j^- \otimes \prod_{j=1}^{\mathrm{N}} \mathsf{f}_j^-\mathsf{f}_j^+ + \dots \tag{8.44}$$

contributes, otherwise the trace over $\mathsf{X}_1^\pm$ is zero. We used here

$$\mathrm{Tr}_{\mathsf{X}_1^\pm}\big(\mathsf{K} \prod_{k \text{ terms}} \mathsf{f}_i^+\mathsf{f}_i^-\big) = \pm\mathrm{i}(-1)^k \delta_{\mathrm{N},k} \ . \tag{8.45}$$

Combining (8.42) and (8.43), we finally get

$$\boldsymbol{\chi}_{\mathsf{X}_1^\pm} = \pm\beta^2 4^N \mathsf{K}\boldsymbol{e}_0 \prod_{j=1}^{\mathrm{N}} \mathsf{f}_j^+\mathsf{f}_j^- + 2^N(\boldsymbol{e}_1^+ - \boldsymbol{e}_1^-) \ . \tag{8.46}$$

□

9. $SL(2,\mathbb{Z})$-action on the centre of Q

In this section, we specialise the projective $SL(2,\mathbb{Z})$-action on the centre $\mathcal{Z}$ of a general factorisable ribbon quasi-Hopf algebra obtained in Section 5.3 to the symplectic fermion quasi-Hopf algebra Q. The result is summarised in Theorem 9.3.

By Theorem 5.19, in general the S- and T-transformations are given by the following invertible linear endomorphisms of $\mathcal{Z}$, for $z \in \mathcal{Z}$,

$$\mathcal{S}_{\mathcal{Z}}(z) \;=\sum_{(\Psi),(\hat\omega_{\mathcal{L}})} \Psi_1\,\boldsymbol{\beta}\,S(\Psi_2)\,(\hat\omega_{\mathcal{L}})_1\,\Psi_3\,\hat\Lambda_{\mathcal{L}}\Big(\hat\Delta_{\mathcal{L}}\big((\hat\omega_{\mathcal{L}})_2\otimes\boldsymbol{\alpha}z\big)\Big)\;, \tag{9.1}$$

$$\mathcal{T}_{\mathcal{Z}}(z) \;=\boldsymbol{v}^{-1}z\;, \tag{9.2}$$

where $\Psi=\Phi^{-1}$, the dual structure maps $\hat\Delta_{\mathcal{L}}$, $\hat\omega_{\mathcal{L}}$ are defined by (8.4). We evaluate (9.1) for the quasi-Hopf algebra Q using the map $\hat\Delta_{\mathcal{L}}$ and the explicit form of $\hat\omega_{\mathcal{L}}$ computed in Proposition 8.1:

$$\mathcal{S}_{\mathcal{Z}}(z) \;=\sum_{(\hat\omega_{\mathcal{L}})}(\hat\omega_{\mathcal{L}})_1\,\hat\Lambda_{\mathcal{L}}\big(z(\hat\omega_{\mathcal{L}})_2\big)=\sum_{(M)}\hat\Lambda_{\mathcal{L}}(M_1z)S(M_2)\;, \tag{9.3}$$

where we also used the fact that in the sum all non-zero summands commute with K, recall Lemma 8.3 and that $\mathcal{S}_{\mathcal{Z}}(z)$ is central. We also recall that $\hat\Lambda_{\mathcal{L}}$ is given in (8.19).

Our aim is to give a decomposition of the $SL(2,\mathbb{Z})$-action (9.2), (9.3) on $\mathcal{Z}=\mathcal{Z}(\mathsf{Q})$. In Proposition 7.15, we described a decomposition of the centre $\mathcal{Z}=Z_0\oplus Z_1$ and a basis in it. In what follows, we will need a slightly different decomposition:

$$\mathcal{Z}(\mathsf{Q})=\mathcal{Z}_P\oplus\mathcal{Z}_{\mathbb{A}} \tag{9.4}$$

with

$$\mathcal{Z}_P \coloneqq \mathrm{span}_{\mathbb{C}}\big\{\,\boldsymbol{\phi}_{\mathsf{P}_0^+}\,,\,\boldsymbol{\phi}_{\mathsf{X}_1^+}\,,\,\boldsymbol{\phi}_{\mathsf{X}_1^-}\big\}\;, \tag{9.5}$$

$$\mathcal{Z}_{\mathbb{A}} \coloneqq \mathrm{span}_{\mathbb{C}}\Big\{\boldsymbol{e}_0\prod_{j=1}^{2k}\mathsf{f}_{i_j}^{\varepsilon_j}\mid k\le \mathrm{N}/2\,,\;i_j\in\{1,\dots,\mathrm{N}\}\,,\;\varepsilon_j=\pm\Big\}\;,$$

where $\mathcal{Z}_P$ is spanned by the internal characters $\boldsymbol{\phi}_V$ for V projective, while $\mathcal{Z}_{\mathbb{A}}$ is the centre of $\mathbb{A}_{2\mathrm{N}}$ from (7.5). The central elements $\boldsymbol{\phi}_{\mathsf{X}_0^\pm}$ are given in (8.28). For $\boldsymbol{\phi}_{\mathsf{P}_0^+}$, recall that the map $V\mapsto\boldsymbol{\phi}_V$ factors through the Grothendieck ring (as reviewed in Section 8.4 and see also

Section 5.1.2) and thus using the Cartan matrix in (7.61) we can write

$$\boldsymbol{\phi}_{\mathsf{P}_0^+} = 2^{2\mathrm{N}-1}\boldsymbol{\phi}_{\mathsf{X}_0^+} + 2^{2\mathrm{N}-1}\boldsymbol{\phi}_{\mathsf{X}_0^-} = \nu\, 2^{3\mathrm{N}}\beta^2 \mathsf{K}\boldsymbol{e}_0 \prod_{i=1}^{\mathrm{N}} \mathsf{f}_i^+\mathsf{f}_i^- \ . \tag{9.6}$$

This shows that (9.4) is indeed a decomposition of the centre $Z(\mathsf{Q})$ as computed in Proposition 7.15.

From [**GR3**, Prop. 7.1 & Cor. 8.5] we know that $\mathcal{Z}_P$ is an invariant subspace of $Z(\mathsf{Q})$ for the $SL(2,\mathbb{Z})$-action. The next lemma gives the action of $\mathcal{S}_Z$ and $\mathcal{T}_Z$ on $\mathcal{Z}_P$.

Lemma 9.1. *The restriction of the linear maps $\mathcal{S}_{\mathcal{Z}}$, $\mathcal{T}_{\mathcal{Z}}$ from* (9.1) *and* (9.2) *to $\mathcal{Z}_P$ in the basis* (9.5) *is given by the matrices*

$$\mathcal{S}_{\mathcal{Z}_P} = \nu \begin{pmatrix} 0 & 2^{-N} & -2^{-N} \\ 2^{N-1} & 1/2 & 1/2 \\ -2^{N-1} & 1/2 & 1/2 \end{pmatrix} \quad , \quad \mathcal{T}_{\mathcal{Z}_P} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \beta^{-1} & 0 \\ 0 & 0 & -\beta^{-1} \end{pmatrix} \ . \tag{9.7}$$

Proof. To compute the S-transformation on $\mathcal{Z}_P$ we use the relation stated in Corollary 5.20,

$$\mathcal{S}_Z(\boldsymbol{\phi}_V) = \boldsymbol{\chi}_V \ , \tag{9.8}$$

where the central elements $\boldsymbol{\chi}_V$ are given by (8.38) and were computed for simple V in Lemma 8.6. Applying this formula for $V = \mathsf{X}_1^\pm$ and using (8.39) together with (9.6), we can write

$$\mathcal{S}_Z(\boldsymbol{\phi}_{\mathsf{X}_1^\pm}) = \boldsymbol{\chi}_{\mathsf{X}_1^\pm} = \nu\big(\pm 2^{-\mathrm{N}}\boldsymbol{\phi}_{\mathsf{P}_0^+} + \frac{1}{2}(\boldsymbol{\phi}_{\mathsf{X}_1^+} + \boldsymbol{\phi}_{\mathsf{X}_1^-})\big) \ . \tag{9.9}$$

This gives the second and third column in (9.7).

As discussed in Section 8.4 (see also Section 5.1.2), the map $V \mapsto \boldsymbol{\chi}_V$ factors through $\mathrm{Gr}_{\mathbb{C}}(\mathbf{Rep}\,\mathsf{Q})$ and therefore we can write $\boldsymbol{\chi}_{\mathsf{P}_0^+} = 2^{2\mathrm{N}-1}\boldsymbol{\chi}_{\mathsf{X}_0^+} + 2^{2\mathrm{N}-1}\boldsymbol{\chi}_{\mathsf{X}_0^-}$. Using (8.39) together with the expression for $\boldsymbol{\phi}_{\mathsf{X}_1^\pm}$ in (8.28), recall that $\boldsymbol{e}_1 = \boldsymbol{e}_1^+ + \boldsymbol{e}_1^-$, the relation (9.8) for $V = \mathsf{P}_0^+$ is

$$\mathcal{S}_Z(\boldsymbol{\phi}_{\mathsf{P}_0^+}) = \boldsymbol{\chi}_{\mathsf{P}_0^+} = \nu\, 2^{\mathrm{N}-1}(\boldsymbol{\phi}_{\mathsf{X}_1^+} - \boldsymbol{\phi}_{\mathsf{X}_1^-}) \ . \tag{9.10}$$

This finally gives the first column in (9.7) and it finishes our calculation of $\mathcal{S}_{\mathcal{Z}_P}$.

For the T-transformation (9.2) we use the expression (7.17) for $\boldsymbol{v}^{-1}$. This immediately gives the diagonal matrix for $\mathcal{T}_Z$ with the entries $\{1, \beta^{-1}, -\beta^{-1}\}$ in the basis $\big\{\boldsymbol{\phi}_{\mathsf{P}_0^+}, \boldsymbol{\phi}_{\mathsf{X}_1^+}, \boldsymbol{\phi}_{\mathsf{X}_1^-}\big\}$. □

Let $U_i \subset \mathsf{Q}_0$ be the subalgebra $U_i := \mathrm{span}_{\mathbb{C}}\{\boldsymbol{e}_0, \mathsf{f}_i^-\boldsymbol{e}_0, \mathsf{f}_i^+\boldsymbol{e}_0, \mathsf{f}_i^+\mathsf{f}_i^-\boldsymbol{e}_0\}$, with $1 \le i \le \mathrm{N}$. Consider the injective linear map $\vartheta : U_1 \otimes \cdots \otimes U_{\mathrm{N}} \to \mathsf{Q}_0$ given by

$$\vartheta(a_1 \otimes \cdots \otimes a_{\mathrm{N}}) := a_1 \cdots a_{\mathrm{N}} \ . \tag{9.11}$$

Note that this is not an algebra map. Write $U_+ \subset U_1 \otimes \cdots \otimes U_\mathrm{N}$ for the subspace spanned by all homogeneous vectors with an even overall number of $\mathsf{f}_i^\pm$'s. In other words, U_+ is the eigenspace of $(\mathsf{K}(-)\mathsf{K}^{-1})^{\otimes \mathrm{N}}$ of eigenvalue $+1$. The direct summand $\mathcal{Z}_\wedge$ of the centre in (9.5) is the image of U_+,

$$\mathcal{Z}_\wedge = \vartheta(U_+) \ . \tag{9.12}$$

In particular, $\vartheta|_{U_+} : U_+ \to \mathcal{Z}_\wedge$ is a bijection and below we use ϑ^{-1} on elements from $\mathcal{Z}_\wedge$. With the help of ϑ we will now describe the S- and T-transformations on $\mathcal{Z}_\wedge$ as a tensor product of linear maps $\mathcal{S}^i_{\mathcal{Z}_\wedge}, \mathcal{T}^i_{\mathcal{Z}_\wedge} : U_i \to U_i$, for $i = 1, \ldots, \mathrm{N}$.

Lemma 9.2. *$\mathcal{S}_\mathcal{Z}$ and $\mathcal{T}_\mathcal{Z}$ from* (9.1)-(9.2) *map $\mathcal{Z}_\wedge$ to itself. The restrictions of $\mathcal{S}_\mathcal{Z}, \mathcal{T}_\mathcal{Z}$ to $\mathcal{Z}_\wedge$ are given by*

$$\mathcal{S}_\mathcal{Z}\big|_{\mathcal{Z}_\wedge} \;=\; \vartheta \circ \Big(\nu\beta^2 \cdot \big(\mathcal{S}^1_{\mathcal{Z}_\wedge} \otimes \cdots \otimes \mathcal{S}^N_{\mathcal{Z}_\wedge}\big)\big|_{U_+}\Big) \circ \vartheta^{-1} \ , \tag{9.13}$$

$$\mathcal{T}_\mathcal{Z}\big|_{\mathcal{Z}_\wedge} \;=\; \vartheta \circ \big(\mathcal{T}^1_{\mathcal{Z}_\wedge} \otimes \cdots \otimes \mathcal{T}^N_{\mathcal{Z}_\wedge}\big)\big|_{U_+} \circ \vartheta^{-1} \ , \tag{9.14}$$

where the individual maps $\mathcal{S}^i_{\mathcal{Z}_\wedge}$, $\mathcal{T}^i_{\mathcal{Z}_\wedge}$ are given in the basis $\{\boldsymbol{e}_0, \mathsf{f}_i^-\boldsymbol{e}_0, \mathsf{f}_i^+\boldsymbol{e}_0, \mathsf{f}_i^+\mathsf{f}_i^-\boldsymbol{e}_0\}$ of U_i by the matrices

$$\mathcal{S}^i_{\mathcal{Z}_\wedge} = \begin{pmatrix} 0 & 0 & 0 & -2^{-1} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 2 & 0 & 0 & 0 \end{pmatrix} \quad , \quad \mathcal{T}^i_{\mathcal{Z}_\wedge} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 2 & 0 & 0 & 1 \end{pmatrix} \ . \tag{9.15}$$

In particular, the restriction of $\mathcal{S}^2_\mathcal{Z}$ to $\mathcal{Z}_\wedge$ is the identity, and $\mathcal{T}_\mathcal{Z}$ has Jordan blocks of maximum rank $N+1$.

Proof. We first recall that the S-transformation $\mathcal{S}_\mathcal{Z}$ for Q was given in (9.3). Using the explicit expression of M from (8.15), $\mathcal{S}_\mathcal{Z}$ on a central element z becomes

$$\mathcal{S}_\mathcal{Z}(z) = \sum_{\substack{n,m,t_1,s_1,\\ \ldots,t_\mathrm{N},s_\mathrm{N}=0}}^{1} (-\beta^2)^{nm}\, 2^{\sum_i(s_i+t_i)} (-1)^{\sum_i(mt_i+s_i)} \hat{\Lambda}_\mathcal{L}\big(z\boldsymbol{e}_n\mathsf{K}^m \prod_{j=1}^{\mathrm{N}} (\tilde{\mathsf{f}}_j^+)^{t_j}(\tilde{\mathsf{f}}_j^-)^{s_j}\big)$$
$$\times\, \boldsymbol{e}_m S\big(\mathsf{K}^n \prod_{j=1}^{\mathrm{N}} (\mathsf{f}_j^-)^{t_j}(\mathsf{f}_j^+)^{s_j}\big) \ . \tag{9.16}$$

Our aim is to calculate $\mathcal{S}_\mathcal{Z}(z)$ on $\mathcal{Z}_\wedge$. Recall the basis elements of $\mathcal{Z}_\wedge$ in (9.5), they are all in $\mathcal{Z}\boldsymbol{e}_0$. Therefore the terms with $n=1$ are zero in the sum (9.16). Then, we recall $\hat{\Lambda}_\mathcal{L}$ from (8.19) and as a basis element of $\mathcal{Z}_\wedge$ has even number of $\mathsf{f}_i^\pm$ the sum $\sum_j(t_j+s_j)$ should be an even number too, otherwise the value of $\hat{\Lambda}_\mathcal{L}$ on the corresponding element is zero. As $\sum_i(t_i+s_i)$ is even, the factors of K in $\boldsymbol{e}_0\tilde{\mathsf{f}}_j^\pm = \boldsymbol{e}_0\mathsf{f}_j^\pm\mathsf{K}$ cancel (resulting in signs), and so the term $\hat{\Lambda}_\mathcal{L}(\cdots)$ in (9.16) is zero for $m=1$. Combining all these observations, for $z \in \mathcal{Z}_\wedge$ we

have
(9.17)
$$\mathcal{S}_{\mathcal{Z}}(z) = \sum_{\substack{t_1,s_1,\\ \dots,t_{\mathrm{N}},s_{\mathrm{N}}=0}}^{1} 2^{\sum_i(s_i+t_i)}(-1)^{\sum_i(s_i+(t_i+s_i)/2)}\hat{\Lambda}_{\mathcal{L}}\Big(z\boldsymbol{e}_0\prod_{j=1}^{\mathrm{N}}(\mathsf{f}_j^+)^{t_j}(\mathsf{f}_j^-)^{s_j}\Big)\boldsymbol{e}_0 S\Big(\prod_{j=1}^{\mathrm{N}}(\mathsf{f}_j^-)^{t_j}(\mathsf{f}_j^+)^{s_j}\Big)\ ,$$

where the sign $(-1)^{\sum_i(t_i+s_i)/2}$ arises when cancelling the factors of K contained in $\tilde{\mathsf{f}}_j^{\pm}$ (it helps to rewrite the product as in (9.18) below to see this). It is clear that we can write any basis element of $\mathcal{Z}_{\mathbb{A}}$ from (9.5) in the form

$$z = \Big(\prod_{j=1}^{2k}\mathsf{f}_{i_j}^{\varepsilon_j}\Big)\Big(\prod_{j=1}^{M}\mathsf{f}_{k_j}^{+}\mathsf{f}_{k_j}^{-}\Big)\boldsymbol{e}_0\ , \tag{9.18}$$

for some $1 \le k \le \mathrm{N}/2$ and $M, i_j, k_j \in \{1,\dots,\mathrm{N}\}$ such that $i_1 < \dots < i_{2k}$, and $\varepsilon_j \in \{\pm\}$. The essential part of the calculation is to analyse the coefficients $\hat{\Lambda}_{\mathcal{L}}\big(z\boldsymbol{e}_0\prod_j(\mathsf{f}_j^+)^{t_j}(\mathsf{f}_j^-)^{s_j}\big)$ in the sum (9.17). For a given z as in (9.18), the 2N-tuple $\{t_1,s_1,\dots,t_{\mathrm{N}},s_{\mathrm{N}}\}$ here is unique because $z\prod_{j=1}^{\mathrm{N}}(\mathsf{f}_j^+)^{t_j}(\mathsf{f}_j^-)^{s_j}$ should be proportional to the support of $\hat{\Lambda}_{\mathcal{L}}$, recall (8.19). We thus get that $\hat{\Lambda}_{\mathcal{L}}(\cdots)$ is non-zero iff, for the same index choice as in (9.18),

$$\prod_{j=1}^{\mathrm{N}}(\mathsf{f}_j^+)^{t_j}(\mathsf{f}_j^-)^{s_j} = \Big(\prod_{j=1}^{2k}\mathsf{f}_{i_j}^{-\varepsilon_j}\Big)\cdot\Big(\prod_{n=1}^{\mathrm{N}-M-2k}\mathsf{f}_{l_n}^{+}\mathsf{f}_{l_n}^{-}\Big)\ , \tag{9.19}$$

where the l_n take the $N-M-2k$ values in $\{1,\dots,\mathrm{N}\}$ which are different from all i_j, k_j in (9.18). Note that this set of l_n's is unique. After reordering the generators $\mathsf{f}_j^{\pm}$, the element within $\hat{\Lambda}_{\mathcal{L}}(\dots)$ is $(-1)^{\frac{1}{2}\sum_{j=1}^{2k}\varepsilon_j}\boldsymbol{e}_0\prod_{j=1}^{\mathrm{N}}\mathsf{f}_j^+\mathsf{f}_j^-$ [1]. We also note that the 2N tuple corresponding to (9.19) gives

$$\sum_i(t_i+s_i) = 2(\mathrm{N}-M-k)\ , \qquad \sum_i s_i = \mathrm{N}-M-2k+\frac{1}{2}\sum_{j=1}^{2k}(\varepsilon_j+1)\ . \tag{9.20}$$

We thus get for (9.17) the following expression, for z as in (9.18),

$$\begin{aligned}\mathcal{S}_{\mathcal{Z}}(z) &= 2^{2(N-M-k)}\hat{\Lambda}_{\mathcal{L}}\Big(\boldsymbol{e}_0\prod_{j=1}^{\mathrm{N}}\mathsf{f}_j^+\mathsf{f}_j^-\Big)\boldsymbol{e}_0 S\bigg(\Big(\prod_{j=1}^{2k}\mathsf{f}_{i_j}^{\varepsilon_j}\Big)\cdot\Big(\prod_{n=1}^{\mathrm{N}-M-2k}\mathsf{f}_{l_n}^{-}\mathsf{f}_{l_n}^{+}\Big)\bigg)\\ &= (-1)^M\nu\beta^2 2^{\mathrm{N}-2(M+k)}\boldsymbol{e}_0\Big(\prod_{j=1}^{2k}\mathsf{f}_{i_j}^{\varepsilon_j}\Big)\cdot\Big(\prod_{n=1}^{\mathrm{N}-M-2k}\mathsf{f}_{l_n}^{+}\mathsf{f}_{l_n}^{-}\Big)\ ,\end{aligned} \tag{9.21}$$

[1] To get the sign, first we write $\prod_{j=1}^{2k}\mathsf{f}_{i_j}^{\varepsilon_j}\cdot\prod_{j=1}^{2k}\mathsf{f}_{i_j}^{-\varepsilon_j} = (-1)^k\prod_{j=1}^{2k}\mathsf{f}_{i_j}^{\varepsilon_j}\mathsf{f}_{i_j}^{-\varepsilon_j}$, where $\mathsf{f}_{i_1}^{-\varepsilon_1}$ is commuted past an odd number of fermions, so we get -1, then $\mathsf{f}_{i_2}^{-\varepsilon_2}$ is commuted past an even number of fermions, so we get $+1$, etc. – this explains $(-1)^k$. Then to reorder $\mathsf{f}_{i_j}^{\varepsilon_j}\mathsf{f}_{i_j}^{-\varepsilon_j}$ we do nothing if $\varepsilon_j = +1$ and get -1 otherwise. Thus we have to multiply by $(-1)^{\sum_{j=1}^{2k}\frac{1-\varepsilon_j}{2}}$. This together with $(-1)^k$ gives the sign $(-1)^{\frac{1}{2}\sum_{j=1}^{2k}\varepsilon_j}$.

where the l_n are as in (9.19), and where we used our assumption that $i_1 < \cdots < i_{2k}$ and thus had to reorder $\mathsf{f}_{i_j}^{\varepsilon_j}$'s after the application of the antipode S. It is now clear that the image of $\mathcal{Z}_\wedge$ under $\mathcal{S}_\mathcal{Z}$ is $\mathcal{Z}_\wedge$ itself. We also note by a direct calculation that $\mathcal{S}_\mathcal{Z}^2(z) = z$.

Our aim is now to check the expression in (9.13) together with (9.15). Recall that the bijection ϑ was defined in (9.11). Then having z as in (9.18), its image under ϑ^{-1} has the form (of course here for a particular choice of i_j, k_j)

$$\vartheta^{-1}(z) = \boldsymbol{e}_0 \otimes \ldots \otimes \boldsymbol{e}_0 \otimes \boldsymbol{e}_0 \mathsf{f}_{i_1}^{\varepsilon_1} \otimes \ldots \otimes \boldsymbol{e}_0 \mathsf{f}_{k_1}^{+}\mathsf{f}_{k_1}^{-} \otimes \ldots \otimes \boldsymbol{e}_0 \ . \tag{9.22}$$

The action of each tensor component $\mathcal{S}_{\mathcal{Z}_\wedge}^i$, as defined in (9.15), replaces $\boldsymbol{e}_0$ in the i'th factor by $2\mathsf{f}_i^+\mathsf{f}_i^-\boldsymbol{e}_0$ and $\mathsf{f}_i^+\mathsf{f}_i^-\boldsymbol{e}_0$ by $-\frac{1}{2}\boldsymbol{e}_0$, while does nothing to the factors $\boldsymbol{e}_0\mathsf{f}_{i_1}^{\varepsilon_1}$. With this, it is now straightforward to see the equality in (9.13) using the final expression in (9.21).

For the T-transformation on $\mathcal{Z}_\wedge$, we use the expression (7.17) that can be written as

$$\boldsymbol{v}^{-1}\boldsymbol{e}_0 = \prod_{k=1}^{\mathrm{N}} (\mathbf{1} + 2\mathsf{f}_k^+\mathsf{f}_k^-)\boldsymbol{e}_0 \ . \tag{9.23}$$

It is therefore clear that $\mathcal{T}_\mathcal{Z}$ acts on $\mathcal{Z}_\wedge$. Recall that the injective linear map (9.11) restricts to an isomorphism $\vartheta\big|_{U_+} : U_+ \to \mathcal{Z}_\wedge$. Note that

$$\vartheta^{-1}(\boldsymbol{v}^{-1}\boldsymbol{e}_0) = (\boldsymbol{e}_0 + 2\mathsf{f}_1^+\mathsf{f}_1^-\boldsymbol{e}_0) \otimes (\boldsymbol{e}_0 + 2\mathsf{f}_2^+\mathsf{f}_2^-\boldsymbol{e}_0) \otimes \ldots \otimes (\boldsymbol{e}_0 + 2\mathsf{f}_\mathrm{N}^+\mathsf{f}_\mathrm{N}^-\boldsymbol{e}_0) \ . \tag{9.24}$$

It is easy to check that, even though $\vartheta\big|_{U_+} : U_+ \to \mathcal{Z}_\wedge$ is itself not an algebra map, when restricted to the subalgebra of Q_0 generated by $\mathsf{f}_i^+\mathsf{f}_i^-$, $1 \le i \le \mathrm{N}$, it does become an algebra map. In particular for $\boldsymbol{v}^{-1}\boldsymbol{e}_0$ we have $\vartheta^{-1}(\boldsymbol{v}^{-1}z) = \vartheta^{-1}(\boldsymbol{v}^{-1}\boldsymbol{e}_0) \cdot \vartheta^{-1}(z)$ for all $z \in \mathcal{Z}_\wedge$. Thus $\mathcal{T}_\mathcal{Z}|_{\mathcal{Z}_\wedge}$ acts on U_+ by multiplication with (9.24). From this, we immediately get (9.14) with (9.15).

Finally, from the $\mathcal{T}_\mathcal{Z}(\boldsymbol{e}_0)$ we see that $\mathcal{T}_\mathcal{Z}$ has Jordan blocks of maximum rank N+1. Indeed, recall that Jordan blocks of a given dimension behave under tensoring like irreducible $sl(2)$-modules of that same dimension. The matrix $\mathcal{T}_{\mathcal{Z}_\wedge}^i$ in (9.15) has one rank-2 Jordan block and two rank-1 blocks (use the basis $\{2\mathsf{f}_i^+\mathsf{f}_i^-\boldsymbol{e}_0, \boldsymbol{e}_0, \mathsf{f}_i^-\boldsymbol{e}_0, \mathsf{f}_i^+\boldsymbol{e}_0\}$). Thus the maximal rank Jordan cell in the tensor product arises from from tensoring the N rank-2 blocks. This corresponds to the tensor product of N fundamental $sl(2)$-modules, which decomposes into a direct sum with the largest irreducible summand being of dimension $\mathrm{N}+1$. □

As a corollary of the two last lemmas we have the following theorem.

Theorem 9.3. *The projective $SL(2,\mathbb{Z})$-action on the centre $\mathcal{Z}$ of Q given by the linear maps $\mathcal{S}_\mathcal{Z}$ and $\mathcal{T}_\mathcal{Z}$ from* (9.1)-(9.2) *is*

$$\mathcal{S}_\mathcal{Z} = \mathcal{S}_{\mathcal{Z}_P} \oplus \mathcal{S}_{\mathcal{Z}_\wedge} \ , \qquad \mathcal{T}_\mathcal{Z} = \mathcal{T}_{\mathcal{Z}_P} \oplus \mathcal{T}_{\mathcal{Z}_\wedge} \ , \tag{9.25}$$

with the constituent maps as defined in Lemmas 9.1 and 9.2. We have $\mathcal{S}_\mathcal{Z}^2 = \mathrm{id}_\mathcal{Z}$ and $\mathcal{T}_\mathcal{Z}$ has Jordan blocks of ranks up to and including $N+1$.

10. Equivalence of the two projective $SL(2,\mathbb{Z})$-actions

In this final section, we review the modular-group action on the symplectic fermion pseudo-trace functions and compare it with the $SL(2,\mathbb{Z})$ action computed in the previous section on the centre of Q. The main result of this paper is that these two actions are projectively equivalent (Theorem 10.9).

10.1. Modular properties of symplectic fermion pseudo-trace functions

Here, we review the computation of the symplectic fermion pseudo-trace functions and of their modular properties carried out in [**GR2**].

We define two affine Lie super-algebras, $\widehat{\mathfrak{h}}$ and $\widehat{\mathfrak{h}}_{\mathrm{tw}}$, in terms of a symplectic $\mathbb{C}$-vector space $\mathfrak{h}$ of dimension 2N with symplectic form $(-,-)$. The underlying super-vector spaces are

$$\widehat{\mathfrak{h}} = \mathfrak{h} \otimes \mathbb{C}[t^{\pm 1}] \oplus \mathbb{C}\hat{k} \quad , \quad \widehat{\mathfrak{h}}_{\mathrm{tw}} = \mathfrak{h} \otimes t^{\frac{1}{2}}\mathbb{C}[t^{\pm 1}] \oplus \mathbb{C}\hat{k} \ , \tag{10.1}$$

where $t^{\pm 1}$ and $\hat{k}$ are parity-even, and $\mathfrak{h}$ is parity-odd. For $u \in \mathfrak{h}$ and $m \in \mathbb{Z}$ (resp. $m \in \mathbb{Z}+\frac{1}{2}$), abbreviate $u_m := u \otimes t^m$. The Lie super-bracket is given by taking $\hat{k}$ central and setting, for $u, v \in \mathfrak{h}$ and $m, n \in \mathbb{Z}$ (resp. $m, n \in \mathbb{Z} + \frac{1}{2}$),

$$[u_m, v_n] \ = \ (u,v)\, m\, \delta_{m+n,0}\, \hat{k} \ . \tag{10.2}$$

By convention for the bracket of a Lie super-algebra, this is an anti-commutator as u_m, v_n are parity odd. We refer to [**Ru**] for more on $\widehat{\mathfrak{h}}_{(\mathrm{tw})}$ and its representations.

To make the connection to Section 6, choose a basis $\{a_i, b_i \,|\, i = 1, \dots, \mathrm{N}\}$ of $\mathfrak{h}$ which satisfies

$$(a_j, b_k) \ = \ \mathrm{i}\pi\delta_{j,k} \ . \tag{10.3}$$

With this basis as generators, we have $\mathbb{A} = \Lambda(\mathfrak{h})$, that is, the $2^{2\mathrm{N}}$-dimensional Graßmann algebra defined in Section 6.1 is the exterior algebra of $\mathfrak{h}$. The reason to put the factor of $\mathrm{i}\pi$ in the above normalisation is that we will also work with a rescaled basis of $\mathfrak{h}$, namely

$$\alpha^1 = a_1 \ , \ \alpha^2 = \tfrac{1}{\mathrm{i}\pi} b_1 \ , \ \dots \ , \ \alpha^{2\mathrm{N}-1} = a_{\mathrm{N}} \ , \ \alpha^{2\mathrm{N}} = \tfrac{1}{\mathrm{i}\pi} b_{\mathrm{N}} \ . \tag{10.4}$$

The α^i are a symplectic basis in the sense that $(\alpha^1, \alpha^2) = 1$, etc. The basis $\{\alpha^i\}$ is a natural choice when working with the affine Lie algebra $\widehat{\mathfrak{h}}_{(\mathrm{tw})}$, and it is used e.g. in [**Ru**] and [**GR2**, Sec. 6].

For later use we recall the action of the Virasoro zero-mode L_0 on highest weight modules of $\widehat{\mathfrak{h}}_{(\mathrm{tw})}$ from [**Ru**, Rem. 2.5 & 2.7]:

$$\begin{aligned}
&\widehat{\mathfrak{h}}\text{-module}: \quad L_0 = \sum_{k\in 0}^{\mathrm{N}} (\alpha_0^{2k}\alpha_0^{2k-1} - \alpha_0^{2k-1}\alpha_0^{2k}) + H\ ,\\
&\widehat{\mathfrak{h}}_{\mathrm{tw}}\text{-module}: \quad L_0 = -\tfrac{\mathrm{N}}{8} + H_{\mathrm{tw}}\ .
\end{aligned} \tag{10.5}$$

where

$$\begin{aligned}
H &= \sum_{m\in\mathbb{Z}_{>0}} \sum_{k\in 0}^{\mathrm{N}} \left(\alpha_{-m}^{2k}\alpha_m^{2k-1} - \alpha_{-m}^{2k-1}\alpha_m^{2k}\right)\ ,\\
H_{\mathrm{tw}} &= \sum_{m\in\mathbb{Z}_{\geq 0}+\frac{1}{2}} \sum_{k\in 0}^{\mathrm{N}} \left(\alpha_{-m}^{2k}\alpha_m^{2k-1} - \alpha_{-m}^{2k-1}\alpha_m^{2k}\right)\ .
\end{aligned} \tag{10.6}$$

One verifies that $H_{(\mathrm{tw})}$ acts as grading operator on a highest-weight module of $\widehat{\mathfrak{h}}_{(\mathrm{tw})}$, assigning grade zero to the space of ground states.

The braiding on $\mathcal{SF}$ was originally computed in [**Ru**] with respect to the basis in (10.4) and, for example, for $X, Y \in \mathcal{SF}_0$ the result was

$$c_{X,Y} = \tau^{\mathrm{s.v.}}_{X,Y} \circ \exp\Big(- \mathrm{i}\pi \sum_{k=1}^{\mathrm{N}} (\alpha^{2k}\otimes\alpha^{2k-1} - \alpha^{2k-1}\otimes\alpha^{2k})\Big)\ , \tag{10.7}$$

see [**Ru**, Eqn. (6.1)]. This formula, and many others, have factors of $\mathrm{i}\pi$, which can be absorbed into a suitably rescaled copairing. Indeed, with respect to the basis $\{a_k, b_k\}$, the above braiding produces the one presented in (6.14).

Given a module $M \in \mathcal{SF}_0 = \mathbf{Rep}_{\mathrm{s.v.}}\Lambda$, we can use induction to construct an $\widehat{\mathfrak{h}}$-module $\widehat{M}$ as follows. We take u_m for $u \in \mathfrak{h}$, $m > 0$, to act as zero on M, $\hat{k}$ to act by 1, and the zero modes $(a_i)_0$ and $(b_i)_0$ to act as a_i and b_i, respectively. Then

$$\widehat{M} := U(\widehat{\mathfrak{h}}) \otimes_{U(\widehat{\mathfrak{h}}_{\geq 0}\oplus\mathbb{C}\hat{k})} M\ , \tag{10.8}$$

where $U(-)$ denotes the universal enveloping algebra (in super-vector spaces) of a Lie superalgebra, and $\widehat{\mathfrak{h}}_{\geq 0}$ is the subalgebra spanned by non-negative modes. Similarly, for $V \in \mathcal{SF}_1 = \mathbf{Svect}$, we consider the induced $\widehat{\mathfrak{h}}_{\mathrm{tw}}$-module

$$\widehat{V} := U(\widehat{\mathfrak{h}}_{\mathrm{tw}}) \otimes_{U((\widehat{\mathfrak{h}}_{\mathrm{tw}})_{>0}\oplus\mathbb{C}\hat{k})} V\ , \tag{10.9}$$

which is defined as above, except that in $\widehat{\mathfrak{h}}_{\mathrm{tw}}$ there are no zero modes to take care of.

The symplectic fermion vertex operator super-algebra $\mathcal{V}$ is defined in [**Ab**], see also [**DR3**, Sec. 3.1] which uses the present notation (except that $\mathcal{V}$ is denoted by $\mathbb{V}(\mathfrak{h})$ there). The underlying super-vector space is $\mathcal{V} = \widehat{\mathbf{1}}$, for $\mathbf{1} \in \mathcal{SF}$ the tensor unit as given in (6.4). Write $\mathcal{V}_{\mathrm{ev}}$ for the parity-even subspace of $\mathcal{V}$ – this is a vertex operator algebra.

It is shown in [**Ab**] (and stated this way in [**GR2**, Cor. 6.4]) that the functor of first inducing a $\widehat{\mathfrak{h}}_{(\mathrm{tw})}$-module and then restricting to its even subspace defines a $\mathcal{V}_{\mathrm{ev}}$-module:

Proposition 10.1. *$\big(\widehat{-}\big)_{\mathrm{ev}}$ is a faithful $\mathbb{C}$-linear functor $\mathcal{SF} \to \mathbf{Rep}\,\mathcal{V}_{\mathrm{ev}}$.*

Next we compute the image of the endomorphisms of the minimal projective generator $G_{\mathcal{SF}}$ of $\mathcal{SF}$ given in (7.63). This requires a bit of preparation. Define

$$\mathcal{G} := \widehat{\mathbb{A}} \oplus \widehat{T} \ . \tag{10.10}$$

Note that we do not take the even part. Instead we consider the super-vector space on the RHS simply as a vector space. In terms of the functor $(\widehat{-})_{\mathrm{ev}}$ this means

$$\mathcal{G} \cong \big(\widehat{G}_{\mathcal{SF}}\big)_{\mathrm{ev}} \ . \tag{10.11}$$

We will need this isomorphism explicitly. We first describe the isomorphism

$$\iota\colon\ \widehat{\mathbb{A}} \xrightarrow{\sim} (\widehat{\mathbb{A} \otimes \mathbb{C}^{1|1}})_{\mathrm{ev}} \tag{10.12}$$

of $\mathcal{V}_{\mathrm{ev}}$-modules. Recall by (10.8) that $\widehat{\mathbb{A}} = U(\widehat{\mathfrak{h}}) \otimes_{U(\widehat{\mathfrak{h}}_{\geq 0} \oplus \mathbb{C}\hat{k})} \mathbb{A}$. We then define for homogeneous $u \in U(\widehat{\mathfrak{h}})$ and $f \in \mathbb{A}$:

$$\iota\colon\ u \otimes f \mapsto u \otimes f \otimes \begin{cases} (1,0)\ , & \text{if } u \otimes f \text{ even}\ , \\ (0,1)\ , & \text{if } u \otimes f \text{ odd}\ , \end{cases} \tag{10.13}$$

with the inverse map given by (note that v has to be homogeneous too)

$$\iota^{-1}\colon\ u \otimes f \otimes v \mapsto v_{|u|+|f|} \cdot u \otimes f\ , \qquad v \in \mathbb{C}^{1|1}\ . \tag{10.14}$$

The isomorphism $\iota_{\mathrm{tw}}\colon\ \widehat{T} \xrightarrow{\sim} (\widehat{T \otimes \mathbb{C}^{1|1}})_{\mathrm{ev}}$ is defined analogously.

Lemma 10.2. *The image $\mathcal{E}$ under $\big(\widehat{-}\big)_{\mathrm{ev}}$ of $\mathrm{End}_{\mathcal{SF}}(G_{\mathcal{SF}})$ in $\mathrm{End}_{\mathcal{V}_{\mathrm{ev}}}(\mathcal{G})$ is generated by*

- *the action of the zero modes α_0^i, $i = 1, \dots, 2N$,*
- *$\widehat{\mathrm{id}_T}$, the idempotent corresponding to $\widehat{T}$, i.e. the identity map on $\widehat{T}$ and zero on $\widehat{\mathbb{A}}$,*
- *the parity involution $\omega_{\mathcal{G}}$ on $\mathcal{G}$.*

Proof. We first refer to Corollary 7.18 where $\mathrm{End}_{\mathcal{SF}}(G_{\mathcal{SF}})$ was described. The functor $\big(\widehat{-}\big)_{\mathrm{ev}}$ maps a morphism g to $\mathrm{id}_{U(\widehat{\mathfrak{h}})} \otimes g$. We first calculate the image (under the functor) of action of $a \in \mathfrak{h}$ from (7.65):

$$u \otimes f \otimes v \mapsto u \otimes f \cdot a \otimes \Pi(v) = (-1)^{|u|+|f|} a_0 \cdot u \otimes f \otimes \Pi(v)\ , \tag{10.15}$$

were the latter equality is by the definition of the induced representation (10.8). Then composing with the isomorphism ι^{-1} from (10.14) we get the corresponding endomorphism on $\widehat{\mathbb{A}}$:

$$v_{|u|+|f|} u \otimes f \mapsto v_{|a_0|+|u|+|f|+1} (-1)^{|u|+|f|} a_0 \cdot u \otimes f = v_{|u|+|f|} a_0 \cdot \omega_{\mathcal{G}}(u \otimes f)\ . \tag{10.16}$$

We finally note that the image of the κ generator is given by $\omega_{\mathcal{G}}$. Therefore the image of $\mathrm{End}_{\mathcal{SF}}(G_{\mathcal{SF}})$ is generated by $\omega_{\mathcal{G}}$, by $a_0 \circ \omega_{\mathcal{G}}$ for $a \in \mathfrak{h}$ (or, equivalently, by a_0), and by the idempotent for $\widehat{T}$. $\square$

We note that the algebra from Lemma 10.2 was also introduced in [**AN**] in order to describe pseudo-trace functions for $\mathcal{V}_{\rm ev}$. Let us briefly review this construction. For a k-algebra A, recall the space of central forms on A from (3.33) which is defined as

$$C(A) \;=\; \big\{\, \varphi : A \to k \,\big|\, \varphi(ab) = \varphi(ba) \text{ for all } a,b \in A \,\big\} \;. \tag{10.17}$$

By definition, an element $\varphi \in C(A)$ induces a symmetric pairing $(a,b) \to \varphi(ab)$ on A.

Recall the algebra $\mathcal{E} \subset \mathrm{End}_{\mathcal{V}_{\rm ev}}(\mathcal{G})$ from Lemma 10.2 and note that $\mathcal{G}$ is an $\mathcal{E}$-module. To each $\varphi \in C(\mathcal{E})$ one can assign a pseudo-trace function (3.34).

The pseudo-trace functions are generalisations of the characters of VOA modules. Indeed, for each simple $\mathcal{V}_{\rm ev}$-module $\mathcal{M}$ there is a unique $\varphi_{\mathcal{M}} \in C(\mathcal{E})$ such that[1]

$$\xi_{\mathcal{G}}^{\varphi_{\mathcal{M}}}(v,\tau) \;=\; \mathrm{Tr}_{\mathcal{M}}\big(\, o(v)\, e^{2\pi \mathrm{i} \tau (L_0 - c/24)}\,\big) \;, \tag{10.18}$$

i.e. the pseudo-trace function for $\varphi_{\mathcal{M}}$ equals the usual trace over $\mathcal{M}$. In particular, there is a unique $\varphi_{\mathcal{V}_{\rm ev}}$ corresponding to the vacuum character. The general background for the above calculation was given in Section 3.4.

Pseudo-trace functions provide a linear map $\xi_{\mathcal{G}} : C(\mathcal{E}) \to C_1(\mathcal{V}_{\rm ev})$. This map is injective [**AN**, Thm. 6.3.2]. We need in addition:

Conjecture 10.3 ([**AN**, Conj. 6.3.5], [**GR2**, Conj. 5.8])**.** $\xi_{\mathcal{G}} : C(\mathcal{E}) \to C_1(\mathcal{V}_{\rm ev})$ *is an isomorphism.*

Under the above conjecture, the following statement is verified for $\mathcal{V}_{\rm ev}$ in [**GR2**, Sec. 6] and Lemma 10.5 below by explicit calculation.

Proposition 10.4. *Assuming Conjecture 10.3, we have*

- *the action of the modular S- and T-transformation defines linear isomorphisms*

$$S_{\mathcal{V}_{\rm ev}}, T_{\mathcal{V}_{\rm ev}} :\; C(\mathcal{E}) \to C(\mathcal{E}) \;; \tag{10.19}$$

- *the element $\delta := S_{\mathcal{V}_{\rm ev}}(\varphi_{\mathcal{V}_{\rm ev}}) \in C(\mathcal{E})$ defines a non-degenerate pairing on $\mathcal{E}$ via $(f,g) \mapsto \delta(f \circ g)$; the assignment $\hat\delta : Z(\mathcal{E}) \to C(\mathcal{E})$, $z \mapsto \delta(z \cdot (-))$ therefore is a linear isomorphism.*

In [**GR2**, Sec. 5] this statement is conjectured to hold in general for the class of vertex operator algebras considered there. As a consequence of Proposition 10.4, we obtain unique linear isomorphisms

$$S_Z = \hat\delta^{-1} \circ S_{\mathcal{V}_{\rm ev}} \circ \hat\delta \;, \qquad T_Z = \hat\delta^{-1} \circ T_{\mathcal{V}_{\rm ev}} \circ \hat\delta \;. \tag{10.20}$$

[1] Existence of $\varphi_{\mathcal{M}}$ is proven in [**GR2**, Prop. 5.5] (which also provides the simple expression $\varphi_{\mathcal{M}} = \mathrm{Tr}_{\tilde{\mathcal{M}}}(-)$ where $\tilde{\mathcal{M}} = \mathrm{Hom}_{\mathcal{V}_{\rm ev}}(\mathcal{G},\mathcal{M})$ is the corresponding $\mathcal{E}$-module). Uniqueness follows from injectivity of $\xi_{\mathcal{G}}$ as proven in [**AN**] for the choice of $\mathcal{G}$ given in (10.10).

Next we give the explicit results for $\hat{\delta}(\varphi_{\mathcal{M}})$ and S_Z, T_Z. Setting $\varphi_X := \varphi_{(\widehat{X})_{\mathrm{ev}}}$, for $X \in \mathcal{SF}$, the calculation in [**GR2**, Sec. 6] can be expressed as

$$
\begin{aligned}
\hat{\delta}^{-1}(\varphi_{\mathbf{1}}) &= (2\pi)^{\mathrm{N}} \alpha_0^1 \cdots \alpha_0^{2\mathrm{N}} (\omega_{\mathcal{G}} + 1) \ , & \hat{\delta}^{-1}(\varphi_T) &= 2^{\mathrm{N}}\, \widehat{\mathrm{id}_T}\, (\omega_{\mathcal{G}} + 1) \ , \\
\hat{\delta}^{-1}(\varphi_{\Pi\mathbf{1}}) &= (2\pi)^{\mathrm{N}} \alpha_0^1 \cdots \alpha_0^{2\mathrm{N}} (\omega_{\mathcal{G}} - 1) \ , & \hat{\delta}^{-1}(\varphi_{\Pi T}) &= 2^{\mathrm{N}}\, \widehat{\mathrm{id}_T}\, (\omega_{\mathcal{G}} - 1) \ .
\end{aligned} \tag{10.21}
$$

The centre of $\mathcal{E}$ is computed in [**AN**] to be

$$
Z(\mathcal{E}) = \mathcal{Z}_P(\mathcal{E}) \oplus \mathcal{Z}_{\mathbb{A}}(\mathcal{E}) \ , \tag{10.22}
$$

where $\mathcal{Z}_{\mathbb{A}}(\mathcal{E})$ is spanned by even monomials in the zero modes α_0^i, and $\mathcal{Z}_P(\mathcal{E})$ has the basis

$$
\mathcal{Z}_P(\mathcal{E}) = \mathrm{span}_{\mathbb{C}} \big\{ \tfrac{1}{2}\hat{\delta}^{-1}(\varphi_{\mathbf{1}} + \varphi_{\Pi\mathbf{1}}) \, , \, \hat{\delta}^{-1}(\varphi_T) \, , \, \hat{\delta}^{-1}(\varphi_{\Pi T}) \ \big\} \ . \tag{10.23}
$$

The form of the centre also follows from Proposition 7.15 via the equivalence in Theorem 7.6.

The linear map S_Z from (10.20) is computed explicitly in [**GR2**]. For T_Z we give the explicit expression in the next lemma.

Lemma 10.5. *For $z \in \mathcal{Z}_{\mathbb{A}}(\mathcal{E})$ we have*

$$
T_Z(z) = e^{2\pi \mathrm{i} N/12} \cdot e^{2\pi \mathrm{i} \sum_{k=1}^{N} \alpha_0^{2k} \alpha_0^{2k-1}} z \ , \tag{10.24}
$$

while on $\mathcal{Z}_P(\mathcal{E})$ in the basis (10.23) *we get the matrix representation*

$$
T_Z\big|_{\mathcal{Z}_P(\mathcal{E})} = e^{2\pi \mathrm{i} N/12} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{-\pi \mathrm{i} N/4} & 0 \\ 0 & 0 & -e^{-\pi \mathrm{i} N/4} \end{pmatrix} \ . \tag{10.25}
$$

Proof. We start by computing $T_{\mathcal{V}_{\mathrm{ev}}}$. By definition, for all $\varphi \in C(\mathcal{E})$,

$$
\xi_{\mathcal{G}}^{\varphi}(v, \tau + 1) \ = \ \xi_{\mathcal{G}}^{T_{\mathcal{V}_{\mathrm{ev}}}(\varphi)}(v, \tau) \ . \tag{10.26}
$$

Recall the direct sum decomposition (10.10) of $\mathcal{G}$. There are no $\mathcal{V}_{\mathrm{ev}}$-intertwiners between $\widehat{\mathbb{A}}$ and $\widehat{T}$, nor between the different parity subspaces of $\widehat{T}$, and so (as we have already seen in the explicit description above)

$$
\mathcal{E} = \mathcal{E}_0 \oplus \mathcal{E}_1^+ \oplus \mathcal{E}_1^- \quad , \quad \mathcal{E}_0 = \mathrm{End}_{\mathcal{V}_{\mathrm{ev}}}(\widehat{\mathbb{A}}) \ , \ \mathcal{E}_1^+ = \mathbb{C}\, \mathrm{id}_{(\widehat{T})_{\mathrm{ev}}} \ , \ \mathcal{E}_1^- = \mathbb{C}\, \mathrm{id}_{(\widehat{T})_{\mathrm{odd}}} \ . \tag{10.27}
$$

From the definition of the pseudo-trace functions in (3.34) and from the explicit form in [**GR2**, Eqn. (6.16) & App. A] it is a straightforward exercise to show that

$$
\xi_{\mathcal{G}}^{\varphi}(v, \tau + 1) \ = \ t_{\mathcal{G}}^{\varphi}\big(e^{2\pi \mathrm{i}(L_0 - c/24)}\, o(v)\, e^{2\pi \mathrm{i}\tau(L_0 - c/24)} \big) \ = \ t_{\mathcal{G}}^{\varphi'}\big(\, o(v)\, e^{2\pi \mathrm{i}\tau(L_0 - c/24)} \, \big) \ , \tag{10.28}
$$

where we used that the zero mode $o(v)$ commutes with L_0 and set

$$
\varphi'(f) = \begin{cases} e^{2\pi \mathrm{i}(L_0 - c/24)} f & ; \ f \in \mathcal{E}_0 \\ e^{2\pi \mathrm{i}(-\mathrm{N}/8 - c/24)} f & ; \ f \in \mathcal{E}_1^+ \\ -e^{2\pi \mathrm{i}(-\mathrm{N}/8 - c/24)} f & ; \ f \in \mathcal{E}_1^- \end{cases} \tag{10.29}
$$

and, for $f \in \mathcal{E}_0$,

$$e^{2\pi\mathrm{i}(L_0-c/24)}f \;=\; \exp\Big\{2\pi\mathrm{i}\Big(\frac{1}{2}\sum_{k=1}^{\mathrm{N}}\big(\alpha_0^{2k}\alpha_0^{2k-1}-\alpha_0^{2k-1}\alpha_0^{2k}\big)+\frac{\mathrm{N}}{12}\Big)\Big\}f \;. \tag{10.30}$$

The above computation makes use of the explicit form of L_0 as given in (10.5), and of the fact that H has eigenvalues in $\mathbb{Z}_{\geq 0}$, and so vanish in $\exp(2\pi\mathrm{i}(\cdots))$. On the other hand, H_{tw} has eigenvalues in $\mathbb{Z}_{\geq 0}$ on the even part of an $\widehat{\mathfrak{h}}_{\mathrm{tw}}$-module, and in $\mathbb{Z}_{\geq 0}+\frac{1}{2}$ on the odd part. $\square$

Similar to the construction in (9.11) we let $W_i \subset \mathcal{E}_0$ be the subalgebra with basis

$$W_i = \mathrm{span}_{\mathbb{C}}\big\{\mathrm{id}_{\mathcal{E}_0}, \alpha_0^{2i-1}, \alpha_0^{2i}, \alpha_0^{2i}\alpha_0^{2i-1}\big\} \;, \tag{10.31}$$

and consider the linear map $\varpi : W_1 \otimes \cdots \otimes W_{\mathrm{N}} \to \mathcal{E}_0$ given by

$$\varpi(f_1 \otimes \cdots \otimes f_{\mathrm{N}}) := f_1 \cdots f_{\mathrm{N}} \;. \tag{10.32}$$

Let $W_+ \subset W_1 \otimes \cdots \otimes W_{\mathrm{N}}$ be the subspace spanned by products with an even total number of α_0^k's. The map ϖ restricts to an isomorphism $\varpi : W_+ \to \mathcal{Z}_\wedge(\mathcal{E})$. Let $\sigma_k, \tau_k : W_k \to W_k$ be the linear maps whose matrix representations in the basis (10.31) are

$$\sigma_k = \begin{pmatrix} 0 & 0 & 0 & (-2\pi)^{-1} \\ 0 & -\mathrm{i} & 0 & 0 \\ 0 & 0 & -\mathrm{i} & 0 \\ -2\pi & 0 & 0 & 0 \end{pmatrix} \quad , \quad \tau_k = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 2\pi\mathrm{i} & 0 & 0 & 1 \end{pmatrix} \quad . \tag{10.33}$$

Theorem 10.6. *Assume Conjecture 10.3 holds. The linear maps $S_Z, T_Z : Z(\mathcal{E}) \to Z(\mathcal{E})$ preserve the direct sum decomposition* (10.22)*:*

$$S_Z = S_{\mathcal{Z}_P} \oplus S_{\mathcal{Z}_\wedge} \quad , \quad T_Z = T_{\mathcal{Z}_P} \oplus T_{\mathcal{Z}_\wedge} \;. \tag{10.34}$$

The linear endomorphisms $S_{\mathcal{Z}_P}, T_{\mathcal{Z}_P}$ of $\mathcal{Z}_P(\mathcal{E})$ are given by the matrices, with respect to the basis (10.23)*,*

$$S_{\mathcal{Z}_P} = \begin{pmatrix} 0 & 2^N & -2^N \\ 2^{-N-1} & 2^{-1} & 2^{-1} \\ -2^{-N-1} & 2^{-1} & 2^{-1} \end{pmatrix} \quad , \quad T_{\mathcal{Z}_P} = e^{2\pi\mathrm{i}N/12}\cdot\begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{-\pi\mathrm{i}N/4} & 0 \\ 0 & 0 & -e^{-\pi\mathrm{i}N/4} \end{pmatrix} \quad . \tag{10.35}$$

The linear endomorphisms $S_{\mathcal{Z}_\wedge}, T_{\mathcal{Z}_\wedge}$ of $\mathcal{Z}_\wedge(\mathcal{E})$ are given by

$$\begin{aligned} S_{\mathcal{Z}_\wedge} &= \varpi \circ \Big(\big(\sigma_1 \otimes \cdots \otimes \sigma_N\big)\big|_{W_+}\Big) \circ \varpi^{-1} \;, \\ T_{\mathcal{Z}_\wedge} &= e^{2\pi\mathrm{i}N/12}\cdot \varpi \circ \Big(\big(\tau_1 \otimes \cdots \otimes \tau_N\big)\big|_{W_+}\Big) \circ \varpi^{-1} \;. \end{aligned} \tag{10.36}$$

Proof. The expression for S_Z is computed in [**GR2**, Cor. 6.10]. The expression for T_Z is immediate from Lemma 10.5. $\square$

We conclude this section by a conjectural refinement of the $\mathbb{C}$-linear inclusion in Proposition 10.1 to a ribbon equivalence which we need in the next section. For details on how to define the necessary coherence isomorphisms we refer to [**DR3**, Conj. 7.4], and for the specific value of β to [**Ru**, Thm. 6.4].

Conjecture 10.7 ([**DR3**, Conj. 7.4]). *For the choice*

$$\beta = e^{-\mathrm{i}\pi N/4} \ , \tag{10.37}$$

$\big(\widehat{-}\big)_{\mathrm{ev}}$ *is an equivalence of* $\mathbb{C}$*-linear ribbon categories.*

10.2. Comparison of $SL(2,\mathbb{Z})$-actions

To compare the results in Theorems 9.3 and 10.6 we first need to correct a mismatch in conventions for the ribbon twist and then relate $Z(\mathsf{Q})$ from (7.53) to $Z(\mathcal{E})$ from (10.22).

We start by remarking that in [**Ly1, Ly2**] – which we use to compute the $SL(2,\mathbb{Z})$-action on $Z(\mathsf{Q})$ – the convention is that the T generator acts by composing with the ribbon twist θ. However, the modular T-transformation acts by composition with θ^{-1}.[2] Below we will show that the agreement with the categorical T-transformation can be achieved by changing the value of β in Conjecture 10.7 to its inverse.

According to Conjecture 10.7, $\big(\mathbf{Rep}\,\mathcal{V}_{\mathrm{ev}}\,, \theta = e^{-2\pi\mathrm{i}L_0}\big) \cong \mathcal{SF}\big(\mathrm{N}\,,\, \beta = e^{-\mathrm{i}\pi\mathrm{N}/4}\big)$ as ribbon categories. This is equivalent to

$$\big(\mathbf{Rep}\,\mathcal{V}_{\mathrm{ev}}\,, \theta = e^{2\pi\mathrm{i}L_0}\big) \ \simeq \ \mathcal{SF}\big(\mathrm{N}\,,\, \beta = e^{-\mathrm{i}\pi\mathrm{N}/4}\big)^{\mathrm{rev}} \ . \tag{10.38}$$

Here, given a ribbon category $\mathcal{C}$, $\mathcal{C}^{\mathrm{rev}}$ denotes the *reversed ribbon category*, where braiding and twist are replaced by their inverses. On the LHS of (10.38) we now have the convention for θ that matches our quasi-Hopf computation of the T-transformation. To reformulate the RHS, we need the following lemma.

Lemma 10.8. *For all* N, β *we have* $\mathcal{SF}(N,\beta)^{\mathrm{rev}} \simeq \mathcal{SF}(N,\beta^{-1})$ *as ribbon categories.*

Proof. For the sake of the proof we will write $\mathcal{SF}(\mathrm{N},\beta;\Lambda^{\mathrm{co}}_{\mathbb{A}}(\beta),C)$, where $\Lambda^{\mathrm{co}}_{\mathbb{A}}(\beta)$ is the cointegral in (6.12) and C refers to the copairing in (6.9).[3] We stress the β-dependence in the notation $\Lambda^{\mathrm{co}}_{\mathbb{A}}(\beta)$ as our normalisation convention for the cointegral is β-dependent. By [**DR1**, Prop. 4.12] there is a ribbon equivalence

$$\mathcal{SF}(\mathrm{N},\beta;\Lambda^{\mathrm{co}}_{\mathbb{A}}(\beta),C)^{\mathrm{rev}} \ \simeq \ \mathcal{SF}(\mathrm{N},\beta^{-1};\Lambda^{\mathrm{co}}_{\mathbb{A}}(\beta),-C) \ , \tag{10.39}$$

[2]The reason for this mismatch is that by the convention chosen in [**Ru**], following e.g. [**FRS**], the ribbon twist acts by $e^{-2\pi\mathrm{i}L_0}$ while for $\mathbf{Rep}\,\mathcal{V}_{\mathrm{ev}}$ the modular T-action is (up to a constant) given by $e^{2\pi\mathrm{i}L_0}$. One of course could redefine the Lyubashenko's $SL(2,\mathbb{Z})$-action, but we find it more convenient to adapt the conventions from [**Ru**] to the present context.

[3] More generally, the ribbon category $\mathcal{SF}$ is constructed from a symplectic vector space $\mathfrak{h}$, a cointegral and a choice of β, see [**DR1**, Sec. 5.2] for details. C is the copairing for the symplectic form. Including the cointegral and the copairing in the notation makes these choices explicit.

whose underlying functor is the identity (with non-trivial coherence isomorphisms). In [**DR1**, Prop. 4.12] the cointegral stays the same, hence $\Lambda^{\mathrm{co}}_{\mathbb{A}}(\beta)$ appears also on the RHS instead of $\Lambda^{\mathrm{co}}_{\mathbb{A}}(\beta^{-1})$.

It remains to show that on the RHS of (10.39) one can replace $\Lambda^{\mathrm{co}}_{\mathbb{A}}(\beta)$ by $\Lambda^{\mathrm{co}}_{\mathbb{A}}(\beta^{-1})$ and $-C$ by C. Let $\varphi : \mathbb{A} \to \mathbb{A}$ be the Hopf algebra isomorphism given on the generators in (6.2) as $\varphi(a_i) = a_i$, $\varphi(b_i) = -b_i$, $i = 1, \dots, \mathrm{N}$. The isomorphism φ satisfies

$$\Lambda^{\mathrm{co}}_{\mathbb{A}}(\beta) \circ \varphi = (-1)^{\mathrm{N}} \Lambda^{\mathrm{co}}_{\mathbb{A}}(\beta) = \Lambda^{\mathrm{co}}_{\mathbb{A}}(\beta^{-1}) \quad , \qquad (\varphi \otimes \varphi)(C) = -C \ . \tag{10.40}$$

We can now apply [**DR1**, Prop. 4.11] to conclude that

$$\mathcal{SF}(\mathrm{N}, \beta^{-1}; \Lambda^{\mathrm{co}}_{\mathbb{A}}(\beta), -C) \ \simeq \ \mathcal{SF}(\mathrm{N}, \beta^{-1}; \Lambda^{\mathrm{co}}_{\mathbb{A}}(\beta^{-1}), C) \tag{10.41}$$

as ribbon categories. The underlying functor of this equivalence is the identity functor in $\mathcal{SF}_1$, and it is given by φ^*, the pullback of representations along φ, on $\mathcal{SF}_0$. Combining (10.39) and (10.41) proves the statement of the lemma. □

Thanks to Lemma 10.8, the equivalence in Conjecture 10.7 can be restated as

$$\big(\mathbf{Rep}\,\mathcal{V}_{\mathrm{ev}}\,, \theta = e^{2\pi \mathrm{i} L_0}\big) \ \simeq \ \mathcal{SF}\big(\mathrm{N}\,,\, \beta = e^{\mathrm{i}\pi\mathrm{N}/4}\big) \ . \tag{10.42}$$

Denote by $\mathcal{J} : \mathbf{Rep}\,\mathsf{Q}(\mathrm{N},\beta) \to \mathcal{SF}(\mathrm{N},\beta)$ the ribbon equivalence from Proposition 7.6 and constructed in Appendices B and C, see also the inverse equivalence in Section C.8. We need to compute the following chain of algebra isomorphisms for $\beta = e^{\mathrm{i}\pi\mathrm{N}/4}$:

$$\Upsilon \ : \ Z(\mathsf{Q}) \xrightarrow{(1)} \mathrm{End}(Id_{\mathbf{Rep}\,\mathsf{Q}}) \xrightarrow{(2)} \mathrm{End}(Id_{\mathcal{SF}(\beta)}) \xrightarrow{(3)} \mathrm{End}(Id_{\mathcal{SF}(\beta^{-1})^{\mathrm{rev}}}) \xrightarrow{(4)} Z(\mathcal{E}) \ , \tag{10.43}$$

and use this to compare the results of Theorems 9.3 and 10.6. Let $z \in Z(\mathsf{Q})$. Arrow (1) is simply given by acting with z on a module. Arrow (2) maps η to the unique η' such that

$$\eta'_{\mathcal{J}(X)} = \mathcal{J}(\eta_X) \qquad \text{for all} \quad X \in \mathbf{Rep}\,\mathsf{Q} \ . \tag{10.44}$$

Arrow (3) is described in Lemma 10.8 and arrow (4) maps ψ to $(\widehat{\psi_G})_{\mathrm{ev}}$.

We compute the composition of these maps separately for $Z(\mathsf{Q}_0)$ and $Z(\mathsf{Q}_1)$. Let us start with $z \in Z(\mathsf{Q}_0)$. The functor $\mathcal{F} : \mathcal{SF}(\mathrm{N},\beta) \to \mathbf{Rep}\,\mathsf{Q}(\mathrm{N},\beta)$ which is inverse to $\mathcal{J}$ is given explicitly in Appendix C.8. Conversely, the functor $\mathcal{J}$, as $\mathbb{C}$-linear functor, is $\mathcal{J} = \mathcal{E} \circ \mathcal{H}$, where $\mathcal{E}$ is given in the proof of Proposition B.3 and $\mathcal{H}$ in the proof of Proposition C.3. Then for $X \in \mathcal{SF}_0$, η'_X is given by the action of z, where now f^+_k acts as a_k, f^-_k acts as b_k and K acts as parity involution ω_X. This gives arrow (2). Arrow (3) is pullback of representations (see the proof of Lemma 10.8) with the result that f^+_k still acts as a_k but now f^-_k acts as $-b_k$. Finally, for arrow (4) the relation between a_k, b_k and the zero modes $(\alpha^j)_0$ is given in (10.4). Altogether, for $\nu_j, \varepsilon_j \in \{0,1\}$, $\sum_{j=1}^{\mathrm{N}}(\nu_j + \varepsilon_j)$ even,

(10.45)

$$\begin{aligned}\Upsilon\big(\boldsymbol{e}_0 (\mathsf{f}^+_1)^{\nu_1} (\mathsf{f}^-_1)^{\varepsilon_1} \cdots (\mathsf{f}^+_{\mathrm{N}})^{\nu_{\mathrm{N}}} (\mathsf{f}^-_{\mathrm{N}})^{\varepsilon_{\mathrm{N}}}\big) &= (a_1)_0^{\nu_1} (-b_1)_0^{\varepsilon_1} \cdots (a_{\mathrm{N}})_0^{\nu_{\mathrm{N}}} (-b_{\mathrm{N}})_0^{\varepsilon_{\mathrm{N}}} \ , \\ &= (-\pi \mathrm{i})^{\varepsilon_1 + \dots + \varepsilon_{\mathrm{N}}} \cdot (\alpha^1_0)^{\nu_1} (\alpha^2_0)^{\varepsilon_1} \cdots (\alpha^{2\mathrm{N}-1}_0)^{\nu_{\mathrm{N}}} (\alpha^{2\mathrm{N}}_0)^{\varepsilon_{\mathrm{N}}} \ ,\end{aligned}$$

where the factors on the RHS are zero modes of $\widehat{\mathfrak{h}}$ acting on $\mathcal{G}$. Furthermore,

$$\Upsilon\big(\mathsf{K}\boldsymbol{e}_0\mathsf{f}_1^+\mathsf{f}_1^-\cdots\mathsf{f}_\mathrm{N}^+\mathsf{f}_\mathrm{N}^-\big) = (a_1)_0(-b_1)_0\cdots(a_\mathrm{N})_0(-b_\mathrm{N})_0\,\omega_\mathcal{G}\ , \\ = (-\pi\mathrm{i})^\mathrm{N}\cdot\alpha_0^1\alpha_0^2\cdots\alpha_0^{2\mathrm{N}}\,\omega_\mathcal{G}\ . \tag{10.46}$$

For $Z(\mathsf{Q}_1)$ one quickly finds that

$$\Upsilon(\boldsymbol{e}_1^+) = \mathrm{id}_{(\widehat{T})_\mathrm{ev}} = \tfrac{1}{2}\cdot\widehat{\mathrm{id}_T}\circ(\mathrm{id}+\omega_\mathcal{G})\ , \\ \Upsilon(\boldsymbol{e}_1^-) = \mathrm{id}_{(\widehat{T})_\mathrm{odd}} = \tfrac{1}{2}\cdot\widehat{\mathrm{id}_T}\circ(\mathrm{id}-\omega_\mathcal{G})\ . \tag{10.47}$$

After these preparations, we can finally compare the elements $\boldsymbol{\phi}_V$ from (8.28) to (10.21) and the action of the generators of $SL(2,\mathbb{Z})$ as given in Theorems 9.3 and 10.6.

Theorem 10.9. *If one chooses the normalisation constant of the integral for the coend $\mathcal{L}$ in (8.19) to be $\nu = 1$, then*

$$\Upsilon(\boldsymbol{\phi}_{\mathsf{X}_0^+}) = \hat\delta^{-1}(\varphi_\mathbf{1})\ , \qquad \Upsilon(\boldsymbol{\phi}_{\mathsf{X}_1^+}) = \hat\delta^{-1}(\varphi_T)\ , \\ \Upsilon(\boldsymbol{\phi}_{\mathsf{X}_0^-}) = \hat\delta^{-1}(\varphi_{\Pi\mathbf{1}})\ , \qquad \Upsilon(\boldsymbol{\phi}_{\mathsf{X}_1^-}) = \hat\delta^{-1}(\varphi_{\Pi T})\ . \tag{10.48}$$

Furthermore, $S_Z\circ\Upsilon = \Upsilon\circ\mathcal{S}_\mathcal{Z}$ and $T_Z\circ\Upsilon = e^{2\pi\mathrm{i}N/12}\cdot\Upsilon\circ\mathcal{T}_\mathcal{Z}$.

Proof. That the identities in (10.48) hold is immediate from comparing (8.28) and (10.21) via the explicit form of Υ given above. For example,

$$\Upsilon(\boldsymbol{\phi}_{\mathsf{X}_0^+}) = \nu\,2^\mathrm{N}\beta^2(-\pi\mathrm{i})^\mathrm{N}(\omega_\mathcal{G}+\mathrm{id})\alpha_0^1\cdots\alpha_0^{2\mathrm{N}}\ , \tag{10.49}$$

which is equal to $\hat\delta(\varphi_\mathbf{1})$ in (10.21) since $\nu = 1$ and β is fixed as in (10.37).

The basis (9.5) gets mapped to

$$\big\{\ \boldsymbol{\phi}_{\mathsf{P}_0^+}\,,\,\boldsymbol{\phi}_{\mathsf{X}_1^+}\,,\,\boldsymbol{\phi}_{\mathsf{X}_1^-}\ \big\} \xmapsto{\ \Upsilon\ } \big\{\ 2^{2\mathrm{N}}\cdot\tfrac{1}{2}\hat\delta^{-1}(\varphi_\mathbf{1}+\varphi_{\Pi\mathbf{1}})\,,\,\hat\delta^{-1}(\varphi_T)\,,\,\hat\delta^{-1}(\varphi_{\Pi T})\ \big\}\ , \tag{10.50}$$

which differs from (10.23) by a coefficient in the first basis vector. Taking this into account one arrives at $S_{\mathcal{Z}_P}\circ\Upsilon = \Upsilon\circ\mathcal{S}_{\mathcal{Z}_P}$ and $T_{\mathcal{Z}_P}\circ\Upsilon = e^{2\pi\mathrm{i}\mathrm{N}/12}\cdot\Upsilon\circ\mathcal{T}_{\mathcal{Z}_P}$.

The basis for U_k used in Lemma 9.2 is transported to a basis of W_k (via $\varpi^{-1}\circ\Upsilon\circ\vartheta$) as

$$\big\{\,\boldsymbol{e}_0\,,\,\mathsf{f}_k^-\boldsymbol{e}_0\,,\,\mathsf{f}_k^+\boldsymbol{e}_0\,,\,\mathsf{f}_k^+\mathsf{f}_k^-\boldsymbol{e}_0\,\big\} \longmapsto \big\{\,\mathrm{id}\,,\,-\pi\mathrm{i}\,\alpha_0^{2k}\,,\,\alpha_0^{2k-1}\,,\,-\pi\mathrm{i}\,\alpha_0^{2k-1}\alpha_0^{2k}\,\big\}\ . \tag{10.51}$$

This differs from the basis used in (10.31), namely $\{\mathrm{id},\alpha_0^{2k-1},\alpha_0^{2k},\alpha_0^{2k}\alpha_0^{2k-1}\}$, in the order of basis factors and by coefficients. Note that for the present choice of β, the factor β^2 in (9.13) is equal to i^N, which can be distributed over the individual factors of $\mathcal{S}^k_{\mathcal{Z}_\wedge}$. One then verifies that the above change of basis transports $\mathrm{i}\,\mathcal{S}^k_{\mathcal{Z}_\wedge}$ to σ^k and $\mathcal{T}^k_{\mathcal{Z}_\wedge}$ to τ^k as given in (10.33). Comparing (9.13) and (10.36), we see that this proves $S_{\mathcal{Z}_\wedge}\circ\Upsilon = \Upsilon\circ\mathcal{S}_{\mathcal{Z}_\wedge}$ and $T_{\mathcal{Z}_\wedge}\circ\Upsilon = e^{2\pi\mathrm{i}\mathrm{N}/12}\cdot\Upsilon\circ\mathcal{T}_{\mathcal{Z}_\wedge}$. $\square$

This also proves that the linear $SL(2,\mathbb{Z})$-action $\pi_{\mathcal{E}}$ on $Z(\mathcal{E})$ and the projective $SL(2,\mathbb{Z})$-action π_{Q} on $Z(\mathsf{Q})$ agree projectively. Namely, there is a function $\gamma : SL(2,\mathbb{Z}) \to \mathbb{C}^{\times}$ such that the linear isomorphism $\Upsilon : Z(\mathsf{Q}) \to Z(\mathcal{E})$ satisfies

$$\pi_{\mathcal{E}}(g) \circ \Upsilon = \gamma(g) \cdot \Upsilon \circ \pi_{\mathsf{Q}}(g) \qquad \text{for all } g \in SL(2,\mathbb{Z}) \ . \tag{10.52}$$

Appendices

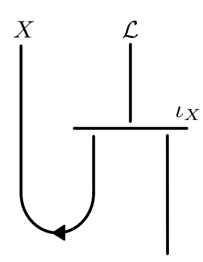
A. Proof of Proposition 4.8

We first prove that the two coalgebra structures are equal. Recall that the coproduct for the universal Hopf algebra $(\mathcal{L}, \varphi)$ is determined by the defining relation (4.4) involving the element from $\mathrm{Nat}(\mathrm{id}, \mathrm{id} \otimes (\mathcal{L} \otimes \mathcal{L}))$ – the right-hand side of (4.4) – while the coproduct for the coend $(\mathcal{L}, \iota)$ is given by the defining relation (4.24) involving the element from $\mathrm{Din}((-)^* \otimes (-), \mathcal{L} \otimes \mathcal{L})$. By Corollary 4.7 and Lemma 4.6, and the universality property of the coend $\mathcal{L}$ we have the commutative diagram (where all arrows are bijections)

(A.1)
$$
\begin{tikzcd}
\mathcal{C}(\mathcal{L}, V) \arrow[r, "\varphi_V"] \arrow[dr, "(-)\circ\iota"'] & \mathrm{Nat}(\mathrm{id}, \mathrm{id} \otimes V) \\
& \mathrm{Din}((-)^* \otimes (-), V) \arrow[u, "\zeta_V"']
\end{tikzcd}
$$

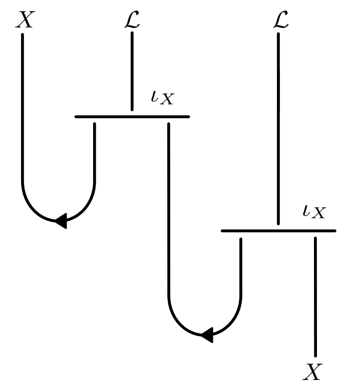
We use this diagram for $V = \mathcal{L} \otimes \mathcal{L}$ to compare the coproducts on $(\mathcal{L}, \iota)$ and on $(\mathcal{L}, \varphi)$ – this is equivalent to comparing the dinatural transformation on RHS of (4.24) with the image of RHS of (4.4) under the bijection $\zeta^{-1}_{\mathcal{L}\otimes\mathcal{L}}$. We begin with rewriting the map $X \xrightarrow{\tilde{\iota}_X} X \otimes \mathcal{L}$ in terms of ι, recall the definition (4.3). Applying the diagram (A.1) for $V = \mathcal{L}$ we get $\tilde{\iota} = \varphi_{\mathcal{L}}(\mathrm{id}) = \zeta_{\mathcal{L}}(\iota)$ or graphically

(A.2) $\tilde{\iota}_X \quad :=$



Then, RHS of (4.4) is

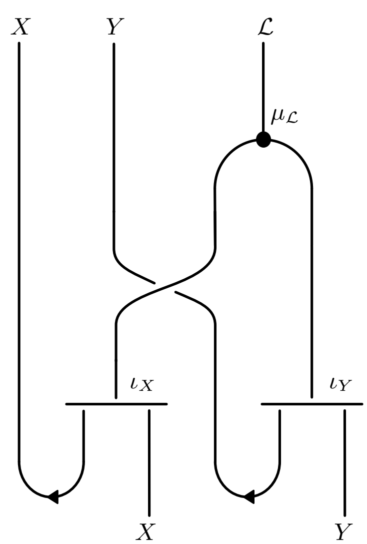
(A.3) $\varphi_{\mathcal{L}\otimes\mathcal{L}}(\Delta_{\mathcal{L}})_X \quad =$



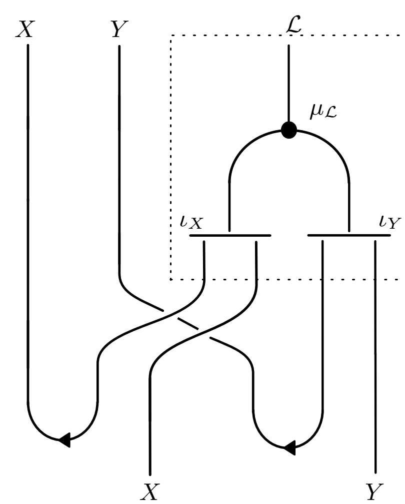
Applying $\zeta^{-1}_{\mathcal{L}\otimes\mathcal{L}}$ on it, recall (4.30), we get then indeed RHS of (4.24) or the dinatural transformation defining the coproduct on the coend $(\mathcal{L}, \iota)$ in Figure 1. For the counit maps, we apply $\zeta^{-1}_{\mathbf{1}}$ on RHS of (4.5) and get indeed RHS of (4.25). Similarly for the antipode S map (though, the algebra structure is discussed below), we apply $\zeta^{-1}_{\mathcal{L}}$ on RHS of (4.10) and get indeed RHS of (4.26), or the corresponding diagram in Figure 1, after an elementary calculation using naturality of the braiding.

To compare the algebra structures, we use a direct calculation instead of the "double" analogue of the diagram (A.1). Recall that the multiplication for $(\mathcal{L}, \varphi)$ is defined in (4.9) by the equality $\varphi^2_{\mathcal{L}}(\mu_{\mathcal{L}})_{X,Y} = \tilde{\iota}_{X\otimes Y}$, where the map φ^2_V is defined in (4.8). In order to show the equality of the multiplications on $(\mathcal{L}, \iota)$ and $(\mathcal{L}, \varphi)$, we compute the image of $\mu_{\mathcal{L}}$ from (4.22) (or graphically in Figure 1) under the map $\varphi^2_{\mathcal{L}}$ and show that it is equal to $\tilde{\iota}_{X\otimes Y}$. We have

(A.4) $\varphi^2_{\mathcal{L}}(\mu_{\mathcal{L}})_{X,Y} \quad =$



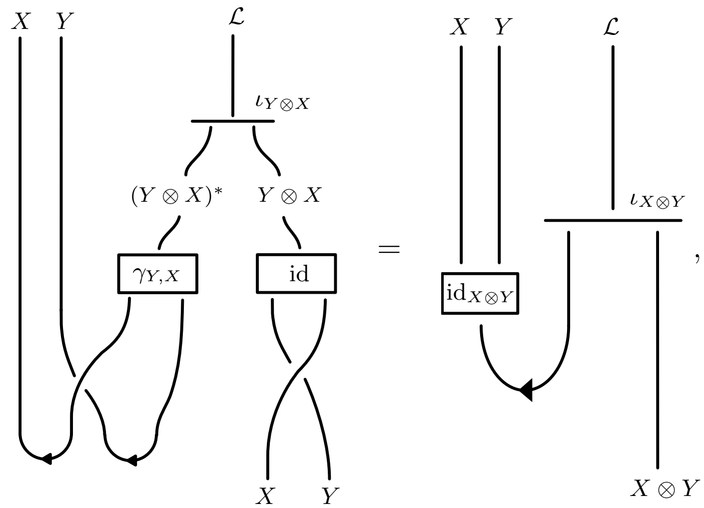
$=$



where we used naturality of the braiding. Using then the defining equality for $\mu_{\mathcal{L}}$ in Figure 1, we can replace the part of the diagram inside the dashed square by RHS of (4.22) that gives

(after an elementary graphical calculus)

(A.5) $\varphi^2_{\mathcal{L}}(\mu_{\mathcal{L}})_{X,Y} =$



where in the last equality we used the dinaturality property (4.19) of $\iota_{Y\otimes X}$ in order to move the braiding from right to left, and then we also used the explicit diagram (2.25) for the isomorphism $\gamma_{Y,X}$, and the zig-zag identity to simplify the diagram. We get thus indeed $\varphi^2_{\mathcal{L}}(\mu_{\mathcal{L}})_{X,Y} = \tilde{\iota}_{X\otimes Y}$, recall (A.2), and therefore the two multiplications are equal. The unit maps $\eta_{\mathcal{L}}$ are compared in a similar way. Finally, for the Hopf pairing $\omega_{\mathcal{L}}$ we calculate $\varphi^2_{\mathbf{1}}(\omega_{\mathcal{L}})_{X,Y}$ along the lines in (A.4) using RHS of (4.28) for the dashed region and simplify it up to RHS of (4.12), as claimed.

B. Equivalence between $\mathcal{SF}$ and $\mathbf{Rep}\,\mathsf{S}$

We give here the first part of the proof of Lemma 7.5. Fix $\mathrm{N} \in \mathbb{N}$ and $\beta \in \mathbb{C}$ with $\beta^4 = (-1)^{\mathrm{N}}$, see (6.1). In this section we introduce a quasi-bialgebra $\mathsf{S} = \mathsf{S}(\mathrm{N}, \beta)$ in **Svect**, define a braiding on its category $\mathbf{Rep}\,\mathsf{S}$ of finite-dimensional representations in **Svect**, and show that $\mathbf{Rep}\,\mathsf{S}$ is equivalent to $\mathcal{SF}(\mathrm{N}, \beta)$ as a braided monoidal category.

B.1. A quasi-bialgebra in Svect

The unital associative algebra $\mathsf{S} = \mathsf{S}(\mathrm{N}, \beta)$ over $\mathbb{C}$ has generators $\mathsf{x}_i^{\pm}$, $i = 1, \dots, \mathrm{N}$ and L, subject to the relations

$$\mathsf{x}_i^{\pm}\mathsf{L} = \mathsf{L}\mathsf{x}_i^{\pm} \ , \quad \{\mathsf{x}_i^{+}, \mathsf{x}_j^{-}\} = \delta_{i,j}\tfrac{1}{2}(1 - \mathsf{L}) \ , \quad \{\mathsf{x}_i^{\pm}, \mathsf{x}_j^{\pm}\} = 0 \ , \quad \mathsf{L}^2 = \mathbf{1} \ . \tag{B.1}$$

We have $\dim \mathsf{S} = 2^{2\mathrm{N}+1}$. Next we turn S into an algebra in **Svect** by giving it a $\mathbb{Z}_2$-grading such that $\mathsf{x}_i^{\pm}$ have odd degree and L has even degree.

Define the central idempotents

$$\boldsymbol{e}_0 := \tfrac{1}{2}(1 + \mathsf{L}) \ , \qquad \boldsymbol{e}_1 := \tfrac{1}{2}(1 - \mathsf{L}) \ . \tag{B.2}$$

Using these, S decomposes as

$$\mathsf{S} = \mathsf{S}_0 \oplus \mathsf{S}_1 \qquad , \qquad \mathsf{S}_0 := \boldsymbol{e}_0\mathsf{S} \quad , \quad \mathsf{S}_1 := \boldsymbol{e}_1\mathsf{S} \ . \tag{B.3}$$

From the defining relations it is immediate that S_0 is a $2^{2\mathrm{N}}$-dimensional Graßmann algebra and S_1 is a $2^{2\mathrm{N}}$-dimensional Clifford algebra.

In the following we will often use the tensor product $A \otimes B$ of algebras A, B in **Svect**. As a super-vector space, $A \otimes B$ is the tensor product $A \otimes_{\mathbf{Svect}} B$ of the underlying super-vector spaces. The unit is $1 \otimes 1$ and the product $\mu_{A\otimes B}$ includes parity signs:

$$\mu_{A\otimes B} := (\mu_A \otimes \mu_B) \circ (\mathrm{id}_A \otimes \tau^{\mathrm{s.v.}}_{B,A} \otimes \mathrm{id}_B) \ : \ A \otimes B \otimes A \otimes B \longrightarrow A \otimes B \ , \tag{B.4}$$

where $\tau^{\mathrm{s.v.}}$ is the symmetric braiding in **Svect**, see (6.7). In terms of homogeneous elements, this reads $(a \otimes b) \cdot (a' \otimes b') = (-1)^{|b||a'|}(aa') \otimes (bb')$.

The notion of a quasi-bialgebra in **vect** is given e.g. in Definition 1.1 (that definition is for quasi-Hopf algebras – just omit the antipode and the corresponding conditions). In **Svect**, the definition is basically the same (and works in general symmetric monoidal categories):

Definition B.1. A quasi-bialgebra in **Svect** is a tuple $(A, \varepsilon, \Delta, \Phi)$, where

- $A \in \mathbf{Svect}$ is a unital associative algebra, such that the unit $\mathbf{1} \in A$ is even and the product respects the $\mathbb{Z}_2$-grading.
- $\varepsilon : A \to \mathbb{C}^{1|0}$ and $\Delta : A \to A \otimes A$ are even linear maps and algebra homomorphisms. (The algebra structure on $A \otimes A$ involves the symmetric braiding in **Svect** as described in (B.4).)
- $\Phi \in A \otimes A \otimes A$ is an even element.

These data is subject to the conditions (1.2)–(1.5) in Definition 1.1, with products in $A\otimes A\otimes A$ involving parity signs as in (B.4).

In presenting the quasi-bialgebra structure for S, we first list the data and will then prove in Proposition B.2 below that these indeed define a quasi-bialgebra in **Svect**.

The (non-coassociative) coproduct $\Delta^{\mathsf{S}} \colon \mathsf{S} \to \mathsf{S} \otimes \mathsf{S}$ and counit $\epsilon \colon \mathsf{S} \to \mathbb{C}$ are given by

$$\begin{aligned} &\Delta^{\mathsf{S}}(\mathsf{x}_i^{\pm}) = \mathsf{x}_i^{\pm} \otimes \mathbf{1} + (\boldsymbol{e}_0 - \mathrm{i}\boldsymbol{e}_1) \otimes \mathsf{x}_i^{\pm} \pm \mathrm{i}\boldsymbol{e}_1 \otimes \boldsymbol{e}_1(\mathsf{x}_i^{+} - \mathsf{x}_i^{-}) \ , \qquad && \varepsilon^{\mathsf{S}}(\mathsf{x}_i^{\pm}) = 0 \ , \\ &\Delta^{\mathsf{S}}(\mathsf{L}) = \mathsf{L} \otimes \mathsf{L} \ , && \varepsilon^{\mathsf{S}}(\mathsf{L}) = 1 \ . \end{aligned} \tag{B.5}$$

It is straightforward to check that Δ^{S} is well-defined on S, i.e. that the above definition in terms of generators is compatible with the relations in (B.1).

The coassociator $\Lambda \in \mathsf{S} \otimes \mathsf{S} \otimes \mathsf{S}$ is given by

$$\begin{aligned} \Lambda = {} & \boldsymbol{e}_0 \otimes \boldsymbol{e}_0 \otimes \boldsymbol{e}_0 \\ & + \boldsymbol{e}_1 \otimes \boldsymbol{e}_0 \otimes \boldsymbol{e}_0 + \Lambda^{010} \boldsymbol{e}_0 \otimes \boldsymbol{e}_1 \otimes \boldsymbol{e}_0 + \boldsymbol{e}_0 \otimes \boldsymbol{e}_0 \otimes \boldsymbol{e}_1 \\ & + \Lambda^{110} \boldsymbol{e}_1 \otimes \boldsymbol{e}_1 \otimes \boldsymbol{e}_0 + \Lambda^{101} \boldsymbol{e}_1 \otimes \boldsymbol{e}_0 \otimes \boldsymbol{e}_1 + \boldsymbol{e}_0 \otimes \boldsymbol{e}_1 \otimes \boldsymbol{e}_1 \\ & + \Lambda^{111} \boldsymbol{e}_1 \otimes \boldsymbol{e}_1 \otimes \boldsymbol{e}_1 \ , \end{aligned} \tag{B.6}$$

where

$$\Lambda^{abc} = \beta^{2abc} \prod_{k=1}^{\mathrm{N}} \Lambda^{abc}_{(k)} \tag{B.7}$$

(i.e. there is a factor of β^2 in sector $\mathbf{111}$) with

(B.8)

$$\begin{aligned} \Lambda^{010}_{(k)} &= \mathbf{1} \otimes \mathbf{1} \otimes \mathbf{1} + (1+\mathrm{i})\mathsf{x}_k^{-} \otimes \mathbf{1} \otimes \mathsf{x}_k^{+} - (1-\mathrm{i})\mathsf{x}_k^{+} \otimes \mathbf{1} \otimes \mathsf{x}_k^{-} - 2\mathsf{x}_k^{+}\mathsf{x}_k^{-} \otimes \mathbf{1} \otimes \mathsf{x}_k^{+}\mathsf{x}_k^{-} \ , \\ \Lambda^{110}_{(k)} &= \mathbf{1} \otimes \mathbf{1} \otimes \mathbf{1} - \mathrm{i}\mathbf{1} \otimes (\mathsf{x}_k^{+} + \mathsf{x}_k^{-}) \otimes (\mathsf{x}_k^{+} + \mathsf{x}_k^{-}) \ , \\ \Lambda^{101}_{(k)} &= \mathbf{1} \otimes \mathbf{1} \otimes \mathbf{1} + \mathrm{i}\mathbf{1} \otimes (\mathsf{x}_k^{+} + \mathrm{i}\mathsf{x}_k^{-}) \otimes (\mathsf{x}_k^{+} + \mathrm{i}\mathsf{x}_k^{-}) + (1-\mathrm{i})(\mathsf{x}_k^{+} - \mathsf{x}_k^{-}) \otimes \mathsf{x}_k^{+}\mathsf{x}_k^{-} \otimes (\mathrm{i}\mathsf{x}_k^{+} - \mathsf{x}_k^{-}) \\ &\quad + (1+\mathrm{i})\mathsf{x}_k^{-} \otimes \mathsf{x}_k^{+} \otimes \mathbf{1} - (1-\mathrm{i})\mathsf{x}_k^{+} \otimes \mathsf{x}_k^{-} \otimes \mathbf{1} + (1+\mathrm{i})\mathbf{1} \otimes \mathsf{x}_k^{+}\mathsf{x}_k^{-} \otimes \mathbf{1} - 2\mathsf{x}_k^{+}\mathsf{x}_k^{-} \otimes \mathsf{x}_k^{+}\mathsf{x}_k^{-} \otimes \mathbf{1} \ , \\ \Lambda^{111}_{(k)} &= (\mathrm{i}-1)\mathbf{1} \otimes \mathbf{1} \otimes \mathbf{1} + (1-\mathrm{i})\mathbf{1} \otimes \mathbf{1} \otimes \mathsf{x}_k^{+}\mathsf{x}_k^{-} + (1+2\mathrm{i})\mathbf{1} \otimes \mathsf{x}_k^{-} \otimes \mathsf{x}_k^{-} + \mathbf{1} \otimes \mathsf{x}_k^{-} \otimes \mathsf{x}_k^{+} \\ &\quad + \mathbf{1} \otimes \mathsf{x}_k^{+} \otimes \mathsf{x}_k^{-} + \mathbf{1} \otimes \mathsf{x}_k^{+} \otimes \mathsf{x}_k^{+} + (1-\mathrm{i})\mathbf{1} \otimes \mathsf{x}_k^{+}\mathsf{x}_k^{-} \otimes \mathbf{1} - (2-2\mathrm{i})\mathbf{1} \otimes \mathsf{x}_k^{+}\mathsf{x}_k^{-} \otimes \mathsf{x}_k^{+}\mathsf{x}_k^{-} \\ &\quad - (1-\mathrm{i})\mathsf{x}_k^{-} \otimes \mathbf{1} \otimes \mathsf{x}_k^{-} + (1+\mathrm{i})\mathsf{x}_k^{-} \otimes \mathbf{1} \otimes \mathsf{x}_k^{+} - (1-\mathrm{i})\mathsf{x}_k^{-} \otimes \mathsf{x}_k^{-} \otimes \mathsf{x}_k^{+}\mathsf{x}_k^{-} \\ &\quad + (1+\mathrm{i})\mathsf{x}_k^{-} \otimes \mathsf{x}_k^{+} \otimes \mathsf{x}_k^{+}\mathsf{x}_k^{-} + (1-\mathrm{i})\mathsf{x}_k^{-} \otimes \mathsf{x}_k^{+}\mathsf{x}_k^{-} \otimes \mathsf{x}_k^{-} - (1+\mathrm{i})\mathsf{x}_k^{-} \otimes \mathsf{x}_k^{+}\mathsf{x}_k^{-} \otimes \mathsf{x}_k^{+} \end{aligned}$$

$$+ (1-\mathrm{i})\mathsf{x}_k^+ \otimes \mathsf{x}_k^- \otimes \mathbf{1} - (1-\mathrm{i})\mathsf{x}_k^+ \otimes \mathsf{x}_k^- \otimes \mathsf{x}_k^+\mathsf{x}_k^- - (1+\mathrm{i})\mathsf{x}_k^+ \otimes \mathsf{x}_k^+ \otimes \mathbf{1}$$
$$+ (1+\mathrm{i})\mathsf{x}_k^+ \otimes \mathsf{x}_k^+ \otimes \mathsf{x}_k^+\mathsf{x}_k^- - (1-\mathrm{i})\mathsf{x}_k^+ \otimes \mathsf{x}_k^+\mathsf{x}_k^- \otimes \mathsf{x}_k^- + (1+\mathrm{i})\mathsf{x}_k^+ \otimes \mathsf{x}_k^+\mathsf{x}_k^- \otimes \mathsf{x}_k^+$$
$$+ 2\mathsf{x}_k^+\mathsf{x}_k^- \otimes \mathbf{1} \otimes \mathbf{1} - 2\mathsf{x}_k^+\mathsf{x}_k^- \otimes \mathbf{1} \otimes \mathsf{x}_k^+\mathsf{x}_k^- - 2\mathrm{i}\mathsf{x}_k^+\mathsf{x}_k^- \otimes \mathsf{x}_k^- \otimes \mathsf{x}_k^- - 2\mathrm{i}\mathsf{x}_k^+\mathsf{x}_k^- \otimes \mathsf{x}_k^+ \otimes \mathsf{x}_k^+$$
$$- 2\mathsf{x}_k^+\mathsf{x}_k^- \otimes \mathsf{x}_k^+\mathsf{x}_k^- \otimes \mathbf{1} + 4\mathsf{x}_k^+\mathsf{x}_k^- \otimes \mathsf{x}_k^+\mathsf{x}_k^- \otimes \mathsf{x}_k^+\mathsf{x}_k^- \ .$$

Note the product of the $\Lambda^{abc}_{(k)}$ is taken in the tensor product of super-algebras $\mathsf{S}\otimes\mathsf{S}\otimes\mathsf{S}$, which includes parity signs, as defined in (B.4). The $\Lambda^{abc}_{(k)}$ is parity-even, and one quickly checks that for $k \neq l$ the elements $\Lambda^{abc}_{(k)}$ and $\Lambda^{abc}_{(l)}$ commute in $S \otimes S \otimes S$, so that the order in which one takes the product does not matter.

Finally, we define an element $r \in \mathsf{S}\otimes\mathsf{S}$ which is given by

$$r = r^{00} \cdot \boldsymbol{e}_0 \otimes \boldsymbol{e}_0 + r^{01} \cdot \boldsymbol{e}_0 \otimes \boldsymbol{e}_1 + r^{10} \cdot \boldsymbol{e}_1 \otimes \boldsymbol{e}_0 + r^{11} \cdot \boldsymbol{e}_1 \otimes \boldsymbol{e}_1 \tag{B.9}$$

where $r^{nm} = \beta^{nm} \prod_{k=1}^{\mathrm{N}} r^{nm}_{(k)}$ and

$$\begin{aligned} r^{00}_{(k)} &= \mathbf{1}\otimes\mathbf{1} - 2\mathsf{x}_k^- \otimes \mathsf{x}_k^+ \ , \\ r^{01}_{(k)} &= \mathbf{1}\otimes\mathbf{1} - (1+\mathrm{i})\mathsf{x}_k^- \otimes \mathsf{x}_k^+ - (1+\mathrm{i})\mathsf{x}_k^+ \otimes \mathsf{x}_k^- - (1+\mathrm{i})\mathsf{x}_k^+\mathsf{x}_k^- \otimes \mathbf{1} + 2\mathrm{i}\mathsf{x}_k^+\mathsf{x}_k^- \otimes \mathsf{x}_k^+\mathsf{x}_k^- \ , \\ r^{10}_{(k)} &= \mathbf{1}\otimes\mathbf{1} - (1-\mathrm{i})\mathsf{x}_k^- \otimes \mathsf{x}_k^+ - (1-\mathrm{i})\mathsf{x}_k^+ \otimes \mathsf{x}_k^- - (1-\mathrm{i})\mathbf{1} \otimes \mathsf{x}_k^+\mathsf{x}_k^- + 2\mathrm{i}\mathsf{x}_k^+\mathsf{x}_k^- \otimes \mathsf{x}_k^+\mathsf{x}_k^- \ , \\ r^{11}_{(k)} &= -\mathrm{i}\mathsf{x}_k^- \otimes \mathsf{x}_k^- - \mathsf{x}_k^- \otimes \mathsf{x}_k^+ - \mathsf{x}_k^+ \otimes \mathsf{x}_k^- + \mathrm{i}\mathsf{x}_k^+ \otimes \mathsf{x}_k^+ + 2\mathsf{x}_k^+\mathsf{x}_k^- \otimes \mathsf{x}_k^+\mathsf{x}_k^- \ . \end{aligned} \tag{B.10}$$

Again, for $k \neq l$ the elements $r^{mn}_{(k)}$ and $r^{mn}_{(l)}$ commute in $\mathsf{S}\otimes\mathsf{S}$. We will use r later to define a braiding in $\mathbf{Rep}\,\mathsf{S}$ (but r is not an R-matrix for S as the braiding will involve an extra parity map).

For $\mathrm{N} = 1$, the quasi-bialgebra algebra S was defined in [**GR1**, Sec. 4]. The proof that $\mathsf{S}(\mathrm{N},\beta)$ is a quasi-bialgebra for general N relies on reducing the problem to the $\mathrm{N} = 1$ case. To do so, choose $\beta_1,\dots,\beta_{\mathrm{N}}$ such that $\beta_i^4 = -1$ and $\beta = \beta_1\cdots\beta_{\mathrm{N}}$. Define A to be the N-fold tensor product

$$\mathsf{A} := \mathsf{S}(1,\beta_1)\otimes\cdots\otimes\mathsf{S}(1,\beta_{\mathrm{N}}) \tag{B.11}$$

of quasi-bialgebras in **Svect**. Thus A is itself a quasi-bialgebra in **Svect**. The product and coproduct of A are defined with parity signs (see (B.4) for the product) and the coassociator Λ^{A} of A is the product in $\mathsf{A}\otimes\mathsf{A}\otimes\mathsf{A}$ of those in the individual factors $\mathsf{S}(1,\beta_i)$. Consider the two-sided ideal I in A generated by

$$I := \left\langle \mathsf{L}_i - \mathsf{L}_j \,\middle|\, 1 \le i,j \le \mathrm{N} \right\rangle \ . \tag{B.12}$$

where L_i stands for the element with L on the i-th tensor factor and $\mathbf{1}$ else. One verifies that this ideal satisfies $\Delta^{\mathsf{A}}(I) \subset I\otimes\mathsf{A} + \mathsf{A}\otimes I$ and $\varepsilon^{\mathsf{A}}(I) = 0$, i.e. it is a (quasi-)bialgebra ideal.

Proposition B.2. *For N, β as in* (6.1)*, the data $(\mathsf{S},\cdot,\mathbf{1},\Delta^{\mathsf{S}},\varepsilon^{\mathsf{S}},\Lambda)$ defines a quasi-bialgebra in* **Svect** *and $\mathsf{S}\cong\mathsf{A}/I$ as quasi-bialgebras.*

PROOF. We write x^+, x^- and L for the generators of $\mathsf{S}(1,\beta_j)$. Consider the surjective even linear map $\varphi : \mathsf{A} \to \mathsf{S}(\mathrm{N},\beta)$ given by

$$\varphi\Big((\mathsf{x}^+)^{\varepsilon_1}(\mathsf{x}^-)^{\delta_1}\mathsf{L}^{m_1} \otimes \cdots \otimes (\mathsf{x}^+)^{\varepsilon_\mathrm{N}}(\mathsf{x}^-)^{\delta_\mathrm{N}}\mathsf{L}^{m_\mathrm{N}} \Big) = (\mathsf{x}_1^+)^{\varepsilon_1}(\mathsf{x}_1^-)^{\delta_1} \cdots (\mathsf{x}_\mathrm{N}^+)^{\varepsilon_\mathrm{N}}(\mathsf{x}_\mathrm{N}^-)^{\delta_\mathrm{N}}\mathsf{L}^{m_1+\cdots+m_\mathrm{N}} \ . \tag{B.13}$$

The map φ satisfies

$$\varphi(\mathbf{1}^{\otimes\mathrm{N}}) = \mathbf{1} \ , \quad \varepsilon^\mathsf{S}\circ\varphi = \varepsilon^\mathsf{A} \ , \quad \varphi(ab) = \varphi(a)\varphi(b) \ , \quad (\varphi\otimes\varphi)\circ\Delta^\mathsf{A} = \Delta^\mathsf{S}\circ\varphi \tag{B.14}$$

Since φ is surjective, this proves that Δ^S is an algebra homomorphism. Furthermore, $\varphi^{\otimes 3}$ maps the coassociator of A to that of S:

$$\varphi^{\otimes 3}(\Lambda^\mathsf{A}) = \Lambda \ . \tag{B.15}$$

This follows from the product form of $\Lambda \in \mathsf{S}^{\otimes 3}$ and the fact that $\beta = \beta_1\cdots\beta_\mathrm{N}$. Using once more that S is surjective, it follows that Λ satisfies the defining relations of a coassociator (see Definition B.1).

Thus S is a quasi-bialgebra. Finally, it is clear from the definition of φ that $\ker(\varphi) = I$. $\square$

B.2. An equivalence from $\mathcal{SF}$ to $\mathbf{Rep}\,\mathsf{S}$

Due to (B.3), $\mathbf{Rep}\,\mathsf{S}$ can be decomposed into two parts:

$$\mathbf{Rep}\,\mathsf{S} = \mathbf{Rep}\,\mathsf{S}_0 \oplus \mathbf{Rep}\,\mathsf{S}_1 \ . \tag{B.16}$$

In this section we construct an equivalence of $\mathbb{C}$-linear categories $\mathcal{D}\colon \mathcal{SF} \to \mathbf{Rep}\,\mathsf{S}$. The functor has two components due to the decompositions in (6.3) and (B.16). On $\mathcal{SF}_0$ we have

$$\mathcal{D}_0\colon \mathcal{SF}_0 \to \mathbf{Rep}\,\mathsf{S}_0 \ , \quad \mathcal{D}_0(U) = U \quad \text{where} \quad \mathsf{x}_i^+ u = a_i u \ , \quad \mathsf{x}_i^- u = b_i u \quad (u\in U) \ , \tag{B.17}$$

and $D_0(f) = f$ on morphisms.

For the second component we need the non-central idempotent

$$\mathsf{b} := \prod_{i=1}^{\mathrm{N}} \mathsf{x}_i^-\mathsf{x}_i^+\boldsymbol{e}_1 \in \mathsf{S}_1 \ . \tag{B.18}$$

To see that indeed $\mathsf{b}\mathsf{b} = \mathsf{b}$ note that for each i separately we have $\mathsf{x}_i^-\mathsf{x}_i^+\mathsf{x}_i^-\mathsf{x}_i^+\boldsymbol{e}_1 = \mathsf{x}_i^-\mathsf{x}_i^+\boldsymbol{e}_1$. The idempotent b generates a S_1-submodule $\mathsf{B}\subset\mathsf{S}_1$:

$$\mathsf{B} := \mathsf{S}\mathsf{b} = \mathsf{S}_1\mathsf{b} = \mathrm{span}\left\{ (\mathsf{x}_\mathrm{N}^+)^{i_\mathrm{N}}\cdots(\mathsf{x}_1^+)^{i_1}\mathsf{b} \mid i_1,\dots,i_\mathrm{N}\in\{0,1\}\right\} . \tag{B.19}$$

Since S_1 is a Clifford algebra, B is a simple (and projective) S_1-module in $\mathbf{Svect}$. There are up to isomorphism two distinct simple S_1-modules, namely B and $\mathsf{B}\otimes_{\mathbf{Svect}}\mathbb{C}^{0|1}$, the parity-shifted copy of B.

The second component of $\mathcal{D}$ is defined as

$$\mathcal{D}_1\colon \mathcal{SF}_1 \to \mathbf{Rep}\,\mathsf{S}_1 \ , \quad \mathcal{D}_1(U) = \mathsf{B}\otimes_{\mathbf{Svect}} U \ , \tag{B.20}$$

where the S_1-action on $\mathcal{D}_1(U)$ is the left multiplication on B and where $U \in \mathcal{SF}_1 \simeq \mathbf{Svect}$. On morphisms we set $\mathcal{D}_1(f) = \mathrm{id}_{\mathsf{B}} \otimes f$.

Proposition B.3. *The functor $\mathcal{D}$ is an equivalence of $\mathbb{C}$-linear categories.*

Proof. We will give an inverse functor $\mathcal{E}\colon \mathbf{Rep}\,\mathsf{S} \to \mathcal{SF}$ to $\mathcal{D}$. The functor $\mathcal{E}$ will again be defined separately in the two sectors.

Given $V \in \mathbf{Rep}\,\mathsf{S}_0$, the $\mathbb{A}$-module $\mathcal{E}(V)$ has V as the underlying super-vector space with the action of a_i and b_i given by the action of x_i^+ and x_i^-, respectively. On morphisms f in $\mathbf{Rep}\,\mathsf{S}_0$ we set $\mathcal{E}(f) = f$. Clearly, $\mathcal{D}$ and $\mathcal{E}$ are inverse to each other as functors between $\mathcal{SF}_0$ and $\mathbf{Rep}\,\mathsf{S}_0$.

For a given object $M \in \mathbf{Rep}\,\mathsf{S}_1$, we set $\mathcal{E}(M) = \mathrm{Hom}^{\mathbf{vect}}_{\mathsf{S}_1}(\mathsf{B}, M)$. This means we consider all S_1-linear maps from B to M, not just the parity-even maps. The decomposition into parity-even and parity-odd maps turns $\mathrm{Hom}^{\mathbf{vect}}_{\mathsf{S}_1}(\mathsf{B}, M)$ into a super-vector space. Since B is simple, we get a canonical isomorphism $\mathrm{Hom}^{\mathbf{vect}}_{\mathsf{S}_1}(\mathsf{B}, \mathsf{B} \otimes_{\mathbf{Svect}} V) \cong V$ of super-vector spaces which is natural in V, that is, $\mathcal{E}(\mathcal{D}(V)) \cong V$. Conversely, since every S_1-module is isomorphic to a direct sum of copies of B and $\mathsf{B} \otimes_{\mathbf{Svect}} \mathbb{C}^{0|1}$, that is $M \cong \mathsf{B} \otimes V$ for some $V \in \mathbf{Svect}$, we get $M \cong \mathsf{B} \otimes \mathrm{Hom}^{\mathbf{vect}}_{\mathsf{S}_1}(\mathsf{B}, M)$. $\square$

B.3. $\mathcal{D}$ as a multiplicative functor

Below we will make an ansatz for isomorphisms

$$\Delta_{U,V} \colon \mathcal{D}(U * V) \to \mathcal{D}(U) \otimes_{\mathbf{Rep}\,\mathsf{S}} \mathcal{D}(V) \ . \tag{B.21}$$

which will be given sector by sector. Both, the expression for $\Delta_{U,V}$ and the proof that it is an isomorphism mimics the corresponding construction for $\mathrm{N} = 1$ in [**GR1**, Sec. 6.3].

The multiplication map $\mathsf{S} \otimes \mathsf{S} \to \mathsf{S}$ is denoted by μ^{S}, and for an S-module U the expression $\rho^U : \mathsf{S} \otimes U \to U$ stands for the S-action on U. We will also need the constants

$$\delta_0 = \prod_{i=1}^{\mathrm{N}} \big(\mathbf{1} \otimes \mathbf{1} + \mathsf{x}_i^- \otimes \mathsf{x}_i^+\big) \in \mathsf{S} \otimes \mathsf{S}, \qquad \delta_1 = \prod_{i=1}^{\mathrm{N}} \big(\mathbf{1} + \mathsf{x}_i^+ \mathsf{x}_i^-\big) \in \mathsf{S}. \tag{B.22}$$

Note that they are multiplicative invertible with inverses

$$\delta_0^{-1} = \prod_{i=1}^{\mathrm{N}} \big(\mathbf{1} \otimes \mathbf{1} - \mathsf{x}_i^- \otimes \mathsf{x}_i^+\big) \in \mathsf{S} \otimes \mathsf{S}, \quad \delta_1^{-1} = \prod_{i=1}^{\mathrm{N}} \big(\mathbf{1} - \mathsf{x}_i^+ \mathsf{x}_i^- \,\big(\boldsymbol{e}_0 + \tfrac{1}{2}\boldsymbol{e}_1\big)\big) \in \mathsf{S}. \tag{B.23}$$

We denote the right multiplication with $a \in \mathsf{S}$ by R_a.

The following list defines $\Delta_{U,V}$ for each of the four possibilities to choose $U \in \mathcal{SF}_i$ and $V \in \mathcal{SF}_j$, $i, j \in \{0, 1\}$, which we refer to as "sector ij". The underlines indicate on which factors S acts. In Figure 1 we give string diagram expressions for $\Delta_{U,V}$.

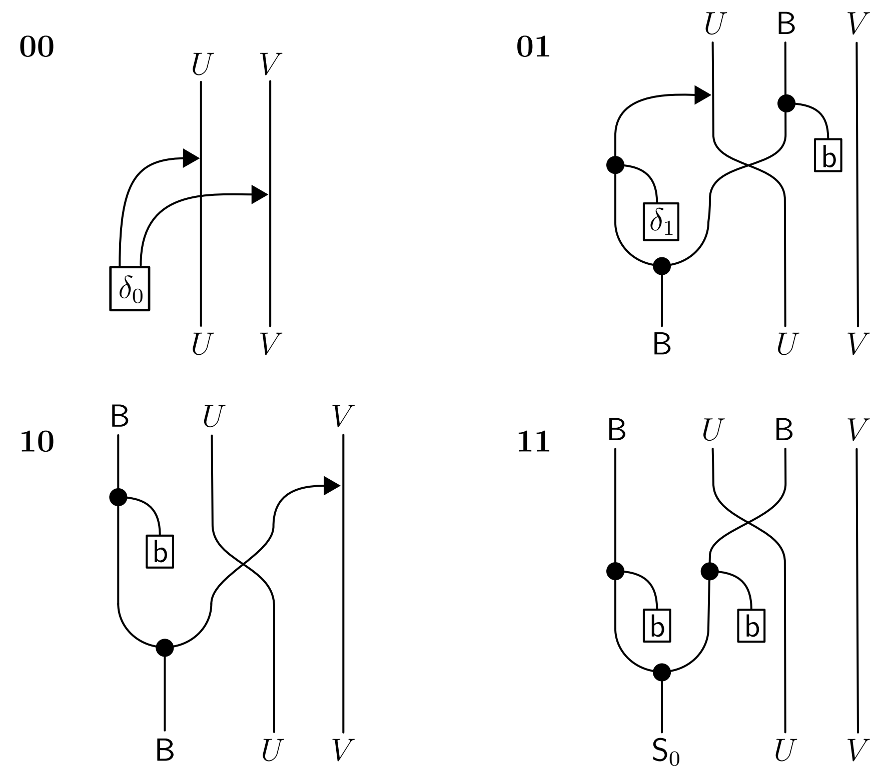


FIGURE 1. String diagram notation for $\Delta_{U,V}$ in the four sectors.

- **00** sector: $\Delta_{U,V} : \underline{U} \otimes \underline{V} \to \underline{U} \otimes \underline{V}$ is given by

$$\Delta_{U,V} = \big(\rho^U \otimes \rho^V\big) \circ \big(\mathrm{id}_{\mathsf{S}} \otimes \tau^{\text{s.v.}}_{\mathsf{S},U} \otimes \mathrm{id}_V\big) \circ \big(\delta_0 \otimes \mathrm{id}_U \otimes \mathrm{id}_V\big) \ , \tag{B.24}$$

where $\tau^{\text{s.v.}}$ is defined in (6.7).

- **01** sector: $\Delta_{U,V} : \underline{\mathsf{B}} \otimes U \otimes V \to \underline{U} \otimes \underline{\mathsf{B}} \otimes V$ is given by

$$\Delta_{U,V} = \big(\rho^U \otimes R_{\mathsf{b}} \otimes \mathrm{id}_V\big) \circ \big(R_{\delta_1} \otimes \tau^{\text{s.v.}}_{\mathsf{S},U} \otimes \mathrm{id}_V\big) \circ \big(\Delta^{\mathsf{S}} \otimes \mathrm{id}_U \otimes \mathrm{id}_V\big) \ . \tag{B.25}$$

The image of R_{b} on S is B, that is, the target is indeed $U \otimes \mathsf{B} \otimes V$.

- **10** sector: $\Delta_{U,V} : \underline{\mathsf{B}} \otimes U \otimes V \to \underline{\mathsf{B}} \otimes U \otimes \underline{V}$ is given by

$$\Delta_{U,V} = \big(R_{\mathsf{b}} \otimes \mathrm{id}_U \otimes \rho^V\big) \circ \big(\mathrm{id}_{\mathsf{S}} \otimes \tau^{\text{s.v.}}_{\mathsf{S},U} \otimes \mathrm{id}_V\big) \circ \big(\Delta^{\mathsf{S}} \otimes \mathrm{id}_U \otimes \mathrm{id}_V\big) \ . \tag{B.26}$$

- **11** sector: $\Delta_{U,V} : \underline{\mathsf{S}_0} \otimes U \otimes V \to \underline{\mathsf{B}} \otimes U \otimes \underline{\mathsf{B}} \otimes V$ is given by

$$\Delta_{U,V} = \big(\mathrm{id}_{\mathsf{B}} \otimes \tau^{\text{s.v.}}_{\mathsf{B},U} \otimes \mathrm{id}_V\big) \circ \big([(R_{\mathsf{b}} \otimes R_{\mathsf{b}}) \circ \Delta^{\mathsf{S}}] \otimes \mathrm{id}_U \otimes \mathrm{id}_V\big) \ . \tag{B.27}$$

Here, in the source S-module we have identified S_0 and $\mathbb{A}$.

Lemma B.4. *The linear maps $\Delta_{U,V}$ are intertwiners of S-modules.*

PROOF. In all sectors beside **00** this clear since Δ^{S} is an algebra map, and since the right-multiplications R_{δ_1} and R_b are left-module intertwiners. In sector **00** we have to check that $(\boldsymbol{e}_0 \otimes \boldsymbol{e}_0) \cdot \Delta^{\mathsf{S}}(a) \cdot \delta_0 = (\boldsymbol{e}_0 \otimes \boldsymbol{e}_0) \cdot \delta_0 \cdot \Delta^{\mathsf{S}}(a)$ holds for all $a \in \mathsf{S}$. This is clear for $a = \mathsf{L}$, and for $a = \mathsf{x}_i^{\pm}$ it is an easy check. $\square$

Lemma B.5. *The $\Delta_{U,V}$ are isomorphisms.*

PROOF. In sectors **00, 01** and **10** the proof is similar as that in [**GR1**, Lemma 6.5]. Indeed, for sector **00**, invertibility of $\Delta_{U,V}$ follows from that of δ_0. For sector **10** and **01** the inverse of $\Delta_{U,V}$ is given by the same string diagram as that in [**GR1**, Lemma 6.5]. The proof that the expressions in [**GR1**, Lemma 6.5] are indeed inverses rests on the two identities

$$(\mathsf{b} \otimes \mathbf{1}) \cdot ((\mathrm{id} \otimes \omega_{\mathsf{S}}) \circ \Delta^{\mathsf{S}}(\mathsf{b})) \cdot (\mathsf{b} \otimes \mathbf{1}) = \mathsf{b} \otimes (\delta_1 \boldsymbol{e}_0) \ , \tag{B.28}$$
$$(\mathsf{b} \otimes \mathbf{1}) \cdot \big((\mathrm{id} \otimes \omega_{\mathsf{S}}) \circ \tau^{\mathrm{s.v.}} \circ \Delta^{\mathsf{S}}(\mathsf{b})\big) \cdot (\mathsf{b} \otimes \mathbf{1}) = \mathsf{b} \otimes (\delta_1^{-1} \boldsymbol{e}_0) \ ,$$

where ω_{S} denotes the parity involution on the super-vector space S as defined in (6.13)[1]. We need to show that these identities remain true for general N, and we will do so for the first identity.

Namely, for $i \neq j$ and by defining $\mathsf{b}^{(i)} = \mathsf{x}_i^- \mathsf{x}_i^+ \boldsymbol{e}_1$ and $\delta_1^{(i)} = \mathbf{1} + \mathsf{x}_i^+ \mathsf{x}_i^-$ we get

$$\begin{aligned}
\mathsf{b}^{(i)}\mathsf{b}^{(j)} \otimes \delta_1^{(i)}\delta_1^{(j)}\boldsymbol{e}_0 &= (\mathsf{b}^{(i)} \otimes (\delta_1^{(i)}\boldsymbol{e}_0)) \cdot (\mathsf{b}^{(j)} \otimes (\delta_1^{(j)}\boldsymbol{e}_0)) \\
&\overset{(*)}{=} \big((\mathsf{b}^{(i)} \otimes \mathbf{1})((\mathrm{id} \otimes \omega_{\mathsf{S}}) \circ \Delta^{\mathsf{S}}(\mathsf{b}^{(i)}))(\mathsf{b}^{(i)} \otimes \mathbf{1})\big) \\
&\qquad \cdot \big((\mathsf{b}^{(j)} \otimes \mathbf{1})((\mathrm{id} \otimes \omega_{\mathsf{S}}) \circ \Delta^{\mathsf{S}}(\mathsf{b}^{(j)}))(\mathsf{b}^{(j)} \otimes \mathbf{1})\big) \\
&\overset{(**)}{=} (\mathsf{b}^{(i)} \otimes \mathbf{1}) \cdot (\mathsf{b}^{(j)} \otimes \mathbf{1}) \cdot ((\mathrm{id} \otimes \omega_{\mathsf{S}}) \circ \Delta^{\mathsf{S}}(\mathsf{b}^{(i)}) \cdot (\mathrm{id} \otimes \omega_{\mathsf{S}}) \circ \Delta^{\mathsf{S}}(\mathsf{b}^{(j)})) \\
&\qquad \cdot (\mathsf{b}^{(i)} \otimes \mathbf{1}) \cdot (\mathsf{b}^{(j)} \otimes \mathbf{1}) \\
&= (\mathsf{b}^{(i)}\mathsf{b}^{(j)} \otimes \mathbf{1}) \cdot ((\mathrm{id} \otimes \omega_{\mathsf{S}}) \circ \Delta^{\mathsf{S}}(\mathsf{b}^{(i)}\mathsf{b}^{(j)})) \cdot (\mathsf{b}^{(i)}\mathsf{b}^{(j)} \otimes \mathbf{1}) \ ,
\end{aligned} \tag{B.29}$$

where we used that $\mathsf{b}^{(i)}$ is an even element and in step (*) that the corresponding identity holds for $\mathrm{N} = 1$, then in step (**) that ω_{S}, Δ^{S} are morphisms in **Svect** and so $(\mathrm{id} \otimes \omega_{\mathsf{S}}) \circ \Delta^{\mathsf{S}}(\mathsf{b}^{(i)})$ is even and moreover does not contain $\mathsf{x}_j^{\pm}$ and therefore it commutes with $\mathsf{b}^{(j)} \otimes \mathbf{1}$. Using then the fact that $\mathsf{b} = \prod_{i=1}^{\mathrm{N}} \mathsf{b}^{(i)}$ and $\delta_1 = \prod_{i=1}^{\mathrm{N}} \delta_1^{(i)}$, this analysis shows the first identity in (B.28). The second identity there is established in a similar way.

In sector **11**, we have to show that

$$\begin{aligned}
\Theta \colon \mathsf{S}_0 &\to \mathsf{B} \otimes_{\mathbf{Svect}} \mathsf{B} \\
a &\mapsto \Delta^{\mathsf{S}}(a) \cdot \mathsf{b} \otimes \mathsf{b}
\end{aligned} \tag{B.30}$$

is an isomorphism. Note that $\Delta^{\mathsf{S}}(\mathsf{x}_i^{\pm}) \cdot \boldsymbol{e}_1 \otimes \boldsymbol{e}_1 = (\mathsf{x}_i^{\pm} \otimes \mathbf{1} - \mathrm{i}\,\mathbf{1} \otimes \mathsf{x}_i^{\mp}) \cdot \boldsymbol{e}_1 \otimes \boldsymbol{e}_1$ and $\mathsf{x}_i^- \mathsf{b} = 0$. We get

$$\Theta(\boldsymbol{e}_0) = \Delta^{\mathsf{S}}(\boldsymbol{e}_0) \cdot \mathsf{b} \otimes \mathsf{b} = \mathsf{b} \otimes \mathsf{b} \tag{B.31}$$

[1]Note that the antipode in [**GR1**] coincides with ω_{S} in sector 0.

$$\begin{aligned}
\Theta(\mathsf{x}_i^+\mathsf{x}_j^+\boldsymbol{e}_0) &= \Delta^{\mathsf{S}}(\mathsf{x}_i^+)\cdot(\mathsf{x}_j^+\mathsf{b})\otimes\mathsf{b} = (\mathsf{x}_i^+\mathsf{x}_j^+\mathsf{b})\otimes\mathsf{b}\\
\Theta(\mathsf{x}_i^-\mathsf{x}_j^-\boldsymbol{e}_0) &= -\Delta^{\mathsf{S}}(\mathsf{x}_i^-)\cdot \mathrm{i}\mathsf{b}\otimes(\mathsf{x}_j^+\mathsf{b}) = (-\mathrm{i})^2\mathsf{b}\otimes(\mathsf{x}_i^+\mathsf{x}_j^+\mathsf{b})\\
\Theta(\mathsf{x}_i^+\mathsf{x}_j^-\boldsymbol{e}_0) &= -\Delta^{\mathsf{S}}(\mathsf{x}_i^+)\cdot \mathrm{i}\mathsf{b}\otimes(\mathsf{x}_j^+\mathsf{b}) = -(\mathrm{i}\mathsf{x}_i^+\mathsf{b})\otimes(\mathsf{x}_j^+\mathsf{b}) + \delta_{i,j}\mathrm{i}\mathsf{b}\otimes(\mathsf{x}_i^-\mathsf{x}_j^+\mathsf{b})\\
&= -(\mathrm{i}\mathsf{x}_i^+\mathsf{b})\otimes(\mathsf{x}_j^+\mathsf{b}) + \delta_{i,j}\mathsf{b}\otimes\mathsf{b}\ .
\end{aligned}$$

Now it is straightforward to see that for $i_1 < \ldots < i_k$ and $j_1 < \ldots < j_m$ we have

$$\begin{aligned}
\Theta(\mathsf{x}_{i_1}^+\cdots\mathsf{x}_{i_k}^+\mathsf{x}_{j_1}^-\cdots\mathsf{x}_{j_m}^-\boldsymbol{e}_0) &= \Delta^{\mathsf{S}}(\mathsf{x}_{i_1}^+\cdots\mathsf{x}_{i_k}^+)\cdot(-\mathrm{i})^m\mathsf{b}\otimes(\mathsf{x}_{j_1}^+\cdots\mathsf{x}_{j_m}^+\mathsf{b})\\
&= (-\mathrm{i})^m(\mathsf{x}_{i_1}^+\cdots\mathsf{x}_{i_k}^+\mathsf{b})\otimes(\mathsf{x}_{j_1}^+\cdots\mathsf{x}_{j_m}^+\mathsf{b}) + \tilde{w},
\end{aligned} \tag{B.32}$$

where $\tilde{w} \in \bigoplus_{i=0}^{k+m-2}\tilde{W}_i$ and $\tilde{W}_i$ is the span of elements $(\mathsf{x}_{i_1}^+\cdots\mathsf{x}_{i_u}^+\mathsf{b})\otimes(\mathsf{x}_{j_1}^+\cdots\mathsf{x}_{j_v}^+\mathsf{b})$ with $i = u + v$. So we can show by induction over $m + k$ that Θ is surjective and since S_0 and $\mathsf{B}\otimes\mathsf{B}$ have dimension $2^{2\mathrm{N}}$, Θ is even bijective. □

From the definition of $\Delta_{U,V}$ it is immediate that these maps are natural in U and V. Together with Lemmas B.4 and B.5 this proves the following proposition:

Proposition B.6. *With the isomorphisms $\Delta_{U,V}$, the functor $\mathcal{D}\colon \mathcal{SF} \to \mathbf{Rep}\,\mathsf{S}$ is multiplicative.*

B.4. Compatibility with associator and unit isomorphisms

In this section we verify that the functor $\mathcal{D}$ from Proposition B.6 is monoidal, i.e. that it is compatible with associators and unit isomorphisms.

In fact, since the unit isomorphisms in $\mathcal{SF}$ and $\mathbf{Rep}\,\mathsf{S}$ are just those of the underlying super-vector spaces, after choosing the isomorphism $\mathbf{1} \to \mathcal{D}(\mathbf{1})$ to be the identity on $\mathbb{C}^{1|0}$, compatibility of $\mathcal{D}$ with the unit isomorphisms is immediate.

The main effort lies in showing that the diagram in (7.24) commutes for $\mathcal{D}\colon \mathcal{SF} \to \mathbf{Rep}\,\mathsf{S}$ and $\Delta_{U,V}$. In a calculation similar to that in [**GR1**, Sec. 7.2] one can evaluate (7.24) for each of the eight possibilities of choosing $U \in \mathcal{SF}_a$, $V \in \mathcal{SF}_b$, $W \in \mathcal{SF}_c$, which we refer to as "sector abc". We define

$$\underline{\Lambda}^{abc} = \Lambda^{abc}\cdot(\boldsymbol{e}_a\otimes\boldsymbol{e}_b\otimes\boldsymbol{e}_c) \tag{B.33}$$

with Λ^{abc} as in (B.6) (it is understood that the Λ^{abc} not spelled out explicitly in (B.6) are set to $\mathbf{1}^{\otimes 3}$). Then (7.24) for $\mathcal{D}$ and $\Delta_{U,V}$ is equivalent to the eight conditions in Table 1.

To verify the eight identities in Table 1, we reduce them to the $\mathrm{N} = 1$ case which has been checked in [**GR1**, Prop. 7.3]. To do so, define $\mathsf{S}^{(k)}$ to be the subalgebra of S generated by x_k^+, x_k^- and L. By definition of S, for $k \neq l$ elements from $\mathsf{S}^{(k)}$ and $\mathsf{S}^{(l)}$ super-commute.

The aim is now to rewrite each condition in Table 1 as a product of terms $k = 1, \ldots, \mathrm{N}$ which use only elements in $\mathsf{S}^{(k)}$ and note that since by [**GR1**, Prop. 7.3] the corresponding identity holds for each k separately, it holds for the product.

$$
\begin{aligned}
\mathbf{000}\ :\quad & \underline{\Lambda}^{000}\cdot(\mathbf{1}\otimes\delta_0)\cdot(\mathrm{id}\otimes\Delta^{\mathsf{S}})(\delta_0)\\
&=(\boldsymbol{e}_0\otimes\boldsymbol{e}_0\otimes\boldsymbol{e}_0)\cdot(\delta_0\otimes\mathbf{1})\cdot(\Delta^{\mathsf{S}}\otimes\mathrm{id})(\delta_0)\\[1ex]
\mathbf{001}\ :\quad & \underline{\Lambda}^{001}\cdot\big((\mathrm{id}\otimes\Delta^{\mathsf{S}})\circ(R_{\delta_1}\otimes R_{\mathsf{b}})\circ\Delta^{\mathsf{S}}(v)\big)\cdot(\mathbf{1}\otimes\delta_1\otimes\mathsf{b})\\
&=(\boldsymbol{e}_0\otimes\boldsymbol{e}_0\otimes\mathbf{1})\cdot(\delta_0\otimes\mathbf{1})\cdot\big((\Delta^{\mathsf{S}}\otimes\mathrm{id})\circ(R_{\delta_1}\otimes R_{\mathsf{b}})\circ\Delta^{\mathsf{S}}(v)\big)\\[1ex]
\mathbf{010}\ :\quad & \underline{\Lambda}^{010}\cdot\big((\mathrm{id}\otimes\Delta^{\mathsf{S}})\circ(R_{\delta_1}\otimes R_{\mathsf{b}})\circ\Delta^{\mathsf{S}}(v)\big)\cdot(\mathbf{1}\otimes\mathsf{b}\otimes\mathbf{1})\\
&=\big((\Delta^{\mathsf{S}}\otimes\mathrm{id})\circ(R_{\mathsf{b}}\otimes\mathrm{id})\circ\Delta^{\mathsf{S}}(v)\big)\cdot(\delta_1\otimes\mathsf{b}\otimes\mathbf{1})\cdot\gamma^{(13)}\\[1ex]
\mathbf{100}\ :\quad & \underline{\Lambda}^{100}\cdot(\mathbf{1}\otimes\delta_0)\cdot\big((\mathrm{id}\otimes\Delta^{\mathsf{S}})\circ(R_{\mathsf{b}}\otimes\mathrm{id})\circ\Delta^{\mathsf{S}}(v)\big)\\
&=\big((\Delta^{\mathsf{S}}\otimes\mathrm{id})\circ(R_{\mathsf{b}}\otimes\mathrm{id})\circ\Delta^{\mathsf{S}}(v)\big)\cdot(\mathsf{b}\otimes\mathbf{1}\otimes\mathbf{1})\\[1ex]
\mathbf{110}\ :\quad & \underline{\Lambda}^{110}\cdot\big((\mathrm{id}\otimes\Delta^{\mathsf{S}})\circ(R_{\mathsf{b}}\otimes R_{\mathsf{b}})\circ\Delta^{\mathsf{S}}(h)\big)\cdot(\mathbf{1}\otimes\mathsf{b}\otimes\mathbf{1})\\
&=\big\{(\Delta^{\mathsf{S}}\otimes\mathrm{id})\big(\delta_0\cdot\Delta^{\mathsf{S}}(h)\big)\big\}\cdot(\mathsf{b}\otimes\mathsf{b}\otimes\mathbf{1})\\[1ex]
\mathbf{101}\ :\quad & \underline{\Lambda}^{101}\cdot\big\{(\mathrm{id}\otimes\Delta^{\mathsf{S}})\big(\Delta^{\mathsf{S}}(h)\cdot\mathsf{b}\otimes\mathsf{b}\big)\big\}\cdot(\mathbf{1}\otimes\delta_1\otimes\mathsf{b})\\
&=(R_{\mathsf{b}}\otimes\mu^{\mathsf{S}}\otimes\mathrm{id})\circ(\Delta^{\mathsf{S}}\otimes\tau^{\mathrm{s.v.}}_{\mathsf{B},\mathsf{S}_0})\circ\big(\big\{(R_{\mathsf{b}}\otimes R_{\mathsf{b}})\circ\Delta^{\mathsf{S}}\circ\mu^{\mathsf{S}}\big\}\otimes\mathrm{id}\big)(h\otimes\gamma)\\[1ex]
\mathbf{011}\ :\quad & \underline{\Lambda}^{011}\cdot\big((\mathrm{id}\otimes\Delta^{\mathsf{S}})(\delta_0)\big)\cdot\big(\mathbf{1}\otimes(\Delta^{\mathsf{S}}(h)\cdot\mathsf{b}\otimes\mathsf{b})\big)\\
&=\big\{\big(\omega_{\mathsf{S}}\circ\mu^{\mathsf{S}}\circ(\mathrm{id}\otimes\omega_{\mathsf{S}})\big)\otimes\mathrm{id}\otimes\mathrm{id}\big\}\\
&\qquad\qquad\circ\big\{\mathrm{id}\otimes\big((R_{\delta_1}\otimes R_{\mathsf{b}})\circ\Delta^{\mathsf{S}}\big)\otimes\mathrm{id}\big\}\circ\big\{\mathrm{id}\otimes\big((R_{\mathsf{b}}\otimes R_{\mathsf{b}})\circ\Delta^{\mathsf{S}}\big)\big\}\circ\Delta^{\mathsf{S}}(h)\\[1ex]
\mathbf{111}:\quad & \underline{\Lambda}^{111}\cdot\big\{\big(\mathrm{id}\otimes(\Delta^{\mathsf{S}}\circ\mu^{\mathsf{S}})\big)\big(\Delta^{\mathsf{S}}(v)\otimes h\big)\big\}\cdot\mathsf{b}\otimes\mathsf{b}\otimes\mathsf{b}\\
&=\big\{\big((R_{\mathsf{b}}\otimes R_{\mathsf{b}})\circ\Delta^{\mathsf{S}}\circ\mu^{\mathsf{S}}\big)\otimes\mathrm{id}\big\}\circ(\mathrm{id}\otimes\tau^{\mathrm{s.v.}}_{\mathsf{B},\mathsf{S}_0})\circ\big\{\big((R_{\delta_1}\otimes R_{\mathsf{b}})\circ\Delta^{\mathsf{S}}(v)\big)\otimes\phi(h)\big\}
\end{aligned}
$$

TABLE 1. Evaluation of the compatibility of associators as in (7.24) for $\mathcal{D}$ and $\Delta_{U,V}$ in each of the eight sectors. The constant γ is defined as $\gamma := \exp(C)\cdot\boldsymbol{e}_0\otimes\boldsymbol{e}_0$ with C as in (6.9) and we use the identification $\mathsf{x}_k^+ = a_k$, $\mathsf{x}_k^- = b_k$ as in (B.17). The map $\phi : \mathsf{S}_0 \to \mathsf{S}_0$ is defined as in (6.11) under the same identification. The above conditions have to hold for all $h \in \mathsf{S}_0$ and $v \in \mathsf{B}$.

For $\underline{\Lambda}^{abc}$ the product decomposition is given in (B.7). The constants b, δ_1 and δ_0 were already defined as products over elements in $\mathsf{S}^{(k)}$ and $\mathsf{S}^{(k)}\otimes\mathsf{S}^{(k)}$ in (B.18) and (B.22). For γ in Table 1 set $C^{(i)} = \mathsf{x}_i^-\otimes\mathsf{x}_i^+ - \mathsf{x}_i^+\otimes\mathsf{x}_i^-$ and use

$$
\gamma = \exp\Big(\sum_{i=1}^{\mathrm{N}} C^{(i)}\Big) = \prod_{i=1}^{\mathrm{N}}\gamma^{(i)} \qquad \text{with} \quad \gamma^{(i)} := \exp C^{(i)}\ . \tag{B.34}
$$

Now sector **000** directly decomposes into products over elements in $(\mathsf{S}^{(k)})^{\otimes 3}$ and hence holds by [**GR1**, Prop. 7.3]. For sector **001** note that, for $c^{(k)}, d^{(k)} \in \mathsf{S}^{(k)}$ even and $v^{(k)} \in \mathsf{B}^{(k)}$ we have, for $i \neq j$,

$$\begin{aligned}
\text{(B.35)} \qquad & (\mathrm{id} \otimes \Delta^{\mathsf{S}}) \circ (R_{c^{(i)}c^{(j)}} \otimes R_{d^{(i)}d^{(j)}}) \circ (\Delta^{\mathsf{S}}(v^{(i)}v^{(j)})) \\
&= (\mathrm{id} \otimes \Delta^{\mathsf{S}}) \circ (R_{c^{(i)}} \otimes R_{d^{(i)}}) \circ (R_{c^{(j)}} \otimes R_{d^{(j)}})\big(\Delta^{\mathsf{S}}(v^{(i)}) \cdot \Delta^{\mathsf{S}}(v^{(j)})\big) \\
&= (\mathrm{id} \otimes \Delta^{\mathsf{S}}) \circ (R_{c^{(i)}} \otimes R_{d^{(i)}})\big(\Delta^{\mathsf{S}}(v^{(i)}) \cdot (R_{c^{(j)}} \otimes R_{d^{(j)}})(\Delta^{\mathsf{S}}(v^{(j)}))\big) \\
&= (\mathrm{id} \otimes \Delta^{\mathsf{S}})\big((R_{c^{(i)}} \otimes R_{d^{(i)}})(\Delta^{\mathsf{S}}(v^{(i)})) \cdot (R_{c^{(j)}} \otimes R_{d^{(j)}})(\Delta^{\mathsf{S}}(v^{(j)}))\big) \\
&= (\mathrm{id} \otimes \Delta^{\mathsf{S}})\big((R_{c^{(i)}} \otimes R_{d^{(i)}})(\Delta^{\mathsf{S}}(v^{(i)}))\big) \cdot (\mathrm{id} \otimes \Delta^{\mathsf{S}})\big((R_{c^{(j)}} \otimes R_{d^{(j)}})(\Delta^{\mathsf{S}}(v^{(j)}))\big) \ .
\end{aligned}$$

This allows one to write the LHS of the equation for sector **001** as a product over elements in $(\mathsf{S}^{(k)})^{\otimes 3}$. For the RHS one uses a similar statement where $\mathrm{id} \otimes \Delta^{\mathsf{S}}$ is replaced by $\Delta^{\mathsf{S}} \otimes \mathrm{id}$.

Analogous arguments work in all sectors, except that in sector **111** we need to deal with $\phi(h)$ separately.

The linear map $\phi \colon \mathsf{S}_0 \to \mathsf{S}_0$ is defined in (6.11), where by abuse of notation we identify S_0 and $\mathbb{\Lambda}$ via $\mathsf{x}_k^+ \boldsymbol{e}_0 = a_k$, $\mathsf{x}_k^- \boldsymbol{e}_0 = b_k$. Following [**DR1**, Sec. 5.2] ϕ can be written explicitly as, for $u \in \mathsf{S}_0$,

$$\text{(B.36)} \qquad \phi(u) = \sum_{m=0}^{2\mathrm{N}} \mathrm{i}^{m(m+1)} \sum_{i_1 \leq \ldots \leq i_m} \varepsilon_{i_1} \cdots \varepsilon_{i_m} \ \Lambda^{\mathrm{co}}_{\mathsf{S}_0}(\mathsf{x}_{i_1}^{\varepsilon_{i_1}} \cdots \mathsf{x}_{i_m}^{\varepsilon_{i_m}} \cdot u) \ \mathsf{x}_{i_1}^{-\varepsilon_{i_1}} \cdots \mathsf{x}_{i_m}^{-\varepsilon_{i_m}} \boldsymbol{e}_0 \ ,$$

where $\varepsilon_{i_n} \in \{\pm 1\}$ such that $i_n = i_{n+1} \Rightarrow \varepsilon_{i_n} > \varepsilon_{i_{n+1}}$. Furthermore, $\Lambda^{\mathrm{co}}_{\mathsf{S}_0} : \mathsf{S}_0 \to \mathbb{C}$ is defined as in (6.12), again using the identification between S_0 and $\mathbb{\Lambda}$. Explicitly, it is zero everywhere on S except in the top degree, where it takes the value

$$\text{(B.37)} \qquad \Lambda^{\mathrm{co}}_{\mathsf{S}_0}\big(\mathsf{x}_1^+\mathsf{x}_1^- \cdots \mathsf{x}_{\mathrm{N}}^+\mathsf{x}_{\mathrm{N}}^- \boldsymbol{e}_0\big) = \beta^{-2} \ .$$

As in the definition of A in (B.11) choose $\beta_1, \ldots, \beta_{\mathrm{N}}$ such that $\beta_i^4 = -1$ and $\beta = \beta_1 \cdots \beta_{\mathrm{N}}$. On each of the subalgebras $\mathsf{S}_0^{(k)}$ we can define the "$\mathrm{N} = 1$ version of ϕ" as follows:

$$\text{(B.38)} \qquad \phi^{(k)}(\boldsymbol{e}_0) = \beta_k^2 \, \mathsf{x}_k^+\mathsf{x}_k^- \boldsymbol{e}_0 \ , \quad \phi^{(k)}(\mathsf{x}_k^{\pm}\boldsymbol{e}_0) = \beta_k^2 \, \mathsf{x}_k^{\pm}\boldsymbol{e}_0 \ , \qquad \phi^{(k)}(\mathsf{x}_k^+\mathsf{x}_k^- \boldsymbol{e}_0) = -\beta_k^2 \, \boldsymbol{e}_0 \ .$$

Indeed, for $\mathrm{N} = 1$ we have $\phi = \phi^{(1)}$ (see also [**GR1**, Eqn. (33)]).

Lemma B.7. *We have, for* $v^{(k)} \in \mathsf{S}_0^{(k)}$,

$$\text{(B.39)} \qquad \phi(v^{(1)} \cdots v^{(N)}) \ = \ \phi^{(1)}(v^{(1)}) \cdots \phi^{(N)}(v^{(N)}) \ .$$

Proof. We start by describing ϕ in a different way: Let $T = \{1, \ldots, \mathrm{N}\} \times \{-1, 1\}$ be a totally ordered set whose ordering is given by, for $(i_1, \varepsilon_1), (i_2, \varepsilon_2) \in T$,

$$\text{(B.40)} \qquad (i_1, \varepsilon_1) < (i_2, \varepsilon_2) \iff i_1 < i_2 \ \vee \ (i_1 = i_2 \ \wedge \ \varepsilon_2 < \varepsilon_1) \ .$$

Let $K \subset T$ be a subset of the form $K = \{(j_1, \varepsilon_{j_1}), \dots, (j_k, \varepsilon_{j_k})\}$, where the indexing agrees with the ordering in the sense that $(j_a, \varepsilon_a) < (j_{a+1}, \varepsilon_{a+1})$. We introduce the elements

$$u_K := \mathsf{x}_{j_1}^{\varepsilon_{j_1}} \cdots \mathsf{x}_{j_k}^{\varepsilon_{j_k}} \boldsymbol{e}_0 \quad , \quad u_K^- := \mathsf{x}_{j_1}^{-\varepsilon_{j_1}} \cdots \mathsf{x}_{j_k}^{-\varepsilon_{j_k}} \boldsymbol{e}_0 \quad , \quad u_\emptyset := \boldsymbol{e}_0 \quad , \quad \beta_K = \beta_{j_1} \cdots \beta_{j_k} \ . \tag{B.41}$$

For two subsets $K \subset L \subset T$ we define the linear map

$$\phi^L(u_K) := \beta_L \, \mathrm{i}^{|L|+2|K|} \, u_{L \setminus K}^- \ . \tag{B.42}$$

For $n = 1, \dots, \mathrm{N}$ let $L_n = \{(n, +), (n, -)\}$. We claim that

$$\begin{aligned} &\text{(a)} \quad \phi^{L_n}(u_K) = \phi^{(n)}(u_K) \quad \text{for } K \subset L_n \ , \\ &\text{(b)} \quad \phi^{\cup_{n=1}^r L_n}(u_K) = \phi^{\cup_{n=1}^{r-1} L_n}(u_{K \setminus L_r}) \cdot \phi^{L_r}(u_{K \cap L_r}) \ , \\ &\text{(c)} \quad \phi^T = \phi \ , \end{aligned}$$

where for (b) we have $r = 1, \dots, \mathrm{N}$ and $K \subset \bigcup_{n=1}^r L_n$. Properties (a)–(c) together imply the statement of the lemma.

Properties (a) and (b) are immediate from the definition. It remains to show property (c). Abbreviate $k = |K|$. The cointegral $\Lambda_{\mathsf{S}_0}^{\mathrm{co}}$ in (B.36) is non-zero only in top-degree, so that for a given u_K, the only non-zero summand in (B.36) is that with $\{(i_1, \varepsilon_{i_1}), \dots, (i_m, \varepsilon_{i_m})\} = T \setminus K$. Thus $\phi(u_K) = \pm u_{L \setminus K}^-$. To determine the sign, we define

$$\begin{aligned} &\eta_1(K) = \mathrm{i}^{(2\mathrm{N}-k)(2\mathrm{N}-k+1)} = (-1)^{\mathrm{N}} \, \mathrm{i}^{(k-1)k} \ , \\ &\eta_2(K) = \prod_{(i,\varepsilon_i) \in T \setminus K} \varepsilon_i \quad , \quad \eta_3(K) = \Big(\prod_{(n,\varepsilon_n) \in K} -\varepsilon_n \Big) \cdot \Big(\prod_{m=1}^{k} (-1)^{2\mathrm{N}-k-m} \Big) \ . \end{aligned} \tag{B.43}$$

The factors in $\phi(u_K)$ from (B.36) correspond to η_1, η_2, η_3 as follows. We have $\beta^2 = \beta_T$ and the product $\beta_T^{-1} \cdot \eta_1$ corresponds to the factor $\mathrm{i}^{m(m+1)}$, for $m = |T \setminus K| = 2\mathrm{N} - k$, times the normalisation of $\Lambda_{\mathsf{S}_0}^{\mathrm{co}}$ from (6.12). The sign η_2 corresponds to $\varepsilon_{i_1} \cdots \varepsilon_{i_m}$ where the i_a run over values in the complement $T \setminus K$, and η_3 corresponds to the sign we get when bringing $u_{T \setminus K} \cdot u_K$ in $\Lambda_{\mathsf{S}_0}^{\mathrm{co}}(\cdots)$ into the form $\mathsf{x}_1^+ \mathsf{x}_1^- \cdots \mathsf{x}_{\mathrm{N}}^+ \mathsf{x}_{\mathrm{N}}^- \boldsymbol{e}_0$. Altogether, the coefficient in front of $u_{L \setminus K}^-$ is $\beta_T^{-1} \, \eta_1(K) \cdot \eta_2(K) \cdot \eta_3(K)$. It is not hard to verify that

$$\eta_2(K) \cdot \eta_3(K) = \Big(\prod_{(i,\varepsilon_i) \in T \setminus K} \varepsilon_i \Big) \Big(\prod_{(n,\varepsilon_n) \in K} -\varepsilon_n \Big) \, \mathrm{i}^{(k-1)k} = \mathrm{i}^{(k-1)k} \cdot (-1)^{\mathrm{N}+k} \ . \tag{B.44}$$

and hence $\eta_1(K) \cdot \eta_2(K) \cdot \eta_3(K) = (-1)^k$. Thus $\phi(u_K) = \beta_T^{-1} (-1)^k u_{T \setminus K}^- = \beta_T (-1)^{\mathrm{N}+k} u_{T \setminus K}^- = \phi^T(u_K)$, as claimed. □

Using the above lemma, one can also write the two sides of sector **111** in Table 1 as products over elements in $(\mathsf{S}^{(k)})^{\otimes 3}$ and conclude its validity from the $\mathrm{N} = 1$ case.

Combining the above result with Propositions B.3 and B.6 we obtain:

Proposition B.8. *The functor $\mathcal{D} \colon \mathcal{SF} \to \mathbf{Rep}\,\mathsf{S}$, together with the isomorphisms $\Delta_{U,V}$ and $\mathrm{id}_{\mathbb{C}^{1|0}} : \mathbf{1} \to \mathcal{D}(\mathbf{1})$, is $\mathbb{C}$-linear monoidal equivalence.*

B.5. Transporting the braiding

Recall the definition of the element $r \in \mathsf{S} \otimes \mathsf{S}$ in (B.9). We use r to define a family of natural isomorphisms

$$\psi_{M,N} : M \otimes N \to N \otimes M \qquad , \quad M, N \in \mathbf{Rep}\,\mathsf{S} \tag{B.45}$$

in $\mathbf{Rep}\,\mathsf{S}$ as a sum over sectors:[2]

$$\psi_{M,N} = \sum_{a,b\in\{0,1\}} \psi^{ab}_{M,N} \qquad , \quad \psi^{ab}_{M,N} = \tau^{\text{s.v.}}_{M,N} \circ \big(r^{ab} \cdot \boldsymbol{e}_a \otimes \boldsymbol{e}_b\big) \circ \big(\mathrm{id}_M \otimes \omega^a_N\big) \ , \tag{B.46}$$

where $\omega^0_N = \mathrm{id}_N$ and ω^1_N is the parity involution, and composition with $r^{ab} \cdot \boldsymbol{e}_a \otimes \boldsymbol{e}_b$ denotes the action of this element of $\mathsf{S} \otimes \mathsf{S}$ on $M \otimes N$ (with the corresponding parity signs resulting from braiding one copy of S past M).

We will show that ψ is the result of transporting the braiding from $\mathcal{SF}$ to $\mathbf{Rep}\,\mathsf{S}$ via the family of isomorphisms $\Delta_{U,V}$ introduced in Section B.3, that is, it is the unique natural family of isomorphisms that makes the diagram (7.25) commute:

$$\begin{tikzcd}
\mathcal{D}(U * V) \arrow[r, "\mathcal{D}(c_{U,V})"] \arrow[d, "\Delta_{U,V}"] & \mathcal{D}(V * U) \arrow[d, "\Delta_{V,U}"] \\
\mathcal{D}(U) \otimes \mathcal{D}(V) \arrow[r, "\psi_{\mathcal{D}(U),\mathcal{D}(V)}"] & \mathcal{D}(V) \otimes \mathcal{D}(U)
\end{tikzcd} \tag{B.47}$$

This then proves that ψ is a braiding on $\mathbf{Rep}\,\mathsf{S}$ and that $\mathcal{D}$ is a braided monoidal functor.

Lemma B.9. *For all $M, N \in \mathbf{Rep}\,\mathsf{S}$, $\psi_{M,N}$ is a morphism in $\mathbf{Rep}\,\mathsf{S}$. Furthermore it is invertible, natural in M, N and makes the diagram* (B.47) *commute.*

Proof. Given morphisms $f : M \to M'$ and $g : N \to N'$ in $\mathbf{Rep}\,\mathsf{S}$, it is immediate from the definition that $\psi_{M',N'} \circ (f \otimes g) = (g \otimes f) \circ \psi_{M,N}$. The lemma follows once we proved that (B.47) commutes. Indeed, since the top path in (B.47) is a morphism in $\mathbf{Rep}\,\mathsf{S}$, so is $\psi_{\mathcal{D}(U),\mathcal{D}(V)}$. And since the top path is invertible, so is ψ.

We now show that (B.47) commutes. In Table 2, the conditions on $\psi_{M,N}$ are given in each of the four sectors. These conditions use the inverse of γ,

$$\gamma^{-1} = \exp(-C) = \prod_{i=1}^{\mathrm{N}} (\mathbf{1} \otimes \mathbf{1} - \mathsf{x}^-_i \otimes \mathsf{x}^+_i + \mathsf{x}^+_i \otimes \mathsf{x}^-_i - \mathsf{x}^+_i \mathsf{x}^-_i \otimes \mathsf{x}^+_i \mathsf{x}^-_i) \cdot \boldsymbol{e}_0 \otimes \boldsymbol{e}_0 \ , \tag{B.48}$$

and the constant

$$\kappa := \exp\big(\tfrac{1}{2}\hat{C}\big) = \prod_{i=1}^{\mathrm{N}} (\mathbf{1} - \mathsf{x}^+_i \mathsf{x}^-_i) \boldsymbol{e}_0 \ . \tag{B.49}$$

---

[2]The appearance of the parity involution is the reason that r is not a universal R-matrix for S.

$$\mathbf{00}\ :\qquad \tau^{\text{s.v.}}_{U,V}\Big(\tau^{\text{s.v.}}_{S,S}(\delta_0)\,.\,\gamma^{-1}\,.\,(u\otimes v)\Big) = \psi^{00}_{U,V}\Big(\delta_0\,.\,(u\otimes v)\Big)$$

$$\begin{aligned}\mathbf{01}\ :\qquad &\tau^{\text{s.v.}}_{U,\mathsf{B}\otimes V}\Big(\big[\tau^{\text{s.v.}}_{\mathsf{B},U}\big(\Delta^{\mathsf{S}}(a)\,.\,(\mathsf{b}\otimes\kappa)\,.\,(\mathbf{1}\otimes u)\big)\big]\otimes v\Big)\\ &= \psi^{01}_{U,\mathsf{B}\otimes V}\Big(\big[\Delta^{\mathsf{S}}(a)\,.\,(\delta_1\otimes\mathsf{b})\,.\,(u\otimes\mathbf{1})\big]\otimes v\Big)\end{aligned}$$

$$\begin{aligned}\mathbf{10}\ :\qquad &\tau^{\text{s.v.}}_{\mathsf{B}\otimes U,V}\circ(\mathrm{id}_{\mathsf{B}}\otimes\mathrm{id}_U\otimes\rho^V)\\ &\qquad\circ(\mathrm{id}_{\mathsf{B}}\otimes\tau^{\text{s.v.}}_{S,U}\otimes\omega_V)\Big(\big[(\tau^{\text{s.v.}}_{S,S}\circ\Delta^{\mathsf{S}}(a))\cdot(\mathsf{b}\otimes(\delta_1\kappa))\big]\otimes u\otimes v\Big)\\ &= \psi^{10}_{\mathsf{B}\otimes U,V}\circ(\mathrm{id}_{\mathsf{B}}\otimes\mathrm{id}_U\otimes\rho^V)\circ(R_{\mathsf{b}}\otimes\tau^{\text{s.v.}}_{\mathsf{S}_0,U}\otimes\mathrm{id}_V)\Big(\Delta^{\mathsf{S}}(a)\otimes u\otimes v\Big)\end{aligned}$$

$$\begin{aligned}\mathbf{11}\ :\qquad &\beta\cdot\tau^{\text{s.v.}}_{\mathsf{B}\otimes U,\mathsf{B}\otimes V}\circ(\mathrm{id}_{\mathsf{B}}\otimes\tau^{\text{s.v.}}_{\mathsf{B},U}\otimes\omega_V)\Big(\tau^{\text{s.v.}}_{\mathsf{B},\mathsf{B}}\big[\Delta^{\mathsf{S}}(h\kappa^{-1})\ \cdot\ (\mathsf{b}\otimes\mathsf{b})\big]\otimes u\otimes v\Big)\\ &= \psi^{11}_{\mathsf{B}\otimes U,\mathsf{B}\otimes V}\circ(\mathrm{id}_{\mathsf{B}}\otimes\tau^{\text{s.v.}}_{\mathsf{B},U}\otimes\mathrm{id}_V)\Big(\big[\Delta^{\mathsf{S}}(h)\ \cdot\ (\mathsf{b}\otimes\mathsf{b})\big]\otimes u\otimes v\Big)\end{aligned}$$

TABLE 2. Conditions on $\psi_{U,V}$ such that (B.47) commutes in each of the four sectors. The conditions have to hold for all $h\in\mathsf{S}_0$, $a\in\mathsf{B}$, $u\in U$, $v\in V$ and all $U\in\mathcal{SF}_i$, $V\in\mathcal{SF}_j$. The element γ is defined as in Table 1 (and its inverse is given in (B.48)) and κ is defined in (B.49).

Substituting the definition of ψ in (B.46), the condition in sector **00** can be rewritten as

$$r^{00}\cdot\boldsymbol{e}_0\otimes\boldsymbol{e}_0\ =\ \tau^{\text{s.v.}}_{S,S}(\delta_0)\cdot\gamma^{-1}\cdot\delta_0^{-1}\ , \tag{B.50}$$

where δ_0 and its inverse are given in (B.22) and (B.23). Note that all ingredients in the above equality are written as products over elements in $\mathsf{S}^{(k)}$ for $k=1,\dots,\mathrm{N}$. This reduces the verification of (B.50) to the case $\mathrm{N}=1$, in which case it is a short calculation combining (B.22), (B.23), (B.48) and the definition of r in (B.9).

In the remaining three sectors the strategy is the same: we show that the required identity can be written as a product over elements in $\mathsf{S}^{(k)}$. This implies that if the equality holds for $\mathrm{N}=1$, it holds for all N. The case $\mathrm{N}=1$ can then be checked by hand or by computer algebra (which is what we did). Below we only explain the reduction to $\mathrm{N}=1$ and we omit the details of the verification for $\mathrm{N}=1$.

The condition on ψ in the **01**-sector can be expressed via r as the following equation in $\mathsf{S}_0\otimes\mathsf{S}_1$:

$$\tau^{\text{s.v.}}_{\mathsf{S}_0,\mathsf{S}_1}\big[r^{01}\cdot\Delta^{\mathsf{S}}(a)\cdot(\mathbf{1}\otimes\mathsf{b})\big] = \Delta^{\mathsf{S}}(a)\cdot(\mathsf{b}\otimes\kappa\delta_1^{-1})\quad ,\qquad a\in\mathsf{B}. \tag{B.51}$$

The elements b, δ_1 and κ are all given in product form, see (B.18), (B.22) and (B.49). This reduces the verification in sector **01** to the case $\mathrm{N}=1$.

We see from (B.46) and Table 2 that the situation in the **10**- and **11**-sectors is different because of the presence of the parity-involution ω. In sector **10** the condition on ψ is expressed in terms of r as the following equation in $\mathsf{S}_1 \otimes \mathsf{S}_0$:

$$\tau^{\text{s.v.}}_{\mathsf{S}_1,\mathsf{S}_0}\big[r^{10} \cdot (\mathrm{id} \otimes \omega)\big(\Delta^{\mathsf{S}}(a)\big) \cdot (\mathsf{b} \otimes \mathbf{1})\big] = \Delta^{\mathsf{S}}(a) \cdot (\delta_1 \kappa \otimes \mathsf{b}) \quad , \qquad a \in \mathsf{B} \ . \tag{B.52}$$

As in sector **01** all elements are given in factorised form, and the above equality thus follows from the fact that it holds for $\mathrm{N} = 1$.

To rewrite the condition on ψ in sector **11** in terms of r, we have to relate $\omega_{\mathsf{B}\otimes V}$, which appears in ψ on the RHS of the condition in sector **11**, recall (B.46), and ω_V, which appears on the LHS of the condition. This can be done as follows. Note that

$$\omega_{\mathsf{B}} \otimes \omega_V = \omega_{\mathsf{B}\otimes V} \ . \tag{B.53}$$

We recall then the basis

$$\mathsf{B} = \mathrm{span}\Big\{ \mathsf{b}_{I=(i_1,\dots,i_{\mathrm{N}})} = \prod_{k=1}^{\mathrm{N}} (\mathsf{x}^-_k)^{i_k} \mathsf{x}^+_k \boldsymbol{e}_1 \mid i_k \in \{0,1\} \Big\} \ , \tag{B.54}$$

with $\omega(\mathsf{b}_I) = (-1)^{\mathrm{N}+\sum_k i_k} \mathsf{b}_I$ and so $\omega_{\mathsf{B}} = \Omega.(-)$ with

$$\Omega = \prod_{i=1}^{\mathrm{N}} (\mathsf{x}^-_i \mathsf{x}^+_i - \mathsf{x}^+_i \mathsf{x}^-_i) \boldsymbol{e}_1 \ . \tag{B.55}$$

The braiding ψ^{11} can be then expressed as

$$\begin{aligned} \psi^{11}_{\mathsf{B}\otimes U,\mathsf{B}\otimes V} &= \tau^{\text{s.v.}}_{\mathsf{B}\otimes U,\mathsf{B}\otimes V} \circ r^{11} \circ \big(\mathrm{id}_{\mathsf{B}\otimes U} \otimes \omega_{\mathsf{B}\otimes V}\big) \\ &= \tau^{\text{s.v.}}_{\mathsf{B}\otimes U,\mathsf{B}\otimes V} \circ (r^{11} \cdot \mathbf{1} \otimes \Omega) \circ (\mathrm{id}_{\mathsf{B}\otimes U} \otimes \mathrm{id}_{\mathsf{B}} \otimes \omega_V) \ . \end{aligned} \tag{B.56}$$

In the last line, the element $r^{11} \cdot \mathbf{1} \otimes \Omega \in \mathsf{S} \otimes \mathsf{S}$ acts on the two tensor factors B in $\mathsf{B} \otimes U \otimes \mathsf{B} \otimes V$. Substituting this into the condition in Table 2 gives the following condition on r^{11}:

$$\tau^{\text{s.v.}}_{\mathsf{S}_1,\mathsf{S}_1}\big[(r^{11} \cdot \mathbf{1} \otimes \Omega) \cdot \Delta^{\mathsf{S}}(h) \cdot (\mathsf{b} \otimes \mathsf{b})\big] = \beta \cdot \Delta^{\mathsf{S}}(h\kappa^{-1}) \cdot (\mathsf{b} \otimes \mathsf{b}) \quad , \qquad h \in \mathsf{S}_0 \ . \tag{B.57}$$

Again one can write this equality as a product over elements in $\mathsf{S}^{(k)}$, $k = 1, \dots, \mathrm{N}$, reducing the verification to $\mathrm{N} = 1$. □

Since $\mathcal{SF}$ is a braided monoidal category, the above lemma shows that ψ defines a braiding on $\mathbf{Rep}\,\mathsf{S}$. Altogether we have shown:

Proposition B.10. *The functor $\mathcal{D}$ in Proposition B.8 is braided monoidal.*

C. Equivalence between $\mathcal{SF}$ and $\mathbf{Rep}\,\mathsf{Q}$

Here, we present the second part of the proof of Lemma 7.5. We begin with introducing a quasi-bialgebra $\hat{\mathsf{Q}}$ in **vect** and show a braided monoidal equivalence between $\mathbf{Rep}\,\mathsf{S}$ and $\mathbf{Rep}\,\hat{\mathsf{Q}}$. Then we present a twisting of $\hat{\mathsf{Q}}$ into Q, and therefore $\mathbf{Rep}\,\mathsf{Q}$ is braided monoidally equivalent to $\mathbf{Rep}\,\mathsf{S}$ and thus to $\mathcal{SF}$. Finally, we use this equivalence to transport the ribbon twist from $\mathcal{SF}$ to $\mathbf{Rep}\,\mathsf{Q}$.

C.1. The quasi-bialgebra $\hat{\mathsf{Q}}$

In this section we introduce the quasi-bialgebra $\hat{\mathsf{Q}} = \hat{\mathsf{Q}}(\mathrm{N}, \beta)$ which is equal to Q as an algebra. It has a different quasi-bialgebra structure, namely

(C.1)
$$\hat{\Delta}(\mathsf{K}) = \mathsf{K} \otimes \mathsf{K} - (1 + (-1)^{\mathrm{N}})\boldsymbol{e}_1 \otimes \boldsymbol{e}_1 \cdot \mathsf{K} \otimes \mathsf{K} \ , \qquad \varepsilon(\mathsf{K}) = 1 \ ,$$
$$\hat{\Delta}(\mathsf{f}_i^{\pm}) = \mathsf{f}_i^{\pm} \otimes \mathbf{1} + \mathsf{K}^{-1} \otimes \mathsf{f}_i^{\pm} - (1 + (-1)^{\mathrm{N}})\boldsymbol{e}_1 \otimes \boldsymbol{e}_1 \cdot \mathsf{K}^{-1} \otimes \mathsf{f}_i^{\pm} \ , \qquad \varepsilon(\mathsf{f}_i^{\pm}) = 0 \ ,$$

where we use the central idempotents $\boldsymbol{e}_i \in \hat{\mathsf{Q}}$ $(i = 0, 1)$ defined as in (7.2). The coassociator for this coproduct is (we will check the axioms below)

(C.2)
$$\hat{\Phi} = \boldsymbol{e}_0 \otimes \boldsymbol{e}_0 \otimes \boldsymbol{e}_0 + \boldsymbol{e}_0 \otimes \boldsymbol{e}_0 \otimes \boldsymbol{e}_1 + \boldsymbol{e}_1 \otimes \boldsymbol{e}_0 \otimes \boldsymbol{e}_0 + \boldsymbol{e}_0 \otimes \boldsymbol{e}_1 \otimes \boldsymbol{e}_1$$
$$+ \boldsymbol{e}_1 \otimes \boldsymbol{e}_1 \otimes \boldsymbol{e}_0 + \hat{\Phi}^{010}\boldsymbol{e}_0 \otimes \boldsymbol{e}_1 \otimes \boldsymbol{e}_0 + \hat{\Phi}^{101}\boldsymbol{e}_1 \otimes \boldsymbol{e}_0 \otimes \boldsymbol{e}_1 + \hat{\Phi}^{111}\boldsymbol{e}_1 \otimes \boldsymbol{e}_1 \otimes \boldsymbol{e}_1 \ ,$$

where the non-trivial components are given by

(C.3)
$$\hat{\Phi}^{010} = \prod_{k=1}^{\mathrm{N}} \hat{\Phi}^{010}_{(k)} \ ,$$
$$\hat{\Phi}^{101} = (-\mathsf{K})^{\mathrm{N}-1} \otimes \mathsf{K}^{\mathrm{N}-1} \otimes \mathbf{1} \cdot \Big(\prod_{k=1}^{\mathrm{N}} \hat{\Phi}^{101}_{(k)} \Big) \cdot \mathsf{K}^{\mathrm{N}-1} \otimes \mathsf{K} \otimes \mathbf{1} \ ,$$
$$\hat{\Phi}^{111} = -\,\mathrm{i}^{\mathrm{N}}\beta^2\, \mathsf{K}^{\mathrm{N}-1} \otimes \mathbf{1} \otimes \mathbf{1} \cdot \Big(\prod_{k=1}^{\mathrm{N}} \hat{\Phi}^{111}_{(k)} \Big) \cdot \mathsf{K}^{\mathrm{N}-1} \otimes \mathsf{K}^{\mathrm{N}} \otimes \mathbf{1} \ ,$$

with

(C.4)
$$\hat{\Phi}^{010}_{(k)} = \Big(\mathbf{1} \otimes \mathbf{1} \otimes \mathbf{1} + (1 + \mathrm{i})\mathsf{f}_k^{+}\mathsf{K} \otimes \mathsf{K} \otimes \mathsf{f}_k^{-} \Big)\Big(\mathbf{1} \otimes \mathbf{1} \otimes \mathbf{1} + (1 - \mathrm{i})\mathsf{f}_k^{-}\mathsf{K} \otimes \mathsf{K} \otimes \mathsf{f}_k^{+} \Big) \ ,$$
$$\hat{\Phi}^{101}_{(k)} = \Big(\mathbf{1} \otimes \mathbf{1} \otimes \mathbf{1} + (1 + \mathrm{i})\mathbf{1} \otimes \mathsf{f}_k^{+}\mathsf{K} \otimes \mathsf{f}_k^{-} + (1 - \mathrm{i})\mathsf{f}_k^{-}\mathsf{K} \otimes \mathsf{f}_k^{+} \otimes \mathbf{1} \Big)$$

$$\times\Big(\mathbf{1}\otimes\mathbf{1}\otimes\mathbf{1}+(1+\mathrm{i})\mathsf{f}_k^+\mathsf{K}\otimes\mathsf{f}_k^-\otimes\mathbf{1}+(1-\mathrm{i})\mathbf{1}\otimes\mathsf{f}_k^-\mathsf{K}\otimes\mathsf{f}_k^+\Big)\ ,$$

$$\hat{\Phi}^{111}_{(k)}=\Big(\mathbf{1}\otimes\mathbf{1}\otimes\mathbf{1}+(\mathrm{i}-1)\big(\mathbf{1}\otimes\mathsf{f}_k^+\mathsf{K}\otimes\mathsf{f}_k^-+\mathsf{f}_k^+\mathsf{K}\otimes\mathsf{K}\otimes\mathsf{f}_k^--\mathsf{f}_k^+\mathsf{K}\otimes\mathsf{f}_k^-\otimes\mathbf{1}+\mathbf{1}\otimes\mathsf{f}_k^-\mathsf{f}_k^+\otimes\mathbf{1}\big)\Big)$$
$$\times\Big(\mathbf{1}\otimes\mathbf{1}\otimes\mathbf{1}-(\mathrm{i}-1)\big(\mathbf{1}\otimes\mathsf{f}_k^-\mathsf{K}\otimes\mathsf{f}_k^++\mathsf{f}_k^-\mathsf{K}\otimes\mathsf{K}\otimes\mathsf{f}_k^+-\mathsf{f}_k^-\mathsf{K}\otimes\mathsf{f}_k^+\otimes\mathbf{1}-\mathbf{1}\otimes\mathsf{f}_k^-\mathsf{f}_k^+\otimes\mathbf{1}\big)\Big)$$
$$\times\Big(\mathbf{1}\otimes\mathbf{1}\otimes\mathbf{1}-2\ \mathbf{1}\otimes\mathsf{f}_k^-\mathsf{f}_k^+\otimes\mathbf{1}\Big)\ .$$

The quasi-bialgebra $\hat{\mathsf{Q}}$ can also be equipped with an R-matrix which is defined as

$$\hat{R}=\hat{R}^{00}\boldsymbol{e}_0\otimes\boldsymbol{e}_0+\hat{R}^{01}\boldsymbol{e}_0\otimes\boldsymbol{e}_1+\hat{R}^{10}\boldsymbol{e}_1\otimes\boldsymbol{e}_0+\hat{R}^{11}\boldsymbol{e}_1\otimes\boldsymbol{e}_1\ , \tag{C.5}$$

where

$$\hat{R}^{0i}=\rho(\mathsf{K})\cdot\prod_{k=1}^{\mathrm{N}}\hat{R}^{0i}_{(k)}\ ,\qquad i=0,1\quad , \tag{C.6}$$

$$\hat{R}^{10}=\rho(\mathsf{K})\cdot\prod_{k=1}^{\mathrm{N}}\hat{R}^{10}_{(k)}\cdot\mathbf{1}\otimes\mathsf{K}\ , \tag{C.7}$$

$$\hat{R}^{11}=(-1)^{\mathrm{N}}\,\mathrm{i}\beta\,\rho(\mathsf{K})\cdot\mathbf{1}\otimes\mathsf{K}^{\mathrm{N}-1}\cdot\Big(\prod_{k=1}^{\mathrm{N}}\hat{R}^{11}_{(k)}\Big)\cdot\mathsf{K}^{\mathrm{N}}\otimes\mathbf{1} \tag{C.8}$$

with the Cartan part

$$\rho(\mathsf{K})=\frac{1}{2}(\mathbf{1}\otimes\mathbf{1}+\omega\otimes\mathbf{1}+\mathbf{1}\otimes\omega-\omega\otimes\omega)\quad ,\quad \omega=(\boldsymbol{e}_0-\mathrm{i}\boldsymbol{e}_1)\mathsf{K}\ , \tag{C.9}$$

and

(C.10)

$$\hat{R}^{00}_{(k)}=\mathbf{1}\otimes\mathbf{1}-2\mathsf{f}_k^-\mathsf{K}\otimes\mathsf{f}_k^+\ ,$$
$$\hat{R}^{01}_{(k)}=\mathbf{1}\otimes\mathbf{1}-(1+\mathrm{i})\mathsf{f}_k^-\mathsf{K}\otimes\mathsf{f}_k^+-(1+\mathrm{i})\mathsf{f}_k^+\mathsf{K}\otimes\mathsf{f}_k^-+(1-\mathrm{i})\mathsf{f}_k^-\mathsf{f}_k^+\otimes\mathbf{1}+2\mathrm{i}\mathsf{f}_k^-\mathsf{f}_k^+\otimes\mathsf{f}_k^-\mathsf{f}_k^+\ ,$$
$$\hat{R}^{10}_{(k)}=\mathbf{1}\otimes\mathbf{1}+(1+\mathrm{i})\mathsf{f}_k^-\mathsf{K}\otimes\mathsf{f}_k^++(1+\mathrm{i})\mathsf{f}_k^+\mathsf{K}\otimes\mathsf{f}_k^-+(1+\mathrm{i})\mathbf{1}\otimes\mathsf{f}_k^-\mathsf{f}_k^+-2\mathrm{i}\mathsf{f}_k^-\mathsf{f}_k^+\otimes\mathsf{f}_k^-\mathsf{f}_k^+\ ,$$
$$\hat{R}^{11}_{(k)}=\mathbf{1}\otimes\mathbf{1}-2\mathrm{i}\mathsf{f}_k^-\mathsf{K}\otimes\mathsf{f}_k^++(\mathrm{i}-1)\mathbf{1}\otimes\mathsf{f}_k^-\mathsf{f}_k^+-(1+\mathrm{i})\mathsf{f}_k^-\mathsf{f}_k^+\otimes\mathbf{1}+2\mathsf{f}_k^-\mathsf{f}_k^+\otimes\mathsf{f}_k^-\mathsf{f}_k^+\ .$$

Remark C.1. For $\mathrm{N}=1$ we have

$$\hat{\Phi}^{010}=\hat{\Phi}^{010}_{(k=1)}\quad ,\quad \hat{\Phi}^{101}=\hat{\Phi}^{101}_{(k=1)}\cdot\mathbf{1}\otimes\mathsf{K}\otimes\mathbf{1}\quad ,\quad \hat{\Phi}^{111}=\frac{\beta^2}{\mathrm{i}}\hat{\Phi}^{111}_{(k=1)}\cdot\mathbf{1}\otimes\mathsf{K}\otimes\mathbf{1} \tag{C.11}$$

and they coincide with the components of the associator in [**GR1**, Sec. 7.4]. So, we have the simple factorisation (C.3) of the nilpotent (off-diagonal) part of the associator into the product of $\mathrm{N}=1$ components. The Cartan part depends only on the parity of N. We also note that the components $\hat{\Phi}^{abc}_{(i)}$ commute with each other: $\hat{\Phi}^{abc}_{(i)}\cdot\hat{\Phi}^{abc}_{(j)}=\hat{\Phi}^{abc}_{(j)}\cdot\hat{\Phi}^{abc}_{(i)}$.
For the $\hat{R}$ element, we note that (C.6) for $\mathrm{N}=1$ is $\hat{R}^{0i}=\rho(\mathsf{K})\hat{R}^{0i}_{(k=1)}$, and it is the **00** and **01** components of the universal R-matrix in [**GR1**, Sec. 7.7]. The Cartan part thus does not change with N while the "off-diagonal" part is just the product of the $\mathrm{N}=1$ components. Also, $\hat{R}^{1i}_{(k=1)}$, for $i\in\{0,1\}$, corresponds to $\mathrm{N}=1$ case: the **10** component of the universal

R-matrix in [**GR1**, Sec. 7.7] is expressed as $\hat{R}^{10} = \rho(\mathsf{K})\cdot \hat{R}^{10}_{(k=1)}\cdot \mathbf{1}\otimes\mathsf{K}$ while the $\mathbf{11}$ component is $\hat{R}^{11} = \frac{\beta}{\mathrm{i}}\rho(\mathsf{K})\cdot \hat{R}^{11}_{(k=1)}\cdot \mathsf{K}\otimes\mathbf{1}$ that agrees with (C.7)-(C.8). Hence, we have again the simple factorisation (C.7)-(C.8) of the nilpotent (off-diagonal) part of $\hat{R}$ into the product of $\mathrm{N}=1$ components.

The following lemma can be easily checked by a direct calculation.

Lemma C.2. *The map* $\hat{\Delta}\colon \hat{\mathsf{Q}} \to \hat{\mathsf{Q}}\otimes\hat{\mathsf{Q}}$ *defined in* (C.1) *is an algebra homomorphism.*

In order to show that $(\hat{\mathsf{Q}}, \cdot, \mathbf{1}, \hat{\Delta}, \varepsilon, \hat{\Phi}, \hat{R})$ is a quasi-triangular quasi-bialgebra we start with the braided monoidal category $\mathbf{Rep}\,\mathsf{S}$ described in Appendix B and verify that $\hat{\Phi}$ and $\hat{R}$ can be obtained via transport along a certain multiplicative functor from $\mathbf{Rep}\,\mathsf{S}$ to $\mathbf{Rep}\,\hat{\mathsf{Q}}$, see Sections C.2–C.5. From this it follows that $\hat{\mathsf{Q}}$ is indeed a quasi-triangular quasi-bialgebra and that $\mathbf{Rep}\,\hat{\mathsf{Q}}$ is braided equivalent to $\mathbf{Rep}\,\mathsf{S}$.

C.2. A $\mathbb{C}$-linear equivalence from $\mathbf{Rep}\,\mathsf{S}$ to $\mathbf{Rep}\,\hat{\mathsf{Q}}$

In this section we present a $\mathbb{C}$-linear functor $\mathcal{G} : \mathbf{Rep}\,\mathsf{S} \to \mathbf{Rep}\,\hat{\mathsf{Q}}$. Recall that ω_U denotes the parity involution on the super-vector space U. For a given $U \in \mathbf{Rep}\,\mathsf{S}$, $\mathcal{G}(U)$ is the underlying vector space U with $\hat{\mathsf{Q}}$-action given by, for $u \in U$,

$$\begin{aligned} \mathsf{K}.u \;&:=\; \mathsf{z}.\omega_U(u) = \omega_U(\mathsf{z}.u)\ , \quad \text{where} \quad \mathsf{z} = \boldsymbol{e}_0 + \mathrm{i}\boldsymbol{e}_1\ , \\ \mathsf{f}^{\pm}_i.u \;&:=\; \mathsf{x}^{\pm}_i.u\ . \end{aligned} \tag{C.12}$$

Note that K^2 acts as L, that is, $\boldsymbol{e}_i \in \hat{\mathsf{Q}}$ $(i=0,1)$ acts as $\boldsymbol{e}_i \in \mathsf{S}$. For a morphism $f : U \to V$ in $\mathbf{Rep}\,\mathsf{S}$ we set $\mathcal{G}(f) = f$.

Proposition C.3. *The functor* $\mathcal{G}$ *is an equivalence of* $\mathbb{C}$*-linear categories.*

Proof. Since the proof is very similar to the proof of [**GR1**, Prop. 5.2] we omit the details here. However, for later reference we introduce a functor $\mathcal{H} : \mathbf{Rep}\,\hat{\mathsf{Q}} \to \mathbf{Rep}\,\mathsf{S}$ which is inverse to $\mathcal{G}$. By inverting the relation in (C.12) we define an involution map on an object $V \in \mathbf{Rep}\,\hat{\mathsf{Q}}$ as

$$\omega_V(v) := (\boldsymbol{e}_0 - \mathrm{i}\boldsymbol{e}_1)\mathsf{K}.v \quad , \quad v \in V\ . \tag{C.13}$$

Then the S-module $\mathcal{H}(V)$ has V as the underlying super-vector space with the $\mathbb{Z}_2$-grading defined by the eigenvalues of ω_V as $\omega_V(v) = (-1)^{|v|}v$, for an eigenvector v of ω_V. Moreover, L acts on $\mathcal{H}(V)$ by $\boldsymbol{e}_0 - \boldsymbol{e}_1 \in \hat{\mathsf{Q}}$ and $\mathsf{x}^{\pm}_i$ acts on $\mathcal{H}(V)$ by $\mathsf{f}^{\pm}_i$. □

Since Q and $\hat{\mathsf{Q}}$ have the same algebra structure we in fact have shown a $\mathbb{C}$-linear equivalence of $\mathbf{Rep}\,\mathsf{S}$ and $\mathbf{Rep}\,\mathsf{Q}$.

C.3. $\mathcal{G}$ as multiplicative functor

In order to show that $\mathcal{G}$ is multiplicative, we define the family of isomorphisms

$$\begin{aligned}\Gamma_{U,V}: \ \mathcal{G}(U \otimes_{\mathbf{Rep}\,\mathsf{S}} V) &\to \mathcal{G}(U) \otimes_{\mathbf{Rep}\,\hat{\mathsf{Q}}} \mathcal{G}(V) \ , \\ u \otimes v &\mapsto u \otimes v + \boldsymbol{e}_1.u \otimes (\xi - 1)\boldsymbol{e}_1.v \ ,\end{aligned} \tag{C.14}$$

where ξ is defined as

$$\xi = \mathrm{i}^{\mathrm{N(N-1)}/2} \prod_{k=1}^{\mathrm{N}} \xi_k \quad \text{with} \quad \xi_k = \mathsf{x}_k^+ + \mathsf{x}_k^- \ . \tag{C.15}$$

Invertibility is easy to see since $(\Gamma_{U,V})^2 = \mathrm{id}_{U \otimes V}$, which follows from $\xi^2 = \boldsymbol{e}_1$. Naturality is also clear. It remains to prove the following lemma.

Lemma C.4. *$\Gamma_{U,V}$ is an intertwiner of $\hat{\mathsf{Q}}$-modules.*

Proof. We need to show that for all $a \in \hat{\mathsf{Q}}$, $u \in U$, $v \in V$ we have

$$\Gamma_{U,V}(a \,\hat{.}\, (u \otimes v)) = a.\Gamma_{U,V}(u \otimes v) \ , \tag{C.16}$$

where the notation $\hat{.}$ emphasises that the action of $\Delta(a) \in \mathsf{S} \otimes \mathsf{S}$ on $U \otimes V$ is in **Svect** and involves parity signs.

Since Δ^{S} and $\hat{\Delta}$ are algebra maps, it is enough to verify this on the generators K, $\mathsf{f}_k^{\pm}$. If $U \notin \mathbf{Rep}\,\mathsf{S}_1$ or $V \notin \mathbf{Rep}\,\mathsf{S}_1$, $\Gamma_{U,V}$ is just the identity, and the verification is straightforward. As an example for the sector **11** case let $a = \mathsf{f}_k^+$ and assume N to be even. Then the two sides of the above identity are

$$\begin{aligned}\mathsf{f}_k^+.\Gamma_{U,V}(u \otimes v) &= (\mathsf{f}_k^+ \otimes \mathbf{1} + \mathsf{K}^{-1} \otimes \mathsf{f}_k^+ - 2\mathsf{K}^{-1}\boldsymbol{e}_1 \otimes \mathsf{f}_k^+\boldsymbol{e}_1).(\boldsymbol{e}_1.u \otimes \xi\boldsymbol{e}_1.v) \\ &= \mathsf{x}_k^+\boldsymbol{e}_1.u \otimes \xi\boldsymbol{e}_1.v + \mathrm{i}(-1)^{|u|}\boldsymbol{e}_1.u \otimes \mathsf{x}_k^+\xi\boldsymbol{e}_1.v \ , \\ \Gamma_{U,V}\big(\mathsf{f}_k^+ \,\hat{.}\, (u \otimes v)\big) &= \Gamma_{U,V}\big(\mathsf{x}_k^+ \,\hat{.}\, (u \otimes v)\big) = \Gamma_{U,V}\big((\mathsf{x}_k^+ \otimes \mathbf{1} - \mathrm{i}\mathbf{1} \otimes \mathsf{x}_k^-) \,\hat{.}\, (u \otimes v)\big) \\ &= \mathsf{x}_k^+\boldsymbol{e}_1.u \otimes \xi\boldsymbol{e}_1.v - \mathrm{i}(-1)^{|u|}\boldsymbol{e}_1.u \otimes \xi\mathsf{x}_k^-\boldsymbol{e}_1.v \ .\end{aligned} \tag{C.17}$$

Since $\mathsf{x}_k^{\pm}\xi\boldsymbol{e}_1 = (-1)^{\mathrm{N}-1}\xi\mathsf{x}_k^{\mp}\boldsymbol{e}_1$ both sides are equal. The calculations for the other generators and for odd N are equally straightforward. $\square$

Altogether, we have shown:

Proposition C.5. *With the isomorphisms $\Gamma_{U,V}$ as in* (C.14)*, the functor $\mathcal{G} : \mathbf{Rep}\,\mathsf{S} \to \mathbf{Rep}\,\hat{\mathsf{Q}}$ is multiplicative.*

C.4. Transporting the associator

In this section we transport the associator from $\mathbf{Rep}\,\mathsf{S}$ to $\mathbf{Rep}\,\hat{\mathsf{Q}}$ along the lines explained around the diagram (7.24). As $\mathbf{Rep}\,\hat{\mathsf{Q}}$ is the category of (finite-dimensional) $\hat{\mathsf{Q}}$-modules in vector spaces, the associator on $\mathbf{Rep}\,\hat{\mathsf{Q}}$ takes the form

$$\alpha^{\mathbf{Rep}\,\hat{\mathsf{Q}}}_{U,V,W}(u\otimes v\otimes w) = \hat{\Phi}.(u\otimes v\otimes w)\ , \tag{C.18}$$

where u, v, w are elements of $U, V, W \in \mathbf{Rep}\,\hat{\mathsf{Q}}$ and for some $\hat{\Phi} \in \hat{\mathsf{Q}}\otimes\hat{\mathsf{Q}}\otimes\hat{\mathsf{Q}}$. In order to compute $\hat{\Phi}$ and to see that it agrees with (C.2)-(C.4), we choose $U = V = W = \hat{\mathsf{Q}}$ and solve the diagram (7.24) with $\mathcal{F}$ replaced by $\mathcal{G}$.

Recall the functor $\mathcal{H}$ inverse to $\mathcal{G}$ from the proof of Proposition C.3. Let us abbreviate $\hat{\mathsf{Q}}_{\mathcal{H}} := \mathcal{H}(\hat{\mathsf{Q}})$. The S-module structure on $\hat{\mathsf{Q}}_{\mathcal{H}}$ is as explained in the proof of Proposition C.3. Commutativity of the diagram (7.24) applied for $\mathcal{F} = \mathcal{G}$ and $\Theta = \Gamma$ then reads for $q \in \hat{\mathsf{Q}}_{\mathcal{H}}^{\otimes 3}$:

$$\big((\Gamma_{\hat{\mathsf{Q}}_{\mathcal{H}},\hat{\mathsf{Q}}_{\mathcal{H}}}\otimes\mathrm{id})\circ\Gamma_{\hat{\mathsf{Q}}_{\mathcal{H}}\otimes\hat{\mathsf{Q}}_{\mathcal{H}},\hat{\mathsf{Q}}_{\mathcal{H}}}\big)\big(\Lambda\,\hat{.}\,q\big) = \hat{\Phi}\cdot\big[(\mathrm{id}\otimes\Gamma_{\hat{\mathsf{Q}}_{\mathcal{H}},\hat{\mathsf{Q}}_{\mathcal{H}}})\circ\Gamma_{\hat{\mathsf{Q}}_{\mathcal{H}},\hat{\mathsf{Q}}_{\mathcal{H}}\otimes\hat{\mathsf{Q}}_{\mathcal{H}}}(q)\big]\ , \tag{C.19}$$

where $\Gamma_{U,V}$ is defined in (C.14) and the notation $\hat{.}$ emphasises that the action of Λ on $\hat{\mathsf{Q}}_{\mathcal{H}}\otimes\hat{\mathsf{Q}}_{\mathcal{H}}\otimes\hat{\mathsf{Q}}_{\mathcal{H}}$ is as in **Svect**, i.e. involves parity signs. During the calculation it is important to be careful with the parity signs. For example, $\Lambda\,\hat{.}\,(\mathbf{1}\otimes\mathbf{1}\otimes\mathbf{1})$ can not be simplified to Λ since $\mathbf{1}\in\hat{\mathsf{Q}}_{\mathcal{H}}$ is not of definite parity. Other examples are the actions of $\Gamma_{\hat{\mathsf{Q}}_{\mathcal{H}},\hat{\mathsf{Q}}_{\mathcal{H}}\otimes\hat{\mathsf{Q}}_{\mathcal{H}}}$ and $\Gamma_{\hat{\mathsf{Q}}_{\mathcal{H}}\otimes\hat{\mathsf{Q}}_{\mathcal{H}},\hat{\mathsf{Q}}_{\mathcal{H}}}$, which are, for $a, b, c \in \hat{\mathsf{Q}}_{\mathcal{H}}$,

$$\begin{aligned}\Gamma_{\hat{\mathsf{Q}}_{\mathcal{H}},\hat{\mathsf{Q}}_{\mathcal{H}}\otimes\hat{\mathsf{Q}}_{\mathcal{H}}}(a\otimes b\otimes c) &= a\otimes b\otimes c + \boldsymbol{e}_1.a\otimes\big[\Delta^{\mathsf{S}}((\xi-1)\boldsymbol{e}_1)\,\hat{.}\,(b\otimes c)\big]\ ,\\ \Gamma_{\hat{\mathsf{Q}}_{\mathcal{H}}\otimes\hat{\mathsf{Q}}_{\mathcal{H}},\hat{\mathsf{Q}}_{\mathcal{H}}}(a\otimes b\otimes c) &= a\otimes b\otimes c + \Delta^{\mathsf{S}}(\boldsymbol{e}_1).(a\otimes b)\otimes\big[(\xi-1)\boldsymbol{e}_1\,.\,c\big]\ .\end{aligned} \tag{C.20}$$

We use below the notations (in the spirit of (B.33))

$$\underline{\hat{\Phi}}^{abc} = \hat{\Phi}^{abc}\cdot(\boldsymbol{e}_a\otimes\boldsymbol{e}_b\otimes\boldsymbol{e}_c) \tag{C.21}$$

with $\hat{\Phi}^{abc}$ as in (C.2) (it is understood that the $\hat{\Phi}^{abc}$ not spelled out explicitly in (C.2) are set to $\mathbf{1}^{\otimes 3}$). We will also use the similar convention for $\underline{\hat{\Phi}}^{abc}_{(k)}$ and $\underline{\Lambda}^{abc}_{(k)}$.

In the following we will present the calculation of $\hat{\Phi}$ in sector **101**. The other cases are similar. For brevity, we write ω instead of ω_{S} for the parity involution in S. Using (C.20) and (C.21), the equality in (C.19) then reduces to

$$\begin{aligned}\underline{\hat{\Phi}}^{101}\,.\,\big(\mathbf{1}_{\mathsf{S}}\otimes\Delta^{\mathsf{S}}(\xi)\,\hat{.}\,(\omega^{\mathrm{N}}\otimes\mathrm{id}\otimes\mathrm{id})(q)\big) &= \mathbf{1}_{\mathsf{S}}\otimes\mathbf{1}_{\mathsf{S}}\otimes\xi\,.\,(\underline{\Lambda}^{101}\,\hat{.}\,q)\\ &= (\omega^{\mathrm{N}}\otimes\omega^{\mathrm{N}}\otimes\mathrm{id})\big(\mathbf{1}_{\mathsf{S}}\otimes\mathbf{1}_{\mathsf{S}}\otimes\xi\,\hat{.}\,(\underline{\Lambda}^{101}\,\hat{.}\,q)\big)\ ,\end{aligned} \tag{C.22}$$

where the parity involutions appear upon converting the action "$\,.\,$" in **vect** to the action "$\hat{.}$" in **Svect**, using that ξ is odd if and only if N is odd. By setting

$$q = \mathbf{1}_{\mathsf{S}}\otimes\Delta^{\mathsf{S}}(\xi)\,\hat{.}\,(\omega^{\mathrm{N}}\otimes\mathrm{id}\otimes\mathrm{id})(p) \tag{C.23}$$

for $p\in\hat{\mathsf{Q}}_{\mathcal{H}}^{\otimes 3}$ (note that since $\xi^2 = \boldsymbol{e}_1$ the above map is a bijection between p's and q's), we get

(C.24)

$$\underline{\hat{\Phi}}^{101}\,.\,p = (\omega^{\mathrm{N}}\otimes\omega^{\mathrm{N}}\otimes\mathrm{id})\Big(\mathbf{1}_{\mathsf{S}}\otimes\mathbf{1}_{\mathsf{S}}\otimes\xi\,\hat{.}\,\underline{\Lambda}^{101}\,\hat{.}\,\mathbf{1}_{\mathsf{S}}\otimes\Delta^{\mathsf{S}}(\xi)\,\hat{.}\,(\omega^{\mathrm{N}}\otimes\mathrm{id}\otimes\mathrm{id})(p)\Big)$$

$$= (\omega^{\mathrm{N}} \otimes \omega^{\mathrm{N}} \otimes \mathrm{id})\Big(\prod_{k=1}^{\mathrm{N}} \big(\mathbf{1}_{\mathsf{S}} \otimes \mathbf{1}_{\mathsf{S}} \otimes \xi_k \,\hat{\cdot}\, \underline{\Lambda}^{101}_{(k)} \,\hat{\cdot}\, \mathbf{1}_{\mathsf{S}} \otimes \Delta^{\mathsf{S}}(\xi_k)\big) \,\hat{\cdot}\, (\omega^{\mathrm{N}} \otimes \mathrm{id} \otimes \mathrm{id})(p)\Big) ,$$

where for the second equality we used the factorisation in (B.7) and (C.15) and the fact that $\underline{\Lambda}^{101}_{(k)}$ are even elements, as well as the equality

$$(\mathbf{1}_{\mathsf{S}} \otimes \mathbf{1}_{\mathsf{S}} \otimes \xi) \,\hat{\cdot}\, (\mathbf{1}_{\mathsf{S}} \otimes \Delta^{\mathsf{S}}(\xi)) = \prod_{k=1}^{\mathrm{N}} \big(\mathbf{1}_{\mathsf{S}} \otimes \mathbf{1}_{\mathsf{S}} \otimes \xi_k \,\hat{\cdot}\, \mathbf{1}_{\mathsf{S}} \otimes \Delta^{\mathsf{S}}(\xi_k)\big) , \tag{C.25}$$

which follows from reordering the parity-odd elements ξ_k and $\Delta^{\mathsf{S}}(\xi_k)$. We now take $\hat{\Phi}^{101}$ in the form (C.3) and check the above equality. First, we know that this equality holds in the $\mathrm{N} = 1$ case [**GR1**, Sec. 7.4], which takes the form

$$\underline{\hat{\Phi}}^{101}_{(k)} \cdot \mathbf{1} \otimes \mathsf{K} \otimes \mathbf{1} \,.\, p = (\omega \otimes \omega \otimes \mathrm{id})\big(\mathbf{1}_{\mathsf{S}} \otimes \mathbf{1}_{\mathsf{S}} \otimes \xi_k \,\hat{\cdot}\, \underline{\Lambda}^{101}_{(k)} \,\hat{\cdot}\, \mathbf{1}_{\mathsf{S}} \otimes \Delta^{\mathsf{S}}(\xi_k)\big) \,\hat{\cdot}\, (\mathrm{id} \otimes \omega \otimes \mathrm{id})(p) \tag{C.26}$$

where $k = 1$ and one has to use (C.4) and the convention in (C.21). We also note that the equality (C.26) holds for general N and k. It can be rewritten as

$$\underline{\hat{\Phi}}^{101}_{(k)} \cdot \mathbf{1} \otimes \mathsf{K} \otimes \mathbf{1} \,.(\mathrm{id} \otimes \omega \otimes \mathrm{id})(p) = (\omega \otimes \omega \otimes \mathrm{id})\big(\mathbf{1}_{\mathsf{S}} \otimes \mathbf{1}_{\mathsf{S}} \otimes \xi_k \,\hat{\cdot}\, \underline{\Lambda}^{101}_{(k)} \,\hat{\cdot}\, \mathbf{1}_{\mathsf{S}} \otimes \Delta^{\mathsf{S}}(\xi_k)\big) \,\hat{\cdot}\, p \,. \tag{C.27}$$

By choosing $p = \mathbf{1}^{\otimes 3}$ and applying (C.27) multiple times on (C.24) it follows that (recall that $\mathbf{1} \in \hat{\mathsf{Q}}_{\mathcal{H}}$ is not of definite parity)

$$\begin{aligned} \underline{\hat{\Phi}}^{101} &= \big(\omega^{\mathrm{N}-1} \otimes \omega^{\mathrm{N}-1} \otimes \mathrm{id}\big) \Big(\underline{\hat{\Phi}}^{101}_{(1)} \cdot \mathbf{1} \otimes \mathsf{K} \otimes \mathbf{1} \,.\, (\mathrm{id} \otimes \omega \otimes \mathrm{id})\Big(\underline{\hat{\Phi}}^{101}_{(2)} \cdot \mathbf{1} \otimes \mathsf{K} \otimes \mathbf{1} \,.\, (\mathrm{id} \otimes \omega \otimes \mathrm{id}) \\ &\quad \cdots \Big(\underline{\hat{\Phi}}^{101}_{(\mathrm{N}-1)} \cdot \mathbf{1} \otimes \mathsf{K} \otimes \mathbf{1} \,.\, (\mathrm{id} \otimes \omega \otimes \mathrm{id}) \Big(\underline{\hat{\Phi}}^{101}_{(\mathrm{N})} \cdot \mathbf{1} \otimes \mathsf{K} \otimes \mathbf{1} \,.\, \omega^{\mathrm{N}-1}(\mathbf{1}) \otimes \mathbf{1} \otimes \mathbf{1}\Big) \Big) \cdots \Big) \,. \end{aligned} \tag{C.28}$$

By identifying ω with the element $(\boldsymbol{e}_0 - \mathbf{i}\boldsymbol{e}_1)\mathsf{K}$, the above expression simplifies to (C.3). Note that $\omega \cdot \boldsymbol{e}_0 = \mathsf{K} \cdot \boldsymbol{e}_0$ and hence $\mathbf{1} \otimes \mathsf{K} \otimes \mathbf{1} \cdot \mathbf{1} \otimes \omega \otimes \mathbf{1} \cdot \boldsymbol{e}_1 \otimes \boldsymbol{e}_0 \otimes \boldsymbol{e}_1 = \boldsymbol{e}_1 \otimes \boldsymbol{e}_0 \otimes \boldsymbol{e}_1$.

Altogether, we have shown:

Proposition C.6. *The natural isomorphism* $\alpha^{\mathbf{Rep}\,\hat{\mathsf{Q}}}$ *from* (C.18) *with* $\hat{\Phi}$ *as in* (C.2)–(C.4) *defines an associator on* $\mathbf{Rep}\,\hat{\mathsf{Q}}$*. With respect to this associator, the equivalence* $\mathcal{G} : \mathbf{Rep}\,\mathsf{S} \to \mathbf{Rep}\,\hat{\mathsf{Q}}$ *with multiplicative structure* $\Gamma_{U,V}$ *defined in* (C.14) *and* $\mathrm{id}_{\mathbb{C}} : \mathbf{1} \to \mathcal{G}(\mathbf{1})$*, is* $\mathbb{C}$*-linear monoidal equivalence.*

C.5. Transporting the braiding

We now similarly transport the braiding along the monoidal equivalence $\mathcal{G} : \mathbf{Rep}\,\mathsf{S} \to \mathbf{Rep}\,\hat{\mathsf{Q}}$ from Proposition C.6, recall the discussion above (7.25).

Lemma C.7. *For all $U, V \in \mathbf{Rep}\,\hat{\mathsf{Q}}$, the isomorphisms $\tau_{U,V} \circ \hat{R}$ with $\hat{R}$ from* (C.5) *are natural in U, V and make the diagram* (7.25) *commute for all $M, N \in \mathbf{Rep}\,\mathsf{S}$:*

$$\begin{tikzcd}
\mathcal{G}(M \otimes_{\mathbf{Rep}\,\mathsf{S}} N) \arrow[r, "\mathcal{G}(\psi_{M,N})"] \arrow[d, "\Gamma_{M,N}"] & \mathcal{G}(N \otimes_{\mathbf{Rep}\,\mathsf{S}} M) \arrow[d, "\Gamma_{N,M}"] \\
\mathcal{G}(M) \otimes_{\mathbf{Rep}\,\hat{\mathsf{Q}}} \mathcal{G}(N) \arrow[r, "\tau_{\mathcal{G}(M),\mathcal{G}(N)} \circ \hat{R}"] & \mathcal{G}(N) \otimes_{\mathbf{Rep}\,\hat{\mathsf{Q}}} \mathcal{G}(M)
\end{tikzcd} \tag{C.29}$$

Proof. It is clear that for morphisms $f : U \to U'$ and $g : V \to V'$ in $\mathbf{Rep}\,\hat{\mathsf{Q}}$ we have $\tau_{U',V'} \circ \hat{R} \circ (f \otimes g) = (g \otimes f) \circ \tau_{U,V} \circ \hat{R}$. The lemma follows once we proved that (C.29) commutes, as it was already argued in the proof of Lemma B.9.

Though the monoidal isomorphisms Γ are non-trivial only in the **11**-sector the transport is non-trivial in all the four sectors due to the passage from **Svect** to **vect**. We recall first the braiding (B.46) in $\mathbf{Rep}\,\mathsf{S}$. The commutativity of the diagram (C.29) in the **00** and **01** sectors then corresponds to the equation

$$\tau_{U,V} \circ \hat{R}^{0i} \,.\, (u \otimes v) = \tau^{\text{s.v.}}_{U,V} \circ r^{0i} \,\hat{.}\, (u \otimes v), \qquad i \in \{0,1\}, \quad u \in U, \quad v \in V, \tag{C.30}$$

where τ is the symmetric braiding in vector spaces. This equation for $\mathrm{N} = 1$ case holds due to [**GR1**], see also Remark C.1. Therefore, using the factorised expression for $\hat{R}^{0i}$ and r^{0i}, the equation (C.30) has the unique solution as in (C.6), where the first factor $\rho(\mathsf{K})$ defined in (C.9) is due to the braiding in super-vector spaces.

Recall then the expression of the braiding in the **10** and **11** sectors of $\mathbf{Rep}\,\mathsf{S}$ in (B.46). The commutativity of the diagram (C.29) corresponds thus to the equations

$$\begin{aligned}
&\tau_{U,V} \circ \hat{R}^{10} \,.\, (u \otimes v) = \tau^{\text{s.v.}}_{U,V} \circ r^{10} \,\hat{.}\, (\mathrm{id} \otimes \omega)\big[u \otimes v\big], \\
&\tau_{U,V} \circ \hat{R}^{11} \,.\, (u \otimes v) = (\mathrm{id} \otimes L_\xi) \circ \tau^{\text{s.v.}}_{U,V} \circ r^{11} \,\hat{.}\, (\mathrm{id} \otimes \omega)\big[u \otimes \xi \,.\, v\big],
\end{aligned} \tag{C.31}$$

with the solutions given in (C.7) and (C.8), correspondingly. The derivation of (C.7) is obvious. To derive (C.8), we note that the equation on $\hat{R}^{11}$ can be rewritten as

$$\hat{R}^{11} \,.\, (u \otimes v) = (-1)^{\mathrm{N}} \rho(\mathsf{K}) \,.(\mathrm{id} \otimes \omega^{\mathrm{N}-1}) \Big[(\mathrm{id} \otimes \omega)\big(\xi \otimes \mathbf{1} \cdot r^{11} \cdot \mathbf{1} \otimes \xi\big) \,\hat{.}\, (\omega^{\mathrm{N}}(u) \otimes v)\Big] \,. \tag{C.32}$$

We use then the factorised expressions for ξ in (C.15) and for r^{11} in (B.9) together with the known solution for $\mathrm{N} = 1$, recall Remark C.1, and it finally gives the expression in (C.8). By the construction of $\hat{R}$, the diagram (C.29) commutes and this finishes the proof. $\square$

Since $\mathbf{Rep}\,\mathsf{S}$ is a braided monoidal category, the above lemma shows that $\hat{R}$ from (C.5) is the R-matrix of $\hat{\mathsf{Q}}$ and by the construction of the transport of the braiding the family $\tau_{U,V} \circ \hat{R}$ defines the braiding in $\mathbf{Rep}\,\hat{\mathsf{Q}}$. Altogether we have shown:

Proposition C.8. *The functor $\mathcal{G}$ in Proposition C.6 is braided monoidal.*

C.6. The quasi-bialgebra Q is the twisting of $\hat{\mathsf{Q}}$

We define the twist ζ – an invertible element in $\hat{\mathsf{Q}} \otimes \hat{\mathsf{Q}}$:

$$\zeta = \boldsymbol{e}_0 \otimes \mathbf{1} + \zeta^{10} \cdot \boldsymbol{e}_1 \otimes \boldsymbol{e}_0 + \zeta^{11} \cdot \boldsymbol{e}_1 \otimes \boldsymbol{e}_1 \ , \tag{C.33}$$

where

$$\zeta^{10} = \left(\prod_{k=1}^{\mathrm{N}} \zeta^{10}_{(k)} \right) \cdot \mathbf{1} \otimes \mathsf{K} \quad , \quad \zeta^{11} = \left(\prod_{k=1}^{\mathrm{N}} \zeta^{11}_{(k)} \right) \cdot \mathbf{1} \otimes \mathsf{K}^{\mathrm{N}-1} \tag{C.34}$$

and

$$\begin{aligned} \zeta^{10}_{(k)} &= \mathbf{1} \otimes \mathbf{1} + (1-\mathrm{i}) \mathbf{1} \otimes \mathsf{f}^+_k \mathsf{f}^-_k + (1-\mathrm{i}) \mathsf{f}^-_k \mathsf{K} \otimes \mathsf{f}^+_k + (1+\mathrm{i}) \mathsf{f}^+_k \mathsf{K} \otimes \mathsf{f}^-_k - 2\mathsf{f}^+_k \mathsf{f}^-_k \otimes \mathsf{f}^+_k \mathsf{f}^-_k \ , \\ \zeta^{11}_{(k)} &= \mathbf{1} \otimes \mathbf{1} - (1+\mathrm{i}) \mathbf{1} \otimes \mathsf{f}^+_k \mathsf{f}^-_k \ . \end{aligned} \tag{C.35}$$

Its inverse is

$$\zeta^{-1} = \boldsymbol{e}_0 \otimes \mathbf{1} + (\zeta^{10})^{-1} \cdot \boldsymbol{e}_1 \otimes \boldsymbol{e}_0 + (\zeta^{11})^{-1} \cdot \boldsymbol{e}_1 \otimes \boldsymbol{e}_1 \tag{C.36}$$

with

$$\begin{aligned} &(\zeta^{10})^{-1} = \mathbf{1} \otimes \mathsf{K}^{-1} \cdot \left(\prod_{k=1}^{\mathrm{N}} (\zeta^{10}_{(k)})^{-1} \right) \quad , \quad (\zeta^{11})^{-1} = \mathbf{1} \otimes \mathsf{K}^{-(\mathrm{N}-1)} \cdot \left(\prod_{k=1}^{\mathrm{N}} (\zeta^{11}_{(k)})^{-1} \right) \ , \\ &(\zeta^{10}_{(k)})^{-1} = \mathbf{1} \otimes \mathbf{1} + (1+\mathrm{i}) \mathbf{1} \otimes \mathsf{f}^+_k \mathsf{f}^-_k - (1-\mathrm{i}) \mathsf{f}^-_k \mathsf{K} \otimes \mathsf{f}^+_k - (1+\mathrm{i}) \mathsf{f}^+_k \mathsf{K} \otimes \mathsf{f}^-_k - 2\mathsf{f}^+_k \mathsf{f}^-_k \otimes \mathsf{f}^+_k \mathsf{f}^-_k \ , \\ &(\zeta^{11}_{(k)})^{-1} = \mathbf{1} \otimes \mathbf{1} - (1-\mathrm{i}) \mathbf{1} \otimes \mathsf{f}^+_k \mathsf{f}^-_k \ . \end{aligned}$$

Since $(\varepsilon \otimes \mathrm{id})(\zeta) = \mathbf{1} = (\mathrm{id} \otimes \varepsilon)(\zeta)$ and ζ is invertible, ζ is indeed a twist. This means (see e.g. [**Dr2**] or [**CP**]) that it defines another quasi-triangular quasi-bialgebra $\hat{\mathsf{Q}}_\zeta$ with the same algebra structure and counit as in $\hat{\mathsf{Q}}$, while the new coproduct Δ_ζ, R-matrix R_ζ and coassociator Φ_ζ are given by

$$\Delta_\zeta(x) = \zeta \, \hat{\Delta}(x) \, \zeta^{-1} \ , \tag{C.37}$$

$$R_\zeta = \zeta_{21} \, \hat{R} \, \zeta^{-1} \ , \tag{C.38}$$

$$\Phi_\zeta = (\zeta \otimes \mathbf{1}) \cdot (\hat{\Delta} \otimes \mathrm{id})(\zeta) \cdot \hat{\Phi} \cdot (\mathrm{id} \otimes \hat{\Delta})(\zeta^{-1}) \cdot (\mathbf{1} \otimes \zeta^{-1}) \ . \tag{C.39}$$

The action of the twist defines a multiplicative structure on the identity functor between the representation categories of both quasi-bialgebras. In particular, the categories are braided monoidally equivalent.

Proposition C.9. *We have $\hat{\mathsf{Q}}_\zeta = \mathsf{Q}$, that is, Q as a quasi-triangular quasi-bialgebra defined in Section 7.1 is obtained from $\hat{\mathsf{Q}}$ by twisting via ζ.*

To prove the proposition, we will need the following lemma.

Lemma C.10. *We have for $a, b, c \in \{0, 1\}$ and $1 \le i \ne j \le N$ the equalities*

$$\left[\hat{\Phi}^{abc}_{(i)}, \hat{\Phi}^{abc}_{(j)}\right] = 0 \ , \qquad \left[(\Delta \otimes \mathrm{id})(\zeta^{11}_{(i)}), \hat{\Phi}^{101}_{(j)}\right] = 0 \ , \tag{C.40}$$

$$\left[\zeta^{ab}_{(i)}, \zeta^{ab}_{(j)}\right] = 0 \ , \qquad \left[\hat{\Phi}^{101}_{(i)}, \mathsf{K} \otimes \mathsf{K} \otimes \mathsf{K}\right] = 0$$

and

$$\left[\zeta^{10}_{(i)} \otimes \mathbf{1}, (\mathrm{id} \otimes \Delta)(\zeta^{11}_{(j)})\right] = 0 \ . \tag{C.41}$$

Proof. This can be easily checked using the anti-commutator relations of Q from (7.3) and by recalling that Q and $\hat{\mathsf{Q}}$ have the same algebra structure. □

Proof of Proposition C.9. Using (C.37)-(C.39), we have to show that $R_\zeta = R$, $\Phi_\zeta = \Phi$ and $\Delta_\zeta = \Delta$ with R, Φ and Δ defined in (7.15), (7.11) and (7.8), respectively. For $\mathrm{N} = 1$ we verified these equalities using computer algebra. The proof for general N will rely on the $\mathrm{N} = 1$ case.

We start with the coproduct and show the statement sector by sector. Clearly, Δ_ζ in (C.37) agrees with the coproduct in Q from (7.8) in the sectors **00** and **01**. Since Δ_ζ is an algebra map, it is enough to show the equality $\Delta_\zeta = \Delta$ on the generators. Below we will use the abbreviation

$$\underline{\zeta}^{ab} = \zeta^{ab} \cdot \boldsymbol{e}_a \otimes \boldsymbol{e}_b \ , \qquad a, b \in \{0, 1\} \ , \tag{C.42}$$

and similarly for $\underline{\zeta}^{ab}_{(k)}$. By applying Lemma C.10 and using that for $\mathrm{N} = 1$ the statement is true we get in sector **10**, for $x_i \in \{\mathsf{K}, \mathsf{f}^+_i, \mathsf{f}^-_i\}$,

$$\begin{aligned} \underline{\zeta}^{10}\, \hat{\Delta}(x_i)\, (\underline{\zeta}^{10})^{-1} &= \left(\prod_{k=1}^{\mathrm{N}} \underline{\zeta}^{10}_{(k)}\right) \mathbf{1} \otimes \mathsf{K} \cdot \hat{\Delta}(x_i) \cdot \mathbf{1} \otimes \mathsf{K}^{-1} \left(\prod_{k=1}^{\mathrm{N}} (\underline{\zeta}^{10}_{(k)})^{-1}\right) \\ &= \left(\prod_{k \ne i} \underline{\zeta}^{10}_{(k)}\right) \cdot \Delta(x_i) \cdot \left(\prod_{k \ne i} (\underline{\zeta}^{10}_{(k)})^{-1}\right) = \Delta(x_i) \ , \end{aligned} \tag{C.43}$$

where we also used that $\underline{\zeta}^{10}_{(k)}$ commutes with $\Delta(x_i)$ for $i \ne k$. For the sector **11**, we prove it similarly.

The twisted R-matrix is $R_\zeta = \zeta_{21}\, \hat{R}\, \zeta^{-1}$ and we begin with the case $\mathrm{N} = 1$. The direct calculation gives (for $\beta^4 = -1$)

$$\begin{aligned} R_\zeta = \frac{1}{2} \sum_{i,j,k=0}^{1} & 2^k \mathrm{i}^{2k(i-j+1)-2ij} (\mathsf{f}^-_1)^k \mathsf{K}^{k+i} \otimes (\mathsf{f}^+_1)^k \mathsf{K}^j \\ & \times (\boldsymbol{e}_0 \otimes \boldsymbol{e}_0 + \mathrm{i}^{-i-k} \boldsymbol{e}_1 \otimes \boldsymbol{e}_0 + \mathrm{i}^j \boldsymbol{e}_0 \otimes \boldsymbol{e}_1 + \mathrm{i}^{-i-k+j} \beta \boldsymbol{e}_1 \otimes \boldsymbol{e}_1) \ . \end{aligned} \tag{C.44}$$

We can rewrite R_ζ in a more compact form involving the $\mathbb{Z}_2$-parity $\omega = \mathsf{K}(\boldsymbol{e}_0 - \mathrm{i}\boldsymbol{e}_1)$:

$$R_\zeta = \sum_{n,m=0}^{1} \beta^{nm} \rho_{n,m} \cdot (\mathbf{1}\otimes\mathbf{1} - 2\mathsf{f}_1^-\omega\otimes\mathsf{f}_1^+) \cdot \boldsymbol{e}_n \otimes \boldsymbol{e}_m \ , \tag{C.45}$$

with the Cartan factor $\rho_{n,m}$ as in (7.14). In the above calculation we used the identities $\rho_{n,0} = \rho(\mathsf{K}) \cdot \boldsymbol{e}_n \otimes \boldsymbol{e}_0$ and $\rho_{n,1} = (-\mathrm{i})^n \mathsf{K}\otimes\mathbf{1}\cdot\rho(\mathsf{K})\cdot\boldsymbol{e}_n\otimes\boldsymbol{e}_1$, with $\rho(\mathsf{K})$ introduced in (C.9). For general N, using the factorised form of ζ we obtain the expression in terms of $\mathrm{N}=1$ terms:

$$R_\zeta \cdot \boldsymbol{e}_n \otimes \boldsymbol{e}_m = \beta^{nm}\rho_{n,m} \cdot \prod_{k=1}^{\mathrm{N}} (\mathbf{1}\otimes\mathbf{1} - 2\mathsf{f}_k^-\omega\otimes\mathsf{f}_k^+)\cdot \boldsymbol{e}_n\otimes\boldsymbol{e}_m, \qquad n,m\in\mathbb{Z}_2, \tag{C.46}$$

where now $\beta^4 = (-1)^{\mathrm{N}}$. This proves $R = R_\zeta$ with the R-matrix for Q given in (7.15) where one has to use $\omega_- = \omega$.

We now turn to the calculation of the twisted coassociator. We prove the equality $\Phi_\zeta = \Phi$ in the sector **101**, the proof for the other sectors is similar. Recall the definition of the coassociators Φ and $\hat{\Phi}$ in (7.11) and (C.2)–(C.4), respectively. We have to show that $\Phi_\zeta^{101} = \mathbf{1}^{\otimes 3}$. The equation (C.39) reduces to

$$\Phi_\zeta^{101} \cdot \boldsymbol{e}_1\otimes\boldsymbol{e}_0\otimes\boldsymbol{e}_1 = \underline{\zeta}^{10}\otimes\mathbf{1}\cdot(\hat{\Delta}\otimes\mathrm{id})(\underline{\zeta}^{11})\cdot\underline{\hat{\Phi}}^{101}\cdot(\mathrm{id}\otimes\hat{\Delta})\big((\underline{\zeta}^{11})^{-1}\big) \ , \tag{C.47}$$

where we used the notations in (C.21) and (C.42), and that $\zeta = \mathbf{1}\otimes\mathbf{1}$ in sector **01**. Using computer algebra for $\mathrm{N}=1$ it turned out that $\Phi_\zeta^{101} = \mathbf{1}^{\otimes 3}$. Taking this into account, the equality (C.47) for $\mathrm{N}=1$ is then equivalent to

$$(\mathrm{id}\otimes\hat{\Delta})(\underline{\zeta}_{(k)}^{11})\cdot\mathbf{1}\otimes\mathsf{K}\otimes\mathbf{1} = \underline{\zeta}_{(k)}^{10}\otimes\mathbf{1}\cdot\mathbf{1}\otimes\mathsf{K}\otimes\mathbf{1}\cdot(\hat{\Delta}\otimes\mathrm{id})(\underline{\zeta}_{(k)}^{11})\cdot\underline{\hat{\Phi}}_{(k)}^{101} \ , \tag{C.48}$$

where $k=1$ and we used that $\mathsf{K}^{-1}\boldsymbol{e}_0 = \mathsf{K}\boldsymbol{e}_0$. We note that the above equation also holds for general k. Together with Lemma C.10 we get for general N:

(C.49)

$$\begin{aligned}
\underline{\zeta}^{10}\otimes\mathbf{1}\cdot(\hat{\Delta}\otimes\mathrm{id})(\underline{\zeta}^{11})\cdot\underline{\hat{\Phi}}^{101} &= \Big(\prod_{i=1}^{\mathrm{N}}\underline{\zeta}_{(i)}^{10}\otimes\mathbf{1}\Big)\cdot\mathbf{1}\otimes\mathsf{K}\otimes\mathbf{1}\cdot\Big(\prod_{i=1}^{\mathrm{N}}(\hat{\Delta}\otimes\mathrm{id})(\underline{\zeta}_{(i)}^{11})\Big)\cdot\mathbf{1}\otimes\mathbf{1}\otimes\mathsf{K}^{\mathrm{N}-1}\\
&\quad\times(-\mathsf{K})^{\mathrm{N}-1}\otimes\mathsf{K}^{\mathrm{N}-1}\otimes\mathbf{1}\cdot\Big(\prod_{i=1}^{\mathrm{N}}\underline{\hat{\Phi}}_{(i)}^{101}\Big)\cdot\mathsf{K}^{\mathrm{N}-1}\otimes\mathsf{K}\otimes\mathbf{1}\\
&= \Big(\prod_{i=1}^{\mathrm{N}-1}\underline{\zeta}_{(i)}^{10}\otimes\mathbf{1}\Big)\cdot\Big(\underline{\zeta}_{(\mathrm{N})}^{10}\otimes\mathbf{1}\cdot\mathbf{1}\otimes\mathsf{K}\otimes\mathbf{1}\cdot(\hat{\Delta}\otimes\mathrm{id})(\underline{\zeta}_{(\mathrm{N})}^{11})\cdot\underline{\hat{\Phi}}_{(\mathrm{N})}^{101}\Big)\\
&\quad\times\Big(\prod_{i=1}^{\mathrm{N}-1}(\hat{\Delta}\otimes\mathrm{id})(\underline{\zeta}_{(i)}^{11})\cdot\underline{\hat{\Phi}}_{(i)}^{101}\Big)\cdot\mathbf{1}\otimes\mathsf{K}^{\mathrm{N}}\otimes\mathsf{K}^{\mathrm{N}-1} \ ,
\end{aligned}$$

where the first equality is by definition of ζ and $\hat{\Phi}^{101}$, while we used the relations in (C.40) at the second equality. Next, by applying (C.48) for $k = \mathrm{N}$ to the previous expression we get

$$\text{(C.50)}\qquad \underline{\zeta}^{10} \otimes \mathbf{1} \cdot (\hat{\Delta} \otimes \mathrm{id})(\underline{\zeta}^{11}) \cdot \underline{\hat{\Phi}}^{101} = \Big(\prod_{i=1}^{\mathrm{N}-1} \underline{\zeta}^{10}_{(i)} \otimes \mathbf{1} \Big) \cdot \Big((\mathrm{id} \otimes \hat{\Delta})(\underline{\zeta}^{11}_{(\mathrm{N})}) \cdot \mathbf{1} \otimes \mathsf{K} \otimes \mathbf{1} \Big) \\ \times \Big(\prod_{i=1}^{\mathrm{N}-1} (\hat{\Delta} \otimes \mathrm{id})(\underline{\zeta}^{11}_{(i)}) \cdot \underline{\hat{\Phi}}^{101}_{(i)} \Big) \cdot \mathbf{1} \otimes \mathsf{K}^{\mathrm{N}} \otimes \mathsf{K}^{\mathrm{N}-1} \ .$$

Finally using first the relation (C.41) and then doing the reordering of the terms in the products as in (C.49) and (C.50) for $i = \mathrm{N}-1, \dots, 1$, we obtain

$$\text{(C.51)}\qquad \text{RHS of (C.50)} = \Big(\prod_{i=1}^{\mathrm{N}} (\mathrm{id} \otimes \hat{\Delta})(\underline{\zeta}^{11}_{(i)}) \Big) \cdot \mathbf{1} \otimes \mathsf{K} \otimes \mathbf{1} \cdot \mathbf{1} \otimes \mathsf{K}^{\mathrm{N}} \otimes \mathsf{K}^{\mathrm{N}-1} \\ = (\mathrm{id} \otimes \hat{\Delta})(\underline{\zeta}^{11}) \ .$$

Hence, $\Phi^{101}_{\zeta} = \mathbf{1}^{\otimes 3} = \Phi^{101}$ which completes the proof. □

C.7. Transporting the ribbon twist

The ribbon twist in $\mathcal{SF}$ is described in Section 6.6. Following the same lines as in [**GR1**, Sec. 7.9] and using that ω_{B} is given by the left-action of Ω in (B.55) we can write, for $M \in \mathbf{Rep}\,\mathsf{Q}$, $m \in M$,

$$\text{(C.52)}\qquad \theta_M(m) = \Big(\boldsymbol{e}_0 \prod_{k=1}^{\mathrm{N}} (\mathbf{1} + 2\mathsf{f}^+_k \mathsf{f}^-_k) - \mathrm{i}\beta^{-1} \boldsymbol{e}_1 \mathsf{K} \prod_{k=1}^{\mathrm{N}} (\mathbf{1} - 2\mathsf{f}^+_k \mathsf{f}^-_k) \Big).m \ .$$

The action of the ribbon element $\boldsymbol{v} \in \mathsf{Q}$ on M is by convention equals the inverse twist. It follows that

$$\text{(C.53)}\qquad \boldsymbol{v} = (\boldsymbol{e}_0 - \beta \mathrm{i} \mathsf{K} \boldsymbol{e}_1) \cdot \prod_{k=1}^{\mathrm{N}} (\mathbf{1} - 2\mathsf{f}^+_k \mathsf{f}^-_k) \ .$$

C.8. Ribbon equivalence $\mathcal{F}\colon \mathcal{SF} \to \mathbf{Rep}\,\mathsf{Q}$

Taking Appendices B and C together (Propositions B.8, B.10 and C.6, C.8, and C.9), we have established a ribbon equivalence $\mathcal{F}\colon \mathcal{SF} \to \mathbf{Rep}\,\mathsf{Q}$ which is the composition

$$\text{(C.54)}\qquad \mathcal{F} = \mathcal{G} \circ \mathcal{D}$$

where the functor $\mathcal{D}$ is defined in (B.17) and (B.20), and the functor $\mathcal{G}$ is in Section C.2. The monoidal structure of $\mathcal{F}$ is defined by

$$\text{(C.55)}\qquad \mathcal{F}_{U,V} \colon \mathcal{F}(U * V) \xrightarrow{\sim} \mathcal{F}(U) \otimes_{\mathbf{Rep}\,\mathsf{Q}} \mathcal{F}(V)$$

with

$$\text{(C.56)}\qquad \mathcal{F}_{U,V} = \big(\zeta.(-)\big) \circ \Gamma_{\mathcal{D}(U),\mathcal{D}(V)} \circ \mathcal{G}(\Delta_{U,V}) \ ,$$

where ζ, $\Gamma_{U,V}$ and $\Delta_{U,V}$ are given in (C.33), (C.14) and in Section B.3, respectively. This monoidal equivalence is by construction braided and ribbon and thus finally proves Lemma 7.5.

D. Proof of Proposition 7.10

In this Appendix, we use for brevity H instead of $H(\mathrm{N})$, for the Hopf algebra introduced in (7.31) and (7.32), and we fix the basis in H:

$$H = \mathrm{span}_{\mathbb{C}} \{ f_{u_1} \cdots f_{u_m} k^v \,|\, 0 \le m \le \mathrm{N}, 1 \le u_1 < u_2 < \ldots < u_m \le \mathrm{N}, v \in \mathbb{Z}_2 \} \ . \tag{D.1}$$

The coproduct Δ on the basis elements (D.1) is given by

$$\begin{aligned} \Delta(f_{u_1} \cdots f_{u_m} k^v) &= \Bigg(\prod_{u=u_1}^{u_m} (f_u \otimes k + \mathbf{1} \otimes f_u) \Bigg) \cdot k^v \otimes k^v \\ &= \Big(\sum_{l=(l_1,\ldots,l_m)\in\mathbb{Z}_2^m} f_{u_1}^{l_1} \cdots f_{u_m}^{l_m} \otimes f_{u_1}^{1-l_1} k^{l_1} \cdots f_{u_m}^{1-l_m} k^{l_m} \Big) \cdot k^v \otimes k^v \\ &= \Big(\sum_{l\in\mathbb{Z}_2^m} \varepsilon_l(m)\, f_{u_1}^{l_1} \cdots f_{u_m}^{l_m} k^v \otimes f_{u_1}^{1-l_1} \cdots f_{u_m}^{1-l_m} k^{v+|l|} \Big) \ . \end{aligned} \tag{D.2}$$

where $\varepsilon_l(m) = (-1)^{\sum_{i=1}^{m-1} l_i((m-i)-\sum_{j=i+1}^{m} l_j)}$ and $|l| = \sum_i l_i \in \mathbb{N}$.

For the Drinfeld double construction we need the dual Hopf algebra

$$(H^{\mathrm{op}})^* = \big(H^*,\, \mu_{H^*} = \Delta^*,\, \mathbf{1}_{H^*} = \varepsilon^*,\, \Delta_{H^*} = \big(\mu^{\mathrm{op}}\big)^*,\, \varepsilon_{H^*} = \eta^*,\, S_{H^*} = (S^{-1})^* \big) \ , \tag{D.3}$$

where μ^{op} is the opposite multiplication, and we used the standard isomorphism of vector spaces $(H\otimes H)^* \cong H^* \otimes H^*$, so $\Delta_{H^*}(\varphi)(a\otimes b) = \varphi(ba)$ and $(\varphi\cdot\psi)(a) = (\varphi\otimes\psi)\big(\Delta(a)\big)$. Using (D.2), we note that the canonical duals of the generators of H do not generate $(H^{\mathrm{op}})^*$ since e.g. $f_i^* \cdot f_j^* = (f_i^* \otimes f_j^*) \circ \Delta = 0$. We then instead use the linear forms

$$\kappa = \mathbf{1}^* - k^* \ , \qquad \varphi_i = (f_i k)^* - f_i^* \ . \tag{D.4}$$

Lemma D.1. *The algebra $(H^{\mathrm{op}})^*$ is generated by κ and φ_i, $1 \le i \le N$, with the defining relations*

$$\kappa^2 = \mathbf{1}_{H^*} \ , \qquad \{\varphi_i, \varphi_j\} = 0 \ , \qquad \{\varphi_i, \kappa\} = 0 \ , \tag{D.5}$$

and the set

$$\{ \varphi_{i_1} \ldots \varphi_{i_m} \kappa^j \mid 0 \le m \le N, 1 \le i_1 < i_2 < \ldots < i_m \le N, j \in \mathbb{Z}_2 \} \tag{D.6}$$

forms a basis. The Hopf-algebra structure of $(H^{\mathrm{op}})^*$ is

$$\begin{aligned} &\Delta_{H^*}(\kappa) = \kappa\otimes\kappa \ , \qquad && \Delta_{H^*}(\varphi_i) = \varphi_i \otimes \mathbf{1}_{H^*} + \kappa\otimes\varphi_i \ , \\ &\varepsilon_{H^*}(\kappa) = 1 \ , && \varepsilon_{H^*}(\varphi_i) = 0 \ , \end{aligned} \tag{D.7}$$

$$S_{H^*}(\kappa) = \kappa\ , \qquad\qquad S_{H^*}(\varphi_i) = \varphi_i\kappa\ .$$

Proof. We begin with the defining relations. The first one from (D.5) is straightforward to check using $\mathbf{1}_{H^*} = \varepsilon = \mathbf{1}^* + k^*$. The next one follows from the calculation:

$$\text{(D.8)}\qquad \begin{aligned}\varphi_i\varphi_j &= \big(((f_ik)^* - f_i^*) \otimes ((f_jk)^* - f_j^*)\big) \circ \Delta = -((f_ik)^* \otimes f_j^* + f_i^* \otimes (f_jk)^*) \circ \Delta \\ &= (f_if_jk)^* + (f_if_j)^* = -((f_jf_ik)^* + (f_jf_i)^*) = -\varphi_j\varphi_i\ ,\end{aligned}$$

where we used the coproduct formula (D.2), and similarly for the third relation.

Now, we construct a basis in $(H^{\mathrm{op}})^*$. Using induction, one can check the relations

$$\text{(D.9)}\qquad \begin{aligned}(-1)^m\varphi_{i_1}\dots\varphi_{i_m} &= (f_{i_1}\dots f_{i_m})^* + (-1)^m(f_{i_1}\dots f_{i_m}k)^* \\ \varphi_{i_1}\dots\varphi_{i_m}\kappa &= (f_{i_1}\dots f_{i_m})^* - (-1)^m(f_{i_1}\dots f_{i_m}k)^*\ .\end{aligned}$$

Since the set of elements $(f_{u_1}\dots f_{u_m}k^v)^*$, with indices as in (D.1), forms a basis in $(H^{\mathrm{op}})^*$, we conclude from the above relations that the set (D.6) is also a basis in $(H^{\mathrm{op}})^*$.

For the coproduct we have by definition $\Delta_{H^*}(\varphi)(a\otimes b) = \varphi(ba)$. This can be written as

$$\text{(D.10)}\qquad \Delta_{H^*}(\varphi_i) = ((f_ik)^* - f_i^*) \circ \mu^{\mathrm{op}}\ .$$

For $f_i^* \circ \mu^{\mathrm{op}}(a\otimes b)$ we should find such pairs (a,b) that $f_i^*(ba)$ is non-zero. There are four such pairs and we get

$$\text{(D.11)}\qquad f_i^* \circ \mu^{\mathrm{op}} = f_i^* \otimes \mathbf{1}^* + \mathbf{1}^* \otimes f_i^* - (f_ik)^* \otimes k^* + k^* \otimes f_ik^*\ ,$$

and we get a similar expression for $(f_ik)^* \circ \mu^{\mathrm{op}}$. Combining the eight total terms in (D.10) we get the coproduct in (D.7). For the antipode we have

$$\text{(D.12)}\qquad S_{H^*}(\varphi_i) = ((f_ik)^* - f_i^*) \circ S^{-1} = f_i^* + (f_ik)^* = \varphi_i\kappa\ .$$

Calculations for the coproduct and antipode for κ, together with the counit, are straightforward. This finally proves the lemma. □

Next we present $D(H)$, the Drinfeld double of H, by following the conventions in [**Ka**, Chapter IX]. As a vector space $D(H)$ is $H^* \otimes H$. It has a Hopf algebra structure with unit $\mathbf{1}_{H^*} \otimes \mathbf{1}$ and counit $\varepsilon(\phi\otimes a) = \varepsilon_{H^*}(\phi)\varepsilon(a)$, with the multiplication defined as

$$\text{(D.13)}\qquad (\phi\otimes a)\cdot(\psi\otimes b) = \sum_{(a)} \phi\cdot\psi(S^{-1}(a''')(-)a') \otimes a''b\ ,$$

where $\psi(S^{-1}(a''')(-)a')$ stands for the map $(x \mapsto \psi(S^{-1}(a''')xa'))$. The coproduct and the antipode are given by

$$\text{(D.14)}\quad \Delta(\phi\otimes a) = \sum_{(\phi),(a)} (\phi'\otimes a')\otimes(\phi''\otimes a'')\ ,\quad S(\phi\otimes a) = \big(\mathbf{1}_{H^*}\otimes S(a)\big)\cdot\big(S_{H^*}(\phi)\otimes\mathbf{1}\big)\ .$$

We will identify an element a in H with $\mathbf{1}_{H^*}\otimes a$ and an element ϕ in $(H^{\mathrm{op}})^*$ with $\phi\otimes\mathbf{1}$, in particular, we write $\phi a = (\phi\otimes\mathbf{1})\cdot(\mathbf{1}_{H^*}\otimes a)$. In this notation, the basis of $D(H)$ is

$$\text{(D.15)}\qquad \{\varphi_{i_1}\cdots\varphi_{i_m}\kappa^u f_{j_1}\cdots f_{j_n}k^v \mid 0\le m,n\le \mathrm{N}, u,v\in\mathbb{Z}_2\}\ ,$$

where as usual we assume that $1 \leq i_1 < i_2 < \ldots < i_m \leq \mathrm{N}$ and similarly for the j's indices. It is well-known that $D(H)$ is quasi-triangular: for any basis $\{b_i | \, i \in I\}$ in H the R-matrix is given by (using the convention above we interpret $b_i, b_i^* \in D(H)$)

$$R_D = \sum_{i \in I} b_i \otimes b_i^* \ . \tag{D.16}$$

Proposition D.2. *$D(H)$ is generated by $k, \kappa, f_i, \varphi_j$ with defining relations* (7.31), (D.5) *and*

$$k\kappa = \kappa k \ , \qquad \varphi_i k = -k\varphi_i \ , \qquad f_i \kappa = -\kappa f_i \ , \qquad [f_i, \varphi_j] = \delta_{i,j}(\kappa - k) \ . \tag{D.17}$$

The Hopf algebra structure is given by (7.32) *and* (D.7). *The R-matrix for $D(H)$ is given by*

$$R_D = \frac{1}{2}(\mathbf{1}\otimes\mathbf{1} + \mathbf{1}\otimes\kappa + k\otimes\mathbf{1} - k\otimes\kappa)\Big(\sum_{\substack{0\leq m\leq N,\\ i_1<\ldots<i_m}} (-1)^m f_{i_1}\ldots f_{i_m}\otimes\varphi_{i_1}\ldots\varphi_{i_m}\Big) \ . \tag{D.18}$$

Proof. Using (D.13) it is straightforward to show the relations in (D.17). For example, we get the last equality since $\sum_{(f)} f_i' \otimes f_i'' \otimes f_i''' = f_i \otimes k \otimes k + \mathbf{1}\otimes f_i \otimes k + \mathbf{1}\otimes\mathbf{1}\otimes f_i$ and then

$$\begin{aligned} f_i\varphi_j &= \varphi_j(k - f_i)k + \varphi_j(k -)f_i + \varphi_j(f_i k -)\mathbf{1} \\ &= -\delta_{i,j}(\mathbf{1}^* + k^*)k + (-f_j^* + (f_j k)^*)f_i + \delta_{i,j}(\mathbf{1}^* - k^*)\mathbf{1} \ . \end{aligned} \tag{D.19}$$

The coproduct, counit, and antipode on the generators were already computed.

For the R-matrix in (D.16), we fix the basis $\{b_i\}$ as in (D.1). Then

$$R_D = \sum_{\substack{0\leq m\leq \mathrm{N},\\ 1\leq i_1<\ldots<i_m\leq\mathrm{N}}} \ \sum_{0\leq j\leq 1} f_{i_1}\ldots f_{i_m}k^j \otimes (f_{i_1}\ldots f_{i_m}k^j)^* \ . \tag{D.20}$$

By applying (D.9) we get

$$\begin{aligned} (f_{i_1}\ldots f_{i_m})^* &= \frac{1}{2}\varphi_{i_1}\ldots\varphi_{i_m}((-1)^m\mathbf{1}_{H^*} + \kappa) \ , \\ (f_{i_1}\ldots f_{i_m}k)^* &= \frac{1}{2}\varphi_{i_1}\ldots\varphi_{i_m}(\mathbf{1}_{H^*} - (-1)^m\kappa)) \end{aligned} \tag{D.21}$$

and therefore

$$R_D = \Big(\sum_{\substack{0\leq m\leq \mathrm{N},\\ i_1<\ldots<i_m}} f_{i_1}\ldots f_{i_m}\otimes\varphi_{i_1}\ldots\varphi_{i_m}\Big)\frac{1}{2}\big((-1)^m\mathbf{1}\otimes\mathbf{1} + \mathbf{1}\otimes\kappa + k\otimes\mathbf{1} - (-1)^m k\otimes\kappa\big) \tag{D.22}$$

which gives (D.18) using the relations between f_i and k, and φ_i and κ. □

Proposition D.3. *For any $N \geq 1$, the linear map*

$$\Psi(\varphi_{i_1}\cdots\varphi_{i_m}\kappa^u f_{j_1}\cdots f_{j_n}k^v) = (-1)^{nu}\, \mathrm{i}^{n(n-1)}2^m \mathsf{f}^+_{i_1}\cdots\mathsf{f}^+_{i_m}\mathsf{f}^-_{j_1}\cdots\mathsf{f}^-_{j_n}\,\omega_+^u\omega_-^{v+n} \ , \tag{D.23}$$

with $\omega_\pm$ from (7.8), *defines an isomorphism of $\mathbb{C}$-algebras $D\big(H(N)\big) \xrightarrow{\sim} \mathsf{Q}(N,\beta)$. Furthermore, for even N this map defines an isomorphism of Hopf algebras between $D\big(H(N)\big)$ and $\mathsf{Q}(N,\pm1)$.*
Moreover, for $\beta = 1$ the R-matrix in Q defined in (7.15) *and the image of R_D under $\Psi\otimes\Psi$ coincide, i.e. Ψ is an isomorphism of quasi-triangular Hopf algebras.*

Proof. In order to show that Ψ is an algebra map it is enough to verify the relations in (7.31), (D.5) and (D.17) for the the image of the generators of $D(H)$ under Ψ. Since we have

$$\Psi(\varphi_i) = 2\mathsf{f}_i^+ \ , \qquad \Psi(\kappa) = \omega_+ \ , \qquad \Psi(f_i) = \mathsf{f}_i^-\omega_- \ , \qquad \Psi(k) = \omega_- \ ,$$

this is an easy check. For example,

$$\begin{aligned}\Psi(f_i)\Psi(\varphi_j) &= \omega_-(\delta_{i,j}(\mathsf{K}^2 - \mathbf{1}) + 2\mathsf{f}_j^+\mathsf{f}_i^-) = \delta_{i,j}(\omega_+ - \omega_-) + 2\mathsf{f}_j^+\mathsf{f}_i^-\omega_- \\ &= \delta_{i,j}(\Psi(\kappa) - \Psi(k)) + \Psi(\varphi_j)\Psi(f_i)\end{aligned} \tag{D.24}$$

Since $\frac{1}{1+\mathrm{i}}\Psi(\kappa + \mathrm{i}k) = \mathsf{K}$ and $\Psi(f_j\kappa) = \mathsf{f}_j^-$ the image of Ψ clearly generates Q. Moreover, since the dimensions of $D(H)$ and Q agree Ψ is bijective.

To show that Ψ is a coalgebra map we use (7.32) and (D.7), and check

(D.25)

$$\begin{aligned}(\Psi\otimes\Psi)(\Delta(\varphi_i)) &= (\Psi\otimes\Psi)(\varphi_i\otimes\mathbf{1} + \kappa\otimes\varphi_i) = 2\mathsf{f}_i^+\otimes\mathbf{1} + 2\omega_+\otimes\mathsf{f}_i^+ = \Delta(2\mathsf{f}_i^+) = \Delta(\Psi(\varphi_i)) \ , \\ (\Psi\otimes\Psi)(\Delta(f_i)) &= (\Psi\otimes\Psi)(f_i\otimes k + \mathbf{1}\otimes f_i) = \mathsf{f}_i^-\omega_-\otimes\omega_- + \mathbf{1}\otimes\mathsf{f}_i^-\omega_- = \Delta(\mathsf{f}_i^-\omega_-) = \Delta(\Psi(f_i)) \ , \\ (\Psi\otimes\Psi)(\Delta(\kappa)) &= (\Psi\otimes\Psi)(\kappa\otimes\kappa) = \omega_+\otimes\omega_+ \overset{(*)}{=} \Delta(\omega_+) = \Delta(\Psi(\kappa)) \ .\end{aligned}$$

Note, the equality $(*)$ is in Q and it holds only if N is even. For the antipode, we have

$$\begin{aligned}\Psi(S(\varphi_i)) &= \Psi(\varphi_i\kappa) = 2\mathsf{f}_i^+\omega_+ = S(2\mathsf{f}_i^+) = S(\Psi(\varphi_i)) \ , \\ \Psi(S(f_i)) &= -\Psi(f_ik) = -\mathsf{f}_i^- = S(\mathsf{f}_i^-\omega_-) = S(\Psi(f_i)) \\ \Psi(S(\kappa)) &= \Psi(\kappa) = \omega_+ = S(\omega_+) = S(\Psi(\kappa)) \ .\end{aligned} \tag{D.26}$$

Recall the R-matrices in Q and $D(H)$ defined in (7.15) and (D.18), respectively. The image of R_D under $\Psi\otimes\Psi$ leads to (using $\beta = 1$)

$$\begin{aligned}(\Psi\otimes\Psi)(R_D) &= \frac{1}{2}(\mathbf{1}\otimes\mathbf{1} + \mathbf{1}\otimes\omega_+ + \omega_-\otimes\mathbf{1} - \omega_-\otimes\omega_-)\Big(\sum(-2)^m\mathsf{f}_{i_1}^-\omega_-\dots\mathsf{f}_{i_m}^-\omega_-\otimes\mathsf{f}_{i_1}^+\dots\mathsf{f}_{i_m}^+\Big) \\ &= \sum_{u,v=0}^{1}\rho_{u,v}\cdot\prod_{k=1}^{\mathrm{N}}(\mathbf{1}\otimes\mathbf{1} - 2\mathsf{f}_k^-\omega_-\otimes\mathsf{f}_k^+)\cdot\boldsymbol{e}_u\otimes\boldsymbol{e}_v = R \ ,\end{aligned}$$

where the sum is taken over $1 \le i_1 < \ldots < i_m \le \mathrm{N}$ with $0 \le m \le \mathrm{N}$. This calculation finishes the proof of the statement. □

Bibliography

[Ab] T. Abe, *A $\mathbb{Z}_2$-orbifold model of the symplectic fermionic vertex operator superalgebra*, Mathematische Zeitschrift **255** (2007) 755–792 [math.QA/0503472]. 3, 120, 121

[ABD] T. Abe, G. Buhl, C. Dong, *Rationality, regularity and C_2-cofiniteness*, Trans. Amer. Math. Soc. **356** (2004), 3391–340 math/0204021 [math.QA]. 38, 39

[AC] D. Altschüler, A. Coste, *Quasi-quantum groups, knots, three-manifolds, and topological field theory*, Commun. Math. Phys. **150** (1992) 83–107 [hep-th/9202047]. 11, 12, 97

[AEG] N. Andruskiewitsch, P. Etingof, S. Gelaki *Triangular Hopf algebras with the Chevalley property*, Michigan Math. J. **49** (2001) no. 2 277–298 [math/0008232 [math.QA]]. 95

[AN] Y. Arike, K. Nagatomo, *Some remarks on pseudo-trace functions for orbifold models associated with symplectic fermions*, Int. J. Math. **24** (2013) 1350008 [1104.0068 [math.QA]]. 3, 40, 41, 122, 123, 169, 170

[BC] D. Bulacu, S. Caenepeel, *On integrals and cointegrals for quasi-Hopf algebras*, J. Algebra **351** (2012) 390–425 1103.2263 [math.QA]. 73

[BDSPV] B. Bartlett, C.L. Douglas, C.J. Schommer-Pries, J. Vicary, *Modular categories as representations of the 3-dimensional bordism 2-category*, 1509.06811 [math.AT]. 1

[BK] B. Bakalov, A. Kirillov Jr., *Lectures on Tensor categories and modular functors*, Amer. Math. Soc., Providence, 2001. 32

[BN1] C.-G. Bontea, D. Nikshych, *On the Brauer-Picard group of a finite symmetric tensor category*, J. Algebra **440** (2015) 187–218 [1408.6445 [math.QA]]. 95

[BN2] D. Bulacu, E. Nauwelaerts, *Quasitriangular and Ribbon Quasi-Hopf Algebras*, Commun. Algebra **31** (2003) 657–672. 20, 25

[Br] A. Bruguières, *Catégories prémodulaires, modularisations et invariants des variétés de dimension 3*, Math. Annal. **316** (2000) 215–236. 53

[BT] D. Bulacu, B. Torrecillas, *Factorizable quasi-Hopf algebras – applications*, J. Pure Appl. Alg. 194 (2004) 39–84 math/0312076 [math.QA] 4, 71, 72, 94, 98

[CG] T. Creuzig, T. Gannon, *Logarithmic conformal field theory, log-modular tensor categories and modular forms*, 1605.04630 [math.QA]. 61, 62

[CP] V. Chari, A. Pressley, *A guide to quantum groups*, CUP, 1994. 9, 43, 156

[De] P. Deligne, *Catgories tannakiennes*, The Grothendieck Festschrift, vol. II, P. Cartier et al., eds. (Birkhuser, Boston 1990), p. 111. 32

[DLM1] C. Dong, H. Li, G. Mason, *Twisted representations of vertex operator algebras*, Math. Ann. **310** (1998) 571–600 [q-alg/9509005]. 39

[DLM2] C. Dong, H. Li, G. Mason, *Modular invariance of trace functions in orbifold theory*, Commun. Math. Phys. **214** (2000) 1–56 [q-alg/9703016]. 39, 40

[DMNO] A. Davydov, M. Müger, D. Nikshych, V. Ostrik, *The Witt group of non-degenerate braided fusion categories*, Crelle **677** (2013) 135–177 [1009.2117 [math.QA]]. 95

[Dr1] V.G. Drinfeld, *On Almost Cocommutative Hopf Algebras*, Leningrad Math. J. **1** (1990) 321–342. 11, 75

[Dr2] V.G. Drinfeld, *Quasi-Hopf algebras*, Leningrad Math. J. **1** (1990), 1419–1457. 9, 10, 11, 20, 22, 74, 96, 156

[DR1] A. Davydov, I. Runkel, *$\mathbb{Z}/2\mathbb{Z}$-extensions of Hopf algebra module categories by their base categories*, Adv. Math. **247** (2013) 192–265 [1207.3611 [math.QA]]. 3, 4, 83, 84, 85, 86, 87, 125, 126, 144, 169, 170

[DR2] A. Davydov and I. Runkel, *A braided monoidal category for symplectic fermions*, [1301.1996 [math.QA]]. 83

[DR3] A. Davydov, I. Runkel, *Holomorphic Symplectic Fermions*, I. Math. Z. (2016) [1601.06451 [math.QA]]. 3, 4, 95, 120, 125

[DSS] Ch. L. Douglas, Ch. Schommer-Pries, N. Snyder, *The balanced tensor product of module categories*, 1406.4204 [math.QA]. 50

[EGNO] P.I. Etingof, S. Gelaki, D. Nikshych, V. Ostrik, *Tensor categories*, Math. Surveys Monographs **205**, AMS, 2015. 2, 14, 15, 16, 17, 32, 92

[EO] P.I. Etingof, V. Ostrik, *Finite tensor categories*, Moscow Math. J. **4** (2004), 627–654, [math.QA/0301027]. 32

[FB] E. Frenkel, D. Ben-Zvi, *Vertex Algebras and Algebraic Curves*, Math. Surveys Monographs **88**, Second Edition, 2004. 37

[FGR2] V. Farsad, A.M. Gainutdinov, I. Runkel, *The symplectic fermion ribbon quasi-Hopf algebra and the $SL(2,\mathbb{Z})$-action on its centre*, in preparation. 61

[FHL] I. Frenkel, Y. Huang, J. Lepowsky., *On axiomatic approaches to vertex operator algebras and modules*, Mem. Amer. Math. Soc.**104**, 1993. 39

[FRS] J. Fuchs, I. Runkel, C. Schweigert, *TFT construction of RCFT correlators. I: Partition functions*, Nucl. Phys. B **646** (2002) 353–497 [hep-th/0204148]. 125

[FS] J. Fuchs, C. Schweigert, *Hopf algebras and finite tensor categories in conformal field theory*, Rev. Union Mat. Argentina **51** (2010) 43–90 [1004.3405 [hep-th]]. 2, 4

[FSS] J. Fuchs, C. Schweigert, C. Stigner, *From non-semisimple Hopf algebras to correlation functions for logarithmic CFT*, J. Phys. A **46** (2013) 494008 [1302.4683 [hep-th]]. 61, 108

[GR1] A.M. Gainutdinov, I. Runkel, *Symplectic fermions and a quasi-Hopf algebra structure on $\overline{U}_{\mathrm{i}}s\ell(2)$*, accepted in J. Algebra [1503.07695 [math.QA]]. 3, 137, 139, 141, 142, 144, 150, 151, 154, 155, 159

[GR2] A.M. Gainutdinov, I. Runkel, *The non-semisimple Verlinde formula and pseudo-trace functions*, 1605.04448 [math.QA]. 3, 40, 41, 61, 62, 119, 121, 122, 123, 124, 169, 170

[GR3] A.M. Gainutdinov, I. Runkel, *Projective objects and modified traces in factorisable finite tensor categories*, 1703.00150 [math.QA]. 63, 110, 114

[GRW] M.R. Gaberdiel, I. Runkel and S. Wood, *Logarithmic Bulk and Boundary Conformal Field Theory and the Full Centre Construction*, Conformal Field Theories and Tensor Categories, Mathematical Lectures from Peking University 2014 (Springer, 2014), pp. 93168 [1201.6273 [hep-th]]. 32

[GS] S. Gelaki, D. Sebbag, *On finite non-degenerate braided tensor categories with a Lagrangian subcategory*, 1703.05787 [math.QA]. 95

[HLZ] Y. Z. Huang, J. Lepowsky and L. Zhang, *A logarithmic generalization of tensor product theory for modules for a vertex operator algebra*, Int. J. Math. **17**, 975 (2006), math/0311235 [math-QA] 40

[HN] F. Hausser, F. Nill, *Integral Theory for Quasi-Hopf Algebras*, math/9904164 [math.QA]. 73, 97, 98

[Hu] Y. Z. Huang, *Rigidity and modularity of vertex tensor categories*, Commun. Contemp. Math. **10**, 871 (2008), math/0502533 [math-QA] 39

[JS] A. Joyal, R. Street, *Braided tensor categories*, Adv. Math. **102** (1993) 20–78. 94

[Ka] Ch. Kassel, *Quantum groups*, Springer, 1995. 16, 162

[Kac] V. G. Kac, *Vertex Algebras for Beginners*, University Lecture Series **10**, Amer. Math. Soc., 1998. 37

[Ke1] T. Kerler, *Mapping class group action on quantum doubles*, Commun. Math. Phys. **168** (1995) 353–388 [hep-th/9402017]. 79

[Ke2] T. Kerler, *Genealogy of nonperturbative quantum-invariants of 3-manifolds: The surgical family*, in: Quantum Invariants and Low-Dimensional Topology, J.E. Andersen et al. (Dekker, New York 1997) 503–547, [math.QA/9601021]]. 2, 28, 52, 64, 65, 107

[KL] T. Kerler, V.V. Lyubashenko, *Non-Semisimple Topological Quantum Field Theories for 3-Manifolds with Corners*, Springer Lecture Notes Math. **1765** (2001). 1, 2, 51, 53, 54, 169, 170

[LM] V.V. Lyubashenko, S. Majid, *Braided Groups and Quantum Fourier Transform*, J. Algebra **166** (1994) 506–528. 3, 71, 79

[Ly1] V.V. Lyubashenko, *Modular transformations for tensor categories*, J. Pure Appl. Alg. **9** (1995) 279–327. 1, 2, 4, 47, 49, 50, 51, 53, 54, 57, 58, 59, 107, 125, 169, 170

[Ly2] V.V. Lyubashenko, *Invariants of three manifolds and projective representations of mapping class groups via quantum groups at roots of unity*, Commun. Math. Phys. **172** (1995) 467–516 [hep-th/9405167]. 1, 2, 50, 64, 65, 71, 125, 169, 170

[Ma1] S. Majid, *Braided groups*, J. Pure Appl. Algebra, **86** (1993) 187–221. 4, 43, 44, 45, 51

[Ma2] S. Majid, *Foundations of Quantum Group Theory*, Cambridge University Press, 1994. 2

[Ma3] S. Majid, *Algebras and Hopf algebras in braided categories*, Lecture Notes in Pure and Appl. Math. 158 (1994) 55–105 [q-alg/9509023]. 25, 26

[McL] S. Mac Lane, *Categories for the working mathematician*, 2nd ed., Springer, 1998. 13, 28, 30

[Mi] M. Miyamoto, *Modular invariance of vertex operator algebras satisfying C(2) cofiniteness*, Duke Math. J. **122** (2004) 51–91 [math.QA/0209101]. 3, 39, 40, 169, 170

[Mü] M. Müger, *From subfactors to categories and topology II. The quantum double of tensor categories and subfactors*, J. Pure Appl. Alg. **180** (2003) 159–219 [math.CT/0111205]. 53

[PO] F. Panaite, F. Van Oystaeyen, *Existence of integrals for finite dimensional quasi-Hopf algebras* Bull. Belg. Math. Soc. Simon Stevin **7** (2000) 261–264. 73

[Ra] D.E. Radford, *The trace function and Hopf algebras*, J. Alg. **163**, 583–622 (1994). 75

[RS] N. Reshetikhin, M.A. Semenov-Tian-Shansky, *Quantum R-matrices and factorization problems*, J. Geom. Phys. **5** (1988) 533–550. 1, 11

[RT1] N. Reshetikhin, V.G. Turaev, *Ribbon graphs and their invariants derived from quantum groups*, Commun. Math. Phys. **127** (1990) 1–26. 18

[RT2] N. Reshetikhin, V.G. Turaev, *Invariants of 3-manifolds via link polynomials and quantum groups*, Inv. Math. **103** (1991) 547–597. 1

[Ru] I. Runkel, *A braided monoidal category for free super-bosons*, J. Math. Phys. **55** 041702 (2014) [1209.5554 [math.QA]]. 3, 4, 83, 84, 86, 119, 120, 125, 169, 170

[Sa] Š. Sakáloš, *On categories associated to a quasi-Hopf algebra*, Commun. Algebra **45** (2017) 722–748 [1402.1393 [math.QA]]. 66

[Sc] M. Schottenloher, *A Mathematical Introduction to Conformal Field Theory*, Lecture Notes in Physics, **759**, Springer-Verlag, Berlin, 2008. 33, 34, 35, 36, 37

[Sh1] K. Shimizu, *The monoidal center and the character algebra*, 1504.01178 [math.QA]. 4, 59, 61, 62, 108

[Sh2] K. Shimizu, *Non-degeneracy conditions for braided finite tensor categories*, 1602.06534 [math.QA]. 53, 86

[So] Y. Sommerhäuser, *On the notion of a ribbon quasi-Hopf algebra*, Revista de la Unión Mat. Argentina **51** (2010) 177–192 [0910.1638 [math.RA]]. 12

[Ta] M. Takeuchi, *Modular Categories and Hopf Algebras*, J. Algebra **243** (2001) 631–643. 1

[Tu] V.G. Turaev, *Quantum Invariants of Knots and 3-Manifolds*, de Gruyter, 1994. 1, 31, 62

[Ul] K.H. Ulbrich, *On Hopf algebras and rigid monoidal categories*, Israel J. Math. **72** (1990), 252–256. 2, 43

[Vi] A. Virelizier, *Kirby elements and quantum invariants*, Proc. London Math. Soc. **93** (2006) 474–514 [math/0312337 [math.GT]]. 71

[Zh] Y.-C. Zhu, *Modular invariance of vertex operator algebras*, J. Amer. Math. Soc. **9** (1996) 237–302. 39, 40

Abstract

This thesis is concerned with "N pairs of symplectic fermions" which are examples of logarithmic conformal field theories in two dimensions. The mathematical language of two-dimensional conformal field theories (on Riemannian surfaces of genus zero) are vertex operator algebras. The representation category of the even part of the symplectic fermion vertex operator super-algebra $\mathbf{Rep}\,\mathcal{V}_{\mathrm{ev}}$ is conjecturally a factorisable finite ribbon tensor category [**DR1, Ru**].

This determines an isomorphism of projective representations between two $SL(2,\mathbb{Z})$-actions associated to $\mathcal{V}_{\mathrm{ev}}$. The first action is obtained by modular transformations on the space of so-called pseudo-trace functions of a vertex operator algebra [**Mi, AN**]. For $\mathcal{V}_{\mathrm{ev}}$ this was developed in [**GR2**]. For the second action one uses that $\mathbf{Rep}\,\mathcal{V}_{\mathrm{ev}}$ is conjecturally a factorisable finite ribbon tensor category and thus carries a projective $SL(2,\mathbb{Z})$-action on a certain Hom-space [**Ly1, Ly2, KL**].

To do so we calculate the $SL(2,\mathbb{Z})$-action on the representation category of a general factorisable quasi-Hopf algebras. Then we show that $\mathbf{Rep}\,\mathcal{V}_{\mathrm{ev}}$ is conjecturally ribbon equivalent to $\mathbf{Rep}\,\mathsf{Q}$, for Q a factorisable quasi-Hopf algebra, and calculate the $SL(2,\mathbb{Z})$-action explicitly on $\mathbf{Rep}\,\mathsf{Q}$.

The result is that the two $SL(2,\mathbb{Z})$-action indeed agree. This poses the first example of such comparison for logarithmic conformal field theories.

Zusammenfassung

In dieser Arbeit befassen wir uns mit “N Paare symplektischer Fermionen”. Dies sind Beispiele für logarithmische konformale Feldtheorien. Die mathematische Sprache von zweidimensionalen konformalen Feldtheorien (auf Riemannflächen mit Geschlecht null) sind Vertex Operatoralgebren. Die Darstellungskategorie des geraden Teils der symplektischen Fermionen Vertex Operatorsuperalgebra $\mathbf{Rep}\,\mathcal{V}_{\mathrm{ev}}$ ist vermutlich eine faktorisierbare, endliche Ribbontensorkategorie [**DR1, Ru**].

Hierdurch wird ein Isomorphismus von projektiven Darstellungen zwischen zwei $SL(2,\mathbb{Z})$-Wirkungen bestimmt, die man mit $\mathcal{V}_{\mathrm{ev}}$ assoziieren kann. Die erste Wirkung erhält man durch modulare Transformation auf dem Raum der sogenannten Pseudo-Spurfunktionen einer Vertex Operatoralgebra [**Mi, AN**]. Diese Wirkung wurde für $\mathcal{V}_{\mathrm{ev}}$ in [**GR2**] berechnet. Für die zweite Wirkung nutzt man, dass $\mathbf{Rep}\,\mathcal{V}_{\mathrm{ev}}$ vermutlich eine faktorisierbare, endliche Ribbontensorkategorie ist und daher eine projektive $SL(2,\mathbb{Z})$-Wirkung auf einen bestimmen Hom-Raum trägt [**Ly1, Ly2, KL**].

Hierfür bestimmen wir die $SL(2,\mathbb{Z})$-Wirkung auf der Darstellungskategorie von beliebigen faktorisierbaren Quasi-Hopfalgebren. Danach zeigen wir, dass unter der oben stehenden Vermutung, $\mathbf{Rep}\,\mathcal{V}_{\mathrm{ev}}$ als Ribbontensorkategorie äquivalent zu $\mathbf{Rep}\,\mathsf{Q}$ ist, wobei Q eine faktorisierbare Quasi-Hopfalgebra ist. Schließlich bestimmen wir exziplit die $SL(2,\mathbb{Z})$-Wirkung auf $\mathbf{Rep}\,\mathsf{Q}$.

Es zeigte sich, dass beide $SL(2,\mathbb{Z})$-Wirkungen in der Tat übereinstimmen. Dies ist das erste Beispiel eines Vergleichs dieser Art für konformale Feldtheorien.