Torus Actions on K-Contact Manifolds: Basic Kirwan Surjectivity, Localization, and Residue Formula

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To Elias and Nynke

Abstract

Given the action of a torus G on a compact K-contact manifold (M, α) and assuming that the action preserves the contact form α, we can consider the contact moment map Ψ. Under the assumption that 0 is a regular value of Ψ, we prove an analogue of Kirwan surjectivity in the setting of equivariant basic cohomology of K-contact manifolds, namely that the inclusion $\Psi^{-1}(0) \to M$ induces a surjective map $H_G^*(M, \mathcal{F}) \to H_G^*(\Psi^{-1}(0), \mathcal{F})$, the basic Kirwan map. If the Reeb vector field induces a free S^1-action, the S^1-quotient is a symplectic manifold and our result reproduces Kirwan's surjectivity for the symplectic manifold M/S^1. We further show that the inclusion of the critical set of Ψ into M induces an injection in equivariant basic cohomology, a result which similarly generalizes the so-called Kirwan injectivity. For the action of a circle $G = S^1$, we also derive a Tolman-Weitsman type description of the kernel of the basic Kirwan map. Furthermore, we show that equivariant formality holds for torus actions on K-contact manifolds if we consider the basic setting, provided 0 is again assumed to be a regular value of Ψ. We further prove an analogue of the Atiyah-Bott-Berline-Vergne localization formula in the setting of equivariant basic cohomology of K-contact manifolds. For this result, it is sufficient to assume that all G-fixed points have a closed Reeb orbit, an assumption that is weaker than assuming 0 to be a regular value of Ψ. As a consequence, we deduce analogues of Witten's non-Abelian localization and the Jeffrey-Kirwan residue formula, which relate integration of equivariant basic forms on the K-contact manifold M to integration of basic forms on the contact quotient $M_0 := \Psi^{-1}(0)/G$. In the special case when the Reeb vector field induces a free S^1-action, these formulae also reduce to the usual ones for the symplectic manifold M/S^1.

Zusammenfassung

Für eine kompakte K-Kontaktmannigfaltigkeit (M, α), auf welcher ein Torus G derart wirkt, dass seine Wirkung die Kontaktform α erhält, können wir die Kontaktimpulsabbildung Ψ betrachten. Unter der Annahme, dass 0 ein regulärer Wert von Ψ ist, beweisen wir eine zur Kirwansurjektivität analoge Aussage in äquivarianter basisartiger Kohomologie von K-Kontaktmannigfaltigkeiten: dass die Inklusion $\Psi^{-1}(0) \to M$ eine surjektive Abbildung $H^*_G(M, \mathcal{F}) \to H^*_G(\Psi^{-1}(0), \mathcal{F})$ induziert, die basisartige Kirwanabbildung. Falls das Reebvektorfeld eine freie S^1-Wirkung erzeugt, ist der S^1-Quotient eine symplektische Mannigfaltigkeit und unser Resultat reproduziert Kirwans Surjektivität für die symplektische Mannigfaltigkeit M/S^1. Weiterhin zeigen wir, dass die Inklusion der kritischen Menge von Ψ in M eine Injektion in äquivarianter basisartiger Kohomologie induziert, ein Resultat, welches auf vergleichbare Weise die sogenannte Kirwaninjektivität verallgemeinert. Für den Fall einer $(G = S^1)$-Wirkung leiten wir eine Tolman-Weitsman-artige Beschreibung des Kernes der basisartigen Kirwanabbildung her. Außderdem zeigen wir, dass die betrachteten Toruswirkungen auf K-Kontaktmannigfaltigkeiten äquivariant formal sind, sofern erneut angenommen wird, dass 0 ein regulärer Wert von Ψ ist, und wir die äquivariante basisartige Kohomologie betrachten. Weiterhin beweisen wir ein Analogon zur Atiyah-Bott-Berline-Vergne-Lokalisierungsformel in äquivarianter basisartiger Kohomologie von K-Kontaktmannigfaltigkeiten. Für dieses Resultat ist es ausreichend, anzunehmen, dass alle G-Fixpunkte einen abgeschlossenen Reeborbit haben, eine Annahme, welche schwächer ist als die Annahme, 0 sei ein regulärer Wert von Ψ. Mit Hilfe dieser Lokalisierungsformel leiten wir Aussagen her, welche analog zu Wittens nicht-abelscher Lokalisierung und der Jeffrey-Kirwan-Residuenformel sind. Diese setzen die Integration von äquivarianten basisartigen Differentialformen auf der K-Kontaktmannigfaltigkeit M mit der Integration von basisartigen Differentialformen auf dem Kontakquotienten $M_0 := \Psi^{-1}(0)/G$ in Beziehung. Im besonderen Fall, dass das Reebvektorfeld eine freie S^1-Wirkung erzeugt, lassen sich auch mit diesen Gleichungen die entsprechenden Aussagen für die symplektische Mannigfaltigkeit M/S^1 herleiten.

Contents

Chapter 1

Introduction

1.1 Motivation

The well-known Kirwan surjectivity asserts that if μ is a moment map for a Hamiltonian action of a compact group K on a compact symplectic manifold N and 0 a regular value thereof, then the Kirwan map $H_K^*(N) \to H_K^*(\mu^{-1}(0))$ induced by the inclusion $\mu^{-1}(0) \subset N$ is an epimorphism (cf. [Kir84, Theorem 5.4]).
This result prompted the question whether a corresponding surjectivity statement holds in other geometries, as well.

Contact manifolds M by definition admit a global 1-form α that satisfies $\alpha \wedge (d\alpha)^n \neq 0$ everywhere. Contact geometry is naturally linked to symplectic geometry. Not only is $(\ker \alpha, d\alpha|_{\ker \alpha})$ a symplectic bundle over M, but a 1-form α on M is a contact form if and only if the 2-form $d(r^2\alpha)$ is a symplectic form on its cone $M \times \mathbb{R}^+$. Furthermore, the special class of regular contact manifolds are total spaces in the Boothby-Wang fibration that has as base space an integral symplectic manifold, where the symplectic form pulls back to the differential of the contact form, and vice versa. Hence, it is understandable that contact geometry is widely referred to as the "twin" or "odd-dimensional analogue" of symplectic geometry and it, thus, seems natural to wonder which results in symplectic geometry allow for an analogous result in the contact case.

Given a contact manifold (M, α) endowed with a G-action that preserves α, the contact moment map is given by $\Psi : M \to \mathfrak{g}^*$, $\Psi(x)(\xi) := \alpha_x(\xi_M(x))$ for $\xi \in \mathfrak{g}$. Then Kirwan surjectivity for contact manifolds is known to no longer hold in general, as the following example by Lerman shows (cf. [Ler04]).

Example 1.1.1. Consider the 3-sphere $S^3 = \{z \in \mathbb{C}^2 \mid |z_1|^2 + |z_2|^2 = 1\} \subset \mathbb{C}^2$ with the S^1-action defined by $\lambda \cdot (z_1, z_2) = (\lambda z_1, \lambda^{-1} z_2)$. Then $X(z_1, z_2) = (iz_1, -iz_2)$ is the fundamental vector field of $1 \in \mathbb{R} \simeq \mathfrak{s}^1$. Considering the ($S^1$-invariant) contact form $\alpha = \frac{i}{2} \sum_{j=1}^{2} (z_j d\bar{z}_j - \bar{z}_j dz_j)$, we compute the contact moment map (see Section 2.3) to be

$$\Psi \colon S^3 \to \mathbb{R}, \qquad (z_1, z_2) \mapsto |z_1|^2 - |z_2|^2.$$

Since the S^1-action is free, the equivariant cohomology is simply the ordinary cohomology of the S^1-quotient and we compute

$$H^*_{S^1}(S^3) = H^*(\mathbb{C}P^1), \qquad H^*_{S^1}(\Psi^{-1}(0)) = H^*(S^1).$$

But there cannot exist an epimorphism from $H^*(\mathbb{C}P^1)$ to $H^*(S^1)$.

This motivates the search for a modification of the Kirwan map in the contact case such that surjectivity does hold. In this dissertation, we follow the approach of considering a certain subcomplex of the Cartan complex of equivariant differential forms.

Let (M, α) be a compact connected contact manifold of dimension $2n + 1$. Then M has a natural foliation $\mathcal{F}$ whose leaves are the orbits of the Reeb vector field R. If R integrates to a free S^1-action, then the space of leaves $M/\mathcal{F}$ is naturally a symplectic manifold of dimension $2n$ and via the pullback of the projection, we can identify differential forms on $M/\mathcal{F}$ with *basic* differential forms $\Omega(M, \mathcal{F}) \subset \Omega(M)$. Usually, however, R does not integrate to a free S^1-action and the space of leaves fails to be a manifold. Nevertheless, we can always consider the subcomplex $\Omega(M, \mathcal{F}) \subset \Omega(M)$ of basic differential forms. The basic cohomology of M is the cohomology of this complex, and it behaves very much like the cohomology of a compact $2n$-dimensional symplectic manifold (at least under the K-contact assumption). Suppose now that in addition a torus G acts on M, preserving the contact form. Then, using the Cartan model of equivariant cohomology, we obtain a subcomplex $C_G(M, \mathcal{F}) \subseteq C_G(M)$ of *Reeb basic equivariant differential forms* and the corresponding cohomology ring $H_G(M, \mathcal{F})$.

This prompted the investigation whether equivariant basic cohomology is the "correct" setting to consider in order to obtain a surjectivity result in contact geometry that corresponds to the known statement in equivariant cohomology of a symplectic manifold.

Kirwan's original proof makes use of the minimal degeneracy of the norm square of the symplectic moment map, a property that is weaker than the Morse-Bott property and which was established in [Kir84, Chapter 4]. The question of minimal degeneracy of the norm square of the contact moment map is still unanswered. Furthermore, Kirwan makes use of the topological definition of equivariant cohomology of a G-manifold N as ordinary cohomology of the space $M \times_G EG$, where EG denotes the total space in the classifying bundle of G. This tool is not available in the basic setting. Hence, Kirwan's approach does not naturally extend to the basic setting on K-contact manifolds. Instead, we want to obtain the epimorphism as a sequence of surjective maps. Goldin introduced the reduction in stages strategy in [Gol02]. She considers a splitting $S^1 \times S^1 \times ... \times S^1$ of a subtorus $K \subset G$. By successively taking S^1-quotients, considering the residual action of the quotient group on the quotient and applying a surjectivity result for the S^1-case, she obtains a sequence of surjections

$$H_G(N) \to H_{G/S^1}(N//S^1) \to H_{G/(S^1 \times S^1)}((N//S^1)//S^1) \to \cdots \to H_{G/K}(N//K).$$

However, the quotient $N//S^1$ is in general an orbifold, not a manifold. Goldin's proof was made rigorous by Baird-Lin in [BL10]. Instead of considering a sequence of quotients, they rather consider a sequence of restrictions, retaining the action of the whole group. This idea was formulated by Ginzburg-Guillemin-Karshon in [GGK02, Section G.2.2] for so-called non-degenerate abstract moment maps. Our approach is based on the proof of [GGK02, Theorem G.13] and a corrected version thereof in [BL10, Proposition B.3.12]. The contact moment map, however, is in general not a non-degenerate abstract moment map (see Remark 2.3.1), and [BL10, Proposition B.3.12] additionally requires a G-invariant almost complex structure. Hence, while providing an alternative proof of Kirwan surjectivity on symplectic manifolds, it does not hold in our case.

1.2 Main Results

Equivariant basic cohomology turned out to be the "natural" setting to consider in the contact case: In equivariant basic cohomology, we not only obtained basic Kirwan surjectivity, but also analogues to other well known results in symplectic geometry.

Our surjectivity result states as follows.

Theorem. *Let (M, α) be a compact K-contact manifold and ξ its Reeb vector field. Let G be a torus that acts on M, preserving α. Denote by $\Psi\colon M \to \mathfrak{g}^*$ the contact moment map and suppose that 0 is a regular value of Ψ. Then the inclusion $\Psi^{-1}(0) \subset M$ induces an epimorphism in equivariant basic cohomology*

$$H^*_G(M, \mathcal{F}) \longrightarrow H^*_G(\Psi^{-1}(0), \mathcal{F}).$$

We call this map the *basic Kirwan map.*

We were further able to prove the following analogue of Kirwan injectivity.

Theorem. *The inclusion* $\mathrm{Crit}\,(\Psi) \subset M$ *induces an injection in equivariant basic cohomology*

$$H^*_G(M, \mathcal{F}) \to H^*_G(\mathrm{Crit}\,(\Psi), \mathcal{F}).$$

If 0 is a regular value of Ψ, the G-action on $\Psi^{-1}(0)$ is locally free and we obtain the contact quotient $M_0 = \Psi^{-1}(0)/G$, a contact orbifold and honest manifold if the G-action is free. Then $H^*_G(\Psi^{-1}(0), \mathcal{F}) = H(M_0, \mathcal{F}_0)$, where $\mathcal{F}_0$ denotes the induced foliation on M_0. In order to completely determine the basic cohomology of the contact quotient, the kernel of the basic Kirwan map is of high interest. In the symplectic setting, Tolman and Weitsman [TW03] found a description of the kernel of the Kirwan map. We also obtained a Tolman-Weitsman type description of the kernel of the basic Kirwan map, at least for the case of an S^1-action.

Theorem. *Let $G = S^1$, $M^{\pm} = \{x \in M \mid \pm\Psi(x) \geq 0\}$ and set*

$$C^{\pm} := \mathrm{Crit}(\Psi) \cap M^{\pm}, \qquad K^{\pm} = \{\sigma \in H^*_G(M, \mathcal{F}) \mid \sigma|_{C^{\pm}} = 0\}.$$

Then the kernel K of the basic Kirwan map $H^*_G(M,\mathcal{F}) \to H^*_G(\Psi^{-1}(0),\mathcal{F})$ is given by

$$K = K^+ \oplus K^-.$$

Another well-known result concerning the equivariant cohomology of a symplectic manifold is the *equivariant formality* of Hamiltonian actions of compact connected Lie groups K on compact symplectic manifolds N, namely that $H_K(N)$ is a free $S(\mathfrak{k}^*)$-module (cf. [Kir84, Proposition 5.8]). We proved that this property also holds for torus actions on K-contact manifolds if we consider the basic setting.

Proposition. *Suppose that 0 is a regular value of Ψ. Then the G-action on $(M,\mathcal{F})$ is equivariantly formal in the basic setting.*

We were also able to obtain an analogue of the Atiyah-Bott-Berline-Vergne localization formula [AB84, BV82]. The following theorem is closely related to results obtained in [Töb14, GNT17].

Theorem. *Suppose a torus G acts on a K-contact manifold (M,α) such that G preserves α, and suppose in addition that the G-fixed points have closed Reeb orbits. Then we have for all $\eta \in H_G(M,\mathcal{F})$ the identity*

$$\int_M \alpha \wedge \eta = \sum_{C_j \subseteq C} \int_{C_j} \frac{i_j^*(\alpha\wedge\eta)}{e_G(\nu C_j,\mathcal{F})},$$

where $C = \operatorname{Crit}\Psi$, $i_j : C_j \hookrightarrow M$ denotes the inclusion of the connected components $C_j \subseteq C$, and $e_G(\nu C_j,\mathcal{F})$ denotes the equivariant basic Euler class of the normal bundle to C_j.

We note that for this result, it is sufficient to assume that all G-fixed points have a closed Reeb orbit, an assumption that is weaker than assuming 0 to be a regular value of Ψ and that is automatically satisfied for total spaces in the Boothby-Wang fibration.

Our next main theorem is an application of this localization formula in the case that 0 is a regular value of the contact moment map Ψ to obtain an integration formula relating integration of equivariant basic forms on M to integration of basic forms on the contact quotient $M_0 := \Psi^{-1}(0)/G$, generalizing the results of Witten

[Wit92] and Jeffrey-Kirwan [JK95] in the symplectic case. For any $\eta \in H_G(M, \mathcal{F})$, with $r = \dim G$, define a function $I^\eta(\epsilon)$ depending on a real parameter $\epsilon > 0$ by

$$I^\eta(\epsilon) = \frac{1}{(2\pi i)^r \mathrm{vol}\,(G)} \int_{M\times\mathfrak{g}} \alpha \wedge \eta(\phi) \wedge e^{id_G\alpha(\phi) - \epsilon|\phi|^2/2} d\phi.$$

We denote by η_0 the image of η under the natural basic Kirwan map $H_G(M, \mathcal{F}) \to H(M_0, \mathcal{F}_0)$, and let α_0 denote the quotient contact form on M_0.

Theorem. *For any $\eta \in H_G(M, \mathcal{F})$, there exists some constant $c > 0$ such that as $\epsilon \to 0^+$, $I^\eta(\epsilon)$ obeys the asymptotic*

$$I^\eta(\epsilon) = \tfrac{1}{n_0} \int_{M_0} \alpha_0 \wedge \eta_0 \wedge e^{\epsilon\Theta + id\alpha_0} + o(\epsilon^{-r/2} e^{-c/\epsilon}), \tag{1.1}$$

where $\Theta \in H^4(M_0, \mathcal{F}_0)$ is the class corresponding to $-\frac{<\phi,\phi>}{2} \in H^4_G(\Psi^{-1}(0), \mathcal{F}) \simeq H^4(M_0, \mathcal{F}_0)$ and n_0 denotes the order of the regular isotropy of the action of G on $\Psi^{-1}(0)$.

A particular consequence of this theorem is the identity

$$\int_{M_0} \alpha_0 \wedge \eta_0 \wedge e^{id\alpha_0} = n_0 \lim_{\epsilon\to 0^+} I^\eta(\epsilon),$$

which expresses intersection pairings on M_0 as limits of equivariant intersection pairings on M.

Consider the distribution $\mathbf{F}(\int_M \alpha \wedge \eta \wedge e^{id_G\alpha})$, where $\mathbf{F}$ denotes Fourier transformation. The main ingredients in the proof of the previous theorem are the result that $\mathbf{F}(\int_M \alpha \wedge \eta \wedge e^{id_G\alpha})$ is piecewise polynomial and smooth near 0, and a particular expression for the polynomial this distribution coincides with near 0. Applying a result of Jeffrey-Kirwan, we then obtain the last of our main theorems.

Theorem. *Let η_0 denote the image of $\eta \in H_G(M, \mathcal{F})$ under the basic Kirwan map. Then we have*

$$\int_{M_0} \alpha_0 \wedge \eta_0 \wedge e^{id\alpha_0} = \frac{n_0}{\mathrm{vol}\,(G)} \mathrm{jkres} \left(\sum_{C_j \subseteq C} e^{-i\langle \mu(C_j), \phi\rangle} \int_{C_j} \frac{i_j^* \left(\alpha \wedge \eta(\phi) e^{id\alpha}\right)}{e(\nu C_j)} [d\phi] \right).$$

Basic Kirwan Surjectivity and Injectivity and the Localization and Residue Formula, to some extent, provide a generalization of the previously known results in ordinary equivariant cohomology. Namely, our examples in Sections 4.2.1 and 6.3.1 show that in the case where the Reeb vector field induces a free S^1-action, our results yield the known statements in ordinary equivariant cohomology for the symplectic manifold $M/\mathcal{F}$. Thus, at least in the case of an integral symplectic form and a Hamiltonian group action that lifts to the S^1-bundle in the Boothby-Wang fibration, the symplectic analogues follow from our results.

1.3 Outline

This thesis is structured as follows. In Chapter 2, we recall fundamentals of (K)-contact geometry and consider actions of a torus G on a compact K-contact manifold (M, α) that leaves the contact form invariant in order to establish preliminary results that will be needed in later chapters to prove our main theorems. We prove an equivariant contact Darboux Theorem (Theorem 2.1.2), which we then apply to obtain a contact Coisotropic Embedding Theorem (Theorem 2.1.3). Under the assumption that 0 is a regular value of the contact moment map Ψ, we show that a basis (X_s) of the Lie algebra of the torus can be chosen in such a way that certain axioms are fulfilled (Proposition 2.4.1). For such a special basis, we derive the Morse-Bott property of the functions $\Psi^{X_{s+1}}|_{Y_s}$, where $Y_s = (\Psi^{X_1}, ..., \Psi^{X_s})^{-1}(0)$ (Proposition 2.4.9). In the remainder of Chapter 2, we apply the contact Coisotropic Embedding Theorem to prove that under the assumption that 0 is a regular value of Ψ, there is an invariant neighborhood of $\Psi^{-1}(0)$ which is equivariantly diffeomorphic to a neighborhood of $\Psi^{-1}(0) \times \{0\}$ in $\Psi^{-1}(0) \times \mathfrak{g}^*$ such that on this neighborhood, the contact form and the moment map are of a specific normal form (Proposition 2.5.4).

We begin Chapter 3 by briefly recalling the concept of basic differential forms and basic cohomology on a contact manifold and then show that the complex of basic differential forms forms a $\mathfrak{g}$-dga (Lemma 3.1.5) and even a G^*-algebra (Lemma 3.1.8). We then briefly describe the Cartan model of equivariant cohomology of a $\mathfrak{k}$-dga and then proceed to discuss equivariant basic cohomology of K-contact manifolds and prove several properties thereof. More precisely, we prove the existence of certain long exact sequences (e.g., the Mayer-Vietoris sequence (Proposition 3.3.6)) and certain isomorphisms in cohomology, as well as the basic equivariant Thom isomorphism (Theorem 3.4.4).

The Kirwan surjectivity result is stated and proved in the first part of Chapter 4. In Section 4.2, we present examples and establish that in the case where the Reeb vector field induces a free S^1-action, our result reproduces the Kirwan surjectivity for the S^1-quotient. The injectivity result is obtained in Section 4.3. In this section, we also derive a Tolman-Weitsman type description of the kernel of the basic Kirwan map for S^1-actions, for which we then also present an example.

In Chapter 5, we prove the equivariant formality (in the basic setting) of the considered torus actions on K-contact manifolds.

In the first section of Chapter 6, we derive a basic Atiyah-Bott-Berline-Vergne type localization formula (Theorem 6.1.9). In Section 6.2, we prove that the parameter dependent integral $I^\eta(\epsilon)$ satisfies certain asymptotics (Theorem 6.2.7). With this result, we then prove the Residue Formula (Theorem 6.2.13). The last section of this chapter is devoted to examples. In particular, in Section 6.3.1, we explain in detail how Theorems 6.1.9 and 6.2.13 may be used to deduce the analogous theorems for symplectic manifolds that occur as $M/\mathcal{F}$ in the case that R induces a free S^1-action.

1.4 Bibliographical Notes

This thesis is based on the results of the following two publications.

[C] **L. Casselmann**.
Basic Kirwan Surjectivity for K-Contact Manifolds.
Annals of Global Analysis and Geometry, 52(2): 157–185, 2017.

[CF] **L. Casselmann and J. M. Fisher**.
Localization for K-Contact Manifolds.
To appear in *Journal of Symplectic Geometry*.

arXiv preprint: 1703.00333, 2017.

The Equivariant contact Darboux Theorem (Theorem 2.1.2) and the contact Coisotropic Embedding Theorem (Theorem 2.1.3) in Section 2.1 are contained in [CF]. [C] contains the construction of a special basis for $\mathfrak{g}$ presented in Section 2.4. Most preliminary results established in Chapter 3 are also taken from this work, with the exception of the more detailed proof of Proposition 3.3.20 and the construction of the basic equivariant Thom isomorphism, which are published in [CF]. All results presented in Chapters 4 and 5 are taken from [C]. While the first proof of Theorem 4.3.7 in Section 4.3 is already contained in [C], we also give an alternative proof of the description of the kernel of the basic Kirwan map. We include it in this work because we believe it might be instructive in finding an analogous description of the kernel for the action of higher rank tori. The results presented in Chapter 6 are all taken from the joint publication [CF].

Details on the contribution of the author of this thesis to the joint publication [CF] can be found on page 117 in the list of publications.

1.5 Notations and Conventions

Albeit they are fairly standard, we briefly state the (notational) conventions we abide by in this thesis. All manifolds are considered to be smooth and connected. We use capital Roman letters to denote Lie groups and the same letters in fraktur font to denote their Lie algebras, e.g., $\mathfrak{g}$ denotes the Lie algebra of the Lie group G. We use $\cdot$ to denote the action of a Lie group G on a manifold M, that is, for $g \in G$ and $x \in M$, $g \cdot x$ denotes the action of g on x and $G \cdot x$ denotes the G-orbit of x. The isotropy group of x is denoted by G_x and its isotropy algebra by $\mathfrak{g}_x$. Furthermore, the superscript $*$ is used to denote the dual of a vector space; in particular, $\mathfrak{g}^*$ is the dual vector space of $\mathfrak{g}$, etc. Furthermore, $S(V)$ denotes the symmetric algebra on a vector space V, where $S(\mathfrak{g}^*)$, the polynomials on $\mathfrak{g}$, is of particular interest to us.

Chapter 2

K-Contact Manifolds

In this chapter, we will establish terminology and notation and recall fundamentals of (K-)contact geometry. We will further prove preliminary results that will be needed in later Chapters.

2.1 Contact Manifolds

In this section, we will recall the definition of contact manifolds and prove an equivariant Contact Darboux Theorem as well as a Contact Coisotropic Embedding Theorem.

We work with the following notion of contact manifolds.

Definition 2.1.1. A *contact manifold* is a pair (M, α), where M is a manifold of dimension $2n+1$, and $\alpha \in \Omega^1(M)$ is a *contact form*, i.e., $\alpha \wedge (d\alpha)^n$ is nowhere zero.

Note that we take the contact form α, and not just the induced hyperplane distribution $\ker \alpha$, as part of the data defining a contact manifold. On any such manifold, there is a distinguished vector field $R \in \mathfrak{X}(M)$, called the *Reeb vector* *field*, which is uniquely determined by the two conditions

$$\iota_R \alpha = 1, \quad \iota_R d\alpha = 0.$$

Note that these conditions imply that $\mathcal{L}_R \alpha = 0$. The contact form gives a direct sum decomposition $TM = \ker \alpha \oplus \mathbb{R}R$, and we note that $\ker \alpha$ is a symplectic vector bundle over M with symplectic form $d\alpha|_{\ker \alpha}$.

The flow of R is denoted by ψ_t and the 1-dimensional foliation it induces by $\mathcal{F}$.

If R induces a free S^1-action, $M/\{\psi_t\}$ is a manifold and $d\alpha$ descends to a symplectic form on $M/\{\psi_t\}$ (Boothby-Wang fibration [BW58]). This, however, is usually not the case.

Later on, we will need a local normal form of the contact moment map Ψ in a neighborhood of $\Psi^{-1}(0)$. In order to obtain this normal (cf. Section 2.5) form, we need to show the uniqueness of certain coisotropic embeddings into contact manifolds. To this end, we first prove an equivariant contact Darboux Theorem for submanifolds. Note that while a contact Darboux Theorem for contact forms in a neighborhood of a point (see, e.g., [Gei06, Theorem 2.24]) is well-known, a contact Darboux Theorem for neighborhoods of submanifolds exists, to our knowledge, so far only for contact structures ([Ler02, Theorem 3.6]) or submanifolds to which the Reeb vector fields are nowhere tangent ([AG90, Theorem B]). We follow Lerman's approach for contact structures. Note that his proof does not generally work for contact *forms* because his function g_t (which is $\varphi_t^*(\dot{\alpha}_t(R_t))$ in the notation of the upcoming proof) might not vanish. It is, however, applicable in our case, because we make the additional assumption that the Reeb vector fields coincide on a neighborhood of the submanifold.

Theorem 2.1.2 (Equivariant Contact Darboux Theorem). *Let Y be a closed submanifold of X and let α^0 and α^1 be two contact forms on X with Reeb vector fields R_i, $i = 0, 1$. Suppose that $\alpha^0_x = \alpha^1_x$ and $d\alpha^0_x = d\alpha^1_x$ for every $x \in Y$ and that there is a neighborhood U of Y in X such that $R_0 = R_1$ on U. Then there exist neighborhoods U_0, U_1 of Y in X and a diffeomorphism $\varphi : U_0 \to U_1$ such that $\varphi|_Y = \mathrm{id}|_Y$ and $\varphi^*\alpha^1 = \alpha^0$.*

Moreover, if a compact Lie group K acts on X, preserving Y, U, and the two contact forms α^0, α^1, then we can choose U_0 and U_1 K-invariant and φ K-equivariant.

Proof. Consider the family of 1-forms $\alpha^t := t\alpha^1 + (1-t)\alpha^0$, $t \in [0,1]$. For every $x \in Y$ and every $t \in [0,1]$, we have $\alpha^t_x = \alpha^1_x = \alpha^0_x$ and $d\alpha^t_x = d\alpha^1_x = d\alpha^0_x$. It follows that α^t are contact forms in a neighborhood of Y for every $t \in [0,1]$: By maximality of the degree, there is a smooth function $f : X \times [0,1] \to \mathbb{R}$ such that $\alpha_t \wedge (d\alpha_t)^n = f\alpha_0 \wedge (d\alpha_0)^n$. $f^{-1}(\mathbb{R} \setminus \{0\})$ is open and contains $Y \times [0,1]$, so for every $(x,t) \in Y \times [0,1]$, there exists a neighborhood $U(x,t)$ of the form $U_t(x) \times (t - \epsilon_{x,t}, t + \epsilon_{x,t}) \cap [0,1]$, $\epsilon_{x,t} > 0$ such that $f|_{U(x,t)} \neq 0$. Since $[0,1]$ is compact, there are $t_1, ..., t_N$: $[0,1] = \cup_{i=1}^N (t_i - \epsilon_{x,t_i}, t_i + \epsilon_{x,t_i}) \cap [0,1]$. Then $\widetilde{U} := \cup_{x\in Y} \left(\cap_{i=1}^N U_{t_i}(x)\right)$ is open, contains Y and f does not vanish on $\widetilde{U} \times [0,1]$. Thus, all α_t are contact forms on $\widetilde{U}$. Without loss of generality, we assume that they are contact forms at least on all of U. α^t are K-invariant because α^0 and α^1 are. Let R_t denote the Reeb vector field of α^t. Since R_t is uniquely determined, R_t is also K-invariant and, on U, we have $R_t = R_0$. Set $\dot{\alpha}_t := \frac{d}{dt}\alpha^t = \alpha_1 - \alpha_0$. $\dot{\alpha}_t$ vanishes on Y and, on U, it is $\dot{\alpha}_t(R_0) = 0$. Define a K-invariant time dependent vector field X_t tangent to the contact distribution $\xi_t := \ker \alpha_t$ by

$$X_t := \left(d\alpha_t|_{\xi_t}\right)^{-1}(-\dot{\alpha}_t|_{\xi_t}).$$

Note that X_t vanishes on Y. By definition of X_t, we have $(\iota_{X_t} d\alpha_t)|_{\xi_t} = -\dot{\alpha}_t|_{\xi_t} = (\dot{\alpha}_t(R_t)\alpha_t - \dot{\alpha}_t)|_{\xi_t}$ and $(\iota_{X_t} d\alpha_t)(R_t) = 0 = (\dot{\alpha}_t(R_t)\alpha_t - \dot{\alpha}_t)(R_t)$. Hence, $\iota_{X_t} d\alpha_t = \dot{\alpha}_t(R_t)\alpha_t - \dot{\alpha}_t$. Since $X_t \in \xi_t$, it follows that

$$\mathcal{L}_{X_t}\alpha_t = \iota_{X_t} d\alpha_t = \dot{\alpha}_t(R_t)\alpha_t - \dot{\alpha}_t.$$

Denote the time dependent flow of X_t by φ_t. φ_t is defined on a neighborhood V of Y since X_t vanishes on Y. Furthermore, φ_t is K-invariant because X_t is K-invariant, and $\varphi_t|_Y = \mathrm{id}_Y$. Then

$$\frac{d}{dt}(\varphi_t^*\alpha_t) = \varphi_t^*(\mathcal{L}_{X_t}\alpha_t + \dot{\alpha}_t) = \varphi_t^*(\dot{\alpha}_t(R_t)\alpha_t).$$

On U, $0 = \dot{\alpha}_t(R_0) = \dot{\alpha}_t(R_t)$. We will find a small neighborhood U_0 of Y with $\varphi_t(U_0) \subset U$ for every t, then we have $\frac{d}{dt}(\varphi_t^*\alpha_t) = 0$ on U_0 and, hence, $\varphi_t^*\alpha_t \equiv \varphi_0^*\alpha_0 = \alpha_0$. $\varphi_1 : U_0 \to \varphi_1(U_0) =: U_1$ hence defines the desired K-invariant contactomorphism. To find U_0, note that for every $(x,t) \in Y \times [0,1]$, there exists a neighborhood $U(x,t)$ of the form $U_t(x) \times (t - \epsilon_{x,t}, t + \epsilon_{x,t}) \cap [0,1]$, $\epsilon_{x,t} > 0$ such that $\varphi(U(x,t)) \subset U$. Since $[0,1]$ is compact, there are $t_1, ..., t_N$: $[0,1] =$

$\cup_{i=1}^{N}(t_i - \epsilon_{x,t_i}, t_i + \epsilon_{x,t_i}) \cap [0,1]$. Then $U_0 := \cup_{x \in Y} \left(\cap_{i=1}^{N} U_{t_i}(x)\right)$ is open, contains Y and $\varphi(U_0 \times [0,1]) \subset U$. $\square$

We will now apply the Equivariant Contact Darboux Theorem in order to obtain a Contact Coisotropic Embedding Theorem.

Theorem 2.1.3 (Contact Coisotropic Embedding Theorem)**.** *Let α be a 1-form on a manifold Z such that $d\alpha$ is of constant rank. Suppose that a compact Lie group K acts on Z, leaving α invariant. Suppose that there are two contact K-manifolds $(X_1, \alpha_1), (X_2, \alpha_2)$ and K-equivariant embeddings $i_j : Z \to X_j$ such that*

(i) $di_j(TZ) \cap \ker \alpha_j$ *is coisotropic in* $(\ker \alpha_j, d\alpha_j|_{\ker \alpha_j})$,

(ii) $i_j^* \alpha_j = \alpha$ *and K preserves α_j,*

(iii) there is a nowhere vanishing K-fundamental vector field X_Z on Z, generated by $X \in \mathfrak{k}$, such that $di_j(X_Z) = R_j$, where R_j denotes the Reeb vector field on X_j, and R_j is the fundamental vector field generated by X on all of X_j. (In particular, the Reeb flow corresponds to the action of a subgroup of K on X_j).

Then there exist K-invariant neighborhoods U_j of $i_j(Z)$ in X_j and a K-equivariant diffeomorphism $\varphi : U_1 \to U_2$ such that $\varphi^* \alpha_2 = \alpha_1$ and $i_2 = \varphi \circ i_1$.

To prove this theorem, we adjust the proof of the well-known Coisotropic Embedding Theorem for symplectic manifolds (see, e.g., [Got82, Section III] or [GS84, Theorem 39.2]) to the contact setting and extend it in order to obtain an equality of contact forms, not only of their differentials. We will need the following two lemmata. The following notation is used. $\xi_j := \ker \alpha_j$, $\zeta_j := di_j(TZ) \cap \ker \alpha_j$, $\omega_j := d\alpha_j|_{\xi_j}$, $\perp := \perp_{d\alpha}$, $\perp_j := \perp_{\omega_j}$. Note that by our assumptions, ζ_j is K-invariant and $\mathbb{R}R_j \subset di_j(TZ)$ and, hence, $di_j(TZ) = \zeta_j \oplus \mathbb{R}R_j$.

Lemma 2.1.4. $\zeta_j^{\perp_j} = di_j(TZ^{\perp}) \cap \xi_j$.

Proof. Let $di_j(v) \in \zeta_j$ be arbitrary. For every $di_j(w) \in di_j(TZ^{\perp}) \cap \xi_j$, we have $\omega_j(di_j(v), di_j(w)) = d\alpha(v,w) = 0$. It follows that $\zeta_j \subset (di_j(TZ^{\perp}) \cap \xi_j)^{\perp_j}$, hence, $\zeta_j^{\perp_j} \supset di_j(TZ^{\perp}) \cap \xi_j$. We now show the reverse inclusion. Since $\zeta_j^{\perp_j} \subset \zeta_j$, an arbitrary $\bar{w} \in \zeta_j^{\perp_j}$ is of the form $\bar{w} = di_j(w)$, $w \in TZ$. For arbitrary $v \in TZ$, it is $d\alpha(v,w) = d\alpha_j(di_j(v), di_j(w))$. Let $\bar{v} \in TZ$ such that $di_j(\bar{v}) \in \zeta_j$ and $di_j(v) - di_j(\bar{v}) \in \mathbb{R}R_j$. Then $d\alpha(v,w) = \omega_j(di_j(\bar{v}), \bar{\omega}) = 0$ since $\bar{\omega} \in \zeta_j^{\perp_j}$, i.e., $w \in TZ^{\perp}$. □

Consider the normal bundles $N_j := TX_j/di_j(TZ)$ of the embeddings i_j.

Lemma 2.1.5. *$N_j := TX_j/di_j(TZ) \simeq (TZ^{\perp}/\mathbb{R}X_Z)^*$ as K-vector bundles over Z.*

Proof. Consider the maps

$$\begin{array}{rrcl} \varphi_j: & TX_j/di_j(TZ) & \to & (di_j(TZ^{\perp})/\mathbb{R}R_j)^* \\ & [v] & \mapsto & d\alpha_j(v,\cdot)|_{di_j(TZ^{\perp})/\mathbb{R}R_j}. \end{array}$$

Since $R_j \in \ker d\alpha_j$ and $di_j(TZ) \perp_{d\alpha_j} di_j(TZ^{\perp})$, the map φ_j is well-defined. By assumption, $di_j(TZ) \cap \xi_j$ is coisotropic. It follows that $di_j(TZ^{\perp})^{\perp_{d\alpha_j}} \subseteq di_j(TZ)$. This, however, yields that φ_j is injective: Let $[v] \in TX_j/di_j(TZ)$ and suppose that $d\alpha_j(v,w) = 0$ for every $[w] \in di_j(TZ^{\perp})/\mathbb{R}R_j$. Then $v \in di_j(TZ^{\perp})^{\perp_{d\alpha_j}} \subseteq di_j(TZ)$ so that $[v] = 0$. By the previous lemma, $\dim \zeta_j^{\perp_j} = \dim TZ^{\perp} - 1$. Since $\dim di_j(TZ) = \dim TX_j - \dim \zeta_j^{\perp_j}$, we obtain $\dim TX_j - \dim di_j(TZ) = \dim di_j(TZ^{\perp}) - 1$, showing that, for dimensional reasons, φ_j has to be surjective, as well. Since i_j is an equivariant embedding, we have K-equivariant isomorphisms $TZ^{\perp}/\mathbb{R}X_Z \simeq di_j(TZ^{\perp})/\mathbb{R}R_j$. □

Proof of the Embedding Theorem 2.1.3. We want to apply Theorem 2.1.2. We will work with a specific realization of the N_j as a K-invariant complement of $di_j(TZ)$ in TX_j such that $\xi_j = \zeta_j \oplus N_j$. This is possible since $\mathbb{R}R_j \subset di_j(TZ)$. Since $X_Z \in TZ^{\perp}$, we can find a K-invariant complement G of $TZ^{\perp}$ in TZ such that $di_1(G) \subset \zeta_1$. Since $i_j^*\alpha_j = \alpha$ by assumption, this means that $di_2(G) \subset \zeta_2$, as well. By injectivity, we have $di_j(TZ^{\perp}) \oplus di_j(G) = di_j(TZ)$. Lemma 2.1.4 then yields $\zeta_j^{\perp_j} \oplus di_j(G) = \zeta_j$, i.e., $di_j(G)$ is a complement of $\zeta_j^{\perp_j}$ in ζ_j. It follows that $di_j(G)$ is a symplectic subbundle of $(\xi_j|_{i_j(Z)}, \omega_j)$. Then $di_j(G)^{\perp_j}$ is also a symplectic subbundle of $(\xi_j|_{i_j(Z)}, \omega_j)$ and

$$\xi_j|_{i_j(X)} = di_j(G) \oplus di_j(G)^{\perp_j}. \tag{2.1}$$

It is $\zeta_j \cap di_j(G)^{\perp_j} = (\zeta_j^{\perp_j} \oplus di_j(G))^{\perp_j} = \zeta_j^{\perp_j}$ and $\zeta_j^{\perp_j} \subset di_j(G)^{\perp_j}$, hence, $\zeta_j^{\perp_j}$ is a Lagrangian subbundle of $di_j(G)^{\perp_j}$. Choose any K-invariant Lagrangian subbundle

W_j in $di_j(G)^{\perp_j}$ complementary to $\zeta_j^{\perp_j}$. In particular, $W_j^{\perp_j} \cap di_j(G)^{\perp_j} = W_j$ so that $W_j \subset W_j^{\perp_j}$. Since $\xi_j|_{i_j(X)} = di_j(G) \oplus \zeta_j^{\perp_j} \oplus W_j = \zeta_j \oplus W_j$ and $di_j(TZ) = \zeta_j \oplus \mathbb{R}R_j$, we can identify the normal bundles N_j with W_j.

By Lemma 2.1.5, we have a canonical K-equivariant vector bundle isomorphism $A : N_1 \to N_2$. Then for $v \in N_1$, $Av \in N_2$ is defined via

$$\omega_1(v, di_1(w)) = \omega_2(Av, di_2(w)) \text{ for every } di_j(w) \in di_j(TZ^\perp) \cap \zeta_j. \tag{2.2}$$

(A neighborhood of the zero section of) N_j can be identified with a K-invariant tubular neighborhood U_j of $i_j(Z)$ in X_j via the exponential maps of K-invariant Riemannian metrics, where Z embeds as the zero section. Then A yields a K-equivariant diffeomorphism $\tilde{A} : U_1 \to U_2$ with $i_2 = \tilde{A} \circ i_1$. Set $\tilde{\alpha}_1 := \tilde{A}^*\alpha_2$. Then $\tilde{\alpha}_1$ is a contact form on U_1. $i_2 = \tilde{A} \circ i_1$ implies that $i_1^*\alpha_1 = \alpha = i_2^*\alpha_2 = i_1^*\tilde{\alpha}_1$. Hence, $(\tilde{\alpha}_1)_{i_1(z)}|_{di_1(TZ)} = (\alpha_1)_{i_1(z)}|_{di_1(TZ)}$. Furthermore, we have $d\tilde{A}|_{N_1} = A$ by construction, so $d\tilde{A}|_{N_1} : N_1 \subset \xi_1 \to N_2 \subset \xi_2$, which yields $(\tilde{\alpha}_1)_{i_1(z)}|_{\xi_1} = 0 = (\alpha_1)_{i_1(z)}|_{\xi_1}$. Thus, $(\tilde{\alpha}_1)_{i_1(z)} = (\alpha_1)_{i_1(z)}$ on all of TX_1.

Since the Reeb vector fields are fundamental vector fields of the same element of $\mathfrak{k}$ and since $\tilde{A}$ is K-invariant, $d\tilde{A}(R_1(p)) = R_2(\tilde{A}(p))$. It follows that $\tilde{\alpha}_1(R_1) = 1$ and $\iota_{R_1} d\tilde{\alpha}_1 = 0$, so R_1 is the Reeb vector field of $\tilde{\alpha}_1$ on U_1.

It remains to show that $(d\tilde{\alpha}_1)_{i_1(z)} = (d\alpha_1)_{i_1(z)}$ on $\xi_1 \times \xi_1$. Since $i_2 = \tilde{A}i_1$, we have $(d\tilde{\alpha}_1)_{i_1(z)} = (d\alpha_1)_{i_1(z)}$ on $\zeta_1 \times \zeta_1$. By construction, $d\tilde{A}|_{N_1} = A$. N_j is ω_j-isotropic and A maps N_1 to N_2, hence, $(d\tilde{\alpha}_1)_{i_1(z)} = (d\alpha_1)_{i_1(z)} = 0$ on $N_1 \times N_1$. Equation (2.2) yields that $(d\tilde{\alpha}_1)_{i_1(z)} = (d\alpha_1)_{i_1(z)}$ on $\zeta_1^{\perp_1} \times N_1$. It remains to show that $(d\tilde{\alpha}_1)_{i_1(z)} = (d\alpha_1)_{i_1(z)}$ on $di_1(G) \times N_1$. Since $N_j \subset di_j(G)^{\perp_j}$, $d\tilde{A}N_1 = N_2$ and $d\tilde{A}di_1(G) = di_2(G)$, both forms vanish on $di_1(G) \times N_1$.

By Theorem 2.1.2, there is a neighborhood U of $i_1(Z)$ and a K-equivariant diffeomorphism g of U into X_1 such that $g_{i_1(Z)} = \mathrm{id}_{i_1(Z)}$ and $g^*\tilde{\alpha}_1 = \alpha_1$. Then $\varphi := \tilde{A} \circ g$, restricted to a small enough neighborhood, satisfies $\varphi^*\alpha_2 = \alpha_1$. □

2.2 K-Contact Manifolds

We now restrict ourselves from general contact manifolds to K-contact manifolds. For this special class of contact manifolds, the behavior of the flow of the Reeb vector field R is restricted. To be more precise, we recall the following definitions.

Definition 2.2.1. Let (M, α) be a contact manifold. A Riemannian metric g on M is called a *contact metric* if

(i) $\ker\alpha \perp_g \ker d\alpha$

(ii) $g|_{\ker d\alpha} = \alpha \otimes \alpha$

(iii) $g|_{\ker\alpha}$ is compatible with the symplectic form $d\alpha$, i.e., there exists a $(1,1)$-tensor field J on $\Gamma(\ker\alpha)$ such that $g = d\alpha(J\cdot, \cdot)$ and $J^2 = -\operatorname{id}$.

(M, α) is called a *K-contact manifold* if there exists a contact metric g on M with $\mathcal{L}_R g = 0$, i.e., such that Reeb vector field R is Killing.

Example 2.2.2. For $n \geq 1$ and $w \in \mathbb{R}^{n+1}, w_j > 0$, consider the sphere

$$S^{2n+1} = \left\{ z = (z_0, ..., z_n) \in \mathbb{C}^{n+1} \mid \sum\nolimits_{j=0}^{n} |z_j|^2 = 1 \right\} \subset \mathbb{C}^{n+1},$$

endowed with the following contact form α_w and corresponding Reeb vector field R_w

$$\alpha_w = \frac{\frac{i}{2}\left(\sum_{j=0}^{n} z_j d\bar{z}_j - \bar{z}_j dz_j\right)}{\sum_{j=0}^{n} w_j |z_j|^2}, \quad R_w = i\left(\sum_{j=0}^{n} w_j (z_j \tfrac{\partial}{\partial z_j} - \bar{z}_j \tfrac{\partial}{\partial \bar{z}_j})\right).$$

(S^{2n+1}, α_w) is called a *weighted Sasakian structure on* S^{2n+1}, cf. [BG08, Example 7.1.12]. In particular, (S^{2n+1}, α_w) is a K-contact manifold with respect to the metric induced by the embedding $S^{2n+1} \hookrightarrow \mathbb{C}^{n+1}$. For $w = (1, ..., 1)$, we obtain the standard contact form on the sphere. Notice that the underlying contact *structure* $\ker\alpha_w$ is independent of the choice of weight w.

From now on, we will always consider a connected, compact K-contact manifold (M, α) with Reeb vector field R and contact metric g, on which a torus G acts in such a way that it preserves the contact form α, i.e., $g^*\alpha = \alpha$ for every $g \in G$. We refer to, e.g., [GNT12, Section 2] or [Bla76] for preliminary considerations. Note that only from Lemma 2.4.7 on we will assume the G-action to be isometric. For a Lie algebra element $X \in \mathfrak{g}$, we denote the fundamental vector field it induces on M by X_M, i.e.,

$$X_M(x) = \left.\frac{d}{dt}\right|_{t=0} \exp(tX) \cdot x.$$

(M, α, g) admits a $(1,1)$-tensor J such that we have the following identities for all $X, Y \in \mathfrak{X}(M)$, where ∇ denotes the Levi-Civita connection of g (see, e.g., [Bla76, pp. 25f, p. 64])

$$JR = 0, \quad J^2 = -\operatorname{id} + \alpha \otimes R, \tag{2.3}$$

$$\alpha(X) = g(R, X), \tag{2.4}$$

$$g(X, JY) = d\alpha(X, Y), \tag{2.5}$$

$$g(JX, JY) = g(X, Y) - \alpha(X)\alpha(Y), \tag{2.6}$$

$$(\nabla_X J)Y = \mathcal{R}(R, X)Y, \tag{2.7}$$

$$\nabla_X R = -JX. \tag{2.8}$$

We set

$$M_R := \{x \in M \mid R(x) \in T_x(G \cdot x)\},$$
$$M_\varnothing := \{x \in M \mid R(x) \notin T_x(G \cdot x)\}.$$

Recall that $T_x(G \cdot x) = \{X_M(x) \mid X \in \mathfrak{g}\}$. For $x \in M_R$, choose any $X^x \in \mathfrak{g}$ such that $X^x_M(x) = R(x)$. This X^x is unique modulo $\mathfrak{g}_x = \{X \in \mathfrak{g} \mid X_M(x) = 0\}$.

Definition 2.2.3. We define the *generalized isotropy algebra* of $x \in M$ by

$$\widetilde{\mathfrak{g}}_x := \{X \in \mathfrak{g} \mid X_M(x) \in \mathbb{R}R(x)\} = \begin{cases} \mathfrak{g}_x \oplus \mathbb{R}X^x & x \in M_R \\ \mathfrak{g}_x & x \in M_\varnothing \end{cases}.$$

Since R is Killing, its flow ψ_t generates a 1-parameter subgroup of the group of isometries of (M, g). Since M is compact, $\operatorname{Iso}(M, g)$ is a compact Lie group (cf. [MS39, Section 5]) and, hence, the closure of ψ_t in $\operatorname{Iso}(M, g)$ is a torus that we denote by T. Note that T as a subgroup in the diffeomorphism group of M is independent of the choice of contact metric. By construction, R is the fundamental vector field of a topological generator of T, which we also denote by R. By definition of the Reeb vector field, it is $0 = \iota_R d\alpha$ and $1 = \iota_R \alpha$, hence

$$\mathcal{L}_R \alpha = d\iota_R \alpha + \iota_R d\alpha = 0. \tag{2.9}$$

It follows that α is invariant under pullback by the Reeb flow

$$\psi_t^* \alpha \equiv \psi_0^* \alpha = \alpha. \tag{2.10}$$

It follows that α is preserved by all of T.

The uniqueness of the Reeb vector field implies that for every $p \in M$ and every $g \in G$, we have

$$dg_p R(p) = R(gp). \tag{2.11}$$

As a result, we obtain that $[X_M, R] = 0$ and, in particular, that the action of G commutes with the flow of the Reeb vector field. As a consequence, the action of G commutes with the T-action. Thus, we can consider the action of the torus $H := G \times T$ on M. Since $\widetilde{\mathfrak{h}}_x = \mathfrak{h}_x \oplus \mathbb{R}R$, we have

$$\widetilde{\mathfrak{g}}_x = \widetilde{\mathfrak{h}}_x \cap (\mathfrak{g} \oplus \{0\}).$$

Note that since M is assumed to be compact, only finitely many different $\mathfrak{g}_x$, $\mathfrak{h}_x$ and, hence, $\widetilde{\mathfrak{g}}_x$ occur.

The group $G \times \{\psi_t\}$ is in general non-compact, which complicates finding, e.g., invariant objects or tubular neighborhoods. The tool to overcome this obstacle is considering the closure T of $\{\psi_t\}$, in particular, we often consider the action of the torus $G \times T$. A closed $G \times \{\psi_t\}$-invariant submanifold $A \subset M$ is automatically $G \times T$-invariant, hence, there exist arbitrarily small $G \times T$-invariant tubular neighborhoods that retract onto A. These retractions are, in particular $G \times \{\psi_t\}$-equivariant.

2.3 The Contact Moment Map

Recall that (M, α) is a connected, compact K-contact manifold with Reeb vector field R, on which a torus G acts in such a way that it preserves the contact form α and that, for any $X \in \mathfrak{g}$, we denoted the corresponding fundamental vector field on M by X_M. The *contact moment map* on M is the map $\Psi\colon M \to \mathfrak{g}^*$, defined by

$$\Psi^X := \Psi(\cdot)(X) := \iota_{X_M}\alpha = \alpha(X_M) \qquad \text{for every } X \in \mathfrak{g}.$$

Remark 2.3.1. Ψ is an abstract moment map according to the definition in [GGK02]: G-invariance (i.e., G-equivariance) stems from the G-invariance of α, and for every closed subgroup $H \subset G$, the map $\Psi^H := \mathrm{pr}_{\mathfrak{h}^*} \circ \Psi\colon M \to \mathfrak{h}^*$ is zero on the points fixed by the H-action, M^H, thus it is in particular constant on the connected components of M^H. In general, however, this map is not a *non-degenerate* abstract moment map, again as defined in [GGK02], since, in general, the inclusion $\{X_M = 0\} \subset Crit(\Psi^X)$ is not an equality, see Equation (2.12) below.

By Cartan's formula, $d\iota_{X_M}\alpha = -\iota_{X_M} d\alpha$ for every $X \in \mathfrak{g}$ since $\mathcal{L}_{X_M}\alpha = 0$. Furthermore, $\ker d\alpha_x = \mathbb{R}R(x)$. This implies that the critical set of the X-component of Ψ is given by

$$Crit(\Psi^X) = \{x \in M \mid X_M(x) \in \mathbb{R}R(x)\} = \{x \in M \mid X \in \widetilde{\mathfrak{g}}_x\}. \tag{2.12}$$

Since $\alpha(R) \equiv 1$ and $(\Psi^X)^{-1}(0) = \{x \in M \mid \alpha_x(X_M(x)) = 0\}$, Equation (2.12) implies

$$Crit(\Psi^X) \cap (\Psi^X)^{-1}(0) = \{x \in M \mid X_M(x) = 0\}. \tag{2.13}$$

Lemma 2.3.2. *Suppose that* 0 *is a regular value of* Ψ. *Then* M *has no* G*-fixed points,* $M^G = \varnothing$.

Proof. Since all fundamental vector fields vanish on M^G, the claim is a consequence of Equation (2.13). □

In analogy to the symplectic setting (cf., e.g., [CdS01, 23.2.1]), we have the following.

Lemma 2.3.3. *Denote the annihilator of* $\widetilde{\mathfrak{g}}_x$ *in* $\mathfrak{g}^*$ *by* $\widetilde{\mathfrak{g}}_x^0$. *The image of* $d\Psi_x$ *is exactly* $\widetilde{\mathfrak{g}}_x^0$.

Proof. The image of the linear map $d\Psi_x$ is the annihilator of the kernel of its transpose. By Equation (2.12), the kernel of $d\Psi_x^t$ is $\widetilde{\mathfrak{g}}_x$. □

2.4 A Special Basis for $\mathfrak{g}$

In this section, we will show that under the assumption that 0 is a regular value of the contact moment map Ψ, the Lie algebra $\mathfrak{g}$ of G admits a basis that fulfills certain axioms. The next proposition and the resulting Proposition 2.4.9 will be crucial to the proof of our surjectivity result Theorem 4.1.1. They are inspired by the idea of the proof of Theorem G.13 in [GGK02] and a corrected version thereof in [BL10, Proposition 3.12, Appendix B]. However, [BL10, Proposition 3.12] requires a non-degenerate abstract moment map and a G-invariant almost complex structure. Hence, while providing an alternative proof of Kirwan surjectivity on symplectic manifolds, it does not hold in our case.

Proposition 2.4.1. *Let (M, α, g) be a compact K-contact manifold and R its Reeb vector field. Let G be a torus that acts on M, preserving α. Denote by $\Psi\colon M \to \mathfrak{g}^*$ the contact moment map and suppose that 0 is a regular value of Ψ. Then there exists a basis $(X_1, ..., X_r)$ of $\mathfrak{g}$ such that for every $s = 1, ..., r$*

(i) $0 \in \mathbb{R}^s$ *is a regular value of* $f_s := (\Psi^{X_1}, ..., \Psi^{X_s})\colon M \to \mathbb{R}^s$.

(ii) $\{x \in M \mid (X_s)_M(x) = 0\} = \varnothing$.

(iii) For all $\mathfrak{g}_x$ of dimension at most $r - s$, the following holds:

$$\mathfrak{g}_x \cap \bigoplus_{j=1}^{s} \mathbb{R}X_j = \{0\}.$$

(iv) For all $\widetilde{\mathfrak{g}}_x$ of dimension at most $r - s$, the following holds:

$$\widetilde{\mathfrak{g}}_x \cap \bigoplus_{j=1}^{s} \mathbb{R}X_j = \{0\}.$$

(v) The critical points C_s of f_s are

$$C_s = \{x \in M \mid \widetilde{\mathfrak{g}}_x \cap \oplus_{j=1}^{s} \mathbb{R}X_j \neq \{0\}\} = \{x \in M \mid \dim \widetilde{\mathfrak{g}}_x > r - s\}.$$

In particular, with $C_0 := \varnothing$,

$$C_s = C_{s-1} \,\dot{\cup}\, \{x \in M \mid \dim \widetilde{\mathfrak{g}}_x = r - s + 1\}.$$

Remark 2.4.2. We remark that a basis with properties *(i)-(iii)* of Proposition 2.4.1 exists on a contact manifold that is not necessarily K-contact, the proof is similar.

Remark 2.4.3. Note that, together with Equation (2.13), Property *(ii)* implies that $Crit(\Psi^{X_s}) \cap (\Psi^{X_s})^{-1}(0)$ is empty.

Proof of Proposition 2.4.1. Recall that there are only finitely many $\mathfrak{g}_x$ and $\widetilde{\mathfrak{g}}_x$, and that $\mathfrak{g}$ does not occur as isotropy algebra (by Lemma 2.3.2). Set

$$\mathfrak{k} = \bigcup \mathfrak{g}_x \ \cup \bigcup_{\widetilde{\mathfrak{g}}_x \neq \mathfrak{g}} \widetilde{\mathfrak{g}}_x$$

and denote its complement by $\mathfrak{a}_0 = \mathfrak{g} \setminus \mathfrak{k}$; as complement of finitely many proper subspaces, $\mathfrak{a}_0$ is open and dense. With Equation (2.13), it follows that *(i)-(v)* hold for $s = 1$ with an arbitrary $X_1 \in \mathfrak{a}_0$.

Now, let us suppose we already found $X_1, ..., X_{s_0}$ such that *(i) - (v)* hold for $s = 1, ..., s_0$; we will construct X_{s_0+1}. Set $W_{s_0} = \oplus_{j=1}^{s_0} \mathbb{R}X_j$. The following set is open and dense in $\mathfrak{g}$ since it is the complement of finitely many proper subspaces:

$$\mathfrak{a}_{s_0} := \mathfrak{g} \setminus \left(\bigcup_{\{x \in M | \dim \mathfrak{g}_x < r - s_0\}} (\mathfrak{g}_x \oplus W_{s_0}) \quad \cup \bigcup_{\{x \in M | \dim \widetilde{\mathfrak{g}}_x < r - s_0\}} (\widetilde{\mathfrak{g}}_x \oplus W_{s_0}) \right).$$

I.e., $\mathfrak{a}_{s_0}$ consists of those X_{s_0+1} such that *(iii)* and *(iv)* hold for $s = s_0 + 1$. Any $X_{s_0+1} \in \mathfrak{a}_0 \cap \mathfrak{a}_{s_0} \neq \varnothing$ will then obviously satisfy *(ii)-(iv)*. To show that the remaining properties are satisfied as well, we need

Lemma 2.4.4. *Set* $M^{\mathfrak{g}_p} = \{x \in M \mid \mathfrak{g}_p \subset \mathfrak{g}_x\}$ *and* $Y_s := f_s^{-1}(0)$. *For every* $\mathfrak{g}_p$ *of dimension* $r - s > 0$ *, the following holds:*

$$M^{\mathfrak{g}_p} \cap Y_s = \varnothing. \tag{2.14}$$

Proof. Let $x \in M^{\mathfrak{g}_p} \cap Y_s$. By *(iii)*, $\mathfrak{g}_p$ and $\oplus_{j=1}^{s} \mathbb{R}X_j$ span all of $\mathfrak{g}$ since their intersection is zero. We have $\oplus_{j=1}^{s} \mathbb{R}X_j \subset \ker \Psi(x)$ by the definition of Y_s and $\mathfrak{g}_p \subset \ker \Psi(x)$ because $\Psi(M^{\mathfrak{g}_p})$ lies in the annihilator of $\mathfrak{g}_p$. $\mathfrak{g}_p$ and $\oplus_{j=1}^{s} \mathbb{R}X_j$ span all of $\mathfrak{g}$ since their intersection is zero by *(iii)*, thus $\Psi(x) = 0$. Lemma 2.3.3 implies, however, that $M^{\mathfrak{g}_p}$ cannot contain a regular point of Ψ, hence, $0 \notin \Psi(M^{\mathfrak{g}_p})$ since 0 is a regular value of Ψ. $\square$

Let us return to the proof of Proposition 2.4.1. We can view f_{s_0+1} as the composition of Ψ and the restriction from $\mathfrak{g}$ to $W_{s_0+1} := \oplus_{j=1}^{s_0+1} \mathbb{R}X_j$. By Lemma 2.3.3, the image of $d\Psi_x$ is $\widetilde{\mathfrak{g}}_x^0$. Composing with the restriction yields that $(df_{s_0+1})_x$ is surjective if and only if $\widetilde{\mathfrak{g}}_x \cap W_{s_0+1} = \{0\}$. Thus, we have

$$C_{s_0+1} = C_{s_0} \ \dot{\cup} \ \{x \in M \mid \widetilde{\mathfrak{g}}_x \cap W_{s_0} = \{0\},\ \widetilde{\mathfrak{g}}_x \cap W_{s_0+1} \neq \{0\}\}\,. \tag{2.15}$$

Since we chose $X_{s_0+1} \in \mathfrak{a}_0 \cap \mathfrak{a}_{s_0}$, we directly obtain the remaining statement of *(v)* for $s = s_0 + 1$, in particular, with $M_\varnothing$ and M_R from page 18:

$$C_{s_0+1} = C_{s_0} \ \dot{\cup} \ \underbrace{\{x \in M_\varnothing \mid \dim \mathfrak{g}_x = r - s_0\}}_{=:A_1} \dot{\cup} \underbrace{\{x \in M_R \mid \dim \widetilde{\mathfrak{g}}_x = r - s\}}_{=:A_2}. \tag{2.16}$$

It remains to show that *(i)* holds for $s = s_0+1$. By assumption, 0 is a regular value of f_{s_0}, thus $C_{s_0} \cap Y_{s_0} = \varnothing$. Lemma 2.4.4 yields that $A_1 \cap Y_{s_0} = \varnothing$. Now, consider an element $x \in A_2 \cap Y_{s_0}$. Then $\widetilde{\mathfrak{g}}_x \cap W_{s_0} = \{0\}$, $\widetilde{\mathfrak{g}}_x \cap W_{s_0+1} \neq \{0\}$. It follows that $X_{s_0+1} \in \widetilde{\mathfrak{g}}_x \oplus W_{s_0}$. For every $X \in W_{s_0}$, $\Psi^X(x) = 0$. Suppose $\Psi^{X_{s_0+1}}(x) = 0$. Then, by definition of Ψ and since $\alpha(R) = 1$, it would follow that $X_{s_0+1} \in \mathfrak{g}_x \oplus W_{s_0}$. However, this contradicts $X_{s_0+1} \in \mathfrak{a}_0 \cap \mathfrak{a}_{s_0}$. We showed that $0 \notin \Psi^{X_{s_0+1}}(C_{s_0+1} \cap Y_{s_0})$, meaning that *(i)* is satisfied for $s = s_0+1$. Hence, we showed that with any choice of $X_{s_0+1} \in \mathfrak{a}_0 \cap \mathfrak{a}^{s_0} \neq \varnothing$, *(i)* - *(v)* hold for $s = s_0 + 1$. □

Recall that we set $f_s := (\Psi^{X_1}, ..., \Psi^{X_s})\colon M \to \mathbb{R}^s$ and $Y_s := f_s^{-1}(0)$.

Lemma 2.4.5. *With (X_s) as in Proposition 2.4.1, we have for every $x \in Y_s$*

$$\{0\} = \widetilde{\mathfrak{g}}_x \cap \oplus_{j=1}^s \mathbb{R}X_j. \tag{2.17}$$

In particular, $\dim \widetilde{\mathfrak{g}}_x \leq r - s$ *and* $\dim \mathfrak{g}_x < r - s$.

Proof. Equation (2.17) follows directly from Proposition 2.4.1, by combining *(i)*, *(iv)* and *(v)*. It directly implies that $\dim \widetilde{\mathfrak{g}}_x \leq r - s$. Since $\dim \mathfrak{g}_x \leq \dim \widetilde{\mathfrak{g}}_x$, and Y_s does not contain a point with isotropy of dimension $r - s$ by Lemma 2.4.4, it follows that $\dim \mathfrak{g}_x < r - s$. □

A main aspect needed for the proof of our main Theorem will be the Morse-Bott property of the functions $\Psi^{X_{s+1}}|_{Y_s}$. As a first step, we now want to compute their critical sets $Crit(\Psi^{X_{s+1}}|_{Y_s})$. Recall that T denotes the closure of the flow of the Reeb vector field R in the isometry group of (M, g), where g is any contact metric, and that T is independent of the choice of g.

Lemma 2.4.6. *With (X_s) as in Proposition 2.4.1, Crit$(\Psi^{X_{s+1}}|_{Y_s})$ is the union of all the minimal $G\times\{\psi_t\}$-orbits, i.e., of all $G\times\{\psi_t\}$-orbits of dimension $s+1$. They coincide with the minimal $G\times T$-orbits. These are exactly the points of Y_s with generalized isotropy algebra of dimension $r-s$. In particular, Crit(Ψ^{X_1}) = Crit(Ψ) is the union of all 1-dimensional $G\times T$-orbits and consists of all points with $\widetilde{\mathfrak{g}}_x = \mathfrak{g}$.*

Proof. Set $(Y_s)_R := Y_s \cap M_R$ and $(Y_s)_\varnothing := Y_s \cap M_\varnothing$, with $M_\varnothing$ and M_R from page 18. We first show that $Crit(\Psi^{X_{s+1}}|_{Y_s}) = \bigcup_{\substack{x\in(Y_s)_R \\ \dim G\cdot x=s+1}} G\cdot x$. Let $x \in Y_s$. By *(i)* of Proposition 2.4.1, $\operatorname{span}\{d\Psi^{X_1}_x, ..., d\Psi^{X_s}_x\}$ is s-dimensional and $T_xY_s = \ker(df_s)_x$. Since the annihilator of T_xY_s in T^*_xM is s-dimensional, it follows that T_xY_s lies in the kernel of a 1-form if and only if that 1-form lies in the span of $\{d\Psi^{X_1}_x, ..., d\Psi^{X_s}_x\}$. Therefore, we obtain

$$Crit(\Psi^{X_{s+1}}|_{Y_s}) = \left\{x \in Y_s \;\middle|\; (d\Psi^{X_{s+1}})_x \in \operatorname{span}\{d\Psi^{X_1}_x, ..., d\Psi^{X_s}_x\}\right\}$$

Using additivity of $d\Psi^X$ in X and applying Equation (2.12), this equation becomes

$$Crit(\Psi^{X_{s+1}}|_{Y_s}) = \{x \in Y_s \mid X_{s+1} \in \widetilde{\mathfrak{g}}_x \oplus W_s\}, \tag{2.18}$$

where $W_s = \oplus_{j=1}^s \mathbb{R}X_j$.

By Lemma 2.4.5, $\dim \mathfrak{g}_x < r-s$ and $\dim \widetilde{\mathfrak{g}}_x \leq r-s$ for every $x \in Y_s$. With *(iv)* of Proposition 2.4.1, the condition in Equation (2.18) can then only be satisfied for $x \in Y_s$ with $\dim \widetilde{\mathfrak{g}}_x = r-s$, thus $x \in (Y_s)_R$. Since in that case, it is $\mathfrak{g} = \widetilde{\mathfrak{g}}_x \oplus W_s$, we automatically obtain that $X_{s+1} \in \widetilde{\mathfrak{g}}_x \oplus W_s$. Hence,

$$Crit(\Psi^{X_{s+1}}|_{Y_s}) = \{x \in (Y_s)_R \mid \dim \widetilde{\mathfrak{g}}_x = r-s\} = \bigcup_{\substack{x\in(Y_s)_R \\ \dim G\cdot x=s+1}} G\cdot x$$

Let $x \in (Y_s)_\varnothing$. From Lemma 2.4.5, we have $\dim \mathfrak{g}_x \leq r-s-1$. Hence, $\dim(G\times T)\cdot x \geq \dim(G\times\{\psi_t\})\cdot x > \dim G\cdot x \geq s+1$, so the $G\times T$- and $G\times\{\psi_t\}$-orbits through x are not minimal. Now, let $x \in (Y_s)_R$ and suppose that $\dim G\cdot x = s+1$ is minimal. By definition of $(Y_s)_R$, $\{\psi_t\}\cdot x \subset G\cdot x$, thus $\dim(G\times\{\psi_t\})\cdot x = s+1$ as well. $G\cdot x$ is closed, hence the same holds for T: $T\cdot x \subset G\cdot x$ and $\dim(G\times T)\cdot x = s+1$ is minimal. $\square$

Lemma 2.4.7. *There exists a contact metric g on M such that all G-fundamental vector fields are Killing vector fields, i.e., such that g is $G\times T$-invariant.*

Proof. Choose any $G \times T$-invariant and $d\alpha$-compatible metric h on $\ker\alpha$, which has to exist since $G \times T$ is compact. Then $g := h \;\oplus\; \alpha \otimes \alpha$ is a $G \times T$-invariant contact metric on M. □

Now, let $N \subset Crit(\Psi^{X_{s+1}}|_{Y_s})$ be a connected component of the critical set. From now on, we will work with a metric according to Lemma 2.4.7, i.e., with an isometric $G \times T$-action.

Lemma 2.4.8. *N is a totally geodesic closed submanifold of even codimension.*

Proof. By Lemma 2.4.6, N is a union of minimal dimensional $G \times T$-orbits. The isotropy group of a point in a tubular neighborhood of an orbit $(G \times T) \cdot p$ is a subgroup of $(G \times T)_p$. By minimality, every point of N in that tubular neighborhood then has to have the same isotropy algebra, so $\{x \in N \mid (\mathfrak{g} \times \mathfrak{t})_x = (\mathfrak{g} \times \mathfrak{t})_p\}$ is open in N. Since N is connected, it follows that the connected component of the isotropy remains the same along N, $(\mathfrak{g} \times \mathfrak{t})_x =: (\mathfrak{g} \times \mathfrak{t})_N$ for all $x \in N$. Since all fundamental vector fields are Killing, we can apply a result of Kobayashi [Kob58, Corollary 1], which directly yields that N is a totally geodesic closed submanifold of even codimension. □

We will denote the g-orthogonal normal bundle of N in Y_s by νN, $T_pY_s = T_pN \oplus_{\perp_g} \nu_pN$. We will now prove the Morse-Bott property of $\Psi^{X_{s+1}}|_{Y_s}$. For a brief introduction to Morse-Bott functions, the reader is referred to Appendix A.

Proposition 2.4.9. *The Hessian H of $\Psi^{X_{s+1}}|_{Y_s}$ along N in normal directions is given by*

$$H_p(v, w) = 2g(w, \nabla_v(JY)) = 2g(w, J\nabla_v Y),$$

where $p \in N$, $Y := (X_{s+1})_{Y_s} - \alpha((X_{s+1})_{Y_s})_p R$, and g is a metric as in Lemma 2.4.7.

Furthermore, the vector $J\nabla_v Y$ is normal and non-zero for every normal vector $v \neq 0$ and H is non-degenerate in normal directions.

In particular, $\Psi^{X_{s+1}}|_{Y_s}$ is a Morse-Bott function.

Proof. Let $p \in N$ and $v, w \in \nu_p N$ be arbitrary. In a sufficiently small neighborhood of p, extend v and w to local vector fields V, W around p such that $(\nabla V)(p) = (\nabla W)(p) = 0$. To shorten notation, let $X := (X_{s+1})_{Y_s}$. Note that since $[X, R] = 0$ by Equation (3.1), we have $\nabla_X R = \nabla_R X$. The first computation in [Ruk99, Section 2] is equally applicable in our case since X is a Killing vector field, hence we obtain at p, applying Equations (2.7) and (2.8),

$$\begin{aligned}
H_p(v,w) &= (V(W(\alpha(X)))\,(p) = (V(W(g(R,X))))\,(p) \\
&= (V(g(\nabla_W R, X) + g(R, \nabla_W X)))\,(p) \\
&= (V(g(-JW, X) - g(\nabla_R X, W)))\,(p) \\
&= (-g(\nabla_V JW, X) - g(JW, \nabla_V X) + V(g(JX, W)))\,(p) \\
&= (-g(\nabla_V JW, X) - g(JW, \nabla_V X) + g(\nabla_V JX, W) + g(JX, \nabla_V W))\,(p) \\
&= (-g((\nabla_V J)W, X) - g(J(\nabla_V W), X) + g(W, J\nabla_V X) + g((\nabla_V J)X, W) \\
&\quad + g(J(\nabla_V X), W) + g(JX, \nabla_V W))\,(p) \\
&= (-g(\mathcal{R}(R,V)W, X) + 2g(W, J\nabla_V X) + g(\mathcal{R}(R,V)X, W))\,(p) \\
&= (2g(\mathcal{R}(R,V)X, W) + 2g(W, J\nabla_V X))\,(p). \qquad (2.19)
\end{aligned}$$

Combining Lemma 2.4.8, Equation (2.8), and the fact that $R(x) \in T_x N$ for all $x \in N$, we obtain that $Jz = -\nabla_z R \in TN$ for all $z \in TN$, hence

$$J : T_p N \to T_p N, \quad J : \nu_p N \to \nu_p N.$$

Set $a := \alpha((X_{s+1})_{Y_s})_p$ and decompose X as $X = aR + Y$. It is

$$(\nabla_V X)(p) = (a\nabla_V R + \nabla_V Y)(p) = -aJv + (\nabla_V Y)_p.$$

Using the tensor properties of the curvature tensor $\mathcal{R}$ and that $\mathcal{R}(R,V)R = -V$ (see [Bla76, p. 65]), we can then continue Equation (2.19) as follows:

$$\begin{aligned}
H_p(v,w) &= (2ag(\mathcal{R}(R,V)R, W) + 2g(\mathcal{R}(R,V)Y, W) + 2g(W, J(-aJV + \nabla_V Y)))\,(p) \\
&= -2ag(v,w) + 2g(\mathcal{R}(R,V)Y, W)(p) + 2ag(v,w) + 2g(W, J\nabla_V Y)(p) \\
&= 2g(\mathcal{R}(R,V)Y, W)(p) + 2g(W, J\nabla_V Y)(p) \\
&= 2g(W, (\nabla_V J)Y + J\nabla_V Y)(p) \\
&= 2g(W, \nabla_V(JY))(p). \qquad (2.20)
\end{aligned}$$

It remains to show that the vector $\nabla_V(JY)(p) = (\mathcal{R}(R,V)Y + J\nabla_V Y)(p)$ is equal to $J\nabla_V Y(p)$, is non-zero, and lies in $\nu_p N$. Let η be an arbitrary vector field in a neighborhood of p that is tangent to N at p. By Lemma 2.4.8, $\nabla_\eta X(p) \in T_pN$. Since X is Killing and $v \in T_pN^{\perp_g}$, we then have $g(\eta, \nabla_V X)_p = -g(\nabla_\eta X, V)_p = 0$. Thus, $\nabla_V X(p) \in \nu_p N$ and, hence, $J\nabla_V X(p) \in \nu_p N$. With Equations (2.8), (2.3) and $\alpha(V)_p = g(R,V)_p = 0$, we obtain

$$g(\eta, J\nabla_V Y)_p = g(\eta, J\nabla_V X)_p - g(\eta, J\nabla_V(aR))_p = -g(\eta, aV)_p = 0,$$

hence $(J\nabla_V Y)(p) \in (T_pN)^{\perp_g} = \nu_p N$. Analogously, we obtain $(\nabla_V Y)(p) \in \nu_p N$. Recall that $\mathcal{L}_X\alpha = 0$. Since N is critical, we obtain on N

$$0 = -d\iota_X\alpha = \iota_X d\alpha = a\iota_R d\alpha + \iota_Y d\alpha = \iota_Y d\alpha.$$

$Y|_N \in \Gamma(\ker\alpha)$, however, since $\alpha(X)|_N \equiv a$, and $d\alpha$ is non-degenerate on $\ker\alpha$. Therefore, it is $Y = 0$ on N and we obtain $\nabla_V(JY)(p) = (\mathcal{R}(R,V)Y + J\nabla_V Y)(p) = (J\nabla_V Y)(p)$.

We now follow the line of argumentation of Rukimbira in [Ruk95, Proof of Lemma 1] to show that $\nabla_v Y$ does not vanish on N. Note that Y is a Killing vector field since X and R are. Let γ be the geodesic through $\gamma(0) = p$ with tangent vector $\dot\gamma(0) = v$. Suppose $(\nabla_v Y)(p) = 0$. Then the Jacobi field $Y \circ \gamma$ satisfies $Y \circ \gamma(0) = 0$ and $\frac{\nabla}{dt}(Y \circ \gamma)(0) = 0$, thus Y vanishes along all of γ. This means that along γ, $X = aR$, though. By Equation (2.12), γ hence consists of critical points of $\Psi^X|_{Y_s}$. Thus, γ lies in N and v has to be tangent to N. This, however, contradicts $v \in \nu_p N$. We conclude that $\nabla_V Y(p)$ is non-zero. Since $\nabla_V Y(p)$ is normal and, hence, lies in $\ker\alpha$, it follows that $J(\nabla_V Y)(p)$ is non-zero. Then we have for every non-zero normal vector $v \in \nu_p N$:

$$H_p(v, J(\nabla_v Y)) = 2g(J(\nabla_v Y), J(\nabla_v Y)) = 2g(\nabla_v Y, \nabla_v Y) \neq 0. \qquad \square$$

Remark 2.4.10. J is skew-symmetric with respect to H: For $v, w \in \nu_p N$, we have

$$\begin{aligned}\tfrac{1}{2}H_p(w, Jv) &= g(w, J\nabla_{Jv}Y) = -g(Jw, \nabla_{Jv}X + aJ^2v) = g(\nabla_{Jw}X, Jv) + ag(Jw, v)\\ &= g(\nabla_{Jw}Y, Jv) + g(-aJ^2w, Jv) - ag(w, Jv)\\ &= -g(J\nabla_{Jw}Y, v) + ag(w, Jv) - ag(w, Jv) = -\tfrac{1}{2}H_p(Jw, v).\end{aligned}$$

In particular, J preserves the positive and negative normal bundle, $J : \nu^\pm N \to \nu^\pm N$.

2.5 A Local Normal Form for the Contact Moment Map

In Section 2.1, we proved a Contact Coisotropic Embedding Theorem (Theorem 2.1.3) by applying an equivariant Contact Darboux Theorem (Theorem 2.1.2). Assume that 0 is a regular value of the contact moment map Ψ. In order to obtain a local normal form for Ψ in a neighborhood of $\Psi^{-1}(0)$, we will now show that certain embeddings satisfy the requirements of Theorem 2.1.3.

Lemma 2.5.1. *The natural embedding $\Psi^{-1}(0) \hookrightarrow M$ satisfies (i)-(iii) of Theorem 2.1.3 with $K = G \times T$.*

Proof. *(ii)* and *(iii)* are obviously satisfied. To show that the distribution $\zeta := T\Psi^{-1}(0) \cap \ker\alpha$ is coisotropic in $(\ker\alpha, d\alpha|_{\ker\alpha} =: \omega)$, recall that 0 is a regular value of Ψ, hence,

$$T_p\Psi^{-1}(0) = \ker d\Psi_p. \tag{2.21}$$

$v \in \ker d\Psi_p$ if and only if $d\Psi_p^X(v) = (d\iota_{X_M}\alpha)_p(v) = 0$ for every $X \in \mathfrak{g}$. Since α is G-invariant, $\mathcal{L}_{X_M}\alpha = 0$, and Cartan's formula yields that $v \in \ker d\Psi_p$ if and only if $d\alpha_p(X_M, v) = 0$ for every $X \in \mathfrak{g}$. It follows that

$$\ker d\Psi_p = (T_pG \cdot p)^{\perp_{d\alpha}} \tag{2.22}$$

since the tangent space to the G-orbit consists of all fundamental vector fields. For $p \in \Psi^{-1}(0)$, it is $0 = \Psi(p)(X) = \alpha_p(X_M(p))$ for every $X \in \mathfrak{g}$. In particular, $T_p(G\cdot p) \subset \ker\alpha_p$. It follows that $(T_pG\cdot p)^{\perp_{d\alpha}} = (T_pG\cdot p)^{\perp_\omega} \oplus \mathbb{R}R_p$. Equations (2.22) and (2.21) yield $T_p\Psi^{-1}(0) \cap \ker\alpha_p = T_p(G\cdot p)^{\perp_\omega} =: \zeta_p$. Then $\zeta_p^{\perp_\omega} = T_p(G\cdot p)$. Ψ is G-invariant, so for every $X \in \mathfrak{g}$, $d\Psi_p(X_M(p)) = 0$. We obtain $\zeta_p^{\perp_\omega} = T_p(G\cdot p) \subset \ker d\Psi_p = (T_pG\cdot p)^{\perp_{d\alpha}}$ and, hence, $\zeta_p^{\perp_\omega} \subset (T_pG\cdot p)^{\perp_{d\alpha}} \cap \ker\alpha_p = (T_pG\cdot p)^{\perp_\omega} = \zeta_p$, ζ is coisotropic. □

Lemma 2.5.2. *The embedding $\Psi^{-1}(0) \cong \Psi^{-1}(0) \times \{0\} \hookrightarrow \Psi^{-1}(0) \times \mathfrak{g}^*$ satisfies (i)-(iii) of Theorem 2.1.3 with $K = G \times T$, where a neighborhood $U = \Psi^{-1}(0) \times V$ of $\Psi^{-1}(0) \times \{0\} \subset \Psi^{-1}(0) \times \mathfrak{g}^*$ is endowed with the contact form $\widetilde{\alpha} := i^*\alpha + z(\theta)$, we denote the inclusion $\Psi^{-1}(0) \hookrightarrow M$ by i, the coordinates on $\mathfrak{g}^*$ by z and θ is a*

G-invariant R-basic connection form on $\Psi^{-1}(0) \to \Psi^{-1}(0)/G$. Furthermore, R is the Reeb vector field of $(U, \widetilde{\alpha})$ and the contact moment map on $(U, \widetilde{\alpha})$ is given by $\widetilde{\Psi}(p, z) = z$.

Remark 2.5.3. Note that a G-invariant R-basic connection form has to exist: By [Mol88, Proposition 2.8], there always exists a connection that is *adapted* to the lifted foliation, i.e., such that the tangent spaces to the leaves are horizontal. Since $G \times T$ is compact, we can obtain a $G \times T$-invariant adapted connection form by averaging over the group. But this connection form then has to be basic, or, as Molino calls it, *projectable*.

Proof. Let $j : \Psi^{-1}(0) \to \Psi^{-1}(0) \times \mathfrak{g}^*$ denote the embedding given by $x \mapsto (x, 0)$. Then $j^*\widetilde{\alpha} = i^*\alpha$ by construction. Choose an orthonormal basis (X_i) of $\mathfrak{g}$ and denote its dual basis by (u_i). Then we can write $\theta = \sum \theta_i X_i$ and $z = \sum z_i u_i$ according to these bases and obtain $d(z(\theta)) = \sum dz_i \wedge \theta_i + z_i d\theta_i = dz(\theta) + z(d\theta)$. With $\Omega = \theta_i \wedge ... \wedge \theta_r$ and $dz = dz_i \wedge ... \wedge dz_r$, at $z = 0$, we have

$$\widetilde{\alpha} \wedge (d\widetilde{\alpha})^n = (-1)^{r(r+1)/2} r!\; i^*(\alpha \wedge (d\alpha)^{n-r}) \wedge \Omega \wedge dz,$$

which is non-degenerate. Therefore there is a neighborhood $U = \Psi^{-1}(0) \times V$ of $\Psi^{-1}(0) \times \{0\}$ in $\Psi^{-1}(0) \times \mathfrak{g}^*$ on which $\widetilde{\alpha}$ is a contact form. θ is R-basic, so $\iota_R\theta = 0$ and $\iota_R\widetilde{\alpha} = \iota_R i^*\alpha = i^*\iota_R\alpha = 1$. $d\theta$ is R-basic, as well, so $\iota_R d\theta = 0$. R is tangent to $\Psi^{-1}(0)$, so $dz_i(R) = 0$. We obtain $\iota_R d\widetilde{\alpha} = \iota_R(i^*d\alpha + dz(\theta) + z(d\theta)) = 0$. The uniqueness of the Reeb vector field yields that R is the Reeb vector field of $(U, \widetilde{\alpha})$. It remains to compute the contact moment map $\widetilde{\Psi}$ on $(U, \widetilde{\alpha})$ and to show that the distribution $\zeta_p := T_p\Psi^{-1}(0) \cap \ker \widetilde{\alpha}_p$ is coisotropic in the symplectic vector bundle $(\ker \widetilde{\alpha}, d\widetilde{\alpha}|_{\ker \widetilde{\alpha}} =: \omega)$. By definition of Ψ, $i^*\alpha$ vanishes on G-fundamental vector fields. For any $X = \sum \lambda_i X_i \in \mathfrak{g}$, we have $z(\theta)(X) = (\sum_i z_i\theta_i)(\sum_j \lambda_j X_j) = \sum_i z_i\lambda_i = z(X)$. Hence, we have $\widetilde{\Psi}(p, z) = z$, which implies $\widetilde{\Psi}^{-1}(0) = \Psi^{-1}(0) \times \{0\} = i(\Psi^{-1}(0))$. Since $d\widetilde{\Psi} = dz$, $\widetilde{\Psi}$ has 0 as a regular value, so that we obtain $T_{(p,0)}(\Psi^{-1}(0) \times \{0\}) = \ker d\widetilde{\Psi}_{(p,0)}$. The rest of the proof works completely analogously to that of Lemma 2.5.1, with α, Ψ replaced by $\widetilde{\alpha}, \widetilde{\Psi}$. □

Applying Theorem 2.1.3 to the two coisotropic embeddings in Lemmata 2.5.1 and 2.5.2, we obtain a local normal form of Ψ around $\Psi^{-1}(0)$.

Proposition 2.5.4. *Suppose that* 0 *is a regular value of* Ψ*. Then there is a* $G \times T$*-invariant neighborhood* U *of* $\Psi^{-1}(0)$ *which is equivariantly diffeomorphic to a neighborhood of* $\Psi^{-1}(0) \times \{0\}$ *in* $\Psi^{-1}(0) \times \mathfrak{g}^*$ *of the form* $\Psi^{-1}(0) \times B_h$*,* $B_h = \{z \in \mathfrak{g} \mid |z| \leq h\}$*, such that in this neighborhood the contact form* α *is equal to* $q^*\alpha_0 + z(\theta)$*, where* $\theta \in \Omega^1(\Psi^{-1}(0), \mathcal{F}, \mathfrak{g})$ *is a* G*-invariant,* $\mathcal{F}$*-basic connection 1-form on* $q : \Psi^{-1}(0) \to \Psi^{-1}(0)/G$*. In particular, on* U*, the moment map is given by* $\Psi(p, z) = z$*.*

Chapter 3

Equivariant Basic Cohomology for K-Contact Manifolds

In this chapter, we define equivariant basic cohomology and give its basic properties. Related constructions for transverse actions of a Lie algebra on a foliated manifold (in particular for the transverse action of $\mathfrak{t}/\mathbb{R}R$ on a K-contact manifold) can be found in [GT16, GNT12, Töb14].

3.1 Basic Cohomology

Recall that we consider a compact K-contact manifold (M, α) with Reeb vector field R and that we denote the foliation induced by R on M by $\mathcal{F}$. Let $\mathfrak{X}(\mathcal{F})$ denote the vector space of vector fields on M that are tangent to the leaves of the foliation $\mathcal{F}$, $\mathfrak{X}(\mathcal{F}) = C^\infty(M) \cdot R$. Differential forms whose contraction with and Lie derivative in the direction of an element of $\mathfrak{X}(\mathcal{F})$ vanish are called $\mathcal{F}$-*basic* (or simply *basic*). Their subspace is denoted by

$$\Omega(M, \mathcal{F}) := \{\omega \in \Omega(M) \mid \mathcal{L}_X\omega = \iota_X\omega = 0 \ \forall\ X \in \mathfrak{X}(\mathcal{F})\}.$$

Cartan's formula directly yields that $\Omega(M, \mathcal{F})$ is differentially closed, i.e., for every $\omega \in \Omega(M, \mathcal{F})$, we have $d\omega \in \Omega(M, \mathcal{F})$, so that $\Omega(M, \mathcal{F})$ is a subcomplex of the de Rham complex of M.

Definition 3.1.1. The *basic cohomology* of the foliated manifold $(M, \mathcal{F})$, denoted by $H^*(M, \mathcal{F})$, is the cohomology of the complex $(\Omega(M, \mathcal{F}), d)$.

For a more elaborate introduction to basic differential forms, also for more general foliations, the reader is referred to [Rei59].

On basic differential forms, there is a natural Poincaré pairing defined by

$$(\xi, \eta) \mapsto \int_M \alpha \wedge \xi \wedge \eta.$$

Lemma 3.1.2. *The Poincaré pairing descends to a well-defined pairing on basic cohomology. If M is a compact K-contact manifold, then the basic cohomology groups are finite-dimensional, $H^r(M, \mathcal{F}) = 0$ for $r > 2n$ and the Poincaré pairing is non-degenerate.*

Proof. See, e.g., [BG08, Proposition 7.2.3]. □

We suppose now that a torus G acts on M, preserving the contact form α. Recall from page 19 that $[X_M, R] = 0$ for every $X \in \mathfrak{g}$. This implies in particular that, for every $f \cdot R \in \mathfrak{X}(\mathcal{F})$,

$$[X_M, f \cdot R] = X_M(f)R \in \mathfrak{X}(\mathcal{F}). \tag{3.1}$$

Remark 3.1.3. Equation (3.1) means that, given $X \in \mathfrak{g}$, for every $Y \in \mathfrak{X}(\mathcal{F})$, the commutator $[X_M, Y]$ is also an element of $\mathfrak{X}(\mathcal{F})$; hence all fundamental vector fields are so called *foliate vector fields* as defined by Molino (see [Mol88, Chapter 2.2]).

Recall the following definition (cf. [GT16, Definition 3.1] or [GS99, Chapter 2.2] for a formulation in the language of superalgebras).

Definition 3.1.4. Let $\mathfrak{k}$ be a finite dimensional Lie algebra and $A = \bigoplus A_k$ a $\mathbb{Z}$-graded algebra. A is called a *differential graded $\mathfrak{k}$-algebra* ($\mathfrak{k}$-dga) or $\mathfrak{k}^*$-algebra, if there exist derivations $d : A \to A$ of degree 1, $\iota_X : A \to A$ of degree -1, and $\mathcal{L}_X A \to A$ of degree 0 for every $X \in \mathfrak{k}$ such that ι_X and $\mathcal{L}_X$ are linear in X and

- $d^2 = 0$,
- $\iota_X^2 = 0$,
- $[\mathcal{L}_X, \mathcal{L}_Y] = \mathcal{L}_{[X,Y]}$,
- $[\mathcal{L}_X, \iota_Y] = \iota_{[X,Y]}$,
- $\mathcal{L}_X = d\iota_X + \iota_X d$.

Lemma 3.1.5. *With the usual differential d inherited from $\Omega(M)$ and $\iota_X \coloneqq \iota_{X_M}$, $\mathcal{L}_X \coloneqq \mathcal{L}_{X_M}$, $\Omega(M, \mathcal{F})$ is a $\mathfrak{g}$-dga.*

Remark 3.1.6. In [GNT12], Goertsches, Nozawa and Töben consider so-called *transverse actions* of Lie algebras on foliated manifolds, especially the action of $\mathfrak{t}/\mathbb{R}R$ on a K-contact manifold. In particular, they show that $\Omega(M, \mathcal{F})$ is a $\mathfrak{t}/\mathbb{R}R$-dga, see [GNT12, Proposition 2, (3.1)].

Proof. The relations of Definition 3.1.4 as well as the degrees of the derivations are inherited from those on $\Omega(M)$. Let $\omega \in \Omega(M, \mathcal{F})$, $X \in \mathfrak{g}$, $Y \in \mathfrak{X}(\mathcal{F})$. For the proof of the $\mathfrak{g}$-dga structure, it remains to show that $\iota_X\omega$ and $\mathcal{L}_X\omega$ are again elements of $\Omega(M, \mathcal{F})$. Note that we have $\iota_{[Y,X_M]}\omega = \mathcal{L}_{[Y,X_M]}\omega = 0$ by Equation (3.1). Then

$$\begin{aligned} \iota_Y \iota_X \omega &= -\iota_X \iota_Y \omega = 0, \\ \mathcal{L}_Y \iota_X \omega &= \iota_{[Y,X_M]}\omega + \iota_X \mathcal{L}_Y \omega = 0, \end{aligned}$$

hence, $\iota_X\omega \in \Omega(M, \mathcal{F})$. Similarly, we obtain $\iota_Y \mathcal{L}_X \omega = 0 = \mathcal{L}_Y \mathcal{L}_X \omega$ and $\mathcal{L}_X\omega \in \Omega(M, \mathcal{F})$. □

As a generalization of the example where a Lie group K acting on a manifold induces the structure of a $\mathfrak{k}$-dga on the differential forms, consider the following definition (cf. [GS99, Definition 2.3.1]).

Definition 3.1.7. Let $\mathfrak{k}$ denote the Lie algebra of an arbitrary Lie group K. A K^*-algebra is a $\mathfrak{k}$-dga A together with a representation ρ of K as automorphisms of A, that is compatible with the derivations in the sense that for all $h \in K, X \in \mathfrak{k}$, it is

- $\frac{d}{dt}\rho(\exp(tX))|_{t=0} = \mathcal{L}_X$,
- $\rho(h)\mathcal{L}_X\rho(h^{-1}) = \mathcal{L}_{Ad_h X}$,
- $\rho(h)\iota_X\rho(h^{-1}) = \iota_{Ad_h X}$,
- $\rho(h)d\rho(h^{-1}) = d$.

For a different formulation in the language of superalgebras, the reader is referred to [GS99, Section 2.3].

Lemma 3.1.8. *The torus action of G on M induces an action on $\Omega(M,\mathcal{F})$ by pullback, i.e., $g^*\omega \in \Omega(M,\mathcal{F})$ for every $g \in G$, $\omega \in \Omega(M,\mathcal{F})$, turning $\Omega(M,\mathcal{F})$ into a G^*-algebra.*

Proof. Let $g \in G$, $\omega \in \Omega(M,\mathcal{F})$, $Y \in \mathfrak{X}(\mathcal{F})$. By Equation (2.11), the vector field $dg(Y)$, defined by $dg(Y)(p) = dg_{g^{-1}p}(Y_{g^{-1}p})$, lies in $\mathfrak{X}(\mathcal{F})$, and, since $\Omega(M,\mathcal{F})$ is differentially closed, we have $d\omega \in \Omega(M,\mathcal{F})$. Hence, we obtain

$$\begin{aligned}\iota_Y g^*\omega &= g^*\iota_{dgY}\omega = 0,\\ \mathcal{L}_Y g^*\omega &= d\iota_Y g^*\omega + \iota_Y dg^*\omega = 0 + \iota_Y g^* d\omega = 0.\end{aligned}$$

The compatibility relations are inherited from $\Omega(M)$. □

Note that since we are considering an Abelian group, the fundamental vector fields satisfy $dg(X_M(p)) = X_M(g \cdot p)$, for every $X \in \mathfrak{g}$, $g \in G$, $p \in M$. Therefore, we obtain by an easy calculation, that, if $\omega \in \Omega(M,\mathcal{F})$ is G-invariant, then so are $\iota_X\omega$ and $\mathcal{L}_X\omega$ for every $X \in \mathfrak{g}$.

3.2 Equivariant Cohomology of a $\mathfrak{k}$-dga

We will briefly review the concept of equivariant cohomology. For a more elaborate introduction, we refer to [GS99], presenting the material from Cartan (cf. [Car50]) in a modern reference; see also [GNT12, Section 4] or [GT16, Section 3.2].

The *Cartan complex* of a $\mathfrak{k}$-dga A is defined as

$$C_{\mathfrak{k}}(A) := (S(\mathfrak{k}^*) \otimes A)^{\mathfrak{k}},$$

where $S(\mathfrak{k}^*)$ denotes the symmetric algebra of $\mathfrak{k}^*$ and the superscript denotes the subspace of $\mathfrak{k}$-invariant elements, i.e., those $\omega \in S(\mathfrak{k}^*) \otimes A$ for which $\mathcal{L}_X\omega = 0$ for every $X \in \mathfrak{k}$. When regarding an element $\omega \in C_{\mathfrak{k}}(A)$ as a $\mathfrak{k}$-equivariant polynomial map $\mathfrak{k} \to A$, i.e., $\omega([X,Y]) = \mathcal{L}_X(\omega(Y))$ for every $X, Y \in \mathfrak{g}$, the differential $d_\mathfrak{k}$ of $C_\mathfrak{k}(A)$ is given by

$$(d_\mathfrak{k}\omega)(X) := d(\omega(X)) - \iota_X(\omega(X)).$$

If $\{X_i\}_{i=1}^r$ is a basis of $\mathfrak{k}$ with dual basis $\{u_i\}_{i=1}^r$, the differential can be written as

$$d_\mathfrak{k}(\omega) = d(\omega) - \sum_{i=1}^r \iota_{X_i}(\omega)u_i.$$

$C_\mathfrak{k}(A)$ can be endowed with the grading $\deg(f \otimes \eta) = 2\deg(f) + \deg(\eta)$. Then $d_\mathfrak{k}$ raises the degree by 1. The *equivariant cohomology* of A (in the Cartan model) is then defined by

$$H^*_\mathfrak{k}(A) := H^*(C_\mathfrak{k}(A), d_\mathfrak{k}).$$

We remark that there are different conventions in the literature concerning the sign in the definition of the differential.

Example 3.2.1. If a compact Lie group K acts on a manifold N, this action induces a $\mathfrak{k}$-dga structure on the algebra of differential forms $\Omega(N)$. This enables us to apply the general construction of the equivariant cohomology of a $\mathfrak{k}$-dga and we obtain the equivariant cohomology of the K-action as

$$H^*_K(N) = H^*_\mathfrak{k}(\Omega(N)).$$

For the following definition compare [GS99, Definition 2.3.4].

Definition 3.2.2. A $\mathfrak{k}$-dga A is called *free*, if, given a basis X_i of $\mathfrak{k}$, there are $\theta_i \in A_1$ (called *connection elements*) such that $\iota_{X_j}(\theta_i) = \delta_{ij}$. If, in addition, the θ_i can be chosen such that their span in A_1 is $\mathfrak{k}$-invariant, then A is said to be *of type (C)*.

Lemma 3.2.3. *A free $\mathfrak{k}$-dga A is automatically of type (C) if the action of $\mathfrak{k}$ on A is induced by an action of a compact Lie group.*

Proof. [GS99, Section 2.3.4]. □

Definition 3.2.4. Let A be a $\mathfrak{k}$-dga. The differentially closed set $A_{\mathrm{bas}} := \{\omega \in A \mid \iota_X\omega = 0 = \mathcal{L}_X\omega$ for every $X \in \mathfrak{k}\}$ is called the *basic subcomplex* of A. An element $\omega \in A$ is called *horizontal* if it satisfies $\iota_X\omega = 0$ for every $X \in \mathfrak{k}$.

If a compact connected Lie group acts locally freely, the equivariant cohomology of its action is the ordinary cohomology of the orbit space (cf. [GGK02, Appendix C.2]). This property can be generalized as follows.

Proposition 3.2.5. *If A is a $\mathfrak{k}$-dga of type (C), then the inclusion $A_{\mathrm{bas}\,\mathfrak{k}} \to C_{\mathfrak{k}}(A)$, $\omega \mapsto 1 \otimes \omega$ induces an isomorphism*

$$H^*(A_{\mathrm{bas}\,\mathfrak{k}}) = H^*_{\mathfrak{k}}(A),$$

whose inverse is induced by the Cartan map*, which is defined as follows. Given a basis X_i of $\mathfrak{k}$ and the dual basis u_i of $\mathfrak{k}^*$, the Cartan map is the composition of the projection $C_{\mathfrak{k}}(A) \to (S(\mathfrak{k}^*) \otimes A_{\mathrm{hor}})^{\mathfrak{k}}$ with the map that maps pure tensors $u^{i_1} \cdots u^{i_l} \otimes \omega \in C_{\mathfrak{k}}(A)$ to $\mu^{i_1} \cdots \mu^{i_l}\omega$. Here, the subscript* hor *denotes the horizontal elements and μ^i are the curvature elements corresponding to the connection elements of A subordinate to the basis X_i.*

Proof. [GS99, Sections 5.1, 5.2]. □

Remark 3.2.6. For the definition of curvature elements for a general $\mathfrak{k}$-dga of type (C), the reader is referred to [GS99, p. 24]. In this thesis, the relevant case where we will need the specific Cartan map is that of a principle bundle with ordinary connection form. Then the forms μ^i are the ordinary curvature forms associated to the given connection. Details on the projection onto the horizontal component can be found, e.g., in [GS99, page 58].

A proof of the following proposition can be found in [GS99, Section 4.6] or [GT16, Proposition 3.9].

Proposition 3.2.7. *Let A be an $(\mathfrak{h} \times \mathfrak{k})$-dga with $A_k = 0$ for $k < 0$, which is of type (C) as an $\mathfrak{h}$-dga. If either $A^{\mathfrak{k}} = A$ or $\mathfrak{k}$ is the Lie algebra of the compact connected Lie group K and the $\mathfrak{k}$-dga structure on A stems from a K^*-algebra structure, then*

$$H^*_{\mathfrak{k}}(A_{\mathrm{bas}\,\mathfrak{h}}) = H^*_{\mathfrak{h}\times\mathfrak{k}}(A)$$

as $S(\mathfrak{k}^*)$-algebras. The isomorphism is induced by the natural inclusion of complexes

$$\left((S(\mathfrak{k}^*) \otimes A_{\mathrm{bas}\,\mathfrak{h}})^{\mathfrak{k}}, d_{\mathfrak{k}}\right) \hookrightarrow \left((S(\mathfrak{k}^*) \otimes S(\mathfrak{h}^*) \otimes A)^{\mathfrak{k}\times\mathfrak{h}}, d_{\mathfrak{k}\times\mathfrak{h}}\right).$$

3.3 Equivariant Basic Cohomology

Recall that we consider a connected, compact K-contact manifold (M, α, g) with Reeb vector field R, on which a torus G acts in such a way that it preserves the contact form α, i.e., $g^*\alpha = \alpha$ for every $g \in G$. We denoted the Reeb flow by $\{\psi_t\}$. We can not only consider the G-action on M, obtaining the equivariant cohomology of the G-action as $H^*_G(M) = H^*_{\mathfrak{g}}(\Omega(M))$, but we can also consider the $G\times\{\psi_t\}$-action on M which induces a $\mathfrak{g} \times \mathbb{R}R$-dga structure on $\Omega(M)$. This yields $H^*_{\mathfrak{g}\times\mathbb{R}R}(M)$. Furthermore, by Lemma 3.1.8, $\Omega(M, \mathcal{F})$ is a G^*-algebra (and especially a $\mathfrak{g}$-dga). The complex of *equivariant basic forms* is given by $C_G(M, \mathcal{F}) = C_{\mathfrak{g}}(\Omega(M, \mathcal{F}))$. Note that this is naturally a subcomplex of $C_G(M)$. We obtain the *equivariant basic cohomology* of the G-action on (M, α) as

$$H_G(M, \mathcal{F}) = H(C_G(M, \mathcal{F}), d_{\mathfrak{g}}).$$

Analogously, we can define equivariant basic cohomology for any open or closed $G \times \{\psi_t\}$-invariant submanifold of M or for any foliated manifold $(N, \mathcal{E})$, acted on by a torus H in such a way that $\Omega(N, \mathcal{E})$ is an H^*-algebra.

Remark 3.3.1. The complexes $C_G(M, \mathcal{F})$, $C_G(M)$, etc. and their cohomologies $H_G(M, \mathcal{F})$, $H_G(M)$, etc. are all naturally modules over $H_G(\text{point}) = S(\mathfrak{g}^*)$.

Remark 3.3.2. More generally, one can define (equivariant) basic cohomology on the category of pairs $(M, \mathcal{F}_M)$ consisting of a manifold M with regular foliation $\mathcal{F}_M$, (acted upon by G such that $\Omega(M, \mathcal{F}_M)$ is a G^*-algebra (cf. Definition 3.1.7), and morphisms $(M, \mathcal{F}_M) \to (N, \mathcal{F}_N)$ given by (equivariant) foliation-preserving smooth maps, i.e. smooth maps which take leaves to leaves. In particular, the $S(\mathfrak{g}^*)$-module structure on $H_G(M, \mathcal{F})$ is induced by the pullback of the map projecting M to the 1-point manifold with trivial foliation.

Since the G-invariant contact form serves as connection element, Proposition 3.2.7 directly gives

Proposition 3.3.3. $H_G(M, \mathcal{F}) = H_{\mathfrak{g}\times\mathbb{R}R}(M)$ *as* $S(\mathfrak{g}^*)$*-algebras. The analogous statement holds for* $G \times \{\psi_t\}$*-invariant submanifolds of* M*.*

Example 3.3.4. Suppose that R induces a free S^1-action. In this case, $\{\psi_t\} = S^1 = T$ and $\pi\colon M \to M/\{\psi_t\} =: B$ is a G-equivariant principal S^1-bundle. The pullback gives an isomorphism $\pi^* : \Omega(B) \to \Omega(M, \mathcal{F})$ and we have $H_G(M, \mathcal{F}) = H_G(B)$ (compare [GT16, Example 3.14]).

Lemma 3.3.5. *Assume G acts on a $G \times T$-invariant submanifold $U \subset M$ with only one $\widetilde{\mathfrak{g}}_x = \widetilde{\mathfrak{g}}_U$; then $H^*_G(U, \mathcal{F}) = S(\widetilde{\mathfrak{g}}^*_U) \otimes H^*_{\mathfrak{k}}(U, \mathcal{F}) = S(\widetilde{\mathfrak{g}}^*_U) \otimes H^*(\Omega(U, \mathcal{F})_{\mathrm{bas}\,\mathfrak{k}})$, where $\mathfrak{k}$ denotes a complement of $\widetilde{\mathfrak{g}}_U$ in $\mathfrak{g}$.*

Proof. Since $\widetilde{\mathfrak{g}}_U$ acts trivially on $\Omega(U, \mathcal{F})$, the Cartan complex can be written as $C_G(U, \mathcal{F}) = S(\widetilde{\mathfrak{g}}^*_U) \otimes S(\mathfrak{k}^*) \otimes \Omega(U, \mathcal{F})^{\mathfrak{k}}$ and $d_G = 1 \otimes d_{\mathfrak{k}}$, hence $H^*_G(U, \mathcal{F}) = S(\widetilde{\mathfrak{g}}^*_U) \otimes H^*_{\mathfrak{k}}(U, \mathcal{F})$. But $\mathfrak{k}$ acts freely and in transversal direction on U, so $\Omega(U, \mathcal{F})$ is a $\mathfrak{k}$-dga of type (C) and $H^*_{\mathfrak{k}}(U, \mathcal{F}) = H^*(\Omega(U, \mathcal{F})_{\mathrm{bas}\,\mathfrak{k}})$ by Proposition 3.2.5. □

The long exact Mayer-Vietoris sequence is well known in ordinary cohomology. The proof presented in [BT13, Proposition 2.3] can be adjusted to the equivariant basic setting so that we obtain an analogous statement.

Proposition 3.3.6 (Mayer-Vietoris sequence)**.** *Let $A \subset M$ be a $G \times T$-invariant submanifold of M and let $U, V \subset A$ be open $G \times T$-invariant subsets such that $U \cup V = A$. Denote the inclusions by $i_U : U \to A$, $i_V : V \to A$, $j_U : U \cap V \to U$, $j_V : U \cap V \to V$. Then there is a long exact sequence*

$$\ldots \to H^*_G(A, \mathcal{F}) \overset{i^*_U \oplus i^*_V}{\to} H^*_G(U, \mathcal{F}) \oplus H^*_G(V, \mathcal{F}) \overset{j^*_U - j^*_V}{\to} H^*_G(U \cap V, \mathcal{F}) \to H^{*+1}_G(A, \mathcal{F}) \to \ldots$$

Proof. We have a short exact sequence

$$0 \to C^*_G(A, \mathcal{F}) \overset{i^*_U \oplus i^*_V}{\to} C^*_G(U, \mathcal{F}) \oplus C^*_G(V, \mathcal{F}) \overset{j^*_U - j^*_V}{\to} C^*_G(U \cap V, \mathcal{F}) \to 0.$$

Exactness on the left is evident. To see exactness on the right, note that since U and V are $G \times T$-invariant and $G \times T$ is compact, we can find a $G \times T$-invariant partition of unity $\{\rho_U, \rho_V\}$ subordinate to the open cover $\{U, V\}$ of A ([GGK02, Corollary B.33]). Then, given $\omega \in C^p_G(U \cap V, \mathcal{F})$, $\rho_U \omega \in C^p_G(V, \mathcal{F})$ and $\rho_V \omega \in C^p_G(U, \mathcal{F})$. It follows that $(\rho_V \omega, -\rho_U \omega)$ lies in $C^p_G(U, \mathcal{F}) \oplus C^p_G(V, \mathcal{F})$ and maps onto ω. Thus, the short sequence is exact and we obtain a long exact sequence in equivariant basic cohomology. □

We will not only work with $G\times T$-invariant submanifolds of $(M,\mathcal{F})$, but also with the (positive/negative) normal bundles of closed invariant submanifolds with lifted $G\times T$-action. For this reason, we now consider the more general case of a foliated manifold $(N,\mathcal{E})$ that is endowed with a $G\times T$-action such that the fundamental vector field of $R\in\mathfrak{t}$ is nowhere vanishing and induces $\mathcal{E}$.

Definition 3.3.7. A subset $A\subset N$ is called *$\mathcal{E}$-saturated* if for every $x\in A$, A contains the whole leaf of $\mathcal{E}$ that runs through x.

To prove our main result, we also need relative and compactly supported equivariant basic cohomology. Our assumption on $(N,\mathcal{E})$ means in particular that for any closed G-invariant, $\mathcal{E}$-saturated submanifold of N, we can find arbitrarily small G-invariant, $\mathcal{E}$-saturated tubular neighborhoods.

Definition 3.3.8. We denote the subcomplex of compactly supported equivariant basic differential forms by $C_{G,c}(N,\mathcal{E})$, and its cohomology by $H_{G,c}(N,\mathcal{E}) = H(C_{G,c}(N,\mathcal{E}),d_{\mathfrak{g}})$.

Analogously to the proof of Proposition 3.3.6, we can adjust the proof of [BT13, Proposition 2.7] to the equivariant basic setting so that we obtain a Mayer-Vietoris sequence for equivariant basic cohomology of compact support.

Proposition 3.3.9 (Mayer-Vietoris sequence for compact supports)**.** *Let $U,V\subset N$ be open $G\times T$-invariant subsets such that $U\cup V=N$. Extending the forms by 0 gives maps $j_U : C_{G,c}(U,\mathcal{F})\to C_{G,c}(N,\mathcal{F})$, $j_V : C_{G,c}(V,\mathcal{F})\to C_{G,c}(N,\mathcal{F})$ and $i_V : C_{G,c}(U\cap V,\mathcal{F})\to C_{G,c}(V,\mathcal{F})$, $i_U : C_{G,c}(U\cap V,\mathcal{F})\to C_{G,c}(U,\mathcal{F})$, which descend to cohomology. Then there is a long exact sequence*

$$\dots\to H^k_{G,c}(U\cap V,\mathcal{F})\overset{i_U\oplus i_V}{\to}H^k_{G,c}(U,\mathcal{F})\oplus H^k_{G,c}(V,\mathcal{F})\overset{j_U-j_V}{\to}H^k_{G,c}(N,\mathcal{F})\to H^{k+1}_{G,c}(U\cap V,\mathcal{F})\to\dots$$

Proof. We have a short exact sequence

$$0\to C^*_{G,c}(U\cap V,\mathcal{F})\overset{i_U\oplus i_V}{\to} C^*_{G,c}(U,\mathcal{F})\oplus C^*_{G,c}(V,\mathcal{F})\overset{j_U-j_V}{\to} C^*_{G,c}(N,\mathcal{F})\to 0.$$

Exactness on the left is evident. To see exactness on the right, note that since U and V are $G\times T$-invariant and $G\times T$ is compact, we can find a $G\times T$-invariant partition of unity $\{\rho_U,\rho_V\}$ subordinate to the open cover $\{U,V\}$ of N

([GGK02, Corollary B.33]). Then $\omega \in C^p_{G,c}(N,\mathcal{F})$ is the image under $(j_U - j_V)$ of $(\rho_U\omega, -\rho_V\omega) \in C^p_{G,c}(U,\mathcal{F}) \oplus C^p_{G,c}(V,\mathcal{F})$. Note that $\rho_U\omega \in C^p_{G,c}(U,\mathcal{F})$ because $\operatorname{supp}\rho_U\omega \subset \operatorname{supp}\rho \cap \operatorname{supp}\omega$, similarly for $\rho_V\omega$. Thus, the short sequence is exact and we obtain a long exact sequence in equivariant basic cohomology with compact support. □

Proposition 3.3.10. *Let $A, B \subset N$ be two G-invariant, $\mathcal{E}$-saturated submanifolds such that there are equivariant maps $f : A \to B$ and $g : B \to A$. If f and g are $G \times \{\psi_t\}$-homotopy inverses, then they induce inverse isomorphisms f^* and g^* in equivariant basic cohomology. If, in addition, the homotopy is proper, the same holds for cohomology with compact support.*

Proof. The proposition is proven analogously to the corresponding statement in ordinary (equivariant) cohomology by constructing a chain homotopy, see, e.g., [BT13, Chapter 4; Cor. 4.1.2] and also [GS99, Section 2.3.3 and Proposition 2.4.1] and the proof of Proposition 3.3.15 below. □

Note that the previous proposition and the corresponding well known statement in ordinary equivariant cohomology apply in particular to the following situation: Let $A \subset M$ be a $G \times T$-invariant submanifold, and let U be a $G \times T$-invariant tubular neighborhood of A in M with projection map $p : U \to A$ and inclusion $i : A \to U$. Then $i^* : H_G(U) \to H_G(A)$ and $i^* : H_G(U,\mathcal{F}) \to H_G(A,\mathcal{F})$ are isomorphisms with inverse p^*.

Definition 3.3.11. Let $A \subset N$ be any G-invariant, $\mathcal{E}$-saturated submanifold. We then consider the complex $C_G(N,A,\mathcal{E}) := C_G(N,\mathcal{E}) \oplus C_G(A,\mathcal{E})$ with the grading $C^k_G(N,A,\mathcal{E}) := C^k_G(N,\mathcal{E}) \oplus C^{k-1}_G(A,\mathcal{E})$ and differential $D(\alpha,\beta) := (d_G\alpha, \alpha|_A - d_G\beta)$. The cohomology of this complex is the *relative equivariant basic cohomology* of (N,A) and denoted by $H^*_G(N,A,\mathcal{E})$.

This definition is based on the definition of ordinary relative de Rham cohomology in [BT13, pp. 78-79] and an equivariant version thereof in [PV07, Section 3.1]. We remark that [PV07, Section 3.1] works analogously for closed submanifolds. Note that a $G \times \{\psi_t\}$-equivariant map of pairs $f : (N,A) \to (\tilde{N},\tilde{A})$, $f(A) \subset \tilde{A}$, induces a map $f^* : C_G(\tilde{N},\tilde{A},\tilde{\mathcal{E}}) \to C_G(N,A,\mathcal{E})$, $f^*(\alpha,\beta) = (f^*\alpha, f|_A^*\beta)$ that descends to

cohomology.
Analogously to the proofs presented in [BT13, PV07], we obtain the following

Proposition 3.3.12. *There is a natural long exact sequence in equivariant basic cohomology*

$$\cdots \xrightarrow{\alpha^*} H^k_G(N,A,\mathcal{E}) \xrightarrow{\beta^*} H^k_G(N,\mathcal{E}) \xrightarrow{\iota_A^*} H^k_G(A,\mathcal{E}) \to \cdots , \tag{3.2}$$

where $\alpha^*(\theta) = (0,\theta)$, $\beta^*(\omega,\theta) = \omega$, and $\iota_A : A \to N$ denotes the inclusion.

Remark 3.3.13. The complex $C^k_G(N,A,\mathcal{E})$ is a special case of the more general concept of a *mapping cone* of a map of chain complexes (cf., e.g., [Wei97, Section 1.5]). In this context, the previous proposition corresponds to [Wei97, 1.5.1].

The considerations of [PV07, Section 3.2] carry over to the basic setting so that we also obtain an excision statement for open submanifolds.

Proposition 3.3.14. *Let $A \subset N$ be a G-invariant, $\mathcal{E}$-saturated* open *submanifold and U a G-invariant, $\mathcal{E}$-saturated open neighborhood of $N \setminus A$. Then the restriction $(\alpha,\beta) \mapsto (\alpha|_U, \beta|_{U\setminus(N\setminus A)})$ induces an isomorphism*

$$H^k_G(N,A,\mathcal{E}) \to H^k_G(U, U\setminus(N\setminus A),\mathcal{E}).$$

Proposition 3.3.15. *Let $A \subset N$ be any G-invariant, $\mathcal{E}$-saturated submanifold. If the equivariant maps $f : (N,A) \to (\tilde{N},\tilde{A})$ and $g : (\tilde{N},\tilde{A}) \to (N,A)$ are $G \times \{\psi_t\}$-homotopy inverses, then they induce inverse isomorphisms f^* and g^* in relative equivariant basic cohomology.*

Proof. Consider an equivariant homotopy $F : N \times I \to N$, $F(\cdot,0) = g\circ f$, $F(\cdot,1) = \mathrm{id}_N$ such that $F(A\times I) \subset A$. Then $F|_{A\times I}$ is a homotopy between $g\circ f|_A$ and id_A. With $Q : C^k_G(N\times I,\mathcal{E}) \to C^{k-1}_G(N,\mathcal{E})$, $\alpha \mapsto \int_0^1 \iota_{\partial_t}\alpha \, dt$, we then obtain (cf. [BT13, Chapter 4] and [GS99, Section 2.3.3])

$$d_G Q F^* + Q F^* d_G = \mathrm{id}_N^* - f^* g^*. \tag{3.3}$$

With Equation (3.3), we can then show that $\mathrm{id}_N^* = f^*g^*$ in relative equivariant basic cohomology. Analogously, we obtain $\mathrm{id}_{\tilde{N}}^* = g^*f^*$ in relative equivariant basic cohomology, which yields that f^* and g^* are isomorphisms in relative equivariant basic cohomology. □

Remark 3.3.16. Alternatively, Proposition 3.3.15 can be proven by applying Propositions 3.3.10 and 3.3.12 and the 5-lemma ([ES15, Lemma 4.3]).

The retraction of a subset $\tilde{A} \subset N$ onto $A \subset \tilde{A}$ might not generally extend to a global map defined on all of N. Even though we, hence, cannot apply Proposition 3.3.15, we still obtain that the relative equivariant basic cohomologies of $(N, \tilde{A})$ and (N, A) are isomorphic. We will not require the following proposition to obtain our main results. Nevertheless, we include the proof because its approach might be of interest to the reader and the result might be helpful for example computations.

Proposition 3.3.17. *Let $A \subset \tilde{A}$ be a $G \times \{\psi_t\}$-equivariant deformation retract, where $A, \tilde{A} \subset N$ are G-invariant, $\mathcal{E}$-saturated submanifolds. Then $H_G^*(N, A, \mathcal{E})$ and $H_G^*(N, \tilde{A}, \mathcal{E})$ are isomorphic.*

Proof. Let $h : \tilde{A} \to A$ be the equivariant map such that $h \circ \iota_A = \mathrm{id}_A$ and $\iota_A \circ h$ is $G \times \{\psi_t\}$-homotopic to $\mathrm{id}_{\tilde{A}}$. Choose a homotopy $F : \tilde{A} \times I \to \tilde{A}$ such that $F(A \times I) \subset A$. Similar to Equation (3.3), we obtain

$$d_G Q F^* + Q F^* d_G = \mathrm{id}_{\tilde{A}}^* - h^* \iota_A^* : C_G(\tilde{A}, \mathcal{E}) \to C_G(\tilde{A}, \mathcal{E}), \tag{3.4}$$

where $Q : C_G^k(\tilde{A} \times I, \mathcal{E}) \to C_G^{k-1}(\tilde{A}, \mathcal{E})$ is given by $\alpha \mapsto \int_0^1 \iota_{\partial_t} \alpha \, dt$. Consider the maps

$$\begin{aligned} \varphi : C_G(N, A, \mathcal{E}) &\to C_G(N, \tilde{A}, \mathcal{E}) \\ (\omega, \theta) &\mapsto (\omega, h^*\theta + QF^*\iota_{\tilde{A}}^*\omega) \\ \text{and} \quad \psi : C_G(N, \tilde{A}, \mathcal{E}) &\to C_G(N, A, \mathcal{E}) \\ (\sigma, \rho) &\mapsto (\sigma, \iota_A^*\rho - \iota_A^* QF^* \iota_{\tilde{A}}^*\sigma). \end{aligned}$$

Note that $\psi \circ \varphi = \mathrm{id}$. Applying Equation (3.4) yields that φ and ψ commute with the relative differential and, hence, induce maps in relative equivariant basic cohomology. If $(\sigma, \rho) \in C_G(N, \tilde{A}, \mathcal{E})$ is closed, then $\iota_{\tilde{A}}^*\sigma = d_G\rho$. With Equation (3.4), it follows that $\varphi \circ \psi(\sigma, \rho) = (\sigma, \rho) + D(0, QF^*\rho - h^*\iota_A^* QF^*\rho)$. Hence, ψ and φ induce inverse maps in cohomology. □

Note that the proofs (cf. also [PV07]) of the previous propositions 3.3.12-3.3.17 carry over to manifolds N with invariant boundary and $A \subset N$ invariant open

submanifold with invariant boundary, as long as the closure of $N \setminus A$ admits arbitrarily small invariant tubular neighborhoods. Propositions 3.3.12, 3.3.15 and 3.3.17 also hold for manifolds N with invariant boundary and $A \subset N$ invariant closed submanifold that is either $A \subset \mathrm{int}N$ or $A = \partial N$.
For compact manifolds N and closed $G\times T$-invariant submanifolds $A \subset N$ (without boundary or with boundary as above), we have an alternative definition of relative cohomology (cf. [GS99, Chapter 11.1]).

Definition 3.3.18. Let $(N, \mathcal{E})$ be a compact foliated manifold with $G \times T$-action such that R is nowhere vanishing and induces $\mathcal{E}$. Let $A \subset N$ be a closed $G \times T$-invariant submanifold. Assume that either N is a manifold without boundary or that N is a manifold with boundary such that ∂N is $G \times T$-invariant, admits arbitrarily small invariant tubular neighborhoods and $A \subset \mathrm{int}N$ or $A = \partial N$. We define the complex $\widetilde{C}_G(N, A, \mathcal{E})$ to be the kernel of the pullback $C_G(N, \mathcal{E}) \to C_G(A, \mathcal{E})$. Since the pullback commutes with the differential, $\widetilde{C}_G(N, A, \mathcal{E})$ is a differential subcomplex of $C_G(N, \mathcal{E})$. We denote its cohomology by $\widetilde{H}_G(N, A, \mathcal{E})$.

Proposition 3.3.19. *There is a natural long exact sequence in equivariant basic cohomology*

$$\cdots \to \widetilde{H}^k_G(N, A, \mathcal{E}) \to H^k_G(N, \mathcal{E}) \to H^k_G(A, \mathcal{E}) \to \cdots$$

Proof. By standard homological algebra, this follows from the existence of the short exact sequence $0 \to \widetilde{C}_G(N, A, \mathcal{E}) \to C_G(N, \mathcal{E}) \to C_G(A, \mathcal{E}) \to 0$. Exactness on the left follows from definition of $\widetilde{C}_G(N, A, \mathcal{E})$. To see exactness on the right, let $\pi : U \to A$ denote a $G \times T$-invariant tubular neighborhood and $f : N \to \mathbb{R}$ an invariant function with $\operatorname{supp} f \subset U$ and $f|_{\widetilde{U}} \equiv 1$ on a smaller invariant neighborhood $\widetilde{U}$ of A. Then $\omega := f\pi^*\theta$ extends θ to M. □

Proposition 3.3.20. *Let $(N, A, \mathcal{E})$ be as in Definition 3.3.18. The natural inclusion map $\Phi : C_{G,c}(N \setminus A, \mathcal{E}) \to \widetilde{C}_G(N, A, \mathcal{E})$ given by extending by 0 induces an isomorphism in cohomology $H_{G,c}(N \setminus A, \mathcal{E}) \cong \widetilde{H}_G(N, A, \mathcal{E})$.*

Proof. We follow the same line of arguments as in the usual equivariant case (see [GS99, Theorem 11.1.1]). First, let $i : A \hookrightarrow U$ be a $G \times T$-invariant tubular neighborhood of A and let $\eta \in \widetilde{C}_G(N, A, \mathcal{E})$ be an equivariantly closed form. Then by Proposition 3.3.10, we can find $\omega \in C_G(U, \mathcal{E})$ so that $\eta|_U = d_G\omega$ since $\eta|_U$ lies in the same cohomology class as 0. Then $i^*\omega$ is equivariantly closed, so $\lambda := \omega - \pi^* i^*\omega$ satisfies $\lambda \in \widetilde{C}_G(U, A, \mathcal{E})$ and $\eta|_U = d_G\lambda$. Let ρ be a $G \times T$-invariant smooth function which is identically 1 on some smaller neighborhood of A and which is compactly supported in U. Then $\eta - d_G(\rho\lambda) \in C_{G,c}(N \setminus A, \mathcal{E})$. This shows surjectivity. Now suppose that $\eta \in C_{G,c}(N \setminus A, \mathcal{E})$ is in the kernel of the induced map on cohomology, i.e., that there exists $\lambda \in C_G(N, A, \mathcal{E})$ such that $\eta = d_G\lambda$. Then since η is compactly supported on $N \setminus A$, there exists a neighborhood U of A on which η is identically zero. Therefore λ is closed on U. Since $i^*\lambda = 0$ by assumption, by Proposition 3.3.10, as above, we have $\lambda = d_G\beta$ for some $\beta \in \widetilde{C}_G(U, A, \mathcal{E})$. Now let ρ be an invariant smooth function which is identically 1 on a neighborhood of A and which has compact support in U. Then $\widetilde{\lambda} := \lambda - d_G(\rho\beta) \in C_{G,c}(N \setminus A, \mathcal{E})$ and we have $\eta = d_G\widetilde{\lambda}$. This shows injectivity. $\square$

Proposition 3.3.21. *The map $\varphi : \widetilde{C}^k_G(N, A, \mathcal{E}) \to C^k_G(N, A, \mathcal{E})$, $\omega \mapsto (\omega, 0)$ induces an isomorphism in cohomology.*

Proof. φ satisfies $D \circ \varphi = \varphi \circ d_G$:

$$D \circ \varphi(\omega) = D(\omega, 0) = (d_G\omega, \omega|_A) = (d_G\omega, 0) = \varphi(d_G\omega).$$

Hence, φ induces a map in cohomology. By Propositions 3.3.12 and 3.3.19, we have the following exact sequences in cohomology, with ι_A^* denoting the pullback to A and $\alpha^*(\theta) = (0, \theta)$, $\beta^*(\omega, \theta) = \omega$:

$$\cdots \to \widetilde{H}^k_G(N, A, \mathcal{E}) \to H^k_G(N, \mathcal{E}) \to H^k_G(A, \mathcal{E}) \to \cdots$$
$$\cdots \xrightarrow{\alpha^*} H^k_G(N, A, \mathcal{E}) \xrightarrow{\beta^*} H^k_G(N, \mathcal{E}) \xrightarrow{\iota_A^*} H^k_G(A, \mathcal{E}) \to \cdots$$

Consider the following diagram, where the two horizontal sequences are, as sections of these two long exact sequences, exact.

$$\begin{tikzcd}
H^{k-1}_G(N,\mathcal{E}) \arrow[r, "\iota_A^*"] \arrow[d, "-\mathrm{id}"] & H^{k-1}_G(A,\mathcal{E}) \arrow[r, "\partial"] \arrow[d, "-\mathrm{id}"] & \widetilde{H}^k_G(N,A,\mathcal{E}) \arrow[r, "\iota^*"] \arrow[d, "\varphi"] & H^k_G(N,\mathcal{E}) \arrow[r, "\iota_A^*"] \arrow[d, "\mathrm{id}"] & H^k_G(A,\mathcal{E}) \arrow[d, "\mathrm{id}"] \\
H^{k-1}_G(N,\mathcal{E}) \arrow[r, "\iota_A^*"] & H^{k-1}_G(A,\mathcal{E}) \arrow[r, "\alpha^*"] & H^k_G(N,A,\mathcal{E}) \arrow[r, "\beta^*"] & H^k_G(N,\mathcal{E}) \arrow[r, "\iota_A^*"] & H^k_G(A,\mathcal{E})
\end{tikzcd}$$

We want to apply the 5-Lemma. The leftmost square and the two squares on the right obviously commute. We show the commutativity of the remaining square;

since $\pm\,\mathrm{id}$ is an isomorphism, the 5-lemma (cf. [ES15, Lemma 4.3]) then yields that φ is an isomorphism, as well. First, we determine the boundary operator ∂. Let θ represent a class in $H_G^{k-1}(A,\mathcal{E})$. By definition of the long exact cohomology sequence, $\partial\theta$ is determined as follows: $\iota_A^* : C_G^k(N,\mathcal{E}) \to C_G^k(A,\mathcal{E})$ is surjective, so there is a form $\omega \in C_G^k(N,\mathcal{E}) : \iota_A^*\omega = \theta$. But $\iota_A^* d_G\omega = d_G\theta = 0$, hence, $d\omega \in \ker(\iota_A^*) = \mathrm{im}(\iota^*)$ and there exists $\gamma \in \widetilde{H}_G^k(N,A,\mathcal{E})$ with $\iota^*\gamma = d_G\omega$. $\iota^* d_G\gamma = d_G d_G\omega = 0$, so we have $d_G\gamma = 0$ by injectivity of ι^*. Then $\partial\theta := \gamma$. Now, let $\pi : U \to A$ denote an invariant tubular neighborhood and $f : N \to \mathbb{R}$ an invariant function with $\operatorname{supp} f \subset U$ and $f|_{\widetilde{U}} \equiv 1$ on a smaller neighborhood of A. Then $\omega := f\pi^*\theta$ extends θ to N. Since $d_G\theta = 0$, we have $d_G\omega = df \wedge \pi^*\theta$. $(df \wedge \pi^*\theta)|_A = 0$ since $df|_{\widetilde{U}} = 0$, hence, $df \wedge \pi^*\theta \in \widetilde{H}_G^k(N,A,\mathcal{E})$. It follows that $\partial\theta = df \wedge \pi^*\theta$. Further, we have $D(f\pi^*\theta, 0) = (df \wedge \pi^*\theta, (f\pi^*\theta)|_A) = (df \wedge \pi^*\theta, \theta)$, so $(df \wedge \pi^*\theta, 0)$ and $(0, -\theta)$ represent the same relative cohomology class. It follows that $\alpha^*(-\mathrm{id}(\theta)) = (0,-\theta) = (df \wedge \pi^*\theta, 0) = \varphi \circ \partial(\theta)$, the diagram commutes. □

3.4 Basic equivariant Thom isomorphism

Let $i : A \hookrightarrow M$ denote the inclusion of a $G \times T$-invariant closed submanifold of codimension d. The goal of this section is to construct a basic equivariant pushforward $i_* : H_G(A,\mathcal{F}) \to H_G(M,\mathcal{F})$ which raises cohomological degree by d. We will follow the presentation in [GS99, Chapter 10] very closely.

To begin, let $p : U \to A$ denote the projection of a $G \times T$-invariant tubular neighborhood. Since U is a $G\times T$-equivariant fiber bundle over A, there is a well-defined pushforward map $p_* : C_{G,c}^k(U) \to C_G^{k-d}(A)$, defined by fiberwise integration. Note that p_* maps equivariant basic forms to equivariant basic forms. From the definition of p_* we immediately obtain the following, which shows that p_* descends to a well-defined map on equivariant (basic) cohomology.

Lemma 3.4.1. *Let $p : U \to A$ be the projection and let $p_* : C_{G,c}(U) \to C_G(A)$ denote fiberwise integration. Then we have for all $\eta \in C_{G,c}(U)$ and for all $\beta \in C_G(A)$*

$$\int_U p^*\beta \wedge \eta = \int_A \beta \wedge p_*\eta.$$

The basic equivariant pushforward i_* will be constructed as follows. An *equivariant basic Thom form* is a closed form $\tau \in C^d_{G,c}(U,\mathcal{F})$ satisfying $p_*\tau = 1$. We will give a construction of equivariant basic Thom forms at the end of this section. Suppose for now that an equivariant basic Thom form has been constructed. Then we define the basic equivariant pushforward as the composition

$$i_* : C^k_G(A,\mathcal{F}) \xrightarrow{p^*} C^k_G(U,\mathcal{F}) \xrightarrow{\wedge\tau} C^{k+d}_{G,c}(U,\mathcal{F}) \to C^{k+d}_G(M,\mathcal{F}), \tag{3.5}$$

where the last arrow denotes extension by zero.

Proposition 3.4.2. *The basic equivariant pushforward satisfies, for all closed forms $\beta \in C_G(A,\mathcal{F})$ and $\eta \in C_G(U,\mathcal{F})$*

$$\int_M \eta \wedge i_*\beta = \int_A i^*\eta \wedge \beta.$$

Proof. $i_*\beta = p^*\beta \wedge \tau$ is a form compactly supported in an invariant neighborhood U of A. Therefore we have

$$\begin{aligned}
\int_M \eta \wedge i_*\beta &= \int_U \eta \wedge p^*\beta \wedge \tau && \text{(by definition of } i_*\text{)}\\
&= \int_U p^*i^*\eta \wedge p^*\beta \wedge \tau && \text{(by Proposition 3.3.10)}\\
&= \int_A i^*\eta \wedge \beta \wedge p_*\tau && \text{(by Lemma 3.4.1)}\\
&= \int_A i^*\eta \wedge \beta && \text{(by } p_*\tau = 1\text{).}
\end{aligned}$$

□

We obviously have $p_* \circ i_* = \mathrm{id}$. Analogously to [GS99, Theorem 10.6.1], we obtain for the induced maps on cohomology:

Theorem 3.4.3 (Basic equivariant Thom isomorphism)**.** *Integration over the fiber defines an isomorphism*

$$p_* : H^k_{G,c}(U,\mathcal{F}) \to H^{k-d}_G(A,\mathcal{F})$$

whose inverse is given by i_*.

It remains to construct the equivariant basic Thom form. We use a variant of the Mathai-Quillen construction based on the presentation in [GS99, Chapter 10] (see also [Töb14, GNT17] for closely related constructions). First we identify U with the normal bundle $\nu A \to A$, equipped with a $G \times T$-invariant metric. Let $P \to A$ denote the bundle of oriented orthonormal frames of νA: it is a $G \times T$-equivariant principal $SO(d)$-bundle over A. Consider the map $P \times \mathbb{R}^d \to \nu A$,

$$(x, (e_1, \ldots, e_d), v) \to (x, v_1 e_1 + \cdots + v_d e_d).$$

It gives a $G \times T$-equivariant diffeomorphism $(P \times \mathbb{R}^d)/SO(d) \cong \nu A$. Equip P with a $G \times T$-invariant basic connection form. Recall that such a form has to exist, see Remark 2.5.3. Using the Cartan model of equivariant basic cohomology, the Cartan map yields isomorphisms

$$\begin{aligned} \phi_{\nu A} &: C_{SO(d)\times G,c}(P \times \mathbb{R}^d, \mathcal{E}) \xrightarrow{\cong} C_{G,c}(\nu A, \mathcal{F}) \\ \phi_A &: C_{SO(d)\times G}(P, \mathcal{E} \times \{*\}) \xrightarrow{\cong} C_G(A, \mathcal{F}), \end{aligned}$$

where $\mathcal{E}$ denotes the foliation induced by R on P. Let $p_2 : P \times \mathbb{R}^d \to \mathbb{R}^d$ be the projection. We define τ by

$$\tau := \phi_{\nu A}(p_2^*(\nu \otimes 1)) \in C_{G,c}(\nu A, \mathcal{F})$$

where $\nu \in C_{SO(d),c}(\mathbb{R}^d)$ is the (modified) universal Thom-Matthai-Quillen form as constructed in [GS99, Section 10.3], $\nu \otimes 1 \in C_{SO(d)\times G,c}(\mathbb{R}^d)$. By analogous arguments to [GS99, Section 10.4], we hence have the following.

Theorem 3.4.4. *The form $\tau \in C^d_{G,c}(U, \mathcal{F})$ as constructed above is a Thom form for the projection $p : U \to A$. Consequently, the basic equivariant pushforward $i_* : H^k_G(A, \mathcal{F}) \to H^{k+d}_G(M, \mathcal{F})$ is well-defined.*

In Section 2.3, we scrutinized the functions $\Psi^{X_{s+1}}|_{Y_s}$ and the connected components N of their critical sets. Recall that every N is a $G \times T$-invariant closed submanifold of even codimension (cf. Lemma 2.4.8) and non-degenerate (cf. Proposition 2.4.9). We will now consider the special case that $A = N$. Denote the Morse index of $\Psi^{X_{s+1}}|_{Y_s}$ on N by λ, the inclusion as the zero section $N \to \nu^\pm N$ by $\iota^\pm$ and the projection by $p^\pm : \nu^\pm N \to N$. For the following definition, compare [GNT17, Section A.1].

Definition 3.4.5. Let k denote the rank of the (positive/negative) normal bundle $\nu^{(\pm)} N$. Then the bundle P of oriented orthonormal frames of $(\nu^{(\pm)} N, \mathcal{F})$ is a foliated $SO(k)$-bundle over N. The equivariant basic Euler form $e_G(\nu^{(\pm)}N, \mathcal{F}) \in C^*_G(N, \mathcal{F})$ of $(\nu^{(\pm)}N, \mathcal{F}) \to (N, \mathcal{F})$ is defined by

$$e_G(\nu^{(\pm)}N, \mathcal{F})(X) = \mathrm{Pf}(F^\theta_G(X)) = \mathrm{Pf}(F^\theta - \iota_X\theta),$$

where $\theta \in \Omega^1(P, \mathcal{F})^G \otimes \mathfrak{so}(k)$ denotes a G-invariant basic connection form on the bundle of oriented orthonormal frames of $(\nu^{(\pm)}N, \mathcal{F})$, $F^\theta_G = d_G\theta + \frac{1}{2}[\theta, \theta]$ its equivariant curvature form and Pf the Pfaffian.

For any $G \times \{\psi_t\}$-invariant connection form, we can analogously define the equivariant Euler form $e_{\mathfrak{g}\times\mathbb{R}R}(\nu^{(\pm)}N) \in C_{\mathfrak{g}\times\mathbb{R}R}(N)$ or, for a $G \times T$-invariant connection form, the equivariant Euler form $e_{G\times T}(\nu^{(\pm)}N) \in C_{G\times T}(N)$.

Note that, while the Euler *form* depends of the choice of connection form, its class (for which we use the same notation) does not. We can think of $e_{\mathfrak{g}\times\mathbb{R}R}(\nu^{(\pm)}N)$ as the restriction of the polynomial map $e_{G\times T}(\nu^{(\pm)}N)$ to $\mathfrak{g} \times \mathbb{R}R$.

Proposition 3.4.6. *Under the $S(\mathfrak{g}^*)$-algebra isomorphism*

$$H_G(N, \mathcal{F}) = H_{\mathfrak{g}\times\mathbb{R}R}(N)$$

of Proposition 3.3.3, $e_G(\nu^{(\pm)}N, \mathcal{F}) = e_{\mathfrak{g}\times\mathbb{R}R}(\nu^{(\pm)}N)$.

Proof. $e_G(\nu^{(\pm)}N, \mathcal{F})$ is, by definition, computed with respect to a G-invariant R-basic connection form θ. Then θ is, in particular, $G \times \{\psi_t\}$-invariant and satisfies $\theta(R) = 0$ so that we can compute $e_{\mathfrak{g}\times\mathbb{R}R}(\nu^{(\pm)}N)$ with respect to the same connection form and obtain $e_{\mathfrak{g}\times\mathbb{R}R}(\nu^{(\pm)}N)(R) = e_G(\nu^{(\pm)}N, \mathcal{F})(0)$ and $e_{\mathfrak{g}\times\mathbb{R}R}(\nu^{(\pm)}N)(X) = e_G(\nu^{(\pm)}N, \mathcal{F})(X)$ for every $X \in \mathfrak{g}$. Since the isomorphism $H_G(N, \mathcal{F}) = H_{\mathfrak{g}\times\mathbb{R}R}(N)$ is induced by the natural inclusion of complexes

$$(C^*_G(N, \mathcal{F}), d_G) \hookrightarrow \left((S(\mathfrak{g}^*) \otimes S((\mathbb{R}R)^*) \otimes \Omega^*(N))^{\mathfrak{g}\times\mathbb{R}R}, d_{\mathfrak{g}\times\mathbb{R}R}\right),$$

we obtain the claim. □

Analogously to Theorem 3.4.3, we obtain

Theorem 3.4.7 (Basic equivariant Thom isomorphism)**.** *Integration over the fiber defines an isomorphism*

$$p^-_* : H^{*+\lambda}_{G,c}(\nu^- N, \mathcal{F}) \to H^*_G(N, \mathcal{F})$$

whose inverse is given by the composition

$$\iota_*^- : H_G^*(N,\mathcal{F}) \xrightarrow{(p^-)^*} H_G^*(\nu^- N,\mathcal{F}) \xrightarrow{\wedge\tau} H_{G,c}^{*+\lambda}(\nu^- N,\mathcal{F}).$$

As in [GS99, Section 10.5], it can be shown that $(\iota^-)^*\tau = e_G(\nu^- N,\mathcal{F})$ *and, hence, that* $(\iota^-)^*\iota_*^- = \wedge e_G(\nu^- N,\mathcal{F})$ *is the multiplication with the basic equivariant Euler class of* $\nu^- N$*.*

The analogous statements hold for the positive and the whole normal bundle, with λ *replaced by* $\operatorname{rank}(\nu N) - \lambda$ *and* $\operatorname{rank}(\nu N)$*, respectively.*

Chapter 4

Basic Kirwan Surjectivity for K-Contact Manifolds

In this Chapter, we state and prove our basic Kirwan surjectivity result. Afterwards, we present examples and establish that in the case where the Reeb vector field induces a free S^1-action, our result reproduces the known Kirwan surjectivity for the S^1-quotient. A Tolman-Weitsman type description of the kernel of the basic Kirwan map for S^1-actions is derived in Section 4.3, for which we then also present an example. In that section, we also obtain an injectivity statement that corresponds to the well-known Kirwan injectivity.

4.1 Basic Kirwan Surjectivity

We will now proceed to state and prove our surjectivity result.

Theorem 4.1.1. *Let (M, α) be a compact K-contact manifold, R its Reeb vector field and $\mathcal{F}$ the foliation that is induced by R. Let G be a torus that acts on M, preserving α. Denote by $\Psi\colon M \to \mathfrak{g}^*$ the contact moment map and suppose that* 0 *is a regular value of Ψ. Then the inclusion $\Psi^{-1}(0) \subset M$ induces an epimorphism in equivariant basic cohomology*

$$H^*_G(M, \mathcal{F}) \longrightarrow H^*_G(\Psi^{-1}(0), \mathcal{F}).$$

Proof. Choose a metric g adapted to α according to Lemma 2.4.7. Let $(X_1, ..., X_r)$ be a basis of $\mathfrak{g}$ according to Proposition 2.4.1. Let again $f_s := (\Psi^{X_1}, ..., \Psi^{X_s})\colon M \to \mathbb{R}^s$, $Y_0 := M$ and $Y_s := f_s^{-1}(0)$ for $s = 1, ..., r$. By Proposition 2.4.9, the functions $\Psi^{X_{s+1}}|_{Y_s}$ are Morse-Bott functions. We will show that the restrictions to the subsets $Y_{s+1} \subset Y_s$ induce the following sequence of epimorphisms:

$$H_G^*(M, \mathcal{F}) = H_G^*(Y_0, \mathcal{F}) \to H_G^*(Y_1, \mathcal{F}) \to ... \to H_G^*(Y_r, \mathcal{F}) = H_G^*(\Psi^{-1}(0), \mathcal{F}).$$

Set $Y_s^c := \left(\Psi^{X_{s+1}}|_{Y_s}\right)^{-1}((-\infty, c])$. Let κ be a critical value of $\Psi^{X_{s+1}}|_{Y_s}$. We denote the connected components of the critical set at level κ by $B_1^\kappa, ..., B_{j_\kappa}^\kappa$ and by λ_i^κ the indices of the non-degenerate critical submanifolds B_i^κ with respect to $\mathrm{Hess}(\Psi^{X_{s+1}}|_{Y_s})$ and with $\nu^\pm B_i^\kappa$ their positive (respective negative) normal bundles.

Let ϵ be small enough such that the interval $[\kappa-\epsilon, \kappa+\epsilon]$ contains no critical values of $\Psi^{X_{s+1}}|_{Y_s}$ besides κ. Since $\Psi^{X_{s+1}}|_{Y_s}$ is a $G \times T$-invariant Morse-Bott function, $Y_s^{\kappa+\epsilon}$ is $(G \times T)$-equivariantly diffeomorphic to $Y_s^{\kappa-\epsilon}$ with j_κ handle bundles of type $(\nu^+ B_i^\kappa, \nu^- B_i^\kappa)$ disjointly attached by Theorem A.4.

$$Y_s^{\kappa+\epsilon} \simeq Y_s^{\kappa-\epsilon} \cup_{\bigcup D^{\nu^+ B_i^\kappa} \oplus S^{\nu^- B_i^\kappa}} \bigcup D^{\nu^+ B_i^\kappa} \oplus D^{\nu^- B_i^\kappa}. \tag{4.1}$$

Here, the $G \times T$-action on $\nu^\pm B_i^\kappa$ is the natural lift of the $G \times T$-action on M. We denote the foliation induced by R on the normal bundle also by $\mathcal{F}$. Let U_i^κ denote an invariant tubular neighborhood of $D^{\nu^+ B_i^\kappa} \oplus D^{\nu^- B_i^\kappa}$. By Diffeomorphism (4.1) and Proposition 3.3.15, we have

$$\begin{aligned}
H_G^*(Y_s^{\kappa+\epsilon}, Y_s^{\kappa-\epsilon}, \mathcal{F}) &= H_G^*(Y_s^{\kappa-\epsilon} \cup_{\cup D^{\nu^+ B_i^\kappa} \oplus S^{\nu^- B_i^\kappa}} D^{\nu^+ B_i^\kappa} \oplus D^{\nu^- B_i^\kappa}, Y_s^{\kappa-\epsilon}, \mathcal{F}) \\
&= H_G^*(\cup\, U_i^\kappa, \cup\, U_i^\kappa \setminus D^{\nu^+ B_i^\kappa} \oplus \mathring{D}^{\nu^- B_i^\kappa}, \mathcal{F}) && \text{(by Proposition 3.3.14)} \\
&= H_G^*(\cup\, D^{\nu^+ B_i^\kappa} \oplus D^{\nu^- B_i^\kappa}, \cup\, D^{\nu^+ B_i^\kappa} \oplus S^{\nu^- B_i^\kappa}, \mathcal{F}) && \text{(by Proposition 3.3.15)} \\
&= H_G^*(\cup\, D^{\nu^- B_i^\kappa}, \cup\, S^{\nu^- B_i^\kappa}, \mathcal{F}) && \text{(by Proposition 3.3.15)} \\
&= \bigoplus H_G^*(D^{\nu^- B_i^\kappa}, S^{\nu^- B_i^\kappa}, \mathcal{F}) \\
&= \bigoplus \widetilde{H}_G^*(D^{\nu^- B_i^\kappa}, S^{\nu^- B_i^\kappa}, \mathcal{F}) && \text{(by Proposition 3.3.21)} \\
&= \bigoplus H_{G,c}^*(\mathring{D}^{\nu^- B_i^\kappa}, \mathcal{F}) && \text{(by Proposition 3.3.20)}
\end{aligned}$$

Consider the $G \times T$-equivariant diffeomorphism $\rho : \mathring{D}^{\nu^- B_i^\kappa} \to \nu^- B_i^\kappa$, $v \mapsto \frac{1}{1-||v||^2} v$. Since ρ is proper, Proposition 3.3.10 yields

$$H_G^*(Y_s^{\kappa+\epsilon}, Y_s^{\kappa-\epsilon}, \mathcal{F}) = \bigoplus H_{G,c}^*(\nu^- B_i^\kappa, \mathcal{F}). \tag{4.2}$$

By the Thom isomorphism (Theorem 3.4.7), we have further

$$H_G^{*-\lambda_i^\kappa}(B_i^\kappa, \mathcal{F}) \xrightarrow{\sim} H_{G,c}^*(\nu^- B_i^\kappa, \mathcal{F}). \tag{4.3}$$

With $(G \times \{\psi_t\})_{B_i^\kappa}$ we denote the isotropy group of $G \times \{\psi_t\}$ on B_i^κ. Since Ψ and g are $G \times T$-invariant, $(G \times \{\psi_t\})_{B_i^\kappa}$ acts fiberwise on $\nu^- B_i^\kappa$ (and $D^{\lambda_i^\kappa} B_i^\kappa$) by restriction of the isotropy representation. We need the following lemmata.

Lemma 4.1.2. *$\nu^{(\pm)} B_i^\kappa$ has no non-zero $(G \times \{\psi_t\})_{B_i^\kappa}$-fixed vectors.*

Proof. For $x \in B_i^\kappa$, let γ_v be the unique geodesic with initial values $\gamma_v(0) = x, \dot\gamma_v(0) = v$, $v \in \nu_x^{(\pm)} B_i^\kappa$. Since $G \times \{\psi_t\}$ acts by isometries, $g \cdot \gamma_v$ is again a geodesic and, by uniqueness, $g \cdot \gamma_v = \gamma_{dg(v)}$ for all $g \in (G \times \{\psi_t\})_{B_i^\kappa}$. Assume v to be a $(G \times \{\psi_t\})_{B_i^\kappa}$-fixed vector. Then $g \cdot \gamma_v = \gamma_{dg(v)} = \gamma_v$ for all $g \in (G \times \{\psi_t\})_{B_i^\kappa}$, hence the isotropy group of all points along γ_v contains $(G \times \{\psi_t\})_{B_i^\kappa}$. By Lemma 2.4.6, however, the critical set is the union of all minimal $G \times \{\psi_t\}$-orbits, hence γ_v lies completely in the connected component B_i^κ. Thus $v = \dot\gamma_v(0) \in T_x B_i^\kappa \perp \nu_x B_i^\kappa$, therefore $v = 0$. $\square$

It follows that $\nu^\pm B_i^\kappa$ has no non-zero $(G \times T)_{B_i^\kappa}$-fixed vectors, therefore, the multiplication with the Euler classes of the negative, positive or whole normal bundle in $H^*_{G\times T}(B_i^\kappa)$ is injective (see [Duf83, Proposition 5] or [AB83, Section 13]). We now show that this also holds for their restriction to $\mathfrak{g} \times \mathbb{R}R$.

Lemma 4.1.3. *Multiplication in $H^*_{\mathfrak{g}\oplus\mathbb{R}R}(B_i^\kappa)$ with the equivariant Euler class of the negative, positive or whole normal bundle of B_i^κ is injective.*

Proof. We present the proof for the case of the negative normal bundle, the other cases work analogously. Denote the Euler class of $\nu^- B_i^\kappa$ by E_i^κ. Let θ denote a $G \times T$-invariant connection 1-form in the bundle P of oriented orthonormal frames of the negative normal bundle over B_i^κ. Then, by definition, for $X \in \mathfrak{g}\oplus\mathbb{R}R$, $E_i^\kappa(X)$ is given by $\mathrm{Pf}(F^\theta - \iota_X\theta)$, where we again denote the Pfaffian $\in S(\mathfrak{so}(\lambda_i^\kappa)^*)^{SO(\lambda_i^\kappa)}$ by Pf. The classification of irreducible torus representations yields that $\nu^- B_i^\kappa$ splits into 2-dimensional subbundles s.t., when written in a basis adapted to the splitting, the $(\mathfrak{g} \oplus \mathfrak{t})_{B_i^\kappa}$-action is given by the standard action of the matrix

$$\begin{pmatrix} 0 & -\alpha_1(X) & & & \\ \alpha_1(X) & 0 & & & \\ & & \ddots & & \\ & & & 0 & -\alpha_{\lambda_i^\kappa/2}(X) \\ & & & \alpha_{\lambda_i^\kappa/2}(X) & 0 \end{pmatrix}, \quad X \in (\mathfrak{g} \oplus \mathfrak{t})_{B_i^\kappa} \tag{4.4}$$

with the weights $\alpha_1, ..., \alpha_{\lambda_i^\kappa/2}$ of the $(\mathfrak{g} \oplus \mathfrak{t})_{B_i^\kappa}$-representation. For every $X \in (\mathfrak{g} \oplus \mathfrak{t})_{B_i^\kappa}$, Matrix (4.4) is an element of $\mathfrak{so}(\lambda_i^\kappa)$. Thus, X_P and the SO-fundamental vectorfield generated by Matrix (4.4) coincide. By the definition of a connection form, $\theta(Y_P) = Y$ for every $Y \in \mathfrak{so}$. Therefore, it holds for every $X \in (\mathfrak{g} \oplus \mathfrak{t})_{B_i^\kappa}$ that

$$\iota_X\theta = \begin{pmatrix} 0 & -\alpha_1(X) & & & \\ \alpha_1(X) & 0 & & & \\ & & \ddots & & \\ & & & 0 & -\alpha_{\lambda_i^\kappa/2}(X) \\ & & & \alpha_{\lambda_i^\kappa/2}(X) & 0 \end{pmatrix}.$$

Since $(\mathfrak{g} \oplus \mathbb{R}R)_{B_i^\kappa} \subset (\mathfrak{g} \oplus \mathfrak{t})_{B_i^\kappa}$, we obtain for every $X \in (\mathfrak{g} \oplus \mathbb{R}R)_{B_i^\kappa}$

$$\mathrm{Pf}(\iota_X\theta) = \frac{1}{(-2\pi)^{\lambda_i^\kappa/2}} \prod_{j=1}^{\lambda_i^\kappa/2} \alpha_j(X). \tag{4.5}$$

Let $\mathfrak{k}$ be a complement of $(\mathfrak{g} \oplus \mathbb{R}R)_{B_i^\kappa}$ in $\mathfrak{g} \oplus \mathbb{R}R$. Then, by the definition of the Cartan complex, we have $C_{\mathfrak{g}\oplus\mathbb{R}R}(B_i^\kappa) = S((\mathfrak{g} \oplus \mathbb{R}R)^*_{B_i^\kappa}) \otimes C_{\mathfrak{k}}(B_i^\kappa)$, $d_{\mathfrak{g}\oplus\mathbb{R}R} = 1 \otimes d_{\mathfrak{k}}$, and $H_{\mathfrak{g}\oplus\mathbb{R}R}(B_i^\kappa) = S((\mathfrak{g} \oplus \mathbb{R}R)^*_{B_i^\kappa}) \otimes H_{\mathfrak{k}}(B_i^\kappa)$. $S((\mathfrak{g} \oplus \mathbb{R}R)^*_{B_i^\kappa})$ is a polynomial ring, so any $\omega_0 \in S((\mathfrak{g} \oplus \mathbb{R}R)^*_{B_i^\kappa})$ with $\omega_0 \neq 0$ is not a zero divisor in $H_{\mathfrak{g}\oplus\mathbb{R}R}(B_i^\kappa)$. More generally, if there is an $\omega_0 \in S((\mathfrak{g} \oplus \mathbb{R}R)^*_{B_i^\kappa})$ such that $\omega \in H_{\mathfrak{g}\oplus\mathbb{R}R}(B_i^\kappa)$ is of the form

$$\omega = \omega_0 \otimes 1 + \text{terms of positive degree in } H_{\mathfrak{k}}(B_i^\kappa),$$

then ω is not a zero divisor in $H_{\mathfrak{g}\oplus\mathbb{R}R}(B_i^\kappa)$ (cf. also [AB83, p. 605]). Hence, for E_i^κ not to be a zero divisor, it suffices to show that its purely polynomial part in $S((\mathfrak{g} \oplus \mathbb{R}R)^*_{B_i^\kappa}) \otimes 1$ is not a zero divisor. Since E_i^κ is a form of degree λ_i^κ, as is $\prod_{j=1}^{\lambda_i^\kappa/2} \alpha_j$, it follows with Equation (4.5) that

$$E_i^\kappa = \frac{1}{(2\pi)^{\lambda_i^\kappa/2}} \prod_{j=1}^{\lambda_i^\kappa/2} \alpha_j \otimes 1 + \text{terms of positive degree in } H_{\mathfrak{k}}(B_i^\kappa),$$

i.e., it suffices to show that $\prod_{j=1}^{\lambda_i^\kappa/2} \alpha_j \not\equiv 0$ on $(\mathfrak{g} \oplus \mathbb{R}R)_{B_i^\kappa}$. Suppose that $\prod_{j=1}^{\lambda_i^\kappa/2} \alpha_j$ vanishes on $(\mathfrak{g} \oplus \mathbb{R}R)_{B_i^\kappa}$. Then, since the linear forms α_j either vanish on $(\mathfrak{g} \oplus \mathbb{R}R)_{B_i^\kappa}$ or have a kernel of codimension 1, there existed an α_{j_0} that vanished on all of $(\mathfrak{g} \oplus \mathbb{R}R)_{B_i^\kappa}$. By (4.4), this meant that a two-dimensional subspace of $\nu^- B_i^\kappa$ vanished under $(\mathfrak{g} \oplus \mathbb{R}R)_{B_i^\kappa}$. This, however, contradicts Lemma 4.1.2. $\square$

Recalling Propositions 3.3.3 and 3.4.6, we also set $E_i^\kappa = e_G(\nu^- B_i^\kappa, \mathcal{F})$ by abuse of notation. We obtain an injective map

$$\oplus(\cdot E_i^\kappa)\colon \bigoplus_i H_G^{*-\lambda_i^\kappa}(B_i^\kappa, \mathcal{F}) \longrightarrow \bigoplus_i H_G^*(B_i^\kappa, \mathcal{F}). \tag{4.6}$$

Now, set $Y_s^\pm := \{\pm \Psi^{X_{s+1}}|_{Y_s} \geq 0\}$. We then have $Y_{s+1} = (\Psi^{X_{s+1}}|_{Y_s})^{-1}(0) = Y_s^+ \cap Y_s^-$. Let $0 < \kappa_0 < \kappa_1 < ... < \kappa_m$ be the critical values of $\Psi^{X_{s+1}}|_{Y_s}$ attained on Y_s^+. Consider the following diagram, in which the top row is the long exact sequence of the pair $((Y_s^+)^{\kappa_j+\epsilon_j}, (Y_s^+)^{\kappa_j-\epsilon_j})$, see Proposition 3.3.12, and the vertical arrow on the right and the diagonal arrow are the restriction to $\cup B_i^{\kappa_j}$.

$$\begin{tikzcd}
\cdots \arrow[r] & H_G^*(Y_s^{\kappa_j+\epsilon_j}, Y_s^{\kappa_j-\epsilon_j}, \mathcal{F}) \arrow[r, "h_j"] \arrow[d, "\cong"'] & H_G^*((Y_s^+)^{\kappa_j+\epsilon_j}, \mathcal{F}) \arrow[r] \arrow[dd] & H_G^*((Y_s^+)^{\kappa_j-\epsilon_j}, \mathcal{F}) \arrow[r] & \cdots \\
& \bigoplus_i H_{G,c}^*(\nu^- B_i^{\kappa_j}, \mathcal{F}) \arrow[d, "\cong"'] \arrow[dr] & & & \\
& \bigoplus_i H_G^{*-\lambda_i^{\kappa_j}}(B_i^{\kappa_j}, \mathcal{F}) \arrow[r, "\oplus(\cdot E_i^{\kappa_j})"'] & \bigoplus_i H_G^*(B_i^{\kappa_j}, \mathcal{F}). & &
\end{tikzcd}$$

The following argument is similar to that in [GT10, Theorem 7.1]. The Isomorphisms (4.2) and (4.3) yield that the two vertical arrows on the left are isomorphisms. Note that $H_G^*(Y_s^{\kappa_j+\epsilon_j}, Y_s^{\kappa_j-\epsilon_j}, \mathcal{F}) = H_G^*((Y_s^+)^{\kappa_j+\epsilon_j}, (Y_s^+)^{\kappa_j-\epsilon_j}, \mathcal{F})$ by excision and homotopy equivalence. The upper part of the diagram commutes, because under the isomorphism $\varphi : H_G(Y_s^{\kappa_j+\epsilon_j}, Y_s^{\kappa_j-\epsilon_j}, \mathcal{F}) \cong \oplus_i H_G(\nu^- B_i^{\kappa_j}, \mathcal{F})$, the submanifolds $B_i^{\kappa_j} \subset Y_s^{\kappa_j+\epsilon_j}$ are preserved, that is, mapped to the zero section of $\nu^- B_i^{\kappa_j}$. Hence, $\oplus|_{B_i^{\kappa_j}} \circ \varphi = \oplus|_{B_i^{\kappa_j}} \circ h_j$, where $h_j : H_G(Y_s^{\kappa_j+\epsilon_j}, Y_s^{\kappa_j-\epsilon_j}, \mathcal{F}) \to H_G(Y_s^{\kappa_j+\epsilon}, \mathcal{F})$ denotes the projection onto the first factor. It follows that the composition of h_j and the right vertical arrow is the restriction to $B_i^{\kappa_j}$ of the first factor. By Theorem 3.4.7, the remainder of the diagram is commutative. Multiplication by $\oplus(\cdot E_i^\kappa)$ is injective by (4.6), therefore h_j has to be injective.

The injectivity of h_j yields that the long exact sequence turns into the short exact sequences

$$0 \to H_G^*(Y_s^{\kappa_j+\epsilon_j}, Y_s^{\kappa_j-\epsilon_j}, \mathcal{F}) \to H_G^*((Y_s^+)^{\kappa_j+\epsilon_j}, \mathcal{F}) \xrightarrow{\iota_j} H_G^*((Y_s^+)^{\kappa_j-\epsilon_j}, \mathcal{F}) \to 0,$$

hence, the natural map ι_j is surjective. Furthermore, we know that the homotopy type does not change before crossing a critical value by Theorem A.2, thus $H_G^*((Y_s^+)^{\kappa_j-\epsilon_j}, \mathcal{F}) = H_G^*((Y_s^+)^{\kappa_{j-1}+\epsilon_{j-1}}, \mathcal{F})$. In particular, $H_G^*((Y_s^+)^{\kappa_0-\epsilon_0}, \mathcal{F}) = H_G^*(Y_{s+1}, \mathcal{F})$ and $H_G^*((Y_s)^+, \mathcal{F}) = H_G^*((Y_s^+)^{\kappa_m+\epsilon_m}, \mathcal{F})$. This yields the following sequence of surjective maps

$$H_G^*((Y_s)^+, \mathcal{F}) = H_G^*((Y_s^+)^{\kappa_m+\epsilon_m}, \mathcal{F}) \to \cdots \to H_G^*((Y_s^+)^{\kappa_0+\epsilon_0}, \mathcal{F}) \to H_G^*(Y_{s+1}, \mathcal{F}).$$

Thus, the natural map $H_G^*(Y_s^+, \mathcal{F}) \to H_G^*(Y_{s+1}, \mathcal{F})$ is surjective.

Analogous reasoning with $-\Psi^{X_{s+1}}|_{Y_s}$ and, hence, the Euler classes of the positive normal bundles yields the surjectivity of $H_G^*(Y_s^-, \mathcal{F}) \to H_G^*(Y_{s+1}, \mathcal{F})$.

We consider the Mayer-Vietoris sequence (see Proposition 3.3.6) of the two open sets $\{x \in Y_s \mid \pm\Psi^{X_{s+1}}(x) > -\delta\} \subset Y_s$. For sufficiently small $\delta > 0$, these sets are $G \times T$-homotopy equivalent to $Y_s^\pm$. The epimorphisms $H_G^*(Y_s^\pm, \mathcal{F}) \to H_G^*(Y_{s+1}, \mathcal{F})$ turn the Mayer-Vietoris sequence into the short exact sequences

$$0 \to H_G^*(Y_s, \mathcal{F}) \overset{(j^+)^* \oplus (j^-)^*}{\to} H_G^*(Y_s^+, \mathcal{F}) \oplus H_G^*(Y_s^-, \mathcal{F}) \overset{(i^+)^* - (i^-)^*}{\to} H_G^*(Y_{s+1}, \mathcal{F}) \to 0, \tag{4.7}$$

where $j^\pm \colon Y_s^\pm \hookrightarrow Y_s$ and $i^\pm \colon Y_{s+1} \hookrightarrow Y_s^\pm$ denote the inclusions. We claim that the composition of these maps induces an epimorphism in equivariant basic cohomology. So let $\omega \in H_G^*(Y_{s+1}, \mathcal{F})$ be arbitrary. We know that $(i^\pm)^*$ are surjective, hence there exist $\eta^\pm \in H_G^*(Y_s^\pm, \mathcal{F})$ such that $(i^\pm)^*(\eta^\pm) = \omega$. But this means that $\eta^+ + \eta^- \in \ker((i^+)^* - (i^-)^*) = \mathrm{im}((j^+)^* + (j^-)^*)$, i.e., there exists $\sigma \in H_G^*(Y_s, \mathcal{F})$ such that $\eta^+ + \eta^- = (j^+)^*(\sigma) + (j^-)^*(\sigma)$. This, however, yields $\omega = (i^+)^* \circ (j^+)^*(\sigma) = (i^-)^* \circ (j^-)^*(\sigma)$ and concludes the proof of the surjectivity

$$H_G^*(Y_s, \mathcal{F}) \twoheadrightarrow H_G^*(Y_{s+1}, \mathcal{F}).$$

Iteration for $s = 0, ..., r-1$ yields the desired sequence of epimorphisms

$$H_G^*(M, \mathcal{F}) = H_G^*(Y_0, \mathcal{F}) \to H_G^*(Y_1, \mathcal{F}) \to ... \to H_G^*(Y_r, \mathcal{F}) = H_G^*(\Psi^{-1}(0), \mathcal{F}). \quad \square$$

Remark 4.1.4. The idea to obtain the Kirwan map as the composition of surjective maps $H^*_G(Y_s, \mathcal{F}) \twoheadrightarrow H^*_G(Y_{s+1}, \mathcal{F})$ stems from the approach used in [GGK02, Proof of Theorem G.13] and [BL10, Proof of Theorem 3.4]. To obtain surjectivity, Euler class arguments were also used in [BL10].

4.2 Examples

4.2.1 Boothby-Wang Fibration

This example shows how, for certain symplectic manifolds, Theorem 4.1.1, reproduces Kirwan's surjectivity result ([Kir84]).

Theorem 4.2.1 (Boothby-Wang [BW58])**.** *Suppose that (N, ω) is a symplectic manifold with integral symplectic form. Then the connection 1-form α on the prequantum circle bundle $M \to N$ is a contact form. Conversely, if (M, α) is a compact contact manifold with Reeb vector field that induces an S^1-action, then there is an integral symplectic manifold (N, ω) such that M is the prequantum circle bundle of N, with connection 1-form given by α.*

We call such a principal S^1-bundle $p : M \to N$ with connection form α a *Boothy-Wang fibration*. Recall that $H(M, \mathcal{F}) \cong H(N)$ via p^*. If a compact Lie group G acts on M, preserving α, then the G-action descends to N and we have $H_G(M, \mathcal{F}) \cong H_G(N)$ via p^* (compare Example 3.3.4 and [GT16, Example 3.14]). Furthermore, we have $\{\psi_t\} = S^1 = T$.

$d\alpha$ descends to a symplectic form ω on N, $d\alpha = p^*\omega$ (see, e.g., [BG08, Theorem 6.1.26]). A symplectic moment map μ on N is defined up to a constant by $d(\mu^X) = \iota_{X_B}\omega$. Since $\mathcal{L}_X\alpha = 0$, however, this equation, when pulled back to M, is equivalent to $-dp^*\mu^X = d\iota_{X_M}\alpha$. $\iota_{X_M}\alpha$ is an S^1-invariant function, so there is a $f^X \in \Omega^0(B)$ such that $p^*f^X = \iota_{X_M}\alpha$. $\mu^X := -f^X$ then defines a moment map for the G-action on (N, ω) and $\mu^{-1}(0) = \Psi^{-1}(0)/S^1$. Suppose that 0 is a regular value of the contact moment map Ψ. Then 0 is also a regular value of the symplectic moment map μ that pulls back to $-\Psi$ and vice versa. Since the inclusion of the zero set of the moment map commutes with the projection onto the S^1-quotient, Theorem 4.1.1 then yields the known Kirwan surjectivity induced by the inclusion $\mu^{-1}(0) \hookrightarrow N$ since we have $H^*_G(N) = H_G(M,\mathcal{F})$ and $H^*_G(\Psi^{-1}(0),\mathcal{F}) = H^*_G(\Psi^{-1}(0)/S^1) = H^*_G(\mu^{-1}(0))$.

Theorem 4.2.2. *Suppose that N is a symplectic manifold with a Hamiltonian G-action such that 0 is a regular value of the moment map. Suppose furthermore that the symplectic form on N is integral and that the G-action lifts to the S^1-bundle (M,α) in the Boothby-Wang fibration $p: M \to N$, preserving α. Let μ denote the symplectic moment map that pulls back to $-\Psi$ and assume that 0 is a regular value of μ. Then the inclusion $\mu^{-1}(0) \hookrightarrow N$ induces a surjective map $H^*_G(N) \twoheadrightarrow H^*_G(\mu^{-1}(0))$ in cohomology.*

4.2.2 S^1-Actions on Odd Spheres with Weighted Sasakian Structure

We will now present an example where $T \neq S^1$.

Consider $(M,\alpha) = (S^{2n+1},\alpha_w)$ from Example 2.2.2 with weight $w \in \mathbb{R}^{n+1}$, $w_j > 0$. If at least two w_j are linearly independent over $\mathbb{Q}$, then T is a torus of rank ≥ 2. Then

$$\alpha_w = \frac{\frac{i}{2}\left(\sum_{j=0}^n z_j d\bar{z}_j - \bar{z}_j dz_j\right)}{\sum_{j=0}^n w_j |z_j|^2}, \qquad R_w = i\left(\sum_{j=0}^n w_j (z_j \tfrac{\partial}{\partial z_j} - \bar{z}_j \tfrac{\partial}{\partial \bar{z}_j})\right).$$

The flow of R_w is given by $\psi_t(z) = (e^{itw_0}z_0, ..., e^{itw_n}z_n)$.

Furthermore, let $G = S^1$ act (freely) on S^{2n+1} with weights $\beta = (\beta_0, ...\beta_n) \in \mathbb{Z}^{n+1}$, that is, by $\lambda \cdot z = (\lambda^{\beta_0}z_0, ..., \lambda^{\beta_n}z_n)$. The fundamental vector field X corresponding to $1 \in \mathbb{R} \simeq \mathfrak{s}^1$ is given by

$$X(z) = i\left(\sum_{j=0}^n \beta_j (z_j \tfrac{\partial}{\partial z_j} - \bar{z}_j \tfrac{\partial}{\partial \bar{z}_j})\right)$$

and we compute the contact moment map to be

$$\Psi(z) = \frac{\sum_{j=0}^n \beta_j |z_j|^2}{\sum_{j=0}^n w_j |z_j|^2}.$$

Lemma 4.2.3. *The equivariant basic cohomology of M is given, as $(S(\mathfrak{g}^*) = \mathbb{R}[u])$-algebra, by*

$$H_G(M,\mathcal{F}) \cong \frac{\mathbb{R}[u,s]}{\langle\prod_{j=0}^n(\beta_j u + w_j s)\rangle},$$

where (u,s) is the basis of $(\mathfrak{g}\oplus\mathbb{R}R_w)^*$ dual to (X,R_w).

Proof. To compute the equivariant basic cohomology of M, consider the diagonal S^1-action on $\mathbb{C}^{n+1}$: $\lambda\cdot z := (\lambda z_0,...,\lambda z_n)$. This action is Hamiltonian with respect to the standard symplectic structure on $\mathbb{C}^{n+1}$ and a (symplectic) moment map is given by $\mu(z) = \frac{1}{2}\sum_j |z_j|^2$. Note that $M = \mu^{-1}(\frac{1}{2})$. The $G\times T$-action and, hence, R_w can be extended to all of $\mathbb{C}^{n+1}$. Set $f := ||\mu - \frac{1}{2}||^2$. μ is $G\times T$-invariant, so the same holds for f. We will compute $H_G(M,\mathcal{F})$ by applying Morse theory with f on $\mathbb{C}^{n+1}$, a technique applied in [Kir84]. The critical set of f is given by $\mathrm{Crit}(f) = \{0\}\ \dot\cup\ M$, and the critical values are $f(0) = 1/4$, $f(M) = 0$. The Hessian H of f at 0 is given by $-\mathrm{id}$, which is nondegenerate and has Morse index $2(n+1)$. For $z\in M$, the normal direction (to M) is spanned by $Y := \sum z_j\partial_{z_j} + \bar z_j\partial_{\bar z_j}$ and $H_z(Y,Y) = 2$, which yields that H_z is non-degenerate in normal direction. It follows that f is a $G\times T$-invariant Morse-Bott function. Recall that $H_G(M,\mathcal{F}) \cong H_{\mathfrak{g}\oplus\mathbb{R}R_w}(M)$ as an $S(\mathfrak{g}^*)$-algebra by Proposition 3.3.3. Note that the relative $\mathfrak{g}\times\mathbb{R}R_w$-equivariant cohomology is constructed as in Definition 3.3.11 and, analogously to Proposition 3.3.12, we obtain the long exact sequence of the pair $(\{z\in\mathbb{C}^{n+1} \mid f(z)\le 1/4+\epsilon\}, \{z\in\mathbb{C}^{n+1} \mid f(z)\le 1/4-\epsilon\})$, where $1/4>\epsilon>0$. Similarly, the relevant isomorphisms of Section 3.3 can be transferred to this setting. Theorem A.4 lets us replace $\{f\le 1/4+\epsilon\}$ by $\{f\le 1/4-\epsilon\}$ with a handle-bundle of type $(0,\nu\{0\})$ attached. As in Equations (4.2) and (4.3), we obtain the isomorphism

$$\begin{aligned}
H^*_{\mathfrak{g}+\mathbb{R}R_w}(\{f\le 1/4+\epsilon\},&\{f\le 1/4-\epsilon\})\\
&\cong H^*_{\mathfrak{g}+\mathbb{R}R_w}(\{f\le 1/4-\epsilon\}\cup_{S^{\nu\{0\}}} D^{\nu\{0\}}, \{f\le 1/4-\epsilon\})\\
&\cong H^*_{\mathfrak{g}+\mathbb{R}R_w}(D^{\nu\{0\}}, S^{\nu\{0\}})\\
&\cong H^*_{\mathfrak{g}+\mathbb{R}R_w,c}(\mathring{D}^{\nu\{0\}})\\
&\cong H^*_{\mathfrak{g}+\mathbb{R}R_w,c}(\nu\{0\})\\
&\cong H^{*-2(n+1)}_{\mathfrak{g}+\mathbb{R}R_w}(\{0\}).
\end{aligned}$$

Note that $\{f \leq 1/4 + \epsilon\}$ is the closed ball of radius $\sqrt{1+\sqrt{1+4\epsilon}}$ and a $G \times T$-equivariant retraction of $\mathbb{C}^{n+1}$, while $\{f \leq 1/4 - \epsilon\}$ is the closed annulus bounded by the spheres of radii $\sqrt{1-\sqrt{1-4\epsilon}}$ and $\sqrt{1+\sqrt{1-4\epsilon}}$ and $G \times T$-equivariantly retracts onto M. With Proposition 3.3.15, we obtain the isomorphism

$$T : H^*_{\mathfrak{g}+\mathbb{R}R_w}(\mathbb{C}^{n+1}, M) \cong H^*_{\mathfrak{g}+\mathbb{R}R_w}(\{f \leq 1/4+\epsilon\}, \{f \leq 1/4-\epsilon\}) \cong H^{*-2(n+1)}_{\mathfrak{g}+\mathbb{R}R_w}(\{0\}).$$

The long exact sequence then looks as follows

$$\begin{tikzcd}
\cdots \arrow[r] & H^*_{\mathfrak{g}+\mathbb{R}R_w}(\mathbb{C}^{n+1}, M) \arrow[r] \arrow[d, "\cong"', "T"] & H^*_{\mathfrak{g}+\mathbb{R}R_w}(\mathbb{C}^{n+1}) \arrow[r] & H^*_{\mathfrak{g}+\mathbb{R}R_w}(M) \arrow[r] & \cdots \\
& H^{*-2(n+1)}_{\mathfrak{g}+\mathbb{R}R_w}(\{0\}) & & &
\end{tikzcd}$$

In this diagram, the combination of T^{-1} with the restriction from $\mathbb{C}^{n+1}$ to $\{0\}$, $H^{*-2(n+1)}_{\mathfrak{g}+\mathbb{R}R_w}(\{0\}) \to H^*_{\mathfrak{g}+\mathbb{R}R_w}(\mathbb{C}^{n+1}) \overset{\cong}{\to} H^*_{\mathfrak{g}+\mathbb{R}R_w}(\{0\})$ is multiplication with the equivariant Euler class e of the normal bundle to $\{0\}$, which is injective (this is seen analogously to Lemma 4.1.3). We obtain short exact sequences

$$0 \to H^{*-2(n+1)}_{\mathfrak{g}+\mathbb{R}R_w}(\{0\}) \overset{\cdot e}{\to} H^*_{\mathfrak{g}+\mathbb{R}R_w}(\{0\}) \to H^*_{\mathfrak{g}+\mathbb{R}R_w}(M) \to 0. \tag{4.8}$$

We will now compute e. The (negative) normal bundle is the trivial bundle $\mathbb{C}^{n+1} \times \{0\}$ which is the product of the line bundles $\nu_j := \mathbb{C}_j \times \{0\}$, where $\mathbb{C}_j$ denotes the j-th coordinate. The bundle of oriented orthonormal frames of ν_j is the trivial bundle $P_j = SO(2) \times \{0\}$. The canonical flat connection form θ_j on P_j is invariant under $G \times \{\psi_t\}$. The vector fields generated by X and R on P_j coincide with the fundamental vector fields of the $SO(2)$-action with weights β_j, w_j, respectively, so $\iota_X\theta_j = \begin{pmatrix} 0 & -\beta_j \\ \beta_j & 0 \end{pmatrix}$ and $\iota_R\theta_j = \begin{pmatrix} 0 & -w_j \\ w_j & 0 \end{pmatrix}$. Since the curvature of θ_j is zero, the Euler class e_j of ν_j is then given by $\mathrm{Pf}(-\iota_X\theta_j u - \iota_{R_w}\theta_j s) = \frac{1}{2\pi}(u\beta_j + sw_j)$, where (u, s) are dual to (X, R_w). We then obtain e as $e = \prod_j e_j = \frac{1}{(2\pi)^{n+1}} \prod_j (u\beta_j + sw_j)$.

The short exact sequence from Equation (4.8) then yields that, as $(\mathbb{R}[u] = S(\mathfrak{g}^*))$-algebra,

$$H_G(M, \mathcal{F}) = \mathbb{R}[u, s]/\langle e\rangle = \mathbb{R}[u, s] \bigg/ \left\langle \prod_j (u\beta_j + sw_j) \right\rangle . \qquad \square$$

Remark 4.2.4. If all w_j are positive integers, the Reeb vector field induces a locally free S^1-action on M and M/S^1 is the weighted projective space $\mathbf{P}(w) =$

$(\mathbb{C}^{n+1} \setminus 0)/\sim$, where $(z_0, \dots, z_n) \sim (\lambda^{w_0} z_0, \dots, \lambda^{w_n} z_n)$ for any $\lambda \in \mathbb{C}^*$ (cf. [BG08, Example 7.1.12; Section 4.5]). Then $H_G(\mathbf{P}(w)) \cong H_G(M, \mathcal{F}) \cong \frac{\mathbb{R}[u,s]}{\langle \prod_{j=0}^{n} (\beta_j u + w_j s) \rangle}$.

Now, consider the special case of a G-action with weight $\beta = (1, ..., 1, -1)$. Then we have

$$\Psi(z) = \frac{\sum_{j=0}^{n-1} |z_j|^2 - |z_n|^2}{\sum_{j=0}^{n} w_j |z_j|^2}$$

and, hence,

$$\Psi^{-1}(0) = S^{2n-1}(\tfrac{1}{\sqrt{2}}) \times S^1(\tfrac{1}{\sqrt{2}}).$$

G acts freely on $\Psi^{-1}(0)$, so $H_G(\Psi^{-1}(0), \mathcal{F}) = H(\Psi^{-1}(0)/G, \mathcal{F}_0)$ by Proposition 3.2.5, where we denote the induced foliation on the quotient by $\mathcal{F}_0$. $\Psi^{-1}(0)/G$ is $\{\psi_t\}$-equivariantly diffeomorphic to $S^{2n-1}(\frac{1}{\sqrt{2}})$ via $[z] \mapsto (\sqrt{2} z_n z_0, ..., \sqrt{2} z_n z_{n-1})$, where ψ_t acts on $S^{2n-1}(\frac{1}{\sqrt{2}})$ by $\psi_t(z) = (e^{it(w_0+w_n)} z_0, ..., e^{it(w_{n-1}+w_n)} z_n)$. This is the Reeb flow of a weighted Sasakian structure on S^{2n-1}, hence, the induced foliation is defined by the Reeb vector field belonging to this Sasakian structure. It follows that the basic cohomology ring $H^*(\Psi^{-1}(0)/G, \mathcal{F}_0)$ is isomorphic to $H^*(\mathbb{C}P^{n-1})$, see [BG08, Proposition 7.5.29].

We will now compute the restriction from $H^*_{\mathfrak{g}+\mathbb{R}R_w}(M)$ to $H^*_{\mathfrak{g}+\mathbb{R}R_w}(\Psi^{-1}(0))$. Since $\iota_X \alpha$ vanishes on $\Psi^{-1}(0)$ and $\iota_{R_w} \alpha_w = 1$, we have $0 = [d_{\mathfrak{g}+\mathbb{R}R_w} \alpha_w] = [d\alpha_w - s]$ in $H^*_{\mathfrak{g}+\mathbb{R}R_w}(\Psi^{-1}(0))$. Similarly, consider the $G \times T$-invariant 1-form

$$\gamma := w_n \alpha_w - i(z_n d\bar{z}_n - \bar{z}_n dz_n).$$

On $\Psi^{-1}(0)$, we have $\gamma(R_w) = 0$ and $\gamma(X) = 1$, so that we obtain $d_{\mathfrak{g}+\mathbb{R}R_w} \gamma = d\gamma + u$. Since $d(i(z_n d\bar{z}_n - \bar{z}_n dz_n))$ vanishes on $\Psi^{-1}(0)$, it is $d\gamma = w_n d\alpha_w$. It follows that $[u] = [w_n d\alpha_w] = [w_n s]$ in $H^*_{\mathfrak{g}+\mathbb{R}R_w}(\Psi^{-1}(0))$. Note that $\alpha_w|_{\Psi^{-1}(0)}$ is G-basic, so $d\alpha_w^{\wedge n}|_{\Psi^{-1}(0)} = 0$. Under $[s] \mapsto [\frac{1}{w_n} u]$, $u^n \mapsto 0$, $H_G(M, \mathcal{F}) = \mathbb{R}[u, s]/\langle e \rangle$ is surjectively mapped to $\mathbb{R}[u]/\langle u^n \rangle \cong H(\mathbb{C}P^{n-1})$.

4.3 The Kernel of the Kirwan Map

In this section, we derive a description of the kernel of the basic Kirwan map for $G = S^1$. Recall that (M, α) denotes a compact K-contact manifold, R its Reeb vector field and Ψ the contact moment map for the action of a torus G on M that preserves α. We assume that 0 is a regular value of Ψ. Throughout this section, (X_s) denotes a basis of $\mathfrak{g}$ according to Proposition 2.4.1. Recall that we set $Y_s = (\Psi^{X_1}, ..., \Psi^{X_s})^{-1}(0)$, $Y_0 = M$, and $Y_s^{\pm} = \{\pm\Psi^{X_{s+1}}|_{Y_s} \geq 0\}$. As in the previous section, denote the inclusions by $\iota_s : Y_{s+1} \to Y_s$, $\iota_s^{\pm} : Y_{s+1} \to Y_s^{\pm}$ and $j_s^{\pm} : Y_s^{\pm} \to Y_s$. Additionally, set $C_s := Crit(\Psi^{X_{s+1}}|_{Y_s})$. Recall that $C_1 = Crit(\Psi)$, see Lemma 2.4.6.

We adjust the computations that Tolman and Weitsman did in the symplectic setting ([TW03, Section 3]) to our case. Note that we apply the results they obtained for S^1-actions to the components $\Psi^{X_{s+1}}|_{Y_s}$ for actions of tori of arbitrary rank. The following Lemma corresponds to [TW03, Lemma 3.1].

Lemma 4.3.1. *Let $f = \Psi^{X_{s+1}}|_{Y_s}$ or $f = -\Psi^{X_{s+1}}|_{Y_s}$ and let κ be any critical value of f. Denote by B_i^κ the connected components of $C_s \cap f^{-1}(\kappa) =: C_s^\kappa$ and with λ_i^κ their indices. Let $\epsilon > 0$ such that $[\kappa - \epsilon, \kappa + \epsilon]$ does not contain a critical value besides κ. Then there exists a short exact sequence*

$$0 \to \oplus_i H_G^{*-\lambda_i^\kappa}(B_i^\kappa,\mathcal{F}) \xrightarrow{\varphi} H_G^*(f^{-1}((-\infty, \kappa + \epsilon]),\mathcal{F}) \to H_G^*(f^{-1}((-\infty, \kappa - \epsilon]),\mathcal{F}) \to 0,$$

such that the composition of the injection φ with the restriction to C_s^κ is the sum of the products with the Euler classes $E_i^\kappa \in H_G^{\lambda_i^\kappa}(B_i^\kappa, \mathcal{F}) \simeq H_{\mathfrak{g}\oplus\mathbb{R}R}^{\lambda_i^\kappa}(B_i^\kappa)$ of the negative normal bundles of the B_i^κ.

Proof. Consider the long exact sequence in relative equivariant basic cohomology (see Proposition 3.3.12) of the pair $(f^{-1}((-\infty, \kappa + \epsilon]), f^{-1}((-\infty, \kappa - \epsilon]))$. The isomorphisms (4.2) and (4.3) yield that $H_G^*(f^{-1}((-\infty, \kappa + \epsilon]), f^{-1}((-\infty, \kappa - \epsilon]), \mathcal{F}) \cong \oplus_i H_G^{*-\lambda_i^\kappa}(B_i^\kappa,\mathcal{F})$. By considering a diagram as in the corresponding part of the proof of Theorem 4.1.1 on page 55, we obtain that the long exact sequence splits into short exact sequences with the claimed properties. □

The symplectic analogue of the following proposition for $G = S^1$ was remarked after [TW03, Theorem 3.2].

Proposition 4.3.2. *Let $f = \Psi^{X_{s+1}}|_{Y_s}$ or $f = -\Psi^{X_{s+1}}|_{Y_s}$. For every regular value a of f, the restriction*

$$H_G^*(f^{-1}((-\infty, a]), \mathcal{F}) \to H_G^*(f^{-1}((-\infty, a]) \cap C_s, \mathcal{F})$$

is injective.

Proof. This proposition is proved by induction on the number k of critical values below a. Let $k = 1$. The Morse-Bott property of f yields that the homotopy type does not change before crossing another critical value, cf. Theorem A.2. By Proposition 3.3.10, the restriction, hence, induces a bijection, in particular, an injection. Now, suppose the claim holds for k. Let a be a regular value of f with $k+1$ critical values below it and denote the highest critical value by κ. Let $\delta > 0$ such that $a - \delta$ is regular and such that there are k critical values below $a - \delta$. Lemma 4.3.1 then yields that the restriction of $H^*_G(f^{-1}((-\infty, a]), \mathcal{F})$ to $H^*_G(f^{-1}((-\infty, a-\delta]), \mathcal{F})$ is surjective and we obtain the commutative diagram

$$\begin{tikzcd} H^*_G(f^{-1}((-\infty, a]), \mathcal{F}) \arrow[r, two heads] \arrow[d] & H^*_G(f^{-1}((-\infty, a-\delta]), \mathcal{F}) \arrow[d, hook] \\ H^*_G(f^{-1}((-\infty, a]) \cap C_s, \mathcal{F}) \arrow[r] & H^*_G(f^{-1}((-\infty, a-\delta]) \cap C_s, \mathcal{F}) \end{tikzcd} .$$

Suppose $\sigma \in H^*_G(f^{-1}((-\infty, a]), \mathcal{F}) : \sigma|_{f^{-1}((-\infty,a]) \cap C_s} = 0$. In particular, we have $\sigma|_{f^{-1}((-\infty,a-\delta]) \cap C_s} = 0$, so by our induction's assumption, it is $\sigma|_{f^{-1}((-\infty,a-\delta])} = 0$. I.e., σ lies in the kernel of the restriction to $f^{-1}((-\infty, a-\delta])$. By Lemmata 4.3.1 and 4.1.3, the restriction of this kernel to C_s^κ is injective. But $\sigma|_{C_s^\kappa} = 0$, hence, $\sigma = 0$. ☐

Since we assumed 0 to be a regular value, we obtain as a direct consequence

Corollary 4.3.3. *The following restrictions are injective:*

$$H^*_G(Y_s^\pm, \mathcal{F}) \to H^*_G(Y_s^\pm \cap C_s, \mathcal{F})$$
$$H^*_G(Y_s, \mathcal{F}) \to H^*_G(C_s, \mathcal{F}).$$

In particular, $H^*_G(M, \mathcal{F}) \to H^*_G(\mathrm{Crit}(\Psi), \mathcal{F})$ *and* $H^*_G(M^\pm, \mathcal{F}) \to H^*_G(\mathrm{Crit}(\Psi) \cap M^\pm, \mathcal{F})$ *are injective.*

Remark 4.3.4. The result that $H^*_G(M, \mathcal{F}) \to H^*_G(\mathrm{Crit}(\Psi), \mathcal{F})$ is injective corresponds to the well-known Kirwan Injectivity in the symplectic setting. Note that if R induces a free S^1-action, then $\mathrm{Crit}(\Psi)/\{\psi_t\}$ consists of the fixed points of the G-action on $M/\{\psi_t\}$ so that our result implies Kirwan's injectivity result for the quotient.

Corollary 4.3.5. *Set* $K_s^\pm := \{\sigma \in H_G^*(Y_s, \mathcal{F}) \mid \sigma|_{Y_s^\pm \cap C_s} = 0\}$. *Then we have* $K_s^\pm = \ker((j_s^\pm)^*)$, *where* $j_s^\pm : Y_s^\pm \to Y_s$ *denotes the inclusion.*

Proof. Obviously $\ker((j_s^\pm)^*) \subset K_s^\pm$. Corollary 4.3.3 yields the reverse inclusion. $\square$

Remark 4.3.6. We also know that the induced maps in equivariant basic cohomology $(j_s^\pm)^* : H_G^*(Y_s, \mathcal{F}) \to H_G^*(Y_s^\pm, \mathcal{F})$ are surjective.

Proof. We know from the proof of Theorem 4.1.1 that $(\iota_s^\pm)^*$ is surjective. So for every $\omega^\pm \in H_G^*(Y_s^\pm, \mathcal{F})$ there exists $\omega^\mp \in H_G^*(Y_s^\mp, \mathcal{F})$ such that $(\iota_s^\pm)^*\omega^\pm = (\iota_s^\mp)^*\omega^\mp$. The exactness of Sequence (4.7) yields that $\omega^+ + \omega^- \in \ker((\iota_s^+)^* - (\iota_s^-)^*) = \mathrm{im}((j_s^+)^* + (j_s^-)^*)$, hence, there exists $\sigma \in H_G^*(Y_s, \mathcal{F}) : \omega^\pm = (j_s^\pm)^*\sigma$. $\square$

As a consequence of the previous corollary, we then obtain the following, which is the contact analogue of [TW03, Theorem 2].

Theorem 4.3.7. *Let* $G = S^1$ *and set*

$$C^\pm := \mathrm{Crit}(\Psi) \cap M^\pm, \qquad K^\pm = \{\sigma \in H_G^*(M, \mathcal{F}) \mid \sigma|_{C^\pm} = 0\}.$$

The kernel K *of the Kirwan map* $H_G^*(M, \mathcal{F}) \to H_G^*(\Psi^{-1}(0), \mathcal{F})$ *is given by*

$$K = K^+ \oplus K^-.$$

Proof. By Corollary 4.3.5, $K^\pm = \ker(j^\pm)^*$. It follows that $K^\pm \subset \ker(i^\pm)^* \circ (j^\pm)^*$, so $K^+ \oplus K^-$ lies in the kernel of the Kirwan map. For the reverse inclusion, consider the Mayer-Vietoris sequence (see Proposition 3.3.6) for (M, M^+, M^-) - or, more precisely, of the two open sets $\{x \in M \mid \pm\Psi(x) < \epsilon\}$ for sufficiently small $\epsilon > 0$ which, by the Morse-Bott property of Ψ, are of the same $G \times T$-homotopy type as $M^\pm$. In (4.7), we saw that it actually consists of the short exact sequences

$$0 \to H_G^*(M, \mathcal{F}) \overset{(j^+)^* \oplus (j^-)^*}{\to} H_G^*(M^+, \mathcal{F}) \oplus H_G^*(M^-, \mathcal{F}) \overset{(i^+)^* - (i^-)^*}{\to} H_G^*(\Psi^{-1}(0), \mathcal{F}) \to 0.$$

Now, suppose η lies in the kernel of the Kirwan map, i.e., $(i^\pm)^*(j^\pm)^*\eta = 0$. This means, however, that $(j^+)^*\eta \oplus 0$ and $0 \oplus (j^-)^*\eta$ lie in the kernel of $(i^+)^* - (i^-)^*$. By exactness of the above sequence, there exist $\eta^\pm \in H_G^*(M, \mathcal{F})$ such that $(j^+)^* \oplus$

$(j^-)^*(\eta^\pm) = (j^\pm)^*\eta$, in particular, $\eta^\pm \in K^\pm$ by Corollary 4.3.5. Then $(j^+)^* \oplus (j^-)^*(\eta^+ + \eta^-) = (j^+)^* \oplus (j^-)^*(\eta)$. Since $(j^+)^* \oplus (j^-)^*$ is injective because the sequence is exact, we obtain $\eta = \eta^+ + \eta^- \in K^+ \oplus K^-$. □

We now present an alternative proof of Theorem 4.3.7, similar to the proof of the corresponding statement in the symplectic setting in [TW03, Theorem 2]. In the symplectic case, this proof generalizes to the setting of the action of higher rank tori, where Morse-Bott theory of the norm square of the symplectic moment map is applied. We believe that, in the contact setting, an analogous description of the kernel holds for the action of tori of higher rank, as well, and deem this second approach also of interest.

For $G = S^1$, the following proposition yields Theorem 4.3.7.

Proposition 4.3.8. *With $K_s^\pm$ as in Corollary 4.3.5 and $\iota_s : Y_{s+1} \to Y_s$ denoting the inclusion, we have* $\ker(\iota_s^*) = K_s^+ \oplus K_s^- =: K_s$.

Proof. Corollary 4.3.5 yields that we have $K_s^+ \oplus K_s^- \subset \ker(\iota_s^*)$ since $\iota_s = j_s^\pm \circ \iota_s^\pm$. For the reverse inclusion, it suffices by Corollary 4.3.3 to show that for every $\sigma \in \ker(\iota_s^*)$, there exists $\widetilde{\sigma} \in K_s$ such that $\sigma|_{C_s} = \widetilde{\sigma}|_{C_s}$. Order the level sets C_s^κ of C_s as C_s^j such that $|\Psi^{X_{s+1}}|_{Y_s}(C_s^i)| \leq |\Psi^{X_{s+1}}|_{Y_s}(C_s^j)|$ for every $i < j$. We prove the claim inductively. It then suffices to show that, given $p > 0$ and $\sigma \in H_G^*(Y_s, \mathcal{F})$ with $\sigma|_{Y_{s+1}} = 0$ and $\sigma|_{C_s^i} = 0$ for all $i < p$, there exists $\widetilde{\sigma} \in K_s$ such that $\widetilde{\sigma}|_{Y_{s+1}} = 0$ and $\widetilde{\sigma}|_{C_s^i} = \sigma|_{C_s^i}$ for all $i \leq p$. Suppose we are given such a σ. Let $\kappa_p := \Psi^{X_{s+1}}(C_s^p)$. Since 0 is regular, it is $\kappa_p \neq 0$. Let us first suppose that $\kappa_p > 0$. By the assumptions on σ, we have, for sufficiently small ϵ, $\sigma|_{(Y_s^+)^{\kappa_p - \epsilon}} = 0$. Hence, $\sigma|_{(Y_s^+)^{\kappa_p + \epsilon}}$ lies in the kernel of the restriction to $Y_s^{\kappa_p - \epsilon}$ and, by Lemma 4.3.1, $\sigma|_{C_s^p}$ is a sum of multiples of the Euler classes E_i^p of the negative normal bundles of the connected components of C_s^p, say, $\sigma|_{C_s^p} = \sum_i \beta_i \wedge E_i^p$. Let β be the image of $\sum \beta_i$ under $\bigoplus_i H_G^{*-\lambda_i^{\kappa_p}}(C_i^{\kappa_p}, \mathcal{F}) \to H_G^*(Y_s^{\kappa_p + \epsilon}, \mathcal{F})$ in the sequence of Lemma 4.3.1. Then $\beta|_{C_s^p} = \sigma|_{C_s^p}$ and $\beta|_{Y_s^{\kappa_p - \epsilon}} = 0$, in particular, $\beta|_{C_s^i} = 0$ for every $i < p$ and $\beta|_{C_s^\kappa} = 0$ for every $\kappa < 0$. By iterating the surjective restrictions of Lemma 4.3.1, β is the restriction of a $\widetilde{\sigma} \in H_G^*(Y_s, \mathcal{F})$. Then it is $\widetilde{\sigma} \in K_s^-$ and $\widetilde{\sigma}$ satisfies the claim.
Similarly, we obtain a $\widetilde{\sigma} \in K_s^+$ that satisfies the claim if we assume $\kappa_p < 0$. □

Set $\widetilde{K}_s^\pm := \{\sigma \in H_G^*(M, \mathcal{F}) \mid \sigma|_{C_s \cap Y_s^\pm} = 0\}$. By surjectivity of the restriction from M to Y_s, we have $K_s^\pm = \widetilde{K}_s^\pm|_{Y_s}$. As a consequence of the previous proposition, we then obtain

Corollary 4.3.9. *The kernel K of the Kirwan map $H_G^*(M, \mathcal{F}) \to H_G^*(\Psi^{-1}(0), \mathcal{F})$ is given by*

$$K = \bigoplus_{s=0}^{r-1} \widetilde{K}_s^+ \oplus \widetilde{K}_s^-.$$

We conclude this section by computing the kernel of the basic Kirwan map for an explicit example.

Example 4.3.10. Let us continue the example presented in Section 4.2.2, with $\beta = (1, ..., 1, -1)$ and $w = (1, ..., 1, w_n)$. Then

$$H_G(M, \mathcal{F}) = \mathbb{R}[u, s]/\langle e \rangle = \mathbb{R}[u, s]/\langle (u + s)^n(-u + w_n s) \rangle$$

by Lemma 4.2.3 and

$$\Psi(z) = \frac{\sum_{j=0}^{n-1} |z_j|^2 - |z_n|^2}{\sum_{j=0}^{n-1} |z_j|^2 + w_n|z_n|^2.}$$

The critical set of Ψ is $S^{2n-1} \times \{0\} \cup \{0\} \times S^1$. For the computation thereof, we refer the reader to Section 6.3.2, Lemma 6.3.5, where the critical sets will be computed for arbitrary β and w. We have $M^+ = \{z \in S^{2n+1} \mid |z_n|^2 \leq \frac{1}{2}\}$ and $M^- = \{z \in S^{2n+1} \mid |z_n|^2 \geq \frac{1}{2}\}$ so that $C^+ = \mathit{Crit}(\Psi) \cap M^+ = S^{2n-1} \times \{0\}$ and $C^- = \mathit{Crit}(\Psi) \cap M^- = \{0\} \times S^1$. Making use of homotopy equivalences, Lemma 4.3.1 with Ψ yields that we have a short exact sequence and a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & H_G^{*-\lambda^+}(C^+, \mathcal{F}) \arrow[r] \arrow[dr, "\cdot e_G(\nu^- C^+, \mathcal{F})"'] & H_G^*(M, \mathcal{F}) \arrow[r] \arrow[d, "i_{C^+}^*", "\cong"'] & H_G^*(C^-, \mathcal{F}) \arrow[r] & 0, \\
& & H_G^*(C^+, \mathcal{F}) & &
\end{tikzcd}$$

where $e_G(\nu^- C^+, \mathcal{F})$ denotes the equivariant basic Euler class of the negative normal bundle $\nu^- C^+$ of C^+ and λ^+ the rank of $\nu^- C^+$ and $i_{C^+} : C^+ \to M$ denotes the inclusion. Similarly, with $-\Psi$, we obtain a short exact sequence and a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & H_G^{*-\lambda^-}(C^-,\mathcal{F}) \arrow[r] \arrow[dr, "\cdot e_G(\nu^+C^-,\mathcal{F})"'] & H_G^*(M,\mathcal{F}) \arrow[r] \arrow[d, "i_{C^-}^*", "\cong"'] & H_G^*(C^+,\mathcal{F}) \arrow[r] & 0, \\
& & H_G^*(C^-,\mathcal{F}) & &
\end{tikzcd}$$

where $e_G(\nu^+C^-,\mathcal{F})$ denotes the equivariant basic Euler class of the positive normal bundle ν^+C^- of C^- and λ^- the rank of ν^+C^- and $i_{C^-}: C^- \to M$ denotes the inclusion. Note that the standard Riemannian metric g on S^{2n+1} is $S^1 \times T$-invariant. The normal bundles of C^+ and C^- are then given by $\nu C^+ = (\{0\}\times\mathbb{C})\times C^+ = \operatorname{span}\{\partial_{x_n}, \partial_{y_n}\}$ and $\nu C^- = (\mathbb{C}^n\times\{0\})\times C_2 = \operatorname{span}\{\partial_{x_j}, \partial_{y_j} \mid j = 0, ..., n-1\}$, respectively, where we used the notation $z_j = x_j + iy_j$. In these bases, the Hessian H of Ψ computes as

$$H|_{\nu C^+} = \begin{pmatrix} -2(1+w_n) & 0 \\ 0 & -2(1+w_n) \end{pmatrix}, \quad H|_{\nu C^-} = \begin{pmatrix} \frac{2(1+w_n)}{w_n^2} & 0 & & & 0 \\ 0 & \frac{2(1+w_n)}{w_n^2} & & & \\ & & \ddots & & \\ & & & \frac{2(1+w_n)}{w_n^2} & 0 \\ 0 & & & 0 & \frac{2(1+w_n)}{w_n^2} \end{pmatrix}.$$

Since $w_n > 0$, it follows that $\nu^-C^+ = \nu C^+$ and $\nu^+C^- = \nu C^-$. Similarly to the computation in Section 4.2.2, we compute $i_{C^+}^* = (s \mapsto d\alpha - u)$, $i_{C^-}^* = (s \mapsto \frac{u}{w_n})$ and the Euler classes

$$\begin{aligned}
e_G(\nu^-C^+,\mathcal{F}) &= \frac{1}{2\pi}(-u + sw_n) = \frac{1}{2\pi}(w_n d\alpha - (1+w_n)u), \\
e_G(\nu^+C^-,\mathcal{F}) &= \frac{1}{(2\pi)^n}(u+s)^n = \left(\frac{1+\frac{1}{w_n}}{2\pi}\right)^n u^n.
\end{aligned}$$

Since the inclusion $C^+ \cup C^- \to M$ induces an injective map in equivariant basic cohomology by Corollary 4.3.3, $K^\pm$ consists exactly of those classes that vanish when restricted to $C^\pm$ and that are a multiple of $e_G(\nu C^\mp,\mathcal{F})$ when restricted to $C^\mp$. Again making use of injectivity, we get

$$\begin{aligned}
K^+ &= \mathbb{R}[u,s]\cdot(u+s)^n/\langle e\rangle \quad \subset \mathbb{R}[u,s]/\langle e\rangle \text{ and} \\
K^- &= \mathbb{R}[u,s]\cdot(-u+sw_n)/\langle e\rangle \subset \mathbb{R}[u,s]/\langle e\rangle.
\end{aligned}$$

Indeed, we see that

$$
\begin{aligned}
H_G(M,\mathcal{F})\big/\big(K^+ + K^-\big) &\cong \mathbb{R}[u,s]\,/(\mathbb{R}[u,s]\cdot(u+s)^n + \mathbb{R}[u,s]\cdot(-u+sw_n)) \\
&\cong \mathbb{R}[u]\,/\langle u^n\rangle \\
&\cong H_G(\Psi^{-1}(0),\mathcal{F}).
\end{aligned}
$$

Chapter 5

Equivariant Formality

Another well-known result concerning the equivariant cohomology of a symplectic manifold is the *equivariant formality* of Hamiltonian actions of compact connected Lie groups H on compact symplectic manifolds N, namely that $H_H(N)$ is a free $S(\mathfrak{h}^*)$-module (cf. [Kir84, Proposition 5.8]). Let us consider the action of a torus G on a compact contact manifold (M, α) such that G preserves α, and assume that 0 is a regular value of the contact moment map. Recall that M then does not contain any G-fixed points (cf. Lemma 2.3.2). This implies that the G-action on M cannot be equivariantly formal: As a result of Borel's localization (cf., e.g., [GGK02, Theorem C.20]), $M^G = \varnothing$ results in $H_G(M)$ being a torsion module.

In this section, we will show that formality does hold for this type of torus actions on K-contact manifolds if we consider the basic setting. For a study of equivariantly formal actions in the setting of equivariant basic cohomology of transverse actions, the reader is referred to [GT16, Section 3.6].

Definition 5.0.11. Let $(N, \mathcal{E})$ be any foliated manifold, acted on by a torus H such that $\Omega(N, \mathcal{E})$ is an H^*-algebra. The H-action on $(N, \mathcal{E})$ is called *equivariantly formal*, if $H^*_H(N, \mathcal{E})$ is a free $S(\mathfrak{h}^*)$-module.

We work with a basis (X_i) of $\mathfrak{g}$ according to Proposition 2.4.1 and we, again, denote the foliation induced by the Reeb vector field R with $\mathcal{F}$ and the 1-dimensional $G \times T$-orbits, i.e., the critical points of Ψ^{X_1} and Ψ, by C, where T denotes the closure of the flow of R, $\{\psi_t\}$.

Lemma 5.0.12. *The G-action on $(C, \mathcal{F})$ is equivariantly formal. More precisely, we have*

$$H^*_G(C, \mathcal{F}) \simeq S(\mathfrak{g}^*) \otimes H^*(C, \mathcal{F}).$$

Proof. We have $C = Crit(\Psi) = \{x \in M \mid \widetilde{\mathfrak{g}}_x = \mathfrak{g}\}$ by Proposition 2.4.1, *(v)*. Lemma 3.3.5 with $\mathfrak{k} = \{0\}$ yields the claim. $\square$

Proposition 5.0.13. *The G-action on $(M, \mathcal{F})$ is equivariantly formal.*

Proof. Consider $X = X_1 \in \mathfrak{g}$ as in Proposition 2.4.1. Recall that Ψ^X is a Morse-Bott function by Proposition 2.4.9. Let $\kappa_1 < ... < \kappa_m$ be the critical values of Ψ^X and denote by $B_1^{\kappa_j}, ..., B_{i_j}^{\kappa_j}$ the connected components of the critical set C at level κ_j and with $\lambda_i^{\kappa_j}$ the indices of the non-degenerate critical submanifolds $B_i^{\kappa_j}$ with respect to Hess(Ψ^X). Set $M^{\kappa_j \pm \epsilon_j} = (\Psi^X)^{-1}((-\infty, \kappa_j \pm \epsilon_j])$. We consider the long exact sequence of the pair $(M^{\kappa_j+\epsilon_j}, M^{\kappa_j-\epsilon_j})$. By Lemma 4.3.1, it turns into the short exact sequences

$$0 \to \oplus_i H_G^{*-\lambda_i^{\kappa_j}}(B_i^{\kappa_j}, \mathcal{F}) \to H^*_G(M^{\kappa_j+\epsilon_j}, \mathcal{F}) \to H^*_G(M^{\kappa_j-\epsilon_j}, \mathcal{F}) \to 0.$$

Inductively, we can now conclude that the G-action on $(M, \mathcal{F})$ is equivariantly formal: Suppose that $H^*_G(M^{\kappa_j+\epsilon_j}, \mathcal{F})$ is a free $S(\mathfrak{g}^*)$-module. By the Morse-Bott property, the homotopy type does not change before crossing a critical value (see Theorem A.2). Proposition 3.3.10 then gives $H^*_G(M^{\kappa_j+\epsilon_j}, \mathcal{F}) = H^*_G(M^{\kappa_{j+1}-\epsilon_{j+1}}, \mathcal{F})$. It follows that $H^*_G(M^{\kappa_{j+1}-\epsilon_{j+1}}, \mathcal{F})$ is a free $S(\mathfrak{g}^*)$-module, as well. By Lemma 5.0.12, $\oplus_i H_G^{*-\lambda_i^{\kappa_{j+1}}}(B_i^{\kappa_{j+1}}, \mathcal{F})$ is also a free $S(\mathfrak{g}^*)$-module. Then the exactness of the sequence yields that $H^*_G(M^{\kappa_{j+1}+\epsilon_{j+1}}, \mathcal{F})$ has to be a free $S(\mathfrak{g}^*)$-module, as well. Hence, induction on j yields that $H^*_G(M, \mathcal{F}) = H^*_G(M^{\kappa_m+\epsilon_m}, \mathcal{F})$ is a free $S(\mathfrak{g}^*)$-module. $\square$

Chapter 6

Localization for K-contact Manifolds

In the first section of this chapter, we derive a basic Atiyah-Bott-Berline-Vergne type localization formula. We will apply this result in the following sections to prove that a specific parameter dependent integral $I^{\eta}(\epsilon)$ satisfies certain asymptotics and to obtain our Residue Formula. The last section of this chapter is devoted to examples. In particular, we will explain in detail how our Localization and Residue Formula may be used to deduce the analogous theorems for symplectic manifolds that occur as $M/\mathcal{F}$ in the case that R induces a free S^1-action.

6.1 The Localization Formula

In this section, we will derive a basic version of an Atiyah-Bott-Berline-Vergne type localization formula. We follow the line of proof in [AB84, Section 3], adjusting it to the basic setting. We assume throughout this section that the G-fixed points have closed Reeb orbits. Then Crit (Ψ), the minimal, 1-dimensional $G \times \{\psi_t\}$-orbits, are the 1-dimensional $G \times T$-orbits. This assumption is obviously satisfied if all Reeb orbits are closed or if there are no G-fixed points. Note that the latter is the case if 0 is a regular value of the contact moment map Ψ.

Recall that for $x \in M$, we denote by $\mathfrak{g}_x$ and $\widetilde{\mathfrak{g}}_x$ the isotropy algebra and the generalized isotropy algebra (cf. Definition 2.2.3), respectively.

Then $\mathrm{Crit}\,(\Psi) = \{x \in M \mid \widetilde{\mathfrak{g}}_x = \mathfrak{g}\}$. By our assumption, $\mathrm{Crit}\,(\Psi)$ is the union of the 1-dimensional $G \times T$-orbits. As in Lemma 2.4.8, it then follows that every connected component is a closed submanifold of even codimension.

Lemma 6.1.1. *$\widetilde{\mathfrak{g}}_x$ is invariant along $G \times T$-orbits.*

Proof. Let $h \in G \times T$, $X \in \mathfrak{g}$. Then, by commutativity of $G \times T$:

$$\begin{aligned} X_M(hx) &= \frac{d}{dt} \exp tX \cdot hx|_{t=0} = \frac{d}{dt} h \cdot \exp tX \cdot x|_{t=0} = dl_h \frac{d}{dt} \cdot \exp tX \cdot x|_{t=0} \\ &= dl_h X_M(x). \end{aligned}$$

l_h is a diffeomorphism, so above equation directly implies $X_M(hx) = 0$ if and only if $X_M(x) = 0$. Recall that the uniqueness of the Reeb vector field implies $dl_g R(x) = R(gx)$ for every $g \in G \times T$. Hence, the previous equation also implies $X_M(hx) \in \mathbb{R}R(hx)$ if and only if $X_M(x) \in \mathbb{R}R(x)$. It follows that $\widetilde{\mathfrak{g}}_x$ remains constant along $G \times T$-orbits. $\square$

Throughout this section, we work with cohomology with complex coefficients. Then $S(\mathfrak{g}^*) = \mathbb{C}[u_1, ..., u_s]$, where the u_i are coordinates of $\mathfrak{g}^* \otimes \mathbb{C}$. We will make use of the notion of the *support* of a finitely generated module. Recall that in the special case of a module H over $\mathbb{C}[u_1, ..., u_l]$, the support is the subset of $\mathbb{C}^l$ defined by:

$$\mathrm{Supp}\, H = \bigcap_{\substack{f \in \mathbb{C}[u_1,\ldots,u_l] \\ fH=0}} V_f,$$

where $V_f = \{u \in \mathbb{C}^l \mid f(u) = 0\}$. In particular, a free module has the whole space $\mathbb{C}^l$ as support. An element $h \in H$ is called a *torsion* element if there is a $0 \neq f \in \mathbb{C}[u_1, ..., u_l]$ with $fh = 0$. If all elements are torsion elements, then H is called a *torsion module*. Note that H is a torsion module if and only if $\mathrm{Supp}\, H$ is a proper subset of $\mathbb{C}^l$. For more details, the reader is referred to [AB84, Section 3] and the reference therein.

Lemma 6.1.2. *Let $O = (G \times T) \cdot x$ be an orbit and suppose that $U \subseteq M$ is a G-invariant $\mathcal{F}$-saturated submanifold admitting a $G \times \{\psi_t\}$-equivariant map p :*

$U \to O$. Then

$$\operatorname{Supp} H_G(U,\mathcal{F}) \subseteq \widetilde{\mathfrak{g}}_x \otimes \mathbb{C}.$$

Proof. The existence of the $G \times \{\psi_t\}$-equivariant map p implies that the $S(\mathfrak{g}^*)$-algebra structure on $H_G(U,\mathcal{F})$ factors as

$$S(\mathfrak{g}^*) \to H_G(O,\mathcal{F}) \to H_G(U,\mathcal{F}),$$

whence we obtain the inclusion of supports $\operatorname{Supp} H_G(U,\mathcal{F}) \subseteq \operatorname{Supp} H_G(O,\mathcal{F})$. Thus, it suffices to show that $\operatorname{Supp} H_G(O,\mathcal{F}) \subseteq \widetilde{\mathfrak{g}}_x \otimes \mathbb{C}$. For all $h \in G \times T$, we have $\widetilde{\mathfrak{g}}_{h\cdot x} = \widetilde{\mathfrak{g}}_x$ by Lemma 6.1.1. In particular, the generalized stabilizer is constant along O. Let $\mathfrak{k}$ be a complement of $\widetilde{\mathfrak{g}}_x$ in $\mathfrak{g}$ such that $\mathfrak{k}$ is the Lie algebra of a subtorus K of G. Since $\widetilde{\mathfrak{g}}_x$ acts trivially on $\Omega(O,\mathcal{F})$, the Cartan complex can be written as $C_G(O,\mathcal{F}) = S(\widetilde{\mathfrak{g}}_x^*) \otimes C_K(O,\mathcal{F})$ and $d_G = 1 \otimes d_K$, hence $H_G(O,\mathcal{F}) = S(\widetilde{\mathfrak{g}}_x^*) \otimes H_K(O,\mathcal{F})$. K acts locally freely and transversally on O, so $\Omega(O,\mathcal{F})$ is a $\mathfrak{k}$-dga of type (C) and $H_K(O,\mathcal{F}) = H(\Omega(O,\mathcal{F})_{\mathrm{bas}\mathfrak{k}})$ by Proposition 3.2.5. It also follows that $K \times \{\phi_t\}$ acts locally freely on O so that the orbits of this action define a foliation $\mathcal{E}$ of O. Since $G \times T$ is compact, we can, in particular, find a metric with respect to which the $K \times \{\phi_t\}$-action is isometric. Hence, $\mathcal{E}$ is a Riemannian foliation (cf. also [Mol88, p. 100]). This, however, means that the basic cohomology $H(O,\mathcal{E}) = H(\Omega(O,\mathcal{F})_{\mathrm{bas}\mathfrak{k}})$ is of finite dimension by [KASH85, Théorème 0]. Therefore, the support of $H_G(O,\mathcal{F})$ is contained in $\widetilde{\mathfrak{g}}_x \otimes \mathbb{C}$. □

Proposition 6.1.3. *Let X be a closed $G \times T$-invariant submanifold of M. Then the supports of $H^*_G(M \setminus X,\mathcal{F})$ and $H^*_{G,c}(M \setminus X,\mathcal{F})$ lie in $\cup_{x \in M\setminus X} \widetilde{\mathfrak{g}}_x. \otimes \mathbb{C}$ Note that since only finitely many different $\widetilde{\mathfrak{g}}_x$ occur on M, this is a finite union.*

Proof. We follow the line of argumentation of [AB84, Proposition 3.4]. See also the proof thereof in [GS99, Theorem 11.4.1]. Let U be a $G \times T$-invariant tubular neighborhood of X. By cohomology equivalence, it suffices to proof the assertion for $H_G(M \setminus \bar{U},\mathcal{F})$. Since $M \setminus U$ is compact, we can cover $M \setminus \bar{U}$ with N tubular neighborhoods U_i of $G \times T$-orbits of points $x_i \in M \setminus U \subset M \setminus X$. Let $V_s = U_1 \cup \ldots \cup U_{s-1}$. Using Lemma 6.1.2 together with the equivariant basic Mayer-Vietoris sequence (for compact supports) for U_s and V_s (cf. Propositions 3.3.6 and 3.3.9), the claim follows by induction, observing that, for any exact sequence $D \to E \to F$ of modules over $\mathbb{C}[u_1, \ldots, u_l]$: $\operatorname{Supp} E \subset \operatorname{Supp} D \cup \operatorname{Supp} F$. □

Let $C := \mathrm{Crit}(\Psi)$. The previous result then immediately yields the following.

Corollary 6.1.4. *The supports of $H^*_G(M \setminus C, \mathcal{F})$ and $H^*_{G,c}(M \setminus C, \mathcal{F})$ lie in the finite union $\bigcup_{\widetilde{\mathfrak{g}}_x \neq \mathfrak{g}} \widetilde{\mathfrak{g}}_x \otimes \mathbb{C}$. In particular, $H^*_G(M \setminus C, \mathcal{F})$ and $H^*_{G,c}(M \setminus C, \mathcal{F})$ are torsion modules over $S(\mathfrak{g}^*)$.*

The same holds for any G-invariant, $\mathcal{F}$-saturated subset of $M\setminus C$ and, by exactness, for the relative equivariant basic cohomology of any pair in $M \setminus C$.

Theorem 6.1.5. *Denote by $i : C \hookrightarrow M$ the inclusion. Then the kernel and cokernel of the induced map $i^* : H^*_G(M, \mathcal{F}) \to H^*_G(C, \mathcal{F})$ have support in $\bigcup_{\widetilde{\mathfrak{g}}_x \neq \mathfrak{g}} \widetilde{\mathfrak{g}}_x \otimes \mathbb{C}$. In particular, both $S(\mathfrak{g}^*)$-modules have the same rank, $\dim H^*(C, \mathcal{F})$, and the kernel of $i^* : H^*_G(M, \mathcal{F}) \to H^*_G(C, \mathcal{F})$ is exactly the module of torsion elements in $H_G(M, \mathcal{F})$.*

Proof. Consider the long exact sequence for the pair (M, C)

$$... \to H^k_G(M, C, \mathcal{F}) \to H^k_G(M, \mathcal{F}) \xrightarrow{i^*} H^k_G(C, \mathcal{F}) \to H^{k+1}_G(M, C, \mathcal{F}) \to ... \,.$$

By exactness, it can immediately be seen that $\ker i^*$ is isomorphic to a quotient module of $H_G(M, C, \mathcal{F})$, and that $\operatorname{coker} i^*$ is a sub-module of $H_G(M, C, \mathcal{F})$. But $H_G(M, C, \mathcal{F})$ is a torsion module with support in $\bigcup_{\widetilde{\mathfrak{g}}_x \neq \mathfrak{g}} \widetilde{\mathfrak{g}}_x \otimes \mathbb{C}$ by Corollary 6.1.4 and Proposition 3.3.20. Since $H^*_G(C, \mathcal{F}) = S^*(\mathfrak{g}^*) \otimes H^*(C, \mathcal{F})$ is a free $S^*(\mathfrak{g}^*)$-module, the rank statement follows and every torsion element has to be mapped to zero under i^*. $\square$

Proposition 6.1.6. *The kernel and cokernel of the push forward $i_* : H_G(C, \mathcal{F}) \to H_G(M, \mathcal{F})$ have support in $\bigcup_{\widetilde{\mathfrak{g}}_x \neq \mathfrak{g}} \widetilde{\mathfrak{g}}_x \otimes \mathbb{C}$ and are therefore torsion.*

Proof. Let $p_j : U_j \to C_j$ denote a sufficiently small invariant tubular neighborhood of the connected component $C_j \subset C$ such that $U_j \cap U_i = \varnothing$ for $i \neq j$ and set $U = \cup U_j$. Then, since U_j can be identified with a disk bundle in the normal bundle over C_j, ∂U_j is a sphere bundle over C_j, in particular, a smooth manifold, and $G \times T$-invariant. Note that Definition 3.3.18 and Propositions 3.3.19 and 3.3.20 extend to include closed subsets that are G-invariant, $\mathcal{F}$-saturated open submanifolds with invariant boundary. $M\setminus U$ is a $G\times T$-invariant open submanifold with boundary and $G\times T$-equivariantly homotopy equivalent to $M\backslash C$. We consider the long exact sequence of the pair $(M, M\setminus U)$.

$$\ldots \to H^k_G(M, M\setminus U, \mathcal{F}) \overset{\text{incl.}}{\to} H^k_G(M,\mathcal{F}) \to H^k_G(M\setminus U,\mathcal{F}) \to \ldots$$

By the Thom isomorphism, we have $H_G(C,\mathcal{F}) \cong H_{G,c}(U,\mathcal{F})$. Furthermore, Proposition 3.3.20 yields $H^*_{G,c}(U,\mathcal{F}) \cong H^*_G(M, M\backslash U,\mathcal{F})$. Combining these isomorphisms with the long exact sequence, we obtain the following commutative diagram.

$$\begin{tikzcd}
\ldots \arrow[r] & H^k_G(M, M\setminus U,\mathcal{F}) \arrow[r] \arrow[dr, phantom, "\circlearrowleft"] & H^k_G(M,\mathcal{F}) \arrow[r] & H^k_G(M\setminus U,\mathcal{F}) \arrow[r] & \ldots \\
& H^k_{G,c}(U,\mathcal{F}) \arrow[u, "\text{extension by }0"', "\cong"] & & & \\
& H^{k-d}_G(C,\mathcal{F}) \arrow[u, "\text{Thom}"', "\cong"] \arrow[uur, "\iota_*"'] & & &
\end{tikzcd}$$

It yields that $\ker \iota_* = \operatorname{im}(H_G(M\setminus U,\mathcal{F}) \to H_G(M, M\setminus U,\mathcal{F}))$ is the image of a torsion module with support in $\bigcup_{\widetilde{\mathfrak{g}}_x\neq\mathfrak{g}} \widetilde{\mathfrak{g}}_x \otimes \mathbb{C}$ and that

$$\begin{aligned}
\operatorname{coker} \iota_* &= H_G(M,\mathcal{F})/\operatorname{im}(\iota_*) = H_G(M,\mathcal{F})/\ker\left(H_G(M,\mathcal{F}) \to H_G(M\setminus U,\mathcal{F})\right) \\
&\cong \operatorname{im}\left(H_G(M,\mathcal{F}) \to H_G(M\setminus U,\mathcal{F})\right) \subset H_G(M\setminus U,\mathcal{F}).
\end{aligned}$$

Hence, $\operatorname{coker}\iota_*$ is isomorphic to the image of $H^*_G(M,\mathcal{F}) \to H^*_G(M\setminus U,\mathcal{F})$, a submodule of a torsion module with support in $\bigcup_{\widetilde{\mathfrak{g}}_x\neq\mathfrak{g}} \widetilde{\mathfrak{g}}_x \otimes \mathbb{C}$. $\square$

From the preceding two statements, it follows that $i^*i_* : H_G(C,\mathcal{F}) \to H_G(C,\mathcal{F})$ is an isomorphism modulo torsion. Exactly as in [GS99, Section 10.5] for ordinary equivariant cohomology, we obtain that i^* maps the equivariant basic Thom class to $e_G(\nu C,\mathcal{F})$. In particular,

Lemma 6.1.7. *$i^*i_* = \wedge e_G(\nu C,\mathcal{F})$ is the multiplication with the basic equivariant Euler class of the normal bundle of C (cf. Definition 3.4.5).*

Hence, $e_G(\nu C,\mathcal{F})$ is invertible in the localized module.

Remark 6.1.8. Alternatively, it can be shown directly that $e_G(\nu C,\mathcal{F})$ is not a zero divisor in $H^*_G(C,\mathcal{F})$, see Lemma 4.1.3.

We are now ready to prove our ABBV-type localization formula, an integration formula. We can consider the integration of $(2n+1)$-forms of the form $\alpha \wedge \omega$, with $\omega \in \Omega^{2n}(M, \mathcal{F})$. By Lemma 3.1.2, the map

$$\int_M \alpha \wedge \cdot : \Omega^{2n}(M, \mathcal{F}) \to \mathbb{R}, \qquad \omega \mapsto \int_M \alpha \wedge \omega$$

descends to a well-defined map on basic cohomology.

Analogously, we can consider the integration of equivariant basic forms and classes. Let η be a form representing a class in $H_G(M, \mathcal{F})$ and denote the basic equivariant pushforward by

$$\Pi_* : H_G(M, \mathcal{F}) \to S(\mathfrak{g}^*), \qquad \Pi_* \eta = \int_M \alpha \wedge \eta.$$

Our Localization Formula then reads as follows.

Theorem 6.1.9. *Suppose a torus G acts on a K-contact manifold (M, α) such that G preserves α, and suppose in addition that the G-fixed points have closed Reeb orbits. Then for all $\eta \in H_G(M, \mathcal{F})$, we have the exact integration formula*

$$\Pi_* \eta = \int_M \alpha \wedge \eta = \sum_{C_j \subseteq C} \int_{C_j} \frac{i_j^*(\alpha \wedge \eta)}{e_G(\nu C_j, \mathcal{F})},$$

where $C_j \subseteq C$ denote the connected components and $i_j : C_j \hookrightarrow M$ their inclusions.

Remark 6.1.10. We note that for this result, it is sufficient to assume that all G-fixed points have a closed Reeb orbit, an assumption that is weaker than assuming 0 to be a regular value of Ψ and that is automatically satisfied for total spaces in the Boothby-Wang fibration.

This theorem is closely related to results obtained in [Töb14, GNT17].

Proof. The inverse of i_* on the localized module is given by $Q := \sum_{C_j \subseteq C} \frac{i_j^*}{e_G(\nu C_j, \mathcal{F})}$. We therefore obtain for every $\eta \in H_G(M, \mathcal{F})$

$$\Pi_* \eta = \int_M \alpha \wedge i_* Q \eta. \tag{6.1}$$

Now, using the definition of i_* in terms of Thom forms we can express η as

$$\eta = i_* Q \eta = \sum_j (i_j)_* \frac{i_j^* \eta}{e_G(\nu C_j, \mathcal{F})} = \sum_j p_j^* \left(\frac{i_j^* \eta}{e_G(\nu C_j, \mathcal{F})} \right) \wedge \tau_j, \tag{6.2}$$

where τ_j is an equivariant basic Thom form compactly supported in a small $G\times T$-invariant tubular neighborhood U_j of C_j, $p_j : U_j \to C_j$ is the projection. By Proposition 3.3.10, we have $p_j^* \circ i_j^* = \mathrm{id}$ on cohomology and

$$\begin{aligned}\int_{U_j} \alpha \wedge p_j^*\left(\frac{i_j^*\eta}{e_G(\nu C_j,\mathcal{F})}\right) \wedge \tau_j &= \int_{U_j} p_j^*\left(\frac{i_j^*(\alpha\wedge\eta)}{e_G(\nu C_j,\mathcal{F})}\right) \wedge \tau_j \\ &= \int_{C_j} \frac{i_j^*(\alpha\wedge\eta)}{e_G(\nu C_j,\mathcal{F})} \wedge (p_j)_*\tau_j.\end{aligned}$$

Since $(p_j)_*\tau_j = 1$, we obtain the desired integration formula by summing over j and using the identites (6.1)-(6.2). □

6.2 Equivariant Integration Formulae

We will assume throughout this section that 0 is a regular value of the contact moment map Ψ. The level set $\Psi^{-1}(0)$ is a smooth $G \times T$-invariant submanifold of M, on which G acts locally freely. We define the contact reduction $M_0 := \Psi^{-1}(0)/G$, which is a contact orbifold and an honest manifold if the action of G on $\Psi^{-1}(0)$ is free (cf., e.g., [BG08, Theorem 8.5.1]). The contact form α on M induces a contact form α_0 on M_0 that pulls back to the restriction of α to $\Psi^{-1}(0)$. Since G and the Reeb flow commute and the Reeb orbits are transversal to the G-orbits along $\Psi^{-1}(0)$, $\Omega(\Psi^{-1}(0),\mathcal{F})$ is a G^*-algebra of type (C) and, hence, we have $H_G(\Psi^{-1}(0),\mathcal{F}) \cong H(\Omega(\Psi^{-1}(0),\mathcal{F})_{\mathrm{bas}\ \mathfrak{g}})$ (cf. Proposition 3.2.5) via the Cartan map. This implies that we have an isomorphism $H_G(\Psi^{-1}(0),\mathcal{F}) \cong H(M_0,\mathcal{F}_0)$, where the later denotes the cohomology of the R-basic differential forms on the contact quotient $M_0 = \Psi^{-1}(0)/G$. Recall from Theorem 4.1.1 that the inclusion $\Psi^{-1}(0) \subset M$ induces a natural surjective map

$$\kappa : H_G(M,\mathcal{F}) \to H_G(\Psi^{-1}(0),\mathcal{F}) \cong H(M_0,\mathcal{F}_0),$$

the basic Kirwan map. In this section, we will derive integration formulae that relate integration of equivariant basic forms on M to integration of basic forms on M_0.

6.2.1 Equivariant Integration

Following an idea of Witten [Wit92], Jeffrey and Kirwan [JK95] proved certain formulae relating integration of equivariant forms on a symplectic manifold to integration of ordinary differential forms on its symplectic quotient. By far the most important ingredient in their proof is the Atiyah-Bott-Berline-Vergne integration formula [AB84, BV82]. Armed with our localization formula (Theorem 6.1.9) and the local normal form of the moment map (Proposition 2.5.4), we will obtain the K-contact analogues, Theorems 6.2.7 and 6.2.13, by the same line of argumentation as Jeffrey-Kirwan. In this section, we establish notation and prove the key prerequisites for these theorems.

Let η be a form representing a class in $H_G(M, \mathcal{F})$ and recall that we denoted the basic equivariant pushforward by

$$\Pi_* \eta = \int_M \alpha \wedge \eta \in S(\mathfrak{g}^*).$$

We will apply Π_* to classes of type $\eta \wedge e^{id_G \alpha}$, which are, strictly speaking, not equivariant basic cohomology classes according to our definition, since they are not polynomial but analytic in $\phi \in \mathfrak{g}$. This is well defined, provided one replaces the codomain with a suitable completion of $S(\mathfrak{g}^*)$. With this in mind, for any closed equivariant basic form η, with $r = \dim \mathfrak{g}$, we consider the integral

$$I^\eta(\epsilon) = \frac{1}{(2\pi i)^r \mathrm{vol}\,(G)} \int_{\mathfrak{g}} e^{-\epsilon|\phi|^2/2} \left(\Pi_* \left(\eta \wedge e^{id_G \alpha}\right)\right)(\phi) d\phi,$$

where $d\phi$ is a measure on $\mathfrak{g}$ corresponding to a metric on $\mathfrak{g}$ that induces a volume form vol_G on G, $\mathrm{vol}\,(G) = \int_G \mathrm{vol}_G$. Then $d\phi/\mathrm{vol}\,(G)$ is independent of that choice.

Remark 6.2.1. Note that $I^\eta(\epsilon)$ is well defined; $\eta \wedge e^{id_G \alpha}$ is only of mild exponential dependence on ϕ so that the factor $e^{-\epsilon|\phi|^2/2}$ ensures convergence of the integral. Indeed, we will see shortly that the Fourier transform $\mathbf{F}\left[\Pi_*(\eta \wedge e^{id_G \alpha})\right]$ is a distribution with compact support, in particular, a tempered distribution. Hence, the same holds for $\Pi_*(\eta \wedge e^{id_G \alpha})$.

Following Jeffrey-Kirwan, we will relate the $\epsilon \to 0$ asymptotics of $I^\eta(\epsilon)$ to intersection pairings on the contact quotient.

First, we must rewrite $I^\eta(\epsilon)$ in a more convenient form. For any tempered distribution on $\mathfrak{g}$, recall the Fourier transform

$$(\mathbf{F}f)(z) = (2\pi)^{-r/2}\int_{\mathfrak{g}} f(\phi)e^{-iz(\phi)}d\phi.$$

By definition, $\mathbf{F}(f)$ is naturally a tempered distribution on $\mathfrak{g}^*$. There are different conventions concerning the constant in the Fourier transform. The normalization we work with in this thesis is chosen so that $\mathbf{F}\mathbf{F}D(y) = D(-y)$.

Set

$$Q^\eta(y) = \mathbf{F}\left[\Pi_*(\eta \wedge e^{id_G\alpha})\right](y).$$

Let g_ϵ denote the Gaußian function $g_\epsilon(\phi) = e^{-\epsilon|\phi|^2/2}$, with Fourier transform $(\mathbf{F}g_\epsilon)(z) = \epsilon^{-r/2}g_{\epsilon^{-1}}(z)$. Note that $I^\eta(\epsilon)$ can be viewed as the L^2 inner product of the functions $g_\epsilon(\phi)$ and $\Pi_*(\eta\wedge e^{id_G\alpha})(\phi)$. Since the Fourier transform is an L^2 isometry (cf., e.g., [Hör90, Theorem 7.1.6]), we have the following identity.

Lemma 6.2.2.

$$I^\eta(\epsilon) = \frac{1}{(2\pi i)^r\epsilon^{r/2}\mathrm{vol}\,(G)}\int_{\mathfrak{g}^*} Q^\eta(y)e^{-|y|^2/2\epsilon}dy.$$

We want to show that the distribution Q^η may be represented by a piecewise polynomial function (cf. Proposition 6.2.6). To this end, some preliminary considerations are necessary.

Lemma 6.2.3. *The distribution $Q^\eta(y)$ can be expressed as follows*

$$Q^\eta(y) = \mathbf{F}\left[\Pi_*(\eta\wedge e^{id_G\alpha})\right](y) = (2\pi)^{s/2}\sum_J i^J\frac{\partial}{\partial y^J}\int_M \alpha\wedge\eta_J\wedge e^{id\alpha}\delta(-\Psi - y),$$

where δ denotes the Dirac delta distribution, and $\eta = \sum_J \eta_J y^J$, summing over multi indices J, with y^j denoting an orthonormal basis of $\mathfrak{g}^*$ and $\eta_J \in \Omega^*(M,\mathcal{F})$. In particular, $Q^\eta(y)$ is supported in the compact set $-\Psi(M)$.

Proof. We make use of the arguments given in [JK95, Sections 5, 7]. With $\eta = \sum_J \eta_J y^J$, we have $\eta(\phi) = \sum_J \eta_J\phi^J$ with ϕ^j denoting the coordinate functions $y^j(\phi)$. Recalling the definition of $Q^\eta(y)$, we have

$$Q^\eta(y) = \mathbf{F}\left[\Pi_*(\eta\wedge e^{id_G\alpha})\right](y) = \frac{1}{(2\pi)^{r/2}}\sum_J\int_M\int_{\mathfrak{g}}\alpha\wedge\eta_J\phi^J\wedge e^{id\alpha - i\langle\Psi,\phi\rangle - i\langle y,\phi\rangle}d\phi$$

$$= \frac{1}{(2\pi)^{r/2}} \sum_J i^J \frac{\partial}{\partial y^J} \int_M \alpha \wedge \eta_J \wedge e^{id\alpha} \int_{\mathfrak{g}} e^{i\langle -\Psi - y, \phi \rangle} d\phi$$
$$= (2\pi)^{r/2} \sum_J i^J \frac{\partial}{\partial y^J} \int_M \alpha \wedge \eta_J \wedge e^{id\alpha} \delta(-\Psi - y).$$

This shows that $Q^\eta(y)$ is the integral over $x \in M$ of a distribution $S(x, y)$ on $M \times \mathfrak{g}$ which is supported on the set $\{(x, y) \mid -\Psi(x) = y\}$. $\square$

The following Proposition 6.2.4 is a part of [JK95, Proposition 3.6] and Lemma 6.2.5 is [JK95, Lemma 4.3]. [JK95, Proposition 3.6 and Lemma 4.3] were derived from results that occur in the works of Guillemin, Lerman, Prato, and Sternberg ([GLS88, GLS96, GP90]).

Proposition 6.2.4. *Let τ, $\beta_1, ..., \beta_N \in \mathfrak{g}^*$ such that the β_j all lie in the interior of some half-space of $\mathfrak{g}^*$ and set $\beta = \{\beta_1, ..., \beta_N\}$. Consider*

$$P(\phi) = \frac{e^{i\tau(\phi)}}{i^N \prod_{j=1}^N \beta_j(\phi)}.$$

Then there is a piecewise polynomial function H_β supported on the cone $C_\beta = \{\sum_j \lambda_j \beta_j \mid \lambda_j \geq 0\}$ such that for ϕ in the complement of the hyperplanes $\{\phi \in \mathfrak{g} \mid \beta_j(\phi) = 0\}$, P is the Fourier transform of h, where $h(y) := H_\beta(y + \tau)$. H_β is smooth at any $y \in U_\beta$, where U_β are the points in $\mathfrak{g}^*$ which are not in any cone spanned by a subset of $\{\beta_1, ..., \beta_N\}$ containing fewer that $r = \dim \mathfrak{g}$ elements.

Lemma 6.2.5. *Let D_1, D_2 be two tempered distributions on $\mathfrak{g}^*$ such that*

1. *$\mathbf{F}D_1 - \mathbf{F}D_2$ is supported on a finite union of hyperplanes.*
2. *There are $\zeta \in \mathfrak{g}^*$ and $k \in \mathbb{R}$ such that the half space $\{y \in \mathfrak{g}^* \mid \langle y, \zeta \rangle > k\}$ contains the support of $D_2 - D_1$.*

Then $D_1 = D_2$

We are now ready to prove the piecewise polynomial property of Q^η.

Proposition 6.2.6. *The distribution $Q^\eta(y)$ may be represented by a piecewise polynomial function.*

Proof. Let C_j denote a connected component of the critical set $C = \operatorname{Crit}\Psi$ of codimension d. By Lemma 2.4.6, $G \times T$ acts in R-direction only and its isotropy $(\mathfrak{g} \times \mathfrak{t})_{C_j}$ has codimension 1. Let θ be a $G \times T$-invariant, basic connection form on the bundle of oriented orthonormal frames of νC_j and denote by F^θ its (ordinary) curvature. Choose a basis (X_i) of $\mathfrak{g} \times \mathfrak{t}$ such that $X_1, ..., X_{N-1}$ is a basis of $(\mathfrak{g} \times \mathfrak{t})_{C_j}$ and $X_N = R$. Denote its dual basis by u_i. Then, since θ is basic, $\iota_{X_N}\theta = 0$. The basic $G \times T$-equivariant Euler form is then given by

$$e_{G\times T}(\nu C_j, \mathcal{F}) = \operatorname{Pf}\left(F^\theta - \sum_i \iota_{X_i}\theta u_i\right) = \operatorname{Pf}\left(F^\theta - \sum_{i=1}^{N-1} \iota_{X_i}\theta u_i\right). \tag{6.3}$$

Denote by $(G \times T)_{C_j} \subset G \times T$ the subtorus that has $(\mathfrak{g} \times \mathfrak{t})_{C_j}$ as Lie algebra. νC_j is a $(G \times T)_{C_j}$-equivariant vector bundle over C_j. By the splitting principle for equivariant bundles, we may assume that the normal bundle splits as a direct sum of line bundles $\nu C_j = \oplus_i L_i$ and $(G\times T)_{C_j}$ acts on L_i with weight β_i^j. Then the basic $(G\times T)_{C_j}$-equivariant Euler form factors as $e_{(G\times T)_{C_j}}(\nu C_j, \mathcal{F}) = \prod_i e_{(G\times T)_{C_j}}(L_i, \mathcal{F})$ and $2\pi e_{(G\times T)_{C_j}}(L_i, \mathcal{F}) = c_i^j + \beta_i^j$, where $c_i^j \in \Omega^2(C_j, \mathcal{F})$ is the (ordinary) basic Euler form of L_i. Hence,

$$(2\pi)^{d/2} e_{(G\times T)_{C_j}}(\nu C_j, \mathcal{F}) = \prod_{i=1}^{d/2}(c_i^j + \beta_i^j). \tag{6.4}$$

We can, however, also compute $e_{(G\times T)_{C_j}}(\nu C_j, \mathcal{F})$ as $\operatorname{Pf}\left(F^\theta - \sum_{i=1}^{N-1} \iota_{Y_i}\theta b_i\right)$, where (Y_i) denotes a basis of $(\mathfrak{g} \times \mathfrak{t})_{C_j}$ and (b_i) its dual basis. Equation (6.3) yields that if we extend $e_{(G\times T)_{C_j}}(\nu C_j, \mathcal{F})$ to all of $\mathfrak{g} \times \mathfrak{t}$ by setting it equal to 0 on $\mathbb{R}R$, we obtain the $G \times T$-equivariant basic Euler form $e_{G\times T}(\nu C_j, \mathcal{F})$. Hence, extending $\beta_i^j \in (\mathfrak{g} \times \mathfrak{t})^*_{C_j}$ and combining Equations (6.3) and (6.4) yields

$$(2\pi)^{d/2} e_{G\times T}(\nu C_j, \mathcal{F}) = \prod_{i=1}^{d/2}(c_i^j + \beta_i^j). \tag{6.5}$$

The definition of the Euler form yields that $e_G(\nu C_j, \mathcal{F})$ is exactly given by the restriction of $e_{G\times T}(\nu C_j, \mathcal{F})$ to $\mathfrak{g}$ so that, by (6.5),

$$(2\pi)^{d/2} e_G(\nu C_j, \mathcal{F}) = \prod_{i=1}^{d/2}(c_i^j + \beta_i^j|_{\mathfrak{g}}). \tag{6.6}$$

By Theorem 6.1.9, we have that

$$\Pi_*(\eta \wedge e^{id_G\alpha}) = \sum_j \int_{C_j} \frac{i_j^*(\alpha \wedge \eta \wedge e^{id_G\alpha})}{e_G(\nu C_j, \mathcal{F})}. \tag{6.7}$$

We now follow the line of argumentation of [JK95, Lemma 2.2]. Equation (6.6) implies that the inverse of $e_G(\nu C_j, \mathcal{F})$ in the localized module is given by

$$\frac{1}{e_G(\nu C_j, \mathcal{F})} = \frac{(2\pi)^{d/2}}{\prod_i \beta_i^j|\mathfrak{g}} \prod_i \left(1 + \frac{c_i^j}{\beta_i^j|\mathfrak{g}}\right)^{-1}.$$

Since $c_i^j \in H^2(C_j, \mathcal{F})$, we know that $\frac{c_i^j}{\beta_i^j|\mathfrak{g}}$ is nilpotent. Thus, $\sum_{l\geq 0}(-1)^l \left(c_i^j/\beta_i^j|\mathfrak{g}\right)^l$ is a finite sum and easily seen to be equal to $\left(1 + c_i^j/\beta_i^j|\mathfrak{g}\right)^{-1}$. We obtain

$$\frac{1}{e_G(\nu C_j, \mathcal{F})} = \frac{(2\pi)^{d/2}}{\prod_i \beta_i^j|\mathfrak{g}} \prod_i \sum_{l_i \geq 0} (-1)^{l_i} \left(\frac{c_i^j}{\beta_i^j|\mathfrak{g}}\right)^{l_i}.$$

It now follows with Equation (6.7) that the pushforward may be written as a sum

$$\Pi_*(\eta \wedge e^{id_G\alpha})(\phi) = \sum_j \sum_{a\in\mathcal{A}_j} \frac{e^{-i\Psi(C_j)(\phi)} \int_{C_j} i_j^*(\alpha \wedge \eta(\phi) \wedge e^{id\alpha}) \wedge c_{j,a}}{\prod_i (\beta_i^j|_{\mathfrak{g}})(\phi)^{n_{j,i}(a)}}, \tag{6.8}$$

where $\mathcal{A}_j$ is a finite indexing set obtained from interchanging summation and multiplication, $c_{j,a} \in H^*(C_j, \mathcal{F})$ is determined by the c_i^j, and $n_{j,i}(a)$ is a non-negative integer. In particular, for every (j, a), the term on the right hand side of Equation (6.8) is given by the product of $\frac{e^{-i\Psi(C_j)(\phi)}}{(\prod_i \beta_i^j|_{\mathfrak{g}})(\phi)^{n_{j,i}(a)}}$ with a polynomial in ϕ, where the polynomial is simply a constant if $\eta = 1$.

Given this description of the pushforward, we can now derive the piecewise polynomial property of $Q^\eta(y)$ for $\eta = 1$, following the approach of [JK95, Theorem 4.2]: Choose a component Λ of the set $\cap_{j,i}\{\phi \in \mathfrak{g} \mid (\beta_i^j|_{\mathfrak{g}})(\phi) \neq 0\}$. Λ is a cone in $\mathfrak{g}$. Denote by $C_{j,\Lambda}$ the component of $\cap_i\{\phi \in \mathfrak{g} \mid (\beta_i^j|_{\mathfrak{g}})(\phi) \neq 0\}$ containing Λ so that $\Lambda = \cap_j C_{j,\Lambda}$. $\beta_i^j|_{\mathfrak{g}}$ lies in the dual cone $C_{j,\Lambda}^* = \{\sum_i \lambda_i \beta_i^j|_{\mathfrak{g}} \mid \lambda_i \geq 0\}$. Pick any $\xi \in \Lambda$. We then set

$$\sigma_{j,i} := \operatorname{sign} \beta_i^j|_{\mathfrak{g}}(\xi), \quad \beta_i^j|_{\mathfrak{g}}^\Lambda := \sigma_{j,i}\beta_i^j|_{\mathfrak{g}}, \quad k_j(a) := \sum_{\substack{i,\\ \sigma_{j,i}=-1}} n_{j,i}(a),$$

for $a \in \mathcal{A}_j$. Let $\gamma_j(a)$ consist of all $\beta_i^j|_{\mathfrak{g}}^{\Lambda}$, occurring with multiplicity $n_{j,i}(a)$. Define the function $D : \mathfrak{g}^* \to \mathbb{C}$ by

$$D(y) = \sum_j \sum_{a \in \mathcal{A}_j} (-i)^{\sum_i n_{j,i}(a)} (-1)^{k_j(a)} H_{\gamma_j(a)}(y + \Psi(C_j)) \int_{C_j} i_j^*(\alpha \wedge e^{id\alpha}) \wedge c_{j,a},$$

where $H_{\gamma_j(a)}$ is the piecewise polynomial function appearing in Proposition 6.2.4. We will show that Q^1 is piecewise polynomial by applying Lemma 6.2.5 to $D_1 = D$ and $D_2 = Q^1$. The first hypothesis is satisfied because Proposition 6.2.4 yields that $\mathbf{F}D(\phi)$ is given by $(\mathbf{F}\mathbf{F}\Pi_* e^{id_G\alpha})(\phi) = (\Pi_* e^{id_G\alpha})(-\phi)$ on the complement of the hyperplanes $\{\phi \in \mathfrak{g} \mid \beta_i^j|_{\mathfrak{g}}^{\Lambda}(\phi) = 0\}$. The second hypothesis is satisfied because D is supported in a half space since all $\beta_i^j|_{\mathfrak{g}}^{\Lambda}$ satisfy $\beta_i^j|_{\mathfrak{g}}^{\Lambda}(\xi) > 0$ for any $\xi \in \Lambda$, and Q^1 is supported in the compact set $-\Psi(M)$ by Lemma 6.2.3. Hence, Q^1 is piecewise polynomial.

Now let η be arbitrary. Note that in the case $\eta = 1$, every (j, a)-summand gave a piecewise polynomial function. $\eta(\phi)$ now contributes a polynomial in ϕ to every summand. Up to a factor of $(-i)$, Fourier transformation interchanges differentiation and multiplication by a coordinate (cf. [Hör90, Lemma 7.1.3]). Hence, for arbitrary η, the piecewise polynomial property of $Q^\eta(y)$ follows from the case $\eta = 1$. $\square$

6.2.2 Asymptotic Analysis

We will prove the following theorem. We denote by η_0 the image of η under the natural basic Kirwan map (cf. Theorem 4.1.1) $H_G(M, \mathcal{F}) \to H(M_0, \mathcal{F}_0)$, and let α_0 denote the quotient contact form on M_0.

Theorem 6.2.7. *For any $\eta \in H_G(M, \mathcal{F})$, there exists some constant $c > 0$ such that as $\epsilon \to 0^+$, $I^\eta(\epsilon)$ obeys the asymptotic*

$$I^\eta(\epsilon) = \frac{1}{n_0} \int_{M_0} \alpha_0 \wedge \eta_0 \wedge e^{\epsilon\Theta + id\alpha_0} + o(\epsilon^{-r/2} e^{-c/\epsilon}), \tag{6.9}$$

where $\Theta \in H^4(M_0, \mathcal{F}_0)$ is the class corresponding to $-\frac{<\phi,\phi>}{2} \in H_G^4(\Psi^{-1}(0), \mathcal{F}) \simeq H^4(M_0, \mathcal{F}_0)$ and n_0 denotes the order of the regular isotropy of the action of G on $\Psi^{-1}(0)$.

A particular consequence of this theorem is the identity

$$\int_{M_0} \alpha_0 \wedge \eta_0 \wedge e^{id\alpha_0} = n_0 \lim_{\epsilon \to 0^+} I^\eta(\epsilon),$$

which expresses intersection pairings on M_0 as limits of equivariant intersection pairings on M.

By Lemma 6.2.2 and Proposition 6.2.6, we are reduced to estimating the asymptotics of an integral of the form $I(\epsilon) = \int e^{-|y|^2/2\epsilon} Q(y)dy$, where $Q(y)$ is piecewise polynomial. Suppose that $Q(y)$ is regular near the origin, and let $Q_0(y)$ denote the polynomial which agrees with $Q(y)$ near the origin. Set

$$I_0(\epsilon) = \frac{1}{(2\pi i)^r \epsilon^{r/2} \mathrm{vol}\,(G)} \int_{\mathfrak{g}^*} Q_0(y) e^{-|y|^2/2\epsilon} dy.$$

Lemma 6.2.8. *Suppose that $Q(y)$ is regular near the origin and define $I(\epsilon)$ and $I_0(\epsilon)$ as above. Then we have the asymptotic*

$$|I(\epsilon) - I_0(\epsilon)| = o(\epsilon^{-r/2} e^{-c/\epsilon})$$

for some constant $c > 0$.

Proof. Let $R(y) = Q(y) - Q_0(y)$. Then $R(y)$ is piecewise polynomial and identically zero in a neighborhood of the origin. Pick $\delta > 0$ so that $R(y)$ is identically zero for $|y| < \delta$. Switching to polar coordinates, we have

$$|I(\epsilon) - I_0(\epsilon)| \le c' \epsilon^{-r/2} \int_{S^{r-1}} \int_\delta^\infty |R(y)| e^{-t^2/2\epsilon} t^{r-1} dt dv_{S^{r-1}},$$

where c' is a constant that does not depend on ϵ. Since $R(y)$ is piecewise polynomial, we can find constants $a_0, \dots, a_N$ so that for $|y| > \delta$, we have $|R(y)| \le \sum_j a_j |y|^j$. Combining this with the previous estimate, we have

$$|I(\epsilon) - I_0(\epsilon)| \le c^{''} \epsilon^{-r/2} \sum_{j=1}^N a_j \int_\delta^\infty t^{j+r-1} e^{-t^2/2\epsilon} dt,$$

where $c^{''}$ is a constant that does not depend on ϵ. This reduces the problem to estimating integrals of the form $\int_\delta^\infty t^\ell e^{-t^2/2\epsilon} dt$ for $\ell \ge 0$. The following Lemma 6.2.9 shows that such an integral is bounded by a function of the form $p(\sqrt{2\epsilon}) e^{-\delta^2/(4\epsilon)}$, where p is a polynomial of degree $\ell + 1$. The result follows. $\square$

Lemma 6.2.9. *The integral $I_n^\delta(a) := \int_\delta^\infty x^n e^{-ax^2} dx$, $a, \delta > 0$, $n \in \mathbb{N}$, is bounded from above by a function of the form $p_n(1/\sqrt{a})e^{-\frac{\delta^2 a}{2}}$, where p_n is a polynomial of degree $n+1$.*

Proof. The claim is shown by induction on n. By substituting $x = \sqrt{a}^{-1} y$, we obtain

$$\begin{aligned}(I_0^\delta(a))^2 &= \left(\sqrt{a}^{-1}\int\limits_{\sqrt{a}\delta}^{\infty} e^{-y^2}dy\right)^2 = \left(\frac{1}{2\sqrt{a}}\int\limits_{\mathbb{R}\setminus[-\sqrt{a}\delta,\sqrt{a}\delta]} e^{-y^2}dy\right)^2 \\ &= \frac{1}{4a}\int\limits_{\mathbb{R}^2\setminus[-\sqrt{a}\delta,\sqrt{a}\delta]^2} e^{-(x^2+y^2)}dxdy \leq \frac{1}{4a}\int\limits_{\mathbb{R}^2\setminus B_{\sqrt{a}\delta}(0)} e^{-(x^2+y^2)}dxdy,\end{aligned}$$

where $B_{\sqrt{a}\delta}(0)$ denotes the ball of radius $\sqrt{a}\delta$, centered at the origin. By passing to polar coordinates, the integral becomes

$$(I_0^\delta(a))^2 \leq \frac{1}{4a}\int\limits_0^{2\pi}\int\limits_{\delta\sqrt{a}}^{\infty}\frac{d}{dr}\left[-\frac{1}{2}e^{-r^2}\right]drd\phi = \frac{1}{4a}\int\limits_0^{2\pi}\frac{1}{2}e^{-\delta^2 a}d\phi = \frac{\pi}{4a}(e^{-\frac{\delta^2 a}{2}})^2.$$

For $n = 1$, we can directly compute

$$I_1^\delta(a) = \int_\delta^\infty xe^{-ax^2}dx = -\frac{1}{2a}\int_\delta^\infty \frac{d}{dx}\left[e^{-ax^2}\right]dx = \frac{1}{2a}e^{-a\delta^2} \leq \frac{1}{2a}e^{-\frac{a\delta^2}{2}}.$$

Thus, the claim holds for $n = 0, 1$. Now, let $n \geq 2$ and suppose the claim holds for $n-2$. We integrate by parts.

$$\begin{aligned}I_n^\delta(a) &= \int_\delta^\infty -\frac{x^{n-1}}{2a}\cdot\frac{d}{dx}\left[e^{-ax^2}\right]dx \\ &= \left[-\frac{x^{n-1}}{2a}\cdot e^{-ax^2}\right]_{x=\delta}^{\infty} + \int_\delta^\infty \frac{(n-1)x^{n-2}}{2a}\cdot e^{-ax^2}dx \\ &= \frac{\delta^{n-1}}{2a}e^{-a\delta^2} + \frac{n-1}{2a}I_{n-2}^\delta(a) \\ &\leq \left(\frac{\delta^{n-1}}{2a} + \frac{n-1}{2a}p_{n-2}(a^{-1/2})\right)e^{-\frac{a\delta^2}{2}}.\end{aligned}$$

Setting $p_n(a^{-1/2}) = \left(\frac{\delta^{n-1}}{2a} + \frac{n-1}{2a}p_{n-2}(a^{-1/2})\right)$ yields the claim. □

We now want to apply Lemma 6.2.8 to Q^η. It remains to show that $Q^\eta(y)$ is regular near 0, and to compute the polynomial $Q^\eta_0(y)$ which agrees with $Q^\eta(y)$ near the origin. We will make use of the local normal form we found in Section 2.5. Analogous statements in the symplectic setting can be found in [JK95, Sections 5, 7, 8].

Proposition 6.2.10. *Suppose that* 0 *is a regular value of* Ψ. *Then* $Q^\eta(y)$ *is regular in some neighborhood of* 0, *and on this neighborhood it coincides with the polynomial* $Q^\eta_0(y)$ *given by*

$$Q^\eta_0(y) = i^r(2\pi)^{r/2}\int_{\Psi^{-1}(0)} q^*(\alpha_0\wedge\eta_0)\wedge e^{iq^*d\alpha_0 - iy(F_\theta)}\Omega,$$

where θ *is a* G*-invariant basic connection form on the* G*-bundle* $q:\Psi^{-1}(0)\to\Psi^{-1}(0)/G$, F_θ *denotes its curvature form,* $\Omega=\theta_1\wedge\ldots\wedge\theta_r$ *is the volume form on the* G*-orbits defined by* θ, $\eta_0\in H(\Psi^{-1}(0)/G,\mathcal{F}_0)$ *represents* $i_0^*\eta\in H_G(\Psi^{-1}(0),\mathcal{F})$, *where the inclusion* $\Psi^{-1}(0)\hookrightarrow M$ *is denoted by* i_0, *and* α_0 *denotes the induced contact form on* $\Psi^{-1}(0)/G$. *Here,* $\Psi^{-1}(0)$ *is endowed with the orientation induced by the volume form* $q^*(\alpha_0\wedge(d\alpha_0)^{n-r})\wedge\Omega$.

In particular, with n_0 *denoting the order of the regular isotropy of the action of* G *on* $\Psi^{-1}(0)$, *we have*

$$\int_{M//G}\alpha_0\wedge\eta_0\wedge e^{id\alpha_0} = \frac{n_0}{i^r(2\pi)^{r/2}\mathrm{vol}\,G}\mathbf{F}\left(\Pi_*(\eta\wedge e^{id_G\alpha})\right)(0).$$

Proof. Recall that

$$Q^\eta(y) = \mathbf{F}\left[\Pi_*(\eta\wedge e^{id_G\alpha})\right](y) = \frac{1}{(2\pi)^{r/2}}\int_M\int_{\mathfrak{g}}\alpha\wedge\eta(\phi)\wedge e^{id_G\alpha - i\langle y,\phi\rangle}d\phi$$

By Lemma 6.2.3, when y is sufficiently small, we may replace the integral over M by an integral over $U\subset M$, where U is a neighborhood of $\Psi^{-1}(0)$. Using the normal form of Proposition 2.5.4, we see that for small y

$$Q^\eta(y) = \frac{1}{(2\pi)^{r/2}}\int\limits_{\mathfrak{g}}\int\limits_{\Psi^{-1}(0)\times B_h}\alpha\wedge\eta(\phi)\wedge e^{id_G\alpha - i\langle y,\phi\rangle}d\phi,$$

where $\Psi^{-1}(0)\times B_h$ is canonically oriented by the contact volume form. Consider the projection $\pi:\Psi^{-1}(0)\times B_h\to\Psi^{-1}(0)\times\{0\}$ and the inclusion $i:\Psi^{-1}(0)\times\{0\}\to\Psi^{-1}(0)\times B_h$. Then $i\circ\pi:\Psi^{-1}(0)\times B_h\to\Psi^{-1}(0)\times B_h$ is $G\times T$-equivariantly homotopic to the identity and, hence, i induces an isomorphism $H_G(\Psi^{-1}(0) \times B_h, \mathcal{F} \times \{\text{pt.}\}) \cong H_G(\Psi^{-1}(0) \times \{0\}, \mathcal{F} \times \{0\}) = H_G(\Psi^{-1}(0), \mathcal{F})$. Since $[q^*\eta_0] = [i_0^*\eta]$ by definition of η_0, it is $[\pi^*q^*\eta_0] = [\eta|_{\Psi^{-1}(0)\times B_h}]$. Therefore, there is a $\gamma \in C_G(\Psi^{-1}(0) \times B_h, \mathcal{F} \times \{\text{pt.}\})$ such that $\eta - \pi^*q^*\eta_0 = d_G\gamma$. Set

$$\begin{aligned}\Delta := Q^\eta(y) - \frac{1}{(2\pi)^{r/2}} \int\limits_{\mathfrak{g}} \int\limits_{\Psi^{-1}(0)\times B_h} \alpha \wedge \pi^*q^*\eta_0 \wedge e^{id_G\alpha - i\langle y,\phi\rangle} d\phi \\ = \frac{1}{(2\pi)^{r/2}} \int\limits_{\mathfrak{g}} \int\limits_{\Psi^{-1}(0)\times B_h} \alpha \wedge d_G\gamma \wedge e^{id_G\alpha - i\langle y,\phi\rangle} d\phi.\end{aligned}$$

Since $d_G d_G \alpha = 0$ and $d_G\phi_j = 0$, we have $d_G\gamma \wedge e^{id_G\alpha - i\langle y,\phi\rangle} = d_G\left(\gamma \wedge e^{id_G\alpha - i\langle y,\phi\rangle}\right)$. The integral over $\Psi^{-1}(0) \times B_h$ picks up only those components of the basic form $d_G\left(\gamma \wedge e^{id_G\alpha - i\langle y,\phi\rangle}\right)$ of degree $2n$, so we can pass to the ordinary differential. We obtain

$$\begin{aligned}(2\pi)^{r/2}\Delta &= \int\limits_{\mathfrak{g}} \int\limits_{\Psi^{-1}(0)\times B_h} \alpha \wedge d(\gamma \wedge e^{id_G\alpha - i\langle y,\phi\rangle}) d\phi \\ &= \int\limits_{\mathfrak{g}} \int\limits_{\Psi^{-1}(0)\times B_h} -d\left(\alpha \wedge \gamma \wedge e^{id_G\alpha - i\langle y,\phi\rangle}\right) + d\alpha \wedge \gamma \wedge e^{id_G\alpha - i\langle y,\phi\rangle} d\phi.\end{aligned}$$

The second summand is basic, hence, its top degree part is zero. Thus, the whole summand vanishes under integration. By Stokes' Theorem, denoting the boundary of B_h by S_h, we obtain

$$(2\pi)^{r/2}\Delta = -\int\limits_{\mathfrak{g}} \int\limits_{\Psi^{-1}(0)\times S_h} \alpha \wedge \gamma \wedge e^{id_G\alpha - i\langle y,\phi\rangle} d\phi.$$

Write $\gamma(\phi) = \sum_J \gamma_J \phi^J$. As in the proof of Lemma 6.2.3, the previous equation becomes

$$(2\pi)^{r/2}\Delta = -(2\pi)^r \sum_J i^J \frac{\partial}{\partial y^J} \int\limits_{\Psi^{-1}(0)\times S_h} \alpha \wedge \gamma_J \wedge e^{id\alpha} \delta(-\Psi - y).$$

Recall that the local normal form of the moment map is given by $\Psi(p,z) = z$. Then, for sufficiently small y, $\delta(-\Psi - y)$ is supported away from S_h and it follows that $\Delta = 0$. This means that, for sufficiently small y,

$$Q^\eta(y) = \frac{1}{(2\pi)^{r/2}} \int\limits_{\mathfrak{g}} \int\limits_{\Psi^{-1}(0)\times B_h} \alpha \wedge \pi^*q^*\eta_0 \wedge e^{id\alpha + i\langle -\Psi - y,\phi\rangle} d\phi$$

$$= (2\pi)^{r/2} \int\limits_{\Psi^{-1}(0)\times B_h} \alpha \wedge \pi^* q^* \eta_0 \wedge e^{id\alpha} \delta(-\Psi - y)$$

$$= (2\pi)^{r/2} \int\limits_{\Psi^{-1}(0)\times B_h} (q^*\alpha_0 + z(\theta)) \wedge q^*\eta_0 \wedge e^{idq^*\alpha_0 + idz(\theta) + iz(d\theta)} \delta(-z - y),$$

where we substituted the normal form of α in the last line. Let j index an orthonormal basis of $\mathfrak{g}$ and the dual basis of $\mathfrak{g}^*$. Denote the according components of θ by θ_j and the coordinate functions of $z \in \mathfrak{g}^*$ by z_j. Set $\Omega = \theta_1 \wedge ... \wedge \theta_r$ and $[dz] = dz_1 \wedge ... \wedge dz_r$. Note that Ω is a volume form on the G-orbits. We only obtain a non-zero contribution from $e^{idz(\theta)}$ from the term containing $(idz(\theta))^r = r! i^r (-1)^{r(r+1)/2} \Omega \wedge [dz]$ since all the factors dz_j must appear. Additional factors of θ will wedge to 0 with Ω, so $z(\theta)$ does not contribute to the integral. We obtain

$$Q^\eta(y) = i^r (-1)^{\frac{r(r+1)}{2}} (2\pi)^{r/2} \int\limits_{\Psi^{-1}(0)\times B_h} q^*(\alpha_0 \wedge \eta_0) \wedge e^{iq^* d\alpha_0 + iz(d\theta)} \Omega \delta(-z - y)[dz]. \tag{6.10}$$

The orientation on $\Psi^{-1}(0) \times B_h$ is canonically given by the contact volume form

$$(q^*\alpha_0 + z(\theta)) \wedge (q^* d\alpha_0 + d(z(\theta)))^n = (-1)^{\frac{r(r-1)}{2}} \tfrac{n!}{(n-r)!} q^*\alpha_0 \wedge (q^* d\alpha_0 + z(d\theta))^{n-r} \wedge \Omega \wedge [dz].$$

For $z = 0$, this volume form differs by a factor of $(-1)^{r(r-1)/2} \frac{n!}{(n-r)!}$ from the volume form $\nu := q^*(\alpha_0 \wedge d\alpha_0^{n-r}) \wedge \Omega \wedge [dz]$. Hence, when changing the orientation of $\Psi^{-1}(0) \times B_h$ in Equation 6.10 to that induced by ν, denoting the thusly oriented manifold by $(\Psi^{-1}(0) \times B_h)^\nu$, we obtain a factor $(-1)^{r(r-1)/2}$ and obtain

$$Q^\eta(y) = i^r (2\pi)^{r/2} \int_{(\Psi^{-1}(0)\times B_h)^\nu} q^*(\alpha_0 \wedge \eta_0) \wedge e^{iq^* d\alpha_0 + iz(d\theta)} \Omega \delta(-z - y)[dz].$$

On B_h, we consider the orientation induced by $[dz]$ and we endow $\Psi^{-1}(0)$ with the orientation induced by $q^*(\alpha_0 \wedge d\alpha_0^{n-s}) \wedge \Omega$ so that their product gives the orientation of $(\Psi^{-1}(0) \times B_h)^\nu$. We continue our computation by integrating over B_h and obtain

$$Q^\eta(y) = i^r (2\pi)^{r/2} \int_{\Psi^{-1}(0)} q^*(\alpha_0 \wedge \eta_0) \wedge e^{iq^* d\alpha_0 - iy(d\theta)} \Omega$$

$$= i^s (2\pi)^{s/2} \int_{\Psi^{-1}(0)} q^*(\alpha_0 \wedge \eta_0) \wedge e^{iq^* d\alpha_0 - iy(F^\theta)} \Omega,$$

where we have replaced the term $d\theta$ by the curvature form $F_\theta = d\theta + \frac{1}{2}[\theta, \theta]$, which, as above, does not change the value of the integral, because the additional factors of θ will wedge to 0 with Ω. Therefore we obtain the claimed expression for $Q_0^\eta(y)$. This is obviously a polynomial in y, since only finitely many terms in the power series expansion of $e^{-iy(F_\theta)}$ are non-zero.

To compute the expression for $y = 0$, we need the following lemma.

Lemma 6.2.11. *Let $x \in \Psi^{-1}(0)$ be a point with regular isotropy. Then $\int_{G \cdot x} \Omega = \mathrm{vol}\,(G)/n_0$. In particular, $\int_{\Psi^{-1}(0)} q^*\omega \wedge \Omega = \mathrm{vol}\,(G)/n_0 \int_{\Psi^{-1}(0)/G} \omega$ for every $\omega \in \Omega(\Psi^{-1}(0)/G)$.*

Proof. Consider the n_0-fold cover $f : G \to G \cdot x$, $g \mapsto g \cdot x$. The connection form θ is uniquely determined on the G-orbits, because it, by definition, maps the fundamental vector field $X_{\Psi^{-1}(0)}$ induced by $X \in \mathfrak{g}$ to X. In particular, with $l_{g^{-1}}$ denoting left multiplication by g^{-1}, we have $(f^*\Omega)_g = l_{g^{-1}}^*[dz]$ so that $\int_G f^*\Omega = \mathrm{vol}\,(G)$. $G \cdot x$ is compact, so there are finitely many open subsets $U_1, ..., U_N \subset G \cdot x$ such that $G \cdot x = \cup_{i=1}^N U_i$ and $f^{-1}(U_i) = \dot{\cup}_{j=1}^{n_0} V_j^i$ for $V_j^i \subset G$ pairwise disjoint and open such that $f|_{V_j^i} : V_j^i \to U_i$ is a diffeomorphism. Let φ_i denote a partition of unity subject to the open cover $\{U_i\}$. Applying the transformation formula yields that, for any choice of j_i, we have

$$\int_{G \cdot x} \Omega = \sum_{i=1}^N \int_{U_i} \varphi_i \Omega = \sum_{i=1}^N \int_{V_{j_i}^i} f|_{V_{j_i}^i}^* (\varphi_i \Omega).$$

In particular, we have

$$n_0 \int_{G \cdot x} \Omega = \sum_{i=1}^N \sum_{j=1}^{n_0} \int_{V_j^i} f|_{V_j^i}^* (\varphi_i \Omega).$$

Since $\mathrm{supp}\,\varphi_i \subset U_i$, it is $\mathrm{supp}\,\varphi_i \circ f \subset \cup_{j=1}^{n_0} V_j^i$ so that we obtain

$$n_0 \int_{G \cdot x} \Omega = \sum_{i=1}^N \int_{\dot{\cup}_j V_j^i} f|_{\dot{\cup}_j V_j^i}^* (\varphi_i \Omega) = \sum_{i=1}^N \int_G f^*(\varphi_i \Omega) = \int_G f^*\Omega = \mathrm{vol}\,(G).$$

The set of regular points in $\Psi^{-1}(0)$ is an open and dense subset such that its complement is a zero set. This yields the remaining claim. $\square$

We resume the proof of Proposition 6.2.10. $\Psi^{-1}(0)/G$ is canonically oriented by $\alpha_0 \wedge d\alpha_0^{n-r}$. Hence, together with above orientation on $\Psi^{-1}(0)$, the projection q induces the same orientation on the fibers as Ω. Ω integrates to $\mathrm{vol}\,(G)/n_0$ over the fiber by Lemma 6.2.11, so, when $y = 0$, the previous equation becomes

$$\mathbf{F}(\Pi_*(\eta \wedge e^{id_G\alpha}))(0) = i^s(2\pi)^{s/2}\mathrm{vol}\,(G)/n_0 \int_{\Psi^{-1}(0)/G} \alpha_0 \wedge \eta_0 \wedge e^{id\alpha_0}. \qquad \square$$

Proposition 6.2.12. *Let $\Theta \in H^4(M_0, \mathcal{F})$ be the class corresponding to the class $-\frac{<\phi,\phi>}{2} \in H^4_G(\Psi^{-1}(0), \mathcal{F}) \simeq H^4(M_0, \mathcal{F}_0)$ under the Cartan map. Then*

$$I_0^\eta(\epsilon) = \tfrac{1}{n_0} \int_{M_0} \alpha_0 \wedge \eta_0 \wedge e^{\epsilon\Theta + id\alpha_0}.$$

Proof. The Cartan map yields $-|F_\theta|^2/2 = q^*\Theta$ in cohomology. By Proposition 6.2.10,

$$\begin{aligned}
I_0^\eta(\epsilon) &= \frac{1}{(2\pi\epsilon)^{r/2}\mathrm{vol}\,(G)} \int_{\Psi^{-1}(0)\times\mathfrak{g}^*} q^*(\alpha_0 \wedge \eta_0) \wedge e^{iq^*d\alpha_0 - iy(F_\theta) - |y|^2/2\epsilon} \wedge \Omega \; dy \\
&= \frac{1}{(2\pi\epsilon)^{r/2}\mathrm{vol}\,(G)} \int_{\Psi^{-1}(0)} q^*(\alpha_0 \wedge \eta_0) \wedge e^{iq^*d\alpha_0} \wedge \Omega \int_{\mathfrak{g}^*} e^{-iy(F_\theta) - |y|^2/2\epsilon} \; dy \\
&= \frac{1}{\mathrm{vol}\,(G)} \int_{\Psi^{-1}(0)} q^*(\alpha_0 \wedge \eta_0) e^{iq^*d\alpha_0 - \epsilon|F_\theta|^2/2} \wedge \Omega \quad \text{by Gaussian integration} \\
&= \tfrac{1}{n_0} \int_{M_0} \alpha_0 \wedge \eta_0 \wedge e^{id\alpha_0 + \epsilon\Theta}
\end{aligned}$$

since Ω integrates to $\mathrm{vol}\,(G)/n_0$ over the fiber, cf. Lemma 6.2.11. $\square$

Combining Lemma 6.2.8 with Proposition 6.2.12, we obtain Theorem 6.2.7.

6.2.3 The Residue Formula

The main ingredients in the proof of Theorem 6.2.7 are the result that $\mathbf{F}(\int_M \alpha \wedge \eta \wedge e^{id_G\alpha})$ is piecewise polynomial and smooth in some neighborhood of 0, and a particular expression for the polynomial this distribution coincides with near 0. We will make use of these established facts and apply a result of Jeffrey-Kirwan in order to obtain the last of our main theorems.

Theorem 6.2.13. *Let (M,α) be a compact K-contact manifold and G a torus that acts on M, preserving α. Assume that 0 is a regular value of the contact moment map Ψ so that the order n_0 of the regular isotropy of the G-action on $\Psi^{-1}(0)$ is finite. Denote by $C_j \subset C = \operatorname{Crit}\Psi$ the connected components of the critical set of Ψ, by $e_G(\nu C_j, \mathcal{F})$ the equivariant basic Euler classes of their normal bundles, by α_0 the induced contact form on the contact quotient M_0, and by $\eta_0 \in H(M_0, \mathcal{F}_0)$ the image of $\eta \in H_G(M, \mathcal{F})$ under the basic Kirwan map. Then we have*

$$\int_{M_0} \alpha_0 \wedge \eta_0 \wedge e^{id\alpha_0} = \frac{n_0}{\operatorname{vol}(G)} \operatorname{jkres}\left(\sum_{C_j \subseteq C} e^{-i\langle \Psi(C_j), \phi\rangle} \int_{C_j} \frac{i_j^*\left(\alpha \wedge \eta(\phi) \wedge e^{id\alpha}\right)}{e_G(\nu C_j, \mathcal{F})} [d\phi] \right). \tag{6.11}$$

Before we prove this theorem, we briefly recall the Jeffrey-Kirwan residue operation.

Proposition 6.2.14 ([JK97, Proposition 3.2])**.** *Let $\Lambda \subset \mathfrak{g}$ be a non-empty open cone and suppose that $\beta_1, \ldots, \beta_N \in \mathfrak{g}^*$ all lie in the dual cone Λ^*. Suppose that $\lambda \in \mathfrak{g}^*$ does not lie in any cone of dimension at most $r-1$ spanned by a subset of $\{\beta_1, \ldots, \beta_N\}$. Let $\{\phi_1, \ldots, \phi_r\}$ be any system of coordinates on $\mathfrak{g}$ and let $d\phi = d\phi_1 \wedge \cdots \wedge d\phi_r$ be the associated volume form. Then there exists a residue operation* $\operatorname{jkres}^\Lambda$ *defined on meromorphic differential forms of the form*

$$h(\phi) = \frac{q(\phi) e^{i\lambda(\phi)}}{\prod_{j=1}^N \beta_j(\phi)} d\phi \tag{6.12}$$

where $q(\phi)$ is a polynomial. The operation $\operatorname{jkres}^\Lambda$ *is linear in its argument and is characterized uniquely by the following properties:*

(i) If $\{\beta_1, \ldots, \beta_N\}$ does not span $\mathfrak{g}^*$ as a vector space then

$$\operatorname{jkres}^\Lambda \left(\frac{\phi^J e^{i\lambda(\phi)}}{\prod_{j=1}^N \beta_j(\phi)} d\phi \right) = 0 \tag{6.13}$$

(ii) For any multi-index J, we have

$$\operatorname{jkres}^\Lambda \left(\frac{\phi^J e^{i\lambda(\phi)}}{\prod_{j=1}^N \beta_j(\phi)} d\phi \right) = \sum_{m\geq 0} \lim_{s\to 0^+} \operatorname{jkres}^\Lambda \left(\frac{\phi^J (i(\lambda(\phi))^m e^{is\lambda(\phi)}}{m! \prod_{j=1}^N \beta_j(\phi)} d\phi \right). \tag{6.14}$$

(iii) *The limit*

$$\lim_{s\to 0^+} \operatorname{jkres}^\Lambda\left(\frac{\phi^J e^{is\lambda(\phi)}}{\prod_{j=1}^N \beta_j(\phi)} d\phi\right) \tag{6.15}$$

is zero unless $N - |J| = r$.

(iv) *If* $N = r$ *and* $\{\beta_1, \dots, \beta_r\}$ *spans* $\mathfrak{g}^*$ *as a vector space, then*

$$\operatorname{jkres}^\Lambda\left(\frac{e^{i\lambda(\phi)}}{\prod_{j=1}^r \beta_j(\phi)} d\phi\right) = 0 \tag{6.16}$$

unless λ *is in the cone spanned by* $\{\beta_1, \dots, \beta_r\}$. *If* λ *is in this cone, then the residue is equal to* $\bar{\beta}^{-1}$, *where* $\bar{\beta}$ *is the determinant of an* $r \times r$ *matrix whose columns are the coordinates of* $\beta_1, \dots, \beta_r$ *with respect to any orthonormal basis defining the same orientation as* $\beta_1, \dots, \beta_r$.

Remark 6.2.15. Note that above axioms do indeed determine $\operatorname{jkres}^\Lambda$ completely. If $\{\beta_1, \dots, \beta_N\}$ spans $\mathfrak{g}^*$ as a vector space, then any coordinate ϕ_k can be written as $\sum_{j=1}^N c_j \beta_j(\phi)$ for some $c_j \in \mathbb{R}$. This means, however, that

$$\frac{\phi_k}{\prod_{j=1}^N \beta_j(\phi)} = \sum_{j=1}^N \frac{c_j}{\prod_{l\neq j}^N \beta_l(\phi)}.$$

Since the residue vanishes by the first axiom if the linear forms in the denominator do not span all of $\mathfrak{g}^*$, we can inductively replace the residue of $\frac{\phi^J e^{i\lambda(\phi)}}{\prod_{j=1}^N \beta_j(\phi)}$ with $N - |J| = r$ by a linear combination of residues of functions of the same form with $J = 0$ and $N = r$, cf. [JK97, Proposition 3.2].

We further recall the following.

Proposition 6.2.16 (Jeffrey-Kirwan)**.** *Consider a function* $h : \mathfrak{g} \to \mathbb{C}$ *and assume that* $\mathbf{F}h$ *is compactly supported. Then* $h[d\phi]$ *lies in the domain of* $\operatorname{jkres}^\Lambda$, *and* $\operatorname{jkres}^\Lambda(h[d\phi])$ *is independent of the cone* Λ. *Suppose further that* $\mathbf{F}h$ *is represented by a smooth function in a neighborhood of 0. Then* $\operatorname{jkres}^\Lambda(h[d\phi]) = i^{-r}\mathbf{F}h(0)/(2\pi)^{r/2}$.

Proof. Consider $\Gamma_{\mathbf{F}h} := \{\xi \in \mathfrak{g} \mid e^{(\cdot,\xi)}\mathbf{F}h \text{ is a tempered distribution}\}$. Since $\mathbf{F}h$ is compactly supported, we have $\Gamma_{\mathbf{F}h} = \mathfrak{g}$, in particular, it contains $-\mathrm{Int}\Lambda$. The proposition is then obtained by combining [JK95, Propositions 8.6, 8.7]. ☐

Theorem 6.2.13 is now a consequence of our localization formula (Theorem 6.1.9) and Proposition 6.2.10.

Proof of Theorem 6.2.13. We know from Proposition 6.2.10 that $\mathbf{F}(\Pi_*\eta \wedge e^{id_G\alpha})$ is smooth near 0. Furthermore, $\mathbf{F}\left(\Pi_*(\eta \wedge e^{id_G\alpha})\right)$ is compactly supported by Lemma 6.2.3. By Proposition 6.2.16, the residue is independent of the cone Λ and can be expressed as $\mathrm{jkres}^\Lambda(\Pi_*(\eta \wedge e^{id_G\alpha})) = i^{-r}(2\pi)^{-r/2}\mathbf{F}\left(\Pi_*(\eta \wedge e^{id_G\alpha})\right)(0)$. By Proposition 6.2.10, we then obtain

$$\int_{M_0} \alpha_0 \wedge \eta_0 \wedge e^{id\alpha_0} = \frac{n_0}{\operatorname{vol} G}\mathrm{jkres}^\Lambda(\Pi_*(\eta \wedge e^{id_G\alpha})).$$

Using the expression for $\Pi_*(\eta\wedge e^{id_G\alpha})$ provided by Theorem 6.1.9, namely, Equation (6.7), we obtain the claimed formula. $\square$

Remark 6.2.17. Recall that we obtained a surjectivity result for the basic Kirwan map, Theorem 4.1.1. Since basic cohomology satisfies Poincaré duality (see Lemma 3.1.2), the Residue Formula in Theorem 6.2.13 in principle provides a method to compute the kernel of the basic Kirwan map, and therefore allows one to compute the basic cohomology ring of the contact quotient.

6.3 Examples

6.3.1 Boothby-Wang Fibrations

We now explain how, for certain symplectic manifolds, the known Localization and Residue Formula may be recovered from our theorems.

As in Section 4.2.1, we consider a Boothy-Wang fibration $p : M \to N$ with connection form α and we assume that a compact Lie group G acts on M, preserving α, and thus inducing an action on N. Recall that p^* gives isomorphisms $H(M, \mathcal{F}) \cong H(N)$ and $H_G(M, \mathcal{F}) \cong H_G(N)$.

Denote the period of the flow ψ_t of the Reeb vector field $R \in \mathfrak{X}(M)$ by $2\pi/\tau$, i.e., $\psi_t = \psi_{t+2\pi/\tau}$ for every $t \in \mathbb{R}$. We then have the following.

Proposition 6.3.1. *α integrates to $2\pi/\tau$ over any Reeb orbit. In particular, for any basic form $p^*\eta \in \Omega(M, \mathcal{F})$, fiberwise integration yields*

$$\int_M \alpha \wedge p^*\eta = 2\pi/\tau \int_N \eta.$$

Proof. Recall that the flow is defined such that $R(\psi_t(x)) = \frac{d}{dt}\psi_t(x)$. Consider the diffeomorphism $F_x : S^1 \to \{\psi_t(x)\}$, $e^{it\tau} \mapsto \psi_t(x)$. Furthermore, $\Phi : (0, 2\pi/\tau) \to S^1$, $t \mapsto e^{it\tau}$ parametrizes $S^1 \setminus \{\text{pt.}\}$. The canonical basis vector on S^1 is given by

$$\tfrac{\partial}{\partial t}(x) = d\Phi_{\Phi^{-1}(x)}(1) = i\tau e^{i\Phi^{-1}(x)\tau}.$$

For $p_0 = e^{it_0\tau} \in S^1$, let γ denote the curve in $(0, 2\pi/\tau)$ given for sufficiently small s by $\gamma(s) = t_0 + s$. Then $\dot\gamma(s) \equiv 1$. We compute

$$\begin{aligned}(dF_x)_{p_0}\left(\tfrac{\partial}{\partial t}(p_0)\right) &= d(F_x \circ \Phi)_{\Phi^{-1}(p_0)}(1) = \tfrac{d}{ds}|_{s=0} F_x \circ \Phi(\gamma(s)) = \tfrac{d}{ds}|_{s=0} F_x(e^{i(t_0+s)\tau}) \\ &= \tfrac{d}{ds}|_{s=0}\psi_{t_0+s}(x) = \tfrac{d}{ds}|_{s=0}\psi_s(\psi_{t_0}(x)) = R(\psi_0(\psi_{t_0}(x))) \\ &= R(\psi_{t_0}(x)) = R(F_x(p_0)).\end{aligned}$$

It follows that $F_x^*\alpha(\frac{\partial}{\partial t}(p_0)) = \alpha(R(F_x(p_0))) = 1$, whence we obtain $F_x^*\alpha = dt$. The transformation formula then yields

$$\int_{\{\psi_t(x)\}} \alpha = \int_{F_x(S^1)} \alpha = \int_{S^1} F_x^*\alpha = \int_0^{2\pi/\tau} dt = 2\pi/\tau. \qquad \square$$

It now follows from Theorem 6.1.9 and Proposition 6.3.1 that we recover the standard localization theorem [AB84] for integral symplectic manifolds.

Theorem 6.3.2. *Suppose that N is a symplectic manifold with a Hamiltonian action of the torus G and suppose furthermore that the symplectic form on N is integral and that the G-action lifts to the S^1-bundle (M, α) in the Boothby-Wang fibration $p : M \to N$, preserving α. Then for any $\eta \in H_G(N)$, with $e(\nu F)$ denoting the (ordinary) equivariant Euler class of a connected component $F \subset N^G$, we have*

$$\int_N \eta = \sum_{F \subseteq N^G} \int_F \frac{i_F^*\eta}{e(\nu F)}.$$

Proof. We again denote the contact moment map by Ψ and the connected components of its critical set by C_j. Note that every C_j is $T = S^1$-invariant so that under the projection p, every C_j is exactly mapped to a connected component F_j of the fixed point set N^G. We can identify νC_j with $p^*\nu F_j = \{(y,v) \in C_j \times \nu F_j \mid p(y) = \pi(v)\}$ via $\nu_y C_j \in v \mapsto (y, dp(v))$. Let P_j denote the bundle of oriented orthonormal frames of νF_j, and set $\bar{p} : p^*P_j \to P_j$, $\bar{p}(y,v) := v$. Given a G-invariant connection form θ on P_j, $\bar{p}^*\theta$ is a basic G-invariant connection form on the bundle of oriented orthonormal frames of $p^*\nu F_j$. For $X \in \mathfrak{g}$, we can then compute $p^*(\mathrm{Pf}(F^\theta - \iota_X\theta)) = \mathrm{Pf}(F^{\bar{p}^*\theta} - \iota_X\bar{p}^*\theta)$ (see, e.g., [Bau09, Satz 6.3] for a detailed computation that the characteristic homomorphism is compatible with pullback in the non-equivariant case that directly carries over to our setting). Thus, we obtain $p^*e_G(\nu F_j) = e_G(\nu C_j, \mathcal{F})$, where the right hand side denotes the equivariant basic Euler class of νC_j. Applying Theorem 6.1.9 and Proposition 6.3.1, we have

$$\begin{aligned}\int_N \eta = \tau/(2\pi)\int_M \alpha\wedge p^*\eta &= \tau/(2\pi)\sum_{C_j\subseteq \mathrm{Crit}\,\mu}\int_{C_j}\frac{i_j^*(\alpha\wedge p^*\eta)}{e_G(\nu C_j,\mathcal{F})}\\ &= \sum_{F\subseteq N^G}\int_F \frac{i_F^*\eta}{e_G(\nu F)}.\end{aligned}$$

□

Suppose that 0 is a regular value of the contact moment map Ψ. Then 0 is also a regular value of the symplectic moment map μ that pulls back to $-\Psi$ and vice versa. Denote by M_0 and N_0 the contact and symplectic quotients, respectively. We have the commutative diagram

$$\begin{array}{ccc} H_G(M,\mathcal{F}) & \overset{\cong}{\to} & H_G(N) \\ \downarrow & & \downarrow \\ H(M_0,\mathcal{F}_0) & \overset{\cong}{\to} & H(N_0) \end{array}$$

In exactly the same manner as the proof of Theorem 6.3.2, we also recover the usual Jeffrey-Kirwan residue theorem [JK95, JK97].

Theorem 6.3.3. *Suppose that N is a symplectic manifold with a Hamiltonian action of a torus G. Suppose furthermore that the symplectic form ω on N is integral and that the G-action lifts to the S^1-bundle (M,α) in the Boothby-Wang fibration $p : M \to N$, preserving α. Let μ denote the symplectic moment map that* *pulls back to* $-\Psi$ *and assume that 0 is a regular value of* μ. *Denote the induced symplectic form on the symplectic quotient* N_0 *by* ω_0. *For* $\eta \in H_G(N)$, *we denote its image under the Kirwan map by* η_0. *We have*

$$\int_{N_0} \eta_0 \wedge e^{i\omega_0} = \frac{n_0}{\mathrm{vol}\,(G)} \mathrm{jkres} \left(\sum_{F \subseteq N^G} e^{i\langle \mu(F), \phi \rangle} \int_F \frac{i_F^* \eta(\phi) \wedge e^{i\omega}}{e_G(\nu F)} [d\phi] \right).$$

Proof. Recall from the proof of Theorem 6.3.2 that $p^* e_G(\nu F_j) = e_G(\nu C_j, \mathcal{F})$, where C_j denotes the connected component of $\mathrm{Crit}\,\Psi$ that is mapped to the connected component $F_j \subset N^G$. Furthermore, we have $p^*\mu = -\Psi$, in particular, $\mu(F_j) = -\Psi(C_j)$, and $p_0^*\omega_0 = d\alpha_0$, where $p_0 : M_0 \to N_0$ and α_0 is the induced contact form on M_0. We apply Proposition 6.3.1, which analogously holds on M_0, and Theorem 6.2.13, and obtain:

$$\begin{aligned}
\int_{N_0} \eta_0 \wedge e^{i\omega_0} &= \tau/(2\pi) \int_{M_0} \alpha_0 \wedge p_0^* \eta_0 \wedge e^{id\alpha_0} \\
&= \tau/(2\pi) \frac{n_0}{\mathrm{vol}\,(G)} \mathrm{jkres} \left(\sum_{C_j \subseteq \mathrm{Crit}\,\Psi} e^{i\langle -\Psi(C_j), \phi \rangle} \int_{C_j} \frac{i_{C_j}^* (\alpha \wedge p^* \eta(\phi) \wedge e^{id\alpha}}{e_G(\nu C_j, \mathcal{F})} [d\phi] \right) \\
&= \frac{n_0}{\mathrm{vol}\,(G)} \mathrm{jkres} \left(\sum_{F \subseteq N^G} e^{i\langle \mu(F), \phi \rangle} \int_F \frac{i_F^* \eta(\phi) \wedge e^{i\omega}}{e_G(\nu F)} [d\phi] \right).
\end{aligned}$$

□

Remark 6.3.4. Note that we obtain the residue formula as stated in [JK95, JK97], without the sign that was added in [JK98] due to an error in [JK95, Section 5]. The situation in [JK95, Section 5] - in the therein defined notation - describes as follows. The only term from $e^{idz'(\theta)}$ that contributes to the integral is $(idz'(\theta))^s/s! = i^s(-1)^{s(s+1)/2}\Omega \wedge [dz']$, which causes a sign to appear in the computation. The integral is taken over a neighborhood $\mathcal{O}$ of $\mu^{-1}(0)$, which is canonically oriented via the symplectic form $q^*\omega_0 + d(z'(\theta))$. The integral is computed by first taking the integral in $\mathfrak{k}^*$-direction, oriented via $[dz']$, followed by fiberwise integration on $\mu^{-1}(0)$, where the fibers are oriented via Ω. An integral over the symplectic quotient $\mathcal{M}_X$ remains; $\mathcal{M}_X$ is canonically oriented via ω_0. The product of these orientations differs from the canonical orientation on $\mathcal{O}$ by a factor $(-1)^{s(s+1)/2}$. Hence, taking into account this change of orientation removes the additional sign (cf. also the proof of Proposition 6.2.10). For this reason, the formula as stated in [JK95, JK97] is the correct formula to consider.

6.3.2 S^1-Actions on Odd Spheres with Weighted Sasakian Structure

We return to the example presented in Section 4.2.2. Namely, consider $(M, \alpha) = (S^{2n+1}, \alpha_w)$ from Example 2.2.2 with weight $w \in \mathbb{R}^{n+1}$, $w_j > 0$, and let $G = S^1$ act (freely) on S^{2n+1} with weights $\beta = (\beta_0, ...\beta_n) \in \mathbb{Z}^{n+1}$, that is, by $\lambda \cdot z = (\lambda^{\beta_0} z_0, ..., \lambda^{\beta_n} z_n)$.

Recall that the flow of R_w is given by $\psi_t(z) = (e^{itw_0} z_0, ..., e^{itw_n} z_n)$, that the fundamental vector field X corresponding to $1 \in \mathbb{R} \simeq \mathfrak{s}^1$ is given by

$$X(z) = i \left(\sum_{j=0}^{n} \beta_j (z_j \tfrac{\partial}{\partial z_j} - \bar{z}_j \tfrac{\partial}{\partial \bar{z}_j}) \right),$$

and that the contact moment map is

$$\Psi(z) = \frac{\sum_{j=0}^n \beta_j |z_j|^2}{\sum_{j=0}^n w_j |z_j|^2}.$$

We already computed in Lemma 4.2.3 that the equivariant basic cohomology of M is given, as $(S(\mathfrak{g}^*) = \mathbb{R}[u])$-algebra, by

$$H_G(M, \mathcal{F}) \cong H_{\mathfrak{g} \oplus \mathbb{R}R_w}(M) = \frac{\mathbb{R}[u, s]}{\langle \prod_{j=0}^n (\beta_j u + w_j s) \rangle},$$

where (u, s) are dual to (X, R_w).

Lemma 6.3.5. *Set* $\lambda_j := \frac{\beta_j}{w_j}$ *and* $J_j := \{l \in \{0, ..., n\} \mid \lambda_l = \lambda_j\}$. $\operatorname{Crit} \Psi$ *consists of at most* $n+1$ *components* D_j*, specified by* $D_j = \{z \in M \mid z_l = 0 \ \forall \ l \in \{0, ..., n\} \setminus J_j\}$.

If the weights β_j *of the* G*-action are such that* $\lambda_j \neq \lambda_l$ *for every* $j \neq l$*, then* $\operatorname{Crit} \Psi$ *consists of* $n+1$ *circles* $C_j = \{z \in M \mid z_l = 0 \ \forall \ l \neq j\}$*, and* $\Psi(C_j) = \beta_j / w_j = \lambda_j$*. Furthermore,* $H_G(C_j, \mathcal{F}) \cong \mathbb{R}[u]$*, and the restriction* $H_G(M, \mathcal{F}) \to H_G(C_j, \mathcal{F})$ *is given by* $s \mapsto -\beta_j u / w_j$*. If we denote the inclusion* $C_j \to M$ *by* i_j*, then* $\int_{C_j} \iota_j^* \alpha_w = \frac{2\pi}{w_j}$*. The equivariant basic Euler class* e_j *of the normal bundle to* C_j *in* M *is given by*

$$e_j = \left(\frac{u}{2\pi}\right)^n \prod_{k \neq j} (\beta_k - \beta_j w_k / w_j).$$

Proof. For every $z \in \cup D_j$, we have $X(z) = \lambda_j R_w(z)$, which yields $\cup D_j \subset \operatorname{Crit}\Psi$. If $z \in M \setminus \cup D_j$, then there are $k \neq j$ such that $z_k, z_j \neq 0$ and $\lambda_k \neq \lambda_j$. It follows that $\beta_j(z_j \frac{\partial}{\partial z_j} - \bar{z}_j \frac{\partial}{\partial \bar{z}_j}) + \beta_k(z_k \frac{\partial}{\partial z_k} - \bar{z}_k \frac{\partial}{\partial \bar{z}_k}) \neq \lambda \Big(w_j(z_j \frac{\partial}{\partial z_j} - \bar{z}_j \frac{\partial}{\partial \bar{z}_j}) + w_k(z_k \frac{\partial}{\partial z_k} - \bar{z}_k \frac{\partial}{\partial \bar{z}_k}) \Big)$ for every $\lambda \in \mathbb{R}$. Since $(\frac{\partial}{\partial z_l}, \frac{\partial}{\partial \bar{z}_l})_{l=0}^n$ form a basis of $T_z\mathbb{C}^{n+1}$, they are linearly independent at z, hence, $X(z) \notin \mathbb{R}R_w(z)$.

Now suppose that $\lambda_j \neq \lambda_l$ for every $j \neq l$. On C_j, it is $R_w = w_j(x_j \partial_{y_j} - y_j \partial_{x_j})$ and $X = \beta_j(x_j \partial_{y_j} - y_j \partial_{x_j}) = \frac{\beta_j}{w_j} R_w$. $d\alpha_w$ is a 2-form, so $\iota_j^* d\alpha_w = 0$. In $H_{\mathfrak{g} \oplus \mathbb{R}R_w}(C_j)$, we compute

$$0 = [d_{\mathfrak{g} \oplus \mathbb{R}R_w} \alpha_w] = [d\alpha_w - \iota_X \alpha_w u - \iota_{R_w} \alpha_w s] = [-\tfrac{\beta_j}{w_j} u - s],$$

thus obtaining the restriction map $s \mapsto -\beta_j u / w_j$.

$\nu C_j = \operatorname{span}\{\partial_{x_k}, \partial_{y_k} \mid k \neq j\} = \mathbb{C}^n \times C_j$ is a trivial bundle that is the product of the line bundles $\operatorname{span}(\partial_{x_k}, \partial_{y_k}) \times C_j$. Denote by θ_j the canonical flat connection on the bundle of oriented orthonormal frames of $\operatorname{span}(\partial_{x_k}, \partial_{y_k}) \times C_j$. The $\mathfrak{g} \oplus \mathbb{R}R_w$-equivariant Euler class of νC_j then is

$$\begin{aligned} e_{\mathfrak{g} \oplus \mathbb{R}R_w}(\nu C_j) &= \prod_{k \neq j} e_{\mathfrak{g} \oplus \mathbb{R}R_w}(\operatorname{span}(\partial_{x_k}, \partial_{y_k})) = \prod_{k \neq j} \operatorname{Pf}(-u \iota_X \theta_k - s \iota_{R_w} \theta_k) \\ &= \prod_{k \neq j} \tfrac{1}{2\pi}(u\beta_k + s w_k) = \left(\tfrac{1}{2\pi}\right)^n \prod_{k \neq j} (u\beta_k + w_k(-\beta_j u / w_j)) \\ &= \left(\frac{u}{2\pi}\right)^n \prod_{k \neq j} (\beta_k - w_k \beta_j / w_j). \end{aligned}$$

On C_j, we have $|z_j|^2 = 1$, $z_l = 0$ for $l \neq j$, so, we can parametrize C_j up to a zero set by $z_j = e^{i\varphi}$, $\varphi \in (0, 2\pi)$. Then $\iota_j^* \alpha_w = \frac{\frac{i}{2}(z_j d\bar{z}_j - \bar{z}_j dz_j)}{w_j} = \frac{d\varphi}{w_j}$ and $\int_{C_j} \iota_j^* \alpha_w = \int_0^{2\pi} \frac{1}{w_j} d\varphi = \frac{2\pi}{w_j}$. □

With our localization formula, we can now compute the contact volume of weighted Sasakian structures on odd spheres.

Remark 6.3.6. This result is known and can also by obtained by combining the observation of Martelli-Sparks-Yau [MSY06] that the volume of a toric Sasakian manifold is related to the volume of the truncated cone over its momentum image and a formula by Lawrence [Law91] for the volume of a simple polytope (cf.

[GNT17, Section 6.2]). Goertsches-Nozawa-Töben also computed the same result via a basic ABBV-type localization formula with respect to the transverse action of $\mathfrak{t}/\mathbb{R}R$, cf. [GNT17, Corollary 6.1].

Proposition 6.3.7. *The contact volume of* $(M, \alpha) = (S^{2n+1}, \alpha_w)$ *is given by*

$$\operatorname{vol}(M, \alpha) = \frac{1}{2^n n!} \int_M \alpha \wedge (d\alpha)^n = \frac{2\pi^{n+1}}{n! w_0 \cdots w_n}.$$

Proof. Recall that $d_G\alpha = d\alpha - \Psi u$. We insert the results of Lemma 6.3.5 into our localization formula. Choose any weights β_j such that $\lambda_j \neq \lambda_j$ for $j \neq l$ so that $\operatorname{Crit}\Psi = \cup_{j=0}^n C_j$. Note that C_j is 1-dimensional, so only the polynomial part of $d_G\alpha$ enters on the right hand side; we need a top degree form on the left hand side when integrating over M, so only $d\alpha$ enters.

$$\begin{aligned}\int_M \alpha \wedge (d\alpha)^n = \int_M \alpha \wedge (d_G\alpha)^n &= \sum_j (-\Psi(C_j)u)^n \int_{C_j} \frac{\iota_j^*\alpha}{e_j} \\ &= (2\pi)^{n+1}(-1)^n \sum_j \left(\frac{\beta_j}{w_j}\right)^n \frac{1}{w_j \prod_{k\neq j}(\beta_k - w_k\beta_j/w_j)} \\ &= \frac{(2\pi)^{n+1}(-1)^n}{w_0 \cdots w_n} \sum_j \frac{\beta_j^n}{\prod_{k\neq j}(w_k^{-1}\beta_k w_j - \beta_j)}.\end{aligned}$$

The right hand side has to be independent of the β_j, so we can take the limit $\beta_0 \to \infty$. Then the $(j = 0)$-summand tends to $(-1)^n$, the others vanish (cf. [GNT17, Corollary 6.1]). □

Now, let us consider the special case of the odd sphere $M = S^3 \subset \mathbb{C}^2$ with Sasakian structure determined by the weight $(w, 1)$ with $w > 0$ irrational. Let $G = S^1$ act on M with weights $\beta = (-1, 1)$. By Lemma 4.2.3, we have $H_G(M, \mathcal{F}) \cong \frac{\mathbb{R}[u,s]}{\langle (ws-u)(s+u)\rangle}$. We obtain from Lemma 6.3.5 for this special case that the critical set is given by $\operatorname{Crit}\Psi = C_0 \,\dot\cup\, C_1$, where $C_0 = S^1 \times \{0\}$ and $C_1 = \{0\} \times S^1$. The equivariant basic cohomology of the connected components is $H_G(C_j, \mathcal{F}) \cong \mathbb{R}[u]$. Furthermore, $\Psi(C_0) = -1/w$, $\Psi(C_1) = 1$, the Euler classes e_j of the normal bundles to C_j in M are $e_0 = \frac{u}{2\pi}\left(1 + \frac{1}{w}\right)$ and $e_1 = -\frac{u}{2\pi}(1 + w)$ and the restrictions $\iota_j^* : H_G(M, \mathcal{F}) \to H_G(C_j, \mathcal{F})$ are given by $\iota_0^* : s \mapsto u/w$ and $\iota_1^* : s \mapsto -u$. Recall that we identified $\mathfrak{s}^1$ with $\mathbb{R}$. If S^1 is parametrized via the angle φ, then this identification corresponds to $\lambda\partial_\varphi \mapsto \lambda$. We determine a metric g on S^1 by $g(\partial_\varphi, \partial_\varphi) = 1$ so that the volume form is given by $\mathrm{vol}_{S^1} = d\varphi$, $\mathrm{vol}(S^1) = 2\pi$. The induced inner product on $\mathbb{R} \simeq \mathfrak{s}^1$ is then multiplication so that the induced measures to consider on $\mathfrak{g}^*$ and $\mathfrak{g}$ are the standard measures du and $d\phi$, respectively.

Let us consider the Mayer-Vietoris sequence (cf. Proposition 3.3.6) of the pair $(M \setminus C_1, M \setminus C_0)$. Note that C_0 is an equivariant retraction of $M \setminus C_1$, C_1 is an equivariant retraction of $M \setminus C_0$, and $\Psi^{-1}(0)$ is an equivariant retraction of $(M \setminus C_1) \cap (M \setminus C_0)$. Basic Kirwan surjectivity (Theorem 4.1.1) yields that the long exact Mayer-Vietoris sequence turns into short exact sequences

$$0 \to H^*_G(M, \mathcal{F}) \overset{\iota_0^* \oplus \iota_1^*}{\to} H^*_G(C_0, \mathcal{F}) \oplus H^*_G(C_1, \mathcal{F}) \to H^*_G(\Psi^{-1}(0), \mathcal{F}) \to 0.$$

Hence, we can write $\eta \in H_G(M, \mathcal{F})$ as $\eta^0 \oplus \eta^1$, with $\eta^j \in H_G(C_j, \mathcal{F}) \cong \mathbb{R}[u]$. If $p = \sum_{l=0}^k p_l u^{k-l} s^l \in H_G(M, \mathcal{F})$ is an arbitrary $2k$-form, then $\iota_0^* p = \sum_{l=0}^k \frac{p_l}{w^l} u^k$ and $\iota_1^* p = \sum_{l=0}^k (-1)^l p_l u^k$. Hence, any given $au^k \in H^{2k}_G(C_0, \mathcal{F})$ and $bu^k \in H^{2k}_G(C_1, \mathcal{F})$, $a, b \in \mathbb{R}$, are both given as the restriction of the same form $p \in H^{2k}_G(M, \mathcal{F})$ if and only if $\sum_{l=0}^k \frac{p_l}{w^l} = a$ and $\sum_{l=0}^k (-1)^l p_l = b$. For $k = 0$, the only solution is $p_0 = a = b$. For $k > 0$, the linear system of equations always has a solution $(p_0, ..., p_k)$, for every $a, b \in \mathbb{R}$. Hence, it becomes evident that $\eta^0 \oplus \eta^1$ lies in the image of $\iota_0^* \oplus \iota_1^*$ if and only if η^0 and η^1 have the same constant term, as polynomials in u.

We compute the argument of jkres in the residue formula to be

$$(2\pi)^2 \left(\frac{e^{i\phi/w} \eta^0(\phi)}{\phi(1+w)} - \frac{e^{-i\phi} \eta^1(\phi)}{\phi(1+w)} \right) d\phi$$

Note that for a rational function g and $\lambda \in \mathbb{R} \setminus \{0\}$, the residue is given as (cf. [JK97, Proposition 3.4])

$$\mathrm{jkres}^{\{t \in \mathbb{R} | t > 0\}} \left(g(\phi) e^{i\lambda\phi} d\phi \right) = \begin{cases} 0 & \lambda < 0 \\ \sum_{b \in \mathbb{C}} \mathrm{Res}_{z=b} \left(g(z) e^{i\lambda z} \right) & \text{else} \end{cases},$$

where Res denotes the ordinary residue.

Thus, we obtain

$$\int_{M//G} \alpha_0 \wedge \eta_0 \wedge e^{id\alpha_0} = \frac{1}{\mathrm{vol}\, G} \mathrm{jkres} \left((2\pi)^2 \left(\frac{e^{i\phi/w} \eta^0(\phi)}{\phi(1+w)} \right) d\phi \right) = \frac{1}{2\pi} \frac{(2\pi)^2 \eta^0(0)}{w+1}$$

$$= \frac{2\pi\eta^0(0)}{w+1}.$$

In particular,

$$\int_{M//G} \alpha_0 \wedge e^{id\alpha_0} = \int_{M//G} \alpha_0 = \frac{2\pi}{1+w}. \tag{6.17}$$

We will now compute the left hand side of Equation (6.17) to see that our formula holds. Note that

$$\Psi^{-1}(0) = S^1\left(\tfrac{1}{\sqrt{2}}\right) \times S^1\left(\tfrac{1}{\sqrt{2}}\right).$$

$\Psi^{-1}(0)/G$ is $\{\psi_t\}$-equivariantly diffeomorphic to S^1 via $[z] \mapsto 2z_1z_0$, where ψ_t acts on S^1 by $\psi_t(z) = e^{it(w+1)}z$. Under this identification, the projection $p : \Psi^{-1}(0) \to M//G$ is given by $(z_0, z_1) \mapsto 2z_1z_0$. Denote the inclusion by $\iota : \Psi^{-1}(0) \hookrightarrow M$. Since $\iota^*|z_j|^2 \equiv 1/2$, we then compute $\iota^*\alpha = \frac{2i}{w+1}(z_0d\bar{z}_0 + z_1d\bar{z}_1)$. Since $p^*(\frac{i}{w+1}zd\bar{z}) = \iota^*\alpha$, we obtain $\alpha_0 = \frac{i}{w+1}zd\bar{z}$.

Up to a zero set, $M//G \simeq S^1$ is parametrized by $\Phi : (0, 2\pi) \to S^1$, $\varphi \mapsto e^{i\varphi}$. In this coordinate, $\alpha_0 = \frac{1}{w+1}d\varphi$. Then $\int_{S^1} \alpha_0 = \int_0^{2\pi} \frac{1}{w+1}d\varphi = \frac{2\pi}{w+1}$, which is exactly the right hand side of Equation 6.17.

Appendix A

Morse-Bott Theory

This Chapter does not contain any original research and was merely added for the convenience of the reader. It contains a very brief introduction to the subject of Morse-Bott functions and their properties, as far as they are used in this thesis. Morse-Bott functions generalize the notion of Morse functions (cf. [Mil63]) and their theory was developed by Bott in [Bot54]. For a more elaborate treatment of the subject, the reader is also referred to, e.g., [Gue02, Nic11].

Consider a compact manifold N and a smooth function $f : M \to \mathbb{R}$ and denote its critical set by $\operatorname{Crit} f$. For $a \in \mathbb{R}$, we further set $N^a = \{p \in N \mid f(p) \leq a\}$. If a is not a critical value of f, then N^a is a smooth manifold with boundary and $f^{-1}(a)$ is a smooth submanifold of N. We recall the following definitions.

Definition A.1. Let p be a critical point of f. Then the *Hessian* of f at p is the symmetric bilinear form $\mathrm{Hess}_p : T_pN \times T_pN \to \mathbb{R}$ defined by

$$\mathrm{Hess}_p(v, w) = V(W(f))_p,$$

where V, W are local extensions of the vectors v, w to vector fields. A connected smooth submanifold $A \subset N$ that consists solely of critical points of f is called a *non-degenerate critical submanifold of* f if, for every $p \in A$, $T_pA = \ker \mathrm{Hess}_p$, that is, Hess is non-degenerate in directions normal to A. If N is endowed with a Riemannian metric, we denote the orthogonal complement of T_pA in T_pN by ν_pA and the subspaces on which Hess_p is positive respective negative definite by $\nu_p^{\pm}A$, the *positive* and *negative normal bundle of* A, respectively. The *index* of a non-degenerate critical submanifold A of f is the rank of the subbundle $\nu^- A \subset TN|_A$. f is called a *Morse-Bott function* if its critical set is a union of non-degenerate critical submanifolds.

Now suppose that a compact Lie group K acts on N and that f is an N-invariant Morse-Bott function. Such functions are a special case of the functions studied by Wasserman in [Was69]. In this situation, N^a is a smooth invariant manifold with boundary and $f^{-1}(a)$ is a smooth invariant submanifold of N. We are interested in how the topology of N^a changes when we change a. It turns out that the K-homotopy type of N^a does not change before a crosses a critical value, more precisely, we have the following.

Theorem A.2 ([Was69, Theorem 4.3])**.** *Suppose that a compact Lie group K acts on the compact manifold N. Let $f : N \to \mathbb{R}$ be a K-invariant Morse-Bott function and suppose that the bounded interval $[a, b]$ does not contain a critical value of f. Then N^a is K-equivariantly diffeomorphic to N^b.*

We can also describe how the topology changes if a critical value is crossed. For this purpose, we need the following definition (cf. [Was69, pp. 146f]).

Definition A.3. Let V, W be Riemannian K-vector bundles over a manifold B. We denote their disk bundles by D, their open unit ball bundles by $\mathring{D}$ and their sphere bundles by S. The bundle $D^V \oplus D^W = \{(v, w) \in V \oplus W \mid ||v|| \leq 1, ||w|| \leq 1\}$ is called a *handle bundle of type* (V, W) with index equal to the rank of W. Let $N \subset \tilde{N}$ be K-manifolds with boundary, and $H \subset \tilde{N}$ a closed subset. We write $\tilde{N} = N \cup_{D^V \oplus S^W} H$ and say that $\tilde{N}$ arises from N by attaching a handle bundle of type (V, W) if

(i) $\tilde{F} : D^V \oplus D^W \to H \subset \tilde{N}$ is an equivariant homeomorphism onto H,

(ii) $\tilde{N} = N \cup H$,

(iii) $\tilde{F}|_{D^V \oplus S^W}$ is an equivariant diffeomorphism onto $H \cap \partial N$,

(iv) $\tilde{F}|_{D^V \oplus \mathring{D}^W}$ is an equivariant diffeomorphism onto $\tilde{N} \setminus N$.

Theorem A.4 ([Was69, Theorem 4.6])**.** *Suppose that a compact Lie group K acts* *on the compact manifold N. Let $f : N \to \mathbb{R}$ be a K-invariant Morse-Bott function and assume that f has a single critical value $a < c < b$ in the bounded interval $[a, b]$. Denote by $B_1, ..., B_k$ the connected components of the critical set of f at level c and by λ_j their respective indices. Then N^b is K-equivariantly diffeomorphic to N^a with k handle-bundles of type $(\nu^+ B_j, \nu^- B_j)$ disjointly attached.*

Outlook

There are still some interesting open questions related to the topics of this thesis which, so far, remain unsolved, and which may be considered in future research.

The symplectic moment map is only determined up to an additive constant that lies in the annihilator of the commutator of the Lie algebra. In particular, for torus actions, any element of the dual Lie algebra can be added. Thus, Kirwan surjectivity for non-zero regular values of the moment map can be obtained from Kirwan surjectivity for 0 as regular value of the moment map. In the contact setting, however, the contact moment map is completely determined so that the question of basic Kirwan surjectivity for non-zero regular values is still open. The construction of the special basis of the Lie algebra (Proposition 2.4.1), for example, which is essential in obtaining basic Kirwan surjectivity, does not work for non-zero regular values.

In the symplectic setting, Kirwan's surjectivity result holds for Hamiltonian actions of compact groups that need not be Abelian. Furthermore, the compactness assumption for the manifold can be weakened; it is sufficient to require that the norm square $||\mu||^2$ of the moment map is *flow closed*, i.e., that every positive time trajectory of $-\mathrm{grad}\ ||\mu||^2$ is contained in a compact set (cf. [Kir84, Chapter 9]). Kirwan's proof makes use of the *minimal degeneracy* of $||\mu||^2$, a property that is weaker than the Morse-Bott property and which was established in [Kir84, Chapter 4]. The question of minimal degeneracy of the norm square of the contact moment map is still unanswered. It is natural to wonder whether basic Kirwan surjectivity on K-contact manifolds does also still hold under similarly weakened assumptions.

As already remarked in Section 4.3, Tolman-Weitsman generalize the proof of their description of the kernel of the Kirwan map (cf. [TW03, Theorem 2]) from the S^1-case to the setting of the action of higher rank tori. As Kirwan in her proof of surjectivity, they apply Morse-Bott theory of the norm square of the symplectic moment map. A Tolman-Weitsman type description of the kernel of the basic Kirwan map for the action of higher rank tori has not been achieved, yet.

All of our considerations in this thesis concerned torus actions only. In [JK95, Theorem 8.1], Jeffrey and Kirwan obtained the Residue Formula not only for torus actions, but also for the action of compact connected Lie groups in general. Among other things, they make use of the fact that integration over the Lie algebra of a compact connected Lie group is related to integration over the Lie algebra of a maximal torus via the Weyl Integration formula. Applying this formula and using our results for torus actions, it remains to investigate further along the line of argument of Jeffrey-Kirwan if our results in Section 6.2 extend to actions of connected non-Abelian compact Lie groups.

In [JMW05], Jeffrey-Mare-Woolf proved the Tolman-Weitsman type description of the kernel of the Kirwan map as an application of the Residue Formula, by considering the so-called residue kernel of the Kirwan map and certain generic circle subgroups of a (maximal) torus. It could be investigated if a similar approach is applicable in the contact setting. However, [JMW05] also requires some Morse-Bott theory for the norm square of the moment map.

In the symplectic setting, if 0 is a singular value of the moment map, Residue Formulae similar to [JK95, Theorem 8.1] were derived in certain situations; namely, for non-singular, connected, complex projective varieties by Jeffrey-Kiem-Kirwan-Woolf (cf. [JKKW03]), for equivariant cohomology classes represented by basic (with respect to the group action) differential forms on cotangent bundles of manifolds that are endowed with an isometric group action by Ramacher (cf. [Ram16]), and for equivariant cohomology classes represented by basic (with respect to the group action) differential forms on general symplectic manifolds by Konstantis-Küster-Ramacher (cf. [KKR16]). If 0 is not a regular value, the quotient of the 0-set of the moment map by the Lie group is in general singular and not even an orbifold. It is an interesting, yet highly speculative question if a Residue Formula in the singular case for K-contact manifolds might hold in certain situations. To begin with, a correct analogue of the basic cohomology of the contact quotient would have to be found in such a singular setting, as well as a normal form for the moment map.

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Publications

This thesis is based on the results of the following two publications.

- **L. Casselmann**.
  Basic Kirwan Surjectivity for K-Contact Manifolds.
  Annals of Global Analysis and Geometry, 52(2): 157–185, 2017.

- **L. Casselmann and J. M. Fisher**.
  Localization for K-Contact Manifolds.
  To appear in *Journal of Symplectic Geometry.*
  arXiv preprint: 1703.00333, 2017.

  The idea for this project stems from discussions by the authors in the GRK 1670 "Mathematics inspired by string theory and quantum field theory". In these discussions, it became evident that the setting of equivariant basic cohomology might provide the correct framework to obtain localization results for K-contact manifolds.
  The introduction and the preliminary Sections 2.1, 2.2, 2.3 were mainly written by Fisher, then revised by me. The first draft of Section 2.4 was also written by Fisher; this section was then made mathematically rigorous by me, in particular, I contributed Theorems 2.14 and 2.15 and Lemmata 2.16 through 2.19, which constitute the proof of Proposition 2.21. Section 3.1 was mainly written by Fisher, then revised by me. The results in Section 3.2 were produced by both of us. Apart from Lemma 4.4, and Proposition 4.7, which Fisher contributed, the results in Section 4 were proven by me. The example computations in Section 5 were a joint effort.

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Eidesstattliche Versicherung

Hiermit erkläre ich an Eides statt, dass ich die vorliegende Dissertationsschrift selbst verfasst und keine anderen als die angegebenen Quellen und Hilfsmittel benutzt habe.

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