Torelli theorems for rational elliptic surfaces and their toric degenerations

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Introduction

This thesis generalizes a Torelli type theorem for certain rational elliptic surfaces to families and toric degenerations of these surfaces.

Torelli type theorems have a long history in algebraic geometry. It started more than hundred years ago with a theorem for the Jacobian variety of a Riemann surface. Later, a Torelli theorem for K3 surfaces was proved in several forms. In its classical form, it says that two K3 surfaces X and X' are isomorphic if and only if there is a Hodge isometry such that $\mathrm{H}^2(X, \mathbb{Z}) \cong \mathrm{H}^2(X', \mathbb{Z})$. Both theorems have in common that a variety or a scheme can be characterized by the image of a 'period map', which relates the original space with some cohomological data. For a Riemann surface these data are given by integrals over 1-cycles which yield a period matrix. For K3 surfaces X the period map uses the class $\mathrm{H}^{2,0}(X) \subset \mathrm{H}^2(X, \mathbb{C})$. Such a result is called a Torelli theorem. For rational elliptic surfaces similar results were known for a long time due to Pinkham [P] and Looijenga (e.g. [HL]).

Here, we treat a special case. We want to study certain rational elliptic surfaces with a smooth effective anticanonical divisor and a section, which are called 'dP_9 surfaces'.

A *dP_9 surface* can be considered as a blow-up of $\mathbb{P}^2$ in the 9 base points of a cubic pencil. This fact implies that its anticanonical divisor is not ample but effective. In this sense, a dP_9 surface is almost a del Pezzo surface. The name 'dP_9 surface' follows the convention for del Pezzo surfaces dP_k used in physics, where k denotes the number of blown-up points rather than the degree $K_X \cdot K_X = 9 - k$ used in algebraic geometry. Contracting an exceptional curve in a dP_9 surface yields an honest del Pezzo surface of degree 1, i.e. in our notation a dP_8 surface.

Another reason why dP_9 surfaces are particularly interesting lies in their close connection to K3 surfaces. First, dP_9 surfaces can be considered as a logarithmic analogue of K3 surfaces, where the anticanonical divisor plays the role of a logarithmic boundary of the surface. Secondly, dP_9 surfaces arise by type II degenerations of

K3 surfaces. In this process, it is possible to degenerate a K3 surface into a union of two dP_9 surfaces glued along anticanonical divisors. In particular, degenerations of a K3 surface leading to dP_9 surfaces were described in [DKW] as fibres of a higher dimensional fibration in the context of the duality between F-theory and heterotic string theory. In [DKW], the question of 'holographic' aspects of such degenerations is also discussed. These 'holographic' aspects correspond to the fact that a dP_9 surface can be recovered from period data on its anticanonical divisor via a Torelli type theorem. This thesis presents proofs of this Torelli type theorem for certain cases.
To understand some degenerations of dP_9 surfaces, we use the language of *toric degenerations*, which was introduced by Gross and Siebert in [GS03, GS, GS1, GS2]. The toric degeneration picture allows for certain toric degenerations of K3 surfaces to be split into two degenerations of dP_9 surfaces glued along their anticanonical divisors. This is analogous to the degenerations described by [DKW] in the context of type II degenerations of K3 surfaces. Looking at toric degenerations yields a possibility of using period integrals within the setup of degenerations and computing them easily applying methods introduced in [RS].
Note that Gross, Hacking and Keel proved a Torelli theorem for Looijenga pairs, i.e. pairs of a smooth projective surface and a connected, singular, nodal anticanonical divisor, in [GHK]. They occur in the context of type III degenerations of K3 surfaces. Although this case looks almost identical to the case which will be treated here, there is a fundamental difference: As we are working with a smooth anticanonical divisor, there are no special points on this divisor, so the methods used in [GHK] do not work in our case.

This thesis is organized in two main parts. In the first part (Chapter 1 and Chapter 2), the Torelli type theorem for dP_9 surfaces is generalized into a Torelli type theorem for families of dP_9 surfaces. The second part (Chapter 3, Chapter 4 and Chapter 5) is concerned with toric degenerations of dP_9 surfaces and shows that the methods used in the first part generalize to this setup.
Chapter 1 starts by recalling well known material on dP_9 surfaces. As mentioned above, rational elliptic surfaces have been studied for a long time, in particular by Looijenga, e.g. [HL, L1, L2]. In [HL], Heckman and Looijenga compare different compactifications of the moduli space of rational elliptic surfaces with a section.

In [L1], Looijenga gives a Torelli type theorem for pairs of a dP_9 surface with a smooth anticanonical divisor using period integrals [L1, p.31] without providing all details of the proof. Chapter 1 of this thesis recalls this theorem and elaborates its proof, reformulating it in terms of an equivalence of categories for surfaces over an algebraically closed field of characteristic 0.
Denote by $\underline{\mathrm{dP}_9}$ the category of pairs (X, D) of a dP_9 surface X with smooth anticanonical divisor D together with an isomorphism $\Lambda_{1,9} \cong \mathrm{Pic}(X)$ fulfilling certain conditions. Here, $\Lambda_{1,9}$ denotes a specific lattice of rank 10 and signature $(1, 9)$. The isomorphism $\Lambda_{1,9} \cong \mathrm{Pic}(X)$ for objects in $\underline{\mathrm{dP}_9}$ is given by a 'geometric marking'. Let $d \in \Lambda_{1,9}$ denote the element which is mapped to the class of the anticanonical divisor by the geometric marking. It holds that an E_8 lattice Q is given by $Q = d^{\perp}/\mathbb{Z}d$.
Morphisms in $\underline{\mathrm{dP}_9}$ are given by isomorphisms of pairs $\varphi\colon (X, D) \to (X', D')$ which induce lattice isomorphisms by composition with $\varphi_*\colon \mathrm{Pic}(X) \to \mathrm{Pic}(X')$.
Let Q be an E_8 lattice and let $\underline{\mathrm{Hom}_Q}$ be the category whose objects are pairs (D, χ_D) of a smooth curve D of genus 1 and a homomorphism $\chi_D\colon Q \to \mathrm{Pic}^0(D)$. We require these homomorphisms to be injective on roots in Q. The morphisms in $\underline{\mathrm{Hom}_Q}$ are given by isomorphisms $\psi\colon D \to D'$ such that for the induced morphisms $\psi_{Pic}\colon \mathrm{Pic}^0(D) \to \mathrm{Pic}^0(D')$ it holds that $\chi_{D'} = \psi_{Pic} \circ \chi_D$.
Then the following theorem is true:

Theorem (Theorem 1.39)**.** *The categories* $\underline{\mathrm{dP}_9}$ *and* $\underline{\mathrm{Hom}_Q}$ *are equivalent.*

In Chapter 2, the theorem which was presented in Chapter 1 for dP_9 surfaces is generalized to families of dP_9 surfaces. The families of dP_9 surfaces $\mathcal{X}$ are endowed with a divisor $\mathcal{D}$ and a marking, which are adapted to the new situation:
Let $\pi\colon (\mathcal{X}, \mathcal{D}) \to S$ be a pair of smooth projective families over an affine Noetherian scheme S such that on geometric fibres, the situation is the same as for the first version of the theorem, i.e. a geometric fibre is given by a pair of a dP_9 surface and a smooth anticanonical divisor. The isomorphism of lattices $\Lambda_{1,9} \cong \mathrm{Pic}(X)$ is replaced by an isomorphism of sheaves $(\Lambda_{1,9})_S \cong R^1\pi_*\mathcal{O}^*_{\mathcal{X}}$, which restricts to a geometric marking on geometric fibres. Pairs $(\mathcal{X}, \mathcal{D}) \to S$ as above, which are endowed with such an isomorphism of sheaves, constitute the objects of a category $\underline{\mathbf{dP_9}}$.
To define the objects of our second category, we have to find an analogue of the

connected component of the identity of the Picard group $\mathrm{Pic}^0(D)$ of a smooth genus 1 curve D. Thus, we consider the connected component of the identity of the relative Picard scheme $\mathbf{Pic}^0_{\mathcal{D}/S}$ of a smooth projective family of genus 1 curves $\mathcal{D} \to S$ with a section over an affine Noetherian scheme S. Pairs of such a family of genus 1 curves $\mathcal{D} \to S$ and a group homomorphism $\chi_{\mathcal{D}/S} \colon Q \to \mathrm{Hom}_S(S, \mathbf{Pic}^0_{\mathcal{D}/S})$ yield objects of our second category $\underline{\mathbf{Hom}_\mathbf{Q}}$.
We prove the following theorem for families of dP_9 surfaces generalizing the proof for surfaces:

Theorem (Theorem 2.24)**.** *The categories* $\underline{\mathbf{dP_9}}$ *and* $\underline{\mathbf{Hom}_\mathbf{Q}}$ *are equivalent.*

Chapter 3 recalls some basics on toric degenerations. Toric degenerations were introduced by Gross and Siebert in [GS1, GS2, GS]. The central fibre of a toric degeneration is given by a union of toric varieties, which are glued along toric divisors, hence the name 'toric degeneration'. Gross and Siebert show that (formal) toric degenerations

$$\mathfrak{X} \to \mathrm{Spf}\, \mathbb{C}[[t]]$$

of logarithmic spaces are closely connected with certain affine data. In particular, the central fibre $X \subset \mathfrak{X}$ of a toric degeneration is encoded in an integral affine manifold B with boundary ∂B, singular locus $\Delta =: B \setminus B_0$, $\iota \colon B_0 \to B$, and an integral tangent sheaf Λ on B_0. Denote the cotangent sheaf on B_0 by $\check{\Lambda}$. The manifold B is endowed with a polyhedral decomposition $\mathscr{P}$ and gluing data $\mathbf{s} \in \mathrm{H}^1(B, \iota_*\check{\Lambda} \otimes \mathbb{C}^*)$. The polyhedra of maximal dimension correspond to toric varieties, which are glued using the gluing data $\mathbf{s}$ to get X. The boundary ∂B corresponds to an anticanonical divisor $D \subset X$. The discrete information contained in the induced logarithmic structure $\mathcal{M}_{(X,D)}$ on the central fibre X corresponds to a multi-valued piecewise linear function φ on B. In [GS], a toric degeneration is constructed starting with affine data $((B, \mathscr{P}), \mathbf{s}, \varphi)$. In [GS1] and [GS2], the other way around is treated, i.e. toric degenerations inducing affine data. Universal toric degenerations

$$\mathfrak{X} \to \mathrm{Spf}\, A[[M]],$$

where A encodes the gluing data and M is a monoid, which is the target space of a universal multi-valued piecewise linear function, were introduced in [GHKS1] and in [GS2]. Note that every toric degeneration over $\mathrm{Spf}\, \mathbb{C}[[t]]$ constructed via [GS] is a pullback of a universal toric degeneration. Here, we want to consider some intermediate picture which is given by toric degenerations

$$\mathfrak{X} \to \operatorname{Spf} A[[t]],$$

i.e. we include all possible gluing data but we fix a multi-valued piecewise linear function φ. Thus, we get deformations of logarithmic spaces over a fixed 1-dimensional base.

In Chapter 4, an example of a toric degeneration of dP_9 surfaces is studied in detail. In particular, we want to compute period integrals on fibres following the approach of [L1] for surfaces. Such integration computations on toric degenerations were introduced in [RS] using 'tropical cycles' defined in [RS] in a similar way as preceded by [Sy]:

Let $X_t \subset \mathfrak{X}$ be a fibre of an analytic extension of a toric degeneration and let $\alpha(t) \in \mathrm{H}_2(X_t, \mathbb{Z})$ be a vanishing cycle. There are tropical cycles $\beta_{trop} \in \mathrm{H}_1(B, \iota_*\Lambda)$ which correspond to other cycles $B(t) \in \mathrm{H}_2(X_t, \mathbb{Z})$. The integration of a relative holomorphic volume form Ω with poles along the anticanonical divisor $D_t \subset X_t$ over $B(t)$ can be computed using β_{trop} and certain affine data, i.e. $\mathbf{s} \in \mathrm{H}^1(B, \iota_*\check{\Lambda} \otimes \mathbb{C}^*)$ and some analogue of a Chern class $c_1(\varphi) \in \mathrm{H}^1(B, \iota_*\check{\Lambda})$. More explicitely, [RS] yields the formula

$$\exp\left(-2\pi i \frac{\int_{B(t)} \Omega}{\int_{\alpha(t)} \Omega}\right) = \langle \mathbf{s}, \beta_{trop} \rangle \cdot t^{\langle c_1(\varphi), \beta_{trop} \rangle}.$$

In particular, we construct certain tropical cycles $(\alpha_0)_{trop}, \ldots, (\alpha_8)_{trop} \in \mathrm{H}_1(B, \iota_*\Lambda)$ corresponding to a root basis of the extended E_8 lattice, i.e. the $\hat{E}_8$ lattice, $d^\perp \subset \Lambda_{1,9}$. Moreover, we identify a cycle $\beta_{\|}^{\partial}$, which does not correspond to a non-zero class in $H_2(X, \mathbb{Z})$. Together, the cycles $(\alpha_0)_{trop}, \ldots, (\alpha_8)_{trop}$ and $\beta_{\|}^{\partial}$ span $\mathrm{H}_1(B, \iota_*\Lambda)$.

We introduce the notation $\mathfrak{R} := \mathit{span}\left((\alpha_0)_{trop}, \ldots, (\alpha_8)_{trop}\right) \subset \mathrm{H}_1\left(B, \iota_*\Lambda\right)$ and $H := \left(\beta_{\|}^{\partial}\right)^{\perp} \subset \mathrm{H}^1(B, \iota_*\check{\Lambda})$. It holds that $\mathfrak{R} \cong Q'$ is an $\hat{E}_8$ lattice. For the pairing used in the integration formula we prove the following proposition by computation:

Proposition (Proposition 4.22)**.** *The pairing*

$$\begin{aligned} \mathfrak{R} \otimes H &\to \mathbb{Z} \\ (\beta_{trop}, \mathbf{s}) &\mapsto \langle \beta_{trop}, \mathbf{s} \rangle \end{aligned}$$

has full rank $r = 9$.

Chapter 5 uses the computations executed in the previous Chapter to generalize the period integrals to toric degenerations of dP_9 surfaces. We show that the period integrals which characterized dP_9 surfaces in the smooth case continue to do so for toric degenerations.
More concretely, let $\mathbb{D} \subset \mathbb{C}$ be an open disk containing the origin. We assume that there exists an analytic log smooth versal deformation of dP_9 surfaces

$$[(\mathcal{Y}, \mathcal{C}), \mathcal{M}] \to \mathcal{S} \to \mathbb{D}$$

of a central fibre $(X, \mathcal{M}_{(X,D)})$ of a toric degeneration of dP_9 surfaces.
We prove the following theorem:

Theorem (Theorem 5.42)**.** *A toric degeneration $(\mathfrak{X}, \mathfrak{D}) \to \operatorname{Spf} A[[t]]$ of (X, D) is the formal completion of $[(\mathcal{Y}, \mathcal{C}), \mathcal{M}]$ at the central fibre $(X, \mathcal{M}_{(X,D)})$ and the period map yields coordinates on the base of any miniversal deformation of the central fibre.*

In formulating this theorem, we assumed the existence of the analytic log smooth versal deformation $(\mathcal{Y}, \mathcal{C})/\mathbb{D}$. This assumption does not impose a very serious restriction on our theorem. The versal deformation is expected to exist in any case, although this cannot be proved in this thesis.
Moreover, we formulate our theorem again in terms of an equivalence of categories. Let $\underline{\mathfrak{dP}}_9$ be the category of toric degenerations of dP_9 surfaces with fixed affine data $((B, \mathscr{P}), \varphi)$. Here, B denotes a suitable integral affine manifold, $\mathscr{P}$ is a polyhedral decomposition of B and φ is a multi-valued piecewise linear function on $(B, \mathscr{P})$. Moreover, we fix a marking Φ on $((B, \mathscr{P}), \varphi)$, which maps $\Lambda_{1,9}$ to tropical cycles on $(B, \mathscr{P})$.
Let $\underline{\mathfrak{Hom}}_Q$ be the category whose objects consist of pairs $(\mathfrak{D} \to \mathfrak{S}, \chi_{\mathfrak{D}})$, where $\mathfrak{D} \to \mathfrak{S}$ denotes a toric degeneration of elliptic curves, which is compatible with $((B, \mathscr{P}), \varphi)$. Again, we keep $((B, \mathscr{P}), \varphi)$ fixed as well as the marking Φ. The second component $\chi_{\mathfrak{D}}$ is given by a homomorphism $\chi_{\mathfrak{D}} \colon Q \to \operatorname{Hom}(\mathfrak{S}, \mathbb{C}^*)$.
We prove the following theorem:

Theorem (Theorem 5.62)**.** *The categories $\underline{\mathfrak{dP}}_9$ and $\underline{\mathfrak{Hom}}_Q$ are equivalent.*

This is is the main result which is proved in this thesis. It confirms the claim of a 'holographic principle' formulated in [DKW] for the case of toric degenerations.

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1. A Torelli theorem for dP_9 surfaces

1.1. Generalities on dP_9 surfaces

In the following section, we want to phrase a Torelli type theorem for dP_9 surfaces. Over $\mathbb{C}$, this theorem is already given in [L1]. As we want to generalize it, we phrase it in terms of an equivalence of categories and we work over an algebraically closed field $\Bbbk$ of characteristic 0. The proof from [L1] using period integrals over $\mathbb{C}$ is elaborated as well as it will become more important in Chapter 5.

Definition 1.1. A *dP_9 surface* is a rational elliptic surface $X \to \mathbb{P}^1$ with irreducible fibres and a section.

Remark 1.2.

1. The fact that the base of the elliptic fibration is isomorphic to $\mathbb{P}^1$ follows from rationality of X (see [ScSh, § 8.1]).
2. As all fibres are irreducible, the singular fibres occurring in the fibration are of Kodaira types I_1 or II.

Proposition 1.3. *Let $X \to \mathbb{P}^1$ be a dP_9 surface. Then the following holds:*

1. *Any fibre $F \subset X$ is an anticanonical divisor, i.e. $-K_X = F$.*
2. *A canonical divisor K_X has vanishing self-intersection, $K_X^2 = 0$.*
3. *A section E is an exceptional curve of the first kind, i.e. it is rational and $E^2 = -1$.*

Proof. The first part follows from Kodaira's canonical bundle formula, [ScSh, p. 39] or [CD, Prop. 5.6.1]. The second part is obvious because we have a fibration. The last part follows from adjunction (see [ScSh, Cor. 6.9] or [GH, p. 471]):

$$\begin{aligned} 0 = g(E) &= \frac{K_X \cdot E + E \cdot E}{2} + 1 \\ &= \frac{(-F) \cdot E + E^2}{2} + 1 \\ &= \frac{-1 + E^2}{2} + 1 \\ \Leftrightarrow E^2 &= -1 \end{aligned}$$

□

Remark 1.4. 1. The first part of Proposition 1.3 explains why a pair (X, D) of a dP_9 surface and a smooth anticanonical divisor D can be considered as some kind of logarithmic analogue of a K3 surface. The anticanonical divisor D is a fibre and serves as a logarithmic boundary of X.

2. The reverse of the third part of Proposition 1.3 holds as well, i.e. any curve $E \subset X$ with $E^2 = -1$ and $E \cdot F = 1$ is a section (see [HL, Prop. 1.2]).

3. The last part of Proposition 1.3 shows that a section E of $X \to \mathbb{P}^1$ can be contracted. This observation leads to Theorem 1.5.

Theorem 1.5. *Any dP_9 surface X is isomorphic to a blow-up of $\mathbb{P}^2$ at the base points of a cubic pencil.*

Proof. [ScSh, Prop. 8.1] or [CD, Thm. 5.6.1]. □

Remark 1.6. The reverse of Theorem 1.5 is not always true. More precisely, a blow-up of $\mathbb{P}^2$ at 9 points is a dP_9 surface as in Definition 1.1 if and only if the following conditions on these points are fulfilled ([L1], Prop. 1.6 and Rem. 1.7):

1. No 3 points lie on a line.

2. No 6 points lie on a conic.

3. No 8 points lie on a cubic that has a singular point at one of the points.

In particular, the following well-known statement (see e.g. [H, Cor. V.4.5]) holds:

Lemma 1.7 (Cayley-Bacharach Theorem)**.** *Any cubic pencil in $\mathbb{P}^2$ is determined by 8 of its base points. Given these 8 points, the ninth base point is already fixed.*

As a Torelli type theorem relies on lattice theoretic facts, the next step is to establish some results on lattices related with dP_9 surfaces.
Let $\Lambda_{1,9}$ denote a lattice of signature $(1,9)$, which is freely generated by $l, e_1, \ldots, e_9$ with $l \cdot l = 1,\ e_i \cdot e_i = -1,\ e_i \cdot e_j = 0$ and $l \cdot e_i = 0$ for $i, j = 1, \ldots, 9,\ i \neq j$. As the surface X is a blow-up of $\mathbb{P}^2$ at 9 points, there exists an isomorphism

$$\phi \colon \Lambda_{1,9} \to \mathrm{H}_2(X, \mathbb{Z}) = \mathrm{H}^2(X, \mathbb{Z}) = \mathrm{Pic}(X).$$

Definition 1.8. Let $[E_i] \in \mathrm{H}_2(X, \mathbb{Z})$ for $i = 1, \ldots, 9$ denote classes of pairwise disjoint exceptional curves, which exist by Theorem 1.5. Let $[L] \in \mathrm{H}_2(X, \mathbb{Z})$ be the class given by the total transform of a general line in $\mathbb{P}^2$ not meeting the blown-up points.
An isomorphism $\phi \colon \Lambda_{1,9} \to \mathrm{H}_2(X, \mathbb{Z})$ is called a *geometric marking* for X if

$$\begin{aligned} \phi(e_i) &= [E_i]\,, \\ \phi(l) &= [L] \text{ and} \\ \phi(d) = \phi(3l - e_1 - \ldots - e_9) &= [-K_X]\,. \end{aligned}$$

Remark 1.9. A geometric marking is far from being unique. For example, the map $\phi' \colon \Lambda_{1,9} \to \mathrm{H}_2(X, \mathbb{Z})$ with $e_1 \mapsto [L] - [E_2] - [E_3]$, $e_2 \mapsto [L] - [E_1] - [E_3]$, $e_3 \mapsto [L] - [E_1] - [E_2]$, $e_4 \mapsto [E_4]$, …, $e_9 \mapsto [E_9]$ and $l \mapsto 2\,[L] - [E_1] - [E_2] - [E_3]$, defines another geometric marking.
Blowing down to $\mathbb{P}^2$ following these two different geometric markings yields two different models of $\mathbb{P}^2$, which are related by a Cremona transformation (see [L1, Ex. 1.5]).

From now on, we will assume the choice of a geometric marking. This motivates us to study the structure of $\Lambda_{1,9}$ in greater detail.

Definition 1.10 ([Kac, p. 1])**.** Let $A = (a_{ij})_{i,j=1,\ldots,n}$ be a $n \times n$ matrix of rank l such that

$$\begin{aligned} a_{ii} &= 2,\ i = 1, \ldots, n, \\ a_{ij} &\in -\mathbb{N},\ i \neq j, \text{ and} \\ a_{ij} = 0 &\Rightarrow a_{ji} = 0. \end{aligned}$$

Then A is called a *generalized Cartan matrix*.

Definition 1.11. A *realization* of a generalized Cartan matrix A of rank l is given by a vector space H and linearly independent sets of vectors $\alpha_1, \ldots, \alpha_n \in H^*$ and $\beta_1, \ldots, \beta_n \in H$. Here, H^* denotes the dual space of H. Using the pairing

$$\langle \cdot, \cdot \rangle \colon H \times H^* \to \mathbb{k}$$

we require that it is possible to reconstruct A via

$$\langle \beta_i, \alpha_j \rangle = a_{ij}, \ i, j = 1, \ldots, n$$
$$n - l = \dim H - n.$$

The set $\{\alpha_1, \ldots, \alpha_n\}$ is called the *root basis*, its span

$$\sum_{i=1}^{n} \mathbb{Z}\alpha_i$$

is the *root lattice*.

Following the discussion in [Kac], one can associate an infinite dimensional Lie algebra to any realization of a generalized Cartan matrix. Here, we do not need the full picture so we return to our concrete situation. We want to construct a root basis and the corresponding generalized Cartan matrix in $\Lambda_{1,9}$. Denote the symmetric bilinear form on the lattice $\Lambda_{1,9}$ by

$$(\cdot, \cdot) \colon \Lambda_{1,9} \times \Lambda_{1,9} \to \mathbb{Z}.$$

Definition 1.12. Using the symmetric bilinear form on $\Lambda_{1,9}$ we get $\Lambda_{1,9}^* \cong \Lambda_{1,9}$ and we define our future *root basis* in $\Lambda_{1,9}$ by

$$\mathcal{B}' := \{\alpha_0 := l - e_1 - e_2 - e_3, \ \alpha_1 := e_1 - e_2, \ \ldots, \ \alpha_7 := e_7 - e_8, \ \alpha_8 := e_8 - e_9\}.$$

Each root $\alpha \in \mathcal{B}'$ induces an automorphism s_α of $\Lambda_{1,9}$ via reflection along the normal plane to α:

$$s_\alpha \colon \Lambda_{1,9} \to \Lambda_{1,9}, \ v \mapsto v + (v, \alpha) \cdot \alpha$$

The group generated by these reflections is the *Weyl group* denoted by

$$\mathfrak{W}' := \langle s_\alpha \mid \alpha \in \mathcal{B}' \rangle \subset O(\Lambda_{1,9}).$$

This reflection group spans an infinite root system $\mathcal{R}'$ via its action on the root basis $\mathcal{B}'$:

$$\mathcal{R}' = \mathfrak{W}' \cdot \mathcal{B}'$$

Lemma 1.13. *The infinite root system $\mathcal{R}'$ spanned by $\mathcal{B}'$ is of type $\hat{E}_8$.*

Proof. Set $\beta_i = -(\alpha_i, \cdot)$, $i = 0, \ldots, 8$. The vectors α_i and β_i are linearly independent and the matrix $A = (a_{ij})_{i,j=0,\ldots,8}$ defined by $a_{ij} := (\beta_i, \alpha_j)$ is a generalized Cartan matrix of type $\hat{E}_8$, i.e. we get

$$A = \begin{pmatrix} 2 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}.$$

□

Remark 1.14. An important property of $\mathcal{B}'$ is the following: Any alternative root basis $\mathcal{B}''$ of $\mathcal{R}'$ is $\mathfrak{W}'$-conjugate to $\mathcal{B}'$, i.e. there exists $w \in \mathfrak{W}'$ such that $\mathcal{B}'' = w\mathcal{B}'$ (see [Kac, Prop. 5.9]).

There is a finite root system associated to each affine infinite root system. In our case, this finite root system of type E_8 is generated by the restricted root basis

$$\mathcal{B} := \{\alpha_0, \ldots, \alpha_7\}.$$

The root basis $\mathcal{B}$ generates a finite root system by [Kac, §6.3] via the action of the reflection group $\mathfrak{W} := \langle s_{\alpha_0}, s_{\alpha_1}, \ldots, s_{\alpha_7} \rangle$. We denote this root system by $\mathcal{R}$:

$$\mathcal{R} := \mathfrak{W} \cdot \mathcal{B}$$

Remark 1.15. Let d be the class of the anticanonical divisor as in Definition 1.8. For every positive element α of the infinite root system $\mathcal{R}'$ there exists a root $\alpha_{E_8} \in \mathcal{R}$ in the associated finite root system and $n \in \mathbb{N}$ such that α can be written in the form

$$\alpha = \alpha_{E_8} + n \cdot d.$$

This is proved in [Kac, Prop. 6.3].

Definition 1.16. Let Q' be the lattice defined by

$$Q' := d^{\perp} \subset \Lambda_{1,9}.$$

The lattice Q' projects onto the lattice

$$Q := \operatorname{im}(d^{\perp}) \subset \Lambda_{1,9}/\mathbb{Z}d.$$

Lemma 1.17. *The vectors $\alpha_0, \dots, \alpha_7$ project into Q and their images span Q. The action of $\mathfrak{W}$ on $\mathcal{B}$ induces an action on Q.*

Proof. This statement follows from the fact that $\alpha_0, \dots, \alpha_7$ are orthogonal to d and thus $s_{\alpha_0}, \dots, s_{\alpha_7}$ act trivially on $\mathbb{Z}d$. Moreover, the vectors $\alpha_0, \dots, \alpha_7$ are linearly independent and span a unimodular lattice of rank $8 = \operatorname{rk} Q$. □

Remark 1.18. It follows from Lemma 1.17 that Q is a root lattice of type E_8. Moreover, the lattice Q' is a root lattice of type $\hat{E}_8$ and spanned by $\alpha_0, \dots, \alpha_8$.

Remark 1.19. Instead of using Definition 1.12 and the following discussion, it is possible to set

$$\begin{aligned} \mathcal{R}' &= \{\alpha \in \Lambda_{1,9} \mid \alpha \cdot \alpha = -2,\ \alpha \cdot d = 0\} \\ \mathcal{R} &= \operatorname{im} \mathcal{R}' \subset Q. \end{aligned}$$

Thus, all elements $\alpha \in Q'$ or $\alpha \in Q$ fulfilling $\alpha \cdot \alpha = -2$ and $\alpha \cdot d = 0$ are roots.

Back on the geometric side, the following proposition holds:

Proposition 1.20 ([HL, Prop. 1.3])**.** *The group of automorphisms* $\operatorname{Aut}(X/\mathbb{P}^1)$ *of X that induce a translation in every smooth fibre acts simply transitively on the set of sections. It is isomorphic to Q.*

As mentioned in Lemma 1.7, the ninth base point of a cubic pencil does not give any additional information. This fact justifies blowing down one of the exceptional curves that are determined by the geometric marking. By Proposition 1.20, the choice of exceptional curve does not change the isomorphism type of the resulting surface.

Proposition 1.21. *Blowing down an exceptional curve E_9 in the dP_9 surface X yields a del Pezzo surface $\bar{X}$ of degree 1.*

Proof. The conditions given in Remark 1.6 also ensure a blow-up of 8 points to be a del Pezzo surface. □

The geometric marking $\phi\colon \Lambda_{1,9} \to \mathrm{H}_2(X,\mathbb{Z})$ induces a geometric marking on $\bar{X}$, i.e. an isomorphism $\bar{\phi}\colon \Lambda_{1,8} = \langle l, e_1, \ldots, e_8\rangle \to \mathrm{H}_2(\bar{X},\mathbb{Z})$ such that

$$\begin{aligned} \bar{\phi}(e_i) &= [E_i]\,, \text{ for } i = 1, \ldots, 8, \\ \bar{\phi}(l) &= [L] \text{ and} \\ \bar{\phi}(\bar{d}) = \bar{\phi}(3l - e_1 - \ldots - e_8) &= [-K_{\bar{X}}]\,. \end{aligned}$$

In fact, the geometric marking $\bar{\phi}$ already determines ϕ.
This motivates an alternative definition of Q as

$$Q := \bar{d}^{\perp} \subset \Lambda_{1,8} = \Lambda_{1,9}/\mathbb{Z}e_9.$$

Next, we want to have a closer look at the relation between the action of $\mathfrak{W}'$, the previous choice of geometric marking and the geometry of our surface. Note that similar statements have also been published recently in [MM].

Theorem 1.22. *Let X be a surface obtained by blowing up 9 points in $\mathbb{P}^2$. Let ϕ be a geometric marking (Definition 1.8) and let $\mathfrak{W}'$ and $\mathcal{R}'$ be as in Definition 1.12. The following statements are equivalent:*

1. *X is a dP_9 surface.*
2. *The composition $\phi \circ w$ is a geometric marking for all $w \in \mathfrak{W}'$.*
3. *Every element $\alpha \in \mathcal{R}'$ can be written as a difference $\phi^{-1}([E]) - \phi^{-1}([E'])$, where E and E' are disjoint exceptional curves.*
4. *No element in $\mathcal{R}'$ represents an effective divisor $A \subset X$.*

Proof. See [L1, Thm. 1.8] for a short version of the proof.

$1 \Rightarrow 2$ We are checking statement 2 for a set of generators of $\mathfrak{W}'$. If all generators map geometric markings to geometric markings, the same is true for finite compositions and therefore for all of $\mathfrak{W}'$.

Having established the definitions one can easily check that $s_{\alpha_i},\ i = 1, \dots, 8$, just interchanges e_i and e_{i+1}. The action of s_{α_0} is given by

$$\begin{aligned} l &\mapsto s_{\alpha_0}(l) = l + 1\alpha_0 = 2l - e_1 - e_2 - e_3 \\ e_1 &\mapsto s_{\alpha_0}(e_1) = e_1 + 1\alpha_0 = l - e_2 - e_3 \\ e_2 &\mapsto s_{\alpha_0}(e_2) = e_2 + 1\alpha_0 = l - e_1 - e_3 \\ e_3 &\mapsto s_{\alpha_0}(e_3) = e_3 + 1\alpha_0 = l - e_1 - e_2 \end{aligned}$$

leaving the other e_i invariant as α_0 is orthogonal to $e_i,\ i > 3$. This construction yields another geometric marking as mentioned in Remark 1.9.

$2 \Rightarrow 3$ Because $\mathcal{R}' := \mathfrak{W}'\mathcal{B}'$ and because the generators $s_{\alpha_i},\ i = 1, \dots, 8$, just interchange basis elements in $\mathcal{B}'$, we can write each root $\alpha \in \mathcal{R}'$ as

$$\alpha = w\alpha_8$$

for some $w \in \mathfrak{W}'$. As statement 3 holds for α_8 and by the analysis above, $\mathfrak{W}'$ maps exceptional classes to exceptional classes, this is true for all $\alpha \in \mathcal{R}'$.

$3 \Rightarrow 4$ Let $\alpha \in \mathcal{R}'$ be the preimage of $[A]$. Assume that $A \sim E - E'$, where E and E' are disjoint exceptional curves. We can complete $\phi^{-1}([E]) = e_1,\ \phi^{-1}([E']) = e_2$ to get an orthogonal basis $l, e_1, e_2, \dots, e_9$ of $\Lambda_{1,9}$. It follows that

$$\phi(l) \cdot [A] = l \cdot \alpha = l \cdot (e_1 - e_2) = 0.$$

As $\phi(l)$ corresponds to a line class in $\mathbb{P}^2$, any effective divisor in $\mathbb{P}^2$ has strictly positive intersection with $\phi(l)$. Therefore, A has to be contained in the fibres of the blow-down of the exceptional curves represented by $e_1, \dots, e_9$. We can make the following general ansatz for α:

$$\phi^{-1}\left([A]\right) = \sum_{i=1}^{9} \lambda_i e_i;\ \lambda_i \in \mathbb{N}$$

Because

$$A \cdot E' = 1 = (\sum_{i=1}^{9} \lambda_i e_i) \cdot e_2 = \lambda_2 e_2 \cdot e_2 = -\lambda_2,$$

it follows that $\lambda_2 = -1$, which contradicts $\lambda_2 \in \mathbb{N}$.

$4 \Leftrightarrow 1$ By [L1, p. 8], all positive roots α_{E_8} in the E_8 lattice Q are up to permutation given by elements of the form

$$\begin{aligned}&e_1 - e_2,\\ &l - e_1 - e_2 - e_3,\\ &2l - e_1 - \ldots - e_6 \text{ and}\\ &3l - 2e_1 - e_2 - \ldots - e_8.\end{aligned}$$

These do not represent positive divisors if and only if the corresponding surface is a dP$_9$ surface by the condition on the blown-up points in $\mathbb{P}^2$ formulated in Remark 1.6. We want to show that statement 4 is equivalent to this restriction on the blown-up points.
As mentioned in Remark 1.15, all positive roots $\alpha \in \mathcal{R}'$ arise as

$$\alpha = \alpha_{E_8} + nd$$

for some $n \in \mathbb{N}$ and α_{E_8} from the list above.
Assume that there is a root $\alpha \in \mathcal{R}'$ which represents a positive divisor A. Such a divisor A is given as the strict transform of a curve $\tilde{A}$ of degree d_A in $\mathbb{P}^2$. Let

$$\phi^{-1}(A) = \lambda_0 l - \sum_{i=1}^{9} \lambda_i e_i$$

be the class representing A. It holds that $d_A = \lambda_0$. Remark 1.19 implies that

$$\phi^{-1}(A) \cdot d = 3\lambda_0 - \sum_{i=1}^{9} \lambda_i = 0.$$

Thus, it follows that $\tilde{A}$ meets any element of the pencil of cubic curves in $\mathbb{P}^2$ in the base points of the cubic pencil exactly $3d_A$ times (counted with multiplicities). This is the number expected by Bézout's Theorem.
As there is a whole pencil of cubic curves through the 9 base points in $\mathbb{P}^2$, we can find at least one such cubic curve $\tilde{D}$ meeting $\tilde{A}$ in the 9 base points plus an additional point. By Bézout's Theorem, $\tilde{A}$ has to factor into the cubic curve $\tilde{D}$ and another curve $\tilde{A}'$ of degree $d_A - 3$.
The strict transform of $\tilde{A}'$ still has zero intersection with d and self-intersection -2, as

$$\begin{aligned}(\alpha - d) \cdot d &= \alpha \cdot d - 0 = 0 \text{ and}\\ (\alpha - d) \cdot (\alpha - d) &= \alpha \cdot \alpha - 0 + 0 = -2.\end{aligned}$$

We can therefore repeat this process k times until $d_A - 3k \leq 3$, which leads us to one of the cases which were excluded above. This proves the equivalence.

$\square$

Remark 1.23. A Q-automorphism $\overline{\Phi}$ induces a unique lattice automorphism

$$\Phi\colon \Lambda_{1,9} \to \Lambda_{1,9},$$

which leaves $d := 3l - e_1 - \cdots - e_9$ invariant (see [L1, p. 8]).

Remark 1.24. Theorem 1.22 also holds if we replace $\mathfrak{W}'$ with $\mathfrak{W}$, $\mathcal{R}'$ with $\mathcal{R}$ and ϕ with $\bar{\phi}$. Heuristically, this is the case because the information given by $\bar{X}$, eight of the exceptional curves or 8 points in $\mathbb{P}^2$, already determines X completely.
As we saw in the last part of the proof, where we argued that it suffices to look at $\mathcal{R}$ instead of $\mathcal{R}'$, this reasoning also applies to the information given by the lattice. The proof of Theorem 1.22 directly translates to this situation when replacing α_8 by α_7 and considering $e_1, \ldots, e_8$ instead of $e_1, \ldots, e_9$. In this case, the last part of the proof becomes trivial.

Moreover, the reverse of statement 2 in Theorem 1.22 holds:

Lemma 1.25. *Having fixed a geometric marking ϕ, any other geometric marking ϕ' arises as ϕw for some $w \in \mathfrak{W}'$.*

Proof. Let ϕ, ϕ' be two markings. They are lattice isomorphisms. Thus, they determine images of the root basis $\mathcal{B}'$ in $\mathrm{H}_2(X, \mathbb{Z})$, which span two $\hat{E}_8$ lattices in $\mathrm{H}_2(X, \mathbb{Z})$. By Remark 1.14 we know that all root bases are related via action of $\mathfrak{W}'$, i.e. there exists an element $w \in \mathfrak{W}'$ such that $\phi'(\mathcal{B}') = w\phi(\mathcal{B}')$. Note that

$$w\phi(\mathcal{B}') = \phi(w\mathcal{B}').$$

We are done, if we can show that we can reconstruct a marking ϕ' on all of $\Lambda_{1,9}$ from its behaviour on a root basis $\mathcal{B}''$. But this fact follows from Remark 1.23. Alternatively, a proof is given in [L2, p. 279]. $\square$

From now on, we will work with a fixed smooth anticanonical divisor $D \subset X$. The embedding into $\mathbb{P}^2$, which is given by blowing down $E_1, \ldots, E_9$, yields a group structure on D via the choice of one of 9 inflection points in $D \subset \mathbb{P}^2$. The other inflection points have order 3. Thus, we fix an inflection point as a marked point $P \in D$. Fixing this point yields an isomorphism $(D, P) \cong \mathrm{Pic}^0(D)$.

1.2. The Torelli theorem for dP_9 surfaces

In the end, we want to phrase our Torelli type theorem in the language of categories. So we pause here to introduce some more definitions.

Definition 1.26. Let $\underline{\mathrm{dP}_9}$ denote the category with pairs (X, D) of a dP_9 surface and a smooth anticanonical divisor D and the choice of a geometric marking $\phi\colon \Lambda_{1,9} \to \mathrm{Pic}(X)$ as objects. Its morphisms are isomorphisms $\varphi\colon (X, D) \to (X', D')$ between objects (X, D), (X', D'), which induce isomorphisms of geometric markings, i.e. $\phi' = \varphi_* \phi$.
Let moreover $\underline{\mathrm{Hom}_Q}$ denote the category whose objects are pairs (D, χ_D) of a smooth curve D of genus 1 and a group homomorphism $\chi_D \in Hom(Q, \mathrm{Pic}^0(D))$, which is injective on roots $\alpha \in \mathcal{R} \subset \Lambda_{1,9}$. The morphisms in $\underline{\mathrm{Hom}_Q}$ are given by isomorphisms $\psi\colon D \to D'$ such that the induced morphism on Picard varieties $\psi_{Pic}\colon \mathrm{Pic}^0(D) \to \mathrm{Pic}^0(D')$ yields a morphism $\chi_D \to \chi_{D'}$ by composition, i.e. $\chi_{D'} = \psi_{Pic} \circ \chi_D$.

Remark 1.27. An automorphism of $\Lambda_{1,9}$ descending to an automorphism of Q has to fix d (see e.g. [L1, p. 8]). For morphisms in $\underline{\mathrm{dP}_9}$ this follows from $D \mapsto D'$ for smooth anticanonical divisors D, D'.

We want to show that the categories $\underline{\mathrm{dP}_9}$ and $\underline{\mathrm{Hom}_Q}$ are equivalent, i.e. we need to construct a functor $F\colon \underline{\mathrm{dP}_9} \to \underline{\mathrm{Hom}_Q}$ and show that it is essentially surjective and fully faithful.
From now on, we proceed in two separate ways. First, we present the more general case for $\Bbbk$ an algebraically closed field of characteristic 0 as before. To this end, we use the theory of Picard groups to construct a morphism $\chi_D\colon Q \to \mathrm{Pic}^0(D)$ starting with a pair (X, D). This part will be generalized for families of dP_9 surfaces in the next chapter.
Secondly, we restrict ourselves to the case $\Bbbk = \mathbb{C}$ and use homology and integration to construct $\chi_D\colon Q \to \mathrm{Pic}^0(D)$. This case was already treated in [L1] and we basically give an elaboration of the proof in [L1]. In Chapter 5, this approach will be generalized to toric degenerations. Note that recently, connected results were published in [MM].
For the general case, i.e. $\Bbbk$ not necessarily equal to $\mathbb{C}$, we get the following proposition:

Construction 1.28. Let (X, D) be a pair of a dP_9 surface X and a smooth anticanonical divisor D over an algebraically closed field $\Bbbk$ of characteristic 0, which is endowed with a geometric marking ϕ. We construct a unique group homomorphism $\chi_D \in \mathrm{Hom}(Q, \mathrm{Pic}^0(D))$ from (X, D) such that no root $\alpha \in \mathcal{R}$ is contained in the kernel of χ_D.
Let $r\colon \mathrm{Pic}(X) \to \mathrm{Pic}(D)$ be the restriction map. It is a group homomorphism. We already know that $\alpha_0, \dots, \alpha_7$ generate Q in $\Lambda_{1,9}/\mathbb{Z}d$. So the geometric marking ϕ induces an isomorphism $Q \cong \mathit{span}(\phi(\alpha_0), \dots, \phi(\alpha_7))/\mathbb{Z}\phi(d)$.
We have to show that $r(\phi(\alpha_0)), \dots, r(\phi(\alpha_7)) \in \mathrm{Pic}^0(D)$ and that $\phi(d) \in \ker r$. Then r descends to a map

$$\bar{r}\colon \mathrm{Pic}(X)/\mathbb{Z}\phi(d) \supset \mathit{span}(\phi(\alpha_0), \dots, \phi(\alpha_7))/\mathbb{Z}\phi(d) \to \mathrm{Pic}^0(D).$$

The composition of $\bar{r}$ with the geometric marking ϕ yields the unique group homomorphism which we denote by χ_D.
First, we show that r maps $\phi(\alpha_0), \dots, \phi(\alpha_7)$ into $\mathrm{Pic}^0(D)$, consisting of the elements of degree 0 in $\mathrm{Pic}(D)$. So an element $\phi(\alpha) \in \mathrm{Pic}(X)$ is mapped to $\mathrm{Pic}^0(D)$ by r if and only if

$$\phi(\alpha) \cdot \phi(d) = 0.$$

As $\alpha_0, \dots, \alpha_7 \in d^\perp$, this condition is fulfilled.
Next, we have to prove that $r(\phi(d)) = 0 \in \mathrm{Pic}(D)$. This holds as D is a smooth fibre in a family of curves, so its normal bundle $\mathcal{O}_D(D) = \mathcal{O}_X(D) \otimes \mathcal{O}_D$ is trivial. Therefore, the restriction map r descends to the map $\bar{r}$. Thus, we have constructed χ_D in a unique way.
To finish our construction we have to show that no root is contained in the kernel of χ_D.
Let $\alpha \in \mathcal{R}$ be a root. Assume that $r(\phi(\alpha)) = 0$. We want to show that this implies that $\phi(\alpha)$ can be represented by an effective divisor A in X. By Theorem 1.22, the fact that X is a dP_9 surface is equivalent to no root α representing an effective divisor.
By the proof of $(3 \Rightarrow 4)$ in Theorem 1.22, the divisor A has to be contained in the exceptional curves of $X \to \mathbb{P}^2$ because $[A] \cdot \phi(l) = 0$. So we can assume

$$A = E - E'$$

for exceptional divisors E, E'. The difference $E - E'$ is contained in the kernel $\ker r$ if and only if $E|_D = E'_D$, which contradicts statement 3 in Theorem 1.22.

For the case $\Bbbk = \mathbb{C}$ we get the same homomorphism using integration. We start with a remark and a technical lemma.

Construction 1.29. Let X be a dP_9 surface and let $D \subset X$ be a smooth anticanonical divisor. We can construct a map

$$j\colon\ \mathrm{H}_1(D, \mathbb{Z}) \to \mathrm{H}_2(X \setminus D, \mathbb{Z})$$

as follows:
Let $[\delta] \in \mathrm{H}_1(D, \mathbb{Z})$ be a 1-cycle which is represented by a closed curve $\delta \subset D \subset X$. Note that D is a smooth elliptic fibre of $X \to \mathbb{P}^1$.
Let $U \subset X$ be an open tubular neighbourhood of $D \subset X$ such that D. Denote by $\pi\colon U \to D$ the corresponding contraction to D. The restriction $\bar{\pi}\colon \partial U \to D$ is an $\mathbb{S}^1$-bundle.
Note that $\partial U \subset X \setminus D$. We define a 2-cycle in $X \setminus D$ by

$$j([\delta]) := \left[\bar{\pi}^{-1}(\delta)\right].$$

This construction extends to a map on all of $\mathrm{H}_1(D, \mathbb{Z})$ via linearity and the existence of generators $\delta_1, \delta_2 \subset D$ of $\mathrm{H}_1(D, \mathbb{Z})$ if we can show that coboundaries are mapped to coboundaries. This holds as coboundaries in $C_1(D, \mathbb{Z})$ are contractible to a point and by composition with the contraction morphism π, their images can be contracted to a 1-dimensional fibre of $\bar{\pi}$.

Lemma 1.30 ([L1, Lemma 2.1])**.** *Let (X, D) be a pair of a dP_9 surface and a smooth anticanonical divisor together with a geometric marking ϕ as in Definition 1.8. The map j from Construction 1.29 fits into the short exact sequence*

$$0 \to \mathrm{H}_1(D, \mathbb{Z}) \xrightarrow{j} \mathrm{H}_2(X \setminus D, \mathbb{Z}) \xrightarrow{q} d^{\perp} \to 0,$$

where $d^{\perp} = \{v \in \Lambda_{1,9} \mid (v, d) = 0\} \subset \Lambda_{1,9}$.

Proof. Relative homology of the pair $(X, X \setminus D)$ yields a long exact sequence

$$\begin{aligned}\mathrm{H}_3(X, \mathbb{Z}) \to \mathrm{H}_3(X, X \setminus D) \to{}& \mathrm{H}_2(X \setminus D, \mathbb{Z}) \to \mathrm{H}_2(X, \mathbb{Z})\\ &\to \mathrm{H}_2(X, X \setminus D) \to \mathrm{H}_1(X \setminus D, \mathbb{Z}) \to \mathrm{H}_1(X, \mathbb{Z}).\end{aligned}$$

Poincaré duality and rationality of X yield $\mathrm{H}_3(X,\mathbb{Z}) = \mathrm{H}^1(X,\mathbb{Z}) = \mathrm{H}_1(X,\mathbb{Z}) = 0$ and thus

$$\begin{aligned} 0 \to \mathrm{H}_3(X, X\setminus D) \to{} & \mathrm{H}_2(X\setminus D,\mathbb{Z}) \to \mathrm{H}_2(X,\mathbb{Z}) \\ & \to \mathrm{H}_2(X, X\setminus D) \to \mathrm{H}_1(X\setminus D,\mathbb{Z}) \to 0. \end{aligned}$$

Let U be a tubular neighbourhood of D as above. In particular, D is a deformation retract of U. Because of this, we have $\mathrm{H}_k(X, X\setminus D) = \mathrm{H}_k(X, X\setminus U)$. By excision it holds that

$$\mathrm{H}_k(X, X\setminus U) = \mathrm{H}_k\left(X\setminus (X\setminus U)^\circ, (X\setminus U)\setminus (X\setminus U)^\circ\right) = \mathrm{H}_k(\bar{U}, \partial U).$$

By Poincaré-Lefschetz duality and retraction, we get that

$$\mathrm{H}_k(\bar{U}, \partial U) = \mathrm{H}^{4-k}(\bar{U},\mathbb{Z}) = \mathrm{H}^{4-k}(D,\mathbb{Z}).$$

Inserting these terms into the long exact sequence yields

$$\begin{aligned} 0 \to \mathrm{H}_1(D,\mathbb{Z}) = \mathrm{H}^1(D,\mathbb{Z}) \to{} & \mathrm{H}_2(X\setminus D,\mathbb{Z}) \to \mathrm{H}_2(X,\mathbb{Z}) \\ & \xrightarrow{\rho} \mathrm{H}^2(D,\mathbb{Z}) \to \mathrm{H}_1(X\setminus D,\mathbb{Z}) \to 0. \end{aligned}$$

Next, we have to show that $d^\perp \cong \ker\left(\rho\colon \mathrm{H}_2(X,\mathbb{Z}) \to \mathrm{H}^2(D,\mathbb{Z})\right)$. Because ρ maps all 2-cycles in X to zero which arise from the inclusion $X\setminus D \hookrightarrow X$, we get $\ker\rho \subset [D]^\perp \subset \mathrm{H}_2(X,\mathbb{Z})$. Moreover, $\ker\rho \subset \mathrm{H}_2(X,\mathbb{Z}) \cong \Lambda_{1,9}$ is free as $\mathrm{H}_2(X,\mathbb{Z})$ and $\mathrm{H}^2(D,\mathbb{Z}) \cong \mathrm{H}_0(D,\mathbb{Z})$ are torsion free. Equality of the two sets follows as we can construct a topological representative for each class in $d^\perp$ which is contained in $X\setminus D$:
A basis of $d^\perp$ is given by $(l-e_2-e_3)-e_1 =: e_0-e_1,\dots,e_8-e_9$. For a basis element e_i-e_j the construction of a representative is explicitly given by (1.2) on page 18 in the proof of Proposition 1.31. Thus, it follows that $d^\perp = \ker\rho$.
By construction, the map $j\colon \mathrm{H}_1(D,\mathbb{Z}) \to \mathrm{H}_2(X\setminus D,\mathbb{Z})$ is given by the composition of maps

$$\mathrm{H}_1(D,\mathbb{Z}) \xrightarrow{j'} \mathrm{H}_3(\bar{U},\partial U) = \mathrm{H}_3(X, X\setminus D) \xrightarrow{\mathrm{d}} \mathrm{H}_2(X\setminus D,\mathbb{Z}).$$

We want to understand j' explicitly. It is a pullback in homology, so it can also be understood as a pullback in cohomology twisted with Poincaré duality, i.e.

$$j'\colon \mathrm{H}_1(D,\mathbb{Z}) \to \mathrm{H}^1(D,\mathbb{Z}) \xrightarrow{\pi^*} \mathrm{H}^1(\bar{U},\mathbb{Z}) \to \mathrm{H}_3(\bar{U},\partial U),$$

where $\pi : \bar{U} \to D$ is the deformation retract.
Let $\delta \in \mathrm{H}_1(D,\mathbb{Z})$ and let $\delta^\vee \in \mathrm{H}^1(D,\mathbb{Z})$ be its dual. Then $\pi^*(\delta^\vee) = (\pi^{-1}(\delta))^\vee$ as π is orientation preserving. By Poincaré duality for manifolds with boundary, $(\pi^{-1}(\delta))^\vee \in \mathrm{H}^1(\bar{U},\mathbb{Z})$ is mapped to $\pi^{-1}(\delta) \in \mathrm{H}_3(\bar{U},\partial U)$ (see e.g. [Kr, p. 131]).
The boundary operator $\mathrm{d}\colon\ \mathrm{H}_3(\bar{U},\partial U) = \mathrm{H}_3(X, X\setminus D) \to \mathrm{H}_2(X\setminus D,\mathbb{Z})$ acts as an intersection of $\pi^{-1}(\delta)$ with ∂U, as this is where the boundary in $\mathrm{H}_3(\bar{U},\partial U)$ sits. This finishes the proof. □

Now we are ready to proceed to the next step.

Proposition 1.31. *Let (X, D) be a pair of a dP_9 surface X and a smooth anticanonical divisor D over $\mathbb{C}$, which is endowed with a geometric marking ϕ. By Construction 1.28, there is a unique group homomorphism $\chi_D \in \mathrm{Hom}(Q,\mathrm{Pic}^0(D))$ which arises via the restriction of line bundles $r\colon\ \mathrm{Pic}(X) \supset \phi(d^\perp) \to \mathrm{Pic}(D)$ and factorization through the quotient $\mathrm{Pic}(X)/\mathbb{Z}\phi(d)$.*
We can construct the same group homomorphism χ_D via integration of a logarithmic 2-form over 2-cycles in $X \setminus D$.

Proof. First, we want to construct a morphism $\chi'_D\colon Q \to \mathrm{Jac}(D)$. In a second step we will show that it is the right one, i.e. $\chi'_D = \chi_D$ up to composition with the Abel-Jacobi map.
Let η be a non-vanishing holomorphic 1-form on D. The Jacobian $\mathrm{Jac}(D)$ fits into the short exact sequence

$$0 \to \mathrm{H}_1(D,\mathbb{Z}) \xrightarrow{\int} \mathrm{H}^0(D,\Omega^1_D)^* \xrightarrow{\eta} \mathrm{Jac}(D) \to 0. \tag{1.1}$$

To make the connection with Lemma 1.30, we need a map connecting the middle terms of the short exact sequences from Lemma 1.30 and (1.1).
By [GH, p. 147], the Poincaré residue map yields a short exact sequence

$$0 \to \Omega^2_X \to \Omega^2_X(\log D) \xrightarrow{\mathrm{res}} \Omega^1_D \to 0.$$

Via the long exact sequence in cohomology and rationality of X there is an isomorphism of cohomology groups:

$$\ldots \to \underbrace{\mathrm{H}^0(X,\Omega^2_X)}_{=0} \to \mathrm{H}^0(X,\Omega^2_X(\log D)) \xrightarrow{\mathrm{res}} \mathrm{H}^0(D,\Omega^1_D) \to \underbrace{\mathrm{H}^1(X,\Omega^2_X)}_{=0} \to \ldots$$

Therefore, there exists a meromorphic 2-form $\omega \in \mathrm{H}^0(X, \Omega_X^2(\log D))$ with $\operatorname{res}\omega = \eta$, whose class is uniquely determined by η. The form ω spans $\mathrm{H}^0(X, \Omega_X^2(\log D)) \cong \mathbb{C}$.
The dual of the residue map yields an isomorphism on dual spaces

$$\operatorname{res}^*\colon \mathrm{H}^0(D, \Omega_D^1)^* \to \mathrm{H}^0(X, \Omega_X^2(\log D))^*.$$

Using the short exact sequence established in Lemma 1.30, we want to show that the following diagram is commutative:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{H}_1(D,\mathbb{Z}) \arrow[r, "j"] \arrow[dd, equal] & \mathrm{H}_2(X \setminus D, \mathbb{Z}) \arrow[r, "q"] \arrow[d, "\frac{1}{2\pi i}\int"] & d^\perp \arrow[r] & 0 \\
& & \mathrm{H}^0(X, \Omega_X^2(\log D))^* \arrow[d, "(\operatorname{res}^*)^{-1}"] & & \\
0 \arrow[r] & \mathrm{H}_1(D,\mathbb{Z}) \arrow[r, "\int"] & \mathrm{H}^0(D, \Omega_D^1)^* \arrow[r, "\eta"] & \operatorname{Jac}(D) \arrow[r] & 0
\end{tikzcd}$$

Choose 1-cycles $\delta_1, \delta_2 \subset D$ such that $\operatorname{span}([\delta_1], [\delta_2]) = \mathrm{H}_1(D, \mathbb{Z})$. For any logarithmic 2-form $\alpha \in \mathrm{H}^0(X, \Omega_X^2(\log D))$ the analysis of the morphism j in Lemma 1.30 and the Residue Theorem (see [Gri, Eq. (2.3)]) yield that

$$\frac{1}{2\pi i}\int_{j(\delta_i)} \alpha = \int_{\delta_i} \operatorname{res}\alpha, \ i = 1, 2.$$

Therefore, our diagram is commutative.
Note that commutativity implies that the composition

$$\mathrm{H}_1(D, \mathbb{Z}) \xrightarrow{j} \mathrm{H}_2(X \setminus D, \mathbb{Z}) \xrightarrow{\frac{1}{2\pi i}\int} \mathrm{H}^0(X, \Omega_X^2(\log D))^*$$

is injective although the second map is not.
By the definition of η, it holds for the evaluation maps that $\frac{1}{2\pi i}\operatorname{ev}_\eta = \operatorname{ev}_\omega \circ \operatorname{res}^*$, which can be restated as

$$\frac{1}{2\pi i}\operatorname{ev}_\omega(\beta) = \operatorname{ev}_\eta\left((\operatorname{res}^*)^{-1}\beta\right), \ \forall \beta \in \mathrm{H}^0(X, \Omega_X^2(\log D))^*,$$

as res^* is an isomorphism.

Thus, we can enlarge our diagram as follows:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{H}_1(D,\mathbb{Z}) \arrow[r, "j"] \arrow[d, equal] & \mathrm{H}_2(X \setminus D,\mathbb{Z}) \arrow[r, "q"] \arrow[d, "\frac{1}{2\pi i}\int"] & d^{\perp} \arrow[r] \arrow[d, dashed, "\chi''_D"] & 0 \\
0 \arrow[r] & \mathrm{H}_1(D,\mathbb{Z}) \arrow[r, "\frac{1}{2\pi i}\int \circ j"] \arrow[d, equal] & \mathrm{H}^0(X,\Omega^2_X(\log D))^* \arrow[r, "\mathrm{ev}_\omega"] \arrow[d, "(\mathrm{res}^*)^{-1}"] & \mathrm{Jac}(D) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathrm{H}_1(D,\mathbb{Z}) \arrow[r, "\int"] & \mathrm{H}^0(D,\Omega^1_D)^* \arrow[r, "\mathrm{ev}_\eta"] & \mathrm{Jac}(D) \arrow[r] & 0
\end{tikzcd}$$

The existence of the morphism

$$\chi''_D\colon d^{\perp} \to \mathrm{Jac}(D)$$

follows from commutativity.

Now we want to show that χ''_D factors into

$$\begin{tikzcd}
d^{\perp} \arrow[rr, "\chi''_D"] \arrow[dr, "/\mathbb{Z}d"'] & & \mathrm{Jac}(D). \\
& Q \arrow[ur, "\chi'_D"'] &
\end{tikzcd}$$

The factorization follows if we can show that

$$\frac{1}{2\pi i}\int\colon\ \mathrm{H}_2(X \setminus D,\mathbb{Z}) \to \mathrm{H}^0(X,\Omega^2_X(\log D))^*$$

maps a representative of $\phi(d)$ to zero.

So let $D' \neq D$ be another elliptic fibre of $X \to \mathbb{P}^1$, i.e. D' is a representative of $\phi(d)$, which is contained in $X \setminus D$ and hence also represents a class in $\mathrm{H}_2(X \setminus D,\mathbb{Z})$. Note that in particular, ω is a generator of $\mathrm{H}^2(X,\Omega^2_X(\log D))$ so it is enough to do the computation for ω. Locally near D, the logarithmic form ω is given as $\omega = \eta \wedge \frac{dx}{x}$ for x the pullback of a coordinate on the base $\mathbb{P}^1$ of X. Without loss of generality, assume that D' is given by $x \equiv c$. This yields

$$\frac{1}{2\pi i}\int_{D' \subset X \setminus D} \omega = \frac{1}{2\pi i}\int_{\{x \equiv c\} \subset X \setminus D} \eta \wedge \frac{dx}{x} = 0,$$

as D' is constant in x.

So $\phi(d)$ and by linearity $\mathbb{Z}\phi(d)$ is contained in the kernel of χ''_D, which therefore factors into the composition of the projection map and a group homomorphism

$$\chi'_D\colon Q \to \mathrm{Jac}(D).$$

Next, we have to show that χ'_D equals χ_D up to the Abel-Jacobi map.
Let $p, p' \in D$ and let $\mathcal{O}_D(p-p') \in \mathrm{Pic}^0(D)$ be the corresponding line bundle. The Abel-Jacobi map ν yields an isomorphism

$$\begin{aligned} \nu\colon\ \mathrm{Pic}^0(D) &\to \mathrm{Jac}(D) \\ \mathcal{O}_D(p-p') &\mapsto \int_{p'}^{p} \eta. \end{aligned}$$

We want to show that

$$\nu \circ \chi_D = \chi'_D$$

by computing χ'_D on generators of Q. Let $\alpha \in \{\alpha_0 = l - e_1 - e_2 - e_3, \alpha_1 = e_1 - e_2, \dots, \alpha_7 = e_7 - e_8\}$ be a set of generators of Q. By Theorem 1.22, it is possible to write

$$\alpha = \phi^{-1}([E]) - \phi^{-1}([E'])$$

for disjoint exceptional curves E and E'. Because $D \cdot E = D \cdot E' = 1$, the exceptional curves meet D in points p and p'. These points are distinct because E and E' are disjoint.
The restriction morphism r maps

$$\begin{aligned} r\colon d^{\perp} &\to \mathrm{Pic}^0(D) \\ \mathcal{O}_X(E - E') &\mapsto \mathcal{O}_D(p - p') \\ \mathcal{O}_X(D) &\mapsto \mathcal{O}_D. \end{aligned}$$

As $\chi''_D(q(A)) = \frac{1}{2\pi i}\int_A \omega$, we want to compute $\frac{1}{2\pi i}\int_A \omega$ for some $A \in \mathrm{H}_2(X \setminus D, \mathbb{Z})$ with $q(A) = \alpha \in d^{\perp}$ and q the morphism from Lemma 1.30.
So we need to find a 2-cycle $A \in \mathrm{H}_2(X \setminus D, \mathbb{Z})$ with $A \sim E - E'$. Let $\gamma\colon\ [0,1] \to D$ be a smooth path in D joining $p =: \gamma(0)$ and $p' =: \gamma(1)$. As before, we take a tubular open neighbourhood U of D such that D is a deformation retract of U and there is a retraction map $\pi\colon U \to D$.
Let $\partial\gamma|_{(0,1)}$ denote the boundary of $\pi^{-1}(\gamma)$ without the relative interiors of $\pi^{-1}(p)$ and $\pi^{-1}(p')$. We claim that it is possible to take A to be of the form

$$A := (E \setminus \pi^{-1}(p)) + \partial\gamma|_{(0,1)} - (E' \setminus \pi^{-1}(p')). \tag{1.2}$$

Now it holds that $A \sim E - E'$ in $\mathrm{H}_2(X \setminus D, \mathbb{Z})$ since

$$\begin{aligned} A - (E - E') &= (E \setminus \pi^{-1}(p)) - E + \partial\gamma|_{(0,1)} - (E' \setminus \pi^{-1}(p')) + E' \\ &= -\pi^{-1}(p) + \partial\gamma|_{(0,1)} + \pi^{-1}(p') \\ &= \partial(\pi^{-1}(\gamma)) \end{aligned}$$

is a boundary.
Using [Gri, Eq. (2.5)], it follows that

$$\begin{aligned} \frac{1}{2\pi i}\int_A \omega &= \frac{1}{2\pi i}\left(\int_{(E\setminus\pi^{-1}(p))}\omega + \int_{\partial\gamma|_{(0,1)}}\omega - \int_{(E'\setminus\pi^{-1}(p'))}\omega\right) \\ &= 0 + \int_\gamma \eta - 0 \\ &= \int_{p'}^{p} \eta \in \mathrm{Jac}(D). \end{aligned} \tag{1.3}$$

It holds that $\int_{(E\setminus\pi^{-1}(p))}\omega = 0$ and $\int_{(E'\setminus\pi^{-1}(p'))}\omega = 0$ because we are integrating a holomorphic 2-form over complex curves.
Thus, the morphism χ''_D is given by

$$\begin{aligned} \chi''_D \colon d^\perp &\to \mathrm{Jac}(D). \\ \alpha &\mapsto \int_{p'}^{p}\eta \\ d &\mapsto 0. \end{aligned}$$

Therefore, it holds that

$$\nu \circ r = \chi''_D.$$

As χ_D and χ'_D are induced by r and χ''_D via factorization through the quotient by $\mathbb{Z}d$, it follows that χ_D and χ'_D are equal up to the fixed isomorphism ν.
Note that in particular, equation (1.3) implies that no roots are contained in the kernel of χ'_D because $p \neq p'$. This finishes the proof. □

From now on, we proceed again with an algebraically closed field $\Bbbk$ of characteristic 0, which is not necessarily equal to $\mathbb{C}$.
Finally, we are ready to define the functor $F \colon \underline{\mathrm{dP}_9} \to \underline{\mathrm{Hom}_Q}$.

Proposition 1.32. *By Proposition 1.31 and Construction 1.28, we can associate with $(X, D) \in \mathbf{ob}(\underline{\mathrm{dP}_9})$ a morphism $\chi_D \in \mathbf{ob}(\underline{\mathrm{Hom}}_Q)$. Setting*

$$F(X, D) := \chi_D,$$

we can define a covariant functor $F\colon \underline{\mathrm{dP}_9} \to \underline{\mathrm{Hom}}_Q$.

Proof. To show that this construction yields a functor, we have to define F on morphisms. Let $\varphi \in \mathbf{mor}(\underline{\mathrm{dP}_9})$ be a morphism from (X, D) to (X', D'). The restriction $\varphi|_D\colon D \to D'$ induces an isomorphism $\varphi_{Pic}\colon \mathrm{Pic}^0(D) \to \mathrm{Pic}^0(D')$ and therefore, we set

$$F(\varphi) := \varphi|_D\colon F(X, D) \to F(X', D'),\ \chi_D \mapsto (\varphi_{Pic} \circ \chi_D\colon Q \to \mathrm{Pic}^0(D')).$$

The two additional conditions,

$$\begin{aligned} F(id_{(X,D)}) &= id_{F(X,D)} \text{ and} \\ F(\varphi_1 \circ \varphi_2) &= F(\varphi_1) \circ F(\varphi_2), \end{aligned}$$

are fulfilled by construction. □

Proposition 1.33. *Let D be a smooth genus 1 curve. With any homomorphism $\chi_D\colon Q \to \mathrm{Pic}^0(D)$ which is injective on roots we can associate a dP_9 surface (X, D). Moreover, we get $F(X, D) \cong \chi_D$, i.e. the functor F from Proposition 1.32 is essentially surjective.*

Proof. We want to embed $\mathrm{Pic}^0(D)$ into a copy of $\mathbb{P}^2$ and fix 9 points $P_1, \ldots, P_9$ in general position on its image such that blowing up $\mathbb{P}^2$ at these points yields a dP_9 surface X with anticanonical divisor isomorphic to D. A geometric marking on X is induced by the exceptional divisors $E_1, \ldots, E_9$ associated to the points $P_1, \ldots, P_9$.
Pick a point $P \in D$. Note that all points on a smooth genus 1 curve are equivalent. The point P defines an isomorphism $\mathrm{Pic}^0(D) \cong (D, P)$. Reading the pair (D, P) as an elliptic curve, the Weierstrass-embedding yields

$$\mathrm{Pic}^0(D) \cong (D, P) \hookrightarrow \mathbb{P}(\mathcal{O}_D(3P)) \cong \mathbb{P}^2. \tag{1.4}$$

By abuse of notation, denote by (D, P) also the image of (D, P) in $\mathbb{P}(\mathcal{O}_D(3P))$. Note that $D \subset \mathbb{P}(\mathcal{O}_D(3P))$ inherits a group structure with P the identity element.

Next, we have to fix points $P_1, \ldots, P_9$.
The morphism χ_D maps the basis elements $\alpha_0, \ldots, \alpha_7 \in Q$ into $\mathrm{Pic}^0(D)$ and composition with the embedding (1.4) yields points in $\mathbb{P}(\mathcal{O}_D(3P))$:

$$\begin{aligned} \alpha_0 = [l - e_1 - e_2 - e_3] &\mapsto a_0 \in \mathbb{P}(\mathcal{O}_D(3P)) \\ \alpha_1 = [e_1 - e_2] &\mapsto a_1 \in \mathbb{P}(\mathcal{O}_D(3P)) \\ \alpha_2 = [e_2 - e_3] &\mapsto a_2 \in \mathbb{P}(\mathcal{O}_D(3P)) \\ &\vdots \\ \alpha_7 = [e_7 - e_8] &\mapsto a_7 \in \mathbb{P}(\mathcal{O}_D(3P)) \end{aligned}$$

We set

$$a_0 + a_1 + 2a_2 =: -3P_3 \tag{1.5}$$

with respect to the group structure on D. Equation (1.5) fixes P_3 up to the addition of arbitrary elements of order 3 in D.
After fixing P_3, the other $P_i,\ i = 1, \ldots, 9$, are defined as follows:

$$\begin{aligned} P_2 &:= a_2 + P_3 \\ P_1 &:= a_1 + P_2 \\ P_4 &:= P_3 - a_3 \\ P_5 &:= P_4 - a_4 \\ &\vdots \\ P_8 &:= P_7 - a_7 \end{aligned} \tag{1.6}$$

The additional point P_9 is fixed by Cayley-Bacharach Theorem (Lemma 1.7).
We want to argue that these points are in general position because χ_D is injective on roots.
Assume that there is a line through P_1, P_2 and P_3. If we insert $a_1 = P_1 - P_2$ and $a_2 = P_2 - P_3$ into equation (1.5), we get that

$$a_0 + P_1 + P_2 + P_3 = 0.$$

The existence of a line through P_1, P_2 and P_3 implies via the induced group law on $D \subset \mathbb{P}(\mathcal{O}_D(3P))$ that $P_1 + P_2 + P_3 = 0$. Thus, $a_0 = 0$, so χ_D is not injective on

roots leading to a contradiction.
The other two cases (a conic through 6 points and a cubic through 8 points, one of them being singular) follow from this discussion because they are in the $\mathfrak{W}$-orbit of α_0. By statement 2 in Theorem 1.22, generality of the position of the base points is independent of the choice of geometric marking. So another choice of geometric marking would yield another set of points in another projective space with a line through 3 of them.
Define X to be the blow-up

$$X \to \mathbb{P}(\mathcal{O}_D(3P)) \tag{1.7}$$

at the points $P_1, \ldots, P_9$ and fix as its anticanonical divisor the strict transform of D in X. This anticanonical divisor is isomorphic to D by construction. By abuse of notation, we denote it by D, too.
Therefore, this construction yields a pair (X, D) with a geometric marking ϕ, which is induced by the exceptional divisors given by preimages of the points $P_1, \ldots, P_9$. The pair (X, D) is unique up to unique isomorphism by Remark 1.34 below.
Next, we have to show that $F(X, D)$ coincides with χ_D.
The functor F acts on (X, D) by associating a morphism

$$F(X, D) \colon Q \to \mathrm{Pic}^0(D).$$

This morphism only depends on the geometric marking ϕ and differences $P_1 - P_2, \ldots, P_7 - P_8$ and another difference $P_0 - P_3$, all understood with respect to the induced group law on D.
Of course, the differences $P_1 - P_2, \ldots, P_7 - P_8$ are base point independent. We have to take a closer look on $P_0 - P_3$ as P_0 was not introduced before. Morally, P_0 corresponds to an exceptional curve in X, which is not associated to one of the basis elements $e_1, \ldots, e_9$ by ϕ. Instead, it is associated to $l - e_1 - e_2$. Denote by $P \in D$ a choice of neutral element in D. The difference

$$P_0 - P_3 = -(P_1 - P) - (P_2 - P) - (P_3 - P) = -P_1 - P_2 - P_3 - 3P$$

is again base point independent, as base points are always of order 3 (see Remark 1.34).
Thus, the choice of base point in D does not affect any of these differences. As the

choice of base point was the only choice we made in our construction, it follows that

$$F(X, D) = \chi_D.$$

Note that this fact does not imply, that our construction leaves D invariant. In contrast, a different choice of base point corresponds to a translation on (D, P), which yields in general a non-trivial automorphism of D. But as we are only considering base point independent differences, the identity morphism id on $\mathrm{Pic}^0(D)$ is enough, i.e. $id \circ F(X, D) = \chi_D$. This finishes the proof. $\square$

Remark 1.34. Note that the pair (X, D) given by Proposition 1.33 is unique up to unique isomorphism. The only choice we made in its construction is involved in fixing P_3 via equation (1.5) on page 21. This choice is unique only up to addition of elements of order 3.
Considering $P_3 + O$ instead of P_3 for O an element of order 3 yields

$$\begin{aligned} P_1 &\mapsto P_1 + O \\ &\vdots \\ P_9 &\mapsto P_9 + O. \end{aligned}$$

We want to argue that the choice of P_3 does not change the isomorphism type of the resulting dP_9 surface as a different choice of 3-torsion point only selects a different inflection point of D in $\mathbb{P}^2$.
In particular, note that translation by an element O on $D \subset \mathbb{P}(\mathcal{O}_D(3P))$ is induced by a unique projective linear automorphism of $\mathbb{P}(\mathcal{O}_D(3P)) \cong \mathbb{P}^2$ if and only if $3O$ is trivial. Uniqueness of the automorphism is implied by the fact that translation on D fixes an isomorphism

$$\mathrm{H}^0(D, \mathcal{O}_D(3O')) \to \mathrm{H}^0(D, \mathcal{O}_D(3O' + 3O)).$$

This statement of course holds for all elements of order 3 (see e.g. Lemma 5 in [Ko]).
Therefore, the pair (X, D) we constructed is unique up to unique isomorphism by the universal property of the blow-up.

Remark 1.35. The condition formulated in equation (1.5) on page 21 is necessary. Let $(X, D) \in \mathbf{ob}(\underline{\mathrm{dP}_9})$ with a geometric marking ϕ. The geometric marking ϕ

induces a contraction morphism for the exceptional divisors $E_1, \dots, E_9$,

$$\begin{aligned} \pi\colon X &\to \mathbb{P}(\phi(l)) = \mathbb{P}(\mathcal{O}_D(3P)) \cong \mathbb{P}^2 \\ E_1 &\mapsto P_1 \\ &\vdots \\ E_9 &\mapsto P_9. \end{aligned}$$

Note that $\phi(l)$ corresponds to a line class. We can choose a divisor in $\mathbb{P}(\phi(l))$ representing the direct image $\pi_*(\phi(l))$ such that it intersects $\pi(D)$ in a triple point, which we denote by P.
Let L denote a line through 2 base points P_1 and P_2 in $\mathbb{P}(\mathcal{O}_D(3P))$. The class $\phi(e_0) = \phi(l - e_1 - e_2)$ corresponds to the third intersection point of $L \cap \pi(D)$. This point is given by $P_0 := -P_1 - P_2$ with respect to the induced group structure on $\pi(D)$ with neutral element $P \in \pi(D)$. There is an induced equation analogous to equation (1.5) via

$$\alpha_0 + \alpha_1 + 2\alpha_2 \mapsto [E_0 - E_3] + [E_1 - E_2] + 2\,[E_2 - E_3] \in \mathrm{Pic}(X)$$

and the restriction to $\pi(D)$, which induces

$$\begin{aligned} ((P_0 - P) - (P_3 - P)) + ((P_1 - P) - (P_2 - P)) + 2((P_2 - P) - (P_3 - P)) \\ &= P_0 + P_1 + P_2 - 3P_3 \\ &= (-P_1 - P_2) + P_1 + P_2 - 3P_3 \\ &= -3P_3. \end{aligned}$$

This explains equation (1.5).

Before we can show that the functor $F\colon \underline{\mathrm{dP}_9} \to \underline{\mathrm{Hom}}_Q$ is also fully faithful, we need another lemma:

Lemma 1.36. *Let D and D' be smooth genus 1 curves and let $\psi\colon D \to D'$ be an isomorphism inducing $\psi_{Pic}\colon \mathrm{Pic}^0(D) \to \mathrm{Pic}^0(D')$. Let $P \in D$ and $P' \in D'$ be arbitrary points. The morphism ψ_{Pic} extends to a morphism*

$$\begin{aligned} \tilde{\psi}\colon \mathbb{P}(\mathcal{O}_D(3P)) &\to \mathbb{P}(\mathcal{O}_{D'}(3P')) \\ D &\mapsto D' \\ P &\mapsto P'. \end{aligned}$$

Proof. Because $\psi_{Pic}\colon \operatorname{Pic}^0(D) \to \operatorname{Pic}^0(D')$ is a group homomorphism, it maps the identity element to the identity element. After a choice of $P \in D$ and $P' \in D'$ it therefore induces an isomorphism

$$(D, P) \to (D', P').$$

Composition with the direct image functor yields a mapping

$$\mathcal{O}_D(3P) \mapsto \mathcal{O}_{D'}(3P').$$

This construction yields an isomorphism $\mathrm{H}^0(D, \mathcal{O}_D(3P)) \to \mathrm{H}^0(D', \mathcal{O}_{D'}(3P'))$ as it is induced by an isomorphism $D \to D'$. Therefore, we get an isomorphism

$$\begin{aligned} \mathbb{P}(\mathcal{O}_D(3P)) &\to \mathbb{P}(\mathcal{O}_{D'}(3P')) \\ D &\mapsto D' \\ P &\mapsto P', \end{aligned}$$

where (D, P) and (D', P') denote, by abuse of notation, the images of (D, P) and (D', P') in $\mathbb{P}(\mathcal{O}_D(3P))$ and $\mathbb{P}(\mathcal{O}_{D'}(3P'))$. $\square$

Proposition 1.37. *The functor $F\colon \underline{\mathrm{dP}_9} \to \underline{\mathrm{Hom}_Q}$ from Proposition 1.32 is full.*

Proof. Let $\psi\colon D \to D'$ be a morphism and $\psi_{Pic}\colon \operatorname{Pic}^0(D) \to \operatorname{Pic}^0(D')$ the associated group homomorphism such that $\psi_{Pic} \circ \chi_D = \chi_{D'}$. Assume that $\chi_D = F((X, D), \phi)$ and $\chi_{D'} = F((X', D'), \phi')$. By Proposition 1.33, the pairs of dP_9 surfaces (X, D) and (X', D') are unique up to unique isomorphism. We want to construct an isomorphism

$$\varphi\colon (X, D) \to (X', D')$$

such that $F(\varphi) = \psi$.
Let $l \in \Lambda_{1,9}$ be the basis element associated to a line class and denote by $E_i \subset X$ and $E_i' \subset X'$ exceptional divisors with $\phi(e_i) = E_i$ and $\phi'(e_i) = E_i'$, $i = 1, \dots, 9$. Recall that the markings ϕ and ϕ' induce contraction morphisms

$$\begin{aligned} X &\to \mathbb{P}(\phi(l)) \\ D &\mapsto D \\ E_i &\mapsto P_i \end{aligned} \qquad \text{and} \qquad \begin{aligned} X' &\to \mathbb{P}(\phi'(l)) \\ D' &\mapsto D' \\ E_i' &\mapsto P_i'. \end{aligned}$$

The contraction morphisms induce isomorphisms on D and D'. Note that the embeddings $D \hookrightarrow \mathbb{P}(\phi(l))$ and $D' \hookrightarrow \mathbb{P}(\phi'(l))$ fix group structures on D and D' up to the choice of a point of order 3, which serves as the identity element. We choose such elements $P \in D$ and $P' \in D'$ of order 3.
Next, we want to use the morphism $\psi\colon D \to D'$. By Lemma 1.36, the morphism ψ induces an isomorphism

$$\begin{aligned} \tilde{\psi}\colon \mathbb{P}(\mathcal{O}_D(3P)) &\to \mathbb{P}(\mathcal{O}_{D'}(3P')) \\ D &\mapsto D' \\ P &\mapsto P'. \end{aligned}$$

Because $\chi_{D'} = \psi_{Pic} \circ \chi_D$, the morphism $\tilde{\psi}$ induces a map on the images of basis elements $\alpha_0, \ldots, \alpha_7 \in Q$ in $\mathbb{P}(\mathcal{O}_D(3P))$ and $\mathbb{P}(\mathcal{O}_{D'}(3P'))$. Note that these elements are just differences of the images of exceptional curves and thus independent of our choice of base point. We get a diagram

$$\begin{tikzcd}
 & Q \arrow[dl, "\chi_D"'] \arrow[dr, "\chi_{D'}"] & \\
\mathrm{Pic}^0(D) \arrow[rr, "\psi_{Pic}"] \arrow[d, hook] & & \mathrm{Pic}^0(D') \arrow[d, hook] \\
\mathbb{P}(\mathcal{O}_D(3P)) \arrow[rr, "\tilde{\psi}"] & & \mathbb{P}(\mathcal{O}_{D'}(3P')) \\
a_0 \arrow[rr, mapsto] & & a_0' \\
 & \vdots & \\
a_7 \arrow[rr, mapsto] & & a_7'.
\end{tikzcd}$$

As in the proof of Proposition 1.33 we construct points $P_1, \ldots, P_9 \in \mathbb{P}(\mathcal{O}_D(3P))$ and $P_1', \ldots, P_9' \in \mathbb{P}(\mathcal{O}_{D'}(3P'))$ via equation (1.5) on page 21. Alternatively, we just take $P_1, \ldots, P_9$ and $P_1', \ldots, P_9'$, which were fixed by the marking above. They fulfil equation (1.5) by construction. Possibly after adjusting our choice of base point, we can assume that

$$\tilde{\psi}\colon P_i \mapsto P_i', \ i = 1, \ldots, 9.$$

The universal property of the blow-up yields a unique morphism

$$\begin{tikzcd} X \arrow[rr, dashed] \arrow[d] & & X' \arrow[d] \\ (\mathbb{P}(\mathcal{O}_D(3P)), P_1, \dots, P_9) \arrow[rr, "\tilde{\psi}"] & & (\mathbb{P}(\mathcal{O}_{D'}(3P')), P'_1, \dots, P'_9). \end{tikzcd} \tag{1.8}$$

It restricts to a morphism $D \to D'$. The induced geometric markings are compatible by construction.
We define this morphism to be

$$\varphi\colon X \to X'.$$

If we can show that the functor F maps φ to ψ, we are done.
We want to argue that this holds by construction as $F(\varphi) = \varphi|_D$.
Recall that $\varphi|_D$ equals $\tilde{\psi}|_D$ up to composition with the contraction morphism and blowing up. Restriction to D yields the desired equality. □

Proposition 1.38. *The functor $F\colon \underline{\mathrm{dP}}_9 \to \underline{\mathrm{Hom}}_Q$ from Proposition 1.32 is faithful.*

Proof. Let $\varphi, \varphi'\colon ((X,D),\phi) \to ((X',D'),\phi')$ be morphisms in $\underline{\mathrm{dP}}_9$ such that

$$F(\varphi) = F(\varphi') = (\psi\colon D \to D') \in \mathbf{mor}\left(\underline{\mathrm{Hom}}_Q\right).$$

We have to show that the equation above implies that $\varphi = \varphi'$.
Note that $\psi = \varphi|_D = \varphi|_{D'}$. Moreover, let

$$\begin{aligned} F(X,D) &=: \left(D, \chi_D\colon Q \to \mathrm{Pic}^0(D)\right) \text{ and} \\ F(X',D') &=: \left(D', \chi_{D'}\colon Q \to \mathrm{Pic}^0(D')\right). \end{aligned}$$

We want to write φ and φ' as maps between blow-ups. Note that we can contract exceptional curves via morphisms

$$\begin{aligned} \pi\colon (X,D) &\to \operatorname{Proj} \mathrm{H}^0(X, \phi(l)) =: \mathbb{P} \text{ and} \\ \pi'\colon (X',D') &\to \operatorname{Proj} \mathrm{H}^0(X', \phi'(l)) =: \mathbb{P}'. \end{aligned}$$

Recall that $l \in \Lambda_{1,9}$ denotes the element which is mapped by ϕ to the class of a line in $\mathbb{P}^2$. These morphisms π and π' are isomorphisms when restricted to D and

D'. Moreover, the direct image maps yield $\varphi_*\colon \phi(l) \mapsto \phi'(l)$ and $\varphi'_*\colon \phi(l) \mapsto \phi'(l)$. Therefore, we get two isomorphisms $\tilde{\varphi}|_D$ and $\tilde{\varphi}'|_D$ which map

$$\begin{aligned} \mathbb{P} = \operatorname{Proj} \mathrm{H}^0(X, \phi(l)) &\to \operatorname{Proj} \mathrm{H}^0(X', \phi'(l)) = \mathbb{P}' \\ D &\mapsto D'. \end{aligned} \tag{1.9}$$

Recall that there are exceptional divisors in X and X', which are identified by $\phi(e_1), \dots, \phi(e_9)$ and $\phi'(e_1), \dots, \phi'(e_9)$. They are contracted to points $P_1, \dots, P_9 \in D \subset \mathbb{P}$ and $P'_1, \dots, P'_9 \in D' \subset \mathbb{P}'$.
We know that $\varphi|_D = \varphi'|_D$ and thus,

$$\varphi|_D(P_i) = \varphi'|_D(P_i),\ i = 1, \dots, 9.$$

As the points $P_1, \dots, P_9$ are in general position, this equation shows that the two isomorphisms from (1.9) are actually the same. We denote this isomorphism by $\tilde{\psi} := \tilde{\varphi}|_D = \tilde{\varphi}'|_D$.
Note that $\tilde{\psi}(P_i),\ i = 1, \dots, 9$, does not necessarily equal P'_i. Denote the image of the divisor fixed by $\phi(\alpha_i)$ in $D \subset \mathbb{P}$ by A_i and the image of the the divisor fixed by $\phi'(\alpha_i)$ in $D' \subset \mathbb{P}'$ by A'_i.
Because ψ yields a map

$$\psi_*\colon A_i \mapsto A'_i,$$

we know that differences of the P_i are mapped to differences of the $P'_i,\ i = 1, \dots, 9$. By the arguments from Remark 1.35 and Remark 1.34, this fixes the $\tilde{\psi}(P_i)$ and P'_i up to the addition of elements of order 3 or equivalently, up to automorphisms of $\mathbb{P}'$ leaving D' invariant.
Therefore, there is a unique isomorphism of $\operatorname{Proj} \mathrm{H}^0(X', \phi'(l)) = \mathbb{P}'$ such that composition with $\tilde{\psi}$ induces an isomorphism

$$\begin{aligned} \mathbb{P} = \operatorname{Proj} \mathrm{H}^0(X, \phi(l)) &\to \operatorname{Proj} \mathrm{H}^0(X', \phi'(l)) = \mathbb{P}' \\ D &\mapsto D' \\ P_1 &\mapsto P'_1 \\ &\vdots \\ P_9 &\mapsto P'_9. \end{aligned}$$

The morphism $\varphi\colon X \to X'$ is the unique map of blow-ups which is induced by this isomorphism. The same is true for φ'. Therefore, it follows that $\varphi = \varphi'$. □

Theorem 1.39. *The categories* $\underline{\mathrm{dP}_9}$ *and* $\underline{\mathrm{Hom}_Q}$ *are equivalent.*

Proof. The functor F defined in Proposition 1.32 is essentially surjective by Proposition 1.33 and moreover, it is fully faithful by Proposition 1.37 and Proposition 1.38. Thus, it induces an equivalence of categories. $\square$

2. A Torelli theorem for families of dP_9 surfaces

In this chapter, we want to generalize the proof from the previous chapter to families of dP_9 surfaces.

2.1. The setup

We will always work with locally Noetherian schemes over an algebraically closed field $\Bbbk$ of characteristic 0.

Definition 2.1. A *family of dP_9 surfaces* over an affine Noetherian scheme S is given by a scheme $\mathcal{X}$ with a divisor $\mathcal{D} \subset \mathcal{X}$ and a smooth, projective morphism

$$\pi\colon (\mathcal{X}, \mathcal{D}) \to S,$$

which restricts to a smooth projective morphism $\pi|_{\mathcal{D}}\colon \mathcal{D} \to S$. Every fibre of the morphism π over a geometric point $\bar{s} \to S$ is given by a pair $(X_{\bar{s}}, D_{\bar{s}})$ of a dP_9 surface $X_{\bar{s}}$ and a smooth anticanonical divisor $D_{\bar{s}}$.

As we want to mimic the definition of a geometric marking from the previous chapter (Definition 1.8), we need the existence of the relative Picard scheme for our families. But first, we want to recall the definition of the étale relative Picard functor.

Definition 2.2 ([Kl, Def. 9.2.2])**.** Let X be a scheme over S and let $T \to S$ be étale. The étale relative Picard functor maps

$$T \mapsto \mathrm{Pic}(X \times_S T)/\,\mathrm{Pic}(T).$$

Lemma 2.3. *The relative Picard scheme* $\mathbf{Pic}_{\mathcal{X}/S}$ *exists and represents the étale relative Picard functor. It is étale over* S*.*

Proof. The map $\pi\colon \mathcal{X} \to S$ is projective and flat as smoothness implies flatness. Moreover, its geometric fibres are integral. By [Kl, Thm. 9.4.8], it follows that $\mathbf{Pic}_{\mathcal{X}/S}$ exists, represents the étale Picard functor and is separated and locally of finite type over S.

Next, we want to show that $\mathbf{Pic}_{\mathcal{X}/S} \to S$ is smooth. By [Kl, Prop. 9.5.19], every point $s \in S$ with $\mathrm{H}^2(X_s, \mathcal{O}_{X_s}) = 0$ has an open neighbourhood $U(s)$ such that $\mathbf{Pic}_{\mathcal{X}/S}$ is smooth over $U(s)$.

Let $s \in S$ be a point and let $\bar{s} \to s$ be a geometric point over s. As the geometric fibre $X_{\bar{s}}$ over $\bar{s}$ is a dP_9 surface and hence rational, we know that

$$\mathrm{H}^2(X_{\bar{s}}, \mathcal{O}_{X_{\bar{s}}}) = 0.$$

Via base change we get a cartesian diagram

$$\begin{tikzcd} X_{\bar{s}} \arrow[r] \arrow[d] & X_s \arrow[d] \\ \bar{s} \arrow[r] & s. \end{tikzcd} \tag{2.1}$$

Because properness commutes with base change, we know that $X_s \to s$ is proper. Note that $\bar{s} \to s$ is flat. Therefore, [G3.1, Prop. 1.4.15] yields that cohomology commutes with base change and thus

$$\mathrm{H}^2(X_s, \mathcal{O}_{X_s}) = 0.$$

As this argument holds for all $s \in S$, the relative Picard scheme $\mathbf{Pic}_{\mathcal{X}/S}$ is smooth over S.

By [Kl, Cor. 9.5.13], the dimension of $\mathbf{Pic}_{\mathcal{X}/S}$ over closed points $s \in S$ can be estimated by

$$\dim \mathbf{Pic}_{X_s/k_s} \leq \dim \mathrm{H}^1(X_s, \mathcal{O}_s).$$

Let $\bar{s} \to s$ be a geometric point over a closed point $s \in S$. By the same reasoning as for diagram (2.1), flat base change yields

$$\dim \mathrm{H}^1(X_s, \mathcal{O}_s) = \dim \mathrm{H}^1(X_{\bar{s}}, \mathcal{O}_{X_{\bar{s}}}) = 0.$$

Thus, $\mathbf{Pic}_{\mathcal{X}/S} \to S$ has fibre dimension 0.

By [G4.4, § 17.10.1], smoothness and fibre dimension 0 imply that $\mathbf{Pic}_{\mathcal{X}/S} \to S$ is étale. □

Definition 2.4. Let $(\Lambda_{1,9})_S$ denote the constant sheaf with stalk the lattice $\Lambda_{1,9}$ on an affine Noetherian scheme S. A *marking* for a family of dP_9 surfaces $\pi\colon (\mathcal{X}, \mathcal{D}) \to S$ is given by an isomorphism of sheaves of lattices

$$\phi\colon (\Lambda_{1,9})_S \to R^1\pi_*\mathcal{O}_{\mathcal{X}}^* \cong \mathbf{Pic}_{\mathcal{X}/S},$$

which maps $d := 3l - e_1 - \cdots - e_9 \in \Lambda_{1,9}$ to the class of $\mathcal{O}_{\mathcal{X}}(\mathcal{D})$.

Remark 2.5. Lemma 2.3 shows that the existence of a marking ϕ on a family $(\mathcal{X}, \mathcal{D}) \to S$ of dP_9 surfaces is an open condition in the étale topology. This means that for all closed points $s \in S$ there is an étale neighbourhood $(U, u) \to (S, s)$ such that there exists a marking $\phi_U\colon (\Lambda_{1,9})_U \to \mathbf{Pic}_{\mathcal{X}_U/U}$ for the base changed family of dP_9 surfaces $(\mathcal{X}_U, \mathcal{D}_U) \to U$.

We will restrict ourselves to families $(\mathcal{X}, \mathcal{D}) \to S$ over affine Noetherian schemes S which do possess a marking ϕ.

Definition 2.6. Denote by $\underline{\mathbf{dP_9}}$ the category with objects families of dP_9 surfaces over affine Noetherian schemes $(\mathcal{X}, \mathcal{D}) \to S$ together with a marking ϕ as in Definition 2.4. The morphisms in this category are given by isomorphisms $\varphi\colon (\mathcal{X}, \mathcal{D})/S \to (\mathcal{X}', \mathcal{D}')/S$ such that we have $\phi' = \varphi_* \circ \phi$ for the corresponding markings ϕ and ϕ'.

Ultimately, we want to recover divisors from certain elements in the relative Picard scheme. This is more complicated than in the case of surfaces.

Proposition 2.7. *Let $\pi\colon (\mathcal{X}, \mathcal{D}) \to S$ be a dP_9 surface together with a fixed marking ϕ. Let $e \in \Lambda_{1,9}$ be a vector with $e^2 = -1$ and $e \cdot d = 1$ and let e_S be the corresponding section in $(\Lambda_{1,9})_S$. Via the marking ϕ, the vector e induces a unique section $\phi(e_S)$ in $\mathbf{Pic}_{\mathcal{X}/S}$.*
Then the section $\phi(e_S)$ lifts to a line bundle $\mathcal{E} \in \mathrm{Pic}(\mathcal{X})$, which is associated to an effective Cartier divisor $\mathbf{E}$. It holds that $\mathbf{E} \times_S \mathcal{D} \to S$ is an isomorphism, i.e. there is a global section $\sigma\colon S \to \mathcal{X}$ of $\pi\colon \mathcal{X} \to S$ which restricts to $\mathbf{E}_{\bar{s}} \cap \mathcal{D}_{\bar{s}}$ on geometric fibres.

Proof. The stalk of $R^1\pi_*\mathcal{O}_{\mathcal{X}}^*$ at a closed point $s \in S$ is isomorphic to $\Lambda_{1,9}$ via ϕ, i.e.

$$(R^1\pi_*\mathcal{O}_{\mathcal{X}}^*)_s = \varinjlim_U \mathrm{Pic}(\pi^{-1}(U))/\mathrm{Pic}(U) \cong \Lambda_{1,9},$$

where U runs over étale open neighbourhoods $(U,u) \to (S,s)$. This follows from the definition of the étale relative Picard functor.
Thus, for each $s \in S$ there exists an étale open neighbourhood $(U,u) \to (S,s)$ such that $e \in \Lambda_{1,9}$ lifts to an element $\mathcal{E}_U \in \mathrm{Pic}(\pi^{-1}(U))$.
We want to show that $\pi_*\mathcal{E}_U$ is locally free of rank 1 on U. Later we will use this fact to argue using global sections of $\mathcal{E}_U$.
As $\mathcal{E}_U$ is a line bundle, it is locally free of rank 1 on $\pi^{-1}(U)$. By the same argument as for diagram (2.1) on page 31, cohomology commutes with base change for a geometric point $\bar{s} \to (U,u) \to (S,s)$. Thus, it follows that

$$\mathrm{H}^i\left(X_u, (\mathcal{E}_U)_u\right) \otimes k_{\bar{s}} \cong \mathrm{H}^i\left(X_{\bar{s}}, (\mathcal{E}_U)_{\bar{s}}\right) \text{ for all } i \geq 0.$$

The geometric fibre $X_{\bar{s}}$ is a dP_9 surface by definition and in particular, it is rational. Denote by $E_{\bar{s}}$ the effective divisor in $X_{\bar{s}}$ corresponding to $(\mathcal{E}_U)_{\bar{s}}$. There is a short exact sequence

$$0 \to \mathcal{O}_{X_{\bar{s}}} \to \mathcal{O}_{X_{\bar{s}}}(E_{\bar{s}}) \to \mathcal{O}_{E_{\bar{s}}}(E_{\bar{s}}) \to 0.$$

Note that $E_{\bar{s}} \subset X_{\bar{s}}$ is an exceptional curve. As $\mathrm{H}^i(X_{\bar{s}}, \mathcal{O}_{E_{\bar{s}}}(E_{\bar{s}})) \cong \mathrm{H}^i(\mathbb{P}^1_{\bar{s}}, \mathcal{O}_{\mathbb{P}^1_{\bar{s}}}(-1))$, the corresponding long exact sequence in cohomology reads

$$0 \to \mathrm{H}^0(X_{\bar{s}}, \mathcal{O}_{X_{\bar{s}}}) \to \mathrm{H}^0(X_{\bar{s}}, \mathcal{O}_{X_{\bar{s}}}(E_{\bar{s}})) \to 0 \to 0 \to \mathrm{H}^1(X_{\bar{s}}, \mathcal{O}_{X_{\bar{s}}}(E_{\bar{s}})) \to \ldots.$$

Let $\bar{s} =: \operatorname{Spec} k_{\bar{s}}$. The long exact sequence in cohomology implies that

$$\mathrm{H}^i\left(X_{\bar{s}}, (\mathcal{E}_U)_{\bar{s}}\right) = \begin{cases} k_{\bar{s}}, & \text{for } i = 0 \\ 0, & \text{for } i \neq 0. \end{cases}$$

By [G3.2, Cor. 7.9.9], this yields that $\pi_*\mathcal{E}_U$ is locally free of rank 1.
So locally, possibly after a restriction of $U \to S$, there is an isomorphism of sheaves

$$\begin{aligned} \mathcal{O}_U &\to \pi_*\mathcal{E}_U \\ 1_U &\mapsto f_U. \end{aligned} \tag{2.2}$$

As $f_U \in \Gamma(U, \pi_*\mathcal{E}_U) = \Gamma(\pi^{-1}(U), \mathcal{E}_U)$ defines a section, we want to show that f_U is regular. If we have done this, it follows that f_U defines an effective Cartier divisor on $\pi^{-1}(U)$ ([G4.4, § 21.1.7] or [St, Tag 0C4S]).
As $\mathcal{E}_U$ is a line bundle, it is locally isomorphic to $\mathcal{O}_{\pi^{-1}(U)}$. Therefore, there are

images $\bar{f}_{U,x} \in \mathcal{O}_{\pi^{-1}(U),x}$ of f_U for $x \in \pi^{-1}(U)$. By § 21.1.7 and Prop. 21.1.8 in [G4.4] or [St, Tag 0C4S], the section f_U is regular if and only if $\bar{f}_{U,x}$ is a non-zero-divisor for all $x \in \pi^{-1}(U)$.
Let $x \in \pi^{-1}(U)$ be a point and $\operatorname{Spec} k_{\bar{x}} := \bar{x} \to x$ a geometric point over x. The composition with π yields a map $\operatorname{Spec} k_{\bar{x}} \to U$, which in turn constitutes a geometric point $\bar{s}_x \to S$. We get a cartesian diagram

$$\begin{tikzcd} X_{\bar{s}_x} \arrow[r, "\iota"] \arrow[d] & \mathcal{X}_U \arrow[d, "\pi"] \\ \bar{s}_x = \operatorname{Spec} k_{\bar{x}} \arrow[r] \arrow[ur] & U. \end{tikzcd}$$

Thus, each geometric point $\bar{x} \to x \in \mathcal{X}_U$ is contained in some geometric fibre $X_{\bar{s}_x} \to \mathcal{X}$.
By [M, Thm. II.3.2(a)], the stalks of a sheaf at geometric points in $\mathcal{X}$ are isomorphic to the stalks of the inverse image sheaf on geometric fibres, i.e. for $\iota\colon X_{\bar{s}} \to \mathcal{X}$ it holds that

$$(\iota^*\mathcal{E}_U)_{\iota^{-1}(x)} \cong (\mathcal{E}_U)_{\bar{x}}.$$

Moreover, henselization is flat and therefore, it is enough to show that $\bar{f}_{U,x}$ is a non-zero-divisor on geometric fibres. But this is the case as the restriction of $\mathcal{E}_U$ to geometric fibres is associated to the exceptional and hence effective Cartier divisor $E_{\bar{s}_x}$.
So we get an effective Cartier divisor

$$\mathbf{E}_U \subset \pi^{-1}(U) \subset \mathcal{X}$$

and $\mathcal{E}_U$ is the associated line bundle.
Next, we want to glue these local pieces. Starting with a different point $s' \in S$ and a different geometric point $\bar{s}' \to s'$, we get a divisor $\mathbf{E}_{U'}$ and an associated line bundle $\mathcal{E}_{U'}$ for another open set U'. Assume that $U \cap U' := V \neq \emptyset$.
We want to show that $\mathbf{E}_U$ and $\mathbf{E}_{U'}$ coincide on V. Let the section f_U be given by the isomorphism (2.2) on page 33. The section f_U generates $\mathbf{E}_U$. Via the same construction let $f_{U'}$ be a section generating $\mathbf{E}_{U'}$. By construction, there is a section $\alpha \in \mathcal{O}_V$ such that

$$f_U = f_{U'} \cdot \alpha.$$

We want to argue that $\alpha \in \mathcal{O}_V^*$. This holds because otherwise f_U would vanish completely on the (closed) zero set of α, which cannot be true as f_U is non-trivial

on geometric fibres. So f_U and $f_{U'}$ define the same effective Cartier divisor on $\pi^{-1}(V)$. In other words, $(\mathbf{E}_U)|_V = (\mathbf{E}_{U'})|_V$ and we get an effective Cartier divisor $\mathbf{E} \subset \mathcal{X}$ and a global line bundle $\mathcal{E} \in \mathrm{Pic}(\mathcal{X})$. Note that we worked étale locally, but as $\mathbf{E}$ is embedded in the scheme $\mathcal{X}$, the subspace $\mathbf{E}$ yields also a divisor in the Zariski topology by [Kn, § II.3.8].
Lastly, we have to construct a section $\sigma\colon S \to \mathcal{X}$. Our candidate for the image of σ is given by

$$\mathbf{E} \cap \mathcal{D} = \mathbf{E} \times_S \mathcal{D}.$$

There is a cartesian diagram

$$\begin{tikzcd} \mathbf{E} \cap \mathcal{D} \arrow[r] \arrow[d] & \mathcal{D} \arrow[d] \\ \mathbf{E} \arrow[r] & S. \end{tikzcd}$$

We know that $\mathcal{D} \to S$ is smooth so $\mathbf{E} \cap \mathcal{D} \to \mathbf{E}$ is also smooth. The morphism $\mathbf{E} \to S$ is flat by [Kl, Lemma 9.3.4(iii)] and the above discussion. Thus, it is smooth by the situation on the geometric fibres and base change. So $\mathbf{E} \cap \mathcal{D} \to S$ is also smooth.
Moreover, $\mathbf{E} \cap \mathcal{D} \to S$ has fibre dimension 0, is locally of finite type and a map between locally Noetherian schemes. Therefore, $\mathbf{E} \cap \mathcal{D} \to S$ is étale.
As $\mathbf{E} \cap \mathcal{D} \to S$ is surjective by construction, it has sections locally in the étale topology. These sections are unique on geometric points and extend to étale neighbourhoods. Thus, they have to glue and we get a global section

$$\sigma\colon S \to \mathbf{E} \cap \mathcal{D} \subset \mathcal{X}.$$

□

Definition 2.8. For a family of dP$_9$ surfaces $(\mathcal{X}, \mathcal{D}) \to S$ together with a fixed marking ϕ we will, without loss of generality, fix the section $\sigma\colon S \to \mathcal{X}$ as the section arising from $e = e_9$ as the choice of primitive element in Proposition 2.7.

Corollary 2.9. *Let $(\mathcal{X}, \mathcal{D}) \to S$ be a family of dP$_9$ surfaces together with a fixed marking ϕ. There is a short exact sequence of sheaves*

$$0 \to \mathbf{Pic}_S \to \mathbf{Pic}_{\mathcal{X}} \to \mathbf{Pic}_{\mathcal{X}/S} \to 0.$$

Proof. By [Kl, Ex. 9.3.11], the equation $\mathcal{O}_S \cong \pi_*\mathcal{O}_\mathcal{X}$ holds universally as π is proper and flat and has reduced, connected geometric fibres. Proposition 2.7 guarantees the existence of a section $\sigma\colon S \to \mathcal{X}$ of $\pi\colon \mathcal{X} \to S$. By [Kl, Rem. 9.2.11], the existence of a section yields the short exact sequence of sheaves. □

Remark 2.10. Note that the short exact sequence constructed in Corollary 2.9 is a short exact sequence only on the level of sheaves which does not necessarily translate to maps of line bundles. Let $e \in \Lambda_{1,9}$ with $e \cdot e = -1$ and $e \cdot d = 1$. By Proposition 2.7, we know that we can lift the image of e to a global line bundle on $\mathcal{X}$. Moreover, $\mathcal{O}_\mathcal{X}(\mathcal{D})$ is given by definition. So we can construct (uniquely fixed) preimages of global sections in $\mathbf{Pic}_{\mathcal{X}/S}$. This argument works for all global sections via linear extension and the marking ϕ because $l - e_1 - e_2$ also fulfils the requirements of Proposition 2.7 and $l - e_1 - e_2, e_1, e_2, \ldots, e_9$ is a basis of $\Lambda_{1,9}$.

We also have to generalize the Jacobian variety of a smooth genus 1 curve:

Proposition 2.11. *Let $(\mathcal{X}, \mathcal{D}) \to S$ be a family of dP_9 surfaces together with a marking ϕ. The relative Picard scheme $\mathbf{Pic}_{\mathcal{D}/S}$ exists and represents the étale relative Picard functor. It is separated and locally of finite type over S.*
The section σ yields an isomorphism

$$(\mathcal{D}, \sigma) \cong \mathbf{Pic}^0_{\mathcal{D}/S},$$

where $\mathbf{Pic}^0_{\mathcal{D}/S}$ denotes the connected component of the identity in $\mathbf{Pic}_{\mathcal{D}/S}$. All global sections of $\mathbf{Pic}^0_{\mathcal{D}/S}/S$ lift to line bundles on $\mathcal{D}$.

Proof. The map $\pi|_\mathcal{D}\colon \mathcal{D} \to S$ is flat as smoothness implies flatness. Moreover, its geometric fibres are integral and it is projective. Therefore, the first part of Proposition 2.11 follows from [Kl, Thm. 9.4.8].
As $\pi|_\mathcal{D}$ is projective and flat with integral curves as geometric fibres, $\mathbf{Pic}^0_{\mathcal{D}/S}$ is open and closed in $\mathbf{Pic}_{\mathcal{D}/S}$. Its fibres over points $\operatorname{Spec} k_s := s \in S$ are given by $\mathbf{Pic}^0_{\mathcal{D}_s/k_s}$ by [Kl, Ex. 9.6.21].
Because $\pi|_\mathcal{D}\colon \mathcal{D} \to S$ is smooth and proper and its fibres are geometrically connected of genus one, the pair $(\mathcal{D}, \sigma)$ is an elliptic curve. Therefore, $\mathbf{Pic}^0_\mathcal{D}/\mathbf{Pic}_S \cong (\mathcal{D}, \sigma)$ follows from [KM, p. 64], where an isomorphism and a group law on $(\mathcal{D}, \sigma)$ are constructed explicitly. The proposition follows via injectivity of $\mathbf{Pic}_\mathcal{D}/\mathbf{Pic}_S \hookrightarrow \mathbf{Pic}_{\mathcal{D}/S}$ by [Kl, Thm. 9.2.5]. □

Before proceeding to maps involving Picard schemes, we have to generalize the fact that a dP_9 surface is elliptically fibered over $\mathbb{P}^1$.

Lemma 2.12. *Let $\pi\colon (\mathcal{X}, \mathcal{D}) \to S$ be a family of dP_9 surfaces together with a marking ϕ. There exists a morphism $p\colon \mathcal{X} \to \mathbf{P}^1_{\mathbf{S}}$ onto a Zariski-locally trivial $\mathbb{P}^1$-bundle $\mathbf{P}^1_{\mathbf{S}}$ on S such that p contracts $\mathcal{D}$.*

Proof. We want to show that $\mathcal{O}_{\mathcal{X}}(\mathcal{D})$ is base point free and that $\pi_*\mathcal{O}_{\mathcal{X}}(\mathcal{D})$ is locally free of rank 2.
The base scheme S is Noetherian and hence quasi-compact. As π is proper, it follows that $\mathcal{X}$ is quasi-compact as well. So the sheaf $\mathcal{O}_{\mathcal{X}}(\mathcal{D})$ is locally generated by finitely many global sections if it is base point free.
The base of any finitely generated subspace in $\Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}}(\mathcal{D}))$ is given by the zero locus of an ideal and therefore, it is closed in $\mathcal{X}$. Thus, it contains a closed point $x \in \mathcal{X}$. The closed point $x \in \mathcal{X}$ is mapped to another closed point $\pi(x) \in S$ by π and therefore, x is contained in a closed fibre $X_{\pi(x)}$.
Let $\operatorname{Spec} k_{\overline{\pi(x)}} := \overline{\pi(x)} \to \pi(x) =: \operatorname{Spec} k_{\pi(x)}$ be a geometric point over $\pi(x) \in S$ and let $X_{\overline{\pi(x)}} = X_{\pi(x)} \times_{k_{\pi(x)}} k_{\overline{\pi(x)}}$ be the geometric fibre over $\overline{\pi(x)}$. There exists a preimage $\overline{x} \in X_{\overline{\pi(x)}}$ of the point $x \in X_{\pi(x)}$.
The restriction $(\mathcal{O}_{\mathcal{X}}(\mathcal{D}))_{X_{\overline{\pi(x)}}}$ is base point free as $X_{\overline{\pi(x)}}$ is a dP_9 surface. This fact contradicts the existence of $\overline{x}$. Thus, $\mathcal{O}_{\mathcal{X}}(\mathcal{D})$ is base point free. We get a morphism

$$p\colon \mathcal{X} \to \operatorname{Proj}\left(\Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}}(\mathcal{D}))\right).$$

This morphism contracts $\mathcal{D}$ by construction. Next, we have to show that $\pi_*\mathcal{O}_{\mathcal{X}}(\mathcal{D})$ is locally free of rank 2.
First, $\pi_*\mathcal{O}_{\mathcal{X}}(\mathcal{D})$ is coherent by [G3.1, Thm. 3.2.1] as it is the direct image of a coherent sheaf under a proper morphism of locally Noetherian schemes. Thus, it is locally free if there exists a cover of étale neighbourhoods where it is free by [H, Ex. III.10.5].
Via restriction to geometric fibres, which is possible by [M, Thm. II.3.2(a)] and [M, Rem. II.3.8], we get that $\pi_*\mathcal{O}_{\mathcal{X}}(\mathcal{D})$ has free stalks of rank 2 on geometric points. This means that each geometric point has an étale neighbourhood where $\pi_*\mathcal{O}_{\mathcal{X}}(\mathcal{D})$ is free by Nakayama's Lemma.
This proves the lemma. □

Now we are ready to formulate an analogue of Construction 1.28.

Proposition 2.13. *Let $(\mathcal{X}, \mathcal{D}) \to S$ be a family of dP_9 surfaces together with a fixed marking ϕ. The elements $l-e_1-e_2-e_3, e_1-e_2, \dots, e_7-e_8 \in \Lambda_{1,9}$ induce global sections of $(\Lambda_{1,9})_S$. There are invertible sheaves $\mathcal{L}'_0, \dots, \mathcal{L}'_7$ on $\mathcal{X}$ corresponding to these sections. Their restriction to $\mathcal{D}$ yields line bundles $\mathcal{L}_0, \dots, \mathcal{L}_7$ on $\mathcal{D}$, which yield sections $l_0, \dots, l_8$ in $\mathbf{Pic}^0_{\mathcal{D}/S}$. This construction descends to a group homomorphism*

$$\chi_{\mathcal{D}/S} \colon Q \to \mathrm{Hom}_S(S, \mathbf{Pic}^0_{\mathcal{D}/S}) \cong \mathrm{Hom}_S(S, (\mathcal{D}, \sigma))$$

such that $\chi_{\mathcal{D}/S}$ does not vanish on roots $\alpha \in \mathcal{R}$ (see Definition 1.12).

Proof. We know that $(\mathcal{D}, \sigma) \cong \mathbf{Pic}^0_{\mathcal{D}/S} \subset \mathbf{Pic}_{\mathcal{D}/S}$ is open and closed by Proposition 2.11.
As the invertible sheaves $\mathcal{L}'_0, \dots, \mathcal{L}'_7$ restrict to elements in $\mathbf{Pic}_{\mathcal{D}/S}$, they induce sections $l_0, \dots, l_8 \colon S \to \mathbf{Pic}_{\mathcal{D}/S}$. Moreover, $l_0|_{k_s}, \dots, l_8|_{k_s}$ have image in $\mathbf{Pic}^0_{D/k_s}$ for any point $s \in S$. Because S is connected, it follows from topology that the morphisms $l_0, \dots, l_8$ also map into the corresponding connected component of $\mathbf{Pic}_{\mathcal{D}/S}$, i.e. into $\mathbf{Pic}^0_{\mathcal{D}/S}$.
We want to show that the line bundle $\mathcal{O}_{\mathcal{X}}(\mathcal{D})$ on $\mathcal{X}$ corresponding to $d := 3l - e_1 - \dots - e_9 \in \Lambda_{1,9}$ restricts to the trivial bundle $\mathcal{O}_{\mathcal{D}}$ on $\mathcal{D}$. This fact implies that the restriction morphism factorizes through the quotient by the image of $\mathbb{Z}d$.
By Lemma 2.12, there is a contraction morphism

$$\begin{aligned} p \colon \mathcal{X} &\to \mathbf{P}^1_{\mathbf{S}} \\ \mathcal{D} &\mapsto \Sigma. \end{aligned}$$

Note that using the section $\sigma \colon S \to \mathcal{D} \subset \mathcal{X}$ we can also write

$$\Sigma := \mathrm{im}(S \xrightarrow{\sigma} \mathcal{X} \xrightarrow{p} \mathbf{P}^1_{\mathbf{S}}).$$

Moreover, the composition $p \circ \sigma \colon S \to \mathbf{P}^1_{\mathbf{S}}$ yields a global section of $\mathbf{P}^1_{\mathbf{S}}/S$. As the morphism p contracts $\mathcal{D}$, we get that

$$\mathcal{O}_{\mathcal{X}}(\mathcal{D}) = p^* \mathcal{O}_{\mathbf{P}^1_{\mathbf{S}}}(\Sigma) \text{ and } \mathcal{O}_{\mathcal{X}} = p^* \mathcal{O}_{\mathbf{P}^1_{\mathbf{S}}}.$$

On $\mathbf{P}^1_{\mathbf{S}}$, there is a short exact sequence

$$0 \to \mathcal{O}_{\mathbf{P}^1_{\mathbf{S}}} \to \mathcal{O}_{\mathbf{P}^1_{\mathbf{S}}}(\Sigma) \to \mathcal{O}_{\Sigma}(\Sigma) \to 0.$$

As p^* is right exact and as there is a corresponding short exact sequence on $\mathcal{X}$, the short exact sequence yields a diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O}_{\mathcal{X}} \arrow[r] \arrow[d, equal] & \mathcal{O}_{\mathcal{X}}(\mathcal{D}) \arrow[r] \arrow[d, equal] & \mathcal{O}_{\mathcal{D}}(\mathcal{D}) \arrow[r] & 0 \\
\dots \arrow[r] & p^*\mathcal{O}_{\mathbf{P}^1_S} \arrow[r] & p^*\mathcal{O}_{\mathbf{P}^1_S}(\Sigma) \arrow[r] & p^*\mathcal{O}_{\Sigma}(\Sigma) \arrow[r] & 0.
\end{tikzcd}$$

Possibly after restriction of S, we get $\mathcal{O}_\Sigma(\Sigma) \cong \mathcal{O}_\Sigma$ and thus,

$$p^*\mathcal{O}_\Sigma(\Sigma) \cong p^*\mathcal{O}_\Sigma \cong \mathcal{O}_{\mathcal{D}} \cong \mathcal{O}_{\mathcal{D}}(\mathcal{D}).$$

This is enough as we are working relative to S, so the restriction of $\mathcal{O}_{\mathcal{X}}(\mathcal{D})$ to $\mathcal{D}$ vanishes in $\mathbf{Pic}_{\mathcal{D}/S}$.
Thus, we have shown that the restriction yields a group morphism $\Lambda_{1,9}/\mathbb{Z}d \to \mathbf{Pic}_{\mathcal{D}/S}$ and that the images of $l - e_1 - e_2 - e_3, e_1 - e_2, \dots, e_7 - e_8$ are contained in $\mathbf{Pic}^0_{\mathcal{D}/S} \cong (\mathcal{D}, \sigma)$.
The images of $l - e_1 - e_2 - e_3, e_1 - e_2, \dots, e_7 - e_8$ in $\Lambda_{1,9}/\mathbb{Z}d$ span Q and therefore, we get an induced group morphism

$$\chi_{\mathcal{D}/S} \colon Q \to \operatorname{Hom}_S(S, \mathbf{Pic}^0_{\mathcal{D}/S}) \cong \operatorname{Hom}_S\left(S, (\mathcal{D}, \sigma)\right).$$

We still have to show that $\chi_{\mathcal{D}/S}$ does not vanish on roots. The image of the section $\sigma \colon S \to \mathcal{X}$ is given by

$$\operatorname{im} \sigma = \mathbf{E}_9 \cap \mathcal{D},$$

i.e. it is the intersection of two effective Cartier divisors and hence is closed in $\mathcal{D}$. By the isomorphism $\mathbf{Pic}^0_{\mathcal{D}/S} \cong (\mathcal{D}, \sigma)$, the identity in $\mathbf{Pic}^0_{\mathcal{D}/S}$ is closed as well. The image of a single element $\alpha \in Q$ is closed in $\mathbf{Pic}^0_{\mathcal{D}/S}$ as it corresponds to a section and, by the same reasoning as in Proposition 2.7, a divisor in $\mathcal{D}$. There are only finitely many roots $\alpha \in \mathcal{R}$. Therefore,

$$V(\chi_{\mathcal{D}/S}) := \operatorname{im} \sigma \cap \chi_{\mathcal{D}/S}(\mathcal{R})$$

is a closed set. Every closed set in $\mathcal{D}$ contains a closed point, which is contained in a closed fibre D_s for some closed point $s \in S$. Let $\bar{s} \to s$ be a geometric point over s and $D_{\bar{s}}$ the associated geometric fibre. By Construction 1.28, the morphism $(\chi_{\mathcal{D}/S})|_{\mathcal{R}}$ cannot vanish there if and only if the corresponding fibre $X_{\bar{s}}$ is a dP_9 surface. So we are done. $\square$

Finally, we can define our second category:

Definition 2.14. Denote by $\underline{\mathbf{Hom_Q}}$ the category with objects given by pairs $(\mathcal{D} \to S, \chi_{\mathcal{D}/S})$, where S denotes an affine Noetherian scheme and $\mathcal{D} \to S$ is a smooth projective family of smooth curves of genus 1 such that a section $\sigma\colon S \to \mathcal{D}$ exists. Moreover, $\chi_{\mathcal{D}/S}\colon Q \to \mathrm{Hom}_S(S, \mathbf{Pic}^0_{\mathcal{D}/S})$ is a group homomorphism, which does not vanish on roots $\alpha \in \mathcal{R}$.
The morphisms in $\underline{\mathbf{Hom_Q}}$ are given by isomorphisms of S-schemes $\psi\colon \mathcal{D}/S \to \mathcal{D}'/S$, which induce isomorphisms $\psi_{Pic}\colon \mathbf{Pic}^0_{\mathcal{D}/S} \to \mathbf{Pic}^0_{\mathcal{D}'/S}$ such that

$$\left(\chi_{\mathcal{D}/S}\colon Q \to \mathrm{Hom}_S(S, \mathbf{Pic}^0_{\mathcal{D}/S})\right) \mapsto \left(\chi_{\mathcal{D}'/S} = \psi_{Pic} \circ \chi_{\mathcal{D}/S}\colon Q \to \mathrm{Hom}_S(S, \mathbf{Pic}^0_{\mathcal{D}'/S})\right).$$

Remark 2.15. Note that a section $\sigma\colon S \to \mathcal{D}$ induces an isomorphism

$$(\mathcal{D}, \sigma) \cong \mathbf{Pic}^0_{\mathcal{D}/S}.$$

2.2. The equivalence of categories

Now we are ready to define a functor $F\colon \underline{\mathbf{dP_9}} \to \underline{\mathbf{Hom_Q}}$. We want to show that F is essentially surjective and fully faithful, i.e. it induces an equivalence of categories.

Proposition 2.16. *By Proposition 2.13, we can associate with* $((\mathcal{X}, \mathcal{D})/S, \phi) \in \mathbf{ob}(\underline{\mathbf{dP_9}})$ *a morphism* $\chi_{\mathcal{D}/S} \in \mathbf{ob}(\underline{\mathbf{Hom_Q}})$. *Setting*

$$F\left(((\mathcal{X}, \mathcal{D})/S, \phi)\right) := (\mathcal{D}/S, \chi_{\mathcal{D}/S}),$$

we can define a covariant functor $F\colon \underline{\mathbf{dP_9}} \to \underline{\mathbf{Hom_Q}}$.

Proof. To show that this construction yields a functor, we have to prove that F behaves well on morphisms. Let $\varphi \in \mathbf{mor}(\underline{\mathbf{dP_9}})$ be a morphism, i.e.

$$\varphi\colon\ ((\mathcal{X}, \mathcal{D})/S, \phi) \to ((\mathcal{X}', \mathcal{D}')/S, \phi').$$

Denote by $r\colon \mathbf{Pic}_{\mathcal{X}/S} \supset \phi(d^{\perp}) \to \mathbf{Pic}^0_{\mathcal{D}/S}$ the restriction morphism used in the proof of Proposition 2.13. By definition, we have $\phi' = \varphi_* \circ \phi$ and moreover, $r \circ \varphi_* = (\varphi|_{\mathcal{D}})_* \circ r$ and therefore,

$$\chi_{\mathcal{D}'/S} = r \circ \phi' = r \circ \varphi_* \circ \phi = (\varphi|_{\mathcal{D}})_* \circ r \circ \phi = (\varphi|_{\mathcal{D}})_* \circ \chi_{\mathcal{D}/S}.$$

Thus, we set

$$F(\varphi) := \varphi|_{\mathcal{D}}.$$

Both additional conditions,

$$F(id_{(\mathcal{X},\mathcal{D})}) = id_{F(\mathcal{X},\mathcal{D})} \text{ and}$$
$$F(\varphi_1 \circ \varphi_2) = F(\varphi_1) \circ F(\varphi_2),$$

are fulfilled by construction. □

Before we can proceed to show that the functor F is essentially surjective, we need some more propositions, which generalize the situation over $\operatorname{Spec}\Bbbk$ from Chapter 1.

Proposition 2.17. *Let* $((\mathcal{X},\mathcal{D})/S,\phi) \in \mathbf{ob}(\underline{\mathbf{dP_9}})$. *There exists a locally trivial* $\mathbb{P}^2$*-bundle* $\mathbf{P^2_S}/S$ *and a morphism*

$$\mathcal{X} \to \mathbf{P^2_S}$$

which contracts the divisors $\mathbf{E}_1,\ldots,\mathbf{E}_9 \subset \mathcal{X}$ *given by Proposition 2.7. Its restriction to these divisors yields sections*

$$\tilde{\sigma}_1,\ldots,\tilde{\sigma}_9\colon S \to \mathbf{P^2_S}.$$

Proof. By Remark 2.10, there exists a line bundle $\mathcal{L}$ on $\mathcal{X}$ corresponding to the section $\phi((l)_S)$ in $\mathbf{Pic}_{\mathcal{X}/S}$. We have to show that $\mathcal{L}$ is generated by global sections and moreover that the space of sections $\Gamma(\mathcal{X},\mathcal{L})$ has rank 3.
First, we want to show that $\mathcal{L}$ is generated by global sections. Because $\mathcal{L}$ is an invertible sheaf, it is enough to show that the stalks of $\mathcal{L}$ do not vanish and that finitely many sections suffice to generate $\mathcal{L}$.
We argue in the same way as in the proof of Lemma 2.12. The fact that the base scheme S is affine and the morphism π is proper implies that $\mathcal{X}$ is quasi-compact. Thus, if locally finitely many sections generate $\mathcal{L}$, the same is true globally.
To show that $\mathcal{L}$ is base point free, take any finitely generated subspace

$$V \subseteq \Gamma(\mathcal{X},\mathcal{L}).$$

Its base locus is closed and by passing to a geometric fibre $X_{\bar{s}}$, $\bar{s} \to s \in S$, we produce a contradiction, as the preimage of the base in $X_{\bar{s}}$ would be non-empty

while $\mathcal{L}$ is base point free on $X_{\bar{s}}$ (see the proof of Lemma 2.12). The restriction $\mathcal{L}|_{X_{\bar{s}}}$ is base point free. This means that $\mathcal{L}$ is base point free by the same reasoning as in the proof of Lemma 2.12.
Base point freeness and quasi-compactness imply that $\mathcal{L}$ is generated by (finitely many) global sections. Thus, $\mathcal{L}$ induces a morphism

$$f\colon \mathcal{X} \to \mathrm{Proj}(\Gamma(\mathcal{X}, \mathcal{L})).$$

We want to show that $\mathrm{Proj}(\Gamma(\mathcal{X}, \mathcal{L}))$ is a $\mathbb{P}^2$-bundle over S or equivalently, that $\pi_*\mathcal{L}$ is locally free of rank 3. We will denote this $\mathbb{P}^2$-bundle by $\mathbf{P^2_S}$.
The line bundle $\mathcal{L}$ is locally free of rank 1. Let $s \in S$ be a point and let $\bar{s} \to s$ be a geometric point over s. By the argument we used for diagram (2.1) on page 31, cohomology commutes with base change. Thus, we get

$$\mathrm{H}^i\left(X_s, \mathcal{L}_s\right) \otimes k_{\bar{s}} \cong \mathrm{H}^i\left(X_{\bar{s}}, \mathcal{L}_{\bar{s}}\right) \text{ for all } i \geq 0.$$

Therefore, we have to compute the cohomology groups $\mathrm{H}^i\left(X_{\bar{s}}, \mathcal{L}_{\bar{s}}\right)$.
We want to use the Nakai-Moishezon criterion to identify an ample line bundle on $X_{\bar{s}}$, on which we will apply the Kodaira vanishing Theorem to compute higher cohomology groups.
To show that the line bundle on $X_{\bar{s}}$ corresponding to the element $4l - e_1 - \cdots - e_9$ is ample by the Nakai-Moishezon criterion, we note that $(4l - e_1 - \cdots - e_9)^2 = 7 > 0$ and irreducible curves in $X_{\bar{s}}$ are given either by the $e_i,\ i = 1, \ldots, 9$, or strict transforms of curves in $\mathbb{P}^2_{k_{\bar{s}}}$.
For $e_i,\ i = 1, \ldots, 9$, it holds that

$$(4l - e_1 - \cdots - e_9) \cdot e_i = 1 > 0.$$

Let $C \subset \mathbb{P}^2_{k_{\bar{s}}}$ be an irreducible curve of degree k and $\bar{C}$ its strict transform in $X_{\bar{s}}$. First, we treat the case $k \geq 3$. If C meets the blown-up points of $X_{\bar{s}} \to \mathbb{P}^2_{k_{\bar{s}}}$ with multiplicity m bigger than 9, it is not irreducible and we can split off a cubic curve by Bezout's Theorem (analogous to the proof of $4 \Leftrightarrow 1$ in Theorem 1.22). Thus, we get for its strict transform

$$(4l - e_1 - \cdots - e_9) \cdot \left[\bar{C}\right] = 4k - m > 0 \text{ for } k \geq 3,\ m \leq 9.$$

Next, we treat the cases $k = 2$ and $k = 1$. In these cases, the fact that the blown-up points are in general position yields further conditions on m. For $k = 2$ we get

$m \leq 5 < 4 \cdot 2$ and for $k = 1$ it follows that $m \leq 2 < 4 \cdot 1$. Thus, for all $k > 0$ it holds that

$$(4l - e_1 - \cdots - e_9) \cdot \left[\bar{C}\right] = 4k - m > 0.$$

So the line bundle corresponding to $(4l - e_1 - \cdots - e_9)$ is ample.
The Kodaira vanishing Theorem states that

$$\mathrm{H}^i(X_{\bar{s}}, [4l - e_1 - \cdots - e_9] \otimes K_{X_{\bar{s}}}) = \mathrm{H}^i(X_{\bar{s}}, [l]) = \mathrm{H}^i(X_{\bar{s}}, \mathcal{L}_{\bar{s}}) = 0, \ \forall\, i > 0.$$

By [G3.2, Cor. 7.9.9], this yields that $\pi_*\mathcal{L}$ is locally free of rank equal to the Euler number $\chi(\mathcal{L})$. Without loss of generality S is connected and therefore, $\chi(\mathcal{L})$ is constant and it is enough to compute it on a geometric fibre $X_{\bar{s}}$.
Let $e = 12$ denote the topological Euler number of a dP_9 surface $X_{\bar{s}}$. Riemann-Roch Theorem for surfaces yields

$$\chi(\mathcal{L}_{\bar{s}}) = \frac{K_{X_{\bar{s}}} \cdot K_{X_{\bar{s}}} + e}{12} + \frac{1}{2}(\mathcal{L}(\mathcal{L} - K_{X_{\bar{s}}})) = 1 + \frac{1}{2}(1+3) = 3$$

and therefore, we are done.
The divisors $\mathbf{E}_1, \ldots, \mathbf{E}_9$ are contracted by construction. $\square$

We also need a generalization of the Cayley Bacharach Theorem (Lemma 1.7).

Proposition 2.18. *Let S be an affine Noetherian scheme. For all 8 sections*

$$\sigma'_1, \ldots, \sigma'_8 \colon S \to \mathbb{P}^2_S$$

whose restriction to any geometric fibre yields points in general position (Remark 1.6), there exists a uniquely fixed ninth section σ'_9 such that any cubic in $\mathbb{P}^2_S$, which passes through $\sigma'_1, \ldots, \sigma'_8$ also passes through σ'_9.

Proof. Denote the structure morphism by $\pi \colon \mathbb{P}^2_S \to S$. Cubics in $\mathbb{P}^2_S$ correspond to global sections in the invertible sheaf $\mathcal{O}_{\mathbb{P}^2_S}(3)$ on $\mathbb{P}^2_S$. We want to single out global sections of $\mathcal{O}_{\mathbb{P}^2_S}(3)$ which correspond to cubics passing through the given sections. Define on S the sheaf

$$\mathcal{F} := \pi_*\mathcal{O}_{\mathbb{P}^2_S}(3).$$

The sheaf $\mathcal{F}$ is coherent by [G3.1, Thm. 3.2.1] because it is the direct image of a proper morphism of locally Noetherian schemes. As S is Noetherian, [H, Ex. II.5.7] implies that $\mathcal{F}$ is locally free if and only if its stalks on points are free. The stalks

on geometric points are free by the situation for dP_9 surfaces (using Thm. II.3.2.(a) and Rem. II.3.8 in [M]). Freeness at geometric points suffices by [H, Ex. III.10.5] and Nakayama's Lemma.
The rank of a locally free sheaf is a locally constant function. Because S is connected, it suffices to check the rank of $\mathcal{F}$ at geometric points. This argument yields $\operatorname{rk} \mathcal{F} = 10$.
Next, we want to define sheaves which record the behaviour of cubic curves on the images of $\sigma_1', \ldots, \sigma_8'$. To this end, we look at the sheaves on S given by

$$\sigma_i'^{*}\mathcal{O}_{\mathbb{P}^2_S}(3),\ i = 1, \ldots, 8.$$

These sheaves are locally free of rank 1.
Define a morphism of sheaves

$$\begin{aligned} \gamma \colon \mathcal{F} &\to \bigoplus_{i=1}^{8} \sigma_i'^{*}\mathcal{O}_{\mathbb{P}^2_S}(3) \\ \mathcal{F}(U) \ni f &\mapsto (f(\sigma_1'), \ldots, f(\sigma_8')) \in (\oplus_{i=1}^{8}\sigma_i'^{*}\mathcal{O}_{\mathbb{P}^2_S}(3))(U),\ \text{for } U \subset S. \end{aligned}$$

We want to show that γ is surjective.
By [M, Prop. II.2.6], a morphism between sheaves is surjective in the Zariski topology if and only if the corresponding map between sheaves in the étale topology is. By [M, Thm. II.2.15c)], a morphism between sheaves in the étale topology is surjective if and only if it is surjective on stalks at geometric points. Our morphism γ is surjective on stalks at geometric points because cubics in $\mathbb{P}^2_{\Bbbk}$ for $\Bbbk$ an algebraically closed field of characteristic 0 are base point free and 10-dimensional and our sections restrict to points in general position on geometric fibres.
The kernel of γ is given by a sheaf $\ker \gamma$. Its sections correspond to cubic curves in $\mathbb{P}^2_S$, which vanish on $\sigma_1', \ldots, \sigma_8'$. As S is Noetherian, the sheaf $\ker \gamma$ is coherent by [H, Prop. II.5.7]. By the situation on stalks, $\ker \gamma$ is locally free of rank 2. At least locally, we can find generators

$$e_1, e_2 \in \ker \gamma.$$

In particular, the generators $e_1, e_2 \in \ker \gamma$ correspond on an open set $U \subset S$ to sections $f_1, f_2 \in \Gamma(U, \mathcal{F})$, whose zero sets yield cubic curves $C_1, C_2 \subset \mathbb{P}^2_U$.
Next, we want to show that the intersection

$$Z := C_1 \cap C_2$$

is étale over S.
To check that Z is étale over S locally, let the coordinate ring of $\mathbb{P}^2_U$ be given by $A\,[x_0, x_1, x_2]$, where A denotes a ring. Without loss of generality assume that $Z \subset \{[x_0, x_1, x_2] \mid x_0 \neq 0\}$. After dehomogenizing and mapping $f_1 \mapsto \bar{f}_1$ and $f_2 \mapsto \bar{f}_2$, a morphism of schemes $Z \to U$ corresponds on the level of rings to a morphism

$$A \to A\,[x_1, x_2]\,/(\bar{f}_1, \bar{f}_2).$$

By [M, Cor. I.3.16], the morphism $Z \to U$ is étale if on Z

$$P := \det \begin{pmatrix} \partial \bar{f}_1/\partial x_1 & \partial \bar{f}_1/\partial x_2 \\ \partial \bar{f}_2/\partial x_1 & \partial \bar{f}_2/\partial x_2 \end{pmatrix}$$

is a unit in $A\,[x_1, x_2]\,/(\bar{f}_1, \bar{f}_2)$.
Look at the base change of the cubics C_1 and C_2 to an algebraically closed field of characteristic 0. The resulting curves intersect transversally. Thus, we know that P does not vanish on geometric points and therefore, it does not vanish on points. For all $s \in S$ let m_s denote the corresponding maximal ideal. We know that

$$P \in \big(A\,[x_1, x_2]\,/(\bar{f}_1, \bar{f}_2)\big) \setminus \left(\bigcup_{s \in S} m_s\right).$$

Therefore, P has to be a unit. So $Z \to U$ is étale.
Note that $\operatorname{im}\sigma'_1, \ldots, \operatorname{im}\sigma'_8 \subset Z$ yield connected components. Therefore,

$$Z \setminus \{\operatorname{im}\sigma'_1, \ldots, \operatorname{im}\sigma'_8\} \to U$$

is also étale and it is surjective. Thus, there are étale locally sections $U \to Z \setminus \{\operatorname{im}\sigma'_1, \ldots, \operatorname{im}\sigma'_8\}$. We want to argue that these sections glue due to their uniqueness:
Let $U' \subset S$ be another suitable open set and assume $V := U \cap U' \neq \emptyset$. On V, the generators $e'_1, e'_2 \in \ker\gamma$ on U' can be written as linear combinations of e_1 and e_2 with coefficients in $\mathcal{O}^*_V$. Let Z' denote the corresponding intersection of cubic curves over U'. It holds that $Z = Z'$ and as $\operatorname{im}\sigma'_1, \ldots, \operatorname{im}\sigma'_8$ glue naturally, the induced sections on $Z \setminus \{\operatorname{im}\sigma'_1, \ldots, \operatorname{im}\sigma'_8\}$ and $Z' \setminus \{\operatorname{im}\sigma'_1, \ldots, \operatorname{im}\sigma'_8\}$ have to glue as well.
So we get a global section $\sigma'_9 \colon S \to \mathbb{P}^2_S$. □

Corollary 2.19. *Proposition 2.18 also holds if we replace* $\mathbb{P}^2_S$ *by a locally trivial* $\mathbb{P}^2$*-bundle* $\mathbf{P^2_S}$ *over* S.

Proof. Choose a local trivialization of $\mathbf{P^2_S}$. As the sections on suitable open subsets are induced by Proposition 2.18, they are unique. Hence, they have to glue. □

Now we can go back to studying the functor $F\colon \underline{\mathbf{dP_9}} \to \underline{\mathbf{Hom}}_{\mathbf{Q}}$, which was introduced in Proposition 2.16. First, we give an analogue to Proposition 1.33. Note that the proof is very similar. But while in the proof of Proposition 1.33 it was no problem to define points in a copy of $\mathbb{P}^2_{\Bbbk}$, once we found the correct condition this construction works only étale locally for families. This poses some new problems. Therefore, we give the proof in full length.

Proposition 2.20. *For all pairs* $(\mathcal{D}/S, \chi_{\mathcal{D}/S}) \in \mathbf{ob}(\underline{\mathbf{Hom}}_{\mathbf{Q}})$ *there exists a family of* dP_9 *surfaces* $(\mathcal{X}, \mathcal{D})/S$ *together with a marking* ϕ *such that*

$$F((\mathcal{X}, \mathcal{D})/S, \phi) \cong (\mathcal{D}/S, \chi_{\mathcal{D}/S}),$$

i.e. the functor F *is essentially surjective.*

Proof. Let $\alpha_0, \dots, \alpha_7 \in Q$ denote the usual basis elements. The morphism $\chi_{\mathcal{D}/S}$ yields

$$\begin{aligned} \chi_{\mathcal{D}/S}\colon Q &\to \mathrm{Hom}_S(S, \mathbf{Pic}^0_{\mathcal{D}/S}) \\ \alpha_0, \dots, \alpha_7 &\mapsto \left(\sigma_0, \dots, \sigma_7\colon S \to \mathbf{Pic}^0_{\mathcal{D}/S} \cong (\mathcal{D}, \sigma)\right), \end{aligned} \tag{2.3}$$

where σ is a section in $\mathcal{D}$ fixing the neutral element of the group structure.
We want to embed $\mathbf{Pic}^0_{\mathcal{D}/S}$ into a $\mathbb{P}^2$-bundle $\mathbf{P^2_S}$ over S and fix étale locally 9 sections $\sigma'_1, \dots, \sigma'_9$ whose restriction to geometric fibres yields points in general position. Blowing up $\mathbf{P^2_S}$ at the images of these sections yields a family of dP_9 surfaces $\mathcal{X} \to S$ with a divisor isomorphic to $\mathcal{D}$. A marking on $\mathcal{X}$ is induced by the exceptional divisors $\mathbf{E}_1, \dots, \mathbf{E}_9$ associated to $\operatorname{im}\sigma'_1, \dots, \operatorname{im}\sigma'_9$.
Note that the choice of σ yields an embedding of S-schemes

$$\mathbf{Pic}^0_{\mathcal{D}/S} \cong (\mathcal{D}, \sigma) \hookrightarrow \mathbb{P}(\mathcal{O}_{\mathcal{D}/S}(3\sigma)) = \mathbf{P^2_S}. \tag{2.4}$$

Here, $\mathbf{P^2_S}$ is a locally trivial $\mathbb{P}^2$-bundle on S. As $(\mathcal{D}, \sigma)/S$ is an elliptic curve, there is a generalized Weierstrass-form for $(\mathcal{D}, \sigma)$ by [KM, p. 67–73]. The embedding (2.4) maps σ to a global section of $\mathbf{P^2_S}$.

By abuse of notation, denote by $(\mathcal{D}, \sigma)$ also the image of $(\mathcal{D}, \sigma)$ in $\mathbb{P}(\mathcal{O}_{\mathcal{D}/S}(3\sigma))$. Note that $\mathcal{D} \subset \mathbb{P}(\mathcal{O}_{\mathcal{D}/S}(3\sigma))$ inherits a group structure with σ as the identity element.

Next, we have to fix sections $\sigma'_1, \dots, \sigma'_9$. Other than in the proof of Proposition 1.33, this is only possible étale locally on S.

By composition of the morphism (2.3) with the embedding (2.4), basis elements $\alpha_0, \dots, \alpha_7 \in Q$ yield sections of $\mathbb{P}(\mathcal{O}_{\mathcal{D}/S}(3\sigma))$:

$$\begin{aligned}
\alpha_0 = [l - e_1 - e_2 - e_3] &\mapsto \left(\sigma_0^{\mathbb{P}} \colon S \to \mathbb{P}(\mathcal{O}_{\mathcal{D}/S}(3\sigma))\right) \\
\alpha_1 = [e_1 - e_2] &\mapsto \left(\sigma_1^{\mathbb{P}} \colon S \to \mathbb{P}(\mathcal{O}_{\mathcal{D}/S}(3\sigma))\right) \\
\alpha_2 = [e_2 - e_3] &\mapsto \left(\sigma_2^{\mathbb{P}} \colon S \to \mathbb{P}(\mathcal{O}_{\mathcal{D}/S}(3\sigma))\right) \\
&\vdots \\
\alpha_7 = [e_7 - e_8] &\mapsto \left(\sigma_7^{\mathbb{P}} \colon S \to \mathbb{P}(\mathcal{O}_{\mathcal{D}/S}(3\sigma))\right)
\end{aligned}$$

We want to set

$$\sigma_0^{\mathbb{P}} + \sigma_1^{\mathbb{P}} + 2\sigma_2^{\mathbb{P}} =: -3\sigma'_3 \tag{2.5}$$

with respect to the group structure on $(\mathcal{D}, \sigma)/S$. This equation would fix σ'_3 up to the addition of arbitrary elements of order 3 in $(\mathcal{D}, \sigma)/S$.

By [KM, Thm. 2.3.1], the kernel of multiplication by 3 in $\mathcal{D}/S$ is only finite and étale over S (and locally isomorphic to $\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$). Therefore, the elements of order 3 in $\mathcal{D}/S$ do not necessarily extend to global sections but only exist étale locally. So we are only able to fix σ'_3 étale locally on S.

Let $\mathcal{U}$ be a suitable étale cover of S. For each $U \in \mathcal{U}$ we get a restricted embedding

$$(\mathcal{D}_U, \sigma_U) \hookrightarrow \mathbb{P}(\mathcal{O}_{\mathcal{D}_U/U}(3\sigma_U)) = \mathbf{P}^{\mathbf{2}}_{\mathbf{S}}|_U$$

and restricted sections $\sigma_{0U}^{\mathbb{P}}, \dots \sigma_{7U}^{\mathbb{P}}$ corresponding to the basis elements $\alpha_0, \dots, \alpha_7 \in Q$. Moreover, we are now able to fix a section σ'_{3U} up to the addition of arbitrary elements of order 3.

After fixing σ'_{3U}, the other σ'_{iU}, $i = 1, 2, 4, \dots, 8$, are determined by (1.6) on page 21 replacing P_i with σ'_{iU} and a_i with $\sigma_{iU}^{\mathbb{P}}$. The additional section σ'_{9U} is fixed by Proposition 2.18.

The restriction of the sections $\sigma'_{1U}, \dots, \sigma'_{9U}$ to geometric fibres $\mathbb{P}^2_{\bar{s}} \subset \mathbf{P}^{\mathbf{2}}_{\mathbf{S}}|_U$, $\bar{s} \to$

$s \in U$, yields points in general position in $\mathbb{P}^2_{\bar{s}}$ because there the situation on a dP$_9$ surface applies (see proof of Proposition 1.33).
So we have 9 sections $\sigma'_{1U}, \ldots, \sigma'_{9U} \colon U \to \mathbb{P}(\mathcal{O}_{\mathcal{D}_U/U}(3\sigma_U)) = \mathbf{P}^\mathbf{2}_\mathbf{S}|_U$. As their images yield divisors in $\mathbf{P}^\mathbf{2}_\mathbf{S}|_U$, we can blow up $\mathbf{P}^\mathbf{2}_\mathbf{S}|_U$ at these divisors to get a scheme $\mathcal{X}_U$ over U:

$$\begin{tikzcd}
 & \mathcal{X}_U \arrow[d] \\
\mathcal{D}_U \arrow[r] \arrow[ur, dashed] \arrow[dr] & \mathbf{P}^\mathbf{2}_\mathbf{S}|_U \arrow[d] \\
 & U
\end{tikzcd}$$

The strict transform of $\mathcal{D}_U$ (defined e.g. in [H, p. 165]) is the image of a unique morphism $\mathcal{D}_U \to \mathcal{X}_U$ because the preimages of $\sigma'_{1U}, \ldots, \sigma'_{9U}$ in $\mathcal{D}_U$ correspond to divisors and therefore, they yield invertible sheaves by [H, Prop. II.7.14]. Denote the image of this morphism by $\mathcal{D}_U$, too. As geometric fibres we get dP$_9$ surfaces because the sections are in general position there and [H, Cor. II.7.15] applies.
We want to show that $\mathcal{X}_U \to U$ is smooth.
Because the geometric fibres are regular and the morphism is of finite type between locally Noetherian schemes by construction, we are done by [H, Thm III.10.2] if we can show that $\mathcal{X}_U \to U$ is flat. Flatness is a local property, so we can work on open subsets of U.
Locally, the closed subscheme $\bigcup_{i=1}^{9} \operatorname{im} \sigma'_{iU}$ is given as the complete intersection of two effective Cartier divisors C_1 and C_2 by the proof of Proposition 2.18. Thus, for a suitable open subset $W \subset U$ we can write

$$\left.\left(\bigcup_{i=1}^{9} \operatorname{im} \sigma'_{iU}\right)\right|_W = (C_1 \cap C_2).$$

Because of this, its ideal sheaf is locally free and also flat. So $\mathcal{X}_U \to (\mathbf{P}^\mathbf{2}_\mathbf{S})_U$ is flat (by the same reasoning as in [Fu, § B.6.7]) and by composition, the same holds for $\mathcal{X}_U \to U$.
Next, we want to construct a marking and argue that $\mathcal{X}_U \to U$ is projective.
As the inverse images of $\operatorname{im} \sigma'_{1U}, \ldots, \operatorname{im} \sigma'_{9U} \subset \mathbf{P}^\mathbf{2}_\mathbf{S}|_U$ yield divisors in $\mathcal{X}_U$, they correspond to line bundles. The same reasoning applies to $\operatorname{im} \sigma^{\mathbb{P}}_{0U}$. We can combine these line bundles on $\mathcal{X}_U$ to construct a line bundle $\mathcal{F}_U$, which restricts to an ample

line bundle on geometric fibres as $\sigma'_{1U}, \ldots, \sigma'_{9U}$ correspond to $e_1, \ldots, e_9 \in \Lambda_{1,9}$ and $\sigma^{\mathbb{P}}_{0U}$ is associated with $l - e_1 - e_2 - e_3 \in \Lambda_{1,9}$. For example, we can take $\mathcal{F}_U$ to be the line bundle corresponding to $4l - e_1 - \ldots - e_9$ following the reasoning in the proof of Proposition 2.17.

The morphism $\mathcal{X}_U \to U$ is locally of finite presentation as it is the blowing up of something of finite type. By [G4.3, Cor. 9.6.4], it follows that the part of $\mathcal{X}_U$, where an invertible sheaf $\mathcal{F}_U$ is relatively ample, is open. More precisely, it is given by the preimage of the set of points $s \in U$ such that the restriction of $\mathcal{F}_U$ to the fibre over $s \in U$ is ample.

Ampleness of $\mathcal{F}_U$ at geometric points translates to all points $s \in U$ via faithfully flat base change $\bar{s} =: \operatorname{Spec} k_{\bar{s}} \to \operatorname{Spec} k_s := s$. Indeed, injectivity of the map to projective space induced by $\mathcal{F}_{\bar{s}}$ implies injectivity over s. Therefore, it follows that $\mathcal{X}_U \to U$ is projective.

If the Picard scheme $\mathbf{Pic}_{\mathcal{X}_U/U}$ exists, the classes of the line bundles we constructed fix a marking ϕ as $l - e_1 - e_2 - e_3, e_1, e_2 \ldots, e_9$ span $\Lambda_{1,9}$. Existence of the Picard scheme follows in particular from smoothness of $\mathcal{X}_U \to U$ (e.g. by [Kl, Thm. 9.4.8]). Therefore, we have shown what we wanted to show locally on U. Next, we have to take care of gluing schemes constructed on different open sets $U, V \in \mathcal{U}$.

Let $V \in \mathcal{U}$ be another open subset of S such that $U \cap V \neq \emptyset$. We want to argue that there exists a unique automorphism of $\mathbf{P}^2_{\mathbf{S}}|_{U\cap V}$ which leaves $\mathcal{D}_{U\cap V}$ invariant and maps

$$
\begin{gathered}
(\sigma'_{1U})|_{U\cap V} \mapsto (\sigma'_{1V})|_{U\cap V} \\
\vdots \\
(\sigma'_{9U})|_{U\cap V} \mapsto (\sigma'_{9V})|_{U\cap V}.
\end{gathered}
$$

Let $\tilde{\sigma}_{U\cap V} \colon U \cap V \to (\mathcal{D}_{U\cap V}, \sigma_{U\cap V})$ be any section of order 3. The existence of an automorphism of $\mathbf{P}^2_{\mathbf{S}}|_{U\cap V}$ is equivalent to the existence of an isomorphism of line bundles $\mathcal{O}_{\mathcal{D}_{U\cap V}}(3\sigma_{U\cap V}) \cong \mathcal{O}_{\mathcal{D}_{U\cap V}}(3\tilde{\sigma}_{U\cap V})$. If there is such an isomorphism, the different embeddings with respect to $\sigma_{U\cap V}$ and $\tilde{\sigma}_{U\cap V}$ just correspond to a different choice of basis in $\Gamma(U \cap V, \mathcal{O}_{\mathcal{D}_{U\cap V}}(3\sigma_{U\cap V}))$ and hence they yield an automorphism of $\mathbf{P}^2_{\mathbf{S}}|_{U\cap V}$ and we are done. We get such an isomorphism because

$$
\mathcal{O}_{\mathcal{D}_{U\cap V}}(3\sigma_{U\cap V}) \cong \mathcal{O}_{\mathcal{D}_{U\cap V}}(3\sigma_{U\cap V} + 3(\tilde{\sigma}_{U\cap V} - \sigma_{U\cap V})) \cong \mathcal{O}_{\mathcal{D}_{U\cap V}}(3\tilde{\sigma}_{U\cap V})
$$

as $3(\tilde{\sigma}_{U\cap V} - \sigma_{U\cap V})$ is trivial with respect to the group structure on $(\mathcal{D}_{U\cap V}, \sigma_{U\cap V})$.

Because addition of an element of order 3 was the only freedom of choice we had in our construction of the blow-up, this is all we need. The induced automorphism of $\mathbf{P}^{\mathbf{2}}_{\mathbf{S}}|_{U\cap V}$ yields an isomorphism of blow-ups

$$((\mathcal{X}_U)_V, (\mathcal{D}_U)_V) \to ((\mathcal{X}_V)_U, (\mathcal{D}_V)_U),$$

which maps the markings ϕ_U and ϕ_V to each other.

So we can glue models $(\mathcal{X}_U, \mathcal{D}_U)$ for $U \in \mathcal{U}$ to get $\pi\colon (\mathcal{X}, \mathcal{D}) \to S$. But as this construction only worked étale locally, up to this point $\mathcal{X}$ is only an algebraic space, not a scheme.

By [Kn, Def. II.7.9], an algebraic space $\pi\colon \mathcal{X} \to S$ is a scheme if it carries a π-ample line bundle. In fact, S is a scheme by definition and a π-ample line bundle yields an embedding of $\mathcal{X}$ into a projective space $\mathbb{P}^n_S$ for some $n > 0$.

Étale locally, on each $U \in \mathcal{U}$, there is a π-ample line bundle $\mathcal{F}_U$ corresponding to a class $f := 4l - e_1 - \ldots - e_9 \in \Lambda_{1,9}$ on geometric points via the marking ϕ_U as above.

We have to show that these sheaves do glue and yield a global sheaf on the algebraic space $\mathcal{X}$. So let $U, V \in \mathcal{U}$ with $U \cap V \neq \emptyset$. The markings ϕ_U and ϕ_V glue on $U \cap V$. The blow-up construction yields fixed exceptional divisors and hence there are line bundles in $\mathrm{Pic}(\mathcal{X}_U)$ corresponding to an element $f \in \Lambda_{1,9}$ via the same argument as in Remark 2.10. Therefore, $\mathcal{F}_U$ and $\mathcal{F}_V$ glue as well.

So $\mathcal{X} \to S$ is a scheme and $((\mathcal{X}, \mathcal{D})/S, \phi)$ really is contained in $\mathbf{ob}(\underline{\mathbf{dP_9}})$.

Lastly, we have to show that $F\left((\mathcal{X}, \mathcal{D}), \phi\right)$ is isomorphic to $(\mathcal{D}/S, \chi_{\mathcal{D}/S})$.

The functor F is defined by associating to $((\mathcal{X}, \mathcal{D}), \phi)$ a morphism

$$F\left((\mathcal{X}, \mathcal{D}), \phi\right) : Q \to \mathrm{Hom}_S(S, \mathbf{Pic}^0_{D/S}).$$

This morphism is basically given by the restriction of line bundles on $\mathcal{X}$ to line bundles on $\mathcal{D}$ and does not depend on the choice of section σ in $\mathcal{D}$. Thus, it holds that

$$F\left((\mathcal{X}, \mathcal{D}), \phi\right) = \chi_{\mathcal{D}/S}.$$

Note that this equality does not mean that we did not change $\mathcal{D}/S$ by automorphisms during our construction, which contained the choice of a section σ. But as we are only considering differences, this choice does not show up on $F\left((\mathcal{X}, \mathcal{D}), \phi\right)$. This finishes the proof. $\square$

Before we can show that the functor $F\colon \underline{\mathbf{dP_9}} \to \underline{\mathbf{Hom}}_{\mathbf{Q}}$ is also fully faithful, we need an analogue of Lemma 1.36:

Proposition 2.21. *Let $\psi\colon \mathcal{D}/S \to \mathcal{D}'/S$ be an isomorphism of smooth genus 1 curves with arbitrary sections $\sigma \in \mathcal{D}/S$ and $\sigma' \in \mathcal{D}'/S$. There is an induced group homomorphism $\psi_{Pic}\colon \mathbf{Pic}^0_{\mathcal{D}/S} \to \mathbf{Pic}^0_{\mathcal{D}'/S}$ and an isomorphism*

$$\begin{gathered}\tilde{\psi}\colon \mathbb{P}(\mathcal{O}_{\mathcal{D}/S}(3\sigma)) \to \mathbb{P}(\mathcal{O}_{\mathcal{D}'/S}(3\sigma')) \\ \mathcal{D} \mapsto \mathcal{D}' \\ \sigma \mapsto \sigma'.\end{gathered}$$

Proof. The morphism ψ_{Pic} induces an isomorphism $\psi'\colon (\mathcal{D}, \sigma) \to (\mathcal{D}', \sigma')$ by composition with $(\mathcal{D}, \sigma) \cong \mathbf{Pic}^0_{\mathcal{D}/S}$ and $(\mathcal{D}', \sigma') \cong \mathbf{Pic}^0_{\mathcal{D}'/S}$. Considering the elliptic curves as groups, the morphism ψ_{Pic} is given by the tensor product with $(s_0) := 3\sigma' - 3\psi(\sigma) \in \mathbf{Pic}^0_{\mathcal{D}'/S}$. This tensor product maps

$$3\psi'(\sigma) \mapsto 3\sigma'.$$

The direct image map ψ'_* yields

$$\begin{gathered}\mathbf{Pic}_{\mathcal{D}/S} \to \mathbf{Pic}_{\mathcal{D}'/S} \\ \mathcal{O}_{\mathcal{D}/S}(3\sigma) \mapsto \mathcal{O}_{\mathcal{D}'/S}(3\sigma').\end{gathered}$$

The proposition follows in the same way as in the proof of Lemma 1.36. □

Now we are ready to generalize Proposition 1.37.

Proposition 2.22. *The functor $F\colon \underline{\mathbf{dP_9}} \to \underline{\mathbf{Hom}}_{\mathbf{Q}}$ is full.*

Proof. The proof proceeds basically in the same way as the proof of Proposition 1.37 replacing (X, D) with $(\mathcal{X}, \mathcal{D})$, (X', D') with $(\mathcal{X}', \mathcal{D}')$ and points with sections. Starting with a morphism $\psi \in \mathbf{mor}(\underline{\mathbf{Hom}}_{\mathbf{Q}})$ we construct a morphism of families of dP_9 surfaces.
The only difference between the proof of Proposition 1.37 and our case is that blowing up does not take place on a copy of $\mathbb{P}^2$ but on a locally trivial $\mathbb{P}^2$-bundle $\mathbf{P^2_S}$, which we blow up at étale locally constructed sections (see the proof of Proposition 2.20). We get blow-ups first étale locally on $\mathbf{P^2_S}$ and $\mathbf{P^2_S}'$ and morphisms between these blow-ups extend via gluing and uniqueness to morphisms of schemes. As everything else carries over exactly, we will not repeat the details here. □

Proposition 2.23. *The functor* $F\colon \underline{\mathbf{dP_9}} \to \underline{\mathbf{Hom_Q}}$ *is faithful.*

Proof. Again, we do not repeat the proof of Proposition 1.38 in detail. As in the proof of Proposition 2.22, the only difference between the present case and the proof of Proposition 1.38 is the fact that the base spaces of the blow-ups are locally trivial $\mathbb{P}^2$-bundles with sections, which are given only étale locally. But this problem is mended via uniqueness and gluing. □

Finally, we are able to give the main theorem of this chapter, which is an analogue to Theorem 1.39.

Theorem 2.24. *The categories* $\underline{\mathbf{dP_9}}$ *and* $\underline{\mathbf{Hom_Q}}$ *are equivalent.*

Proof. The functor F defined in Proposition 2.16 is essentially surjective by Proposition 2.20 and moreover fully faithful by Proposition 2.22 and Proposition 2.23. Thus, it induces an equivalence of categories. □

3. The setup of toric degenerations

In the following chapter, we want to give a short overview of the toric degeneration toolkit using the cone picture. Whereas everything works in arbitrary dimension, we restrict ourselves to $n = 2$ dimensions where it simplifies the presentation.

3.1. Toric degenerations and their central fibres

Toric degenerations of Calabi-Yau pairs

First, we have to define toric degenerations of Calabi-Yau pairs following the lines of [GS]. The central fibre of such a degeneration is given by a totally degenerate Calabi-Yau pair:

Definition 3.1 ([GS, Def. 1.6]/ [GS1, Def. 4.1])**.** A *totally degenerate Calabi-Yau pair* (X, D) is given by a reduced Gorenstein variety X and a reduced divisor $D \subset X$ with the following properties: Let

$$\nu \colon \tilde{X} \to X$$

be the normalization of X. The normalized variety

$$\tilde{X} =: \coprod_i X_i$$

is given by a disjoint union of algebraically convex toric varieties X_i. Moreover, the conductor locus $C \subset \tilde{X}$ is a reduced divisor and the divisor class

$$[C] + \nu^* [D]$$

equals the sum of all toric prime divisors in $\tilde{X}$. The restriction of the normalization to C, i.e. the map $\nu|_C \colon C \to \nu(C)$, is unramified and generically two to one. Lastly, the commutative diagram

$$\begin{tikzcd} C \arrow[r, hook] \arrow[d, "\nu|_C"'] & \tilde{X} \arrow[d, "\nu"] \\ \nu(C) \arrow[r, hook] & X \end{tikzcd}$$

is cartesian and cocartesian.

Definition 3.2. A *polarized* totally degenerate Calabi-Yau pair $(X, D, \mathcal{L})$ consists of a totally degenerate Calabi-Yau pair (X, D) and an ample line bundle $\mathcal{L}$ on X.

Definition 3.3 ([GS1, Def. 4.3])**.** A *toric log Calabi-Yau pair* is given by a polarized totally degenerate Calabi-Yau pair (X, D) with a logarithmic structure $\mathcal{M}_{(X,D)}$ on X fulfilling the following conditions:
In the singular locus $Sing(X) \subset X$, there is a closed subset $Z \subsetneq Sing(X)$, $\operatorname{codim} Z \geq 2$, not containing any toric stratum, such that $(X \setminus Z, \mathcal{M}_{(X,D)}|_{X \setminus Z})$ is fine and log smooth over the standard log point $O^\dagger$.
Let $\rho \in \Gamma(X, \overline{\mathcal{M}}_{(X,D)})$ be the image of $1 \in \mathbb{N}$ under the structure map, i.e.

$$\begin{aligned} (X, \mathcal{M}_{(X,D)}) &\to O^\dagger \\ \overline{\mathcal{M}}_{(X,D)} &\leftarrow \mathbb{N} \\ \rho &\leftarrow\!\shortmid 1. \end{aligned}$$

Let $\bar{x} \hookrightarrow X \setminus Z$ be a geometric point. The monomial $z^\rho \in \mathbb{C}[\overline{\mathcal{M}}_{(X,D),\bar{x}}]$ vanishes once along each toric Weil divisor of $\operatorname{Spec} \mathbb{C}[\overline{\mathcal{M}}_{(X,D),\bar{x}}]$. Moreover, we require the logarithmic structure $\mathcal{M}_{(X,D)}$ to be *positive* (see [GS1, Def. 4.17]). This condition will be stated more explicitly in Remark 3.31.

Remark 3.4. By [GS, Thm. 1.30], totally degenerate Calabi-Yau pairs (X, D) fulfilling some additional conditions arise as central fibres of so called toric degenerations $(\mathfrak{X}, \mathfrak{D}) \to T$ for a scheme T. Note that there is a divisorial log structure on $\mathfrak{X}$ induced by $X \cup \mathfrak{D} \subset \mathfrak{X}$. Restricting this divisorial log structure to (X, D) turns the central fibre (X, D) into a toric log Calabi-Yau pair by [GS1, Prop. 4.6].
The existence of a polarization ensures projectivity, which will be important later on.

Definition 3.5 ([GS, Def. 1.8])**.** Let T be the spectrum of a discrete valuation ring with closed point $O \in T$. A *toric degeneration of Calabi-Yau pairs* is a flat morphism $\pi\colon \mathfrak{X} \to T$ with a reduced divisor $\mathfrak{D} \subset \mathfrak{X}$ such that

1. the total space $\mathfrak{X}$ is normal;
2. the central fibre $X := \pi^{-1}(O)$ together with $D := \mathfrak{D} \cap X$ forms a totally degenerate Calabi-Yau pair (Definition 3.1);
3. there is a codimension 2 locus $\mathfrak{Z} \subset \mathfrak{X}$ not containing any toric stratum of X such that
$$\pi|_{\mathfrak{X}\setminus\mathfrak{Z}}\colon (\mathfrak{X}, \mathfrak{D}) \setminus \mathfrak{Z} \to (T, O)$$
is a log smooth morphism. Denote the intersection $\mathfrak{Z} \cap X$ by Z.

Analogously, one can define a *formal toric degeneration of Calabi-Yau pairs* (see [GS, Def. 1.9]) as a morphism $\hat{\pi}\colon \hat{X} \to \hat{O}$, where $\hat{O}$ denotes the completion of T at the closed point $O \in T$. The other properties of toric degenerations of Calabi-Yau pairs translate accordingly.
In [GS1], it is shown that a toric degeneration of Calabi-Yau pairs induces certain affine data. These data encode the central fibre (X, D) as well as the discrete part of a log structure $\mathcal{M}_{(X,D)}$ on X. In [GS], the same kind of affine data are used to construct a formal toric degeneration of Calabi-Yau pairs, proving that it is possible to go back and forth between affine data and (formal) toric degenerations.
Here, instead of looking at toric degenerations over the completions of spectra of discrete valuation rings, we want to use the closely related setup from [GHKS1] to construct a more universal family. But before doing this, we have to review some facts about the affine data used in [GS] and [GS1].

Integral affine manifolds with integral polyhedral decompositions

The structure of the central fibre X given by the normalization $\nu\colon \tilde{X} = \coprod X_i \to X$ is recorded by a polyhedral decomposition of an affine manifold with singularities.

Definition 3.6. An *affine manifold* B of dimension n is a topological manifold with an affine structure, i.e. an atlas with transition maps in

$$\mathrm{Aff}(\mathbb{R}^n) := \mathbb{R}^n \rtimes \mathrm{Gl}_n(\mathbb{R}).$$

We allow for B to contain a singular locus $\Delta \subset B$ of codimension 2, i.e. the atlas is only defined on $B_0 := B \setminus \Delta$. Denote the inclusion map by

$$\iota\colon B_0 \to B.$$

The manifold B is an *integral affine manifold* if the transition maps are contained in

$$\mathrm{Aff}(\mathbb{Z}^n) := \mathbb{Z}^n \rtimes \mathrm{Gl}_n(\mathbb{Z}).$$

Remark 3.7. We will always allow the affine manifold B to have non-empty boundary ∂B. The boundary ∂B will correspond to an anticanonical divisor $D \subset X$.

Definition 3.8 ([GHKS1, p. 9])**.** Let B be an integral affine manifold. An *integral polyhedral decomposition* $\mathscr{P}$ of B is a set of integral polyhedra together with integral affine maps $e\colon \omega \to \tau,\ \omega, \tau \in \mathscr{P}$. These integral affine morphisms map faces to faces and for any proper face $\omega \subsetneq \tau \in \mathscr{P}$ there is a morphism $e\colon \omega \to \tau$. These morphisms turn $\mathscr{P}$ into a category $\underline{Cat}(\mathscr{P})$. We assume that for each $\tau \in \mathscr{P}$ the set of cells $\{\sigma \in \mathscr{P} \mid \mathrm{Hom}(\tau, \sigma) \neq \emptyset\}$ is finite.
Moreover, we require for the underlying topological manifold B_{top} of B that

$$B_{top} = \varinjlim_{\tau \in \mathscr{P}} \tau$$

and that for all $\tau \in \mathscr{P}$ the induced map $\tau \to B$ is injective.

By abuse of notation, we will treat cells $\tau \in \mathscr{P}$ as subsets of B.
Using these basic definitions, we can fix some more notation.

Definition 3.9. The set of maximal cells in the integral polyhedral decomposition $\mathscr{P}$ of an integral affine manifold B is denoted by $\mathscr{P}_{max}$.
In general, we use the notation $\mathscr{P}^{[k]}$ for the set of k-dimensional cells in $\mathscr{P}$.
Moreover, the subset of cells covering the interior of B is denoted by $\mathscr{P}_{int}$. The boundary of B is covered by $\mathscr{P}_\partial := \mathscr{P} \setminus \mathscr{P}_{int}$.

Definition 3.10. Let B be an integral affine manifold with singularities. We denote by $\Lambda_{\mathbb{R}} \cong \mathbb{R}^n$ the *tangent sheaf* of B_0.
Because the holonomy of B_0 is contained in $\mathbb{Z}^n \rtimes \mathrm{Gl}_n(\mathbb{Z})$, we can define a locally constant sheaf of integral lattices as a subsheaf of the tangent sheaf of B_0. It is denoted by Λ (see [GS1, Def. 1.9]).
As any integral polyhedron $\omega \in \mathscr{P}^{[k]}$ is an integral affine manifold itself, we define Λ_ω as the corresponding subsheaf of its tangent sheaf with stalk $\mathbb{Z}^k$. Note that $\Lambda_\omega \neq \Lambda|_\omega$ for $\omega \notin \mathscr{P}_{max}$.

The last structure we need on an integral affine manifold B ensures compatibility of $\mathscr{P}$ with the affine structure on B. In [GS1, Def. 1.22], this compatibility is contained in the definition of a (toric) integral polyhedral decomposition. In [GS, Def. 1.1], it appears as an additional fan structure and in [GHKS1], it is described in the course of the construction of a polyhedral affine manifold. We will use the formulation in terms of a fan structure.
Let $\mathscr{P}$ be an integral polyhedral decomposition. The *open star* of a cell $\tau \in \mathscr{P}$ is given by the open set

$$U_\tau := \bigcup_{\{\sigma \in \mathscr{P} \mid \mathrm{Hom}(\tau,\sigma)\neq\emptyset\}} \mathrm{Int}\,\sigma. \tag{3.1}$$

Definition 3.11 ([GS, Def. 1.1])**.** A *fan structure* for $(B, \mathscr{P})$ is a collection of continuous maps $S_\tau \colon U_\tau \to \mathbb{R}^k$ for all cells $\tau \in \mathscr{P}^{[n-k]}$, $k = 0, \dots, n$, satisfying the following conditions:

1. $S_\tau^{-1}(0) = \mathrm{Int}\,\tau$.
2. There exists a vector subspace $W \subseteq \mathbb{R}^k$ for any morphism $e \colon \tau \to \sigma$ in $\underline{Cat}(\mathscr{P})$ such that the restriction $S_\tau|_{\mathrm{Int}\,\sigma}$ is induced by an surjective morphism $\Lambda_\sigma \to W \cap \mathbb{Z}^k$.
3. Let $\tau \in \mathscr{P}$ and let $\Sigma_\tau \subset \mathbb{R}^k$ be the fan induced by cones $K_e := \mathbb{R}_{\geq 0} \cdot S_\tau(\sigma \cap U_\tau)$ for $e \colon \tau \to \sigma$. It has compact support.
4. For each $e \colon \tau \to \sigma$, $\sigma \in \mathscr{P}^{[n-l]}$, $l \leq k$, the map S_τ induces S_σ up to an integral linear transformation via the composition

$$U_\sigma \hookrightarrow U_\tau \xrightarrow{S_\tau} \mathbb{R}^k \to \mathbb{R}^k / \operatorname{span}(S_\tau(\mathrm{Int}\,\sigma)) \cong \mathbb{R}^l.$$

Definition 3.12. As already mentioned in Definition 3.11, a fan structure induces for each $\tau \in \mathscr{P}$ a *fan*

$$\Sigma_\tau := \bigcup_{e\colon \tau \to \sigma} K_e,\ K_e := \mathbb{R}_{\geq 0} \cdot S_\tau(\sigma \cap U_\tau).$$

We will also use an open cover of an integral affine manifold B which is finer than the cover induced by the open stars (3.1) of cells:
Let $\mathscr{P}_{bar}$ be the barycentric subdivision of $\mathscr{P}$ for an n-dimensional integral affine manifold B. The open stars of vertices $\tilde{v} \in \mathscr{P}_{bar}^{[0]}$ given by (3.1) define an open cover

$$\mathcal{W} := \{U_{\tilde{v}} \mid \tilde{v} \in \mathscr{P}_{bar}^{[0]}\} \tag{3.2}$$

of B. The open cover $\mathcal{W}$ is a good cover and therefore, it will be used in calculations later on. Note that open sets $U_{\tilde{v}} \in \mathcal{W}$ can be labelled by the cells $\tau \in \mathscr{P}$ such that $\tilde{v}$ is given as the barycenter of τ. For example, if $\tilde{v}$ is the barycenter of $\sigma \in \mathscr{P}_{max}$ the open set $U_{\tilde{v}} =: U_\sigma$ is simply given by the interior of σ.
Next, we want to study the singular locus $\Delta \subset B$ in more detail.
We will assume from now on that the singular locus Δ is given by a small perturbation of the closure

$$\mathrm{cl}\left(\bigcup_{\tau \in T} \tau\right) \subset B,\ T := \{\tau \in \mathscr{P}_{bar}^{[n-2]} \backslash \mathscr{P}^{[n-2]} \mid \tau \not\subseteq \operatorname{Int}\sigma,\ \sigma \in \mathscr{P}_{max} \text{ or } \sigma \in \mathscr{P}_{\partial}^{[n-1]}\}.$$

A detailed construction of Δ is given in [GS, p. 1310–1312].
Now we can describe in more detail what happens near the singular locus Δ following the description in [GS, p. 1312]. It is possible to do this in all generality, but here we are restricting ourselves to $n = 2$ for the first time.
Let $\sigma_\pm \in \mathscr{P}_{max} = \mathscr{P}^{[2]}$, $\rho \in \mathscr{P}^{[1]}$ and $v_\pm \in \mathscr{P}^{[0]}$ such that $v_\pm \subset \rho \subset \sigma_\pm$. Parallel transport from v_+ through σ_+ to v_- and back to v_+ via σ_- as indicated in Figure 3.1 yields a monodromy transformation T_ρ of the stalk $\Lambda|_{v_+}$. Let $d_\rho \in \Lambda_\rho|_{v_+} \subset \Lambda|_{v_+}$ and $\check{d}_\rho \in \left(\Lambda_\rho|_{v_+}\right)^\perp \subset \Lambda|_{v_+}$ be primitive vectors with $\check{d}_\rho$ pointing into σ_+ and d_ρ pointing from v_+ to v_-. There exists an integer $k_\rho \in \mathbb{Z}$ such that the monodromy transformation for each $m \in \Lambda|_{v_+}$ is given by

$$T_\rho(m) = m + k_\rho \langle m, \check{d}_\rho \rangle d_\rho.$$

This formula describes the local effect of the singular locus Δ on parallel transport around a 1-cell ρ.

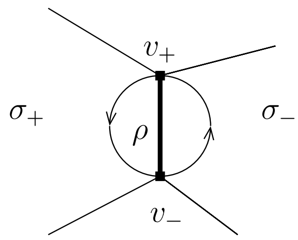

Figure 3.1.: A monodromy transformation

Definition 3.13 ([GS, Def. 1.4])**.** An integral affine manifold B with singular locus Δ is called *positive* if $k_\rho \geq 0$ for all bounded cells $\rho \in \mathscr{P}^{[1]}$ with $\rho \notin \mathscr{P}_\partial$.

From now on, we will only consider positive integral affine manifolds with an integral polyhedral decomposition and a fan structure. We also assume that our manifolds satisfy a more technical condition called *simplicity* (see [GS1, Def. 1.60]).

Definition 3.14 ([GS1, Ex. 1.62])**.** We call an integral affine manifold of dimension 2 *simple* if and only if monodromy around each singular point $P \in \Delta$ is up to the choice of coordinates given by the matrix $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$.

Construction of a central fibre

Using the setup introduced up to this point, we can construct a preliminary version of a central fibre $X(B, \mathscr{P})$ induced by an integral affine variety B with singular locus Δ and an integral polyhedral decomposition $\mathscr{P}$. We follow the *cone picture*, presented in [GS, § 1.2].

Remark 3.15. In the following section, we will work over a ring A. For the time being, set $A = \mathbb{C}$ and $\mathbb{G}_m(A^\times) = \mathbb{C}^*$. Later on, we will switch to a more universal point of view replacing A with a more general ring. By using this seemingly artificial notation now, we will not have to repeat anything then.

Let $v \subseteq \tau \in \mathscr{P}$ be a vertex and let Σ_v be the fan introduced in Definition 3.12. We can define another fan with convex cones by

$$\tau^{-1}\Sigma_v := \{K_e + \Lambda_{\tau,\mathbb{R}} \mid K_e \in \Sigma_v,\ e\colon v \to \sigma,\ v \subseteq \tau \subseteq \sigma\}.$$

The integral points of $\tau^{-1}\Sigma_v \cup \infty$ can be endowed with the following monoid structure:

$$m+n = \begin{cases} m+n, \text{ if } m \text{ and } n \text{ are contained in a common cone in } \tau^{-1}\Sigma_v \\ \infty, \text{ otherwise} \end{cases}$$

The definition of $\tau^{-1}\Sigma_v$ is independent of the choice of vertex v via parallel transport along τ. Thus, it justifies the definition of a scheme

$$V(\tau) := \operatorname{Spec} A\left[\tau^{-1}\Sigma_v\right]. \tag{3.3}$$

We want to use these schemes to construct a contravariant functor $F\colon \underline{Cat}(\mathscr{P}) \to \underline{schemes}$. On objects $\tau \in \mathbf{ob}\,(\underline{Cat}(\mathscr{P}))$, define

$$\begin{aligned} F\colon \underline{Cat}(\mathscr{P}) \to & \underline{schemes} \\ \mathscr{P} \ni \tau \mapsto & F(\tau) := V(\tau). \end{aligned} \tag{3.4}$$

For a morphism $e\colon \omega \to \tau$ in $\underline{Cat}(\mathscr{P})$ there is a map of fans $\omega^{-1}\Sigma_v \to \tau^{-1}\Sigma_v$. This map induces a morphism $F(e)$ of schemes

$$F(e)\colon V(\tau) = \operatorname{Spec} A\left[\tau^{-1}\Sigma_v\right] \to \operatorname{Spec} A\left[\omega^{-1}\Sigma_v\right] = V(\omega).$$

Now we can construct the scheme $X(B,\mathscr{P})$ as the colimit of the contravariant functor $F\colon \underline{Cat}(\mathscr{P}) \to \underline{schemes}$:

$$X(B,\mathscr{P}) := \varinjlim F \tag{3.5}$$

Remark 3.16. There is an equivalent construction of $X(B,\mathscr{P})$ starting not with fans $\tau^{-1}\Sigma_v$, $v \in \tau \in \mathscr{P}$, but with cones. This construction explains the name 'cone picture' for the construction. For each $\sigma \in \mathscr{P}^{[k]}$ with $v \in \sigma$ a vertex define a cone over σ by

$$C(\sigma) := \{(rp, r) \in \mathbb{R}^k \times \mathbb{R} \mid r \in \mathbb{R}_{\geq 0}, p \in \sigma \subset \mathbb{R}^k\}.$$

The cone $C(\sigma)$ induces a toric monoid

$$P(\sigma) := C(\sigma) \cap (\Lambda_\sigma \oplus \mathbb{Z}).$$

The monoid $P(\sigma)$ inherits a grading from the projection $\Lambda_\sigma \oplus \mathbb{Z} \to \mathbb{Z}$. Hence it induces a toric variety

$$X_\sigma = \operatorname{Proj}\left(\mathbb{Z}\left[P\left(\sigma\right)\right]\right).$$

The central fibre $X(B,\mathscr{P})$ can be constructed via gluing these toric varieties along their toric divisors ([GS, Ex. 1.13] and [GS1, § 2.1]). This alternative construction allows for some important observations:

1. The toric varieties X_σ for $\sigma \in \mathscr{P}$ inherit a line bundle $\mathcal{O}_{X_\sigma}(1)$. These line bundles glue to form a line bundle $\mathcal{O}_{X(B,\mathscr{P})}(1)$ on $X(B,\mathscr{P})$. If $\mathscr{P}$ is finite, it follows that $X(B,\mathscr{P})$ is projective and $\mathcal{O}_{X(B,\mathscr{P})}(1)$ is ample (see [GS1, § 2]).

2. The normalization of $X(B,\mathscr{P})$ is given by $\coprod_{\sigma\in\mathscr{P}_{max}} X_\sigma$. Thus, it fulfills the conditions in Definition 3.1 for $D := \bigcup_{\tau\in\mathscr{P}_\partial} X_\tau \subset X(B,\mathscr{P})$.

Remark 3.17. The gluing of toric varieties X_σ used in Remark 3.16 and the gluing of the $V(\tau)$ in the definition of the functor F from (3.4) on page 60 is not unique. It is possible to compose it with automorphisms to construct a different scheme.

These automorphisms are given by *gluing data* s, which are locally given by *piecewise multiplicative maps*:

Definition 3.18 ([GS, p. 1321])**.** Let $\tau \in \mathscr{P}$ be a cell with a vertex $v \in \tau$. A *piecewise multiplicative map* with respect to $\tau^{-1}\Sigma_v$ is a continuous map

$$\mu\colon \Lambda|_v \cap \left|\tau^{-1}\Sigma_v\right| \to \mathbb{G}_m(A^\times).$$

We require that its restriction μ^σ to any cone in $\tau^{-1}\Sigma_v$ induced by $\tau \subseteq \sigma$ is a homomorphism of monoids.

Definition 3.19 ([GS, Def. 1.18])**.** Let $(B,\mathscr{P})$ be an integral affine manifold with a polyhedral decomposition and a fan structure. *Open gluing data* consist of a collection $s = (s_e)_{e\in\mathbf{mor}(\underline{Cat}(\mathscr{P}))}$ of piecewise multiplicative maps satisfying the following conditions:

1. The map s_e is a piecewise multiplicative map with respect to $\tau^{-1}\Sigma_v$ for $e\colon \omega \to \tau$.

2. For the identity map $id\colon \tau \to \tau$ it holds that $s_{id} = 1$.

3. For morphisms $e_1\colon \tau_1 \to \tau_2$ and $e_2\colon \tau_2 \to \tau_3$ inducing $e_3 = e_2 \circ e_1\colon \tau_1 \to \tau_3$ the open gluing maps satisfy

$$s_{e_3} = s_{e_1} \cdot s_{e_2}$$

where they are defined.

Open gluing data s and s' are *cohomologous* if for all $\tau \in \mathscr{P}$ there exist piecewise multiplicative maps t_τ with respect to $\tau^{-1}\Sigma_v$ such that $s'_e = t_\tau \cdot t_\omega^{-1} \cdot s_e$ for any $e\colon \omega \to \tau$.

It is also possible to phrase the gluing data in form of a Čech 1-cycle $\mathbf{s} \in H^1(\mathcal{W}, \iota_* \check{\Lambda} \otimes \mathbb{G}_m(A^\times))$ called *lifted gluing data*. This follows from the theorem:

Theorem 3.20 (Thm. 5.2 and Thm. 5.4 in [GS1], also [GHKS1, p. 100])**.** *If $(B, \mathscr{P})$ is positive and simple, open gluing data $s = (s_e)_e$ are equivalent to lifted gluing data*

$$\mathbf{s} \in \mathrm{H}^1(B, \iota_* \check{\Lambda} \otimes \mathbb{G}_m(A^\times)).$$

Now we can modify the functor $F\colon \underline{Cat}(\mathscr{P}) \to \underline{schemes}$ we constructed in (3.4) on page 60 to get a new functor

$$F_{\mathbf{s}}\colon \underline{Cat}(\mathscr{P}) \to \underline{schemes}.$$

On objects $\tau \in \mathbf{ob}\,(\underline{Cat}(\mathscr{P}))$, we set

$$F_{\mathbf{s}}(\tau) := F(\tau).$$

On morphisms, the definition of F is modified. Let $\omega \subseteq \tau$, i.e. there is a morphism $e\colon \omega \to \tau$. We define a morphism $F_{\mathbf{s}}(e)\colon V(\tau) \to V(\omega)$ via

$$\begin{aligned} F_{\mathbf{s}}(e)^*\colon\ \operatorname{Spec} A\left[\omega^{-1}\Sigma_v\right] &\to \operatorname{Spec} A\left[\tau^{-1}\Sigma_v\right] \\ z^m &\mapsto s_e^{-1}(m) \cdot F(e)^*(z^m). \end{aligned}$$

In the same way as in equation (3.5) on page 60, this functor allows us to define $X(B, \mathscr{P}, \mathbf{s})$ as the colimit

$$X(B, \mathscr{P}, \mathbf{s}) := \varprojlim F_{\mathbf{s}}.$$

Remark 3.21. As in Remark 3.16 the scheme $X(B, \mathscr{P}, \mathbf{s})$ inherits a line bundle $\mathcal{O}_{X(B,\mathscr{P})}(1)$. If $\mathscr{P}$ is finite, $X(B, \mathscr{P}, \mathbf{s})$ is projective and $\mathcal{O}_{X(B,\mathscr{P},\mathbf{s})}(1)$ is ample.

3.2. The log structure and the universal point of view

The multi-valued piecewise linear function

For the construction of a formal degeneration with central fibre given by $X(B, \mathscr{P}, \mathbf{s})$ we need additional data in form of a log structure, which is smooth on the part of $X(B, \mathscr{P}, \mathbf{s})$ corresponding to B_0. This data is in part recorded by a multi-valued piecewise linear function $\varphi\colon B \to \mathbb{R}$.

Remark 3.22. In the following construction, we consider multi-valued piecewise linear functions with values in $\mathbb{R}$. There is a universal picture described in [GHKS1] using a toric monoid Q instead of $\mathbb{N}$, Q^{gp} instead of $\mathbb{Z}$, $Q^{gp}_{\mathbb{R}} := Q^{gp} \otimes_{\mathbb{Z}} \mathbb{R}$ instead of $\mathbb{R}$ and $Q_{\mathbb{R}}$ instead of $\mathbb{R}_{\geq 0}$. But for reasons of simplicity we will not introduce the full picture here.

Definition 3.23 ([GHKS1, Def. 1.4])**.** A $\mathbb{Z}$-valued *piecewise affine function* is a continuous map

$$\varphi : B \setminus \Delta \to \mathbb{R},$$

which is on each cell $\sigma \in \mathscr{P}_{max}$ given by an integral affine function with values in $\mathbb{R}$. The sheaf of such functions is denoted by $\mathcal{PA}(B, \mathbb{Z})$.

Definition 3.24. Let $\varphi \in \mathcal{PA}(B, \mathbb{Z})$ be a piecewise affine function. On each cell $\sigma \in \mathscr{P}_{max}$, it has a *slope*

$$n \in \operatorname{Hom}(\Lambda_\sigma, \mathbb{Z}) = \check{\Lambda}_\sigma \otimes \mathbb{Z}.$$

Let σ, σ' be maximal cells adjacent to a cell $\rho \in \mathscr{P}^{[n-1]}$. Assume that ψ has slopes n on σ and n' on σ'.
Let moreover $\delta : \Lambda|_x \to \mathbb{Z}$ for $x \in \operatorname{Int}\rho$ be given by the quotient map sending Λ_ρ to 0 and being positive on vectors pointing from ρ into σ'. As φ is continuous along ρ, it follows that

$$(n - n')(\Lambda_\rho) = 0.$$

So we can define the *kink* $\kappa_\rho(\varphi) \in \mathbb{Z}$ of φ along ρ via

$$n - n' = \delta \cdot \kappa_\rho(\varphi).$$

If it is clear with which piecewise affine function we are dealing, we will sometimes write κ_ρ instead of $\kappa_\rho(\varphi)$.

Remark 3.25. Note that being a well defined continuous function on $B \setminus \Delta$ implies that for any codimension 1 cell ρ with $\rho \cap \Delta \neq \emptyset$ it holds that the slope along ρ vanishes on Λ_ρ. This is due to the fact that the function has to be invariant under the monodromy action.

Definition 3.26 ([GHKS1, Ex. 1.11]). The sheaf of *piecewise linear functions* $\mathcal{PL}(B, \mathbb{Z})$ is defined as a quotient sheaf by

$$0 \to \underline{\mathbb{Z}} \to \mathcal{PA}(B, \mathbb{Z}) \to \mathcal{PL}(B, \mathbb{Z}) \to 0.$$

Locally, a piecewise linear function φ has representatives $\tilde{\varphi}$ which are piecewise affine functions and fulfil the following additional conditions:
Let $\rho_1, \ldots, \rho_k$ be the codimension 1 cells meeting in a point $v \in \mathscr{P}^{[0]}$ and let κ_{ρ_i} be the corresponding kinks of the piecewise affine function $\tilde{\varphi}$ on an open neighbourhood of v. Note that the kinks are independent of the choice of representative $\tilde{\varphi}$. Let moreover $n_1, \ldots, n_k$ be (oriented) primitive normal vectors with respect to the ρ_i. Then the *balancing condition* is satisfied, i.e.

$$\sum_{i=1}^{k} \kappa_{\rho_i} \otimes n_i = 0.$$

Definition 3.27. The sheaf of $\mathbb{Z}$-valued *multi-valued piecewise linear functions* on $B_0 = B \setminus \Delta$ is given by

$$\mathcal{MPL}(B, \mathbb{Z}) := \mathcal{PL}(B, \mathbb{Z}) / \mathcal{H}om(\iota_* \Lambda, \underline{\mathbb{Z}}).$$

A global section $\varphi \in \mathrm{MPL}(B, \mathbb{Z}) := \Gamma\left(B_0, \mathcal{MPL}(B, \mathbb{Z})\right)$ is locally represented by a piecewise affine function $\tilde{\varphi}$, whose kinks satisfy the balancing condition, i.e. with the same notation as in Definition 3.26

$$\sum_{i=1}^{k} \kappa_{\rho_i} \otimes n_i = 0.$$

Note that an element in $\mathcal{MPL}(B, \mathbb{Z})$ is determined by fixing a kink κ_ρ for each cell $\rho \in \mathscr{P}^{[n-1]}$ with $\rho \notin \mathscr{P}_\partial$.
A multi-valued piecewise linear function is *convex* if and only if all kinks of representatives $\kappa_\rho(\tilde{\varphi})$ are contained in $\mathbb{N} \subset \mathbb{Z}$. It is *strictly convex* if and only if all kinks are strictly positive.

Remark 3.28. We will always work with polarized totally degenerate Calabi-Yau pairs inducing an intersection complex $(B, \mathscr{P})$ on which there is a strictly convex piecewise linear multi-valued function φ.

As mentioned before, the primary use of the strictly convex multi-valued piecewise linear function φ is to fix the discrete part of the log structure on the central fibre $X(B, \mathscr{P}, \mathbf{s})$. But it also provides an alternative construction of $X(B, \mathscr{P}, \mathbf{s})$ following the cone picture (see [GS, § 1.2]). Other than the first construction, it involves the multi-valued piecewise linear function φ although φ does not affect the isomorphism class of $X(B, \mathscr{P}, \mathbf{s})$. We present this construction here, because it allows for an introduction of the corresponding log structure.
As the open patches $V(v)$ (see equation (3.3) on page 60) induced by fans near vertices $v \in \mathscr{P}^{[0]}$ already cover $X(B, \mathscr{P}, \mathbf{s})$, we restrict ourselves to describing the situation near the vertices.
Let $(B, \mathscr{P})$ be a pair of an affine integral manifold B of dimension n with singularities as above and an integral polyhedral decomposition $\mathscr{P}$ of B. Let φ be a strictly convex multi-valued piecewise linear function on B.
Let $v \in \mathscr{P}^{[0]}$ be a vertex. There exists a representative $\tilde{\varphi} \in \mathcal{PA}(U_v, \mathbb{Z})$ of φ near v. Without loss of generality, the graph $\Gamma(\tilde{\varphi})$ of $\tilde{\varphi}$ is contained in $(\Lambda_{\mathbb{R}})|_v \oplus \mathbb{R}_{\geq 0}$. We define a strictly convex rational polyhedral cone

$$C_v = \operatorname{conv}(\Gamma(\tilde{\varphi})),$$

which induces a monoid

$$P_v := C_v \cap (\Lambda|_v \oplus \mathbb{N})$$

with a grading inherited from the projection $\Lambda|_v \oplus \mathbb{N} \to \mathbb{N}$.
Let $K_\sigma \subset C_v$ denote the facets of C_v corresponding to maximal cells $\sigma \in \mathscr{P}_{max}$ with $v \in \sigma$. The local patch of $X(B, \mathscr{P}, \mathbf{s})$ is given by

$$V'(v) := \bigcup_{K_\sigma, v \in \sigma} \operatorname{Spec} A\left[K_\sigma \cap (\Lambda|_v \oplus \mathbb{N})\right]. \tag{3.6}$$

In the above formula, a gluing process along toric divisors corresponding to faces is tacitly assumed. Note that it is possible to define in the same way patches $V'(\tau)$ for other cells $\tau \in \mathscr{P}$ but we will not give details here.
At this point, it becomes obvious why the form of φ does not play any role for the isomorphism type of $X(B, \mathscr{P}, \mathbf{s})$. The definition of $V'(v)$ only depends on the

fan Σ_v (Definition 3.12). The open patches obtained by equation (3.6) equal the schemes $V(v)$ given by equation (3.3) on page 60 and can be glued in the same way using gluing data.

The log structure

In this context, we will not go into detail about log structures in general. An elaborated treatment of the topic can be found in [GS1, § 3]. Here, we will just try to explain the role of the strictly convex multi-valued piecewise linear function φ. Let $v \in \mathscr{P}^{[0]}$ be a vertex. Recall from equation (3.6) on page 65 that

$$V(v) \cong V'(v) \cong \operatorname{Spec}\left(A\left[P_v/\mathbb{N}\right]\right),$$

where the monoid structure on $P_v/\mathbb{N}$ is the same as the one on $v^{-1}\Sigma_v = \Sigma_v$ inducing $V(v)$.
This construction is supposed to endow the patches $V(v)$ with a log structure $\mathcal{M}_{V(v)}$ induced by the monoid P_v. Unfortunately, P_v does not determine the log structure uniquely. It does only fix $\mathcal{M}_{V(v)}$ up to the choice of an extension

$$1 \to \mathcal{O}^{\times}_{V(v)} \to \mathcal{M}_{V(v)} \to P_v \to 0.$$

Remark 3.29. For a moment suppose that we settled the choice of $\mathcal{M}_{V(v)}$. We want to argue, that we can construct (non-unique) morphisms of logarithmic spaces.
Note that we can turn $\operatorname{Spec} A$ into a logarithmic space $(\operatorname{Spec} A)^{\dagger} := (\operatorname{Spec} A, \mathbb{N} \oplus A^{\times})$. For $A = \mathbb{C}$ this yields simply the standard log point $O^{\dagger}$. There are induced morphisms $V(v) \to \operatorname{Spec} A$ and these morphisms induce log morphisms

$$\begin{aligned} &(V(v), \mathcal{M}_{V(v)}) \to (\operatorname{Spec} A, \mathbb{N} \oplus A^{\times}) \\ &\Lambda|_v \oplus \mathbb{N} \supset P_v \leftarrow \mathbb{N}. \end{aligned}$$

We require this morphism to be independent of the choice of extension for $\mathcal{M}_{V(v)}$. This requirement fixes the preimage of $\mathbb{N}$ in the ghost sheaf $\overline{\mathcal{M}}_{V(v)}$, while it does not fix the morphism as a whole.

But until now, we have not fixed the choice of extension locally.
And there is another problem: In general, the log structures $\mathcal{M}_{V(v)}$ on the schemes $V(v)$ do not glue to give a (smooth) log structure on $X(B, \mathscr{P}, \mathbf{s})$. But they do

glue on $X(B, \mathscr{P}, \mathbf{s}) \setminus Z$, where Z is a codimension 2 locus corresponding to the singular locus Δ of B. The discrete part of the gluing is exactly the information we get from the multi-valued piecewise linear function φ.
The remaining freedom of choice of extensions can be phrased in terms of a sheaf $\mathcal{LS}_{X^g}$ on $X(B, \mathscr{P}, \mathbf{s})$.

Definition 3.30 ([GS1, Def. 3.19])**.** Define $\mathcal{LS}_{X^g}$ as the subsheaf of the sheaf of extensions on $X(B, \mathscr{P}, \mathbf{s})$ which are locally extensions of P_v by $\mathcal{O}^{\times}_{V(v)}$ and leave the preimage of $\mathbb{N}$ fixed.

Therefore, any section of $\mathcal{LS}_{X^g}$ yields a log structure $\mathcal{M}_{(X,D)}$ on $X(B, \mathscr{P}, \mathbf{s}) \setminus Z$, which is induced by gluing choices of log structures on the patches $V(v)$ via φ.

Remark 3.31 ([GS1, Def. 4.17]). Let again $v \in \mathscr{P}^{[0]}$ be a vertex. We will describe a section $f \in \Gamma(\mathcal{LS}_{X^g})$ on the open patch $V(v) \setminus Z$. For any $\rho \in \mathscr{P}^{[n-1]}$ with $e \colon v \to \rho$ we define a closed subset

$$V(e) := \operatorname{Spec} A\,[K_e] \subset V(v).$$

Then $f \in \Gamma(V(v) \setminus Z, \mathcal{LS}_{X^g})$ is given by a family

$$f = (f_e)_{e\colon v \to \rho,\ \rho \in \mathscr{P}^{[n-1]}},\ f_e \in \mathcal{O}_{V(e)},$$

satisfying some compatibility condition.
A log structure is *positive* if and only if the functions f_{e_ρ} defining it have no poles. Positivity is a necessary condition for a log structure to belong to a toric degeneration. For this reason, it was mentioned in the definition of a toric log Calabi-Yau pair, Definition 3.3, although we were not able to state this condition in more detail there.

The remaining choice of a section of $\mathcal{LS}_{X^g}$ is settled by the following theorem:

Theorem 3.32 ([GS1, Thm. 5.2])**.** *Let $(B, \mathscr{P})$ be positive and simple with fixed lifted gluing data* $\mathbf{s}$*. Then there exists a unique normalized log structure of type fixed by φ.*

Thus, after fixing $\mathbf{s}$ and φ there exists a uniquely determined choice of smooth log structure on $X(B, \mathscr{P}, \mathbf{s}) \setminus Z$ over $O^{\dagger}$.
Summarizing, the affine data given by

1. a simple, positive integral affine manifold B with singularities Δ and a boundary ∂B,
2. a finite polyhedral decomposition $\mathscr{P}$ of B with a fan structure
3. open or equivalently lifted gluing data s or $\mathbf{s}$ and
4. a strictly convex multi-valued piecewise linear function φ

is enough to form a totally degenerate Calabi-Yau pair with a polarization and a log structure. In [GS], it was shown that it is even enough to construct a formal toric degeneration of Calabi-Yau pairs explicitly.
Here, we want to introduce a more universal point of view following [GHKS1].

The universal point of view

In the following paragraph, we will introduce a more general version of the gluing data $\mathbf{s}$.
As mentioned in Theorem 3.20, the open gluing data s are equivalent to lifted gluing data

$$\mathbf{s} \in \mathrm{H}^1(B, \iota_*\check{\Lambda} \otimes \mathbb{G}_m(A^\times)).$$

As we announced earlier, we will replace $A = \mathbb{C}$ by a more general ring ([GS2, § 5.2] and [GHKS1, § A.2]):

Definition 3.33. From now on, set

$$A := \mathbb{C}\left[\mathrm{H}^1(B, \iota_*\check{\Lambda})^*\right] = \mathbb{C}\left[\mathrm{H}^1(B, \iota_*\check{\Lambda})^*_f\right],$$

where $\mathrm{H}^1(B, \iota_*\check{\Lambda})^*_f$ denotes the free part of $\mathrm{H}^1(B, \iota_*\check{\Lambda})^*$ and equality follows from divisibility of $\mathbb{C}$.

Note that the group of units $\mathbb{G}_m(A^\times)$ of $A^\times$ is given by

$$\mathbb{G}_m(A^\times) = \mathbb{C}^* \times \mathrm{H}^1(B, \iota_*\check{\Lambda})^*_f.$$

If we use this decomposition of $\mathbb{G}_m(A^\times)$ to describe the group containing the lifted gluing data, we get

$$\begin{aligned}\mathrm{H}^1(B, \iota_*\check{\Lambda} \otimes \mathbb{G}_m(A^\times)) &= \mathrm{H}^1(B, \iota_*\check{\Lambda} \otimes (\mathbb{C}^* \times \mathrm{H}^1(B, \iota_*\check{\Lambda})^*_f) \\ &= \mathrm{H}^1(B, \iota_*\check{\Lambda} \otimes \mathbb{C}^*) \times \left(\mathrm{H}^1(B, \iota_*\check{\Lambda}) \otimes \mathrm{H}^1(B, \iota_*\check{\Lambda})^*_f\right).\end{aligned}$$

From now on, set

$$\mathbf{s} = (s_0, \sigma) \in \mathrm{H}^1(B, \iota_*\check{\Lambda} \otimes \mathbb{C}^*) \times \left(\mathrm{H}^1(B, \iota_*\check{\Lambda}) \otimes \mathrm{H}^1(B, \iota_*\check{\Lambda})^*_f\right).$$

The second component σ can be considered as a map $\sigma\colon\ \mathrm{H}^1(B, \iota_*\check{\Lambda})_f \to \mathrm{H}^1(B, \iota_*\check{\Lambda})$. From now on, we set

$$S := \operatorname{Spec} A.$$

Lemma 3.34. *We can view* $\mathbf{s} \in \mathrm{H}^1(B, \iota_*\check{\Lambda} \otimes \mathbb{G}_m(A^\times))$ *as a function*

$$\mathbf{s}\colon S_{an} \to \mathrm{H}^1(B, \iota_*\check{\Lambda} \otimes \mathbb{C}^*).$$

Proof. Note that $\mathrm{H}^1(B, \iota_*\check{\Lambda}) \otimes \mathbb{C}^*$ embeds into $\mathrm{H}^1(B, \iota_*\check{\Lambda} \otimes \mathbb{C}^*)$. Let

$$\xi \in \mathrm{H}^1(B, \iota_*\check{\Lambda}) \otimes \mathbb{C}^* = \mathrm{H}^1(B, \iota_*\check{\Lambda})_f \otimes \mathbb{C}^*$$

be a closed point in $S = \operatorname{Spec} A$. We define a map $\mathbf{s}$ by

$$\mathbf{s}\colon \xi \mapsto \sigma(\xi) \cdot s_0 \in \mathrm{H}^1(B, \iota_*\check{\Lambda} \otimes \mathbb{C}^*).$$

□

Lemma 3.34 implies that lifted gluing data $\mathbf{s}$ for $A = \mathbb{C}\left[\mathrm{H}^1(B, \iota_*\check{\Lambda})^*\right]$ associate with every point of S lifted gluing data for $A = \mathbb{C}$ as in Theorem 3.20.
After all ([GS2, § 5.2]), this construction induces a family

$$X_0(B, \mathscr{P}) \to S = \operatorname{Spec} \mathbb{C}\left[\mathrm{H}^1(B, \iota_*\check{\Lambda})^*\right].$$

The next step is to endow S and $X_0(B, \mathscr{P})$ with suitable log structures (see [GHKS1, p. 108]).
Recall that we endowed $S = \operatorname{Spec} A$ with a log structure to turn it into a log scheme

$$S^\dagger = (S, \mathcal{M}_S = \mathbb{N}_S \oplus \mathcal{O}_S^\times).$$

The morphism $\alpha_S\colon \mathcal{M}_S \to \mathcal{O}_S$ is induced by the map

$$\mathcal{M}_S = \mathbb{N}_S \oplus \mathcal{O}_S^\times \ni (n, s) \mapsto \begin{cases} 0, & \text{if } n \neq 0 \\ s, & \text{if } n = 0. \end{cases}$$

As above there are morphisms $V(v) \to S$ and by Remark 3.29, these morphisms induce log morphisms

$$(V(v), \mathcal{M}_{V(v)}) \to (S, \mathcal{M}_S).$$

This construction yields a log structure $\mathcal{M}_{(X_0,D_0)}$ such that there is an induced morphism

$$\big(X_0(B,\mathscr{P}), \mathcal{M}_{(X_0,D_0)}\big) \to (S, \mathcal{M}_S).$$

This morphism is not universal with respect to the log structure on $X_0(B,\mathscr{P}) \to S$ as we are still using the monoid $\mathbb{N}$ for the construction of the log structure ([GHKS1, p. 110]). Nevertheless, it is universal with respect to the gluing data $\mathbf{s}$ on $(B,\mathscr{P})$ with fixed multi-valued piecewise linear function φ.
The morphism $\big(X_0(B,\mathscr{P}), \mathcal{M}_{(X_0,D_0)}\big) \to (S, \mathcal{M}_S)$ can be used as a central fibre to construct ([GHKS1, p. 110]) a formal degeneration of Calabi-Yau pairs

$$\mathfrak{X} \to \operatorname{Spf} A[[\mathbb{N}]].$$

The central fibre of this degeneration is given by $X_0(B,\mathscr{P}) \to S$ and the completion was taken with respect to the ideal $I_0 \subset A_{(0)}[\mathbb{N}]$ generated by $\mathbb{N}_{>0}$.

Remark 3.35. An interesting feature of the degeneration $\mathfrak{X} \to \operatorname{Spf} A[[\mathbb{N}]]$ is the fact that it carries a *torus action*.

Denote the dual space of the space of global multi-valued piecewise linear functions by $\mathrm{MPL}(B,\mathbb{Z})^*$ and the dual space of global piecewise linear functions by $\mathrm{PL}(B,\mathbb{Z})^*$. There is a commutative diagram corresponding to the diagram [GHKS1, (4.8)], which in our case (see [GHKS1, § A.3]) reads

$$\begin{tikzcd} \mathrm{MPL}(B,\mathbb{Z})^* \arrow[r] \arrow[d] & \mathrm{PL}(B,\mathbb{Z})^* \arrow[d] \\ \mathbb{N} \arrow[r] & \Gamma. \end{tikzcd} \tag{3.7}$$

There holds the following proposition ([GHKS1, Prop. A.6]):

Proposition 3.36. *The relative torus* $\operatorname{Spec}(A[\Gamma]) \subset \operatorname{Aut}_A(X_0(B,\mathscr{P}))$ *acts on* $X_0(B,\mathscr{P})$ *and this action extends canonically to an action on* $\mathfrak{X} \to \operatorname{Spf} A[[\mathbb{N}]]$.

Moreover, there is by [GHKS1, § A.3]* a second torus action, which acts non-trivially on $\operatorname{Spec} A \subset \operatorname{Spf} A\,[[\mathbb{N}]]$.
Recall that we can map global piecewise linear functions to global multi-valued piecewise linear functions, i.e. there is a morhism

$$\kappa\colon\ \mathrm{PL}(B,\mathbb{Z}) \to \mathrm{MPL}(B,\mathbb{Z}).$$

Moreover, recall that there is an induced map

$$c_1\colon\ \mathrm{MPL}(B,\mathbb{Z}) \to \mathrm{H}^1(B,\iota_*\check{\Lambda}).$$

Denote the image of this map by H. By [GHKS1, Lemma A.9], it holds that

$$\mathrm{MPL}(B,\mathbb{Z}) \cong \kappa\,(\mathrm{PL}\,(B,\mathbb{Z})) \oplus H.$$

Moreover, we can write

$$A = \operatorname{Spec}[\mathrm{H}^1(B,\iota_*\check{\Lambda})^*] = \operatorname{Spec}[H^* \oplus F^*] \subset \operatorname{Spec}[H^* \oplus \tilde{F}^*] =: \tilde{A},$$

where $\tilde{F}^*$ denotes the dual of the saturation of the complement of H in $\mathrm{H}^1(B,\iota_*\check{\Lambda})$. It is possible to pull back $\mathfrak{X} \to \operatorname{Spf} A[[\mathbb{N}]]$ to a family $\tilde{\mathfrak{X}} \to \operatorname{Spf} \tilde{A}[[\mathbb{N}]]$. There holds the following proposition ([GHKS1, Prop. A.12]):

Proposition 3.37. *Projection to H^* followed by composition with the dual of c_1 induces an action of $\operatorname{Spec}\mathbb{C}[H]$ on $\operatorname{Spf}\tilde{A}[[\mathbb{N}]]$ which lifts to the family $\tilde{\mathfrak{X}} \to \operatorname{Spf}\tilde{A}[[\mathbb{N}]]$.*

We will use the torus action later, as we have to take it into consideration when talking about isomorphism classes of deformations of $X_0(B,\mathscr{P})$.

*To appear in an updated version.

4. Example: A toric degeneration of dP_9 surfaces

In this chapter, we will look at an example of a toric degeneration of dP_9 surfaces in detail. We want to compute period integrals for this example explicitly.
A (formal) toric degeneration $\pi\colon (\mathfrak{X}, \mathfrak{D}) \to \operatorname{Spf}\mathbb{C}[[t]]$ with central fibre (X, D) induces lifted gluing data $\mathbf{s} \in \mathrm{H}^1(B, \iota_*\check{\Lambda} \otimes \mathbb{C}^*)$ and a strictly convex multi-valued piecewise linear function φ. We can associate with φ a class $c_1(\varphi) \in \mathrm{H}^1(B, \iota_*\check{\Lambda})$. It is possible to define a pairing

$$\begin{aligned} \mathrm{H}_1(B, \iota_*\Lambda) \otimes \mathrm{H}^1(B, \iota_*\check{\Lambda} \otimes \mathbb{C}^*) &\to \mathbb{C}^* \\ (\beta_{trop}, \mathbf{s}) &\mapsto \langle \beta_{trop}, \mathbf{s} \rangle. \end{aligned}$$

As (X, D) is compact, there exists an analytic extension of $(\mathfrak{X}, \mathfrak{D}) \to \operatorname{Spf}\mathbb{C}[[t]]$ by the discussion in [RS, § 2.1]. We denote this analytic extension by $(\mathcal{X}, \mathcal{D}) \to T$ with central fibre given by $(X, D) \to O$.
Note that monodromy around the critical locus $O \in T$ yields a vanishing cycle $\alpha(t) \in \mathrm{H}_2(X_t, \mathbb{Z})$ in a fibre $X_t \subset \mathcal{X}$ for $t \neq O$ (see [RS, p. 2]).
By [RS, § 3], there are n-cycles $B(t) \in \mathrm{H}_n(X_t, \mathbb{Z})$, $n = \dim X$, which can be associated to a so called tropical cycle $\beta_{trop} \in \mathrm{H}_1(B_0, \Lambda) \cong \mathrm{H}_1(B, \iota_*\Lambda)$. The cycles $B(t)$ are first constructed in the central fibre X using the moment map and then continued to X_t. Thus, the vanishing cycle $\alpha(t)$ does not appear in this picture.
Let Ω be the non-vanishing relative logarithmic volume form on $\mathcal{X}$ with poles along $X \cup \mathcal{D}$ coming with the construction which can by [RS, Lemma 4.1] be fixed uniquely by requiring that $\int_{\alpha(t)} \Omega = (2\pi i)^n$, $n = \dim B = 2$.
There is the following formula introduced in [RS]* which computes the period

*The preprint [RS], which is online now, contains a slightly different version of this formula. An updated version is to appear.

integrals over $B(t)$ by purely linear means:

$$\exp\left(-2\pi i \frac{\int_{B(t)} \Omega}{\int_{\alpha(t)} \Omega}\right) = \exp\left(-\frac{1}{2\pi i}\int_{B(t)} \Omega\right) = \langle \beta_{trop}, \mathbf{s}\rangle \cdot t^{\langle \beta_{trop}, c_1(\varphi)\rangle} \tag{4.1}$$

We want to use this formula to generalize the period computation used in the proof of Proposition 1.31. To this end, we start with an integral affine manifold B with singular locus Δ, which was introduced in [CPS, Fig. 6.4].

4.1. Computation of $\mathrm{H}^1(B, \iota_*\check{\Lambda})$

First, we have to examine the manifold B in more detail and compute a basis of $\mathrm{H}^1(B, \iota_*\check{\Lambda})$.
The affine manifold B is topologically isomorphic to a closed 2-dimensional disk with 12 focus-focus singularities.

Remark 4.1. Note that the toric degeneration is projective as B is contractible and thus

$$\mathrm{H}^2(B, \mathbb{C}^*) = 0.$$

Projectivity is a necessary condition for the computation of the period integrals to be valid (see [RS, § 2]).

Next, we want to describe the polyhedral decomposition $\mathscr{P}$ of B introducing notation for its cells and an associated open cover of B.
The polyhedral decomposition $\mathscr{P}$ of B consists of

1. vertices, $v_0, v_1, \ldots, v_6 \in \mathscr{P}^{[0]}$,
2. inner radial 1-cells, $\rho_1, \ldots, \rho_6 \in \mathscr{P}^{[1]}$,
3. outer radial 1-cells, $\mu_1, \ldots, \mu_6 \in \mathscr{P}^{[1]}$,
4. cells parallel to the boundary $\omega_1, \ldots, \omega_6 \in \mathscr{P}^{[1]}$,
5. the boundary cells, which are left unlabelled,
6. inner maximal cells, $\sigma_1, \ldots, \sigma_6 \in \mathscr{P}^{[2]}$, and
7. outer maximal cells, $\tau_1, \ldots, \tau_6 \in \mathscr{P}^{[2]}$.

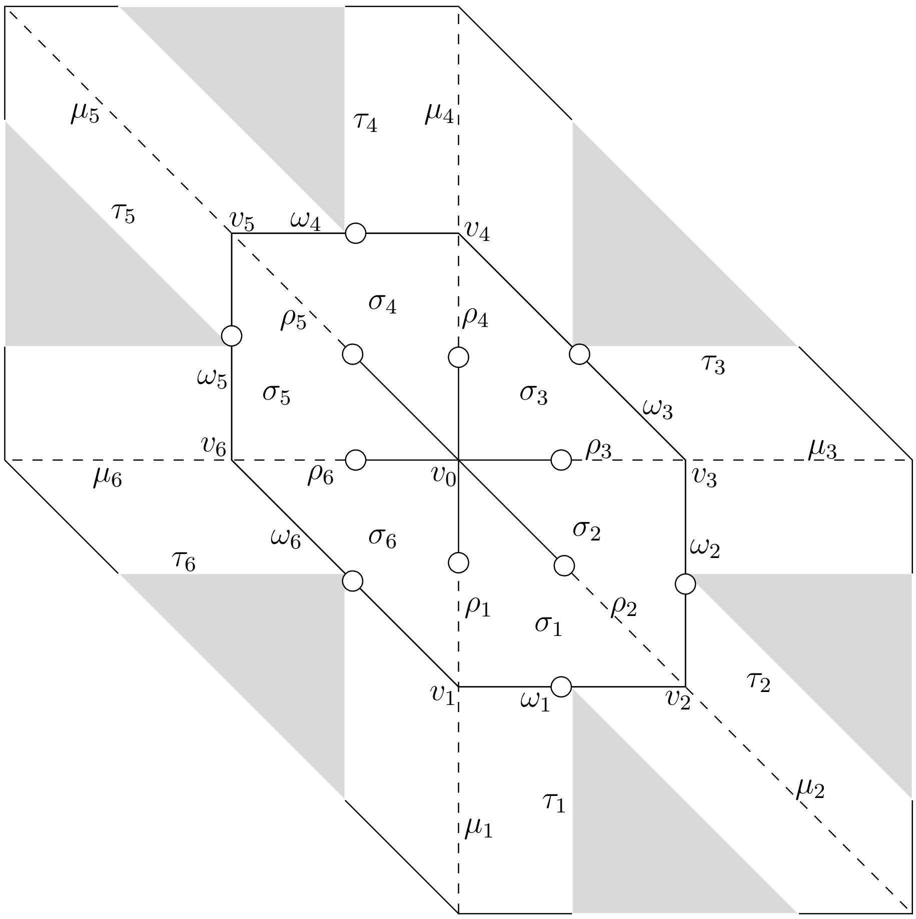

Figure 4.1.: The polyhedral decomposition of B

The singular locus Δ consists of 12 points each of which is located near the barycenter of one of the cells $\rho_1, \dots, \rho_6, \omega_1, \dots, \omega_6$.

The manifold B with its polyhedral decomposition and the singular points is depicted in Figure 4.1. The little circles correspond to focus-focus singularities. They cause non-trivial monodromy, which is described by a gluing process. The grey areas are cut out and the maximal outer cells $\tau_1, \dots, \tau_6$ are glued along the cuts such that they end up square-shaped. A similar process takes place along the dashed lines, depicting the 1-cells $\mu_1, \dots, \mu_6$ and part of $\rho_1, \dots, \rho_6$. At these dashed lines B is cut and glued after a linear transformation, which leaves Λ_{ρ_i} and Λ_{μ_i} invariant and maps $\Lambda_{\omega_{i-1}}$ to Λ_{ω_i}. Near a vertex v_i, the manifold B looks like

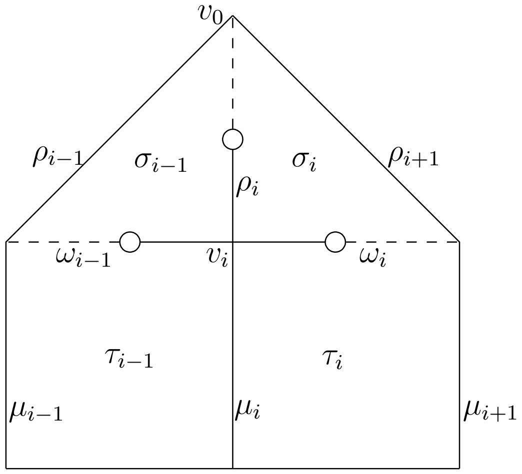

Figure 4.2.: Detail of B near v_i

Figure 4.2, i.e. the outer cells form a flat cylinder with straight boundary. Note again that the boundary cells go unlabelled, they are contained in the boundary of the maximal cells $\tau_1, \ldots, \tau_6$.

Remark 4.2. From now on, we will always use the following convention: When talking about cells of index i in B, e.g. σ_i, we are thinking modulo 6 to make up for the rotational symmetry of B. For example, if $i = 1$ we set $i - 1 = 6$ etc.

We want to compute $\mathrm{H}^1(B, \iota_*\check{\Lambda})$ using Čech cohomology. As a good open cover of B we take

$$\mathcal{W} = \{U_{\tilde{v}} \mid \tilde{v} \in \mathscr{P}_{bar}^{[0]}\},$$

which was introduced in equation (3.2) on page 58. Recall that we introduced the shorthand notation U_τ for $U_{\tilde{v}} \in \mathcal{W}$ such that $\tilde{v}$ is the barycenter of τ. To simplify notation further, we set

$$\begin{aligned} U_\tau \cap U_\rho &=: U_{\tau\rho} (= U_{\rho\tau}) \\ U_v \cap U_\rho \cap U_\sigma &=: U_{v\rho\sigma}. \end{aligned}$$

Next, we want to fix charts on the open sets in $\mathcal{W}$ and transition maps on the intersections.
Note that we can use the same transition maps for B and for changes of coordinates for $\iota_*\Lambda$.

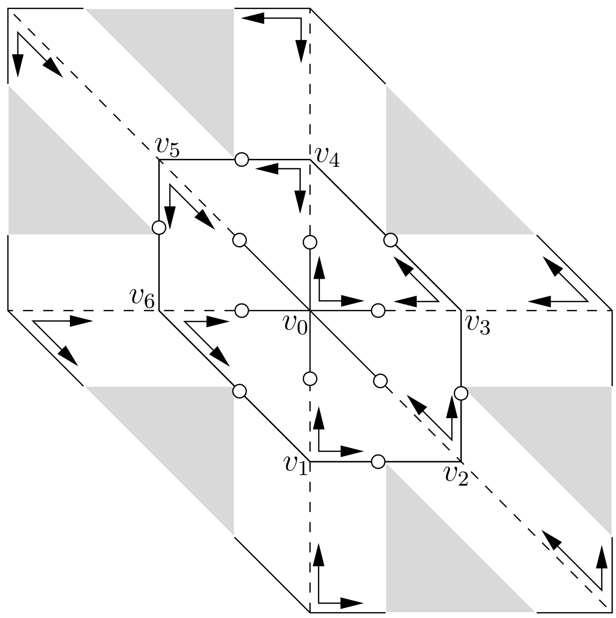

Figure 4.3.: Coordinates on B

On open sets which contain the singular points, i.e. $U_{\rho_1},\dots,U_{\rho_6},U_{\omega_1},\dots,U_{\omega_6}$, the space of global sections of $\iota_*\Lambda$ is only 1-dimensional. It consists of vector fields which restrict to vector fields in the tangent spaces $\iota_*\Lambda_{\rho_i}$ and $\iota_*\Lambda_{\omega_i}$ of the 1-cells ρ_i and ω_i, $i=1,\dots,6$.

The situation on the inner cells $U_{v_0},U_{\sigma_1},\dots,U_{\sigma_6}$ is more complicated. We fix coordinates as indicated in Figure 4.3. On $U_{v_1},\dots,U_{v_6}$ the first basis vector is induced by parallel transport from $\Lambda_{\omega_1},\dots,\Lambda_{\omega_6}$, the second basis vector is induced by $\Lambda_{\rho_1},\dots,\Lambda_{\rho_6}$. On U_{v_0} we choose the same coordinates as on U_{v_1} via parallel transport through U_{σ_1}.

On U_{ρ_i},U_{ω_i} and U_{σ_i} we use the same coordinates as on U_{v_i} via direct parallel transport from U_{v_i}. Therefore, the transition maps on $U_{v_i\sigma_i},U_{v_i\rho_i},U_{\rho_i\sigma_i},U_{\omega_i\sigma_i}$ and $U_{v_i\omega_i}$ are trivial.

Recall that B has straight boundary, i.e. monodromy along ∂B is trivial. This means that the transition maps on $U_{\mu_i\tau_i}$ and $U_{\mu_i\tau_{i-1}}$ as well as on $U_{v_i\mu_i}$, $U_{v_i\tau_i}$ and $U_{v_i\tau_{i-1}}$ are trivial as well, i.e. they are given by $\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$.

On intersections involving one of the U_{v_i}, $i=1,\dots,6$, i.e. $U_{v_i\omega_{i-1}}$ and $U_{v_i\sigma_{i-1}}$, we consider the coordinates coming from U_{v_i}. On $U_{v_0\rho_i}$ and $U_{v_0\sigma_i}$ we use coordinates

coming from U_{v_0}. Otherwise, the maximal cell involved in the intersection determines the chart we use. This applies on $U_{\rho_i\sigma_{i-1}}$.
On triple intersections we use the coordinates induced by U_{v_i}, which yields non-trivial transition functions on $U_{v_0\rho_i\sigma_i}$, $U_{v_0\rho_{i+1}\sigma_i}$, $U_{v_{i+1}\omega_i\sigma_i}$ and $U_{v_{i+1}\rho_{i+1}\sigma_i}$.
We want to write down all non-trivial transition matrices A_{ij} explicitly. The matrix A_{ij} describes the transition map from coordinates which are induced by coordinates on U_{v_i} to coordinates which are parallel to coordinates on U_{v_j}:

$$\begin{aligned}
A_{01} &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & A_{10} &= A_{01}^{-1} & A_{02} &= \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix} & A_{20} &= A_{02}^{-1} \\
A_{03} &= \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} & A_{30} &= A_{03}^{-1} & A_{04} &= \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} & A_{40} &= A_{04}^{-1} \\
A_{05} &= \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix} & A_{50} &= A_{05}^{-1} & A_{06} &= \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix} & A_{60} &= A_{06}^{-1} \\
A_{i(i+1)} &= \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} := A_{12} & A_{(i+1)i} &= A_{i(i+1)}^{-1} & i &= 1, \dots, 6
\end{aligned}$$

As we want to compute cohomology groups for the dual sheaf $\iota_*\check{\Lambda}$ instead of $\iota_*\Lambda$, these matrices are transformed accordingly. Let therefore B_{ij} be the transition matrices from dual vector fields on open sets with coordinates induced by U_{v_i} to dual vector fields on open sets with coordinates induced by U_{v_j}. These matrices are given by

$$B_{ij} = (A_{ij}^{-1})^T.$$

Having established notation, we can now introduce boundary maps

$$\begin{aligned}
&\partial^0 \colon C^0(\mathcal{W}, \iota_*\check{\Lambda}) \to C^1(\mathcal{W}, \iota_*\check{\Lambda}) \text{ and} \\
&\partial^1 \colon C^1(\mathcal{W}, \iota_*\check{\Lambda}) \to C^2(\mathcal{W}, \iota_*\check{\Lambda}).
\end{aligned}$$

They are given explicitly in the appendix, Section A.1. Note that we omit cells in $\mathscr{P}_\partial$ for the sake of reducing the number of components.

Proposition 4.3. *In the example, the cohomology groups of* $\iota_*\check{\Lambda}$ *are given by*

$$\begin{aligned}
&\mathrm{H}^0(B, \iota_*\check{\Lambda}) = 0, \\
&\mathrm{H}^1(B, \iota_*\check{\Lambda}) = \mathbb{Z}^{10} \textit{ and} \\
&\mathrm{H}^2(B, \iota_*\check{\Lambda}) = 0.
\end{aligned}$$

Proof. We use the matrices ∂^0 and ∂^1 to compute the cohomology groups in Maple. As $\mathrm{H}^0(B, \iota_*\check{\Lambda})$ is given by the kernel of ∂^0, it is easily determined.
The cohomology group

$$\mathrm{H}^1(B, \iota_*\check{\Lambda}) = \ker(\partial^1)/\operatorname{im}(\partial^0)$$

was computed in Maple by importing the matrices ∂^0, ∂^1 and executing the command

```
IntersectionBasis([NullSpace(Transpose(∂⁰)),NullSpace(∂¹)]):
```

This computation yields basis vectors of $\mathrm{H}^1(B, \iota_*\check{\Lambda})$.
The fact that $\mathrm{H}^2(B, \iota_*\check{\Lambda})$ vanishes is implied by ∂^1 having full rank. □

Remark 4.4. Note that our computation of $\mathrm{H}^1(B, \iota_*\check{\Lambda})$ yields 10 basis vectors with 192 entries. These vectors can be combined into a matrix which contains these vectors as columns. We denote this (192×10)-matrix by R.

4.2. Tropical cycles

Next, we have to find tropical cycles $(\alpha_0)_{trop}, \ldots, (\alpha_8)_{trop} \in \mathrm{H}_1(B, \iota_*\Lambda)$, which correspond to our basis $\alpha_0, \ldots, \alpha_8$ of $d^\perp \subset \Lambda_{1,9}$ and generate $D^\perp \subset \mathrm{H}_2(X_t, \mathbb{Z})$.

Tropical cycles corresponding to exceptional divisors

We want to determine tropical cycles which correspond to exceptional curves in $\mathrm{H}_2(X_t, \mathbb{Z}),\ t \in T$, following the lines of [RS]. Note that these tropical cycles are not contained in $\mathrm{H}_1(B, \iota_*\Lambda)$ as they have non-empty intersection with the boundary.
In our example, we look at a toric degeneration over a base B in two dimensions. As mentioned before, 2-cycles in X are constructed from tropical cycles using the moment map. On the interior of a cell $\sigma \in \mathscr{P}$ the moment map yields (see [RS, § 3]):

$$\mu_\sigma \colon X_\sigma \supset \mathrm{Hom}(\Lambda_\sigma, U(1)) \to \sigma$$

The group $\mathrm{Hom}(\Lambda_\sigma, U(1)) \cong (U(1))^n \cong (\mathbb{S}^1)^n$ acts simply transitively on fibres. A fibre $(X_\sigma)_b$ over a point $b \in B$ in the interior of σ is given by T_b^*B/Λ_b^*. The cotangent space at a point $x \in (X_\sigma)_b \subset X_\sigma$ is isomorphic to

$$T_bB \times T_bB \cong \mathbb{R}^4,$$

where the first factor accounts for the fibre directions and the second factor is given by the directions within the base B.

Definition 4.5 ([RS, Def. 0.2], also [Sy])**.** A *tropical cycle* γ is a graph with oriented edges $\gamma_i \subset \gamma,\ i \in \mathbb{N}$, in $B \backslash \Delta$ carrying a non-trivial section $\xi_i \in \Gamma(\gamma_i, \Lambda|_{\gamma_i})$. Its vertices are not contained in cells $\tau \in \mathscr{P}^{[n-1=1]}$ and $\gamma_i \cap \tau$ is given by an isolated point in the interior of τ for $\tau \in \mathscr{P}^{[n-1=1]}$. Moreover, let $v \in \gamma$ be a vertex where 3 or more edges $\gamma_1, \ldots, \gamma_k \subset \gamma$ directed towards v and carrying vectors $\xi_1, \ldots, \xi_k$ meet. We require that these vectors fulfil the balancing condition

$$\xi_1 + \cdots + \xi_k = 0. \tag{4.2}$$

If a component $\gamma_i,\ i = 1, \ldots, k$, is not directed towards v, we use $-\xi_i$ instead of ξ_i in the balancing condition (4.2).

Before we proceed towards the construction of tropical cycles, we need some more lemmas. First, we want to analyse a tropical cycle surrounding a focus-focus singularity on a 1-cell ρ:

Lemma 4.6. *Consider a tropical cycle which surrounds a focus-focus singularity on a cell $\rho \in \mathscr{P}^{[1]}$. It meets itself at a point $p \notin \rho$ and then is continued leaving the situation (see Figure 4.4). It holds that on the edge leading away from the singularity, it carries a vector ζ obtained by parallel transport from Λ_ρ. Assuming this vector is primitive, the cycle around the singularity is started carrying a generator $\xi \in \Lambda/\Lambda_\rho$.*

Proof. We want to fix a basis of the integral tangent sheaf Λ with an orientation given by Figure 4.4, i.e. there is a generator $e_\rho \in \Lambda_\rho$ and a primitive vector $\check{d}_\rho$ such that $\left(\check{d}_\rho\right)^\perp = \Lambda_\rho$. The monodromy around the focus-focus singularity acts on a vector $\xi \in \Lambda$ via

$$\xi \mapsto \xi + \langle \check{d}_\rho, \xi \rangle \cdot e_\rho.$$

Let ζ denote the vector which the edge leading away from the singularity is carrying. Recall that 3 edges of our tropical cycle meet at the point p. The balancing condition (4.2) at p implies that

$$\xi - (\xi + \langle \check{d}_\rho, \xi \rangle \cdot e_\rho) + \zeta = 0.$$

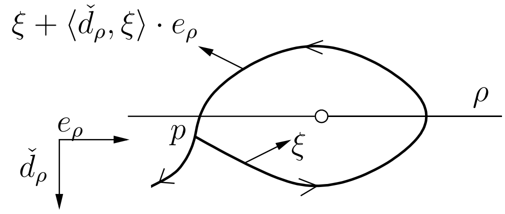

Figure 4.4.: A 1-cycle surrounding a focus-focus singularity

This equation is equivalent to $\zeta = c_1 \cdot e_\rho$ for $c_1 = \langle \check{d}_\rho, \xi \rangle \in \mathbb{Z}$.
Setting $\zeta := e_\rho$ the balancing condition (4.2) yields

$$\xi - (\xi + \langle \check{d}_\rho, \xi \rangle \cdot e_\rho) + e_\rho = 0$$
$$\iff \xi = -\check{d}_\rho + c_2 \cdot e_\rho, \; c_2 \in \mathbb{Z}.$$

□

For the sake of simplicity we will from now on assume primitivity (i.e. $c_2 = 0$ in the notation of the proof of Lemma 4.6) without further comment.
By Lemma 4.6, tropical cycles in our picture can end in a singular point. The other possibility for tropical cycles to end is by meeting the boundary ∂B:

Lemma 4.7. *A tropical cycle γ ending in ∂B or a singular point $P \in \rho \in \mathscr{P}^{[1]}$ carries a vector $\xi \in \Lambda_\rho$ or $\xi \in \Lambda_{\partial B}$ respectively.*

Proof. By [RS, p. 19], the fibre of the 2-cycle in $\mathrm{H}_2(X, \mathbb{Z})$ associated to $\gamma \subset B$ carrying a vector ξ over a point $b \in \sigma \in \mathscr{P}$ is given by

$$\mathrm{Hom}(\xi^\perp, U(1)) \subset \mathrm{Hom}\left(\Lambda_b, U(1)\right).$$

Therefore, this fibre is shrinking to a point for γ approaching ∂B if and only if $\xi \in \Lambda_{\partial B}$ as in this case $\xi^\perp = 0 \in \Lambda_{\partial B}$ (see [RS, § 3]). Note that this reasoning only works because ∂B induces a toric degeneration itself corresponding to an anticanonical divisor within smooth fibres of our degeneration. Other toric divisors in the conductor locus of X get smoothed in X_t and therefore do not display this behaviour any more, i.e. there are no cycles shrinked to a point on their preimages within smooth fibres X_t. In contrast, the boundary ∂B keeps corresponding to the

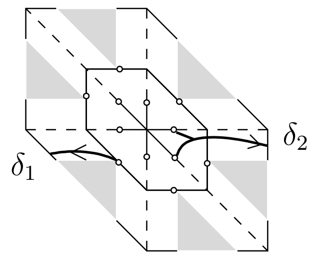

Figure 4.5.: Types of cycles

anticanonical divisor in X_t, i.e. cells in $\mathscr{P}_\partial^{[2-1]}$ keep corresponding to a codimension 1 locus in X_t. Therefore, cycles which shrink to a point when approaching the boundary keep this property in smooth fibres X_t.
For the singularity P the vanishing cycle approaching $P \in \rho$ is by the Picard-Lefschetz formula given by a cycle which is invariant with respect to monodromy. In our case, the only cycles which are invariant under the monodromy action are given by tropical cycles carrying vectors $\xi \in \Lambda_\rho$ as these vectors are the only ones which are invariant with respect to the monodromy on B. This finishes the proof of the lemma. □

In the example, we want to look at two types of cycles, which are indicated in Figure 4.5. The first one, δ_1, connects the singular point within ω_i with the boundary. It carries a primitive integral vector which is mapped to a primitive element in $\Lambda_{\partial B}$ by parallel transport along δ_1.
The second one, δ_2, is more complicated. It starts in a singular point contained in ρ_i. By the balancing condition at the middle vertex, it follows that the edge of δ_2 connecting this vertex and the singular point contained ρ_{i+1} has an opposite direction. Moreover, the cycle δ_2 also ends at the boundary.
Both types of cycles can be rotated, i.e. there are 6 cycles of type δ_1 and 6 cycles of type δ_2 for $i = 1, \ldots, 6$. They correspond to classes in $\mathrm{H}_2(X_t, \mathbb{Z})$ which can be represented by rational curves.

Remark 4.8. Note that until now, we have not specified the sign of the integral vectors ξ_1 and ξ_2 carried by δ_1 and δ_2 when running into ∂B. But as we want them to correspond to classes in $\mathrm{H}_2(X_t, \mathbb{Z})$ which can be represented by divisors (rather than by (-1) times a divisor), we require ξ_1 and ξ_2 to be positive with respect to the orientation on ∂B.

We want to compute all intersection numbers for homological cycles associated to tropical cycles of types δ_1 and δ_2.

Remark 4.9. Let $\gamma_1, \gamma_2 \subset B$ be tropical cycles, whose edges e_{γ_1} and e_{γ_2} intersect without loss of generality transversally in $b \in B$. The edges are directed and carry non-trivial sections $\xi_{\gamma_1} \in \Gamma(e_{\gamma_1}, \Lambda|_{e_{\gamma_1}})$ and $\xi_{\gamma_2} \in \Gamma(e_{\gamma_2}, \Lambda|_{e_{\gamma_2}})$. Let $\dot{e}_{\gamma_1}$ and $\dot{e}_{\gamma_2}$ denote generators of $T_b e_{\gamma_1} \subset \Lambda_{e_{\gamma_1}}$ and $T_b e_{\gamma_2} \subset \Lambda_{e_{\gamma_2}}$ respectively with sign determined by the direction of the edge. Let *or* represent the choice of orientation on B. We can always assume that

$$(\dot{e}_{\gamma_1}, \dot{e}_{\gamma_2}) = \pm 1 \cdot or.$$

Without loss of generality, we will always assume $\dot{e}_{\gamma_1} \wedge \dot{e}_{\gamma_2} = \det(\dot{e}_{\gamma_1}, \dot{e}_{\gamma_2}) = \pm 1$ tacitly from now on. This implies that with respect to the given orientation and directions of e_{γ_1} and e_{γ_2}, it holds that

$$(\dot{e}_{\gamma_1}, \dot{e}_{\gamma_2}) = \det(\dot{e}_{\gamma_1}, \dot{e}_{\gamma_2}) \cdot or.$$

Lemma 4.10. *The intersection number of homological cycles C_1 and C_2 corresponding to tropical cycles γ_1 and γ_2 intersecting transversally in a point $b \in \sigma \subset B$ is given by*

$$\gamma_1 \cdot \gamma_2 := \det(\dot{e}_{\gamma_1}, \dot{e}_{\gamma_2}) \cdot \det(\xi_{\gamma_1}, \xi_{\gamma_2}).$$

Proof. We want to compute the intersection number using a choice of orientation which is compatible on B and X_σ.
As before we factorize the tangent space $T_x(X_\sigma) = T_b^* B \times T_b^* B \cong \mathbb{R}^2 \times \mathbb{R}^2$. Next, we have to look at the tangent spaces of the cycles C_1 and C_2. By construction, the tangent space $T_x C_1$ is spanned by the dual basis vectors $\dot{e}_{\gamma_1}^*$ and $\xi_{\gamma_1}^\perp$. For $T_x C_2$ holds an analogous result.
Now, we want to compare $(T_x C_1, T_x C_2)$ with the orientation on X_σ. First, we look at the first factor of $T_x X_\sigma$, which corresponds to the (co-)tangent space of B. By Remark 4.9, this yields

$$(\dot{e}_{\gamma_1}, \dot{e}_{\gamma_2}) = \det(\dot{e}_{\gamma_1}, \dot{e}_{\gamma_2}) \cdot or_B.$$

Next, we have to take care of the tangent directions within the fibre $(X_\sigma)_b$. Thus, we want to compare $(\xi_{\gamma_1}^\perp, \xi_{\gamma_2}^\perp)$ with the induced orientation on $(X_\sigma)_b$. In a similar

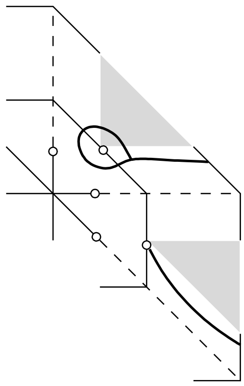

Figure 4.6.: A cycle replacing a vanishing thimble

way as in Remark 4.9, it holds that

$$(\xi_{\gamma_1}^{\perp}, \xi_{\gamma_2}^{\perp}) = \det\left(\xi_{\gamma_1}^{\perp}, \xi_{\gamma_2}^{\perp}\right) \cdot or_{(X_\sigma)_b}.$$

By invariance of the determinant under action of $\mathrm{SL}(2,\mathbb{Z})$ and transposition, it holds that

$$\det\left(\xi_{\gamma_1}^{\perp}, \xi_{\gamma_2}^{\perp}\right) = \det\left(\xi_{\gamma_1}, \xi_{\gamma_2}\right).$$

Thus, it follows that

$$(T_xC_1, T_xC_2) = \det(\dot{e}_{\gamma_1}, \dot{e}_{\gamma_2}) \cdot \det(\xi_{\gamma_1}, \xi_{\gamma_2}).$$

□

We want to compute the self-intersection of such a cycle ending at a singular point of B:
Let A be the transformation matrix corresponding to a focus-focus singularity. To compute the self-intersection of a curve at a singular point, we replace the vanishing thimble with a homologous curve surrounding the singular point as indicated by Figure 4.6 to simplify computations. This process does not affect the homology class of the corresponding cycle. Note that by Lemma 4.7, this modification yields exactly the case treated in Lemma 4.6. In a next step, we disturb the modified curve such that we get a transversal self-intersection. The modified curve and its disturbed version are illustrated in the left part of Figure 4.7 in solid and dotted lines.

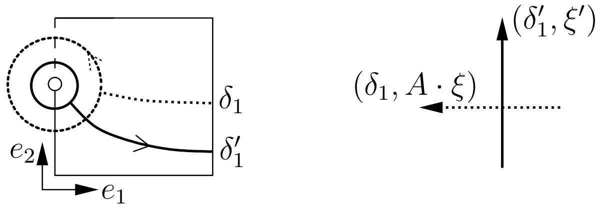

Figure 4.7.: Self-intersection at a focus-focus singularity

Lemma 4.11. *All tropical cycles of type δ_1 have self-intersection number -1.*

Proof. We want to use Lemma 4.10. First, we fix a basis as indicated by Figure 4.7. We get a factor

$$\det\left(\dot{e}_{\delta_1}, \dot{e}_{\delta_1'}\right) = \det\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = -1$$

from the directions of the edges, which are singled out, again in solid and dotted lines, in the right part of Figure 4.7.
Monodromy around the singular point leaves vectors parallel to the 1-cell invariant. In Figure 4.7, the monodromy matrix has the form $A = \left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right)$. At the intersection point, the disturbed cycle δ_1' carries a vector $\xi' = \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)$ parallel to the invariant direction. The cycle δ_1 carries a vector $A \cdot \xi$ for ξ such that $\xi' + \xi - A \cdot \xi = 0$, i.e. $\xi = \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$ and $A \cdot \xi = \left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right)$. Therefore, we get for the fibre direction a factor

$$\det\left(\xi, \xi'\right) = \det\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = 1,$$

which yields

$$\delta_1 \cdot \delta_1' = (-1) \cdot 1 = -1$$

as self-intersection number at the singularity. Because δ_1' can be arranged such that this is the only intersection point of δ_1 and δ_1', this finishes the proof. $\square$

Note that due to symmetry Lemma 4.11 holds for all curves of type δ_1 and for curves of type δ_2 at singular points. For the curves of type δ_2 we need to compute another number, which yields by the same argument:

Lemma 4.12. *All tropical cycles of type δ_2 have self-intersection number -1.*

Proof. At each of the two singularities, we get a contribution of -1 by Lemma 4.11. We will not repeat this computation. Moreover, there is an additional intersection point with intersection number $+1$ as can easily be seen. Therefore, the self-intersection number is given by

$$\delta_2 \cdot \delta_2 = (-1) + (-1) + 1 = -1.$$

□

Lemma 4.13. *Two neighbouring tropical cycles of type δ_2, i.e. a cycle ending in ρ_{i-1} and ρ_i and an cycle ending in ρ_i and ρ_{i+1}, have intersection number* 1.

Proof. The intersection of two neighbouring curves of type δ_2 is the same as the self-intersection at a singularity but with the orientation of one of the curves reversed. Therefore the intersection number equals $-(-1) = +1$. □

Ultimately, we want to understand homology of a pair (X, D), so we also need a tropical cycle corresponding to the anticanonical divisor.

Lemma 4.14. *The tropical cycle $\beta_\perp^\partial$ corresponding to the anticanonical class d is given by a small perturbation of ∂B carrying a primitive vector ξ, which is a primitive generator of $\Lambda|_{\partial B}/\Lambda_{\partial B}$ pointing inward into B.*

Proof. We can modify $\beta_\perp^\partial$ by pushing it into ∂B, i.e. $\beta_\perp^\partial = \partial B$. Note that the fibre of the homological cycle corresponding to $\beta_\perp^\partial$ over a point $b \in \partial B$ is given by

$$\mathrm{Hom}(\xi^\perp, U(1)) = \mathrm{Hom}\left((\Lambda_{\partial B})\,|_b, U(1)\right).$$

The latter corresponds to the fibre of the anticanonical divisor D over $b \in \partial B$. Note that the induced orientation on $\Lambda_{\partial B}$ is positive. □

Using Lemma 4.14, a straightforward computation yields:

Lemma 4.15. *The intersection number of curves corresponding to tropical cycles of type δ_1 and δ_2 with the fixed smooth anticanonical divisor is 1.*

As we are looking at a degeneration of dP_9 surfaces, we know that all curves in $\mathrm{H}_2(X_t, \mathbb{Z})$ can be identified with elements in $\Lambda_{1,9} = \mathit{span}\,(l, e_1, \ldots, e_9)$. We want to show that our tropical cycles can be assumed to represent elements in $\Lambda_{1,9}$ in

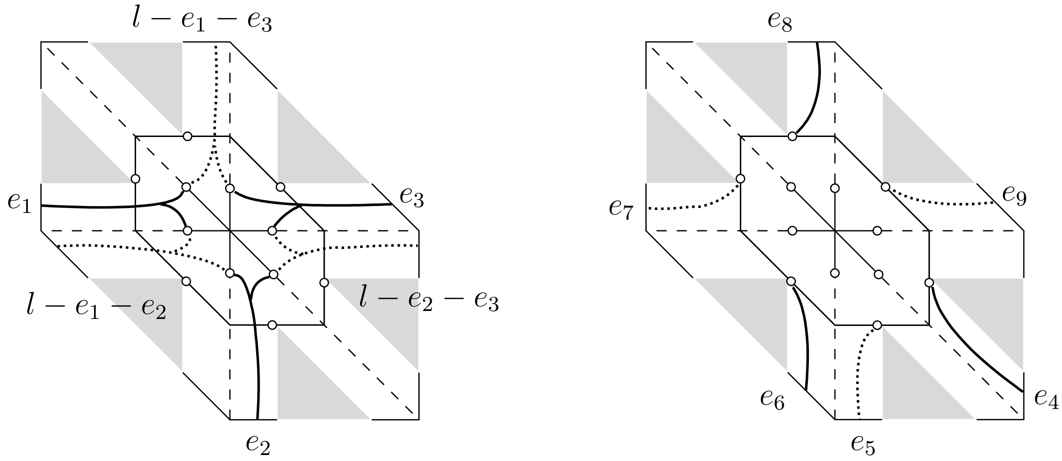

Figure 4.8.: Tropical cycles corresponding to exceptional curves

the way indicated in Figure 4.8.
The labelling of the curves of type δ_1 corresponds to an arbitrary choice of orthogonal exceptional curves.
The labelling of the curves of type δ_2 is unique up to rotation, as we can deduce from $\delta_2 \cdot \delta_2 = -1$ and $\delta_2 \cdot \beta_\perp^\partial = 1$, requiring that the coefficients of δ_2 are in $\mathbb{Z}$:

Lemma 4.16. *The intersection numbers of our tropical cycles are in tune with the correspondence indicated in Figure 4.8.*

Proof. First, note that we can split off the cycles corresponding to curves of type δ_1 from $\Lambda_{1,9}$ because they are orthogonal (-1)-curves. We can assume them to correspond to $e_4, \ldots, e_9$.
For the attribution of the other curves we look at a toy model sketched in Figure 4.9. This is possible as all affine models have the same linear monodromy representation. Note that $e_4, \ldots, e_9$ are already taken. Moreover, we can choose e_3 to be one of the cycles of type δ_2 as it has zero intersection with $e_4, \ldots, e_9$. In Figure 4.9, it corresponds to the cycle opposite β_2, which is not indicated. The same reasoning cannot be applied to the other cycles of type δ_2, as we get non-vanishing intersections. So we use our toy model, where we have for each cycle of type δ_2 the ansatz

$$\beta_i \sim a \cdot l + b \cdot e_1 + c \cdot e_2, \; i = 1, 2, 3.$$

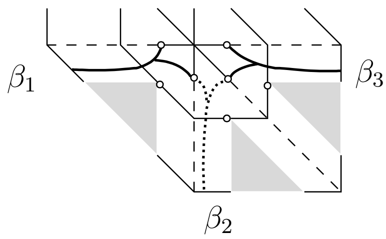

Figure 4.9.: A toy model

Therefore, there hold equations

$$\beta_i^2 = -1 \text{ and } \beta_i \cdot \beta_\perp^\partial = (a \cdot l + b \cdot e_1 + c \cdot e_2) \cdot (3l - e_1 - e_2) = 1.$$

These equations translate to

$$\begin{aligned} a^2 - b^2 - c^2 &= -1 \text{ and} \\ 3a + b + c &= 1, \end{aligned}$$

which yields the quadratic equation $4a^2 + 3a(b-1) + b(b-1) = 0$. Solving this quadratic equation for a produces the discriminant $-\frac{7}{16}b^2 - \frac{2}{16}b + \frac{9}{16}$. As the discriminant has to be non-negative, we have that $b \in \{-1, 0, 1\}$. By plugging $b \in \{-1, 0, 1\}$ into the other equations we get the following cases:

$$\begin{aligned} &\beta_i \sim e_1 \\ &\beta_i \sim e_2 \\ &\beta_i \sim l - e_1 - e_2 \end{aligned}$$

The attribution of these solutions is determined by the fact that $\beta_1 \cdot \beta_2 = 1 = \beta_2 \cdot \beta_3$ and $\beta_1 \cdot \beta_3 = 0$. $\square$

Lemma 4.17. *There is a tropical cycle $\beta_\parallel^\partial$ which does not correspond to a non-zero class in $\mathrm{H}_2(X_t, \mathbb{Z})$. It is given by a small perturbation of ∂B carrying a primitive vector ξ, which is obtained via parallel transport from $\Lambda_{\partial B} \subsetneq \Lambda|_{\partial B}$.*

Proof. As in the proof of Lemma 4.7 we note that the fibre of the homological cycle in X over each point $b \in \beta_\parallel^\partial$ shrinks to a point if we push $\beta_\parallel^\partial$ into ∂B. By this reasoning, the homological 2-cycle corresponding to $\beta_\parallel^\partial$ shrinks into a 1-cycle. Hence it cannot correspond to a non-vanishing class of 2-cycles. $\square$

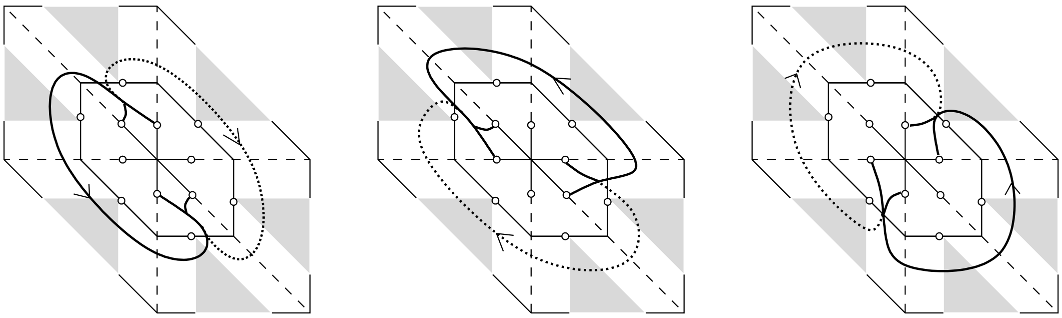

Figure 4.10.: Cycles representing α_0

Remark 4.18. Note that Lemma 4.17 relies on the existence of the boundary ∂B. If we worked with an open manifold $B \setminus \partial B$ instead, Lemma 4.17 would not be true.

Cycles corresponding to a root basis of $\hat{E}_8$

An $\hat{E}_8$ lattice is generated by $\alpha_0 = l - e_1 - e_2 - e_3,\ \alpha_1 = e_1 - e_2,\ \ldots,\ \alpha_8 = e_8 - e_9$. We construct tropical cycles $(\alpha_0)_{trop}, \ldots, (\alpha_8)_{trop} \in \mathrm{H}_1(B, \iota_* \Lambda)$ corresponding to these basis elements by taking two of the tropical cycles from Lemma 4.16 and gluing them. As the tropical cycles representing $e_1, \ldots, e_9$ differ from each other in form, we get several cases:

1) The cycle $(\alpha_0)_{trop}$:

There are three (or six, counting symmetric versions) different tropical cycles representing $\alpha_0 = l - e_1 - e_2 - e_3$. This ambiguity is caused by the three different ways to split α_0, i.e.

$$\alpha_0 = (l - e_1 - e_2) - e_3 = (l - e_1 - e_3) - e_2 = (l - e_2 - e_3) - e_1.$$

All possibilities are depicted in Figure 4.10.

Remark 4.19. Note that we always have two possibilities to join the cycles representing exceptional curves: by surrounding B in either positive or negative direction. In Figure 4.10, this is indicated by solid and dotted lines. The choice of direction corresponds to adding or subtracting the tropical cycle $\beta_{\|}^{\partial}$ from Lemma 4.17, which induces a trivial homological cycle class in $\mathrm{H}_2(X_t,\mathbb{Z})$. We can mend this ambiguity by considering only lifted gluing data $\mathbf{s}$ which are contained in the orthogonal part

$$\left(\beta_{\|}^{\partial}\right)^{\perp} = \{\mathbf{s} \in \mathrm{H}^1(B,\iota_*\check{\Lambda}\otimes\mathbb{C}^*) \mid \mathbf{s}\left(\beta_{\|}^{\partial}\right)=1\}.$$

In Remark 5.39, we will explain why this procedure does not impose any serious restriction. In Maple this construction is implemented by defining

```
β_∥^∂:=Matrix(192,1):
for k from 0 to 5 do β_∥^∂[(13+k· 32),1]:=1; β_∥^∂[(27+k· 32),1]:=-1:  od:
```

and for R the matrix from Remark 4.4

```
P:=ProjectionMatrix(IntersectionBasis[R,NullSpace(Transpose(β_∥^∂))]):
```

The matrix P projects vectors in $\mathrm{H}^1(B,\iota_*\check{\Lambda})$ into the subspace $(\beta_{\|}^{\partial})^{\perp}$.

Remark 4.20. Our notation for $\mathrm{H}^1(B,\iota_*\check{\Lambda}\otimes\mathbb{C}^*)$ is a little unconventional. Note that there is a product of additive, free, finitely generated groups

$$\mathrm{H}_1(B,\iota_*\Lambda)\otimes\mathrm{H}^1(B,\iota_*\check{\Lambda}).$$

Taking the spectra $\operatorname{Spec}\mathbb{C}[\mathrm{H}_1(B,\iota_*\Lambda)]$ and $\operatorname{Spec}\mathbb{C}[\mathrm{H}^1(B,\iota_*\check{\Lambda})]$ yields algebraic groups whose $\mathbb{C}$-valued points are given by

$$\mathrm{H}_1(B,\iota_*\Lambda)^*\otimes\mathbb{C}^* \cong \operatorname{Hom}\left(\mathrm{H}_1(B,\iota_*\Lambda),\mathbb{C}^*\right)$$
$$\text{and}\ \ \mathrm{H}^1(B,\iota_*\check{\Lambda})^*\otimes\mathbb{C}^* \cong \operatorname{Hom}\left(\mathrm{H}^1(B,\iota_*\check{\Lambda}),\mathbb{C}^*\right).$$

Note that because $\mathrm{H}^2(B,\iota_*\check{\Lambda})=0$, the universal coefficient theorem implies that

$$\mathrm{H}^1(B,\iota_*\check{\Lambda}\otimes\mathbb{C}^*) = \mathrm{H}^1(B,\iota_*\check{\Lambda})\otimes\mathbb{C}^*.$$

The pairing of gluing data $\mathbf{s}\in\mathrm{H}^1(B,\iota_*\check{\Lambda})$ with a tropical cycle $\beta\in\mathrm{H}_1(B,\iota_*\Lambda)$ yields

$$\mathrm{H}_1(B,\iota_*\Lambda)\otimes\left(\mathrm{H}^1(B,\iota_*\check{\Lambda})\otimes\mathbb{C}^*\right)\to\mathbb{C}^*. \tag{4.3}$$

Choosing a basis $(\mathbf{s_1},\ldots,\mathbf{s_{10}})$ of $\mathrm{H}^1(B,\iota_*\check{\Lambda})\otimes\mathbb{C}^*$ and a basis $(\beta_1,\ldots,\beta_{10})$ of $\mathrm{H}_1(B,\iota_*\Lambda)$ fixes isomorphisms

$$\mathrm{H}^1(B,\iota_*\check{\Lambda})\otimes\mathbb{C}^*\cong(\mathbb{C}^*)^{10}\ \text{and}\ \mathrm{H}_1(B,\iota_*\Lambda)\cong\mathbb{Z}^{10}.$$

If these bases are related to dual bases of the underlying additive groups, the pairing (4.3) looks like

$$(\mathbf{s_1}, \ldots, \mathbf{s_{10}}) \otimes (\beta_1, \ldots, \beta_{10}) \longrightarrow (\mathbf{s_1}^{\beta_1} \cdot \ldots \cdot \mathbf{s_{10}}^{\beta_{10}}).$$

Thus our notation does make sense as we are implicitly referring to the underlying additive structure on $\mathrm{H}^1\left(B, \iota_*\check{\Lambda}\right)^*$.

We want to fix vectors corresponding to tropical cycles in $\mathrm{H}_1(B, \iota_*\Lambda)$ using the Čech cover $\mathcal{W}$ and coordinates from Figure 4.3. To simplify this procedure, we deform the tropical cycles until they only contain straight edges joining vertices $v \in \mathscr{P}_{bar}$ such that $v \in \mathscr{P}^{[0]}$ or v is the barycenter of $\sigma \in \mathscr{P}_{max}$. Any such edge carrying a vector ξ yields a contribution ξ on $U_{v\sigma}$.
Thus, the pairing is given by matrix multiplication with a vector with 192 entries in the Čech cohomology group $\mathrm{H}^1(B, \iota_*\check{\Lambda})$ because for each set $U_{v\sigma}$ we multiply ξ with an element in $\iota_*\check{\Lambda}|_{U_{v\sigma}}$.
Let β be a vector representing a tropical cycle. The pairing

$$\begin{aligned} \mathrm{H}_1(B, \iota_*\Lambda) \otimes \left(\beta_{\|}^{\partial}\right)^{\perp} &\to \mathbb{Z} \\ (\beta, \mathbf{s}) &\mapsto \langle \beta, \mathbf{s} \rangle \end{aligned}$$

is implemented as

```
Multiply(Transpose(Multiply(P,R)),β);
```

For example, a cycle representing α_0 can be modified in the way indicated in Figure 4.11. A cycle surrounding the singular point on the 1-cell ρ_n, $n = 1, \ldots, 6$, is implemented as

```
W:=proc(n)
local w1:=Matrix(32,1), w2:=Matrix(32,1), k:=modp(n-2,6),
l:=modp(n-1,6), m:=modp(n,6), M:=Matrix(192,1):
w1[3..4,1]:=Multiply(InverseMatrix(A0l,Vector([1,0]));
w1[25..26,1]:=-Multiply(A12, Vector([1,0]));
w2[3..4,1]:=
  -Multiply(InverseMatrix(A0m),Multiply(A12, Vector([1,0])));
w2[11..12,1]:=Multiply(A12,Vector([1,0]));
M[(1+32· k)..(32+32· k),1]:=w1;
```

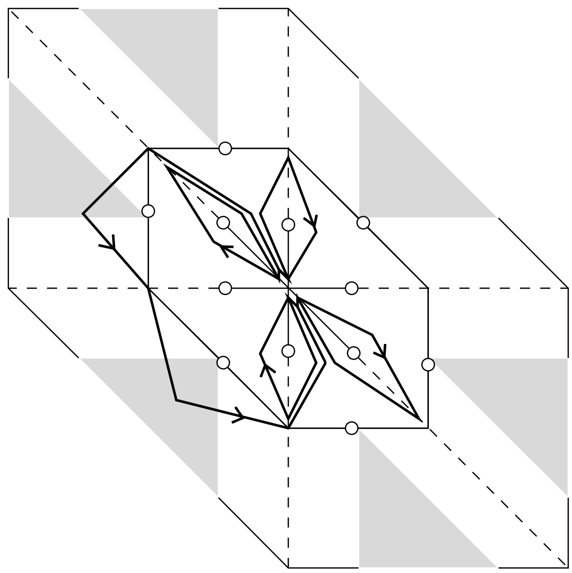

Figure 4.11.: A modified tropical cycle representing α_0

```
M[(1+32· l)..(32+32· l),1]:=w2;
M
end:
```

The tropical cycle we need to connect the barycenter of σ_n with the barycenter of σ_m is given by

```
Path1:=proc(n,m)
local M:=Matrix(192,1), k:=modp(n-1,6), l:=modp(m-1,6), i;
M[(25+k· 32),1]:=-1;
M[(3+k· 32)..(4+k· 32),1]:=
  Multiply(InverseMatrix(A0 modp(n,6)), Vector([1,0]));
M[(11+l· 32),1]:=1;
M[(3+l· 32)..(4+l· 32),1]:=
  -Multiply(InverseMatrix(A0 modp(m,6)),Vector([1,0]));
if n+1>m then wrong input else for i from n+i to m-1 do
M[(13+32·modp(i-1,6)),1]:=M[(13+32·modp(i-1,6)),1]+1;
M[(27+32·modp(i-1,6)),1]:=M[(27+32·modp(i-1,6)),1]-1;
od fi; M
```

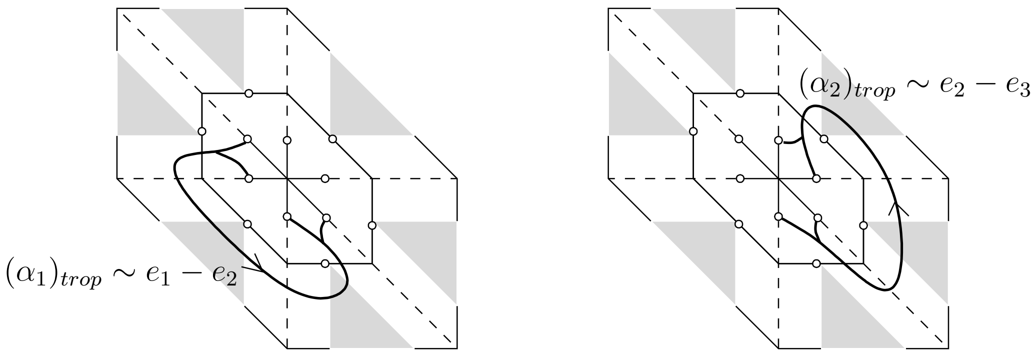

Figure 4.12.: Tropical cycles representing α_1 and α_2

```
end:
```

Together, this implements a vector corresponding to the difference of two opposite cycles of type δ_2, the first of them starting at the singular point on ρ_n:

```
(δ2 − δ2'):=proc(n)
local k:=modp(n,6), l:=modp(n+3,6), k1:=modp(n+1,6),
l1:=modp(n+4,6);
W(k)-W(k1)+Path1(n,n+3)-W(l)+W(l1)
end:
```

To verify that all three tropical cycles representing α_0 are equivalent with respect to the pairing introduced above, we compute

```
Rank(Multiply(Transpose(Multiply(P,R)),(δ2 − δ2')(6)-(δ2 − δ2')(4)));
Rank(Multiply(Transpose(Multiply(P,R)),(δ2 − δ2')(4)-(δ2 − δ2')(2)));
```

This computation yields zero in both cases. It also serves as an additional test for the implementation we performed in Maple up to this point.

2) The cycles $(\alpha_1)_{trop}$ and $(\alpha_2)_{trop}$:

The tropical cycles representing $\alpha_1 = e_1 - e_2$ and $\alpha_2 = e_2 - e_3$ look very similar. They are indicated in Figure 4.12. We implement these two cycles in Maple, using the work we already did in order to implement $(\alpha_0)_{trop}$. The vector corresponding to $(\alpha_1)_{trop}$ is given by

```
(α1)trop:= W(5)-W(6)-W(1)+W(2)+Path1(5,7):
```

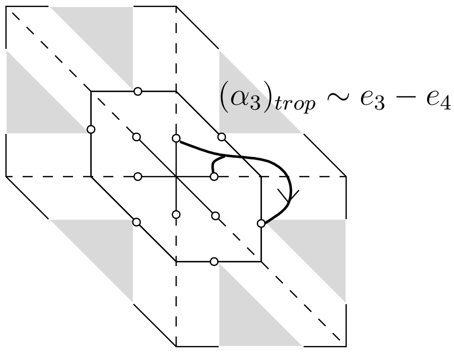

Figure 4.13.: A cycle representing α_3

and the cycle $(\alpha_2)_{trop}$ similarly by

$(\alpha_2)_{trop}$:= W(1)-W(2)-W(3)+W(4)+Path1(1,3):

3) The cycle $(\alpha_3)_{trop}$:

The cycle representing $\alpha_3 = e_3 - e_4$ is the only one mixing the two types of cycles (see Figure 4.13). We first implement a cycle surrounding the singular point on ω_n:

```
V:=proc(n)
local m:=modp(n-1,6), k:=1+m· 32, l:=32+m· 32,
v:=Matrix(32,1);
v(14,1):=-1; v(28,1):=1; v(26,1):=-1; v(12,1):=1; v(11,1):=-1;
Matrix(192,1)(k..l,1):=v
end:
```

Moreover, we need an auxiliary path connecting the barycenter of σ_3 with the starting point of `V(2)`. It is defined by

```
Path2:=Matrix(192,1):
Path2[(2· 32+11),1]:=-1:
Path2[(2· 32+3)..(2· 32+4),1]:= Vector([-1,1]):
Path2[(13+32),1]:=1;
Path2[(27+32),1]:=-1;
```

Using this implementation we can fix $(\alpha_3)_{trop}$ as

$(\alpha_3)_{trop}$:= W(3)-W(4)-V(2)+Path2:

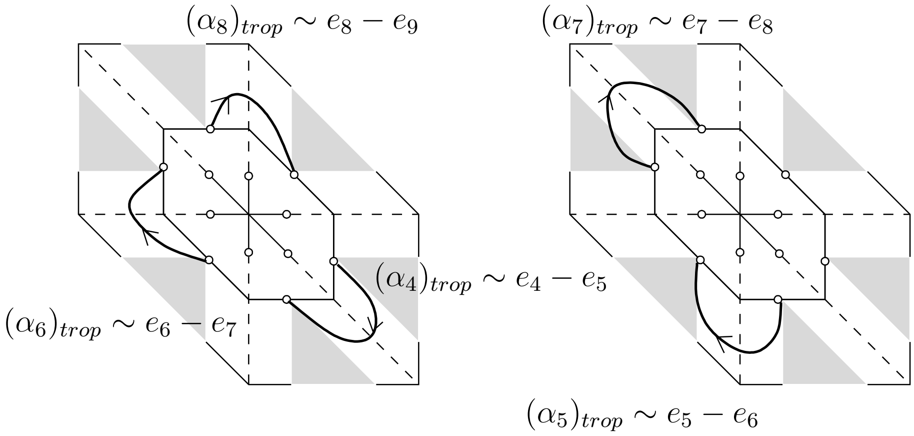

Figure 4.14.: Cycles representing $\alpha_4, \ldots, \alpha_8$

4) The cycles $(\alpha_4)_{trop}, \ldots, (\alpha_8)_{trop}$:

The remaining five cycles are again rotationally symmetric to each other. They are sketched in Figure 4.14. First, we implement an auxiliary path corresponding to the part of $\beta^{\partial}_{\|}$ contained in $\tau_n, \ldots, \tau_m$:

```
Path3:=proc(n,m)
local M:=Matrix(192,1),i;
if m>n then wrong input else for i from n to m do
M[(13+32·modp(i-1,6)),1]:=M[(13+32·modp(i-1,6)),1]+1;
M[(27+32·modp(i-1,6)),1]:=M[(27+32·modp(i-1,6)),1]-1;
od fi end:
```

This allows us to define a difference of cycles of the first type via

$(\delta_1 - \delta_1')$`:=proc(n)`
`V(n)-Path3(n-1,n-1)-V(n-1)`
`end:`

Thus we get

$(\alpha_4)_{trop}$`:=`$(\delta_1 - \delta_1')$`(2):` $(\alpha_5)_{trop}$`:=`$(\delta_1 - \delta_1')$`(1):` $(\alpha_6)_{trop}$`:=`$(\delta_1 - \delta_1')$`(6):`
$(\alpha_7)_{trop}$`:=`$(\delta_1 - \delta_1')$`(5):` $(\alpha_8)_{trop}$`:=`$(\delta_1 - \delta_1')$`(4):`

The intersection matrix:

To check our construction of $(\alpha_0)_{trop}, \ldots, (\alpha_8)_{trop}$, we compute the intersection matrix of $(\alpha_0)_{trop}, \ldots, (\alpha_8)_{trop}$ by hand. We know that $D^\perp \subset \mathrm{H}_2(X_t, \mathbb{Z})$, $t \neq O$, is isomorphic to the lattice $\hat{E}_8$ (or rather $(-1) \cdot \hat{E}_8$). If our construction is correct, the tropical cycles should reproduce the corresponding intersection matrix.

Lemma 4.21. *The intersection matrix of* $(\alpha_0)_{trop}, \ldots, (\alpha_8)_{trop}$ *is given by*

$$\begin{pmatrix} -2 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 \end{pmatrix}.$$

Proof. The intersection numbers are computed using the intersection numbers of the exceptional curves (Lemma 4.11, Lemma 4.12 and Lemma 4.13) and the fact that additional intersection points do not add anything. This is the case because the parts of $(\alpha_0)_{trop}, \ldots, (\alpha_8)_{trop}$ connecting the cycles corresponding to exceptional curves are given by parts of $\beta^\partial_\parallel$. Thus, all of them carry parallel vectors. $\square$

Cross check and conclusion

We want to check our implementation in Maple by reconstructing the class of the anticanonical divisor.
Algebraically, the class of the anticanonical divisor D is given by

$$d := 3l - e_1 - \ldots - e_9.$$

It has a unique decomposition into the basis elements $\alpha_0, \ldots, \alpha_8$, which is invariant under the action of the Weyl group. This decomposition is given by

$$d = 3\alpha_0 + 2\alpha_1 + 4\alpha_2 + 6\alpha_3 + 5\alpha_4 + 4\alpha_5 + 3\alpha_6 + 2\alpha_7 + \alpha_8.$$

We want to reconstruct this decomposition using Maple. To do this, we implement the cycle $\beta_\perp^\partial$ corresponding to d following Lemma 4.14:

```
β⊥∂:=Matrix(192,1):
for k from 0 to 5 do
β⊥∂[(14+k· 32),1]:=1;
β⊥∂[(28+k· 32),1]:=-1;
od:
```

We can decompose $\beta_\perp^\partial$ by computing

```
solve(Equate(Multiply(Transpose(Multiply(P,R)),β⊥∂),
Multiply(Transpose(Multiply(P,R)),
a0·(α0)trop+a1·(α1)trop+a2·(α2)trop + a3·(α3)trop + a4·(α4)trop
+ a5·(α5)trop + a6·(α6)trop + a7·(α7)trop + a8·(α8)trop)));
```

This computation yields the expected result.
We close this subsection with a result which will become more important later on.

Proposition 4.22. *Let* $\mathfrak{R} := span\left((\alpha_0)_{trop}, \ldots, (\alpha_8)_{trop}\right) \subset \mathrm{H}_1\left(B, \iota_*\Lambda\right)$. *The pairing*

$$\begin{aligned} \mathfrak{R} \otimes \left(\beta_\parallel^\partial\right)^\perp &\to \mathbb{Z} \\ (\beta_{trop}, \mathbf{s}) &\mapsto \langle \beta_{trop}, \mathbf{s} \rangle \end{aligned}$$

has full rank $r = 9$.

Proof. In Maple, the rank of the pairing can be computed by

```
 Rank(Multiply(Transpose(Multiply(P,R)),
<(α0)trop|(α1)trop|(α2)trop|(α3)trop|(α4)trop|(α5)trop|(α6)trop|(α7)trop|(α8)trop>));
```

This computation yields the expected result. □

4.3. Piecewise linear functions and cohomology

Let β_{trop} be a tropical cycle and φ a strictly convex multi-valued piecewise linear function. The second factor of the period integrals given by equation (4.1) on page 73 is determined by the same pairing as we used in Proposition 4.22 via

$$\begin{aligned} \mathrm{H}_1(B, \iota_*\check{\Lambda}) \otimes \mathrm{H}^1(B, \iota_*\Lambda) &\to \mathbb{Z} \\ (\beta_{trop}, c_1(\varphi)) &\mapsto \langle \beta_{trop}, c_1(\varphi) \rangle. \end{aligned}$$

To understand $c_1(\varphi) \in \mathrm{H}^1(B, \iota_*\check{\Lambda})$, we compute a long exact sequence of cohomology groups for sheaves of piecewise linear functions.
Recall from Definition 3.27 that there is a short exact sequence of sheaves

$$0 \to \iota_*\check{\Lambda} \to \mathcal{PL} \to \mathcal{MPL} \to 0. \tag{4.4}$$

The short exact sequence (4.4) induces a long exact sequence in cohomology:

$$\begin{aligned} 0 \to \mathrm{H}^0(B, \iota_*\check{\Lambda}) \to PL \to MPL \to \mathrm{H}^1(B, \iota_*\check{\Lambda}) \to \mathrm{H}^1(B, \mathcal{PL}) \\ \to \mathrm{H}^1(B, \mathcal{MPL}) \to \mathrm{H}^2(B, \iota_*\check{\Lambda}) \to \dots \end{aligned} \tag{4.5}$$

We want to compute this long exact sequence for our example.
Recall from Proposition 4.3 that

$$\mathrm{H}^i(B, \iota_*\check{\Lambda}) = \begin{cases} 0, & i = 0, 2, \\ \mathbb{Z}^{10}, & i = 1. \end{cases}$$

Thus, the long exact sequence (4.5) is reduced to

$$0 \to PL \to MPL \to \mathbb{Z}^{10} \to \mathrm{H}^1(B, \mathcal{PL}) \to \mathrm{H}^1(B, \mathcal{MPL}) \to 0.$$

Cohomology groups of $\mathcal{PL}$

To compute cohomology groups of $\mathcal{PL}$, we choose an open cover of $B \setminus \partial B$, which is adapted to the polyhedral decomposition. Let $v_0, \dots, v_6$ be the dimension zero cells of $\mathscr{P}$ as indicated in Figure 4.1. We define open sets via the open star (3.1) from page 57:

$$U_i := U_{v_i}$$

These open sets yield an open cover

$$\mathcal{U} = \{U_0, \dots, U_6\} \tag{4.6}$$

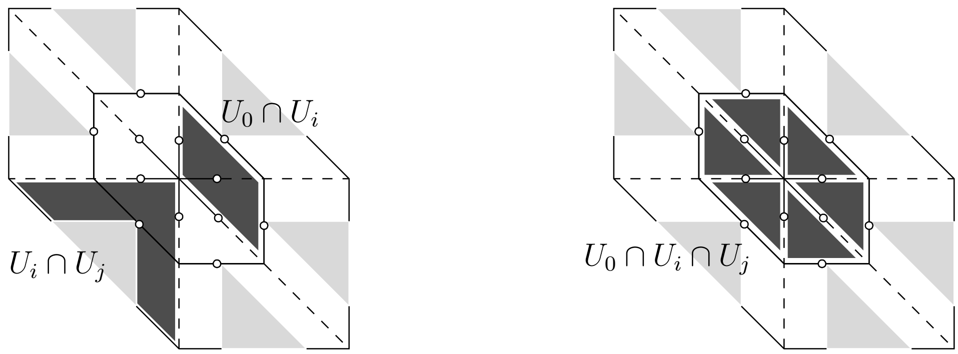

Figure 4.15.: Intersections and triple intersections

of $B \setminus \partial B$. But the covering of $B \setminus \partial B$ suffices for our purpose.
The open sets $U_0, \dots, U_6$ are contractible. Moreover, they are much easier to use than the finer cover (3.2) on page 58, which was induced by the barycentric subdivision.
Non-empty intersections $U_i \cap U_j$ are open stars of a 1-cell and triple intersections $U_i \cap U_j \cap U_k$ are open stars of a 2-cell i.e. in our situation they consist of the interiors of maximal cells $\sigma_1, \dots, \sigma_6$.
The two different types of intersections and all of the triple intersections are indicated in Figure 4.15. Locally, on any maximal cell σ with coordinates x, y we can write an integral linear function as $\varphi(x, y) = \lambda_1 x + \lambda_2 y$, $\lambda_1, \lambda_2 \in \mathbb{Z}$.
In our example, Definition 3.26 yields $\varphi|_{U_0} \equiv 0$. Moreover, the balancing condition holds on the U_i.
With a suitable choice of basis ($x \in \Lambda_{\omega_i}$, $y \in \Lambda_{\rho_i}$) via parallel transport, we can write $\left(\varphi|_{U_i \cap U_j}\right)|_{U_i} = \lambda^i_{ij} y$ and $(\varphi|_{U_0 \cap U_i})|_{U_k} = \lambda^k_{0i} x$ for $k = 0, i$ and $\lambda^i_{ij}, \lambda^k_{0i} \in \mathbb{Z}$. Moreover, we can set $(\varphi|_{U_i})|_{U_0} = 0$ and $(\varphi|_{U_i})|_{U_i \setminus U_0} = \lambda_i y$ for $\lambda_i \in \mathbb{Z}$.
By this reasoning, we get the following terms:

$$C^0(\mathcal{U}, \mathcal{PL}) = \mathcal{PL}|_{U_0} \times \mathcal{PL}|_{U_1} \times \cdots \times \mathcal{PL}|_{U_6} = 0 \times \mathbb{Z}^6$$

$$C^1(\mathcal{U}, \mathcal{PL}) = \prod_{i=1}^{6} \mathcal{PL}|_{U_i \cap U_{i+1}} \times \prod_{i=1}^{6} \mathcal{PL}|_{U_0 \cap U_i} = \mathbb{Z}^{24}$$

$$C^2(\mathcal{U}, \mathcal{PL}) = \prod_{i=1}^{6} \mathcal{PL}|_{\sigma_i} = \mathbb{Z}^{12}$$

Note that the cover $\mathcal{U}$ is acyclic for the sheaf $\mathcal{PL}$, as it is flasque on triple intersections and higher cohomology groups vanish on intersections and the open sets $U_0, \ldots, U_6$.
The latter can be proved by computing Čech cohomology e.g. on $U_0 \cap U_i$ using the cover consisting of two open sets V_1 and V_2, each arising as the union of the interiors of σ_{i-1} and σ_i together with one component of $\rho_i \setminus \Delta$ respectively. The sets V_1 and V_2 are acyclic for $\mathcal{PL}$, e.g. by looking at the short exact sequence induced by Definition 3.27 and using the fact that $\check{\Lambda}$ and $\mathcal{MPL}$ are flasque on V_1 and V_2. A similar argument holds for the other sets in question. Thus, we can proceed to calculating cohomology groups for all of B.
Because $\varphi|_{U_0} \equiv 0$, a global $\mathcal{PL}$-function is determined by its kink at ω_i, which is the same for $i = 1, \ldots, 6$, so

$$\mathrm{H}^0(B, \mathcal{PL}) = \mathbb{Z}.$$

Moreover, this reasoning yields that

$$\operatorname{im}(\partial^0 \colon C^0(\mathcal{U}, \mathcal{PL}) \to C^1(\mathcal{U}, \mathcal{PL})) = \mathbb{Z}^5.$$

The differential $\partial^1 \colon C^1(\mathcal{U}, \mathcal{PL}) \to C^2(\mathcal{U}, \mathcal{PL})$ is surjective and

$$\ker(\partial^1 \colon C^1(\mathcal{U}, \mathcal{PL}) \to C^2(\mathcal{U}, \mathcal{PL})) = \mathbb{Z}^{12}.$$

This fact implies that

$$\mathrm{H}^1(B, \mathcal{PL}) = \mathbb{Z}^7.$$

Cohomology groups of $\mathcal{MPL}$

We take the open cover $\mathcal{U}$ of B introduced in equation (4.6) on page 97. A $\mathcal{MPL}$-function can be described purely by its kinks along the rays, which have to fulfil the balancing condition by Definition 3.27. By this reasoning, we get the following terms:

$$\begin{aligned} C^0(\mathcal{U}, \mathcal{MPL}) &= \mathbb{Z}^4 \times \prod_{i=1}^{6} \mathbb{Z}^2 = \mathbb{Z}^{16} \\ C^1(\mathcal{U}, \mathcal{MPL}) &= \mathbb{Z}^{12} \\ C^2(\mathcal{U}, \mathcal{MPL}) &= 0 \end{aligned}$$

For a global $\mathcal{MPL}$-function we get 4 degrees of freedom from U_0, which fixes one radial degree of freedom in each of the $U_1, \ldots, U_6$. Moreover, there is one degree of freedom which determines the kink at all of the $\omega_i,\ i = 1, \ldots, 6$. Therefore, we get the following cohomology groups:

$$\begin{aligned} \mathrm{H}^0(B, \mathcal{MPL}) &= \mathbb{Z}^5 \\ \mathrm{H}^1(B, \mathcal{MPL}) &= \mathbb{Z} \\ \mathrm{H}^2(B, \mathcal{MPL}) &= 0 \end{aligned} \tag{4.7}$$

Inserting all terms into the long exact sequence (4.5) from page 97 yields:

$$\begin{tikzcd} 0 \arrow[r] & PL \arrow[r] \arrow[d, equal] & MPL \arrow[r] \arrow[d, equal] & \mathrm{H}^1(B, \iota_*\check{\Lambda}) \arrow[r] \arrow[d, equal] & \mathrm{H}^1(B, \mathcal{PL}) \arrow[r] \arrow[d, equal] & \mathrm{H}^1(B, \mathcal{MPL}) \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & \mathbb{Z} \arrow[r] & \mathbb{Z}^5 \arrow[r] & \mathbb{Z}^{10} \arrow[r] & \mathbb{Z}^7 \arrow[r] & \mathbb{Z} \arrow[r] & 0 \end{tikzcd} \tag{4.8}$$

The pairing with $c_1(\varphi)$

The multi-valued piecewise linear function φ on B is an element in MPL in the exact sequence (4.8) on page 100. The class $c_1(\varphi)$ is given by its image

$$\begin{aligned} MPL &\to \mathrm{H}^1(B, \iota_*\check{\Lambda}) \\ \varphi &\mapsto c_1(\varphi). \end{aligned}$$

We want to use [GHKS2, Lemma 13.7] to compute $\langle \beta_{trop}, c_1(\varphi) \rangle$ for a cycle $\beta_{trop} \in \mathrm{H}_1(B, \iota_*\Lambda)$ by means of a pairing with a cycle $\delta_{trop}(\varphi)$. The cycle $\delta_{trop}(\varphi)$ is specified as follows:
The multi-valued piecewise linear function φ is determined by its kinks $\kappa_\rho \in \mathbb{Z}$ along 1-cells $\rho \in \mathscr{P}^{[1]}$ by Definition 3.27. Let d_ρ be a generator of $\Lambda_\rho,\ \rho \in \mathscr{P}^{[1]}$. The cycle $\delta_{trop}(\varphi)$ is given by a 1-cycle in B_0, which is a small perturbation of the 1-skeleton of $\mathscr{P}$ with direction given by the d_ρ. It carries the vector

$$\kappa_\rho \cdot d_\rho$$

on the part corresponding to $\rho \in \mathscr{P}^{[1]}$. The balancing condition on tropical cycles, (4.2) on page 79, is fulfilled by the balancing condition on piecewise linear functions in Definition 3.26.

Lemma 4.23 ([GHKS2, Lemma 13.7]). *Let $\beta_{trop} \in \mathrm{H}_1(B_0, \Lambda)$ and $\varphi \in MPL$. Using the cycle $\delta_{trop}(\varphi) \in \mathrm{H}_1(B_0, \Lambda) \cong \mathrm{H}_1(B, \iota_*\Lambda)$ specified above, it holds that*

$$\beta_{trop} \cdot \delta_{trop}(\varphi) = \langle \beta_{trop}, c_1(\varphi) \rangle.$$

Remark 4.24. The product of two cycles $\gamma_1, \gamma_2 \in \mathrm{H}_1(B, \iota_*\Lambda)$ used in Lemma 4.23 is computed as follows:
Denote the intersection points by

$$\gamma_1 \cap \gamma_2 = \{p_1, \dots, p_k\} \in B_0.$$

Without loss of generality we can assume transversality. Moreover, for each p_i we can choose $\epsilon_i \in \{-1, 1\}$ such that for positively directed tangent vectors $\lambda_1 \in T_{p_i}\gamma_1$ and $\lambda_2 \in T_{p_i}\gamma_2$ of the cycles γ_1 and γ_2 at p_i, it holds that

$$\det(\lambda_1 | \lambda_2) = \epsilon_i \cdot c, \ c > 0.$$

Let $(\xi_1)_{p_i}$ and $(\xi_2)_{p_i}$ be the vectors in $\Lambda|_{p_i}$ carried by γ_1 and γ_2. The intersection product is given by

$$\gamma_1 \cdot \gamma_2 = \sum_{i=1}^{k} \epsilon_i \cdot \det\Big((\xi_1)_{p_i}, (\xi_2)_{p_i}\Big).$$

Note that by the reasoning on MPL leading to the cohomology groups (4.7) on page 100, we can set

$$\begin{aligned} \kappa &:= \kappa_{\omega_1} = \dots = \kappa_{\omega_6} \\ \text{and } \kappa_i &:= \kappa_{\rho_i} = \kappa_{\mu_i}. \end{aligned}$$

The balancing condition on tropical cycles, (4.2) on page 79, at v_0 yields

$$\kappa_{\rho_1}\begin{pmatrix}1\\0\end{pmatrix} + \kappa_{\rho_2}\begin{pmatrix}0\\1\end{pmatrix} + \kappa_{\rho_3}\begin{pmatrix}-1\\1\end{pmatrix} + \kappa_{\rho_4}\begin{pmatrix}-1\\0\end{pmatrix} + \kappa_{\rho_5}\begin{pmatrix}0\\-1\end{pmatrix} + \kappa_{\rho_6}\begin{pmatrix}1\\-1\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}.$$

This equation is equivalent to

$$\begin{aligned} \kappa_1 - \kappa_4 &= \kappa_3 - \kappa_6 \\ \text{and } \kappa_2 - \kappa_5 &= \kappa_6 - \kappa_3. \end{aligned}$$

Remark 4.25. The pairing with the cycle $\beta_{\parallel}^{\partial}$ from Lemma 4.17 does not vanish for non-trivial $c_1(\varphi)$. We get

$$\beta_{\parallel}^{\partial} \cdot \delta_{trop}(\varphi) = \kappa_1 + \kappa_2 + \kappa_3 + \kappa_4 + \kappa_5 + \kappa_6.$$

In contrast, we get $\beta_{\perp}^{\partial} \cdot \delta_{trop}(\varphi) = 0$ for the cycle $\beta_{\perp}^{\partial}$ from Lemma 4.14. We will give an interpretation of these facts in Chapter 5.

Note that a cycle $\beta_{trop} \in \mathrm{H}_1(B, \iota_*\Lambda)$ does not receive any contributions from $c_1(\varphi)$ while surrounding a singularity as in Lemma 4.6. The only contributions we get in our example arise from parts of $(\alpha_0)_{trop}, \ldots, (\alpha_8)_{trop}$ which coincide with $\beta_{\parallel}^{\partial}$:

$$\begin{aligned}
(\alpha_0)_{trop} \cdot \delta_{trop}(\varphi) &= \kappa_1 + \kappa_2 + \kappa_3 \\
(\alpha_1)_{trop} \cdot \delta_{trop}(\varphi) &= \kappa_6 + \kappa_1 \\
(\alpha_2)_{trop} \cdot \delta_{trop}(\varphi) &= \kappa_2 + \kappa_3 \\
(\alpha_3)_{trop} \cdot \delta_{trop}(\varphi) &= -\kappa_3 \\
(\alpha_4)_{trop} \cdot \delta_{trop}(\varphi) &= -\kappa_2 \\
(\alpha_5)_{trop} \cdot \delta_{trop}(\varphi) &= -\kappa_1 \\
(\alpha_6)_{trop} \cdot \delta_{trop}(\varphi) &= -\kappa_6 \\
(\alpha_7)_{trop} \cdot \delta_{trop}(\varphi) &= -\kappa_5 \\
(\alpha_8)_{trop} \cdot \delta_{trop}(\varphi) &= -\kappa_4
\end{aligned}$$

Moreover, none of the cycles $(\alpha_0)_{trop}, \ldots, (\alpha_8)_{trop}$ yields a contribution of the kink κ, so this part of φ is invisible to us. This fact fits nicely into the general picture, as κ corresponds to the image of PL in MPL, which is mapped to zero by

$$c_1 \colon MPL \to \mathrm{H}^1(B, \iota_*\check{\Lambda}).$$

So, we do not lose any information using the cycles $(\alpha_0)_{trop}, \ldots, (\alpha_8)_{trop}$ for our Torelli type theorem.

5. Toric degenerations of dP_9 surfaces

5.1. Simple log dP_9 pairs

We want to restrict the toric degeneration setup from Section 3.1 to toric degenerations of dP_9 surfaces. First, we introduce some notation.

Definition 5.1. We define
$$\mathfrak{O}^\dagger := \operatorname{Spf} \mathbb{C}[[t]]^\dagger.$$
The log structure on $\mathfrak{O}$ is the divisorial log structure induced by
$$\begin{aligned} \mathbb{N} &\to \mathbb{C}[[t]] \\ 1 &\mapsto t. \end{aligned}$$
Moreover, we denote the standard log point by $O^\dagger$ and we set
$$\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}.$$
Note that $\mathbb{D}$ can be endowed with a divisorial log structure for $0 \in \mathbb{D}$. This construction yields inclusions of logarithmic spaces $O^\dagger \subset \mathfrak{O}^\dagger \subset \mathbb{D}^\dagger$.

We want to strengthen Definition 3.3 to get only central fibres of degenerations of dP_9 pairs:

Definition 5.2. A *simple log dP_9 pair* is a polarized toric log Calabi-Yau pair (X, D) (Definition 3.3) with $\dim X = 2$ such that there exists a strictly convex piecewise linear function on the induced affine manifold $(B, \mathscr{P})$. Moreover, $D \neq \emptyset$ is an effective anticanonical divisor with trivial (non-logarithmic) normal bundle
$$\mathcal{N}_{D/X} = \mathcal{O}_D$$

such that $(D, \emptyset)$ is a degenerate Calabi-Yau pair itself.
We require $(B, \mathscr{P})$ to be positive (see Definition 3.13) and simple (see Definition 3.14). Topologically, it has to hold that $\partial B \cong \mathbb{S}^1$ and $B \cong cl(\mathbb{D})$ and moreover, the singular locus Δ consists of 12 points.

Remark 5.3. The condition $\partial B \cong \mathbb{S}^1$ in Definition 5.2 is necessary because we want the anticanonical divisor within smooth geometric fibres to be a smooth curve of genus 1, in particular compact and connected, which fixes the topological space underlying ∂B. The necessity of the condition that the singular locus Δ consists of 12 points is proved in Lemma 5.11 below.

Lemma 5.4. *The logarithmic structure $\mathcal{M}_{(X,D)}$ of a simple log dP_9 pair (X, D) contains a subsheaf which constitutes another logarithmic structure $\mathcal{M}_X \subset \mathcal{M}_{(X,D)}$. The logarithmic structure $\mathcal{M}_X$ restricts to a logarithmic structure $\mathcal{M}_D$ on D.*

Proof. Note that a section $f \in \Gamma\left(U, (\mathcal{M}_{(X,D)})|_U\right)$, $U \subset X$ can locally be split into a function which does not vanish on $U \setminus D$ and a section of the ghost sheaf. We define a morphism of sheaves

$$\beta\colon \mathcal{M}_{(X,D)} \to \underline{\mathbb{N}}_D$$

by mapping a section $f \in \Gamma\left(U, (\mathcal{M}_{(X,D)})|_U\right)$, $U \subset X$, to $\nu_D(f)$, the sum of the vanishing orders of the corresponding function along $D \cap U$ (see e.g. [GS1, Ex. 3.2]). Define the new logarithmic structure $\mathcal{M}_X$ on X by

$$\mathcal{M}_X := \ker\left(\beta\colon \mathcal{M}_{(X,D)} \to \underline{\mathbb{N}}_D\right) \subset \mathcal{M}_{(X,D)}.$$

Let $i\colon D \hookrightarrow X$ denote the inclusion. On D, we get an induced logarithmic structure

$$\mathcal{M}_D := i^*\mathcal{M}_X.$$

□

Remark 5.5. Note that $(D, \mathcal{M}_D) \hookrightarrow (X, \mathcal{M}_X)$ is strict by definition.

Remark 5.6. The generic fibre (X_η, D_η) of a formal toric degeneration $(\mathfrak{X}, \mathfrak{D})/\mathfrak{O}^\dagger$ of dP_9 surfaces is smooth by Prop. 1.6 and the discussion on p. 16 in [Ts] or analogously, by Prop. 2.2 and Rem. 3.25 in [GS2] as $\dim X_\eta = 2$ and we assume positivity and simplicity. Thus, it follows that away from the central fibre, we get locally the situation from Chapter 2.

Next, we want to introduce divisorial deformations, which constitute the right category for deformation theory of degenerated Calabi-Yau spaces.

Definition 5.7 ([GS2, Def. 2.7], [Ts, Def. 3.7])**.** Let $(X, D)/O^\dagger$ be a toric log Calabi-Yau pair. Assume that the associated affine data $(B, \mathscr{P})$ are positive and simple. Let A be an Artin local $\mathbb{C}[[t]]$-algebra with $\operatorname{Spec} A$ turned into a logarithmic space via pull-back of the log structure from $\mathfrak{O}^\dagger$. A *divisorial log deformation* of X over $\operatorname{Spec} A^\dagger$ is given by a morphism of logarithmic spaces

$$\pi\colon \mathcal{X}^\dagger \to \operatorname{Spec} A^\dagger$$

fulfilling the following conditions:

1. There is an isomorphism $\mathcal{X}^\dagger \times_{\operatorname{Spec} A^\dagger} O^\dagger \cong X$.

2. The morphism π is flat and $\pi|_{\mathcal{X}\setminus Z}$ is log smooth.

3. Let $\bar{x} \to Z$ be a geometric point, which lies over X_τ. Recall from equation (3.6) on page 65 that there are monoids P_τ such that X can be covered by patches $V(\tau) = \operatorname{Spec} \mathbb{C}[\partial P_\tau]$. Let $Y = \operatorname{Spec} \mathbb{C}[P_\tau]$ instead. This construction induces
$$\mathcal{Y} := Y \times_{O^\dagger} \operatorname{Spec} A^\dagger.$$
For each geometric point $\bar{x} \to Z$ there is an étale neighbourhood $\mathcal{V}^\dagger$ such that there exists a diagram with strict étale maps over $\operatorname{Spec} A^\dagger$
$$\begin{tikzcd} & \mathcal{V}^\dagger \arrow[dl] \arrow[dr] & \\ \mathcal{X}^\dagger & & \mathcal{Y}^\dagger. \end{tikzcd}$$

Definition 5.8. A *deformation of dP_9 pairs* is a pair of divisorial log deformations $(\mathcal{X} \supset \mathcal{D})$ of a simple log dP_9 pair $(X, D)/O^\dagger$ over an Artin local $\mathbb{C}[[t]]$-algebra A. Moreover, we require that $\mathcal{D} \subset \mathcal{X}$ is an effective divisor and that geometric fibres $(X_{\bar{\eta}}, D_{\bar{\eta}})$, $\bar{\eta} \to \eta \neq O \in \operatorname{Spec} A$, which do not lie over the central fibre, are given by a pair of a smooth dP_9 surface with smooth anticanonical divisor (as in Definition 1.26).

Lemma 5.9. *If there exists a polarization $\mathcal{L}$ on a simple log dP_9 pair (X, D), it always can be continued to yield a polarization on general fibres of a divisorial deformation of dP_9 pairs with central fibre (X, D).*

Proof. This follows from [GHKS1, Thm. A.6]. □

Remark 5.10. Recall that projectivity is necessary for our computation of the period integrals via [RS]. The existence of a polarization yields projectivity. So Lemma 5.9 ensures that given a polarization on the central fibre, we can apply the period computations from [RS].

Next, we need some facts and definitions concerning the affine situation.

5.2. The integral affine geometry

We want to show that we can generalize the computations we executed in Chapter 4.

Lemma 5.11. *Let B be a positive, simple integral affine manifold with singularities, which is associated to a toric degeneration $(\mathfrak{X}, \mathfrak{D}) \to \mathfrak{O}^\dagger$ of dP_9 surfaces with a smooth anticanonical divisor.*
The boundary $\partial B \subset B$ has a neighbourhood which is isomorphic to a flat cylinder and the integral affine manifold B is topologically isomorphic to a closed 2-disk in $\mathbb{R}^2$ with 12 focus-focus singularities.

Proof. As on the central fibre, $D \subset X$ has trivial normal bundle, the integral affine manifold $B \setminus \partial B$ can be modified to form a Landau-Ginzburg model as follows: Replace any cell $\sigma \subset B$ with $\partial B \cap \sigma \neq \emptyset$ by an unbounded cell $\tilde{\sigma}$ containing σ such that $\partial\sigma \backslash \partial B \subset \partial\tilde{\sigma}$ and σ and $\tilde{\sigma}$ have the same asymptotic direction. Thus if $\sigma \subset \mathbb{R}^2$ and $\xi \in \Lambda_\sigma = \mathbb{Z}^2$ spans the asymptotic direction of σ then $\tilde{\sigma} = \sigma + \mathbb{R}_{\geq 0} \cdot \xi$. These new, unbounded cells form an integral affine manifold which is homeomorphic with B. This modified integral affine manifold yields a Landau-Ginburg model.
By [CPS, Prop. 2.1], the unbounded edges in $B \setminus \partial B$ have to be parallel.
By [CPS, Prop. 2.2] and because ∂B induces a toric degeneration of elliptic curves, the boundary ∂B of B is straight.
Next, denote by X_η the generic fibre of $\mathfrak{X}$. Note that $\dim \mathrm{H}^1(X_\eta, \Omega^1_{X_\eta}) = 10$

follows from base change (2.1), page 31, and the fact that for any geometric fibre $X_{\bar\eta} \hookrightarrow \mathcal{X}$, $\eta \neq O$, it holds that

$$\dim \mathrm{H}^1(X_{\bar\eta}, \Omega^1_{X_{\bar\eta}}) = 10$$

as we are looking at a degeneration of dP_9 surfaces. By [CPS, Prop. 6.11], B is homeomorphic to $\mathbb{D}^2$ with

$$\dim \mathrm{H}^1(X_\eta, \Omega^1_{X_\eta}) + 2 = 10 + 2 = 12$$

singular points.
Lastly, we want to argue that all 12 singular points in B are given by focus-focus singularities. But this is immediate as focus-focus singularities are the only positive, simple singularities in dimension 2. □

To generalize our results, we will use the following lemma of Moishezon-Livne:

Lemma 5.12 ([LM, Lemma 8])**.** *Let* $x := \left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right)$, $y := \left(\begin{smallmatrix} 1 & -1 \\ 0 & 1 \end{smallmatrix}\right)$ *and let* $A_1, \ldots, A_\mu \in SL(2,\mathbb{Z})$ *be such that if* $\Theta_i := A_i^{-1} x A_i$, $i = 1, \ldots, \mu$, *then* $\Theta_1 \cdot \ldots \cdot \Theta_\mu = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. *It follows that* $\mu = 0$ *(mod 2). Moreover, there exists a finite sequence of elementary transformations starting with some elementary transformation of* $(\Theta_1, \ldots, \Theta_\mu)$ *such that for the resulting* μ*-tuple* $(\underline{\Theta}_1, \ldots, \underline{\Theta}_\mu)$ *in* $SL(2,\mathbb{Z})$ *it holds that*

$$\underline{\Theta}_1 = x,\ \underline{\Theta}_2 = y, \ldots,\ \underline{\Theta}_{\mu-1} = x,\ \underline{\Theta}_\mu = y.$$

Elementary transformations are transformations of μ-tuples of the form

$$(x_1, \ldots, x_\mu) \to (x_1, \ldots, x_{i-1}, x_{i+1}, x_{i+1}^{-1} x_i x_{i+1}, x_{i+2}, \ldots, x_\mu) \text{ and}$$
$$(x_1, \ldots, x_\mu) \to (x_1, \ldots, x_{i-1}, x_i x_{i+1} x_i^{-1}, x_i, x_{i+2}, \ldots, x_\mu).$$

In our case, an elementary transformation corresponds to changing the choice of generators of $\pi_1(B_0)$. Moreover, it holds that $\mu = 12$.

Lemma 5.13. *Let* B *be an integral affine manifold with singularities associated to a toric degeneration of* dP_9 *surfaces. The monodromy representation for* Λ *and also for* $\check\Lambda$ *is equivalent, up to an elementary transformation, to the one we used in Chapter 4.*

Proof. Recall that B is topologically given by a 2-disc with 12 focus-focus singularities at points $P_1, \ldots, P_{12}$. Thus, we can fix a base point $p \in B_0$ and paths $\gamma_1, \ldots, \gamma_{12}$ from p to $P_1, \ldots, P_{12}$ such that the monodromy matrices $\Theta_1, \ldots, \Theta_{12}$, which are associated to small loops along these paths, fulfil the conditions of Lemma 5.12.
This is possible because the boundary ∂B has an open neighbourhood which is isomorphic to a flat cylinder by Lemma 5.11. This fact implies that

$$\Theta_1 \cdot \cdots \cdot \Theta_{12} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Moreover, monodromy around a focus-focus singularity P_i is always computed via orientation preserving base change from $A = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$.
Therefore, the monodromy representation on Λ is for each choice of B equivalent up to elementary transformation. The same holds for its dual sheaf $\check{\Lambda}$. □

Proposition 5.14. *Let $(\mathfrak{X}, \mathfrak{D}) \to T$ be an analytic extension of $(\mathfrak{X}, \mathfrak{D}) \to \mathfrak{O}^\dagger$. There are cycles $(\alpha_0)_{trop}, \ldots, (\alpha_8)_{trop} \subset \mathrm{H}_1(B_0, \Lambda)$ such that for $t \neq O \in T$ there is an isomorphism*

$$\Phi\colon \hat{E}_8 \cong span((\alpha_0)_{trop}, \ldots, (\alpha_8)_{trop}) \to [D_t]^\perp \subset \mathrm{H}_2(X_t, \mathbb{Z}).$$

Proof. Recall that there are tropical cycles in $\mathrm{H}_1(B, \iota_* \Lambda)$ corresponding to cycles in $\mathrm{H}_2(X_t, \mathbb{Z})$. The intersection pairing on these cycles depends only on the linear part of the affine holonomy representation $\pi_1(B_0) \to Aff(\mathbb{Z}^2)$, which equals the monodromy of the sheaf Λ (see [GS1, Def. 1.9]).
By Lemma 5.13, the monodromy of the sheaf Λ is fixed up to an elementary transformation. Note that elementary transformations leave the intersection pairing invariant. Starting from our example (Chapter 4), we get for each choice of B tropical cycles $(\alpha_0)_{trop}, \ldots, (\alpha_8)_{trop}$ such that

$$\hat{E}_8 \cong span((\alpha_0)_{trop}, \ldots, (\alpha_8)_{trop}) \subset \mathrm{H}_1(B_0, \Lambda).$$

The morphism

$$\begin{aligned} \mathrm{H}_1(B_0, \Lambda) &\to \mathrm{H}_2(X_t, \mathbb{Z}) \\ \gamma &\mapsto C(t) \end{aligned}$$

defined in [RS] is compatible with intersection numbers by construction. The image of $(\alpha_0)_{trop}, \ldots, (\alpha_8)_{trop}$ is contained in $[D_t]^{\perp}$, as the intersection of each $(\alpha_i)_{trop}$ with the image of

$$d = 3\alpha_0 + 2\alpha_1 + 4\alpha_2 + 6\alpha_3 + 5\alpha_4 + 4\alpha_5 + 3\alpha_6 + 2\alpha_7 + \alpha_8$$

is trivial and intersection numbers for $\mathrm{H}_1(B, \iota_*\Lambda)$ are preserved. This finishes the proof. $\square$

Lemma 5.15. *Let B be the affine manifold associated to a simple log dP_9 pair as in Definition 5.2. Then it holds that*

$$\mathrm{H}^i(B, \iota_*\check{\Lambda}) = \begin{cases} 0, \ i = 0, \\ \mathbb{Z}^{10}, \ i = 1, \\ 0, \ i = 2. \end{cases}$$

Proof. This lemma follows from the proof of [CPS, Prop. 6.11], Lemma 5.13 and Proposition 4.3. $\square$

There is another affine sheaf, which we will use later on:

Definition 5.16. Let $U \subset B$ be an open subset. We want to define a sheaf $\iota_*\check{\Lambda}_N \subset \iota_*\check{\Lambda}$. If $U \cap \partial B = \emptyset$, we set

$$(\iota_*\check{\Lambda}_N)|_U := \iota_*\check{\Lambda}|_U.$$

If $U \cap \partial B \neq \emptyset$, we define

$$(\iota_*\check{\Lambda}_N)|_U := \{\lambda \in \iota_*\check{\Lambda}|_U \mid \lambda(\Lambda_{\partial B}) \equiv 0\}.$$

Lemma 5.17. *Let B be as in Lemma 5.15. Let $\iota_*\check{\Lambda}_N$ be the sheaf introduced in Definition 5.16. It holds that*

$$\mathrm{H}^i(B, \iota_*\check{\Lambda}_N) = \begin{cases} 0, \ i = 0, \\ \mathbb{Z}^{10}, \ i = 1, \\ 0, \ i = 2. \end{cases}$$

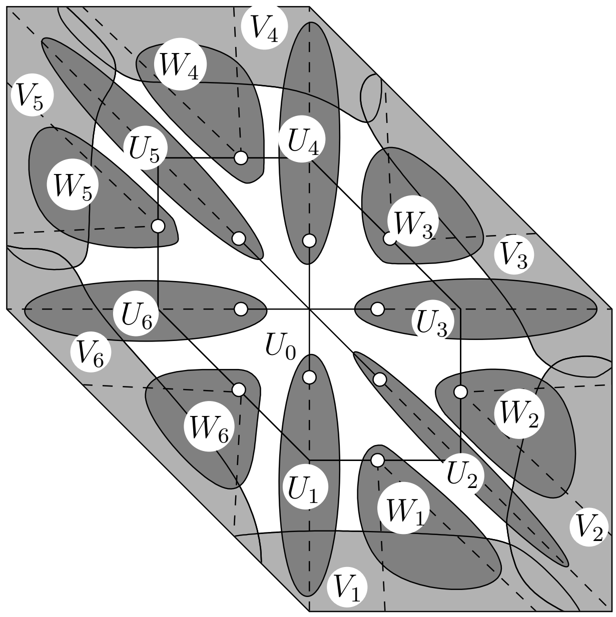

Figure 5.1.: Another open cover of B

Proof. If we can compute the cohomology groups in our example, the computation generalizes by Lemma 5.13, as $\iota_*\check{\Lambda}_N \subset \iota_*\check{\Lambda}$ only depends on monodromy and the fact that B has straight boundary ∂B.
On B, we choose a new open cover

$$\mathcal{U} = \{U_0, U_1, \ldots, U_6, W_1, \ldots, W_6, V_1, \ldots, V_6\}.$$

The open sets U_i, V_i and W_i are indicated in Figure 5.1. The open set U_0 is assumed to be a small open neighbourhood of the central star-shaped region, which is left white in Figure 5.1. Its boundary is not marked separately.
Note that $U_0 \cap V_i$ has three components, $(U_0V_i)_1, (U_0V_i)_2, (U_0V_i)_3$, and the triple intersections $U_0 \cap U_i \cap V_i$ and $U_0 \cap W_i \cap V_i$ have two components, $(U_0U_iV_i)_1, (U_0U_iV_i)_2$ and $(U_0W_iV_i)_1, (U_0W_iV_i)_2$.
We use the same coordinates as indicated in Figure 4.3 on page 76. The corres-

ponding matrices for

$$\partial^0\colon C^0(\mathcal{U}, \iota_*\check{\Lambda}_N) \to C^1(\mathcal{U}, \iota_*\check{\Lambda}_N) \text{ and}$$
$$\partial^1\colon C^1(\mathcal{U}, \iota_*\check{\Lambda}_N) \to C^2(\mathcal{U}, \iota_*\check{\Lambda}_N)$$

are given in the appendix, Section A.2.
A computation, which is completely analogous to the proof of Proposition 4.3, yields the cohomology groups for $\iota_*\check{\Lambda}_N$. □

5.3. Toric degenerations and periods

We have to fix some notation for the log geometric picture:

Definition 5.18. Recall that by Lemma 5.4, there are logarithmic structures $\mathcal{M}_X$, $\mathcal{M}_{(X,D)}$ and $\mathcal{M}_D$ on X and D, which turn them into logarithmic spaces over $O^\dagger$. To simplify notation for sheaves of logarithmic differentials and logarithmic tangent sheaves, we set

$$\begin{aligned} \Omega^1_X &:= \Omega^1_{(X,\mathcal{M}_X)}/O^\dagger, & \Theta_X &:= \Theta_{(X,\mathcal{M}_X)}/O^\dagger, \\ \Omega^1_D &:= \Omega^1_{(X,\mathcal{M}_D)}/O^\dagger, & \Theta_D &:= \Theta_{(D,\mathcal{M}_D)}/O^\dagger, \\ \Omega^1_X(\log D) &:= \Omega^1_{(X,\mathcal{M}_{(X,D)})}/O^\dagger, & \Theta_X(\log D) &:= \Theta_{(X,\mathcal{M}_{(X,D)})}/O^\dagger. \end{aligned}$$

Next, we want to introduce a short exact sequence of sheaves, which is related to the vertical long exact sequence occurring on the left side of the diagram [CFGK, (11)].

Proposition 5.19. *Let $(X, D)/O^\dagger$ be a simple log dP_9 pair (Definition 5.2). There is a short exact sequence of sheaves*

$$0 \to \Theta_X(-D) \xrightarrow{\phi} \Theta_X(\log D) \to \Theta_D \to 0.$$

Proof. First, we want to define a map

$$\phi\colon \Theta_X(-D) \to \Theta_X(\log D)$$

and show that it is injective.
For open sets $U \subset X$ with $U \cap D = \emptyset$ we can take $\phi|_U := id_U$ as $\Theta_X(-D)(U) =$

$\Theta_X(\log D)(U)$.
Let $I := \mathcal{O}_X(-D)$ denote the ideal sheaf of the divisor $D \subset X$. Let $U \subset X$ with $U \cap D \neq \emptyset$ be an open set. Note that Θ_X is the sheaf of log derivations with values in $\mathcal{O}_X$ on X. Multiplying with I yields $\Theta_X(-D)$. Let $(\theta, \xi) \in \Theta_X(-D)(U)$ be a log derivation on U and denote by $\alpha\colon \mathcal{M}_X \to \mathcal{O}_X$ the structure map of the log structure on X.
In particular, θ preserves I. Therefore, we set

$$\phi(\theta,\xi)\colon f \mapsto \left(\theta\left(\alpha\left(f\right)\right), \alpha\left(f\right)^{-1} \cdot \theta\left(\alpha\left(f\right)\right)\right); \ \forall f \in \mathcal{M}_X$$

following [GS2, Ex. 1.4]. By [GS2, Prop. 1.3], we can restrict ourselves to analysing only the derivation θ. Moreover, the map ϕ is injective.
Next, we have to show that the quotient $\Theta_X(\log D)/\Theta_X(-D)$ is isomorphic to Θ_D. Denote the inclusion by $i\colon D \to X$. Note that as $D \subset X$ is a closed subscheme, there is a short exact sequence

$$0 \to I \to \mathcal{O}_X \xrightarrow{pr} i_*\mathcal{O}_D \to 0. \tag{5.1}$$

We claim that composition of derivations in $\Theta_X(\log D)$ with pr factors through $\Theta_X(\log D)/\Theta_X(-D)$ and that $\Theta_X(-D)$ is its kernel. Let $U \subset X$ be an open subset and let $\theta \in \Theta_X(\log D)(U)$ be a derivation.
It holds that

$$\theta(f) \in I(U), \ \forall f \in \mathcal{O}_X(U) \Longleftrightarrow \theta \in \phi(\Theta_X(-D)(U)).$$

By the short exact sequence (5.1), sections $f \in \mathcal{O}_X(U)$ are contained in the kernel of pr if and only if $f \in I(U)$. This fact implies that

$$pr \circ \theta = 0 \Longleftrightarrow \theta \in \phi\left(\Theta_X\left(-D\right)\left(U\right)\right).$$

So a derivation in $\Theta_X(\log D)$ is mapped to zero by composition with pr if and only if it is contained in the image of $\Theta_X(-D)$. Thus, we get that composition with pr factors through $\Theta_X(\log D)/\Theta_X(-D)$. Moreover, a derivation θ in $\Theta_X(\log D)$ descends to a derivation on $i_*\mathcal{O}_D$ by factorization. Thus, it holds that composition of a derivation in $\Theta_X(\log D)$ with pr yields a derivation which is contained in Θ_D. More explicitely, recall that all derivations $\theta \in \Theta_X(\log D)(U)$ preserve $I(U)$. Therefore, it holds that

$$pr \circ \theta \circ pr = pr \circ \theta$$

and thus, $pr \circ \theta$ acts as a derivation on $i_*\mathcal{O}_D$, which is isomorphic to $\mathcal{O}_D$.
We still have to show surjectivity of $\Theta_X(\log D) \to \Theta_D$.
Surjectivity follows because there is an injection $\Theta_D \hookrightarrow i^*\Theta_X(\log D)$ as $\mathcal{M}_{(X,D)}$ maps surjectively onto $\mathcal{M}_D$. The same argument is used in the proof of [O, Prop. III.2.3.2]. Thus, every section in Θ_D is is induced by a section of $\Theta_X(\log D)$ locally. Therefore, the map $\Theta_X(\log D) \to \Theta_D$ is surjective. □

Remark 5.20. By [GS2, Thm. 2.11], it holds that $\mathrm{H}^1(X, \Theta_X(\log D))$ is the tangent space of the divisorial log deformation functor of $\left((X, D), \mathcal{M}_{(X,D)}\right)$. Similarly, $\mathrm{H}^1(X, \Theta_X)$ and $\mathrm{H}^1(D, \Theta_D)$ yield the tangent spaces for divisorial log deformations of $(X, \mathcal{M}_X)$ and $(D, \mathcal{M}_D)$ respectively.
By [GS2, Thm. 2.11], the cohomology group $\mathrm{H}^2(X, \Theta_X(\log D))$ is the obstruction space for $\left(X, \mathcal{M}_{(X,D)}\right)$. Similarly, we get the obstruction space $\mathrm{H}^2(D, \Theta_D)$ for deformations of $(D, \mathcal{M}_D)$. All of these obstruction spaces are trivial, so all first order deformations do lift.

We want to analyse the relationship between the tangent space of deformations of $\left(X, \mathcal{M}_{(X,D)}\right)$ and the tangent space of deformations of $(D, \mathcal{M}_D)$. To this end, we use the short exact sequence from Proposition 5.19.

Proposition 5.21. *There is an induced exact sequence of cohomology groups*

$$0 \to \mathrm{H}^0(D, \Theta_D) \to \mathrm{H}^1(X, \Theta_X(-D)) \to \mathrm{H}^1(X, \Theta_X(\log D)) \to \mathrm{H}^1(D, \Theta_D) \to 0.$$

Proof. The short exact sequence from Proposition 5.19 induces a long exact sequence in cohomology. Its terms can be computed using integral affine geometry. By [Ts, Prop. 3.15] (analogous to [GS2, Thm. 3.23]) and Lemma 5.15, it holds that

$$\mathrm{H}^i(X, \Theta_X(\log D)) \cong \mathrm{H}^i(B, \iota_*\check{\Lambda} \otimes \mathbb{C}) = \begin{cases} 0, \ i = 0, \\ \mathbb{C}^{10}, \ i = 1, \\ 0, \ i = 2. \end{cases}$$

Note that in the integral affine picture, there is a short exact sequence of sheaves

$$0 \to \iota_*\check{\Lambda}_N \to \iota_*\check{\Lambda} \to \check{\Lambda}_{\partial B} \to 0.$$

As above it follows by [Ts, Prop. 3.15] that

$$\mathrm{H}^i(D, \Theta_D) \cong \mathrm{H}^i(\partial B, \check{\Lambda}_{\partial B}) \cong \mathrm{H}^i(\mathbb{S}^1, \underline{\mathbb{Z}}).$$

We want to show that

$$\mathrm{H}^i(B,\iota_*\check{\Lambda}_N) \cong \mathrm{H}^i(X,\Theta_X(-D)).$$

Note that there is a commutative diagram with exact rows such that all vertical arrows are isomorphisms:

$$\begin{tikzcd}
\mathrm{H}^{i-1}(B,\iota_*\check{\Lambda}\otimes\mathbb{C}) \arrow[r] \arrow[d] & \mathrm{H}^{i-1}(\partial B,\check{\Lambda}_{\partial B}\otimes\mathbb{C}) \arrow[r] \arrow[d] & \mathrm{H}^i(B,\iota_*\check{\Lambda}_N\otimes\mathbb{C}) \arrow[r] & \mathrm{H}^i(B,\iota_*\check{\Lambda}\otimes\mathbb{C}) \arrow[r] \arrow[d] & \mathrm{H}^i(\partial B,\check{\Lambda}_{\partial B}\otimes\mathbb{C}) \arrow[d] \\
\mathrm{H}^{i-1}(X,\Theta_X(\log D)) \arrow[r] & \mathrm{H}^{i-1}(D,\Theta_D) \arrow[r] & \mathrm{H}^i(X,\Theta_X(-D)) \arrow[r] & \mathrm{H}^i(X,\Theta_X(\log D)) \arrow[r] & \mathrm{H}^i(D,\Theta_D)
\end{tikzcd}$$

We want to argue that there exists an isomorphism

$$f^i\colon\ \mathrm{H}^i(B,\iota_*\check{\Lambda}_N\otimes\mathbb{C}) \to \mathrm{H}^i(X,\Theta_X(-D))$$

fitting into the commutative diagram above. The existence of this isomorphism follows by the argument which was used in the proof of [Ts, Thm. 3.18], applying exactness and the existence of vertical isomorphisms. Hence the long exact sequence in cohomology reduces accordingly by inserting the terms we computed for the affine case. In particular, Lemma 5.17 yields that

$$\mathrm{H}^2(X,\Theta_X(-D)) = \mathrm{H}^2(B,\iota_*\check{\Lambda}_N) = 0.$$

□

Remark 5.22. The forgetful morphism from deformations of (X,D) to deformations of D induces on tangent spaces of the respective deformation functors the morphism

$$\mathrm{H}^1(X,\Theta_X(\log D)) \to \mathrm{H}^1(D,\Theta_D)$$

occurring in Proposition 5.21. This holds by construction of $\Theta_X(\log D)\to\Theta_D$ in the proof of Proposition 5.19:
For an open subset $U\subset X$ a derivation $\theta\in\Theta_X(\log D)(U)$ was mapped to a derivation $pr\circ\theta\in\Theta_D(U\cap D)$ with pr denoting the projection map $\mathcal{O}_X\to\mathcal{O}_D$. This map corresponds to the forgetful morphism as derivations generate infinitesimal automorphisms and log automorphisms preserve D.
Therefore, the tangent space of divisorial log deformations of the pair (X,D) relative to deformations of D is given by

$$\ker\left(\mathrm{H}^1\left(X,\Theta_X\left(\log D\right)\right) \to \mathrm{H}^1\left(D,\Theta_D\right)\right) \cong \mathrm{H}^1\left(X,\Theta_X\left(-D\right)\right)/\,\mathrm{H}^0(D,\Theta_D).$$

Recall that we only want to consider deformations of pairs (X, D) leaving the normal bundle $\mathcal{N}_{D/X}$ trivial. Thus, we have to analyse the relation between $\mathcal{N}_{D/X}$ and e.g. Θ_D and Θ_X in more detail.

Remark 5.23. As $(D, \mathcal{M}_D) \subset (X, \mathcal{M}_X)$ is strict, the logarithmic normal bundle and the classical normal bundle coincide by [O, § III.3.4].

Lemma 5.24. *The normal sheaf $\mathcal{N}_{D/X}$ is invertible and there is a short exact sequence of sheaves*

$$0 \to \Theta_D \to \Theta_X|_D \to \mathcal{N}_{D/X} \to 0.$$

Proof. Denote the ideal sheaf of D by $I := \mathcal{O}_X(-D)$ and the inclusion by

$$i \colon (D, \mathcal{M}_D) \to (X, \mathcal{M}_X).$$

By [O, Prop. III.2.3.2], there is an exact sequence

$$I/I^2 \to i^*\Omega^1_X \to \Omega^1_D \to 0.$$

Note that I/I^2 is invertible because $D \subset X$ is a local complete intersection and X is reduced (e.g. [Liu, Cor. 6.3.8]). We want to show that the first map is injective. This can be checked on a generator $f \in I$ of I/I^2. It holds that $dlog(f) \neq 0$, either again by [O, Prop. III.2.3.2] or because this holds for the underlying non-logarithmic spaces and this translates to logarithmic differentials by [K1, Def. 5.1]. Thus, we get a short exact sequence.
Dualizing the short exact sequence

$$0 \to I/I^2 \to i^*\Omega^1_X \to \Omega^1_D \to 0$$

with respect to $\mathcal{O}_D$ yields

$$0 \to \Theta_D \to \Theta_X|_D \to \mathcal{N}_{D/X} \to \mathcal{E}xt^1(\Omega^1_D, \mathcal{O}_D) \to \mathcal{E}xt^1(\Omega^1_X, \mathcal{O}_D) \to \ldots.$$

As $\mathcal{E}xt^1(\Omega^1_D, \mathcal{O}_D) = \mathcal{E}xt^1(\mathcal{O}_D, \Theta_D) = 0$, e.g. by [H, Prop. III.6.3], this finishes the proof. □

Remark 5.25. As is well-known, see e.g. [S, Thm. 3.3.1], the tangent space of deformations of $\mathcal{N}_{D/X}$, as well as of any line bundle on D, is given by

$$\mathrm{H}^1(D, \mathcal{O}_D).$$

Next, we want to construct a map from the tangent space of deformations of $(X, \mathcal{M}_{(X,D)})$ relative to deformations of $(D, \mathcal{M}_D)$ to the tangent space of deformations of $\mathcal{N}_{D/X}$. Ultimately, our aim is to single out elements which generate deformations of X leaving the normal bundle $\mathcal{N}_{D/X}$ trivial. The kernel of the morphism we construct in the following proposition will be our candidate for this.

Construction 5.26. By restriction of $\Theta_X(-D)$ to D we want to construct a morphism

$$\left(\mathrm{H}^1\left(X, \Theta_X\left(-D\right)\right) / \, \mathrm{H}^0\left(D, \Theta_D\right)\right) \to \mathrm{H}^1(D, \mathcal{O}_D).$$

We construct the desired morphism by first restricting $\Theta_X(-D)$ to D and then mapping

$$\Theta_X(-D)|_D \to \mathcal{O}_D.$$

The latter map is constructed as follows:
By Lemma 5.24, there is a short exact sequence

$$0 \to \Theta_D \to \Theta_X|_D \to \mathcal{N}_{D/X} \to 0. \tag{5.2}$$

As $\mathcal{N}_{D/X}$ is invertible, there is a trace map

$$\mathcal{N}_{D/X} \otimes \mathcal{N}^*_{D/X} \cong \mathcal{O}_D.$$

By tensoring with $\mathcal{N}^*_{D/X} = I/I^2$, the short exact sequence (5.2) yields

$$0 \to \Theta_D \otimes \mathcal{N}^*_{D/X} \to \Theta_X|_D \otimes I/I^2 \to \mathcal{O}_D \to 0. \tag{5.3}$$

We want to show that

$$\Theta_X|_D \otimes I/I^2 \cong \Theta_X\left(-D\right)|_D.$$

Note that

$$\begin{aligned} \Theta_X(-D)|_D &= \left(\Theta_X \otimes \mathcal{O}_X\left(-D\right)\right) \otimes \mathcal{O}_D \text{ and} \\ \Theta_X|_D \otimes I/I^2 &= \left(\Theta_X \otimes \mathcal{O}_D\right) \otimes \left(\mathcal{O}_X\left(-D\right) \otimes \mathcal{O}_D\right). \end{aligned}$$

By associativity of the tensor product and the existence of a canonical isomorphism $A \otimes B \cong B \otimes A$ we get the desired isomorphism.

Using this isomorphism, the long exact sequence in cohomology induced by (5.3) yields a map

$$\mathrm{H}^1\left(D, \Theta_X\left(-D\right)|_D\right) \to \mathrm{H}^1(D, \mathcal{O}_D).$$

To finish the proof, we have to show that the map

$$\mathrm{H}^1\left(X, \Theta_X\left(-D\right)\right) \to \mathrm{H}^1\left(D, \Theta_X\left(-D\right)|_D\right) \to \mathrm{H}^1(D, \mathcal{O}_D) \tag{5.4}$$

factorizes through the quotient by $\mathrm{H}^0(D, \Theta_D)$.

We want to consider the situation locally. Let $U \subset X$ be an affine open subset with $U \cap D \neq \emptyset$. Let $w \in I(U)$ be a generator of the ideal sheaf $I := \mathcal{O}_X(-D)$. Then $\mathrm{d}w$ generates $\mathcal{N}^*_{D/X} = I/I^2$ and let ∂_w be the generator of $\mathcal{N}_{D/X}$ dual to $\mathrm{d}w$. By [Fr, § 3], there is a local uniformizer z of $\mathcal{O}_{U_D}$, $U_D := U \cap D$, such that Θ_X is locally generated by ∂_z and ∂_w over $\mathcal{O}_U$.

Note that near nodes of D, we have to consider a pair of generators $(\partial_z, \frac{\partial_z}{z})$ instead of ∂_z alone. As this fact does not change anything, we will not spell out both cases, i.e. near a node and within the regular part of D, but stick to the latter situation. Summarizing, the sheaves we need can be written in terms of local generators, i.e.

$$\begin{aligned} (\Theta_X)|_U &= \mathcal{O}_U \cdot \partial_z \oplus \mathcal{O}_U \cdot \partial_w, \\ \Theta_X(\log D)|_U &\cong \mathcal{O}_U \cdot (w\partial_w) \oplus \mathcal{O}_U \cdot \partial_z \text{ and} \\ (\Theta_D)|_{U_D} &\cong \mathcal{O}_{U_D} \cdot \partial_z. \end{aligned} \tag{5.5}$$

We want to proceed by analysing the short exact sequence we constructed above. Recall that $\Theta_X(-D)|_D = \Theta_X \otimes \mathcal{N}^*_{D/X}$. Thus, locally on a suitable open set $U_D \subset D$ the short exact sequence (5.3), page 116, looks like

$$0 \to \mathcal{O}_{U_D} \cdot (\partial_z \otimes \mathrm{d}w) \to \mathcal{O}_{U_D} \cdot (\partial_w \otimes \mathrm{d}w) \oplus \mathcal{O}_{U_D} \cdot (\partial_z \otimes \mathrm{d}w) \to \mathcal{O}_{U_D} \cdot (\partial w \otimes \mathrm{d}w) \to 0.$$

This means that

$$\ker\left(\left(\Theta_X\left(-D\right)|_D\right)|_{U_D} \to \left(\mathcal{N}_{D/X} \otimes \mathcal{N}^*_{D/X}\right)|_{U_D}\right) \cong \mathcal{O}_{U_D} \cdot (\partial_z \otimes \mathrm{d}w). \tag{5.6}$$

Hence the kernel of $\Theta_X(-D)|_D \to \mathcal{O}_D$ is locally generated by $\partial_z \otimes \mathrm{d}w$.

We want to trace the contribution of $\mathrm{H}^0(D, \Theta_D) \to \mathrm{H}^1(D, \Theta_X(-D)|_D)$. To do this, we follow the snake lemma using diagram chase on the diagram

$$\begin{tikzcd} & C^0(X, \Theta_X(-D)) \arrow[r] \arrow[d] & C^0(X, \Theta_X(\log D)) \arrow[r] \arrow[d] & C^0(D, \Theta_D) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & C^1(X, \Theta_X(-D)) \arrow[r] & C^1(X, \Theta_X(\log D)) \arrow[r] & C^1(D, \Theta_D). & \end{tikzcd}$$

We start with a representative of a class in $\mathrm{H}^0(D, \Theta_D)$ and trace it through the diagram. Note that the image of $C^0(X, \Theta_X(-D))$ vanishes in $\mathrm{H}^1(X, \Theta_X(-D))$. Therefore, the image of $\mathrm{H}^0(D, \Theta_D)$ is contained in the span of $\partial_z \otimes \mathrm{d}w$ and thus, in the kernel of $(\Theta_X(-D)|_D)|_{U_D} \to (\mathcal{N}_{D/X} \otimes \mathcal{N}^*_{D/X})|_{U_D}$ by equation (5.6).
It follows that the morphism (5.4) factorizes, i.e. the dashed arrows exist in the commutative diagram

$$\begin{tikzcd}
\mathrm{H}^1(X, \Theta_X(-D)) \arrow[r] \arrow[d] & \mathrm{H}^1(D, \Theta_X(-D)|_D) \arrow[r] \arrow[d, dashed] & \mathrm{H}^1(D, \mathcal{O}_D). \\
\mathrm{H}^1(X, \Theta_X(-D)) / \mathrm{H}^0(D, \Theta_D) \arrow[r] & \mathrm{H}^1(D, \Theta_X(-D)|_D) / \mathrm{H}^0(D, \Theta_D) \arrow[ur, dashed] &
\end{tikzcd}$$

This finishes the construction.

Remark 5.27. Note that the local generators given by the equations in (5.5) on page 117 yield in particular, that

$$\ker\big(\Theta_X(\log D)|_{U_D} \to (\Theta_D)|_{U_D}\big) \cong \mathcal{O}_{U_D} \cdot (w\partial_w) \oplus \mathcal{O}_{U_D} \cdot (w\partial_z). \tag{5.7}$$

We will need this description later.

We are working relative to deformations of D to simplify the situation. Nevertheless, we have to understand toric degenerations of elliptic curves.

Remark 5.28. A classical Tate curve is given by a singular cubic curve $E_0 \to \operatorname{Spec}\mathbb{C}$, which is isomorphic to a singular fibre of type I_n, $n \geq 1$, in Kodaira's classification of singular fibres. It can be embedded into an analytic family of elliptic curves $\mathcal{E} \to \mathbb{D}$ with central fibre E_0, which we call the Tate family.
From the point of view of toric degenerations, a Tate curve of type I_n arises as the central fibre of a toric degeneration with associated affine manifold topologically isomorphic to $\mathbb{S}^1$ divided into n intervals. Note that viewing the Tate curve as a central fibre of a toric degeneration endows it with a logarithmic structure, i.e. $E_0^\dagger \to O^\dagger$ is the central fibre of a toric degeneration

$$(\mathfrak{E}, \mathcal{M}_{\mathfrak{E}}) \to \mathfrak{O}^\dagger.$$

A toric degeneration of a Tate curve can be lifted to a family, as the obstruction space vanishes, i.e.

$$\mathrm{H}^2(E_0, \Theta_{E_0}) = \mathrm{H}^2(\mathbb{S}^1, \underline{\mathbb{Z}}) = 0.$$

Its base is at first 1-dimensional over $\mathfrak{O}^\dagger$ as

$$\mathrm{H}^1(E_0, \Theta_{E_0}) = \mathrm{H}^1(\mathbb{S}^1, \underline{\mathbb{Z}}) = \mathbb{Z}.$$

Considering the torus action on the toric degeneration (see Proposition 3.37), the dimension is reduced by one so we get a diagram

$$\mathfrak{E} \to \mathfrak{O}^\dagger_{\mathfrak{E}} \to \mathfrak{O}^\dagger.$$

This toric degeneration $\mathfrak{E} \to \mathfrak{O}^\dagger$ embeds into a Tate family $\mathcal{E} \to \mathbb{D}$. Thus, the Tate family inherits a logarithmic structure, i.e. we can write $(\mathcal{E}, \mathcal{M}_{\mathcal{E}}) \to \mathbb{D}^\dagger$.
In the end, this reasoning yields a diagram of logarithmic spaces with injective horizontal arrows

$$\begin{tikzcd} E_0 \arrow[r] \arrow[d] & \mathfrak{E} \arrow[r] \arrow[d] & \mathcal{E} \arrow[d] \\ O^\dagger \arrow[r] & \mathfrak{O}^\dagger \arrow[r] & \mathbb{D}^\dagger. \end{tikzcd}$$

Remark 5.29. From now on, we do not want to consider analytic families any more. Note that we can take the completion of the analytic Tate family $\mathcal{E} \to \mathbb{D}$ at the central fibre which yields a formal family

$$(\hat{\mathcal{E}}, E_0) \to \operatorname{Spf} \mathbb{C}[[t]].$$

Moreover, we make the following *assumption*:
Let $(X, \mathcal{M}_{(X,D)})/O^\dagger$ be as in Definition 5.2. Assume that there exists an analytic log smooth versal divisorial deformation of dP_9 pairs

$$\left([(\mathcal{Y}, \mathcal{C}), \mathcal{M}] \to \mathcal{S}\right)/\mathbb{D}$$

of $(X, \mathcal{M}_{(X,D)})/O^\dagger$. As before, we can take the completion of the analytic family $((\mathcal{Y}, \mathcal{C}) \to \mathcal{S})/\mathbb{D}$ at the central fibre to get a formal family

$$\left(\left((\hat{\mathcal{Y}}, Y_0), (\hat{\mathcal{C}}, C_0)\right) \to \left(\hat{\mathcal{S}}, s_0\right)\right) / \operatorname{Spf} \mathbb{C}[[t]].$$

Lemma 5.30. *A Tate family is log versal but the log deformation functor of a Tate curve has no hull.*

Proof. By [GS2, Rem. 2.19], the log deformation functor of a Tate curve satisfies $\mathrm{H}^1_{\log}$, $\mathrm{H}^2_{\log}$ and $\mathrm{H}^3_{\log}$ in [K2]. Thus, it has a pseudo-hull by [K2, Thm. 3.16]. By

[GS2, Rem. 2.19], a Tate family is a pseudo-hull. By [K2, Lemma 3.8], this yields log versality.
Note that a Tate curve is not rigid in the first order as

$$\mathrm{H}^0(E_0, \Theta_{E_0}) \cong \mathrm{H}^0(\mathbb{S}^1, \underline{\mathbb{Z}} \otimes \mathbb{C}) \neq 0.$$

So the log deformation functor of a Tate curve has no hull. $\square$

Lemma 5.31. *Let $(X, \mathcal{M}_{(X,D)})/O^\dagger$ be as in Definition 5.2. Let*

$$\Big(\big((\hat{\mathcal{Y}}, Y_0), (\hat{\mathcal{C}}, C_0)\big) \to \big(\hat{\mathcal{S}}, s_0\big)\Big) \,/\, \mathrm{Spf}\,\mathbb{C}[[t]]$$

denote the formal completion of an analytic log smooth versal divisorial deformation of dP_9 pairs as in Remark 5.29. Let moreover $(\hat{\mathcal{E}}, E_0) \to \mathrm{Spf}\,\mathbb{C}[[t]]$ denote a formal Tate family with $E_0 = D$ as in Remark 5.29. Then the map $(\hat{\mathcal{C}}, C_0) \to (\hat{\mathcal{S}}, s_0)/\,\mathrm{Spf}\,\mathbb{C}[[t]]$ induces a surjective morphism onto the base of the formal Tate curve

$$(\hat{\mathcal{S}}, s_0) \to \mathrm{Spf}\,\mathbb{C}[[t]].$$

Proof. The formal Tate family $(\hat{\mathcal{E}}, E_0) \to \mathrm{Spf}\,\mathbb{C}[[t]]$ is log versal for $E_0 = D$. Therefore, there is a commutative diagram

$$\begin{tikzcd}
(\hat{\mathcal{C}}, C_0) \arrow[r] \arrow[d] & (\hat{\mathcal{E}}, E_0) \arrow[d] \\
(\hat{\mathcal{S}}, s_0) \arrow[r, "\phi"] & \mathrm{Spf}\,\mathbb{C}[[t]].
\end{tikzcd}$$

So a morphism $\phi\colon (\hat{\mathcal{S}}, s_0) \to \mathrm{Spf}\,\mathbb{C}[[t]]$ exists. We have to show that it is surjective. This is true locally by the exact sequence from Lemma 5.19, which implies that the corresponding morphism of tangent spaces of deformations

$$\mathrm{H}^1(X, \Theta_X(\log D)) \to \mathrm{H}^1(D, \Theta_D)$$

is surjective. $\square$

Finally, we are ready to determine the tangent space of deformations of (X, D) relative to deformations of D leaving the normal bundle of $D \subset X$ invariant.

Theorem 5.32. *Let $(X, \mathcal{M}_{(X,D)})/O^\dagger$ be as before. Again, let*

$$\left(\left((\hat{\mathfrak{Y}}, Y_0), (\hat{\mathcal{C}}, C_0)\right) \to \left(\hat{\mathcal{S}}, s_0\right)\right) / \operatorname{Spf} \mathbb{C}[[t]]$$

denote the formal completion of an analytic log smooth versal divisorial deformation of dP_9 pairs. The relative tangent space of $(\hat{\mathcal{S}}, s_0)/\operatorname{Spf}\mathbb{C}[[t]]$ at the central fibre is isomorphic to

$$\ker\left(\left(\mathrm{H}^1\left(X, \Theta_X(-D)\right) / \mathrm{H}^0\left(D, \Theta_D\right)\right) \to \mathrm{H}^1\left(D, \mathcal{O}_D\right)\right).$$

Proof. Recall that by Proposition 5.21 and Remark 5.22, the tangent space of deformations of (X, D) relative to deformations of D is given by

$$\ker\left(\mathrm{H}^1(X, \Theta_X(\log D)) \to \mathrm{H}^1(D, \Theta_D)\right) \cong \left(\mathrm{H}^1(X, \Theta_X(-D))/\,\mathrm{H}^0(D, \Theta_D)\right).$$

We need to consider deformations which leave $\mathcal{N}_{D/X}$ trivial. By Remark 5.25, the tangent space of deformations of $\mathcal{N}_{D/X}$ is given by $\mathrm{H}^1(D, \mathcal{O}_D)$.
The kernel of the map

$$\left(\mathrm{H}^1\left(X, \Theta_X(-D)\right) / \mathrm{H}^0\left(D, \Theta_D\right)\right) \to \mathrm{H}^1(D, \mathcal{O}_D)$$

constructed in Construction 5.26 is our candidate for generating the right deformations.
Recall from equation (5.6) on page 117 that the kernel of the underlying map of sheaves is locally, on an open set $U_D \subset D$, given by

$$\ker\left(\left(\Theta_X(-D)|_D\right)|_{U_D} \to \left(\mathcal{N}_{D/X} \otimes \mathcal{N}^*_{D/X}\right)|_{U_D}\right) \cong \mathcal{O}_{U_D} \cdot (w\partial_z). \tag{5.8}$$

We want to use a Čech point of view to show that as local representatives on a Čech cover, the vector fields of the form $\mathcal{O}_{U_D} \cdot (\partial_z \otimes \mathrm{d}w)$ are precisely the vector fields which generate first order deformations of (X, D) such that $\mathcal{N}_{D/X}$ (and by construction also D) are left invariant. To this end, we use the standard setup for locally trivial first order deformations.
Note that we can restrict ourselves to studying the non-logarithmic normal bundle $\mathcal{N}_{D/X}$ as we are not interested in triviality of the logarithmic normal bundle. They do not coincide here, as $(D, \mathcal{M}_D) \to (X, \mathcal{M}_{(X,D)})$ is not strict.
We start by recalling the general picture, given e.g. in [S, § 1.2.4]. Later, we will restrict ourselves first to deformations, which leave D invariant and secondly, to

deformations which leave $\mathcal{N}_{D/X}$ trivial.
In general, a first order deformation of an algebraic variety is given by a diagram

$$\begin{tikzcd}
(X, D) \arrow[r] \arrow[d] & (\mathcal{X}, \mathcal{D}) \arrow[d] \\
\operatorname{Spec} \mathbb{C} \arrow[r] & \operatorname{Spec} \mathbb{C}[\epsilon]/\epsilon^2.
\end{tikzcd}$$

Choose an affine open cover $\mathcal{U}$ of X such that for each $U_i \in \mathcal{U}$ there is a local trivialization, i.e. an isomorphism

$$\psi_i \colon \left(U_i \times \operatorname{Spec} \mathbb{C}[\epsilon]/\epsilon^2\right) \to \mathcal{X}|_{U_i}.$$

On intersections $U_i \cap U_j =: U_{ij}$ with $U_i,\ U_j \in \mathcal{U}$ this yields automorphisms

$$\psi_{ij} = \psi_i^{-1} \circ \psi_j \colon \left(U_{ij} \times \operatorname{Spec} \mathbb{C}[\epsilon]/\epsilon^2\right) \to \left(U_{ij} \times \operatorname{Spec} \mathbb{C}[\epsilon]/\epsilon^2\right).$$

The family of such automorphisms $\{\psi_{ij}\}_{ij}$ fixes a Čech cocycle of Θ_X and a class in $\mathrm{H}^1(X, \Theta_X)$.
In turn, the deformation $\mathcal{X}$ can be obtained by gluing the patches $U_i \times \operatorname{Spec} \mathbb{C}[\epsilon]/\epsilon^2$ in a way that leaves X invariant using the ψ_{ij}.
Next, we want to look at the situation on the level of rings. Let $U_{ij} := U_i \cap U_j$ be an open affine subset in X with $U_{ij} \cap D \neq \emptyset$.
Without loss of generality, we can write $U_{ij} = \operatorname{Spec} R$ for a ring R. An automorphism $\psi_{ij} \colon (U_{ij} \times \operatorname{Spec} \mathbb{C}[\epsilon]/\epsilon^2) \to (U_{ij} \times \operatorname{Spec} \mathbb{C}[\epsilon]/\epsilon^2)$ corresponds to an automorphism of rings

$$\psi_{ij}^* \colon R[\epsilon]/\epsilon^2 \to R[\epsilon]/\epsilon^2.$$

Note that via triviality on X it holds that

$$\psi_{ij}^*|_R = id_R + \epsilon \cdot \theta_{ij}$$

for a derivation $\theta_{ij} \colon R \to R$, which can be written as an element in $\mathcal{O}_{U_{ij}} \cdot \partial_z \oplus \mathcal{O}_{U_{ij}} \cdot \partial_w$ by the first equation in (5.5) on page 117.
So far, we have not restricted our construction in any way. As a first step towards our situation, we want to focus on deformations which leave D invariant.
Let $I := \mathcal{O}_X(-D)$ be the ideal sheaf of D and let w be a generator of I on U_{ij}.
Recall that we are only considering the situation relative to deformations of D,

thus we want to neglect deformations of D. On the level of rings, this means that our morphisms ψ_{ij}^* on $R[\epsilon]/\epsilon^2$ have to descend to morphisms

$$\psi_{ij}^*|_D \colon R/I[\epsilon]/\epsilon^2 \to R/I[\epsilon]/\epsilon^2$$

and moreover, have to be trivial up to coboundary on R/I.
So firstly, the ideal generated by w has to be invariant under ψ_{ij}^*, i.e. for any derivation θ there exists some $f \in R$ such that

$$\psi_{ij}^* \colon w \mapsto w + \epsilon \cdot \theta(w) = w + \epsilon \cdot w \cdot f = w \cdot (1 + \epsilon \cdot f)$$

and $(1 + \epsilon \cdot f) \in R[\epsilon]/\epsilon^2$ is a unit.
Secondly, the $\psi_{ij}^*|_D \colon R/I[\epsilon]/\epsilon^2 \to R/I[\epsilon]/\epsilon^2$ have to equal the identity map up to a Čech coboundary, i.e. the derivation θ_{ij}/w can be written as $\theta_{ij} = \alpha_j - \alpha_i$ for derivations α_i and α_j on U_i and U_j. Thus, it is locally generated by $w\partial_w$ and $w\partial_z$.
In a second step, we now also require $\mathcal{N}_{D/X}$ to stay trivial for our deformation. Thus, we finally are dealing with the situation we want to consider. This is a little bit more complicated to achieve than the previous step.
The fact that $\mathcal{N}_{D/X}$ stays trivial is equivalent to $\mathcal{N}_{D/X}^* = I/I^2$ staying trivial. Thus, the sheaf I/I^2 is supposed to stay trivial, i.e. it is supposed to have a global section. But ψ_{ij}^* acts on its generator $\mathrm{d}w$ as follows:

$$\psi_{ij}^* \colon \mathrm{d}w \mapsto (1 + \epsilon \cdot f)\mathrm{d}w$$

Therefore, I/I^2 stays invariant if and only if there exists $\{a_{U_i} \mid U_i \in \mathcal{U}\} \in \mathcal{C}^0(\mathcal{U}, \mathcal{O}^*)$ with

$$(1 + \epsilon \cdot f) = a_{U_j}/a_{U_i}.$$

Using the ansatz $a_{U_j} = c_j + \epsilon \cdot d_j$ and $a_{U_i} = c_i + \epsilon \cdot d_i$ we solve the equation above separating coefficients of ϵ which yields

$$c_j + \epsilon \cdot d_j = c_i + \epsilon \cdot f \cdot c_i + \epsilon \cdot d_i \Longrightarrow c_j = c_i \text{ and } f = \frac{d_j - d_i}{c_i} = \frac{d_j}{c_j} - \frac{d_i}{c_i}.$$

Thus, we can write the function $\theta_{ij}(w) = f$ on U_{ij} as a difference of a function on U_j and a function on U_i. This means that the cocycle $\theta_{ij}(w)$ has to vanish up to a coboundary.
Recall that by equation (5.7) on page 118, we know that

$$\ker\left((\Theta_X(\log D))\,|_{U_D} \to (\Theta_D)\,|_{U_D}\right) \cong \mathcal{O}_{U_D} \cdot (w\partial_w) \oplus \mathcal{O}_{U_D} \cdot (w\partial_z),\ U_D \subset D.$$

By the analysis above, the coefficient of $(w\partial_w)$ has to vanish up to a coboundary in this kernel which translates to a map of cohomology classes, as we were compatible coboundaries in every step.
This finishes the proof as it agrees with equation (5.6) on page 117 and equation (5.8) on page 121. □

Corollary 5.33. *Let the situation be as in Theorem 5.32. There is a digram as follows, where numbers denote relative dimensions:*

$$\left((\hat{\mathcal{Y}}, Y_0), (\hat{\mathcal{C}}, C_0)\right) \xrightarrow{2} \left(\hat{\mathcal{S}}, s_0\right) \xrightarrow{8} \operatorname{Spf} \mathbb{C}[[t]]$$

Proof. The dimensions follow from Theorem 5.32 and the fact that we can compute the involved cohomology groups using integral affine geometry, i.e.

$$\begin{aligned}\dim \mathrm{H}^1(X, \Theta_X(\log D)) &= \dim \mathrm{H}^1(B, \iota_* \check{\Lambda} \otimes \mathbb{C}) = 10,\\ \dim \mathrm{H}^0(D, \Theta_D) &= \dim \mathrm{H}^0(\mathbb{S}^1, \underline{\mathbb{Z}} \otimes \mathbb{C}) = 1 \text{ and}\\ \dim \mathrm{H}^1(D, \mathcal{O}_D) &= \dim \mathrm{H}^1(\mathbb{S}^1, \mathbb{C}) = 1.\end{aligned}$$

□

Next, we want to understand the situation for toric degenerations in more detail.

Lemma 5.34. *Let $D \subset X$ be as in Definition 5.2 and let $\mathbf{s} \in \mathrm{H}^1(B, \iota_* \check{\Lambda} \otimes \mathbb{C}^*)$ be lifted gluing data on the induced affine manifold B. Because we want to work relative to deformations of D, assume that the induced gluing data for D, i.e. $\mathbf{s}_D \in \mathrm{H}^1(\partial B, \check{\Lambda}_{\partial B} \otimes \mathbb{C}^*)$, are trivial.*
Then the normal bundle $\mathcal{N}_{D/X}$ is trivial if and only if

$$\mathbf{s}|_{\partial B} = 1 \in \mathrm{H}^1(\partial B, \iota_* \check{\Lambda}|_{\partial B} \otimes \mathbb{C}^*).$$

Proof. We want to construct a global trivialization of the ideal sheaf $I := \mathcal{O}_X(-D)$. Removing the boundary $\partial B \subset B$, we get an affine manifold $B \setminus \partial B$ without boundary. Denote by U the union of open stars (3.1) from page 57 of cells $\tau \in \mathscr{P}_\partial$ without ∂B:

$$U := \left(\left(\bigcup_{\tau \in \mathscr{P}_\partial} U_\tau\right) \setminus \partial B\right) \subset B$$

Note that we can rescale U by shrinking it in the direction of ∂B, which is uniquely determined via parallel transport. The open set U forms part of a cylinder with one boundary given by ∂B. We are reducing the diameter of this cylinder using parallel transport of ∂B. This process leaves us with a unique asymptotic vector $-\xi$ in $(\Lambda_B)|_U$ pointing into the non-circle direction of the cylinder. We get a theta function $\theta_{-\xi}$ on the central fibre X by [GHKS1, p. 90] if

$$\mathbf{s}|_{\partial B} = 1 \in \mathrm{H}^1(\partial B, \iota_* \check{\Lambda}|_{\partial B} \otimes \mathbb{C}^*).$$

As we are only interested in the situation at the central fibre, any higher order terms in $\theta_{-\xi}$ are irrelevant to us.
By the form of $\theta_{-\xi}$ given in [GHKS1, Thm. 3.19], the order zero part of $\theta_{-\xi}$ has a pole of first order along D and

$$\theta_{-\xi} \in I$$

is a global generator of I.
Equivalence follows from the fact that theta functions generate the ring of functions on $X \setminus Z$ by [GHKS1, Thm. 4.12]. □

Remark 5.35. We want to interpret Lemma 5.34 with respect to our example from Chapter 4. Recall that we introduced a tropical cycle $\beta_\perp^\partial$ corresponding to D in Lemma 4.14. The condition formulated in Lemma 5.34 is equivalent to

$$\mathbf{s} \in \left(\beta_\perp^\partial\right)^\perp \Leftrightarrow \mathbf{s}\left(\beta_\perp^\partial\right) = 1.$$

Lemma 5.36. *Let $(X, D)/O^\dagger$ be a simple log dP_9 pair as in Definition 5.2. There is always a diagram*

$$(X_0, D_0) \xrightarrow{2} \mathfrak{S}^0 \xrightarrow{9} \mathfrak{T}^0 \xrightarrow{1} O^\dagger$$

such that (X, D) is the fibre of (X_0, D_0) over some point $s_0 \in S =: \mathfrak{S}^0$. Taking the canonical family from [GS] yields

$$(\mathfrak{X}, \mathfrak{D}) \xrightarrow{2} \mathfrak{S} \xrightarrow{9} \mathfrak{T} \xrightarrow{1} \mathfrak{O}^\dagger.$$

The numbers above the arrows indicate relative dimensions.

Proof. By Rem. 2.19 and § 5.2 in [GS2] and [GHKS1, § A2], we get $(X, D) \subset (X_0, D_0)/\mathfrak{S}^0$, where

$$\mathfrak{S}^0 := S = \operatorname{Spec} \mathbb{C}\left[\mathrm{H}^1\left(B, \iota_* \check{\Lambda}\right)^*\right] \times O^\dagger \cong (\mathbb{C}^*)^{10} \times O^\dagger.$$

The space $\mathfrak{T}^0$ is the base of a toric deformation of the Tate curve. By the same reasoning as in Remark 5.28, we get

$$\mathfrak{T}^0 := \operatorname{Spec} \mathbb{C} \left[\mathrm{H}^1 \left(\partial B, \iota_* \check{\Lambda}_{\partial B} \right)^* \right] \times O^\dagger \cong \mathbb{C}^* \times O^\dagger.$$

Dimensions are given by $\dim \mathrm{H}^1(B, \iota_* \check{\Lambda}) = 10$ and $\mathrm{H}^1(\partial B, \check{\Lambda}_{\partial B}) = \mathrm{H}^1(\mathbb{S}^1, \check{\Lambda}_{\mathbb{S}^1}) \cong \mathbb{Z}$. The morphism $\mathfrak{S}^0 \to \mathfrak{T}^0$ is induced by the restriction map to ∂B and projection to $\check{\Lambda}_{\partial B} \subset \iota_* \check{\Lambda}|_{\partial B}$.
Taking the canonical family (see [GHKS1, § A.5]) yields

$$\mathfrak{S} := \operatorname{Spf} \mathbb{C} \left[\mathrm{H}^1 \left(B, \iota_* \check{\Lambda} \right)^* \right] [[t]] \text{ and}$$
$$\mathfrak{T} := \operatorname{Spf} \mathbb{C} \left[\mathrm{H}^1 \left(\partial B, \check{\Lambda}_{\partial B} \right)^* \right] [[t]].$$

□

Proposition 5.37. *There is a $\mathbb{C}^*$-action on the diagrams from Lemma 5.36. In particular, it acts transitively on $\mathfrak{T}$.*

Proof. Proposition 3.37 recalled the existence of a torus action from [GHKS1, § 5.2]. In our case, we work with a fixed multivalued piecewise linear function φ and thus, this torus action is a $\mathbb{C}^*$-action. We want to analyse this $\mathbb{C}^*$-action in more detail. Recall from the long exact sequence (4.5) on page 97 that

$$c_1(\varphi) \in \mathrm{H}^1(B, \iota_* \check{\Lambda}).$$

The weight of the torus action on $\mathfrak{S}^0$ is determined by $c_1(\varphi)$.* Recall from Remark 4.25 that $c_1(\varphi)(\beta_{\|}^{\partial}) \neq 0$ as φ is strictly convex. This fact generalizes from the example by the same reason. We can split, not necessarily orthogonally on the underlying lattice,

$$\mathrm{H}^1(B, \iota_* \check{\Lambda} \otimes \mathbb{C}^*) = \mathrm{H}^1(B, \iota_* \check{\Lambda}) \otimes \mathbb{C}^* = \left(\beta_{\|}^{\partial}\right)^{\perp} \times H_D \tag{5.9}$$

with $\dim H_D = 1$ and $\dim(\beta_{\|}^{\partial})^{\perp} = 9$. Without loss of generality we can assume $c_1(\varphi) \in H_D$. Note that $\mathbf{s} \in (\beta_{\|}^{\partial})^{\perp}$ is equivalent to $\mathbf{s}$ being trivial on D, so H_D projects surjectively onto $\mathfrak{T}^0$ while $(\beta_{\|}^{\partial})^{\perp}$ is contained in the kernel of the projection. Thus, $\mathbb{C}^*$ acts transitively on $\mathfrak{T}$. □

*This is to appear in an updated version of [GHKS1].

Remark 5.38. The splitting given in equation (5.9) is not necessarily orthogonal with respect to the underlying lattice. Thus, it does not imply that $\mathbb{C}^*$ acts trivially on the part of $\mathfrak{S}^0$ corresponding to $(\beta_{\|}^{\partial})^{\perp}$. Nevertheless, as $\mathbb{C}^*$ acts transitively on the part corresponding to H_D and by extension on $\mathfrak{T}^0$, it follows that the rank of $(\beta_{\|}^{\partial})^{\perp}$ is preserved.

Remark 5.39. Recall that in Remark 4.19, we restricted ourselves to lifted gluing data with

$$\mathbf{s} \in \left(\beta_{\|}^{\partial}\right)^{\perp} \subset \mathrm{H}^1(B, \iota_*\check{\Lambda} \otimes \mathbb{C}^*).$$

We were not able to explain this restriction properly back then. Proposition 5.37 shows that $\mathbf{s} \in (\beta_{\|}^{\partial})^{\perp}$ is equivalent to $\mathbf{s}$ restricting to trivial gluing data on D. Working relative to deformations of D, this restriction is well justified, even more so as taking the quotient by the $\mathbb{C}^*$-action collapses deformations of D, which are encoded in H_D (see equation (5.9)).

Lemma 5.40. *Restriction to toric degenerations which leave $\mathcal{N}_{D/X}$ and D invariant yields a diagram*

$$(\mathfrak{X}', \mathfrak{D}') \xrightarrow{2} \mathfrak{S}' \xrightarrow{8} \mathfrak{O}^{\dagger}.$$

Proof. By Remark 5.35 and Remark 5.39, we want to consider the 8-dimensional subspace of $\mathfrak{S}/\mathfrak{O}^{\dagger}$ which corresponds to $(\beta_{\|}^{\partial})^{\perp} \cap (\beta_{\perp}^{\partial})^{\perp}$. By the splitting (5.9) on page 126, this subspace translates to an 8-dimensional subset $(\mathfrak{S}' \subset \mathfrak{S})/\mathfrak{O}^{\dagger}$ as it corresponds to a transversal section of the torus action.
We define $(\mathfrak{X}', \mathfrak{D}') \subset (\mathfrak{X}, \mathfrak{D})$ to be the preimage of $\mathfrak{S}' \subset \mathfrak{S}$ under the structure morphism $(\mathfrak{X}, \mathfrak{D}) \to \mathfrak{S}$. □

Proposition 5.41. *Let $(\mathcal{X}', \mathcal{D}') \to \mathcal{S}' \to T$ be an analytic extension of the diagram from Lemma 5.40. Denote by $\mathbb{B}$ a suitable open neighbourhood of $0 \in \mathbb{C}^8 \cong \mathrm{H}^0\left(X, \Omega^2_{X/D}(\log D)\right)^*$. The period map $\exp\left(-\frac{1}{2\pi i}\int \Omega(t)\right) : \mathrm{H}_2(X_t, \mathbb{Z}) \to \mathbb{C},\ t \in T$, from equation (4.1) on page 73 induces an injective morphism over T,*

$$\mathcal{S}' \to (\mathbb{B} \times T).$$

Proof. Recall that we constructed an $\hat{E}_8$ basis $(\alpha_0)_{trop}, \ldots, (\alpha_8)_{trop} \in \mathrm{H}_1(B, \iota_*\Lambda)$ and a cycle $\beta_{\|}^{\partial} \in \mathrm{H}_1(B, \iota_*\Lambda)$, which does not correspond to a non-zero class in

$\mathrm{H}_2(X_t, \mathbb{Z})$, $t \in T$, in Chapter 4. All these facts generalize by Proposition 5.14. As in Remark 4.19 we use the notation

$$\left(\beta_{\parallel}^{\partial}\right)^{\perp} \subset \mathrm{H}^1(B, \iota_* \check{\Lambda} \otimes \mathbb{C}^*) \text{ and } \mathfrak{R} := \mathrm{span}((\alpha_0)_{trop}, \ldots, (\alpha_8)_{trop}) \subset \mathrm{H}_1(B, \iota_* \Lambda).$$

By Proposition 4.22 and Proposition 5.14, there is a pairing of rank 9

$$\begin{aligned} \mathfrak{R} \otimes \left(\beta_{\parallel}^{\partial}\right)^{\perp} &\to \mathbb{Z} \\ (\gamma, \mathbf{s}) &\mapsto \langle \gamma, \mathbf{s} \rangle. \end{aligned}$$

Let $C(t)$ be the cycle in $\mathrm{H}_2(X_t, \mathbb{Z})$, $t \in T$, corresponding to $\gamma \in \mathrm{H}_1(B, \iota_* \Lambda)$. The period integrals are given by

$$\exp\left(-\frac{1}{2\pi i}\int_{C(t)} \Omega\right) = \langle \gamma, \mathbf{s} \rangle \cdot t^{\langle \gamma, c_1(\varphi) \rangle}.$$

As we want the period integrals to be defined and non-zero for $t = 0$, we need to restrict ourselves to the subspace

$$c_1(\varphi)^{\perp} \subset \mathrm{H}_1(B, \iota_* \Lambda).$$

Moreover, this restriction ensures that the rank of the pairing stays invariant when turning on the $\mathbb{C}^*$-action from Proposition 3.37 as $c_1(\varphi) \in H_D$[†].
In $\mathrm{H}^1(B, \iota_* \check{\Lambda} \otimes \mathbb{C}^*)$, we restrict ourselves to the 8-dimensional subspace

$$H^S := \left(\beta_{\parallel}^{\partial}\right)^{\perp} \cap \left(\beta_{\perp}^{\partial}\right)^{\perp}.$$

We want to argue that we get a pairing of rank 8,

$$\begin{aligned} c_1(\varphi)^{\perp} \otimes H^S &\to \mathbb{Z} \\ (\gamma, \mathbf{s}) &\mapsto \exp\left(-\frac{1}{2\pi i}\int_{C(t)} \Omega\right) = \langle \gamma, \mathbf{s} \rangle \cdot t^{\langle \gamma, c_1(\varphi) \rangle} = \langle \gamma, \mathbf{s} \rangle. \end{aligned}$$

This holds as $c_1(\varphi) \in H_D$ in the splitting (5.9) on page 126 and

$$H^S \subset \left(\beta_{\parallel}^{\partial}\right)^{\perp} \subset \left(\beta_{\parallel}^{\partial}\right)^{\perp} \times H_D = \mathrm{H}^1(B, \iota_* \check{\Lambda} \otimes \mathbb{C}^*).$$

Moreover, $\langle \beta_{\parallel}^{\partial}, c_1(\varphi) \rangle \neq 0$ as φ is strictly convex and thus, $\beta_{\parallel}^{\partial} \notin c_1(\varphi)^{\perp}$. Thus, the period map has rank 8 relative to T. This finishes the proof. □

[†]See also the discussion which is to appear in [RS, § 5.1] for an explanation why this restriction is necessary and suffices.

Theorem 5.42. *Let $(X, \mathcal{M}_{(X,D)})/O^\dagger$ be a simple log dP_9 pair as in Definition 5.2. Assume that there exists an analytic log smooth versal divisorial deformation of dP_9 pairs*

$$[(\mathcal{Y}, \mathcal{C}), \mathcal{M}] \to \mathcal{S} \to \mathbb{D}$$

of $(X, \mathcal{M}_{(X,D)})/O^\dagger$.
Then the toric degeneration $((\mathfrak{X}', \mathfrak{D}') \to \mathfrak{S}')/\mathfrak{O}^\dagger$ of (X, D) from Lemma 5.40 is a formal completion of $([(\mathcal{Y}, \mathcal{C}), \mathcal{M}] \to \mathcal{S})/\mathbb{D}$ at the central fibre and the period map produces coordinates locally on the base $\mathcal{S}/\mathbb{D}$.

Proof. For most of the proof we follow the lines of the first part of the proof of [GHKS2, Prop. 13.6].
Recall that $\mathfrak{O} = \operatorname{Spf} \mathbb{C}[[t]]$. We want to argue that the reduction $(\mathfrak{X}')^k$ of $(\mathfrak{X}', \mathfrak{D}') \to \mathfrak{S}' \to \mathfrak{O}^\dagger$ modulo t^{k+1} is a complex analytic space.
We know that X_0/S is a complex analytic space. Thus, the same holds for the pullback to $(\mathfrak{S}')^0$. So $(\mathfrak{X}')^k$ is a complex analytic space as it is a subset of a finite order deformation of a complex analytic space.
The same argument holds for the reductions $(\mathfrak{D}')^k$ and $\mathfrak{S}'^k$. As $[(\mathcal{Y}, \mathcal{C}), \mathcal{M}]$ is versal, there is a map ψ_k such that the diagram

$$\begin{tikzcd} ((\mathfrak{X}')^k, (\mathfrak{D}')^k) \arrow[r] \arrow[d] & (\mathcal{Y}, \mathcal{C}) \arrow[d] \\ \mathfrak{S}'^k \arrow[r, "\psi_k"] & \mathcal{S}. \end{tikzcd}$$

commutes. Denote by $\mathfrak{S}'^0/O^\dagger$ the reduction of $\mathfrak{S}'/\mathfrak{O}^\dagger$ modulo t, i.e. the base of the central fibre over $O^\dagger$. On rings at the central fibre, ψ_k corresponds to a morphism

$$\psi_k^* \colon \mathcal{O}_{\mathcal{S},0} \to \mathcal{O}_{\mathfrak{S}'^0,0}\,[t]\,/(t^{k+1}).$$

We can assume that the morphisms ψ_k are compatible for increasing $k \in \mathbb{N}$. Thus, we can form the limit, which yields a morphism

$$\psi^* \colon \mathcal{O}_{\mathcal{S},0} \to \mathcal{O}_{\mathfrak{S}',0}.$$

As the period maps are injective by Proposition 5.41 and the dimensions of $\mathcal{S}$ and $\mathfrak{S}'$ coincide, the period maps yield local coordinates for $\mathcal{S}/\mathbb{D}$.
Therefore, $(\mathfrak{X}', \mathfrak{D}')$ is locally a completion of $[(\mathcal{Y}, \mathcal{C}), \mathcal{M}]$ at $0 \in \mathbb{D}$. □

5.4. The equivalence of categories

We want to deduce an equivalence of categories for toric degenerations of dP_9 surfaces analogous to Theorem 1.39 and Theorem 2.24.
Note that the central fibre of a toric degeneration is not determined by the generic fibre. There are many possible transformations of the central fibre which leave the generic fibre invariant. Thus, we have to work relative to affine data fixing some structure on the central fibre:

Definition 5.43. Let B be an affine manifold with singularities Δ and a polyhedral decomposition $\mathscr{P}$ which is induced by a simple log dP_9 pair. We also fix a strictly convex multi-valued piecewise linear function φ on $(B, \mathscr{P})$. We call the fixed data $((B, \mathscr{P}), \varphi)$ a *choice of affine data.*

As before we have to fix a marking, which determines a choice of exceptional curves.

Definition 5.44. A *marking* Φ on $((B, \mathscr{P}), \varphi)$ is given by a map

$$\begin{aligned} \Phi\colon \Lambda_{1,9} &\to \{\beta \mid \beta \text{ is a tropical cycle in } B\} \\ \text{with } \Phi(d) &= \beta_{\perp}^{\partial}. \end{aligned}$$

A marking is determined by a choice of tropical cycles $\epsilon_0, \ldots, \epsilon_9$ in B which generate exceptional curves as in Lemma 4.16 and have intersection numbers corresponding to classes $e_0 := l - e_1 - e_2, e_1, \ldots, e_9 \in \Lambda_{1,9}$ because these classes form a basis of $\Lambda_{1,9}$.
Note that elements of $Q' = d^{\perp} \subseteq \Lambda_{1,9}$ are mapped to tropical cycles in $\mathrm{H}_1(B, \iota_* \Lambda)$.

Lemma 5.45. *For every choice of affine data $((B, \mathscr{P}), \varphi)$ there exists a marking Φ.*

Proof. Recall that we constructed a marking for our example in Lemma 4.16. Using Lemma 5.12, the existence of a marking for the general case follows in the same way as Proposition 5.14. $\square$

Lemma 5.46. *A marking Φ on $((B, \mathscr{P}), \varphi)$ induces a geometric marking on smooth fibres X_t of an analytic extension of a toric degeneration of dP_9 surfaces $(\mathfrak{X}, \mathfrak{D}) \to \mathfrak{O}^{\dagger}$ and vice versa.*

Proof. By Lemma 5.45, we can always find at least one marking Φ on $((B,\mathscr{P}),\varphi)$. The marking Φ fixes tropical cycles $\epsilon_0,\ldots,\epsilon_9$ on B. These tropical cycles $\epsilon_0,\ldots,\epsilon_9$ induce exceptional curves $E_0(t),\ldots,E_9(t)$ on smooth fibres X_t by [RS, § 3]. The exceptional curves $E_0(t),\ldots,E_9(t)\subset X_t$ fix a geometric marking ϕ on X_t.
The proof of the other way around is more difficult. Assume that we fixed a geometric marking ϕ on a smooth fibre X_t. By the argument above, there exists a marking Φ_0 on $((B,\mathscr{P}),\varphi)$ inducing a geometric marking ϕ_0 on X_t. By Lemma 1.25, there exists a unique element of the Weyl group $w_0\in\mathfrak{W}'$ such that

$$\phi=\phi_0 w_0\colon \Lambda_{1,9}\to \mathrm{H}_2(X_t,\mathbb{Z}).$$

Recall from Definition 1.12 that the Weyl group acts via addition of integer multiples of roots $\alpha\in Q'$. As each root $\alpha\in Q'$, represented by a tropical cycle using the marking Φ_0, we can modify Φ_0 to get a new marking

$$\Phi := \Phi_0 w_0$$

on $((B,\mathscr{P}),\varphi)$. By construction, the marking Φ induces the geometric marking ϕ on smooth fibres. □

As before we want to phrase our theorem in the language of categories. We start by defining the category of toric degenerations of dP_9 surfaces.

Definition 5.47. Let $((B,\mathscr{P}),\varphi)$ denote some fixed choice of affine data. Moreover, we fix a marking Φ on $((B,\mathscr{P}),\varphi)$.
Let $\underline{\mathfrak{dP}}_9$ denote the category whose objects consist of toric degenerations

$$(\mathfrak{X},\mathfrak{D})\to\mathfrak{S}\to\mathfrak{O}=\mathrm{Spf}\,\mathbb{C}[[t]]$$

of dP_9 surfaces over affine formal schemes $\mathfrak{S}=:\mathrm{Spf}\,\mathcal{A}$ such that the central fibre (X,D) and its logarithmic structure are of type $((B,\mathscr{P}),\varphi)$. We require $\mathcal{A}$ to be of finite type over $\mathbb{C}[[t]]$.
Morphisms in this category are given by cartesian diagrams over $\mathfrak{O}^\dagger$, i.e.

$$\begin{tikzcd}
(\mathfrak{X}_1,\mathfrak{D}_1) \arrow[r] \arrow[d] & (\mathfrak{X}_2,\mathfrak{D}_2) \arrow[d] \\
\mathfrak{S}_1 \arrow[r] & \mathfrak{S}_2.
\end{tikzcd}$$

Recall that we are using a (restricted) universal setting. This yields a universal element in $\underline{\mathfrak{dP}}_9$, which we will discuss next.

Definition 5.48. Recall that we introduced a toric degeneration $(\mathfrak{X}', \mathfrak{D}') \to \mathfrak{S}'$ in Lemma 5.40. For reasons of simplicity we write

$$((\mathfrak{Y}, \mathfrak{C}) \to \mathfrak{U})$$

for $((\mathfrak{X}', \mathfrak{D}') \to \mathfrak{S}')$. This notation is justified by Theorem 5.42.

Definition 5.49. To fix some more notation, recall from the proof of Proposition 5.41

$$H^S := \left(\beta_{\parallel}^{\partial}\right)^{\perp} \cap \left(\beta_{\perp}^{\partial}\right)^{\perp} \subset \mathrm{H}^1(B, \iota_* \check{\Lambda}) \times \mathbb{C}^*.$$

We want to consider the underlying lattice, i.e. we set

$$H_S := \left(\beta_{\parallel}^{\partial}\right)^{\perp} \cap \left(\beta_{\perp}^{\partial}\right)^{\perp} \subset \mathrm{H}^1(B, \iota_* \check{\Lambda}).$$

Keep in mind that $(\beta_{\parallel}^{\partial})^{\perp}$ and $(\beta_{\perp}^{\partial})^{\perp}$ are always understood with respect to the corresponding group, i.e. $(\mathbb{C}^*, 1)$ or $(\mathbb{Z}, 0)$.

Lemma 5.50. *It holds that*

$$\mathfrak{U} \cong \operatorname{Spf} \mathbb{C}\left[H_S^*\right][[t]] \subset \operatorname{Spf} \mathbb{C}\left[\mathrm{H}^1\left(B, \iota_* \check{\Lambda}\right)^*\right][[t]].$$

Proof. We claim that $\operatorname{Spec} \mathbb{C}[H_S^*]$ is a transversal section of the $\mathbb{G}_m$-orbits in $\operatorname{Spec} \mathbb{C}[\mathrm{H}^1(B, \iota_* \check{\Lambda})^*]$.
Proposition 5.37 yields that the torus action is transitive on the part of $\mathrm{H}^1(B, \iota_* \check{\Lambda})$ which corresponds to the base $\mathfrak{T}$ of the degeneration of elliptic curves, i.e. on $\mathrm{H}^1(\partial B, \check{\Lambda}_{\partial B})$. Thus, without loss of generality we can assume that the gluing data are trivial on ∂B. This assumption yields a transversal section of the torus orbits. Gluing data which are trivial on ∂B are contained in $(\beta_{\parallel}^{\partial})^{\perp}$. The lemma then follows from Lemma 5.40. □

Proposition 5.51. *The toric degeneration* $(\mathfrak{Y}, \mathfrak{C}) \to \mathfrak{U}$ *is universal for toric degenerations in* $\underline{\mathfrak{dP}}_9$.

Proof. This holds by [GHKS1, Thm A.5][‡] and Theorem 5.42. □

[‡]To appear in a future version of [GHKS1].

Remark 5.52. Universality of $((\mathfrak{Y}, \mathfrak{C}) \to \mathfrak{U})$ means that for every $((\mathfrak{X}, \mathfrak{D}) \to \mathfrak{S}) \in \mathbf{ob}(\underline{\mathfrak{dP}}_9)$, there is a unique homomorphism

$$\mathfrak{S} \to \mathfrak{U}$$

inducing $(\mathfrak{X}, \mathfrak{D}) \to (\mathfrak{Y}, \mathfrak{C})$. As $\mathfrak{S}$ is affine by definition, it holds that $\mathfrak{S} = \operatorname{Spf} \mathcal{A}$ for a finitely generated $\mathbb{C}[[t]]$-algebra $\mathcal{A}$. Thus, we get an induced homomorphism of finitely generated $\mathbb{C}[[t]]$-algebras, i.e.

$$\mathbb{C}[H_S^*][[t]] \to \mathcal{A}.$$

Next, we will introduce some intermediate category $\underline{\mathrm{Hom}}_{\mathbb{C}[H_S^*][[t]]}$. Strictly speaking, it is not necessary for our purpose, but it does make our discussion more transparent. In the end, we want to show that $\underline{\mathfrak{dP}}_9$ is equivalent to another category $\underline{\mathfrak{Hom}}_Q$. We will show that both of them are anti-equivalent to $\underline{\mathrm{Hom}}_{\mathbb{C}[H_S^*][[t]]}$.

Definition 5.53. Fix a choice of affine data $((B, \mathscr{P}), \varphi)$ and a marking Φ. We define the category $\underline{\mathrm{Hom}}_{\mathbb{C}[H_S^*][[t]]}$ as follows:
Its objects are pairs of a finitely generated $\mathbb{C}[[t]]$-algebra $\mathcal{A}$ together with a homomorphism of finitely generated $\mathbb{C}[[t]]$-algebras

$$\psi_{\mathcal{A}} \colon \mathbb{C}[H_S^*][[t]] \to \mathcal{A}.$$

Let $(\mathcal{A}_1, \psi_{\mathcal{A}_1}), (\mathcal{A}_2, \psi_{\mathcal{A}_2}) \in \mathbf{ob}(\underline{\mathrm{Hom}}_{\mathbb{C}[H_S^*][[t]]})$ be objects. A homomorphism in $\underline{\mathrm{Hom}}_{\mathbb{C}[H_S^*][[t]]}$ from $(\mathcal{A}_1, \psi_{\mathcal{A}_1})$ to $(\mathcal{A}_2, \psi_{\mathcal{A}_2})$ is given by a homomorphism of finitely generated $\mathbb{C}[[t]]$-algebras $\psi \colon \mathcal{A}_1 \to \mathcal{A}_2$ such that

$$\psi_{\mathcal{A}_2} = \psi \circ \psi_{\mathcal{A}_1}.$$

We can easily prove the following proposition:

Proposition 5.54. *The categories* $\underline{\mathfrak{dP}}_9$ *and* $\underline{\mathrm{Hom}}_{\mathbb{C}[H_S^*][[t]]}$ *are anti-equivalent.*

Proof. By Remark 5.52, each object $((\mathfrak{X}, \mathfrak{D}) \to \mathfrak{S} =: \operatorname{Spf} \mathcal{A}) \in \mathbf{ob}(\underline{\mathfrak{dP}}_9)$ induces an object $(\mathcal{A}, \psi_{\mathcal{A}}) \in \mathbf{ob}(\underline{\mathrm{Hom}}_{\mathbb{C}[H_S^*][[t]]})$. The object $(\mathcal{A}, \psi_{\mathcal{A}})$ is uniquely determined by universality of $(\mathfrak{Y}, \mathfrak{C}) \to \mathfrak{U}$. To define a contravariant functor

$$\begin{aligned} F \colon \underline{\mathfrak{dP}}_9 &\to \underline{\mathrm{Hom}}_{\mathbb{C}[H_S^*][[t]]} \\ ((\mathfrak{X}, \mathfrak{D}) \to \mathfrak{S}) &\mapsto (\mathcal{A}, \psi_{\mathcal{A}} \colon \mathbb{C}[H_S^*][[t]] \to \mathcal{A}), \end{aligned}$$

it remains to define the functor F on morphisms. Let

$$\begin{tikzcd} (\mathfrak{X}_1, \mathfrak{D}_1) \arrow[r, "\xi_{\mathfrak{X}}"] \arrow[d] & (\mathfrak{X}_2, \mathfrak{D}_2) \arrow[d] \\ \mathfrak{S}_1 \arrow[r, "\xi_{\mathfrak{S}}"] & \mathfrak{S}_2 \end{tikzcd}$$

be a morphism in $\underline{\mathfrak{d}\mathfrak{P}}_9$. Set moreover,

$$\begin{aligned} F((\mathfrak{X}_1, \mathfrak{D}_1) \to \mathfrak{S}_1) &=: (\mathcal{A}_1, \psi_{\mathcal{A}_1} \colon \mathbb{C}[H_S^*][[t]] \to \mathcal{A}_1) \text{ and} \\ F((\mathfrak{X}_2, \mathfrak{D}_2) \to \mathfrak{S}_2) &=: (\mathcal{A}_2, \psi_{\mathcal{A}_2} \colon \mathbb{C}[H_S^*][[t]] \to \mathcal{A}_2). \end{aligned}$$

The morphism of schemes $\xi_{\mathfrak{S}} \colon \mathfrak{S}_1 \to \mathfrak{S}_2$ induces a morphism of finitely generated $\mathbb{C}[[t]]$-algebras

$$(\xi_{\mathfrak{S}})_* \colon \mathcal{A}_2 \to \mathcal{A}_1.$$

We set $F(\xi_{\mathfrak{X}}, \xi_{\mathfrak{S}}) =: ((\xi_{\mathfrak{S}})_* : \mathcal{A}_2 \to \mathcal{A}_1)$. As $\psi_{\mathcal{A}_2}$ is the unique map which induces $(\mathfrak{X}_2, \mathfrak{D}_2)$ via pullback, it holds that

$$\psi_{\mathcal{A}_1} = (\xi_{\mathfrak{S}})_* \circ \psi_{\mathcal{A}_2}.$$

The functor F is essentially surjective because each object $(\mathcal{A}, \psi_{\mathcal{A}})$ of $\underline{\mathrm{Hom}}_{\mathbb{C}[H_S^*][[t]]}$ induces an object of $\underline{\mathfrak{d}\mathfrak{P}}_9$ by taking the pullback

$$\begin{tikzcd} \operatorname{Spf} \mathcal{A} \times_{\mathfrak{U}} (\mathfrak{Y}, \mathfrak{C}) \arrow[r] \arrow[d] & (\mathfrak{Y}, \mathfrak{C}) \arrow[d] \\ \operatorname{Spf} \mathcal{A} \arrow[r] & \mathfrak{U}. \end{tikzcd}$$

By construction, it holds that

$$F \colon \big((\operatorname{Spf} \mathcal{A} \times_{\mathfrak{U}} (\mathfrak{Y}, \mathfrak{C})) \to \operatorname{Spf} \mathcal{A} \big) \mapsto (\mathcal{A}, \psi_{\mathcal{A}}).$$

We still have to show that F is fully faithful. This follows because morphisms of rings and morphisms of the associated spectra of rings are equivalent. $\square$

Now we are ready to define the category $\underline{\mathfrak{Hom}}_Q$.

Remark 5.55. Note that a choice of affine data $((B, \mathscr{P}), \varphi)$ induces on $\partial B \cong \mathbb{S}^1$ a polyhedral decomposition $\mathscr{P}_\partial$ and a multi-valued piecewise linear function $\varphi|_{\partial B}$.

This data are enough to fix a log versal toric degeneration $\mathfrak{C} \to \mathfrak{O}^\dagger$ of a Tate curve C. We take the pullback

$$\begin{tikzcd} \mathfrak{C}' \arrow[r] \arrow[d] & \mathfrak{C} \arrow[d] \\ \mathfrak{U} \arrow[r] & \mathfrak{O}^\dagger \end{tikzcd}$$

for $\mathfrak{U} = \operatorname{Spf} \mathbb{C}[H_S^*][[t]]$ as in Definition 5.49. The family $\mathfrak{C}' \to \mathfrak{U}$ is a toric degeneration of elliptic curves with central fibre $\mathfrak{C}^0 = C \times \operatorname{Spec} \mathbb{C}[H_S^*]$.

Definition 5.56. Fix again some choice of affine data $((B, \mathscr{P}), \varphi)$ and a marking Φ. We define a category $\underline{\mathfrak{Hom}}_Q$ as follows:
Its objects are given by pairs $(\mathfrak{D} \to \mathfrak{S}, \chi_{\mathfrak{D}})$, whose first component is a toric degeneration of elliptic curves, which is compatible with $((\partial B, \mathscr{P}_\partial), \varphi|_{\partial B})$ and given by a pullback diagram

$$\begin{tikzcd} \mathfrak{D} \arrow[r] \arrow[d] & \mathfrak{C} \arrow[d] \\ \mathfrak{S} \arrow[r] & \mathfrak{O}^\dagger. \end{tikzcd}$$

Here, $\mathfrak{S} =: \operatorname{Spf} \mathcal{A}$ denotes an affine scheme such that $\mathcal{A}$ is a finitely generated $\mathbb{C}[[t]]$-algebra. The second component is a group homomorphism

$$\chi_{\mathfrak{D}} \colon Q \to \operatorname{Hom}(\mathfrak{S}, \mathbb{C}^*)\,,$$

such that $\chi_{\mathfrak{D}}(\alpha) \colon \mathfrak{S} \to \mathbb{C}^*$ is constant with respect to $\mathfrak{D}$ (see Remark 5.57).
The morphisms in $\underline{\mathfrak{Hom}}_Q$ are given by cartesian diagrams over $\mathfrak{O}^\dagger$, i.e.

$$\begin{tikzcd} \mathfrak{D}_1 \arrow[r, "\xi_{\mathfrak{D}}"] \arrow[d] & \mathfrak{D}_2 \arrow[d] \\ \operatorname{Spf} \mathcal{A}_1 := \mathfrak{S}_1 \arrow[r, "\xi_{\mathfrak{S}}"] & \mathfrak{S}_2 =: \operatorname{Spf} \mathcal{A}_2. \end{tikzcd}$$

We require that

$$(\chi_{\mathfrak{D}_1}(\alpha) : \mathfrak{S}_1 \to \mathbb{C}^*) = ((\chi_{\mathfrak{D}_2}(\alpha) \circ \xi_{\mathfrak{S}}) : \mathfrak{S}_1 \to \mathbb{C}^*) \text{ for all } \alpha \in Q.$$

Remark 5.57. Note that the homomorphism $\chi_{\mathfrak{D}}(\alpha) \colon \operatorname{Spf} \mathcal{A} \to \mathbb{C}^*$ for $\alpha \in Q$ corresponds on rings to a homomorphism

$$(\chi_{\mathfrak{D}}(\alpha))_* : \mathbb{C}[\mathbb{Z}] \to \mathcal{A}.$$

Moreover, there is the structure morphism $\pi\colon \operatorname{Spf}\mathcal{A} \to \mathfrak{O}$, which yields a morphism of finitely generated $\mathbb{C}[[t]]$-algebras

$$\pi_*\colon \mathbb{C}[[t]] \to \mathcal{A}.$$

The condition that $\chi_{\mathfrak{D}}(\alpha)$ is constant with respect to $\mathfrak{O}$ is equivalent to the fact that $(\chi_{\mathfrak{D}}(\alpha))_*$ does not map into the maximal ideal of $\mathcal{A}$, which is generated by $\operatorname{im}\pi_*$.

We want to define a contravariant functor $G\colon \underline{\mathrm{Hom}}_{\mathbb{C}[H_S^*][[t]]} \to \underline{\mathfrak{Hom}}_Q$ starting with the following lemma:

Lemma 5.58. *We can associate with an object in* $\underline{\mathrm{Hom}}_{\mathbb{C}[H_S^*][[t]]}$ *an object in* $\underline{\mathfrak{Hom}}_Q$ *in a unique way.*

Proof. We start with an object $(\mathcal{A}, \psi_{\mathcal{A}}\colon \mathbb{C}[H_S^*][[t]] \to \mathcal{A}) \in \mathbf{ob}(\underline{\mathrm{Hom}}_{\mathbb{C}[H_S^*][[t]]})$. Note that by the fixed choice of affine data, there is a fixed versal toric degeneration of elliptic curves $\mathfrak{C} \to \mathfrak{O}^\dagger$ (see Remark 5.55). We take the pullback

$$\begin{array}{ccc} \mathfrak{D} := \mathfrak{C} \times_{\mathfrak{O}} \operatorname{Spf}\mathcal{A} & \longrightarrow & \mathfrak{C} \\ \downarrow & & \downarrow \\ \operatorname{Spf}\mathcal{A} & \longrightarrow & \mathfrak{O}^\dagger. \end{array}$$

This diagram defines a toric degeneration of elliptic curves.
Next, we have to fix a morphism $\chi_{\mathfrak{D}}\colon Q \to \operatorname{Hom}(\operatorname{Spf}\mathcal{A}, \mathbb{C}^*)$.
Recall that the marking Φ maps each element $\alpha \in Q' = d^\perp$ to a uniquely determined linear combination of tropical cycles in $\mathrm{H}_1(B, \iota_*\Lambda)$. Recall that $\Phi(d) = \beta_\perp^\partial$, which is mapped to zero by H_S. Since the induced pairing $Q' \otimes H_S$ descends to a pairing of full rank

$$\begin{aligned} d^\perp/d = Q \otimes H_S &\to \mathbb{Z} \\ (\alpha, \mathbf{s}) &\mapsto \langle \alpha, \mathbf{s}\rangle, \end{aligned}$$

it follows that $Q \cong H_S^*$. An element $\alpha \in Q$ induces a morphism

$$\begin{aligned} \chi_{\mathfrak{U}}(\alpha)\colon \operatorname{Spf}\mathbb{C}[H_S^*][[t]] = \mathfrak{U} &\longrightarrow \mathbb{C}^* = \operatorname{Spec}\mathbb{C}[\mathbb{Z}] =: \operatorname{Spec}\mathbb{C}[x, x^{-1}] \\ H_S^* \ni \langle \alpha, \cdot\rangle &\longleftarrow x. \end{aligned}$$

Composition of $(\chi_{\mathfrak{U}}(\alpha))_*$ with $\psi_{\mathcal{A}}$ yields a morphism of schemes

$$\begin{aligned} \chi_{\mathfrak{D}}(\alpha)\colon \operatorname{Spf}\mathcal{A} &\longrightarrow \operatorname{Spec}\mathbb{C}[\mathbb{Z}] =: \operatorname{Spec}\mathbb{C}[x,x^{-1}] = \mathbb{C}^* \\ \psi_{\mathcal{A}}(\langle\alpha,\cdot\rangle) &\longleftarrow x. \end{aligned} \tag{5.10}$$

Note that $\chi_{\mathfrak{D}}(\alpha)$ is constant with respect to $\mathfrak{O}$, because $\chi_{\mathfrak{U}}(\alpha)$ is. □

Proposition 5.59. *We can define a contravariant functor $G\colon \underline{\mathrm{Hom}}_{\mathbb{C}[H_S^*][[t]]} \to \underline{\mathfrak{Hom}}_Q$ via Lemma 5.58.*

Proof. On objects, we use Lemma 5.58 to define G. But we also have to fix G on morphisms. So let $(\psi\colon \mathcal{A}_1 \to \mathcal{A}_2) \in \mathbf{mor}(\underline{\mathrm{Hom}}_{\mathbb{C}[H_S^*][[t]]})$ be a morphism. Set $G(\mathcal{A}_1, \psi_{\mathcal{A}_1}) =: (\mathfrak{D}_1, \chi_{\mathfrak{D}_1})$ and $G(\mathcal{A}_2, \psi_{\mathcal{A}_2}) =: (\mathfrak{D}_2, \chi_{\mathfrak{D}_2})$. Note that ψ induces a pullback diagram

$$\begin{tikzcd} \mathfrak{D}_2 \arrow[r, "\xi_{\mathfrak{D}}"] \arrow[d] & \mathfrak{D}_1 \arrow[d] \\ \mathfrak{S}_2 \arrow[r, "\xi_{\mathfrak{S}}"] & \mathfrak{S}_1, \end{tikzcd}$$

where $(\xi_{\mathfrak{S}})_* = \psi$. We have to show that $\chi_{\mathfrak{D}_2}(\alpha) = \chi_{\mathfrak{D}_1}(\alpha) \circ \xi_{\mathfrak{S}}$ for all $\alpha \in Q$. By Lemma 5.58, we know that

$$(\chi_{\mathfrak{D}_i}(\alpha))_* = \psi_{\mathcal{A}_i} \circ (\chi_{\mathfrak{U}}(\alpha))_*, \; i = 1, 2.$$

Moreover, we know by definition that

$$\psi_{\mathcal{A}_2} = \psi \circ \psi_{\mathcal{A}_1}.$$

Thus, it follows that

$$(\chi_{\mathfrak{D}_2}(\alpha))_* = \psi_{\mathcal{A}_2} \circ (\chi_{\mathfrak{U}}(\alpha))_* = \psi \circ \psi_{\mathcal{A}_1} \circ (\chi_{\mathfrak{U}}(\alpha))_* = \psi \circ (\chi_{\mathfrak{D}_1}(\alpha))_*.$$

This finishes the proof as all maps are associative and respect the identity. □

Next, we have to show that the functor G induces a duality of categories.

Proposition 5.60. *Let $(\mathfrak{D} \to \mathfrak{S}, \chi_{\mathfrak{D}}) \in \mathbf{ob}(\underline{\mathfrak{Hom}}_Q)$. There is an associated element in $\underline{\mathrm{Hom}}_{\mathbb{C}[H_S^*][[t]]}$ and the functor G is essentially surjective.*

Proof. Let $\mathfrak{S} = \operatorname{Spf} \mathcal{A}$ for a finitely generated $\mathbb{C}[[t]]$-algebra $\mathcal{A}$ over $\mathbb{C}[[t]]$. Recall that $\chi_{\mathfrak{D}} \colon Q \to \operatorname{Hom}(\operatorname{Spf} \mathcal{A}, \mathbb{C}^*)$. Note that the 8 basis elements $\alpha_0, \dots, \alpha_7 \in Q$ yield 8 morphisms

$$\left((\chi_{\mathfrak{D}}(\alpha_0))_* =: h_*^0\right), \dots, \left((\chi_{\mathfrak{D}}(\alpha_7))_* =: h_*^7\right) \in \operatorname{Hom}(\mathbb{C}[\mathbb{Z}], \mathcal{A}).$$

We want to use these morphisms to construct a morphism $\mathbb{C}[H_S^*][[t]] \to \mathcal{A}$, which will induce $\psi_{\mathcal{A}}$ in an object $(\mathcal{A}, \psi_{\mathcal{A}}) \in \mathbf{ob}(\underline{\operatorname{Hom}}_{\mathbb{C}[H_S^*][[t]]})$.
Recall from the proof of Lemma 5.58 that by the choice of marking Φ there is a pairing $Q \otimes H_S \to \mathbb{Z}$. This pairing has full rank, i.e. there is an isomorphism

$$H_S^* \cong Q \cong \mathit{span}_{\mathbb{Z}}(\alpha_0, \dots, \alpha_7).$$

Thus, we get an isomorphism

$$\zeta \colon \operatorname{Spf} \mathbb{C}[\mathbb{Z}\alpha_0 \times \ldots \times \mathbb{Z}\alpha_7][[t]] = \operatorname{Spf} \mathbb{C}[Q][[t]]] \longrightarrow \operatorname{Spf} \mathbb{C}[H_S^*][[t]].$$

Composition of ζ_* with $h_*^0, \dots, h_*^7$ yields a morphism

$$\mathbb{C}[H_S^*][[t]] \xrightarrow{\zeta_*} \mathbb{C}[\mathbb{Z}\alpha_0 \times \ldots \times \mathbb{Z}\alpha_7][[t]] \xrightarrow{(h_*^0, \dots, h_*^7)} \mathcal{A}.$$

We take the induced map to be $\psi_{\mathcal{A}} \colon \mathbb{C}[H_S^*][[t]] \to \mathcal{A}$. This yields an element $(\mathcal{A}, \psi_{\mathcal{A}}) \in \mathbf{ob}(\underline{\operatorname{Hom}}_{\mathbb{C}[H_S^*][[t]]})$.
We still have to show that $G(\mathcal{A}, \psi_{\mathcal{A}}) = (\mathfrak{D}, \chi_{\mathfrak{D}})$. Set

$$G(\mathcal{A}, \psi_{\mathcal{A}}) =: (\mathfrak{D}_G, \chi_{\mathfrak{D}}^G).$$

Recall that we associated with $(\mathcal{A}, \psi_{\mathcal{A}})$ the pullback $\mathfrak{D}_G := \mathfrak{C} \times_{\mathfrak{O}} \operatorname{Spf} \mathcal{A}$. Thus, it holds that $\mathfrak{D} = \mathfrak{D}_G$ by uniqueness of the pullback.
For all $\alpha \in Q$ it holds that

$$\left(\chi_{\mathfrak{D}}^G(\alpha)\right)_* = \psi_{\mathcal{A}} \circ (\chi_{\mathfrak{U}}(\alpha))_* = (\chi_{\mathfrak{D}}(\alpha))_* \circ \zeta_*|_{\operatorname{im}(\chi_{\mathfrak{U}}(\alpha))_*} \circ (\chi_{\mathfrak{U}}(\alpha))_* = (\chi_{\mathfrak{D}}(\alpha))_* .$$

The first part follows from the morphism (5.10) on page 137. The last equality follows because $(\chi_{\mathfrak{U}}(\alpha))_* : \mathbb{C}[\mathbb{Z}] \to \mathbb{C}[H_S^*][[t]]$ maps a generator in $\mathbb{Z}$ to $\langle \alpha, \cdot \rangle \in H_S^*$ and $\zeta_*|_{\operatorname{im}(\chi_{\mathfrak{U}}(\alpha))_*}$ maps $\langle \alpha, \cdot \rangle \in H_S^*$ back to a generator in $\mathbb{Z}\alpha$. So their composition yields the identity on $\mathbb{C}[\mathbb{Z}]$.
This finishes the proof. $\square$

Proposition 5.61. *The categories $\underline{\mathfrak{Hom}}_Q$ and $\underline{\mathrm{Hom}}_{\mathbb{C}[H_S^*][[t]]}$ are anti-equivalent.*

Proof. We have to show that the functor G is fully faithful. We start by showing that it is full.
Let $G(\mathcal{A}_1, \psi_{\mathcal{A}_1}) = (\mathfrak{D}_1, \chi_{\mathfrak{D}_1})$ and $G(\mathcal{A}_2, \psi_{\mathcal{A}_2}) = (\mathfrak{D}_2, \chi_{\mathfrak{D}_2})$. Let

$$\begin{tikzcd} \mathfrak{D}_1 \arrow[r, "\xi_{\mathfrak{D}}"] \arrow[d] & \mathfrak{D}_2 \arrow[d] \\ \mathfrak{S}_1 \arrow[r, "\xi_{\mathfrak{S}}"] & \mathfrak{S}_2 \end{tikzcd}$$

be a morphism in $\underline{\mathfrak{Hom}}_Q$. It yields a morphism $(\xi_{\mathfrak{S}})_* =: \psi\colon \mathcal{A}_2 \to \mathcal{A}_1$ on base rings.
We have to show that for corresponding morphisms $\psi_{\mathcal{A}_1}\colon \mathbb{C}[H_S^*][[t]] \to \mathcal{A}_1$ and $\psi_{\mathcal{A}_2}\colon \mathbb{C}[H_S^*][[t]] \to \mathcal{A}_2$ it holds that

$$\psi \circ \psi_{\mathcal{A}_2} = \psi_{\mathcal{A}_1}.$$

This follows from the compatibility of $\chi_{\mathfrak{D}_1}$ and $\chi_{\mathfrak{D}_2}$, i.e. from

$$\chi_{\mathfrak{D}_1} = \chi_{\mathfrak{D}_2} \circ \xi_{\mathfrak{S}}$$

with $(\xi_{\mathfrak{S}})_* = \psi$ and the fact that $\chi_{\mathfrak{D}_i}$ determines $\psi_{\mathcal{A}_i}$, $i = 1, 2$. Thus, we get that G is full.
Faithfulness follows as $\operatorname{Spf} \mathcal{A}_2 \to \operatorname{Spf} \mathcal{A}_1$ already fixes $\psi\colon \mathcal{A}_1 \to \mathcal{A}_2$ and the corresponding morphism on $\chi_{\mathfrak{D}}$ and $\chi_{\mathfrak{D}'}$ is uniquely determined. □

This discussion implies our main theorem:

Theorem 5.62. *The categories $\underline{\partial\mathfrak{P}}_9$ and $\underline{\mathfrak{Hom}}_Q$ are equivalent.*

Proof. This is an immediate consequence of Proposition 5.61 and Proposition 5.54. □

Intrinsic meaning of $\underline{\mathfrak{Hom}}_Q$

We want to give a more intrinsic picture of $\underline{\mathfrak{Hom}}_Q$ to explain why it corresponds to the categories $\underline{\mathrm{Hom}}_Q$ and $\underline{\mathbf{Hom}}_{\mathbf{Q}}$ from Theorem 1.39 and Theorem 2.24.
To this end, we want to have a closer look at $((\partial B, \mathscr{P}_\partial), \varphi|_{\partial B})$.

As $\partial B \cong \mathbb{S}^1$, the polyhedral decomposition $\mathscr{P}_\partial$ of ∂B consists of a cycle of intervals $I_i,\ i = 1, \ldots, n$, of integral length glued along their endpoints.
Denote these vertices by $v_1, \ldots, v_n$. We assume that they are labelled in a cyclical order and we use indices cyclically, i.e. for $i = n$ we set $v_{i+1} = v_1$. We assume that the interval I_i is given by $[v_i, v_{i+1}]$.
The multi-valued piecewise linear function $\varphi|_{\partial B}$ endows each vertex v_i with a kink $\kappa_i \in \mathbb{N}$.

Lemma 5.63. *Recall that we constructed classes of exceptional curves in* $\mathrm{H}_2(\mathrm{X_t}, \mathbb{Z})$ *using tropical cycles on* $(B, \mathcal{P})$ *where* X_t *denotes a fibre of an analytic extension of a toric degeneration of* dP_9 *surfaces. Assume that* $\mathcal{C} \to T$ *is the restricted analytic extension of the toric degeneration of the boundary divisor of this toric degeneration.*
Let β *denote such a tropical cycle corresponding to an exceptional class* $E_t \in \mathrm{H}_2(\mathrm{X_t}, \mathbb{Z})$*. Intersection with* $\mathcal{C}$ *yields a unique section* $\sigma\colon T \to \mathcal{C}$.

Proof. Note that exceptional curves wihtin a degeneration of dP_9 surfaces deform uniquely. Moreover, the exceptional class E_t is monodromy invariant and therefore, it can be viewed as the restriction of a class $\mathcal{E} \in \mathrm{H}_2(\mathcal{X}, \mathbb{Z})$.
The intersection point $E_t \cap C_t$ with the fixed anticanonical divisor is fixed as well within each fibre. These points yield a curve within the total space $\mathcal{C}$ and thus a unique section $\sigma\colon T \to \mathcal{C}$. $\square$

Lemma 5.64. *Let as above* $\mathfrak{C} \to \mathfrak{O}^\dagger$ *be a toric degeneration of elliptic curves with central fibre* C *and analytic extension* $\mathcal{C} \to T$ *which is the restriction of the analytic extension of a toric degeneration of* dP_9 *surfaces. Let* Ω *denote the non-vanishing relative logarithmic volume form which was introduced on page 73.*
Let $\sigma, \sigma'\colon T \to \mathcal{C}$ *be sections which are given by intersections of exceptional curves within the degeneration of* dP_9 *surfaces with the anticanonical divisor as in Lemma 5.63. Note that* $\sigma(0) =: p$ *and* $\sigma'(0) =: p'$ *are contained in* C^{reg}*. Let there be a path* $\gamma_t \subset C_t$ *for all* $t \in T$ *leading from* $\sigma'(t)$ *to* $\sigma(t)$*. Then* σ *and* σ' *or equivalently* p *and* p' *can be represented by elements in* $\mathbb{C}^* \times \mathbb{Z}$ *via integration over* γ_t *, i.e.*

$$(p, p') \mapsto \exp\left(-\int_{\gamma_t} \Omega\right) \mapsto c \cdot t^k \mapsto (c, k) \in \mathbb{C}^* \times \mathbb{Z}.$$

Here, k *denotes the sum over the kinks* κ_i *belonging to vertices* v_i *which have to be crossed when joining* p *and* p'*. Note that all connected components of the* C^{reg}

are isomorphic to $\mathbb{C}^*$ *in a fixed way by the framing induced by the toric structure. Thus, we can treat* p *and* p' *as points in* $\mathbb{C}^*$ *and fix the factor* $c := \frac{p'}{p}$*.*

Proof. Recall that the central fibre C is a Tate curve, i.e. it consists of a cycle of n projective lines $\mathbb{P}^1_0, \ldots, \mathbb{P}^1_{n-1}$. As these projective lines are given torically, they come with a framing. This means that we can assume that they are glued at points $0_l, \infty_l \in \mathbb{P}^1_l$, $l = 0, \ldots, n-1$. Recall that the multi-valued piecewise linear function $\varphi|_{\partial B}$ yields a kink $\kappa_l \in \mathbb{N}$ at each vertex $v_l \in \partial B$.
Let $p \in \mathbb{P}^1_i \subset C$ and $p' \in \mathbb{P}^1_j \subset C$. Without loss of generality, we can assume that $i \geq j$. Note that by the framing, we can fix $p, p' \in \mathbb{C}^*$. We define a map by

$$(p, p') \mapsto \begin{cases} (p'/p, \sum_{l=i+1}^{j} \kappa_l), & i \neq j \\ (p'/p, 0), & i = j. \end{cases}$$

We want to reconstruct this mapping via integration.
Recall that $p, p' \in C$ induce sections $\sigma, \sigma' \colon T \to \mathcal{C}$ in a unique way. For all $t \in T$ let $\gamma_t \colon [0,1] \to C_t$ denote a path in C_t such that $\gamma_t(0) = \sigma'(t)$ and $\gamma_t(1) = \sigma(t)$. We want to integrate fibrewise over $\gamma_t \subset C_t$ and take the exponential of this integral. Let Ω denote a relative logarithmic volume form on $\mathcal{C}$. We want to argue that by the treatment of the Tate curve in [RS], we get

$$\exp\left(-\int_{\gamma_t} \Omega\right) = \frac{p'}{p} \cdot t^{\sum_{l=i+1}^{j} \kappa_l}.$$

Note that as $p, p' \in C^{reg}$ and each component of C^{reg} can be embedded into $\mathbb{C}^*$ in a fixed way by the existence of the framing, the fraction in our formula does make sense.
As in [RS, p. 9], let $r_t \colon C_t \to C$ denote an retraction map and $\pi \colon C \to \partial B \cong \mathbb{S}^1$ the moment map. Without loss of generality, we can choose r_t such that $r_t(\sigma(t)) = p$ and $r_t(\sigma'(t)) = p'$ to simplify our picture. Both, p and p' are mapped to certain points in ∂B, which we will also denote by p and p' by abuse of notation. Note that $\pi(\gamma_0)$ yields a section of ∂B which joins p and p' in ∂B.
We choose a family of open sets $\mathcal{W}$ in ∂B which are pairwise disjoint and whose closures cover ∂B. The preimages of open sets $U \in \mathcal{W}$ can be intersected with γ_0 and γ_t to induce open patches on γ_0 and γ_t respectively. It holds that

$$\int_{\gamma_t} \Omega = \sum_{U \in \mathcal{W}} \int_{(r_t \circ \pi)^{-1}(U)} \Omega.$$

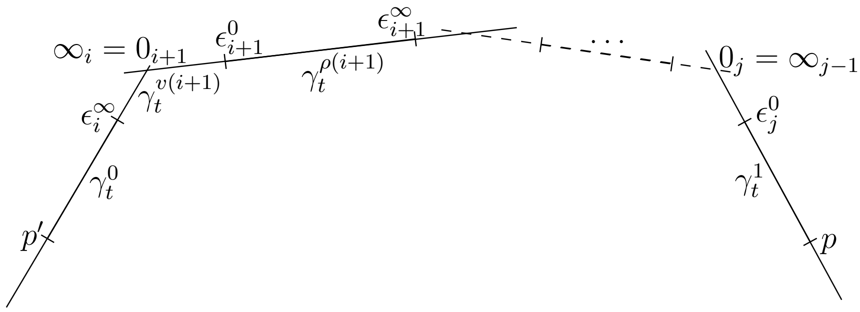

Figure 5.2.: Splitting of γ_0

The induced patching of γ_0 can be viewed as follows: We split γ_0 into $2(j-i)+1$ parts. Let $\epsilon > 0$ and for a point $p_l \in \mathbb{P}^1_l$ let $B_\epsilon(p_l) \subset \mathbb{P}^1_l$ be a sphere of radius ϵ around p_l.
Denote by γ_0^0 the part of γ_0 leading from p' to $\gamma_0 \cap B_\epsilon(\infty_i) =: \epsilon_i^\infty$.
Denote by $\gamma_0^{v(l+1)}$ the part of γ_0 from $\gamma_0 \cap B_\epsilon(\infty_l) =: \epsilon_l^\infty$ to $\gamma_0 \cap B_\epsilon(0_{l+1}) =: \epsilon^0_{l+1}$ and by $\gamma_0^{\rho l}$ the part of γ_0 from $\gamma_0 \cap B_\epsilon(0_l) =: \epsilon_l^0$ to $\gamma_0 \cap B_\epsilon(\infty_l) =: \epsilon_l^\infty$ for $l \in i+1, \dots, j$.
Moreover, let γ_0^1 be the part of γ_0 from $\gamma_0 \cap B_\epsilon(0_j) := \epsilon_j^0$ to p.
In Figure 5.2, we give a sketch idea of the splitting of γ_0 and our notation.
Adopting conventions and arguments from [RS, p. 10], we get

$$\int_{\gamma_t^{v(l+1)}} \Omega = \int_{(\epsilon_l^\infty)^{-1} t^{\kappa_l}}^{\epsilon^0_{l+1}} \frac{du}{u} = \log \epsilon_l^\infty + \log \epsilon^0_{l+1} - \log t^{\kappa_l} \text{ and}$$
$$\int_{\gamma_t^{\rho l}} \Omega = \int_{\epsilon_l^0}^{(\epsilon_l^\infty)^{-1}} \frac{du}{u} = -\log \epsilon_l^0 - \log \epsilon_l^\infty.$$

By the same reasoning, we compute

$$\int_{\gamma_t^0} \Omega = \int_{p'}^{(\epsilon_i^\infty)^{-1}} \frac{du}{u} = -\log p' - \log \epsilon_i^\infty \text{ and}$$
$$\int_{\gamma_t^1} \Omega = \int_{\epsilon_j^0}^{p} \frac{du}{u} = \log p - \log \epsilon_j^0.$$

These formulas imply

$$\int_{\gamma_t} \Omega = \int_{\gamma_t^0} \Omega + \sum_{l=i+1}^{j} \left(\int_{\gamma_t^{vl}} \Omega + \int_{\gamma_t^{\rho l}} \Omega \right) + \int_{\gamma_t^1} \Omega$$
$$= -\log p' + \sum_{l=i+1}^{j} \log t^{\kappa_l} + \log p.$$

Taking exponentials and a minus sign yields the desired formula.
We then map

$$\left(p'/p \cdot t^{\sum_{l=i+1}^{j} \kappa_l}\right) \mapsto (p'/p, \sum_{l=i+1}^{j} \kappa_l) \in \mathbb{C}^* \times \mathbb{Z}.$$

$\square$

This observation makes the connection with the integrals we computed for the Torelli theorem on surfaces.

Recall that we can associate with an object $(\mathfrak{D} \to \mathfrak{S}, \chi_{\mathfrak{D}}) \in \mathbf{ob}(\underline{\mathfrak{Hom}}_Q)$ an object $(\mathcal{A}, \psi_{\mathcal{A}}) \in \mathbf{ob}(\underline{\mathrm{Hom}}_{\mathbb{C}[H_S^*][[t]]})$. The morphism $\chi_{\mathfrak{D}}(\alpha)\colon \mathfrak{S} \to \mathbb{C}^*$ maps a point $s \in \mathfrak{S}$ to the pullback of the $\mathbb{C}^*$-part in the formula of Lemma 5.64 under the map $\mathfrak{S} \to T$, which is induced by $\psi_{\mathcal{A}}$ and the projection map $\mathfrak{U} \to T$.
This holds because roots $\alpha \in Q$ correspond to differences of distinct tropical cycles ϵ, ϵ', which generate exceptional curves. On smooth fibres, the proof of Proposition 1.31 computes exactly the same integrals as Lemma 5.64 but using the Residue Theorem.

Remark 5.65. In Theorem 1.39 and Theorem 2.24, we did not fix a marking for the category of homomorphisms from Q into some space related with an elliptic curve.
We can circumvent fixing Φ by using the fact that we get an induced geometric marking on smooth fibres. By Lemma 5.46, a geometric marking on smooth fibres induces a marking Φ on $((B, \mathscr{P}), \varphi)$.
The induced marking on smooth fibres is fixed by $\chi_{\mathfrak{D}}$ and $\langle \cdot, c_1(\varphi) \rangle$ as the integral from Lemma 5.64, maps into the Jacobian variety of the anticanonical divisor and the situation is the same as for Theorem 1.39. There, the map of Q into the Jacobian variety was enough to reconstruct a geometric marking.

A. Appendix

A.1. Matrices from Chapter 4

We will use the shorthand notation

$$\mathbf{1} := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \ \mathbf{0} := \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \ \vec{e}_1 := \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \ \vec{e}_2 := \begin{pmatrix} 0 \\ 1 \end{pmatrix} \text{ and } \vec{0} := \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The map $\partial^0 \colon C^0(\mathcal{W}, \iota_* \check{\Lambda}) \to C^1(\mathcal{W}, \iota_* \check{\Lambda})$ is given by a (192×62)-matrix with non-zero blocks distributed as follows:

$$\partial^0 = \begin{pmatrix} v & C_1 & D_1 & 0 & 0 & 0 & 0 \\ v & 0 & C_2 & D_2 & 0 & 0 & 0 \\ v & 0 & 0 & C_3 & D_3 & 0 & 0 \\ v & 0 & 0 & 0 & C_4 & D_4 & 0 \\ v & 0 & 0 & 0 & 0 & C_5 & D_5 \\ v & D_6 & 0 & 0 & 0 & 0 & C_6 \end{pmatrix}.$$

Here, v denotes a (32×2) matrix while C_i and D_i are (32×10) matrices. The matrices v, C_i and D_i are given below.

$$
\left(v|C_i|D_i\right) =
\begin{array}{cc}
\check{\Lambda}|_U & \begin{array}{c|cccccc|cccccc} v_0 & v_i & \rho_i & \mu_i & \omega_i & \sigma_i & \tau_i & v_{i+1} & \rho_{i+1} & \mu_{i+1} & \omega_{i+1} & \sigma_{i+1} & \tau_{i+1} \end{array} \\
\begin{array}{c}
v_0\rho_i \\ v_0\sigma_i \\ v_i\rho_i \\ v_i\mu_i \\ v_i\omega_i \\ v_i\sigma_i \\ v_i\tau_i \\ \rho_i\sigma_i \\ \mu_i\tau_i \\ \omega_i\sigma_i \\ \omega_i\tau_i \\ v_{i+1}\omega_i \\ v_{i+1}\sigma_i \\ v_{i+1}\tau_i \\ \rho_{i+1}\sigma_i \\ \mu_{i+1}\tau_i
\end{array}
&
\left(\begin{array}{c|cccccc|cccccc}
\mathbf{-1} & \mathbf{0} & B_{i0}\cdot\vec{e}_1 & \mathbf{0} & \vec{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \vec{0} & \mathbf{0} & \vec{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{-1} & \mathbf{0} & \vec{0} & \mathbf{0} & \vec{0} & B_{i0} & \mathbf{0} & \mathbf{0} & \vec{0} & \mathbf{0} & \vec{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{-1} & \vec{e}_1 & \mathbf{0} & \vec{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \vec{0} & \mathbf{0} & \vec{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{-1} & \vec{0} & \mathbf{1} & \vec{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \vec{0} & \mathbf{0} & \vec{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{-1} & \vec{0} & \mathbf{0} & \vec{e}_2 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \vec{0} & \mathbf{0} & \vec{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{-1} & \vec{0} & \mathbf{0} & \vec{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \vec{0} & \mathbf{0} & \vec{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{-1} & \vec{0} & \mathbf{0} & \vec{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \vec{0} & \mathbf{0} & \vec{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & -\vec{e}_1 & \mathbf{0} & \vec{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \vec{0} & \mathbf{0} & \vec{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \vec{0} & \mathbf{-1} & \vec{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \vec{0} & \mathbf{0} & \vec{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \vec{0} & \mathbf{0} & -\vec{e}_2 & \mathbf{1} & \mathbf{0} & \mathbf{0} & \vec{0} & \mathbf{0} & \vec{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \vec{0} & \mathbf{0} & -\vec{e}_2 & \mathbf{0} & \mathbf{1} & \mathbf{0} & \vec{0} & \mathbf{0} & \vec{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \vec{0} & \mathbf{0} & \vec{e}_2 & \mathbf{0} & \mathbf{0} & \mathbf{-1} & \vec{0} & \mathbf{0} & \vec{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \vec{0} & \mathbf{0} & \vec{0} & B_{12} & \mathbf{0} & \mathbf{-1} & \vec{0} & \mathbf{0} & \vec{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \vec{0} & \mathbf{0} & \vec{0} & \mathbf{0} & \mathbf{1} & \mathbf{-1} & \vec{0} & \mathbf{0} & \vec{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \vec{0} & \mathbf{0} & \vec{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & -B_{21}\cdot\vec{e}_1 & \mathbf{0} & \vec{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \vec{0} & \mathbf{0} & \vec{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \vec{0} & \mathbf{-1} & \vec{0} & \mathbf{0} & \mathbf{0}
\end{array}\right)
\end{array}
$$

The map $\partial^1 \colon C^1(\mathcal{W}, \iota_*\check{\Lambda}) \to C^2(\mathcal{W}, \iota_*\check{\Lambda})$ is given by a (120×192)-matrix:

$$
\partial^1 = \begin{pmatrix}
F_1 & G_1 & 0 & 0 & 0 & 0 \\
0 & F_2 & G_2 & 0 & 0 & 0 \\
0 & 0 & F_3 & G_3 & 0 & 0 \\
0 & 0 & 0 & F_4 & G_4 & 0 \\
0 & 0 & 0 & 0 & F_5 & G_5 \\
G_6 & 0 & 0 & 0 & 0 & F_6
\end{pmatrix}
$$

Each block is a (20×32) matrix. The matrices F_i and G_{i-1} are given explicitly below with the same shorthand notation as before.

$$F_i = \begin{array}{c|cccccccccccccccc}
\check{\Lambda}|_U & v_0\rho_i & v_0\sigma_i & v_i\rho_i & v_i\mu_i & v_i\omega_i & v_i\sigma_i & v_i\tau_i & \rho_i\sigma_i & \mu_i\tau_i & \omega_i\sigma_i & \omega_i\tau_i & v_{i+1}\omega_i & v_{i+1}\sigma_i & v_{i+1}\tau_i & \rho_{i+1}\sigma_i & \mu_{i+1}\tau_i \\
\hline
v_0\rho_i\sigma_i & \mathbf{1} & \mathbf{-1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & B_{i0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
v_0\rho_{i+1}\sigma_i & \mathbf{0} & \mathbf{-1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & B_{i0} & \mathbf{0} \\
v_i\rho_i\sigma_i & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{-1} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
v_{i+1}\rho_{i+1}\sigma_i & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{-1} & \mathbf{0} & B_{12} & \mathbf{0} \\
v_i\omega_i\sigma_i & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{-1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
v_i\omega_i\tau_i & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{-1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
v_{i+1}\omega_i\sigma_i & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & B_{12} & \mathbf{0} & \mathbf{1} & \mathbf{-1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
v_{i+1}\omega_i\tau_i & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{1} & \mathbf{0} & \mathbf{-1} & \mathbf{0} & \mathbf{0} \\
v_i\mu_i\tau_i & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{-1} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
v_{i+1}\mu_{i+1}\tau_i & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{-1} & \mathbf{0} & \mathbf{1}
\end{array}$$

$$G_{i-1} = \begin{array}{c|cccccccccccccccc}
\check{\Lambda}|_U & v_0\rho_i & v_0\sigma_i & v_i\rho_i & v_i\mu_i & v_i\omega_i & v_i\sigma_i & v_i\tau_i & \rho_i\sigma_i & \mu_i\tau_i & \omega_i\sigma_i & \omega_i\tau_i & v_{i+1}\omega_i & v_{i+1}\sigma_i & v_{i+1}\tau_i & \rho_{i+1}\sigma_i & \mu_{i+1}\tau_i \\
\hline
v_0\rho_i\sigma_i & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
v_0\rho_{i+1}\sigma_i & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
v_i\rho_i\sigma_i & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
v_{i+1}\rho_{i+1}\sigma_i & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
v_i\omega_i\sigma_i & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
v_i\omega_i\tau_i & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
v_{i+1}\omega_i\sigma_i & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
v_{i+1}\omega_i\tau_i & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
v_i\mu_i\tau_i & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
v_{i+1}\mu_{i+1}\tau_i & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0}
\end{array}$$

A.2. Matrices from Lemma 5.17

We will denote natural numbers by $\mathtt{0}, \mathtt{1}, \mathtt{2}, \mathtt{3} \ldots \in \mathbb{N}$.
First, there is the (90×20)-matrix ∂^0

$$\partial^0 = \begin{pmatrix} w_1 & M_1 & N_1 & 0 & 0 & 0 & 0 \\ w_2 & 0 & M_2 & N_2 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\ w_6 & N_6 & 0 & 0 & 0 & 0 & M_6 \end{pmatrix}.$$

The components w_i, M_i and N_i are a (15×2)- and two (15×3)-matrices. They are given by

$$\langle w_i|M_i|N_i\rangle = \begin{array}{r@{}c} & \begin{array}{ccccccc} U_0 & U_i & W_i & V_i & U_{i+1} & W_{i+1} & V_{i+1} \end{array} \\ \begin{array}{r} U_0U_i \\ U_0W_i \\ (U_0V_i)_1 \\ (U_0V_i)_2 \\ (U_0V_i)_3 \\ U_iV_i \\ W_iV_i \\ V_iV_{i+1} \end{array} & \left(\begin{array}{c|ccc|ccc} \mathbf{-1} & B_{i0}\cdot\vec{e}_1 & \vec{0} & \vec{0} & \vec{0} & \vec{0} & \vec{0} \\ \mathbf{-1} & \vec{0} & B_{i0}\cdot\vec{e}_2 & \vec{0} & \vec{0} & \vec{0} & \vec{0} \\ -B_{0i-1} & \vec{0} & \vec{0} & \vec{e}_2 & \vec{0} & \vec{0} & \vec{0} \\ -B_{0i} & \vec{0} & \vec{0} & \vec{0} & \vec{e}_2 & \vec{0} & \vec{0} \\ -B_{0i} & \vec{0} & \vec{0} & \vec{0} & \vec{e}_2 & \vec{0} & \vec{0} \\ \mathbf{0} & -\vec{e}_1 & \vec{0} & \vec{e}_2 & \vec{0} & \vec{0} & \vec{0} \\ \mathbf{0} & \vec{0} & -\vec{e}_2 & \vec{e}_2 & \vec{0} & \vec{0} & \vec{0} \\ \vec{0}^T & 0 & 0 & -1 & 0 & 0 & 1 \end{array}\right) \end{array}.$$

We get a (60×90)-matrix ∂^1 with non-zero components distributed as follows:

$$\partial^1 = \begin{pmatrix} P_1 & Q_1 & 0 & 0 & 0 & 0 \\ 0 & P_2 & Q_2 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ Q_6 & 0 & 0 & 0 & 0 & P_6 \end{pmatrix}$$

The components P_i and Q_i are

$$P_i = \begin{array}{r@{}c} & \begin{array}{cccccccc} U_0U_i & U_0W_i & (U_0V_i)_1 & (U_0V_i)_2 & (U_0V_i)_3 & U_iV_i & W_iV_i & V_iV_{i+1} \end{array} \\ \begin{array}{r} (U_0U_iV_i)_1 \\ (U_0U_iV_i)_2 \\ (U_0W_iV_i)_1 \\ (U_0W_iV_i)_2 \\ U_0V_iV_{i+1} \end{array} & \left(\begin{array}{cccccccc} B_{0i-1} & \mathbf{0} & \mathbf{-1} & \mathbf{0} & \mathbf{0} & B_{21} & \mathbf{0} & \vec{0} \\ B_{0i} & \mathbf{0} & \mathbf{0} & \mathbf{-1} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \vec{0} \\ \mathbf{0} & B_{0i} & \mathbf{0} & \mathbf{-1} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \vec{0} \\ \mathbf{0} & B_{0i} & \mathbf{0} & \mathbf{0} & \mathbf{-1} & \mathbf{0} & B_{12} & \vec{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \vec{e}_2 \end{array}\right) \end{array}$$

and

$$Q_i = \begin{array}{r@{}c} & \begin{array}{cccccccc} U_0U_{i+1} & U_0W_{i+1} & (U_0V_{i+1})_1 & (U_0V_{i+1})_2 & (U_0V_{i+1})_3 & U_{i+1}V_{i+1} & W_{i+1}V_{i+1} & V_{i+1}V_{i+2} \end{array} \\ \begin{array}{r} (U_0U_iV_i)_1 \\ (U_0U_iV_i)_2 \\ (U_0W_iV_i)_1 \\ (U_0W_iV_i)_2 \\ U_0V_iV_{i+1} \end{array} & \left(\begin{array}{cccccccc} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \vec{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \vec{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \vec{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \vec{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{-1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \vec{0} \end{array}\right) \end{array}.$$

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Abstract

This thesis consists of two parts. It starts with a known Torelli theorem, which holds for pairs (X, D) of a rational elliptic surface X with a section and irreducible fibres and a fixed smooth anticanonical divisor $D \subset X$. The surface X is called a 'dP$_9$ surface'. Moreover, the surface X carries a 'geometric marking', i.e. a fixed isomorphism ϕ between a lattice $\Lambda_{1,9}$ of signature $(1, 9)$ and the Picard group $\mathrm{Pic}(X)$. The category of such triples $((X, D), \phi)$ is equivalent to the category of homomorphisms from an E_8 lattice Q into the Jacobi variety $\mathrm{Pic}^0(D)$ of some smooth elliptic curve D. The homomorphism associated with a triple $((X, D), \phi)$ is determined by certain periods on the anticanonical divisor D.
This Torelli theorem is generalized to the case of a smooth projective family $\mathcal{X} \to S$ with an effective divisor $\mathcal{X} \supset \mathcal{D} \to S$ over an affine, Noetherian scheme S. The geometric fibres of $(\mathcal{X}, \mathcal{D}) \to S$ are given by pairs of a dP$_9$ surface and a smooth anticanonical divisor. Moreover, $\mathcal{X}$ is endowed with an analogue of the the geometric marking, i.e. there is a fixed isomorphism of sheaves between the constant sheaf $(\Lambda_{1,9})_S$ with stalk $\Lambda_{1,9}$ and the sheaf $R^1\pi_*\mathcal{O}^*_{\mathcal{X}}$, which is represented by the relative Picard scheme of $\mathcal{X}/S$. It is shown that the category of such families with generalized marking is equivalent to the category whose objects are pairs $(\mathcal{D}/S, \chi_{\mathcal{D}/S})$ of a smooth projective family of elliptic curves $\mathcal{D} \to S$ with a section over an affine Noetherian scheme S and a homomorphism $\chi_{\mathcal{D}/S} \colon Q \to \mathrm{Hom}(S, \mathbf{Pic}^0_{\mathcal{D}/S})$. Here, Q denotes an E_8 lattice and $\mathbf{Pic}^0_{\mathcal{D}/S}$ is the connected component of the identity of the relative Picard scheme of $\mathcal{D} \to S$.
In the second part of this thesis, we first consider an example of a toric degeneration of dP$_9$ surfaces in detail. In particular, the periods used in the original Torelli theorem are computed explicitly. Based on these computations the Torelli theorem from the first part of this thesis is generalized to toric degenerations of dP$_9$ surfaces. Assuming that there exists an analytic log smooth versal divisorial deformation of a torically degenerated dP$_9$ surface we show that the associated

toric degeneration is a completion of this deformation along the central fibre.
After fixing some affine data, we prove that the category of toric degenerations of dP_9 surfaces $(\mathfrak{X}, \mathfrak{D}) \to \mathfrak{S}$ is equivalent to the category whose objects are pairs of a toric degeneration of elliptic curves $\mathfrak{D} \to \mathfrak{S}$ and a homomorphism $\chi_{\mathfrak{D}} \colon Q \to \mathrm{Hom}(\mathfrak{S}, \mathbb{C}^*)$.

Earlier publications derived from this dissertation: –

Zusammenfassung

Die vorliegende Arbeit besteht aus zwei Teilen. Ausgangspunkt ist ein bereits bekanntes Torelli Theorem. Dieses Theorem gilt für Paare (X, D) bestehend aus einer rationalen elliptischen Fläche X mit einem Schnitt und irreduziblen Fasern, einer sogenannten dP_9-Fläche, und einem fixierten glatten antikanonischen Divisor D. Darüber hinaus trägt die dP_9-Fläche X eine „geometrische Markierung", d.h. es gibt einen fest gewählten Isomorphismus ϕ zwischen einem Gitter $\Lambda_{1,9}$ der Signatur $(1, 9)$ und der Picardgruppe $\mathrm{Pic}(X)$. Die Kategorie solcher Tripel $((X, D), \phi)$ ist äquivalent zu der Kategorie, deren Objekte durch Homomorphismen von einem E_8-Gitter Q in die Jacobi Varietät $\mathrm{Pic}^0(D)$ einer glatten elliptischen Kurve D gegeben sind. Der einem Tripel $((X, D), \phi)$ zugeordnete Homomorphismus wird durch Perioden auf dem antikanonischen Divisor bestimmt.
Dieses Torelli Theorem wird in der vorliegenden Arbeit auf den Fall einer glatten, projektiven Familie $\mathcal{X} \to S$ mit effektivem Divisor $\mathcal{X} \supset \mathcal{D} \to S$ über einem affinen, Noetherschen Schema S verallgemeinert, deren geometrische Fasern aus Paaren einer dP_9 Fläche mit glattem antikanonischen Divisor bestehen. Diese Familien werdem mit einem Analogon der geometrischen Markierung versehen, einem Isomorphismus zwischen der konstanten Garbe $(\Lambda_{1,9})_S$ mit Halm $\Lambda_{1,9}$ und der Garbe $R^1\pi_*\mathcal{O}^*_{\mathcal{X}}$, die durch das relative Picardschema repräsentiert wird. Es wird gezeigt, dass die Kategorie, deren Objekte solche Familien mit verallgemeinerter Markierung sind, äquivalent ist zu der Kategorie deren Objekte durch Paare $(\mathcal{D}/S, \chi_{\mathcal{D}/S})$ einer glatten, projektiven Familie elliptischer Kurven $\mathcal{D} \to S$ mit einem Schnitt über einem affinen Noetherschen Schema S und einem Homomorphismus $\chi_{\mathcal{D}/S} \colon Q \to \mathrm{Hom}(S, \mathbf{Pic}^0_{\mathcal{D}/S})$ gegeben ist. Dabei bezeichnet Q ein E_8-Gitter und $\mathbf{Pic}^0_{\mathcal{D}/S}$ die Zusammenhangskomponente der Identität des relativen Picardschemas von $\mathcal{D} \to S$.
Im zweiten Teil der Arbeit wird erst ein Beispiel einer torischen Entartung von dP_9 Flächen im Detail betrachtet. Dabei werden insbesondere die Perioden, die im

ursprünglichen Torelli Theorem auftauchen, explizit berechnet. Darauf aufbauend wird das Torelli Theorem aus dem ersten Teil der Arbeit auf torische Entartungen von dP_9 Flächen verallgemeinert. Unter der Annahme, dass eine analytische, log glatte, verselle divisoriale Deformation einer torisch entarteten dP_9 Fläche existiert, wird gezeigt, dass die zugehörige torische Entartung eine Vervollständigung dieser Deformation an der zentralen Faser darstellt.
Nachdem gewisse affine Daten fixiert wurden, wird bewiesen, dass die Kategorie torischer Entartungen von dP_9-Flächen $(\mathfrak{X}, \mathfrak{D}) \to \mathfrak{S}$ äquivalent zu der Kategorie ist, deren Objekte als Paare einer torischen Entartung elliptischer Kurven $\mathfrak{D} \to \mathfrak{S}$ und eines Homomorphismus $\chi_{\mathfrak{D}} \colon Q \to \mathrm{Hom}(\mathfrak{S}, \mathbb{C}^*)$ gegeben sind.
Aus dieser Dissertation hervorgegangene Vorveröffentlichungen: –

Eidesstattliche Versicherung

Hiermit erkläre ich an Eides statt, dass ich die vorliegende Dissertationsschrift selbst verfasst und keine anderen als die angegebenen Quellen und Hilfsmittel benutzt habe.

Hamburg, den ____________________ ____________________

Unterschrift