The KSBA compactification of the moduli space of degree 2 K3 pairs: a toroidal interpretation

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vorgelegt von
Carsten Liese
aus Marburg

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Vorsitzender der Prüfungskommission: Jens Struckmeier

Erstgutachter: Bernd Siebert

Zweitgutachter: Paul Hacking

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Hiermit erkläre ich an Eides statt, dass ich die vorliegende Diessertationsschrift selbst verfasst und keine anderen als die angegebenen Quellen und Hilfsmittel benutzt habe.

I hereby declare, on oath, that I have written the present dissertation on my own and have not used other than the acknowledged resources and aids.

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Dedicated to my family.

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Abstract

The aim of the current thesis a comparison of the KSBA and the GHKS partial compactification of the moduli space of K3 pairs of degree 2. We check a conjecture of Sean Keel that predicts the boundary of these spaces to be the same. We first calculate the required input data, the Dolgachev-Nikulin-Voisin mirror family, and the show that indeed all surfaces that appear in the boundary of the KSBA compactification appear in the GHKS compactification.

Abstract

Ziel der vorliegenden Arbeit ist der Vergleich zweier (partieller) Kompaktifizierungen des Raumes stabiler K3 Paare des Grades 2. Wir überprüfen eine Vermutung Sean Keels, die besagt, dass beide Kompaktifizierungen übereinstimmen. Dazu berechnen wir projektive Modelle der Dolgachev-Nikulin-Voisin Spiegelfamilie und zeigen dann, dass alle entarteten K3 Flächen der KSBA Kompaktifizierung in der GHKS Kompaktifizierung auftreten.

Contents

Introduction

K3 surfaces are a fundamental class of compact complex surfaces [BHPVdV15] with applications in compactifications of string theories and in particular on string dualities [Asp96]. As complex manifolds they form one family (of dimension 20) and, in particular, they are mutually all diffeomorphic. Not every K3 surface is algebraic, that is, can be defined by polynomial equations. Those that can are *projective*, that is, they can be defined by homogeneous polynomials as a subset $X \subset \mathbb{CP}^n$ of some complex projective space. The intersection with a general hyperplane $H \subset \mathbb{CP}^n$ defines a smooth complex curve $C = H \cap X$ in X. The pair consisting of X together with the class of C in $H^2(X, \mathbb{Z})$ (or the corresponding holomorphic line bundle $L = \mathcal{O}(C)$) is called a *polarized K3 surface.* The genus of C as a closed surface is then called the genus of the polarized K3 surface X. For example, a hyperplane section of a quartic surface $X \subset \mathbb{CP}^3$ is a plane curve of degree 4, which has genus 3.

For any genus $g \geq 2$ polarized K3 surfaces of genus g also form one connected family, each of dimension 19. There is a theory describing this family as a quotient of an open set in some homogeneous space by a discrete (arithmetic) group. In modern terms, one describes this quotient as a moduli space (stack), denoted $\mathscr{F}_g$. Somewhat more generally, one considers pseudo-polarized K3 surfaces, i.e. K3 surfaces with a polarization given by a nef and big bundle. Instead of considering line bundles, one can also look at the pair (X, C), which leads to a moduli space $\mathscr{P}_g$ of dimension $19 + g$.

One fundamental feature of these moduli spaces is that they are non-compact. The non-compactness arises by the fact that holomorphic families of polarized K3 surfaces, say over the punctured unit disk $D^* = \{z \in \mathbb{C} \,|\, 0 < |z| < 1\}$, typically can not be extended over the origin as a smooth family. Rather, the fibre over the origin will be some kind of singular K3 surface. By work of Kulikov and Persson-Pinkham [Kul77],[PP81], the candidate singular K3 surfaces have been known for a long time. It is also well-understood that the extension is not unique, it can be modified by a sequence of explicit operations on the central fibre.

For many applications, and also because the moduli spaces $\mathscr{F}_g$ are natural objects of study by themselves, it would be desirable to compactify the moduli space $\mathscr{F}_g$ in such a fashion that the universal family of (now singular, in general) $K3$ surfaces also extends. Such compactifications are referred to as *modular*. For $\mathscr{P}_g$ a modular compactification is the KSBA compactification (after the work of Kollár and Shepherd-Barron [KSB88] and Alexeev [Ale96], see also [Laz12]), which is natural from the point of view of the minimal model program. Both for $\mathscr{F}$ and $\mathscr{P}_g$ there are many other compactifications known, for instance the Bailey-Borel compactification or toroidal compactifications, but none of these are known to be modular.

However, there is a distinguished toroidal partial compactification. Hacking and Keel suggested in 2007 to use mirror symmetry to build a modular toroidal compactification. The details are joint work with Gross and Siebert, whose mirror [GS11] construction provides the technical core of this modular compactification, see [GHKS]. The project is currently nearing a certain first finished form [GHKS]. In this form the compactification is only partial (over the so-called type-III locus, the remaining type-II locus may be amenable to ad hoc methods), and while the family of K3 surfaces indeed extends, it is not clear how to characterize the occurring singular K3 surfaces intrinsically.

The GHKS construction produces a $(19+g)$-dimensional family $\mathfrak{X} \to \bar{\mathbb{P}}^g$ locally of pairs (X, C), which is then cut down to 19 dimensions. Sean Keel conjectured and sketched a proof that the $(19+g)$-dimensional family indeed describes the KSBA compactification. This conjecture is the motivation for our work.

For K3 surfaces with pseudo ample polarization of degree 2, Laza [Laz12] constructs the KSBA compactification. He also gives an explicit description of the (generic) limit surfaces appearing in the components of the type III boundary of the KSBA compactification. While the GHKS construction is not yet available in full generality in the genus 2 case, it can be constructed over the type III locus. We show that over this locus, all (generic) limit surfaces of the KSBA compactification appear - for a precise statement, see below.

One crucial input of the GHKS construction is the Dolgachev-Nikulin-Voisin mirror family [Dol96]. While the general form of this family is certainly known to experts, we give a construction and show that its Mori fan is finite.

We now give a more detailed overview of the thesis with precise results.

In chapter 1 we review different approaches to compactifying the moduli space of smooth complex $K3$ surfaces. We define a moduli functor of $K3$ with ADE singularities and show that it defines a Deligne-Mumford moduli stack F_g. This is, of course, all well known to experts; nevertheless, we spell out the argument. Also, we briefly review the construction of the coarse moduli spaces as period domains together with the standard compactifications and mirror symmetry of these domains. We recall relevant aspects of the KSBA compactification of the moduli space of degree 2 K3 pairs.

In chapter 2, we give a brief review of the main ingredients of the construction of [GHKS].

In chapter 3 we will, following ideas of Paul Hacking, identify models of the Dolgachev-Nikulin-Voisin mirror family .

In chapter 4, we proof that all generic limit surfaces of components of the type III boundary of the KSBA compactification occur as fibres of the GHKS family $\bar{\mathbb{P}}^g$.

Theorem 0.0.1. *Let $\mathfrak{X}$ be the GHKS family over the type III boundary stratum. Then all generic limit surfaces of the type III boundary of the KSBA compactification $\bar{\mathrm{P}}_2$ appear as fibres over strata Z of* $\mathrm{Morifan}(\mathcal{Y})$.

We will also show how to obtain the type II limit surfaces.

Theorem 0.0.2. *Let $\mathfrak{X}$ be the GHKS family over the type III boundary stratum. Then the generic limit surfaces of type II_i, $i = 1 \dots 4$ of the KSBA compactification $\bar{\mathrm{P}}_2$ appear as fibres over strata Z of* $\mathrm{Morifan}(\mathcal{Y})$.

We remark that the missing boundary components of type II should also be fibres of $\mathfrak{X}$. The missing ingredient are smoothings of n-vertices to elliptic singularities.

Chapter 1

Moduli Spaces of K3 surfaces

Introduction

In this section we review aspects of the moduli theory of K3 surfaces. We begin with the smooth case, i.e. smooth K3 surfaces with polarization and then move on to K3 surfaces with ADE singularities. We show that the moduli stack of such K3 surfaces is a Deligne-Mumford stack. Then we introduce the moduli space of degree d K3 pairs P_d and also show it is a Deligne-Mumford stack. For moduli of smooth K3 surfaces, we follow the excellent reference [Huy]. There are several other excellent surveys available. The article [Laz16] is a comprehensive exposition of different approaches to the compactification problem in algebraic geometry. Compactifications of locally symmetric varieties are reviewed in [Loo03]. Also, there is an exhaustive treatment in [BJ06].

1.1 Moduli stacks of K3 surfaces

The moduli functor $\mathcal{M}_g$ of smooth K3 surfaces over a noetherian base S is defined by the mapping

$$(\mathrm{Sch}/S)^{op} \to (\mathrm{Sets}), T \mapsto (f\colon X \to T, \mathcal{L})$$

sending a scheme T of finite type over S to a smooth proper family $X \to T$ together with a line bundle $\mathcal{L} \in \mathrm{Pic}(X/T)$ such that the geometric fibres X_s are polarized K3 surfaces with polarization $\mathcal{L}_{|X_s}$ that squares to $2g-2$. Here we define to such families to be equivalent if there exists an T-isomorphism $\psi\colon X \to X'$ and a line bundle $\mathcal{L}_0$ on T such that $\psi^*\mathcal{L}' \cong \mathcal{L} \otimes f^*\mathcal{L}_0$.

For an S-morphism $h\colon T' \to T$, we define $\mathcal{M}_g(T) \to \mathcal{M}_g(T')$ as the pullback

$$(X \to T, \mathcal{L}) \mapsto (X \times_T T' \to T', h_X^*\mathcal{L}).$$

This defines the moduli functor of smooth polarized K3 surfaces of degree g. The moduli functor $\mathcal{M}_g$ can also be viewed as a category fibred in groupoids. We have the following theorem:

Theorem 1.1.1. *Over a noetherian base S, $\mathcal{M}_g$ is a Deligne-Mumford stack. It is coarsly represented by an algebraic space M_g.*

Proof. The proof is implicit in [DM69] and can be found in [Riz05]. The statement about coarse representation is [KM97]. □

Over the complex numbers there is the following theorem by Pyatetskii-Shapiro and Shafarevich [PSS71].

Theorem 1.1.2. *Over $\mathbb{C}$, the moduli stack $\mathcal{M}_g$ is coarsly represented by a quasi-projective variety.*

We now change perspective from smooth K3 surfaces to $K3$ surfaces with ADE singularities, i.e. normal complete surfaces X over $\mathbb{C}$ with trivial canonical bundle, $H^1(X, \mathcal{O}_X) = 0$ and at most ADE singularities. Similarly to the smooth case, we obtain a moduli functor $\mathcal{F}_g$. As a first step towards showing that $\mathcal{F}_g$ is a Deligne-Mumford stack, we show the following.

Proposition 1.1.3. *Let $(X, \mathcal{L})$ be polarized K3 surface with ADE singularities. Then the group scheme $Aut(X, \mathcal{L})$ is reduced and finite.*

Proof. First, ADE singularities are isolated singularities. Hence there is an open cover $\{U_y\}_{y \in S}$, where S denotes the set of ADE singularites and U_y contains exactly one singular point y. As X is quasi-compact this means that X has only finitely many singular points. Then, as all singularities are rational double points, the minimal resolution of $\pi \colon Y \to X$ is a smooth surface Y with $H^1(Y, \mathcal{O}_Y) = H^1(X, \mathcal{O}_X) = 0$ and trivial canonical bundle, cf [[Dol12], 8.1.2], i.e. a K3 surface. The pullback of L is big and nef, making Y into a pseudo polarized K3 surface. Every automorphism of X preserving the polarization lifts to a unique automorphism of Y by the universal property of the blow up. Also, the induced morphism preserves the pseudo polarization, so there is a map

$$\mathrm{Aut}(X, L) \to \mathrm{Aut}(Y, \pi^* L).$$

This map is injective, as two maps $X \to X$ inducing the same morphism $Y \to Y$ agree on an open dense subset of X, and thus on all of X, because X is separated. Hence the automorphism group of X is finite, as $\mathrm{Aut}(Y, \pi^* L)$ is. As we work over $\mathbb{C}$, there are no infinitesimal automorphisms, so $\mathrm{Aut}(X, \mathcal{L})$ is reduced. □

Note that the result also follows from separatedness, boundedness and locally closedness of the moduli functor.

Following the theory outlined in [Vie95], we prove that the moduli functor $\mathcal{F}_g$ defines an algebraic stack which we will also denote by $\mathcal{F}_g$. This will be a Deligne-Mumford stack due to the finite automorphism groups.

Proposition 1.1.4. *The fibered category defined by $\mathcal{F}_g$ is Deligne-Mumford stack (in the étale topology). It is coarsly represented by an algebraic space F_g.*

Proof. We show that the functor F_g is separated, locally closed and bounded. By the general theory of moduli functors, this implies that the fibered category F_g is the quotient stack $[H/G]$ with H a subscheme of a Hilbert scheme ang G an algebraic group.

First, we show F_g is separated. We do this in the larger class of normal surfaces with rational singularities, as in [Kol85]. Following the proof of Corollary 13.24 in [HK11] but replacing the theorem of Mumford-Matsuka by Proposition 3.3.1 in [Kol85], it follows that the moduli functor for this larger class is separated. Hence it is also separated for the smaller class of K3 surfaces with ADE singularities. Also, Theorem 2.1.2 in [Kol85] shows that F_g is bounded. By Lemma 1.18 in [Vie95], the moduli functor $\mathfrak{M}$ of normal polarized surfaces with ADE singularities is open. Hence it is locally closed. The geometric condition to be a K3 is also locally closed. Hence F_g is locally closed. This shows that F_g is the algebraic stack given by a quotient of some subscheme H of some Hilbertscheme by an algebraic group, see [[Vie95], p.295]. By Proposition 1.1.3, F_g is a Deligne-Mumford stack, compare [[Edi00], section 2].

The statement on the coarse representation follows once we show that the action of G on H is proper and has finite stabilizers. Following the proof of Lemma 7.6 in [Vie95], we apply the valuative criterion for properness. So let T be a DVR, K its quotient field, $S = \operatorname{Spec} T$ and $U = \operatorname{Spec} K$. Suppose there is a commutative diagram

$$\begin{tikzcd} U \arrow[r, "\delta_0"] \arrow[d] & \mathbb{P}G \times H \arrow[d, "\bar{\psi}"] \\ S \arrow[r, "\delta"] & H \times H. \end{tikzcd}$$

We need to find a morphism $\delta' \colon S \to \mathbb{P}G \times H$ such that the diagram commutes. Let $(f_i \colon X_i \to S, \mathcal{L}_i, \phi_i \colon \mathbb{P}(f_{i*}\mathcal{L}^\mu) \cong \mathbb{P}^N \times S)$, $i = 1, 2$ be two families from $\mathfrak{H}(S)$ obtained by pulling back the universal family by $\mathrm{pr}_i \circ \delta$, with μ the number such that $\mathcal{L}^\mu$ is very ample from the boundedness of F_g and ϕ_i the isomorphisms corresponding to the embeddings $X_i \to \mathbb{P}^N$ given by the bundles $\mathcal{L}_i^\mu$. By commutativity of the diagram, the restriction to U of the families are isomorphic. Then by separatedness of F_g, $(f_1, \mathcal{L}_1)$ is isomorphic to $(f_2, \mathcal{L}_2)$.

This means there is an isomorphism $\tau\colon X_1 \to X_2$ and an isomorphism $\theta\colon \mathcal{L}_1 \cong \tau^*\mathcal{L}_2$. As τ is an S-isomorphism, we have $f_{1*}\mathcal{L}_1 \cong f_{2*}\mathcal{L}_2^{\mu}$. Hence there is an isomorphism

$$\theta'\colon \mathbb{P}(f_{1*}\mathcal{L}_1^{\mu}) \cong \mathbb{P}(f_{2*}\mathcal{L}_2^{\mu}).$$

Set $\gamma = \phi_2 \circ \theta' \circ \phi_1^{-1}$. As in [Vie95], the lifting of δ is now given by $(\gamma, \mathrm{pr}_2 \circ\delta)$. □

One can employ the methods of Viehweg to obtain the following result.

Proposition 1.1.5. *The coarse moduli space F_g is a quasi-projective scheme.*

Proof. This follows from Theorem 8.23 in [Vie95] using the functor $\mathcal{F}_g$ with index 1 and considering the Hilbert polynomial as element of $\mathbb{Q}[T_1, T_2]$. □

1.2 Periods and Mirror Symmetry

In the previous section we have seen that there are moduli stacks $\mathcal{M}_g$ and $\mathcal{F}_g$ of smooth polarized K3 surfaces and polarized K3 surfaces with ADE singularities. They both admit coarse moduli spaces M_g and F_g that - over $\mathbb{C}$ - are quasi-projective varieties. In this section we will review the theory of period domains for K3 to surfaces show that the moduli spaces M_g and F_g can be understood in these terms. As the period spaces are locally symmetric varieties, they admit the Baily-Borel compactification and toroidal compactifications of [AMRT10]. We follow the exposition in [Dol96].

Let Λ be a lattice, i.e. a finitely generated abelian group with bilinear form $(,)$ of signature (n_+, n_-). As the bilinear form is non-degenerate, its zero locus in $\mathbb{P}(\Lambda_{\mathbb{C}})$ is smooth and we define an open subset D (in the classical topology) by

$$D := \{x \in \mathbb{P}(\Lambda_{\mathbb{C}}) : (x)^2 = 0 \text{ and } (x, \bar{x}) > 0\} \subset \mathbb{P}(\Lambda_{\mathbb{C}}).$$

This is called the period domain associated with Λ.

Let $H^2(X, \mathbb{Z})$ be the integral cohomology of a K3 surface X. The quadratic form induced by the cap-product gives $H^2(X, \mathbb{Z})$ the structure of a lattice with signature $(3, 19)$. More precisely, on shows that

$$\mathrm{H}^2(X, \mathbb{Z}) \cong U^{\oplus 3} \oplus E_8^{\oplus 2}.$$

Here, U is the hyperbolic plane and E_8 the lattice defined by the negative of the Cartan matrix of a E_8 root system. We define the K3 lattice $\Lambda_{K3} = U^{\oplus 3} \oplus E_8^{\oplus 2}$.

Also, define

$$V(X) = \{x \in H^{1,1}_{\mathbb{R}}(X) \cap H^2(X, \mathbb{R} : x^2 > 0\}.$$

This cone has two connected components. Let $V^+(X)$ denote he component that contains the class of a Kähler form. Moreover, set

$$\triangle(X) = \{\delta \in \mathrm{Pic}(X) : (\delta, \delta) = -2\}$$

and let $\triangle(X)^+$ denote the effective classes in $\triangle(X)$. Define

$$C(X) = \{x \in V(X)^+ : (x, \delta) \geq 0 \forall \delta \in \triangle(X)^+\},$$

and let $C(X)^+$ be the set of interior points. The we set

$$\mathrm{Pic}(X)^+ = C(X) \cap H^2(X, \mathbb{Z}), \qquad \mathrm{Pic}(X)^{++} = C(X)^+ \cap H^2(X, \mathbb{Z}).$$

Note that the elements of these spaces are pseudo-ample and ample divisor classes respectively.

For a given lattice M of signature $(1, t)$, we similarly have the cone $V(M)$ of square positive elements of $M_{\mathbb{R}}$. We fix one coomponents and denote it by $V(M)^+$. Again, $\triangle(M)$ denotes the square -2 elements and we fix a subset $\triangle(M)^+$ of positive roots. This defines

$$C(M)^+ = \{h \in V(M)^+ : (h, \delta) > 0 \quad \forall \delta \in \triangle(M)^+\}.$$

We have the following notion

Definition 1.2.1. An M-polarized K3 surface is a pair (X, j) where X is a K3 surface and $j \colon M \to \mathrm{Pic}(X)$ is a primitive lattice embedding. An M-polarized K3 surface X is pseudo ample if $j(C(M)^+) \cap \mathrm{Pic}(X)^+ \neq \emptyset$ and ample if $j(C(M)^+) \cap \mathrm{Pic}(X)^{++} \neq \emptyset$.

For a lattice M with embedding into Λ_{K3} as above, we set $N = M_{\Lambda_{K3}^{\perp}}$. There is the following definition.

Definition 1.2.2. A marked M-polarized K3 surface is a pair (X, θ) where X is a K3 surface and $\theta \colon H^2(X, \mathbb{Z}) \cong \Lambda_{K3}$ an isomorphism of lattices such that $\theta(M)^{-1} \subset \mathrm{Pic}(X)$.

Defining $j_0 = \phi^{-1}_{|M}$, a marked M-polarized K3 surface X is a M-polarized K3 surface. Define

$$D_M := \{x \in \mathbb{P}(\Lambda_{\mathbb{C}}) : (x)^2 = 0, (x, \bar{x}) > 0\} \subset \mathbb{P}(N_{\mathbb{C}}).$$

By the local Torelli theorem, there is a holomorphic map $S \to D_M$ that is locally an isomorphism - the period map - where S is the local moduli space of a K3 surface X. Let K_M be the fine moduli space of K3 surfaces with a polarization as above. The local moduli spaces can be glued together to a holomorphic map $K_M \to D_M$. By Theorem 3.1 of [Dol96], the restriction to of this map to the space of K3 surfaces with pseudo-ample polarization as above remains surjective. Let $\triangle := \{x \in M^{\perp} : (x)^2 = -2\}$. For any $\delta \in \triangle$, set $H_\delta = \{z \in M^{\perp}_{\mathbb{C}} : (z, \delta) = 0\}$. Then define

$$D^{\circ}_M = D_M \backslash (\cup_{\delta \in \triangle} H_\delta \cap D_M)$$

. Let Γ denote the orthogonal transformations of Λ_{K3} fixing M. Suppose also that M is such that any two embeddings $M \to \Lambda_{K3}$ differ by an isometry. We have the following results, cf [Dol96]

Theorem 1.2.3. *Let $\mathbf{K}_l$ be the space of isomorphism classes of pseudo-ample M-polarized K3 surfaces X. Then $\mathbf{K}_M$ is a quasi-projective variety and*

$$\mathbf{K}_M \cong \Gamma \backslash D_M.$$

Also, $\mathbf{K}_M \cong F_g$, i.e. $\mathbf{K}_M$ is a coarse moduli space for pseudo-ample polarized K3 surfaces.

For ample M-polarizations, one similarly has the following.

Theorem 1.2.4. *Let $\mathbf{K}^a_M$ be the space of isomorphism classes ample M- polarized K3 surfaces. Then $\mathbf{K}^a_M$ is a quasi-projective variety and*

$$\mathbf{K}^a_M \cong \Gamma \backslash D^{\circ}_M.$$

Also, $\mathbf{K}^a_M \cong F_g$, i.e. $\mathbf{K}^a_M$ is a coarse moduli space for ample polarized K3 surfaces.

We are interested in degree 2 case. Thus we choose $M = \langle 2 \rangle = \mathbb{Z}l$ with $(l, l) = 2$. Then the embedding $M \to \Lambda_{K3}$ is unique by [[Dol96], Cor. 5.2] and thus the moduli spaces from above exist. Also, for this choice of polarization, both moduli spaces are 19 dimensional, see [[Dol96], Cor. 5.2].

Our next aim is to define a mirror family for a family of M-polarized K3 surfaces. We will define a mirror lattice $\check{M}$. Then we obtain the mirror moduli space $\mathbf{K}_{\check{M}}$, and for a given (complete) family of pseudo-ample M- polarized K3 surfaces, any (complete) family of pseudo-ample $\check{M}$ polarized K3 surfaces will be called a mirror family. Of course, this construction fits into the picture of mirror symmetry for Calabi-Yau threefolds, see [Dol96]. We do not repeat these general notions here as we are only interested in the construction of mirror family and not its properties. Also, we only consider the degree 2 case.

For a primitive isotropic vector f in a non-degenerate even lattice S , let div f denote the positive generator of the image of the induced map $S \to \mathbb{Z}$.

Definition 1.2.5. A primitive isotropic vector $f \in S$ is called m-admissible if div $f = m$ and there is a primitive isotropic vector g with $(f, g) = m$ and div $g = m$.

Given an m-admissible isotropic vector f in $M^\perp$, set

$$\mathbb{Z}f^\perp_{M^\perp}/\mathbb{Z}f = \check{M}$$

.

Definition 1.2.6. The moduli space $\mathbf{K}_{\check{M}}$ is called the mirror moduli space of $\mathbf{K}_M$.

Let S be the local deformation space of a $\check{M}$-polarized K3 surface X. There is a family of $\check{M}$ polarized K3 surfaces $X_0 \to S$ and the composition with the period map $X \to \mathbf{K}_{\check{M}}$ gives a family over the mirror moduli space. In particular, this means that such a family has $\mathrm{Pic}(X_s) = \check{M}$ for each fibre X_s.

In the next section, we will recall the Baily-Borel compactification. With this in mind, we define

Definition 1.2.7. The mirror family $X \to \mathbf{K}_{\check{M}}$ over a pointed neighbourhood U of the unique 1-cusp - i.e. U does not contain the cusp - will be called the Dolgachev-Nikulin-Voisin mirror family.

To conclude, for $g = 2$, we find a 1-admissible vector f and then

$$\check{M} = U \oplus E_8^2 \oplus \langle -2 \rangle$$

with $\langle -2 \rangle$ the lattice generated by an element of square -2, see [[Dol96], §7].

1.3 Compactifications

We have seen that the moduli space of pseudo-polarized K3 surfaces can be identified with an arithmetic quotient of the period domain $D := D_M$, where $M = \langle 2 \rangle$. The period domain D can be represented as a Grassmannian and we note that the quotient $\Gamma \backslash D$ is a locally symmetric variety: a connected component D^+ of the Grassmanian is a bounded Hermitian domain of type IV and taking the quotient by the subset Γ' of isometries that preserve the component yields a representation

$$\Gamma \backslash D \cong \Gamma' \backslash D^+$$

as locally symmetric variety. As such, it has a canonical minimal compactification, the Baily-Borel compactification that is constructed as follows.

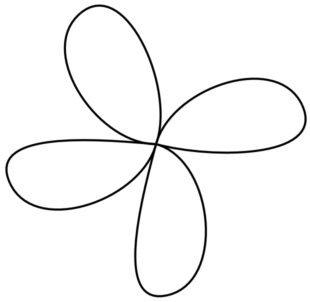

Figure 1.1: The boundary of the Baily-Borel compactification of $\mathcal{F}_2$

1.3.1 The Baily-Borel compactification

As a bounded Hermitian domain, D^+ has an embedding into its compact dual $\check{D}$. The closure of D^+ in $\check{D}$ decomposes into irreducible components, i.e. connected complex analytic submanifolds. Let G be the group of automorphisms of D^+. A boundary component is called rational if its G-stabilizer is defined over $\mathbb{Q}$. There is a topology on the set of boundary components due to Satake that was shown by Baily and Borel [BB66] to give a projective variety $(\Gamma\backslash D)^*$ that extends the analytic structure on $\Gamma\backslash D$. This is the Baily-Borel compactification. The boundary components are of high codimension, other than the space D^+ itself there are only 1-dimensional (type II components) or singletons (type III components). The number of boundary components has been calculated by Scattone in [Sca87, §6.1].

Theorem 1.3.1 ([Sca87])**.** *The boundary of* $(\Gamma\backslash D)^*$ *consists of* 4 *curves meeting in a single point.*

Also, note that the type II boundary components, i.e. the curves are in $1:1$ correspondence with rank 2 isotropic sublattices of Λ_{K3} modulo the action of Γ. In degree 2, the 4 type II components correspond to root lattices of type $2E_8 + A_1, E_7 + D_{10}, D_{16} + A_1$ and A_{17}.

While the Baily-Borel compactification is a canonical compactification, there is no modular interpretation known. Also, it is almost always singular. This was improved upon in the degree 2 case by the GIT compactification of Shah [Sha80].

1.3.2 Shah's moduli space

Note that by Mayer's theorem, [May72], degree two K3 surfaces have a special form: they come as double covers. More precisely, we distinguish two cases, cf [Laz12]

(NU) (non-unigonal case) $|H|$ is base point free, in which case X is a double cover of $\mathbb{P}^2$ branched along a sextic C with at worst ADE singularities.

(U) (unigonal case) $|H|$ has a base curve R. Then $H = 2E + R$ where E is elliptic and R smooth rational. The free part $2E$ maps X to a plane conic, and gives an elliptic fibration on X. On the other hand, $|2H|$ is basepoint free and maps X two-to-one onto the cone Σ_4^0 over the rational normal curve in $\mathbb{P}^4$. The map

$$X \to \Sigma_4^0$$

is ramified at the vertex and in a degree 12 curve B, which does not pass through the vertex. B has at most ADE singularities.

Recall the moduli space of plane sextic curves: it is the GIT quotient

$$\overline{\mathcal{M}} := \mathbb{P}H^0(\mathbb{P}^2, \mathcal{O}(6)^2_{\mathbb{P}}) /\!\!/ \mathrm{SL}(3).$$

By [Sha80], this GIT quotient has the following description

Theorem 1.3.2 ([Sha80]). *Let $\overline{\mathcal{M}}$ be the GIT quotient of plane sextics.*

(i) A sextic with ADE singularities is GIT stable. Thus there exists an open subset $\mathcal{M} \subset \overline{\mathcal{M}}$, which is a coarse moduli space for sextics with ADE singularities.

(ii) $\overline{\mathcal{M}} \backslash \mathcal{M}$ consists of 7 strata:

(type II) Z_1, Z_2, Z_3 and Z_4 with dimension $2, 1, 2$ and 1 respectively.

(type III) τ and ζ of dimension 1 and 0.

(type IV) a point ω.

(iii) The adjacencies of the boundary strata are as follows:

(a) $\zeta \in \overline{Z}_i$ for all i

(b) $\overline{\tau} \in \overline{Z}_1 \cap \overline{Z}_3$

(c) $\overline{\tau} = \tau \cup \{\zeta\} \cup \{\omega\}$.

By Meyer's theorem, item (i) means that $\mathcal{M}$ is a moduli space of non-unigonal degree 2 K3 surfaces. By [Sha80], the strata of the boundary $\overline{\mathcal{M}} \backslash \mathcal{M}$ correspond to certain GIT models, see Table 1.1.

All unigonal K3 surfaces are mapped to the point ω. To remedy this, [Sha80] constructs a moduli space $\hat{\mathcal{M}}$ by blowing up ω. The boundary strata of $\hat{\mathcal{M}}$ are the strict transforms of the $\overline{Z}_i$, we denote these by $\hat{Z}_i$. Shah's result is the following:

Stratum	Sextic curve
Z_1	$\Pi_{i=1}^3(x_0x_2 + a_ix_1^2), a_i \in \mathbb{C}$
Z_2	$x_2^2 f_4(x_0, x_1) = 0,\ f_4$ no multiple factors
Z_3	$(x_0x_2 + x_1^2)^2 f_2(x_0, x_1, x_2) = 0,$ induced quadrics intersect in 4 distinct points
Z_4	$f_3(x_0, x_1, x_2)^2 = 0,\ f_3$ smooth
τ	$(x_0x_2 + x_1^2)^2(x_0x_2 + ax_1^2) = 0,\ a \neq 0$
ζ	$x_0^2x_1^2x_2^2 = 0$
ω	$(x_0x_2 + x_1^2)^3 = 0$

Table 1.1: Boundary strata of $\overline{\mathcal{M}}$.

Theorem 1.3.3 ([Sha80])**.** *The blow up of $\overline{\mathcal{M}}$ in ω gives a projective compactification $\hat{\mathcal{M}}$ of the moduli space $\mathcal{F}_2$ of degree 2 $K3$ surfaces. The boundary strata of $\mathcal{F}_2 \subset \hat{\mathcal{M}}$ are the strict transforms of the boundary strata of $\overline{\mathcal{M}}$. Moreover, the boundary points of $\hat{\mathcal{M}}$ correspond to degenerations of $K3$ surfaces that are double covers of $\mathbb{P}^2$ or Σ_4^0 and have at worst slc singularities.*

The moduli space $\hat{\mathcal{M}}$ comes with a period map $\hat{\mathcal{M}} \to (\Gamma\backslash D)^*$. More precisely, there is the following Theorem due to Looijenga [Loo86].

Theorem 1.3.4. *There is an extension of open embeddings $F_2 \subset \hat{\mathcal{M}}$ and $F_2 \subset (\Gamma\backslash D)^*$ such that the following diagram commutes.*

$$\begin{tikzcd}
 & \hat{\mathcal{M}} \arrow[dl] \arrow[dr] & \\
\overline{\mathcal{M}} \arrow[rr, dashed] & & (\Gamma\backslash D)^*
\end{tikzcd}$$

The morphism $\hat{\mathcal{M}} \to \overline{\mathcal{M}}$ is Shah's blow-up and $\hat{\mathcal{M}} \to (\Gamma\backslash D)^*$ is the Looijenga modification associated to a certain hyperplane arrangement.

While the space $\hat{\mathcal{M}}$ from a certain perspective is an improvement over the Baily-Borel compactification, it still is not a modular compactification, being in a sense to degenerate. However, we will see presently that it will play a prominent role in the analysis of the KSBA compactification. A different attempt of improving the properties of the Baily-Borel compactification are the toroidal compactifications of Mumford et al. [AMRT10]. This is discussed in a later section.

1.3.3 The KSBA compactification

In this subsection we recall Lazas' KSBA compactification of K3 pairs of degree 2, [Laz12]. Instead of considering K3 surfaces with a polarization $(X, \mathcal{L})$ of degree d, one considers *K3 pairs* (X, L) of $K3$ surfaces X together with an ample divisors L of degreee d and instead of $\mathcal{F}_g$ considers the moduli stack $\mathcal{P}_g$ of such pairs. There is a forgetful functor

$$\mathcal{P}_g \to \mathcal{F}_g$$

that is smooth and proper and in particular $\mathcal{P}_g$ is a smooth Deligne-Mumford stack.

The limit objects are then the K3 stable pairs of degree d:

Definition 1.3.5. Let X be a surface, L an effective divisor on X and $d = 2g-2$ an even positive integer. The pair (X, L) is a *stable K3 pair of degree* d if

(i) X is Gorenstein with $\omega_X \cong \mathcal{O}_X$.

(ii) L is an ample Cartier divisor.

(iii) The pair $(X, \epsilon L)$ is semi-log canonical (slc) for all small $\epsilon > 0$.

(iv) There exists a flat deformation $(\mathcal{X}, \mathcal{L})/T$ of (X, L) over the germ of a smooth curve such that $\mathcal{L}$ is an effective relative Cartier divisor and such that the general fibre (X_t, L_t) is a degree d K3 pair.

The moduli stack of stable K3 pairs is denoted by $\bar{\mathcal{P}}_d$. The following result is due to Laza, [Laz12]:

Proposition 1.3.6. *The stack $\bar{\mathcal{P}}_d$ is a Deligne-Mumford stack. Its coarse moduli space $\bar{\mathcal{P}}_d$ is a proper algebraic space containing the moduli space of degree d K3 pairs and is a geometric compactification of $\mathcal{P}_d$.*

Due to results of Shepherd-Barron [SB83], the stable K3 pairs of degree d are given by log canonical models of central fibres of degenerations of K3 surfaces. For $g = 2$, the possible surfaces have been classified by [Tho10]. These are the surfaces that appear in the boundary of $\bar{\mathcal{P}}_2$.

Laza constructs a auxiliary compactification $\hat{\mathcal{P}}_2$ to study the KSBA compactification. This is done by means of a GIT quotient of sextic pairs, i.e. pairs (C, L) of sextic curves C together with a line L. One obtains a map

$$\hat{\mathcal{P}}_2 \to \hat{\mathcal{M}},$$

where the space $\hat{\mathcal{M}}$ is the compactification of [Sha80]. This means that at least outside the locus where the GIT semistable locus is replaced by KSBA stable pairs, one can use the GIT models of [Sha80] to understand the type II boundary components and thereby their limits, the type III boundary.

Theorem 1.3.7 ([Laz12], Thm 4.1). *The GIT quotient $\hat{\mathcal{P}}_2$ compactifies the moduli space of degree 2 pairs $\mathcal{P}_2$ and has the following properties:*

i) *$\hat{\mathcal{P}}_2$ has a natural forgetful map $\hat{\mathcal{P}}_2 \to \hat{\mathcal{M}}$;*

ii) *the GIT stable locus $\mathcal{P}_2^s \subset \hat{\mathcal{P}}_2$ is a moduli space of KSBA stable degree 2 pairs (X, H) such that X is a double cover of $\mathbb{P}^2$ (or Σ_4^0). Thus $\mathcal{P}_2^s$ is a common open subset of $\hat{\mathcal{P}}_2$ and $\overline{\mathcal{P}}$;*

iii) *the stricly semistable locus $\hat{\mathcal{P}}_2 \backslash \mathcal{P}_2^s$ is a surface $\tilde{Z}_1$ that maps one-to-one to the closure of the stratum $\hat{Z}_1 \subset \hat{\mathcal{M}}$.*

In the remainder of this section, we give a description of the type III boundary components and their generic points. Figure 1.2 shows the incidence relation and generic surfaces of the components.

The A_{17} stratum: $\mathrm{III}_\zeta, \mathrm{III}_\alpha$ and III_1

The Baily-Borel boundary component A_{17} is isomorphic to the stratum Z_4 of Shah. The latter is GIT stable and thus the map

$$\hat{\mathcal{P}}_2 \to \hat{\mathcal{M}}$$

is a $\mathbb{P}^2$ fibration over Z_4. The GIT models in the type II boundary over the A_{17} component are given by

$$z^2 = f_3(x_i)^2$$

where f_3 is a smooth cubic. The normalization is two copies of $\mathbb{P}^2$ with double curve $E = V(f_3)$. As the number of components in a degeneration can only go up and $\overline{Z}_4 = \{\zeta\} \cup Z_4$, the corresponding type III boundary component is the locus where the cubic becomes nodal. The deepest degeneration is III_ζ, where f_3 is a triangle. The locus III_α is where the generic point is such that $V(f_3)$ is reducible and finally for the generic point of III_1, $V(f_3)$ is irreducible. See Figure 1.2.

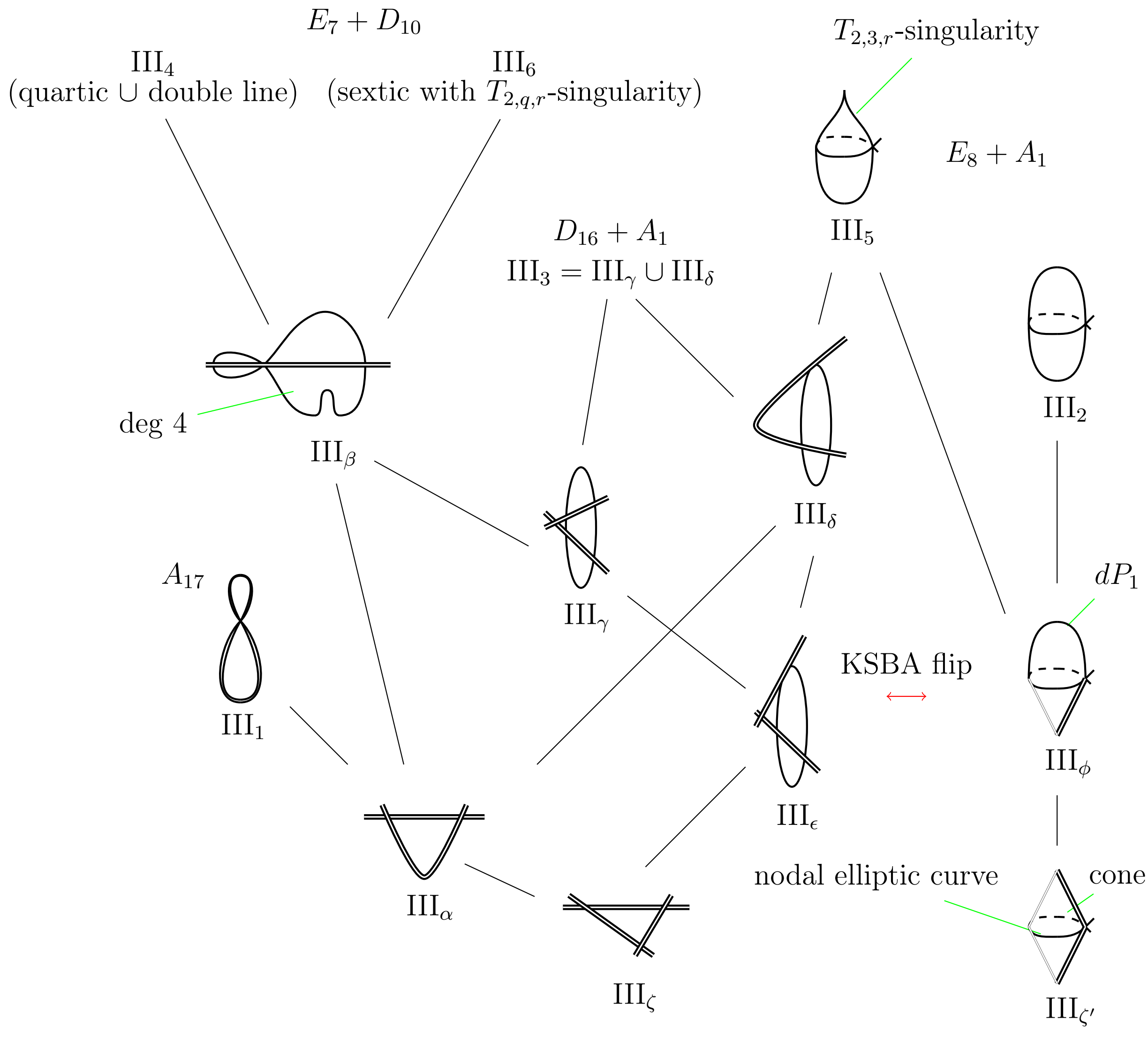

Figure 1.2: The type III boundary of the KSBA compactification.

The $E_7 + D_{10}$ stratum: $\mathrm{III}_\beta, \mathrm{III}_4$ **and** III_6.

To describe these surfaces, we first recall the surfaces in the corresponding loci of the type II boundary. Over the stratum Z_2, the space $\hat{\mathcal{P}}_2$ agrees with the KSBA compactification $\overline{\mathcal{P}}_2$. This is because after a choice of a polarizing divisor, all sextics C mapping to Z_2 are either ϵ-stable or ϵ-unstable[1]. Furthermore, over Z_2, ϵ-stability is the same as KSBA-stability. Thus by Theorem 1.3.7, both compactifications agree. Therefore, in a stable pair (C, L) of a sextic C with divisor L, the sextic is given by the GIT model of Table 1.1. Moreover, as a central fibre of a type II degeneration, it has to be of the form specified by the classification of [Tho10]. Hence, there are three geometric possibilities for the type II boundary over the $E_7 + D_{10}$ component, see [[Tho10], Table 4].

(i) C is a sextic containing a double line,

$$x_0^2 f_4(x_1, x_2) = 0, \quad f_4 \text{ smooth}.$$

After normalization, the double cover given by this sextic is a Del Pezzo surface of degree 2. The line gives the anticanonical section D.

(ii) The sextic C is reduced with unique $\tilde{E}_7$ singularity.

(iii) A sextic containing a double line and an $\tilde{E}_7$ singularity.

Letting $j \to \infty$ gives the type III models. By the matching of [Tho10] with the GIT analysis of [Sha80], one obtains, as $\zeta \subset \overline{Z}_2$, the limiting cases

III_4 The sextic of (ii) degenerates to a quartic with tangent line: $l^2(x_i) f_4(x_i)$, Thompson's model III.III.1.

III_6 The $\tilde{E}_7$ singularity degenerates to a cusp singularity $T_{2,q,r}$, $q \geq 4, r \geq 5$, Thompson's model III.0h.

Moreover, the above strata contain:

III_β The model of this stratum is a double cover of $\mathbb{P}^2$ branched in a nodal quartic with a double line passing through the node. This is a specialization of III.1.

III_γ The model here is a double cover of $\mathbb{P}^2$ branched along two double lines and a conic. The intersection of the lines gives a degenerate cusp singularity.

[1] This is one place where the choice of a divisor versus a line bundle is essential: Z_2 is stricly semi-stable, fixing a line bundle results in (ϵ-) strictly semi-stable points.

The $D_{16}+A_1$ stratum: III_3, III_ϵ **and** III_δ**.**

The preimage of the $D_{16}+A_1$ Baily Borel boundary component $\mathrm{II}_{D_{16}+A_1}$ in $\hat{\mathcal{M}}$ is given by the fibration

$$\hat{Z}_3 \backslash \hat{\tau} \to \mathrm{II}_{D_{16}+A_1},$$

where $\hat{Z}_3$ and $\hat{\tau}$ are the strict transforms of $\overline{Z}_3$ and $\overline{\tau}$. The points of $\hat{Z}_3\backslash\hat{\tau}$ are GIT stable and therefore give stable points in $\hat{\mathcal{P}}_2$ and therefore in $\overline{\mathcal{P}}_2$. The GIT model corresponding to this stratum are the sextics given by

$$(x_0x_2 + x_1^2)^2 f_2(x_0, x_1, x_2) = 0,$$

such that the quadric $(x_0x_2 + x_1^2)$ and f_2 intersect in 4 distinct points. The model matched is Thompson's $(II.2)$, i.e. the double cover

$$q^2(x_i)f_2(x_i) = z^2 \subset \mathbb{P}(1,1,1,3).$$

Here, q is a smooth quartic, f_2 is reduced and $|q \cap f_2| = 4$. The double curve is the double cover of $V(q)$ branched at the 4 intersection points. The type III limits are then given as follows.

(III_γ) As above, the model here is a double cover of $\mathbb{P}^2$ branched along two double lines and a generic quadric with a degenerate cusp singularity given by the intersection of the two lines.

(III_δ) The model is $z^2 = q_0^2 q$, with conics that are tangent.

These to strata form the component III_3, i.e $\mathrm{III}_3 = \mathrm{III}_\gamma \cup \mathrm{III}_\delta$. The intersection of the two components is the stratum given by

(III_ϵ) Double covers of $\mathbb{P}^2$ branched in double lines plus a conic that is tangent to one of the lines.

The E_8+A_1 stratum: $\mathrm{III}_{\zeta'}, \mathrm{III}_\phi, \mathrm{III}_5$ **and** III_2**.**

We look at the type II boundary components first. The GIT models are given by the models of the stratum $\hat{Z}_1$. Hence the GIT models are those corresponding to

$$\Pi_{i=1}^3(x_0x_2 + a_i x_1^2), a_i \in \mathbb{C}.$$

The corresponding geometric possibilities are sextic curves with one or two $\tilde{E}_8$ singularities.

Now, if the hyperplane section of the polarising divisor does not pass through the singularity, a pair (X, L) of this locus is GIT stable and thus KSBA stable. If it does pass through the singularity, the pair is GIT semistable and KSBA unstable. Hence one applies the ' KSBA flip ' and replaces the pair with a surface

$$X = V_1 \cup_E V_2$$

such that both components are degree 1 del Pezzos glued along an elliptic curve. This can further degenerate to cases where one or both components become elliptic ruled surfaces with $\tilde{E}_8$ singularity.

The type III limits are then as follows:

(III_2) The section along which the del Pezzos are glued becomes nodal.

(III_5) The $\tilde{E}_8$ degenerates to a $T_{2,3,r}$ singularity with $r \geq 7$.

(III_ϕ) One of the del Pezzo surfaces in III_2 degenerates to a conesover the nodal curve, i.e surfaces $X = X_1 \cup X_2$ with X_1 a del Pezzo of degree 1 and normalisation $X_0^\nu = \mathbb{P}^2$.

($\mathrm{III}_{\zeta'}$) Both del Pezzo surfaces degenerate to cones. This is the pillow surface given by the triangulation $\mathcal{T}$ depicted in Figure 3.1 in chapter 3.

1.3.4 Toroidal compactifications

For a locally symmetric variety, there may be very many toroidal compactifications. The crucial point is that, in contrast to the Baily-Borel compactification, these depend on choices. For moduli spaces of K3 surfaces, one has to choose a fan structure for each Type III boundary component. We follow the description of Looijenga [Loo03].

Recall that a Type III boundary component corresponds to a rank 1 isotropic sublattice in $M^\perp \subset \Lambda_{K3}$. Let $I \subset M^\perp$ be such a sublattice. Consider $(I^\perp/I)_\mathbb{R}$. The intersection form on $M^\perp$ descends and we let C denote the cone given by a connected component of square positive elements. Let C_+ denote the convex hull of $\bar{C} \cap I^\perp/I$. With $G \subset \mathrm{Aut}(I^\perp/I)$, we have the following notion, see [GHKS].

Definition 1.3.8. A Mumford-Looijenga fan $\triangle$ on C for G is a collection of strictly convex rational polyhedral cones σ in $(I^\perp/I)_\mathbb{R}$, closed under pairwise intersection and taking faces, such that

i. $g(\sigma) \in \triangle$ for all $g \in G, \sigma \in \triangle$,

ii. $|\triangle| := \cup_{\sigma\in\triangle}\sigma = C^{rc}$,

iii. There are only finitely many G orbits of cones in $\triangle$.

We have seen that in the degree 2 case, there is only one Type III boundary component. While in order to construct toroidal compactifications of general locally symmetric varieties, Mumford-Looijenga fans must be chosen for all boundary components in a compatible way, for K3 surfaces, one only needs to choose Mumford-Looijenga fans for the Type III boundary components. Hence, for the moduli space of K3 surfaces of degree 2 - or more generally, for degree $2n$ with n square free - the choice of a single Mumford-Looijenga fan determines a toroidal compactification. The following theorem is Looijenga's version of [AMRT10].

Theorem 1.3.9. *Let $\triangle$ be a Mumford-Looijenga fan for Γ. Then there is a normal analytic space $X^{\triangle}$ and a proper and birational morphism $X^{\triangle} \to (\Gamma\backslash D)^*$.*

In the degree 2 case we have $\Gamma = O^*(\check{M})$ and $C = V(\check{M})$, where $\check{M}$ is the mirror lattice from above and $O^*(\check{M})$ is the group of automorphisms of Λ_{K3} fixing l with l the generator of $M = \mathbb{Z}l$ with $l^2 = 2$. Thus, in order to construct a toroidal compactification, we need a Mumford-Looijenga fan on a connected component of the square positive cone $V(\check{M})$. Let $\mathcal{Y}$ be the Dolgachev-Nikulin-Voisin family as above. Then $\mathrm{Pic}(Y^{gen}) = \check{M}$, and $V(\check{M})$ is the connected component containing the ample cone of the generic fibre. Keel and Hacking suggested to use the Mori fan of the threefold $\mathcal{Y}$ to construct a toroidal compactification.

1.3.5 Mori fan and rational maps

We recall the Mori fan of a projective variety. See [KH00] and [GHKS] for more information. Also, Chapter 7 of [Dol12] contains an excellent review of some of the notions used here.

A *contraction* is a rational map between proper normal varieties such that the inverse rational map has no exceptional divisors.

Definition 1.3.10. A resolution of a rational map $f\colon X \to Y$ of projective varieties is a pair of regular projective morhisms $\pi\colon X' \to X$ and $\sigma\colon X' \to Y$ such that $f = \sigma \circ \pi^{-1}$ and π is an isomorphism over $\mathrm{dom}(f)$.

$$\begin{tikzcd} & X' \arrow[dl, "\pi"'] \arrow[dr, "\sigma"] & \\ X \arrow[rr, dashed, "f"] & & Y \end{tikzcd}$$

This allows us to define pullbacks for rational maps. Given a rational map $f\colon X \to Y$, resolve the map and for $L \in \mathrm{Pic}(Y)$ set $f^*(L) = \pi_*\sigma^*L \in A^1(X)$. This is independent of the resolution, cf Lemma 10.1 [GHKS].

Now, let $\mathcal{Y} \to \delta$ be a $\mathbb{Q}$-factorial, projective variety. Let $f\colon \mathcal{Y} \dashrightarrow Z$ be a rational map (and everything is assumed projective over δ). Let $N \subset \operatorname{Pic}(Z)_{\mathbb{R}}$ be the cone of positive real combinations of basepoint free divisors on Z. Then set

$$C(f) := \{f^*(N)\}.$$

Let $\operatorname{Morifan}(\mathcal{Y}) \subset \operatorname{Pic}(\mathcal{Y})_{\mathbb{R}}$ denote the collection of all cones $C(f)$ for all contractions $f\colon \mathcal{Y} \dashrightarrow Z$. For threefolds, this gives a well defined fan structure with support the Moving cone $\operatorname{Mov}(\mathcal{Y})$ of $\mathcal{Y}$.

Chapter 2

The GHKS family

Introduction

We review some aspects of the construction of Gross, Siebert, Hacking and Keel [GHKS]. There, the authors produce a (partial) toroidal compactification of F_g using the Morifan of the Dolgachev-Nikulin-Voisin family. The starting point of the GHKS construction is the Dolgachev-Nikulin-Voisin family. To a semi-stable model $\mathcal{Y} \to S$ of the DNV family, they associate an affine manifold B with singularities by taking the dual intersection complex of the central fibre Y and endowing it with an affine structure. For each cone in (a Mumford-Looijenga refinement δ of) Morifan($\mathcal{Y}$), they produce a family over a toric stratum of the toric variety $TV(\delta)$, by constructing a homogeneous coordinate ring. This construction - minus the scattering diagram- can also be found in [GHKS16]. The families are glued to a polarized family

$$(\mathfrak{X}, \mathcal{O}(1)) \to T\hat{V}(\delta)$$

over the completion of $TV(\delta)$ along the type III part of the toric boundary. They obtain a toroidal partial compactification $\bar{\mathbb{P}}^g$ of the moduli space of triples

$$\mathbb{P}(\pi_*(\mathcal{O}_{\mathcal{X}}(1)) = \{(S, \mathcal{O}(1), C) : (S, \mathcal{O}(1)) \in F_g, C \in |\mathcal{O}(1)|\}.$$

The family $(\mathfrak{X}, \mathcal{O}(1)) \to T\hat{V}(\delta)$ can be viewed as a family of triples as over a formal neighbourhood of the type III boundary, it comes with theta functions, i.e. a canonical basis of sections.

It can be shown that $(\mathfrak{X}, \mathcal{O}(1))$ glues to the universal family over $\mathbb{P}(\pi_*\mathcal{O}(1)) \to F_g$. Hence one obtains a family over the type III locus of stable pairs. One feature of the construction is that to each contraction of the DNV model $\mathcal{Y} \to S$, it associates a toric stratum in $TV(\delta)$ and the family is trivial over these strata. Hence, to each contraction

$\mathcal{Y} \to Z$, there is an associated degenerate K3 surface. The GHKS construction is not yet available for genus 2 in its full generality. However, this only concerns the gluing to the universal family over the interior of the moduli space. Over the type III boundary, it can be constructed for all degrees.

We now recall some of the ingredients of the construction. We will be brief and follow the excellent exposition [GHKS16]. We relate the construction of the K3 family to the Looijenga pairs of [GHK15a]. Appropriate Looijenga pairs give local description of the K3 family which we will later use to study strata. An example illustrating these concepts is given at the end of the section.

2.1 Affine manifolds, Wall Structures and Mirror families

Definition 2.1.1. An affine manifold B_0 is a differentiable manifold with an equivalence class of charts with transition functions in $\mathbb{R}^n \rtimes \mathrm{GL}(\mathbb{R}^n)$. If the transition functions are in $\mathbb{Z}^n \times \mathrm{GL}(\mathbb{R}^n)$, B_0 is integral. An affine manifold comes with sheaves $\Lambda, \check{\Lambda}$ of integral tangent and cotangent vectors.

We assume we have a space B and a decomposition $\mathscr{P}$ of B into integral polyhedra. Then $B_0 = B \backslash \triangle$ can be made into an affine manifold, with $\triangle$ a codimension 2 subset. A pair $(B, \mathscr{P})$ is a *polyhedral affine pseudomanifold.*

We also assume the existence of a $\mathscr{P}$-piecewise affine function ϕ on B_0 that takes values in the Grothendieck group Q^{gp} of some toric monoid Q. This means ϕ is, on each open set U in B_0, a continuous map

$$U \to Q^{gp}_{\mathbb{R}}$$

that restricts to a integral affine function on each maximal cell of $\mathscr{P}$. Such a function is *convex* if it takes values in Q.

Remark 2.1.2. One can show that to characterize such a function, it is enough specify the kinks on each ray of B, see [GHKS16, Definition 1.6].

Assume the data of a polyhedral pseudomanifold $(B, \mathscr{P})$ together with a convex piecewise affine function ϕ. The crucial input for the construction of [GHKS16] is the existence of a *wall structure* $\mathscr{S}$. This is a set of polynomial functions associated to rays on the affine manifold that take values in certain rings. We do not need the precise definitions here, these can be found in [GHKS16], Definition 2.11. The point of these structures is that if they fullfill an additional property - consistency - they induce a *theta functions*, which give a canonical basis of the (homogeneous) coordinate ring of the family defined by $(B, \mathscr{P})$.

We now indicate how this is used to produce the K3 mirror family.

Construction 2.1.3. Let $\mathcal{Y} \to S$ be a projective semi-stable model of the Dolgachev-Nikulin-Voisin family of degree 2. There is a cone σ in Morifan($\mathcal{Y}$) corresponding to $\mathcal{Y} \to S$, i.e. $\sigma = f^* \operatorname{Nef}(Y/S)$ for some marking f. By the Kulikov classification, the dual intersection complex of the central fibre $\mathcal{Y}_0$ is a triangulation $\mathscr{G}$ of the sphere $B = S^2$. In [GHKS, Construction 1.15], an affine structure is given, with $\triangle$ the vertices of $\mathscr{G}$, defining a pair $(B, \mathscr{P})$. Also, there is a $\mathscr{G}$-piecewise affine section ϕ that is defined by setting the change of slope on a edge e of $\mathscr{G}$ to be equal to the self-intersection of the component of the double locus of $\mathcal{Y}_0$ corresponding to e. This defines starting data for the construction of a mirror family over $TV(\operatorname{Nef}(Y))$. The remaining datum for the construction of a family $\mathcal{X} \to TV(\sigma)$ is a *scattering diagram* giving a wall structure, see[GHKS, §3]. Running this construction for every cone in the Morifan and completing along the type III boundary gives the family see [GHKS, Theorem 6.2]. Note that in the degree 2 situation, no refinement of the Mori fan is necessary, as it is a rational polyhedral fan.

2.2 Mirror families for log Calabi-Yau surfaces

In this section we outline the GHKS mirror theory in the situation of rational surfaces with anticanonical cycles.

Definition 2.2.1 ([GHK15a], Def 1.1)**.** A *Looijenga pair* is a smooth rational projective surface Y together with a reduced nodal curve $D \in |-K_Y|$ with at least one singular point.

In particular, the normalised components of any model of the Dolgachev-Nikulin-Voisin family are Looijenga pairs. The construction of [GHK15a] produces mirror families for Looijenga pairs. For technical reasons, one prefers working with pairs (Y, D) such that D has at least three components. The tool to obtain these are toric blow-ups.

Definition 2.2.2 ([GHK15a], Def 1.2)**.** Let (Y, D) be a Looijenga pair.

(i) A toric blow-up of (Y, D) is a birational morphism $\pi \colon \tilde{Y} \to Y$ such that if $\tilde{D}$ is the reduced scheme structure on $\pi^{-1}(D)$, then $(\tilde{Y}, \tilde{D})$ is a Looijenga pair.

(ii) A toric model of (Y, D) is a birational morphism $(Y, D) \to (\tilde{Y}, \tilde{D})$ to a smooth toric surface $\overline{Y}$ with toric boundary $\overline{D}$ such that $D \to \overline{D}$ is an isomorphism.

If (Y, D) is a Looijenga pair, blowing up a node of the anticanonical cycle is a toric blow-up. Indeed,

Lemma 2.2.3. *Let (Y, D) be a Looijenga pair.*

(i) Let p be a node of D. Then

$$(Bl_p(Y), D') \to (Y, D)$$

is a toric blow-up, where D' is the sum of the strict transforms of the components of D and the exceptional curve of the blow-up.

(ii) Let p be a smooth point meeting exactly one component of D. Then $(Bl_p(Y), D')$ is a Looijenga pair, where D' is the strict transform of D.

Proof. Let D_0, D_1 be the components of $D = \sum_i D_i$ with $p \in D_0 \cap D_1$. Write $X = \mathrm{Bl}_p(Y)$. Denote the blow up by $\pi \colon X \to Y$. Let E be the exceptional curve of the blow up, $\tilde{D}_i$ the strict transforms of the D_i. Then

$$K_X = \pi^* K_Y + E.$$

Hence

$$-K_X = -\pi^* K_Y - E = \pi^* \sum_i D_i - E = \sum_i \tilde{D}_i + 2E - E = D'.$$

For (ii), assume $p \in D_0$. Let E be the exceptional curve of the blow up. Then, similarly, with D' the sum of the strict transforms of the D_i

$$\begin{aligned} -K_X &= -\pi^* K_Y - E = \pi^* \sum_i D_i - E = \pi^* D_0 + \sum_{i>0} D_i - E \\ &= \tilde{D}_0 + E + \sum_{i>0} D_i - E = \sum_i \tilde{D}_i - E = D'. \end{aligned}$$

□

A Looijenga pair (Y, D), with $D = D_1 + \dots D_n$, induces an affine manifold $B :$ For each node $p_{i,i+1} = D_i \cap D_{i+1}$, let $M_{i,i+1}$ be the rank 2 lattice with basis v_i, v_{i+1} and the cone $\sigma_{i,i+1} = \mathrm{cone}(v_i, v_{i+1})$. We glue these along the rays $\rho_i = \mathbb{R} v_i$. This defines a manifold B with polyhedral decomposition Σ given by the cones. An integral affine structure is defined by charts

$$\psi_i \colon U_i := \mathrm{int}(\sigma_{i-1,i} \cup \sigma_{i,i+1}) \to \mathbb{Z}^2_{\mathbb{R}},$$

with

$$\psi(v_{i-1}) = (1, 0), \quad \psi_i(v_i) = (0, 1), \quad \psi_i(v_{i+1}) = (-1, -D_i^2)$$

on the closure of U_i and linear on the cones.

Construction 2.2.4. We relate the construction of the K3 mirror family to the Looijenga pairs in [GHK15a]. Consider the data $(B, \mathscr{G})$ as above, i.e. B the 2-sphere with triangulation $\mathscr{G}$ together with the convex piecewise affine function ϕ. By construction, a vertex y of $\mathscr{G}$ corresponds to a component $Y_0 \subset \mathcal{Y}_0$. Let Y denote the normalisation of Y_0. The double locus of $\mathcal{Y}_0$ defines an anticanonical cycle D and the pair (Y, D) is a Looijenga pair. The affine manifold with polyhedral decomposition $B_Y, \mathscr{G}_Y$ induced by the pair (Y, D) is the tangent wedge of B at y. The function ϕ induces a piecewise affine function $\bar{\phi}$, using the map $\mathrm{NE}(Y) \to \mathrm{NE}(\mathcal{Y})$. This is starting data for the construction of [GHK15a].

The remaing input is the *canonical scattering diagram* of a pair (Y, D). It is defined in [GHK15a, §3]. We recall the definition.

Definition 2.2.5. Let $\tilde{Y}, \tilde{D}$ be Looijenga pair. Let C be an irreducible component of $\tilde{D}$. Let $\beta \in A_1(\tilde{Y}, \mathbb{Z})$ be a class such that

$$\beta.\tilde{D}_i = \begin{cases} k_\beta & \tilde{D}_i = C \\ 0 & \tilde{D}_i \neq C \end{cases}$$

for some $k_\beta > 0$.

In the situation of the definiton, let F be the closure of $\tilde{D} \backslash C$ and set $\tilde{Y}^o = \tilde{Y} \backslash F$ and $C^o = C \backslash F$. There is a moduli space $\mathfrak{M}(\tilde{Y}^o / C^o)$ of stable relative maps of genus zero curves representing the class β with tangency order k_β at an unspecified point of C^o. It is proper over $\mathbb{C}$ [GHK15a, Lemma 3.2], and thus one defines

$$N_\beta = \int_{[\mathfrak{M}(\tilde{Y}^o/C^o)]^{vir}} 1.$$

Construction 2.2.6. [GHK15a, Definition 3.3] Let (Y, D) be a Looijenga pair with affine manifold B, fan Σ and piecewise affine function ϕ. Fix a ray $\mathfrak{d} \subset B$ with endpoint the origin. If $\mathfrak{d}$ coincides with with a ray of Σ, set $\Sigma' = \Sigma$. Otherwise, let Σ' be the refinement of Σ obtained by adding the ray δ and a number of other rays such that each cone of Σ' is integral affine isomorphic to the first quadrant of $\mathbb{R}^2$. By [GHK15a, Lemma 1.6], this gives a toric blow-up $\pi \colon \tilde{(Y)} \to Y$. Let $C \subset \pi^{-1}(D)$ be the irreducible component corresponding to $\mathfrak{d}$.

Let $\tau_\mathfrak{d} \in \Sigma$ be the smallest cone containing $\mathfrak{d}$. Let $m_\mathfrak{d} \in \Lambda_{\tau_\mathfrak{d}}$ be a primitive generator of the tangent space to $\mathfrak{d}$, pointing away from the origin. Define

$$f_\mathfrak{d} = \exp\left[\Sigma_\beta k_\beta N_\beta z^{\pi_*(\beta) - \phi_{\tau_\mathfrak{d}}(k_\beta m_\mathfrak{d})}\right].$$

Here, β runs over all classes as above and $\phi_\mathfrak{d}$ is the localisation of ϕ as in [GHK15a, Construction 2.2]

Definition 2.2.7. The collection $S_c(Y) := \{(\mathfrak{d}, f_{\mathfrak{d}}) | \mathfrak{d} \subset B$ is a ray of rational slope$\}$ is called the *canonical scattering diagram* of (Y, D).

Theorem 2.2.8. *[GHK15a, Thm 3.8] The canonicial scattering diagram is consistent.*

Remark 2.2.9. Scattering diagrams are only constructed for anticanonical cycles of length at least 3. To obtain scattering diagrams for shorter cycles, one resricts in an apprmedian way, see [GHK15a, §6.2].

Let (Y, D) be a Looijenga pair. The wall structure given by the canonical scattering diagram is consistent and hence one can construct a mirror family. For the statement of the result, we need the notion of an *n-vertex* $\mathbb{V}_n$. For $n \geq 3$, this is the reduced cyclic union of coordinate $\mathbf{A}^2$'s:

$$\mathbb{V}_n = \mathbf{A}^2_{x_1,x_2} \cup \mathbf{A}^2_{x_2,x_3} \cup \cdots \cup \mathbf{A}^2_{x_n,x_1} \subset \mathbf{A}^2_{x_1,\ldots x_n}.$$

We also define n-vertices for $n \leq 2$, by setting

$$\mathbb{V}_1 = \operatorname{Spec} \mathbb{C}[x, y, z]/(xyz - x^2 - z^3),$$

and

$$\mathbb{V}_2 = \operatorname{Spec} \mathbb{C}[x, y, z]/(y^2 - x^2 z^2).$$

Assume that the Looijenga pair (Y, D) is such that the cone of curves $\bar{\mathrm{NE}}(Y)$ is a rational polyhedral cone. The following result is a simplified version of [GHK15a, Theorem 0.1].

Theorem 2.2.10. *Let (Y, D) be a Looijenga pair with $\bar{\mathrm{NE}}(Y)$ rational polyhedral. Suppose $D = D_1 + \cdots + D_n$. Let $\mathfrak{m}$ be the maximal ideal of the monoid ring $R = \mathbb{C}[NE(Y)]$. There is a formal flat family*

$$\mathcal{X}_m \to \operatorname{Spf} \hat{R},$$

with $\hat{R}$ the completion with respect to $\mathfrak{m}$.

Now, let (Y, D) be the Looijenga pair defined in Construction 2.2.4. One has the canonical scattering diagram $S_c(Y)$. The scattering diagram produced in [GHKS] also induces a scattering diagram $S_{K3}(Y)$ on (Y, D). Up to the addition of certain rays - higher order *incoming* rays, see e.g [GHK15a, Defiinition 2.3] - these diagrams agree. It is expected that the scattering diagrams $S_c(Y)$ and $S_{K3}(Y)$ are equivalent and thus produce isomorphic families. We assume this in the following.

Assumption 2.2.11. *Let (Y, D) be a Looijenga pair obtained by localisation as in Construction 2.2.4. The scattering diagrams $S_c(Y)$ and $S_{K3}(Y)$ are equivalent.*

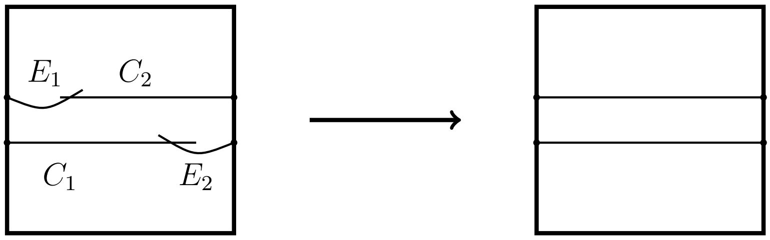

Figure 2.1: Blow-up of $\mathbb{P}^1 \times \mathbb{P}^1$

Let $(B, \mathscr{G}, \phi)$ be the combinatorial data of the $K3$ mirror family. The mirror family defined by a Looijenga pair (Y, D) obtained by localisation at a vertex of v of B together with $S_{K3}(Y)$ determines the deformation of the K3 mirror family locally around the n-vertex singularity defined by v. Under Assumption 2.2.11, we can study the deformation theory of

$$(\mathfrak{X}, \mathcal{O}(1)) \to TV(\widehat{\text{Morifan}}(\mathcal{Y}))$$

around its n-vertex singularities by studying the deformation theory of the corresponding Looijenga pairs in the framework of [GHK15a].

We close with an example.

Example 2.2.12. Take (Y, D) the blow up of $\mathbb{P}^1 \times \mathbb{P}^1$ in two points on opposing components of the toric boundary, see Figure 2.1. Here, $D = D_1 + D_2 + D_3 + D_4$ is an anticanonical cycle.

To this surface, we associate a affine pseudo manifold as follows, see [GHK15a]. For each node $p_{i,i+1} = D_i \cap D_{i+1}$, we define a rank two $M_{i,i+1}$ lattice with basis v_i, v_{i+1} and the cone $\sigma_{i,i+1} = \text{cone}(v_i, v_{i+1})$. We glue these along the rays $\rho_i = \mathbb{R}v_i$. This defines a manifold B with polyhedral decomposition Σ given by the cones. An integral affine structure is defined by charts

$$\psi_i \colon U_i := \text{int}(\sigma_{i-1,i} \cup \sigma_{i,i+1}) \to \mathbb{Z}^2_{\mathbb{R}},$$

with

$$\psi(v_{i-1}) = (1, 0), \quad \psi_i(v_i) = (0, 1), \quad \psi_i(v_{i+1}) = (-1, -D_i^2)$$

on the closure of U_i and linear on the cones. After a suitable modification on can write $B \backslash \rho_2$ in a chart such that

$$\psi(v_2) = (-1, 1), \qquad \psi(v_3) = (-1, 0), \qquad \psi(v_4) = (0, -1), \qquad \psi(v_1) = (1, 0),$$

see Figure 2.2.

We let ϕ be the function with kink the class of D_i on ρ_i, where we write D_i for the components of the anticanonical cycle. The canonical wall structure i is given by the

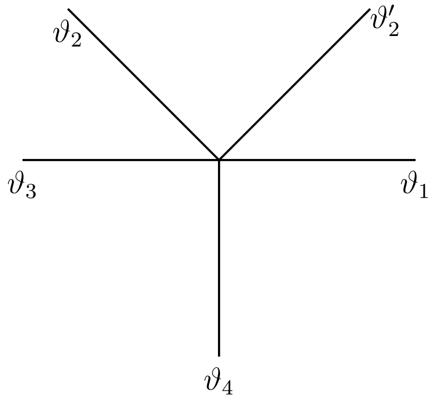

Figure 2.2: The affine manifold B.

following structure:

$$\mathscr{S} := \{(\rho_1, (1 + z^{E_1} z_1^{-1})(1 + z^{C_1} z_2^{-}1)), (\rho_3, (1 + z^{E_2} z_3^{-1})(1 + z^{C_2} z_3^{-}1))\}.$$

With this data, one calculates the theta functions as

$$\begin{aligned}
\vartheta_1 &= z_1 \\
\vartheta_2 &= z_2 \cdot (1 + z_1 z^{-E_1})(1 + z_1^{-1} z^{C_1}) \\
\vartheta_3 &= z_3 = z_1^{-1} z^{D_4} \\
\vartheta_4 &= z_4.
\end{aligned}$$

This defines the mirror family.

Chapter 3

The Dolgachev-Nikulin-Voisin family

Introduction

In this section we compute models of the Dolgachev-Nikulin-Voisin mirror family for K3 surfaces of degree 2 in (-1)-form. This will be the starting point for our investigaton in chapter 4. We also show that the Mori fan of the DNV family is a polyhedral fan precisely for genus 2.

3.1 d-semi-stable K3 surfaces

Central fibres of semi-stable degenerations of K3 surfaces are the so called *d-semi-stable K3 surfaces*, [Fri83]. Below, we will see that it is enough to construct a central fibre with certain properties to obtain a model for the Dolgachev-Nikulin-Voisin family $\mathcal{Y} \to S$. The property that governs the smoothability of a degenerate K3 surface is the *d-semi-stability*, introduced in the next definition.

Definition 3.1.1. Let $X = \cup X_i$ be a variety with normal crossings, $D_i = X_i \cap U_{j \neq i} X_j$. Let I_D be the ideal sheaf of D in X, I_{X_i} the ideal sheaf of X_i in X. Set

$$\mathcal{O}_D(-X) = \bigotimes_i I_{X_i}/I_{X_i} I_D,$$

the product taken over $\mathcal{O}_D$. Let $\mathcal{O}_D(X)$ be the dual. Then X is d-semi-stable, if

$$\mathcal{O}_D(X) \cong \mathcal{O}_D.$$

The next definition characterizes the class of degenerate K3 surfaces.

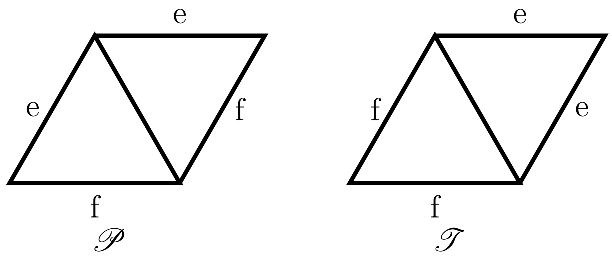

Figure 3.1: The triangulations $\mathscr{P}$ and $\mathscr{T}$.

Definition 3.1.2 ([Fri83]). Let X be a compact complex surface with normal crossings. X is a d-semi-stable K3 surface of type III if

(i) X is d-semi-stable,

(ii) $\omega_X = \mathcal{O}_X$,

(iii) $X = \cup_i X_i$ with X_i rational and the double curves $X_i \cap X_j$ are cycles of rational curves,

(iv) The dual graph is a triangulation of the sphere S^2.

For a not nececessarily d-semi-stable K3 surface X, let $[X]$ denote its locally trivial deformation class. In particular, we can consider the deformation classes of d-semi-stable K3 surface in -1 form, i.e. such that all self intersection numbers of components $D_{ij} = X_i \cap X_j$ equal to -1 (in the normalisation X_i^ν). By [Laz08], these deformation classes correspond to the two non-isotopic triangulations of the sphere with 2 triangles that are given by two triangles glued along the boundary and two triangles glued along one side to each other, with the remaining sides identified. We shall denote the first of these triangulations by $\mathscr{P}$ and the latter by $\mathscr{T}$. For precise definitions see [Laz08], but see Figure 3.1.

We recall the follwing construction, see e.g. [Laz08], Section 3.1. Let X be a d-semi-stable K3 surface. Writing $X = \cup_i X_i$ and D_{ij} for the double curves, there is a map

$$\oplus_i H^2(X_i, \mathbb{Z}) \to \oplus_{ij} H^2(D_{ij}, \mathbb{Z}). \tag{3.1.1}$$

As the precise construction is unimportant for us, we refer to [[KK98], 151ff] for details. Let L denote the kernel of 3.1.1. All the X_i are rational surfaces, we have the Betti numbers given by

$$b_2(X_i) = 10 - K_{X_i}^2.$$

Also, X is d-semi-stable and thus smoothable by [[Fri83], 5.10]. Hence from [[Per77], 2.4.4], we obtain the triple point formula

$$D_{ij}^2 + D_{ji}^2 = t_{ij},$$

where t_{ij} is the number of triple points on D_{ij}. As X is assumed to be of type III, $t_{ij} = -2$, unless D_{ij} is nodal, in which case $t = 0$. Here D_{ij} is considered to be a divisor on X_i. Also, let n,e and t be the numbers of components, double curves and triple points of the dual graph of X. Then, by Euler's formula, $n - e + t = 2$. As the double curves are smooth rational and the general fibre is a K3 surface, we also have the formula [[Per77], 2.4.6]

$$-\sum_i K_{X_i}^2 = -6n + 12$$

which, combined with the formula for the Betti numbers shows that the rank of $\oplus_i H^2(X_i, \mathbb{Z})$ is $4n + 12$. Also, Friedman shows that over $\mathbb{Q}$ the map is surjective. The sublattice L is primitive, as it is the Kernel of a map between torsion free lattices. As $e = 3n - 6$, this implies the rank if L is $n + 18$.
There is Carlson's extension map

$$c_X \colon L \to \mathbb{C}^*.$$

Again, we will not need the explicit homomorphism, instead we refer to [Car79] for details. The important fact is that the kernel of c_X can be identified with the Picard group $\mathrm{Pic}(X)$, [[Car79], 7E],[FS86].
Now, note that the preceding statements about cohomology groups are purely topological and hence also valid for for degenerate K3 surfaces that are not necessarily d-semi-stable. The point is that if $L = \mathrm{Pic}(X)$, X is d-semi-stable, by [FS86], p.25.

We summarize:

Lemma 3.1.3. *Let $X_0 = \cup_i X_i$ be a not necessarily d-semi-stable K3 surface of type III. Let $\sqcup_i X_i \to \cup_i X_i$ be the normalization map. Then*

(i) there is an injective morphism

$$\mathrm{Pic}(X) \to \mathrm{Pic}(\sqcup_i X_i),$$

(ii) X is d-semi-stable if $L = \mathrm{Pic}(X)$.

Below we will construct a d-semi-stable K3 surface by gluing rational surfaces. Recall that gluing schemes X_i along closed subschemes Z_{ij} amounts to constructing the push-out and that this push-out is indeed a scheme, cf. [Sch05]. While the details of this construction are not important for us, the following property of the construction is essential.

Lemma 3.1.4. *Let X be the pushout of finitely many integral schemes X_i glued along finitely many closed subschemes $Z_{ij} \subset X_i$. Then* $\mathrm{DivCl}(X) = \mathrm{Pic}(X)$.

Proof. First, X is a scheme by iterating [[Sch05], 3.7]. By definition, for an open set U in X, $\Gamma(U, \mathcal{O}_X)$ is a certain subring of the product $\prod_i \Gamma(\alpha_i^{-1}(U), \mathcal{O}_{X_i})$, where α_i is the canonical morphism $X_i \to X$. As the X_i are reduced, this means that X is reduced. For each component X_i, choose an affine open U_i in $X_i \backslash \cup Z_{ij}$. Then $\coprod U_i$ is an dense open affine subset of X. As X is reduced, U is schematically dense. Hence $\mathrm{DivCl}(X) = \mathrm{Pic}(X)$ by [[GW10], 11.27].

□

The next result follows from the deformation theory of [FS86].

Proposition 3.1.5. *Let $[Y]$ be the locally trivial deformation class of a d-semi-stable K3 surface Y with Y having $t = 2d$ triple points. Let $Y_0 \in [Y]$ be such that $c_{Y_0} = 1$, i.e. the Carlson map is trivial. There is a unique one-parameter smoothing $\mathcal{Y} \to S = \mathrm{Spec}\,\mathbb{C}[[t]]$ of Y_0 such that the restriction $\mathrm{Pic}(\mathcal{Y}) \to \mathrm{Pic}(Y_0)$ is an isomorphism and $\mathrm{Pic}(\mathcal{Y}_\eta) \cong \check{M}$.*

Proof. If $c_Y = 1$, $\mathrm{Pic}(Y) = L$. Let n be the number of components of Y. Consider the divisors $\xi_i = \sum_j D_{ij} - D_{ji}$, $i = 1, \ldots, n$.. They span a primitive sublattice K.

We also pick linear independent divisors $L_1, \ldots L_{19}$ that generate $\mathrm{Pic}(Y)$ mod K. If $\mathcal{Y} \to S$ is a deformation with $\mathrm{Pic}(\mathcal{Y}) \cong \mathrm{Pic}(Y)$ via restriction, then by definition $\mathcal{Y} \to S$ is a deformation of Y together with the L_i. We shall show that there is a unique such 1-parameter deformation, up to automorphisms on the base.

Let $\mathfrak{X} \to V$ be the semiuniversal deformation of [FS86]. By the calculation in [FS86], the locus V' in the smoothing component of V where the L_i deform is 1 dimensional and smooth. Let $\mathfrak{X}_0 \to V'$ be the restriction of the semiuniversal family. By [FS86], this is a smoothing of Y. Let R be the analytic algebra defining the germ, let R' be the completion of R with respect to the maximal ideal. Then $R' \cong \mathbb{C}[[t]]$. This defines a formal scheme $\hat{\mathcal{Y}} \to \mathrm{Spf}\,\mathbb{C}[[t]]$, and by the condition that all L_i deform, there is a $\mathcal{L} \in \mathrm{Pic}(\hat{\mathcal{Y}})$ restricting to an ample bunde on Y and thus by Grothendieck's existence theorem a deformation $\mathcal{Y} \to S$ with $S = \mathrm{Spec}\,\mathbb{C}[[t]]$. By construction, $\mathrm{Pic}(\mathcal{Y}) \cong \mathrm{Pic}(Y)$ via restriction.

Also, for the degeneration $\mathcal{Y} \to S$, it follows from [Kaw97], using the fact that S is a DVR, that we have an exact sequence

$$0 \to \mathbb{Z}^{\mathcal{Y}} \to \mathrm{Pic}(\mathcal{Y}) \to \mathrm{Pic}(\mathcal{Y}_\eta) \to 0 \tag{3.1.2}$$

with $\mathbb{Z}^{\mathcal{Y}}$ the abelian group generated by the components Y_i of the central fibre modulo the relation $\sum Y_i = 0$.

The statement about the Picard group of the generic fibre follows from [Laz08], Proposition 4.3 and Corollary 4.6 and the Sequence 3.1.2.

We show that $\mathcal{Y} \to S$ is a smoothing of Y. The Kodaira Spencer class of $\mathcal{Z} \to S'$ is by construction a class in the smoothing component V, so the analytic deformation $\mathfrak{X}_0 \to R$ is smooth by [Fri83, Proposition]. In particular, its local rings in closed points are regular, and thus by [Mat89, Theorem 23.7] the local rings in all $y \in Y$ of the formal smoothing are regular. By the same theorem, this implies that the stalks of closed points of the central fibre of $\mathcal{Y} \to S$ are regular local rings. This implies that $\mathcal{Y}$ is regular by [GW10, Remark 6.25]. In particular, the generic fibre is a smooth K3 surface. Also, by adjunction, $\mathcal{Y}$ has trivial canonical bundle. So $\mathcal{Y} \to S$ is indeed a semistable model.

Now, suppose $\mathcal{Y}' \to S$ is a second such model. By formal semiuniversality, $\mathcal{Y}' \to S$ is pulled back from $\mathcal{Y} \to S$ via a homomorphism $\mathbb{C}[[t]] \to \mathbb{C}[[t]]$. Because $\mathcal{Y}' \to S$ is regular, the uniformizing parameter t maps to at with a a unit. Hence, $\mathcal{Y}' \to S$ is canonically isomorphic to $\mathcal{Y} \to S$. This proves the result.

□

3.2 Models of the DNV family in genus 2

We now construct models of the DNV family in genus 2 in (-1)-form. To get such a model, we need to find a d-semistable K3 surface Y_0 in -1 form with trivial Carlson map such that deformation induced by Proposition 3.1.5 has generic fibre Y_η with $\mathrm{Pic}(Y_\eta) = \check{M}$. As $g-1$ is square free, this means that the number of triple points is 2, so in particular a central fibre sitting in such a degeneration will have 3 components. Hence a surface as required can be found in the locally trivial deformation class of degenerate K3 surfaces in -1 form defined by the triangulations $\mathscr{P}$ and $\mathscr{T}$. These deformation classes can be expicitely described, see [Laz08, Proposition 5.2]. For example, the surfaces in the class defined by $\mathscr{P}$ are triples of (weak) del Pezzo surfaces of degree 2 glued along the anticanonical divisor. The required triviality of the Carlson map forces the components of Y_0 to have a maximal configuration of rational double points. We construct these surfaces and then glue them in a way such that the Carlson map is trivial, inducing a model of the DNV family of genus 2. We recall some facts about weak del Pezzo surfaces first, a reference is [Dol12].

Definition 3.2.1. A weak del Pezzo surface is a nonsingular surface S with big and nef anticanonical divisor.

Similar to usual del Pezzo surfaces, weak del Pezzo can be constructed by blowing up points on $\mathbb{P}^2$ and in fact, a blow up of $\mathbb{P}^2$ in 8 or less point is a weak del Pezzo surface if

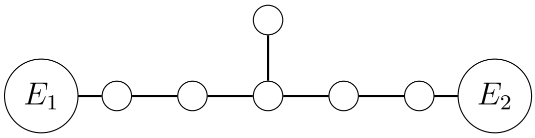

Figure 3.2: The E_6 root system and the exceptional curves E_1, E_2.

and only if the points are in almost general position, cf. [Dol12, Chapter §]. Note that this entails blowing up points several times. Also, we have the following.

Proposition 3.2.2. *Let S be a weak del Pezzo surface.*

(i) Let $S \to S'$ be a blowing down of a (-1)-curve E. Then S' is a weak del Pezzo surface.

(ii) Let $S' \to S$ be a blowing-up with center a point not lying on any (-2) curve. Assume $K_S^2 > 1$. Then S' is a weak del Pezzo surface.

There is some control over the (-2) curves on a weak del Pezzo.

Proposition 3.2.3. *Let S be a weak del Pezzo of degree $d = 9 - N$. Then the number r of (-2) curves is less than or equal to N and the sublattice generated by them is a root lattice of rank r.*

Definition 3.2.4. Let (Y, D) be a anticanonical pair. Let $D = \sum D_i$ and p be a smooth point of exactly one D_i. If $n = 1$, the *n-fold blow up* of Y in p is the usual blow up, if $n > 1$, the n-fold blow up of Y in p is the blow up of the $n-1$-fold blow up $\pi\colon Y' \to Y$ in the point $\mathrm{ex}(\pi) \cap \pi_*^{-1} D_i$. More generally, if $(p_1, \ldots, p_k)$ is an ordered set of points $p_i \in Y$ such that each component D_i contains at most one p_i as a smooth point, we define by the obvious generalisation the *$(n_1, \ldots, n_k)$-blow up of Y in $(p_1, \ldots, p_k)$*.

Construction 3.2.5. We now construct the component surfaces. For each such surface, we als define a *special point*. This point will play a role in the gluing of the components. We need weak del Pezzo surfaces of degree $d = 1, 2, 4$.

d=1: Let $Q = \mathbb{P}^1 \times \mathbb{P}^1$ with toric boundary $D = D_1 + D_2 + D_3 + D_4$, ordered cyclically. Let $p_i \in D_i$, $i = 1 \ldots 4$, be points in the smooth part of D such that p_i, p_{i+2} are in the same fibre of one of the two rulings, see Figure 3.3. Let $\tilde{Q}$ be the $(1, 5, 1, 3)$-blow-up of Q in (p_1, p_2, p_3, p_4). The strict transforms of the D_1, D_3 have self-intersection -1 on $\tilde{Q}$. Blowing down these yields a surface such that the strict transform of D_4 has self intersection (-1). Then, blowing this down yields a surface $\mathfrak{Y}_1$ with boundary divisor an anticanonical cycle $\tilde{D}$ of self intersection 1 and an E_8 root system of

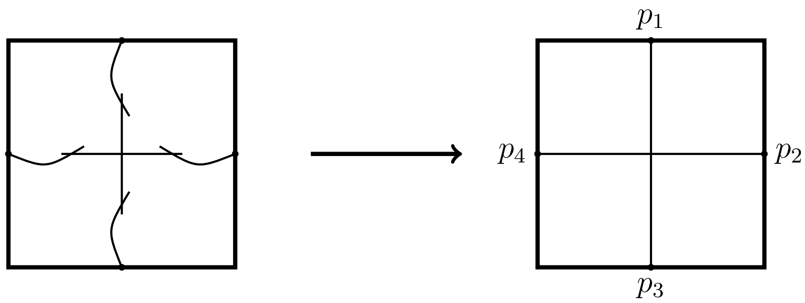

Figure 3.3: The blow up of $\mathbb{P}^1 \times \mathbb{P}^1$.

effective (-2)-curves. This is a weak del Pezzo of degree 1, see Figure 3.4. There is a unique -1 curve E meeting $\tilde{D}$. The special points are two copies of the node on $\tilde{D}$ and the point in $E \cap \tilde{D}$.

d=2: For degree 2, we take $\mathbb{P}^2$ together with its toric boundary $(xyz = 0)$. We can fix three collinear points, one on each boundary divisor, say p, q, r. Let Y' be the $(3,3,2)$-blow up of $\mathbb{P}^2$ in (p,q,r). This yields a weak del Pezzo surface of degree 1, as we have blown up 8 points that are not on (-2) curves. Now, blow down the strict transform of the toric divisor that is a (-1)-curve. Then the resulting surface $\mathfrak{Y}_2$ is also a weak del Pezzo surface of degree 2 with anticanonical cycle $D = D_1 + D_2$. It carries an E_6 configuration of effective (-2)-curves by construction. There are also 2 exceptional curves E_1, E_2 of the first kind each meeting a long end of the root system and a component of the anticanonical divisor, see Figure 3.2. The special points of D_i are the points $D_i \cap E_i$ and the two points in $D_i \cap D_{i+1}$.

d=4: Again, let $Q = \mathbb{P}^1 \times \mathbb{P}^1$ with toric boundary $D = \mathrm{D}_1 + D_2 + D_3 + D_4$. Let p_i, $i = 1 \dots 4$, be points on the intersection of the fibres of the two ruling with the toric boundary components. Blow up Q in the once in each p_i, the resulting surface $\mathfrak{Y}_4$ is a weak del Pezzo of degree 4, with an A_2 root system of effective (-2)-curves and an anticanonical cycle $\tilde{D} = \tilde{D}_1 + \tilde{D}_2 + \tilde{D}_3 + \tilde{D}_4$ where the $\tilde{D}_i$ are the strict transforms of the D_i. There are 4 (-1) curves E_i, $i = 1 \dots 4$ on $\mathfrak{Y}_4$ that are not components of $\tilde{D}$. Each E_i meets exactly one of the D_i transversally. The special points of $\tilde{D}_i$ are the points $\tilde{D}_i \cap E_i$, $\tilde{D}_i \cap \tilde{D}_{i+1}$ and $\tilde{D}_i \cap \tilde{D}_{i-1}$, indices considered cyclically.

Let $\mathscr{G} \in \{\mathscr{P}, \mathscr{T}\}$. The triangulation $\mathscr{G}$ defines a locally trivial deformation class $[Y]_{\mathscr{G}}$ of d-semistable K3 surfaces, and we can take the member $Y_{\mathscr{G}}$ of $[Y]_{\mathscr{G}}$ such that the (normalization) of each component is from the above list. The gluing is such that the nodes of the anticanonical divisors and the the special points are identified. We show that

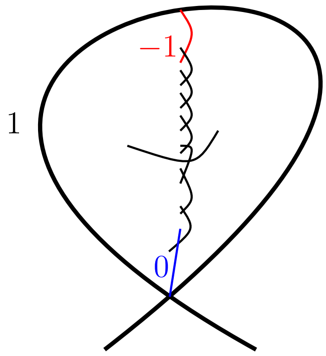

Figure 3.4: The (weak) dP_1 components.

the gluing specified in the construction is optimal, i.e. the resulting surfaces have trivial Carlson map. We learnt this condition from [GHKS].

Proposition 3.2.6. *Let $\mathscr{G} \in \{\mathscr{P}, \mathscr{T}\}$. The surfaces $Y_{\mathscr{G}}$ have trivial Carlson map, i.e. $c_{Y_{\mathscr{G}}} = 1$.*

Proof. Each of the surfaces $\mathfrak{Y}_i$ has a $\mathbb{Q}$-basis of $\mathrm{Pic}(\mathfrak{Y}_i)$ given by the -2-curves and the interior -1-curves, i.e. those that are not components of the anticanonical divisor. For the normalisation Y_i^ν of a component Y_i, of $Y_{\mathscr{G}}$, we denote this basis by B_i. The Picard group of $Y_{\mathscr{G}}$ is given by the kernel of the Carlson map. Let Y_{simp} be a semi-simplicial resolution of $Y_{\mathscr{G}}$, see [Car79],[KK98, §4.2.2]. We assume $Y_{\mathscr{G}} = \cup Y_i$ to have simple normal crossings. Then the one takes Y_{simp} to be

$$Y_p = \coprod Y_{i_0} \cap \cdots \cap Y_{i_p} \qquad (i_0 < \dots i_p)$$

for $p = 0, 1, 2$. Face maps are given by maps $\delta_i \colon Y_p \to Y_{p-1}$ such that δ_i has the structure of an inclusion on the components of Y_p, see [KK98, §4.2.2]. In the normal crossing case one has to modify this by including appropriate normalisations and normalisation maps. Let L_i be a tuple of divisors written in the basis B_i, with i running throught the non-hexagonal components, such that the degrees on double curves agree. Also, on the hexagonal components H, for each divisor L there is a linearly equivalent divisor that restricts to $\deg L_{|D_i} m_i$ with m_i denoting the interior special point on $D_i \subset H$ by (the proof of) [GHK15b, Lemma 2.8]. By the gluing condition, the support of the pullback of L_i to any component of the double curve $Y_i \cap Y_j$ is equal, hence trivial under the Carlson map, cf [Car79], page 8 ff. This implies the result. $\square$

We obtain the following result.

Proposition 3.2.7. *Let $\mathscr{G} \in \{\mathscr{P}, \mathscr{T}\}$. The surface $Y_{\mathscr{G}}$ is a d-semistable K3 surface. The associated smoothing $\mathcal{Y}_{\mathscr{G}} \to S$ is a model in (-1)-form of the Dolgachev-Nikulin-Voisin family, i.e.* $\operatorname{Pic}(\mathcal{Y}_{\mathscr{G}\eta}) = \check{M}_{2d}$ *and* $\operatorname{Pic}(\mathcal{Y}_{\mathscr{G}}) \cong \operatorname{Pic}(Y_{\mathscr{G}})$. *The threefold $\mathcal{Y}_{\mathscr{G}}$ is projective over the base S.*

Proof. $Y_{\mathscr{G}}$ is a d-semistable K3 surface in (-1)-form by construction. The Carlson map is trivial, i.e. $c_{Y_{\mathscr{G}}} = 1$, because the gluing is optimal. It is also projective: for $\mathscr{G} = \mathscr{T}$, any triple of ample divisors on the component surfaces can be, after taking multiples such that the degree on components of the double curves agrees, be glued to an ample bundle on $Y_{\mathscr{T}}$. If $\mathscr{G} = \mathscr{P}$, it is easy to see by the symmetry of $\mathfrak{Y}_2$, that there is an amle divisor A on $\mathfrak{Y}_2$ with degree agreeing on both components of the anticanonical curve. For example, take the bundle given by the tuple $(7, 11, 13, 16, 13)$ on the (-2)-curves, i.e. the i-th number corresponds to the i-th root in the ordering of [Dol12], p. 404, and by adding $10E_i$ for E_i, $i = 1, 2$ the -1 curves in the exceptional locus of the blow up in the above construction. Taking this bundle on each copy of $\mathfrak{Y}_2$ in $Y_{\mathscr{P}}$ defines a ample bundle. Because ampleness is an open condition and as we work over the spectrum of a DVR, the lift of an ample bundle to $\mathcal{Y}_{\mathscr{G}} \to S$ defines a ample bundle over S, and thus $\mathcal{Y}_{\mathscr{G}}$ is projective over the base S.

Write $Y_{\mathscr{G}} = \cup_{i=1}^{3} Y_i$. The condition $\operatorname{Pic}(\mathcal{Y}_{\mathscr{G}\eta}) = \check{M}$ follows from the exact sequence

$$0 \to G \to \operatorname{Pic}(\mathcal{Y}) \to \operatorname{Pic}(Y^{gen}) \to 0,$$

where G is the free abelian group generated by the components of $Y_{\mathscr{G}}$ modulo $\sum_i Y_i = 0$: We have $\operatorname{Pic}(Y_{\mathscr{G}}) = \operatorname{Pic}(\mathcal{Y})$. Also, we can identify the images of the components Y_i with the classes ξ of [[Laz08], 3.1] in $L = \operatorname{Pic}(Y_{\mathscr{P}})$, and thus obtain

$$\operatorname{Pic}(Y^{gen}) \cong L/A = \bar{L}$$

with $\bar{L}$ the lattice from Laza [Laz08] and A the span of the ξ_i. By Example 5.11 in [Laz08], this shows

$$\operatorname{Pic}(Y^{gen})) = \check{M}.$$

Hence, the degeneration $\mathcal{Y}_{\mathscr{G}} \to S$ is indeed a model of the Dolgachev-Nikulin-Voisin mirror family in genus 2.

□

$Y_{\mathscr{P}}$ $Y_{\mathscr{T}}$

Figure 3.5: The surfaces $Y_{\mathscr{P}}$ and $Y_{\mathscr{T}}$.

3.3 The Morifan of the Dolgachev-Nikulin-Voisin family

Let $\mathcal{Y} \to S$ be a model of the DNV family. We have the following result.

Proposition 3.3.1. *Let $\mathcal{Y} \to S$ be the Dolgachev-Nikulin-Voisin family of genus g. Let Y^{gen} denote the generic fibre.*

i. If $g = 2$, $\mathrm{Aut}(Y^{gen})$ is finite: $\mathrm{Aut}(Y^{gen}) = \mathbb{Z}/2 \times S_3$.

ii. If $g \geq 3$, $\mathrm{Aut}(Y^{gen})$ is not finite.

Proof. Let $\check{M}_g$ denote the Picard lattice of the generic fibre of $\mathcal{Y} \to S$. These lattices are given in [Dol96]. It follows from [Nik83], that the automorphism group is finite only for $g = 2$, as only that lattice appears in Nikulin's classification. The concrete description for $g = 2$ is due to [Kon89]. ☐

We have the following exact sequence,

$$0 \to \mathbb{Z}^g \to \mathrm{Pic}(\mathcal{Y}) \xrightarrow{r} \mathrm{Pic}(Y^{gen}) \to 0,$$

the homomorphism r being restriction.

Lemma 3.3.2. *[[GHKS], 9.2] Let $\mathcal{Y} \to S$ be the Dolgachev-Nikulin-Voisin family. Then*

$$r(\mathrm{Mov}(\mathcal{Y})) = \mathrm{Mov}(Y^{gen}) = \mathrm{Nef}(Y^{gen})$$

.

There is the following result, see [[GHKS], 9.5].

Theorem 3.3.3. *Let $\triangle \subset |\operatorname{Nef}(Y^{gen}|$ be a rational polyhedral cone. Then*

i. $r^{-1}(\triangle)$ and $r^{-1}(\triangle) \cap \operatorname{Mov}(\mathcal{Y})$ are rational polyhedral cones.

ii. The collection

$$\{r^{-1}(\triangle) \cap \gamma : \gamma \in \operatorname{Morifan}(\mathcal{Y})\}$$

is a finite set of rational polyhedral cones with support $r^{-1}(\triangle)$.

iii. The collection

$$\{r^{-1}(\triangle) \cap \gamma : \gamma \in \operatorname{Morifan}(\mathcal{Y}) \cap \operatorname{Mov}(\mathcal{Y})\}$$

is a finite set of rational polyhedral cones with support $r^{-1}(\triangle) \cap \operatorname{Mov}(\mathcal{Y})$.

iv. We have

$$|\operatorname{Morifan}(\mathcal{Y})| = \operatorname{Mov}(\mathcal{Y})$$

Recall the following notion.

Definition 3.3.4. Let $\mathcal{Y} \to S$ be a projective morphism, $\mathcal{Y}$ smooth. We say $\mathcal{Y} \to S$ is a *Mori dream space* if

i. The relative Picard group $\operatorname{Pic}(\mathcal{Y}/S)$ is a finitely generated abelian group,

ii. There exist finitely many rational maps (over S), $f_i \colon \mathcal{Y} \dashrightarrow Y_i$, that are isomorphisms in codimension 1 such that if D is a movable $\mathbb{R}$-divisor, there exist f_i and a semiample divisor D_i with $D = f_i^* D_i$.

It turns out that the Dolgachev-Nikulin-Voisin family is a Mori dream space precisely when $g = 2$.

Theorem 3.3.5. *The Dolgachev-Nikulin-Voisin family $\mathcal{Y} \to S$ is a Mori dream space if and only if the genus g is 2.*

Proof. Let $p \colon \mathcal{Y} \to S$ denote the DNV family. The Picard group $\operatorname{Pic}\mathcal{Y} = \operatorname{Pic}(\mathcal{Y}/S)$ is a finitely generated abelian group, as it it equal to the $\mathrm{N}^1(\mathcal{Y}/S)$. Moreover, by Theorem 3.3.3, the support of $\operatorname{Morifan}(\mathcal{Y})$ is $\operatorname{Mov}(\mathcal{Y})$ and by Lemma 3.3.2, the latter is contained in the preimage under r of the Nef cone of the generic fibre, i.e. $\operatorname{Morifan}(\mathcal{Y}) \subset r^{-1}(\operatorname{Nef}(Y^{gen})$. In particular, $\gamma \cap r^{-1}(\operatorname{Nef}(Y^{gen}) = \gamma$ for all $\gamma \in \operatorname{Morifan}(\mathcal{Y})$. If $g = 2$, $\operatorname{Nef}(Y^{gen})$ is a rational polyhedral cone by [[Huy],4.8], as $\operatorname{Aut}(Y^{gen}) = \mathbb{Z}/2 \times S_3$ by Proposition 3.3.1. Hence we can take $\triangle = \operatorname{Nef}(Y^{gen})$ in Theorem 3.3.3 and get

$$\{r^{-1}(\operatorname{Nef}(Y^{gen}) \cap \gamma : \gamma \in \operatorname{Morifan}(\mathcal{Y}) \cap \operatorname{Mov}(\mathcal{Y})\} = \{\gamma : \gamma \in \operatorname{Morifan}\mathcal{Y}\}$$

is a finite set of polyhedral cones, i.e. Morifan($\mathcal{Y}$) is rational polyhedral.

Thus, by definition of the Morifan, there are finitely many contractions $f_i\colon \mathcal{Y} \dashrightarrow Y_i$ such that for every movable $\mathbb{R}$-divisor D there is a f_i with $D = f_i^* D_i$ for some D_i semiample. This means that $\mathcal{Y} \to S$ is a Mori dream space.

If $g \geq 3$, the automorphism group of Y^{gen} is infinite by Lemma 3.3.1, and thus Nef(Y^{gen}) is not rational polyhedral, again by [[Huy],4.8]. This means that Morifan($\mathcal{Y}$) cannot be rational polyhedral as else the generators would give a finite set of generators of Nef(Y^{gen}), and thus there is no finite collection of contractions $f_i\colon \mathcal{Y} \to Y_i$ such that every movable divisor is a pullback of a semiample divisor of Y_i. So in this case, $\mathcal{Y}$ is not a Mori dream space. $\square$

Chapter 4

Deformations

Introduction

Having constructed semi-stable models $\mathcal{Y}_{\mathscr{P}}$ and $\mathcal{Y}_{\mathscr{T}}$ of the Dolgachev-Nikulin-Voisin family, we now study deformations. In total, we show the following theorem. For the statement, let $TV(\widehat{\text{Morifan}}(\mathcal{Y}))$ be the completion of the toric variety given by $\text{Morifan}(\mathcal{Y})$ along the type III part of the toric boundary.

Theorem 4.0.1. *Let $\mathfrak{X}$ be the 21 dimensional GHKS family over the locus of stable pairs. Then all generic limit surfaces of the type III boundary of the KSBA compactification $\bar{\mathrm{P}}_2$ appear as fibres over strata Z of $TV(\widehat{\text{Morifan}}(\mathcal{Y}))$.*

Recall the relation between contractions of semi-stable models and deformations of the pillow surfaces: Let $\mathcal{Y}$ be a projective semi-stable model of the Dolgachev-Nikulin-Voisin family. By definition, a contraction defines a face F of $\text{Morifan}(\mathcal{Y})$, so F defines a toric stratum in the base of the family induced by the central fibre Y_0 of $\mathcal{Y}$. Restriction of the local model to this stratum then defines a deformation of the pillow surface corresponding to Y_0. By construction, the family $\mathcal{X}$ is trivial over toric strata, i.e. there is a degenerate surface associated to each contraction. This is the approach taken here and in the following chapter.

We will also show how to obtain the type II limit surfaces.

Theorem 4.0.2. *Let $\mathfrak{X}$ be the GHKS family. Then limit surfaces of type II_i, $i = 1 \ldots 4$, of the KSBA compactification $\bar{\mathrm{P}}_2$ appear as fibres over strata Z of $TV(\widehat{\text{Morifan}}(\mathcal{Y}))$.*

The constructions in section 4.1 build on ideas of Paul Hacking, the deformation to III_ϵ is due to him.

4.1 Deformations of n-vertices

The fibre of the genus 2 GHKS family $\mathfrak{X}$ is, over each maximal subcone of the Morifan of the Dolgachev-Nikulin-Voisin family, given by three deformations of n vertices glued together. For example, we have constructed a semi-stable model $\mathcal{Y}_{\mathscr{P}} \to S$ with dual graph the triangulation $\mathscr{P}$. This pillow surface $X_{\mathscr{P}}$ is three 2-vertices glued together, i.e. the fibre of the GHKS family over the (maximal) toric stratum corresponding to $\mathcal{Y}_{\mathscr{P}}$. We want to (partially) smooth $X_{\mathscr{P}}$ (and the pillow surface $X_{\mathscr{T}}$ given by the triangulation $\mathscr{T}$) to other surfaces of Laza's list. We will deform the 2-vertices individually, using the theory of [GHK15a]. We will call the deformation $\mathcal{X}$ of an n-vertex as in [[GHK15a], Thm 0.1] the *mirror family.*

To proceed, note that the mirror family of an anticanonical surface with cycle D consisting of 2 or less components is constructed via a toric blow up $\tilde{Y}$. This blow-up is necessary if one wants to use toric models for the construction of the mirror, as in [GHK15a]. We now construct a toric blow up and a toric model for the components of the central fibre of model $\mathcal{Y}_{\mathscr{P}}$.

Construction 4.1.1. Let $Y_0 \subset Y_{\mathscr{P}}$ be a component of the central fibre of the Dolgachev-Nikulin-Voisin family in (-1)-form, with deformation class $\mathscr{P}$. Let $D = D_1 + D_2$ be the anticanonical cycle of Y given by the double curve, with $D_i^2 = -1$.

Set $\bar{Y} = \mathbb{P}^1 \times \mathbb{P}^1$ with toric boundary a cycle $\bar{D}_i$, $i = 1 \dots 4$ and let $\bar{F}_1, \bar{F}_2$ denote interior fibres of the two rulings of $\bar{Y}$. Let p_j and q_j, $j = 1, 2$, denote the points of intersection of $\bar{F}_1$ and $\bar{F}_2$ with the components of the boundary, respectively. Assume the p_j lie on $\bar{D}_2$ and $\bar{D}_4$, see Figure 4.1. Blow up the p_j three times and the q_j once. Denote the resulting surface by $\tilde{Y}$. Repeated application of Lemma 2.2.3 shows that the strict transforms $\tilde{D}_i$ of the $\bar{D}_i$ form an anticanonical cycle $\tilde{D}$. Hence, $(\tilde{Y}, \tilde{D})$ is a Looijenga pair. The strict transforms $\tilde{D}_1$ and $\tilde{D}_3$ are (-1) curves. Contracting these defines a morphism $\tau\colon \tilde{Y} \to Y$ to an anticanonical pair (Y, D) with D the image of $\tilde{D}$. By construction, Y has an E_6 root system of effective (-2)-curves, giving a basis of the subspace of $\mathrm{Pic}(Y)$ that is orthogonal to K_Y.

Also, the anticanonical divisor has degree $D^2 = 2$ and is big and nef: there are only finitely many (-1) and (-2)-curves on Y and thus the Mori cone $\mathrm{NE}(Y)$ is polyhedral and generated by curves C with $C^2 < 0$, by Proposition 5.1.1.6 and Theorem 5.1.3.1 in [ADHL14]. This shows Y is a weak del Pezzo surface of degree 2. Hence, by the Global Torelli Theorem of [GHK15b], Y is isomorphic Y_0.

The E_6 system is depicted in Figure 4.2. Note that the cycle $E_1 + F_1 + F_2 + F_3 + F_4 + F_5 + E_2$ determines a ruling $Y \to \mathbb{P}^1$. The components of the anticanonical divisor are

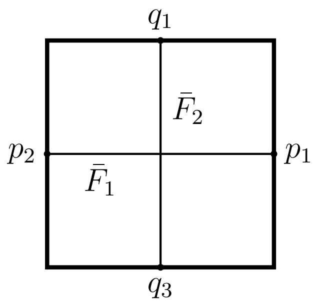

Figure 4.1: The surface $\bar{Y}$ with rulings and marked points.

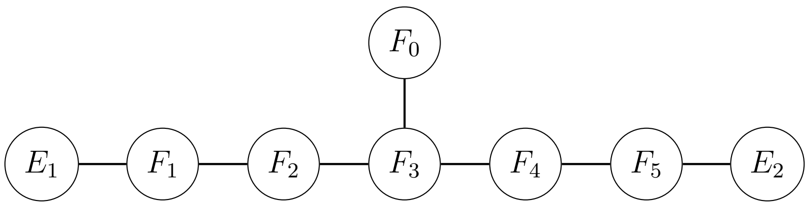

Figure 4.2: The E_6 root system given by the F_i and the (-1)curves E_1 and E_2.

sections of this ruling.

The discussion implies the following.

Lemma 4.1.2. *Let $Y \subset Y_{\mathscr{P}}$ be a component of the DNV-family in (-1) form. Then there is a toric blow up $\tilde{Y} \to Y$, a ruling $r_1 \colon Y \to \mathbb{P}^1$, a ruling $r_2 \colon \tilde{Y} \to \mathbb{P}^1$ and a toric model $\pi \colon \tilde{Y} \to \bar{Y}$ such that the following diagram commutes*

$$\begin{tikzcd}
\tilde{Y} \arrow[r, "\pi"] \arrow[d] \arrow[dr, "r_2"'] & \mathbb{P}^1 \times \mathbb{P}^1 \arrow[d] \\
Y \arrow[r, "r_1"'] & \mathbb{P}^1.
\end{tikzcd}$$

Note that the diagram of Lemma 4.1.2 shows that the Nef cone of $\mathbb{P}^1$ defines a ray in $\mathrm{Nef}(Y)$, and therefore a divisor Z in $TV(\mathrm{Nef}(Y))$. The latter sits in $TV(\mathrm{Nef}(\tilde{Y}))$ by Lemma 4.1.4 below..

The divisor Z defined by the ruling $r_2 \colon \tilde{Y} \to \mathbb{P}^1$ defines a deformation of the 2-vertex to a normal crossing surface. More precisely:

Proposition 4.1.3. *The fibre of the mirror family $\mathcal{X}$ restricted to the divisor Z defined by the ruling $r_1 \colon Y \to \mathbb{P}^1$ is a normal crossing surface.*

Proof. Let $\tilde{Y}$ be the toric blow up of Y that admits the toric model $\pi \colon \tilde{Y} \to \mathbb{P}^1 \times \mathbb{P}^1$, as above. By Lemma 4.1.2 the ruling r_1 defining Z factors through π.

Over the locus defined by the vanishing of the exceptional divisors of the toric model, $\mathfrak{X}$ is isomorphic to the family defined by $\mathbb{P}^1 \times \mathbb{P}^1$ with plt function φ induced by pullback, and fan Σ the fan of $\mathbb{P}^1 \times \mathbb{P}^1$, see e.g the discussion in chapter 3 of [GHK15a], or conclude by strong uniqueness of scattering diagrams. The remaining deformation direction is then given by the morphism $\mathrm{NE}(\mathbb{P}^1 \times \mathbb{P}^1) \to \mathrm{NE}(\mathbb{P}^1)$ contracting the toric boundary divisors $\bar{D}_1, \bar{D}_3$ in the fibre $\mathbb{P}^1 \times \mathbb{P}^1$. This is the degeneration given by erasing the rays corresponding to $\bar{D}_1, \bar{D}_3$, by [GHK15a, 1.3]. The fan $\bar{\Sigma}$ of convex cones of maximal domains of linearity of the induced map $\bar{\varphi}$ is given by to half spaces, see figure 1.1. Hence the fibre is given by the algebra

$$\mathbb{C}[\bar{\Sigma}] = \oplus \mathbb{C} z^m$$

with m running through the integral points of Σ. This defines normal crossing surface: there is an morphism

$$\mathbb{C}[X_1, X_2, X_3, X_4] \to \mathbb{C}[\bar{\Sigma}]$$

sending X_i to e_i, with kernel $(X_1X_3 - 1, X_2X_4)$ Hence we find

$$\mathbb{C}[X_1^{\pm}, X_2, X_4]/(X_2X_4) \cong \mathbb{C}[\bar{\Sigma}].$$

This is the localization of $\mathbb{C}[X_1, X_2, X_4]/(X_2X_4)$ in X_1 and thus an open subscheme of a normal crossing scheme. Hence, $\operatorname{Spec} \mathbb{C}[\bar{\Sigma}]$ is a normal crossing surface.

□

Lemma 4.1.4. *Let $\pi\colon Y' \to Y$ be a birational morphism of smooth rational varieties. Then there is a toric morphism $TV(\mathrm{Nef}(Y)) \to TV(\mathrm{Nef}(Y'))$.*

Proof. Let L be a nef line bundel on Y and C an integral curve in Y'. Then

$$\pi^*L.C = L.\pi_*C$$

by the projection formula. As both varieties are projective, π is proper so $\pi_*C = d[D]$ with D the image of C under π, so π_*C is a positive multiple of an integral curve, thus $L.\pi_*C \geq 0$ and therefore π^* restricts to a morphism

$$\pi^*\colon \mathrm{Nef}(Y) \to \mathrm{Nef}(Y').$$

This is linear, so by [[CLS11], Thm 3.3.4], π^* induces the required morphism of toric varieties. □

Let Y be a component of $Y_{\mathscr{P}}$ as above. Let $D = D_1 + D_2$ be the anticanonical cycle. Let E_1, E_2 be the irreducible interior (-1)-curves with numbering such that $E_i.D_i = 1$. There is the E_6 root system of (-2)-curves F_i, $i = 1 \ldots 6$, where we use the same numbering as in

Figure 4.1. Contraction of D_2 yields a map $\gamma\colon Y \to Y'$ to a weak del Pezzo surface Y'. The image of E_2 under π_1 is curve E with $E^2 = 0$. Also, the curve $C := \gamma_*(E_1+F_1+F_2+F_3+F_4)$ is an exceptional curve on Y', being a tree of an irreducibel (-1)-curve and (-2)-curves. Hence there is a blow-down

$$\pi_C\colon Y' \to Y''.$$

By Proposition 3.2.2, Y'' is a weak del Pezzo surface of degree 8. It contains the (-1)-curve $\pi_{C*}\gamma_*(F_5)$ and no (-2) curves. Therefore $Y'' \cong \mathcal{H}_1$, where $\mathcal{H}_1$ is the Hirzebruch surface of degree 1. There is a ruling $Y'' \to \mathbb{P}^1$ given by contracting $\pi_{C*}(E)$

Proposition 4.1.5. *Let Y be a component of the central fibre $Y_{\mathscr{P}}$ of $\mathcal{Y}_{\mathscr{P}}$. Let $\gamma\colon Y \to Y'$ be the contraction of a component D_1 of the anticanonical cycle. Let $r_3\colon Y'' \to \mathbb{P}^1$ be the composition of the contraction $\pi_C\colon Y' \to Y''$ and the ruling $Y'' \to \mathbb{P}^1$.*

(i) The contraction $\gamma\colon Y \to Y'$ deforms the 2-vertex defined by Y to a 1-vertex

(ii) The ruling $r_3\colon Y \to \mathbb{P}^1$ deforms the 2-vertex to $\{z^2 = y^2(x^2-y+1)\} \subset \operatorname{Spec}\mathbb{C}[x,y,z]$.

Proof. This is essentially the calculation in [GHK15a, 6.2]. In order to calculate the fibre of the mirror family over the stratum defined by $\gamma\colon Y \to Y'$, we need to work on a toric blow up $\tilde{Y}$ of Y, which can be obtained by blowing up a node of the anticanonical divisor. The affine manifold B of $\tilde{Y}$ is depicted in figure 4.3. Here, $w, w+w'$, are the rays corresponding to the components of the anticanonical divisor D while $w+2w'$ is the refinement corresponding to the toric blow-up, see [GHK15a], 1.2 for the construction of B. The gluing is so that w is identifyed with w'. Let m be the asymptotic monomial with $\bar{m} = 2w + w'$, see [GHKS16], Definition 3.1. Contraction of D_1 means that D_1 is set to 0 in all decoration functions. The the corresponding ray is $w + w'$. This means that over the maximal monomial ideal $\mathrm{NE}(Y)$, broken lines can pass the ray $w+w'$, but do not pick up any monomials. In particular, broken lines passing $w+w'$ do not bend. Also, the ray $w+2w'$ can be passed by broken lines as the corresponing divisor is set to 0 in the calculation of the family induced by Y. Denoting the theta functions with asymptotic monomial n by ϑ_n, the multiplication rule for broken lines yields the following equations.

$$\begin{aligned} \vartheta_w\vartheta_{w+w'} &= \vartheta_{2w+w'} + \vartheta_{w+2w'} \\ \vartheta_{2w+w'}\vartheta_{w+2w'} &= t\vartheta^3_{w+w'}. \end{aligned}$$

Hence $\vartheta_w\vartheta_{w+w'} - \vartheta_{2w+w'} = \vartheta^2_{w+2w'}$ and thus

$$\vartheta_{2w+w'}(\vartheta_w\vartheta_{w+w'} - \vartheta_{2w+w'}) = \vartheta^3_{w+w'}$$

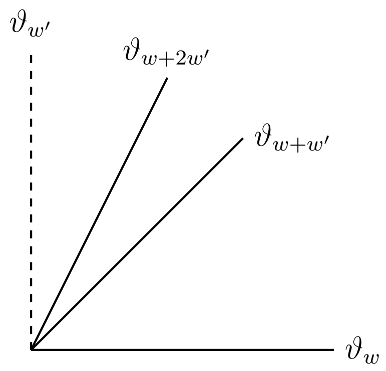

Figure 4.3: The affine manifold B

The change of coordinates $\vartheta_{w+w'} \mapsto y, \vartheta_w \mapsto 2x$ and $\vartheta_{2w+w'} \mapsto z+xy$ gives the equation $(z+xy)(2xy-(z+xy)) = y^3$, so

$$z^2 = y^2(x^2+y)$$

Next, we show the second item. Note that contracting $\mathcal{Y} \to \mathbb{P}^1$ then means that now the ray $w+w'$ carries a term $1+tz^n$ with $n=(-1,-1)$ on any maximal cone of the polyhedral decomposition of B, because all exceptional curves but E meeting the (contracted) component D_2 contain a component that is not contracted under $Y \to Y'$ and hence their contribution in the multiplication rule is modded out. This yields the following set of equations.

$$\begin{aligned} \vartheta_w \vartheta_{w+w'} &= \vartheta_{2w+w'} + \vartheta_{w+2w'} \\ \vartheta_{2w+w'} \vartheta_{w+2w'} &= \vartheta_{w+w'}^3 + \vartheta_{w+w'}^2. \end{aligned}$$

Under a the same change of coordinates as above, this yields the equation

$$z^2 = y^2(x^2-y+1)$$

□

Remark 4.1.6. Note that this proposition is true for all Looijenga pairs (Y,D) that are toric blow ups of a pair (Y',D') with D' a nodal curve and which have a ruling as in the proposition.

We shall also need morphisms corresponding to 'do nothing' on the 2-vertex. If Y is a component of $Y_{\mathscr{P}}$, application of the morphism defined by the anticanonical sheaf $\mathcal{O}(-K_Y)$ does the job.

Proposition 4.1.7. *Let Y be a component of $Y_{\mathscr{P}}$. Let $\mathcal{X}$ be the mirror family of Y. Let $\tau_k\colon Y \to Y_k$ be the morphism defined by the anicanonical system. The fibre of $\mathcal{X}$ over the stratum defined by τ_k is a 2-vertex.*

Proof. Let $Y \to Y_k$ be the morphism defined by $\mathcal{O}(-K_Y)$. Consider the toric blow-up given above, i.e. blow up Y once in each of the nodes of the anticanonical cycle to obtain $\tilde{Y} \to Y$. There is a morphism $\tilde{Y} \to \tilde{Y}_k$ defined by $\mathcal{O}(-K_{\tilde{Y}})$. The images of the exceptional curves C_1, C_2 of the toric blow up are still (-1) curves. This defines a commutative diagram

$$\begin{tikzcd} \tilde{Y} \arrow[r] \arrow[d] & \tilde{Y}_k \arrow[d] \\ Y \arrow[r] & Y_k \end{tikzcd}$$

where the morphism $\tilde{Y}_k \to Y_k$ is the contraction of the images of C_1 and C_2. Hence we need to show that the family given by $\tilde{Y}_k$ is the same as that of $\tilde{Y}$, as then its further restriction (toric blow down) is the family defined by Y_k.

Equality of the of the fibres over the maximal strata follows from the fact that $\tilde{Y} \to \tilde{Y}_k$ contracts precisely the -2 curves. Any $\mathbb{A}_1$ class β necessarily contains a component C that is not a (-2), as it intersects the anticanonical divisor. So C is contained in the maximal monomial ideal $\mathfrak{m}$ of $\mathrm{NE}(\tilde{Y}_k)$, and thus β vanishes modulo $\mathfrak{m}$.

□

We also need the following proposition.

Proposition 4.1.8. *Let $\pi\colon \mathcal{Y} \to Z$ be a birational contraction with $\mathcal{Y}$ a semi-stable model of the Dolgachev-Nikulin-Voisin family with triangulation $\mathscr{P}$. Then the fibre X of $\mathfrak{X} \to TV(\widehat{\mathrm{Morifan}}(\mathcal{Y}))$ over the stratum S defined by π is a 2:1 cover of $\mathbb{P}^2$ branched in a sextic.*

Proof. Let B_0 be the sextic curve that is the double curve of $X_{\mathscr{P}}$. The deformation of the algebra structure induces deformations of the 2 copies of $\mathbb{P}^2$ and of B_0, as the defining equations are given in terms of theta functions, and these deform. As $\mathbb{P}^2$ is rigid there are still 2 copies of $\mathbb{P}^2$ contained in X, defining a surjection $\mathbb{P}^2 \cup \mathbb{P}^2 \to X$. By flatness, Euler numbes are constant and thus the $\mathbb{P}^2$'s intersect in a (sextic) curve B, which is the deformation of B_0. Denote the two copies of $\mathbb{P}^2$ by X_1 and X_2. Let $B \to X_1$ be the inclusion and let ϕ: $B \to X_2$ the morphism giving the copy of B in X_2. This defines a diagram

$$\begin{tikzcd} B \arrow[r] \arrow[d, "\phi"] & X_1 \arrow[d, "\sigma_2"] \\ X_2 \arrow[r, "\sigma_2"] & \mathbb{P}^2 \end{tikzcd}$$

with σ_i the coordinate change bringing the copies of B into normal form. This defines a 2:1 cover of $\mathbb{P}^2$ branched in B. $\square$

4.2 Surfaces of type $\mathrm{III}_\zeta, \mathrm{III}_\alpha$ and III_1

We return to the deformations of the pillow surface $X_{\mathscr{P}}$. We note the obvious result.

Proposition 4.2.1. *The GHKS family* $\mathfrak{X} \to TV(\widehat{\mathrm{Morifan}}(\mathcal{Y}))$ *has a fibre that is the surface* III_ζ.

Proposition 4.1.3 shows that we can smooth the vertices of $X_{\mathscr{P}}$, thereby obtaining certain surfaces from Laza's list. More precisely, we have the following.

Proposition 4.2.2 (The A_{17} case). *The GHKS family* $\mathfrak{X} \to TV(\widehat{\mathrm{Morifan}}(\mathcal{Y}))$ *has strata with fibres*

(i) A union of 2 *copies of* $\mathbb{P}^2$*'s glued along a reducible cubic.*

(ii) A union of 2 *copies of* $\mathbb{P}^2$*'s glued along an irreducible nodal cubic.*

In particular, $\mathfrak{X} \to TV(\widehat{\mathrm{Morifan}}(\mathcal{Y}))$ *has fibres given by surfaces of type* III_α *and* III_1.

Proof. This is an application of the previous section: we can smooth one or two of the 2-vertices. By construction of the mirror family, the rulings of the components discussed there give a partial smoothing of $X_{\mathscr{P}}$. In particular over the locus Z from Proposition 4.1.3, the fibre of $\mathcal{X}$ is given by a smoothing of a single vertex 2-vertex singularity v to a normal crossing singularity. Here, $\mathcal{X}$ is the mirror family of the Looijenga pair associated to v.

In order to only trivially deform the remaining vertices u, w, we use the maps given by Proposition 4.1.7. Note that the degrees of the respective bundles on the double curves of the central fibre agree so the bundles indeed induce a morphism ϕ on the family $\mathcal{Y}_{\mathscr{P}}$. Concretely, for a suitable numbering of the components Y_i $i = 1, 2, 3$ of the components of the central fibre, ϕ is the morphism given by $\mathcal{O}(-K_{Y_1}) \otimes \mathcal{O}(-K_{Y_2}) \otimes \mathcal{O}(F)$ with F the fibre as in Proposition 4.1.3.

Over the stratum $F(\phi)$ defined by ϕ the deformations of u, w are trivial by Proposition 4.1.7, i.e. the fibres are 2-vertices and as already noted above, the remaining vertex deforms to a normal crossing singularity. In total, the restriction to Ω yields a deformation of the pillow surface $X_{\mathscr{P}}$ to a $2:1$ cover of $\mathbb{P}^2$ branched in a conic and a line. This is a surface of the desired type.

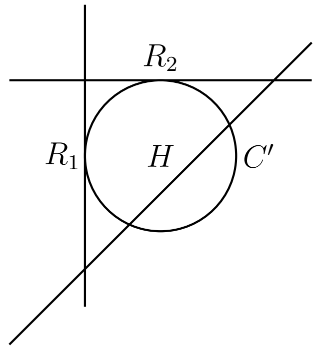

Figure 4.4: The two rulings R_1 and R_2, the smooth section H and the inverse image $C'$of C in Q.

A similar argument shows (ii): The morphism ψ given by $\mathcal{O}(-K_{Y_1}) \otimes \mathcal{O}(F) \otimes \mathcal{O}(F)$ in a suitable numbering of the components of $Y_{\mathscr{P}}$ lifts to a morphism on $\mathcal{Y}_{\mathscr{P}}$. By the Proposition 4.1.3, the two 2-vertices corresponding to the components that are contracted are smoothed to normal crossing singularities while the remaining vertex deforms trivially. Thus, $X_{\mathscr{P}}$ deforms to a surface S_{III_1} that is a $2:1$ cover of $\mathbb{P}^2$ branched in a nodal cubic. □

4.3 Surfaces of type III_γ, III_δ and III_ϵ

In this section we show how to obtain the generic surfaces of the components III_γ, III_δ and III_ϵ. All these have only 1 component, so in addition to the smoothing we saw in the previous section, we shall deform $X_{\mathscr{P}}$ such that the resulting surface is irreducible.

The generic surface X of the III_ϵ component is a double cover of $\mathbb{P}^2$ branched over a conic and two double lines, one tangent to the conic. The normalization of X is a smooth quadric $Q = \mathbb{P}^1 \times \mathbb{P}^1$ in $\mathbb{P}^3$. The quadric Q has two rulings, denoted by R_1 and R_2. Choose a point x not on Q and project to a plane. This gives a $2:1$ map

$$Q \to \mathbb{P}^2.$$

that is branched along a conic $C \subset \mathbb{P}^2$, cf [Har92, p.286]. Let L be a line in $\mathbb{P}^2$ that is tangent to C. Then the inverse image of L consists of 1 line of each of the rulings R_1, R_2. Also, let M be a line that is not tangent to C. The inverse image of M is a smooth hyperplane section, denoted by H, see Figure 4.4.

One can construct X by gluing H to itself using the $2:1$ map $H \to M$ and identifying the two rulings R_1 ans R_2. This gives back the $2:1$ cover X of $\mathbb{P}^2$ branched in a conic and two double lines.

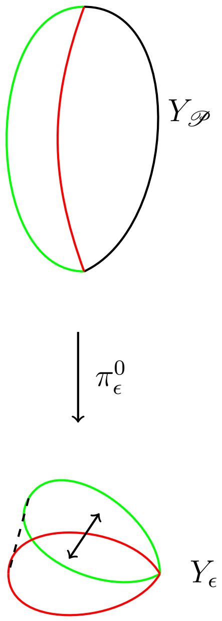

Figure 4.5: The surface Y_ϵ: a component of the double curve of $Y \subset \mathcal{Y}_{\mathscr{P}}$ is contracted. The double arrow indicates gluing resulting from further contracting Y.

Now consider the DNV family $\mathcal{Y}_{\mathscr{P}}$ in (-1)-form. Let Y be a component of the central fibre $Y_{\mathscr{P}}$. Let D_1 be a component of the anticanonical divisor D of Y. There is a (-1)-curve E_1 with $E_1.D_1 = 1$. As above, the divisor $D_1 + E_1$ induces a ruling

$$Y \to \mathbb{P}^1.$$

This ruling induces a morphism

$$\pi_\epsilon^0 \colon Y_{\mathscr{P}} \to Y_\epsilon,$$

where Y_ϵ is given by contracting D_1 and E_1, so Y_ϵ is the surface with Y contracted to a $\mathbb{P}^1$ and an isomorphism on $Y_{\mathscr{P}} \backslash Y_\epsilon$, see Figure 4.5. The morphism π_ϵ^0 extends to a morphism $\pi_\epsilon \colon \mathcal{Y}_{\mathscr{P}} \to \mathcal{Y}_\epsilon$ to a threefold $\mathcal{Y}_\epsilon$ with central fibre Y_ϵ.

Proposition 4.3.1. *Let Y be a component of $Y_{\mathscr{P}}$, $D_1 \subset Y$ a component of the double curve of Y and E_1 a (-1)-curve with $E_1.D_1 = 1$. Let $\pi_\epsilon \colon \mathcal{Y}_{\mathscr{P}} \to Y_\epsilon$ be the induced morphism. Over the stratum $F(\pi_\epsilon)$ determined by π_ϵ, the fibre of the family $\mathfrak{X} \to TV(\widehat{\mathrm{Morifan}(\mathcal{Y})})$ is a double cover of $\mathbb{P}^2$ branched in a conic and two double lines, a surface of type* III_ϵ.

Proof. Write $Y_{\mathscr{P}} = Y_1 \cup Y_2 \cup Y_3$, with $Y = Y_1$. Suppose $D_1 = Y_1 \cap Y_2$. By Proposition 4.1.5, the contraction of D_1 deforms the central fibre of the mirror family induced by Y_2 to a 1-vertex, while π_ϵ^0 restricted to Y is a ruling as in Proposition 4.1.5, and thus deforms the corresponding 2-vertex to $z^2 = y^2(x^2 - y + 1)$. Hence, by Proposition 4.1.8, the deformation of $X_{\mathscr{P}}$ is a $2:1$ cover of $\mathbb{P}^2$ branched in a conic and two double lines, meeting the conic tangentially. This is the surface Q from above. □

One can continue along this line to obtain a surface of type III_γ. Recall that this is given by the double cover of $\mathbb{P}^2$ branched in two double lines and a generic quadric. As above, the normalization of this is the smooth quadric $Q = \mathbb{P}^1 \times \mathbb{P}^1$ in $\mathbb{P}^3$ with two rulings, again denoted by R_1 and R_2. As before, there is a $2:1$ map

$$Q \to \mathbb{P}^2.$$

that is branched along a conic $C \subset \mathbb{P}^2$. Now, let L and M be lines in $\mathbb{P}^2$ that are not tangent to C. Then inverse images of L and M are smooth hyperplane sections, denoted by H and K. One can construct X by gluing H and K to itself using the $2:1$ map. This gives back the $2:1$ cover X of $\mathbb{P}^2$ branched in a conic and two double lines.

We give a morphism that deforms $X_{\mathscr{P}}$ to this surface. Let Y_1, Y_2 and Y_3 be the components of $Y_{\mathscr{P}}$. Then $D = Y_1 \cap Y_2$ is a component of the anticanonical divisor of both Y_1 and Y_2. As we have seen, there exists rulings π_1, π_2 of Y_1 and Y_2 defined by D and a (-1)-curve. There are bundles bundles L_i defining the morphisms $\pi_i\colon Y_1\mathbb{P}^1$. These have degree agreeing on D and have the same (positive) degree on the remaining component of the anticanonical cycle of Y_i. Thus, there is a number n such that $L_1 \otimes L_2 \otimes \mathcal{O}(nK_{Y_3})$ defines a bundle on $Y_{\mathscr{P}}$ and thus on $\mathcal{Y}_{\mathscr{P}}$. Hence there is a morphism

$$\pi_\gamma\colon \mathcal{Y}_{\mathscr{P}} \to \mathcal{Y}_\gamma$$

with $\mathcal{Y}_\gamma$ a threefold with restriction to $Y_{\mathscr{P}}$ the contraction of Y_1 and Y_2.

Proposition 4.3.2. *The morphism $\pi_\gamma\colon \mathcal{Y}_{\mathscr{P}} \to \mathcal{Y}_\gamma$ defines a stratum of the GHKS-family $\mathfrak{X} \to TV(\widehat{\mathrm{Morifan}}(\mathcal{Y}))$, with fibre a surface of type* III_γ.

Proof. Again, the surface $\mathbb{P}^1 \times \mathbb{P}^1$ has toric boundary divisors D_1, D_2, D_3 and D_4, in cyclic order. As before, the contraction defined by D deforms the pillow $X_{\mathscr{P}}$ to a surface Y' given by $\mathbb{P}^1 \times \mathbb{P}^1$, with D_1 identified with D_2 and D_3 identified with D_4 using the $2:1$ map $\mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^2$, i.e. this is a 2:1 covering of $\mathbb{P}^2$ branched in a conic and two double lines tangent to the conic. The further contraction of the exceptional curves E, E' then defines, by Proposition 4.1.5, a smoothing of both 1-vertices of Y' such that the double lines meet the conic in 4 points. This is the surface Q from above, a double cover of $\mathbb{P}^2$ branched in a conic and a two double lines.

□

The generic surface of the component III_δ is obtained as follows. We first take a contraction as defined by the smoothing of the 2-vertex in proposition 4.1.3. Then, there are two components Y_1 and Y_2 of $Y_{\mathscr{P}}$ that are not modified. Hence, we can apply a contraction π as before, i.e contract $Y_1 \cap Y_2$ and a (-1)-curve E. Let π_δ^0 be the composition of these morphisms. It induces a morphism $\pi_\delta \colon \mathcal{Y}_{\mathscr{P}} \to \mathcal{Y}_\delta$.

Proposition 4.3.3. *The morphism π_δ defines a stratum over which the fibre of $\mathfrak{X}$ is a $2:1$ cover of $\mathbb{P}^2$ branched in a conic C and a (double) conic C' that is tangent to C, a surface of type III_δ.*

Proof. The partial smoothing of the 2-vertex given by contracting one component of the central fibre Y_0 via Proposition 4.1.3 deforms the corresponding 2-vertex to a double conic. The remaining contraction is given by contracting the intersection D of the remaining components and then further contracting one component along the ruling. By the same analysis as above, this results in a double cover branched in a double conic meeting a conic tangentially. □

The models of III_δ also smooth to cusp singularities of type $T_{2,3,r}$ with $r \geq 7$. These cannot be obtained with the semi-stable model we have been using so far. We will show how to get such cusp singularities below.

4.4 Surfaces of type $\mathrm{III}_{\zeta'}$, $\mathrm{III}_\phi, \mathrm{III}_2$ and III_5

This locus contains the pillow surface $X_{\mathscr{T}}$ as its deepest degeneration. As in the $X_{\mathscr{P}}$ case, the surface $X_{\mathscr{T}}$ is readily available.

Proposition 4.4.1. *The GHKS family $\mathfrak{X} \to TV(\widehat{\mathrm{Morifan}}(\mathcal{Y}))$ has a fibre that is the surface III_ζ.*

Proof. This is by construction: $X_{\mathscr{T}}$ is the fibre over the 0-stratum given by a projective semi-stable model of the DNV family with dual intersection complex $\mathscr{T}$, say $\mathcal{Y}_{\mathscr{T}}$. □

We now want to deform $X_{\mathscr{T}}$ to the surfaces of type $E_8 + A_1$. The remaining surfaces of this type are given by surfaces $X_1 \cup X_2$ with at least one X_i a del Pezzo of degree 1, the other a cone over a nodal curve, i.e $\mathbb{P}^2$ with two toric boundary divisors identified.

To obtain del Pezzo surfaces, consider the following. Let $Y_{\mathscr{T}} = V_1 \cup V_2 \cup V_3$, where V_1, V_3 are the smooth components. Take a smooth component V_1, weak del Pezzo surfaces with an E_8 root system of effeective (-2)-curves. There is an element c of this root system that meets 3 other roots. Let F_0 be the element among these three that only meets c. Also, there is an element F_7 that meets a unique element of the root system and the 0-curve E_0 meeting the anticanonical divisor given by the double locus in a node, compare

Figure 3.4. Then, contraction of the unique irreducible (-1)-curve and the curves in the E_8 system different from F_0, F_7 and then further contraction along the ruling E_0 defines a morphism $\phi_0\colon Y_{\mathscr{T}} \to Y'_\phi$ contracting V_1. The image of $V_1 \cap V_2$ under ϕ_0 is a (-1)-curve on the normalisation of the image of V_2 and can be contracted. Let Y_ϕ denote the resulting surface. This construction lifts to the threefold $\mathcal{Y}_{\mathscr{T}}$, as shown by the following lemma.

Lemma 4.4.2. *There is a morphism $\phi\colon \mathcal{Y}_{\mathscr{T}} \to \mathcal{Y}_\phi$ lifting the morphism $\cup V_i \to Z_0$ given by ϕ_0 followed by a contraction of the curve $\phi_0(V_1 \cap V_2)$.*

Proof. The composition is a morphism from the central fibre of $\mathcal{Y}_{\mathscr{P}}$. On components, it is given by line bundles $\mathcal{L}_i$ with degree agreeing on the double curves because the morphism $\cup_i V_i \to Y_\phi$ is induced by a morphism from the normalization descending to the push-out. This means there is a lift $\mathcal{Y}_{\mathscr{T}} \to \mathcal{Y}_\phi$ as required.

□

Thus we have a contraction

$$\phi\colon \mathcal{Y}_{\mathscr{T}} \to \mathcal{Y}_\phi.$$

The stratum $F(\phi)$ define by ϕ gives the generic surface of type III_ϕ. To show this, we first analyse the contraction $V_1 \to \mathbb{P}^1 \to \mathrm{Spec}(\mathbb{C})$. Recall from [GHK15a] that (V_1, D), where D is the anticanonical cycle induced by the double curve of $Y_{\mathscr{T}}$, defines a deformation of a 1-vertex.

Proposition 4.4.3. *Let V be a smooth component of $Y_{\mathscr{T}}$ with anticanonical cycle D induced from the double curve of $Y_{\mathscr{T}}$. Let E_0 be the (0)-curve meeting D in a node. Let $\phi_V\colon V \to \mathbb{P}^1 \to \mathrm{Spec}(\mathbb{C})$ be the contraction along the ruling defined by E_0, followed by contraction to a point. The morphism ϕ_V deforms the 1-vertex to a smooth point.*

Proof. Set Let $\tilde{V}$ be a toric blow up with toric model $\tilde{V} \to \bar{V}$ such that $\bar{Y}$ is a ruled surface. This exists by construction of V_2: blowing up the node of D gives a surface V' and anticanonical cycle given by the exceptional curve of $V' \to V$ and the strict transform of D, blowing up the nodes of of this cycle yields a Looijenga pair $(\tilde{V}, \tilde{D})$ with $\tilde{D}$ a cycle of length 4. There is a toric model $\tilde{V} \to \mathbb{P}^1 \times \mathbb{P}^1$.

This defines a commutative diagram

$$\begin{tikzcd}
\tilde{V} \arrow[r] \arrow[d] & \mathbb{P}^1 \times \mathbb{P}^1 \arrow[dr] & \\
Y \arrow[r] & \mathbb{P}^1 \arrow[r] & \mathrm{Spec}(\mathbb{C}).
\end{tikzcd}$$

The toric model $\tilde{Y} \to \mathbb{P}^1 \to \mathbb{P}^1$ defines a trivial deformation of the 4-vertex. As before, over the locus defined by the toric model, we are in a purely toric situation, i.e. the affine manifold B is $\mathbb{R}^2$ with and the fan structure σ is the fan of $\mathbb{P}^1 \times \mathbb{P}^1$. The remaining deformation given by $\mathbb{P}^1 \to \mathbb{P}^1 \to \mathrm{Spec}(\mathbb{C})$ is the Mumford degeneration given by erasing all rays of Σ, see section 1.3 of [GHK15a]. Hence, the algebra structure is given by (1.5) of [GHK15a]. Hence the 4-vertex defined by $\tilde{V}$ deforms to $\mathrm{Spec}(\mathbb{Z}^2)$. As $\tilde{V} \to \mathrm{Spec}(\mathbb{C})$ factors through the contraction $\tilde{V} \to V$, the same is true for the 1-vertex defined by V.

□

Proposition 4.4.4. *The fibre over the stratum defined by ϕ is a surface $X = X_0 \cup_D X_1$ with X_0 a del Pezzo surface of degree 1 and $X_1^{\mu} \cong \mathbb{P}^2$. The gluing curve D is a nodal curve. In particular, surfaces of type III_{ϕ} appear as fibres of the GHKS-family.*

Proof. First, the restriction of ϕ to V_3 is the identity and thus the corresponding component is still a 1-vertex, so it is a cone given by

$$z^2 = y^2(y + x^2).$$

The restriction of ϕ to V_1 smoothes the singularity given by V_1, by Proposition 4.4.3.

The restriction of ϕ to V_2 is corresponds to the contraction of a component of the anticanonical cycle D_2^{ν} of V_2^{ν}, where D_2 is the anticanonical cycle on V_2 induced by the double curve. Hence we have a contraction $V_2^{\nu} \to V'$, a toric blow-up of V'. The surface V' has an anticanonical cycle D' of length 3 induced by $V_2^{\nu} \to V'$. By definition of mirror families of Looijenga pairs, this means that the 4-vertex defined by V_2 deformes to the 3-vertex $\mathbb{V}_3$. In terms of polyhedra, this corresponds to erasing the ray connecting the 4-vertex with the 1-vertex defined by V_1.

Hence, the resulting surface X has two components, one smooth and one a cone as above. Let X_0 be the smooth component. The double curve D defined by the rays of $\mathbb{V}_3$ that lie on both components of X is a degree 1 curve as locally around $\mathbb{V}_3$ it is a union of lines. Also, D is an anticanonical divisor, as X is a model of a central fibre of a type III degeneration of K3 surfaces. Hence (X_0, D) is an anticanonical surface and D is big and nef. Therefore X_0 is a (weak) del Pezzo of degree 1.

□

The same construction can be applied to deform both both components to del Pezzo surfaces.

Proposition 4.4.5. *There is a contraction of $Y_{\mathscr{T}}$ such that the fibre of $\mathfrak{X} \to TV(\widehat{\mathrm{Morifan}(\mathcal{Y})})$ over the induced stratum is a surface of type III_2.*

Proof. By construction, both curves $\phi_0(V_1 \cap V_2)$ and $\phi_0(V_2 \cap V_3)$ can be contracted on the level of central fibres. By the proof of 4.4.2, the morphism contracting both these curves lifts to a morphism $\mathcal{Y}_{\mathscr{T}} \to Z'$. Let Z_0 be the image of $\phi_0(V_2)$. The normalization of Z_0 is the anticanonical surface obtained by blowing down the preimages of $\phi_0(V_1 \cap V_2)$ and $\phi_0(V_2 \cap V_3)$ in the normalization of V_2. Hence, V_2^ν is a toric blow-up of Z_0^ν. Thus, over the central fibre of the stratum induced by $\mathcal{Y}_{\mathscr{T}} \to Z'$, the 4-vertex deforms to a 2-vertex. Also, both 1-vertices deform to $\operatorname{Spec} \mathbb{C}[\mathbb{Z}^2]$, by Proposition 4.4.3. Let X denote the resulting surface. Again, from the polyhedral description of X it follows that $X = X_0 \cup X_1$. Both components are indeed smooth because the 1-vertices are smoothed. As above, the components are anticanonical surfaces glued along a big and nef anticanonical curve of degree 1, i.e. del Pezzo surfaces of degree 1.

□

Now, we show how to obtain cusp singularities of type $T_{2,q,r}$ with $q \geq 3$ and $r \geq 7$. In order to get such a smoothing, we need to change the semi-stable model of the Dolgachev-Nikulin-Voisin family to one that has a central fibre with a component of negative boundary, i.e. for a component $S \subset Y_0$, the intersection matrix $\{D_i.D_j\}$ of the anticanonical cycle has to be negative definite and no component D_i is allowed to be exceptional. We can construct such a model from the $Y_{\mathscr{T}}$ model by flopping enough curves.

Construction 4.4.6. Let $Y_{\mathscr{T}}$ have components V_1, V_2 and V_3, with V_1, V_3 the degree 1 del Pezzos. By construction, there is a a cycle $E_1 + F + E_2$ of 3 curves meeting D_{21} and D_{23} with $E_i^2 = -1$ and $F^2 = -2$ - these are the images of the exceptional curves meeting the components of the toric boundary of the $\mathbb{P}^1 \times \mathbb{P}^1$ that are not identified and the strict transform of the corresponding fibre. Also by construction, there is a (-1)-curve $C = C_1 + C_2 + C_3$ of length 3 on V_3 meeting D_{32} in $E_2 \cap D_{23}$. Hence one can flop C to V_2. The resulting semi-stable model Y_1 is again projective: denote the components of Y_1 by W_1, W_2 and W_3, with W_i the image of V_i.

Let D_{21} be $W_1 \cap W_2$ considered as a curve on W_2. There is a blow down $W_2^\nu \to \mathbb{P}^1 \times \mathbb{P}^1$ with D_{21}^ν the strict transform of a component $\bar{D}_1$ of the toric boundary $\bar{D}$. On $\mathbb{P}^1 \times \mathbb{P}^1$, there is an automorphism defined by the involution on $\bar{D}_1$ that fixes the interior special point and interchanges the nodes of $\bar{D}$ lying on $\bar{D}_1$. This automorphism lifts to W_2^ν by the universal property of blow-ups. Using this automorphism, it follows that there is an ample bundle A_2 on W_2 such that the degree of A_2 is the same on the components of the anticanonical cycle that are identified under $W_2^\nu \to W_2$. Let A_1, and A_3 be ample bundles on V_1 and on V_3.

Because W_1 and W_3 are glued to W_2 along distinct curves high enough multiples of A_1, A_2 and A_3 glue to an ample bundle on Y_1. C. Hence Y_1 carries an ample line bundle that, as we have seen above, lifts to a bundle on $\mathcal{Y}_1$ and this threefold thus is quasi-projective. As it is also proper, it is projective.

Now one can successively flop the (strict transforms of the) curves $E_1 + F + E_2$ and the flopped curves C_1^+ and C_2^+ of C_1 and C_2 to V_1. By the same arguments as above, the resulting threefold $\mathcal{Y}_D$ is projective. Let Y_D be the central fibre. Denoting the transforms of components $V_i \subset Y_{\mathscr{T}}$ again by V_i, we note the following: The boundary of V_1 is negative, with $D_{12}^2 = -4$. The normalisation of V_2 has two -1 curves C_1, C_2 that meet each of the components of the anticanonical with self intersection -1 and the image of C is a (0)-curve. Contracting C_1, C_2 thus gives a degree 8 surface without -1 curves, i.e. $\mathbb{P}^1 \times \mathbb{P}^1$. The component V_3 can be contracted similar as above.

We have the following proposition.

Proposition 4.4.7. *There is a contraction $\mathcal{Y}_D \to \mathcal{S}$ with such that the image of Y_D is a cusp singularity of $\mathcal{S}$. The induced deformation defines a smoothing to a $T_{2,3,r}$ singularity with $r \geq 7$.*

Proof. Take the anticanonical bundle $\mathcal{O}(-K_{V_1})$ on V_1, let $\mathcal{O}(F)$ be the bundle defining the contraction along the ruling on V_2 and $\mathcal{O}(G)$ define the contraction of V_3. After choosing suitable multiples, these bundles define a projective morphism on $\mathcal{Y}_{\mathcal{D}}$ contracting the components V_2 and V_3. Denote this morphism by $\pi_T \colon Y_{\mathcal{D}} \to \mathcal{Y}_{V_1}$. The threefold $\mathcal{Y}_{V_1}$ is projective.

The component V_1 has negative boundary. Thus there is a contraction $V_1 \to S$ with S a cusp singularity. By an extension of Theorem 7.5 of [GHK15a] to the case of anticanonical cycles with length $n = 1, 2$, there is a stratum I over which the mirror family of V_1 smooths to the dual cusp $\check{S}$ of S. By the Hirzebruch-Zagier algorithm, $\check{S}$ has type $T_{2,3,r}$ with $r \geq 7$. Let $V_1 \to S_0$, for some surface S_0, denote the corresponding contraction. Post-composing the restriction of the morphism $\pi_T \colon Y_{\mathcal{D}} \to \mathcal{Y}_{V_1}$ to the central fibre with this contraction then induces a contraction

$$\pi_D \colon Y_{\mathcal{D}} \to \mathcal{Y}_{V_1} \to \mathcal{S}$$

with $\mathcal{S}$ a threefold with central fibre S_0. By construction, the fibre of the deformation induced by π_D is a surface with a cusp singularity of type $T_{2,3,r}$ with $r \geq 7$.

□

4.5 Surfaces of type III_4, III_β and III_6

We show how to obtain generic models of the strata III_4, III_β and III_6. To begin, consider again the model giving the III_γ model: there, the surface is obtained by deleting an edge connecting two 2-vertices and smoothing the 1-vertices thus obtained. One can then delete one of the remaining edges, thereby obtaining a 1-vertex and a double line with two pinch points on it. Let $\pi_\gamma\colon \mathcal{Y}_{\mathscr{P}} \to \mathcal{Y}_\gamma$ be the contraction defining the model III_γ. Let σ be the morphism obtained from this by post-composing with the contraction of a component of the anticanonical cycle of the remaining component of the central fibre Y_0. By similar arguments as above, the composition lifts to a morphism of threefolds

$$\mathcal{Y}_{\mathscr{P}} \to Z,$$

which we again denote by σ.

Proposition 4.5.1. *The morphism $\sigma\colon \mathcal{Y}_{\mathscr{P}} \to Z$ defines a stratum over which the fibre of the GHKS family is given by a surface of type* III_4.

Proof. Let $\pi_\gamma\colon \mathcal{Y}_{\mathscr{P}} \to \mathcal{Y}_\gamma$ be the contraction defining the model III_γ. Considering the normalisation, the remaining component V has an anticanonical cycle $D = D_1 + D_2$ of length 2 with intersection numbers $D_i^2 = -1$. Contracting -say- D_1 then by Proposition 4.1.5 yields a branch locus given by a double line and a conic, intersecting tangentially. Let Y_0 be the component of $Y_{\mathscr{P}}$ that is contracted to a point. By Proposition 4.4.3, the $2-$vertex associated to Y_0 is smoothed to $\operatorname{Spec}\mathbb{C}[\mathbb{Z}^2]$. The resulting surface X is a 2:1 cover of $\mathbb{P}^2$, with double curve given by a quartic meeting a double line tangentially. This is a surface of type III_4. □

The next surface we want to obtain is type III_β. We again use the model $\mathcal{Y}_{\mathscr{P}}$. Take a component V_1 of $Y_{\mathscr{P}}$. Consider an irreducible -1-curve C. Flop it to the adjacent component, say V_0. Let V_2 be the third component. Let C' be the exceptional curve of the first kind on V_2 meeting $V_2 \cap V_0$. Flop it to V_0. This yields a projective semi-stable model $\mathcal{Y}_\beta$ of the Dolgachev-Nikulin-Voisin family, as follows.

Proposition 4.5.2. *The model $\mathcal{Y}_\beta$ is projective.*

Proof. We can write the last flop as

$$\begin{tikzcd} Y'_{\mathscr{P}} \arrow[rr, dashed] \arrow[dr] & & Y' \arrow[dl] \\ & Y & \end{tikzcd}$$

with $Y'_{\mathscr{P}}$ the result of the first flop and $Y = \cup_i V'_i$, $i = 0,1,2$ with V'_1 the blow down of V_1, and $V'_2 \cong V_2$ and V'_0 the blow up of V_0. Here, V'_1, V'_2 and V_0 are weak del Pezzo surface of degree 3, 3 and 1, respectively. For each $i = 0,1,2$, the cone of curves of V'_i is finitely polyhedral by Theorem 5.1.3.1 in [ADHL14], as V'_i has only finitely many (-1)- and (-2)-curves, as it is a weak del Pezzo. Also, being an anticanonical surface, there are no curves with $C^2 < -2$ that are not components of the anticanonical divisor $D = D_1 + D_2$.

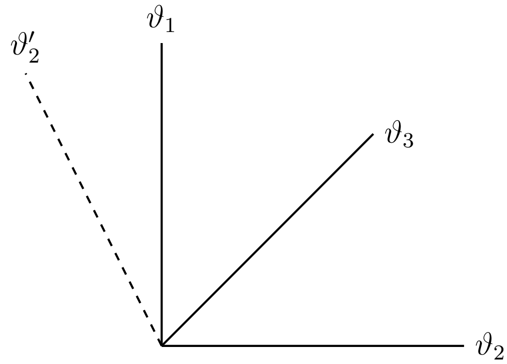

Figure 4.6: The affine manifold B.

Also, the components of the boundary have non-positive selfintersection, and thus either generate extremal rays by Lemma 1.22 in [KM98], or have self-intersection 0. Hence, by Lemma 4.5.5 below, we can fix an ample bundle A_i on V_i' with degree n_i on each boundary component. Then the bundles $n_1n_2A_3$, $n_2n_3A_1$ and $n_1n_3A_2$ glue to an ample bundle on Y. As the morphism $Y' \to Y$ is projective, this means that Y' is projective and thus carries an ample bundle that extends to $\mathcal{Y}_\beta$, by 1.2.17 in [Laz04].

□

Write the central fibre of $\mathcal{Y}_\beta$ as $Y_0 \cup Y_1 \cup Y_2$. It has a component, say, Y_0, with anticanonical cycle $D = D_1 + D_2$,with $D_i^2 = -2$. The remaining components Y_1, Y_2 are toric blow-ups of Looijenga pairs with anticanonical cycle of length 1. Let ϕ_β be the morphism contracting Y_1 and Y_2 via their ruling. Take a component of the boundary of Y_0, say $D_1 = Y_0 \cap Y_1$. For any curve C on Y_0 not equal to D_1, $D_1.C \geq 0$. Hence there is a morphism contracting precisely D_1. Post-compose the contraction of Y_1, Y_2 with this morphism. As above, the resulting morphism lifts to a morphism Φ_β on the threefold.

Proposition 4.5.3. *There is a contraction $\Phi_\beta \colon \mathcal{Y}_\beta \to Z$ such that the fibre of the GHKS-mirror family over the stratum $F(\Phi_\beta)$ is a surface of type* III_β.

Proof. By Proposition 4.1.5, contraction of Y_2 along the ruling deforms the branch curve of the fibre of the mirror family locally to a double line through a conic. The contraction of Y_1 to a point smoothes the branch curve locally around the associated vertex. The local picture around the vertex associated to Y_0 is calculated as follows. The affine manifold induced by Y_0 is depicted in Figure 4.6, with $v_2 = (1,0), v_1 = (0,1), v_3 = (1,1)$ and $v_2' = (-1,2)$. The ray v_3 is a toric blow up. By the multiplication rule for theta functions, noting that contraction of D_1 means that broken lines can cross the rays v_1, v_3, but do not pick up anything, we have, with $m = (-1, 3)$ on the interior of any maximal cell,

$$\begin{aligned} \vartheta_1\vartheta_2 &= \vartheta_3 + \vartheta_m \\ \vartheta_3\vartheta_m &= \vartheta_1^4. \end{aligned}$$

After a coordinate transformation as in the proof of Proposition 4.1.5, this yields

$$z^2 = y^2(x^2 - y^2),$$

a double line meeting a node in its singularity. In total, application of Φ_β deforms the pillow $X_{\mathscr{P}}$ to a $2:1$ cover with branch curve a nodal quartic and a double line through the node. This is a surface of type III_β. □

The components of the central fibre of $\mathcal{Y}_\beta$ that are weak del Pezzo surfaces of degree 3 can be contracted along a ruling. This induces a deformation to a $T_{2,q,r}$ singularity with $q \geq 4, r \geq 5$.

Proposition 4.5.4. *There is a contraction* $\tau\colon \mathcal{Y}_\beta \to \mathcal{S}$ *such that the induced deformation defines a smoothing to a* $T_{2,q,r}$ *singularity with* $q \geq 4, r \geq 5$.

Proof. The components Y_1 and Y_2 of the central fibre Y_β of $\mathcal{Y}_\beta$ have a ruling similar to the components in the model $Y_{\mathscr{P}}$. This gives a morphism $\psi_T\colon Y_\beta \to \mathcal{Y}_0$ where $\mathcal{Y}_0$ has central fibre Y_0' that is Y_0 glued to itself along the boundary. Y_0 has negative boundary. Thus there is a contraction $Y_0 \to S_0$ factoring through Y_0' with S_0 a cusp singularity. Again, by an extension of Theorem 7.5 of [GHK15a] to the case of anticanonical cycles with length $n = 1, 2$, there is a stratum I over which the mirror family of Y_0 smooths to the dual cusp $\check{S}$ of S. Post-composing the restriction of the morphism $Y_\beta \to \mathcal{Y}_0$ to the central fibre with this contraction then induces a contraction

$$\psi_D\colon \mathcal{Y}_\beta \to \mathcal{Y}_0 \to \mathcal{S}$$

with $\mathcal{S}$ a threefold with central fibre S_0. The (normalization of the) fibre of the deformation induced by ψ_D is a surface with a cusp singularity of type $T_{2,q,r}$, $q \geq 4, r \geq 5$, where the type follows because the cusp is a smoothing of a surface of type III_β. □

Lemma 4.5.5. *Let* (Y, D) *be an anticanonical pair with anticanonical cycle* $D = D_1 + D_2$ *and suppose* $\overline{\mathrm{NE}}(Y)$ *is locally polyhedral. Let* A *be an ample divisor such that* $A.D_1 > A.D_2$. *Suppose* D_1 *is extremal or* $D_1^2 \in \{0, 1\}$. *Then there is an ample divisor* A' *on* Y *such that* $A'.D_1 = A'.D_2$.

Proof. Let $A.D_1 = m_1$ and $A.D_2 = m_2$ and suppose $m_1 > m_2$. Assume first the ray generated by D_1 is extremal. Let E denote the set of extremal rays of $\overline{\mathrm{NE}}(Y)$. Consider the convex hull C of $E \backslash \{\mathbb{R}_+[D_1]\}$. This is a convex cone contained in the closed cone $\overline{\mathrm{NE}}(Y)$. Let $\{z_i\}_i$ be a convergent sequence with limit $z \in \mathrm{NE}(Y)$.

Take a neighbourhood of z such that $\overline{\mathrm{NE}}(Y)$ is polyhedral, with generators $E_1, \ldots, E_m$ for some extremal rays E_i such that $z_i \in \mathrm{conv}(\{\mathbb{R}_+ E_j\}_j)$ for all i. Then $z_i = \sum_{j=1}^m a_j E_j$, and if $E_j = \mathbb{R}_+[D_1]$, then $a_j = 0$. As the convex hull of finitely many rays is closed, this implies z is in C. So C is a proper closed convex subcone of $\overline{\mathrm{NE}}(Y)$. By Lemma 6.7 of [Deb13], there is a linear form positive on $C - 0$ and vanishing on $\mathbb{R}_+[D_1]$. By the same Lemma, this means that the interior of C^* intersects the rational hyperplane $(\mathbb{R}_+[D_1])^\perp$ and thus there is a divisor M on $\mathcal{Y}'$ that is positive on C and zero on D_1. In particular, M is nef. Let $M.D_2 = n > 0$. Then $nA + n(m_1 - m_2)M$ is ample and

$$(nA + n(m_1 - m_2)M).X = \begin{cases} nm_1 & \text{if } X = D_1 \\ nm_2 + nm_1 - nm_2 & \text{if } X = D_2. \end{cases}$$

This proves the lemma if D_1 is extremal. If instead $D_1^2 \in \{0, 1\}$, one takes $M := D_1$. □

4.6 Surfaces of type II_1, II_2, II_3 and II_4

We show how to obtain certain type II limit surfaces. In all cases considered, we smooth the branch curves by contracting all components of the central fibre of a model of the Dolgachev-Nikulin-Voisin family. As the procedure is by now familiar, we will be somewhat sketchy.

Consider the model $\mathcal{Y}_{\mathscr{P}}$ with central fibre $Y_{\mathscr{P}}$. We have seen that each of the components can be contracted along a ruling. Also, the sheaves giving these contractions glue to a sheaf on $Y_{\mathscr{P}}$. Hence there is a contraction

$$\Phi_1 \colon \mathcal{Y}_{\mathscr{P}} \to \mathcal{Z}_1$$

to a projective threefold.

Proposition 4.6.1. *The stratum determined by Φ_1 has fibre given by a double cover of $\mathbb{P}^2$ branched in a smooth cubic, a model of type II_1.*

Proof. This follows as before: each of the components is contracted along the ruling of Proposition 4.1.3, and thus the branch curve is smoothed at each of the vertices. Hence the nodal cubic deforms to a smooth cubic, and the resulting surface is indeed a 2:1 cover of $\mathbb{P}^2$ ramified in a smooth cubic. □

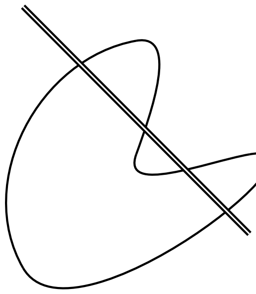

Figure 4.7: The generic surface of type II_4

Next, consider the model $\mathcal{Y}_{\mathscr{T}}$. We can contract the del Pezzo surfaces of degree 1 along a ruling as before. The normalization of the remaining component V has, by construction, a ruling with fibre the double curves meeting the other components. This contraction descends to V. These morphisms compose to a morphism

$$\Phi_2 \colon \mathcal{Y}_{\mathscr{T}} \to \mathcal{Z}_2.$$

Proposition 4.6.2. *The stratum determined by Φ_2 has fibre given by two del Pezzo surfaces of degree 1 glued along an elliptic curve, a surface of type II_2.*

Proof. This is the same proof as Proposition 4.4.4, only now the 4-vertex is smoothed via Proposition 4.1.3. The resulting surface thus consists of two del Pezzo surfaces of degree 1 glued along a smooth elliptic curve. □

Consider the morphism $\sigma \colon \mathcal{Y}_{\mathscr{P}} \to \mathcal{Z}_\delta$ giving the surface of type III_δ i.e. a double cover of $\mathbb{P}^2$ branched in a double conic meeting a conic tangentially. The central fibre of Z_δ has component a dP_3 with a ruling as in Proposition 4.1.5. Post-composing $\sigma_{|Y_{\mathscr{P}}}$ with this ruling then defines a morphism

$$\Phi_3 \colon \mathcal{Y}_{\mathscr{P}} \to \mathcal{Z}_3.$$

Proposition 4.6.3. *The stratum determined by Φ_3 has fibre given by a $2:1$ cover of $\mathbb{P}^2$ branched in two conics, a surface of type II_3.*

Proof. The fibre is a deformation of the surface mirror to $\sigma \colon \mathcal{Y}_{\mathscr{P}} \to \mathcal{Z}_\delta$. Because we contract the component corresponding to the locus where the mirror family is locally a double conic tangent to a conic along a ruling as above, by Proposition 4.1.5, the branch curve is (locally) deformed to two conics intersecting transversally. In total, this gives a surface of the desired type. □

Consider the morphism $\sigma\colon \mathcal{Y}_{\mathscr{P}} \to Z$ giving the surface of type III_4. The central fibre of Z has component a dP_3 with a ruling as in Proposition 4.1.5. Post-composing $\sigma_{|Y_{\mathscr{P}}}$ with this ruling then defines a morphism

$$\Phi_4\colon \mathcal{Y}_{\mathscr{P}} \to \mathcal{Z}_4.$$

Proposition 4.6.4. *The stratum determined by Φ_4 has fibre given by a double cover of $\mathbb{P}^2$ branched in a quartic and a double line through it. This is a model of type* II_4.

Proof. This is similar to how surfaces of type III_4 are obtained. Contracting the component that locally gives the 2:1 cover branched in a conic tangent to a double line deforms the branch curve of that model further. By Proposition 4.1.5, the double line tangent to a conic (locally) gets deformed to a double line through a conic. In total, the branch curve is a double line through a quartic. $\square$

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