A study of infinite graphs of a certain symmetry and their ends

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Chapter 0

Overview and structure

The topic of this thesis is to study infinite graphs which have some sort of symmetry. Most of the graphs studied in this thesis are Cayley graphs of groups, quasi-transitive or transitive graphs. The main goal of this thesis is to use the symmetry of those graphs to extend known results of finite graphs to infinite graphs. We hereby focus mostly on two-ended graphs.

This thesis consists of five major parts: The first part consists of Chapter 1 and Chapter 2. Chapter 1 gives insight into the studied problems, their history and our results. Chapter 2 presents most of the general definitions and notations we use. It is split into three sections. Section 2.1 recalls the definitions and notations related to the topology used in this thesis. Section 2.2 is used to remind the reader of the most important definitions and notations used for graphs. The final section of Chapter 2, Section 2.3, displays the commonly used group theoretic notations. We will already use those notations in Chapter 1.

The second major part of this thesis is Chapter 3, which studies Hamilton circles of two-ended Cayley graphs. We expand our studies of Hamilton circles in Cayley graphs in Chapter 4 in which we extend a variety of known Hamiltonicity results of finite Cayley graphs to infinite Cayley graphs.

Chapter 5 makes up the third major part of this thesis. As Chapter 3 and Chapter 4 have mostly studied two-ended groups and their Cayley graphs we expand our knowledge about two-ended groups further. We collect and prove characterizations of two-ended groups, their Cayley graphs and even two-ended transitive graphs which need not be Cayley graphs of any group.

The last major part of this thesis is Chapter 6 in which we show that for transitive graphs there exists a way of splitting those graphs in manner similar to Stallings theorem.

Chapter 1

Introduction and motivation

In 1959 Elvira Rapaport Strasser [59] proposed the problem of studying the existence of Hamilton cycles in Cayley graphs for the first time. In fact the motivation of finding Hamilton cycles in Cayley graphs comes from the "bell ringing" and the "chess problem of the knight". Later, in 1969, Lovász [2] extended this problem from Cayley graphs to vertex-transitive graphs. He conjectured that every finite connected transitive graph contains a Hamilton cycle except only five known counterexamples, see [2].

As the Lovász conjecture is still open, one might instead try to solve the, possibly easier, Lovász conjecture for finite Cayley graphs which states: Every finite Cayley graph with at least three vertices contains a Hamilton cycle. Doing so enables the use of group theoretic tools. Moreover, one can ask for what generating sets a particular group contains a Hamilton cycle. There are a vast number of papers regarding the study of Hamilton cycles in finite Cayley graphs, see [18, 23, 42, 73, 74] and for a survey of the field see [75].

We focus on Hamilton cycles in infinite Cayley graphs in Chapter 3 and Chapter 4. As cycles are always finite, we need a generalization of Hamilton cycles for infinite graphs. We follow the topological approach of Diestel and Kühn [14, 15, 17], which extends the notion of a Hamilton cycle in a sensible way by using the circles in the Freudenthal compactification $|\Gamma|$ of a locally finite graph Γ as "infinite cycles", also see Section 2.1. There are already results on Hamilton circles in general infinite locally finite graphs, see [28, 32, 35, 36].

It is worth remarking that the weaker version of the Lovasz's conjecture does not hold for infinite groups. For example, it is straightforward to check that the Cayley graph of any free group with the standard generating set does not contain Hamilton circles, as it is a tree.

It is a known fact that every locally finite graph needs to be 1-tough to contain a Hamilton circle, see [28]. Futherhmore, Geogakopoulos [28] showed that the weak Lovász's conjecture cannot hold for infinite groups which can be written as a free product with amalgamation of more than k groups over a finite subgroup of order k. Geogakopoulos also proposed the following problem:

Problem 1. [28, Problem 2] *Let Γ be a connected Cayley graph of a finitely generated group. Then Γ has a Hamilton circle unless there is a $k \in \mathbb{N}$ such that the Cayley graph of Γ is the amalgamated product of more than k groups over a subgroup of order k.*

In Section 3.2.1 we give a counterexample to Problem 1. Hamann conjectured that the weak Lovász's conjecture for infinite groups holds for infinite groups with at most two ends except when the Cayley graph is the double ray.

Conjecture. [33] *Any Cayley graph of a finitely generated group with at most two ends is Hamiltonian except the double ray.*

Stallings [67] showed in 1971 that finitely generated groups with more than one end split over a finite subgroup. We show that there is a way of splitting transitive graphs, not necessarily Cayley graphs, with more than one end over some finite subgraphs. This is possible by using nested separation systems. Nested separation systems have been of great use in recent time. Carmesin, Diestel, Hundertmark and Stein used nested separation systems in finite graphs to show that every connected graph has a tree-decomposition which distinguishes all its k-blocks [10]. Addtionally, Carmesin, Diestel, Hamann and Hundertmark showed that every connected graph even has canonical tree-decomposition distinguishing its k-profiles [8, 9]. With the help of the tree amalgamation defined by Mohar in 2006 [49] we are now able to extend theorem of Stallings to locally finite transitive graphs, and furthermore even to quasi-transitive graphs, see Section 2.2 for the definitions.

Chapter 2

Definitions and notations

In this chapter we recall the definitions and notations used in this thesis. Our notation and the terminologies of group theory and topology and graph theory follows [62], [55] and [14], respectively. Please note the following: As Chapter 3, Chapter 4 and Chapter 5 are mostly group based G will be reserved for groups in those chapters. In those chapters Γ will be reserved for graphs. As Chapter 6 is more strongly related to graph theory, this is reversed for Chapter 6. In Chapter 6 G will denote graphs and not groups. In Chapter 6 we will denote groups, mostly groups acting on graphs, by Γ. As the majority of this thesis is written such that G is a group and Γ is a graph, this is also true for Chapter 2.

2.1 On topology

A brief history

End theory plays a very crucial role in topology, graph theory and group theory, see the work of Hughes, Ranicki, Möller and Wall [38, 50, 51, 71]. In 1931 Freudenthal [25] defined the concept of ends for topological spaces and topological groups for the first time. Let X be a locally compact Hausdorff space. In order to define ends of the topological space X, he looked at infinite sequence $U_1 \supseteq U_2 \supseteq \cdots$ of non-empty connected open subsets of X

such that the boundary of each U_i is compact and such that $\bigcap \overline{U_i} = \emptyset$.[1] He called two sequences $U_1 \supseteq U_2 \supseteq \cdots$ and $V_1 \supseteq V_2 \supseteq \cdots$ to be equivalent, if for every $i \in \mathbb{N}$, there are $j, k \in \mathbb{N}$ in such a way that $U_i \supseteq V_j$ and $V_i \supseteq U_k$. The equivalence classes of those sequences are the ends of X. The ends of groups arose from ends of topological spaces in the work of Hopf [37]. Halin [30], in 1964, defined vertex-ends for infinite graphs independently as equivalence classes of rays, one way infinite paths. Diestel and Kühn [17] showed that if we consider locally finite graphs as one dimensional simplicial complexes, then these two concepts coincide. Dunwoody [20] showed that in an analogous way, we can define the number of vertex-ends for a given finitely generated group G as the number of ends of a Cayley graph of G. By a result of Meier we know that this is indeed well-defined as the number of ends of two Cayley graphs of the same group are equal, as long the generating sets are finite, see [45]. Freudenthal [26] and Hopf [37] proved that the number of ends for infinite groups G is either 1,2 or ∞. This is exactly one more than the dimension of the first cohomology group of G with coefficients in $\mathbb{Z}G$. Subsequently Diestel, Jung and Möller [16] extended the above result to arbitrary (not necessarily locally finite) transitive graphs. They proved that the number of ends of an infinite arbitrary transitive graph is either 1,2 or ∞. In 1943 Hopf [37] characterized two-ended finitely generated groups. Then Stallings [67, 66] characterized all finitely generated groups with more than one end. Later, Scott and Wall [61] gave another characterization of two-ended finitely generated groups. Cohen [12] studied groups of cohomological dimension one and their connection to two-ended groups. Afterwards Dunwoody in [21] generalized his result. In [41] Jung and Watkins studied groups acting on two-ended transitive graphs.

The definition

Let X be a locally compact Hausdorff space X. Consider an infinite sequence $U_1 \supseteq U_2 \supseteq \cdots$ of non-empty connected open subsets of X such that the boundary of each U_i is compact and $\bigcap \overline{U_i} = \emptyset$. Two such se-

[1] In Section 2.1 $\overline{U_i}$ defines the closure of U_i.

quences $U_1 \supseteq U_2 \supseteq \cdots$ and $V_1 \supseteq V_2 \supseteq \cdots$ are *equivalent* if for every $i \in \mathbb{N}$, there are $j, k \in \mathbb{N}$ in such a way that $U_i \supseteq V_j$ and $V_i \supseteq U_k$. The equivalence classes[2] of those sequences are *topological ends* of X. The *Freudenthal compactification* of the space X is the set of ends of X together with X. A neighborhood of an end $[U_i]$ is an open set V such that $V \supsetneq U_n$ for some n. We denote the Freudenthal compactification of the topological space X by $|X|$.

We use the following application of the Freudenthal compactification. For that we have to anticipate two-definitions from Section 2.2. A *ray* in a graph, is a one-way infinite path. The subrays of a ray are it's *tails*. We say two rays R_1 and R_2 of a given graph Γ are equivalent if for every finite set of vertices S of Γ there is a component of $\Gamma \setminus S$ which contains both a tail of R_1 and of R_2. The classes of the equivalent rays is called *vertex-ends* and just for abbreviation we say *ends*. If considering the locally finite graph Γ as a one dimensional complex and endowing it with the one complex topology then the topological ends of Γ coincide with the vertex-ends of Γ. For a graph Γ we denote the Freudenthal compactification of Γ by $|\Gamma|$. The ends of a graph Γ are denoted by $\Omega(\Gamma)$.

A homeomorphic image of $[0, 1]$ in the topological space $|\Gamma|$ is called *arc*. A *Hamilton arc* in Γ is an arc including all vertices of Γ. By a *Hamilton circle* in Γ, we mean a homeomorphic image of the unit circle in $|\Gamma|$ containing all vertices of Γ. Note that Hamilton arcs and circles in a graph always contain all ends of the graph. A Hamilton arc whose image in a graph is connected, is a *Hamilton double ray*. It is worth mentioning that an uncountable graph cannot contain a Hamilton circle. To illustrate, let C be a Hamilton circle of graph Γ. Since C is homeomorphic to S^1, we can assign to every edge of C a rational number. Thus we can conclude that $V(C)$ is countable and hence Γ is also countable.

[2] We denote the equivalence class of U_i by $[U_i]$.

2.2 On graphs

Let Γ be a graph with vertex set V and edge set E. For a set $X \subseteq V$ we set $\Gamma[X]$ to be the induced subgraph of Γ on X. The neighbourhood of a set of vertices X of a graph Γ are all vertices in $V \setminus X$ which are adjacent to X, we denote this set by $N(X)$. The set of edges between X and $N(X)$ is denoted by $\delta(X)$ and we call it the *co-boundary* of X. For a graph Γ let the induced subgraph on the vertex set X be called $\Gamma[X]$. A path between two vertices is called *geodesic* if it is a shortest path between them.

Let $P\Gamma$ ($F\Gamma$) be the set of all subsets (finite subsets) of V. Furthermore we set $Q\Gamma = \{A \in P\Gamma \mid |\delta(A)| < \infty\}$. It is worth mentioning that $P\Gamma$ can be regarded as a $\mathbb{Z}_2$-vector space with the symmetric difference and so we are able to talk about the dimension of $Q\Gamma/F\Gamma$.

A *ray* is a one-way infinite path in a graph, the infinite sub-paths of a ray are its *tails*. An *end* of a graph is an equivalence class of rays in which two rays are equivalent if and only if there exists no finite vertex set S such that after deleting S those rays have tails completely contained in different components. We say an end ω *lives* in a component C of $\Gamma \setminus X$, where X is a subset of $V(\Gamma)$ or a subset of $E(\Gamma)$, when a ray of ω has a tail completely contained in C, and we denote C by $C(X, \omega)$. We say a component of a graph is *big* if there is an end which lives in that component. Components which are not big are called *small*. A slightly weaker version of ends living in a vertex set is the following: An end ω is *captured* by a set of vertices X is every ray of ω has infinite intersection with X. An end ω of a graph Γ is *dominated by a vertex* v if there is no finite set S of vertices $S \setminus v$ such that $v \notin C(S, \omega) \cup S$. Note that this implies that v has infinite degree. An end is *dominated* if there exists a vertex dominating it. A sequence of vertex sets $(F_i)_{i \in \mathbb{N}}$ is a *defining sequence* of an end ω if $C_{i+1} \subsetneq C_i$, with $C_i := C(F_i, \omega)$ and $\bigcap C_i = \emptyset$. We define the *degree of an end* ω as the supremum over the number of edge-disjoint rays belonging to the class which corresponds to ω, see the work of Bruhn and Stein [7]. If an end does not have a finite degree we say that this end has infinite vertex degree and call such an end a *thick end*. Analogously, an end with finite vertex degree is a *thin end*. If a graph only has thin ends, then this graph is *thin*.

A graph is called *Hamiltonian* if it contains either a Hamilton cycle or its closure in the Freudenthal compactification contains a Hamilton circle. In slight abuse of notation we omit the closure when talking about a graph containing a Hamilton circle.

Thomassen [68] defined a Hamilton cover of a finite graph Γ to be a collection of mutually disjoint paths $P_1, \ldots, P_m$ such that each vertex of Γ is contained in exactly one of the paths. For easier distinction we call this a *finite Hamilton cover*. An *infinite Hamilton cover* of an infinite graph Γ is a collection of mutually disjoint *double rays*, two way infinite paths, such that each vertex of Γ is contained in exactly one of them. The *order* of an infinite Hamilton cover is the number of disjoint double rays in it.

A locally finite quasi-transitive graph[3] is *accessible* if and only if there exists a natural number k such that every pair of two ends of that graph can be separated by at most k edges. Note that for graphs with bounded maximal degree the definition of accessibility is equivalent to the following: A graph of bounded maximal degree is accessible if and only if there exists a natural number k' such that every pair of two ends of that graph can be separated by at most k' vertices. As the maximum degree in a locally finite quasi-transitive graphs is bounded, we may use "vertex accessibility" for those graphs.

Cuts and separations

A finite set $C = E(A, A^*) \subseteq E$ is a *finite cut* if (A, A^*) is a partition of the vertex set and if $|E(A, A^*)|$ is finite. We say a cut $C = E(A, A^*)$ is induced by the partition (A, A^*). We denote the set of all finite cuts by $\mathcal{B}_{\text{fin}}(\Gamma)$. A finite cut $E(A, A^*)$ is called *k-tight* if $|E(A, A^*)| = k$ and if moreover $G[A]$ and $G[A^*]$ are connected. We note that $\mathcal{B}_{\text{fin}}(\Gamma)$ with the symmetric difference forms a vector space over $\mathbb{Z}_2$. We note that if $C = E(A, A^*)$ is a cut, then the partition (gA, gA^*) induces a cut for every $g \in \mathsf{Aut}(\Gamma)$. For the sake of simplicity we denote this new cut only by gC.

[3]See Section 2.3 for the definition of quasi-transitive graphs.

In the following we give an ordering on $\mathcal{B}_{\text{fin}}(\Gamma)$ to make it a poset. Suppose that $C_1 = E(A, A^*)$ and $C_2 = E(B, B^*)$ are two finite cuts. Then $C_1 \leq C_2$ if and only if $A \subseteq B$ and $A^* \supseteq B^*$ or $A \subseteq B^*$ and $A^* \supseteq B$. Two cuts are called *comparable* if $C_1 \leq C_2$ or $C_2 \leq C_1$. Dunwoody [22] proved that if a graph Γ has at least two ends, then there exists a cut $C \in \mathcal{B}_{\text{fin}}(\Gamma)$ such that C and gC are comparable for every $g \in \mathsf{Aut}(\Gamma)$. As a consequence of the above mentioned result he characterized all groups acting on those graphs.

A concept similar to cuts is the concept of separations. Let Γ be a graph. A *separation* of Γ is an ordered pair (A, A^*) with $A, A^* \subseteq V(\Gamma)$ such that $\Gamma = \Gamma[A] \cup \Gamma[A^*]$.[4] For a separation (A, A^*) we call $A \cap A^*$ the *separator* of this separation. A k-separation of Γ is a separation (A, A^*) such that the size of $A \cap A^*$ is k. We call a separation (A, A^*) *tight* if there exists a component of $\Gamma \setminus (A \cap A^*)$ such that each vertex of $A \cap A^*$ has a neighbor in that component. A separation (A, A^*) is *splitting separation* if it separates ends, i.e there are ends ω and ω' such that ω lives in $\Gamma[A \setminus A^*]$ and such that ω' lives in $\Gamma[A^* \setminus A]$.

We define a partial order $\leq$ on the set of all separations of Γ. For two separations (A, A^*) and (B, B^*) let $(A, A^*) \leq (B, B^*)$ if and only if $A \subseteq B$ and $A^* \supseteq B^*$. Two separations (A, A^*) and (B, B^*) are *nested* if one of the following is true:

$$(A, A^*) \leq (B, B^*), (A, A^*) \leq (B^*, B), (A^*, A) \leq (B, B^*), (A^*, A) \leq (B^*, B).$$

We denote this by $(A, A^*) \parallel (B, B^*)$. Otherwise we say that the separations (A, A^*) and (B, B^*) are *crossing.* We denote crossing separations by $(A, A^*) \nparallel (B, B^*)$. A set $\mathcal{O}$ of separations is called *nested* if each pair of elements of $\mathcal{O}$ are comparable. For two separations (A, A^*) and (B, B^*) we call the sets

$$A \cap B, A \cap B^*, A^* \cap B \text{ and } A^* \cap B^*$$

the *corners* of these separations. Corners give rise to four possible *corner*

[4]This implies that there is no edge from $A \setminus A^*$ to $A^* \setminus A$ in Γ.

separations which consist of a "corner vs. the rest", i.e.:

$$(A \cap B, A^* \cup B^*), (A \cap B^*, A^* \cup B), (A^* \cap B, A \cup B^*) \text{ and } (A^* \cap B^*, A \cup B).$$

The corners $A \cap B$ and $A^* \cap B^*$ are *opposite*, as are the corners $A \cap B^*$ and $A^* \cap B$.

A set $\mathcal{O}$ of separations is *symmetric* if for every separation $(A, A^*) \in \mathcal{O}$, the separation (A^*, A) also is in $\mathcal{O}$.

The *order* of a separation is the size of its separator. In this thesis we only consider separations of finite order, thus from here on, any separation will always be a separation of finite order.

For two-ended graphs we strengthen the definition of tight separations. Let $k \in \mathbb{N}$ and let Γ be a two-ended graph with a separation (A, A^*). We call (A, A^*) *k-tight* if the following holds:

1. $|A \cap A^*| = k$.
2. There is an end ω_A living in a component C_A of $A \setminus A^*$.
3. There is an end ω_{A^*} living in a component C_A^* of $A^* \setminus A$.
4. Each vertex in $A \cap A^*$ is adjacent to vertices in both C_A and C_{A^*}.

If a separation (A, A^*) of a two-ended graph is k-tight for some k, then this separation is just called *tight*. We use this stronger definition of tight or k-tight separations only in Chapter 5. Note that finding tight separations is always possible for two-ended graphs. In an analogous matter to finite cuts, one may see that (gA, gA^*) is a tight separation for $g \in \mathsf{Aut}(\Gamma)$ whenever (A, A^*) is a tight separation. Note that this is true for both definitions of tight.

A separation (A, A^*) is *connected* if $\Gamma(A \cap A^*)$ is connected. See the work of Carmesin, Diestel, Hundertmark and Stein [10] for applications and results on separations.

Tree-decomposition

A *tree-decomposition* of a graph Γ is a pair $(T, \mathcal{V})$ such that T is a tree and such that $\mathcal{V} = (V_t)_{t \in V(T)}$ is a family of vertex sets of Γ with the additional following conditions:

(T1) $V(\Gamma) = \bigcup_{t \in V(T)} V_t$.

(T2) For every edge $e = xy$ of Γ there is a $t \in V(T)$ such that $x \in V_t$ and $y \in V_t$.

(T3) $V_{t_1} \cap V_{t_2} \subseteq V_{t_3}$ whenever t_3 lies on the path in T between t_1 and t_2.

The sets V_t are also called parts of a tree-decomposition. The vertices of a tree T in a tree-decomposition will be called nodes. Please note that if $e = t_1 t_2$ is an edge of a tree T of a tree-decomposition then $V_{t_1} \cap V_{t_2}$ is a separator of G unless $V_{t_1} \cap V_{t_2} = V_{t_i}$ for $i \in \{1, 2\}$. We also call all the sets of the form $V_{t_1} \cap V_{t_2}$ the *adhesion sets* of the tree-decomposition.

A tree-decomposition $(T, \mathcal{V})$ of finite adhesion distinguishes two ends ω_1 and ω_2 if there is an adhesion set $V_{t_1} \cap V_{t_2}$ such that ω_1 lives in a different components of $\Gamma \setminus (V_{t_1} \cap V_{t_2})$ than ω_2.

Tree amalgamation

Next we recall the defitinition of the *tree amalgamation* for graphs which was first defined by Mohar in [49]. We use the tree amalgamation to obtain a generalization of factoring quasi-transitive graphs in a similar manner to the HNN-extensions or free-products with amalgamation over finite groups.[5]

For that let us recall the definition of a semiregular tree. A tree T is (p_1, p_2)-*semiregular* if there exist $p_1, p_2 \in \{1, 2, \ldots\} \cup \infty$ such that for the canonical bipartition $\{V_1, V_2\}$ of $V(T)$ the vertices in V_i all have degree p_i for $i = 1, 2$.

In the following let T be the (p_1, p_2)-semiregular tree. Suppose that there is a mapping c which assigns to each edge of T a pair

[5]See Section 2.3 for details about the HNN-extension or the free-product with amalgamation.

$$(k, \ell),\ 0 \le k < p_1,\ 0 \le \ell < p_2,$$

such that for every vertex $v \in V_1$, all the first coordinates of the pairs in $\{c(e) \mid v \text{ is incident with } e\}$ are distinct and take all values in the set $\{k \mid 0 \le k < p_1\}$, and for every vertex in V_2, all the second coordiantes are distinct and exhaust all values of the set $\{\ell \mid 0 \le \ell < p_2\}$.

Let Γ_1 and Γ_2 be graphs. Suppose that $\{S_k \mid 0 \le k < p_1\}$ is a family of subsets of $V(\Gamma_1)$, and $\{T_\ell \mid 0 \le \ell < p_2\}$ is a family of subsets of $V(\Gamma_2)$. We shall assume that all sets S_k and T_ℓ have the same cardinality, and we let $\phi_{k\ell}\colon S_k \to T_\ell$ be a bijection. The maps $\phi_{k\ell}$ are called *identifying maps*.

For each vertex $v \in V_i$, take a copy Γ_i^v of the graph $\Gamma_i, i = 1, 2$. Denote by S_k^v (if $i = 1$) and T_ℓ^v (if $i = 2$) the corresponding copies of S_k or T_ℓ in $V(\Gamma_i^v)$. Let us take the disjoint union of graphs $\Gamma_i^v, v \in V_i, i = 1, 2$. For every edge $st \in E(T)$, with $s \in V_1, t \in V_2$ and such $c(st) = (k, \ell)$ we identify each vertex $x \in S_k^s$ with the vertex $y = \phi_{k\ell}(x)$ in T_ℓ^t. The resulting graph Y is called the *tree amalgamation* of the graphs Γ_1 and Γ_2 over the *connecting tree* T. We denote Y by $\Gamma_1 *_T \Gamma_2$. In the context of tree amalgamations the sets $\{S_k \mid 0 \le k < p_1\}$ and $\{T_\ell \mid 0 \le \ell < p_2\}$ are also called *the sets of adhesion sets* and a single S_k or T_ℓ might be called an *adhesion set* of this tree amalgamation. In particular the set $\{S_k\}$ is said to be the set of adhesion sets of Γ_1 and $\{T_\ell\}$ to be the set of adhesion sets of Γ_2. In the case that $\Gamma_1 = \Gamma_2$ and that $\phi_{k\ell}$ is the identity for all k and ℓ we may say that $\{S_k\}$ is the set of adhesion sets of this tree amalgamation. If the adhesion sets of a tree amalgamation are finite, then this tree amalgamation is *thin*.

Alternative notations for graphs

As this thesis considers Cayley graphs on several occasions it is very useful to be able to consider edges as labeled by the corresponding generators. For that we use the following notation originally used by [42, 75].

In addition to the notation of paths and cycles as sequences of vertices such that there are edges between successive vertices we use the following notation: For that let g and $s_i,\ i \in \mathbb{Z}$, be elements of some group and $k \in \mathbb{N}$.

In this notation $g[s_1]^k$ denotes the concatenation of k copies of s_1 from the right starting from g which translates to the path $g, (gs_1), \ldots, (gs_1^k)$ in the usual notation. Analogously $[s_1]^k g$ denotes the concatenation of k copies of s_1 starting again from g from the left. We use $g[s_1, \ldots, s_n]^k$ to denote the following path

$$g, g(s_1), \ldots, g(s_1 \cdots s_n), g(s_1 \cdots s_n)s_1, \ldots, g(s_1 \cdots s_n)^2, \ldots, g(s_1 \cdots s_n)^k$$

In addition $g[s_1, s_2, \ldots]$ translates to be the ray $g, (gs_1), (gs_1s_2), \ldots$ and

$$[\ldots, s_{-2}, s_{-1}]g[s_1, s_2, \ldots]$$

translates to be the double ray

$$\ldots, (gs_{-1}s_{-2}), (gs_{-1}), g, (gs_1), (gs_1s_2), \ldots$$

When discussing rays we extend the notation of $g[s_1, \ldots, s_n]^k$ to k being countably infinite and write $g[s_1, \ldots, s_2]^{\mathbb{N}}$ and the analogue for double rays. By

$$g[s_1]^{k_1}[s_2]^{k_2} \cdots$$

we mean the ray

$$g, gs_1, gs_1^2, \ldots, gs_1^{k_1}, gs_1^{k_1}s_2, \ldots, gs_1^{k_1}s_2^{k_2}, \ldots$$

and analogously

$$\cdots [s_1]^{k_{-1}} g[s_1]^{k_1} \cdots$$

defines the double ray

$$\ldots, gs_{-1}^{k_{-1}}, \ldots, gs_{-1}, g, gs_1, gs_1^2, \ldots, gs_1^{k_1}, \ldots$$

Sometimes we will use this notation also for cycles. Stating that $g[c_1, \ldots, c_k]$ is a cycle means that $g[c_1, \ldots, c_{k-1}]$ is a path and that the edge c_k joins the vertices $gc_1 \cdots c_{k-1}$ and g.

2.3 On groups

As we only consider groups with locally finite Cayley graphs in this thesis, we assume that all generating sets are finite.

For a subset A of a set X we denote the complement of A by A^c. We denote the disjoint union of two sets A and B by $A \sqcup B$.

Let $G = \langle S \rangle$. The Cayley graph associated with (G, S) is a graph having one vertex associated with each element of G and edges (g_1, g_2) whenever $g_1 g_2^{-1}$ lies in S. For a set $T \subseteq G$ we set $T^{\pm} := T \cup T^{-1}$. Throughout this thesis we assume that all generating sets are symmetric, i.e. whenever $s \in S$ then $s^{-1} \in S$. Thus if we add an element s to a generating set S, we always also add the inverse of s to S as well.

We denote the Cayley graph of G with respect to S with $\Gamma(G, S)$. A finite group G is a *p-group* if the order of each element of G is a power of p, where p is a prime number. Let A and B be two subsets of G. Then AB denotes the set $\{ab \mid a \in A, b \in B\}$. We use this to also define A^2 as AA. Let $H \leq G$, then for $g \in G$ and $h \in H$ we denote $g^{-1}Hg$ and $g^{-1}hg$ by H^g and h^g, respectively. An important subgroup of H is $\mathsf{Core}(H) := \cap_{g \in G} H^g$ which is always normal in G and moreover if $[G : H] = n$, then the index $\mathsf{Core}(H)$ in G is at most $n!$, see the work of Scott [62, Theorem 3.3.5]. We denote the order of the element g by $o(g)$. We denote the *centralizer* of the element g by $C_G(g) := \{h \in G \mid hg = gh\}$ and the *commutator subgroup* of G by G'. Furthermore, $N_G(H)$, $C_G(H)$ and $Z(G)$ denote the normalizer subgroup of H in G, the centralizer subgroup of H in G and the center of G, respectively. If H is a characteristic subgroup of G, then we write $H \mathsf{char} G$.

Assume that H and K are two groups. Then G is called an *extension* of H by K if there is a short exact sequence:

$$1 \to H \to G \to K \to 1$$

For a group $G = \langle S \rangle$ we define $e(G) := |\Omega(\Gamma(G, S)|$. We note that this definition is independent of the choice of S as

$$|\Omega(\Gamma(G, S))| = |\Omega(\Gamma(G, S'))|$$

as long as S and S' are finite, see the work of Meier [45, Theorem 11.23]. Let H be a normal subgroup of $G = \langle S \rangle$. In Chapters 4 and Chapter 5 we denote the set $\{sH \mid s \in S\}$ by $\overline{S}$. We notice that $\overline{S}$ generates $\overline{G} := G/H$. A subgroup H of G is called *characteristic* if any automorphism ϕ of G maps H to itself and we denote it by $H \mathsf{char} G$.

A *finite dihedral group* is defined with the presentation $\langle a, b \mid b^2, a^n, (ba)^2 \rangle$, where $n \in \mathbb{N}$ and denote the finite dihedral groups by D_{2n}. The infinite dihedral group is a group with the presentation $\langle a, b \mid b^2 = 1, bab = a^{-1} \rangle$ which is denoted by D_∞. It is worth remarking that it is isomorphic to $\mathbb{Z}_2 * \mathbb{Z}_2$.

A group G is called a *planar group* if there exists a generating set S of G such that $\Gamma(G, S)$ is a planar graph.

Suppose that G is an abelian group. A finite set of elements $\{g_i\}_{i=1}^n$ of G is called *linear dependent* if there exist integers λ_i for $i = 1, \ldots, n$, not all zero, such that $\sum_{i=1}^n \lambda_i g_i = 0$. A system of elements that does not have this property is called *linear independent*. It is an easy observation that a set containing elements of finite order is linear dependent. The *rank* of an abelian group is the size of a maximal independent set. This is exactly the rank the torsion free part, i.e if $G = \mathbb{Z}^n \oplus G_0$ then the rank of G is n, where G_0 is the torsion part of G.

Let R be a unitary ring. Then we denote the group ring generated by R and G by RG. In this thesis we only deal with the group rings $\mathbb{Z}_2 G$ and $\mathbb{Z}G$. We denote the group of all homomorphisms from the group ring RG to an abelian group A by $\mathsf{Hom}_{\mathbb{Z}}(RG, A)$.

Free product with amalgamation

Let G_1 and G_2 be two groups with subgroups H_1 and H_2 respectively such that there is an isomorphism $\phi\colon H_1 \to H_2$. The *free product with amalgamation* is defined as

$$G_1 *_{H_1} G_2 := \langle S_1 \cup S_2 \mid R_1 \cup R_2 \cup H_1 \phi^{-1}(H_1) \rangle.$$

A way to present elements of a free product with amalgamation is the Britton's Lemma:

Lemma 2.3.1. [4, Theorem 11.3] *Let G_1 and G_2 be two groups with subgroups $H_1 \cong H_2$ respectively. Let T_i be a left transversal* [6] *of H_i for $i = 1, 2$. Any element $x \in G_1 *_H G_2$ can be uniquely written in the form $x = x_0 x_1 \cdots x_n$ with the following:*

(i) $x_0 \in H_1$.

(ii) $x_j \in T_1 \backslash 1$ *or* $x_i \in T_2 \backslash 1$ *for* $j \geq 1$ *and the consecutive terms* x_j *and* x_{j+1} *lie in distinct transversals.*

This unique form is the normal form *for x.* □

A generating set S of $G_1 *_H G_2$ is called *canonical* if S is a union of S_i for $i = 1, \ldots, 3$ such that $\langle S_i \rangle = G_i$ for $i = 1, 2$ and $H = \langle S_3 \rangle$. We note that when $H = 1$, then we assume that $S_3 = \emptyset$. When we write $G = G_1 *_H G_2$ we always assume that $G_1 \neq 1 \neq G_2$.

HNN-extension

Let $G = \langle S \mid R \rangle$ be a group with subgroups H_1 and H_2 in such a way that there is an isomorphism $\phi \colon H_1 \to H_2$. We now insert a new symbol t not in G and we define the *HNN-extension* of $G *_{H_1}$ as follows:

$$G *_{H_1} := \langle S, t \mid R \cup \{t^{-1} h t \phi(h)^{-1} \mid \text{ for all } h \in H_1\} \rangle.$$

Ends of Cayley graphs

As we are studying the Hamiltonicity of Cayley graphs throughout this thesis, it will be important to pay attention to the generating sets involved, see Chapter 3 and Chapter 4. Throughout this thesis, whenever we discuss Cayley graphs we assume that any generating set $S = \{s_1, \ldots, s_n\}$ is *minimal* in the following sense: Each $s_i \in S$ cannot be generated by $S \setminus \{s_i\}$, i.e. we have that $s_i \notin \langle s_j \rangle_{j \in \{1,\ldots,n\} \setminus \{i\}}$. We may do so because say $S' \subseteq S$ is a minimal generating set of G. If we can find a Hamilton circle C in $\Gamma(G, S')$, then this

[6] A *transversal* is a system of representatives of left cosets of H_i in G_i and we always assume that 1 belongs to it.

circle C will still be a Hamilton circle in $\Gamma(G, S)$. For this it is important to note that the number of ends of G and thus of $\Gamma(G, S')$ does not change with changing the generating set to S by [45, Theorem 11.23], as long as S is finite, which will always be true in this thesis.

We now cite a structure for finitely generated groups with two ends.

Theorem 2.3.2. [61, Theorem 5.12] *Let G be a finitely generated group. Then the following statements are equivalent.*

(i) *The number of ends of G is 2.*

(ii) *G has an infinite cyclic subgroup of finite index.*

(iii) *$G = A *_C B$ and C is finite and $[A : C] = [B : C] = 2$ or $G = C *_C$ with C is finite.* □

Throughout this thesis we use Theorem 2.3.2 to characterize the structure of two-ended groups, see Section 3.1 for more details.

To illustrate that considering different generating sets can make a huge difference let us consider the following two examples. Take two copies of $\mathbb{Z}_2$, with generating sets $\{a\}$ and $\{b\}$, respectively. Now consider the free product of them. It is obvious that this Cayley graph with generating set $\{a, b\}$ does not contain a Hamilton circle, see Figure 2.1. Again consider $\mathbb{Z}_2 * \mathbb{Z}_2$ with generating set $\{a, ab\}$ which is isomorphic to $D_\infty = \langle x, y \mid x^2 = (xy)^2 = 1\rangle$. It is easy to see that the Cayley graph of D_∞ with this generating set contains a Hamilton circle, see Figure 2.2.

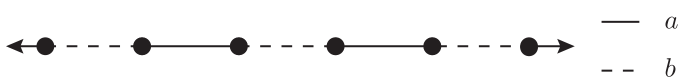

Figure 2.1: The Cayley graph of $\mathbb{Z}_2 * \mathbb{Z}_2$ with the generating set $\{a, b\}$ which does not contain a Hamilton circle.

The action of groups

A group G *acts on a set* X if there exists a function $f : G \times X \to X$ with $f(g, x) := gx$ such that the following is true:

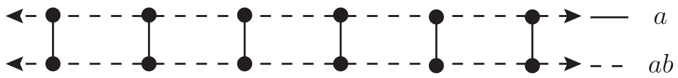

Figure 2.2: The Cayley graph of $\mathbb{Z}_2 * \mathbb{Z}_2$ with the generating set $\{a, ab\}$ in which the dashed edges form a Hamilton circle.

(i) $g_1(g_2x) = (g_1g_2)x$,

(ii) $1x = x$.

The action of a group G on a set X is called *trivial* if $gx = x$ for all $g \in G$ and all $x \in X$. In this thesis we assume that no action we consider is the trivial action.

Let a group G act on a set X. For every element of $x \in X$ we denote the orbit containing x by Gx. The *quotient set* $G \setminus X$ is the set of all orbits. In particular whenever we consider the automorphism group G of a graph Γ, the *quotient graph* $G \setminus \Gamma$ is a graph with the vertices $\{v_i\}_{i \in I} \subseteq V(\Gamma)$ such that v_i's are the representatives of the orbits, and the vertices v_i and v_j are adjacent if and only if there are $h_1, h_2 \in G$ such that h_1v_i is adjacent to h_2v_j. Now let Y be a subset of X. Then we define the *set-wise stabilizer* of Y with respect to G as

$$G_Y := \{h \in G \mid hy \in Y, \forall y \in Y\}.$$

If G acts on X with finitely many orbits, i.e. $G \setminus X$ is finite, then we say the action is *quasi-transitive*. A graph Γ is called *transitive* if $\mathsf{Aut}(\Gamma)$ acts transitively and if the action of $\mathsf{Aut}(\Gamma)$ on the set of vertices of Γ has only finitely many orbits, then we say Γ is *quasi-transitive*.

One of the strongest tools in studying groups acting on graphs is the Bass-Serre Theory. This theory enables us to characterize groups acting on trees in terms of fundamental groups of graphs of groups.

Lemma 2.3.3. [64] *Let G act without inversion of edges on a tree that thas no vertices of degree one and suppose G acts transitively on the set of (undirected) edges. If G acts transitively on the tree then G is an HNN-extension of the stabilizer of a vertex over the pointwise stabilizer of an edge. If there are* *two orbits on the vertices of the tree then G is the free product of the stabilizers of two adjacent vertices with amalgamation over the pointwise stabilizer of an edge.* $\square$

Geometric group theory

Let (X, d_X) and (Y, d_Y) be two metric spaces and let $\phi\colon X \to Y$ be a map. The map ϕ is a *quasi-isometric embedding* if there is a constant $\lambda \geq 1$ such that for all $x, x' \in X$:

$$\frac{1}{\lambda} d_X(x, x') - \lambda \leq d_Y(\phi(x), \phi(x')) \leq \lambda d_X(x, x') + \lambda.$$

The map ϕ is called *quasi-dense* if there is a λ such that for every $y \in Y$ there exists $x \in X$ such that $d_Y(\phi(x), y) \leq \lambda$. Finally ϕ is a *quasi-isometry* if it is both quasi-dense and a quasi-isometric embedding. If X is quasi-isometric to Y, then we write $X \sim_{QI} Y$. Remember that $G = \langle S \rangle$ can be equipped by the word metric induced by S. Thus any group can be turned to a topological space by considering its Cayley graph and so we are able to talk about quasi-isometric groups and it would not be ambiguous if we use the notation $G \sim_{QI} H$ for two groups H and G. A result of Meier reveals the connection between Cayley graphs of a group with different generating sets.

Lemma 2.3.4. [45, Theorem 11.37] *Let G be a finitely generated group and let S and S' be two finite generating sets of G. Then $\Gamma(G, S) \sim_{QI} \Gamma(G, S')$.* $\square$

By Lemma 2.3.4 we know that any two Cayley graphs of the same group are quasi-isometric if the corresponding generating sets are finite. Let G be a finitely generated group with generating set S. Let $B(u, n)$ be the ball of radius n around the vertex u of $\Gamma(G, S)$ i.e.:

$$B(u, n) = \{v \in \Gamma(G, S) \mid d(u, v) < n\}.$$

Suppose that $c(n)$ is the number of infinite components of $\Gamma(G, S) \setminus B(u, n)$. It is important to notice that since $\Gamma(G, S)$ is a transitive graph, it does not matter where we pick u up. Thus the definition of $c(n)$ is well-defined. Now we are ready to define the number of ends of G. We set $e(n) = \lim_{n\to\infty} c(n)$. Please note that the number of ends of a group G coincides $|\Omega(\Gamma(G, S)|$ for any finitely generated group as long as S is finite.

Lemma 2.3.5. [5, Corollary 2.3] *Finitely generated quasi-isometric groups all have the same number of ends.* $\square$

Now by Lemma 2.3.5 we can conclude the following Corollary 2.3.6.

Corollary 2.3.6. [45, Theorem 11.23] *The number of ends of a finitely generated group G is independent of the chosen generating set.* $\square$

Lemma 2.3.7. [45, Proposition 11.41] *Let H be a finite-index subgroup of a finitely generated group G. Then $H \sim_{QI} G$.* $\square$

Lemma 2.3.5 and Lemma 2.3.7 together imply the following corollary.

Corollary 2.3.8. *Let G be a finitely generated group with a subgroup H is of finite index, then the numbers of ends of H and G are equal.*

Chapter 3

Hamilton circles in Cayley graphs

3.1 Hamilton circles

In this section we prove sufficient conditions for the existence of Hamilton circles in Cayley graphs. In Section 3.1.1 we take a look at abelian groups. Section 3.1.2 contains basic lemmas and structure theorems used to prove the main results of Chapter 3 which we prove in the Section 3.1.3.

3.1.1 Abelian groups

In the following we will examine abelian groups as a simple starting point for studying Hamilton circles in infinite Cayley graphs. Our main goal in this section is to extend a well-known theorem of Nash-Williams from one-ended abelian groups to two-ended abelian groups by a simple combinatorial argument. First, we cite a known result for finite abelian groups.

Lemma 3.1.1. [65, Corollary 3.2] *Let G be a finite abelian group with at least three elements. Then any Cayley graph of G has a Hamilton cycle.* □

Next we state the theorem of Nash-Williams.

Theorem 3.1.2. [56, Theorem 1] *Let G be a finitely generated abelian group with exactly one end. Then any Cayley graph of G has a Hamilton circle.* □

It is obvious that the maximal class of groups to extend Theorem 3.1.2 to cannot contain $\Gamma(\mathbb{Z}, \{\pm 1\})$, as this it cannot contain a Hamilton circle. In Theorem 3.1.3 we prove that this is the only exception.

Theorem 3.1.3. *Let G be an infinite finitely generated abelian group. Then any Cayley graph of G has a Hamilton circle except $\Gamma(\mathbb{Z}, \{\pm 1\})$.*

Proof. By the fundamental theorem of finitely generated abelian groups [62, 5.4.2], one can see that $G \cong \mathbb{Z}^n \oplus G_0$ where G_0 is the torsion part of G and $n \in \mathbb{N}$. It follows from [61, lemma 5.6] that the number of ends of $\mathbb{Z}^n$ and G are equal. We know that the number of ends of $\mathbb{Z}^n$ is one if $n \geq 2$ and two if $n = 1$. By Theorem 3.1.2 we are done if $n \geq 2$. So we can assume that G has exactly two ends.

Since $\Gamma(\mathbb{Z}, \{\pm 1\})$ is not allowed, we may assume that S contains at least two elements. Now suppose that $S = \{s_1, \ldots, s_k\}$ generates G such that S is minimal in the sense of generating of G. Without loss generality assume that the order of s_1 is infinite. Let i be the smallest natural number such that $s_2^{i+1} \in \langle s_1 \rangle$. Since the rank of G is one, we can conclude that $\{s_1, s_2\}$ are dependent and thus such an i exists. In the following we define a sequence of double rays. We start with the double ray $R_1 = [s_1^{-1}]^{\mathbb{N}} 1 [s_1]^{\mathbb{N}}$. Now we replace every other edge of R_1 by a path to obtain a double ray spanning $\langle s_1, s_2 \rangle$. The edge $1s_1$ will be replaced by the path $[s_2]^i [s_1] [s_2^{-1}]^i$. We obtain the following double ray:

$$R_2 = \cdots [s_2^{-1}]^i [s_1^{-1}] [s_2]^i [s_1^{-1}] 1 [s_2]^i [s_1] [s_2^{-1}]^i [s_1] \cdots$$

Note that R_2 spans $\langle s_1, s_2 \rangle$. We will now repeat this kind of construction for additional generators building double rays R_ℓ such that R_ℓ spans the subgroup generated by the first ℓ generators. For simplicity we denote R_ℓ by

$$[\ldots, y_{-2}, y_{-1}] 1 [y_1, y_2, \ldots]$$

with

$$y_m \in \{s_1, s_2, \ldots, s_\ell\}^{\pm} \text{ for every } m \in \mathbb{Z} \setminus \{0\}.$$

As above let $i \in \mathbb{N}$ be minimal such that $s_{\ell+1}^{i+1} \in \langle s_1, s_2, \ldots, s_j \rangle$. We now define the double ray

$$R_{\ell+1} = \cdots [s_{\ell+1}^{-1}]^i [y_{-2}][s_{\ell+1}]^i [y_{-1}] 1 [s_{\ell+1}]^i [y_1][s_{\ell+1}^{-1}]^i [y_2] \cdots .$$

We now repeat the process until we have defined the double ray R_{k-1}, say

$$R_{k-1} = [\ldots, x_{-2}, x_{-1}] 1 [x_1, x_2, \ldots]$$

with $x_m \in \{s_1, \ldots, s_{k-1}\}^{\pm}$ for every $m \in \mathbb{Z} \setminus \{0\}$. Now let i be the smallest natural number such that $s_k^{i+1} \in \langle s_1, \ldots, s_{k-1} \rangle$. Now, put

$$\mathcal{P}_1 = \cdots [s_k^{-1}]^{i-1} [x_{-2}][s_k]^{i-1} [x_{-1}] 1 [s_k]^{i-1} [x_1][s_k^{-1}]^{i-1} [x_2] \cdots$$

and

$$\mathcal{P}_2 = [\ldots, x_{-2}, x_{-1}] s_k^i [x_1, x_2, \ldots].$$

It is not hard to see that $\mathcal{P}_1 \cup \mathcal{P}_2$ is a Hamilton circle of $\Gamma(G, S)$. $\square$

Remark 3.1.4. *One can prove Theorem* 3.1.2 *by the same arguments used in the above proof of Theorem* 3.1.3.

3.1.2 Structure tools

In this section we assemble all the most basic tools to prove our main results of Chapter 3. Our most important tools are Lemma 3.1.6 and Lemma 3.1.7 which we also use in Chapter 4. In both lemmas we prove that a given graph Γ contains a Hamilton circle if Γ admits a partition of its vertex set fulfilling the following nice properties. All partition classes are finite and of the same size. Each partition class contains some special cycle and between two consecutive partition classes there are edges in Γ connecting those cycles in a useful way, see Lemma 3.1.6 and 3.1.7 for details.

But first we state a well known Lemma about the structure of Hamilton circles in two-ended graphs.

Lemma 3.1.5. [Folklore] *Let $\Gamma = (V, E)$ be a two-ended graph and let R_1*

and R_2 be two doubles rays such that the following holds:

(i) $R_1 \cap R_2 = \emptyset$

(ii) $V = R_1 \cup R_2$

(iii) *For each $\omega \in \Omega(\Gamma)$ both R_i have a tail that belongs to ω.*

Then $R_1 \sqcup R_2$ is a Hamilton circle of Γ. □

Lemma 3.1.6. *Let Γ be a graph that admits a partition of its vertex set into finite sets X_i, $i \in \mathbb{Z}$, fulfilling the following conditions:*

(i) *$\Gamma[X_i]$ contains a Hamilton cycle C_i or $\Gamma[X_i]$ is isomorphic to K_2.*

(ii) *For each $i \in \mathbb{Z}$ there is a perfect matching between X_i and X_{i+1}.*

(iii) *There is a $k \in \mathbb{N}$ such that for all $i, j \in \mathbb{Z}$ with $|i - j| \geq k$ there is no edge in Γ between X_i and X_j.*

Then Γ has a Hamilton circle.

Proof. By (i) we know that each X_i is connected and so we conclude from the structure given by (ii) and (iii) that Γ has exactly two ends. In addition note that $|X_i| = |X_j|$ for all $i, j \in \mathbb{Z}$. First we assume that $\Gamma[X_i]$ is just a K_2. It follows directly that Γ is spanned by the double ladder, which is well-known to contain a Hamilton circle. As this double ladder shares its ends with Γ, this Hamilton circle is also a Hamilton circle of Γ.

Now we assume that $|X_i| \geq 3$. Fix an orientation of each C_i. The goal is to find two disjoint spanning doubles rays in Γ. We first define two disjoint rays belonging to the same end, say for all the X_i with $i \geq 1$. Pick two vertices u_1 and w_1 in X_1. For R_1 we start with u_1 and move along C_1 in the fixed orientation of C_1 till the next vertex on C_1 would be w_1. Then, instead of moving along C_1, we move to X_2 by the given matching edge. We take this to be a the initial part of R_1. We do the analogue for R_2 by starting with w_1 and moving also along C_1 in the fixed orientation till the next vertex would be u_1, then move to X_2. We repeat the process of starting with two vertices u_i and w_i contained in some X_i, where u_i is the first vertex of R_1 on X_i and w_i the analogue for R_2. We follow along the fixed orientation on C_i till the next vertex would be u_i or w_i, respectively. Then we move to X_{i+1} by the giving matching edges. One can easily see that each vertex of X_i for $i \geq 1$ is contained exactly either in R_1 or R_2. By moving from u_1 and w_1 to X_0 by the matching edges and then using the same process but moving from X_i to X_{i-1} extents the rays R_1 and R_2 into two double rays. Obviously those double rays are spanning and disjoint. As Γ has exactly two ends it remains to show that R_1 and R_2 have a tail in each end, see Lemma 3.1.5. By (iii) there is a k such that there is no edge between any X_i and X_j with $|i-j| \geq k$. The union $\bigcup_{i=\ell}^{\ell+k} X_i$, $\ell \in \mathbb{Z}$, separates Γ into two components such that R_i has a tail in each component, which is sufficient. $\square$

Next we prove a slightly different version of Lemma 3.1.6. In this version we split each X_i into an "upper" and "lower" part, X_i^+ and X_i^-, and assume that we only find a perfect matching between upper and lower parts of adjacent partition classes, see Lemma 3.1.7 for details.

Lemma 3.1.7. *Let Γ be a graph that admits a partition of its vertex set into finite sets $X_i, i \in \mathbb{Z}$ with $|X_i| \geq 4$ fulfilling the following conditions:*

(i) $X_i = X_i^+ \cup X_i^-$, *such that* $X_i^+ \cap X_i^- = \emptyset$ *and* $|X_i^+| = |X_i^-|$

(ii) $\Gamma[X_i]$ *contains an Hamilton cycle C_i which is alternating between X_i^- and X_i^+.*[1]

(iii) *For each $i \in \mathbb{Z}$ there is a perfect matching between X_i^+ and X_{i+1}^-.*

(iv) *There is a $k \in \mathbb{N}$ such that for all $i, j \in \mathbb{Z}$ with $|i - j| \geq k$ there is no edge in Γ between X_i and X_j.*

Then Γ has a Hamilton circle.

Even though the proof of Lemma 3.1.7 is very closely related to the proof of Lemma 3.1.6, we still give the complete proof for completeness.

[1]Exactly every other element of C_i is contained in X_i^-.

Proof. By (i) we know that each X_i is connected and so we conclude from the structure given by (ii) and (iii) that Γ has exactly two ends. In addition note that $|X_i| = |X_j|$ for all $i, j \in \mathbb{Z}$.

Fix an orientation of each C_i. The goal is to find two disjoint spanning doubles rays in Γ. We first define two disjoint rays belonging to the same end, say for all the X_i with $i \geq 0$. Pick two vertices u_1 and w_1 in X_1^-. For R_1 we start with u_1 and move along C_1 in the fixed orientation of C_1 till the next vertex on C_1 would be w_1, then instead of moving along C_1 we move to X_2^- by the given matching edge. Note that as w_1 is in X_1^- and because each C_i is alternating between X_i^- and X_i^+ this is possible. We take this to be a the initial part of R_1. We do the analog for R_2 by starting with w_1 and moving also along C_1 in the fixed orientation till the next vertex would be u_1, then move to X_2^-. We repeat the process of starting with some X_i in two vertices u_i and w_i, where u_i is the first vertex of R_1 on X_i and w_i the analog for R_2. We follow along the fixed orientation on C_i till the next vertex would be u_i or w_i, respectively. Then we move to X_{i+1} by the giving matching edges. One can easily see that each vertex of X_i for $i \geq 1$ is contained exactly either in R_1 or R_2. By moving from u_1 and w_1 to X_0^+ by the matching edges and then using the same process but moving from X_i^- to X_{i-1}^+ extents the rays R_1 and R_2 into two double rays. Obviously those double rays are spanning and disjoint. As Γ has exactly two ends it remains to show that R_1 and R_2 have a tail in each end, see Lemma 3.1.5. By (iv) there is a k such that there is no edge between any X_i and X_j with $|i-j| \geq k$ the union $\bigcup_{i=\ell}^{\ell+k} X_i$, $\ell \in \mathbb{Z}$ separates Γ into two components such that R_i has a tail in each component, which is sufficient. $\square$

Remark 3.1.8. *It is easy to see that one can find a Hamilton double ray instead of a Hamilton circle in Lemma* 3.1.6 *and Lemma* 3.1.7*. Instead of starting with two vertices and following in the given orientation to define the two double rays, one just starts in a single vertex and follows the same orientation.*

The following lemma is one of our main tools in proving the existence of Hamilton circles in Cayley graphs. It is important to note that the restriction, that $S \cap H = \emptyset$, which looks very harsh at first glance, will not be as restrictive in the later parts of this thesis. In most cases we can turn the case $S \cap H \neq \emptyset$ into the case $S \cap H = \emptyset$ by taking an appropriate quotient.

Lemma 3.1.9. *Let $G = \langle S \rangle$ and $\widetilde{G} = \langle \widetilde{S} \rangle$ be finite groups with non-trivial subgroups $H \cong \widetilde{H}$ of indices two such that $S \cap H = \emptyset$ and such that $\Gamma(G, S)$ contains a Hamilton cycle. Then the following statements are true.*

(i) $\Gamma(G *_H \widetilde{G}, S \cup \widetilde{S})$ *has a Hamilton circle.*

(ii) $\Gamma(G *_H \widetilde{G}, S \cup \widetilde{S})$ *has a Hamilton double ray.*

To prove Lemma 3.1.9 we start by finding some general structure given by our assumptions. This structure will make it possible to use Lemma 3.1.7 and Remark 3.1.8 to prove the statements (i) and (ii).

Proof. First we define $\Gamma := \Gamma(G *_H \widetilde{G}, S \cup \widetilde{S})$. Let $s \in S \setminus H$ and let $\widetilde{s}$ be in $\widetilde{S} \setminus \widetilde{H}$. By our assumptions $\Gamma(G, S)$ contains a Hamilton cycle. Say this cycle is $C_0 = 1[c_1, \ldots, c_k]$. It follows from $S \cap H = \emptyset$ that C_0 is alternating between H and the right coset Hs. For each $i \in \mathbb{Z}$ we now define the graph Γ_i.

$$\text{For } i \geq 0 \text{ we define } \Gamma_i := \Gamma[H(s\widetilde{s})^i \cup H(s\widetilde{s})^i s]$$
$$\text{and for } i \leq -1 \text{ we define } \Gamma_i := \Gamma[H\widetilde{s}(s\widetilde{s})^{-i-1} \cup H(\widetilde{s}s)^{-i}].$$

Note that the Γ_i partition the vertices of Γ. By our assumptions we know that C_0 is a Hamilton cycle of Γ_0. We now define Hamilton cycles of Γ_i for all $i \neq 0$.

$$\text{For } i \geq 1 \text{ we define } C_i := (s\widetilde{s})^i[c_1, \ldots, c_k]$$
$$\text{and for } i \leq -1 \text{ we define } C_i := (\widetilde{s}s)^{-i}[c_1, \ldots, c_k].$$

To show that C_i is a Hamilton cycle of Γ_i it is enough to show that C_i is a cycle and that C_i contains no vertex outside of Γ_i, because all cosets of H have the same size and because C_0 is a Hamilton cycle of $\Gamma_0 = \Gamma(G, S)$.

For $i \geq 1$ we first show that C_i is a cycle. It follows directly from the fact that C_0 is a cycle that in Γ each C_i is closed.[2] Assume for a contraction that $(s\widetilde{s})^i c_0 \cdots c_j = (s\widetilde{s})^i c_0 \cdots c_\ell$ for some $j < \ell$. This contracts that C_0 is a cycle as it is equivalent to $1 = c_{j+1} \cdots c_\ell$.

It remains to show that every vertex of C_i is contained in Γ_i. Since H is a normal subgroup of both G and $\widetilde{G}$, the elements s and $\widetilde{s}$ commute with H. As each vertex $v := c_0 \ldots c_j$ is contained in either H or Hs we can conclude that $(s\widetilde{s})^i v \in (s\widetilde{s})^i H = H(s\widetilde{s})^i$ or $(s\widetilde{s})^i v \in (s\widetilde{s})^i Hs = H(s\widetilde{s})^i s$.

Next we note some easy observations on the structure of the C_i's. First note that $C_i \cap C_j = \emptyset$ for $i \neq j$ and also that the union of all C_i's contains all the vertices of Γ. In addition note that each C_i is alternating between two copies of H as C_0 was alternating between cosets of Γ_0. Finally note that by the structure of Γ there is no edge between any Γ_i and Γ_j with $|i - j| \geq 2$ in Γ. By the structure of Γ for $i \geq 0$ we get a perfect matching between $C_i \cap H(s\widetilde{s})^i s$ and $C_{i+1} \cap H(s\widetilde{s})^{i+1}$ by $\widetilde{s}$.

By a similar argument one can show that for $i < 0$ we get a similar structure and the desired perfect matchings.

The statement (i) now follows by Lemma 3.1.7. Analog statement (ii) follows by Remark 3.1.8. □

We now recall two known statements about Hamilton cycles on finite groups, which we then will first combine and finally generalize to infinite groups. For that let us first recall some definitions. A group G is called *Dedekind*, if every subgroup of G is normal in G. If a Dedekind group G is also non-abelian, it is called a *Hamilton group*.

Lemma 3.1.10. [11] *Any Cayley graph of a Hamilton group G has a Hamilton cycle.* □

In addition we know that all finite abelian groups also contain Hamilton cycles by Lemma 3.1.1. In the following remark we combine these two facts.

Remark 3.1.11. *Any Cayley graph of a finite Dedekind group of order at least three contains a Hamilton cycle.*

[2] Γ contains the edge between the image of c_1 and c_k for each C_i.

3.1.3 Main results of Chapter 3

In this section we prove our main results of Chapter 3. For that let us recall that by Theorem 2.3.2 we know that every two-ended group is either a free product with amalgamation over a finite subgroup of index two or an HNN-extension over a finite subgroup. Now we prove our first main result, Thereom 3.1.12, which deals with the first type of groups. To be more precise we use Remark 3.1.11 to prove that there is a Hamilton circle in the free product with amalgamation over the subgroup of index two of a Dedekind group and an arbitrary group.

Theorem 3.1.12. *Let $G = \langle S\rangle$ and $\widetilde{G} = \langle\widetilde{S}\rangle$ be two finite groups with non-trivial subgroups $H \cong \widetilde{H}$ of indices two and such that G is a Dedekind group. Then $\Gamma(G *_H \widetilde{G}, S \cup \widetilde{S})$ has a Hamilton circle.*

Proof. First, it follows from Remark 3.1.11 that $\Gamma(G, S)$ has a Hamilton cycle. If all generators of $S = \{s_1, \ldots, s_n\}$ lie outside H, then Lemma 3.1.9 completes the proof. So let $s_n \in S \setminus H$ and let $\widetilde{s} \in \widetilde{S} \setminus \widetilde{H}$. Let us suppose that $S' := \{s_1, \ldots, s_i\}$ is a maximal set of generators of S contained in H and set $L := \langle S'\rangle$. First note that L is a normal subgroup of G. We now have two cases, either $H = L$ or $L \neq H$. We may assume that $L \neq H$ as otherwise we can find a Hamilton circle of $\Gamma(G *_H \widetilde{G}, S \cup \widetilde{S})$ by Lemma 3.1.6 as H is a Dedekind group and thus $\Gamma(H, S')$ contains a Hamilton cycle. Because $L \subsetneq H$ and $H \cong \widetilde{H}$ we conclude that there is a subgroup of $\widetilde{H}$ that is corresponding to L, call this $\widetilde{L}$.

Let Λ be the Cayley graph of the group $G/L *_{H/L} \widetilde{G}/\widetilde{L}$ with the generating set $\overline{S} \cup \overline{\widetilde{S}}$, where $\overline{S}$ and $\overline{\widetilde{S}}$ the corresponding generating sets of G/L and $\widetilde{G}/\widetilde{L}$, respectively. Note that every generator of the quotient group G/L lies outside of H/L. Hence it follows from Lemma 3.1.9, that we can find a Hamilton double ray in Λ, say $\mathcal{R}$. Now we are going to use $\mathcal{R}$ and construct a Hamilton circle for $\Gamma := \Gamma(G *_H \widetilde{G}, S \cup \widetilde{S})$. Since L is a subgroup of H, we can find a Hamilton cycle in the induced subgroup of L, i.e. $\Gamma(L, S')$. We denote this Hamilton cycle in $\Gamma(L, S')$ by $C = [x_1, \ldots, x_n]$. We claim that the induced subgraph of any coset of L of $G *_H \widetilde{G}$ contains a Hamilton cycle. Let Lx be an arbitrary coset of $G *_H \widetilde{G}$. If we start with x and move along the edges given by C, then we obtain a cycle. We will show that this cycle lies in Lx. Since L is a normal subgroup of both G and $\widetilde{G}$ it implies that L is a normal subgroup of $G *_H \widetilde{G}$. Since L is normal, the element x commutates with the elements of L and so $x[C]$ lies in Lx and the claim is proved. It is important to notice that $\mathcal{R}$ gives a perfect mating between each two successive cosets. Thus we are ready to invoke the Lemma 3.1.6 and this completes the proof. $\square$

The following Theorem 3.1.14 proves that the second type of two-ended groups also contains a Hamilton circle, given some conditions.

Remark 3.1.13. *Let us have a closer look at an HNN extension of a finite group C. Let $C = \langle S \mid R \rangle$ be a finite group. It is important to notice that every automorphism $\phi\colon C \to C$ gives us an HNN-extension $G = C *_C$. In particular every such HNN-extension comes from an automorphism $\phi\colon C \to C$. Therefore C is a normal subgroup of G with the quotient $\mathbb{Z}$, as the presentation of HNN-extension $G = C *_C$ is*

$$\langle S, t \mid R,\, t^{-1}ct = \phi(c)\, \forall c \in C \rangle.$$

Hence G can be expressed by a semidirect product $C \rtimes \mathbb{Z}$ which is induced by ϕ. To summarize; every two-ended group with a structure of HNN-extension is a semidirect product of a finite group with the infinite cyclic group.

Theorem 3.1.14. *Let $G = (H \rtimes F, X \cup Y)$ with $F = \mathbb{Z} = \langle Y \rangle$ and $H = \langle X \rangle$ and such that H is finite and H contains a Hamilton cycle. Then G has a Hamilton circle.*

Proof. Let $C = [c_1, \ldots, c_t]$ be a Hamilton cycle in $\Gamma(H, X)$. We now make a case study about the size of Y.

Case I : If $|Y| = 1$, then $F = \mathbb{Z} = \langle y \rangle$. Since H is a normal subgroup of G, it follows that $gH = Hg$ for each $g \in G$. Thus the vertices of the set Cg form a cycle for every $g \in G$. Let C_g be the cycle of Hg for all $g \in \mathbb{Z}$, and let $\mathcal{C}$ be the set of all those cycles. We show that for every pair of $g, h \in \mathbb{Z}$ we either have $C_h \cap C_g = \emptyset$ or $C_h = C_g$. Suppose that $C_g \cap C_h \neq \emptyset$. This means that

$$\begin{aligned} c_i y^g &= c_j y^h \\ \Leftrightarrow c_j^{-1} c_i &= y^{h-g}. \end{aligned}$$

The order of the left hand side is finite while the order of the right hand side is infinite. Thus we conclude that $y^{h-g} = 1$ which in turn yields that $g = h$ thus we get $C_g = C_h$. We claim that every vertex is contained in $\mathcal{C}$. Suppose that $g \in G$. Since $G = H \rtimes \mathbb{Z}$, we deduce that $G = H\mathbb{Z}$. In other words, there is a natural number i and an $h \in \mathbb{Z}$ such that $g = c_i h$ and so g lies in the cycle C_h. These conditions now allow the application of Lemma 3.1.6, which concludes this case.

Case II : Assume that $|Y| \geq 2$. By Theorem 3.1.3 there are two disjoint double rays

$$\mathcal{R}_1 = [\ldots, x_{-2}, x_{-1}]1[x_1, x_2, \ldots]$$

and

$$\mathcal{R}_2 = [\ldots, y_{-2}, y_{-1}]x[y_1, y_2, \ldots]$$

where $x_i, y_i, x \in Y^{\pm}$ such that the vertices of $\mathcal{R}_1 \cup \mathcal{R}_2$ cover all elements $\mathbb{Z}$. Since H is a normal subgroup of G, we can conclude that $gH = Hg$. Thus the vertices of the set gC form a cycle for every $g \in G$. Now consider the double rays

$$P_1 = \cdots [x_{-2}][c_1, \ldots, c_{t-1}][x_{-1}]1[c_1, \ldots, c_{t-1}][x_1][c_1, \ldots, c_{t-1}] \cdots$$

and

$$P_2 = \cdots [y_{-2}][c_1, \ldots, c_{t-1}][y_{-1}]x[c_1, \ldots, c_{t-1}][y_1]][c_1, \ldots, c_{t-1}] \cdots .$$

For easier notation we define $a := c_1 \cdots c_{t-1}$. We claim that $P_1 \cap P_2 = \emptyset$. There are 4 possible cases of such intersections. We only consider this one case, as the others are analog. So assume to the contrary

$$x \cdot ay_1 \cdots ay_{\ell_1} \cdot c_1 \cdots c_{\ell'_1} = ax_1 \cdots ax_{\ell_2} \cdot c_1 \cdots c_{\ell'_2}.$$

Since H is a normal subgroup of G, for every $g \in G$ we have $ag = gh$ for some $h \in H$. It follows that

$$\begin{aligned} x \cdot ay_1 \cdots ay_{\ell_1} \cdot c_1 \cdots c_{\ell'_1} &= ax_1 \cdots ax_{\ell_2} \cdot c_1 \cdots c_{\ell'_2} \\ \Leftrightarrow x \cdot y_1 \cdots y_{\ell_1} h \cdot c_1 \cdots c_{\ell'_1} &= x_1 \cdots x_{\ell_2} h' \cdot c_1 \cdots c_{\ell'_2} \text{ for some } h, h' \in H \\ \Leftrightarrow x \cdot y_1 \cdots y_{\ell_1} \bar{h} &= x_1 \cdots x_{\ell_2} \bar{h}' \text{ for some } \bar{h}, \bar{h}' \in H \\ \Leftrightarrow (x_1 \cdots x_{\ell_2})^{-1} x \cdot y_1 \cdots y_{\ell_1} &= \bar{h}' \bar{h}^{-1} \end{aligned}$$

The left side of this equation again has finite order, but the right side has infinite order. It follows that

$$\begin{aligned} (x_1 \dots x_i)^{-1} x y_1 \cdots y_j &= 1 \\ x y_1 \cdots y_j &= x_1 \dots x_i \end{aligned}$$

But this contradicts our assumption that $\mathcal{R}_1$ and $\mathcal{R}_2$ were disjoint. Therefore, as $V(\mathcal{P}_1 \cup \mathcal{P}_2) = V(\Gamma(G, X \cup Y))$, the double rays $\mathcal{P}_1$ and $\mathcal{P}_2$ form the desired Hamilton circle. □

3.2 Multiended groups

In this section we give a few insights into the problem of finding Hamilton circles in groups with more than two ends, as well as showing a counterexample for Problem 1. We call a group to be a *multiended group* if is has more than two ends. Please recall that Diestel, Jung and Möller [16] proved that any transitive graph with more than two ends has infinitely many ends[3] and as all Cayley graphs are transitive it follows that the number of ends of any group is either zero, one, two or infinite. This yields completely new challenges for finding a Hamilton circle in groups with more than two ends. In the

[3] In this case the number of ends is uncountably infinite.

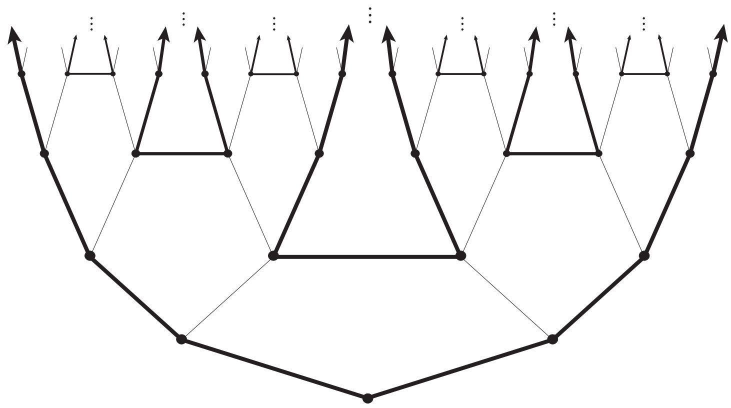

Figure 3.1: Hamilton circle in the Wild Circle.

following we provide the reader with an example to illustrate the problems of finding a Hamilton circles in an infinite graph with uncountably many ends. In Figure 3.1 we illustrate the graph which is known as the Wild Circle, for more details see [14, Figure 8.5.1]. The thick edges of this locally finite connected graph form a Hamilton circle which uses only countably many edges and vertices while visiting all uncountably many ends. Thus studying graphs with more than two ends to find Hamilton circles is more complicated than just restricting one-self to two-ended graphs.

3.2.1 A counterexample of Problem 1

We now give a counterexample to Problem 1. Define $G_1 := G_2 := \mathbb{Z}_3 \times \mathbb{Z}_2$. Let $\Gamma := \Gamma(G_1 *_{\mathbb{Z}_2} G_2)$. Let $G_1 = \langle a, b\rangle$ and $G_2 = \langle a, c\rangle$ where the order of a is two and the orders of b and c, respectively, are three. In the following we show that the assertion of Problem 1 holds for Γ and we show that $|\Gamma|$ does not contain a Hamilton circle.

For that we use the following well-known lemma and theorem.

Lemma 3.2.1. [14, Lemma 8.5.5] *If Γ is a locally finite connected graph, then a standard subspace* [4] *of $|\Gamma|$ is topologically connected (equivalently: arc-connected) if and only if it contains an edge from every finite cut of Γ of which*

[4] A standard subspace of $|\Gamma|$ is a subspace of $|\Gamma|$ that is a closure of a subgraph of Γ.

it meets both sides. □

Theorem 3.2.2. [15, Theorem 2.5] *The following statements are equivalent for sets $D \subseteq E(\Gamma)$:*

(i) *Every vertex and every end has even degree in D.*

(ii) *D meets every finite cut in an even number of edges.* □

Assume for a contradiction that there is a Hamilton circle in Γ and let D be its edge set. Clearly D contains precisely two edges incident to every vertex. Theorem 3.2.2 tells us that D meets every finite cut in an even number and every vertex twice. Since circles are connected and arc-connected we can, by Lemma 3.2.1, conclude that D meets every finite cut in at least one edge. We will now show that there is no set $D \subseteq E$ with these properties. For this purpose we study two cases: In each case we will consider a few finite cuts in Γ that show that such a D cannot exist. Figures 3.2 and 3.3 display induced subgraphs of Γ. The relevant cuts in those figures are the edges that cross the thick lines. The cases we study are that D contains the dashed edges of the appropriate figure corresponding to the case, see Figures 3.2 and 3.3. For easier reference we call the two larger vertices the *central vertices*.

Case 1: We now consider Figure 3.2, so we assume that the edges from the central vertices into the 'upper' side are one going to the left and the other to the right. First we note that the cut F ensures that the curvy edge between the central vertices is not contained in D. Also note that F ensures that the remaining two edges leaving the central vertices must go to the 'lower' side of Figure 3.2. As the cuts B and C have to meet an even number of edges of D we may, due to symmetry, assume that the dotted edge is also contained in D. This yields the contraction that the cut A now cannot meet any edge of D.

Case 2: This case is very similar to Case 1. Again we may assume that the there are two edges leaving the central into the 'upper' and the 'lower' side, each. The cut C ensures that D must contain both dotted edges. But this again yields the contraction that A cannot meet any edge in D.

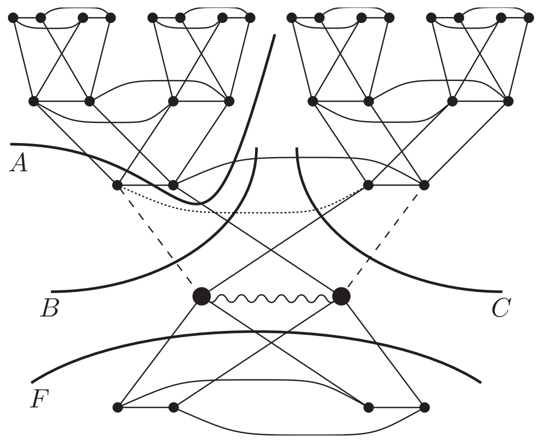

Figure 3.2: Case 1

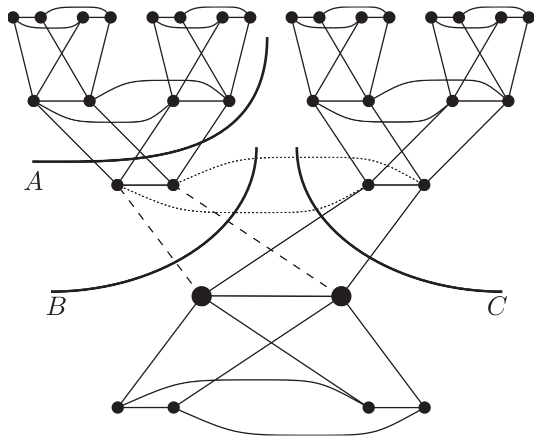

Figure 3.3: Case 2

It remains to show that $G_1 *_{\mathbb{Z}_2} G_2$ cannot be obtained as a free product with amalgamation over subgroups of size k of more than k groups. If $G_1 *_{\mathbb{Z}_2} G_2$ were fulfilling the premise of Problem 1 then there would be a finite $W \subset V(\Gamma)$, say $|W| = k$, such that $\Gamma \setminus W$ has more than k components.

We will now use induction on the size of W. For a contraction we assume that such a set W exists. For that we now introduce some notation to make the following arguments easier. In the following we will consider each group element as its corresponding vertex in Γ. As Γ is transitive we may assume that 1 is contained in W. Furthermore we may even assume that no vertex which has a representation starting with c is contained in W. Let X_i

be the set of vertices in Γ that have distance exactly i from $\{1, a\}$. We set $W_i := X_i \cap W$. For $x_i \in W_i$ let x_i^- be its neighbour in X_{i-1}, note that this is unique. For a vertex $x \in X_i$ let $\bar{x}$ be the neighbour of x in X_i which is not xa, note this will always be either xb or xc. For a set Y of vertices of Γ let C_Y be the number of components of $\Gamma \setminus Y$.

As Γ is obviously 2-connected the induction basis for $|W| = 0$ or $|W| = 1$ holds trivially.

We now assume that $|W| = k$ and that for each W' with $|W'| \leq |W| - 1$ we know that $C_{W'} \leq |W'|$. In our argument we will remove sets of vertices of size ℓ from W while decreasing C_W by at most ℓ.

Let $x \in W$ be a vertex with the maximum distance to $\{1, a\}$ in Γ, say $x \in X_i$.

Suppose that $xa \notin W$. The set $\{xb, xb^2\}$ intersects at most one component of $\Gamma \setminus W$, as the two vertices are connected by an edge. We can use the same argument for $\{xc, xc^2\}$. If $xa \notin W$, then it lies in one of these components as well. If is xb further away from $\{1, a\}$, then it is connected to xb by the path $xb, xba = xab, xa$, otherwise we can argue analogously with c instead of b. Hence x has neighbors in at most two components of $\Gamma \setminus W$, so removing x reduces C_W by at most one.

So we may assume that $xa \in W$. Let us consider the eight neighbors of x and xa. We know that four of those neighbors are in X_{i+1}. We may assume that those four vertices are xb, xab, xb^2 and xab^2. By our choice of x we know that all those vertices belong to the same component of $\Gamma \setminus W$. We may assume that xc and xac^2 are in X_i. By our above arguments for the case that $x_a \notin W$ we may assume that either x_c and xac^2 are both in W or both not in W. If xc and xac^2 are both in W, then $C_{W \setminus \{x, xa\}} \leq C_W - 1$ and we are done. If xc and xac^2 are both not in W, then $C_{W \setminus \{x, xa\}} \leq C_W - 2$ and we are done.

3.2.2 Closing Chapter 3

We still believe that it should be possible to find a condition on the size of the subgroup H to amalgamate over relative to the index of H in G_1 and G_2

such that the free product with amalgamation of G_1 and G_2 over H contains a Hamilton circle for the standard generating set. In addition it might be necessary to require some condition on the group G_1/H. We conjecture the following:

Conjecture 1. *There is a function $f : \mathbb{N} \to \mathbb{N}$ and let $G = G_1 *_H G_2$ where $G_1 = \langle S_1 \rangle$ and $G_2 = \langle S_2 \rangle$ are finite groups with following properties:*

(i) $[G_1 : H] = k$ *and* $[G_2 : H] = 2$.

(ii) $|H| \geq f(k)$.

(iii) *Each subgroup of H is normal in G_1 and G_2.*

(iv) $\Gamma(G_1/H, S/H)$ *contains a Hamilton cycle.*

Then $\Gamma(G_1 *_H G_2, S_1 \cup S_2)$ contains a Hamilton circle.

Chapter 4

From circles to cycles

4.1 Groups with Hamilton circles

One of the strongest results about the Lovász's conjecture is the following theorem which has been proved by Witte.

Theorem 4.1.1. [74, Theorem 6.1] *Every connected Cayley graph on any finite p-group is Hamiltonian.* □

In this section we are trying to present a generalization for Theorem 4.1.1 for infinite groups. First of all we need to show that two-ended groups always contain a subgroup of index two.

Lemma 4.1.2. *Let G be a finitely generated two-ended group. Then G contains a subgroup of index two.*

Proof. It follows from [45, Lemma 11.31] and [45, Theorem 11.33] that there exists a subgroup H of index at most 2 together with a homomorphism $\phi\colon H \to \mathbb{Z}$ with finite kernel. Now if G is equal to H, then we deduce that G/K is isomorphic to $\mathbb{Z}$ where K is the kernel of ϕ. Let L/K be the subgroup of G/K corresponding to $2\mathbb{Z}$. This implies that the index of L in G is 2, as desired. □

Now by Lemma 4.1.2 we know that G always possesses a subgroup H of index 2. In Theorem 4.1.5 we show that if any Cayley graph of H is Hamiltonian, then $\Gamma(G, S)$ contains a Hamilton circle if $S \cap H = \emptyset$.

For two-ended graphs we say $R_1 \sqcup R_2$ is a Hamilton circle if the double rays R_1 and R_2 fulfill the conditions of Lemma 3.1.5. Lemma 3.1.5 directly implies the following corollary.

Corollary 4.1.3. *Let G be a two-ended group with a subgroup H of index two. If any Cayley graph of H contains a connected Hamilton arc, then any Cayley graph $\Gamma(G,S)$ of G contains a Hamilton circle if $H = \langle S \cap H \rangle$.* $\square$

The problem of finding Hamilton circles in graphs with more than two ends is a harder problem than finding Hamilton circles in graphs with one or two ends, as we have seen in Section 3.2.1. For graphs with one or two ends the goal is to find one or two double rays containing all the vertices which behave nicely with the ends. For graphs with uncountalby many ends, it is not so straightforward to know what this desired structure could be. But the following powerful lemma by Bruhn and Stein helps us by telling us what such a structure looks like.

Lemma 4.1.4. [7, Proposition 3] *Let C be a subgraph of a locally finite graph Γ. Then the closure of C is a circle if and only if the closure of C is topologically connected and every vertex or end of Γ in this closure has degree two in C.* $\square$

Theorem 4.1.5. *Let $G = \langle S \rangle$ be a two-ended group with a subgroup H of index 2 such that $H \cap S = \emptyset$ and such that $|S| > 2$. If any Cayley graph of H is Hamiltonian, then $\Gamma(G,S)$ is also Hamiltonian.*

Proof. First we notice that H is two-ended, see [61, Lemma 5.6]. Let $g \in S$. We claim that gS generates H. Since the index H in G is 2, we conclude that S^2 generates H. So it is enough to show that $\langle gS \rangle = \langle S^2 \rangle$. In order to verify this we only need to show that $s_i s_j \in \langle gS \rangle$, where $s_i, s_j \in S$. Since the both of gs_i^{-1} and gs_j lie in gS, we are able to conclude that $s_i s_j$ belongs to $\langle gS \rangle$. We now suppose that $\mathcal{R}_1 \sqcup \mathcal{R}_2$ is a Hamilton circle in $\Gamma(H, gS)$. Let

$$\mathcal{R}_i = [\ldots, gs_{i_{-2}}, gs_{i_{-1}}]g_i[gs_{i_1}, gs_{i_2}, \ldots],$$

where $s_{i_j} \in S$ for $i = 1,2$ and $j \in \mathbb{Z} \setminus \{0\}$. Without loss of generality we can assume that $g_1 = 1$. We will now "expand" the double rays $\mathcal{R}_i$ to double rays in $\Gamma(G,S)$. So we define

$$\mathcal{R}_i' := [\ldots, g, s_{i_{-2}}, g, s_{i_{-1}}]g_i[g, s_{i_1}, g, s_{i_2}, \ldots]$$

for $i = 1,2$. We note that $S \cap H = \emptyset$. First we show that $\mathcal{R}_i'$ really is a double ray. This follows directly from the definition of $\mathcal{R}_i'$ and the fact that $\mathcal{R}_i$ is a double ray. It remains to show that $\mathcal{R}_1'$ and $\mathcal{R}_2'$ are disjoint and moreover their union covers each vertex of $\Gamma(G,S)$. Suppose that $\mathcal{R}_1'$ and $\mathcal{R}_2'$ meet. Let $v \in \mathcal{R}_1' \cap \mathcal{R}_2'$ with the minimal distance in $\mathcal{R}_1'$ from the vertex 1. Now we have the case that $v \in H$ or $v \notin H$. Both cases directly give a contradiction. From $v \in H$ we can conclude that $\mathcal{R}_1$ and $\mathcal{R}_2$ meet, which contradicts our assumptions. Assume that $v \notin H$. Without loss of generality assume that $v \neq 1$. Suppose that the path from 1 to v in $\mathcal{R}_1'$ used s_{1_1}. This implies that $vg^{-1} \in H$ and $vg^{-1} \in \mathcal{R}_1', \mathcal{R}_2'$. But this contradicts both the minimality of the distance of v from 1 and the fact that $vg^{-1} \in \mathcal{R}_1, \mathcal{R}_2$. If the path from 1 to v in $\mathcal{R}_1'$ does not use s_{1_1} then it must contain $s_{1_{-1}}$. This implies that we can use $g^{-1}v$ instead of vg^{-1} to get the same contradictions as in the above case.

It remains to show that $\mathcal{R}_1'$ and $\mathcal{R}_2'$ each have a tail in each of the two ends of $\Gamma(G,S)$. Let ω and ω' be the two ends of $\Gamma(G,S)$ and let X be a finite vertex set such that $C(X,\omega) \cap C(X,\omega') = \emptyset$. It remains to show that $\mathcal{R}_i'$ has a tail in both $C(X,\omega)$ and $C(X,\omega')$. By symmetry it is enough to show that $\mathcal{R}_i'$ has a tail in $C := C(X,\omega)$. Let C_H be the set of vertices in C which are contained in H. By construction of $\mathcal{R}_i'$ we know that $\mathcal{R}_i' \cap C_H$ is infinite. And as $\Gamma(G,S)$ is infinite and as $\mathcal{R}_i'$ is connected, we can conclude that C contains a tail of $\mathcal{R}_i'$. □

With an analogous method of the proof of Theorem 4.1.5, one can prove the following theorem.

Theorem 4.1.6. *Let $G = \langle S \rangle$ be a two-ended group with a subgroup H of index 2 such that $H \cap S = \emptyset$. If any Cayley graph of H contains a Hamilton double ray, then so does $\Gamma(G,S)$.* □

Corollary 4.1.7. *Let H be a two-ended group such that any Cayley graph* *of H is Hamiltonian. If $G = \langle S \rangle$ is any extension of H by $\mathbb{Z}_2$ in such a way that $S \cap H = \emptyset$, then $\Gamma(G, S)$ has a Hamilton double ray.* □

Lemma 4.1.8. *Any Cayley graph of $\mathbb{Z}$ contains a Hamilton double ray.*

Proof. Let $\mathbb{Z} = \langle S \rangle$. We proof Lemma 4.1.8 by induction on $|S|$. There is nothing to show for $|S| = 2$. So we may assume that $|S| > 2$ and any Cayley graph of $\mathbb{Z}$ with less than $|S|$ generators contains a Hamilton double ray. Let $s \in S$ and define $H := \langle S \setminus s \rangle$. Because H is a subgroup of $\mathbb{Z}$ we know that H is cyclic. By the induction hypothesis we know that there is a Hamilton double ray of H, say $R_H = [\ldots x_{-2}, x_{-1}]1[x_1, x_2, \ldots]$. Let $k := [\mathbb{Z} : H]$, note that $k \in \mathbb{N}$. So we have $G = \bigsqcup_{i=0}^{k-1} Hs^i$. We define

$$R := \cdots [s^{-1}]^{-(k-1)}[x_{-2}][s]^{k-1}[x_{-1}]1[s]^{k-1}[x_1][s^{-1}]^{-(k-1)}[x_2] \cdots$$

As $\mathbb{Z}$ is abelian we can conclude that R covers all vertices of $\Gamma(G, S)$. It remains to show that R has tails in both ends of $\Gamma(G, S)$ which follows directly from the fact that R_H is a Hamilton arc of H and the fact that the index of H in G is finite. □

We now give two lemmas which show that we can find normal subgroups in certain free-products with amalgamations or HNN-extensions.

Lemma 4.1.9. *Let $G = G_1 *_H G_2$ be a finitely generated 2-ended group, then H is normal in G.*

Proof. As G is two-ended we know that $[G_i : H] = 2$ for $i \in \{1, 2\}$. Let $g \in G$ be any element. Let $f \in H$. We have to show that $gfg^{-1} \in H$. It is sufficient to check the case when g is a generator of G. But this case is obvious. □

Lemma 4.1.10. *Let G be a two-ended group which splits over $\mathbb{Z}_p$ as an HNN-extension. i.e. $G = \langle k, t \mid k^p = 1, tkt^{-1} = \phi(k) \rangle$, with $\phi \in \mathsf{Aut}(\mathbb{Z}_p)$. Then $\mathbb{Z}_p$ is normal in G.*

Proof. Let $g \in G$. We have to show that $gf = f^r g$ for $g \in G$ and $f \in \mathbb{Z}_p$ and some $r \in \mathbb{Z}$. By our presentation of G we know that $g = k^{i_1}t^{j_1} \cdots k^{i_n}t^{j_n}$.

From $tkt^{-1} = \phi(k) = k^\ell$ for some $\ell \in \mathbb{Z}$ we conclude the following:

$$\begin{aligned} tkt^{-1} &= k^\ell \\ \Rightarrow t^2kt^{-2} &= tk^\ell t^{-1} \\ &= (tkt^{-1})^\ell \\ &= k^{\ell^2} \\ \Rightarrow t^2k &= k^{\ell^2}t^2 \end{aligned}$$

By induction we obtain $t^x k = k^{\ell^x} t^x$ for $x \in \mathbb{N}$ and we can extend this by replacing t with t^{-1} to all $x \in \mathbb{Z}$. This implies

$$\begin{aligned} t^x k t^{-x} &= k^{\ell^x} t^x t^{-x} = k^{\ell^x} \text{ for all } x \in \mathbb{Z} \\ \Rightarrow (t^x k t^{-x})^m &= (k^{\ell^x})^m = k^y \text{ for some } y \in \mathbb{Z} \\ \Rightarrow t^x k^m t^{-x} &= k^{y'} \text{ for some } y' \in \mathbb{Z} \\ t^x k^m &= k^{y'} t^x \end{aligned}$$

This implies that we have a presentation of each $g \in G$ as $g = k^y t^x$ for some $x, y \in \mathbb{Z}$. Let $f \in \mathbb{Z}_p$, say $f = k^u$, be given. We conclude

$$\begin{aligned} gf = k^y t^x k^u &= k^y k^{y'} t^x \text{ for some } y' \in \mathbb{Z} \\ &= k^{y''} k^u t^x \text{ for some } y'' \in \mathbb{Z} \\ &= k^{y''} g \end{aligned}$$

This finishes the proof. □

Witte has shown that any Cayley graph of a finite dihedral group contains a Hamilton path.

Lemma 4.1.11. [73, Corollary 5.2] *Any Cayley graph of the finite dihedral group contains a Hamilton path.* □

Next we extend the above mentioned lemma from a finite dihedral group to the infinite dihedral group.

Lemma 4.1.12. *Any Cayley graph of D_∞ contains a Hamilton double ray.*

Proof. Let S be an arbitrary generating set of $D_\infty = \langle a, b \mid b^2 = (ab)^2 = 1\rangle$. Let S_1 be a maximal subset of S in a such way that $S_1 \subseteq \langle a \rangle$ and define $S_2 := S \setminus S_1$. We note that each element of S_2 can be expressed as $a^j b$ which has order 2 for every $j \in \mathbb{Z}$. First we consider the case that S_1 is not empty. Assume that $H = \langle a^i \rangle$ is the subgroup generated by S_1. We note that $H \mathsf{char} \langle a \rangle \trianglelefteq D_\infty$ and so we infer that $H \trianglelefteq D_\infty$. It follows from Lemma 4.1.8 that we have the following double ray $\mathcal{R}$:

$$[\ldots, s_{-2}, s_{-1}]1[s_1, s_2, \ldots],$$

spanning H with each $s_i \in S_1$ for $i \in \mathbb{Z} \setminus \{0\}$. We notice that $D_\infty/H = \langle \overline{S_2} \rangle$ is isomorphic to D_{2i} for some $i \in \mathbb{N}$ and by Lemma 4.1.11 we are able to find a Hamilton path of D_∞/H, say $[x_1 H, \ldots, x_{2i-1} H]$, each $x_\ell \in S_2$ for $\ell \in \{1, \ldots, 2i-1\}$. On the other hand, the equality $bab = a^{-1}$ implies that $ba^t b = a^{-t}$ for every $t \in \mathbb{Z}$ and we deduce that $xa^t x = a^{-t}$ for every $t \in \mathbb{Z}$ and $x \in D_\infty \setminus \langle a \rangle$.[1] In other words, we can conclude that $xs_i x = s_i^{-1}$ for each $s_i \in S_1$ and $x \in D_\infty \setminus \langle a \rangle$. We now define a double ray $\mathcal{R}'$ in D_∞ and we show that it is a Hamiltonian double ray. In order to construct $\mathcal{R}'$, we define a union of paths. Set

$$P_j := p_j[x_1, \ldots, x_{2i-1}, s_{j+1}^{-1}, x_{2i-1}, \ldots, x_1, s_{j+2}],$$

where $p_j := s_1 \cdots s_j$ whenever $j > 0$, $p_j := s_{-1} \cdots s_j$ whenever $j < 0$ and finally $p_0 := 1$. It is straightforward to see that P_{2j} and $P_{2(j+1)}$ meet in exactly one vertex. We claim that the collection of all P_{2j}'s are pairwise edge disjoint for $j \in \mathbb{Z}$. We only show the following case and we leave the other cases to the reader. Assume that $p_{2j} x_1 \cdots x_\ell$ meets with $p_{2j'} x_1 \cdots x_{2i-1} s_{2j'+1}^{-1} x_{2i-1} \cdots x_{\ell'}$, where $j < j'$ and $\ell \leq \ell'$. It is enough to verify $\ell = \ell'$. It is not hard to see that $p_{2j} x_1 \cdots x_\ell = p_{2j'} s_{2j'+1}^{-1} x_1 \cdots x_{\ell'}$. We can see that the left hand side of the equality belongs to the coset $Hx_1 \cdots x_\ell$ and the other lies in $Hx_1 \cdots x_{\ell'}$ and so we conclude that $\ell = \ell'$. We are now ready to define our desired

[1] This follows as every element of $D_\infty \setminus \langle a \rangle$ can be presented by $a^i b$ for $i \in \mathbb{Z}$.

double ray. We define

$$\mathcal{R}' := \bigcup_{j\in\mathbb{Z}} P_{2j}.$$

It is straightforward to check $\mathcal{R}'$ contains every element of D_∞, thus we conclude that $\mathcal{R}'$ is a Hamilton ray, as desired.

If S_1 is empty, then $S \cap \langle a \rangle = \emptyset$ and Theorem 4.1.6 completes the proof.

$\square$

With a slight change to the proof of Lemma 4.1.12 we can obtain a Hamilton circle for D_∞.

Theorem 4.1.13. *The Cayley graph of D_∞ is Hamiltonian for any generating set S with $|S| \geq 3$.*

Proof. As this proof is a modification of the proof of Theorem 4.1.12, we continue to use the notations of that proof here. We may again assume that $S_1 \neq \emptyset$. Otherwise $X := \langle S_1 \rangle \subseteq \langle a \rangle$ which implies that $X \cong \mathbb{Z}$. In this case using Lemma 4.1.8 and applying Theorem 4.1.5 finishes the proof.

Since $|S| \geq 3$, each of P_j has length at least one. Now we define new paths

$$P'_j := p_j[x_1, \ldots, x_{2i-2}, s_{j+1}^{-1}, x_{2i-2}, \ldots, x_1, s_{j+2}],$$

where $p_j := s_1 \cdots s_j$ whenever $j > 0$, $p_j := s_{-1} \cdots s_j$ whenever $j < 0$ and finally $p_0 := 1$.

$$\mathcal{R}_1 := \bigcup_{j\in\mathbb{Z}} P'_{2j} \text{ and } \mathcal{R}_2 := [\ldots, s_{-2}s_{-1}]x_{2i-1}[s_1, s_2, \ldots].$$

Now $\mathcal{R}_1 \sqcup \mathcal{R}_2$ is a Hamilton circle. $\square$

Theorem 4.1.14. *Let $G = \langle S \rangle$ be a two-ended group which splits over $\mathbb{Z}_p$ such that $S \cap \mathbb{Z}_p \neq \emptyset$, where p is a prime number. Then $\Gamma(G, S)$ has a Hamilton circle.*

Proof. First we notice that S and $\mathbb{Z}_p$ meet in exactly one element and its inverse, say $S \cap \mathbb{Z}_p = \{k, k^{-1}\}$. By Theorem 2.3.2 we already know that G is isomorphic to $G_1 *_{\mathbb{Z}_p} G_2$ or an HNN-extension of $\mathbb{Z}_p$, where $|G_1| = |G_2| = 2p$.

Let us first assume that $G \cong G_1 *_{\mathbb{Z}_p} G_2$, where G_i is a finite group such that $[G_i : \mathbb{Z}_p] = 2$ for $i = 1, 2$. Since $\mathbb{Z}_p$ by Lemma 4.1.9 is a normal subgroup of G, we deduce that $G/\mathbb{Z}_p \cong \mathbb{Z}_2 * \mathbb{Z}_2$ which is isomorphic to D_∞. We set $S' := S \setminus \{k, k^{-1}\}$ and now the subgroup generated by S' has only trivial intersection with $\mathbb{Z}_p$. Otherwise $\mathbb{Z}_p \ni x \in \langle S' \rangle$ yields that $k \in \langle S' \rangle$, which cannot happen as S was minimal. We denote this subgroup by H. Note that $H\mathbb{Z}_p = G$ because $\mathbb{Z}_P$ is normal.[2] So we can conclude that H is isomorphic to $D_\infty \cong \mathbb{Z}_2 * \mathbb{Z}_2$ as:

$$\mathbb{Z}_2 * \mathbb{Z}_2 \cong (G_1 *_{\mathbb{Z}_p} G_2)/\mathbb{Z}_P = G/\mathbb{Z}_p = (H\mathbb{Z}_P)/\mathbb{Z}_p \cong H/(H \cap \mathbb{Z}_p p) = H$$

It follows from Lemma 4.1.12 that there exists the following Hamilton double ray $\mathcal{R}$ in H:

$$[\ldots, s_{-2}, s_{-1}]1[s_1, s_2, \ldots],$$

with $s_i \in S'$. We notice that $\mathcal{R}$ gives a transversal of the subgroup $\mathbb{Z}_p$. Set $x_i := \Pi_{j=1}^{i} s_j$ for $i \geq 1$ and $x_i := \Pi_{j=-1}^{i}$ for $i \leq -1$. There is a perfect matching between two consecutive cosets $\mathbb{Z}_p x_i$ and $\mathbb{Z}_p x_{i+1}$.[3] It is important to note that $\mathbb{Z}_p = \langle k \rangle$ is a normal subgroup of G. We use this to find a cycle in each coset of $\mathbb{Z}_p$.[4]

We now are ready to apply Lemma 3.1.6 to obtain a Hamilton circle.

Now assume that G is an HNN-extension which splits over $\mathbb{Z}_p$. We recall that G can be represented by $\langle k, t \mid k^p = 1, t^{-1}kt = \phi(k) \rangle$, with $\phi \in \mathsf{Aut}(\mathbb{Z}_p)$. Since $\mathbb{Z}_p$ is a normal subgroup, we conclude that $G = \mathbb{Z}_p \langle t \rangle = \langle k \rangle \langle t \rangle$. Again set $S' := S \setminus \{k, k^{-1}\}$ and $H := \langle S' \rangle$.

$$\langle S' \rangle = H = H/(H \cap \mathbb{Z}_p) \cong \mathbb{Z}_p H/\mathbb{Z}_p = G/\mathbb{Z}_p = \mathbb{Z}_p \langle t \rangle / \mathbb{Z}_p \cong \langle t \rangle.$$

Hence we conclude that S' generates $\langle t \rangle$. It follows from Lemma 4.1.8

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[2] To illustrate: Consider the generating sets. Because $\langle k \rangle$ is normal in G we can conclude that $G = \langle S \rangle = \langle S' \rangle \langle k \rangle$.

[3] This matching is given by s_{i+1} for $i \geq 1$ and s_{i-1} for $i \leq -1$.

[4] To illustrate consider the following case, all other cases are analogous: By normality of $\mathbb{Z}_p$ in G we know that $x_i k^\ell = k^{\ell'} x_i$. And as we have a cycle in $\mathbb{Z}_p$ given by k we have such a cycle coset of $\mathbb{Z}_p$.

that $\Gamma(\langle t\rangle, S')$ contains a Hamilton double ray. By the same argument as in the other case we can find the necessary cycles and the matchings between them to use Lemma 3.1.6 to find the desired Hamilton circle. $\square$

In the following theorem we are able to drop the condition of $S \cap H \neq \emptyset$ if $p = 2$.

Theorem 4.1.15. *Let G be a two-ended group which splits over $\mathbb{Z}_2$. Then any Cayley graph of G is Hamiltonian.*

Proof. Suppose that $G = \langle S\rangle$. If S meets $\mathbb{Z}_2 = \{1, k\}$, then we can use Theorem 4.1.14 and we are done. So we can assume that S does not intersect $\mathbb{Z}_2$. We note that $\mathbb{Z}_2$ is a normal subgroup of G either by Lemma 4.1.9 or Lemma 4.1.10 and we deduce from Theorem 2.3.2 that $\overline{G} = G/\mathbb{Z}_2$ is isomorphic to $\mathbb{Z}$ or D_∞. In either case we can find a Hamilton double ray in $\Gamma(\overline{G}, \overline{S})$ by either Lemma 4.1.8 or Lemma 4.1.12, say

$$\overline{\mathcal{R}} = [\ldots, \bar{s}_{-1}]1[\bar{s}_1, \ldots].$$

This double ray induces a double ray in $\Gamma(G, S)$, say

$$\mathcal{R} = [\ldots, s_{-1}]1[s_1, \ldots].$$

We notice that $\mathcal{R}$ meets every coset of $\mathbb{Z}_2$ in G exactly once. We now define the following double ray

$$\mathcal{R}' := [\ldots, s_{-1}]k[s_1, \ldots].$$

It is important to note that $\mathcal{R}$ and $\mathcal{R}'$ do not intersect each other. Otherwise there would be a vertex adjacent to two different edges with the same label and this yields a contradiction. Now it is not hard to see that $\mathcal{R} \sqcup \mathcal{R}'$ forms a Hamilton circle. $\square$

Remark 4.1.16. *The assumption that G is two-ended is necessary and it cannot be extended to multi-ended groups, see Section 3.2.1 in which we study $G = \mathbb{Z}_6 *_{\mathbb{Z}_2} \mathbb{Z}_6$. We proved that there is a generating set S of G of size 3 such that $\Gamma(G, S)$ is not Hamiltonian.*

4.2 Generalization of Rapaport Strasser

In this section we take a look at the following famous theorem about Hamilton cycles of Cayley graphs of finite groups which is known as Rapaport Strasser's Theorem and generalize the case of connectivity two to infinite groups in Theorem 4.2.4.

Theorem 4.2.1. [59] *Let G be a finite group, generated by three involutions a, b, c such that $ab = ba$. Then the Cayley graph $\Gamma(G, \{a, b, c\})$ is Hamiltonian.* $\square$

In the following, we will try to extend Theorem 4.2.1 to infinite groups. But we need to be careful. There are nontrivial examples of infinite groups such that their Cayley graphs do not possess any Hamilton circle, as we have seen in Section 3.2.1. Here we have an analogous situation. For instance let us consider $\mathbb{Z}_2 * (\mathbb{Z}_2 \times \mathbb{Z}_2)$ with a canonical generating set. Suppose that a is the generator of the first $\mathbb{Z}_2$. Then every edge with the label a in this Cayley graph is a cut edge. Hence we only consider Cayley graphs of connectivity at least two. On the other hand our graphs are cubic and so their connectivities are at most three.

We note that by Bass-Serre theory, we are able to characterize groups with respect to the low connectivity as terms of fundamental groups of graphs. It has been done by Droms, see Section 3 of [19]. But what we need here is a presentation of these groups. Thus we utilize the classifications of Georgakopoulos [29] to find a Hamilton circle. First we need the following crucial lemma which has been proved by Babai.

Lemma 4.2.2. [3, Lemma 2.4] *Let Γ be any cubic Cayley graph of any one-ended group. Then Γ is 3-connected.* $\square$

By the work of Georgakopoulos in [29] we have the following lemma about the generating sets of 2-connected cubic Cayley graphs.

Lemma 4.2.3. [29, Theorem 1.1 and Theorem 1.2] *Let $G = \langle S \rangle$ be a group, where $S = \{a, b, c\}$ is a set of involutions and $ab = ba$. If $\kappa(\Gamma(G, S)) = 2$, then G is isomorphic to one of the following groups:*

(i) $\langle a, b, c \mid a^2, b^2, c^2, (ab)^2, (abc)^m \rangle$, $m \geq 1$.

(ii) $\langle a, b, c \mid a^2, b^2, c^2, (ab)^2, (ac)^m \rangle$, $m \geq 2$. □

With the help of the lemmas above we are able to prove the extension of Theorem 4.2.1 for 2-connected graphs.

Theorem 4.2.4. *Let $G = \langle S \rangle$ be a group, where $S = \{a, b, c\}$ is a set of involutions such that $ab = ba$. If $\kappa(\Gamma(G, S)) = 2$, then $\Gamma(G, S)$ is Hamiltonian.*

Proof. Using Lemma 4.2.3 we can split the proof in two cases:

(i) $G \cong \langle a, b, c \mid a^2, b^2, c^2, (ab)^2, (abc)^m \rangle$, $m \geq 1$.
If $m = 1$, then G is finite and we are done with the use of Theorem 4.2.1. So we can assume that $m \geq 2$. We set $\Gamma := \Gamma(G, \{a, b, c\})$. Let $\mathcal{R}$ be the graph spanned by all the edges with labels a or c. It is obvious that $\mathcal{R}$ spans Γ as every vertex is incident with an edge with the label a and an edge with the label c. We want to apply Lemma 4.1.4. Obviously $\mathcal{R}$ induces degree two on every vertex of Γ. It follows from transitivity, that for any end ω there is a defining sequence $(F_i)_{i \in \mathbb{N}}$ such that $|F_i| = 2$ and such that the label of each edge in each F_i is c.

To illustrate, consider the following: The cycle $C := 1[a, b, a, b]$ separates Γ into two non-empty connected graphs, say Γ_1 and Γ_2. Let e_1 and e_2 be the two edges of Γ between C and Γ_1. Note that the label of both of those edges is c, additionally note that $F := \{e_1, e_2\}$ separates Γ_1 from $\Gamma[\Gamma_2 \cup C]$. Let R' be any ray in Γ belonging to an end ω. There is an infinite number of edges contained in R' with the label c as the order of a, b, ab and ba is two, let D be the set of those edges. We can now pick images under some automorphisms of F which meet D to create the defining sequence $(F_i)_{i \in \mathbb{N}}$.

Each such F_i is met by exactly two double rays in $\mathcal{R}$. It is straightforward to check that $\mathcal{R}$ meets every finite cut of Γ. This implies that the closure of $\mathcal{R}$ is topologically connected and that each end of Γ has degree two in this closure.

(ii) $G \cong \langle a,b,c \mid a^2, b^2, c^2, (ab)^2, (ac)^m\rangle$, $m \geq 2$

The proof of (ii) is very similar to (i). But we use the edge with labels b and c and the defining sequence consists of two edges both with label b instead of c.[5] □

In the following we give an outlook on the problem of extending Theorem 4.2.1 to infinite groups with 3-connected Cayley graphs. Similar to the Lemma 4.2.3 there is a characterization for 3-connected Cayley graphs which we state in Lemma 4.2.5. Note that the items (i) and (ii) have at most one end.

Lemma 4.2.5. [27] *Let $G = \langle S \rangle$ be a planar group, where $S = \{a,b,c\}$ is a set of involutions and $ab = ba$. If $\kappa(\Gamma(G,S)) = 3$, then G is isomorphic to one of the following groups:*

(i) $\langle a,b,c \mid a^2, b^2, c^2, (ab)^2, (acbc)^m\rangle, m \geq 1$.

(ii) $\langle a,b,c \mid a^2, b^2, c^2, (ab)^2, (bc)^m, (ca)^p\rangle, m,p \geq 2$.

(iii) $\langle a,b,c \mid a^2, b^2, c^2, (ab)^2, (bcac)^n, (ca)^{2m}\rangle, n,m \geq 2$ □

Lemma 4.2.5 gives us hope that the following Conjecture 2 might be a good first step to prove Conjecture 3 of Georgakopoulos and Mohar, see [27].

Conjecture 2. *Let G be a group, generated by three involutions a,b,c such that $ab = ba$ and such that $\Gamma(G,\{a,b,c\})$ is 2-connected. Then the Cayley graph $\Gamma(G,\{a,b,c\})$ is Hamiltonian.*

Conjecture 3. [27] *Every finitely generated 3-connected planar Cayley graph admits a Hamilton circle.*

We hope that methods used to prove Conjecture 2, and then possibly Conjecture 3, would open the possibility to also prove additional results like the extension of Theorem 4.2.6 of Rankin, which we propose in Conjecture 4.

[5] One could also show that Γ is outer planar as it does not contain a K_4 or $K_{2,3}$ minor and thus contains a unique Hamilton circle, see the work of Heuer [34] for definitions and the proof.

Theorem 4.2.6. [58] *Let G be a finite group, generated by two elements a, b such that $(ab)^2 = 1$. Then the Cayley graph $\Gamma(G, \{a, b\})$ has a Hamilton cycle.* $\square$

Conjecture 4. *Let $G =$ be a group, with a generating set $S = \{a^{\pm}, b^{\pm}\}$ such that $(ab)^2 = 1$ and $\kappa(\Gamma(G, S)) \geq 2$. Then $\Gamma(G, S)$ contains a Hamilton circle.*

4.3 Finding generating sets admitting Hamilton circles

This section has two parts. In the first part we study the Hamiltonicity of Cayley graphs obtained by adding a generator to a given generating sets of a group. In the second part, we discuss an important theorem called the Factor Group Lemma which plays a key role in studying Hamiltonianicity of finite groups.

4.3.1 Adding generators

Fleischner proved in [24] that the square of every 2-connected finite graph has a Hamilton cycle. Georgakopoulos [28] has extended this result to Hamilton circles in locally finite 2-connected graphs. This result implies the following corollary:

Corollary 4.3.1. [28] *Let $G = \langle S \rangle$ be an infinite group such that $\Gamma(G, S)$ is 2-connected then $\Gamma(G, S \cup S^2)$ contains a Hamilton circle.* $\square$

In the following we extend the idea of adding generators to obtain a Hamilton circle in the following manner. We show in Theorem 4.3.2 that under certain conditions, it suffices to add just a single new generator instead of adding an entire set of generators to obtain a Hamilton circle in the Cayley graph.

Theorem 4.3.2. *Let $G = \langle S \rangle$ be a group with a normal subgroup H which is isomorphic to the infinite cyclic group, i.e. $H = \langle a \rangle$, such that $\Gamma(\overline{G}, \overline{S} \setminus \{H\})$ has a Hamilton cycle. Then $\Gamma(G, S \cup \{a^{\pm}\})$ is Hamiltonian.*

Proof. We first notice that because $\overline{G}$ contains a Hamilton cycle, G contains a cyclic subgroup of finite index and Theorem 2.3.2 implies that G is two-ended. We set $\Gamma := \Gamma(G, S \cup \{a^{\pm}\})$. Let $C = H[x_1, \ldots, x_n]$ be the Hamilton cycle of $\Gamma(\overline{G}, \overline{S} \setminus \{H\})$. As G is two-ended, we only need to find two disjoint double rays which together span Γ such that for every finite set $X \subset V(\Gamma)$ each of those rays has a tail in each infinite component of $\Gamma \setminus X$. By the structure of G we can write

$$G = \langle a \rangle \sqcup \bigsqcup_{i=1}^{n-1} \left(\left(\Pi_{j=1}^{i} x_j \right) \langle a \rangle \right).$$

Let Γ' be the subgraph of Γ induced by $\bigsqcup_{i=1}^{n-1} \left(\Pi_{j=1}^{i} x_j \right) \langle a \rangle$. We now show that there is a double ray R spanning Γ' that has a tail belonging to each end. Together with the double ray generated by a this yields a Hamilton circle. To find R we will show that there is a "grid like" structure in Γ'. One might picture the edges given by a as horizontal edges and we show that the edges given by the x_i are indeed vertical edges yielding a "grid like" structure.

We claim that each x_i either belongs to $C_G(a)$, i.e. $ax_i = x_i a$, or that we have the equality $ax_i = x_i a^{-1}$. By the normality of $\langle a \rangle$, we have $a^g \in \langle a \rangle$ for all $g \in G$. In particular we can find $\ell, k \in \mathbb{Z} \setminus \{0\}$ such that $a^{(x_i^{-1})} = a^k$ and $a^{x_i} = a^{\ell}$.[6] Hence we deduce that $1 = a^{\ell k - 1}$. It implies that $k = \ell = \pm 1$ for each i. For the sake of simplicity, we assume that $k = \ell = 1$ for all i. The other cases are totally analogous, we only have to switch from using a to a^{-1} in the appropriate coset in the following argument.

Now we are ready to define the two double rays, say R_1 and R_2, which yield the desired Hamilton circle. For R_1 we take $\langle a \rangle$. To define R_2 we first define a ray R_2^+ and R_2^- which each starting in x_1. Let

$$R_2^+ := x_1[x_2, \ldots, x_{n-1}, a, x_{n-1}^{-1}, \ldots, x_2^{-1}, a]^{\mathbb{N}}$$
$$R_2^- := x_1[a^{-1}, x_2, \ldots, x_{n-1}, a^{-1}, x_{n-1}^{-1}, \ldots, x_2^{-1}, a^{-1}]^{\mathbb{N}}$$

[6] $a^{(x_i^{-1})} = x_i a x_i^{-1} = a^k \Rightarrow a = x_i^{-1} a^k x_i = (x_i^{-1} a x_i)^k$ and with $x_i^{-1} a x_i = a^{x_i} = a^{\ell}$ this implies $a = (a^{\ell})^k = a^{\ell k} \Rightarrow 1 = a^{\ell k - 1}$.

By our above arguments, all those edges exist and we define $R_2 := R_2^+ \cup R_2^-$. By construction it is clear that $R_1 \cap R_2 = \emptyset$ and $V(\Gamma) \subseteq R_1 \cup R_2$. It also follows directly from construction that for both ends of G there is a tail of R_i that belongs to that end. $\square$

Under the assumption that the weak Lovász's conjecture holds true for finite Cayley graphs, we can reformulate Theorem 4.3.2 in the following way:

Corollary 4.3.3. *For any two-ended group $G = \langle S \rangle$ there exists an $a \in G$ such that $\Gamma(G, S \cup \{a^{\pm}\})$ contains a Hamilton circle.*[7]

Proof. It follows from Theorem 2.3.2 that G has a subgroup of finite index which is isomorphic to $\mathbb{Z}$. We denote this subgroup by H. If H is not normal, then we substitute H with $\mathsf{Core}(H)$ which has a finite index as well. Now we are ready to invoke Theorem 4.3.2 and we are done. $\square$

Corollary 4.3.4. *Let $G = \langle S \rangle$ be a group and let $G' \cong \mathbb{Z}$ have a finite index. Then there exists an element $a \in G$ such that $\Gamma(G, S \cup \{a^{\pm}\})$ has a Hamilton circle.* $\square$

One might be interested in finding a small generating set for a group such that the Cayley graph with respect to this generating set is known to contain a Hamilton cycle or circle. For finite groups this was done by Pak and Radoiĉiċ.

Theorem 4.3.5. [57, Theorem 1] *Every finite group G of size $|G| \geq 3$ has a generating set S of size $|S| \leq \log_2 |G|$, such that $\Gamma(G, S)$ contains a Hamilton cycle.* $\square$

A problem with extending Theorem 4.3.5 to infinite groups is that having a generating set of size at most $\log_2$ of the size of the group is no restriction if the group is infinite. We only consider context-free groups and prevent the above problem by considering the index of the free subgroups in those context-free groups[8] to obtain a finite bound for the size of the generating

[7] This remark remains true even if we only assume that every finite group contains a Hamilton path instead of a Hamilton cycle.

[8] A group G is called *context-free* if G contains a free subgroup with finite index.

sets, see Theorem 4.3.9 for the details. Before we extend Theorem 4.3.5 to infinite graphs we need some more lemmas. In the following we give an extension of Lemma 3.1.6 from two-ended graphs to graphs with arbitrary many ends.

Lemma 4.3.6. *Let Γ' be an infinite graph and let $\mathcal{C}'$ be a Hamilton circle of Γ'. Let Γ be a graph fulfilling the following conditions:*

(i) Γ'_i *with* $i \in \{1, \ldots, k\}$*, are* k *pairwise disjoint copies of* Γ' *such that*

   (a) $V(\Gamma) = \bigsqcup_{i=1}^{k} V(\Gamma'_i)$.

   (b) $\bigsqcup_{i=1}^{k} E(\Gamma'_i) \subseteq E(\Gamma)$.

(ii) *Let* Φ *be the natural projection of* $V(\Gamma)$ *to* $V(\Gamma')$ *and set* $[v]$ *to be the set of vertices in* Γ *such that* Φ *maps them to* v*. Then for each vertex* v' *of* Γ' *there is*

   (a) *an edge between the two vertices in* $[v]$ *if* $k = 2$*, or*

   (b) *a cycle* C_v *in* Γ *consisting exactly of the vertices* $[v]$ *if* $k \geq 3$.

(iii) *There is a* $j \in \mathbb{N}$ *such that in* Γ *there is no edge between vertices* v *and* w *if* $d_{\Gamma'}(\Phi(v), \Phi(w)) \geq j$.

Then Γ *has a Hamilton circle.*

Proof. The proof of Lemma 4.3.6 consists of two parts. First we extend the collection of double rays that $\mathcal{C}'$ induces on Γ' to a collection of double rays spanning $V(\Gamma)$ by using the cycles C_v. Note that if $k = 2$, we consider the edge between the two vertices in each $[v]$ as C_v as the circles found by (ii) (b) only are used to collect all vertices in $[v]$ in a path, which is trivial if there are only two vertices in $[v]$. In the second part we show how we use this new collection of double rays to define a Hamilton circle of Γ. Let v' and w' be two vertices in Γ' and let v_i and w_i be the vertices corresponding to v' and w' in Γ_i. If $v'w'$ is an edge of Γ' then by assumption (ii) we know that v_iw_i is an edge of Γ for each i. This implies that there is a perfect matching between the cycles C_v and C_w.

The Hamilton circle $\mathcal{C}'$ induces a subgraph of Γ', say $\mathcal{R}'$. As Γ' is infinite, we know that $\mathcal{R}'$ consists of a collection of double rays. Let

$$R' = \ldots, r_{-1}, r_0, r_1, \ldots$$

be such a double ray. Let $R'_1, \ldots, R'_k$ be the copies of R' in Γ given by assumption (i). Let r_i^j be the vertex of R_j corresponding to the vertex r_i. We now use R' to construct a double ray R in Γ that contains all vertices of Γ which are contained in any R'_j. We first build two rays R^+ and R^- which together will contain all vertices of the copies of R'.

For R^+ we start in the vertex r_0^1 and take the edge $r_0^1 r_1^1$. Now we follow the cycle C_{r_1} till the next vertex would be r_1^1, say this vertex is r_1^ℓ and now take the edge $r_1^\ell r_2^\ell$. We repeat this process of moving along the cycles C_v and then taking a matching edge for all positive i. We define R^- analogously for all the negative i by also starting in r_0^1 but taking the cycle C_{r_0} before taking matching edges. Finally we set R to be the union of R^+ and R^-. As $R^+ \cap R^- = r_0^1$ we know that R is indeed a double ray. Let $\mathcal{R}$ be the set of double rays obtained by this method from the set of $\mathcal{R}'$.

In the following we show that the closure of $\mathcal{R}$ is a Hamilton circle in $|\Gamma|$. By Lemma 4.1.4 it is enough to show the following three conditions.

1. $\mathcal{R}$ induces degree two at every vertex of Γ,

2. the closure of $\mathcal{R}$ is topologically connected and

3. every end of Γ is contained in the closure of $\mathcal{R}$ and has degree two in $\mathcal{R}$.

1. follows directly by construction. We can conclude 2. directly from the following three facts: First: Finite paths are topologically connected, secondly: there is no finite vertex separator separating any two copies of Γ' in Γ and finally: $\mathcal{R}'$ was a Hamilton circle of Γ', and thus $\mathcal{R}'$ meets every finite cut of Γ' and hence $\mathcal{R}$ meets every finite cut of Γ. It is straightforward to check that by our assumptions there is a natural bijection between the ends of Γ and Γ'.[9] This, together with the assumption that the closure of $\mathcal{R}'$ is a

[9]Assumption (iv) implies that no two ends of Γ' get identified and the remaining parts are trivial or follow from the Jumping Arc Lemma, see [14, 15].

Hamilton circle of Γ', implies 3. and thus the proof is complete. $\square$

Now we want to invoke Lemma 4.3.6 in order to study context-free groups. First of all let us review some basic notations and definitions regarding context-free groups. Let us have a closer look at context-free groups. In the following, F will always denote a free group and F_r will denote the free group of rank r. So let F be a free subgroup of finite index of G. If $F = F_1$, then G is two-ended, see Theorem 2.3.2. Otherwise G has infinitely many ends, as the number of ends of G is equal to the number of ends of F by Lemma 2.3.8. To extend Theorem 4.3.5 to infinite groups we first need to introduce the following notation. Let G be a context-free group with a free subgroup F_r with finite index.

It is known that $\mathsf{Core}(F_r)$ is a normal free subgroup of finite index, see [4, Corollary 8.4, Corollary 8.5]. Here we need two notations. For that let G be a fixed group. By m_H we denote the index of a subgroup H of G, i.e. $[G : H]$. We set

$$n_G := \min\{m_H \mid H \text{ is a normal free subgroup of } G \text{ and } [G : H] < \infty\}$$

and

$$r_G := \min\{\mathsf{rank}(H) \mid H \text{ is a normal free subgroup of } G \text{ and } n_G = m_H\}.$$

It is worth remarking that $n_G \leq n!(r-1)+1$, because we already know that $\mathsf{Core}(F_r)$ is a normal subgroup of G with finite index at most $n!$. On the other hand, it follows from the Nielsen-Schreier Theorem, see [4, Corollary 8.4], that $\mathsf{Core}(F_r)$ is a free group as well and by Schreiers formula (see [4, Corollary 8.5]), we conclude that the rank of $\mathsf{Core}(F_r)$ is at most $n!(r-1)+1$.

We want to apply Corollary 4.3.1 to find a generating set for free groups such that the corresponding Cayley graph contains a Hamilton circle. By a theorem of Geogakopoulos [28], one could obtain such a generating set S of F_r by starting with the standard generating set, say S', and then defining $S := S' \cup S'^2 \cup S'^3$. Such a generating set has the size $8r^3 + 4r^2 + 2r$. In Lemma 4.3.7 we find a small generating set such that F_r with this generating set is 2-connected and obtain in Corollary 4.3.8 a generating set of size $6r(r+1)$ such that the Cayley graph of F_r with this generating set contains a Hamilton circle.

Lemma 4.3.7. *There exists a generating set S of F_r of size less than $6r$ such that $\Gamma(F_r, S)$ is 2-connected.*

Proof. Let $\{s_1, \ldots, s_r\}^{\pm}$ be the standard generating set of F_r. We set

$$T := \{s_1, \ldots, s_r, s_1^2, \ldots, s_r^2, s_1 s_2, s_1 s_3, \ldots s_1 s_r\}.$$

Finally we define $S := T^{\pm}$. It is straightforward to see that $|T| = 3r - 1$ and hence $|S| = 6r - 2$. We now claim that $\Gamma := \Gamma(F_r, S)$ is 2-connected. For that we consider $\Gamma \setminus \{1\}$ where 1 is the vertex corresponding to the neutral element of F_r. It is obvious that the vertices s_i and s_i^{-1} are contained in the same component of $\Gamma \setminus \{1\}$ as they are connected by the edge s_i^2. Additionally the edges of the form $s_1 s_i$ imply that s_1 and s_i are always in the same component. This finishes the proof. □

Using Lemma 4.3.7 and applying Corollary 4.3.1 we obtain the following corollary.

Corollary 4.3.8. *For every free group F_r there exists a generating set S of F_r of size at most $6r(6r+1)$ such that $\Gamma(F_r, S)$ contains a Hamilton circle.* □

We are now able to find a direct extension of Theorem 4.3.5 for context-free groups.

Theorem 4.3.9. *Let G be a context-free group with $n_G \geq 2$. Then there exists a generating set S of G of size at most $\log_2(n_G) + 1 + 6r_G(6r_G + 1)$ such that $\Gamma(G, S)$ contains a Hamilton circle.*

Proof. Suppose that G is a context-free group. Furthermore let F_r be a free subgroup of G with finite index n, where $r \geq 1$. We split our proof into two cases.

First assume that $r = 1$. This means that G contains a subgroup isomorphic to $\mathbb{Z}$ with finite index and thus G is two-ended. Let $H = \langle g \rangle$ be the normal free subgroup of G such that $m_{\langle g\rangle} = n_G$. Let $\overline{G}:= G/H$. We may assume $|\overline{G}| \geq 3$. By the assumptions we know that $|\overline{G}| \geq 2$, so if $|\overline{G}| = 2$ then we choose an element $f \notin H$ and obtain a Hamilton circle of $\Gamma := \Gamma(G, S^{\pm})$ with $S := \{f, g\}$ as Γ is isomorphic to the double ladder. Our assumptions imply that $\overline{G}$ is a group of order n_G. As n_G is finite, we can apply Theorem 4.3.5 to $\overline{G}$ to find a generating set $\overline{S}$ of $\overline{G}$ such that $\Gamma(\overline{G}, \overline{S})$ contains a Hamilton cycle. For each $\bar{s} \in \overline{S}$ we now pick a representative s of $\bar{s}$. Let S' be the set of all those representatives. We set $S := S' \cup \{g, g^{-1}\}$. By construction we know that $G = \langle S\rangle$. It is straightforward to check that $\Gamma(G, S)$ fulfills the conditions of Lemma 3.1.6 and thus we are done as $|S| = \log_2(n_G) + 2$.

Now suppose that $r \geq 2$. Let H be a normal free subgroup of G such that $\mathsf{rank}(H) = r_G$. By Corollary 4.3.8 we know that there is a generating set S_H of size at most $6r_G(6r_G + 1)$ such that $\Gamma_H := \Gamma(H, S_H)$ contains a Hamilton circle.

If $n_G = 2$ then, like in the above case, we can just choose an $f \in G \setminus H$ and a set of representatives for the elements in S_H, say S', and set $S := S' \cup f^{\pm}$ to obtain a generating set such that $\Gamma(G, S)$ fulfills the condition of Lemma 4.3.6.

So let us assume that $n_G \geq 3$. We define $\overline{G} := G/H$. As $\overline{G}$ is a finite group we can apply Theorem 4.3.5 to obtain a generating set $\overline{S}$ for $\overline{G}$ of size at most $\log_2(n_G)$ such that $\Gamma(\overline{G}, \overline{S})$ contains a Hamilton cycle. Again choose representatives of $\overline{S}$ to obtain S'. Let $S := S' \cup S_H$. Note that

$$|S| \leq 6r_G(6r_G + 1) + \log_2(n_G).$$

By construction we know that $G = \langle S\rangle$. Again it is straightforward to check that $\Gamma := \Gamma(G, S)$ fulfills the conditions of Lemma 4.3.6 and thus we are done. $\square$

Corollary 4.3.10. *Let G be a two-ended group. Then there exists a generating set S of G of $\log_2(n_G) + 3$ such that $\Gamma(G, S)$ contains a Hamilton circle.* $\square$

Remark 4.3.11. *We note that it might not always be best possible to use Theorem 4.3.9 to obtain a small generating set for a given context-free group.*

The advantage about Theorem 4.3.9 compared to just applying Corollary 4.3.1 is that one does not need to "square" the edges between copies of the underlying free group. This is a trade-off though, as the following rough calculation shows. Suppose that $\Gamma := \Gamma(G, S)$ *where* G *is a context-free group. Additionally assume that* Γ *is 2-connected, which is the worst for Theorem 4.3.9 when comparing Theorem 4.3.9 with a direct application of Corollary 4.3.1. Applying Corollary 4.3.1 to* Γ *we obtain that* $\Gamma(G, S \cup S^2)$ *is Hamiltonian. For instance, let* F_r *be a normal free subgroup of* G *with* $r_G = r$ *and* $[G : F_r] = n_G$*. We now define* S_F *as the standard generating set of* F_r *and* S_H *as the representative of the cosets of* F_r*. Then set* $S := S_F \cup S_H$*. We have*

$$\begin{aligned} |S_F^2| &= 4r^2 = 4r_G^2 \\ |S_H S_F| = |S_F S_H| &= 2r_G = 2n_G r_G \\ |S_H^2| &= n_G^2. \end{aligned}$$

Applying Corollary 4.3.1 yields a generating set of size $4r_G^2 + 4r_G n_G + n_G^2$ *while a a direct application of Theorem 4.3.9 yields a generating set of size at most* $\log_2(n_G) + 1 + 6r_G(6r_G + 1)$*. Thus which result is better depends the rank of the underlying free group and* n_G*.*

4.3.2 Factor Group Lemma

In this section we study extensions of the finite Factor Group Lemma to infinite groups. For that we first cite the Factor Group Lemma:

Theorem 4.3.12. [42, Lemma 2.3] *Let* $G = \langle S \rangle$ *be finite and let* N *be a cyclic normal subgroup of* G*. If* $[\bar{x_1}, \ldots, \bar{x_n}]$ *is a Hamilton cycle of* $\Gamma(G/N, \overline{S} \setminus \{N\})$ *and the product* $x_1 \cdots x_n$ *generates* N*, then* $\Gamma(G, S)$ *contains a Hamilton cycle.* □

To be able to extend Theorem 4.3.12, we have to introduce some notation. Let G be a group with a generating set S such that G acts on a set X. The vertex set of the *Schreier graph* are the elements of X. We join two vertices x_1 and x_2 if and only if there exists $s \in \{S\}$ such that $x_1 = sx_2$. We denote the Schreier graph by $\Gamma(G, S, X)$.

Suppose that X is the set of right cosets of a subgroup H of G. It is an easy observation that G acts on X. Now we are ready to generalize the Factor Group Lemma without needing the cyclic normal subgroup. It is worth remarking that if we consider the trivial action of G on G, we have the Cayley graph of G with respect to the generating S, i.e. $\Gamma(G, S, G) = \Gamma(G, S)$.

Theorem 4.3.13. *Let $G = \langle S \rangle$ be a group and let H be a subgroup of G and let X be the set of left cosets of H. If $1 < [G : H] < \infty$ and $[x_1, \ldots, x_n]$ is a Hamilton cycle of $\Gamma(G, S, X)$ and the product $x_1 \cdots x_n$ generates H, then we have the following statements.*

(i) *If G is finite, then $\Gamma(G, S)$ contains a Hamilton cycle.*

(ii) *If G is infinite, then $\Gamma(G, S)$ contains a Hamilton double ray.*

Proof. (i) Let us define $a := x_1 \cdots x_n$. Assume that $[G : H] = \ell$. We claim that $C := 1[x_1, \ldots, x_n]^\ell$ is the desired Hamilton cycle of G. It is obvious that C contains every vertex of H at least once. Suppose that there is a vertex $v \neq 1$ in C which is contained at least twice in C. Say

$$v = a^{i_1}[x_1, \ldots, x_{i_2}] = a^{j_1}[x_1, \ldots, x_{j_2}] \text{ with } i_1 \leq j_1 < l \text{ and } i_2, j_2 < n.$$

This yields that

$$x_1 \cdots x_{i_2} = a^k x_1 \cdots x_{j_2} \text{ with } k := j_1 - i_1 \geq 0.$$

As 1 and a^k are contained in H, we may assume that $i_2 = j_2$. Otherwise $x_1 \cdots x_{i_2}$ would belong to a different right coset of H as $a^k x_1 \cdots x_{j_2}$ which would yield a contradiction. Thus we can now write

$$x_1 \cdots x_{i_2} = a^k x_1 \cdots x_{j_2}$$

and it implies that $k = 0$. We conclude that C is indeed a cycle. It remains to show that every vertex of $\Gamma(G, S)$ is contained in C. So let $v \in V(\Gamma(G, S))$ and let $Hx_1 \cdots x_k$ be the coset that contains v. So we can write $v = hx_1 \cdots x_k$ with $h \in H$. As a generates H we know that $h = a^j$. So we can conclude that $v = a^j x_1 \cdots x_i \in C$. So C is indeed a Hamilton cycle of G.

(ii) The proof of (ii) is analogous to the above proof. First notice that since G has a cyclic subgroup of finite index, we can conclude that G is two-ended by Theorem 2.3.2. We now repeat the above construction with one small change. Again define $a := x_1 \cdots x_n$. As the order of a in H is infinite, we define C to be a double ray. So let

$$C := [x_1^{-1}, \ldots, x_n^{-1}]^{\mathbb{N}} 1 [x_1, \ldots, x_n]^{\mathbb{N}}.$$

It is totally analogously to the above case to show that no vertex of $\Gamma(G, S)$ is contained more than once in C, we omit the details here. It remains to show that every vertex of $\Gamma(G, S)$ is contained in C. This is also completely analogue to the above case. □

Let us have a closer look at the preceding theorem. As we have seen in the above proof the product $x_1 \cdots x_n$ plays a key role. In the following we want to investigate a special case. Suppose that $G = \langle S \rangle$ is an infinite group with a normal subgroup $H = \langle a \rangle$ of finite index and moreover assume that G/H contains a Hamilton cycle $1[x_1, \ldots, x_n]$. Depending on the element $x = x_1 \cdots x_n$, the following statements hold:

(i) If $x = a$, then $\Gamma(G, S)$ has a Hamilton double ray.

(ii) If $x = a^2$, then $\Gamma(G, S)$ has a Hamilton circle.

(iii) If $x = a^k$ and $k \geq 3$, then $\Gamma(G, S)$ has an infinite Hamilton cover of order k.

This yields the following conjecture:

Conjecture 5. *There exists a finite Hamilton cover for every two-ended transitive graph.*

In 1983 Durnberger [23] proved the following theorem:

Theorem 4.3.14. [23, Theorem 1] *Let G be a finite group with $G' \cong \mathbb{Z}_p$. Then any Cayley graph of G contains a Hamilton cycle.* $\square$

This yields the following natural question: What does an infinite group G with $G' \cong \mathbb{Z}_p$ look like?

Lemma 4.3.15. *Let G be a finitely generated group such that $|G'| < \infty$. Then G has at most two ends.*

Proof. Since G/G' is a finitely generated abelian group, by [62, 5.4.2] one can see that $G/G' \cong \mathbb{Z}^n \oplus Z_0$ where Z_0 is a finite abelian group and $n \in \mathbb{N} \cup \{0\}$. As the number of ends of $\mathbb{Z}^n \oplus Z_0$ is at most two we can conclude that the number of ends of G is at most two by [61, Lemma 5.7]. $\square$

We close this chapter with a conjecture. We propose an extension of Theorem 4.3.14. Please note that the methods of the proof of Theorem 4.1.14 can be used to show the special case of Conjecture 6 if the generating set does not have empty intersection with the commutator subgroup.

Conjecture 6. *Let G be an infinite group with $G' \cong \mathbb{Z}_p$. Then any Cayley graph of G contains a Hamilton circle.*

Chapter 5

Two-ended graphs and groups

5.1 Two-ended graphs

This section is split into two parts. In Section 5.1.1 we characterize quasi-transitive two-ended graphs without dominated ends. In Section 5.1.2 we characterize groups acting on those graphs with finitely many orbits.

5.1.1 Characterization

We characterize quasi-transitive two-ended graphs without dominated ends in Theorem 5.1.1 which is similar to the characterization of two-ended groups, see the item (iv) of Theorem 5.2.1. The second theorem in this section is Theorem 5.1.7, which states that for quasi-transitive two-ended graphs without dominated ends each end is thin. We give a direct proof of Theorem 5.1.7 here but one can deduce Theorem 5.1.7 from Theorem 5.1.1.

Theorem 5.1.1. *Let Γ be a connected quasi-transitive graph without dominated ends. Then the following statements are equivalent:*

(i) Γ *is two-ended.*

(ii) $\Gamma = \bar{\Gamma} *_T \bar{\Gamma}$ *fulfills the following properties:*

a) $\bar{\Gamma}$ *is a connected rayless graph of finite diameter.*

b) *All adhesion sets of the tree amalgamation contained in $\bar{\Gamma}$ are finite and connected and pairwise disjoint.*

c) *The identification maps are all the identity.*

d) *T is a double ray.*

(iii) *Γ is quasi-isometric to the double ray.*

In Theorem 5.1.1 we characterize graphs which are quasi-isometric to the double ray. It is worth mentioning that Krön and Möller [43] have studied arbitrary graphs which are quasi-isometric to trees.

Before we can prove Theorem 5.1.1 we have to collect some tools used in its proof. The first tool is the following Lemma 5.1.2 which basically states that in a two-ended quasi-transitive graph Γ we can find a separation fulfilling some nice properties. For that let us define a *type 1 separation* of Γ as a separation (A, A^*) of Γ fulfilling the following conditions:

(i) $A \cap A^*$ contains an element from each orbit.

(ii) $\Gamma[A \cap A^*]$ is a finite connected subgraph.

(iii) Exactly one component of $A \setminus A^*$ is big.

Lemma 5.1.2. *Let Γ be a connected two-ended quasi-transitive graph. Then there exists a type 1 separation of Γ.*

Proof. As the two ends of Γ are not equivalent, there is a finite S such that the ends of Γ live in different components of $\Gamma \setminus S$. Let C be a big component of $\Gamma \setminus S$. We set $\bar{A} := C \cup S$ and $\bar{A}^* := \Gamma \setminus C$ and obtain a separation $(\bar{A}, \bar{A}^*)$ fulfilling the condition (iii). Because $\bar{A} \cap \bar{A}^* = S$ is finite, we only need to add finitely many finite paths to $\bar{A} \cap \bar{A}^*$ to connect $\Gamma[\bar{A} \cap \bar{A}^*]$. As Γ is quasi-transitive there are only finitely many orbits of the action of $\mathsf{Aut}(\Gamma)$ on $V(\Gamma)$. Picking a vertex from each orbit and a path from that vertex to $\bar{A} \cap \bar{A}^*$ yields a separation (A, A^*) fulfilling all the above listed conditions. $\square$

In the proof of Lemma 5.1.2 we start by picking an arbitrary separation which we then extend to obtain type 1 separation. The same process can be used when we start with a tight separation, which yields the following corollary:

Corollary 5.1.3. *Let* Γ *be a two-ended quasi-transitive graph and let* $(\bar{A}, \bar{A}^*)$ *be a tight separation of* Γ*. Then there is an extension of* $(\bar{A}, \bar{A}^*)$ *to a type 1 separation* (A, A^*) *such that* $\bar{A} \cap \bar{A}^* \subseteq A \cap A^*$*.* □

Every separation (A, A^*) which can be obtained by Corollary 5.1.3 is a *type 2 separation*. We also say that the tight separation $(\bar{A}, \bar{A}^*)$ induces the type 2 separation (A, A^*).

In Lemma 5.1.4 we prove that in a quasi-transitive graph without dominated ends there are vertices which have arbitrarily large distances from one another. This is very useful as it allows to map separators of type 1 separations far enough into big components, such that the image and the preimage of that separation are disjoint.

Lemma 5.1.4. *Let* Γ *be a connected two-ended quasi-transitive graph without dominated ends, and let* (A, A^*) *be a type 1 separation. Then for every* $k \in \mathbb{N}$ *there is a vertex in each big component of* $\Gamma \setminus (A \cap A^*)$ *that has distance at least* k *from* $A \cap A^*$*.*

Proof. Let Γ and (A, A^*) be given and set $S := A \cap A^*$. Additionally let ω be an end of Γ and set $C := C(S, \omega)$. For a contradiction let us assume that there is a $k \in \mathbb{N}$ such that every vertex of C has distance at most k from S. Let $R = r_1, r_2, \ldots$be a ray belonging to ω. We now define a forest T as a sequence of forests T_i. Let T_1 be a path from r_1 to S realizing the distance of r_1 and S, i.e.: T_1 is a shortest path between r_1 and S. Assume that T_i is defined. To define T_{i+1} we start in the vertex r_{i+1} and follow a shortest path from r_{i+1} to S. Either this path meets a vertex contained in T_i, say v_{i+1}, or it does not meet any vertex contained in T_i. In the first case let P_{i+1} be the path from r_{i+1} to v_{i+1}. In the second case we take the entire path as P_{i+1}. Set $T_{i+1} := T_i \cup P_{i+1}$. Note that all T_i are forests by construction. For a vertex $v \in T_i$ let $d_i(v, S)$ be the length of a shortest path in T_i from v to any vertex in S. Note that, as each component of each T_i contains at exactly one vertex of S by construction, this is always well-defined. Let $P = r_i, x_1, x_2, \ldots, x_n, s$ with $s \in S$ be a shortest path between r_i and S. As P is a shortest path between r_i and S the subpath of P starting in x_j and going to s is a shortest $x_j - s$ path. This implies that for v of any T_i we have $d_i(v, S) \leq k$. We now conclude that the diameter of all components of T_i is at most $2k$ and hence each component of $T := \bigcup T_i$ also has diameter at most $2k$, furthermore note that T is a forest. As S is finite there is an infinite component of T, say T'. As T' is an infinite tree of bounded diameter it contains a vertex of infinite degree, say u. So there are infinitely many paths from u to R which only meet in u. But this implies that u is dominating the ray R, a contradiction. □

Our next tool used in the proof of Theorem 5.1.1 is Lemma 5.1.5 which basically states that small components have small diameter.

Lemma 5.1.5. *Let Γ be a connected two-ended quasi-transitive graphs without dominated ends. Additionally let $S = S_1 \cup S_2$ be a finite vertex set such that the following holds:*

(i) $S_1 \cap S_2 = \emptyset$.

(ii) $\Gamma[S_i]$ *is connected for* $i = 1, 2$.

(iii) S_i *contains an element from of each orbit for* $i = 1, 2$.

Let H be a rayless component of $\Gamma \setminus S$. Then H has finite diameter.

Proof. Let Γ, S and H be given. Assume for a contradiction that H has unbounded diameter. We are going to find a ray inside of H to obtain a contradiction. Our first aim is to find a $g \in \mathsf{Aut}(\Gamma)$ such that the following holds:

(i) $gS_i \subsetneq H$

(ii) $gH \subsetneq H$.

Let d_m be the maximal diameter of the S_i, and let d_d be the distance between S_1 and S_2. Finally let $d_S = d_d + 2d_m$.

First assume that H only has neighbors in exactly one S_i. This implies that $\Gamma \setminus H$ is connected. Let w be a vertex in H of distance greater than $2d_S$ from S and let $g \in \mathsf{Aut}(\Gamma)$ such that $w \in gS$. This implies that $gS \subsetneq H$. But as $\Gamma \setminus H$ contains a ray, we can conclude that $gH \subsetneq H$. Otherwise gH would contain a ray, as $\Gamma \setminus H$ contains a ray and is connected.

So let us now assume that H has a neighbor in both S_i. Let P be a shortest $S_1 - S_2$ path contained in $H \bigcup (S_1 \cup S_2)$, say P has length k. We pick a vertex $w \in H$ of distance at least $2d_S + k + 1$ from S, and we pick a $g \in \mathsf{Aut}(\Gamma)$ such that $w \in gS$. Obviously we know that $gP \subseteq (gH \cup gS)$. By the choice of g we also know that $gP \subseteq H$. This yields that $gH \subseteq H$, as gH is small. We can conclude that $gH \neq H$ and hence $gS_i \subsetneq H$ follows directly by our choice of g.

Note that as gH is a component of $\Gamma \setminus gS$ fulfilling all conditions we had on H we can iterate the above defined process with gH instead of H. We can now pick a vertex $v \in S$. Let U be the images of v. As H is connected we apply the Star-Comb lemma, see [14], to H and U. We now show, that the result of the Star-Comb lemma cannot be a star. So assume that we obtain a star with center x. Let $\ell := |S|$. Let d_X be the distance from S to x. By our construction we know that there is a step in which we use a $g_x \in \mathsf{Aut}(G)$ such that $d(S, g_x S) > d_x$. Now pick $\ell + 1$ many leaves of the star which come from steps in the process after we used g_x. This implies that in the star, all the paths from those $\ell + 1$ many leaves to x have to path through a separator of size ℓ, which is a contradiction. So the Star-Comb lemma yields a comb and hence a ray. $\square$

Lemma 5.1.6. *Let Γ be a two-ended connected quasi-transitive graph without dominated ends and let (A, A^*) be a type 1 separation and let C be the big component of $A \setminus A^*$. Then there is a $g \in \mathsf{Aut}(\Gamma)$ such that $g(C) \subsetneq C$.*

Proof. Let Γ be a two-ended connected quasi-transitive graph without dominated ends and let (A, A^*) be a type 1 separation of Γ. Set $d := \mathsf{diam}(A \cap A^*)$. Say the ends of Γ are ω_1 and ω_2 and set $C_i := C(A \cap A^*, \omega_i)$. Our goal now is to find an automorphism g such that $g(C_1) \subsetneq C_1$.

To find the desired automorphism g first pick a vertex v of distance $d + 1$

from $A \cap A^*$ in C_1. As (A, A^*) is a type 1 separation of the quasi-transitive graph Γ there is an automorphism h of Γ that maps a vertex of $A \cap A^*$ to v. Because $\Gamma[A \cap A^*]$ is connected and because $d(v, A \cap A^*) \geq d + 1$ we can conclude that $(A \cap A^*)$ and $h(A \cap A^*)$ are disjoint. If $h(C_1) \subsetneq C_1$ we can choose g to be h, so let us assume that $h(C_1) \supseteq C_2$. Now pick a vertex w in C_1 of distance at least $3d + 1$ from $A \cap A^*$, which is again possible by Lemma 5.1.4. Let f be an automorphism such that $w \in f(A \cap A^*)$. Because $d(w, A \cap A^*) \geq 3d + 1$ we can conclude that

$$A \cap A^*,\ h(A \cap A^*) \text{ and } f(A \cap A^*)$$

are pairwise disjoint and hence in particular $f \neq h$. Again if $f(C_1) \subsetneq C_1$ we may pick f as the desired g, so assume that $f(C_1) \supseteq C_2$.

This implies in particular that $fC_2 \subsetneq hC_2$ which yields that

$$h^{-1}f(C_2) \subsetneq C_2$$

which concludes this proof. □

Note that the automorphism in Lemma 5.1.6 has infinite order. Now we are ready to prove Theorem 5.1.1.

Proof of Theorem 5.1.1. We start with **(i)** $\Rightarrow$ **(ii)**.
So let Γ be a graph fulfilling the conditions in Theorem 5.1.1 and let Γ be two-ended. Additionally let (A, A^*) be a type 1 separation of Γ given by Lemma 5.1.2 and let d be the diameter of $\Gamma[A \cap A^*]$. Say the ends of Γ are ω_1 and ω_2 and set $C_i := C(A \cap A^*, \omega_i)$. By Lemma 5.1.6 we know that there is an element $g \in \mathsf{Aut}(\Gamma)$ such that $g(C_1) \subsetneq C_1$.

We know that either $A \cap gA^*$ or $A^* \cap gA$ is not empty, without loss of generality let us assume the first case happens. Now we are ready to define the desired tree amalgamation. We define the two graphs Γ_1 and Γ_2 like follows:

$$\Gamma_1 := \Gamma_2 := \Gamma[A^* \cap gA].$$

Note that as $A \cap A^*$ is finite and because any vertex of any ray in Γ with distance greater than $3d + 1$ from $A \cap A^*$ is not contained in Γ_i we can conclude Γ_i is a rayless graph.[1] The tree T for the tree amalgamation is just a double ray. The families of subsets of $V(\Gamma_i)$ are just $A \cap A^*$ and $g(A \cap A^*)$ and the identifying maps are the identity. It is straightforward to check that this indeed defines the desired tree amalgamation. The only thing remaining is to check that Γ_i is connected and has finite diameter. It follows straight from the construction and the fact that Γ is connected that Γ_i is indeed connected.

It remains to show that Γ_i has finite diameter. We can conclude this from Lemma 5.1.5 by setting $S := g^{-1}(A \cap A^*) \bigcup g^2(A \cap A^*)$. As Γ_i is now contained in a rayless component of $\Gamma \setminus S$.

(ii) $\Rightarrow$ (iii) Let $\Gamma = \bar{\Gamma} *_T \bar{\Gamma}$ where $\bar{\Gamma}$ is a rayless graph of diameter λ and T is a double ray. As T is a double ray there are exactly two adhesion sets, say S_1 and S_2, in each copy of $\bar{\Gamma}$. We define $\hat{\Gamma} := \bar{\Gamma} \setminus S_2$. Note that $\hat{\Gamma} \neq \emptyset$. Let $T = \ldots, t_{-1}, t_0, t_1, \ldots$. For each $t_i \in T$ we now define Γ_i to be a copy of $\hat{\Gamma}$. It is not hard to see that $V(\Gamma) = \bigsqcup_{i \in \mathbb{Z}} V(\Gamma_i)$, where each Γ_i isomorphic to $\hat{\Gamma}$. We now are ready to define our quasi-isometric embedding between Γ and the double ray $R = \ldots, v_1, v_0, v_1, \ldots$ Define $\phi \colon V(\Gamma) \to V(R)$ such that ϕ maps every vertex of Γ_i to the vertex v_i of R. Next we show that ϕ is a quasi-isomorphic embedding. Let v, v' be two vertices of Γ. We can suppose that $v \in V(\Gamma_i)$ and $v' \in V(\Gamma_j)$, where $i \leq j$. One can see that

$$d_\Gamma(v, v') \leq (|j - i| + 1)\lambda$$

and so we infer that

$$\frac{1}{\lambda} d_\Gamma(v, v') - \lambda \leq d_R(\phi(v), \phi(v')) = |j - i| \leq \lambda d_\Gamma(v, v') + \lambda.$$

As ϕ is surjective we know that ϕ is quasi-dense. Thus we proved that ϕ is

[1] Here we use that any ray belongs to an end in the following manner: Since $A \cap B$ and $g(A \cap B)$ are finite separators of Γ separating Γ_1 from any C_i, no ray in Γ_i can be equivalent to any ray in any C_i and hence Γ would contain at least three ends.

a quasi-isometry between Γ and R.

(iii) $\Rightarrow$ (i) Suppose that ϕ is a quasi-isometry between Γ and the double ray, say R, with associated constant λ. We shall show that Γ has exactly two ends, the case that Γ has exactly one end leads to a contradiction in an analogous manner. Assume to the contrary that there is a finite subset of vertices S of Γ such that $\Gamma \setminus S$ has at least three big components. Let $R_1 := \{u_i\}_{i\in\mathbb{N}}$, $R_2 := \{v_i\}_{i\in\mathbb{N}}$ and $R_3 := \{r_i\}_{i\in\mathbb{N}}$ be three rays of Γ, exactly one in each of those big components. In addition one can see that $d_R(\phi(x_i), \phi(x_{i+1})) \leq 2\lambda$, where x_i and x_{i+1} are two consecutive vertices of one of those rays. Since R is a double ray, we deduce that two infinite sets of $\phi(R_i) := \{\phi(x) \mid x \in R_i\}$ for $i = 1, 2, 3$ converge to the same end of R. Suppose that $\phi(R_1)$ and $\phi(R_2)$ converge to the same end. For a given vertex $u_i \in R_1$ let v_{j_i} be a vertex of R_2 such that the distance $d_R(\phi(u_i), \phi(v_{j_i}))$ is minimum. We note that $d_R(\phi(u_i), \phi(v_{j_i})) \leq 2\lambda$. As ϕ is a quasi-isometry we can conclude that $d_\Gamma(u_i, v_{j_i}) \leq 3\lambda^2$. Since S is finite, we can conclude that there is a vertex dominating a ray and so we have a dominated end which yields a contradiction. $\square$

Theorem 5.1.7. *Let Γ be a two-ended quasi-transitive graph without dominated ends. Then each end of Γ is thin.*

Proof. By Lemma 5.1.2 we can find a type 1 separation (A, A^*) of Γ. Suppose that the diameter of $\Gamma[A \cap A^*]$ is equal to d. Let C be a big component of $\Gamma \setminus A \cap A^*$. By Lemma 5.1.4 we can pick a vertex r_i of the ray R with distance greater than d from S. As Γ is quasi-transitive and $A \cap A^*$ contains an element from of each orbit we can find an automorphism g such that $r_i \in g(A \cap A^*)$. By the choice of r_i we now have that

$$(A \cap A^*) \cap g(A \cap A^*) = \emptyset.$$

Repeating this process yields a defining sequence of vertices for the end living in C each of the same finite size. This implies that the degree of the end living in C is finite. $\square$

For a two-ended quasi-transitive graph Γ without dominated ends let $s(\Gamma)$ be the maximal number of disjoint double rays in Γ. By Theorem 5.1.7 this is always defined. With a slight modification to the proof of Theorem 5.1.7 we obtain the following corollary:

Corollary 5.1.8. *Let Γ be a two-ended quasi-transitive graphs without dominated ends. Then the degree of each end of Γ is at most $s(\Gamma)$.*

Proof. Instead of starting the proof of Theorem 5.1.7 with an arbitrary separation of finite order we now start with a separation (B, B^*) of order $s(\Gamma)$ separating the ends of Γ which we then extend to a connected separation (A, A^*) containing an element of each orbit. The proof then follows identically with only one additional argument. After finding the defining sequence as images of (A, A^*), which is too large compared to $s(\Gamma)$, we can reduce this back down to the separations given by the images of (B, B^*) because $(B \cap B^*) \subseteq (A \cap A^*)$ and because (B, B^*) already separated the ends of Γ. $\square$

It is worth mentioning that Jung [40] proved that if a connected locally finite quasi-transitive graph has more than one end then it has a thin end.

5.1.2 Groups acting on two-ended graphs

In this section we investigate the action of groups on two-ended graphs without dominated ends with finitely many orbits. We start with the following lemma which states that there are only finitely many k-tight separations containing a given vertex. Lemma 5.1.9 is a separation version of a result of Thomassen and Woess for vertex cuts [70, Proposition 4.2] with a proof which is quite closely related to their proof.

Lemma 5.1.9. *Let Γ be a two-ended graph without dominated ends then for any vertex $v \in V(\Gamma)$ there are only finitely many k-tight separations containing v.*

Proof. We apply induction on k. The case $k = 1$ is trivial. So let $k \geq 2$ and let v be a vertex contained in the separator of a k-tight separation (A, A^*). Let C_1 and C_2 be the two big components of $\Gamma \setminus (A \cap A^*)$. As (A, A^*) is a k-tight separation we know that v is adjacent to both C_1 and C_2. We now consider the graph $\Gamma^- := \Gamma - v$. As v is not dominating any ends we can find a finite vertex set $S_1 \subsetneq C_1$ and $S_2 \subsetneq C_2$ such that S_i separates v from the end living in C_i for $i \in \{1,2\}$.[2] For each pair x, y of vertices with $x \in S_1$ and $y \in S_2$ we now pick a $x - y$ path P_{xy} in Γ^-. This is possible as $k \geq 2$ and because (A, A^*) is k-tight. Let $\mathcal{P}$ be the set of all those paths and let V_P be the set of vertices contained in the path contained in $\mathcal{P}$. Note that V_P is finite because each path P_{xy} is finite and both S_1 and S_2 are finite. By the hypothesis of the induction we know that for each vertex in V_P there are only finitely $(k-1)$-tight separations meeting that vertex. So we infer that there are only finitely many $(k-1)$-tight separations of Γ^- meeting V_P. Suppose that there is a k-tight separation (B, B^*) such that $v \in B \cap B^*$ and $B \cap B^*$ does not meet V_P. As (B, B^*) is k-tight we know that v is adjacent to both big components of $\Gamma \setminus B \cap B^*$. But this contradicts our choice of S_i. Hence there are only finitely many k-tight separations containing v, as desired. $\square$

In the following we extend the notation of diameter from connected graphs to not necessarily connected graphs. Let Γ be a graph. We denote the set of all subgraphs of Γ by $\mathcal{P}(\Gamma)$. We define the function $\rho\colon \mathcal{P}(\Gamma) \to \mathbb{Z} \cup \{\infty\}$ by setting $\rho(X) = \mathsf{sup}\{\mathsf{diam}(C) \mid C \text{ is a component of } X\}$.[3]

Lemma 5.1.10. *Let Γ be a quasi-transitive two-ended graph without dominated ends with $|\Gamma_v| < \infty$ for every vertex v of Γ and let (A, A^*) be a tight separation of Γ. Then for infinitely many $g \in \mathsf{Aut}(\Gamma)$ either the number $\rho(A \Delta gA)$ or $\rho(A \Delta gA)^c$ is finite.*

Proof. Let (A, A^*) be a tight separation of Γ. It follows from Lemma 5.1.9 and $|\Gamma_v| < \infty$ that there are only finitely $g \in \mathsf{Aut}(\Gamma)$ such that

$$(A \cap A^*) \cap g(A, A^*) \neq \emptyset.$$

[2] A finite vertex set S separates a vertex $v \notin S$ from an end ω_1 if v is not contained in the component $G \setminus S$ which ω_1 lives.

[3] If the component C does not have finite diameter, we say its diameter is infinite.

This implies that there are infinitely many $g \in \mathsf{Aut}(\Gamma)$ such that

$$(A \cap A^*) \cap g(A \cap A^*) = \emptyset.$$

So let $g \in \mathsf{Aut}(G)$ with $(A \cap A^*) \cap g(A \cap A^*) = \emptyset$.

By definition we know that either $A\Delta gA$ or $(A\Delta gA)^c$ contains a ray. Without loss of generality we may assume the second case. The other case is analogous. We now show that the number $\rho(A\Delta gA)$ is finite. Suppose that C_1 is the big component of $\Gamma \setminus (A \cap A^*)$ which does not meet $g(A \cap A^*)$ and C_2 is the big component of $\Gamma \setminus g(A \cap A^*)$ which does not meet $(A \cap A^*)$. By Lemma 5.1.4 we are able to find type 1 separations (B, B^*) and (C, C^*) in such a way that $B \cap B^* \subsetneq C_1$ and $C \cap C^* \subsetneq C_2$ and such that the $B \cap B^*$ and $C \cap C^*$ each have empty intersection with $A \cap A^*$ and $g(A \cap A^*)$. Now it is straightforward to verify that $A\Delta gA$ is contained in a rayless component X of $\Gamma \setminus ((B \cap B^*) \bigcup (C \cap C^*))$. Using Lemma 5.1.5 we can conclude that X has finite diameter and hence $\rho(A\Delta gA)$ is finite. $\square$

Assume that an infinite group G acts on a two-ended graph Γ without dominated ends with finitely many orbits and let (A, A^*) be a tight separation of Γ. By Lemma 5.1.10 we may assume $\rho(A\Delta gA)$ is finite for infinitely many $g \in \mathsf{Aut}(\Gamma)$. We set

$$H := \{g \in G \mid \rho(A\Delta gA) < \infty\}.$$

We call H the *separation subgroup* induced by (A, A^*).[4] In the sequel we study separations subgroups. We note that we infer from Lemma 5.1.10 that H is infinite.

Lemma 5.1.11. *Let G be an infinite group acting on a two-ended graph Γ without dominated ends with finitely many orbits such that $|\Gamma_v| < \infty$ for every vertex v of Γ. Let H be the separation subgroup induced by a tight separation (A, A^*) of Γ. Then H is a subgroup of G of index at most 2.*

Proof. We first show that H is indeed a subgroup of G. As automorphisms

[4]See the proof of Lemma 5.1.11 for a proof that H is indeed a subgroup.

preserve distances it is that for $h \in H, g \in G$ we have

$$\rho(g(A\Delta hA)) = \rho(A\Delta hA) < \infty.$$

As this is in particular true for $g = h^{-1}$ we only need to show that H is closed under multiplication and this is straightforward to check as one may see that

$$\begin{aligned} A\Delta h_1h_2A &= (A\Delta h_1A)\Delta(h_1A\Delta h_1h_2A) \\ &= (A\Delta h_1A)\Delta h_1(A\Delta h_2A). \end{aligned}$$

Since $\rho(A\Delta h_iA)$ is finite for $i = 1, 2$, we conclude that h_1h_2 belongs to H.

Now we only need to establish that H has index at most two in G. Assume that H is a proper subgroup of G and that the index of H is bigger than two. Let H and Hg_i be three distinct cosets for $i = 1, 2$. By Lemma 5.1.10 we know that there are only finitely many $g \in \mathsf{Aut}(\Gamma)$ such that both $\rho(A\Delta g_iA$ and $\rho((A\Delta g_iA)^c$ are infinite. As H is infinite we may therefore assume that $\rho((A\Delta g_iA)^c)$ is finite for $i = 1, 2$ as $g_1, g_2 \notin H$. Note that

$$A\Delta g_1g_2^{-1}A = (A\Delta g_1A)\Delta g_1(A\Delta g_2^{-1}A).$$

On the other hand we already know that

$$A\Delta g_1g_2^{-1}A = (A\Delta g_1A)^c\Delta(g_1(A\Delta g_2^{-1}A))^c.$$

We notice that the diameter of $A\Delta g_iA$ is infinite for $i = 1, 2$. Since $g_2 \notin H$ we know that $g_2^{-1} \notin H$ and so $\rho(g_1(A\Delta g_2^{-1}A))$ is infinite. By Lemma 5.1.10 we infer that $\rho(g_1(A\Delta g_2^{-1}A)^c)$ is finite. Now as the two numbers $\rho((A\Delta g_1A)^c)$ and $\rho(g_1(A\Delta g_2^{-1}A)^c)$ are finite we conclude that $\rho A\Delta g_1g_2^{-1}A < \infty$. Thus we conclude that $g_1g_2^{-1}$ belongs to H. It follows that $H = Hg_1g_2^{-1}$ and multiplying by g_2 yields $Hg_1 = Hg_2$ which contradicts $Hg_1 \neq Hg_2$. $\square$

Theorem 5.1.12. *Let G be a group acting with only finitely many orbits on a two-ended graph Γ without dominated ends such that $|\Gamma_v| < \infty$ for every vertex v of Γ. Then G contains an infinite cyclic subgroup of finite index.*

Proof. Let (A, A^*) be a tight separation and let $(\bar{A}, \bar{A}^*)$ be the type 2 separation given by Corollary 5.1.3. Additionally let H be the separation subgroup induced by (A, A^*). We now use Lemma 5.1.6 on $(\bar{A}, \bar{A}^*)$ to find an element $h \in G$ of infinite order. It is straightforward to check that $h \in H$. Now it only remains to show that $L := \langle h \rangle$ has finite index in H.

Suppose for a contradiction that L has infinite index in H and for simplicity set $Z := A \cap A^*$. This implies that $H = \bigsqcup_{i \in \mathbb{N}} Lh_i$. We have the two following cases:

Case I: There are infinitely many $i \in \mathbb{N}$ and $j_i \in \mathbb{N}$ such that $h_i Z = h^{j_i} Z$ and so $Z = h^{-j_i} h_i Z$. It follows from Lemma 5.1.9 that there are only finitely many f-tight separations meeting Z where $|Z| = f$. We infer that there are infinitely many $k \in \mathbb{N}$ such that $h^{-j_\ell} h_\ell Z = h^{-j_k} h_k Z$ for a specific $\ell \in \mathbb{N}$. Since the size of Z is finite, we deduce that there is $v \in Z$ such that for a specific $m \in \mathbb{N}$ we have $h^{-j_m} h_m v = h^{-j_n} h_n v$ for infinitely many $n \in \mathbb{N}$. So we are able to conclude that the stabilizer of v is infinite which is a contradiction. Hence for $n_i \in \mathbb{N}$ where $i = 1, 2$ we have to have

$$(h^{-j_m} h_m^{-1}) h^{-j_{n_1}} h_{n_1} = (h^{-j_m} h_m)^{-1} h^{-j_{n_2}} h_{n_2}.$$

The above equality implies that $Lh_{n_1} = Lh_{n_2}$ which yields a contradiction.

Case II: We assume that are only finitely many $i \in \mathbb{N}$ and $j_i \in \mathbb{N}$ such that $h_i Z = h^{j_i} Z$. We define the graph $X := \Gamma[A \Delta hA]$. We can conclude that $\Gamma = \cup_{i \in \mathbb{Z}} h^i X$. We can assume that $h_i Z \subseteq h^{j_i} X$ for infinitely many $i \in N$ and $j_i \in \mathbb{N}$ and so we have $h^{-j_i} h_i Z \subseteq X$. Let p be a shortest path between Z and hZ. For every vertex v of p, by Lemma 5.1.9 we know that there are finitely many tight separation gZ for $g \in G$ meeting v. So we infer that there are infinitely many $k \in \mathbb{N}$ such that $h^{-j_\ell} h_\ell Z = h^{-j_k} h_k Z$ for a specific $\ell \in \mathbb{N}$. Then with an analogue method we used for the preceding case, we are able to show that the stabilizer of at least one vertex of Z is infinite and again we conclude that $(h^{-j_m} h_m^{-1}) h^{-j_{n_1}} h_{n_1} = (h^{-j_m} h_m)^{-1} h^{-j_{n_2}} h_{n_2}$ for $n_1, n_2 \in \mathbb{N}$. Again it yields a contradiction. Hence each case gives us a contradiction and it proves our theorem as desired. □

5.2 Applications

In this section we use the results of the preceding section in order to study two-ended groups. We split this section into two parts. In Section 5.2.1 we investigate the characterization of two-ended groups. In Section 5.2.2 we study subgroups of those groups.

5.2.1 Two-ended groups

In the following we use the results of Section 5.1.2 to give an independent proof of some known characterizations of two-ended groups as well as a new characterization, see Theorem 5.2.1. It is worth mentioning that the equivalence of the items (i − iv) has been shown in by Scott and Wall [61]. The equivalence between the item (vi) and (i) has been proved by Dick and Dunwoody [13]. Finally Cohen in [12] proved that the item (vii) is equivalent to (i).

Theorem 5.2.1. *Let G be a finitely generated group. Then the following statements are equivalent:*

(i) *G is a two-ended group.*

(ii) *G has an infinite cyclic subgroup of finite index.*

(iii) *G has a finite normal subgroup K such that $G/K \cong D_\infty$ or $\mathbb{Z}$.*

(iv) *G is isomorphic to either $A *_C B$ and C is finite and $[A : C] = [B : C] = 2$ or $*_\phi C$ with C is finite and $\phi \in \mathsf{Aut}(C)$.*

(v) *Any Cayley graph of $G \sim_{QI} \Gamma(\mathbb{Z}, \pm 1)$.*

(vi) *There is an action of G on the double ray with finite stabilizers and one edge orbit.*

(vii) *The dimension of $H^1(G, \mathbb{Z}_2 G)$ is one.*[5]

[5] $H^i(G, X)$ denotes the ith cohomolgy group of the group G with coefficients in the ring X.

The above theorem with conjunction of Theorem 5.1.12 implies the following corollary immediately:

Corollary 5.2.2. *Let G be an infinite group acting with only finitely many orbits on a two-ended graph Γ without dominated ends. Then G is two-ended.* □

Before we can prove Theorem 5.2.1 we have to collect some tools and concepts used in the proof of Theorem 5.2.1. For the sake of simplicity, we introduce the following shorthand. We call

$$\mathsf{Hom}_{\mathbb{Z}}(\mathbb{Z}G, \mathbb{Z}_2) \text{ and } \mathsf{Hom}_{\mathbb{Z}}(\mathbb{Z}G, \mathbb{Z}_2)/\mathbb{Z}_2G$$

by $\overline{\mathbb{Z}_2G}$ and $\widetilde{\mathbb{Z}_2G}$, respectively. We notice that those groups can be regarded as $\mathbb{Z}_2$-vector spaces. We start with the following lemma which is known as Shapiro's Lemma.

Lemma 5.2.3. [6, Proposition 6.2] *Let H be a subgroup of a group G and let A be an RH-module. Then $H^i(H, A) = H^i(G, \mathsf{Hom}_{RH}(RG, A))$.*

Lemma 5.2.4. *Let G be a finitely generated group. Then*

$$dim H^0(G, \widetilde{\mathbb{Z}_2G}) = 1 + dim H^1(G, \mathbb{Z}_2G).$$

Proof. First of all, we note that the short exact sequence

$$0 \to \mathbb{Z}_2G \hookrightarrow \overline{\mathbb{Z}_2G} \twoheadrightarrow \widetilde{\mathbb{Z}_2G} \to 0$$

gives rise to the following long sequence:

$$0 \to H^0(G, \mathbb{Z}_2G) \to H^0(G, \overline{\mathbb{Z}_2G}) \to H^0(G, \widetilde{\mathbb{Z}_2G}) \to H^1(G, \mathbb{Z}_2G) \to 0$$

We notice that G acts on $\overline{\mathbb{Z}_2G}$ by $g.f(x) := gf(g^{-1}x)$ and it follows from Lemma 5.2.3 that $H^i(G, \overline{\mathbb{Z}_2G}) = 0$ for every $i \geq 1$. But $H^0(G, A)$ is an invariant subset of A under the group action of G. Thus we deduce that

$$H^0(G, \mathbb{Z}_2G) = 0 \text{ and } H^0(G, \overline{\mathbb{Z}_2G}) = \mathbb{Z}_2.$$

Hence we have

$$dimH^0(G,\widetilde{\mathbb{Z}_2G}) = 1 + dimH^1(G,\mathbb{Z}_2G). \qquad \square$$

Lemma 5.2.5. *Let $G = \langle S \rangle$ be a finitely generated group and $\Gamma := \Gamma(G,S)$. Then the spaces $P\Gamma$ and $F\Gamma$ can be identified by $\overline{\mathbb{Z}_2G}$ and $\mathbb{Z}_2G$, respectively.*

Proof. Suppose that $f \in \overline{\mathbb{Z}_2G}$. We define $A_f := \{g \in G \mid f(g) = 1\}$. Now it is straightforward to check that there is a one to one correspondence between $\overline{\mathbb{Z}_2G}$ and $P\Gamma$. The second case is obvious. $\square$

Lemma 5.2.5 directly yields the following corollary.

Corollary 5.2.6. *Let $G = \langle S \rangle$ be a finitely generated group and let Γ be the Cayley graph of G with respect to S. Then dimension of $Q\Gamma/F\Gamma$ is equal to $dimH^0(\Gamma,\widetilde{\mathbb{Z}_2G})$.* $\square$

Before we can start the proof of Theorem 5.2.1 we cite some well known facts we use proof of Theorem 5.2.1.

Lemma 5.2.7. [62, Theorem 15.1.13] *Let G be a finitely generated group such that $[G : Z(G)]$ is finite. Then G' is finite.* $\square$

Lemma 5.2.8. [39, Proposition 4.8] *Let G be a finitely generated group and let H and K be subgroups of G such that HK is also a subgroup of G. Then $[HK : H] = [K : H \cap K]$.* $\square$

Lemma 5.2.9 (N/C Theorem)**.** [62, Theorem 3.2.3] *Let G be a group and let $H \leq G$ then $N_G(H)/C_G(H)$ is isomorphic to a subgroup of $\mathsf{Aut}(H)$.* $\square$

Lemma 5.2.10. [70, Proposition 4.1] *Let Γ be an infinite graph, let e be an edge of Γ, and let k be a natural number. Then G has only finitely many k-tight cuts containing e.* $\square$

Lemma 5.2.11. [22, Theorem 1.1] *Let Γ be a connected graph with more than one end. Then there exists a k-tight cut (A,A^*) such that for any $g \in Aut(\Gamma)$ either $(A,A^*) \leq g(A,A^*)$ or vice versa.* $\square$

Let us now have a precise look at an HNN-extension.

Remark 5.2.12. *Let $C = \langle S \mid R \rangle$ be a finite group. Every automorphism ϕ of C gives us an HNN-extension $G = *_\phi C$. We can build an HNN-extension from an automorphism $\phi \colon C \to C$. Therefore C is a normal subgroup of G with the quotient $\mathbb{Z}$, as the presentation of HNN-extension $G = *_\phi C$ is*

$$\langle S, t \mid R,\, t^{-1}ct = \phi(c)\, \forall c \in C \rangle.$$

Hence G can be expressed by a semidirect product $C \rtimes \mathbb{Z}$ which is induced by ϕ.

We now are in the position to prove the main theorem of this section. Theorem 5.2.1. We illustrate the strategy to proof Theorem 5.2.1 in the following diagram, see Figure 5.1.

$$\begin{array}{ccccccc}
\text{(i)} & & \Longleftarrow & & \text{(vii)} & & \\
\Downarrow & & & & \Uparrow & & \\
\text{(ii)} & \Rightarrow & \text{(iii)} & \Rightarrow & \text{(iv)} & \Rightarrow & \text{(v)} \\
 & & & & \Uparrow & \swarrow & \\
 & & & & \text{(vi)} & &
\end{array}$$

Figure 5.1: Structure of the proof of Theorem 5.2.1

Proof of Theorem 5.2.1. (i) $\Rightarrow$ (ii) Let Γ be a Cayley graph of G and thus G acts on Γ transitively. Now it follows from Theorem 5.1.12 that G has an infinite cyclic subgroup of finite index.

(ii) $\Rightarrow$ (iii) Suppose that $H = \langle g \rangle$ and we may assume that H is normal, otherwise we replace H by $\mathsf{Core}(H)$. Let $K = C_G(H)$ and since $[G : H]$ is finite, we deduce that $[K : Z(K)]$ is finite , because H is contained in $Z(K)$ and the index of H in G is finite. In addition, we can assume that K is a finitely generated group, as $[G : K] < \infty$ we are able to apply Lemma 2.3.7. We now invoke Lemma 5.2.7 and conclude that K' is a finite subgroup. On the other hand K/K' must be a finitely generated abelian group. Since K is infinite, one may see that $K/K' \cong \mathbb{Z}^n \oplus K_0$, where K_0 is a finite abelian group and $n \geq 1$.

We now claim that $n = 1$. Since $[G : H] < \infty$ and $H \subseteq K$, we infer that $[K : H] < \infty$. But Lemma 2.3.7 implies that $e(K) = e(H \cong \mathbb{Z})$. Thus K is two-ended and if $n \geq 2$, then $\mathbb{Z}^n \oplus R$ is one-ended which is a contradiction. Hence the claim is proved. Next we define a homomorphism $\psi\colon K \to \mathbb{Z}$ with the finite kernel K_0. Since K_0 is finite subgroup of K such that $K/K_0 \cong \mathbb{Z}$, we deduce that $K_0 \mathsf{char} K$. It follows from Lemma 5.2.9, that $G/C_G(H)$ is isomorphic to a subgroup of $\mathsf{Aut}(\mathbb{Z}) \cong \mathbb{Z}_2$ and so we may assume that K is a normal subgroup of G. If $K = G$, then we are done. We suppose that $K < G$. We notice that $K_0 \mathsf{char} K \lhd G$ and so K_0 is a finite normal subgroup of G.

We claim that G/K_0 is not an abelian group. Since K is a proper subgroup of G, we are able to find $g \in G \setminus K$ such that g does not commutate with $h \in H \subseteq K$ and we have $h^{-1}ghg^{-1} \in H$. So gK_0 and hK_0 do not commutate and the claim is proved. Let aK_0 generate $K/K_0 \cong \mathbb{Z}$ and we pick up an element bK_0 in $(G/K_0) \setminus (K/K_0)$. We can see that $G/K_0 = \langle aK_0, bK_0 \rangle$. We note that $K/K_0 \trianglelefteq G/K_0$ and so $bab^{-1}K_0 = a^i K_0$ for some $i \in \mathbb{Z}$. Since K_0 is a finite group and G/K_0 is not abelian, we conclude that $bab^{-1}K_0 = a^{-1}K_0$. We already know that $[G : K] = 2$ and so $b^2K_0 \in K/K_0$. We assume that $b^2K_0 = a^jK_0$ for some $j \in \mathbb{Z}$. With $bab^{-1}K_0 = a^{-1}K_0$ and we deduce that $j = 0$. Thus $b^2K_0 = K_0$ and we conclude that $G/K_0 = K/K_0\langle bK_0 \rangle$. In other words one can see that $G/K_0 = \mathbb{Z}\mathbb{Z}_2$, where $\mathbb{Z}$ is a normal subgroup.

(iii) $\Rightarrow$ (iv) Let $G = KN$ such that N is a finite normal subgroup of G and $K \cong \mathbb{Z}$ or $K \cong D_\infty$ and moreover $K \cap N = 1$. If $K \cong \mathbb{Z}$, then by Remark 5.2.12 we get an HNN-extension of $*_\psi N$ where $\psi \in \mathsf{Aut}(N)$. So we may assume that $\phi\colon G/N \to \langle a \rangle * \langle b \rangle$, where $\langle a \rangle \cong \langle b \rangle \cong \mathbb{Z}_2$. Let A and B be the pull-backs of $\langle a \rangle$ and $\langle b \rangle$ by h, respectively. We note that the index of K in both of A and B is two. Let us define a homomorphism $\Phi\colon A *_C B \to G$, by setting $\Phi(X) = X$, where $X \in \{A, B\}$. It is not hard to see that Φ is an isomorphism.

(iv) $\Rightarrow$ (v) Assume that G is isomorphic to either $A *_C B$ where C is finite and $[A : C] = [B : C] = 2$ or $*_\phi C$ with C is finite and $\phi \in \mathsf{Aut}(C)$. If we consider a canonical generating set S for G, then one may see that $\Gamma(G, S)$ is a two-ended graph. So by Theorem 5.1.1 we are done.

(v) ⇒ (vi) Since the Cayley graph is quasi-isometric to the double ray, we conclude that G is a two-ended group. We choose a generating set S for G and consider $\Gamma := \Gamma(G, S)$. We now construct a "structure tree"[6] R of Γ, which will be the double ray, in such a way that G acts on R and all stabilizers are finite with exactly one edge orbit. It follows from Lemma 5.2.11 that there is a finite cut $C = (A, A^*)$ of Γ such that the set $\mathcal{S} := \{g(A, A^*) \mid g \in G\}$ is a nested set. As $\mathcal{S}$ is nested, we can consider $\mathcal{S}$ as a totally ordered set. Let $g \in G$ be such that $g(A, A^*)$ is the predecessor of (A, A^*) in this order. We may assume that $A \subsetneq gA$. This implies that $\Gamma \setminus (A \cup gA^*)$ is finite. Let $g' \in G$ such that $g'(A, A^*)$ is the predecessor of $g(A, A')$. We can conclude that $g^{-1}g'(A, A^*)$ is the predecessor of (A, A^*) and as predecessors are unique we can conclude that $g' = g^2$. Hence we can decompose Γ by g into infinitely many finite subgraphs such that between any two of these subgraphs there are finitely many edges. We now contract each finite subgraph to a vertex and for every finite cut between two consecutive subgraphs we consider an edge. Thus we obtain the double ray R in such way that G acts on R. It is straightforward to check that there is only one edge orbit. So we only need to establish that the stabilizers are finite. Let e be an edge of R. Then e corresponds to a k-tight cut C. It follows from Lemma 5.2.10 that there are finitely many k-tight cuts meeting C. So it means that the edge stabilizer of R is finite. With an analogous argument one can show that the vertex stabilizer of R is finite as well.

(vi) ⇒ (iv) Since G acts on the double ray, we are able to apply the Bass-Serre theory. So it follows from Lemma 2.3.3 that G is either a free product with amalgamation over a finite subgroup or an HNN-extension of finite subgroup. More precisely, the group G is isomorphic to $G_1 *_{G_2} G_3$ or $*_\phi G_1$, where G_i is finite subgroup for $i = 1, 2, 3$ and $\phi \in \mathsf{Aut}(G_2)$. On the other hand, Theorem 5.1.12 implies that G must be two-ended. Now we show that $[G_1 : G_2] = [G_1 : G_3] = 2$. We assume to contrary $[G_i : G_2] \geq 3$ for some $i \in \{1, 3\}$. Then $G_1 *_{G_2} G_3$ has infinitely many ends which yields a contradiction. One may use a similar argument to show that $G_1 = G_2$ for

[6] For more details about the structure tree see [52].

the HNN-extension.

(vi) ⇒ (vii) Since $\Gamma = \Gamma(G, S) \sim_{QI} R$, where R is the double ray, we conclude that G is a two-ended group. It follows from Lemma 5.2.4 that we only need to compute $dimH^0(G, \widetilde{\mathbb{Z}_2G})$ in order to calculate $dimH^1(G, \mathbb{Z}_2G)$. By Corollary 5.2.6, it is enough to show that the dimension of $Q\Gamma/F\Gamma$ is two. Let $\{e_1, \ldots, e_n\}$ be an independent vector of $Q\Gamma$. Since the co-boundary of each e_i is finite, we are able to find finitely many edges of G containing all co-boundaries, say K. We note that Γ is a locally finite two-ended graph and so we have only two components C_1 and C_2 of $\Gamma \setminus K$. Every e_i corresponds to a set of vertices of Γ. We notice that each e_i takes the same value on each C_i. In other words, e_i contains both ends of an edge $e \in C_i$ or none of them. We first assume that $2 \leq n$. Then there are at least two vectors of $\{e_1, \ldots, e_n\}$ which take the same value on a component C_1 and it yields a contradiction with independence of these vectors. Hence we have shown that $n \geq 2$. Let K be a finite set of vertices of Γ such that C_1 and C_2 are the infinite components of $\Gamma \setminus K$. Since the co-boundary of each C_i is finite, each C_i can be regarded as an element of $Q\Gamma/F\Gamma$ and it is not hard to see that they are independent.

(vii) ⇒ (i) As we have seen in the last part the dimension of $Q\Gamma/F\Gamma$ is exactly the number of ends. Hence Lemma 5.2.4 and Corollary 5.2.6 complete the proof. □

Remark 5.2.13. *It is worth remarking that by Part (iii) of Theorem* 5.2.1 *every two-ended group can be expressed by a semi-direct product of a finite group with* $\mathbb{Z}$ *or* D_∞.

5.2.2 Subgroups of two-ended groups

In this section we give some new results about subgroups of two-ended groups. It is known that every subgroup of finite dihedral is isomorphic to a cyclic group of another dihedral group. Next we prove this result for the infinite dihedral group.

Lemma 5.2.14. *Every subgroup of D_∞ is isomorphic to either a cyclic group or to D_∞.*

Proof. By the definition of D_∞ we know that each element of D_∞ can be expressed by $a^i b^j$ where $i \in \mathbb{Z}$ and $j \in \mathbb{Z}_2$. Let H be an arbitrary infinite subgroup of D_∞. We have a natural homomorphism from $f\colon H \to D_\infty/\langle a\rangle$. If the map f is trivial, then H is cyclic and we are done. So we can assume that f is surjective. We note that $K := \mathsf{Ker}(f)$ has index 2 in H and moreover $K = \langle a^i\rangle$ for some $i \geq 2$. Thus we deduce that H contains an element $a^j b$ where $j \in \mathbb{Z}$. It is straightforward to verify that the homomorphism $\psi\colon H \to D_\infty$ where ψ carries a^i to x and $a^j b$ to y is an isomorphic map, as desired. $\square$

Corollary 5.2.15. *Let H be an infinite subgroup of D_∞, then the index of H in D_∞ is finite.*

Proof. Assume that H is an arbitrary infinite subgroup of D_∞. Let us have a look at $H_1 := H \cap \langle a\rangle$. If H_1 is trivial, then since $\langle a\rangle$ is a maximal subgroup, one may see that $H\langle a\rangle = D_\infty$. So we infer that $H \cong D_\infty/\langle a\rangle \cong \mathbb{Z}_2$ which yields a contradiction. Thus H_1 is not trivial. Suppose that $H_1 = \langle a^i\rangle$ where $i \geq 1$. Because the index H_1 in D_∞ is finite, we have that $[G : H] < \infty$. $\square$

Theorem 5.2.16. *If G is a two-ended group and H is an infinite subgroup of G, then the following statements hold:*

(i) *H has finite index in G.*

(ii) *H is two-ended.*

Proof. It follows from part (iii) of Theorem 5.2.1 that there is a finite normal subgroup K such that G/K is isomorphic either to $\mathbb{Z}$ or to D_∞. First assume that H contains an element of K. In this case, H/K is isomorphic to a subgroup of $\mathbb{Z}$ or D_∞. By Corollary 5.2.15 we infer that $[G/K : H/K]$ is finite and so we deduce that $[G : H]$ is finite. Thus suppose that $K \nsubseteq H$. Since K is a normal subgroup of H, we know that HK is a subgroup of G. With an analogous argument of the preceding case we can see that $[G/K : HK/K]$ is finite and so $[G : HK]$ is finite. By Lemma 5.2.8 we have equality

$$[HK : K] = [K : H \cap K]$$

and so $[HK : K]$ is finite. On the other hand one can see that

$$[G : H] = [G : HK][HK : H].$$

Hence $[G : H] < \infty$, as desired. □

If we suppose that an infinite group G has more than one end, then the converse of the above theorem is also correct.

Theorem 5.2.17. *Let G be a finitely generated group with $e(G) > 1$ and the index of every infinite subgroup is finite, then G is two-ended.*

Proof. First we claim that G is not a torsion group. By Stallings theorem we know that we can express G as either free-product with amalgamation over finite subgroup or an HNN-extension over a finite subgroup. Thus we are able to conclude that G contains an element of infinite order, say g and the claim is proved. By assumption the index of $\langle g \rangle$ in G is finite. Thus the equivalence of (i) and (ii) in Theorem 5.2.1 proves that G is two-ended. □

The following example shows that we cannot drop the condition $e(G) > 1$ in the Theorem 5.2.17. For that let us recall some definition: An infinite group T is a *Tarski Monster group* if each nontrivial subgroup of T has p elements, for some fixed prime p. It is well known that such a group exists for large enough primes p.

Example 5.2.18. *Let T be a Tarski monster group for a large enough prime p. Note that it is well known that T is a finitely generated group. By the well known theorem of Stallings we know that $e(T) = 1$. We set $G := T \times \mathbb{Z}_2$. Note that G is also one-ended, as the index T in G is finite. In the following we show that the only infinite subgroup of G is T. Now let H be an infinite* *subgroup of G. It is obvious that $H \nsubseteq T$ as that would imply that H is finite. As T is a maximal subgroup of G we know that $TH = G$.*

$$2 = [G : T] = [TH : T] = [H : H \cap T].$$

For the last equality in the statement above we used Lemma 5.2.8. As $H \cap T$ is a subgroup of T we conclude it is finite. Thus we know that H is finite giving us a contradiction.

Theorem 5.2.19. *Let G be an infinite finitely generated solvable group[7] such that the index of every infinite subgroup is finite. Then G is two-ended.* □

Proof. First we show that G is not torsion. Assume to contrary that G is a torsion group. It is known that any finitely generated solvable torsion group is finite, see [60, Theorem 5.4.11]. This implies that G is finite and it yields a contradiction. Hence G has an element g of infinite order. Again by assumption we know that the index $\langle g \rangle$ is finite in G. Thus the equivalence of (i) and (ii) in Theorem 5.2.1 proves that G is two-ended. □

In the sequel, we are going to study the commutator subgroup of two-ended groups.

Theorem 5.2.20. *Let G be a two-ended group which splits over a subgroup C of order n. Then either $4 \leq [G : G'] \leq 4n$ or $|G'| \leq n$.*

Proof. If G is an HNN-extension, then $G = C\mathbb{Z}$. So G/C is an abelian group and we infer that G' is a subgroup of C and we are done. So we assume that G is a free product with amalgamation over C. In this case, $G/C \cong D_\infty$. It is not hard to see that the commutator subgroup of D_∞ is generated by $\langle a^2 \rangle$. thus we deduce that $G'K/K$ has index 4 in G/K. In other words, one can see that $[G : G'K] = 4$. On the other hand, we have $G'K/G' \cong K/G' \cap K$. Hence we can conclude that $[G : G']$ does not exceed $4n$. □

We close Chapter 5 with the following example.

[7] A group G is *solvable* if the derived series terminates, i.e. there exists a k such that $G^{(k)} = 1$ with $G^{(0)} = G$ and $G^{(n)} = [G^{(n-1)}, G^{(n-1)}]$.

Example 5.2.21. *For instance, suppose that G is a semi-direct product of $\mathbb{Z}_n$ by $\mathbb{Z}$. It is straightforward that $G' \cong \mathbb{Z}_n$. For the other case let $G = D_\infty \times A_5$, where A_5 is the alternating group on the 5 letters. We note that A_5 is a perfect group and so $A_5' = A_5$. Now we can see that $[G : G'] = 240$.*

Chapter 6

Splitting graphs

We want to remind the reader, that in Chapter 6 the symbols of groups and graphs change, see Chapter 2 for the reasoning. In this chapter we denote groups by Γ and graphs by G.

6.1 Finding tree-decompositions

We start this section by studying separations and separation systems. Our goal is to show that we can separate any two given ends of a graph by separations which behave nicely.

So let G be a locally finite graph. For two different given ends ω_1 and ω_2 let (A, A^*) be a splitting separation such that its separator is the minimum size among all separator of splitting separations separating ω_1 and ω_2. We define $\mathcal{S}(\omega_1, \omega_2)$ as the set of all separations (B, B^*) separating ω_1 and ω_2 such that $|B \cap B^*| = |A \cap A^*|$, i.e.

$$\mathcal{S}(\omega_1, \omega_2) = \{(B, B^*) \mid (B, B^*) \text{ separates } \omega_1 \text{ and } \omega_2; |A \cap A^*| = |B \cap B^*|\}.$$

We notice that with this notation, ω_1 and ω_2 live in B and B^*, respectively.

For a given graph G let $\mathcal{S}_k$ be the set of all tight splitting k-separations of G. We denote the set of all tight k-separations by $\mathcal{S}_k(G)$.

It will be important to our arguments that we can limit the number of some special type of separations meeting a given finite vertex set S. For this we cite a lemma by Thomasen and Woess.

Lemma 6.1.1. [70, Corollary 4.3] *Let $S \subseteq V(G)$ be a finite set of a locally finite graph G. Then there are only finitely many $(A, A^*) \in \mathcal{S}_k(G)$ such that their separators meet S.* $\square$

For two given ends ω_1 and ω_2 of G, we can find a tight m-separation which separates ω_1 and ω_2. Now for a separation $(A, A^*) \in \mathcal{S}(\omega_1, \omega_2)$, we associate to the separation (A, A^*), a set containing all crossing tight ℓ-separations where $\ell \leq k$ and we denote it by $N(A, A^*)$ i.e.

$$N_k(A, A^*) := \{(B, B^*) \in \bigcup_{\ell \leq k} \mathcal{S}_\ell(G) \mid (A, A^*) \nparallel (B, B^*)\}$$

It follows from Lemma 6.1.1 that the size of $N_k(A, A^*)$ for a separation (A, A^*) is finite. We denote this size by $n_k(A, A^*)$. We call this number the *crossing number* of the separation (A, A^*). We set $n(\omega_1, \omega_2)$ to be the minimum number among all numbers $n_k(A, A^*)$ for all elements of $\mathcal{S}(\omega_1, \omega_2)$, i.e.

$$n_k(\omega_1, \omega_2) := \min\{n_k(A, A^*) \mid (A, A^*) \in \mathcal{S}(\omega_1, \omega_2)\}.$$

A separation in $\mathcal{S}(\omega_1, \omega_2)$ is called *narrow separation of type* (ω_1, ω_2, k) if its crossing number is equal to $n_k(\omega_1, \omega_2)$ and if additionally $n_k(\omega_1, \omega_2) \geq 1$. We denote the set of all narrow separations of type (ω_1, ω_2, k) by $\mathcal{N}_k(\omega_1, \omega_2)$.

Let us define $\mathcal{N}^k$ as the set of separations which are narrow for a pair two different ends, i.e. $\mathcal{N}^k := \bigcup \mathcal{N}_k(\omega_1, \omega_2)$, for all $\omega_1 \neq \omega_2 \in \Omega(G)$. Let $\mathcal{N}^k_\ell \subseteq \mathcal{N}^k$ be the set of all the separations in $\mathcal{N}^k$ with separators of size at most ℓ for $\ell \in \mathbb{N}$. Please note that $\mathcal{N}^k_\ell$ and $\mathcal{N}^k$ are symmetric.

Theorem 6.1.2. *Let Γ be a group acting on a locally finite graph G with finitely many orbits. Then the action Γ on $\mathcal{N}_\ell$ has finitely many orbits.*

Proof. Assume that $U \subseteq V(G)$ is finite such that $\Gamma U = V(G)$. It follows from Lemma 6.1.1 that there are only finitely many narrow separations whose separators meet U, say (A_i, A_i^*) for $i = 1, \ldots, m$. Suppose that (A, A^*) is an arbitrary separation in $\mathcal{N}_\ell$. Let $v \in A \cap A^*$ be an arbitrary vertex. By the definition of U we can now map x into U by some $g \in \Gamma$. We can conclude that $g(A \cap A^*)$ is a separator of a separation that meets U, as it contains gx. Thus we can conclude that $g(A, A^*)$ is one of the (A_i, A_i^*)'s. $\square$

Next we are going to show that $\mathcal{N}^k$ is a nested set. In order to show this, we have to verify some facts and lemmas. Let $(A, A^*) \in \mathcal{N}_k(\omega_1, \omega_2)$ and $(B, B^*) \in \mathcal{N}_k(\omega_1', \omega_2')$ be two crossing narrow separations. Let W be defined as $W := \{\omega_1, \omega_2, \omega_1', \omega_2'\}$. Then we have the two following cases:

- There is exactly one corner separation of $\{(A, A^*), (B, B^*)\}$ that does not capture an end in W.
- Every corner separation of $\{(A, A^*), (B, B^*)\}$ captures an end of W.

We study each case independently. The aim is to show that there are always two opposite corners capturing the ends ω_1 and ω_2 which belong to $\mathcal{S}(\omega_1, \omega_2)$.

Lemma 6.1.3. *Let $(A, A^*) \in \mathcal{S}(\omega_1, \omega_2)$ and $(B, B^*) \in \mathcal{S}(\omega_1', \omega_2')$ be two crossing separations and let $W = \{\omega_1, \omega_2, \omega_1', \omega_2'\}$. If there is exactly one corner separation of $\{(A, A^*), (B, B^*)\}$ that does not capture an end in W, then there are two opposite corners capturing ends of W which belong to $\mathcal{S}(x, y)$ for suitable $x, y \in W$.*

Proof. Let $(A, A^*) \in \mathcal{S}(\omega_1, \omega_2)$ and $(B, B^*) \in \mathcal{S}(\omega_1', \omega_2')$ be two crossing separations and let $W = \{\omega_1, \omega_2, \omega_1', \omega_2'\}$. Such that there is exactly one corner separation of $\{(A, A^*), (B, B^*)\}$ that does not capture an end in W, then there are two opposite corners capturing ends of W. Either there are exactly two or exactly three corners capturing ends of W. If there are exactly two corners capturing ends of W, then those corners are opposite corners and we are done. So we may assume that there are exactly three corners capturing ends of W. Without loss of generality, let us assume that $(A \cap B^*, A^* \cup B)$ does not capture an end of W. In the following we assume that the ends of W are distributed as shown in the Figure 6.1. We denote the numbers of vertices in various subsets of the separators with the letters *a*-*e* as indicated in Figure 6.1.

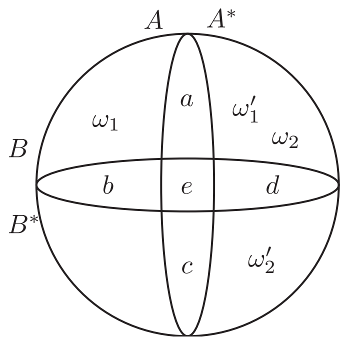

Figure 6.1: Crossing separations with one corner without an end.

Note that the separation $(A \cap B, A^* \cup B^*)$ separates ω_1 and ω_2. Furthermore note that $(A^* \cap B^*, A \cup B)$ separates ω_1' and ω_2'. This implies that

$$a + b + e \geq a + e + c \text{ and } c + e + d \geq b + e + d.$$

Thus one can see that $b = c$ and we deduce that $(A \cap B, A^* \cup B^*) \in \mathcal{S}(\omega_1, \omega_2)$ and $(A^* \cap B^*, A \cup B) \in \mathcal{S}(\omega_1', \omega_2')$, as desired. With analogous methods one can easily verify the other possible distributions of the ends of W, we omit this here. $\square$

Lemma 6.1.4. *Let $(A, A^*) \in \mathcal{S}(\omega_1, \omega_2)$ and $(B, B^*) \in \mathcal{S}(\omega_1', \omega_2')$ be two crossing separations and let $W = \{\omega_1, \omega_2, \omega_1', \omega_2'\}$. Then if every corner separation of $\{(A, A^*), (B, B^*)\}$ captures an end of W, then every corner belongs to $\mathcal{S}(x, y)$ for suitable $x, y \in W$.*

Proof. As every corner separation of $\{(A, A^*), (B, B^*)\}$ captures an end of W, we know that (A, A^*) separates ω_1' and ω_2' and moreover ω_1 and ω_2 are separated by (B, B^*). Thus $|A \cap A^*| = |B \cap B|$ and so $(B, B^*) \in \mathcal{S}(\omega_1, \omega_2)$ and $(A, A^*) \in \mathcal{S}(\omega_1', \omega_2')$. Now let the ends of W be distributed as shown in Figure 6.2.

We shall show that the size of separator $(A \cap B, A^* \cup B^*)$ is exactly the same as the size of separator (A, A^*). Since the separation $(A \cap B, A^* \cup B^*)$ separates ω_1 and ω_2, we can conclude that

$$a + b + e \geq a + e + c.$$

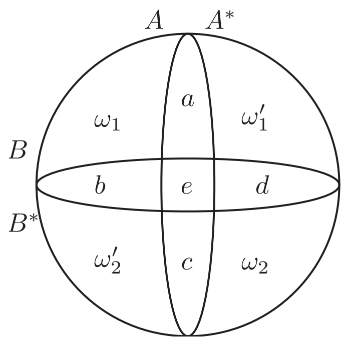

Figure 6.2: Crossing separations where an end lives in every corner.

Analogously ω_1 and ω_2 can be separated by the separation

$$(A^* \cap B^*, A \cup B) \text{ and so } c + e + d \geq a + e + c.$$

We deduce that $b = c$ and this means that the separation $(A \cap B, A^* \cup B^*)$ belongs to $\mathcal{S}(\omega_1, \omega_2)$. With the similar method, one can verify that $a = d$ and show an analogous result for the other corners. □

The next lemma we need shows that when dealing with nested separations the corner separations behave in a nice way. For this we need an infinite version of a lemma in [10] which has been proved by Carmesin, Diestel, Hundertmark and Stein.

Lemma 6.1.5. *Let $(A, A^*), (B, B^*)$ and (C, C^*) be splitting separations. Additionally let $(A, A^*) \nparallel (B, B^*)$. Then the following statements hold:*

(i) *If $(C, C^*) \parallel (A, A^*)$ and $(C, C^*) \parallel (B, B^*)$, then (C, C^*) is nested with every corner separation of $\{(A, A^*), (B, B^*)\}$.*

(ii) *If $(C, C^*) \parallel (A, A^*)$ or $(C, C^*) \parallel (B, B^*)$, then (C, C^*) is nested with any two opposite corner separations of $\{(A, A^*), (B, B^*)\}$.*

Proof. For the proof of the (i), see [10, Lemma 2.2].[1] In the following we prove the second part here. Assume to the contrary that (C, C^*) is neither

[1]Even though the proof in [10] is just for finite graphs, it works totally analogously.

nested with $(A \cap B, A^* \cup B^*)$ nor with $(A^* \cap B^*, A \cup B)$. Without loss of generality, we can suppose that

$$C \subseteq B \text{ and so } B^* \subseteq C^*.$$

So we conclude that

$$C \cap (A \cup B) = C \text{ and we conclude that } A \cup B \supseteq C.$$

On the other hand, we have

$$C^* \cap (A^* \cap B^*) = A^* \cap B^* \text{ and it yields that } C^* \supseteq (A^* \cap B^*).$$

Hence we found that $(A \cup B, A^* \cap B^*) \leq (C, C^*)$ and it yields a contradiction. The other cases are similar to the above case. □

In Theorem 6.1.6 we now prove our aim, i.e. we show that $\mathcal{N}^k$ is a nested set.

Theorem 6.1.6. *Let G be a locally finite graph. Then the set $\mathcal{N}^k$ is a nested set.*

Proof. Assume for a contradiction that

$$(A, A^*) \in \mathcal{N}_k(\omega_1, \omega_2) \text{ and } (B, B^*) \in \mathcal{N}_k(\omega_1', \omega_2')$$

are two crossing narrow separations. Set $W := \{\omega_1, \omega_2, \omega_1', \omega_2'\}$.

Let (X, X^*) and (Y, Y^*) be two opposite corner separations such that exactly one end in W lives in X and Y, respectively. Now we need the following two claims:

$$\textbf{Claim I: } N_k(X, X^*) \cap N_k(Y, Y^*) \subseteq N_k(A, A^*) \cap N_k(B, B^*).$$

Let $(C, C^*) \in N_k(X, X^*) \cap N_k(Y, Y^*)$. Then we have

$$(C, C^*) \nparallel (X, X^*) \text{ and } (C, C^*) \nparallel (Y, Y^*)$$

So it follows from part (ii) of Lemma 6.1.5 that

$$(C, C^*) \nparallel (A, A^*) \text{ and } (C, C^*) \nparallel (B, B^*)$$

and we are done.

Claim II: $N_k(X, X^*) \cup N_k(Y, Y^*) \subsetneq N_k(A, A^*) \cup N_k(B, B^*)$.

To show the inclusion suppose that

$$(C, C^*) \in N_k(X, X^*), \text{ but}$$

$$(C, C^*) \notin N_k(A, A^*) \text{ and } (C, C^*) \notin N_k(B, B^*).$$

So $(C, C^*) \parallel (A, A^*)$ and (B, B^*). By first part of Lemma 6.1.5 we conclude that (C, C^*) is nested with every corner of $\{(A, A^*), (B, B^*)\}$. Therefore we get a contradiction, as $(C, C^*) \in N_k(X, X^*)$.

As (A, A^*) is assumed to be crossing (B, B^*) we know

$$(A, A^*) \in N_k(A, A^*) \cup N_k(B, B^*).$$

We know that (A, A^*) is nested with both (X, X^*) and (Y, Y^*). Thus Claim II is proved.

By symmetry and by renaming the ends and the sides we only have two cases:
Case I: ω_1 lives in $A \cap B$ and ω_2' lives in $A^* \cap B^*$.
By Lemma 6.1.3 we conclude that

$$(A \cap B, A^* \cup B^*) \in \mathcal{S}(\omega_1, \omega_2) \text{ and } (A^* \cap B^*, A \cup B) \in \mathcal{S}(\omega_1', \omega_2').$$

As both (A, A^*) and (B, B^*) are narrow, we know that

$$n_k(A \cap B, A^* \cup B^*) \geq n_k(\omega_1, \omega_2) \text{ and } n_k(A^* \cap B^*, A \cup B) \geq n_k(\omega_1', \omega_2').$$

Claim I yields

$$\begin{aligned}&|N_k(A\cap B, A^*\cup B^*)\cap N_k(A^*\cap B^*, A\cup B)|\\ &\qquad \le |N_k(A,A^*)\cap N_k(B,B^*)|\end{aligned}$$

Claim II yields

$$\begin{aligned}&|N_k(A\cap B, A^*\cup B^*)\cup N_k(A^*\cap B^*, A\cup B)|\\ &\qquad < |N_k(A,A^*)\cup N_k(B,B^*)|\end{aligned}$$

Now we have a simple calculation.

$$\begin{aligned}n_k(\omega_1,\omega_2)+n_k(\omega_1',\omega_2') &\le n_k(A\cap B, A^*\cup B^*)+n_k(A^*\cap B^*, A\cup B)\\ &= |N_k(A\cap B, A^*\cup B^*)\cup N_k(A^*\cap B^*, A\cup B)|\\ &\quad +|N_k(A\cap B, A^*\cup B^*)\cap N_k(A^*\cap B^*, A\cup B)|\\ &< |N_k(A,A^*)\cup N_k(B,B^*)|+|N_k(A,A^*)\cap N_k(B,B^*)|\\ &= |N_k(A,A^*)|+|N_k(B,B^*)| = n_k(A,A^*)+n_k(B,B^*)\\ &= n_k(\omega_1,\omega_2)+n_k(\omega_1',\omega_2').\end{aligned}$$

In other words, we found that

$$n_k(\omega_1,\omega_2)+n_k(\omega_1',\omega_2') < n_k(\omega_1,\omega_2)+n_k(\omega_1',\omega_2')$$

and this yields a contradiction.
Case II: ω_1 lives in $A\cap B$ and ω_2 lives in $A^*\cap B^*$.
By switching the names of ω_1' and ω_2' we can assume that ω_1' lives in $A\cap B^*$.
By Lemma 6.1.4 we conclude that

$$\begin{aligned}(A\cap B, A^*\cup B^*)\in\mathcal{S}(\omega_1,\omega_2) \text{ and } (A\cap B^*, A^*\cup B)\in\mathcal{S}(\omega_1',\omega_2')\\ \text{and } (A^*\cap B, A\cup B^*)\in\mathcal{S}(\omega_1',\omega_2') \text{ and } (A^*\cap B^*, A\cup B)\in\mathcal{S}(\omega_1,\omega_2)\end{aligned}$$

In the same manner to the above calculation we now obtain:

$$\begin{aligned}
2n_k(\omega_1,\omega_2)+2n_k(\omega_1',\omega_2') \leq{}& n_k(A\cap B, A^*\cup B^*)+n_k(A^*\cap B^*, A\cup B)\\
&+n_k(A^*\cap B, A\cup B^*)+n_k(A\cap B^*, A^*\cup B)\\
&=2|N_k(A\cap B, A^*\cup B^*)\cup N_k(A^*\cap B^*, A\cup B)|\\
&+2|N_k(A\cap B, A^*\cup B^*)\cap N_k(A^*\cap B^*, A\cup B)|\\
&<2|N_k(A,A^*)\cup N_k(B,B^*)|+2|N_k(A,A^*)\cap N_k(B,B^*)|\\
&=2|N_k(A,A^*)|+2|N_k(B,B^*)|=2n_k(A,A^*)+2n_k(B,B^*)\\
&=2n_k(\omega_1,\omega_2)+2n_k(\omega_1',\omega_2').
\end{aligned}$$

This is again a contradiction and hence we are done. $\square$

It is known that every symmetric nested system of separations of a finite graph can be used to define a tree-decomposition. See the work of Carmesin, Diestel, Hundertmark and Stein [10].

We will use the same tools in order to define a tree-decomposition of an infinite quasi-transitive graph G. We define a relation between separations of a system of nested separations. Let $\mathcal{O}$ be a symmetric system of nested separations. Assume that (A,A^*) and (B,B^*) belong to $\mathcal{O}$.

$$(A,A^*)\sim(B,B^*) :\Leftrightarrow \begin{cases} (A,A^*)=(B,B^*) \text{ or}\\ (A^*,A) \text{ is a predessor}^2\text{of } (B,B^*) \text{ in } (\mathcal{O},\leq)\end{cases}$$

It follows from [10, Lemma 3.1] that $\sim$ is an equivalence relation. We denote the equivalence class of (A,A^*) by $[(A,A^*)]$. We now are ready to define a tree-decomposition $(T,\mathcal{V})$ of G. We define the nodes for the tree T of the tree-decomposition $(T,\mathcal{V})$ as the equivalence classes. More precisely

$$V_{[A,A^*]}:=\bigcap\{B\mid (B,B^*)\in[(A,A^*)]\}$$

Now put $\mathcal{V}:=\{V_{[A,A]}\}$. For every $[(A,A^*)]$ we add the edge $[(A,A^*)][(A^*,A)]$ and so $(T,\mathcal{V})$ is a tree-decomposition of G.

[2] In a partial order $(P,\leq)$, an element $x\in P$ is a predecessor of an element $z\in P$ if $x<z$ but there is no $y\in P$ with $x<y<z$.

A symmetric nested systems of separations $\mathcal{O}$ is *sparse* if for every

$$(A, A^*), (B, B^*) \in \mathcal{O}$$

there are only finitely many $(C, C^*) \in \mathcal{O}$ such that

$$(A, A^*) \leq (C, C^*) \leq (B, B^*).$$

By [10, Lemma 3.2, Lemma 3.3, Theorem 3.4] we get the following lemma:[3]

Lemma 6.1.7. [10] Let G be a locally finite graph, and let $\mathcal{O}$ be a sparse symmetric nested systems of separations, then $\mathcal{O}$ defines a tree-decomposition of G. □

Using Lemma 6.1.1 we obtain the following corollary to Theorem 6.1.6.

Corollary 6.1.8. *Let G be a quasi-transitive graph then $\mathcal{N}_\ell$ is a sparse symmetric nested system of separations for each $\ell \in \mathbb{N} \cup \{0\}$.*

Proof. By Theorem 6.1.6 we know that $\mathcal{N}_\ell$ is nested as $\mathcal{N}_\ell^k \subseteq \mathcal{N}^k$. Let (A, A^*) and (B, B^*) be two separations in $\mathcal{N}_\ell^k$. Let x be a vertex in a shortest path between a vertex v in $A \cap A^*$ and a vertex w in $B \cap B^*$. By Lemma 6.1.1 we know there are only finitely many separators in $\mathcal{N}_e^k ll$ which contain x. As there are only a finite number of pairs of vertex v, w with $v \in A \cap A^*$ and $w \in B \cap B^*$ we are done. □

Let Γ be a group acting on a locally finite graph G with at least two ends. A tree-decomposition $(T, \mathcal{V})$ for G with the following properties is a *type 0 tree-decomposition with respect to* Γ:

(i) $(T, \mathcal{V})$ distinguishes at least two ends.

(ii) $(T, \mathcal{V})$ has finite adhesion.

(iii) Γ acts transitively on the edges of T.

[3] The proofs in [10] are just for finite graphs. But with the additional assumption that the system is sparse the proofs are identical.

If the group acting on G is obvious in the context we just omit naming the group and just say $(T, \mathcal{V})$ a type 0 tree-decomposition of G.

Theorem 6.1.9. *Let* Γ *be a group acting on a locally finite graph* G *with at least two ends. Then there is a type 0 tree-decomposition* $(T, \mathcal{V})$ *for* G.

Proof. By Lemma 6.1.7 it is enough to find a sparse symmetric nested set of splitting separations that is invariant under Γ. Assume that $(A, A^*) \in \mathcal{N}^k$ and let $\mathcal{O}$ be the orbit of (A, A^*) under Γ.[4] As Γ is acting on G we know that $g(A, A^*) \in \mathcal{N}^k$ for each $g \in \Gamma$. So it follows from Theorem 6.1.6 that $\mathcal{O}$ is nested. By Corollary 6.1.8 we know that $\mathcal{O}$ is sparse. It is obvious that making $\mathcal{O}$ symmetric by adding (A^*, A) to $\mathcal{O}$ whenever $(A, A^*) \in \mathcal{O}$ does not change $\mathcal{O}$ being nested nor sparse, hence by the method mentioned above, we are done. $\square$

Let Γ be a group acting on a locally finite graph G with at least two ends. A type 0 tree-decomposition $(T, \hat{\mathcal{V}})$ with additional properties that each adhesion set is connected is a *type 1 tree-decomposition with respect to* Γ. As with type 0 tree-decomposition we omit 'with respect to Γ' if the group acting on the graph is clear.

In the following Theorem 6.1.10 we modify $(T, \mathcal{V})$ given by Theorem 6.1.9 in order to obtain a type 1 tree-decomposition.

Theorem 6.1.10. *Let* Γ *be a group acting on a locally finite graph* G. *There is a type 1 tree-decomposition of* G *with respect to* Γ.

Proof. We use Theorem 6.1.9 to find a type 0 tree-decomposition $(T, \mathcal{V})$ of G. Let u and v be two vertices of an adhesion set $V_t \cap V_{t'}$. Assume that $\mathcal{P}$ is the set of all geodesics between u and v and assume that V_1 is the set of all vertices of G which are contained in a geodesic in $\mathcal{P}$. Now we add all vertices of V_1 to the adhesion set $V_t \cap V_{t'}$. We continue for each pair of vertices in any adhesion set. We denote a new decomposition by $(T, \hat{\mathcal{V}})$ and the part obtained from V_t is called $\hat{V}_t$.

[4]Note that all separators of separations in $\mathcal{O}$ have the same size and hence $\mathcal{O} \subseteq \mathcal{N}_e^k ll$ for some k, ℓ.

Now we show that $(T, \hat{\mathcal{V}})$ is a type 1 tree-decomposition. For that we first show, that $(T, \hat{\mathcal{V}})$ is indeed a tree-decomposition. As $(T, \mathcal{V})$ is a tree-decomposition it suffices to show that if there is a vertices x such that $x \in \hat{V}_t$ and $x \in \hat{V}_{t'}$ then x is also in all $\hat{V}_{t''}$ for all t'' on the $t - t'$ path in T. As we have not removed any vertices from any part, it suffices to check this for vertices which were contained in a geodesic in the process of connecting the adhesion sets. So let x_1 and x_2 be to distinct vertices in an adhesion set and let P be a geodesic between x_1 and x_2. Additionally let c be a different than x_1 or x_2 on P. Say $x_1, x_2 \in \hat{V}_t$ and $c \in \hat{V}_{t'} \setminus V_t$ for some $t' \neq t$. Assume that there is a t'' which is on a $t - t'$ path such that $t \neq t'' \neq t'$. We may assume that $c \in V_{t'} \setminus V_{t''}$. We have to show that $c \in \hat{V}_{t''}$. Let S be the adhesion set of $(T, \mathcal{V})$ corresponding to the edge of T that separates t'' from t'. Let $P' = p_1, \ldots p_n$ be the subpath of P such that p_1 is the first vertex that P has in S and p_n is the last vertex P has in S. As P is a geodesic, this implies that P' is a $p_1 - p_n$ geodesic. By our assumptions we know that $c \in P'$. This implies that $c \in V_{t''}$.

Now we show that $(T, \hat{\mathcal{V}})$ still distinguishes at least two ends, has a finite adhesion set and Γ acts on $(T, \hat{\mathcal{V}})$. There are two ends ω_1 and ω_2 which are separated by $(T, \mathcal{V})$. It means that there exist two rays $R_i \in \omega_i$ for $i = 1, 2$ and $t_1t_2 \in E(T)$ such that $V_{t_1} \cap V_{t_2}$ separates ω_1 and ω_2. Assume that T_i is the component of $T - t_1t_2$ containing the node t_i for $i = 1, 2$. Without loss of generality we can assume that $\bigcup_{t \in T_i} V_t$ contains a tail of R_i. So this yields that $\hat{V}_{t_1} \cap \hat{V}_{t_2}$ separates tails of R_1 and R_2 where $\hat{V}_{t_i}$ is induced part by V_{t_i} for $i = 1, 2$ as $(V_{t_1} \cap V_{t_2}) \subseteq (\hat{V}_{t_1} \cap \hat{V}_{t_2})$.

To see that all the adhesion sets of $(T, \hat{\mathcal{V}})$ are finite, one might note the following: Let P be a geodesic and $v, w \in P$. This implies that vPw[5] is a geodesic between v and w. This directly implies that we only added finitely many vertices to each adhesion set as G is locally finite. Since we added all vertices of geodesics between vertices of adhesion sets, the construction of $(T, \hat{\mathcal{V}})$ implies that Γ acts on $(T, \hat{\mathcal{V}})$. Thus $(T, \hat{\mathcal{V}})$ is a type 1 tree-decomposition with respect to Γ, as desired. $\square$

[5] For a path P and two vertices $v, w \in P$ we define the path from v to w contained in P as vPw.

By the proof of Theorem 6.1.10 we get the following corollary which will be useful in Section 6.2.

Corollary 6.1.11. *Let $(T, \mathcal{V})$ be a type 0 tree-decomposition of a locally finite graph G with respect to a group Γ. Then $(T, \mathcal{V})$ can be extended to a type 1 tree-decomposition $(T, \hat{\mathcal{V}})$ of G with respect to Γ.*[6] □

We call a tree-decomposition of a graph G *connected* if all parts are connected. In the following lemma we show that any tree-decomposition of a connected graph is connected if all of its adhesion sets are connected. The proof of Lemma 6.1.12 is a little bit technical but the intuition is quite easy. We pick two arbitrary vertices in the same part. As our graph is connected we can pick a path connecting those vertices in the entire graph. Such a path must leave and later reenter that part through an adhesion set. Even stronger it must leave and reenter any part through the same adhesion set. As we assume every adhesion set to be connected we can change the path to instead of leaving the part to be rerouted inside that adhesion set.

Lemma 6.1.12. *A tree-decomposition of a connected graph G is connected if all its adhesion sets are connected.*

Proof. Suppose that u and w are two vertices of V_t for some $t \in V(T)$. Since G is connected, there is a path $P = p_1, \ldots, p_n$ between u and w and lets say $p_1 = u$ and $p_n = w$. If $P \subseteq V_t$ then we are done. So we may assume that P leaves V_t. Let $p_i \in V_t$ such that $p_{i+1} \notin V_t$ and let p_{i^+} be the first vertex of P that comes after p_i such that $p_{i^+} \in V_t$. We say the vertex p_{i^+} corresponds to the vertex p_i. As $u = p_1$ and $p_n = w \in V_t$ we know that such a vertex must always exist. Let X be the set of all vertices $p_i \in V_t$ such that $p_{i+1} \notin V_t$ and let X^+ be the set of all vertices p_{i^+} corresponding to vertices in X. By the definition of a tree-decomposition we know that for each i such that $p_i \in X$ there is an adhesion set S_i such that $p_i \in S$ and $p_{i^+} \in S$. Now we are ready to change the path P to be completely

[6] Extending here is meant in the sense of the proof of Theorem 6.1.9. I.e. we extend a tree-decomposition by, for each part, adding a finite number of vertices to that parts whilst keeping it a tree-decomposition

contained in V_t. Let i be the smallest integer such that $p_i \in V_t$ and let S_i be the adhesion set containing both p_i and p_{i^+}. We pick a path Q_i from p_i to p_{i^+} contained in S_i. Let k be the largest natural number such that p_k is contained in Q_i. We change the path P to go to p_i and then to use Q_i till the vertex p_k and then continue on along P. It is straightforward to see that the new path P contains less vertices outside of V_t. Iterating this process yields a $u-w$ path completely contained in V_t. $\square$

Theorem 6.1.13. *Let Γ be a group acting on a locally finite graph with finitely many orbits. Additionally let $(T, \hat{\mathcal{V}})$ be a type 1 tree-decomposition of G. Then there exists $H \leq \Gamma$ whose action on each part of $(T, \hat{\mathcal{V}})$ has finitely many orbits.*

Proof. Let $\hat{V}_t = [(A, A^*)]$ be an arbitrary part of $(T, \hat{\mathcal{V}})$. We claim that the stabilizer of $\hat{V}_t$ in Γ satisfies the assumption of H. We define

$$K_B := \{g \in \Gamma \mid g(B, B^*) \sim (B, B^*)\} \text{ for every } (B, B^*) \sim (A, A^*).$$

It is not hard to see that K_B is a subgroup of Γ and moreover $K_B \subseteq \Gamma_{\hat{V}_t}$ for each $(B, B^*) \sim (A, A^*)$. Let $g \in \Gamma$ such that $g(B, B^*) \sim (B, B^*)$ and let (C, C^*) be a separation such that $g(B, B^*) \sim (C, C^*)$, then we know that $(B, B^*) \sim (C, C^*)$ and so $g \in \Gamma_{\hat{V}_t}$.

We now show that $\Gamma_{\hat{V}_t}$ acts on the set $\{B \mid (B, B^*) \sim (A, A^*)\}$ with only two orbits. As $(T, \hat{\mathcal{V}})$ is type 1 tree-decomposition we know that Γ acts on the sides of the separations with only two orbits. Assume for a contradiction that there are at least three orbits $\{B_i\}_{i \in \{1,2,3\}}$ on $\{B \mid (B, B^*) \sim (A, A^*)\}$ where $(A, A^*) \sim (B_i, B_i^*)$ for every $i \in \{1, 2, 3\}$. There are an element $g \in \Gamma$ and $i, j \in \{1, 2, 3\}$ in such a way that $B_i = gB_j$. On the other hand, we have $(B_i, B_i^*) \sim (A, A^*)$ which yields a contradiction. We use the fact that $g(B_j, B_j^*) \sim (B_j, B_j^*)$ to infer that $g \in K_{B_j} \subseteq \Gamma_{\hat{V}_t}$, but we know that B_i and B_j belong to different orbits under the action $\Gamma_{\hat{V}_t}$.

Next we show that the action of $\Gamma_{\hat{V}_t}$ on the adhesion sets of $\hat{V}_t$ has only two orbits. Assume to contrary that the action $\Gamma_{\hat{V}_t}$ has at least three orbits $\{B_i \cap B_i^* \mid (B_i, B_i^*) \sim (A, A^*)\}_{i \in \{1,2,3\}}$. Since the group $\Gamma_{\hat{V}_t}$ acts with only two orbits on $\{B \mid (B, B^*) \sim (A, A^*)\}$, there exist $i, j \in \{1, 2, 3\}$ and $g \in \Gamma_{\hat{V}_t}$ such that $gB_i = B_j$ and so $gB_i^* = B_j^*$. We deduce that $g(B_i \cap B_i^*) = B_j \cap B_j^*$ where $g \in \Gamma_{\hat{V}_t}$ and this yields a contradiction, as they lie in different orbits.

We now claim that there exists $d \in \mathbb{N}$ in such a way that for every vertex of $v \in \hat{V}_t$ there is an adhesion set $B \cap B^*$ of $\hat{V}_t$ such that $d(v, B \cap B^*) \leq d$. Thus we deduce that the action $\Gamma_{\hat{V}_t}$ on the set of $\{B \cap B^* \mid (B, B^*) \sim (A, A^*)\}$ has finitely many orbits. For every $u \in \hat{V}_t$, suppose that $B_u \cap B_u^*$ has the minimum distance d_u from u among all adhesion sets. Assume to contrary that the set $\{d_u \mid u \in V_t\}$ is not bounded. Without loss of generality suppose that there is an increasing sequence $d_{v_1} < d_{v_2} < \cdots$. Since the action of Γ on G has finitely many orbits, there is a $g \in \Gamma$ such that there are $i, j \in \mathbb{N}$ with $j > i$ and $gv_i = v_j$. Therefore it yields a contradiction, as we have

$$d_{v_i} = d(v_i, B_{v_i} \cap B_{v_i}^*) = d(gv_i, g(B_{v_i} \cap B_{v_i}^*)) = d(v_j, g(B_{v_i} \cap B_{v_i}^*)) \geq d_{v_j}.$$

Since every vertex of $\hat{V}_t$ has a distance less than d from an adhesion set of $\hat{V}_t$ and because the action of $\Gamma_{\hat{V}_t}$ on the set $\{B \cap B^* \mid (B, B^*) \sim (A, A^*)\}$ has finitely many orbits, we deduce that $\Gamma_{\hat{V}_t}$ acts on $\hat{V}_t$ with finitely many orbits. □

Corollary 6.1.14. *Let Γ be a group acting on a locally finite graph G with finitely many orbits and $(T, \hat{\mathcal{V}})$ be a type 1 tree-decomposition. Then the stabilizer of each part $\hat{V}_t$ of $(T, \hat{\mathcal{V}})$ acts on $\hat{V}_t$ with finitely many orbits, in particular every part is quasi-transitive.* □

Theorem 6.1.15. *Let Γ be a group acting on locally finite graph G and let $(T, \hat{\mathcal{V}})$ be a type 1 tree-decomposition of G with respect to Γ. Then the degree of each node $t \in V(T)$ is finite if and only if $\hat{V}_t$ is finite.*

Proof. If $\hat{V}_t$ is finite, then it is a straightforward argument to show that the degree of t is finite.

So assume that the degree of t is finite. Suppose that $\hat{V}_t = \bigcap_{i=1}^n B_i$ and we denote the corresponding adhesion sets by $B_i \cap B_i^*$ for $i = 1, \ldots, n$. By Corollary 6.1.14, we find a finite subset U of vertices $\hat{V}_t$ such that $\mathsf{Aut}(\hat{V}_t)U = \hat{V}_t$. Let now $v \in U$ be an arbitrary vertex which is not in any adhesion set. Then we are able to find an adhesion set $B_j \cap B_j^*$ in such a way that any geodesic from $(B_j \cap B_j^*)$ to v is the shortest among all geodesics between $(B_i \cap B_i^*)$ and v for $i = 1, \dots, n$. Since U is a finite set, we deduce that there exists $k \in \mathbb{N}$ such that for every $v \in V_t$ there is an adhesion set $A_i \cap B_i$ in such a way that $d(v, B_i \cap B_i^*) \leq k$. Therefore $\hat{V}_t$ is finite, as G is a locally finite graph, as desired. $\square$

Corollary 6.1.16. *Let G be a locally finite graph and let $(T, \hat{\mathcal{V}})$ be a type 1 tree-decomposition of G with respect to $\mathsf{Aut}(G)$. Then the degree of each t with $t \in V(T)$ is finite if and only if $\hat{V}_t$ is finite.* $\square$

Theorem 6.1.17. *Let G be a locally finite graph and additionally let $(T, \mathcal{V})$ be a tree-decomposition of G such that the maximal size of the adhesion sets is finite and furthermore bounded. Then any thick end of G is captured by a part $V_t \in \mathcal{V}$.*

Proof. Suppose that ω is a thick end of G. Let k be the maximal size of the adhesion sets of $(T, \mathcal{V})$ of G. Suppose for a contradiction that ω is not captured by any part. As ω is a thick end, we can chose $k+1$ vertex disjoint rays belonging to ω. Let those rays be $R_1, \dots, R_{k+1}$.

We first show that each ray R_i must leave every part V_t eventually.[7] For a contradiction assume that there is a ray R_i which does not eventually leave a part V_t. As ω is not captured by any part, it is not captured by V_t and hence there exists a ray that only meets V_t finitely many times. Let us call that ray R and let R^+ be a tail of R such that R^+ does not meet V_t. We now have the contradiction that R^+ and R_i belong to ω but there exists a finite adhesion set separating R^+ and R_i.

For each ray R_i let X_i be the set of nodes $t \in T$ such that R_i contains a vertex of V_t. Let $T_i := T[X_i]$.[8] By the axioms of tree-decompositions we know that T_i is connected. As each ray R_i has to leave each part eventually we know that T_i contains a ray, say R_i^T. Let us now consider R_i^T and R_j^T for $i \neq j$.

[7] There is a vertex in R_i such that no later vertex of R_i is contained in V_t.
[8] $T[X]$ is the subgraph of T induced by X.

First suppose that R_i^T and R_j^T do not meet. This implies that there is an adhesion set S such that R_i and R_j have tails in different components of $G \backslash S$. This contradicts that R_i and R_j belong to the same end. Let $Z_{ij} := R_i^T \cap R_j^T$. We claim that $Z_{ij}^T := T[Z_{ij}]$ is a ray. We have just seen that Z_{ij}^T is not empty. If Z_{ij}^T is not a ray, then we may assume that there is a vertex x_i of R_i^T such $x \in Z_{ij}^T$ and $x_{i+1} \notin Z_{ij}^T$. But this also implies that there is an adhesion set separating a tail of R_i from R_j. So we conclude that Z_{ij}^T is ray.

Let $Z := \bigcap_{j=2}^{k+1} Z_{1j}$ and $Z^T := T[Z]$. By our argument above we can conclude that Z^T is also a ray. Let $Z^T = z_1, z_2, \dots$ This implies that the part V_{z_0} contains a vertex from each of $k+1$ rays $R_1, \dots, R_{k+1}$. As each of those rays also contains a vertex in V_{z_2} we have a contradiction. There are $k+1$ disjoint rays going through a separator of size at most k. $\square$

Corollary 6.1.18. *Let G be a locally finite graph and Γ be a group acting on G with finitely orbits. Then any thick end of Γ is captured by a part any type 1 tree-decomposition with respect to Γ.* $\square$

We obtain the following nice theorem by just using the tools proved so far. Let G be a locally finite graph and let $(T, \mathcal{V})$ be a tree-decomposition of G. Suppose that ω_1 and ω_2 are two ends of G and furthermore assume that ω_1 is captured by V_1 and ω_2 is captured by V_2. We say $(T, \mathcal{V})$ distinguishes ω_1 and ω_2 *efficiently* if the following conditions are fulfilled:

(i) $|V_i \cap V_j| < \infty$ for all $i \neq j$.

(ii) $V_1 \neq V_2$.

(iii) If the minimal size of a separator separating ω_1 from ω_2 is k then there exists an adhesion set $V_i \cap V_j$ of size k separating ω_1 from ω_2.

Finally we say that $(T, \mathcal{V})$ *distinguishes* $\Omega(G)$ *efficiently* if $(T, \mathcal{V})$ distinguishes each pair ω_1, ω_2 of $\Omega(G)$ efficiently.

Theorem 6.1.19. *Let G be a locally finite graph. For each $k \in \mathbb{N}$ there exists a tree-decomposition of G that distinguishes all ends of G which can be separated by at most k vertices efficiently.*

Proof. Let k be given. Now consider $\mathcal{N}_k^k$. By Corollary 6.1.8 we know that $\mathcal{N}_k^k$ is a sparse symmetric nested system of separations. By Lemma 6.1.7 we obtain a tree-decomposition $(T, \mathcal{V})$ of G. That $(T, \mathcal{V})$ separates all ends of G which can be separated by at most k vertices efficiently follows directly from the definition of $\mathcal{N}_k^k$. $\square$

6.2 Splitting of graphs

We start this section by showing that we use nice type 1 tree-decompositions to obtain tree-amalgamations.

Lemma 6.2.1. *Let Γ be a group acting on a locally finite graph G with finitely many orbits. Then any type 1 tree-decomposition $(T, \hat{\mathcal{V}})$ of G with respect to Γ induces a tree amalgamation $G = V_t *_T V_{t'}$ with V_t and $V_{t'}$ in $\hat{\mathcal{V}}$.*

Proof. We already know that $\Gamma \setminus T$ is the K_2. In other words, the vertices of $\Gamma \setminus T$ are $\{V_t, V_{t'}\}$, where V_t and $V_{t'}$ are parts of $(T, \hat{\mathcal{V}})$ and such that $tt' \in E(T)$. We now show that G is the tree amalgamation $V_t *_T V_{t'}$. Because $\Gamma \setminus T$ is the K_2 we can conclude that T is a (p_1, p_2)-semiregular tree where p_1 and p_2 are the numbers of adhesion sets in V_t and $V_{t'}$, respectively. We set V_t as G_1 and $V_{t'}$ as G_2 in the above definition of tree amalgamation. The adhesion sets contained in V_t and $V_{t'}$ play the role of the sets $\{S_k\}$ and $\{T_\ell\}$, respectively. As all adhesion sets in V_t and V_t' are isomorphic we can find the desired bijections $\phi_{k\ell}$. It is obvious that we can find a mapping c so we conclude that $G = V_t *_T V_{t'}$. $\square$

Any tree amalgamation of a locally finite graph with a quasi-transitive action which can be obtained by Lemma 6.2.1 is called a *tree amalgamation with respect to* Γ.
Finally we are ready to give the graph-theoretical version of Stallings' theorem.

Theorem 6.2.2. *If G is a locally finite quasi-transitive graph with more than one end, then G is a thin tree amalgamation of quasi-transitive graphs.*

Proof. Since G is a locally finite quasi-transitive graph with more than one end there is a type 1 tree-decomposition $(T, \hat{\mathcal{V}})$ of G by Corollary 6.1.12. Using Lemma 6.2.1 together with Corollary 6.1.14 means that we are done. □

6.3 Accessible graphs

In this section we first define the process of splitting of a locally finite quasi-transitive graph and then define an algorithm of splitting a locally finite quasi-transitive graph which terminates after finitely many steps if and only if the graph is accessible, see Theorem 6.3.2.

We say that we *split* a locally finite quasi-transitive G with more than one end if we write G as a thin tree amalgamation $G = G_1 *_T G_2$ with respect to some group Γ. In this case we call G_1 and G_2 the *factors* of this split. If the G_i have more than one end each, we can split the G_i by a tree amalgamation with respect to a group Γ'. An iteration of such a process is called a *splitting process* of G. We say *a process of splitting terminates* if there is a step in which all the factors contain at most one end each.

Algorithm 1. Given a locally finite quasi-transitive graph G with more then one end we define a splitting process in the following:

For the first step we do the following: Assume that i is the smallest integer such that $\mathcal{N}_i^i$ is not empty. Let Ω_i be the set of ends of G which can be split by separations in $\mathcal{N}_i^i$. We pick a separation $(A, A^*) \in \mathcal{N}_i^i$ such that $n(\omega_1, \omega_2)$ is minimal among all ends in Ω_i.

Let $\mathcal{O}$ be the orbit of (A, A^*) under $\mathsf{Aut}(G)$. By Theorem 6.1.6 we know that $\mathcal{O}$ is nested. By making $\mathcal{O}$ symmetric and using Lemma 6.1.7 and Corollary 6.1.8 we obtain a tree-decomposition of G, say $(T, \mathcal{V})$. Note $(T, \mathcal{V})$ is a type 0 tree-decomposition of G. By Corollary 6.1.11 we can extend $(T, \mathcal{V})$ to a type 1 tree-decomposition $(T, \hat{\mathcal{V}})$. By Lemma 6.2.1 we can split G. Say $G = G_1 *_T G_2$.

Let us now assume that we have split G at least once. Let G_j be a factor which captures at least two ends of G. We now check if there is a separation in $\mathcal{N}_i^i$ that separates any two ends of G captured by G_j. If there is no such separation we increase i until the new $\mathcal{N}_i^i$ contains a separation which separates two ends of G which are captured by G_j. For each separation (A, A^*) in $\mathcal{N}_i^i$ we now consider the separation $(\bar{A}, \bar{A}^*)$ induced by (A, A^*) on G_j such that (A, A^*) separates two ends captured by G_j. Among all such separations $(\bar{A}, \bar{A}^*)$ we now pick all those such that $A \cap A^*$ is minimal, let the set of those be X. Let us now pick a separation $(\bar{B}, \bar{B}^*) \in X$ such that its crossing number is minimal among all separations in X. Let $\mathcal{O}$ be the orbit of $(\bar{B}, \bar{B}^*)$ under the action of $\mathsf{Aut}(G)_{G_j}$. Note that $\mathcal{O}$ is a sparse nested system of separations. Making $\mathcal{O}$ symmetric in the usual way we can obtain a type 0 tree-decomposition of G_j by Lemma 6.1.7. By Corollary 6.1.11 we make it to a type 1 tree-decomposition of G_j under the action $\mathsf{Aut}(G)_{G_j}$. So by Theorem 6.2.1 we can find a thin tree amalgamation of G_j with respect to $\mathsf{Aut}(G)_{G_j}$. We now repeat this process for each factor G_j for $j = 1, 2$.

To summarize, we start with a narrow separation of which the separator has the minimal size and we consider the type 1 tree-decomposition induced by this separation. This type 1 tree-decomposition gives us a thin tree-amalgamation of two new graphs, say G_1 and G_2. Let us assume that G_1 has more than one end. We know consider the narrow separations of G that separates ends captured in G_1. We pick one outside of the orbit of the first one of minimal size which is also crossing the minimal number of tight separations of G. We are considering the separation of G_1 which is induced by this chosen separation. We note finding those separations is possible. We now consider the orbit of this induced separation. Note that we are first looking for separations in $\mathcal{N}_i^i$ which separate ends in G_1 here. If we have to increase i we still look for the separations with the smallest order. This has the consequence that we are first using all separations in $\mathcal{N}_y^x$ with $y \leq x$ before we increase x.

Again we repeat the process and we are able to express G_1 as a thin tree amalgamation $G_{11} *_{T_1} G_{12}$ with respect to $\mathsf{Aut}(G)_{G_1}$. If G_2 has more than one end, then we can express G_2 as a thin tree amalgamation $G_{21} *_{T_2} G_{22}$. Afterwards, we repeat this process for each G_{ij} where $i, j \in \{1, 2\}$ and continue so on. We notice that we are able to repeat the process as long as each factor has more than one end.

Theorem 6.3.1. *Let G be a locally finite quasi-transitive graph. Then for every two ends ω_1 and ω_2 of G Algorithm 1 splits ω_1 and ω_2.*

Proof. Let ω_1 and ω_2 be two ends of G and let k be the smallest integer such that there is a separation in $\mathcal{N}_k^k$ that separates those two ends. We assume that ℓ is the smallest integer such that $\mathcal{N}_\ell^\ell$ is not empty. We start Algorithm 1 with $\mathcal{N}_\ell^\ell$. First we claim that after finitely many steps we are forced to move to $\mathcal{N}_{\ell+1}^{\ell+1}$. It follows from Theorem 6.1.2 that $\mathsf{Aut}(G)$ acts with finitely many orbits on $\mathcal{N}_\ell^{\ell+1}$. So we suppose that X_i, for $i = 1, \ldots, t$, are the orbits of $\mathcal{N}_\ell^\ell$ under action $\mathsf{Aut}(G)$. Additionally assume that

$$|A \cap A^*| \leq |B \cap B^*| \text{ and } n_\ell(A, A^*) \leq n_\ell(B, B^*)$$

for $(A, A^*) \in X_i$ and $(B, B^*) \in X_j$ if $t \geq j > i \geq 1$.

Due to Algorithm 1 we need to start with X_1 and let $G_1 *_{T_1} G_2$ be a thin tree-amalgamation of G obtained from X_1. Then suppose that $(A, A^*) \in X_2$ separates two ends living in G_1. We continue Algorithm 1 and we find a type 1 tree-decomposition of G_1 with respect to $\mathsf{Aut}(G)_{G_1}$. We show that all elements of X_2 separating two ends of G_1 are used in the second step of our Algorithm. We know that $\mathsf{Aut}(G)$ acts on T_1. In other words, if $(T_1, \mathcal{V})$ is the type 1 tree-decomposition of $G_1 *_{T_1} G_2$, then $g\hat{V_t} = \hat{V_{t'}}$ for every $g \in \mathsf{Aut}(G)$ where $t, t' \in T_1$. Thus if $(B, B^*) \in X_2$ separates two ends of G_1, then there a $g \in \mathsf{Aut}(G)$ such that $g(B, B^*) = (A, A^*)$ and furthermore we deduce that $gG_1 = G_1$ and so $g \in \mathsf{Aut}(G)_{G_1}$. Hence (B, B^*) is used in the second step. Now we are able to conclude that after finitely many steps we can move to $\mathcal{N}_{\ell+1}^{\ell+1}$, as the action of $\mathsf{Aut}(G)$ has finitely many orbits on N_ℓ. With an analogous method we can show that Algorithm 1 has finitely many steps between two consecutive $\mathcal{N}_n$ and $\mathcal{N}_{n+1}$. Thus after finitely many steps we are able to reach to $\mathcal{N}_k^k$, as desired. $\square$

Theorem 6.3.2. *If G is a locally finite quasi-transitive graph, then the process of splitting of G defined in Algorithm 1 terminates if and only if G is accessible.*

Proof. First suppose that the process of splitting of G terminates. We need to show that there is a k such that we can separate any two different ends ω and ω' of G by at most k edges. As G is quasi-transitive, the maximum degree of G is bounded and hence it suffices to show that there is k such that each pair of ends of G can be separated by at most k vertices.

We now show that there is a k such that we can extend any separation obtained in some step of the splitting process to a separation of the entire G with an adhesion set of size at most k. Let G_1 and G_2 be two graphs obtained during the splitting process in such a way that $G_2 \subsetneq G_1$.

We now use a separation (A, A^*) used to define G_2 to define a separation (B, B^*) of G_2. If (A, A^*) is a separation of G_2 we are done. So let us assume that $A \cap A^*$ meets some adhesion sets contained in G_1. We know from Lemma 6.1.1 that each vertex in $A \cap A^*$ only meets finitely many adhesion sets of tight separations of G_1. Since $A \cap A^*$ is finite, we know that $A \cap A^*$ only meets finitely many adhesion sets of tight separations of G_1 . Thus the union of $A \cap A^*$ with all adhesion sets of tight separations meeting $A \cap A^*$ gives us a separation of G_2. Note that we only need that $A \cap A^*$ is a finite set. This union now gives an adhesion set $B \cap B^*$ of a separation (B, B^*) of finite order. We can do this for every step in the splitting process. Since we have finitely many steps, we are able to take the maximum among all sizes of those $B \cap B^*$, say this maximum is k. So we can separate each two ends of G with at most k vertices as each end of G lives in a part of some finite step.

For the backward implication, we assume that we can separate each two ends with at most k vertices. This implies Algorithm 1 never considers a $\mathcal{N}_\ell^\ell$ for $\ell > k$. By Theorem 6.3.1 we already know that for each pair of ends, Algorithm 1 distinguishes these two ends. On the other hand we can separate every pair of ends by an element in $\mathcal{N}_k^k$. Hence we infer that our algorithm stops after finitely many steps and as result the splitting process terminates. □

We close the section by remarking that we can strengthen Theorem 6.1.19 for accessible quasi-transitive graphs.

Remark 6.3.3. *Let G be an accessible quasi-transitive graph, then there exists a tree-decomposition of G that distinguishes all ends of G efficiently.*

6.4 Applications

Let G be a locally finite graph. Krön and Möller [43] have shown that thin graphs are quasi-isometric to trees for arbitrary graph. We start with the following crucial lemma.

Lemma 6.4.1. [72, Theorem 3.1 and Theorem 3.3] *Suppose that G is a locally finite graph and let $x, y \in V(G) \cup \Omega(G)$ be two distinct points. There is a geodesic arc between x and y.* □

The following Theorem 6.4.2 is a generalization from transitive to quasi-transitive graphs of a theorem of Thomassen and Woess [70, Theorem 5.3]. The proof here uses the same general strategy as the proof by Thomassen and Woess.

Theorem 6.4.2. *Let G be a locally finite quasi-transitive graph which is thin. Then G is accessible.*

Proof. In order to show that G is accessible it is enough to show that the size of splitting separations has an upper bound. Assume for a contradiction that this is not true and let (A_i, A_i^*) be a sequence of minimal separations of G in such a way that $|A_i \cap A_i^*| > |A_j \cap A_j^*|$ for $i > j$ and suppose that ω_i and ω_i' live in a component of A_i and A_i^*, respectively. By Lemma 6.4.1, we are able to find geodesic double rays R_i between ω_i and ω_i' for $i \geq 1$. Let $S := \{v_1, \ldots, v_n\}$ be a set of representatives of all orbits. We may assume that each R_i meets S, otherwise we can switch R_i with gR_i for a suitable automorphism g of G. Since we have infinitely many double rays, we can infer that there exists an infinite subsequence $\{R_{i_j}\}_{j\in\mathbb{Z}}$ meeting S in the same vertex. We may assume that this vertex is v_0, otherwise we just relabel the vertices in S. Let P_{i_j} and Q_{i_j} be $v_1R_{i_j}$ and $R_{i_j}v_1$ which are two geodesic rays belonging of ω_{i_j} and ω_{i_j}' respectively. Since the degree of v_1 is finite and we have infinitely many rays $\{P_{i_j}\}_{j\in\mathbb{Z}}$, we deduce that $\{P_{i_j}\}_{j\in\mathbb{Z}}$ is convergent to a ray P. With an analogous method we may assume that $\{Q_{i_j}\}_{j\in\mathbb{Z}}$ is convergent to a geodesic ray Q. Suppose that ω and ω' are ends containing the rays P and Q respectively. Let (A, A^*) be a minimal separation for ω and ω', where ω and ω' live in A and A^* respectively. It follows from definition of convergence that there is $N \in \mathbb{N}$ such that the geodesic double ray R_{i_k} contains a subpath $u_k(P \cup Q)v_k$ of the geodesic double ray $P \cup Q$, where $k > N$. We may assume that $u_k \in A$ and $v_k \in A^*$. We already know that a separation $(A_{i_k}, A^*_{i_k})$ with $|A_{i_k} \cap A^*_{i_k}| > |A \cap A^*|$ separates ω_{i_k} and ω'_{i_k}. On the other hand the separation (A, A^*) separates ω_{i_k} and ω'_{i_k} and it yields a contradiction, as $|A_{i_k} \cap A^*_{i_k}|$ is minimum among separators which separates ω_{i_k} and ω'_{i_k}. □

In proof the next theorem we use the following result of Thomassen.

Lemma 6.4.3. [69, Proposition 5.6.] *If G is an infinite locally finite connected quasi-transitive graph with only one end, then that end is thick.* □

Theorem 6.4.4. *Let G be a locally finite quasi-transitive graph. Then G is thin if and only if the splitting process of G ends up with finite graphs.*

Proof. First assume that G is thin. It follows from Theorem 6.4.2 that G is accessible and so Theorem 6.3.2 implies that the process of splitting terminates after finitely many steps. Thus it is enough to show that all graphs in the final steps are finite. Assume to contrary that there is an infinite graph in a final step, say H. Since G is a thin graph, the graph H possesses exactly one thin end ω. We know by Corollary 6.1.14 that H is a quasi-transitive graph. Hence Lemma 6.4.3 implies that ω is thick, a contradiction. For the backward implication, suppose that G has a thick end ω. It follows from Corollary 6.1.18 that ω was captured by a part and so this end remained in a part in the splitting process in each step and hence the part containing this end is infinite in each step. Thus we found a contradiction, as desired. □

Virtually free groups have been intensively studied in computer science and mathematics, see [1, 53, 54]. A group Γ is called *virtually free* if it contains a free subgroup of finite index. There are some characterizations of those groups, see [1]. In particular Woess [76] has shown that G is a finitely generated virtually free group if and only if every end of any Cayley graph of G is thin.

Using our splitting process we obtain another characterization for finitely generated virtually free groups and as an application of this characterization we infer the well-known result that finitely generated virtually free groups are accessible. Indeed, in 1983 Linnell [44] proved that any finitely generated group with only finitely many conjugacy classes of finite subgroups is accessible. In 1993 Sénizergues [63] has shown that if G is a finitely generated virtually free group then there are only finitely many conjugacy classes of finite subgroups of G. Both results combined show that any finitely generated virtually free group is accessible.

Theorem 6.4.5. *Let Γ be a finitely generated group. Then G is a virtually free group if and only if the splitting process of a Cayley graph of G ends up with finite graphs.* $\square$

As an immediate consequence of the above theorem we have the following corollary.

Corollary 6.4.6. *Finitely generated virtually free groups are accessible.* $\square$

Appendix A

We summarize the results shown in this thesis in the following very briefly. We first give a summery in German then in English.

A.1 Zusammenfassung

In Chapter 3 zeigen wir, dass Cayley-Graphen von Gruppen, welche als freies Produkt mit Amalgamation über einer endlichen Untergruppe oder als HNN-Erweiterung einer endlichen Gruppe geschrieben werden können, einen topologischen Hamiltonkreis besitzen, falls einer der Faktoren eine Dedekind-Gruppe ist. In Chapter 4 untersuchen wir weitere Cayley-Graphen auf topologische Hamiltonkreise. Unter anderem verallgemeinern wir das berühmte Resultat von Rapaport Strasser welches besagt: Jeder Cayley-Graph einer endlichen Gruppe, welche von drei Involutionen erzeugt wird, von denen zwei kommutieren, enthält einen Hamiltonkreis. Wir verallgemeinern dies zu unendlichen Gruppen deren Cayley Graph Zusammenhang 2 hat. Zusätzlich zeigen wir, dass, wenn eine Gruppe über einer Untergruppe zerfällt, welche isomorph zu einer zyklischen Gruppe von Primordnung ist, dann jeder Cayley-Graph dieser Gruppe einen topologischen Hamiltonkreis hat, sofern das benutzte Erzeugenendensystem diese Untergruppe trifft.

In Chapter 5 erweitern wir unsere Studien von zweiendigen Gruppen und Graphen und geben eine detaillierte Liste von Charakterisierungen dieser Objekte. Chapter 6 zeigt, dass man den Prozess des Teilens von Gruppen im Sinne von Stallings auf mehrendinge quasi-transitive Graphen erweitern kann. Es ist bekannt, dass ein solcher Prozess des Teilens von Gruppen genau für erreichbare Gruppen terminiert. Wir zeigen, dass es einen Prozess gibt quasi-transitive Gruppen zu teilen, welcher genau für erreichbare Graphen terminiert.

A.2 Summary

Chapter 3 shows that Cayley graphs of groups which are either a free product with amalgamation over a finite subgroup of index two or an HNN-extension over a finite subgroup contain a Hamilton circle if at least one of the factors is a Dedekind group. Chapter 4 further explores Hamilton circles on Cayley graphs. Among other things we extend the famous result of Rapaport Strasser which states every Cayley graph of a finite group which is generated by three involutions, two of which commute, contains a Hamilton cycle to infinite groups in the 2-connected case. Additionally, we show that if a two-ended group splits over a subgroup isomorphic to a finite cycle group of prime order, then any Cayley graph of that group contains a Hamilton circle as long as the generating set used to generate that Cayley graph does meet that subgroup.

In Chapter 5 we extend our studies of two-ended groups and graphs and give a detailed list of characterizations of those objects. Chapter 6 shows that the process of splitting groups defined by Stallings can be extended to quasi-transitive graphs. It is known that such a process of splitting groups terminates exactly for accessible groups. We show there is a process of splitting quasi-transitive graphs that terminates exactly for accessible graphs.

A.3 My contribution

My co-authors and I share an equal work in the papers on which this thesis is based. Highlights of my contributions are finding, formulating and proving the structure tools used throughout in Chapter 3 and Chapter 4. In particular Lemma 3.1.6, Lemma 3.1.7 and Lemma 4.3.6 are mine. Furthermore, the proof of Theorem 3.1.12, one of the main results in Chapter 3 mostly based on those tools. Additionally, the proof for the counterexample to Problem 1 is done by me. The proof Theorem 4.2.4 is also mostly done by me. The charaterization of connected quasi-transitive graphs without dominated ends in Chapter 5 is also done by me. Algorithm 1 in Chapter 6 was also formulated and proved by me. Note that this was inspired by an algorithm which was obtained in a discussion between Lehner, Miraftab and me.

This thesis is based on the following papers: Chapter 3 on [46], Chapter 4 on [47], Chapter 5 on [48], Chapter 6 on [31].

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Eidesstattliche Versicherung

Hiermit versichere ich an Eides statt, die vorliegende Dissertation selbst verfasst und keine anderen als die angegebenen Hilfsmittel benutzt zu haben. Ich versichere, dass diese Dissertation nicht in einem früheren Promotionsverfahren eingereicht wurde.