On Tangles and Trees

Dissertation

zur Erlangung des Doktorgrades
an der Fakultät für Mathematik, Informatik
und Naturwissenschaften
der Universität Hamburg

vorgelegt im Fachbereich Mathematik
von

Daniel Weißauer

Hamburg, 2018

Erster Gutachter: Prof. Dr. Reinhard Diestel

Zweiter Gutachter: Prof. Mathias Schacht, Ph.D.

Die Disputation erfolgte am 21. August 2018.

Contents

1 Introduction

This thesis is a contribution to structural graph theory and the theory of graph minors. We begin with a broad description of the topic of this thesis. Subsequently, we give a summary of our results and explain how they fit into and expand the existing body of research.

We only consider finite and undirected graphs without loops or parallel edges. Definitions of graph theoretic concepts used throughout the thesis are gathered in Section 1.9.

1.1 The timeless tussle of tangles and trees

The main topic of this thesis is the interplay between highly cohesive substructures, which we figuratively refer to as 'tangles', and decompositions over a (graph-theoretic) tree, which we simply call 'trees'. Neither of these is intended as a precise technical term at this point. We will conceive various objects of different nature as tangles and do not attempt to give a formal definition encompassing all of these. Instead, we deliberately leave the meanings of these expressions undefined and rely solely on the connotations they may carry.

We are primarily interested in situations in which a certain pair of a tangle and a tree cannot coexist: the existence of the specified substructure makes a decomposition of the desired kind impossible and vice versa. For example, both a clique on more than k vertices and a $(k \times k)$-grid-minor are *obstructions* to tree-decompositions of width $< k$ in that a graph cannot contain either of these two tangles and have such a tree-decomposition.

The tangle and the tree are *dual* to one another if the converse holds as well, that is, if every graph either contains this substructure or has the desired decomposition, but not both. The occurence of the tangle is then a sufficient and necessary obstruction for the specified decomposition. Such dualities are not only aesthetically pleasing and intrinsically interesting, but often a useful tool, as they allow to deduce from the *absence* of one structure, the tangle or the tree, the *presence* of the other. For instance, Seymour and Thomas [61] proved the following duality theorem for tree-width:

Theorem 1.1 (Tree-width Duality Theorem [61]). *For every positive integer k, a graph has tree-width $\geq k$ if and only if it contains a bramble of order greater than k.*

For the two examples given earlier, and many others to come, duality does not hold: there exist triangle-free graphs of tree-width $\geq k$ which do not contain the $(k \times k)$-grid as a minor. In some of these cases, there is a partial remedy, which may come in two different flavours or even a combination of the two. First, it might be that duality does hold for a restricted class of graphs. For example, it is well-known (see [18]) that every *chordal* graph without a clique of order greater than k has tree-width $< k$. Thus for chordal graphs, large cliques are dual to tree-decompositions of small width. Second, it might be that duality holds 'qualitatively', that is, if we allow for some numerical trade-off. We refer to such statements as *structure theorems*. For instance, there is no duality between grid-minors and tree-decompositions of small width, but Robertson and Seymour [57] proved the following:

Theorem 1.2 (Grid Minor Theorem [57]). *For every positive integer k there exists a w such that every graph of tree-width $\geq w$ contains the $(k \times k)$-grid as a minor.*

Simpler proofs with better bounds on $w = w(k)$ were later discovered by Diestel, Gorbunov, Jensen and Thomassen [25] and by Robertson, Seymour and Thomas [55]. A recent breakthrough by Chekuri and Chuzhoy [15] shows that there exists a polynomial p such that every graph of tree-width at least $p(k)$ contains the $(k \times k)$-grid as a minor.

In each of our chapters, except for Chapter 4, we study a different pair of a tangle and a tree and possible duality or structure theorems for these.

1.2 Connected tree-decompositions

Intuitively, a tree-decomposition $(T, \mathcal{V})$ of a graph G can be regarded as giving a bird's-eye view on the graph's global structure, represented by T, while each part represents local information about the graph. But this interpretation can be misleading: the tree-decomposition may have disconnected parts, containing vertices which lie at great distance in G, and so this intuitively appealing distinction between local and global structure cannot be maintained.

This can be remedied if we require every part to be connected. We call such a tree-decomposition *connected*. The *connected tree-width* $\mathrm{ctw}(G)$ is defined accordingly as the minimum width of a connected tree-decomposition of the graph G. Trivially, the connected tree-width of a graph is at least as large as its tree-width and, as Jegou and Terrioux [42] observed, long cycles are examples of graphs of small tree-width but large connected tree-width. Diestel and Müller [26] showed that, more generally, the existence of long *geodesic* cycles, that is, cycles in a graph G that contain a shortest path in G between any two of their vertices, raises the connected tree-width. Furthermore, they proved that these two obstructions to small connected tree-width, namely, large tree-width and long geodesic cycles, are qualitatively the only obstructions:

Theorem 1.3 ([26])**.** *There is a function $f : \mathbb{N}^2 \to \mathbb{N}$ such that the connected tree-width of any graph of tree-width at most k and without geodesic cycles of length greater than ℓ is at most $f(k, \ell)$.*

They also showed that $f(k, \ell) = \mathcal{O}(k^3\ell)$. In fact, their proof does not only work with geodesic cycles, but with any collection of cycles that generate the cycle space of the graph G. Given a graph G, we define $\ell(G)$ to be the smallest integer $\ell \geq 3$ such that the cycles of length at most ℓ generate the cycle space of G. Our main result improves the bound of Diestel and Müller significantly:

Theorem 1.4 ([41])**.** *The connected tree-width of a graph G is at most* $\mathrm{tw}(G)(\ell(G)-2)$.

We also discuss an example that demonstrates that this bound is best possible up to a constant factor.

Diestel and Müller [26] conjectured that a duality theorem analogous to Theorem 1.1 holds for connected tree-width and the maximum *connected order* of a bramble: the minimum size of a connected vertex set meeting every element of the bramble. We disprove their conjecture by giving an infinite family of counterexamples.

The study of connected tree-decompositions is the topic of Chapter 2.

1.3 Algebraically grid-like graphs

As above, we denote by $\ell(G)$ the smallest integer $\ell \geq 3$ such that the cycles of length at most ℓ generate the cycle space of G. This is at most the length of a longest geodesic cycle, but can be significantly smaller: consider e. g. an $(n \times n)$-grid where every edge except for those on the boundary is subdivided once. Then the boundary is a geodesic cycle of length $4(n-1)$, while the cycle space is generated by the collection of subdivided squares, each of length at most 8. It is no coincidence that the graph in this example has large tree-width: Diestel and Müller [26] proved that the connected tree-width of a graph containing a geodesic cycle of length k is at least $k/2$, so we obtain the following unexpected corollary of Theorem 1.4:

Corollary 1.5. *Every graph G containing a geodesic cycle of length k has tree-width at least $k/2(\ell(G) - 2)$.*

It really is crucial here that the cycle is geodesic: the wheel-graph contains a long cycle, its cycle space is generated by triangles, but its tree-width is only 3. One might interpret Corollary 1.5 roughly as asserting that the only way to generate a long cycle using short cycles and without distorting its metric, is to arrange the short cycles in a grid-like manner.

Besides the numerical trade-off between tree-width and grid-minors, this interpretation has a more fundamental shortcoming: in Corollary 1.5, we have to assume that the whole cycle space of G is generated by short cycles, while our interpretation would suggest a 'local version' where only the long cycle itself needs to be generated by short cycles.

In Chapter 3, we prove that such a local version of Corollary 1.5 is indeed true:

Theorem 1.6 ([66]). *Let G be a graph containing a geodesic cycle of length at least $2kp$ which can be generated by cycles of length at most p. Then G has tree-width at least k.*

1.4 Steiner trees and higher geodecity

In Chapter 4, we introduce a concept of higher geodecity based on the Steiner distance of a set of vertices, which was introduced by Chartrand, Oellermann, Tian and Zou [14]

Let G be a graph and $\ell : E(G) \to \mathbb{R}^+$ a function that assigns to every edge $e \in E(G)$ a positive *length* $\ell(e)$. This extends naturally to subgraphs $H \subseteq G$ as $\ell(H) := \sum_{e \in E(H)} \ell(e)$. The *Steiner distance* $\mathrm{sd}_G(A)$ of a set $A \subseteq V(G)$, which is defined as the minimum length of a connected subgraph of G containing A, where $\mathrm{sd}_G(A) := \infty$ if no such subgraph exists. Every such minimizer is necessarily a tree and we call it a *Steiner tree for A in G*. In the case where $A = \{x, y\}$, the Steiner distance of A is the ordinary distance $\mathrm{dist}_G(x, y)$ between x and y. Hence this definition yields a natural extension of the notion of "distance" for sets of more than two vertices. Corresponding notions of radius, diameter and convexity have been studied in the literature [14, 2, 1, 3, 37, 8]. Here, we initiate the study of *Steiner geodecity*, with a focus on structural assumptions that cause a collapse in the naturally arising hierarchy, which we now describe.

Let $H \subseteq G$ be a subgraph of G, equipped with the length-function $\ell|_{E(H)}$. It is clear that for every $A \subseteq V(H)$ we have $\mathrm{sd}_H(A) \geq \mathrm{sd}_G(A)$. For a natural number k, we say that H is *k-geodesic in G* if $\mathrm{sd}_H(A) = \mathrm{sd}_G(A)$ for every $A \subseteq V(H)$ with $|A| \leq k$. We call H *fully geodesic in G* if it is k-geodesic for every $k \in \mathbb{N}$.

By definition, a k-geodesic subgraph is m-geodesic for every $m \leq k$. In general, this hierarchy is strict: we provide, for every $k \in \mathbb{N}$, examples of graphs $H \subseteq G$ with a length-function $\ell : E(G) \to \mathbb{R}^+$ such that H is k-geodesic, but not $(k+1)$-geodesic. On the other hand, it is easy to see that if $H \subseteq G$ is a 2-geodesic *path*, then it is necessarily fully geodesic, because the Steiner distance of any $A \subseteq V(H)$ in H is equal to the maximum distance between two vertices $a, b \in A$. Our first result extends this to all trees:

Theorem 1.7 ([69]). *Let G be a graph with length-function ℓ and $T \subseteq G$ a tree. If T is 2-geodesic in G, then it is fully geodesic.*

Here, it really is necessary for the subgraph to be acyclic. The natural follow-up question is what happens in the case where the subgraph is a cycle.

Theorem 1.8 ([69]). *Let G be a graph with length-function ℓ and $C \subseteq G$ a cycle. If C is 6-geodesic in G, then it is fully geodesic.*

We present examples showing that the number 6 cannot be replaced by any smaller integer. We then lay the foundations for a more general theory, aiming at a deeper understanding of the phenomenon displayed in Theorem 1.7 and Theorem 1.8. Our notion of *shortcut trees*, which is central to our proofs of these two theorems as well as to our more abstract endeavours, leads us to a family of minor-closed classes of graphs, one for every tree, and numerous open questions remain. We also take the opportunity to present the short and easy proof that in any graph G with length-function ℓ, the cycle space of G is generated by the set of fully geodesic cycles.

1.5 On the block-number of graphs

Given $k \in \mathbb{N}$, a set X of at least k vertices of a graph G is $(<k)$-*inseparable* if no two vertices in X can be separated in G by deleting fewer than k vertices. A maximal such set is a k-*block*. The maximum integer k for which G contains a k-block is the *block number* of G, denoted by $\beta(G)$.

The k-blocks of a graph are particularly natural and concrete instances of our concept of a tangle. They were first studied by Mader [51] and have recently attracted substantial attention in the theory of decompositions of graphs, see [12, 10, 11, 9].

Following a question raised in [12], the study of graphs which do not contain k-blocks was initiated by Carmesin, Diestel, Hamann and Hundertmark [13], with a focus on degree-conditions. In Chapter 5, we continue this line of research, but with an emphasis on structural properties of graphs without k-blocks and on the relation of the block number to other width-parameters of graphs.

Diestel, Eberenz and Erde [19] proved a duality theorem for k-blocks and described a class $\mathcal{T}_k$ of tree-decompositions such that a graph has no k-block if and only if it has a tree-decomposition in $\mathcal{T}_k$. The only downside is that $\mathcal{T}_k$ is given rather abstractly and thus seems difficult to work with.

We prove a structure theorem for graphs without k-blocks that involves a simpler class of tree-decompositions which are an obvious obstruction to the existence of a k-block:

Theorem 1.9 ([68])**.** *Let G be a graph and $k \geq 2$ an integer.*

(i) *If G has no $(k+1)$-block, then G has a tight tree-decomposition of adhesion at most k in which every torso has at most k vertices of degree at least $2k(k-1)$.*

(ii) *If G has a tree-decomposition in which every torso has at most k vertices of degree at least k, then G has no $(k+1)$-block.*

This yields a qualitative duality: Every graph either has a $(k+1)$-block or a tree-decomposition which demonstrates that it has no $2k^2$-block.

We also study the block number of graphs in classes of graphs that exclude some fixed graph as a topological minor. Dvořák [28] implicitly characterized those classes $\mathcal{G}$ for which there exists an upper bound on the block number of graphs in $\mathcal{G}$. We make this characterization explicit.

The absence of an absolute bound does not have to be the end of the story, however. For instance, while the tree-width of planar graphs cannot be bounded by a constant, the seminal Planar Separator Theorem of Lipton and Tarjan [49] implies that n-vertex planar graphs have tree-width at most $c\sqrt{n}$ for some constant $c > 0$. We prove a bound on the block number in the same spirit. Note that the Planar Separator Theorem can be extended to arbitrary minor-closed classes of graphs, as shown by Alon, Seymour and Thomas [4], but not to classes excluding a topological minor.

Theorem 1.10 ([68])**.** *Let $\mathcal{G}$ be a class of graphs excluding some fixed graph as a topological minor. There exists a constant $c = c(\mathcal{G})$ such that every $G \in \mathcal{G}$ satisfies $\beta(G) \leq c\sqrt[3]{|G|}$.*

Finally, we relate the block number of a graph to its tree-width. It is easy to see that $\beta(G) \leq \mathrm{tw}(G) + 1$, so the existence of a k-block forces large tree-width. However, a graph can have arbitrarily large tree-width and yet have no 5-block: $(k \times k)$-grids are such graphs. Since tree-width does not increase when taking minors, the tree-width of G (plus one) is even an upper bound for the block number of *every minor* of G. We prove a converse to this statement, namely that a graph with large tree-width must have a minor with large block number.

Theorem 1.11 ([68])**.** *Let $k \geq 1$ be an integer and G a graph. If $\mathrm{tw}(G) \geq 2k^2 - 2$, then some minor of G contains a k-block. This bound is optimal up to a constant factor.*

1.6 In the absence of long chordless cycles, tree-width is a local parameter

In an effort to make the statement of the title precise, let us call a graph parameter P *global* if there is a constant c such that for all k and r there exists a graph G for which every subgraph H of order at most r satisfies $P(H) < c$, while $P(G) > k$. The intention here is that P can be arbitrarily large on G even if it is bounded by the constant c on all its subgraphs of some bounded order.

Tree-width (with $c = 2$) and the chromatic number (with $c = 3$) are global parameters. Indeed, it is a classic result of Erdős [30] that for all k and r there exists a graph of chromatic number $> k$ (and hence tree-width $\geq k$) for which every subgraph on at most r vertices is a forest (and hence has tree-width <2 and chromatic number <3). It is well-known (see [18]) that the situation changes when we restrict ourselves to *chordal graphs*, graphs without chordless cycles of length ≥ 4:

$$\forall k: \text{ Every } K_{k+1}\text{-free chordal graph has tree-width } {<}k. \tag{1.1}$$

Hence the only obstruction for a chordal graph to have small tree-width or chromatic number is the presence of a large clique. In particular, tree-width and chromatic number are *local* parameters for the class of chordal graphs.

In 1985, Gyárfás [40] made a conjecture which implies that chromatic number is a local parameter for the larger class of ℓ*-chordal* graphs, those which have no chordless cycle of length $> \ell$:

$$\forall \ell, r\ \exists k: \text{ Every } K_r\text{-free } \ell\text{-chordal graph is } k\text{-colourable.} \tag{1.2}$$

This conjecture remained unresolved for 30 years and was proved only recently by Chudnovsky, Scott and Seymour [16]. In view of (1.1), it is tempting to think that an analogue of (1.2) might hold with tree-width in place of chromatic number. Complete bipartite graphs $K_{s,s}$, however, are examples of triangle-free 4-chordal graphs of large tree-width. Therefore a verbatim analogue of (1.2) is not possible and any graph whose presence as a subgraph we can hope to force by assuming ℓ-chordality and large tree-width will be bipartite.

On the positive side, Bodlaender and Thilikos [7] showed that every *star* can be forced as a subgraph in ℓ-chordal graphs by assuming large tree-width. However, since stars have tree-width 1, this does not establish locality of tree-width. In Chapter 6, we will prove that in fact *any* bipartite graph can be forced as a subgraph:

Theorem 1.12 ([67])**.** *Let $\ell \geq 4$ be an integer and F a bipartite graph. Then there exists an integer k such that every ℓ-chordal graph of tree-width $\geq k$ contains F as a subgraph.*

This shows that tree-width is local for ℓ-chordal graphs: Given any integer c, there exists an integer k such that every ℓ-chordal graph of tree-width $\geq k$ has a subgraph isomorphic to $K_{c,c}$, which has order $2c$ and tree-width c.

Theorem 1.12 also has an immediate application to an Erdős-Pósa type problem. Kim and Kwon [43] showed that chordless cycles of length > 3 have the Erdős-Pósa property:

Theorem 1.13 ([43])**.** *For every integer k there exists an integer m such that every graph G either contains k vertex-disjoint chordless cycles of length > 3 or a set X of at most m vertices such that $G - X$ is chordal.*

They also constructed, for every integer $\ell \geq 4$, a family of graphs showing that the analogue of Theorem 1.13 for chordless cycles of length $> \ell$ fails. We complement their negative result by proving that the Erdős-Pósa property *does* hold when restricting the host graphs to graphs not containing $K_{s,s}$ as a subgraph.

Corollary 1.14 ([67])**.** *For all ℓ, s and k there exists an integer m such that every $K_{s,s}$-free graph G either contains k vertex-disjoint chordless cycles of length $> \ell$ or a set X of at most m vertices such that $G - X$ is ℓ-chordal.*

1.7 The structure of graphs excluding a topological minor

One of the landmark results of the graph minors series of Robertson and Seymour is a structure theorem for graphs excluding a fixed graph as a minor [60]. It is easy to see that G cannot contain H as a minor if there is a surface in which G can be embedded but H cannot. Loosely speaking, the structure theorem of Robertson and Seymour asserts an approximate converse to this, thereby revealing the deep connection between topological graph theory and the theory of graph minors:

Theorem 1.15 ([60] (informal))**.** *For every integer p, every graph excluding K_p as a minor has a tree-decomposition in which every torso is almost embeddable in a surface in which K_p does not embed.*

Consider now the relation of topological minors. It is easy to see that if G contains K_p as a topological minor, then G also contains K_p as a minor. The converse is not true, as there exist cubic graphs with arbitrarily large complete minors. For topological minors, we thus have an additional degree-based obstruction, which is fundamentally different from the topological obstruction of surface-embeddings for graph minors. Grohe and Marx [39] proved a result in a similar spirit to Theorem 1.15 for graphs excluding a fixed graph as a topological minor:

Theorem 1.16 ([39] (informal))**.** *For every integer r, every graph excluding K_r as a topological minor has a tree-decomposition in which every torso either*

(i) *has a bounded number of vertices of high degree, or*

(ii) *is almost embeddable in a surface of bounded genus.*

The proof of Theorem 1.16 given in [39] uses Theorem 1.15 as a black box, and the torsos satisfying condition (ii) can be seen as those regions of the graph which contain no large complete minor. Similarly, in light of Theorem 1.9, our structure theorem for graphs without k-blocks, the torsos satisfying condition (i) of Theorem 1.16 seem to correspond to the regions containing no large block.

Indeed, if G contains a subdivision of a clique, then it contains both a clique minor and a block. Or, put differently, the absence of a clique minor or a block are both obstructions to the existence of a complete topological minor. Theorem 1.16 implies that these are, in a local sense, the only obstructions: any graph without a large topological minor has a tree-decomposition into parts whose torsos either do not contain a large minor, or do not contain a large block. Furthermore, by Theorem 1.15 and Theorem 1.9, the converse is also true: if we can decompose a graph into parts whose torsos either do not contain a large minor or do not contain a large block, then we can refine this tree-decomposition into one satisfying the requirements of Theorem 1.16.

In Chapter 7, we take this as a starting point for a new proof of Theorem 1.16 based on the theory of *profiles* and distinguishing tree-decompositions. We believe that this viewpoint leads to a natural and conceptually clearer proof.

1.8 Tangles in abstract separation systems

Chapter 8 is a contribution to a comprehensive project currently pursued at Hamburg which utilizes the concept of tangles developed by Robertson and Seymour [58] to capture highly cohesive substructures in various contexts within graph theory and beyond.

Up to this point, we have used the word 'tangle' as a synonym for 'highly cohesive substructure', deliberately leaving its precise meaning unspecified. As a minimum requirement, it should be hard to split a 'highly cohesive substructure' into (roughly) equal parts. Consequently, for every low-order separation there is going to be one side which contains ‘most’ of the tangle. The tangle thus induces an orientation of all low-order separations ‘towards it’.

In the tenth paper of their Graph Minors series, Robertson and Seymour [58] turned this upside down by taking it as a defining property of a tangle: a way of consistently orienting all low-order separations. Here, ‘consistency’ has a precise technical sense and the tangles defined that way do not encompass all the substructures we have previously considered as ‘tangles’.

This change of perspective has been nothing short of a change of paradigm and brought with it conceptual advantages as well as some of a practical and technical kind. The aim of the current project has been to push these ideas further and develop an abstract theory of tangles, now again in the broad, unspecific sense of ‘highly cohesive substructures’, based on the fundamental idea underlying the tangles of Robertson and Seymour. Following this thought, we now *define* tangles as consistent orientations of a set of separations. The nature of these ‘separations’ and the meaning of ‘consistency’ will depend on the context, and this flexibility allows us to interpret a wide variety of ‘highly cohesive substructures’ as tangles in this abstract formal sense.

In pursuit of this aim, an axiomatic framework of abstract separation systems and tangles has been developed which allows the generalisation of two fundamental results from [58] about tangles in graphs. One of these is the *tangle-tree theorem*. It says that any set of distinguishable tangles can in fact be distinguished pairwise by a nested set of separations: for every pair of tangles there is a separation in this nested collection that distinguishes them. Since these separations are nested, they split the underlying structure in a tree-like way, and every tangle ‘points’ towards a different node of this tree.

The other fundamental result from [58], the *tangle duality theorem*, asserts that if there are no tangles of a given order, then the entire underlying structure can be split in a tree-like way so that every region corresponding to a node of the structure tree is ‘too small to host a tangle’.

These two theorems have been generalized to the abstract setting by Diestel, Hundertmark and Lemanczyk [21] and by Diestel and Oum [23], respectively. However, the tangle-tree theorem in [21] borrows another aspect of tangles in graphs which is not inherent to separation systems: a submodular order function defined on an ambient universe of separations. The tangle duality theorem in [23], on the other hand, rests entirely on concepts defined within the domain of separation systems. However, every known application of this theorem to concrete instances of separation systems, such as those given by Diestel and Oum [22] and by Diestel and Whittle [24], makes use of such a submodular order function. This raises the question of whether abstract separation systems are structurally rich enough to allow for an interesting theory by themselves or whether they are necessarily dependent on submodular functions.

We show that appealing to the extraneous concept of a submodular order function is indeed unnecessary. To this end, we isolate a structural consequence of the existence of a submodular order function and use it to define what it means for a separation system to be ('structurally') *submodular*. We then show that this condition is sufficient to prove a tangle-tree theorem for *abstract tangles*:

Theorem 1.17 ([20])**.** *Every submodular separation system $\vec{S}$ contains a tree set of separations that distinguishes all the abstract tangles of S.*

We also apply the tangle duality theorem of [23] to show the following:

Theorem 1.18 ([20])**.** *Let $\vec{S}$ be a submodular separation system without degenerate elements in a distributive universe $\vec{U}$. Then exactly one of the following holds:*

(i) *S has an abstract tangle.*

(ii) *There exists an S-tree over $\mathcal{T}^*$.*

1.9 Notation and terminology

Our notation and terminology are mostly standard and follow [18], where the reader may also find some helpful background material. We only give a brief account of those instances where either there does not seem to be a universal norm or we deviate from it.

In most situations, we are only interested in the isomorphism type of a graph and not its concrete set of vertices. Therefore, we will identify isomorphic graphs when there is no danger of confusion. We call a set A of vertices of a graph G *connected* if the subgraph $G[A]$ induced by A is connected. A *model* of a graph H in a graph G is a family $(B_v)_{v\in V(H)}$ of non-empty, disjoint connected sets of vertices of G, indexed by $V(H)$, such that G contains an edge between B_u and B_v whenever $uv \in E(H)$. The sets B_v are called the *branchsets* of the model. The graph H is a *minor* of G if G contains a model of H. Equivalently, H is a minor of G if (a graph isomorphic to) H can be obtained from a subgraph of G by contracting some edges.

A *separation* of G is a tuple (A, B) of sets of vertices such that $V(G) = A \cup B$ and there are no edges between $A \setminus B$ and $B \setminus A$. The set $A \cap B$ is called the *separator* and the *order* of the separation is $|A, B| := |A \cap B|$. We call the separation *tight* if there exists a component K of $G[A \setminus B]$ such that every vertex in $A \cap B$ has a neighbor in K.

Given $k \in \mathbb{N}$, a set X of at least k vertices of a graph G is $(<k)$-*inseparable* if no two vertices in X can be separated in G by deleting fewer than k vertices. Equivalently, for every separation (A, B) of order $<k$, either $X \subseteq A$ or $X \subseteq B$, but not both. A maximal $(<k)$-inseparable set is a k-*block* and can be thought of as a highly connected part of the graph, although it may draw its connectivity from the ambient graph G rather than just the subgraph induced by X itself. The maximum integer k for which G contains a k-block is the *block number* of G, denoted by $\beta(G)$.

A *tree-decomposition* of G is a pair $(T, \mathcal{V})$ of a tree T and a family $\mathcal{V} = (V_t)_{t \in T}$ of sets $V_t \subseteq V(G)$, indexed by the nodes of T, such that

(T 1) every vertex of G lies in some set V_t,

(T 2) for every edge $uv \in E(G)$ there exists a $t \in T$ with $\{u, v\} \subseteq V_t$,

(T 3) $V_{t_1} \cap V_{t_3} \subseteq V_{t_2}$ whenever $t_2 \in t_1 T t_3$.

Equivalenty, (T 1) and (T 3) may be subsumed by demanding that, for every $v \in V(G)$, the set of all $t \in T$ with $v \in V_t$ induces a non-empty subtree of T.

For a subtree $S \subseteq T$, we write $V_S := \bigcup_{s \in S} V_s$. The sets V_t are called the *parts* of $(T, \mathcal{V})$. The *width* of $(T, \mathcal{V})$ is the maximum size of a part minus one. The *tree-width* $\mathrm{tw}(G)$ of G is the minimum width of a tree-decomposition of G. The sets $V_s \cap V_t$ for adjacent nodes $s, t \in T$ are called *adhesion sets*. The maximum size of an adhesion set is the *adhesion* of $(T, \mathcal{V})$.

A tree-decomposition gives the graph a rough tree-like structure. The key property of tree-decompositions is that the (oriented) edges of the tree T induce separations of G, as explained in the following:

Lemma 1.19 ([18])**.** *Let $(T, \mathcal{V})$ be a tree-decomposition of G, let $t_1 t_2 \in E(T)$ and let T_1, T_2 be the components of $T - t_1 t_2$, where $t_1 \in T_1, t_2 \in T_2$. Then (V_{T_1}, V_{T_2}) is a separation of G with separator $V_{t_1} \cap V_{t_2}$.*

We will use this fact freely throughout the thesis. We call (V_{T_1}, V_{T_2}) the separation of G *induced by* (t_1, t_2). Note that (t_2, t_1) induces (V_{T_2}, V_{T_1}). In this way, every edge $st \in E(T)$ has the *order* $|V_s \cap V_t|$, which is the order of the separation it induces in G. We call a tree-decomposition *tight* if every separation of G induced by an edge of T is tight.

The *torso* of $t \in T$ is the graph obtained from $G[V_t]$ by adding an edge between any pair of non-adjacent vertices of V_t which lie in a common adhesion set. We can define the *torso* of a subtree $S \subseteq T$ similarly as the graph obtained from $G[V_S]$ by adding an edge between any pair of non-adjacent vertices of V_S which lie in a common adhesion set.

A tree-decomposition of a torso can be used to *refine* the original tree-decomposition, as follows.

Let $S \subseteq T$ be a subtree and let $(R, \mathcal{W})$ be a tree-decomposition of the torso of S, where we relabel the nodes so that $V(R) \cap V(T) = \emptyset$. Obtain the tree T' from $(T - S) \cup R$ as follows. For every edge $st \in E(T)$ with $s \in S, t \notin S$, the adhesion set $V_s \cap V_t$ is a clique in the torso of S. It follows from Lemma 1.19 that there exists an $r \in R$ with $V_s \cap V_t \subseteq W_r$. In T', add an edge between r and t. Then T' is a tree. Let $V'_t := V_t$ for $t \in T \setminus S$ and $V'_r := W_r$ for $r \in R$. It is easy to verify that $(T', \mathcal{V}')$ is indeed a tree-decompositionof G.

Tree-decompositions also allow *contractions*. Contracting $st \in E(T)$ yields the tree-decomposition $(T', \mathcal{V}')$, where T' is obtained from T by contracting st to a single new node x, $V'_u := V_u$ for $u \in V(T) \setminus \{s, t\}$ and $V_x := V_s \cup V_t$. Observe that contractions do not affect the separations induced by edges $e \in E(T) \setminus \{st\}$ or the torsos of nodes $u \in T \setminus \{s, t\}$.

The *fatness* of a tree-decomposition of an n-vertex graph G is the tuple $(a_0, \dots, a_n)$, where a_i denotes the number of parts of size $n - i$. If $(T, \mathcal{V})$ has lexicographically minimum fatness among all tree-decompositions of adhesion $<k$, we call $(T, \mathcal{V})$ *k-atomic*.

A tree-decomposition is *k-lean* if it has adhesion less than k and for any $s, t \in T$, not necessarily distinct, and any $A \subseteq V_s$, $B \subseteq V_t$ with $|A| = |B| \leq k$, either there is a set of $|A|$ disjoint A-B-paths in G or there is an edge $uw \in E(sTt)$ with $|V_u \cap V_w| < |A|$. Equivalently, $(T, \mathcal{V})$ is k-lean if it has adhesion $<k$ and for all $s, t \in T$ and $1 \leq p \leq k$, either sTt contains an edge of order $<p$ or every separation (U, W) of G with $|U \cap V_s| \geq p$ and $|W \cap V_t| \geq p$ has order at least p.

Carmesin, Diestel, Hamann and Hundertmark [13] observed that the short proof of Thomas' Theorem [64] given by Bellenbaum and Diestel [5] shows the following:

Theorem 1.20 ([5])**.** *Every k-atomic tree-decomposition is k-lean.*

(In fact, Thomas' Theorem [64] treats the case $k = |G|$.)

We are also going to use the following easy observation.

Lemma 1.21 ([68])**.** *Every k-atomic tree-decomposition is tight.*

Proof. Let $t_1t_2 \in E(T)$ and let T_1, T_2 be the components of $T - t_1t_2$, where $t_1 \in T_1$ and $t_2 \in T_2$. Consider the induced separation (V_{T_1}, V_{T_2}) of G, let $X := V_{t_1} \cap V_{t_2}$ be the separator and $H := G[V_{T_1}]$.

Let $C_1, \dots, C_m$ be the components of $H - X$. Assume for a contradiction that $N(C_i) \subsetneq X$ for every $1 \leq i \leq m$. Obtain the tree T' from the disjoint union of m copies $T_1^1, \dots, T_1^m$ of T_1, where each $t \in T_1$ corresponds to m vertices $t^i \in T_1^i$ for $i \in [m]$, and one copy of T_2 by joining t_2 to every t_1^i, $i \in [m]$. For $t \in T_2$ let $V'_t = V_t$, and for $t \in T_1$ let

$V'_{t^i} = V_t \cap (C_i \cup N(C_i))$. Observe that the adhesion of $(T', \mathcal{V}')$ is less than k. Let the fatness of $(T, \mathcal{V})$ be $a = (a_i)_i$ and let the fatness of $(T', \mathcal{V}')$ be $a' = (a'_i)_i$.

Clearly $|V'_{t^i}| \leq |V_t|$ for every $t \in T_1$. If $|V'_{t^i}| = |V_t|$ for some $i \in [m]$, then $V_t \subseteq C_i \cup N(C_i)$ and for all $j \neq i$ we have $|V'_{t^j}| \leq |N(C_i)| < |X|$. Choose $t \in T_1$ with $r := |G| - |V_t|$ minimum under the condition that there is no i with $|V_{t^i}| = |V_t|$. Since $N(C_i) \subsetneq X$ for every $i \in [m]$, the node t_1 satisfies this condition. Thus $r \leq |G| - |V_{t_1}| \leq |G| - |X|$. Then $a_s = a'_s$ for all $s < r$ and $a_r > a'_r$, so that a' is lexicographically smaller than a, a contradiction. $\square$

2 Connected tree-decompositions

Recall that a tree-decomposition is *connected* if every part induces a connected subgraph. The *connected tree-width* of a graph G is then the minimum width of a connected tree-decomposition. Given a graph G, we define $\ell(G)$ to be the smallest natural number ℓ such that the cycles of length at most ℓ generate the cycle space of G. The main result of this chapter is the following bound on the connected tree-width:

Theorem 2.1. *The connected tree-width of any graph G is at most* $\mathrm{tw}(G)(\ell(G) - 2)$.

This will be proved in Sections 2.2 and 2.3. In Section 2.5, we discuss an example that demonstrates that this bound is best possible up to a constant factor.

Diestel and Müller [26] observed that every bramble in a graph G has connected order at most $\mathrm{ctw}(G) + 1$, so that Theorem 2.1 immediately yields an upper bound on the connected order of any bramble in G. Here, the *connected order* of a bramble is the minimum size of a connected vertex-set meeting every element of the bramble. Using the techniques developed in the proof of Theorem 2.1, we will give a slightly stronger bound on the maximum connected order of a bramble in Section 2.4.

Diestel and Müller [26] conjectured an analogue of the tree-width duality theorem of Seymour and Thomas [61] for connected tree-width that is, that every graph G contains a bramble of connected order $\mathrm{ctw}(G) + 1$. In Section 2.6, we construct an infinite family of counterexamples to this conjecture. Section 2.7 contains some concluding remarks.

2.1 Preliminaries

In our proof of Theorem 2.1 we will make use of an explicit procedure that transforms a given tree-decomposition into a connected tree-decomposition by iteratively adding paths to a disconnected part of the decomposition. For this to work efficiently, we will restrict ourselves to paths of a particular kind.

A *rooted tree* is a tree T with a specified *root* $r \in V(T)$. A node $s \in T$ is a *descendant* of $t \in T$ (and t is an *ancestor* of s) if t lies on the path between s and r in T. The descendants of t then form a subtree $D_t \subseteq T$. A descendant s of t is a *child* of t if $st \in E(T)$.

Let $(T, \mathcal{V})$ be a *rooted* tree-decomposition of G, i. e. T is rooted. For $t \in T$, a path P in G is *t-admissible* if it lies entirely in V_{D_t}, joins different components of V_t and is shortest possible with these properties. Note that t-admissible paths have precisely two vertices in V_t:

Lemma 2.2. *Let $(T, \mathcal{V})$ be a rooted tree-decomposition of a graph G, $t \in T$ and P a t-admissible path. Then there is a unique child s of t such that all internal vertices of P lie in $V_{D_s} \setminus V_t$.* □

In general, t-admissible paths need not exist. However, as we shall see, we can easily confine ourselves to tree-decompositions that always have t-admissible paths.

We call a tree-decomposition $(T, \mathcal{V})$ *stable* if for every edge $t_1 t_2 \in E(T)$ both V_{T_1} and V_{T_2} are connected in G, where for $i \in \{1, 2\}$ we denote by T_i the component of $T - t_1 t_2$ containing t_i. (Later, we will use this naming convention without further mention.)

Lemma 2.3. *Let $(T, \mathcal{V})$ be a rooted stable tree-decomposition of a connected graph G. Then every $t \in T$ with disconnected V_t has a t-admissible path.* □

Stable tree-decompositions were also studied in [32], where they are called 'connected tree-decompositions'. In that article, an explicit algorithm is presented that turns a tree-decomposition of a connected graph into a stable tree-decomposition without increasing its width. For our purposes it suffices to know that every connected graph has a stable tree-decomposition of minimum width. This can also be deduced from [26, Corollary 3.4].

Proposition 2.4. *Every connected graph G has a stable tree-decomposition of width $\mathrm{tw}(G)$.* □

If we add a t-admissible path P to a part V_t in order to join two of its components, we might not obtain a tree-decomposition. The following lemma shows how it can be patched.

Lemma 2.5. *Let $(T, \mathcal{V})$ be a rooted tree-decomposition of a graph G, $t \in T$ and P a t-admissible path. For $u \in T$ let*

$$W_u := \begin{cases} V_u \cup (V(P) \cap V_{D_u}), & \text{if } u \in T_t, \\ V_u, & \text{if } u \notin T_t. \end{cases} \tag{$*$}$$

Let $\mathcal{W} = (W_u)_{u \in T}$. Then $(T, \mathcal{W})$ is a tree-decomposition of G. For all $u \in T$, every component of W_u contains a vertex of V_u. If $(T, \mathcal{V})$ is stable, so is $(T, \mathcal{W})$.

Proof. Since $V_u \subseteq W_u$ for all $u \in T$, every vertex and every edge of G is contained in some part W_u.

Let I be the set of internal vertices of P. By Lemma 2.2 there is a unique child s of t such that $I \subseteq V_{D_s} \setminus V_t$. For $x \notin I$, the set of parts containing x has not changed. For $x \in I$, the set $A_x := \{u \in T\colon x \in V_u\}$ induces a subtree of D_s and $x \in W_u$ if and only if $u \in A_x$ or u lies on the path joining t to A_x. So $\{u\colon x \in W_u\}$ is also a subtree of T.

Note that every component of $P \cap V_{D_u}$ is a path with ends in V_u. Therefore every $x \in W_u \setminus V_u$ is joined to two vertices in V_u and thus every component of W_u contains vertices from V_u.

Suppose now $(T, \mathcal{V})$ is stable, let $t_1t_2 \in E(T)$ and $i \in \{1, 2\}$. Then V_{T_i} is connected. For $x \in W_{T_i} \setminus V_{T_i}$ there is a $u \in T_i \cap D_t$ with $x \in W_u \setminus V_u$. But then, by the above, W_u contains a path joining x to V_{T_i}. As $V_{T_i} \subseteq W_{T_i}$, also W_{T_i} is connected. $\square$

2.2 The construction

We now describe a construction that turns a stable tree-decomposition $(T, \mathcal{V})$ of a connected graph into a connected tree-decomposition. First, choose a root r for T and keep it fixed. It will be crucial to our analysis that the nodes of T are processed in an order compatible with the tree-order, i. e. we enumerate the nodes $t_1, t_2, \ldots$ so that each node precedes its descendants and we process the nodes in this order.

Initially we set $W_t = V_t$ for all $t \in T$. Throughout the construction, we maintain the invariant that $(T, \mathcal{W})$ is a stable tree-decomposition *extending* $(T, \mathcal{V})$, by which we mean that they are tree-decompositions over the same rooted tree, satisfying $V_t \subseteq W_t$ for all $t \in T$.

When processing a node $t \in T$ with disconnected part W_t, we use the stability of $(T, \mathcal{W})$ to find a t-admissible path by Lemma 2.3 and update $\mathcal{W}$ as in $(*)$. By Lemma 2.5, this does not violate stability and it clearly reduces the number of components of W_t by one. We iterate this until W_t is connected. Once that is achieved, we continue with the next node in our enumeration.

Observe that each 'update' only affects descendants of the current node. Once a node $t \in T$ has been processed, so have all of its ancestors. Hence, no further changes are made to W_t afterwards. In particular, W_t remains connected. It thus follows that, when every node has been processed, the resulting tree-decomposition is indeed connected.

In order to control the size of each part W_u, we will use a *bookkeeping graph* Q_u to keep track of what we have added. Initially, Q_u is the empty graph on V_u, and in each step Q_u is a graph on the vertices of W_u. Whenever something is added to W_u, we are considering a t-admissible path P for some ancestor t of u and P contains vertices of W_{T_u}. Every component of $P \cap W_{D_u}$ is a path with ends (and possibly also some internal vertices) in W_u. We then add $P \cap W_{D_u}$ to Q_u, that is, we add all the vertices not contained in W_u and all the edges of $P \cap W_{D_u}$.

Lemma 2.6. *During every step of the procedure, Q_u is acyclic.*

Proof. This is certainly true initially. Suppose now that at some step a cycle is formed in Q_u. By definition, it must be that an ancestor t of u is being processed and a t-admissible path P is added such that two vertices $a, b \in W_u$, which were already connected in Q_u, lie in the same component of $P \cap W_{D_u}$.

The vertices a, b being connected in Q_u by a path $a = a_0 a_1 \ldots a_n = b$ means that for every $0 \leq j \leq n-1$ there has been an ancestor t_j of u that added a path P_j such that a_j, a_{j+1} were consecutive vertices on a segment S_j of $P_j \cap W_{D_u}$. By the order in which the nodes are processed and by $(*)$, these t_j are also ancestors of t. Therefore when P_j was added to W_{t_j}, the segment S_j was contained in a segment of $P_j \cap W_{D_t}$, since $W_{D_t} \supseteq W_{D_u}$. Therefore, at the time P is added to W_t, all these segments are contained in W_t and, in particular, $a, b \in W_t$.

By Lemma 2.2, P does not have internal vertices in W_t, so a and b must in fact be the ends of P. But W_t already contains a walk from a to b, consisting of the segments S_j, so that the two do not lie in different components of W_t, contradicting the t-admissibility of P. □

We now show how the sparse structure of Q_u reflects the efficiency of our procedure.

Lemma 2.7. *The number of components of Q_u never increases. Whenever $|W_u|$ increases, the number of components of Q_u decreases.*

Proof. Suppose that in an iteration a change is made to Q_u. Then an ancestor t of u is being processed and the chosen t-admissible path P meets W_{D_u}. Every component of $P \cap W_{D_u}$ is a path with ends in W_u. Therefore, every newly introduced vertex is joined by a path to a vertex already contained in Q_u and no new components are created.

If a vertex from $P \cap (W_{D_u} \setminus W_u)$ is added to W_u, the segment containing it has length at least two and its endpoints a and b are already contained in Q_u. By Lemma 2.6, Q_u must remain acyclic, so that a and b in fact lie in different components of Q_u, which are now joined by a path, thus reducing the number of components of Q_u. □

The previous lemma allows us to control the number of iterations that affect a fixed node $t \in T$. The second key ingredient for the proof of Theorem 2.1 will be to bound the length of each of the paths used, see Section 2.3.

Proposition 2.8. *Let G be a connected graph, $(T, \mathcal{V})$ a rooted stable tree-decomposition of G. For $t \in T$ let $m_t \geq 1$ be such that for every stable tree-decomposition $(T, \mathcal{W})$ extending $(T, \mathcal{V})$ and every ancestor t' of t, the length of a t'-admissible path in $(T, \mathcal{W})$ does not exceed m_t. Then the construction produces a connected tree-decomposition $(T, \mathcal{U})$ in which for all $t \in T$*

$$|U_t| \leq m_t(|V_t| - 1) + 1.$$

Proof. We have already shown that $(T, \mathcal{U})$ is connected. By Lemma 2.7, every time $|W_t|$ increased, the number of components of Q_t decreased and it never increased. Since initially Q_t had precisely $|V_t|$ components, this can only have happened at most $|V_t| - 1$ times. In each such iteration we added some internal vertices of a t'-admissible path in a stable tree-decomposition extending $(T, \mathcal{V})$ for some ancestor t' of t, thus we added at most $m_t - 1$ vertices. In total, we have

$$|U_t| \leq |V_t| + (m_t - 1)(|V_t| - 1) = m_t(|V_t| - 1) + 1. \qquad \square$$

2.3 Bounding the length of admissible paths

We will now use ideas from [26] to bound the length of t-admissible paths in stable tree-decompositions. Together with Proposition 2.8, this will imply our main result.

Lemma 2.9. *Let G be a graph and Γ a set of cycles generating the cycle space of G. Let $(T, \mathcal{V})$ be a stable tree-decomposition of G and $t_1 t_2 \in E(T)$. Suppose that $V_{t_1} \cap V_{t_2}$ meets two distinct components of V_{t_1}. Then there is a cycle $C \in \Gamma$ such that some component of $C \cap V_{T_2}$ meets V_{t_1} in two distinct components.*

Proof. As V_{T_2} is connected, we can choose a shortest path P in V_{T_2} joining two components of V_{t_1}. Let $x, y \in V_{t_1}$ be its ends and note that all internal vertices of P lie in $V_{T_2} \setminus V_{t_1}$. As V_{T_1} is connected as well, we also find a path $Q \subseteq V_{T_1}$ joining x and y, which is then internally disjoint from P. By assumption, there is a subset $\mathcal{C}$ of Γ such that $P + Q = \bigoplus \mathcal{C}$. We subdivide $\mathcal{C}$ as follows: $\mathcal{C}_1$ comprises all those cycles which are entirely contained in $V_{T_1} \setminus V_{T_2}$, $\mathcal{C}_2$ those in $V_{T_2} \setminus V_{T_1}$ and $\mathcal{C}_X$ those that meet $X := V_{t_1} \cap V_{t_2}$.

Assume now for a contradiction that for every $C \in \mathcal{C}_X$ and every component S of $C \cap V_{T_2}$ there is a unique component D_S of V_{t_1} met by S. Note that S is a cycle if $C \subseteq V_{T_2}$ and a path with ends in X otherwise. Either way, the number of edges of S between X and $V_{T_2} \setminus X$, denoted by $|E_S(X, V_{T_2} \setminus X)|$, is always even. It thus follows that for any component D of V_{t_1}

$$|E_C(D, V_{T_2} \setminus X)| = \sum_{S \subseteq C \cap V_{T_2}} |E_S(D, V_{T_2} \setminus X)| = \sum_{S\colon D_S = D} |E_S(X, V_{T_2} \setminus X)|$$

is even. But then also the number of edges in $\bigoplus \mathcal{C}_X$ between D and $V_{T_2} \setminus X$ is even. Since the edges of $\bigoplus \mathcal{C}_1$ and $\bigoplus \mathcal{C}_2$ do not contain vertices from X, we have

$$E_{\bigoplus \mathcal{C}_X}(X, V_{T_2} \setminus X) = E_{P+Q}(X, V_{T_2} \setminus X) = \{xx', yy'\},$$

where x' and y' are the neighbours of x and y on P, respectively. Due to parity, x and y need to lie in the same component of V_{t_1}, which is a contradiction. □

Proof of Theorem 2.1. Since both parameters appearing in the bound do not increase when passing to a component of G and as we can combine connected tree-decompositions of the components to obtain a connected tree-decomposition of G, it suffices to consider the case that G is connected.

We use Lemma 2.9 to bound the length of t-admissible paths in any stable tree-decomposition of G. Let $\ell = \ell(G)$ and Γ be the set of all cycles of length at most ℓ, which by definition generates the cycle space of G. Let $(T, \mathcal{W})$ be a rooted stable tree-decomposition, $t \in T$ and P a t-admissible path. By Lemma 2.2 there is a child s of t such that all internal vertices of P lie in $V_{D_s} \setminus V_t$. By Lemma 2.9 we find a cycle $C \in \Gamma$ and a path $S \subseteq C \cap V_{D_s}$ joining distinct components of V_t. Since $S \subseteq V_{D_t}$ and P was chosen to be a shortest such path, we have $|P| \leq |S|$. The ends of S lie in distinct components of V_t and are therefore, in particular, not adjacent, so that overall

$$|V(P)| \leq |V(S)| \leq |V(C)| - 1 \leq \ell - 1.$$

By Proposition 2.4, G has a stable tree-decomposition $(T, \mathcal{V})$ of width $\operatorname{tw}(G)$. Proposition 2.8 then guarantees that we find a connected tree-decomposition of width at most $(\ell - 2)\operatorname{tw}(G)$. □

2.4 Brambles

Recall that a *bramble* is a collection of connected vertex sets of a given graph such that the union of any two of them is again connected. A *cover* of a bramble is a set of vertices that meets every element of the bramble. The aim of this section is to derive a strengthened upper bound on the *connected order* of a bramble, the minimum size of a connected cover.

Lemma 2.10. *Suppose $(T, \mathcal{V})$ is a tree-decomposition of a graph G and $k \in \mathbb{N}$ an integer such that for every $t \in T$ there is a connected set of size at most $k+1$ containing V_t. Then G has no bramble of connected order greater than $k+1$.*

Proof. Let $\mathcal{B}$ be a bramble of G. By a standard argument, one of the parts V_t of $(T, \mathcal{V})$ covers $\mathcal{B}$ and thus so does any connected set containing V_t. $\square$

Let us call the smallest integer k such that there is a tree-decomposition satisfying the hypothesis of Lemma 2.10 the *weak connected tree-width* $\operatorname{wctw}(G)$ of the graph G. Clearly $\operatorname{wctw}(G) \leq \operatorname{ctw}(G)$, as any connected tree-decomposition of minimum width satisfies the hypothesis.

Theorem 2.11. *The weak connected tree-width of any graph G is at most* $\operatorname{tw}(G)\lfloor\frac{\ell(G)}{2}\rfloor$.

Proof. It suffices to consider the case where G is connected, since all three parameters involved are simply their respective maxima over the components of G. Let $\ell = \ell(G)$ and let Γ be the set of all cycles of G of length at most ℓ, which by definition generates the cycle space of G. By Proposition 2.4, G has a stable tree-decomposition $(T, \mathcal{V})$ of width $\operatorname{tw}(G)$. We now show that every part V_t of $(T, \mathcal{V})$ is contained in a connected set of size at most $(|V_t| - 1)\lfloor\frac{\ell}{2}\rfloor + 1$.

Let now $t \in T$ be fixed. Root T at t and apply the construction from Section 2.2. As t does not have any ancestors other than itself, the statement follows from Proposition 2.8 once we have verified that all t-admissible paths in a stable tree-decomposition $(T, \mathcal{W})$ extending $(T, \mathcal{V})$ have length at most $\ell/2$. So let $(T, \mathcal{W})$ be a stable tree-decomposition of G extending $(T, \mathcal{V})$ and let P be a t-admissible path. By Lemma 2.2, all its internal vertices lie in $W_{D_s} \setminus W_t$ for some child s of t. By Lemma 2.9 we find a cycle $C \in \Gamma$ that meets W_t in two vertices x, y from distinct components of W_t. Either segment of C between x and y lies in W_{D_t} and joins two components of W_t, so by minimality P has length at most $\lfloor\ell/2\rfloor$. $\square$

Corollary 2.12. *Let G be a graph containing a cycle. Then G has no bramble of connected order greater than* $\operatorname{tw}(G)\lfloor\frac{\ell(G)}{2}\rfloor + 1$.

Proof. Immediate from Theorem 2.11 and Lemma 2.10. $\square$

2.5 A graph of large connected tree-width

In this section we discuss an example that shows that our upper bound on connected tree-width is tight up to a constant factor. Given $n, k \in \mathbb{N}$, $n \geq 3$, obtain G from the complete graph on n vertices by subdividing every edge with k newly introduced vertices. As subdivision does not alter tree-width, we have $\mathrm{tw}(G) = n - 1$. The cycle space of G is generated by the collection of all subdivisions of triangles of the underlying complete graph, so $\ell(G) = 3(k + 1)$. We will now show that the connected tree-width of G is precisely $r := (n - 1)(k + 1) - \lfloor (k+1)/2 \rfloor$. The bound of Theorem 2.1 is therefore asymptotically tight up to a factor of 3.

Let $A \subseteq V(G)$ denote the n vertices of the complete graph we started with. The graph G thus consists of A and, for any two $a, b \in A$, a path P_{ab} of length $k+1$ between them. We first describe a bramble that cannot be covered with any connected set of size at most r. The lower bound on the connected tree-width of G then follows from Lemma 2.10. Let $X \subseteq V(G)$ be a connected set and $j := |A \cap X|$. Call $x \in X$ *essential* if either $x \in A$ or $x \in P_{a,b}$ for some $a, b \in A \cap X$ and *inessential* otherwise. Since the essential vertices contain a subdivision of a tree on $A \cap X$, it is clear that X has at least $(j - 1)(k + 1) + 1$ essential vertices.

Let X be a connected set of size at most r. Then, by the above, X cannot contain all the vertices of A and, moreover, all vertices of $A \setminus X$ lie in the same component $C(X)$ of $G - X$: If $a, b \in A \setminus X$, then by connectedness either $X \cap P_{ab} = \emptyset$ or $X \subseteq P_{ab}$, in which case a and b can be joined through some other $c \in A$. Let $\mathcal{B}$ be the collection of all these components $C(X)$ for $X \subseteq V(G)$ connected of size at most r.

Clearly, $\mathcal{B}$ cannot be covered by any connected set of size at most r, so it only remains to verify that $\mathcal{B}$ is indeed a bramble. Let $X_1, X_2 \subseteq V$ be two connected sets of size at most r, containing j_1, j_2 vertices of A, respectively. Suppose that $C(X_1) \cup C(X_2)$ was not connected. Then for every pair (a, b) with $a \in A \setminus X_1$, $b \in A \setminus X_2$, the sets X_1 and X_2 must have a common vertex on P_{ab}. By definition, all these are inessential vertices for both sets, so

$$\begin{aligned}|X_1| + |X_2| &\geq (j_1 - 1)(k + 1) + (j_2 - 1)(k + 1) + 2 + (n - j_1)(n - j_2)(k + 1)\\ &= (k + 1)(n^2 - (n - 1)(j_1 + j_2) + j_1 j_2 - 2) + 2.\end{aligned}$$

This expression, seen as a function of j_1 and j_2, assumes its minimum for $j_1 = j_2 = n-1$. We thus conclude $|X_1| + |X_2| \geq 2(k + 1)(n - 1) - k + 1$, hence the larger of the two sets has size at least $r + 1$, a contradiction.

We now describe a connected tree-decomposition of width r. Fix two $a, b \in A$ and let $A^- := A \setminus \{a, b\}$. Let T be a star with root s and leaves $t, u_1, \ldots, u_m$ with $m = \binom{n-2}{2}$. Each V_{u_i} consists of a different path P_{cd} with $c, d \in A^-$. Let V_s consist of the union of all P_{bc} with $c \in A^-$ and the first $\lceil (k + 1)/2 \rceil$ vertices from P_{ba}. Define V_t similarly.

2.6 A counterexample to duality

In this section we present a graph whose connected tree-width is larger than the largest connected order of any of its brambles. Hence, we disprove the duality conjecture of Diestel and Müller [26] for connected tree-width.

Let $n \geq 4$ be an integer. For $i = 0, 1, 2$, let $P_i = x^i_1 \dots x^i_{2n}$ be three pairwise disjoint paths and $Q = y_1 \dots y_{4n}$ another path disjoint from each P_i. Between every two vertices x^i_j, y_k we add a new internally disjoint path $P^i_{j,k}$ of length $5n$, except for $k = n+j$, where they have length n. Let G' be the resulting graph. Let G be the disjoint union of G' with a cycle C of length $16n+2$, where we choose two antipodal vertices a, b, i. e. vertices of C with $d_C(a, b) = 8n + 1$, and add the edges ax^0_1, ay_1 and bx^0_{2n}, by_{4n}. Figure 2.1 shows the graph G without P_2 and its attachment paths to Q.

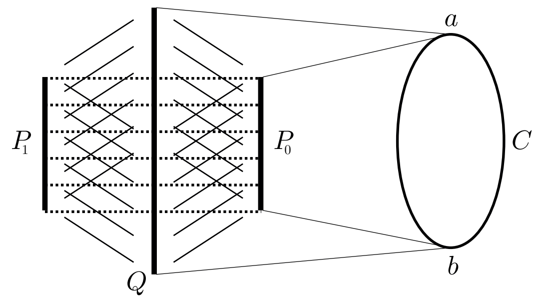

Figure 2.1: The graph G without P_2 and its attachment paths.

We claim that the connected order of any of its brambles is at most $5n + 3$ and that its connected tree-width is at least $6n$. Thus, up to additive constants, these parameters differ at least by a factor of $6/5$.

We will now give a tree-decomposition demonstrating that $\mathrm{wctw}(G) \leq 5n+2$, which is sufficient to prove the upper bound on the connected order of any bramble by Lemma 2.10. Start with $V_{t_0} := V(Q) \cup \{a, b\}$, which is connected and of size $4n + 2$. Clearly, $G - V_{t_0}$ consists of five components: each of the P_i along with their attachments to V_{t_0} and the two arcs of C. Accordingly, we add five branches to t_0, each decomposing one of the components, as follows.

For $i = 0, 1, 2$, attach a path $t^i_1 \dots t^i_{2n-1}$ to t_0 and put $V_{t^i_j} = V_{t_0} \cup \{x^i_j, x^i_{j+1}\}$. Each of these is contained in a connected set of size $5n + 3$, as x^i_j is joined to Q by a path of length n. To each t^i_j attach $4n$ leaves, each consisting of some $P^i_{j,k}$, $k \in [4n]$, which obviously does not exceed the prescribed size. To t^i_{2n-1} we add another $4n$ leaves consisting of all the $P^i_{2n,k}$. To decompose C, we attach two more paths $s^1_0 s^1_1$ and $s^2_0 s^2_1$ to t_0, one for each arc S^j of $C - \{a, b\}$. For $j = 1, 2$, $V_{s_0^j}$ contains $\{a, b\}$ and the $3n$ vertices of S^j which lie closest to a, while $V_{s_1^j}$ contains b and its closest $5n + 1$ vertices on S^j. Both of these sets are contained in connected sets of size $5n + 2$. Figure 2.2 shows our decomposition tree.

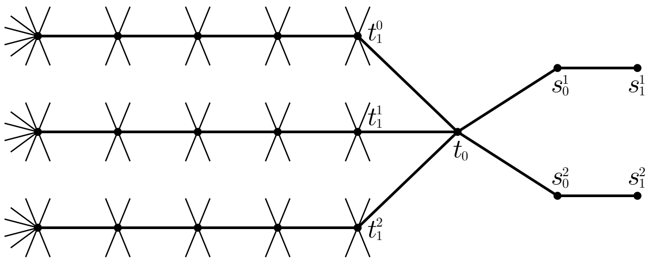

Figure 2.2: The decomposition tree of G.

To show $\mathrm{ctw}(G) \geq 6n$, let us assume for a contradiction that G had a connected tree-decomposition $(T, \mathcal{V})$ of width less than $6n$. We shall show that some part V_t contains Q and some other part $V_{t'}$ contains P_0. To see that some part contains Q, we define a bramble as follows: For all i, j, k, let $B^i_{j,k}$ be the union of all the paths from x^i_j to Q with all end vertices except for y_k deleted. It is easy so see that the collection $\mathcal{B}_1$ of all these sets $B^i_{j,k}$ is a bramble. Therefore, some part V_t of $(T, \mathcal{V})$ must cover $\mathcal{B}_1$. If some vertex $y_k \in Q$ is not included in V_t, then V_t must contain at least one vertex from each of the $6n$ pairwise disjoint sets $B^i_{j,k} \setminus \{y_k\}$ with $i \in \{0, 1, 2\}$ and $j \in [2n]$. Since no such selection of vertices is connected without the addition of further vertices, this contradicts our assumption that $|V_t| \leq 6n$.

We now show that some part contains P_0. Let C^* be the cycle of length $12n + 2$ consisting of one of the a–b paths on C together with Q and let $\mathcal{B}_2$ be the bramble consisting of all segments of C^* of length $6n + 1$. Again, there must be a part $V_{t'}$ covering $\mathcal{B}_2$. Assume for a contradiction that some vertex $x^0_j \in P_0$ was not contained in $V_{t'}$. Observe, crucially, that C^* is geodesic in $G^* := G - x^0_j$ and hence $\mathcal{B}_2$ has connected order $6n + 2$ in G^* (see [26, Lemma 7.1]). As $V_{t'}$ is a cover of $\mathcal{B}_2$ in G^*, it follows that $V_{t'} \geq 6n + 2$, which is a contradiction.

So we have found parts $V_t, V_{t'}$ containing Q and P_0, respectively. Choose two such parts at minimum distance in T. Note first that $t \neq t'$, because $P_0 \cup Q$ has size $6n$ and we need at least one further vertex, for example a, to connect these two paths. We now distinguish two cases. Suppose first that another node s of T lies between V_t and $V_{t'}$. By our choice of t, t', there must be some $x^0_j, y_k \notin V_s$. But V_s separates V_t and $V_{t'}$, so it must contain some vertex from $P^0_{j,k}$. Being connected, V_s is actually contained in this path. But then it cannot separate any other two vertices of P and Q, which is a contradiction. Suppose now that t and t' are neighbours in T. Pick any $x^0_p \in P_0 \setminus V_t$ and $y_q \in Q \setminus V_{t'}$. Since $V_t \cap V_{t'}$ separates the two, it contains some vertex of $P^0_{p,q}$, and thus at least one of V_t, $V_{t'}$ contains at least half the vertices of $P^0_{p,q}$. We may assume that this applies to V_t; the other case follows symmetrically. For every $1 \leq j \leq 2n$ consider $R_j := \bigcup_{k=1}^{4n} P^0_{j,k} \setminus Q$, the subdivision of a star with root x^0_j. These are pairwise disjoint and disjoint from Q, and since V_t contains at least $\lfloor n/2 \rfloor \geq 2$ vertices from R_p, there is some $m \in [2n]$ with $V_t \cap R_m = \emptyset$, by our assumption on the width. As $V_t \cap V_{t'}$ separates x^0_m from V_t, we must have $Q \subseteq V_{t'}$, contradicting $t \neq t'$.

2.7 Concluding remarks and open problems

Define the *connected bramble number* $\mathrm{cbn}(G)$ of a graph G to be the maximum connected order of any bramble in G. In Section 2.4 we observed that

$$\mathrm{cbn}(G) - 1 \leq \mathrm{wctw}(G) \leq \mathrm{ctw}(G) \qquad (\dagger)$$

holds for any graph G. Diestel and Müller [26] conjectured that $\mathrm{cbn}(G) - 1 = \mathrm{ctw}(G)$, but our example in Section 2.6 shows that the second of the two inequalities in (†) cannot be replaced by an equality. We do suspect, however, that the first inequality is in fact an equality:

Conjecture 1. Let k be a positive integer and G a graph. Then G has a tree-decomposition in which every part is contained in a connected set of at most k vertices if and only if every bramble of G can be covered by a connected set of size at most k.

It seems that neither the proof techniques of ordinary tree-width duality nor the ideas underlying our counterexample to connected tree-width duality are apt to solve this problem; hence we are confident that an inquiry into this problem is going to provide new ideas and insights.

The second problem concerns the second inequality of (†). The proof of [26, Theorem 1.2] combined with the improved bound of Theorem 2.1 shows that

$$\mathrm{ctw}(G) \leq 2(\mathrm{cbn}(G) - 1)(\mathrm{cbn}(G) - 2),$$

unless G is a forest in which case $\mathrm{ctw}(G) = \mathrm{tw}(G)$. This implies a locality principle for connected tree-width: if there is a tree-decomposition in which every part, individually, can be wrapped in a connected set of size at most k, then there is a tree-decomposition with connected parts of size at most $2(k-1)(k-2)$. It would be interesting to get a better understanding of this dependency.

Question 1. Is there a constant $\alpha > 0$ such that for every graph G

$$\mathrm{ctw}(G) \leq \alpha \ \mathrm{wctw}(G)?$$

Our example in Section 2.6 shows that this is not true for any $\alpha < 6/5$.

3 Algebraically grid-like graphs

By the Grid Minor Theorem of Robertson and Seymour [57], every graph of sufficiently large tree-width contains a large grid as a minor. Tree-width may therefore be regarded as a measure of 'grid-likeness' of a graph. The grid contains a long cycle on the perimeter, which is the $\mathbb{F}_2$-sum of the rectangles inside. Moreover, the grid distorts the metric of the cycle only by a factor of two. We prove that every graph that resembles the grid in this algebraic sense has large tree-width.

In order to incorporate this factor of two and to allow for more flexibility, we equip the edges of our graphs with lengths. For a graph G, a *length-function on G* is simply a map $\ell : E(G) \to \mathbb{R}^+$. We then define the *$\ell$-length* $\ell(H)$ of a subgraph $H \subseteq G$ as the sum of the lengths of all edges of H. This naturally induces a notion of distance between two vertices of G, where we define d_G^ℓ as the minimum ℓ-length of a path containing both. A subgraph $H \subseteq G$ is *geodesic with respect to ℓ* if it contains a path of length $d_G^\ell(a, b)$ between any two vertices $a, b \in V(H)$.

When no length-function is specified, the notions of length, distance and geodecity are to be read with respect to $\ell \equiv 1$ constant.

Let G be the $(n \times n)$-grid and let C be the cycle of length $4(n-1)$ on the perimeter. Consider the length-function ℓ which is equal to 1 on $E(C)$ and assumes the value 2 elsewhere. Then C is geodesic in G of ℓ-length $\ell(C) = 4(n-1)$. Moreover, C is the $\mathbb{F}_2$-sum of all squares, each of which has ℓ-length at most 8. We show that any graph which shares this algebraic feature of the grid has large tree-width:

Theorem 3.1. *Let k be a positive integer and $r > 0$. Let G be a graph with rational-valued length-function ℓ. Suppose G contains a geodesic cycle C with $\ell(C) \geq 2rk$, which is the $\mathbb{F}_2$-sum of cycles of ℓ-length at most r. Then the tree-width of G is at least k.*

This yields a cycle space based criterion for large tree-width. The proof of Theorem 3.1 will be given in Section 3.1. Section 3.2 contains some remarks and a qualitative converse to Theorem 3.1, showing that this algebraic property of the grid really does capture tree-width.

3.1 Separating the cycle

The relation to tree-width is established via a well-known separation property of graphs of bounded tree-width, due to Robertson and Seymour [56].

Lemma 3.2 ([56])**.** *Let k be a positive integer, G a graph and $A \subseteq V(G)$. If the tree-width of G is less than k, then there exists $X \subseteq V(G)$ with $|X| \leq k$ such that every component of $G - X$ contains at most $|A \setminus X|/2$ vertices of A.*

It is not hard to see that Theorem 3.1 can be reduced to the case where $\ell \equiv 1$. This case is treated in the next theorem.

Theorem 3.3. *Let k, p be positive integers. Let G be a graph containing a geodesic cycle C of length at least $4\lfloor p/2 \rfloor k$, which is the $\mathbb{F}_2$-sum of cycles of length at most p. Then for every $X \subseteq V(G)$ of order at most k, some component of $G - X$ contains at least half the vertices of C.*

Proof of Theorem 3.1, assuming Theorem 3.3. Let $\mathcal{D}$ be a set of cycles of length at most r with $C = \bigoplus \mathcal{D}$. Since ℓ is rational-valued, we may assume that $r \in \mathbb{Q}$, as the premise also holds for r' the maximum ℓ-length of a cycle in $\mathcal{D}$. Take an integer M so that rM and $\ell'(e) := M\ell(e)$ are natural numbers for every $e \in E(G)$.

Obtain the subdivision G' of G by replacing every $e \in E(G)$ by a path of length $\ell'(e)$. Denote by C', D' the subdivisions of C and $D \in \mathcal{D}$, respectively. Then $C' = \bigoplus_{D \in \mathcal{D}} D'$ is geodesic in G'. Moreover $|C'| = M\ell(C) \geq 2(Mr)k$, while $|D'| = M\ell(D) \leq Mr$ for every $D \in \mathcal{D}$. By Theorem 3.3, for every $X \subseteq V(G')$ with $|X| \leq k$ there exists a component of $G' - X$ that contains at least half the vertices of C'. By Lemma 3.2, G' has tree-width at least k. Since tree-width is invariant under subdivision, the tree-width of G is also at least k. □

Our goal is now to prove Theorem 3.3. The proof consists of two separate lemmas. The first lemma involves separators and $\mathbb{F}_2$-sums of cycles.

Lemma 3.4. *Let G be a graph, $C \subseteq G$ a cycle and $\mathcal{D}$ a set of cycles in G such that $C = \bigoplus \mathcal{D}$. Let $\mathcal{R}$ be a set of disjoint vertex-sets of G such that for every $R \in \mathcal{R}$, $R \cap V(C)$ is either empty or induces a connected subgraph of C. Then either some $D \in \mathcal{D}$ meets two distinct $R, R' \in \mathcal{R}$ or there is a component Q of $G - \bigcup \mathcal{R}$ with $V(C) \subseteq V(Q) \cup \bigcup \mathcal{R}$.*

Proof. Suppose that no $D \in \mathcal{D}$ meets two distinct $R, R' \in \mathcal{R}$. Then C has no edges between the sets in $\mathcal{R}$: any such edge would have to lie in at least one $D \in \mathcal{D}$. Let $Y := \bigcup \mathcal{R}$ and let $\mathcal{Q}$ be the set of components of $G - Y$.

Let $Q \in \mathcal{Q}$, $R \in \mathcal{R}$ and $D \in \mathcal{D}$ arbitrary. If D has an edge between Q and R, then D cannot meet $Y \setminus R$. Therefore, all edges of D between Q and $V(G) \setminus Q$ must join Q to R. As D is a cycle, it has an even number of edges between Q and $V(G) \setminus Q$ and thus between Q and R. As $C = \bigoplus \mathcal{D}$, we find

$$e_C(Q, R) \equiv \sum_{D \in \mathcal{C}} e_D(Q, R) \equiv 0 \mod 2.$$

For every $R \in \mathcal{R}$ which intersects C, there are precisely two edges of C between R and $V(C) \setminus R$, because $R \cap C$ is connected. As mentioned above, C contains no edges between R and $Y \setminus R$, so both edges join R to $V(G) \setminus Y$. But C has an even number of edges between R and each component of $V(G) \setminus Y$, so it follows that both edges join R to the same $Q(R) \in \mathcal{Q}$.

Since every component of $C - (C \cap Y)$ is contained in a component of $G - Y$, it follows that there is a $Q \in \mathcal{Q}$ containing all vertices of C not contained in Y. $\square$

To deduce Theorem 3.3, we want to apply Lemma 3.4 to a suitable family $\mathcal{R}$ with $\bigcup \mathcal{R} \supseteq X$ to deduce that some component of $G - X$ contains many vertices of C. Here, $\mathcal{D}$ consists of cycles of length at most ℓ, so if the sets in $\mathcal{R}$ are at pairwise distance $> \lfloor \ell/2 \rfloor$, then no $D \in \mathcal{D}$ can pass through two of them. The next lemma ensures that we can find such a family $\mathcal{R}$ with a bound on $|\bigcup \mathcal{R}|$, when the cycle C is geodesic.

Lemma 3.5. *Let d be a positive integer, G a graph, $X \subseteq V(G)$ and $C \subseteq G$ a geodesic cycle. Then there exists a family $\mathcal{R}$ of disjoint sets of vertices of G with $X \subseteq \bigcup \mathcal{R} \subseteq X \cup V(C)$ and $|\bigcup \mathcal{R} \cap V(C)| \leq 2d|X|$ such that for each $R \in \mathcal{R}$, the set $R \cap V(C)$ induces a (possibly empty) connected subgraph of C and the distance between any two sets in $\mathcal{R}$ is greater than d.*

Proof. Let $Y \subseteq V(G)$ and $y \in Y$. For $j \geq 0$, let $B^j_Y(y)$ be the set of all $z \in Y$ at distance at most jd from y. Since $|B^0_Y(y)| = 1$, there is a maximum number j for which $|B^j_Y(y)| \geq 1 + j$, and we call this $j = j_Y(y)$ the *range of y in Y*. Observe that every $z \in Y \setminus B^{j_Y(y)}$ has distance greater than $(j_Y(y) + 1)d$ from y.

Starting with $X_1 := X$, repeat the following procedure for $k \geq 1$. If $X_k \cap V(C)$ is empty, terminate the process. Otherwise, pick an $x_k \in X_k \cap V(C)$ of maximum range in X_k. Let $j_k := j_{X_k}(x_k)$ and $B_k := B^{j_k}_{X_k}(x_k)$. Let $X_{k+1} := X_k \setminus B_k$ and repeat.

Since the size of X_k decreases in each step, there is a smallest integer m for which $X_{m+1} \cap V(C)$ is empty, at which point the process terminates. By construction, the distance between B_k and X_{k+1} is greater than d for each $k \leq m$. For each $1 \leq k \leq m$, there are two edge-disjoint paths $P^1_k, P^2_k \subseteq C$, starting at x_k, each of length at most $j_k d$, so that $B_k \cap V(C) \subseteq S_k := P_k^1 \cup P_k^2$. Choose these paths minimal, so that the endvertices of S_k lie in B_k. Note that every vertex of S_k has distance at most $j_k d$ from x_k. Therefore, the distance between $R_k := B_k \cup S_k$ and X_{k+1} is greater than d.

We claim that the distance between R_k and $R_{k'}$ is greater than d for any $k < k'$. Since $B_{k'} \subseteq X_{k+1}$, it is clear that every vertex of $B_{k'}$ has distance greater than d from R_k. Take a vertex $q \in S_{k'} \setminus R_{k'}$ and assume for a contradiction that its distance to R_k was at most d. Then the distance between x_k and q is at most $(j_k + 1)d$. Let $a, b \in B_{k'}$ be the endvertices of $S_{k'}$. If $x_k \notin S_{k'}$, then one of a and b lies on the shortest path from x_k to q within C and therefore has distance at most $(j_k + 1)d$ from x_k. But then, since j_k is the range of x_k in X_k, that vertex would already lie in B_k, a contradiction. Suppose now that $x_k \in S_{k'}$. Then x_k lies on the path in $S_{k'}$ from $x_{k'}$ to one of a or b, so the distance between x_k and $x_{k'}$ is at most $j_{k'}d$. Since $x_{k'} \in X_k \cap V(C)$, it follows from our choice of x_k that

$$j_k = j_{X_k}(x_k) \geq j_{X_k}(x_{k'}) \geq j_{X_{k'}}(x_{k'}) = j_{k'},$$

where the second inequality follows from the fact that $X_{k'} \subseteq X_k$ and $j_Y(y) \geq j_{Y'}(y)$ whenever $Y \supseteq Y'$. But then $x_{k'} \in B_k$, a contradiction. This finishes the proof of the claim.

Finally, let $\mathcal{R} := \{R_k \colon 1 \leq k \leq m\} \cup \{X_{m+1}\}$. The distance between any two sets in $\mathcal{R}$ is greater than d. For $k \leq m$, $R_k \cap V(C) = S_k$ is a connected subgraph of C, while $X_{m+1} \cap V(C)$ is empty. Moreover,

$$\begin{aligned}
|\bigcup \mathcal{R} \cap V(C)| = \sum_{k=1}^{m} |S_k| &\leq \sum_{k=1}^{m} (1 + 2j_k d) \\
&\leq \sum_{k=1}^{m} (1 + 2(|B_k| - 1)d) \\
&\leq \sum_{k=1}^{m} 2|B_k|d \leq 2d|X|.
\end{aligned}$$

□

Proof of Theorem 3.3. Let $X \subseteq V(G)$ of order at most k and let $d := \lfloor p/2 \rfloor$. By Lemma 3.5, there exists a family $\mathcal{R}$ of disjoint sets of vertices of G with $X \subseteq \bigcup \mathcal{R} \subseteq X \cup V(C)$ and $|\bigcup \mathcal{R} \cap V(C)| \leq 2dk$ so that for each $R \in \mathcal{R}$, the set $R \cap V(C)$ induces a (possibly empty) connected subgraph of C and the distance between any two sets in $\mathcal{R}$ is greater than d.

Let $\mathcal{D}$ be a set of cycles of length at most p with $C = \bigoplus \mathcal{D}$. Then no $D \in \mathcal{D}$ can meet two distinct $R, R' \in \mathcal{R}$, since the diameter of D is at most d. By Lemma 3.4, there is a component Q of $G - \bigcup \mathcal{R}$ which contains every vertex of $C \setminus \bigcup \mathcal{R}$. This component is connected in $G - X$ and therefore contained in some component Q' of $G - X$, which then satisfies

$$|Q' \cap V(C)| \geq |C| - |\bigcup \mathcal{R} \cap V(C)| \geq |C| - 2dk.$$

Since $|C| \geq 4dk$, the claim follows. $\square$

3.2 Remarks and a converse

We have described the content of Theorem 3.1 as an *algebraic* criterion for a graph to have large tree-width. The reader might object that the cycle C being geodesic is a metric property and not an algebraic one. Karl Heuer has pointed out to us, however, that geodecity of a cycle can be expressed as an algebraic property after all. This is a consequence of a more general lemma of Gollin and Heuer [36], which allowed them to introduce a meaningful notion of geodecity for cuts.

Proposition 3.6 ([36]). *Let G be a graph with length-function ℓ and $C \subseteq G$ a cycle. Then C is ℓ-geodesic if and only if there do not exist cycles D_1, D_2 with $\ell(D_1), \ell(D_2) < \ell(C)$ such that $C = D_1 \oplus D_2$.*

Finally, we'd like to point out that Theorem 3.1 does not only offer a 'one-way criterion' for large tree-width, but that it has a qualitative converse. First, we recall the Grid Minor Theorem of Robertson and Seymour [57], phrased in terms of walls. For a positive integer t, an *elementary t-wall* is the graph obtained from the $(2t \times t)$-grid as follows. Delete all edges with endpoints $(i, j), (i, j+1)$ when i and j have the same parity. Delete the two resulting vertices of degree one. A *t-wall* is any subdivision of an elementary t-wall. Note that the $(2t \times 2t)$-grid has a subgraph isomorphic to a t-wall.

Theorem 3.7 (Grid Minor Theorem [57]). *For every t there exists a k such that every graph of tree-width at least k contains a t-wall.*

Here, then, is our qualitative converse to Theorem 3.1, showing that the algebraic condition in the premise of Theorem 3.1 in fact captures tree-width.

Corollary 3.8. *For every L there exists a k such that for every graph G of tree-width at least k the following holds. There exists a rational length-function on G so that G contains a geodesic cycle C with $\ell(C) \geq L$ which is the $\mathbb{F}_2$-sum of cycles of ℓ-length at most* 1.

Proof. Let $s := 3L$. By the Grid Minor Theorem, there exists an integer k such that every graph of tree-width at least k contains an s-wall. Suppose G is a graph of tree-width at least k. Let W be an elementary s-wall so that G contains some subdivision W' of W, where $e \in E(W)$ has been replaced by some path $P^e \subseteq G$ of length $m(e)$.

The outer cycle C of W satisfies $d_C(u,v) \leq 3d_W(u,v)$ for all $u,v \in V(C)$. Moreover, C is the $\mathbb{F}_2$-sum of cycles of length at most six.

Define a length-function ℓ on G as follows. Let $e \in E(G)$. If $e \in P^f$ for $f \in E(C)$, let $\ell(e) := 1/m(f)$. Then $\ell(P^f) = 1$ for every $f \in E(C)$. If $e \in P^f$ for $f \in E(W) \setminus E(C)$, let $\ell(e) := 3/m(f)$. Then $\ell(P^f) = 3$ for every $f \in E(W) \setminus E(C)$. If $e \notin E(W')$, let $\ell(e) := 10s^3$, so that $\ell(e) > \ell(W')$.

It is easy to see that the subdivision $C' \subseteq G$ of C is geodesic in G. It has length $\ell(C') = |C| \geq 6s$ and is the $\mathbb{F}_2$-sum of the subdivisions of 6-cycles of W. Each of these satisfies $\ell(D) \leq 18$. Rescaling all lengths by a factor of $1/18$ yields the desired result.

$\square$

4 Steiner trees and higher geodecity

Let G be a graph and $\ell : E(G) \to \mathbb{R}^+$ a function that assigns to every edge $e \in E(G)$ a positive *length* $\ell(e)$. This naturally extends to subgraphs $H \subseteq G$ as $\ell(H) := \sum_{e \in E(H)} \ell(e)$. The *Steiner distance* $\mathrm{sd}_G(A)$ of a set $A \subseteq V(G)$ is defined as the minimum length of a connected subgraph of G containing A, where $\mathrm{sd}_G(A) := \infty$ if no such subgraph exists. Every such minimizer is necessarily a tree, called a *Steiner tree for A in G*. In the case where $A = \{x, y\}$, the Steiner distance of A is the ordinary distance $\mathrm{dist}_G(x, y)$ between x and y. Hence this definition yields a natural extension of the notion of "distance" for sets of more than two vertices.

Let $H \subseteq G$ be a subgraph of G, equipped with the induced length-function $\ell|_{E(H)}$. It is clear that for every $A \subseteq V(H)$ we have $\mathrm{sd}_H(A) \geq \mathrm{sd}_G(A)$. For a natural number k, we say that H is *k-geodesic in G* if $\mathrm{sd}_H(A) = \mathrm{sd}_G(A)$ for every $A \subseteq V(H)$ with $|A| \leq k$. We call H *fully geodesic in G* if it is k-geodesic for every $k \in \mathbb{N}$.

By definition, a k-geodesic subgraph is m-geodesic for every $m \leq k$. In general, this hierarchy is strict: In Section 4.5 we provide, for every $k \in \mathbb{N}$, examples of graphs $H \subseteq G$ with a length-function $\ell : E(G) \to \mathbb{R}^+$ such that H is k-geodesic, but not $(k+1)$-geodesic. On the other hand, it is easy to see that if $H \subseteq G$ is a 2-geodesic *path*, then it is necessarily fully geodesic, because the Steiner distance of any $A \subseteq V(H)$ in H is equal to the maximum distance between two $a, b \in A$. Our first result extends this to all trees.

Theorem 4.1. *Let G be a graph with length-function ℓ and $T \subseteq G$ a tree. If T is 2-geodesic in G, then it is fully geodesic.*

Here, it really is necessary for the subgraph to be acyclic (see Corollary 4.17). Hence the natural follow-up question is what happens in the case where the subgraph is a cycle.

Theorem 4.2. *Let G be a graph with length-function ℓ and $C \subseteq G$ a cycle. If C is 6-geodesic in G, then it is fully geodesic.*

We will show that the number 6 cannot be replaced by any smaller integer.

In Section 4.1 we introduce notation and terminology needed in the remainder of this chapter. Section 4.2 contains observations and lemmas that will be used later. We then prove Theorem 4.1 in Section 4.3. In Section 4.4 we prove Theorem 4.2 and provide an example showing that the number 6 is optimal. Section 4.5 contains an approach towards a general theory, aiming at a deeper understanding of the phenomenon displayed in Theorem 4.1 and Theorem 4.2. Finally, in Section 4.6 we take the opportunity to present the short and easy proof that in any graph G with length-function ℓ, the cycle space of G is generated by the set of fully geodesic cycles.

4.1 Preliminaries

We use additive notation for adding or deleting vertices and edges. Specifically, let G be a graph, H a subgraph of G, $v \in V(G)$ and $e = xy \in E(G)$. Then $H + v$ is the graph with vertex-set $V(H) \cup \{v\}$ and edge-set $E(H) \cup \{vw \in E(G) \colon w \in V(H)\}$. Similarly, $H + e$ is the graph with vertex-set $V(H) \cup \{x, y\}$ and edge-set $E(H) \cup \{e\}$.

Let G be a graph with length-function ℓ. A *walk* in G is an alternating sequence $W = v_1 e_1 v_2 \dots e_k v_{k+1}$ of vertices v_i and edges e_i such that $e_i = v_i v_{i+1}$ for every $1 \leq i \leq k$. The walk W is *closed* if $v_1 = v_{k+1}$. Stretching our terminology slightly, we define the *length* of the walk as $\operatorname{len}_G(W) := \sum_{i=1}^{k} \ell(e_i)$. The *multiplicity* $\operatorname{mult}_W(e)$ of an edge $e \in E(G)$ is the number of times it is traversed by W, that is, the number of indices $1 \leq j \leq k$ with $e = e_j$. It is clear that

$$\operatorname{len}_G(W) = \sum_{e \in E(G)} \operatorname{mult}_W(e) \ell(e). \tag{4.1}$$

Let G be a graph and C a cycle with $V(C) \subseteq V(G)$. We say that a (closed) walk W in G is *traced by* C in G if it can be obtained from C by choosing a starting vertex $x \in V(C)$ and an orientation $\overrightarrow{C}$ of C and replacing every directed edge $\overrightarrow{ab} \in E(\overrightarrow{C})$ by a shortest path in G from a to b. A cycle may trace several walks, but they all have the same length: Every walk W traced by C satisfies

$$\operatorname{len}_G(W) = \sum_{ab \in E(C)} \operatorname{dist}_G(a, b). \tag{4.2}$$

Even more can be said if the graph G is a tree. Then the shortest a-b-path is unique for every $ab \in E(C)$ and the walks traced by C differ only in their starting vertex and/or orientation. In particular, every walk W traced by C in a tree T satisfies

$$\forall e \in E(T) \colon \ \operatorname{mult}_W(e) = |\{ab \in E(C) \colon e \in aTb\}|. \tag{4.3}$$

Let T be a tree and $X \subseteq V(T)$. Let $e \in E(T)$ and let T_1^e, T_2^e be the two components of $T-e$. In this manner, e induces a bipartition $X = X_1^e \cup X_2^e$ of X, given by $X_i^e = V(T_i^e) \cap X$ for $i \in \{1, 2\}$. We say that the bipartition is *non-trivial* if neither of X_1^e, X_2^e is empty. The set of leaves of T is denoted by $L(T)$. If $L(T) \subseteq X$, then every bipartition of X induced by an edge of T is non-trivial.

Let G be a graph with length-function ℓ, $A \subseteq V(G)$ and T a Steiner tree for A in G. Since $\ell(e) > 0$ for every $e \in E(G)$, every leaf x of T must lie in A, for otherwise $T - x$ would be a tree of smaller length containing A.

In general, Steiner trees need not be unique. If G is a tree, however, then every $A \subseteq V(G)$ has a unique Steiner tree given by $\bigcup_{a,b \in A} aTb$.

4.2 The toolbox

The first step in all our proofs is a simple lemma that guarantees the existence of a particularly simple substructure that witnesses the failure of a subgraph to be k-geodesic.

Let H be a graph, T a tree and ℓ a length-function on $T \cup H$. We call T a *shortcut tree for H* if the following hold:

(SCT 1) $V(T) \cap V(H) = L(T)$,

(SCT 2) $E(T) \cap E(H) = \emptyset$,

(SCT 3) $\ell(T) < \mathrm{sd}_H(L(T))$,

(SCT 4) For every $B \subsetneq L(T)$ we have $\mathrm{sd}_H(B) \leq \mathrm{sd}_T(B)$.

Conditions (SCT 3) and (SCT 4) may be subsumed by saying that $L(T)$ is inclusion-minimal with the property $\mathrm{sd}_T(L(T)) < \mathrm{sd}_H(L(T))$. Note that, by definition, H is not $|L(T)|$-geodesic in $T \cup H$.

Lemma 4.3. *Let G be a graph with length-function ℓ, k a natural number and $H \subseteq G$. If H is not k-geodesic in G, then G contains a shortcut tree for H with at most k leaves.*

Proof. Among all $A \subseteq V(H)$ with $|A| \leq k$ and $\mathrm{sd}_G(A) < \mathrm{sd}_H(A)$, choose A such that $\mathrm{sd}_G(A)$ is minimum. Let $T \subseteq G$ be a Steiner tree for A in G. We claim that T is a shortcut tree for H.

Claim 1: $L(T) = A = V(T) \cap V(H)$.

The inclusions $L(T) \subseteq A \subseteq V(T) \cap V(H)$ are clear. We show $V(T) \cap V(H) \subseteq L(T)$. Assume for a contradiction that $x \in V(T) \cap V(H)$ had degree $d \geq 2$ in T. Let $T_1, \ldots, T_d$ be the components of $T - x$ and for $j \in [d]$ let $A_j := A \cap V(T_j) \cup \{x\}$. Since $L(T) \subseteq A$, every tree T_i contains some $a \in A$ and so $A \not\subseteq A_j$. In particular $|A_j| \leq k$. Moreover

$\mathrm{sd}_G(A_j) \leq \ell(T_j + x) < \ell(T)$, so by our choice of A and T it follows that $\mathrm{sd}_G(A_j) = \mathrm{sd}_H(A_j)$. Therefore, for every $j \in [d]$ there exists a connected $S_j \subseteq H$ with $A_j \subseteq V(S_j)$ and $\ell(S_j) \leq \ell(T_j + x)$. But then $S := \bigcup_j S_j \subseteq H$ is connected, contains A and satisfies

$$\ell(S) \leq \sum_{j=1}^{d} \ell(S_j) \leq \sum_{j=1}^{d} \ell(T_j + x) = \ell(T),$$

which contradicts the fact that $\mathrm{sd}_H(A) > \ell(T)$ by choice of A and T.

Claim 2: $E(T) \cap E(H) = \emptyset$.

Assume for a contradiction that $xy \in E(T) \cap E(H)$. By Claim 1, $x, y \in L(T)$ and so T consists only of the edge xy. But then $T \subseteq H$ and $\mathrm{sd}_H(A) \leq \ell(T)$, contrary to our choice of A and T.

Claim 3: $\ell(T) < \mathrm{sd}_H(L(T))$.

We have $\ell(T) = \mathrm{sd}_G(A) < \mathrm{sd}_H(A)$. By Claim 1, $A = L(T)$.

Claim 4: For every $B \subsetneq L(T)$ we have $\mathrm{sd}_H(B) \leq \mathrm{sd}_T(B)$.

Let $B \subsetneq L(T)$ and let $T' := T - (A \setminus B)$. By Claim 1, T' is the tree obtained from T by chopping off all leaves not in B and so

$$\mathrm{sd}_G(B) \leq \ell(T') < \ell(T) = \mathrm{sd}_G(A).$$

By minimality of A, it follows that $\mathrm{sd}_H(B) = \mathrm{sd}_G(B) \leq \mathrm{sd}_T(B)$. □

Our proofs of Theorem 4.1 and Theorem 4.2 proceed by contradiction and follow a similar outline. Let $H \subseteq G$ be a subgraph satisfying a certain set of assumptions. The aim is to show that H is fully geodesic. Assume for a contradiction that it was not and apply Lemma 4.3 to find a shortcut tree T for H. Let C be a cycle with $V(C) \subseteq L(T)$ and let W_H, W_T be walks traced by C in H and T, respectively. If $|L(T)| \geq 3$, then it follows from (4.2) and (SCT 4) that $\mathrm{len}(W_H) \leq \mathrm{len}(W_T)$.

Ensure that $\mathrm{mult}_{W_T}(e) \leq 2$ for every $e \in E(T)$ and that $\mathrm{mult}_{W_H}(e) \geq 2$ for all $e \in E(S)$, where $S \subseteq H$ is connected with $L(T) \subseteq V(S)$. Then

$$2\,\mathrm{sd}_H(L(T)) \leq 2\,\ell(S) \leq \mathrm{len}(W_H) \leq \mathrm{len}(W_T) \leq 2\,\ell(T),$$

which contradicts (SCT 3).

The first task is thus to determine, given a tree T, for which cycles C with $V(C) \subseteq V(T)$ we have $\mathrm{mult}_W(e) \leq 2$ for all $e \in E(T)$, where W is a walk traced by C in T. Let $S \subseteq T$ be the Steiner tree for $V(C)$ in T. It is clear that W does not traverse any edges $e \in E(T) \setminus E(S)$ and $L(S) \subseteq V(C) \subseteq V(S)$. Hence we can always reduce to this case and may for now assume that $S = T$ and $L(T) \subseteq V(C)$.

Lemma 4.4. *Let T be a tree, C a cycle with $L(T) \subseteq V(C) \subseteq V(T)$ and W a walk traced by C in T. Then $\mathrm{mult}_W(e)$ is positive and even for every $e \in E(T)$.*

Proof. Let $e \in E(T)$ and let $V(C) = V(C)_1^e \cup V(C)_2^e$ be the induced bipartition. Since $L(T) \subseteq V(C)$, this bipartition is non-trivial. By (4.3), $\mathrm{mult}_W(e)$ is the number of $ab \in E(C)$ such that $e \in aTb$. By definition, $e \in aTb$ if and only if a and b lie in different sides of the bipartition. Every cycle has a positive even number of edges across any non-trivial bipartition of its vertex-set. □

Lemma 4.5. *Let T be a tree and C a cycle with $L(T) \subseteq V(C) \subseteq V(T)$. Then*

$$2\ell(T) \leq \sum_{ab \in E(C)} \mathrm{dist}_T(a, b).$$

Moreover, there is a cycle C with $V(C) = L(T)$ for which equality holds.

Proof. Let W be a walk traced by C in T. By Lemma 4.4, (4.1) and (4.2)

$$2\ell(T) \leq \sum_{e \in E(T)} \mathrm{mult}_W(e)\ell(e) = \mathrm{len}(W) = \sum_{ab \in E(C)} \mathrm{dist}_T(a, b).$$

To see that equality can be attained, let $2T$ be the multigraph obtained from T by doubling all edges. Since all degrees in $2T$ are even, it has a Eulerian trail W, which may be considered a closed walk in T with $\mathrm{mult}_W(e) = 2$ for all $e \in E(T)$. This walk traverses the leaves of T in some cyclic order, which yields a cycle C with $V(C) = L(T)$. It is easily verified that W is traced by C in T and so

$$2\ell(T) = \sum_{e \in E(T)} \mathrm{mult}_W(e)\ell(e) = \mathrm{len}(W) = \sum_{ab \in E(C)} \mathrm{dist}_T(a, b).$$

□

We have now covered everything needed in the proof of Theorem 4.1, so the curious reader may skip ahead to Section 4.3.

In general, not every cycle C with $V(C) = L(T)$ achieves equality in Lemma 4.5. Consider the tree T from Figure 4.2 and the following three cycles on $L(T)$

$$C_1 = abcda,\ C_2 = acdba,\ C_3 = acbda.$$

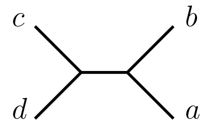

Figure 4.1: A tree with four leaves

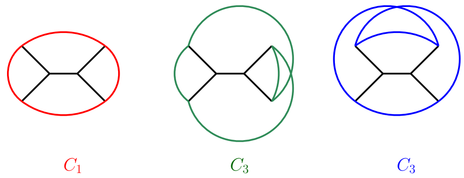

Figure 4.2: Three cycles on T

For the first two, equality holds, but not for the third one. But how does C_3 differ from the other two? It is easy to see that we can add C_1 to the planar drawing of T depicted in Figure 4.2: There exists a planar drawing of $T \cup C_1$ extending this particular drawing. This is not true for C_2, but it can be salvaged by exchanging the positions of a and b in Figure 4.2. Of course, this is merely tantamount to saying that $T \cup C_i$ is planar for $i \in \{1, 2\}$. On the other hand, it is easy to see that $T \cup C_3$ is isomorphic to $K_{3,3}$ and therefore non-planar.

Indeed, we have the following topological characterization of the cycles achieving equality in Lemma 4.5, which may be of independent interest:

Lemma 4.6. *Let T be a tree and C a cycle with $V(C) = L(T)$. Let W be a walk traced by C in T. The following are equivalent:*

(a) *$T \cup C$ is planar.*

(b) *For every $e \in E(T)$, both $V(C)^e_1, V(C)^e_2$ are connected in C.*

(c) *W traverses every edge of T precisely twice.*

Proof. (a) $\Rightarrow$ (b): Fix a planar drawing of $T \cup C$. The closed curve representing C divides the plane into two regions and the drawing of T lies in the closure of one of them. By symmetry, we may assume that it lies within the closed disk inscribed by C. Let $A \subseteq V(C)$ disconnected and choose $a, b \in A$ from distinct components of $C[A]$. Then C is the union of two edge-disjoint a-b-paths S_1, S_2 and both of them must meet $C \setminus A$, say $c \in V(S_1) \setminus A$ and $d \in V(S_2) \setminus A$.

The curves representing aTb and cTd lie entirely within the disk and so they must cross. Since the drawing is planar, aTb and cTd have a common vertex. In particular, A cannot be the set of leaves within a component of $T - e$ for any edge $e \in E(T)$.

(b) $\Rightarrow$ (c): Let $e \in E(T)$. By assumption, there are precisely two edges $f_1, f_2 \in E(C)$ between $V(C)_1^e$ and $V(C)_2^e$. These edges are, by definition, the ones whose endpoints are separated in T by e. By (4.3), $m_W(e) = 2$.

(c) $\Rightarrow$ (a): For $ab \in E(C)$, let $D_{ab} := aTb + ab \subseteq T \cup C$. The set

$$\mathcal{D} := \{D_{ab} \colon ab \in E(C)\}$$

of all these cycles is the fundamental cycle basis of $T \cup C$ with respect to the spanning tree T. Every edge of C occurs in only one cycle of $\mathcal{D}$. By assumption and (4.3), every edge of T lies on precisely two cycles in $\mathcal{D}$. Covering every edge of the graph at most twice, the set $\mathcal{D}$ is a *sparse basis* of the cycle space of $T \cup C$. By MacLane's Theorem (see [18]), $T \cup C$ is planar. $\square$

4.3 Shortcut trees for trees

Proof of Theorem 4.1. Assume for a contradiction that $T \subseteq G$ was not fully geodesic and let $R \subseteq T$ be a shortcut tree for T. Let $T' \subseteq T$ be the Steiner tree for $L(R)$ in T. By Lemma 4.5, there is a cycle C with $V(C) = L(R)$ such that

$$2\ell(R) = \sum_{ab \in E(C)} \operatorname{dist}_R(a, b).$$

Note that T' is 2-geodesic in T and therefore in G, so that $\operatorname{dist}_{T'}(a, b) \leq \operatorname{dist}_R(a, b)$ for all $ab \in E(C)$. Since every leaf of T' lies in $L(R) = V(C)$, we can apply Lemma 4.5 to T' and C and conclude

$$2\ell(T') \leq \sum_{ab \in E(C)} \operatorname{dist}_{T'}(a, b) \leq \sum_{ab \in E(C)} \operatorname{dist}_R(a, b) = 2\ell(R),$$

which contradicts (SCT 3). $\square$

4.4 Shortcut trees for cycles

By Lemma 4.3, it suffices to prove the following.

Theorem 4.7. *Let T be a shortcut tree for a cycle C. Then $T \cup C$ is a subdivision of one of the five (multi-)graphs in Figure 4.3. In particular, C is not 6-geodesic in $T \cup C$.*

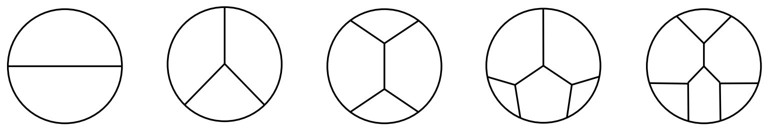

Figure 4.3: The five possible shortcut trees for a cycle

Theorem 4.7 is best possible in the sense that for each of the graphs in Figure 4.3 there exists a length-function which makes the tree inside a shortcut tree for the outer cycle, see Figure 4.4. These length-functions were constructed in a joint effort with Pascal Gollin and Karl Heuer in an ill-fated attempt to prove that a statement like Theorem 4.2 could not possibly be true.

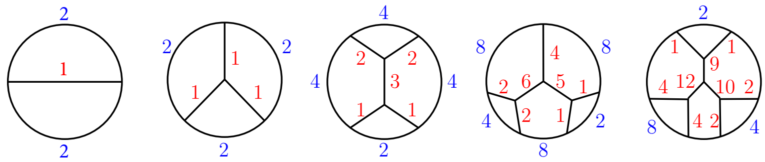

Figure 4.4: Shortcut trees for cycles

This section is devoted entirely to the proof of Theorem 4.7. Let T be a shortcut tree for a cycle C with length-function $\ell : E(T \cup C) \to \mathbb{R}^+$ and let $L := L(T)$.

The case where $|L| = 2$ is trivial, so we henceforth assume that $|L| \geq 3$. By suppressing any degree-2 vertices, we may assume without loss of generality that $V(C) = L(T)$ and that T contains no vertices of degree 2.

Lemma 4.8. *Let $T_1, T_2 \subseteq T$ be edge-disjoint trees. For $i \in \{1, 2\}$, let $L_i := L \cap V(T_i)$. If $L = L_1 \cup L_2$ is a non-trivial bipartition of L, then both $C[L_1], C[L_2]$ are connected.*

Proof. By (SCT 4) there are connected $S_1, S_2 \subseteq C$ with $\ell(S_i) \leq \mathrm{sd}_T(L_i) \leq \ell(T_i)$ for $i \in \{1, 2\}$. Assume for a contradiction that $C[L_1]$ was not connected. Then $V(S_1) \cap L_2$ is non-empty and $S_1 \cup S_2$ is connected, contains L and satisfies

$$\ell(S_1 \cup S_2) \leq \ell(S_1) + \ell(S_2) \leq \ell(T_1) + \ell(T_2) \leq \ell(T),$$

which contradicts (SCT 3). $\square$

Lemma 4.9. *$T \cup C$ is planar and 3-regular.*

Proof. Let $e \in E(T)$, let T_1, T_2 be the two components of $T-e$ and let $L = L_1 \cup L_2$ be the induced (non-trivial) bipartition of L. By Lemma 4.8, both $C[L_1], C[L_2]$ are connected. Therefore $T \cup C$ is planar by Lemma 4.6.

To see that $T \cup C$ is 3-regular, it suffices to show that no $t \in T$ has degree greater than 3 in T. We just showed that $T \cup C$ is planar, so fix some planar drawing of it. Suppose for a contradiction that $t \in T$ had $d \geq 4$ neighbors in T. In the drawing, these are arranged in some cyclic order as $t_1, t_2, \dots, t_d$. For $j \in [d]$, let $R_j := T_j + t$, where T_j is the component of $T - t$ containing t_j. Let T_{odd} be the union of all R_j for odd $j \in [d]$ and T_{even} the union of all R_j for even $j \in [d]$. Then $T_{\text{odd}}, T_{\text{even}} \subseteq T$ are edge-disjoint and yield a nontrivial bipartition $L = L_{\text{odd}} \cup L_{\text{even}}$ of the leaves. But neither of $C[L_{\text{odd}}], C[L_{\text{even}}]$ is connected, contrary to Lemma 4.8.

□

Lemma 4.10. *Let $e_0 \in E(C)$ arbitrary. Then for any two consecutive edges e_1, e_2 of C we have $\ell(e_1) + \ell(e_2) > \ell(e_0)$. In particular $\ell(e_0) < \ell(C)/2$.*

Proof. Suppose that $e_1, e_2 \in E(C)$ are both incident with $x \in L$. Let $S \subseteq C$ be a Steiner tree for $B := L \setminus \{x\}$ in C. By (SCT 4) and (SCT 3) we have

$$\ell(S) \leq \mathrm{sd}_T(B) \leq \ell(T) < \mathrm{sd}_C(L).$$

Thus $x \notin S$ and $E(S) = E(C) \setminus \{e_1, e_2\}$. Thus $P := C - e_0$ is not a Steiner tree for B and we must have $\ell(P) > \ell(S)$. □

Let $t \in T$ and let N be its set of neighbors in T. For every $s \in N$, the set L_s of leaves x with $s \in tTx$ is connected in C. For each s, there exist two edges $f_s^1, f_s^2 \in E(C)$ with precisely one endpoint in L_s.

Lemma 4.11. *There is a $t \in T$ such that for every $s \in N$ and any $f \in \{f_s^1, f_s^2\}$ we have $\ell(C[L_s] + f) < \ell(C)/2$.*

Proof. We construct a directed graph D with $V(D) = V(T)$ as follows. For every $t \in T$, draw an arc to any $s \in N$ for which $\ell(C[L_s] + f_s^i) \geq \ell(C)/2$ for some $i \in \{1, 2\}$.

Claim: If $\overrightarrow{ts} \in E(D)$, then $\overrightarrow{st} \notin E(D)$.

Assume that there was an edge $st \in E(T)$ for which both $\overrightarrow{st}, \overrightarrow{ts} \in E(D)$. Let T_s, T_t be the two components of $T - st$, where $s \in T_s$, and let $L = L_s \cup L_t$ be the induced bipartition of L. By Lemma 4.8, both $C[L_s]$ and $C[L_t]$ are connected paths, say with endpoints a_s, b_s and a_t, b_t (possibly $a_s = b_s$ or $a_t = b_t$) so that $a_s a_t \in E(C)$ and $b_s b_t \in E(C)$ (see Figure 4.5). Without loss of generality $\ell(a_s a_t) \leq \ell(b_s b_t)$. Since $\overrightarrow{ts} \in E(D)$ we

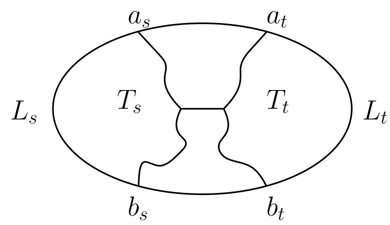

Figure 4.5: The setup in the proof of Lemma 4.11

have $\ell(C[L_t] + b_s b_t) \geq \ell(C)/2$ and therefore $C[L_s] + a_s a_t$ is a shortest a_t-b_s-path in C. Similarly, it follows from $\overrightarrow{st} \in E(D)$ that $\mathrm{dist}_C(a_s, b_t) = \ell(C[L_t] + a_s a_t)$.

Consider the cycle $Q := a_t b_s a_s b_t a_t$ and let W_T, W_C be walks traced by Q in T and in C, respectively. Then $\mathrm{len}(W_T) \leq 2\,\ell(T)$, whereas

$$\mathrm{len}(W_C) = 2\,\ell(C - b_s b_t) \geq 2\ \mathrm{sd}_C(L).$$

By (SCT 4) we have $\mathrm{dist}_C(x, y) \leq \mathrm{dist}_T(x, y)$ for all $x, y \in L$ and so $\mathrm{len}(W_C) \leq \mathrm{len}(W_T)$. But then $\mathrm{sd}_C(L) \leq \ell(T)$, contrary to (SCT 3). This finishes the proof of the claim.

Since every edge of D is an orientation of an edge of T and no edge of T is oriented both ways, it follows that D has at most $|V(T)| - 1$ edges. Since D has $|V(T)|$ vertices, there is a $t \in V(T)$ with no outgoing edges. □

Fix a node $t \in T$ as guaranteed by the previous lemma. If t was a leaf with neighbor s, say, then $\ell(f_s^1) = \ell(C) - \ell(C[L_s] + f_s^2) > \ell(C)/2$ and, symmetrically, $\ell(f_s^2) > \ell(C)/2$, which is impossible. Hence by Lemma 4.9, t has three neighbors $s_1, s_2, s_3 \in T$ and we let $L_i := C[L_{s_i}]$ and $\ell_i := \ell(L_i)$. There are three edges $f_1, f_2, f_3 \in E(C) \setminus \bigcup E(L_i)$, where f_1 joins L_1 and L_2, f_2 joins L_2 and L_3 and f_3 joins L_3 and L_1. Each L_i is a (possibly trivial) path whose endpoints we label a_i, b_i so that, in some orientation, the cycle is given by

$$C = a_1 L_1 b_1 + f_1 + a_2 L_2 b_2 + f_2 + a_3 L_3 b_3 + f_3.$$

Hence $f_1 = b_1 a_2$, $f_2 = b_2 a_3$ and $f_3 = b_3 a_1$ (see Figure 4.6).

The fact that $\ell_1 + \ell(f_1) \leq \ell(C)/2$ means that $L_1 + f_1$ is a shortest a_1-a_2-path in C and so $\mathrm{dist}_C(a_1, a_2) = \ell_1 + \ell(f_1)$. Similarly, we thus know the distance between all pairs of vertices on C with just one segment L_i and one edge f_j between them.

If $|L_i| \leq 2$ for every $i \in [3]$, then $T \cup C$ is a subdivision of one the graphs depicted in Figure 4.3 and we are done. Hence from now on we assume that at least one L_i contains at least 3 vertices.

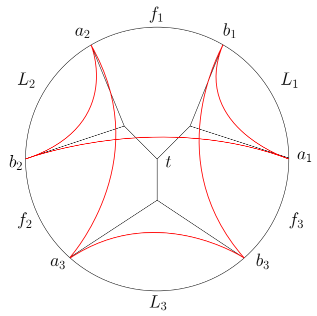

Figure 4.6: The cycle Q

Lemma 4.12. *Suppose that* $\max\{|L_s|\colon s \in N\} \geq 3$. *Then there is an* $s \in N$ *with* $\ell(f_s^1 + C[L_s] + f_s^2) \leq \ell(C)/2$.

Proof. For $j \in [3]$, let $r_j := \ell(f_{s_j}^1 + L_j + f_{s_j}^2)$. Assume wlog that $|L_1| \geq 3$. Then L_1 contains at least two consecutive edges, so by Lemma 4.10 we must have $\ell_1 > \ell(f_2)$. Therefore

$$r_2 + r_3 = \ell(C) + \ell(f_2) - \ell_1 < \ell(C),$$

so the minimum of r_2, r_3 is less than $\ell(C)/2$. □

By the previous lemma, we may assume without loss of generality that

$$\ell(f_2) + \ell_3 + \ell(f_3) \leq \ell(C)/2, \tag{4.4}$$

so that $f_2 + L_3 + f_3$ is a shortest a_1-b_2-path in C. We now combine this with the inequalities from Lemma 4.11 to obtain the final contradiction.

Consider the cycle $Q = a_1b_2a_2a_3b_3b_1a_1$ (see Figure 4.6). Let W_T be a walk traced by Q in T. Every edge of T is traversed at most twice, hence

$$\ell(W_T) = \sum_{ab \in E(Q)} \mathrm{dist}_T(a,b) \leq 2\ell(T). \tag{4.5}$$

Let W_C be a walk traced by Q in C. Using (4.4) and the inequalities from Lemma 4.11, we see that

$$\ell(W_C) = \sum_{ab \in E(Q)} \mathrm{dist}_C(a,b) = 2\ell(C) - 2\ell(f_1).$$

But by (SCT 4) we have $\mathrm{dist}_C(a,b) \leq \mathrm{dist}_T(a,b)$ for all $a, b \in L(T)$ and therefore

$\ell(W_C) \leq \ell(W_T)$. Then by (4.5)

$$2\ell(C) - 2\ell(f_1) = \ell(W_C) \leq \ell(W_T) \leq 2\ell(T).$$

But then $S := C - f_1$ is a connected subgraph of C with $L(T) \subseteq V(S)$ satisfying $\ell(S) \leq \ell(T)$. This contradicts (SCT 3) and finishes the proof of Theorem 4.7. $\square$

4.5 Towards a general theory

We have introduced a notion of higher geodecity based on the concept of the Steiner distance of a set of vertices. This came as a hierarchy of properties: Every k-geodesic subgraph is, by definition, also m-geodesic for any $m < k$. This hierarchy is strict in the sense that for every k there exists graphs G and $H \subseteq G$ and a length-function ℓ on G such that H is k-geodesic in G, but not $(k+1)$-geodesic.

To see this, let G be a complete graph with $V(G) = [k+1] \cup \{0\}$ and let H be the subgraph induced by $[k+1]$. Define $\ell(0j) := k-1$ and $\ell(ij) := k$ for all $i, j \in [k+1]$. Assume H was not k-geodesic. Then G contains a shortcut tree T for H with $|L(T)| \leq k$. By (SCT 1) and (SCT 2), T must be a star with center 0. Therefore

$$\ell(T) = (k-1)|L(T)| \geq k(|L(T)| - 1) = \mathrm{sd}_H(L(T)),$$

contrary to (SCT 3). Hence H is a k-geodesic subgraph of G. However, the star S with center 0 and $L(S) = [k+1]$ shows that

$$\mathrm{sd}_G(V(H)) \leq (k+1)(k-1) < k^2 = \mathrm{sd}_H(V(H)),$$

so H is not $(k+1)$-geodesic in G.

Theorem 4.1 and Theorem 4.2 show that trees and cycles exhibit a strange behaviour in the sense that the hierarchy collapses for these subgraphs.

We now attempt to capture this phenomenon more systematically. For a given natural number $k \geq 2$, let us denote by $\mathcal{H}_k$ the class of all graphs H with the property that whenever G is a graph with $G \supseteq H$ and ℓ is a length-function on G such that H is k-geodesic in G, then H is already fully geodesic.

By definition, this yields an ascending sequence $\mathcal{H}_2 \subseteq \mathcal{H}_3 \subseteq \ldots$ of classes of graphs. By Theorem 4.1, $\mathcal{H}_2$ contains all trees. By Theorem 4.2, all cycles are contained in $\mathcal{H}_6$. The example above shows that $K_{k+1} \notin \mathcal{H}_k$.

Proposition 4.13. *Let $k \geq 2$ be an integer and H a graph. Then $H \in \mathcal{H}_k$ if and only if every shortcut tree for H has at most k leaves.*

Proof. Suppose first that $H \in \mathcal{H}_k$ and let T be a shortcut tree for H. By (SCT 3), H is not $|L(T)|$-geodesic in $T \cup H$. Let m be the minimum integer such that H is not m-geodesic in $T \cup H$. By Lemma 4.3, $T \cup H$ contains a shortcut tree S for H with at most m leaves. But then by (SCT 1) and (SCT 2), S is the Steiner tree in T of $B := L(S) \subseteq L(T)$. If $B \subsetneq L(T)$, then $\ell(S) = \mathrm{sd}_T(B) \geq \mathrm{sd}_H(B)$ by (SCT 4), so we must have $B = L(T)$ and $m \geq |L(T)|$. Thus H is $(|L(T)| - 1)$-geodesic in $T \cup H$, but not $|L(T)|$-geodesic. As $H \in \mathcal{H}_k$, it must be that $|L(T)| - 1 < k$.

Suppose now that every shortcut tree for H has at most k leaves and let $H \subseteq G$ k-geodesic with respect to some length-function $\ell : E(G) \to \mathbb{R}^+$. If H was not fully geodesic, then G contained a shortcut tree T for H. By assumption, T has at most k leaves. But then $\mathrm{sd}_G(L(T)) \leq \ell(T) < \mathrm{sd}_H(L(T))$, so H is not k-geodesic in G. □

For a tree T, let $\mathcal{G}_T$ be the class of all graphs which admit a shortcut tree isomorphic to T. By Proposition 4.13, $\mathcal{H}_k$ is the intersection of $\mathcal{G}_T$ over all trees T with more than k leaves. In the sequel, we will therefore study these classes $\mathcal{G}_T$. Observe that $\mathcal{G}_T$ is empty when T has only one node and that $\mathcal{G}_P$ is the class of all graphs with at least 2 vertices when P is a non-trivial path. It is easy to see that $\mathcal{G}_T = \mathcal{G}_{T'}$ when T' is a subdivision of T. Our example above actually shows that $K_k \notin \mathcal{G}_{K_{1,k}}$.

Theorem 4.14. *$\mathcal{G}_T$ is minor-closed for every tree T.*

Proof. This is trivial if $|L(T)| \leq 2$, so we now assume that T has at least 3 leaves. Let G be a graph and $H \preceq G$ a minor of G. Suppose that $H \notin \mathcal{G}_T$. After renaming the vertices of T, we may assume that there is a length-function $\ell : E(H \cup T) \to \mathbb{R}^+$ such that T is a shortcut tree for H. By adding a sufficiently small positive real number to every $\ell(e)$, $e \in E(T)$, we may assume that the inequalities in (SCT 4) are strict, that is, there exists some $\epsilon > 0$ such that

$$\mathrm{sd}_H(B) \leq \mathrm{sd}_T(B) - \epsilon$$

for every $B \subseteq L(T)$ with $2 \leq |B| < |L(T)|$.

Since H is a minor of G, we find a family $(B_v)_{v \in V(H)}$ of connected subsets of $V(G)$ and a family $(b_e)_{e \in E(H)}$ of edges of G such that b_{uv} joins B_u and B_v for every $uv \in E(H)$. For every $t \in L(T) \subseteq V(H)$, pick an arbitrary vertex $x_t \in B_t$. For the remaining $t \in V(T) \setminus L(T)$, introduce new vertices $x_t \notin V(G)$. Denote by T' the tree on $\{x_t \colon t \in T\}$ where $x_s x_t \in E(T')$ if and only if $st \in E(T)$. Observe that $t \mapsto x_t$ is an isomorphism between T and T' and that T' satisfies (SCT 1) and (SCT 2) for G.

We now define a length-function $\ell' : E(G \cup T') \to \mathbb{R}^+$. For $st \in E(T)$ we let $\ell'(x_s x_t) := \ell(st)$. We define the length of $f \in E(G)$ as follows. If $f \in G[B_v]$ for some $v \in V(H)$, then f receives the length $\delta := \epsilon/e(G)$. If $f = b_e$ for some $e \in E(H)$, then we let $\ell'(f) := \ell(e)$. If neither holds, we let $\ell'(f) := \ell(T) + 1$ and call f *external.*

We first show that (SCT 3) holds, that is, $\mathrm{sd}_{T'}(L(T')) \leq \mathrm{sd}_G(L(T'))$. Let $S \subseteq G$ connected with $L(T') \subseteq V(S)$. If S contains an external edge, then

$$\ell'(S) > \ell(T) = \mathrm{sd}_{T'}(L(T')).$$

Let $R \subseteq H$ be the subgraph where $v \in V(R)$ if and only if $V(S) \cap B_v$ is non-empty and $e \in E(R)$ if and only if $b_e \in E(S)$. Since S is connected, so is R. Moreover $L(T) \subseteq V(R)$, since $x_t \in V(S) \cap B_t$ for every $t \in L(T)$. But T is a shortcut tree for H, so it follows that

$$\ell'(S) \geq \ell(R) > \mathrm{sd}_T(L(T)) = \mathrm{sd}_{T'}(L(T')).$$

Since S was arbitrary, we have shown that $\mathrm{sd}_G(L(T)) > \mathrm{sd}_{T'}(L(T'))$.

We now verify that (SCT 4) holds. Let $B' \subseteq L(T')$ with $2 \leq |B'| < |L(T')|$ arbitrary and let B be the set of all $t \in L(T)$ with $x_t \in B'$. By assumption, there exists a connected $R \subseteq H$ with $B \subseteq V(R)$ and $\ell(R) \leq \mathrm{sd}_T(B) - \epsilon$. Let

$$S := \bigcup_{r \in V(R)} G[B_r] + \{b_e \colon e \in E(R)\} \subseteq G.$$

Then S is connected, $B' \subseteq V(S)$ and

$$\ell'(S) \leq \delta e(G) + \ell(R) \leq \mathrm{sd}_T(B).$$

□

Proposition 4.15. *Let T be a tree with at least 3 leaves and G a graph. Then $G \in \mathcal{G}_T$ if and only if every component of G is in $\mathcal{G}_T$.*

Proof. Let K be a component of G and suppose K had a shortcut tree $T' \cong T$ with length-function ℓ; after renaming we may assume $V(T') \cap V(G) = L(T')$. Extend ℓ to G by defining $\ell(e) := \ell(T') + 1$ for every $e \in E(G) \setminus E(K)$. It is easy to see that T' is then a shortcut tree for G as well.

Suppose now that G had a shortcut tree $T' \cong T$ with length-function ℓ. Let $x, y \in L(T')$. Since T' has at least 3 leaves, it follows from (SCT 4) that

$$\mathrm{sd}_G(\{x, y\}) \leq \mathrm{sd}_{T'}(\{x, y\}) < \infty,$$

so x and y lie in the same component of G. Hence there is a component K of G with $L(T') \subseteq K$. But then T' is in fact a shortcut tree for K. $\square$

Corollary 4.16. *Let $k \geq 2$ be an integer. Then $\mathcal{H}_k$ is minor-closed and $G \in \mathcal{H}_k$ if and only if every component of G is in $\mathcal{H}_k$.* $\square$

Corollary 4.17. *$\mathcal{H}_2$ is the class of forests.*

Proof. By Theorem 4.1, every tree is in $\mathcal{H}_2$. By the above, $\mathcal{H}_2$ thus contains all forests. If G is not a forest, then G contains the triangle K_3 as a minor. We have seen in Section 4.4 that K_3 has a shortcut tree isomorphic to the star with 3 leaves, so $K_3 \notin \mathcal{H}_2$. Since $\mathcal{H}_2$ is minor-closed, it follows that $G \notin \mathcal{H}_2$. $\square$

It is easy to see that if T_2 is obtained from T_1 by contracting an edge which is not incident to a leaf of T_1, then $\mathcal{G}_{T_1} \subseteq \mathcal{G}_{T_2}$: whenever T_2 occurs as a shortcut tree, assign a sufficiently small positive length to the contracted edge to obtain T_1 as a shortcut tree.

In particular, $\mathcal{G}_{K_{1,k}}$ contains $\mathcal{G}_T$ for every tree T with k leaves. In general, this inclusion is strict: we have seen in Section 4.4 that for each $2 \leq k \leq 6$, there is (up to subdivision) only one tree with k leaves which can occur as a shortcut tree for a cycle on at least k vertices.

Proposition 4.18. *$\mathcal{G}_{K_{1,4}}$ is the class of outerplanar graphs.*

Proof. Let G be an outerplanar graph and assume for a contradiction $T \cong K_{1,4}$ was a shortcut tree for G under the length-function ℓ. Since G is outerplanar, G can be drawn in a disk D with all vertices on the boundary ∂D. The leaves of T appear on ∂D in some cyclic order as v_1, v_2, v_3, v_4. For $B \subsetneq L(T)$, G contains a connected subgraph $R_B \subseteq G$ with $B \subseteq V(R_B)$ and $\ell(R) \leq \mathrm{sd}_T(B)$. By the way G is drawn in the disk, $R_{\{1,3\}}$ and $R_{\{2,4\}}$ intersect. Hence $R := R_{\{1,3\}} \cup R_{\{2,4\}}$ is connected, $L(T) \subseteq R$ and

$$\ell(R) \leq \ell(R_{\{1,3\}}) + \ell(R_{\{2,4\}}) = \ell(T),$$

contrary to (SCT 3).

Now let G be a graph which is not outerplanar. Then G contains either K_4 or $K_{2,3}$ as a minor. By Theorem 4.14, it suffices to show that $K_4, K_{2,3} \notin \mathcal{G}_{K_{1,4}}$. We have already seen before that $K_k \notin \mathcal{G}_{K_{1,k}}$ for every integer $k \geq 2$.

We now show that $K_{2,k} \notin \mathcal{G}_{K_{1,k+1}}$ for integer $k \geq 2$. Let $G \cong K_{2,k}$ with $V(G) = \{a, b\} \cup [k]$, where $xy \in E(G)$ if and only if $x \in \{a, b\}$ and $y \in [k]$ (or vice versa). Let $T \cong K_{1,k+1}$ with $V(T) = \{a, r\} \cup [k]$ and edges rx for all $x \in \{a\} \cup [k]$. Let $\ell(aj) := k$ for all $j \in [k]$ and define $\ell(e) := k - 1$ for all other $e \in E(G \cup T)$. It is easy to verify that T is indeed a shortcut tree for G. $\square$

Consider now an arbitrary tree T with $k \geq 3$ leaves. What can be said about the graphs in $\mathcal{G}_T$? We have already seen that $K_k, K_{2,k-1} \notin \mathcal{G}_{K_{1,k}} \subseteq \mathcal{G}_T$, so no graph in $\mathcal{G}_T$ contains any of these two as a minor. However, $\mathcal{G}_T$ is not determined by the number k of leaves alone, so an obstruction more closely related to the structure of T is desirable.

Proposition 4.19. *Let T be a tree with at least 3 leaves. Then the line graph of T is not in $\mathcal{G}_T$.*

Proof. Let G be the line graph of T, that is, the graph with $V(G) = E(T)$, where two edges of T are adjacent in G if and only if they are incident in T. Identify each leaf x of T with the unique edge $xx' \in E(T) = V(G)$. For $t \in T$, let $K_T(t)$ be the set of all $e \in E(T)$ which are incident to t. Note that $e, f \in E(T)$ are adjacent in G if and only if there exists a $t \in T$ with $e, f \in K_T(t)$ and that this t, if it exists, is unique.

Fix some $0 < \epsilon < 1/e(T)$. To each edge $st \in E(T)$ we assign the length

$$\ell(st) := d_T(s) + d_T(t) - (3 - \epsilon),$$

where $d_T(x)$ denotes the degree of a vertex x in T. We assign to every edge of $K_T(t)$ the length $d_T(t) - 1$. This defines a map $\ell : E(G) \cup E(T) \to \mathbb{R}^+$.

We claim that T is a shortcut tree for G. Let $L \subseteq L(T)$ and let $S \subseteq T$ be the Steiner tree for L in T. Then

$$\ell(S) = \sum_{st \in E(S)} (d_T(s) + d_T(t) - (3 - \epsilon)) = \sum_{s \in S} d_S(s) d_T(s) - (3 - \epsilon) e(S). \tag{4.6}$$

Let $H \subseteq G$ be an inclusion-minimal connected subgraph with $L \subseteq V(H)$. Then $E(S) \subseteq V(H)$, because every edge of S separates two $a, b \in L$ in T and therefore also in G. Since H is inclusion-minimal, it follows that $E(S) = V(H)$, because $H[E(S)]$ is already connected and contains L. We claim that $H[K_S(s)]$ is connected for every $s \in S$. Let (A, B) be a non-trivial bipartition of $K_S(s)$. Let $S_A \subseteq S$ be the component of $S - B$ containing s and let $S_B \subseteq S$ be the component of $S - A$ containing s. Note that both S_A and S_B meet L. This yields a non-trivial bipartition $E(S) = E(S_A) \cup E(S_B)$ of the vertices of H. Since H is connected, there exist $e_A \in E(S_A)$ and $e_B \in E(S_B)$ which are adjacent in H. Hence there exists some $t \in S$ with $e_A, e_B \in K_S(s)$. But then we must have $t = s$ and $e_A \in A$, $e_B \in B$.

Conversely, it is easy to see that any $H \subseteq G$ with $V(H) = E(S)$ for which $H[K_S(s)]$ is connected for every $s \in S$ is already connected and satisfies $L \subseteq V(H)$. Hence we obtain all inclusion-minimal connected $H \subseteq G$ with $L \subseteq V(H)$ by choosing, for every $s \in S$, some spanning tree on $G[K_S(s)]$, and taking their union.

By definition of our length-function, every spanning tree on $G[K_S(s)]$ has the same length $(d_S(s)-1)(d_T(s)-1)$. Therefore

$$\mathrm{sd}_G(L) = \sum_{s\in S}(d_S(s)-1)(d_T(s)-1) = \sum_{s\in S} d_S(s)d_T(s) - e(S) - 2e(T) + 1. \tag{4.7}$$

Comparing (4.6) and (4.7), we see that $\mathrm{sd}_T(L) < \mathrm{sd}_G(L)$ if and only if $e(S) = e(T)$, that is, if and only if $L = L(T)$. Therefore (SCT 3) and (SCT 4) are satisfied and T is indeed a shortcut tree for G. $\square$

Corollary 4.20. *The class of C_4-minor free graphs is not contained in any $\mathcal{H}_k$, $k \geq 2$.*

Proof. Given $k \geq 2$, let T be a subcubic tree with more than k leaves. Then the line graph G of T has no C_4-minor. By Proposition 4.19, G has a shortcut tree isomorphic to T. By Proposition 4.13, $G \notin \mathcal{H}_k$. $\square$

Note that $C_4 \cong K_{2,2}$ and that every C_4-minor free graph is outerplanar. In view of Proposition 4.18, we ask the following:

Question 2. Does there exist, for every integer $k \geq 1$, an integer $m = m(k)$ such that every graph in $\mathcal{G}_{K_{1,m}}$ contains $K_{2,k}$ as a minor?

We showed in Section 4.4 that $\mathcal{H}_6$ contains all cycles. Since cycles are subdivisions of K_3, an affirmative answer to the following would be a generalization of our result:

Question 3. Does there exist, for every graph G, an integer $k = k(G)$ such that every subdivision of G is in $\mathcal{H}_k$?

4.6 Generating the cycle space

Let G be a graph with length-function ℓ. It is a well-known fact (see e.g. [18, Chapter 1, exercise 37]) that the set of 2-geodesic cycles generates the cycle space of G. This extends as follows, showing that fully geodesic cycles abound.

Proposition 4.21. *Let G be a graph with length-function ℓ. The set of fully geodesic cycles generates the cycle space of G.*

We remark, first of all, that the proof is elementary and does not rely on Theorem 4.2, but only requires Lemma 4.3 and Lemma 4.4.

Let $\mathcal{D}$ be the set of all cycles of G which cannot be written as a 2-sum of cycles of smaller length. The following is well-known.

Lemma 4.22. *The cycle space of G is generated by $\mathcal{D}$.*

Proof. It suffices to show that every cycle is a 2-sum of cycles in $\mathcal{D}$. Assume this was not the case and let $C \subseteq G$ be a cycle of minimum length that is not a 2-sum of cycles in $\mathcal{D}$. In particular, $C \notin \mathcal{D}$ and so there are cycles $C_1, \ldots, C_k$ with $C = C_1 \oplus \ldots \oplus C_k$ and $\ell(C_i) < \ell(C)$ for every $i \in [k]$. By our choice of C, every C_i can be written as a 2-sum of cycles in $\mathcal{D}$. But then the same is true for C, which is a contradiction. $\square$

Proof of Proposition 4.21. We show that every $C \in \mathcal{D}$ is fully geodesic. Indeed, let $C \subseteq G$ be a cycle which is not fully geodesic and let $T \subseteq G$ be a shortcut tree for C. Then $L(T)$ splits C into segments and there is a cycle D with $V(D) = L(T)$ such that C is a union of edge-disjoint $L(T)$-paths P_{ab} joining a and b for $ab \in E(D)$.

For $ab \in E(D)$ let $C_{ab} := aTb + P_{ab}$. Every edge of C lies in precisely one of these cycles. An edge $e \in E(T)$ lies in C_{ab} if and only if $e \in aTb$. By Lemma 4.4 and (4.3), every $e \in E(T)$ lies in an even number of cycles C_{ab}. Therefore $C = \bigoplus_{ab \in E(D)} C_{ab}$.

For every $ab \in E(D)$, C contains the path $S = C - E(P_{ab})$ with $L(T) \subseteq V(S)$. Since T is a shortcut tree for C, it follows from (SCT 3) that

$$\ell(C_{ab}) \leq \ell(T) + \ell(P_{ab}) < \ell(S) + \ell(P_{ab}) = \ell(C).$$

In particular, $C \notin \mathcal{D}$. $\square$

The fact that 2-geodesic cycles generate the cycle space has been extended to the topological cycle space of locally finite graphs graphs by Georgakopoulos and Sprüssel [35]. Does Proposition 4.21 have a similar extension?

5 On the block number of graphs

Recall that a set X of at least k vertices of a graph G is $(< k)$-*inseparable* if no two vertices in X can be separated in G by deleting fewer than k vertices. A maximal such set is a k-*block* and can be thought of as a highly connected part of the graph, although it may draw its connectivity from the ambient graph G rather than just the subgraph induced by X itself. The maximum integer k for which G contains a k-block is the *block number* of G, denoted by $\beta(G)$. In this chapter, we study how the block number relates to other notions of "width" of graphs.

The main result of this chapter is a structure theorem for graphs without k-blocks:

Theorem 5.1. *Let G be a graph and $k \geq 2$ an integer.*

(i) *If G has no $(k+1)$-block, then G has a tight tree-decomposition of adhesion at most k in which every torso has at most k vertices of degree at least $2k(k-1)$.*

(ii) *If G has a tree-decomposition in which every torso has at most k vertices of degree at least k, then G has no $(k+1)$-block.*

This yields a qualitative duality: Every graph either has a $(k+1)$-block or a tree-decomposition that demonstrates that it has no $2k^2$-block. Theorem 5.1 will be proved in Section 5.2.

We then study the block number of graphs from classes excluding some fixed topological minor. Dvořák [28] implicitly characterized those classes $\mathcal{G}$ for which the block number of graphs in $\mathcal{G}$ is bounded. As k-blocks are not mentioned in [28], we make this characterization explicit in Section 5.3. For classes for which no absolute bound on β exists, we prove the following bound relative to the number of vertices:

Theorem 5.2. *Let $\mathcal{G}$ be a class of graphs excluding some fixed graph as a topological minor. There exists a constant $c = c(\mathcal{G})$ such that every $G \in \mathcal{G}$ satisfies $\beta(G) \leq c\sqrt[3]{|G|}$.*

Finally, we relate the block number to tree-width. It is easy to see that $\beta(G) \leq \mathrm{tw}(G) + 1$, so the existence of a k-block forces large tree-width. However, a graph can have arbitrarily large tree-width and yet have no 5-block: $k \times k$-grids are such graphs. Since tree-width does not increase when taking minors, the tree-width of G (plus one) is even an upper bound for the block number of *every minor* of G. We prove a converse to this statement, namely that a graph with large tree-width must have a minor with large block number.

Theorem 5.3. *Let $k \geq 1$ be an integer and G a graph. If $\mathrm{tw}(G) \geq 2k^2 - 2$, then some minor of G contains a k-block. This bound is optimal up to a constant factor.*

A more precise version of this theorem will be proved in Section 5.5.

5.1 Preliminaries

For a vertex $v \in V$ and integer k, a *k-fan from v* is a collection $\mathcal{Q}$ of k paths which all have v as a common starting vertex and are otherwise disjoint. It is a k-fan *to* some $U \subseteq V$ if the end-vertex of every path in $\mathcal{Q}$ lies in U. We explicitly allow a fan to contain the trivial path consisting only of the vertex v itself. The end-vertex of this path is v. A set $X \subseteq V$ is a *k-fan-set* if from every $x \in X$ there exists a k-fan to X. The *∞-admissibility* $\mathrm{adm}_\infty(G)$ is the maximum k for which G contains a k-fan-set.

Lemma 5.4. *Let G be a graph and $X \subseteq V(G)$ a $(<k)$-inseparable set of vertices for some $k \in \mathbb{N}$. Then X is a k-fan-set.*

Proof. Suppose there was an $x \in X$ with no $(k-1)$-fan from x to $X \setminus \{x\}$. By Menger's Theorem there is a set $S \subseteq V \setminus \{x\}$ with $|S| < k - 1$ separating x from $X \setminus (S \cup \{x\})$. Since $|X| \geq k$, there must be some $y \in X \setminus (S \cup \{x\})$. Since X is $(<k)$-inseparable, S cannot separate x and y, a contradiction. □

The converse is not true: If G is a disjoint union of cliques of order k, then $V(G)$ is a k-fan-set, but not even a 1-block.

Let G be a graph and $X \subseteq V(G)$ a $(<k)$-inseparable set of vertices. Then every $x \in X$ has degree at least $k-1$ in G. Therefore G must have at least $k(k-1)/2$ edges. Since any minor of G has at most $e(G)$ edges, it follows that

$$\max_{H \preceq G} \beta(H) \leq 1 + \sqrt{2e(G)}. \tag{5.1}$$

Similarly, the tree-width of G can place a bound on the block number of every minor of G. The following is well-known, see [18, Lemma 12.3.4].

Lemma 5.5 ([18])**.** *Let G be a graph, $X \subseteq V(G)$ a $(<k)$-inseparable set of vertices for some $k \in \mathbb{N}$ and let $(T, \mathcal{V})$ be a tree-decomposition of G of adhesion $< k$. Then there exists a $t \in T$ with $X \subseteq V_t$.*

Therefore the tree-width of G must then be at least $k-1$. Since every minor of G has tree-width at most $\mathrm{tw}(G)$, we have $\beta(H) \leq \mathrm{tw}(G)+1$ for every $H \preceq G$.

5.2 The structure of graphs without k-blocks

Perhaps the most trivial reason a graph G can fail to contain a $(k+1)$-block is if G has at most k vertices of degree at least k. These graphs can be used as building blocks for graphs of block number at most k

Proof of Theorem 5.1 (ii). Let $(T, \mathcal{V})$ be a tree-decomposition in which every torso has at most k vertices of degree at least k. Assume that G contained a $(k+1)$-block X. Every adhesion-set $V_s \cap V_t$, $st \in E(T)$, is a clique in the torso of t, so $|V_s \cap V_t| \leq k$ by assumption on the degrees.

Since X is a $(k+1)$-block, it follows from Lemma 5.5 that $X \subseteq V_t$ for some $t \in T$. We will show that every vertex of X has degree at least k in the torso of t, which is a contradiction.

Let $x \in X$ arbitrary and let $A \subseteq V_t$ be the set of all neighbors of x in the torso of t. If $|A| \geq k$, we are done. Otherwise, let $y \in X \setminus (A \cup \{x\})$. In particular, x and y are non-adjacent in G, so by Menger's Theorem there is a set $\mathcal{P}$ of $k+1$ internally disjoint x-y-paths in G. Since every $P \in \mathcal{P}$ has both end-vertices in V_t, it has a vertex $z_P \in V(P) \cap V_t \setminus \{x\}$ which lies closest to x along P. Then x and this vertex z_P must either be adjacent or lie in a common adhesion-set $V_s \cap V_t$. Hence $z_P \in A$ and, in particular, $z_P \neq y$. As the paths in $\mathcal{P}$ are internally disjoint, all these vertices z_P are distinct. Thus the degree of x in the torso of t is at least $k+1$. □

The converse, decomposing a graph with no $(k+1)$-block into graphs of almost bounded degree, is more intricate.

Lemma 5.6. *Let $(T, \mathcal{V})$ a tight tree-decomposition of G of adhesion $<k$ for $k \geq 3$. Let $m \geq 1$ be an integer, $t \in T$ and $x \in V_t$. If x has degree at least $(m-1)(k-2)$ in the torso of t, then there exists an m-fan from x to V_t.*

Proof. Let A be the set of vertices of V_t which are adjacent to x in G and let B be the set of vertices that are adjacent to x in the torso of t, but not in G. Let $\mathcal{Q}_A$ be the fan consisting of the trivial path $\{x\}$ and single edges to each $a \in A$. We now construct a fan $\mathcal{Q}_B$ from x to B consisting of V_t-paths.

For every $b \in B$ there is an edge $st \in E(T)$ with $\{b, x\} \subseteq V_s \cap V_t$. Let R be the set of all neighbors s of t in T with $x \in V_s$. Let $S \subseteq R$ minimal such that $B \subseteq \bigcup_{s \in S} V_s$. For $s \in S$, let $B_s := B \cap V_s$. By minimality of S, every B_s contains some vertex $b_s \notin \bigcup_{s' \neq s} B_{s'}$.

Let T_s be the component of $T - st$ containing s and $G_s := G[V_{T_s}]$. Since $(T, \mathcal{V})$ is tight, there is a component of $G_s - V_t$ that contains neighbors of both x and b_s. We therefore find a path P_s from x to b_s in G_s that meets V_t only in its endpoints. The set $\mathcal{Q}_B := \{P_s \colon s \in S\}$ is an $|S|$-fan from x to B in which every path is internally disjoint from V_t. Thus $\mathcal{Q} := \mathcal{Q}_A \cup \mathcal{Q}_B$ is a fan from x to V_t.

It remains to show $|\mathcal{Q}| \geq m$. As $B \subseteq \bigcup_s B_s$, we have

$$|A| + \sum_{s \in S} |B_s| \geq |A| + |B| \geq (m-1)(k-2).$$

Note that $x \in V_s \cap V_t$ for every $s \in S$ and $B_s \subseteq V_s \cap V_t \setminus \{x\}$. Since $(T, \mathcal{V})$ has adhesion less than k, it follows that $|B_s| \leq k-2$. Therefore

$$|\mathcal{Q}| = 1 + |A| + |S| \geq 1 + \frac{|A| + (k-2)|S|}{k-2} \geq m.$$

□

Lemma 5.7. *Let $(T, \mathcal{V})$ be a k-lean tree-decomposition of G, $t \in T$ and $u, v \in V(G)$. If from both u and v there are $(2k-1)$-fans to V_t, then u and v cannot be separated by deleting fewer than k vertices.*

Proof. Suppose there was some $S \subseteq V(G) \setminus \{u, v\}$, $|S| < k$, separating u and v. We find a set of k paths of the fan from u to V_t which are disjoint from S and let $R_u \subseteq V_t$ be their endvertices. Note that all vertices in R_u lie in the component of $G - S$ containing u. Define $R_v \subseteq V_t$ similarly for v.

Since $(T, \mathcal{V})$ is k-lean, we find k vertex-disjoint paths from R_u to R_v. All of these paths must pass through S, a contradiction. □

Combining these lemmas, we can prove the following refinement of Theorem 5.1 (i):

Theorem 5.8. *Let G be a graph, $k \geq 3$ an integer, $(T, \mathcal{V})$ a k-atomic tree-decomposition of G and $t \in T$. If the torso at t contains at least k vertices of degree at least $2(k-1)(k-2)$, then G has a k-block B with $B \subseteq V_t$.*

Proof. Let $X_t \subseteq V_t$ be the set of vertices of degree at least $2(k-1)(k-2)$ in the torso of t. By Lemma 5.6, every $x \in X_t$ has a $(2k-1)$-fan to V_t. By Lemma 5.7, no two vertices of X_t can be separated by deleting fewer than k vertices.

If $|X_t| \geq k$, then X_t is $({<}k)$-inseparable and G has a k-block $Y \supseteq X$. By Lemma 5.5, there exists a node $s \in T$ with $Y \subseteq V_s$. Since $X \subseteq V_s \cap V_t$ and $(T, \mathcal{V})$ has adhesion ${<}k$, we must have $s = t$. □

5.3 Topological minors and k-blocks

When considering k-blocks, the topological minor relation is more natural than the ordinary minor relation. For example, it is easy to see that a $(k+1)$-block in a graph H yields a $(< k)$-inseparable set in any graph containing H as a topological minor. No such statement is true when considering minors: It is easy to construct a triangle-free graph G of maximum degree 3 that contains the complete graph of order k as a minor. This graph G has no 4-block.

In this section we study the block number of graphs from classes of graphs $\mathcal{G}$ that exclude some fixed graph as a topological minor. Examples of such classes include graphs of bounded genus, bounded tree-width or bounded degree. In general, there exists no upper bound on the block number of graphs in $\mathcal{G}$. In fact, we can explicitly describe a planar graph with block number k: take a rectangular $rk \times k$-grid, add $2(r+1)$ vertices to the outer face and join each of these to k vertices on the perimeter of the grid (see Figure 5.1). If $2(r+1) \geq k$, these new vertices are $(<k)$-inseparable.

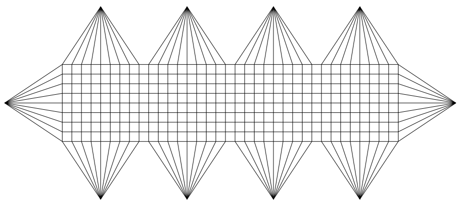

Figure 5.1: A planar graph with a 9-block of order 10.

We are thus faced with two tasks: First, to characterize those classes for which there exists an upper bound on the block number. Second, to obtain a relative upper bound on the block number of graphs in $\mathcal{G}$ when no absolute upper bound exists.

5.3.1 The bounded case

As indicated in the introduction, Dvořák [28] implicitly characterized the classes for which there exists an upper bound on the block number. Since k-blocks are not mentioned in [28], we make this characterization explicit here without introducing any ideas not present in [28].

A small modification of the graph depicted in Figure 5.1 yields a planar graph H_k with roughly $k^3/2$ vertices and block number k which can be drawn in the plane so that every vertex of degree greater than 3 lies on the outer face: Essentially, replace the square grid by a hexagonal grid and join the 'new' vertices only to degree-2 vertices on the perimeter.

Suppose that H is a graph with the property that every graph G that does not contain H as a topological minor satisfies $\beta(G) < s$ for some constant $s = s(H)$. Then H is a topological minor of H_s and therefore planar. Moreover, H "inherits" a drawing in the plane in which all vertices of degree greater than 3 lie on the outer face.

The simplest case of a deep structure theorem for graphs excluding a fixed graph as a topological minor [28, Theorem 3] asserts a converse to this in a strong form.

Theorem 5.9 ([28])**.** *Let H be a graph drawn in the plane so that every vertex of degree greater than 3 lies on the outer face. Then there exists an $r = r(H)$ such that every graph that does not contain H as a topological minor has a tree-decomposition in which every torso contains at most r vertices of degree at least r.*

It is now easy to characterize the graphs whose exclusion as a topological minor bounds the block number.

Corollary 5.10. *Let H be a graph. The following are equivalent:*

(i) *There is an integer $s = s(H)$ such that every graph G that does not contain H as a topological minor satisfies $\beta(G) < s$.*

(ii) *H can be drawn in the plane such that every vertex of degree greater than 3 lies on the outer face.*

Proof. (i) $\to$ (ii): By assumption, the graph H_s contains H as a topological minor. The desired drawing of H can then be obtained from the drawing of H_s.

(ii) $\to$ (i): By Theorem 5.9 and Theorem 5.1 (ii). $\square$

Note that every graph that contains H_k as a topological minor necessarily has a k-block. Theorem 5.9 thereby implies a qualitative version of Theorem 5.1 (i), but without explicit bounds.

Corollary 5.11. *Let $\mathcal{G}$ be a class of graphs. The following are equivalent:*

(i) *There is a $k \in \mathbb{N}$ such that $\beta(G) \leq k$ for every $G \in \mathcal{G}$.*

(ii) *There is an $m \in \mathbb{N}$ such that no $G \in \mathcal{G}$ contains H_m as a topological minor.*

(iii) *There is an $r \in \mathbb{N}$ such that every graph in $\mathcal{G}$ has a tree-decomposition in which every torso has at most r vertices of degree at least r.*

5.3.2 The unbounded case

We now turn to the case where $\mathcal{G}$ is a class of graphs excluding some fixed graph as a topological minor for which there exists no upper bound on the block number of graphs in $\mathcal{G}$. If $\mathcal{G}$ is closed under taking topological minors, then by Corollary 5.11 this implies $H_k \in \mathcal{G}$ for all $k \in \mathbb{N}$. Since $|H_k| \leq \beta(H_k)^3$, the bound in Theorem 5.2 is optimal up to a constant factor.

Our aim now is to prove Theorem 5.2. In light of Lemma 5.4, it clearly suffices to show the following.

Theorem 5.12. *Let $\mathcal{G}$ be a class of graphs excluding some fixed graph as a topological minor. There exists a constant $c = c(\mathcal{G})$ such that every $G \in \mathcal{G}$ containing a k-fan-set X has at least $c|X|k^2$ vertices.*

This immediately yields the following strengthening of Theorem 5.2.

Corollary 5.13. *Let $\mathcal{G}$ be a class of graphs excluding some fixed graph as a topological minor. Let $G \in \mathcal{G}$ and let X be the set of all vertices of G that lie in some k-block of G. Then $|X| \leq |G|/(ck^2)$, where $c = c(\mathcal{G})$ is the constant from Theorem 5.12.*

Proof. By Lemma 5.4, every k-block of G is a k-fan-set. It is easy to see that a union of k-fan-sets is again a k-fan-set. Since X is the union of all k-blocks, it is therefore a k-fan-set. By Theorem 5.12 we have $|G| \geq c|X|k^2$ for $c = c(\mathcal{G})$. □

We now turn to the proof of Theorem 5.12 above. Excluding a topological minor ensures that our graph and all its topological minors are sparse. The following is well-known, see [18, Chapter 7].

Lemma 5.14. *Let $\mathcal{G}$ be a class of graphs excluding some fixed graph as a topological minor. There exist constants $\alpha, d > 0$ such that every topological minor G of a graph in $\mathcal{G}$ has at most $d|G|$ edges and an independent set of order at least $\alpha|G|$.*

Proof of Theorem 5.12. Let $G \in \mathcal{G}$ and $k \in \mathbb{N}$. To ease notation, we assume that $X \subseteq V(G)$ is a $(k+1)$-fan-set instead of just a k-fan-set. This only has an effect on the constant c.

For every $x \in X$ let $\mathcal{Q}_x$ be a k-fan from x to $X \setminus \{x\}$. Taking subpaths, if necessary, we may assume that no $Q \in \mathcal{Q}_x$ has an internal vertex in X. We use initial segments of the paths in $\mathcal{Q}_x$ to construct a subdivision of a star with center x. Lemma 5.14 will enable us to find many disjoint such subgraphs.

We adopt an idea from [38]. For some integer r that we are going to choose later, let $\mathcal{P}$ be a maximal set of internally disjoint X-paths of length at most $2r$ such that for any two $x, y \in X$ there is at most one path in $\mathcal{P}$ joining them. The paths in $\mathcal{P}$ will be used as barriers to separate the subdivided stars. Let $B := X \cup \bigcup \mathcal{P}$.

For $x \in X$ and $Q \in \mathcal{Q}_x$, let $Q' \subseteq Q$ be the maximal subpath of length at most r with $Q' \cap B = \{x\}$. If the length of Q' is less than r, then the next vertex along Q lies in B. We say that this vertex *stops* the path Q'. Define $\mathcal{Q}'_x := \{Q' : Q \in \mathcal{Q}_x\}$ and $S_x := \bigcup \mathcal{Q}'_x$.

The paths in $\mathcal{P}$ provide us control on the overlap of the stars and allow us to separate them. Let $H = H(\mathcal{P})$ be the auxiliary graph with vertex-set X where $xy \in E(H)$ if and only if some $P \in \mathcal{P}$ joins x and y.

$$\text{If } \bigcup \mathcal{Q}'_x \cap \bigcup \mathcal{Q}'_y \neq \emptyset, \text{ then } xy \in E(H). \tag{5.2}$$

Indeed, if $\bigcup \mathcal{Q}'_x \cap \bigcup \mathcal{Q}'_y \neq \emptyset$ then we can find an X-path P of length at most $2r$ between x and y which is internally disjoint from all paths in $\mathcal{P}$. By maximality of $\mathcal{P}$, there must already be some $R \in \mathcal{P}$ joining x and y. Similarly

$$\text{If } x \in X \text{ stops some } Q' \in \mathcal{Q}'_y, \text{ then } xy \in E(H). \tag{5.3}$$

The graph H is clearly a topological minor of G. It follows from Lemma 5.14 that $|\mathcal{P}| = |E(H)| \leq d|X|$ and that H contains an independent set $Y \subseteq X$ with $|Y| \geq \alpha|X|$. By (5.2), the stars with centers in Y are pairwise disjoint and $z \in Y$ does not stop any $Q' \in \mathcal{Q}'_y$ for $y \in Y$. We will show that, on average, many paths in $\mathcal{Q}'_y$, $y \in Y$, have length r.

For $y \in Y$ let q_y be the number of $Q' \in \mathcal{Q}'_y$ that were stopped. Extending each $Q' \in \mathcal{Q}'_y$ that was stopped by a single edge, we obtain a path Q'' from y to the vertex $v \in B$ that stopped Q'. Note that if v stops some $Q' \in \mathcal{Q}'_y$, then $v \in B \setminus Y$. We therefore obtain a bipartite graph J with $V(J) = Y \cup (B \setminus Y)$ as a topological minor of G, where $yv \in E(J)$ if and only if v stops some $Q' \in \mathcal{Q}'_y$. It follows from Lemma 5.14 that

$$e(J) \leq d|J| \leq d(|X| + (2r-1)|\mathcal{P}|) \leq 2rd^2|X|.$$

Since the paths in $\mathcal{Q}_y$ intersect only in y, no vertex can stop more than one $Q' \in \mathcal{Q}'_y$. Therefore

$$\sum_{y \in Y} q_y \leq e(J) \leq 2rd^2|X|.$$

It follows that

$$\begin{aligned}|G| &\geq \sum_{y\in Y}|S_y| > \sum_{y\in Y} r(k-q_y) \geq r|Y|k - 2r^2d^2|X| \\ &\geq r|X|(\alpha k - 2rd^2).\end{aligned}$$

Setting $r := \lfloor \frac{\alpha k}{4d^2} \rfloor$ yields the desired result. $\square$

5.4 Admissibility and k-blocks

We now look more closely at the relation between block number and ∞-admissibility. By Lemma 5.4, every k-block is a k-fan-set, so $\beta(G) \leq \mathrm{adm}_\infty(G)$ is trivial. We prove that the two parameters are within a constant multiplicative factor of one another.

Proposition 5.15. *For every graph G*

$$\lfloor \frac{\mathrm{adm}_\infty(G)+1}{2} \rfloor \leq \beta(G) \leq \mathrm{adm}_\infty(G).$$

It only remains to show $\beta(G) \geq \lfloor(\mathrm{adm}_\infty(G)+1)/2\rfloor$. Lemma 5.7 provides a sufficient condition for a set of vertices to be $({<}\,k)$-inseparable. Our proof is an adaptation of the proof of [13, Theorem 4.2], where it is shown that $\beta(G) \geq \lfloor \delta(G)/2 \rfloor + 1$. This is also a consequence of our result, since $V(G)$ itself is a $(\delta(G)+1)$-fan-set.

Proof of Proposition 5.15. The inequality $\beta(G) \leq \mathrm{adm}_\infty(G)$ follows from Lemma 5.4.

Suppose now that $\mathrm{adm}_\infty(G) \geq 2k-1$ and let $X \subseteq V(G)$ be a $(2k-1)$-fan-set. We will show that X contains a $({<}\,k)$-inseparable set.

Let $(T, \mathcal{V})$ be a k-lean tree-decomposition of G. Let $S \subseteq T$ be a minimal subtree such that $X \subseteq \bigcup_{s\in S} V_s$. Let $t \in S$ be a leaf of S. If $S = \{t\}$, then $X \subseteq V_t$ and from every $x \in X$ there is a $(2k-1)$-fan to V_t. By Lemma 5.7, X itself is already $({<}\,k)$-inseparable.

Otherwise, let t' be the unique neighbor of t in S and let $W := X \cap V_t \setminus V_{t'}$. Note that $W \neq \emptyset$, for otherwise $S - t$ would violate the minimality of S. Let $w \in W$ arbitrary and let $\mathcal{Q}_w$ be a $(2k-1)$-fan from w to X. Every $Q \in \mathcal{Q}_w$ whose endvertex is not in W must meet $V_t \cap V_{t'}$. Thus at most $|V_t \cap V_{t'}| < k$ paths from $\mathcal{Q}_w$ have endvertices outside W. In particular, $|W| \geq k$. Furthermore, by stopping every $Q \in \mathcal{Q}_w$ when it hits $V_t \cap V_{t'}$ (if it does), we obtain a $(2k-1)$-fan from w to V_t. By Lemma 5.7, the vertices of W cannot be separated by deleting fewer than k vertices. $\square$

5.5 Tree-width and k-blocks

This section is devoted to the relation between tree-width and the occurrence of k-blocks in a minor. By considering random graphs, one can show that there are graphs G_n on n vertices with $2n$ edges and tree-width at least γn for some absolute constant $\gamma > 0$ (see [44, Corollary 5.2]). By (5.1) we have $\beta(H) \leq 1 + \sqrt{4n}$ for every $H \preceq G_n$. Hence the bound in Theorem 5.3 is best possible up to constant factors.

We now show that every graph of tree-width at least $2(k^2-1)$ has a minor with block number at least k. In fact, this follows easily from a lemma in the proof of the Grid Minor Theorem given by Diestel, Jensen, Gorbunov and Thomassen [25]. To state their result, we need to introduce some terminology.

Let G be a graph. Call a set X of vertices *externally k-linked in G* if for any $Y, Z \subseteq X$, $|Y| = |Z| \leq k$, there are $|Y|$ disjoint X-paths joining Y and Z. A *k-mesh of order m* is a separation (A, B) with $|A \cap B| = m$ such that $A \cap B$ is externally k-linked in $G[B] - E(A \cap B)$ and there is a tree $T \subseteq G[A]$ with $\Delta(T) \leq 3$ such that every vertex of $A \cap B$ lies in T and has degree at most 2 in T.

Lemma 5.16 ([25, Lemma 4])**.** *Let G be a graph and $m \geq k \geq 1$ integers. If the tree-width of G is at least $k + m - 1$, then G has a k-mesh of order m.*

Lemma 5.17. *Let $p \geq 0, k \geq 2$ be integers and let T be a tree with $\Delta(T) \leq 3$ and $X \subseteq V(T)$ a set of at least $(2p+1)(k-1)$ vertices of degree at most 2. Then there are disjoint subtrees $T_1, \ldots, T_p \subseteq T$ such that $|T_i \cap X| \geq k$ for every $i \in [p]$.*

Proof. By induction on p. The case where $p \in \{0, 1\}$ is trivial. In the inductive step, declare a leaf r of T as the root and thus introduce an order on T. Choose $t \in T$ maximal in the tree-order such that D_t, the subtree containing t and all its descendants, contains at least k vertices of X. Note that $t \neq r$, since $|X| > k$ and r has degree 1.

If $t \in X$, then $|D_t \cap X| = k$ because t has only one successor s and $|D_s \cap X| < k$. If $t \notin X$, then similarly $|D_t \cap X| \leq 2(k-1)$. Let $S := T - D_t$ and note that $|S \cap X| \geq |X| - 2(k-1)$. By the inductive hypothesis applied to S and $S \cap X$ we find disjoint $S_1, \ldots, S_{p-1} \subseteq S$ with $|S_i \cap X| \geq k$ for all $i \in [p-1]$. For $1 \leq i < p$ let $T_i := S_i$ and put $T_p := D_t$. These subtrees of T are as desired. $\square$

We thus obtain the following more precise version of Theorem 5.3.

Theorem 5.18. *Let G be a graph and $p \geq k \geq 2$ integers. If the tree-width of G is at least $2(k-1)(p+1)$, then some minor of G contains a $({<}k)$-inseparable independent set of size p.*

Proof. Let $m := \mathrm{tw}(G) - k + 1$. By Lemma 5.16 above, G has a k-mesh (A, B) of order m. Let $T \subseteq G[A]$ be the tree guaranteed by the definition.

Since $m \geq (k-1)(2p+1)$, we can apply the lemma above to find disjoint subtrees $T_1, \ldots, T_p \subseteq T$ such that each contains at least k vertices of $A \cap B$. Let

$$W := (B \setminus A) \cup \bigcup_{i=1}^{p} V(T_i)$$

and obtain H from $G[W]$ by deleting all edges between T_i and T_j for $i \neq j$. Given $1 \leq i, j \leq p$, the graph H contains k disjoint paths between $T_i \cap (A \cap B)$ and $T_j \cap (A \cap B)$ with no internal vertices or edges in A, since $A \cap B$ is externally k-linked in $G[B] - E(A, B)$.

Contracting each T_i to a single vertex thus yields the desired $({<}\,k)$-inseparable independent set in a minor of H and thus of G. $\square$

Taking $p = k$ clearly yields Theorem 5.3.

5.6 Concluding remarks

From Theorem 5.3 and Lemma 5.4 we deduce the following.

Corollary 5.19. *Let $k \geq 1$ be an integer. Every graph of tree-width at least $2k^2 - 2$ has a minor with ∞-admissibility at least k.*

Richerby and Thilikos [54] proved the existence of a function g such that graphs of tree-width at least $g(k)$ have a minor with ∞-admissibility $\geq k$. In their proof, $g(k)$ is the minimum N such that graphs of tree-width at least N have the $k^{3/2} \times k^{3/2}$-grid as a minor. The existence of such an N is guaranteed by the *Grid Minor Theorem* of Robertson and Seymour [55]. In comparison, our proof is short and simple: the only non-trivial step was a lemma from [25], whose proof is about a page long and in fact the first step in their proof of the Grid-Minor Theorem. Moreover, we have provided an explicit quadratic bound on $g(k)$, while even the existence of a polynomial upper bound for N is a recent breakthrough-result of Chekuri and Chuzhoy [15].

Dvořák proved that for every k there are integers m and d such that every graph with ∞-admissibility at most k has a tree-decomposition in which every torso contains at most m vertices of degree at least d ([28, Corollary 5]). The proof is based on a deep structure theorem for graphs excluding a topological minor [28, Theorem 3] and does not yield explicit bounds for m and d. Combining Theorem 5.1 with Lemma 5.4, we obtain a much simpler proof that avoids the use of advanced graph minor theory and moreover provides explicit values for the parameters involved.

Corollary 5.20. *Let $k \geq 1$ be an integer. If G has ∞-admissibility at most k, then G has a tree-decomposition of adhesion at most k in which every torso contains at most k vertices of degree at least $2k(k-1)$.*

It seems challenging to obtain stronger estimates: What is the minimum $N = N(k)$ such that every graph without a k-block has a tree-decomposition in which every torso contains at most N vertices of degree at least N? Can we always find a tree-decomposition in which every torso has a bounded number of vertices of degree at least αk for some constant $\alpha > 0$?

Admissibility of graphs has primarily been studied with a length-restriction imposed. We call the maximum length of a path in a fan $\mathcal{Q}$ the *radius* of $\mathcal{Q}$. A (k, r)-fan-set is a set $X \subseteq V(G)$ such that from every $x \in X$ there is a k-fan of radius at most r to X. The *r-admissibility* $\mathrm{adm}_r(G)$ is the maximum k for which G has a (k, r)-fan-set. In particular

$$1 + \max_{H \subseteq G} \delta(H) = \mathrm{adm}_1(G) \leq \mathrm{adm}_2(G) \leq \ldots \leq \mathrm{adm}_{|G|}(G) = \mathrm{adm}_\infty(G).$$

Note that for every integer $r \geq 1$ trivially

$$\mathrm{adm}_r(G) > \mathrm{adm}_\infty(G) - \frac{|G|}{r+1}, \tag{5.4}$$

since a fan cannot contain $|G|/(r+1)$ paths of length $> r$.

Grohe, Kreutzer, Rabinovich, Siebertz and Stavropoulos [38] showed that for every class $\mathcal{G}$ of graphs excluding a topological minor we have $\mathrm{adm}_r(G) = \mathcal{O}(r)$ for every $G \in \mathcal{G}$. Taking (5.4) into account, we obtain the trivial estimate $\mathrm{adm}_\infty(G) = \mathcal{O}(\sqrt{|G|})$ for $G \in \mathcal{G}$, which also follows from a simple edge-count and Lemma 5.14. On the other hand, Theorem 5.12 shows that $\mathrm{adm}_r(G) = \mathcal{O}(\sqrt[3]{|G|})$ for every r. Hence for values of r which are large with respect to $|G|$, namely for $r \geq K\sqrt[3]{|G|}$ for some constant $K > 0$, our result is a substantial improvement of the estimate of Grohe et al.

Let $\mathcal{G}$ be a class of graphs excluding a topological minor. For $n, r \in \mathbb{N}$ let

$$F(n, r) := \max\{\mathrm{adm}_r(G) \colon G \in \mathcal{G}, |G| = n\}.$$

We know by Theorem 5.12 that $F(n, r) = \mathcal{O}(\sqrt[3]{n})$ for all $r \in \mathbb{N}$, while Grohe et al [38] showed $F(n, r) = \mathcal{O}(r)$ for all $n \in \mathbb{N}$. It appears to be an interesting problem to try to obtain a unified bound.

6 In the absence of long chordless cycles, tree-width is a local parameter

In general, the tree-width of a graph may be arbitrarily large even when every subgraph of bounded order is a tree. We show that tree-width is a “local” parameter for the class of *ℓ-chordal graphs*, those without chordless cycles of length $> \ell$, in the sense that if all subgraphs of G of order at most $2s$ have tree-width less than s, then the tree-width of G is bounded by some constant depending on ℓ and s. More specifically, we prove the following:

Theorem 6.1. *For all ℓ and s, every graph of sufficiently large tree-width contains either a complete bipartite graph $K_{s,s}$ or a chordless cycle of length greater than ℓ.*

Note that this implies Theorem 1.12 from the introduction, because every bipartite graph F is a subgraph of $K_{s,s}$ for $s = |F|$. Theorem 6.1 will be proved in Section 6.1. It also has an immediate application to an Erdős-Pósa type problem. Kim and Kwon [43] recently showed that chordless cycles of length > 3 have the Erdős-Pósa property:

Theorem 6.2 ([43])**.** *For every integer k there exists an integer m such that every graph G either contains k vertex-disjoint chordless cycles of length > 3 or a set X of at most m vertices such that $G - X$ is chordal.*

They also constructed, for every integer $\ell \geq 4$, a family of graphs showing that the analogue of Theorem 6.2 for chordless cycles of length $> \ell$ fails. We complement their negative result by proving that the Erdős-Pósa property *does* hold when restricting the host graphs to graphs not containing $K_{s,s}$ as a subgraph:

Corollary 6.3. *For all ℓ, s and k there exists an integer m such that every $K_{s,s}$-free graph G either contains k vertex-disjoint chordless cycles of length $> \ell$ or a set X of at most m vertices such that $G - X$ is ℓ-chordal.*

In Section 6.2, we formally introduce the Erdős-Pósa property, restate Corollary 6.3 in that language and give a proof thereof. Section 6.3 then closes with some open problems.

6.1 Proof of Theorem 6.1

Our proof is a cascade with three steps. First, we show that sufficiently large tree-width forces the presence of a k-block.

Lemma 6.4. *Let ℓ, k and $w \geq 2(\ell-2)(k-1)^2$ be positive integers. Then every ℓ-chordal graph of tree-width $\geq w$ contains a k-block.*

We then prove that the existence of a k-block yields a bounded-length subdivision of a complete graph.

Lemma 6.5. *Let ℓ, m and $k \geq 5m^2\ell/4$ be positive integers. Then every ℓ-chordal graph that contains a k-block contains a $(\leq 2\ell-3)$-subdivision of K_m.*

In the last step, we show that such a subgraph gives rise to a copy of $K_{s,s}$.

Lemma 6.6. *For all integers ℓ and s there exists a $q > 0$ such that the following holds. Let m, r be positive integers with $m \geq qr$. Then every ℓ-chordal graph that contains a $(\leq r)$-subdivision of K_m contains $K_{s,s}$ as a subgraph.*

It is immediate that Theorem 6.1 follows once we have established these three lemmas.

6.1.1 Proof of Lemma 6.4

A trivial obstacle to our search for a copy of $K_{s,s}$ is the absence of vertices of high degree. Bodlaender and Thilikos [7] showed, however, that ℓ-chordal graphs of bounded degree have bounded tree-width. Their exponential bound was later improved by Kosowski, Li, Nisse and Suchan [45] and by Seymour [62].

Theorem 6.7 ([62])**.** *Let ℓ and Δ be positive integers and G a graph. If G is ℓ-chordal and has no vertices of degree greater than Δ, then the tree-width of G is at most $(\ell-2)(\Delta-1)+1$.*

By demanding large tree-width, we can therefore guarantee a large number of vertices of high degree. We now show that these are not all just scattered about the graph. Recall the structure theorem for graphs without k-blocks from Chapter 5:

Theorem 5.1. *Let $k \geq 3$ be a positive integer and G a graph. If G has no k-block, then there is a tight tree-decomposition of G of adhesion $< k$ such that every torso has fewer than k vertices of degree at least $2(k-1)(k-2)$.*

Now let ℓ, k and $w \geq 2(\ell-2)(k-1)^2$ be positive integers. Let G be an ℓ-chordal graph with no k-block. For $k = 2$, this means that G is acyclic and therefore has tree-width 1. Suppose from now on that $k \geq 3$. We show that the tree-width of G is less than w.

By Theorem 5.1, there is a tight tree-decomposition $(T, \mathcal{V})$ of G such that every torso has fewer than k vertices of degree at least $d := 2(k-1)(k-2)$. Let $t \in T$ arbitrary, let N be the set of neighbors of t in T and let H be the torso at t. We claim that H is ℓ-chordal.

Let $C \subseteq H$ be a chordless cycle with $|C| \geq 4$. For every edge $xy \in E(C) \setminus E(G)$, there is some $s \in N$ with $x, y \in V_s \cap V_t$. Let (W_s, W_t) be the separation of G induced by (s, t). Since $(T, \mathcal{V})$ is tight, there exists an x-y-path P^{xy} in $G[W_s]$ which meets V_t only in its endpoints. Observe that for every $s \in N$, the adhesion set $V_s \cap V_t$ is a clique in H and therefore C contains at most two vertices of V_s and these are adjacent in C. Hence we can replace every edge $xy \in E(C) \setminus E(G)$ by P^{xy} and obtain a chordless cycle C' of G with $|C'| \geq |C|$. Since G is ℓ-chordal, it follows that $|C| \leq \ell$. This proves our claim.

Now, let $A \subseteq V(H)$ be the set of all vertices of degree $\geq d$ in H. Then $H - A$ is ℓ-chordal and has no vertices of degree $> d-1$. By Theorem 6.7, the tree-width of $H - A$ is at most $(\ell-2)(d-2)+1$. Therefore

$$\mathrm{tw}(H) \leq |A| + \mathrm{tw}(H - A) \leq k + (\ell-2)(d-2) < w.$$

We have shown that every torso has tree-width $< w$. We can then take a tree-decomposition of width $< w$ of each torso and combine all these to a tree-decomposition of G of width $< w$. $\square$

6.1.2 Proof of Lemma 6.5

As we have seen in Chapter 5, the presence of a k-block does not guarantee the existence of a subdivision of K_m for any $m \geq 5$: consider, for example, the graph in Figure 5.1. Our aim in this section is to show that for ℓ-chordal graphs, sufficiently large blocks do indeed yield bounded-length subdivisions of complete graphs.

Let ℓ, m and $k \geq 5m^2\ell/4$ be positive integers. Let G be an ℓ-chordal graph and $X \subseteq V(G)$ a k-block of G. Let $L := 2\ell - 3$. Assume for a contradiction that G contained no $(\leq L)$-subdivision of K_m. Let $x, y \in X$ non-adjacent. Then G contains a set $\mathcal{P}^{xy}$ of k internally disjoint x-y-paths. Taking subpaths, if necessary, we may assume that each path in $\mathcal{P}^{xy}$ is induced. Let $p_0 := m + m^2(\ell-2)$.

Claim: Fewer than p_0 paths in $\mathcal{P}^{xy}$ have length $> \ell/2$.

Proof of Claim. Let $\mathcal{P}_0$ be the set of all paths in $\mathcal{P}^{xy}$ of length $> \ell/2$ and $p := |\mathcal{P}_0|$. Assume for a contradiction that $p \geq p_0$. Let $P, Q \in \mathcal{P}_0$. Then $P \cup Q$ is a cycle of length $> \ell$. Since G is ℓ-chordal, $P \cup Q$ has a chord. This chord must join an internal vertex of P to an internal vertex of Q. Choose such vertices $v_P^Q \in P$ and $v_Q^P \in Q$ so that the cycle $D := xPv_P^Q v_Q^P Qx$ has minimum length. Note that D is an induced cycle and therefore has length at most ℓ. In particular, the segment of P joining x to v_P^Q has length at most $\ell - 2$ and similarly for Q and v_Q^P.

For $P \in \mathcal{P}_0$, let P' be a minimal subpath of P containing every vertex v_P^Q, $Q \in \mathcal{P}_0 \backslash \{P\}$. Then $\mathcal{P} := \{P' \colon P \in \mathcal{P}_0\}$ is a family of p disjoint paths, each of length at most $\ell - 3$, and G contains an edge between any two of them. Fix an arbitrary $\mathcal{Q} \subseteq \mathcal{P}$ with $|\mathcal{Q}| = m$. Since $p \geq p_0$, every $Q \in \mathcal{Q}$ contains a vertex u_Q which has neighbors on at least m^2 different paths in $\mathcal{P} \setminus \mathcal{Q}$.

Let $U := \{u_Q \colon Q \in \mathcal{Q}\}$. We iteratively construct a $(\leq L)$-subdivision of K_m with branchvertices in U. Let $t := \binom{m}{2}$ and enumerate the pairs of vertices of U arbitrarily as $e_1, \ldots, e_t$. In the j-th step, we assume that we have constructed a family $\mathcal{R}^j = (R_i)_{i<j}$ of internally disjoint U-paths of length at most L, so that R_i joins the vertices of e_i and meets at most two paths in $\mathcal{P} \setminus \mathcal{Q}$. We now find a suitable path R_j.

Let $Q^1, Q^2 \in \mathcal{Q}$ with $e_j = u_{Q^1} u_{Q^2}$. At most $2(j-1) < m^2$ paths in $\mathcal{P} \setminus \mathcal{Q}$ meet any of the paths in $\mathcal{R}^j$. Since u_{Q^1} is adjacent to vertices on at least m^2 different paths in $\mathcal{P} \setminus \mathcal{Q}$, there is a $P^1 \in \mathcal{P} \setminus \mathcal{Q}$ which is disjoint from every R_i, $i < j$, and contains a neighbor of u_{Q^1}. We similarly find a path $P^2 \in \mathcal{P} \setminus \mathcal{Q}$ for u_{Q^2}. Since either $P^1 = P^2$ or G has an edge between P^1 and P^2, $P^1 \cup P^2 \cup \{u_{Q^1}, u_{Q^2}\}$ induces a connected subgraph of G and therefore contains a u_{Q^1}-u_{Q^2}-path R_j of length at most L, which meets at most two paths in $\mathcal{P} \setminus \mathcal{Q}$.

Proceeding like this, we find the desired subdivision of K_m after t steps. This contradiction finishes the proof of the claim. $\square$

Let $Y \subseteq X$ with $|Y| = m$. For any two non-adjacent $x, y \in Y$, let $\mathcal{Q}^{xy} \subseteq \mathcal{P}^{xy}$ be the set of all $P \in \mathcal{P}^{xy}$ of length at most $\ell/2$ which have no internal vertices in Y. By the claim above, we have

$$|\mathcal{Q}^{xy}| > k - p_0 - (m-2) \geq \binom{m}{2} \frac{\ell}{2}.$$

Pick one path $P \in \mathcal{Q}^{xy}$ for each pair of non-adjacent vertices $x, y \in Y$ in turn, disjoint from all previously chosen paths. Since $|\mathcal{Q}^{xy}| \geq \binom{m}{2}\frac{\ell}{2}$ and each path has fewer than $\ell/2$ internal vertices which future paths need to avoid, we can always find a suitable such path P. Together with all edges between adjacent vertices of Y, this yields a $(\leq \ell/2)$-subdivision of K_m in G with branchvertices in Y. $\square$

We would like to point out that a modification of the above argument can be used to produce a $(\leq \ell/2)$-subdivision of K_m if k is significantly larger.

Indeed, suppose we find a family $\mathcal{P}$ of p disjoint paths, each of length at most $\ell - 3$, such that G contains an edge between any two of them. Then the subgraph H induced by $\bigcup_{P \in \mathcal{P}} V(P)$ has at most $(\ell - 2)p$ vertices and at least $\binom{p}{2}$ edges. One can then use a classic result of Kövari, Sós and Turán [46] to show that H contains a copy of K_{m,m^2} if p is sufficiently large. Since K_{m,m^2} contains a (≤ 2)-subdivision of K_m, this establishes an upper bound on the number of paths of length $> \ell/2$ in any $\mathcal{P}^{xy}$. The rest of the proof remains the same.

6.1.3 Proof of Lemma 6.6

The combination of Lemma 6.4 and Lemma 6.5 already establishes that tree-width is a local parameter for ℓ-chordal graphs. The purpose of Lemma 6.6 is to narrow the set of bounded-order obstructions down as far as possible. We will use the following theorem of Kühn and Osthus [48].

Theorem 6.8 ([48])**.** *For every integer s and every graph H there exists a d such that every graph with average degree at least d either contains $K_{s,s}$ as a subgraph or contains an induced subdivision of H.*

In fact, we only need the special case $H = C_{\ell+1}$. This special case has a simpler proof which can be found in Kühn's PhD-thesis [47]. Fix an integer d such that every ℓ-chordal graph of average degree at least d contains $K_{s,s}$ as a subgraph. We prove the assertion of Lemma 6.6 with $q := d^2 \frac{\ell^\ell}{4(\ell-3)!}$.

Let m, r be positive integers with $m \geq qr$ and let G be an ℓ-chordal graph containing a $(\leq r)$-subdivision of K_m. Let X be the set of branchvertices and $(P^{xy} \colon x, y \in X)$ the family of paths of the subdivision. Taking subpaths, if necessary, we may assume that every path is induced.

Assume for a contradiction that G contained no copy of $K_{s,s}$. By Theorem 6.8, every subgraph of G contains a vertex of degree $< d$. In particular, there is an independent set $Y \subseteq X$ with $|Y| \geq m/d$. Let H be the subgraph of G induced by $\bigcup_{x,y \in Y} V(P^{xy})$. Note that $|H| \leq r\binom{|Y|}{2}$.

Call an edge of H *red* if it joins a vertex $x \in Y$ to an internal vertex of a path P^{yz} with $x \notin \{y, z\}$. Call an edge of H *blue* if it joins an internal vertex of a path P^{wx} to an internal vertex of a path P^{yz} with $\{w, x\} \neq \{y, z\}$. We will show that H must contain many edges which are either red or blue, so that the average degree of H is at least d.

Fix an arbitrary cycle R with $V(R) = Y$. For any $Z \subseteq Y$ with $|Z| = \ell$, obtain the cycle R_Z with $V(R_Z) = Z$ by contracting every Z-path of R to a single edge. We then get a cycle $C_Z \subseteq H$ by replacing every edge $xy \in R_Z$ with the path P^{xy}. Since each path P^{xy} has length at least 2 and H is ℓ-chordal, the cycle C_Z must have a chord. Since Y is independent and every path P^{xy} is induced, the chord must be a red or blue edge of H.

Consider a red edge $xv \in E(H)$ with $x \in Y$, $v \in P^{yz}$ and $x \notin \{y, z\}$. If this edge is a chord for a cycle C_Z, then $\{x, y, z\} \subseteq Z$. Hence it can only occur as a chord for at most

$$\binom{|Y|-3}{\ell-3} \leq \frac{|Y|^{\ell-3}}{(\ell-3)!}$$

choices of Z. Similarly, every blue edge $uv \in E(H)$ with $u \in P^{wx}$, $v \in P^{yz}$ and $\{w, x\} \neq \{y, z\}$ can only be a chord of C_Z if $\{w, x, y, z\} \subseteq Z$. This also happens for at most

$$\binom{|Y|-3}{\ell-3} \leq \frac{|Y|^{\ell-3}}{(\ell-3)!}$$

choices of Z. Let f be the number of edges of H which are either red or blue. Since every $Z \subseteq Y$ with $|Z| = \ell$ gives rise to a chord, it follows that

$$\frac{|Y|^\ell}{\ell^\ell} \leq \binom{|Y|}{\ell} \leq f \frac{|Y|^{\ell-3}}{(\ell-3)!}.$$

This shows that the average degree of H is

$$d(H) \geq \frac{2f}{|H|} \geq \frac{4(\ell-3)!}{r\ell^\ell}|Y| \geq d.$$

By Theorem 6.8, H contains a copy of $K_{s,s}$. □

6.2 Erdős-Pósa for long chordless cycles

A classic theorem of Erdős and Pósa [31] asserts that for every integer k there is an integer r such that every graph either contains k disjoint cycles or a set of at most r vertices meeting every cycle. This result has been the starting point for an extensive line of research, see the survey by Raymond and Thilikos [53].

Let $\mathcal{F}, \mathcal{G}$ be classes of graphs and $\leq$ a containment relation between graphs. We say that $\mathcal{F}$ *has the Erdős-Pósa property for* $\mathcal{G}$ *with respect to* $\leq$ if there exists a function f such that for every $G \in \mathcal{G}$ and every integer k, either there are disjoint $Z_1, \dots, Z_k \subseteq V(G)$ such that for every $1 \leq i \leq k$ there is an $F_i \in \mathcal{F}$ with $F_i \leq G[Z_i]$, or there is a $X \subseteq V(G)$

with $|X| \leq f(k)$ such that $F \not\leq G - X$ for every $F \in \mathcal{F}$. When $\mathcal{G}$ is the class of all graphs, we simply say that $\mathcal{F}$ *has the Erdős-Pósa property with respect to* $\leq$. We write $F \subseteq G$ if F is isomorphic to a subgraph of G and $F \subseteq_{\mathrm{i}} G$ if F is isomorphic to an *induced* subgraph of G.

The theorem of Erdős and Pósa then asserts that the class of cycles has the Erdős-Pósa property with respect to $\subseteq$. This implies that cycles also have the Erdős-Pósa property with respect to $\subseteq_{\mathrm{i}}$. It is known that for every ℓ, the class of cycles of length $> \ell$ has the Erdős-Pósa property with respect to $\subseteq$, see [65, 6, 52]. Recently, Kim and Kwon [43] proved that cycles of length > 3 possess the Erdős-Pósa property with respect to $\subseteq_{\mathrm{i}}$:

Theorem 6.9 ([43]). *There exists a constant c such that for every integer k, every graph G either contains k vertex-disjoint chordless cycles of length > 3 or a set X of at most $ck^2 \log k$ vertices such that $G - X$ is chordal.*

In contrast, Kim and Kwon [43] showed that cycles of length $> \ell$ do *not* have the Erdős-Pósa property with respect to $\subseteq_{\mathrm{i}}$ if $\ell \geq 4$. For any given n, they constructed a graph G_n with no two disjoint chordless cycles of length $> \ell$, for which no set of fewer than n vertices meets every chordless cycle of length $> \ell$ in G_n. This graph G_n contains a copy of $K_{n,n}$. We show that this is essentially necessary:

Corollary 6.10. *For all integers ℓ and s, the class of cycles of length $> \ell$ has the Erdős-Pósa property for the class of $K_{s,s}$-free graphs with respect to $\subseteq_{\mathrm{i}}$.*

This follows from Theorem 6.1 by a standard argument. Since the proof is quite short, we provide it for the sake of completeness. First, recall the following consequence of the Grid Minor Theorem of Robertson and Seymour [57].

Theorem 6.11 ([57]). *For all positive integers p and q there exists an r such that for every graph G with tree-width $\geq r$, there are disjoint $Z_1, \dots, Z_p \subseteq V(G)$ such that $G[Z_i]$ has tree-width $\geq q$ for every $1 \leq i \leq p$.*

Proof of Corollary 6.10. Let k be an integer. By Theorem 6.1, there exists an integer t such that every ℓ-chordal graph with tree-width $\geq t$ contains $K_{s,s}$. By Theorem 6.11, there exists an r such that every graph with tree-width $> r$ has k vertex-disjoint subgraphs of tree-width $\geq t$.

Let G be a $K_{s,s}$-free graph. We show that either G contains k disjoint chordless cycles of length $> \ell$ or there is a set of at most $r(k-1)$ vertices whose deletion leaves an ℓ-chordal graph.

Suppose first that the tree-width of G was greater than r. Let $Z_1, \dots, Z_k$ be disjoint sets of vertices such that $G[Z_i]$ has tree-width $\geq t$ for every i. Then, by Theorem 6.1, every $G[Z_i]$ must contain a chordless cycle of length $> \ell$, since $K_{s,s} \not\subseteq G[Z_i]$. Therefore G contains k disjoint chordless cycles of length $> \ell$.

Suppose now that G had a tree-decomposition $(T, \mathcal{V})$ of width $< r$. For every chordless cycle $C \subseteq G$ of length $> \ell$, let $T_C \subseteq T$ be the subtree of all $t \in T$ with $V_t \cap V(C) \neq \emptyset$. If there are k disjoint such subtrees $T_{C^1}, \ldots, T_{C^k}$, then $C^1, \ldots, C^k$ are also disjoint and we are done. Otherwise, there exists $S \subseteq V(T)$ with $|S| < k$ which meets every subtree T_C. Then $Z := \bigcup_{s \in S} V_s$ meets every chordless cycle of length $> \ell$ in G and $|Z| \leq r(k-1)$.

□

6.3 Open problems

A large amount of research is dedicated to the study of χ-*boundedness* of graph classes, introduced by Gyárfás [40]. Here, a class $\mathcal{G}$ of graphs is called χ-*bounded* if there exists a function f such that for every integer k and $G \in \mathcal{G}$, either G contains a clique on $k+1$ vertices or G is $f(k)$-colourable. This is a strengthening of the statement that chromatic number is a local parameter for $\mathcal{G}$, with cliques being the only bounded-order subgraphs to look for.

As we have seen, cliques are not the only reasonable local obstruction to having small tree-width. Nontheless, we may still ask

1. For which classes of graphs is tree-width a local parameter?
2. What kind of bounded-order subgraphs can we force on these classes?
3. For which classes can we force large cliques by assuming large tree-width?

We have seen in Section 6.2 that long chordless cycles have the Erdős-Pósa property for the class of $K_{s,s}$-free graphs. For which other classes is this true? Kim and Kwon [43] raised this question for the class of graphs without chordless cycles of length four.

7 The structure of graphs excluding a topological minor

Grohe and Marx [39] proved the following structure theorem for graphs excluding a topological minor:

Theorem 7.1 ([39]). *For every positive integer r there exists an integer p such that every graph which does not contain K_r as a topological minor has a tree-decomposition in which every torso either*

(i) *has at most p vertices of degree greater than p, or*

(ii) *does not contain K_p as a minor.*

In fact, Grohe and Marx [39] give an algorithm which computes such a tree-decomposition in time $f(r)|V(G)|^c$ for some computable function f and constant c.

We give an independent, non-algorithmic and conceptually simpler proof of Theorem 7.1. We are going to show the following:

Theorem 7.2. *For every positive integer r, every graph which does not contain K_r as a topological minor has a tree-decomposition in which every torso either*

(i) *does not contain an r^4-block, or*

(ii) *does not contain K_{2r^2} as a minor.*

Using Theorem 5.1, our structure theorem for graphs without k-blocks, we can refine this tree-decomposition by decomposing every torso which does not contain an r^2-block. This yields the following:

Corollary 7.3. *For every positive integer r, every graph which does not contain K_r as a topological minor has a tree-decomposition in which every torso either*

(i) *has at most r^4 vertices of degree greater than $2r^8$, or*

(ii) *does not contain K_{2r^2} as a minor.* □

The idea of our proof is as follows. Both large minors and large blocks point towards a 'big side' of every separation of low order. A subdivision of a clique simultaneously gives rise to both a complete minor and a block and, what's more, the two are hard to separate in the sense that they choose the same 'big side' for every low-order separation. A qualitative converse to this observation is already implicit in previous work on graph minors and linkage problems: if a graph contains a large complete minor and a large block which cannot be separated from that minor, then the graph contains a subdivision of a complete graph.

Therefore, if we that assume our graph does not contain a subdivision of K_r, then we can separate any large minor from every large block. It then follows from the *tangle tree theorem* of Robertson and Seymour [58] (or rather its extension to *profiles* [21, 10]) that there exists a tree-decomposition which separates every block from every minor. Hence each part is either free of large minors or free of large blocks.

However, the aim is to control the *torsos*, and not every tree-decomposition will provide this control. We therefore contract some parts of our tree-decomposition and use the minimality of the remaining set of separations to exclude blocks and minors in the torsos.

In Section 7.3, we give a slightly different proof of Theorem 7.1 using k-atomic tree-decompositions to strengthen the bounds on the degrees.

7.1 Profiles of a graph

Throughout the following, k denotes a positive integer and G a graph. We endow the set of all separations of G with the partial order $\leq$ given by

$$(A, B) \leq (C, D) \;:\Leftrightarrow\; A \subseteq C,\; B \supseteq D.$$

This turns the set of separations into a *lattice* with meet $\wedge$ and join $\vee$ given by

$$(A, B) \wedge (C, D) = (A \cap C, B \cup D),\; (A, B) \vee (C, D) = (A \cup C, B \cap D).$$

Note that $(A, B) \mapsto (B, A)$ is an order-reversing involution on this lattice. The subset of all separations of G of order $< k$ is denoted by $S_k(G)$. It carries the induced partial order, but is not necessarily a lattice.

Intuitively, an orientation of $S_k(G)$ points towards one side of each separation. We achieve this formally by defining an *orientation* of $S_k(G)$ as a subset of $S_k(G)$ which contains exactly one element from each pair $\{(A, B), (B, A)\} \subseteq S_k(G)$. The intended meaning is that an orientation containing (A, B) 'points towards' B.

As a sort of minimum requirement for an orientation to 'point somewhere' in the graph, rather than orienting each separation arbitrarily, we demand that it does not contain two separations which clearly point away from each other. Therefore, we call an orientation O *consistent* if for any two distinct $(A, B), (C, D) \in O$ we do not have $(D, C) \leq (A, B)$. The restriction that $(A, B) \neq (C, D)$ allows O to include a *cosmall* separation (A, B) satisfying $(B, A) \leq (A, B)$. Note that, in this case, $A = A \cup B = V(G)$. Declaring B as the large side then seems rather counterintuitive, so we call orientations not containing any cosmall separations *regular.* Regular, consistent orientations are down-closed with respect to the ordering.

Carmesin, Diestel, Hamann and Hundertmark [10] introduced profiles as both an abstraction of k-blocks and a generalization of the tangles of Robertson and Seymour [58]. A *profile* of $S_k(G)$, or *k-profile* for short, is a consistent orientation O[1] of $S_k(G)$ which satisfies the following:

$$\text{if } (A, B), (C, D) \in O, \text{ then } (B \cap D, A \cup C) \notin O. \tag{P}$$

Note that $(A \cup C, B \cap D)$ is the supremum of (A, B) and (C, D) in the lattice of separations of G – but, in general, $(A \cup C, B \cap D)$ need not be contained in $S_k(G)$. More explicitly, the property (P) asserts that *if* the supremum $(A \cup C, B \cap D)$ of two separations $(A, B), (C, D) \in O$ has order $< k$, then it must also be contained in O.

By definition, two orientations O, O' of $S_k(G)$ are distinct if and only if there is some $(A, B) \in O \setminus O'$. Informally, O and O' disagree on which of A and B is the 'large side' of the separation. We then say that (A, B) *distinguishes* O and O'. Observe that $(B, A) \in O' \setminus O$ also distinguishes them. We say that (A, B) distinguishes them *efficiently* if it has minimum order among all separations distinguishing them.

As a natural extension, we say that a set $\mathcal{S}$ of separations *(efficiently) distinguishes* a set $\mathcal{O}$ of orientations if for any two orientations in $\mathcal{O}$ there is a separation in $\mathcal{S}$ which (efficiently) distinguishes them.

7.1.1 Profiles induced by blocks and models

Let $B \subseteq V(G)$ be $(< k)$-inseparable. Then for every separation $(U, W) \in S_k(G)$ either $B \subseteq U$ or $B \subseteq W$, but not both. Hence B induces an orientation $O(B)$ of $S_k(G)$ given by

$$O(B) := \{(U, W) \in S_k(G) \colon B \subseteq W\}.$$

[1] In [10], a slightly more restrictive notion of 'consistency' is used. As a consequence, only *regular* profiles are considered in [10]

Note that every k-block containing B induces the same orientation of $S_k(G)$.

As the vertices of a complete graph are indistinguishable (formally: the complete graph is *vertex-transitive*), we ease the notation for models slightly when dealing with models of complete graphs. Given an integer m, a *model of* K_m is a family $\mathcal{X}$ of m pairwise disjoint connected sets of vertices of G, the *branchsets* of $\mathcal{X}$, such that G contains an edge between any two of them. If $m \geq k$, then for every separation $(U, W) \in S_k(G)$ precisely one of $W \setminus U$ and $U \setminus W$ contains a branchset of $\mathcal{X}$. That way, $\mathcal{X}$ induces an orientation $O(\mathcal{X})$ of $S_k(G)$ via

$$O(\mathcal{X}) := \{(U, W) \in S_k(G)\colon X \cap W \text{ is non-empty for every } X \in \mathcal{X}\}.$$

Note that both $O(B)$ and $O(\mathcal{X})$ depend implicitly on the integer k and not on the set B or the model $\mathcal{X}$ alone.

Lemma 7.4. *Let m be a positive integer with $2(m+1) \geq 3k$ and let O be an orientation of $S_k(G)$. If*

(i) *$O = O(B)$ for a $(<k)$-inseparable set B, or*

(ii) *$O = O(\mathcal{X})$ for a model $\mathcal{X}$ of K_m,*

then O is a regular k-profile.

Proof. (i) Suppose $O = O(B)$ for a $(<k)$-inseparable set $B \subseteq V(G)$. To verify consistency, let $(U_1, W_1), (U_2, W_2) \in S_k(G)$ and suppose that $(W_2, U_2) \leq (U_1, W_1) \in O$. Then $B \subseteq W_1 \subseteq U_2$, so $(W_2, U_2) \in O$ and $(U_2, W_2) \notin O$. Regularity and (P) are trivial.

(ii) Suppose $O = O(\mathcal{X})$ for a model $\mathcal{X}$ of K_m. We first check that O is consistent. Let $(U_1, W_1), (U_2, W_2) \in S_k(G)$ and suppose that $(W_2, U_2) \leq (U_1, W_1) \in O$. Let $X \in \mathcal{X}$ arbitrary. Then $X \cap U_2 \supseteq X \cap W_1$, which is non-empty. Thus $(W_2, U_2) \in O$ and $(U_2, W_2) \notin O$. Again, regularity of O is trivial.

To show (P), let $(R_1, S_1), (R_2, S_2) \in O$ and suppose that $(R, S) := (R_1 \cup R_2, S_1 \cap S_2)$ has order $<k$. Consider the set $\mathcal{Y}$ of all $X \in \mathcal{X}$ which are contained in $S_1 \setminus R_1$. Note that $|\mathcal{Y}| \geq m - |R_1 \cap S_1|$. If some $Y \in \mathcal{Y}$ is contained in $S_2 \setminus R_2$, then $(R, S) \in O(\mathcal{X})$ and $(S, R) \notin O(\mathcal{X})$. Otherwise, since $(R_2, S_2) \in O(\mathcal{X})$, every $Y \in \mathcal{Y}$ meets $(R_2 \cap S_2) \setminus R_1$. As the branchsets are disjoint, it follows that

$$|(R_2 \cap S_2) \setminus R_1| \geq m - |R_1 \cap S_1|.$$

Symmetrically, we find that $|(R_1 \cap S_1) \setminus R_2| \geq m - |R_2, S_2|$. But then

$$|R, S| \geq |(R_2 \cap S_2) \setminus R_1| + |(R_1 \cap S_1) \setminus R_2| \geq 2(m - (k-1)) \geq k,$$

which contradicts our initial assumption on the order of (R,S). $\square$

We will show that if a $(<k)$-inseparable set and a model of K_m, with $m \geq 2k-1$, induce the same profile of $S_k(G)$, then G contains the complete graph on $r \sim \sqrt{k}$ vertices as a topological minor. To prove this, we will make use of a lemma of Robertson and Seymour [59] that allows us to 'pull' the branchsets of a model of K_m onto a specified (somewhat smaller) set of vertices.

The *completion* of G at a set $Z \subseteq V(G)$ is the graph G^Z obtained from G by making the vertices of Z pairwise adjacent. Note that Z is $(<|Z|)$-inseparable in G^Z. A *Z-based model* is a model of $K_{|Z|}$ in which every branchset contains one vertex of Z.

Lemma 7.5 ([59])**.** *Let G be a graph, $Z \subseteq V(G)$ and $p := |Z|$. Let $q \geq 2p-1$ and let $\mathcal{X}$ be a model of K_q in G^Z. If $\mathcal{X}$ and Z induce the same orientation of $S_p(G^Z)$, then G has a Z-based model.*

Lemma 7.6. *Let r, $k \geq r(r-1)$ and $m \geq 2r(r-1)$ be positive integers. Let B be a k-block and $\mathcal{X}$ a model of K_m in G. If B and $\mathcal{X}$ induce the same orientation of $S_{r(r-1)}$, then G contains a subdivision of K_r with arbitrarily prescribed branchvertices in B.*

Lemma 7.6 is similar to [39, Lemma 6.16], and the proof is basically the same.

Proof. Let $q := r(r-1)$. Suppose B and $\mathcal{X}$ induce the same orientation of S_q and let $B_0 \subseteq B$ of order r arbitrary. Let H be the graph obtained from G by replacing every $b \in B_0$ by an independent set J_b of order $(r-1)$, where every vertex of J_b is adjacent to every neighbor of b in G and to every vertex of J_c if b, c are adjacent. Let $J := \bigcup_b J_b$ and note that $|J| = q$. We regard G as a subgraph of H by identifying each $b \in B$ with one arbitrary vertex in J_b. This makes $\mathcal{X}$ a model of K_m in H.

Assume for a contradiction that there was a separation (U,W) of H of order $< q$ such that $J \subseteq U$ and $X \subseteq W \setminus U$ for some $X \in \mathcal{X}$. We may assume without loss of generality that for every $b \in B_0$, either $J_b \subseteq U \cap W$ or $J_b \cap (U \cap W) = \emptyset$: If there is a $z \in J_b \setminus (U \cap W)$, then $z \in U \setminus W$, and we can delete any $z' \in J_b \cap W$ from W and maintain a separation (because z and z' have the same set of neighbours in H) with the desired properties. In particular, for every $b \in B_0$ we find $b \in W$ if and only if $J_b \subseteq W$. Since $|U \cap W| < |J|$, it follows that there is at least one $b_0 \in B_0$ with $J_{b_0} \subseteq (U \setminus W)$. Let $(U', W') := (U \cap V(G), W \cap V(G))$ be the induced separation of G. Then $X \subseteq W' \setminus U'$ and $b_0 \in U' \setminus W'$. Since $|U' \cap W'| \leq |U \cap W| < q$ and B is a k-block, we have $B \subseteq U'$. But then (U', W') distinguishes B and $\mathcal{X}$, which is a contradiction to our initial assumption.

We can now apply Lemma 7.5 to H and find a J-based model $\mathcal{Y} = (Y_j)_{j \in J}$ in H. For each $b \in B_0$, label the vertices of J_b as $(v_c^b)_{c \in B_0 \setminus \{b\}}$. For $b \neq c$, H contains a v_c^b-v_b^c-path $P'_{b,c} \subseteq Y_{v_c^b} \cup Y_{v_b^c}$. These paths are pairwise disjoint, because the Y_j are, and

$P'_{b,c} \cap J = \{v_c^b, v_b^c\}$. For each such path $P'_{b,c}$, obtain $P_{b,c} \subseteq G$ by replacing v_c^b by b and v_b^c by c. The collection of these paths $(P_{b,c})_{b,c \in B_0}$ gives a subdivision of K_r with branchvertices in B_0. □

We now study the separations which efficiently distinguish blocks and minors. Recall that a separation (A, B) is *tight* if $G[A \setminus B]$ has a component K in which every vertex of $A \cap B$ has a neighbor. The separation is *generic* if $G[A]$ has an $A \cap B$-based model.

Lemma 7.7. *Let O_1, O_2 be consistent regular orientations of $S_k(G)$ and let $(A, B) \in O_2 \setminus O_1$ efficiently distinguish them. Then:*

(i) *If O_1 is a profile, then (A, B) is tight.*

(ii) *If $O_1 = O(\mathcal{X})$ for a model $\mathcal{X}$ of K_m, $m \geq 2k - 1$, then (A, B) is generic.*

Proof. (i): Let $Q := A \cap B$. By repeatedly applying (P), it follows that there exists a (unique) component K of $G[A \setminus B]$ such that $(C, D) := (V \setminus K, Q \cup K) \in O_1$. Note that $(D, C) \leq (A, B)$, so $(D, C) \in O_2$ by consistency.

Assume for a contradiction that there was some $q \in Q$ with no neighbor in K. Then $(C', D') := (C, D \setminus \{q\})$ is also a separation of G. As $(C, D) \leq (C', D')$, we find $(D', C') \in O_2$. Since (A, B) has minimum order in $O_2 \setminus O_1$, it must be that $(D', C') \in O_1$ as well. But then

$$(C \cup D', D \cap C') = (V, Q) \in O_1,$$

since $|Q| < k$ and O_1 is a profile. This contradicts our assumption that O_1 is regular.

(ii): Let $Q := A \cap B$ and $\mathcal{Y} := (X \cap A)_{X \in \mathcal{X}}$. Since $(B, A) \in O_{\mathcal{X}}$, $\mathcal{Y}$ is a model of K_m in $G[A]^Q$. We wish to apply Lemma 7.5 to Q and $\mathcal{Y}$ in the graph $G[A]$. Suppose Q and $\mathcal{Y}$ did not induce the same orientation of $S_{|Q|}(G[A]^Q)$. That is, there is a separation (U, W) of $G[A]^Q$ with $|U \cap W| < |Q|$ and $Q \subseteq U$ such that $Y \cap U = \emptyset$ for some $Y \in \mathcal{Y}$. There exists an $X \in \mathcal{X}$ such that $Y = X \cap A$. But X cannot meet B, since it is connected and does not meet Q. Therefore $X = Y$.

Now $(U', W') := (U \cup B, W)$ is a separation of G. Note that $X \subseteq W' \setminus U'$, so $(U', W') \in O_1$. Since $(W', U') \leq (A, B)$, it follows from the consistency of O_2 that $(W', U') \in O_2$. But $|U' \cap W'| = |U \cap W| < |Q|$, which contradicts the fact that (A, B) efficiently distinguishes O_1 and O_2.

Hence Q and $\mathcal{Y}$ do induce the same orientation of $S_{|Q|}(G[A]^Q)$. By Lemma 7.5, $G[A]$ has a Q-based model, making (A, B) generic. □

7.1.2 Profiles and tree-decompositions

In Chapter 5, we pointed out that if $(T, \mathcal{V})$ is a tree-decomposition of adhesion $<k$ and B is a k-block, then there exists a unique $t \in T$ with $B \subseteq V_t$. In fact, profiles also 'inhabit' parts of tree-decompositions, in the following sense.

Let $(T, \mathcal{V})$ be a tree-decomposition of adhesion $< k$ and let O be an orientation of $S_k(G)$. Every pair (s, t) of adjacent vertices of T induces a separation $(W_s, W_t) \in S_k(G)$, and the pair (t, s) induces its 'inverse' (W_t, W_s). As O contains precisely one of these two separations, it induces an orientation of the edges of T by directing st from s towards t if and only if $(W_s, W_t) \in O$. When O is consistent, this orientation will direct the edges of T towards some node $t_O \in T$, which we call the *home node* of O in T. If $O = O(B)$ for a k-block B or $O = O(\mathcal{X})$ for a model $\mathcal{X}$ of K_m, we abbreviate this to $t_B := t_{O(B)}$ and $t_{\mathcal{X}} := t_{O(\mathcal{X})}$, respectively.

We say that an edge $e \in E(T)$ *(efficiently) distinguishes* two orientations O, O' if one (and then both) of the separations of G induced by e does. When O, O' are consistent, this is the case if and only if $e \in t_O T t_{O'}$. This observation often allows us to reduce problems about separations distinguishing orientations of $S_k(G)$ to much simpler problems within trees. The tree-decomposition $(T, \mathcal{V})$ *(efficiently) distinguishes* a set $\mathcal{O}$ of orientations if for any two orientations on $\mathcal{O}$ there is an edge of T which (efficiently) distinguishes them. If every orientation in $\mathcal{O}$ is consistent, then $(T, \mathcal{V})$ distinguishes $\mathcal{O}$ if and only if $O \mapsto t_O$ is an injective map $\mathcal{O} \to V(T)$.

Theorem 7.8 ([10, Theorem 4.5])**.** *Every graph has a tree-decomposition $(T, \mathcal{V})$ of adhesion $<k$ which efficiently distinguishes all its k-profiles.*

For suitable values of r, k, m, it thus follows from Lemma 7.6 and Theorem 7.8 that every graph excluding K_r as a topological minor has a tree-decomposition which efficiently distinguishes every k-block from every model of K_m. Then no node is simultaneously the home node of a k-block and of a model of K_m. Our aim now is to use this to control the torsos. In order to achieve this, we will need to take a somewhat coarser tree-decomposition obtained by a suitable contraction.

Let T be a tree with a weight-function $\mu : E(T) \to \mathbb{R}^+$ and let D be a graph with $V(D) \subseteq V(T)$. The idea is that D prescribes which pairs of nodes of T have to be separated. In our application, $V(D)$ is going to a complete bipartite graph with the set of home nodes of blocks on one side and the set of home nodes of models on the other.

A set $F \subseteq E(T)$ is a *D-barrier* if F contains an edge of aTb for any $ab \in E(D)$ and for every $f \in F$ there is an $ab \in E(D)$ such that f has minimum weight in aTb and no other edge of F lies in aTb.

Lemma 7.9. *Let T be a tree with a weight-function $\mu : E(T) \to \mathbb{R}^+$ and let D be a graph with $V(D) \subseteq V(T)$. Then there exists a D-barrier.*

Proof. Order the edges of T as $e_1, \dots, e_n$ such that $\mu(e_1) \geq \dots \geq \mu(e_n)$. The *index* of $e \in E(T)$ is the integer j with $e_j = e$. Call $e \in E(T)$ *smooth for $ab \in E(D)$* if e is the edge of maximum index in aTb. Observe that precisely one edge of T is smooth for each $ab \in E(D)$ and this edge has minimum weight in aTb.

We iteratively construct a barrier F. Starting with $F_0 := \emptyset$, do the following for $j = 1, \dots, n$. If there exists $ab \in E(D)$ for which e_j is smooth and F_{j-1} contains no edge of aTb, let $F_j := F_{j-1} \cup \{e_j\}$. Else, let $F_j := F_{j-1}$. Finally, define $F := F_n$.

We now verify that F is indeed a barrier. First, let $ab \in E(D)$ arbitrary. Let $e_j \in E(T)$ be smooth for ab. If F_{j-1} contains an edge of aTb, then so does $F \supseteq F_{j-1}$. Otherwise, $e_j \in aTb$ lies in $F_j \subseteq F$. Either way, F contains an edge of aTb.

Let now $f \in F$ and let j be its index. By construction, there exists $ab \in E(D)$ for which f is smooth such that F_{j-1} contains no edge of aTb. Since $F \subseteq F_j \cup \{e_{j+1}, \dots, e_n\}$ and no edge e_k, $k > j$, lies in aTb, it follows that f is indeed the only edge of F in aTb. □

The following lemmas will help us control the torsos after contracting the components of $T - F$ for a suitable barrier F. A set τ of separations is called a *star* if $(A, B) \leq (D, C)$ holds for all distinct $(A, B), (C, D) \in \tau$. We define $J(\tau) := \bigcap_{(A,B)\in\tau} B$ and $\operatorname{tor}(\tau)$, the *torso of τ*, as the graph obtained from $G[J(\tau)]$ by taking the completion of each set $A \cap B$ for $(A, B) \in \tau$.

Lemma 7.10. *Let $m \geq k$ and let $\tau \subseteq S_k(G)$ be a star of generic separations. If* $\operatorname{tor}(\tau)$ *contains a K_m-minor, then there exists a model $\mathcal{X}$ of K_m in G with $\tau \subseteq O_{\mathcal{X}}$.*

Proof. Contracting, for each $(U, W) \in \tau$, every branchset of some fixed $U \cap W$-based model in $G[U]$ onto the single vertex of $U \cap W$ it contains, we obtain $\operatorname{tor}(\tau)$ as a minor of G. If $\operatorname{tor}(\tau)$ contains a K_m-minor, then this yields a model $\mathcal{X}$ of K_m in G in which every branchset meets $J(\tau)$. But then $\tau \subseteq O_{\mathcal{X}}$, because $X \cap W \supseteq X \cap J(\tau)$ is non-empty for every $X \in \mathcal{X}$ and $(U, W) \in \tau$.

□

The analogous statement for blocks instead of models only comes with a numerical trade-off. We also need the following elementary observation:

Lemma 7.11. *Let B be a $(< k)$-inseparable set of at least $k + 1$ vertices. Then G contains k internally disjoint x-y-paths for any two $x, y \in B$.*

Proof. Let $x, y \in B$. If $xy \notin E(G)$, then the existence of these paths follows from Menger's theorem. Suppose now that $xy \in E(G)$ and let $H := G - xy$.

If H contains a set $\mathcal{P}$ of $k-1$ internally disjoint x-y-paths, then we can add to $\mathcal{P}$ the path consisting of the single edge xy to obtain our desired set of paths. Otherwise, by Menger's Theorem, there exists a set $S \subseteq V(H) \setminus \{x, y\}$ of fewer than $k-1$ vertices which separates x and y.

Since $|B| \geq k+1$, there exists a $z \in B \setminus (S \cup \{x, y\})$. Then S must separate z in H from at least one of x and y, say from y. But then $S \cup \{x\}$ separates z from y in G, contrary to our assumption that B was $({<}k)$-inseparable. □

Lemma 7.12. *Let $\tau \subseteq S_k(G)$ be a star of tight separations. If* $\operatorname{tor}(\tau)$ *contains a k^2-block, then G has a k-block B with $\tau \subseteq O_B$.*

Proof. Suppose that $B \subseteq J(\tau)$ is a k^2-block of $\operatorname{tor}(\tau)$. We show that B is $(< k)$-inseparable in G. It then follows that there exists a k-block $B' \supseteq B$ and trivially $\tau \subseteq O(B) = O(B')$. For $k \leq 2$, the torso of τ is a subgraph of G and the assertion is trivial. We now consider the case $k \geq 3$.

Let $a, b \in B$ and suppose there was a set $X \subseteq V(G) \setminus \{a, b\}$ with $|X| < k$ which separates a and b. For every $x \in X \setminus J(\tau)$ there exists a unique $(U, W) \in \tau$ with $x \in U \setminus W$. We say that (U, W) is *blocked* by x and let σ be the set of all separations in τ which are blocked by some vertex in X. Let

$$Y := (X \cap J(\tau)) \cup \bigcup_{(U,W) \in \sigma} U \cap W \setminus \{a, b\}.$$

Note that $Y \subseteq J(\tau)$ and $|Y| \leq |X|(k-1)$. By Lemma 7.11, $\operatorname{tor}(\tau)$ contains a set $\mathcal{P}$ of $k^2 - 1$ internally disjoint x-y-paths. Since $k \geq 3$ and $|Y| \leq (k-1)^2$, there is a path $P \in \mathcal{P}$ of length at least 2 which does not meet Y.

For every 'virtual' edge $e \in E(P) \setminus E(G)$, there is some $(U, W) \in \tau$ with $e \subseteq U \cap W$. Since $e \neq ab$, it must be that (U, W) has not been blocked, for otherwise one endvertex of e would lie in Y. But the separation (U, W) is tight, so there exists a path $P_e \subseteq U$ with the same endvertices as e with no internal vertices in $J(\tau)$. No vertex of P_e lies in X, for otherwise that vertex would block (U, W). Replacing every virtual edge e by P_e, we obtain an X-avoiding connected subgraph $R \subseteq G$ containing a and b, contrary to our assumption. □

7.2 A proof of the structure theorem

We now combine everything to give a proof of Theorem 7.2.

Suppose G does not contain K_r as a topological minor. Let $k := r(r-1)$ and $m := 2k$. By Theorem 7.8, there exists a tree-decomposition $(T, \mathcal{V})$ which efficiently distinguishes all k-profiles of G. By Lemma 7.6, $O(B) \neq O(\mathcal{X})$ for every k-block B and every model $\mathcal{X}$ of K_m. Let T_b be the set of all nodes t_B for k-blocks B and let T_m be the set of all nodes $t_{\mathcal{X}}$ for models $\mathcal{X}$ of K_m. Note that $T_b \cap T_m = \emptyset$.

Every edge of T carries a weight μ given by the order of the two separations it induces. Note that for k-profiles O and O', every edge of minimum weight in $t_O T t_{O'}$ efficiently distinguishes O and O'. Let D be the complete bipartite graph with bipartition (T_b, T_m) and let $F \subseteq E(T)$ be a D-barrier (Lemma 7.9). Let S be a component of $T - F$. Since F is a D-barrier, S does not meet both T_b and T_m.

Let F_S be the set of all pairs (t, s) of adjacent vertices with $s \in S, t \notin S$ and let

$$\tau := \{(W_t, W_s) \colon (t, s) \in F\}$$

be the star of separations these pairs induce in G. Note that the torso of S is precisely $\mathrm{tor}(\tau)$. Let $(t, s) \in F$ arbitrary. There exist $t_B \in T_b$ and $t_{\mathcal{X}} \in T_m$ for which st has minimum weight in $t_B T t_{\mathcal{X}}$ and is the unique edge of F along $t_B T t_{\mathcal{X}}$. Hence precisely one of $t_B, t_{\mathcal{X}}$ lies in S and (W_t, W_s) efficiently distinguishes $O(B)$ and $O(\mathcal{X})$. By Lemma 7.7, (W_t, W_s) is tight and if $t_B \in S$, then (W_t, W_s) is generic. Hence we have shown that τ consists of tight separations and if $S \cap T_m$ is empty, then every separation in τ is generic.

We now show that the torso of S either contains no k^2-block or no K_m-minor. Suppose that $\mathrm{tor}(\tau)$ contained a k^2-block. By Lemma 7.12, there exists a k-block B of G with $\tau \subseteq O(B)$. Then $t_B \in S$ and $S \cap T_m$ must be empty. Therefore τ is a star of generic separations. If $\mathrm{tor}(\tau)$ then contained a K_m-minor, then by Lemma 7.10 there'd be a model $\mathcal{X}$ of K_m in G with $\tau \subseteq O(\mathcal{X})$. But then $t_{\mathcal{X}} \in S$, a contradiction.

Now, contract each component of $T - F$ to a single node. The torso of each node in this new tree-decomposition $(T', \mathcal{V}')$ is the same as the torso in $(T, \mathcal{V})$ of the subtree it originated from. As we have seen, each of these either contains no k^2-block or no K_m-minor. $\square$

As indicated at the beginning, Corollary 7.3 follows immediately from Theorem 7.2: if a torso contains no r^4-block, then it can be decomposed further via Theorem 5.1, so that each new torso has at most r^4 vertices of degree greater than $2r^8$. Doing this for each part whose torso contains K_{2r^2} as a minor, we obtain the desired tree-decomposition.

7.3 Stronger structure theorems

In our proof of Corollary 7.3, the bound on the degrees within the torsos was established in two steps. First, if a part contains no k-block of G, then its torso has no k^2-block. Second, if a torso contains no k^2-block, then by Theorem 5.1 it can be decomposed so that each new torso has at most k^2 vertices of degree greater than $2k^4$. Each of the two steps of this argument doubled the exponent of the parameter bounding the degrees. By merging the two steps, we improve the bound on the degrees.

Recall our refined structure theorem for k-blocks from Chapter 5:

Theorem 5.8. *Let G be a graph, $k \geq 3$ an integer, $(T, \mathcal{V})$ a k-atomic tree-decomposition of G and $t \in T$. If the torso at t contains at least k vertices of degree at least $2(k-1)(k-2)$, then G has a k-block B with $B \subseteq V_t$.*

Instead of starting with a distinguishing tree-decomposition and then refining it using a k-atomic tree-decomposition of each torso, we now begin with a k-atomic tree-decomposition of the whole graph. As it turns out, such tree-decompositions also efficiently distinguish the orientations induced by blocks and models.

Lemma 7.13. *Let $r, k \geq r(r-1), m \geq 2k$ be positive integers, let G be a graph containing no subdivision of K_r and let $(T, \mathcal{V})$ be a k-lean tree-decomposition of G. Then $(T, \mathcal{V})$ efficiently distinguishes the orientations $O(B)$ and $O(\mathcal{X})$ of $S_k(G)$ induced by any k-block B and any model $\mathcal{X}$ of K_m.*

Proof. Let $(T, \mathcal{V})$ be a k-lean tree-decomposition of G. We know by Lemma 7.6 that $O(B) \neq O(\mathcal{X})$ for every k-block B and every model $\mathcal{X}$ of K_m. Let us call an orientation O of $S_k(G)$ *anchored* if for every $(U, W) \in O$, there are at least k vertices in $W \cap V_{t_O}$.

Note that every orientation $O = O(B)$ induced by a k-block B is trivially anchored, since $B \subseteq V_{t_B}$. But the same is true for the orientation $O = O(\mathcal{X})$ induced by a model $\mathcal{X}$ of K_m. Indeed, let $(U, W) \in O(\mathcal{X})$. Then every branchset of $\mathcal{X}$ meets $V_{t_\mathcal{X}}$. At least k branchsets of $\mathcal{X}$ are disjoint from $U \cap W$, say $X_1, \ldots, X_k$, and they all lie in $W \setminus U$. For $1 \leq i \leq k$, let $x_i \in X_i \cap V_{t_\mathcal{X}}$ and note that $R := \{x_1, \ldots, x_k\} \subseteq W \cap V_{t_\mathcal{X}}$.

We now show that $(T, \mathcal{V})$ efficiently distinguishes all anchored orientations of $S_k(G)$. Let O_1, O_2 be two anchored orientations of $S_k(G)$ and let their home nodes be t_1 and t_2 respectively. If $t_1 \neq t_2$, let p be the minimum order of an edge along $t_1 T t_2$, and put $p := k$ otherwise. Choose some $(U, W) \in O_2 \setminus O_1$ of minimum order. Since O_1 and O_2 are anchored, we have $|U \cap V_{t_1}| \geq k$ and $|W \cap V_{t_2}| \geq k$. As $(T, \mathcal{V})$ is k-lean, it follows that $|U \cap W| \geq p$. Hence $t_1 \neq t_2$ and $(T, \mathcal{V})$ efficiently distinguishes O_1 and O_2. $\square$

If we now use a k-atomic tree-decomposition in place of an arbitrary tree-decomposition which efficiently distinguishes blocks and models, we do not have to process nodes containing no k-blocks any further, because Theorem 5.8 already places a bound on the degrees in their torsos. The rest of the proof is the same as the proof of Theorem 7.2.

Theorem 7.14. *For every positive integer r, every graph which does not contain K_r as a topological minor has a tree-decomposition in which every torso either*

(i) *has fewer than r^2 vertices of degree at least $2r^4$, or*

(ii) *does not contain K_{2r^2} as a minor.*

Proof. Let $k := r(r-1)$ and $m := 2k$. Let $(T, \mathcal{V})$ be a k-atomic tree-decomposition of G. By Lemma 7.13, $(T, \mathcal{V})$ efficiently distinguishes the orientations of $S_k(G)$ induced by k-blocks from those induced by models of K_m. Let $T_b \subseteq V(T)$ be the set of all nodes t_B for k-blocks B and T_m the set of all $t_{\mathcal{X}}$, $\mathcal{X}$ a model of K_m. By Theorem 5.8, the torso of each $t \in T \setminus T_b$ has fewer than k vertices of degree at least $2k^2$.

Let the weight of an edge of T be the order of the separation it induces in G and let D be the complete bipartite graph with bipartition (T_b, T_m). By Lemma 7.9, there exists a D-barrier F. Let S be a component of $T - F$ that meets T_b. As in the proof of Theorem 7.2, it follows that the torso of S does not contain K_m as a minor.

Obtain the tree-decomposition $(T', \mathcal{V}')$ by contracting every component of $T - F$ that meets T_b to a single node. The torso of such a new node coincides with the torso of the subtree it originated from, and therefore does not contain a K_m-minor. The torso of every other node remained the same. Each of these nodes lies in $T \setminus T_b$, so its torso contains fewer than k vertices of degree at least $2k^2$.

□

Using methods from graph minor theory and topological graph theory, Dvořák [28] recently refined the embeddability condition in Theorem 1.16 to reflect more closely the topology of embeddings of an arbitrary graph H which is to be excluded as a topological minor. Building upon Dvořák's theorem, Liu and Thomas [50] then improved the bounds on the degrees. These refinements are beyond the scope of our methods.

8 Tangles in abstract separation systems

In this chapter, we leave the realm of graphs and set out to explore the concept of a tangle from the abstract point of view of separation systems.

The two main theorems of Robertson and Seymour [58] about tangles in graphs, the *tangle tree theorem* and the *tangle duality theorem*, have been generalized to the abstract setting by Diestel, Hundertmark and Lemanczyk [21] and by Diestel and Oum [23], respectively. However, the tangle tree theorem in [21] and the applications of the tangle duality theorem of [23] given in [22] and [24] relied on a concept borrowed from tangles in graphs, which is foreign to the world of separation systems: a submodular order function defined on an ambient universe of separations. This might make it seem as if the general notion of abstract separation systems was not strong enough to support a rich theory on its own and thus necessarily relied on the support of such order functions.

We show that this is not the case. The existence of a submodular order function has an immediate structural consequence on the separation system under consideration, and this consequence can serve to give a purely structural definition of what it means for a separation system to be *submodular*. In Section 8.2, we prove the following tangle tree theorem for *abstract tangles* in submodular separation systems:

Theorem 8.1. *Every submodular separation system $\vec{S}$ contains a tree set of separations that distinguishes all the abstract tangles of S.*

(See Section 8.1 for the relevant definitions.)

Based on this new notion of structural submodularity, we apply the tangle duality theorem [23] to prove a duality theorem for abstract tangles:

Theorem 8.2. *Let $\vec{S}$ be a submodular separation system without degenerate elements in a distributive universe $\vec{U}$. Then exactly one of the following holds:*

(i) S has an abstract tangle.

(ii) There exists an S-tree over $\mathcal{T}^*$.

The proof of Theorem 8.2 is contained in Section 8.3. As a key step in the proof, we show that every submodular separation system is *separable*, which is crucial for every application of the tangle duality theorem [23].

At this point, the sceptical reader might raise an eyebrow and ask whether this condition of submodularity might not merely be a submodular order function in disguise, working its magic in the background. Our methodological and perhaps unsatisfying reply to this objection is that nonetheless we have a purely structural hypothesis which allows for a proof contained entirely within the language of separation systems, even if its range of application should be limited to cases which had already been covered previously. However, hopefully more convincing, we present in Section 8.4 a natural family of submodular separation systems which do not come with a submodular order function that witnesses their submodularity.

8.1 Abstract Separation Systems

For a gentle yet thorough introduction to abstract separation systems, we refer the reader to [17]. Any terminology not defined here can be found there. We also recommend the introductory sections of [21, 23, 22] and [24] to get an idea of the expressive strength of abstract separation systems and the broad range of applications. In the following, we provide a self-contained account of just the definitions and basic facts about abstract separation systems that we need in this chapter.

A *separation system* $(\vec{S}, \leq, ^*)$ is a partially ordered set with an order-reversing involution $^*\colon \vec{S} \to \vec{S}$. The elements of $\vec{S}$ are called *(oriented) separations*. The *inverse* of $\vec{s} \in \vec{S}$ is $\vec{s}^{\,*}$, which we usually denote by $\overleftarrow{s}$. An *(unoriented) separation* is a set $s = \{\vec{s}, \overleftarrow{s}\}$ consisting of a separation and its inverse and we then refer to $\vec{s}$ and $\overleftarrow{s}$ as the two *orientations* of s. Note that it may occur that $\vec{s} = \overleftarrow{s}$, we then call $\vec{s}$ *degenerate*. The set of all separations is denoted by S. When the context is clear, we often refer to oriented separations simply as separations in order to improve the flow of text.

If the partial order $(\vec{S}, \leq)$ is a lattice with join $\vee$ and meet $\wedge$, then we call $(\vec{S}, \leq, ^*, \vee, \wedge)$ a *universe* of (oriented) separations. It is *distributive* if it is distributive as a lattice. Typically, the separation systems we are interested in are contained in a universe of separations. In most applications, one starts with a universe $(\vec{U}, \leq, ^*, \vee, \wedge)$ and then defines $\vec{S}$ as the set of all separations of *low order* with respect to some order function $|\cdot|\colon \vec{U} \to \mathbb{R}^+$ that is *symmetric* and *submodular*, that is, $|\vec{s}| = |\overleftarrow{s}|$ and

$$|\vec{s} \vee \vec{t}| + |\vec{s} \wedge \vec{t}| \leq |\vec{s}| + |\vec{t}|$$

holds for all $\vec{s}, \vec{t} \in \vec{U}$. Submodularity of the order function in fact plays a crucial role in several arguments. One immediate consequence is that whenever both $\vec{s}$ and $\vec{t}$ lie in $\vec{S}_k := \{\vec{u} \in \vec{U} : |\vec{u}| < k\}$, then at least one of $\vec{s} \vee \vec{t}$ and $\vec{s} \wedge \vec{t}$ again lies in $\vec{S}_k$.

In order to avoid recourse to the external concept of an order function if possible, let us turn this last property into a definition that uses only the language of lattices. Let us call a subset M of a lattice $(L, \vee, \wedge)$ *submodular* if for all $x, y \in M$ at least one of $x \vee y$ and $x \wedge y$ lies in M. A separation system $\vec{S}$ contained in a given universe $\vec{U}$ of separations is *submodular* if it is submodular as a subset of the lattice underlying $\vec{U}$.

We say that $\vec{s} \in \vec{S}$ is *small* (and $\overleftarrow{s}$ is co-small) if $\vec{s} \leq \overleftarrow{s}$. An element $\vec{s} \in \vec{S}$ is *trivial* in $\vec{S}$ (and $\overleftarrow{s}$ is co-trivial) if there exists $t \in S$ whose orientations $\vec{t}, \overleftarrow{t}$ satisfy $\vec{s} < \vec{t}$ as well as $\vec{s} < \overleftarrow{t}$. Notice that trivial separations are small.

Two separations $s, t \in S$ are *nested* if there exist orientations $\vec{s}$ of s and $\vec{t}$ of t such that $\vec{s} \leq \vec{t}$. Two oriented separations are nested if their underlying separations are. We say that two separations *cross* if they are not nested. A set of (oriented) separations is *nested* if any two of its elements are. A nested separation system without trivial or degenerate elements is a *tree set*. A set σ of non-degenerate oriented separations is a *star* if for any two distinct $\vec{s}, \vec{t} \in \sigma$ we have $\vec{s} \leq \overleftarrow{t}$. A family $\mathcal{F} \subseteq 2^{\vec{U}}$ of sets of separations is *standard for* $\vec{S}$ if for any trivial $\vec{s} \in \vec{S}$ we have $\{\overleftarrow{s}\} \in \mathcal{F}$. Given $\mathcal{F} \subseteq 2^{\vec{U}}$, we write $\mathcal{F}^*$ for the set of all elements of $\mathcal{F}$ that are stars.

An *orientation* of S is a set $O \subseteq \vec{S}$ which contains for every $s \in S$, exactly one of $\overleftarrow{s}, \vec{s}$. An orientation O of S is *consistent* if whenever $r, s \in S$ are distinct and $\vec{r} \leq \vec{s} \in \vec{S}$, then $\overleftarrow{r} \notin \vec{S}$. The idea behind this is that two separations $\overleftarrow{r}$ and $\vec{s}$ are thought of as pointing away from each other if $\vec{r} \leq \vec{s}$. If we wish to orient r and s towards some common region of the structure which they are assumed to 'separate', as is the idea behind tangles, we should therefore not orient them as $\overleftarrow{r}$ and $\vec{s}$.

Tangles in graphs also satisfy another, more subtle, consistency requirement: they never orient three separations r, s, t so that the union of their sides to which they do *not* point is the entire graph. This can be mimicked in abstract separation systems by asking that three oriented separations in an 'abstract tangle' must never have a co-small supremum; see [17, Section 5]. So let us implement this formally.

Given a family $\mathcal{F} \subseteq 2^{\vec{U}}$, we say that O *avoids* $\mathcal{F}$ if there is no $\sigma \in \mathcal{F}$ with $\sigma \subseteq O$. A consistent $\mathcal{F}$-avoiding orientation of S is called an $\mathcal{F}$*-tangle* of S. An $\mathcal{F}$-tangle for $\mathcal{F} = \mathcal{T}$ with

$$\mathcal{T} := \{\{\vec{r}, \vec{s}, \vec{t}\} \subseteq \vec{U} \colon \; \vec{r} \vee \vec{s} \vee \vec{t} \text{ is co-small}\}$$

is an *abstract tangle*.

A separation $s \in S$ *distinguishes* two orientations O_1, O_2 of S if $O_1 \cap s \neq O_2 \cap s$. Likewise, a set N of separations *distinguishes* a set $\mathcal{O}$ of orientations if for any two $O_1, O_2 \in \mathcal{O}$, there is some $s \in N$ which distinguishes them.

Let us restate our tangle-tree theorem for abstract tangles:

Theorem 8.1. *Every submodular separation system $\vec{S}$ contains a tree set of separations that distinguishes all the abstract tangles of S.*

We now introduce the structural dual to the existence of abstract tangles. An *S-tree* is a pair (T, α) consisting of a tree T and a map $\alpha : \vec{E}(T) \to \vec{S}$ from the set $\vec{E}(T)$ of orientations of edges of T to $\vec{S}$ such that $\alpha(y, x) = \alpha(x, y)^*$ for all $xy \in E(T)$. Given $\mathcal{F} \subseteq 2^{\vec{U}}$, we call (T, α) an *S-tree over $\mathcal{F}$* if $\alpha(F_t) \in \mathcal{F}$ for every $t \in T$, where

$$F_t := \{(s, t) \colon st \in E(T)\}.$$

It is easy to see that if S has an abstract tangle, then there can be no S-tree over $\mathcal{T}$. Our duality theorem, which we now re-state, asserts a converse to this. Recall that $\mathcal{T}^*$ denotes the set of stars in $\mathcal{T}$.

Theorem 8.2. *Let $\vec{S}$ be a submodular separation system without degenerate elements in a distributive universe $\vec{U}$. Then exactly one of the following holds:*

(i) *S has an abstract tangle.*

(ii) *There exists an S-tree over $\mathcal{T}^*$.*

Here, it really is necessary to exclude degenerate separations: a single degenerate separation will make the existence of abstract tangles impossible, although there might still be $\mathcal{T}^*$-tangles (and therefore no S-trees over $\mathcal{T}^*$). We will actually prove a duality theorem for $\mathcal{T}^*$-tangles without this additional assumption and then observe that $\mathcal{T}^*$-tangles are in fact already abstract tangles, unless $\vec{S}$ contains a degenerate separation.

In applications, we do not always wish to consider all the abstract tangles of a given separation system. For example, if $\vec{S}$ consists of the bipartitions of some finite set X, then every $x \in X$ induces an abstract tangle

$$\theta_x := \big\{(A, B) \in \vec{S} \colon x \in B\big\},$$

the *principal tangle* induced by x. In particular, abstract tangles trivially exist in these situations. In order to exclude principal tangles, we could require that every tangle θ of S must satisfy $(\{x\}, X \setminus \{x\}) \in \theta$ for every $x \in X$.

More generally, we might want to prescribe for some separations s of S that any tangle of S we consider must contain a particular one of the two orientations of s rather than the other. This can easily be done in our abstract setting, as follows. Given $Q \subseteq \vec{U}$, let us say that an abstract tangle θ of S *extends* Q if $Q \cap \vec{S} \subseteq \theta$. It is easy to see that θ

extends Q if and only if θ is $\mathcal{F}_Q$-avoiding, where

$$\mathcal{F}_Q := \{\{\overleftarrow{s}\}\colon\ \vec{s} \in Q \text{ non-degenerate}\}.$$

We call $Q \subseteq \vec{U}$ *down-closed* if $\vec{r} \leq \vec{s} \in Q$ implies $\vec{r} \in Q$ for all $\vec{r}, \vec{s} \in \vec{U}$.

Here, then, is our refined duality theorem for abstract tangles.

Theorem 8.3. *Let $\vec{S}$ be a submodular separation system without degenerate elements in a distributive universe $\vec{U}$ and let $Q \subseteq \vec{U}$ be down-closed. Then exactly one of the following assertions holds:*

(i) *S has an abstract tangle extending Q.*

(ii) *There exists an S-tree over $\mathcal{T}^* \cup \mathcal{F}_Q$.*

Observe that Theorem 8.3 implies Theorem 8.2 by taking $Q = \emptyset$.

8.2 A tangle-tree theorem

In this section we will prove Theorem 8.1. In fact, we are going to prove a slightly more general statement. Let $\mathcal{P} := \{\{\vec{s}, \vec{t}, (\vec{s} \vee \vec{t})^*\}\colon\ \vec{s}, \vec{t} \in \vec{U}\}$. The $\mathcal{P}$-tangles are known as *profiles*[1].

Theorem 8.4. *Let $\vec{S}$ be a submodular separation system and Π a set of profiles of S. Then $\vec{S}$ contains a tree set that distinguishes Π.*

This implies Theorem 8.1, by the following easy observation.

Lemma 8.5. *Every abstract tangle is a profile.*

Proof. Let $\vec{s}, \vec{t} \in \vec{U}$ and $\vec{r} := \vec{s} \vee \vec{t}$. Then

$$\vec{s} \vee \vec{t} \vee \overleftarrow{r} = \vec{r} \vee \overleftarrow{r}$$

is co-small, so $\{\vec{s}, \vec{t}, \overleftarrow{r}\} \in \mathcal{T}$. Therefore $\mathcal{P} \subseteq \mathcal{T}$ and every $\mathcal{T}$-tangle is a $\mathcal{P}$-tangle. $\square$

We first recall a basic fact about nestedness of separations. For two separations s, t in a universe of separations, we define the four *corners* $\vec{s} \wedge \vec{t}$, $\vec{s} \wedge \overleftarrow{t}$, $\overleftarrow{s} \wedge \vec{t}$ and $\overleftarrow{s} \wedge \overleftarrow{t}$.

Lemma 8.6 ([17])**.** *Let $\vec{U}$ be a universe of separations and let $s, t \in U$ cross. Then every separation that is nested with both s and t is nested with every corner of s and t.*

[1]These are, of course, an abstraction of the *k-profiles of graphs* that we encountered in Chapter 7.

In the proof of Theorem 8.4, we take a nested set $\mathcal{N}$ of separations that distinguishes some set Π_0 of regular profiles and we want to exchange one element of $\mathcal{N}$ by some other separation while maintaining that Π_0 is still distinguished. The following lemma simplifies this exchange.

Lemma 8.7. *Let $\vec{S}$ be a separation system, $\mathcal{O}$ a set of consistent orientations of S and $\mathcal{N} \subseteq S$ an inclusion-minimal nested set of separations that distinguishes $\mathcal{O}$. Then for every $t \in \mathcal{N}$ there is a unique pair of orientations $O_1, O_2 \in \mathcal{O}$ that are distinguished by t and by no other element of $\mathcal{N}$.*

Proof. It is clear that at least one such pair must exist, for otherwise $\mathcal{N} \setminus \{t\}$ would still distinguish $\mathcal{O}$, thus violating the minimality of $\mathcal{N}$.

Suppose there was another such pair, say O_1', O_2'. After relabeling, we may assume that $\vec{t} \in O_1 \cap O_1'$ and $\overleftarrow{t} \in O_2 \cap O_2'$. By symmetry, we may further assume that $O_1 \neq O_1'$. Since $\mathcal{N}$ distinguishes $\mathcal{O}$, there is some $r \in \mathcal{N}$ with $\vec{r} \in O_1$, $\overleftarrow{r} \in O_1'$.

As t is the only element of $\mathcal{N}$ distinguishing O_1, O_2, it must be that $\vec{r} \in O_2$ as well, and similarly $\overleftarrow{r} \in O_2'$. We hence see that for any orientation τ of $\{r, t\}$, there is an $O \in \{O_1, O_2, O_1', O_2'\}$ with $\tau \subseteq O$. Since $\mathcal{N}$ is nested, there exist orientations of r and t pointing away from each other. But then one of O_1, O_2, O_1', O_2' is inconsistent, which is a contradiction. □

Proof of Theorem 8.4. Note that it suffices to show that there is a nested set $\mathcal{N}$ of separations that distinguishes Π: Every consistent orientation contains every trivial and every degenerate element, so any inclusion-minimal such set $\mathcal{N}$ gives rise to a tree-set.

We prove this by induction on $|\Pi|$, the case $|\Pi| = 1$ being trivial.

For the induction step, let $P \in \Pi$ be arbitrary and $\Pi_0 := \Pi \setminus \{P\}$. By the induction hypothesis, there exists a nested set $\mathcal{N}$ of separations that distinguishes Π_0. If some such set $\mathcal{N}$ distinguishes Π, there is nothing left to show. Otherwise, for every nested $\mathcal{N} \subseteq S$ which distinguishes Π_0 there is a $P' \in \Pi_0$ which $\mathcal{N}$ does not distinguish from P. Note that P' is unique. For any $s \in S$ that distinguishes P and P', let $d(\mathcal{N}, s)$ be the number of elements of $\mathcal{N}$ which are not nested with s.

Choose a pair $(\mathcal{N}, s)$ so that $d(\mathcal{N}, s)$ is minimum. Clearly, we may assume $\mathcal{N}$ to be inclusion-minimal with the property of distinguishing Π_0. If $d(\mathcal{N}, s) = 0$, then $\mathcal{N} \cup \{s\}$ is a nested set distinguishing Π and we are done, so we now assume for a contradiction that $d(\mathcal{N}, s) > 0$.

Since $\mathcal{N}$ does not distinguish P and P', we can fix an orientation of each $t \in \mathcal{N}$ such that $\vec{t} \in P \cap P'$. Choose a $t \in \mathcal{N}$ such that t and s cross and $\vec{t}$ is minimal. Let (P_1, P_2) be the unique pair of profiles in Π_0 which are distinguished by t and by no other element of $\mathcal{N}$, say $\overleftarrow{t} \in P_1$, $\vec{t} \in P_2$. Let us assume without loss of generality that $\overleftarrow{s} \in P_1$. The situation is depicted in Figure 8.1. Note that we do not know whether $\vec{s} \in P_2$ or $\overleftarrow{s} \in P_2$. Also, the roles of P and P' might be reversed, but this is insignificant.

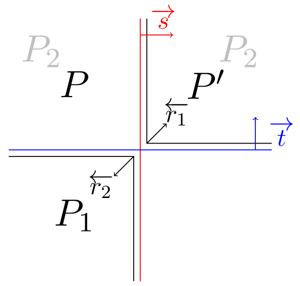

Figure 8.1: Crossing separations

Suppose first that $\vec{r_1} := \vec{s} \vee \vec{t} \in \vec{S}$. Let $Q \in \{P, P'\}$. If $\vec{s} \in Q$, then $\vec{r_1} \in Q$, since $\vec{t} \in P \cap P'$ and Q is a profile. If $\vec{r_1} \in Q$, then $\vec{s} \in Q$ since Q is consistent and $\vec{s} \leq \vec{r_1} \in Q$: it cannot be that $\vec{s} = \overleftarrow{r_1}$, since then s and t would be nested. Hence each $Q \in \{P, P'\}$ contains $\vec{r_1}$ if and only if it contains $\vec{s}$. In particular, r_1 distinguishes P and P'. By Lemma 8.6, every $u \in \mathcal{N}$ that is nested with s is also nested with r_1. Moreover, t is nested with r_1, but not with s, so that $d(\mathcal{N}, r_1) < d(\mathcal{N}, s)$. This contradicts our choice of s.

Therefore $\vec{s} \vee \vec{t} \notin \vec{S}$. Since $\vec{S}$ is submodular, it follows that $\vec{r_2} := \vec{s} \wedge \vec{t} \in \vec{S}$. Moreover, r_2 is nested with every $u \in \mathcal{N} \setminus \{t\}$. This is clear if $\vec{t} \leq \vec{u}$ or $\vec{t} \leq \overleftarrow{u}$, since $\vec{r_2} \leq \vec{t}$. It cannot be that $\overleftarrow{u} \leq \vec{t}$, because $\vec{u}, \vec{t} \in P$ and P is consistent. Since $\mathcal{N}$ is nested, only the case $\vec{u} < \vec{t}$ remains. Then, by our choice of $\vec{t}$, u and s are nested and it follows from Lemma 8.6 that u and r_2 are also nested. Hence $\mathcal{N}' := (\mathcal{N} \setminus \{t\}) \cup \{r_2\}$ is a nested set of separations.

To see that $\mathcal{N}'$ distinguishes Π_0, it suffices to check that r_2 distinguishes P_1 and P_2. We have $\vec{r_2} \in P_2$ since P_2 is consistent and $\vec{r_2} \leq \vec{t} \in P_2$: if $\vec{r_2} = \overleftarrow{t}$, then s and t would be nested. Since $\overleftarrow{r_2} = \overleftarrow{s} \vee \overleftarrow{t}$ and $\overleftarrow{s}, \overleftarrow{t} \in P_1$, we find $\overleftarrow{r_2} \in P_1$. Any element of $\mathcal{N}'$ which is not nested with s lies in $\mathcal{N}$. Since $t \in \mathcal{N} \setminus \mathcal{N}'$ is not nested with s, it follows that $d(\mathcal{N}', s) < d(\mathcal{N}, s)$, contrary to our choice of $\mathcal{N}$ and s. $\square$

8.3 A tangle duality theorem

Our goal in this section is to prove Theorem 8.3. The proof will be an application of a more general duality theorem of Diestel and Oum. We first need to introduce the central notion of separability.

A separation $\vec{s} \in \vec{S}$ *emulates* $\vec{r}$ *in* $\vec{S}$ if $\vec{s} \geq \vec{r}$ and for every $\vec{t} \in \vec{S} \setminus \{\overleftarrow{r}\}$ with $\vec{t} \geq \vec{r}$ we have $\vec{s} \vee \vec{t} \in \vec{S}$. For $\vec{s} \in \vec{S}$, $\sigma \subseteq \vec{S}$ and $\vec{x} \in \sigma$, define

$$\sigma_{\vec{x}}^{\vec{s}} := \{\vec{x} \vee \vec{s}\} \cup \{\vec{y} \wedge \overleftarrow{s} :\ \vec{y} \in \sigma \setminus \{\vec{x}\}\}.$$

Lemma 8.8. *Suppose $\vec{s} \in \vec{S}$ emulates a non-trivial $\vec{r}$ in $\vec{S}$, and let $\sigma \subseteq \vec{S}$ be a star such that $\vec{r} \leq \vec{x} \in \sigma$. Then $\sigma_{\vec{x}}^{\vec{s}} \subseteq \vec{S}$ is a star.*

Proof. Note that for every $\vec{y} \in \sigma \setminus \{\vec{x}\}$ we have $\vec{r} \leq \overleftarrow{y}$. It is clear that for any two distinct $\vec{u}, \vec{v} \in \sigma_{\vec{x}}^{\vec{s}}$ we have $\vec{u} \leq \overleftarrow{v}$, so we only need to show that every element of $\sigma_{\vec{x}}^{\vec{s}}$ is non-degenerate and lies in $\vec{S}$. For every $\vec{u} \in \sigma_{\vec{x}}^{\vec{s}}$ there is a non-degenerate $\vec{t} \in \vec{S}$ with $\vec{r} \leq \vec{t}$ such that either $\vec{u} = \vec{t} \vee \vec{s}$ or $\overleftarrow{u} = \vec{t} \vee \vec{s}$.

Let $\vec{t} \in \vec{S}$ be non-degenerate with $\vec{r} \leq \vec{t}$. Since $\vec{s}$ emulates $\vec{r}$ in $\vec{S}$, we find $\vec{t} \vee \vec{s} \in \vec{S}$. Assume for a contradiction that $\vec{t} \vee \vec{s}$ was degenerate. Since $\vec{t}$ is non-degenerate, we find that $\vec{t} < \vec{t} \vee \vec{s}$, so that $\vec{t}$ is trivial. But then so is $\vec{r}$, because $\vec{r} \leq \vec{t}$. This contradicts our assumption on $\vec{r}$. □

Given some $\mathcal{F} \subseteq 2^{\vec{U}}$, we say that $\vec{s}$ *emulates* $\vec{r}$ *in* $\vec{S}$ *for* $\mathcal{F}$ if $\vec{s}$ emulates $\vec{r}$ in $\vec{S}$ and for every star $\sigma \subseteq \vec{S} \setminus \{\overleftarrow{r}\}$ with $\sigma \in \mathcal{F}$ and every $\vec{x} \in \sigma$ with $\vec{x} \geq \vec{r}$ we have $\sigma_{\vec{x}}^{\vec{s}} \in \mathcal{F}$.

The separation system $\vec{S}$ is *$\mathcal{F}$-separable* if for all non-trivial and non-degenerate $\vec{r}_1, \overleftarrow{r}_2 \in \vec{S}$ with $\vec{r}_1 \leq \vec{r}_2$ and $\{\overleftarrow{r}_1\}, \{\vec{r}_2\} \notin \mathcal{F}$ there exists an $\vec{s} \in \vec{S}$ which emulates $\vec{r}_1$ in $\vec{S}$ for $\mathcal{F}$ while simultaneously $\overleftarrow{s}$ emulates $\overleftarrow{r}_2$ in $\vec{S}$ for $\mathcal{F}$.

Theorem 8.9 ([23, Theorem 4.3])**.** *Let $\vec{U}$ be a universe of separations and $\vec{S} \subseteq \vec{U}$ a separation system. Let $\mathcal{F} \subseteq 2^{\vec{U}}$ be a set of stars, standard for $\vec{S}$. If $\vec{S}$ is $\mathcal{F}$-separable, then exactly one of the following holds:*

(i) *There exists an $\mathcal{F}$-tangle of S.*

(ii) *There exists an S-tree over $\mathcal{F}$.*

Since the family $\mathcal{F}$ in Theorem 8.9 is assumed to be a set of stars, we cannot work directly with $\mathcal{T} \cup \mathcal{F}_Q$. We thus keep only those sets in $\mathcal{T}$ which happen to be stars and apply Theorem 8.9 with $\mathcal{F} = \mathcal{T}_Q$, where $\mathcal{T}_Q := \mathcal{T}^* \cup \mathcal{F}_Q$. As it turns out, this does not make a difference as long as $\vec{S}$ has no degenerate elements, see Lemma 8.15 below.

We start with a simple observation that will be useful later.

Lemma 8.10. *Let $\vec{U}$ be a distributive universe of separations. Let $\vec{u}, \vec{v}, \vec{w} \in \vec{U}$. If $\vec{u} \leq \vec{v}$ and $\vec{v} \vee \vec{w}$ is co-small, then $\vec{v} \vee (\vec{w} \wedge \overleftarrow{u})$ is co-small.*

Proof. Let $\vec{x} := \vec{v} \vee (\vec{w} \wedge \overleftarrow{u})$. By distributivity of $\vec{U}$

$$\vec{x} = (\vec{v} \vee \vec{w}) \wedge (\vec{v} \vee \overleftarrow{u}) \geq (\vec{v} \vee \vec{w}) \wedge (\vec{u} \vee \overleftarrow{u}).$$

Let $\vec{s} := \vec{v} \vee \vec{w}$ and $\vec{t} := \vec{u} \vee \overleftarrow{u}$. Then $\overleftarrow{s} \leq \vec{s}$ by assumption and $\overleftarrow{s} \leq \overleftarrow{v} \leq \vec{t}$. Further $\overleftarrow{t} \leq \vec{u} \leq \vec{t}$ and $\overleftarrow{t} \leq \vec{u} \leq \vec{v}$. Therefore

$$\overleftarrow{x} \leq \overleftarrow{s} \vee \overleftarrow{t} \leq \vec{s} \wedge \vec{t} \leq \vec{x}. \qquad \square$$

We now prove that $\vec{S}$ is $\mathcal{T}_Q$-separable in a strong sense.

Let $(L, \vee, \wedge)$ be a lattice and let $M \subseteq L$. Given $x, y \in M$, we say that x *pushes* y if $x \leq y$ and for any $z \in M$ with $z \leq y$ we have $x \wedge z \in M$. Similarly, we say that x *lifts* y if $x \geq y$ and for any $z \in M$ with $z \geq y$ we have $x \vee z \in M$. Observe that both of these relations are reflexive and transitive: Every $x \in M$ pushes (lifts) itself and if x pushes (lifts) y and y pushes (lifts) z, then x pushes (lifts) z. We say that M is *strongly separable* if for all $x, y \in M$ with $x \leq y$ there exists a $z \in M$ that lifts x and pushes y.

The definitions of lifting, pushing and strong separability extend verbatim to a separation system within a universe of separations when regarded as a subset of the underlying lattice. The notions of lifting and emulating are of course closely related: If $\vec{s} \in \vec{S}$ lifts $\vec{r} \in \vec{S}$, then $\vec{s}$ emulates $\vec{r}$ in $\vec{S}$. Observe also that $\vec{s}$ pushes $\vec{r}$ if and only if $\overleftarrow{s}$ lifts $\overleftarrow{r}$.

We call a set $\mathcal{F} \subseteq 2^{\vec{U}}$ *closed under shifting* if whenever $\vec{s} \in \vec{S}$ emulates in $\vec{S}$ a non-trivial and non-degenerate $\vec{r} \in \vec{S}$ with $\{\overleftarrow{r}\} \notin \mathcal{F}$, then it does so for $\mathcal{F}$.

The following is immediate from the definitions:

Lemma 8.11. *Let $\vec{U}$ be a universe of separations, $\vec{S} \subseteq \vec{U}$ a separation system and $\mathcal{F} \subseteq 2^{\vec{U}}$ a set of stars. If $\vec{S}$ is strongly separable and $\mathcal{F}$ is closed under shifting, then $\vec{S}$ is $\mathcal{F}$-separable.* $\square$

Lemma 8.12. *If $Q \subseteq \vec{U}$ is down-closed and $\vec{U}$ is distributive, then $\mathcal{T}_Q$ is closed under shifting.*

Proof. Let $\vec{r} \in \vec{S}$ non-trivial and non-degenerate with $\{\overleftarrow{r}\} \notin \mathcal{F}$. Let $\vec{s} \in \vec{S}$ emulate $\vec{r}$ in $\vec{S}$, let $\mathcal{T}_Q \ni \sigma \subseteq \vec{S} \setminus \{\overleftarrow{r}\}$ and $\vec{r} \leq \vec{x} \in \sigma$. We have to show that $\sigma^{\vec{s}}_{\vec{x}} \in \mathcal{T}_Q$. From Lemma 8.8 we know that $\sigma^{\vec{s}}_{\vec{x}}$ is a star, so we only need to verify that $\sigma^{\vec{s}}_{\vec{x}} \in \mathcal{T} \cup \mathcal{F}_Q$.

Suppose first that $\sigma \in \mathcal{T}^*$. Let $\vec{w} := \bigvee(\sigma \setminus \{\vec{x}\})$. Applying Lemma 8.10 with $\vec{u} = \vec{s}$ and $\vec{v} = \vec{x} \vee \vec{s}$, we see that

$$\bigvee \sigma^{\vec{s}}_{\vec{x}} = (\vec{x} \vee \vec{s}) \vee (\vec{w} \wedge \overleftarrow{s})$$

is cosmall. Since $\sigma^{\vec{s}}_{\vec{x}}$ has at most three elements, it follows that $\sigma^{\vec{s}}_{\vec{x}} \in \mathcal{T}$.

Suppose now that $\sigma \in \mathcal{F}_Q$. Then $\sigma = \{\vec{x}\}$ and $\overleftarrow{x} \in Q$. As Q is down-closed, we have $\overleftarrow{x} \wedge \overleftarrow{s} \in Q$. Since $\sigma^{\vec{s}}_{\vec{x}}$ is a star, $\overleftarrow{x} \wedge \overleftarrow{s}$ is non-degenerate and therefore

$$\sigma^{\vec{s}}_{\vec{x}} = \{\vec{x} \vee \vec{s}\} = \{(\overleftarrow{x} \wedge \overleftarrow{s})^*\} \in \mathcal{F}_Q. \qquad \square$$

Virtually all applications of Theorem 8.9 given in [23] involve a separation system of the form $\vec{S} = \vec{S}_k$ consisting of all separations of order $< k$ within some ambient universe $\vec{U}$, endowed with a symmetric and submodular order function. In most situations, submodularity is only used to ensure that at least one of any two opposite corners of two separations s, t of order $< k$ again has order $< k$ – which is tantamount to saying that $\vec{S}_k$ is (structurally) submodular, a fact that motivated this abstract notion in the first place.

The proof that $\vec{S}_k$ is separable (see [22, Lemma 3.4]), however, requires a more subtle use of the submodularity of the order function: the orders of the two corners are not compared to the fixed value k, but to the orders of s and t. This kind of argument is naturally difficult, if not impossible, to mimic in our set-up. As a consequence, separability was added as an additional assumption on the submodular separation system in [19, Theorem 3.9].

However, we can prove that every submodular separation system is in fact separable, thereby showing that this additional assumption may be removed:

Lemma 8.13. *Let L be a finite lattice and $M \subseteq L$ submodular. Then M is strongly separable.*

This lemma allows further applications of Theorem 8.9 beyond the present context of abstract tangles. For instance, we will make use of it in Section 8.4.2 to show that the separation system of clique separations of a graph is separable.

Proof. Call a pair $(a, b) \in M \times M$ *bad* if $a \le b$ and there is no $x \in M$ that lifts a and pushes b. Assume for a contradiction that there was a bad pair and choose one, say (a, b), such that $I(a, b) := \{u \in M : a \le u \le b\}$ is minimal.

We claim that a pushes every $z \in I(a, b) \setminus \{b\}$. Indeed, assume for a contradiction that there was some such z which a did not push. Since $I(a, z) \subsetneq I(a, b)$, it follows from our choice of the pair (a, b) that the pair (a, z) is not bad. Hence there exists some $x \in M$ which lifts a and pushes z. By assumption, $x \neq a$, so that $I(x, b) \subsetneq I(a, b)$. Again, by choice of (a, b), the pair (x, b) is not bad, yielding a $y \in M$ which lifts x and pushes b. By transitivity, it follows that y lifts a. But then (a, b) is not a bad pair, which is a contradiction. An analogous argument establishes that b lifts every $z \in I(a, b) \setminus \{a\}$.

Since (a, b) is bad, a does not push b, so there is some $x \in M$ with $x \leq b$ for which $a \wedge x \notin M$. Similarly, there is a $y \in M$ with $y \geq a$ for which $b \vee y \notin M$. Since M is submodular, it follows that $a \vee x, b \wedge y \in M$. Note that $a \vee x, b \wedge y \in I(a, b)$. Furthermore, $x \leq a \vee x$ and $a \wedge x \notin M$, so a does not push $a \vee x$. We showed that a pushes every $z \in I(a, b) \setminus \{b\}$, so it follows that $a \vee x = b$. Similarly, we find that $b \wedge y = a$. But then

$$x \vee y = x \vee (a \vee y) = b \vee y \notin M,$$
$$x \wedge y = (x \wedge b) \wedge y = x \wedge a \notin M.$$

This contradicts the submodularity of M. $\square$

Theorem 8.14. *Let $\vec{S}$ be a submodular separation system in a distributive universe $\vec{U}$ and let $Q \subseteq \vec{U}$ down-closed. Then exactly one of the following holds:*

(i) *There exists a $\mathcal{T}_Q$-tangle of S.*

(ii) *There exists an S-tree over $\mathcal{T}_Q$.*

Proof. By Lemmas 8.12 and 8.13, $\vec{S}$ is $\mathcal{T}_Q$-separable. Since every trivial element is small and non-degenerate, $\mathcal{T}_Q$ is standard for $\vec{S}$. Hence Theorem 8.9 applies and yields the desired duality. $\square$

Our original aim was a duality theorem for abstract tangles, not for $\mathcal{T}^*$-tangles. However, as long as $\vec{S}$ contains no degenerate elements, these notions coincide:

Lemma 8.15. *Let $\vec{U}$ be a distributive universe of separations and let $\vec{S} \subseteq \vec{U}$ be a submodular separation system without degenerate elements. Then the $\mathcal{T}^*$-tangles are precisely the abstract tangles.*

Proof. Since $\mathcal{T}^* \subseteq \mathcal{T}$, every abstract tangle is also a $\mathcal{T}^*$-tangle. We only need to show that, conversely, every $\mathcal{T}^*$-tangle in fact avoids $\mathcal{T}$.

For $\sigma \in \mathcal{T}$, let $d(\sigma)$ be the number of pairs $\vec{s}, \vec{t} \in \sigma$ which are not nested. Let O be a consistent orientation of S and suppose O was not an abstract tangle. Choose $\mathcal{T} \ni \sigma \subseteq O$ such that $d(\sigma)$ is minimum and, subject to this, σ is inclusion-minimal. We will show that σ is indeed a star, thus showing that O is not a $\mathcal{T}^*$-tangle.

If σ contained two comparable elements, say $\vec{s} \leq \vec{t}$, then $\sigma' := \sigma \setminus \{\vec{s}\}$ satisfies $\sigma' \in \mathcal{T}$, $\sigma' \subseteq O$ and $d(\sigma') \leq d(\sigma)$, violating the fact that σ is inclusion-minimal. Hence σ is an antichain. Since $\vec{S}$ has no degenerate elements, it follows from the consistency of O that any two nested $\vec{s}, \vec{t} \in \sigma$ satisfy $\vec{s} \leq \overleftarrow{t}$. To show that σ is a star, it thus suffices to prove that any two elements are nested.

Suppose that σ contained two crossing separations, say $\vec{s}, \vec{t} \in \sigma$. By submodularity of $\vec{S}$, at least one of $\vec{s} \wedge \overleftarrow{t}$ and $\overleftarrow{s} \wedge \vec{t}$ lies in $\vec{S}$. By symmetry we may assume that $\vec{r} := \vec{s} \wedge \overleftarrow{t} \in \vec{S}$. Let $\sigma' := (\sigma \setminus \{\vec{s}\}) \cup \{\vec{r}\}$. Since O is consistent, $\vec{r} \leq \vec{s}$ and $r \neq s$, it follows that $\vec{r} \in O$ and so $\sigma' \subseteq O$ as well.

Let $\vec{w} = \bigvee(\sigma \setminus \{\vec{t}\})$. As $\vec{v} \vee \vec{w} = \bigvee \sigma$ is co-small, we can apply Lemma 8.10 with $\vec{u} = \vec{v} = \vec{t}$ to deduce that $\vec{t} \vee (\vec{w} \wedge \overleftarrow{t})$ is co-small as well. But

$$\vec{t} \vee (\vec{w} \wedge \overleftarrow{t}) = \vec{t} \vee \bigvee_{\vec{x} \in \sigma \setminus \{\vec{t}\}} (\vec{x} \wedge \overleftarrow{t}) \leq \bigvee \sigma',$$

so $\bigvee \sigma'$ is also co-small and $\sigma' \in \mathcal{T}$.

We now show that $d(\sigma') < d(\sigma)$. Since s and t cross, while r and t do not, it suffices to show that every $\vec{x} \in \sigma \setminus \{\vec{s}\}$ which is nested with $\vec{s}$ is also nested with $\vec{r}$. But for every such $\vec{x}$ we have $\vec{s} \leq \overleftarrow{x}$. Since $\vec{r} \leq \vec{s}$, we get $\vec{r} \leq \overleftarrow{x}$ as well, showing that r and x are nested. So in fact $d(\sigma') < d(\sigma)$, which is a contradiction. This completes the proof that σ is nested and therefore a star. $\square$

When $\vec{S}$ has no degenerate elements, the abstract tangles extending Q are precisely the $\mathcal{T}^*$-tangles extending Q (by Lemma 8.15), which are exactly the $\mathcal{T}_Q$-tangles. Therefore, Theorem 8.3 is an immediate consequence of Theorem 8.14.

8.4 Special cases and applications

8.4.1 Tangles in graphs and matroids

We briefly indicate how tangles in graphs and matroids can be seen as special cases of abstract tangles in separation systems. Tangles in graphs and hypergraphs were introduced by Robertson and Seymour in [58], but a good deal of the work is done in the setting of connectivity systems. Geelen, Gerards and Whittle [34] made this more explicit and defined tangles as well as the dual notion of branch-decompositions for connectivity systems, an approach that we will follow.

Let X be a finite set and $\lambda : 2^X \to \mathbb{Z}$ a map assigning integers to the subsets of X such that $\lambda(X \setminus A) = \lambda(A)$ for all $A \subseteq X$ and

$$\lambda(A \cup B) + \lambda(A \cap B) \leq \lambda(A) + \lambda(B)$$

for all $A, B \subseteq X$. The pair (X, λ) is then called a *connectivity system*.

Both graphs and matroids give rise to connectivity systems. For a given graph G, we can take $X := E(G)$ and define $\lambda(F)$ as the number of vertices of G incident with edges in both F and $E \setminus F$. Given a matroid M with ground-set X and rank-function r, we take λ to be the connectivity function $\lambda(A) := r(A) + r(X \setminus A) - r(X)$.

Now consider 2^X as a universe of separations with set-inclusion as the partial order and $A^* = X \setminus A$ as involution. For an integer k, the set $\vec{S}_k$ of all sets A with $\lambda(A) < k$ is then a submodular separation system. Let $Q := \{\emptyset\} \cup \{\{x\} \colon x \in X\}$ consist of the empty-set and all singletons of X and note that Q is down-closed.

A *tangle of order k* of (X, λ), as defined in [34], is then precisely an abstract tangle extending Q. It is easy to see that (X, λ) has a *branch-decomposition* of width $<k$ if and only if there exists an S_k-tree over $\mathcal{T}^* \cup \mathcal{F}_Q$. Theorem 8.3 then yields the classic duality theorem for tangles and branch-decompositions in connectivity systems, see [58, 34].

8.4.2 Clique Separations

We now describe a submodular separation system that is not derived from a submodular order function, and provide a natural set of stars for which Theorem 8.9 applies.

Let $G = (V, E)$ be a finite graph and $\vec{U}$ the universe of all separations of G, that is, pairs (A, B) of subsets of V with $V = A \cup B$ such that there is no edge between $A \setminus B$ and $B \setminus A$. Here the partial order is given by $(A, B) \leq (C, D)$ if and only if $A \subseteq C$ and $B \supseteq D$, and the involution is simply $(A, B)^* = (B, A)$. For $(A, B) \in \vec{U}$, we call $A \cap B$ the *separator* of (A, B). It is an *a-b-separator* if $a \in A \setminus B$ and $b \in B \setminus A$. We call $A \cap B$ a *minimal separator* if there exist $a \in A \setminus B$ and $b \in B \setminus A$ for which $A \cap B$ is an inclusion-minimal a-b-separator.

Recall that a *hole* in a graph is an induced cycle on more than three vertices. A graph is *chordal* if it has no holes.

Theorem 8.16 (Dirac [27])**.** *A graph is chordal if and only if every minimal separator is a clique.*

Let $\vec{S}$ be the set of all $(A, B) \in \vec{U}$ for which $G[A \cap B]$ is a clique. We call these the *clique separations*. Note that $\vec{S}$ is closed under involution and therefore a separation system. To avoid trivialities, we will assume that the graph G is not itself a clique. In particular, this implies that $\vec{S}$ contains no degenerate elements.

Lemma 8.17. *Let $s, t \in S$. At least three of the four corners of s and t are again in $\vec{S}$. In particular, $\vec{S}$ is submodular.*

Proof. Let $\vec{s} = (A, B)$ and $\vec{t} = (C, D)$. Since $G[A \cap B]$ is a clique and (C, D) is a separation, we must have $A \cap B \subseteq C$ or $A \cap B \subseteq D$, without loss of generality $A \cap B \subseteq C$.

Similarly, it follows that $C \cap D \subseteq A$ or $C \cap D \subseteq B$; we assume the former holds. For each corner other than $\vec{s} \wedge \vec{t} = (A \cap C, B \cup D)$, the separator is a subset of either $A \cap B$ or $C \cap D$ and therefore a clique. This proves our claim. $\square$

Suppose that the graph G contains a hole H. Then for every $(A, B) \in \vec{S}$, either $H \subseteq A$ or $H \subseteq B$. In this way, every hole H induces an orientation

$$O_H := \{(A, B) \in \vec{S} \colon H \subseteq B\}$$

of $\vec{S}$. We now describe these orientations as tangles over a suitable set of stars.

Let $\mathcal{F} \subseteq 2^{\vec{U}}$ be the set of all sets $\{(A_1, B_1), \dots (A_n, B_n)\} \subseteq \vec{U}$ for which $G[\bigcap B_i]$ is a clique (note that the graph without any vertices is a clique). As usual, we denote by $\mathcal{F}^*$ the set of all elements of $\mathcal{F}$ which are stars.

Theorem 8.18. *Let O be an orientation of S. Then the following are equivalent:*

(i) *O is an $\mathcal{F}^*$-tangle.*

(ii) *O is an $\mathcal{F}$-tangle.*

(iii) *There exists a hole H with $O = O_H$.*

It is easy to see that every orientation O_H induced by a hole H is an $\mathcal{F}$-tangle. To prove that, conversely, every $\mathcal{F}$-tangle is induced by a hole, we use Theorem 8.16 and an easy observation about clique-separators, Lemma 8.19 below. The proof that every $\mathcal{F}^*$-tangle is already an $\mathcal{F}$-tangle, the main content of Lemma 8.20 below, is similar to the proof of Lemma 8.15, but some care is needed to keep track of the separators of two crossing separations.

For a set $\tau \subseteq \vec{U}$, let $J(\tau) := \bigcap_{(A,B)\in\tau} B$ be the intersection of all the right sides of separations in τ, where $J(\emptyset) := V(G)$.

Lemma 8.19. *Let τ be a set of clique separations, $J = J(\tau)$ and $K \subseteq J$. Let $a, b \in J \setminus K$. If K separates a and b in $G[J]$, then it separates them in G.*

Proof. We prove this by induction on $|\tau|$, the case $\tau = \emptyset$ being trivial. Suppose now $|\tau| \geq 1$ and let $(X, Y) \in \tau$ arbitrary. Put $\tau' := \tau \setminus \{(X, Y)\}$ and $J' := J(\tau')$. Note that $J = J' \cap Y$. Let $G' := G[J']$ and $(X', Y') := (X \cap J', Y \cap J')$.

Then $K \subseteq J'$ and $a, b \in J' \setminus K$. Suppose K did not separate a and b in G' and let $P \subseteq J'$ be an induced a-b-path avoiding K. Since $G'[X' \cap Y']$ is a clique, P has at most two vertices in $X' \cap Y'$ and they are consecutive vertices along P. As $a, b \in Y'$ and (X', Y') is a separation of G', it follows that $P \subseteq Y'$. But then K does not separate a and b in $J = J' \cap Y$, contrary to our assumption.

Hence K separates a and b in G'. By inductive hypothesis applied to τ', it follows that K separates a and b in G. $\square$

Lemma 8.20. *Every $\mathcal{F}^*$-tangle is an $\mathcal{F}$-tangle and a regular profile.*

Proof. Let P be an $\mathcal{F}^*$-tangle. It is clear that P contains no co-small separation, since $\{(V, A)\} \in \mathcal{F}^*$ for every co-small $(V, A) \in \vec{S}$. Since P is consistent, it follows that P is in fact down-closed.

We now show that P is a profile. Let $(A, B), (C, D) \in P$ and assume for a contradiction that $(E, F) := ((A, B) \vee (C, D))^* \in P$. Recall that either $C \cap D \subseteq A$ or $C \cap D \subseteq B$.

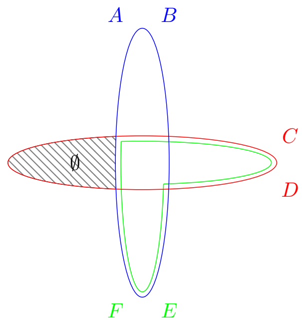

Figure 8.2: The case $C \cap D \subseteq B$

Suppose first that $C \cap D \subseteq B$; this case is depicted in Figure 8.2. Let $(X, Y) := (A, B) \wedge (D, C)$ and note that $X \cap Y \subseteq A \cap B$, so that $(X, Y) \in \vec{S}$. It follows from the consistency of P that $(X, Y) \in P$. Let $\tau := \{(C, D), (E, F), (X, Y)\}$ and observe that $\tau \subseteq P$ is a star. However

$$J(\tau) = D \cap (A \cup C) \cap (B \cup C) = (D \cap B) \cap (A \cup C),$$

which is the separator of (E, F). Since $(E, F) \in \vec{S}$, the latter is a clique, thereby contradicting the fact that P is an $\mathcal{F}^*$-tangle.

Suppose now that $C \cap D \subseteq A$. Let $(X, Y) := (B, A) \wedge (C, D)$ and note that $X \cap Y \subseteq A \cap B$, so that $(X, Y) \in \vec{S}$. Since P is down-closed, it follows that $(X, Y) \in P$. Therefore $\tau := \{(A, B), (E, F), (X, Y)\} \subseteq P$. But τ is a star and

$$J(\tau) = B \cap (A \cup C) \cap (A \cup D) = B \cap (A \cup (C \cap D)) = B \cap A$$

is a clique, which again contradicts our assumption that P is an $\mathcal{F}^*$-tangle. This contradiction shows that P is indeed a profile.

We now prove that for any $\tau \subseteq P$ there exists a star $\sigma \subseteq P$ with $J(\sigma) = J(\tau)$. It follows then, in particular, that P is an $\mathcal{F}$-tangle.

Given $\tau \subseteq P$, choose $\sigma \subseteq P$ with $J(\sigma) = J(\tau)$ such that $d(\sigma)$, the number of crossing pairs of elements of σ, is minimum and, subject to this, σ is inclusion-minimal. Then σ is an antichain: If $(A, B) \leq (C, D)$ and both $(A, B), (C, D) \in \sigma$, then $\sigma' := \sigma \setminus \{(A, B)\}$ satisfies $J(\sigma') = J(\sigma)$, thus violating the minimality of σ. Since $\sigma \subseteq P$ and P is consistent, no two elements of σ point away from each other. Therefore, any two nested elements of σ point towards each other. To verify that σ is a star, it suffices to check that σ is nested.

Assume for a contradiction that σ contained two crossing separations (A, B) and (C, D). If $(E, F) := (A, B) \vee (C, D) \in \vec{S}$, obtain σ' from σ by deleting (A, B) and (C, D) and adding (E, F). We have seen above that P is a profile, so $\sigma' \subseteq P$. By Lemma 8.6, every element of $\sigma \setminus \{(A, B), (C, D)\}$ that is nested with both (A, B) and (C, D) is also nested with (E, F). Since σ' misses the crossing pair $\{(A, B), (C, D)\}$, it follows that $d(\sigma') < d(\sigma)$. But $J(\sigma') = J(\sigma)$, contradicting the minimality of σ.

Hence it must be that $(E, F) \notin \vec{S}$, so $A \cap B \nsubseteq C$ and $C \cap D \nsubseteq A$. Therefore $(X, Y) := (A, B) \wedge (D, C) \in \vec{S}$. Let $\sigma' := (\sigma \setminus \{(A, B)\}) \cup \{(X, Y)\}$. Note that $(X, Y) \leq (A, B) \in P$, so $\sigma' \subseteq P$. Moreover $Y \cap D = (B \cup C) \cap D = B \cap D$, since $C \cap D \subseteq B$. Therefore $J(\sigma') = J(\sigma)$. As mentioned above, any $(U, W) \in \sigma \setminus \{(A, B)\}$ that is nested with (A, B) satisfies $(A, B) \leq (W, U)$. Therefore $(X, Y) \leq (A, B) \leq (W, U)$, so (X, Y) is also nested with (U, W). It follows that $d(\sigma') < d(\sigma)$, which is a contradiction. This completes the proof that σ is nested and therefore a star.

$\square$

Proof of Theorem 8.18. (i) $\to$ (ii): See Lemma 8.20.

(ii) $\to$ (iii): Let O be an $\mathcal{F}$-tangle and $J := J(O)$. We claim that there is a hole H of G with $H \subseteq J$. Such a hole then trivially satisfies $O_H = O$.

Assume there was no such hole, so that $G[J]$ is a chordal graph. Since O is $\mathcal{F}$-avoiding, $G[J]$ itself cannot be a clique, so there exists a minimal set $K \subseteq J$ separating two vertices $a, b \in J \setminus K$ in $G[J]$. By Theorem 8.16, K induces a clique in G. By Lemma 8.19, K separates a and b in G, so there exists a separation $(A, B) \in \vec{S}$ with $A \cap B = K$, $a \in A \setminus B$ and $b \in B \setminus A$. As O orients $\vec{S}$, it must contain one of $(A, B), (B, A)$, say without loss of generality $(A, B) \in O$. But then $J \subseteq B$, contrary to $a \in J$. This proves our claim.

(iii) $\to$ (i): We have $H \subseteq J(O_H)$, so $J(O_H)$ does not induce a clique.

$\square$

The upshot of Theorem 8.18 is that a hole in a graph, although a very concrete substructure, can be regarded as a tangle. This is in line with the idea, set forth in [24], that tangles arise naturally in very different contexts, and underlines the expressive strength of abstract separation systems and tangles.

What does our abstract theory then tell us about the holes in a graph? The results we will derive are well-known and not particularly deep, but it is nonetheless remarkable that the theory of abstract separation systems, emanating from the theory of highly connected substructures of a graph or matroid, is able to express such natural facts about holes.

First, by Lemma 8.20, every hole induces a profile of S. Hence Theorem 8.4 applies and yields a nested set $\mathcal{N}$ of clique-separations distinguishing all holes which can be separated by a clique. This is similar to, but not the same as, the decomposition by clique separators of Tarjan [63]: the algorithm in [63] essentially produces a maximal nested set of clique separations and leaves "atoms" that do not have any clique separations, whereas our tree set merely distinguishes the holes and leaves larger pieces that might allow further decomposition.

Second, we can apply Theorem 8.9 to find the structure dual to the existence of holes. It is clear that $\mathcal{F}^*$ is standard, since $\mathcal{F}^*$ contains $\{(V, A)\}$ for every $(V, A) \in \vec{S}$.

Lemma 8.21. *$\vec{S}$ is $\mathcal{F}^*$-separable.*

Proof. By Lemma 8.17 and Lemma 8.13, $\vec{S}$ is strongly separable. We show that $\mathcal{F}^*$ is closed under shifting.

So let $(X, Y) \in \vec{S}$ emulate a non-trivial $(U, W) \in \vec{S}$ with $\{(W, U)\} \notin \mathcal{F}^*$, let $\sigma = \{(A_i, B_i) \colon 0 \leq i \leq n\} \subseteq \vec{S}$ with $\sigma \in \mathcal{F}^*$ and $(U, W) \leq (A_0, B_0)$. Then

$$\sigma' := \sigma^{(X,Y)}_{(A_0,B_0)} = \{(A_0 \cup X, B_0 \cap Y)\} \cup \{(A_i \cap Y, B_i \cup X) \colon 1 \leq i \leq n\}.$$

By Lemma 8.8, $\sigma' \subseteq \vec{S}$ is a star; it remains to show that $J(\sigma')$ is a clique. We mimic the proof of [22, Lemma 6.1].

Let $(A, B) := \bigvee_{i \geq 1}(A_i, B_i)$ and note that $(A, B) \leq (B_0, A_0)$, since σ is a star. Then

$$(B, A) \wedge (V, B_0) = (B, B_0) \in \vec{U}.$$

But $B \cap B_0 = J(\sigma)$ is a clique, so in fact $(B, B_0) \in \vec{S}$. Since $(U, W) \leq (A_0, B_0) \leq (B, A)$, we see that $(U, W) \leq (B, B_0)$. As (X, Y) emulates (U, W) in $\vec{S}$, we find that $(E, F) := (X, Y) \vee (B, B_0) \in \vec{S}$. It thus follows that

$$J(\sigma') = (X \cup B) \cap (Y \cap B_0) = E \cap F$$

is indeed a clique. Therefore $\sigma' \in \mathcal{F}^*$. $\square$

Theorem 8.22. *Let G be a graph. Then the following are equivalent:*

(i) *G has a tree-decomposition in which every part is a clique.*

(ii) *There exists an S-tree over $\mathcal{F}^*$.*

(iii) *S has no $\mathcal{F}^*$-tangle.*

(iv) *G is chordal.*

Proof. (i) $\to$ (ii): Let $(T, \mathcal{V})$ be a tree-decomposition of G in which every part is a clique. For adjacent $s, t \in T$, let $\alpha(s,t) \in \vec{U}$ be the separation of G induced by (s,t). This defines a map $\alpha : \vec{E}(T) \to \vec{U}$ with $\alpha(s,t) = \alpha(t,s)^*$. The separator of $\alpha(s,t)$ is $V_s \cap V_t$, which is a clique by assumption. Hence (T, α) is in fact an S-tree. It is easy to see that $\alpha(F_t)$ is a star for every $t \in T$ and that $J(\alpha(F_t)) = V_t$. Therefore (T, α) is an S-tree over $\mathcal{F}^*$.

(ii) $\to$ (i): Given an S-tree (T, α) over $\mathcal{F}^*$, define $V_t := J(\alpha(F_t))$ for $t \in T$. It is easily verified that $(T, \mathcal{V})$ is a tree-decomposition of G. Each V_t is then a clique, since $\alpha(F_t) \in \mathcal{F}$.

(ii) $\leftrightarrow$ (iii): Follows from Theorem 8.9, since $\mathcal{F}^*$ is standard for $\vec{S}$ and $\vec{S}$ is $\mathcal{F}^*$-separable by Lemma 8.21.

(iii) $\leftrightarrow$ (iv): Follows from Theorem 8.18. $\square$

The equivalence of (i) and (iv) is a well-known characterization of chordal graphs that goes back to a theorem Gavril [33] which identifies chordal graphs as the intersection graphs of subtrees of a tree.

Bibliography

[1] P. Ali. “The Steiner diameter of a graph with prescribed girth”. *Discrete Math.* 313.12 (2013), pp. 1322–1326.

[2] P. Ali, P. Dankelmann, and S. Mukwembi. “Upper bounds on the Steiner diameter of a graph”. *Discrete Appl. Math.* 160.12 (2012), pp. 1845–1850.

[3] P. Ali, S. Mukwembi, and P. Dankelmann. “Steiner diameter of 3, 4 and 5-connected maximal planar graphs”. *Discrete Appl. Math.* 179 (2014), pp. 222–228.

[4] N. Alon, P. Seymour, and R. Thomas. “A separator theorem for nonplanar graphs”. *J. Amer. Math. Soc.* 3.4 (1990), pp. 801–808.

[5] P. Bellenbaum and R. Diestel. “Two short proofs concerning tree-decompositions”. *Comb., Probab. Comput.* 11 (2002), pp. 1–7.

[6] E. Birmelé, J. A. Bondy, and B. A. Reed. “The Erdős-Pósa property for long circuits”. *Combinatorica* 27.2 (2007), pp. 135–145.

[7] H. L. Bodlaender and D. M. Thilikos. “Treewidth for graphs with small chordality”. *Discrete Appl. Math.* 79.1-3 (1997). 4th Twente Workshop on Graphs and Combinatorial Optimization (Enschede, 1995), pp. 45–61.

[8] J. Cáceres, A. Márquez, and M. L. Puertas. “Steiner distance and convexity in graphs”. *European J. Combin.* 29.3 (2008), pp. 726–736.

[9] J. Carmesin and P. Gollin. “Canonical tree-decompositions of a graph that display its k-blocks”. *J. Combin. Theory Ser. B* 122 (2017), pp. 1–20.

[10] J. Carmesin, R. Diestel, M. Hamann, and F. Hundertmark. “Canonical tree-decompositions of finite graphs I. Existence and algorithms”. *J. Combin. Theory Ser. B* 116 (2016), pp. 1–24.

[11] J. Carmesin, R. Diestel, M. Hamann, and F. Hundertmark. “Canonical tree-decompositions of finite graphs II. Essential parts”. *J. Combin. Theory Ser. B* 118 (2016), pp. 268–283.

[12] J. Carmesin, R. Diestel, F. Hundertmark, and M. Stein. “Connectivity and tree structure in finite graphs”. *Combinatorica* 34.1 (2014), pp. 1–35.

[13] J. Carmesin, R. Diestel, M. Hamann, and F. Hundertmark. "k-Blocks: a connectivity invariant for graphs". *SIAM J. Discrete Math.* 28 (2014), pp. 1876–1891.

[14] G. Chartrand, O. R. Oellermann, S. L. Tian, and H.-B. Zou. "Steiner distance in graphs". *Časopis Pěst. Mat.* 114.4 (1989), pp. 399–410.

[15] C. Chekuri and J. Chuzhoy. "Polynomial bounds for the grid-minor theorem". *STOC'14—Proceedings of the 2014 ACM Symposium on Theory of Computing.* ACM, New York, 2014, pp. 60–69.

[16] M. Chudnovsky, A. Scott, and P. Seymour. "Induced Subgraphs of Graphs with Large Chromatic Number. III. Long Holes". *Combinatorica* 37.6 (2017), pp. 1057–1072.

[17] R. Diestel. "Abstract Separation Systems". *ArXiv e-prints* (2014). To appear in *Order*. arXiv: `1406.3797`.

[18] R. Diestel. *Graph Theory.* 5th. Springer, 2017.

[19] R. Diestel, J. Erde, and P. Eberenz. "Duality theorem for blocks and tangles in graphs". *SIAM J. Discrete Math.* 31.3 (2017), pp. 1514–1528.

[20] R. Diestel, J. Erde, and D. Weißauer. "Structural submodularity and tangles in abstract separation systems". *ArXiv e-prints* (2018). arXiv: `1805.01439`.

[21] R. Diestel, F. Hundertmark, and S. Lemanczyk. "Profiles of separations: in graphs, matroids and beyond". *ArXiv e-prints* (2011). To appear in *Combinatorica*. arXiv: `1110.6207`.

[22] R. Diestel and S.-i. Oum. "Tangle-tree duality: in graphs, matroids and beyond". *ArXiv e-prints* (2017). arXiv: `1701.02651`.

[23] R. Diestel and S. Oum. "Tangle-tree duality in abstract separation systems". *ArXiv e-prints* (2017). arXiv: `1701.02509`.

[24] R. Diestel and G. Whittle. "Tangles and the Mona Lisa". *ArXiv e-prints* (2016). arXiv: `1603.06652`.

[25] R. Diestel, K. Gorbunov, T. Jensen, and C. Thomassen. "Highly connected sets and the excluded grid theorem". *J. Combin. Theory (Series B)* 75 (1999), pp. 61–73.

[26] R. Diestel and M. Müller. "Connected tree-width". *Combinatorica* 38.2 (2018), pp. 381–398.

[27] G. A. Dirac. "On rigid circuit graphs". *Abh. Math. Sem. Univ. Hamburg* 25 (1961), pp. 71–76.

[28] Z. Dvořák. "A stronger structure theorem for excluded topological minors". *ArXiv e-prints* (2012). arXiv: 1209.0129.

[29] J. Erde and D. Weißauer. "A short derivation of the structure theorem for graphs with excluded topological minors". *ArXiv e-prints* (2018). arXiv: 1807.01119.

[30] P. Erdős. "Graph theory and probability". *Canad. J. Math.* 11 (1959), pp. 34–38.

[31] P. Erdős and L. Pósa. "On independent circuits contained in a graph". *Canad. J. Math.* 17 (1965), pp. 347–352.

[32] P. Fraigniaud and N. Nisse. "Connected Treewidth and Connected Graph Searching". *Proceedings of the 7th Latin American Conference on Theoretical Informatics.* LATIN'06. Springer-Verlag, 2006, pp. 479–490.

[33] F. Gavril. "The intersection graphs of subtrees in trees are exactly the chordal graphs". *J. Combinatorial Theory Ser. B* 16 (1974), pp. 47–56.

[34] J. Geelen, B. Gerards, N. Robertson, and G. Whittle. "Obstructions to branch-decomposition of matroids". *J. Combin. Theory (Series B)* 96 (2006), pp. 560–570.

[35] A. Georgakopoulos and P. Sprüssel. "Geodetic topological cycles in locally finite graphs". *Electronic J. Comb.* 16 (2009), #144.

[36] P. Gollin and K. Heuer. *Geodesic cuts.* In preparation.

[37] T. Gologranc. "Graphs with 4-Steiner convex balls". *Taiwanese J. Math.* 19.5 (2015), pp. 1325–1340.

[38] M. Grohe, S. Kreutzer, R. Rabinovich, S. Siebertz, and K. Stavropoulos. "Colouring and Covering Nowhere Dense Graphs". *ArXiv e-prints* (2016). arXiv: 1602.05926.

[39] M. Grohe and D. Marx. "Structure Theorem and Isomorphism Test for Graphs with Excluded Topological Subgraphs". *SIAM Journal on Computing* 44.1 (2015), pp. 114–159.

[40] A. Gyárfás. "Problems from the world surrounding perfect graphs". *Proceedings of the International Conference on Combinatorial Analysis and its Applications (Pokrzywna, 1985).* Vol. 19. 3-4. 1987, 413–441 (1988).

[41] M. Hamann and D. Weißauer. "Bounding connected tree-width". *SIAM J. Discrete Math.* 30.3 (2016), pp. 1391–1400.

[42] P. Jegou and C. Terrioux. "Tree-decompositions with connected clusters for solving constraint networks". *CP 2014*. Ed. by B. O'Sullivan. Vol. 8656. Springer Lecture Notes in Computer Science, 2014, pp. 403–407.

[43] E. J. Kim and O. Kwon. "Erdős-Pósa property of chordless cycles and its applications". *Proceedings of the Twenty-Ninth Annual ACM-SIAM Symposium on Discrete Algorithms.* SIAM. 2018, pp. 1665–1684.

[44] T. Kloks and H. L. Bodlaender. *Only few graphs have bounded treewidth.* Department of Computer Science, Utrecht University, 1992.

[45] A. Kosowski, B. Li, N. Nisse, and K. Suchan. "k-chordal graphs: from cops and robber to compact routing via treewidth". *Automata, languages, and programming. Part II.* Vol. 7392. Lecture Notes in Comput. Sci. Springer, Heidelberg, 2012, pp. 610–622.

[46] T. Kövari, V. T. Sós, and P. Turán. "On a problem of K. Zarankiewicz". *Colloquium Math.* 3 (1954), pp. 50–57.

[47] D. Kühn. "Cycles, minors and trees". PhD thesis. Hamburg University, 2001.

[48] D. Kühn and D. Osthus. "Induced subdivisions in $K_{s,s}$-free graphs of large average degree". *Combinatorica* 24.2 (2004), pp. 287–304.

[49] R. J. Lipton and R. E. Tarjan. "A separator theorem for planar graphs". *SIAM J. Appl. Math.* 36.2 (1979), pp. 177–189.

[50] C.-H. Liu and R. Thomas. "Excluding subdivisions of bounded degree graphs". *ArXiv e-prints* (2014). arXiv: `1407.4428`.

[51] W. Mader. "Über n-fach zusammenhängende Eckenmengen in Graphen". *J. Combin. Theory (Series B)* 25 (1978), pp. 74–93.

[52] F. Mousset, A. Noever, N. Škorić, and F. Weissenberger. "A tight Erdős-Pósa function for long cycles". *J. Combin. Theory Ser. B* 125 (2017), pp. 21–32.

[53] J.-F. Raymond and D. M. Thilikos. "Recent techniques and results on the Erdős-Pósa property". *Discrete Appl. Math.* 231 (2017), pp. 25–43.

[54] D. Richerby and D. M. Thilikos. "Searching for a visible, lazy fugitive". *SIAM J. Discrete Math.* 25.2 (2011), pp. 497–513.

[55] N. Robertson, P. Seymour, and R. Thomas. "Quickly Excluding a Planar Graph". *J. Combin. Theory (Series B)* 62 (1994), pp. 323–348.

[56] N. Robertson and P. Seymour. "Graph Minors. II. Algorithmic Aspects of Tree-Width". *J. Algorithms* 7 (1986), pp. 309–322.

[57] N. Robertson and P. Seymour. "Graph Minors. V. Excluding a Planar Graph". *J. Combin. Theory (Series B)* 41 (1986), pp. 92–114.

[58] N. Robertson and P. Seymour. "Graph Minors. X. Obstructions to Tree-Decomposition". *J. Combin. Theory (Series B)* 52 (1991), pp. 153–190.

[59] N. Robertson and P. Seymour. "Graph Minors. XIII. The Disjoint Paths Problem". *J. Combin. Theory (Series B)* 63 (1995), pp. 65–110.

[60] N. Robertson and P. Seymour. "Graph Minors. XVI. Excluding a non-planar graph". *J. Combin. Theory (Series B)* 89 (2003), pp. 43–76.

[61] P. Seymour and R. Thomas. "Graph Searching and a Min-Max Theorem for Tree-Width". *J. Combin. Theory (Series B)* 58.1 (1993), pp. 22–33.

[62] P. Seymour. "Tree-chromatic number". *J. Combin. Theory (Series B)* 116 (2016), pp. 229–237.

[63] R. Tarjan. "Decomposition by clique separators". *Discrete mathematics* 55.2 (1985), pp. 221–232.

[64] R. Thomas. "A Menger-like property of tree-width; the finite case". *J. Combin. Theory (Series B)* 48 (1990), pp. 67–76.

[65] C. Thomassen. "On the presence of disjoint subgraphs of a specified type". *J. Graph Theory* 12.1 (1988), pp. 101–111.

[66] D. Weißauer. "Algebraically grid-like graphs have large tree-width". *ArXiv e-prints* (2018). arXiv: `1802.05158`.

[67] D. Weißauer. "In absence of long chordless cycles, large tree-width becomes a local phenomenon". *ArXiv e-prints* (2018). arXiv: `1803.02703`.

[68] D. Weißauer. "On the block number of graphs". *ArXiv e-prints* (2017). arXiv: `1702.04245`.

[69] D. Weißauer. "Steiner trees and higher geodecity". *ArXiv e-prints* (2017). arXiv: `1703.09969`.

Appendix

Summary

In this dissertation, we study various notions of tangles and decompositions over trees and prove structure theorems for graphs excluding specific types of tangles. Each type of tangle or tree considered may be regarded as a way of giving a precise meaning to intuitive concepts of structural complexity and cohesion of a graph. The thesis may be loosely divided into three parts.

The first part, consisting of Chapters 2-4, highlights metric aspects of graphs in relation to their structural complexity. In the second part, which comprises Chapters 5-7, we expand the theory of k-blocks and use it to derive structure theorems for three types of tangles. In Chapter 8, which constitutes the third and last part, we go beyond the realm of graphs and enter the sphere of abstract separation systems, taking steps towards an axiomatic theory of tangles. We now give a brief summary of each chapter.

In Chapter 2 we study tree-decompositions of small width in which every part induces a connected subgraph. We strengthen a structure theorem of Diestel and Müller [26], which identifies long geodesic cycles and large tree-width as the essential obstructions to such decompositions, and prove that our bound is best possible up to a constant factor. Diestel and Müller [26] conjectured an analogue of the tree-width duality theorem. We refute their conjecture by presenting an infinite family of counterexamples.

Chapter 3 starts with an observation concerning an immediate corollary of our work in Chapter 2. We strengthen this by proving a 'local' version of this corollary, which is in line with our geometric intuition. This leads to a notion of being 'algebraically grid-like' and shows that this property is essentially equivalent to having large tree-width.

In Chapter 4 we introduce a notion of higher geodecity based on the Steiner distance of a set of vertices. We prove that the naturally arising hierarchy collapses in certain situations. Our fundamental concept of shortcut trees then leads us to a family of classes of graphs, one for each tree. Since each of these classes is minor-closed, we thus reenter the realm of tangles which we left at the beginning of the chapter.

In Chapter 5 we pick up on a question raised by Carmesin, Diestel, Hundertmark and Stein [12] and study the relation between the block number and other parameters of

width. Our most fundamental result is a structure theorem for graphs without k-blocks. We also prove an upper bound on the block number of n-vertex graphs excluding a fixed topological minor. We then establish a close relationship between tree-width and the occurrence of blocks in some minor.

Proving a well-known conjecture of Gyárfás [40], Chudnovsky, Scott and Seymour [16] showed that the chromatic number of K_r-free ℓ-chordal graphs is bounded. In Chapter 6, we prove an analogue for tree-width in place of chromatic number, showing that every ℓ-chordal graph of sufficiently large tree-width contains a complete bipartite graph $K_{s,s}$. We apply this to prove an Erdős-Pósa type theorem for long chordless cycles in $K_{s,s}$-free graphs, complementing a negative result of Kim and Kwon [43].

In Chapter 7 we give a short and conceptually intuitive proof of a structure theorem of Grohe and Marx [39] for graphs excluding a fixed topological minor. Our proof is based on the theory of profiles of graphs and our structure theorem from Chapter 5 for graphs without k-block.

In Chapter 8, we take steps towards an abstract theory of tangles and prove a tangle-tree theorem and a tangle duality theorem in abstract separation systems based on a notion of structural submodularity, which replaces the common assumption of the existence of a suitable submodular order function.

Zusammenfassung

In dieser Dissertation untersuchen wir verschiedene Arten von Knäueln und Zerlegungen entlang von Bäumen und beweisen Struktursätze für Graphen, die bestimmte Arten von Knäueln nicht enthalten. Jede Art von Knäuel kann als Möglichkeit aufgefasst werden, intuitiven Konzepten struktureller Komplexität und Kohäsion eines Graphen eine präzise Bedeutung zu verleihen. Diese Arbeit lässt sich grob in drei Teile unterteilen.

Der erste Teil, bestehend aus Kapitel 2-4, handelt von metrischen Eigenschaften von Graphen in Relation zu ihrer strukturellen Komplexität. Im zweiten Teil, der Kapitel 5-7 umfasst, erweitern und vertiefen wir die Theorie der k-Blöcke und verwenden diese, um Struktursätze für drei verschiedene Arten von Knäueln zu beweisen. Kapitel 8, welches den dritten und letzten Teil darstellt, enthält weitere Schritte hin zu einer abstrakten, axiomatischen Theorie von Knäueln, losgelöst von ihrem graphentheoretischen Ursprung. Wir geben nun eine kurze Zusammenfassung der einzelnen Kapitel.

In Kapitel 2 untersuchen wir Baumzerlegungen geringer Weite, in denen jeder Teil einen zusammenhängenden Teilgraphen induziert. Wir verschärfen einen Struktursatz von Diestel und Müller [26], welcher lange isometrische Kreise und große Baumweite als die essentiellen Hindernisse für solche Zerlegungen identiziert, und beweisen, dass unsere Schranke bis auf einen konstanten Faktor bestmöglich ist. Diestel und Müller [26] vermuteten, dass für diese Baumzerlegungen ein Dualitätssatz analog zu Theorem 1.1 gelte. Wir präsentieren eine unendliche Familie von Gegenbeispielen zu dieser Vermutung.

Kapitel 3 beginnt mit der Beobachtung einer unmittelbaren Folge unserer Arbeit in Kapitel 2. Wir beweisen eine 'lokale' Version dieses Korollars, welche unsere geometrische Intuition bestätigt. Dies führt zu einem algebraischen Begriff von Ähnlichkeit zu einem Gittergraphen und zeigt, dass diese Eigenschaft im Wesentlichen äquivalent zu großer Baumweite ist.

In Kapitel 4 führen wir einen Begriff der höheren Isometrie ein. Wir beweisen, dass die auf natürliche Weise entstehende Hierarchie in gewissen Situationen kollabiert. Unser fundamentales Konzept eines Abkürzungsbaums führt uns dann zu einer Familie von Klassen von Graphen, je eine für jeden Baum. Da jede dieser Klassen unter Minorenbildung abgeschlossen ist, betreten wir damit wieder die Sphäre der Knäuel, welche wir zu Beginn des Kapitels verließen.

In Kapitel 5 greifen wir eine von Carmesin, Diestel, Hundertmark und Stein [12] aufgeworfene Frage auf und untersuchen die Beziehung der Blockzahl zu anderen Weiteparametern. Unser wichtigstes Resultat ist ein Struktursatz für Graphen ohne k-Block. Wir beweisen außerdem eine obere Schranke für die Blockzahl von Graphen, welche einen fixierten topologischen Minor nicht enthalten, in Abhängigkeit von ihrer Eckenzahl. Zuletzt decken wir den engen Zusammenhang auf zwischen Baumweite und dem Auftreten eines k-Blocks in einem Minor.

Chudnovsky, Scott und Seymour [16] haben kürzlich bewiesen, dass die chromatische Zahl K_r-freier ℓ-chordaler Graphen beschränkt ist, und damit eine Vermutung von Gyárfás [40] bestätigt. In Kapitel 6 beweisen wir ein Analogon für Baumweite anstelle von chromatischer Zahl und zeigen, dass jeder ℓ-chordale Graph hinreichend großer Baumweite einen vollständig bipartiten Graphen $K_{s,s}$ als Teilgraph enthält. Wir verwenden dies, um die Erdős-Pósa Eigenschaften langer induzierter Kreise in $K_{s,s}$-freien Graphen zu beweisen. Dies ergänzt ein negatives Resultat von Kim und Kwon [43].

In Kapitel 7 geben wir einen kurzen und konzeptuell intuitiven Beweis eines Struktursatzes von Grohe und Marx [39] für Graphen, die einen fixierten Graphen nicht als topologischen Minor enthalten. Unser Beweis basiert auf der Theorie von Profilen und unserem Struktursatz aus Kapitel 5 für Graphen ohne k-Block.

Kapitel 8 ist ein Schritt hin zu einer abstrakten Theorie von Knäueln. Wir beweisen einen Knäuel-Baum Satz und einen Knäuel-Dualität Satz für abstrakte Teilungssysteme. Zentral ist hierbei ein Begriff struktureller Submodularität, welcher die übliche Annahme der Existenz einer submodularen Ordnungsfunktion ersetzt.

Publications related to this thesis

The following articles are related to this thesis:

- Chapter 2 is based on [41].
- Chapter 3 is based on [66].
- Chapter 4 is based on [69].
- Chapter 5 is based on [68].
- Chapter 6 is based on [67].
- Chapter 7 is based on [29].
- Chapter 8 is based on [20].

Declaration on my contributions

The content of Chapter 2 is joint work with Matthias Hamann [41]. We developed the proof of Theorem 1.4 and the discussion of the example in Section 2.5 together; Matthias discovered the proof of Lemma 2.9, whereas the idea of the bookkeeping graphs was mine. I came up with a template of the counterexample presented in Section 2.6, but assigning the correct lengths to the numerous paths has been a non-trivial effort by both of us.

Chapter 3 grew out of a corollary of the work with Matthias Hamann [41]; this corollary had appeared in [41]. I discussed the possible 'local' version of this corollary with Matthias Hamann and together we unsuccessfully tried to come up with a proof in 2015. In 2017, I revisited the problem and proved Theorem 1.6 on my own.

Concerning Chapter 4, I am grateful to Pascal Gollin and Karl Heuer for spending a very strange week with me constructing the examples displayed in Figure 4.4 and trying desperately to come up with a construction for seven leaves. A few weeks afterwards, I proved Theorem 1.8, showing that no such construction exists. Sorry guys. Except for the examples in Figure 4.4, Chapter 4 is entirely my own work.

Chapter 5 is my own work, but I thank Joshua Erde for pointing out during lunch that k-lean tree-decompositions might help in a proof of Proposition 5.15.

Chapter 6 is entirely my own work.

Chapter 7 is joint work with Joshua Erde [29]. I came up with the idea of a proof of the structure theorem for graphs excluding a topological minor and discussed it with Josh. Many helpful discussions with Josh on the topic and several failed joint attempts helped to find the core of the problem. I finally discovered the two proofs presented here. Only the shorter of the two, given in Section 7.3, appeared in [29].

Chapter 8 is joint work with Reinhard Diestel and Joshua Erde [20]. When they invited me to join the project, they had already proved the tangle tree theorem (Theorem 1.17) and the only missing piece for a proof of the tangle duality theorem (Theorem 1.18) was separability of the separation system. I proved that every submodular separation system is separable (Lemma 8.13), thereby completing their proof. Moreover, I came up with my own proof of the tangle tree theorem, which is the one given in this thesis as well as in [20]. The application to clique separations in Section 8.4 is my own work.

Eidesstattliche Erklärung

Hiermit erkläre ich an Eides statt, dass ich die vorliegende Dissertation selbst verfasst und keine anderen als die angegebenen Hilfsmittel benutzt habe.
Darüber hinaus versichere ich, dass diese Dissertation nicht in einem früheren Promotionsverfahren eingereicht wurde.

Ort	Datum	Unterschrift