Local density properties of Andrásfai graphs and powers of Hamiltonian cycles in hypergraphs

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1 Introduction

1.1 Overview

In this thesis we will investigate some extremal and probabilistic questions, which we take a closer look at after a brief introduction to these fields. Throughout this thesis we will consider finite simple undirected graphs and hypergraphs $G = (V, E)$, where V is the vertex set and E is the edge set of G. We assume that the reader is familiar with basic notations and concepts of graph theory, like $\delta(G)$ and $\Delta(G)$ signifying the minimum and maximum degree. For an introduction to graph theory and any notation not defined in this thesis we refer to the textbooks [10, 11, 17].

Extremal graph theory and Mantel's theorem

Extremal graph theory studies the quantitative aspects of the dependence between structural graph properties and graph invariants. The origin of extremal graph theory is usually set in 1941 with the well known result by Turán [83], who investigated the dependence between the edge density and the existence of a clique of certain order as a subgraph. He showed that among all K_{r+1}-free n-vertex graphs the complete (almost) balanced r-partite graph $T(n, r)$ has the largest number of edges. The graph $T(n, r)$ is called Turán graph.

To study the structural properties of graphs that do not contain a certain subgraph is a central aspect of extremal graph theory. Turán's theorem is one example. Here we want to take a closer look to a special case of this theorem also known as Mantel's theorem [57]. In 1907 Mantel proved that any triangle-free graph on n vertices contains at most $\lfloor n^2/4 \rfloor$ edges. The extremal graph is the complete bipartite graph with partition classes of size $\lfloor n/2 \rfloor$ and $\lceil n/2 \rceil$. This fundamental statement of extremal graph theory is the result the local density problem investigated in this thesis traces back to.

Dirac's theorem and Pósa's conjecture

Other structural graph properties that are often studied concern the existence of certain spanning subgraphs, i.e. subgraphs that cover all vertices. Unlike before a bound on the edge density is usually not very informative, because a graph can have $\binom{n}{2} - (n-1)$ edges and still contain an isolated vertex. Instead we could study which minimum vertex degree conditions imply the existence of a certain kind of spanning subgraph.

One of the first results of this type was proven by G. A. Dirac [18] in 1952. He showed that every graph $G = (V, E)$ with $|V| \geqslant 3$ and minimum vertex degree $\delta(G) \geqslant |V|/2$ contains a Hamiltonian cycle, that is a cycle containing all vertices. Since on any set V of at least three vertices there are graphs G with minimum degree $\delta(G) = \lceil |V|/2 \rceil - 1$, which do not contain a Hamiltonian cycle, this is an optimal result.

Another spanning structure that was studied is the k-th power of a Hamiltonian cycle. The k-th power of a Hamiltonian cycle C is obtained from C by adding all edges between distinct vertices of distance at most k in C.

In 1962 Pósa [23] conjectured that every graph $G = (V, E)$ with $|V| \geqslant 5$ and minimum degree $\delta(G) \geqslant 2|V|/3$ contains the square of a Hamiltonian cycle, that is the 2-nd power of a Hamiltonian cycle. This conjecture was generalised further by Seymour to the so-called Pósa-Seymour conjecture [78], asking for the k-th power of a Hamiltonian cycle in graphs G with $\delta(G) \geqslant \frac{k}{k+1}|V|$.

A proof of this generalised conjecture for large graphs was obtained by Komlós, Sárközy, and Szemerédi [46]. Their proof is based on the regularity method for graphs and uses the so-called blow-up lemma [45] that was developed by the same authors shortly before.

Regularity method

A conjecture of Erdős and Turán [27] about the upper density of subsets of the integers which contain no arithmetic progression of fixed length stimulated a lot of research in various fields of mathematics. First results concerning this conjecture were obtained by Roth [75, 76] for arithmetic progressions of length 3, his result is a special case of Theorem 1 with $\ell = 3$. The case $\ell = 4$, that means Szemerédi's theorem for arithmetic progressions of length 4, was established by Szemerédi [79] in 1969. Finally in 1975 Szemerédi [80] solved the conjecture, showing the following theorem.

Theorem 1 (Szemerédi's theorem). *For every $\ell \geqslant 3$ and $\delta > 0$ there exists $n_0 = n_0(\ell, \delta)$ such that if $A \subseteq [n] = \{1, \dots, n\}$ with $n \geqslant n_0$ and $|A| \geqslant \delta n$, then A contains an arithmetic progression of length ℓ.*

Later alternative proofs with different mathematical background appeared by Furstenberg [29], Tao [82], and Gowers [31].

Szemerédi established in his proof a lemma analysing the structure of dense bipartite graphs that later gave rise to the development of a powerful tool in extremal graph theory called Szemerédi's Regularity Lemma [81]. Under appropriate circumstances it can be used to show the existence of a fixed subgraph in a graph. The lemma shows that the edge set of any graph can be decomposed into constantly many "blocks" such that almost all are "quasirandom". We will make this precise in the following.

For a graph $G = (V, E)$ and two disjoint sets $A, B \subseteq V$, let $e(A, B)$ denote the number of edges in G with one vertex in A and one in B. Moreover, we call $d(A, B) = e(A, B)/(|A||B|)$ the *density* of the bipartite subgraph $G[A, B]$ of G consisting of all edges in G with one vertex in A and one in B. Given a graph $G = (V, E)$ and $\varepsilon > 0$ we say two non-empty disjoint subsets $X, Y \subseteq V$ are *ε-regular* if

$$|d_G(X, Y) - d_G(X', Y')| < \varepsilon$$

holds for all subsets $X' \subseteq X$, $Y' \subseteq Y$ with $|X'||Y'| > \varepsilon|X||Y|$. Szemerédi's Lemma is then stated as follows.

Theorem 2 (Szemerédi's Regularity Lemma). *For every $\varepsilon > 0$ and integer t_0, there exists integers $T_0 = T_0(\varepsilon, t_0)$ and $n_0 = n_0(\varepsilon, t_0)$ such that for every graph $G = (V, E)$ with $|V| = n \geqslant n_0$ the following holds.*

There exists a vertex partition $V_1 \cup \ldots \cup V_t = V$, $t_0 \leqslant t \leqslant T_0$, satisfying

(i) $|V_1| \leqslant \ldots \leqslant |V_t| \leqslant |V_1| + 1$, *and*

(ii) *all but at most $\varepsilon\binom{t}{2}$ pairs (V_i, V_j), $1 \leqslant i < j \leqslant t$, are ε-regular.*

Often Szemerédi's Regularity Lemma is used together with the so-called Counting Lemma we state in the following.

Theorem 3 (Counting Lemma)**.** *For all $d > 0$, $\gamma > 0$ and every positive integer ℓ, there exist $\varepsilon > 0$ and n_0 so that whenever G is an ℓ-partite graph with ℓ-partition $V_1 \cupdot \ldots \cupdot V_\ell$, and $|V_1| = \ldots = |V_\ell| = n \geqslant n_0$, satisfying for all $1 \leqslant i < j \leqslant \ell$*

(i) $d_G(V_i, V_j) = d \pm \varepsilon$, and

(ii) (V_i, V_j) is ε-regular

then the number $|\mathcal{K}_\ell(G)|$ of ℓ-cliques in G satisfies $|\mathcal{K}_\ell(G)| = d^{\binom{\ell}{2}} n^\ell (1 \pm \gamma)$

The joint application of Szemerédi's Regularity Lemma and the Counting Lemma is called the regularity method. The original proof of Roth's theorem about arithmetic progressions of length 3 by Ruzsa and Szemerédi [77] used Szemerédi's precursor of the Regularity Lemma in an iterative way, which can nowadays be replaced by a single application of the regularity method. They showed that every graph G_n on n vertices having $o(n^3)$ triangles contains a triangle-free subgraph G'_n having only $o(n^2)$ edges less. This result is known as the Triangle Removal Lemma and implies Roth's theorem.

A *k-uniform hypergraph*, or short *k-graph*, $H = (V, E)$ consists of a finite set $V(H)$ of vertices and a family $E = E(H)$ of k-element subsets of V, which are called *(hyper)edges*. A Regularity Lemma for 3-graphs has been developed by Frankl and Rödl [28] and also extensions to k-graphs were obtained by Gowers [32, 33] and by Rödl and Skokan [74]. Moreover, a Counting Lemma for k-graphs was proven by Nagle, Rödl, and Schacht [61]. In Chapter 3 we will use the regularity method to find an almost spanning squared cycle and therefore we will introduce the hypergraph regularity method in more detail in Section 3.5.2.

The random graph

The random graph on n vertices, where each edge is included with probability p, is denoted by $\mathbb{G}(n, p)$. In the beginning of probabilistic combinatorics this graph was only used as a tool in proofs, but later on it evolved to a subject studied on its own. For comprehensive accounts of random graph theory we refer to the textbooks [9] and [39]. One of the first uses of $\mathbb{G}(n, p)$ is due to Erdős [22], who used random graphs to show the existence of graphs that contain no short cycle and have a high chromatic number. In the context of random graphs we will often say that an event happens *asymptotically almost surely*, or a.a.s., if it happens with probability tending to 1 as $n \to \infty$. Besides studying the random graph on its own also randomly perturbed graphs, that are graphs obtained by adding random edges to a fixed graph, became a subject of research. First research concerning randomly perturbed graphs can be found in the work of Bohman, Frieze, and Martin [8]. We will study perturbed hypergraphs and their Hamiltonicity in Section 1.4.

1.2 Local density problems

Remember that Mantel's theorem asked for the maximum number of edges a triangle-free graph can have. Generalising this question it has been asked for a "local" density condition that guarantees the existence of a triangle. We will make this precise in the following.

We say an n-vertex graph G is (α, β)-*dense* if every subset of $\lfloor \alpha n \rfloor$ vertices spans more than βn^2 edges. Given $\alpha \in (0, 1]$ Erdős, Faudree, Rousseau, and Schelp [26] asked for the minimum $\beta = \beta(\alpha)$ such that every (α, β)-dense graph contains a triangle. For example, Mantel's theorem asserts that $\beta(1) = 1/4$.

For $1/2 < \alpha \leqslant 1$ the balanced complete bipartite graph gives the lower bound

$$\beta(\alpha) \geqslant \frac{1}{4}(2\alpha - 1)\,,$$

by taking one of the parts completely and the missing $\alpha n - n/2$ vertices from the other part (see Figure 1.1). The next graph one considers for obtaining lower bound on the function $\beta(\cdot)$ is the so-called balanced blow-up of a 5-cycle. The general definition of this concept reads as follows.

Definition 4. *A homomorphism from a graph G into a graph F is a mapping of the vertex sets $\varphi\colon V(G) \to V(F)$ with the property $\{\varphi(x), \varphi(y)\} \in E(F)$ whenever $\{x, y\} \in E(G)$. If such a homomorphism exist, we say that G is homomorphic to F. Moreover, we say that G is a blow-up of a graph F if there exists a surjective homomorphism φ from G to F, but for any proper supergraph of G on the same vertex set the mapping φ is not a homomorphism into F anymore. A blow-up is* balanced *if the preimages $\varphi^{-1}(v)$ of all vertices $v \in V(F)$ have the same size.*

The balanced blow-up of the 5-cycle gives for $2/5 < \alpha \leqslant 3/5$ the lower bound

$$\beta(\alpha) \geqslant \frac{1}{25}(5\alpha - 2)\,,$$

by taking two mutually independent parts of the blow-up completely and the missing $\alpha n - 2n/5$ vertices from one part, which has only edges to one of the parts that we already chose. Furthermore, for $3/8 < \alpha \leqslant 1/2$ the balanced blow up of the Andrásfai graph F_3 (see Figure 1.2) gives the lower bound

$$\beta(\alpha) \geqslant \frac{1}{64}(8\alpha - 3)\,,$$

by taking three mutually independent parts of the blow-up completely and the missing $\alpha n - 3n/8$ vertices from one part, which has only edges to one of the parts that we already chose.

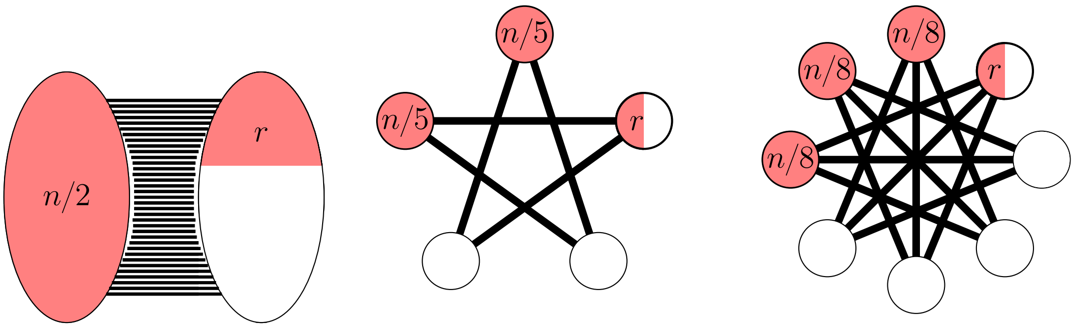

Figure 1.1: Balanced complete bipartite graph with $r = \alpha n - n/2$, balanced blow-up of a 5-cycle and the graph F_3 where $r = \alpha n - 2n/5$ in the first case and $r = \alpha n - 3n/8$ in the second.

Since

$$\frac{1}{4}(2\alpha - 1) \geqslant \frac{1}{25}(5\alpha - 2)$$

is only true for $\alpha \geqslant 17/30$, Erdős et al. conjectured that for $\alpha \geqslant 17/30$ the balanced complete bipartite graph gives the best lower bound for the function $\beta(\alpha)$, which leads to

$$\beta(\alpha) = \frac{1}{4}(2\alpha - 1)\,. \tag{1.1}$$

The same authors verified this conjecture for $\alpha \geqslant 0.648$ and the best result in this direction is due to Krivelevich [49], who verified it for every $\alpha \geqslant 3/5$. For $\alpha < 17/30$ balanced blow-ups of the 5-cycle yield a better lower bound for $\beta(\alpha)$ and Erdős et al. conjectured

$$\beta(\alpha) = \frac{1}{25}(5\alpha - 2) \tag{1.2}$$

for $\alpha \in [53/120, 17/30]$, since

$$\frac{1}{25}(5\alpha - 2) \geqslant \frac{1}{64}(8\alpha - 3)$$

is only true for $\alpha \geqslant 53/120$.

The special case $\beta(1/2) = 1/50$ was considered before by Erdős [24] (see also [25] for a monetary bounty for this problem).

Conjecture 5 (Erdős). *Every* $(1/2, 1/50)$*-dense graph contains a triangle.*

Currently, the best known upper bound on $\beta(1/2)$ is $1/36$ and was obtained by Krivelevich [49]. Besides the balanced blow-up of the 5-cycle Simonovits (see, e.g., [25]) noted that balanced blow-ups of the Petersen graph yield the same lower bound for Conjecture 5 and, more generally, for (1.2) in the corresponding range.

Conjecture 5 asserts that every triangle-free n-vertex graph G contains a subset of $\lfloor n/2 \rfloor$ vertices that spans at most $n^2/50$ edges. Our first result (see Theorem 6 below) verifies this for graphs G that are homomorpic to a triangle-free graph from a special class.

1.2.1 Andrásfai graphs

A well studied family of triangle-free graphs, which appear in the lower bound constructions for the function $\beta(\alpha)$ above, are the so-called *Andrásfai graphs*. For an integer $d \geqslant 1$ the Andrásfai graph F_d is the d-regular graph with vertex set

$$V(F_d) = \{v_1, \dots, v_{3d-1}\},$$

where $\{v_i, v_j\}$ forms an edge if

$$d \leqslant |i - j| \leqslant 2d - 1. \tag{1.3}$$

Note that $F_1 = K_2$ and $F_2 = C_5$ (see Figure 1.2). It is easy to check that Andrásfai graphs are triangle-free and balanced blow-ups of these graphs play a prominent rôle in connection with extremal problems for triangle-free graphs (see, e.g., [1, 15, 34, 40]).

Our first result validates Conjecture 5 (stated in the contrapositive) for graphs homomorphic to some Andrásfai graph.

Theorem 6. *If a graph G is homomorphic to an Andrásfai graph F_d for some integer $d \geqslant 1$, then G is not* $(1/2, 1/50)$*-dense.*

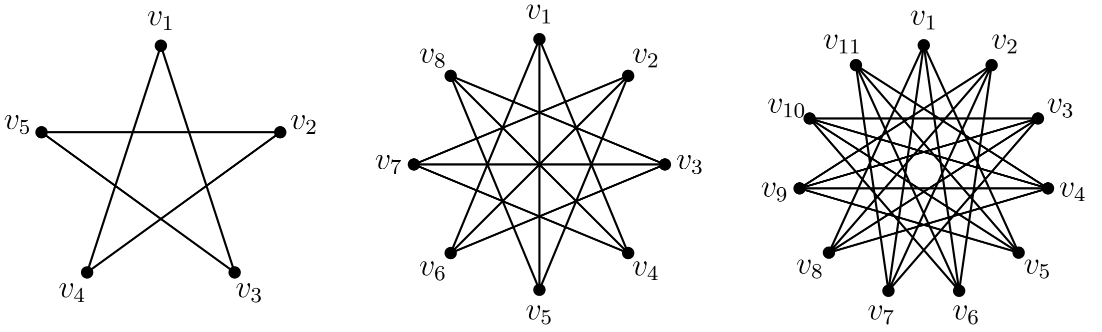

Figure 1.2: Andrásfai graphs F_2, F_3, and F_4.

Since F_d is homomorphic to $F_{d'}$ if and only if $d' \geqslant d$, Theorem 6 extends recent work of Norin and Yepremyan [63], who obtained such a result for n-vertex graphs G homomorphic to F_5 with the additional minimum degree assumption $\delta(G) \geqslant 5n/14$.

Owing to the work of Chen, Jin, and Koh [15], which asserts that every triangle-free 3-chromatic n-vertex graph G with minimum degree $\delta(G) > n/3$ is homomorphic to some Andrásfai graph, we deduce from Theorem 6 that Conjecture 5 holds for all such graphs G. Similarly, combining Theorem 6 with a result of Jin [40], which asserts that triangle-free graphs G with $\delta(G) > 10n/29$ are homomorphic to F_9, implies Conjecture 5 for those graphs as well. We summarise these direct consequences of Theorem 6 in the following corollary.

Corollary 7. *Let G be a triangle-free graph on n vertices.*

(*a*) *If $\delta(G) > 10n/29$, then G is not $(1/2, 1/50)$-dense.*

(*b*) *If $\delta(G) > n/3$ and $\chi(G) \leqslant 3$, then G is not $(1/2, 1/50)$-dense.*

We remark that part (*a*) slightly improves earlier results of Krivelevich [49] and of Norin and Yepremyan [63] (see also [44] where an average degree condition was considered).

1.2.2 Generalised Andrásfai graphs of higher odd-girth

We consider the following straightforward variation of Andrásfai graphs of *odd-girth* at least $2k+1$, i.e., graphs without odd cycles of length at most $2k-1$. For integers $k \geqslant 2$ and $d \geqslant 1$ let F_d^k be the d-regular graph with vertex set

$$V(F_d^k) = \{v_1, \ldots, v_{(2k-1)(d-1)+2}\},$$

where $\{v_i, v_j\}$ forms an edge if

$$(k-1)(d-1)+1 \leqslant |i-j| \leqslant k(d-1)+1\,. \tag{1.4}$$

In particular, for $k = 2$ we recover the definition of the Andrásfai graphs from (1.3) and for general $k \geqslant 2$ we have $F_1^k = K_2$, $F_2^k = C_{2k+1}$ and for every $d \geqslant 2$ the graph F_d^k has odd-girth $2k+1$ (see Figure 1.3).

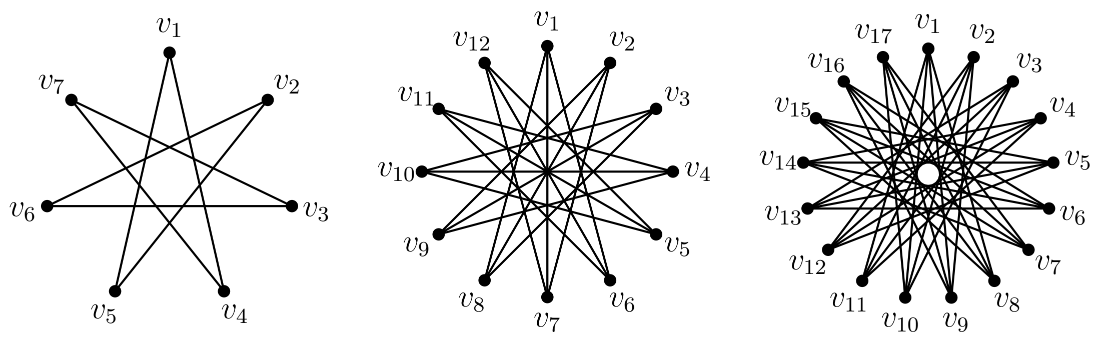

Figure 1.3: Generalised Andrásfai graphs F_2^3, F_3^3, and F_4^3 of odd-girth 7.

Our main result generalises Theorem 6 for graphs of odd-girth at least $2k+1$. In fact, the constant $\frac{1}{2(2k+1)^2}$ appearing in Theorem 8 is best possible as balanced blow-ups of C_{2k+1} show. One can attain this bound by taking k mutually independent parts of the blow-up completely and the missing $n/2 - kn/(2k+1)$ vertices from one part, which has only edges to one of the parts that we already chose.

Theorem 8. *If a graph G is homomorphic to a generalised Andrásfai graph F_d^k for some integers $k \geqslant 2$ and $d \geqslant 1$, then G is not $(\frac{1}{2}, \frac{1}{2(2k+1)^2})$-dense.*

Analogous to the relation between Conjecture 5 and Theorem 6 one may wonder if every $(\frac{1}{2}, \frac{1}{2(2k+1)^2})$-dense graph contains an odd cycle of length at most $2k-1$. Letzter and Snyder [55] showed that a graph G on n vertices with $\delta(G) > \frac{n}{5}$ and odd-girth at least 7 is homomorphic to F_k^3, for some k. Therefore combining this result with Theorem 8 we get the following.

Corollary 9. *Let G be a graph with odd-girth at least 7 on n vertices. If $\delta(G) > \frac{n}{5}$, then G is not $(\frac{1}{2}, \frac{1}{98})$-dense.*

A similar question for even holes is not interesting, because every dense graph contains a 4-cycle.

For $k = 2$ Theorem 8 reduces to Theorem 6 and the rest of this work concerns the proof of Theorem 8. The proof is given in Section 2.2 and makes use of a geometric representation of graphs homomorphic to generalised Andrásfai graphs, which we introduce in Section 2.1.

1.3 Squares of Hamiltonian cycles in 3-uniform hypergraphs

Recall Pósa's conjecture, which asked for a minimum degree condition that implies the existence of a 2-nd power of a Hamiltonian cycle in a graph. We study an analogous Pósa-type problem for 3-uniform hypergraphs, i.e., what minimum pair-degree condition guarantees the existence of a squared Hamiltonian cycle?

A 3-*uniform hypergraph* $H = (V, E)$ consists of a finite set $V(H)$ of vertices and a family $E = E(H)$ of 3-element subsets of V, which are called *(hyper)edges*. Throughout this section and in Chapter 3 if we talk about hypergraphs we will always mean 3-uniform hypergraphs. We will write xy and xyz instead of $\{x, y\}$ and $\{x, y, z\}$ for edges and hyperedges. Similarly, we shall say that $wxyz$ is a *tetrahedron* or a $K_4^{(3)}$ in a hypergraph H if the triples wxy, wxz, wyz, and xyz are edges of H.

There are at least two concepts of minimum degree and several notions of cycles like tight, loose and Berge cycles [6] (see also [7]). Here we will only introduce some of them.

Let $H = (V, E)$ be a hypergraph and $v \in V$ a vertex of H then we denote by

$$d_H(v) = |\{e \in E \colon v \in e\}|$$

the *degree of* v and by

$$\delta_1(H) = \min\{d_H(v) \colon v \in V\}$$

the *minimum vertex degree* of H taken over all $v \in V$.
Similarly, for two vertices $u, v \in V$ we denote by

$$d_H(u, v) = |N_H(u, v)| = |\{e \in E \colon u, v \in e\}|$$

the *pair-degree of* u *and* v and by

$$\delta_2(H) = \min\{d_H(u, v) \colon uv \in V^{(2)}\}$$

the *minimum pair-degree* of H taken over all pairs of vertices of H.

We call a hypergraph P a *tight path of length* ℓ, if $|V(P)| = \ell + 2$ and there exists an ordering of the vertices $V(P) = \{v_1, \dots, v_{\ell+2}\}$ such that a triple e forms a hyperedge of P iff $e = \{v_i, v_{i+1}, v_{i+2}\}$ for some $i \in [\ell]$. A *tight cycle C of length $\ell \geqslant 4$* consists of a path $v_1 \dots v_\ell$ of length $\ell - 2$ and the additional hyperedges $\{v_{\ell-1}, v_\ell, v_1\}$ and $\{v_\ell, v_1, v_2\}$. Moreover, we call a hypergraph P' a *squared path of length $\ell \geqslant 2$*, if $|V(P')| = \ell + 2$ and there exists an ordering of the vertices $V(P') = \{v_1, \dots, v_{\ell+2}\}$ such that a triple e forms a hyperedge iff $e \subset \{v_i, v_{i+1}, v_{i+2}, v_{i+3}\}$ for some $i \in [\ell - 1]$. Similarly, a *squared cycle C' of length $\ell \geqslant 5$* consists of a squared path $v_1 \dots v_\ell$ of length $\ell - 2$ and the additional hyperedges e, which are 3-subsets of at least one of the sets $\{v_{\ell-2}, v_{\ell-1}, v_\ell, v_1\}$, $\{v_{\ell-1}, v_\ell, v_1, v_2\}$ or $\{v_\ell, v_1, v_2, v_3\}$.

Thus an n-vertex hypergraph H contains a *spanning* squared cycle if its vertices can be arranged on a circle in such a way that every triple of vertices contained in an interval of length 4 is an edge of H. Such spanning squared cycles will be called *squared Hamiltonian cycles* in this article. Clearly this is a natural analogue of the concept of squared Hamiltonian cycles in graphs, where any pair contained in an interval of length 3 is required to be an edge.

The first asymptotically optimal Dirac-type result for 3-uniform hypergraphs was obtained by Rödl, Ruciński, and Szemerédi, who proved in [70] that every n-vertex hypergraph H with $\delta_2(H) \geqslant (\frac{1}{2} + o(1))n$ contains a Hamiltonian cycle. In [72] they showed this for large n under the optimal assumption $\delta_2(H) \geqslant \lfloor n/2 \rfloor$. Moreover, it was proved in [67] that a minimum vertex degree condition of $\delta_1(H) \geqslant (\frac{5}{9} + o(1))\frac{n^2}{2}$ guaranties the existence of a Hamiltonian cycle as well, where the constant $5/9$ is again best possible. We will study which pair-degree condition implies a squared Hamiltonian cycle in 3-uniform hypergraphs and we will prove the following theorem in Chapter 3.

Theorem 10. *For every $\alpha > 0$ there exists an integer n_0 such that every 3-uniform hypergraph H with $n \geqslant n_0$ vertices and with minimum pair-degree $\delta_2(H) \geqslant (\frac{4}{5} + \alpha)n$ contains a squared Hamiltonian cycle.*

We will denote by $K_4^{(3)}$ the complete 3-uniform hypergraph on 4 vertices. Note that any four consecutive vertices in a squared Hamiltonian cycle span a copy of $K_4^{(3)}$. Therefore, if n is divisible by 4, a squared Hamiltonian cycle contains a $K_4^{(3)}$-tiling, i.e., $\frac{n}{4}$ vertex disjoint copies of $K_4^{(3)}$. The problem to enforce $K_4^{(3)}$-tilings by an appropriate pair-degree condition was studied by Pikhurko [65], who exhibited for every n divisible by 4 a hypergraph H on n vertices with $\delta_2(H) = \frac{3}{4}n - 3$ not containing a $K_4^{(3)}$-tiling. Moreover, he proved that every n-vertex hypergraph H with $\delta_2(H) \geqslant \big(\frac{3}{4} + o(1)\big)n$ contains vertex-disjoint copies of $K_4^{(3)}$ covering all but at most 14 vertices. We remark that based on Pikhurko's work [65] the pair-degree problem for $K_4^{(3)}$-tilings was solved by Keevash and Mycroft in [43]. They showed that all 3-uniform hypergraphs H of sufficiently large order n with $4 \mid n$ and minimum pair-degree

$$\delta_2(H) \geqslant \begin{cases} 3n/4 - 2 & \text{if } 8 \mid n, \\ 3n/4 - 1 & \text{otherwise} \end{cases}$$

contain a perfect $K_4^{(3)}$-tiling.

Notice that in view of Pikhurko's example the constant $\frac{4}{5}$ occuring in Theorem 10 cannot be replaced by anything below $\frac{3}{4}$ in case $4 \mid n$. In order to extend this observation to all congruence classes modulo 4 we take a closer look at the construction from [65]. Partition the vertex set $V = A_0 \mathbin{\dot\cup} A_1 \mathbin{\dot\cup} A_2 \mathbin{\dot\cup} A_3$ such that $\big||A_i| - |A_j|\big| \leqslant 1$ for $0 \leqslant i < j \leqslant 3$. Let H be the hypergraph (see Figure 1.4) consisting of all the triples that satisfy one of the following properties

- have exactly two vertices in A_0,
- intersect each of A_0, A_i, A_j for some $1 \leqslant i < j \leqslant 3$,
- have three vertices inside some A_i with $1 \leqslant i \leqslant 3$,
- have two vertices in A_j and on vertex in A_i for $ij \in [3]^{(2)}$.

Every $K_4^{(3)}$ intersecting A_0 has exactly 2 vertices in A_0, since A_0 spans no edge and if a $K_4^{(3)}$ would intersect A_0 in only one vertex, then its remaining three vertices must come from A_1, A_2, A_3 (one from each set), but three such vertices do not span an edge in H. A squared Hamiltonian cycle $C \subseteq H$ needs to contain at least one $K_4^{(3)}$ that intersects A_0, but then each $K_4^{(3)} \subseteq C$ needs to intersect A_0 in two vertices. This implies $|A_0| \geqslant n/2$, which contradicts our assumption and shows that H is indeed not containing a squared Hamiltonian cycle.

The proof of Theorem 10 is based on the *absorption method* developed by Rödl, Ruciński, and Szemerédi in [72]. In Section 3.1 we will discuss the general structure of the proof.

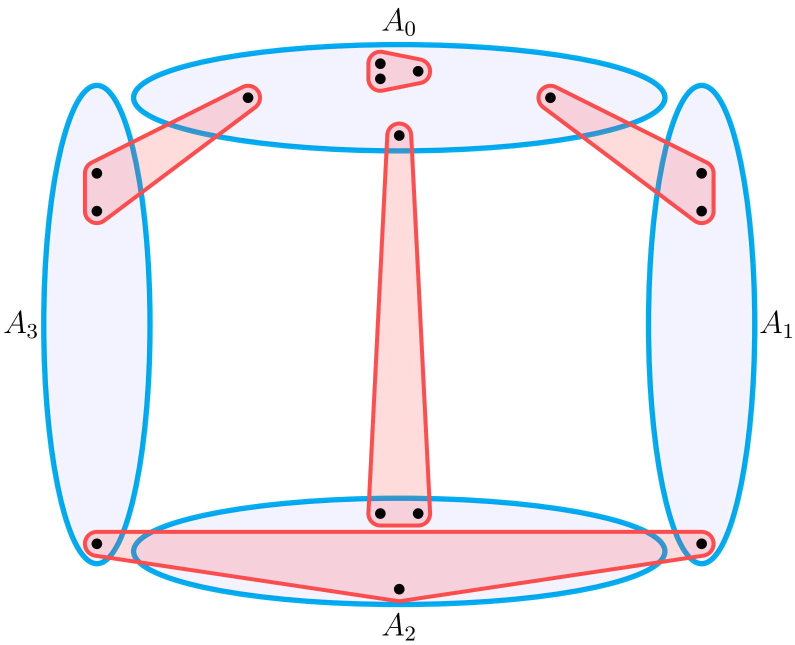

Figure 1.4: Complement of the hypergraph H, where the existing kinds of edges are indicated in red, e.g. all tripels with 3 vertices in A_0 span an edge in the complement of H.

1.4 Powers of tight Hamiltonian cycles in randomly perturbed hypergraphs

1.4.1 Hamiltonian cycles

The study of Hamiltonicity (the existence of a cycle as a spanning subgraph) has been a central and fruitful area in graph theory. It is likely that good characterizations of graphs with Hamiltonian cycles do not exist, and it becomes natural to study sufficient conditions that guarantee Hamiltonicity. Among a large variety of such results, recall that we already stated the most famous one, Dirac's theorem, which shows that every n-vertex graph ($n \geqslant 3$) with minimum degree at least $n/2$ is Hamiltonian.

Moreover, recall the binomial random graph $\mathbb{G}(n,p)$, which is another well-studied object in graph theory. Pósa [66] and Korshunov [48] independently determined the threshold for Hamiltonicity in $\mathbb{G}(n,p)$, which is $(\log n)/n$. This implies that almost all dense graphs are Hamiltonian. In this sense the degree constraint in Dirac's theorem is very strong. In fact, Bohman, Frieze, and Martin [8] studied the random graph model that starts with a given, dense graph and adds m random edges. In particular, they showed that for every $\alpha > 0$ there is $c = c(\alpha)$ such that if we start with a graph with minimum degree at least αn and we add cn random edges, then the resulting graph is Hamiltonian a.a.s.. By considering the complete bipartite graph with vertex classes of sizes αn and $(1-\alpha)n$, one sees that the result above is tight up to the value of c.

It is natural to study Hamiltonicity problems in uniform hypergraphs. Given a k-graph H with a set S of d vertices (where $1 \leqslant d \leqslant k-1$) we define $N_H(S)$ to be the collection of $(k-d)$-sets T such that $S \cup T \in E(H)$, and let $\deg_H(S) := |N_H(S)|$ (the subscript H is omitted whenever H is clear from the context). The *minimum d-degree* $\delta_d(H)$ of H is the minimum of $\deg_H(S)$ over all d-vertex sets S in H. We refer to $\delta_{k-1}(H)$ as the *minimum codegree* of H.

In the last two decades, there has been growing interest in extending Dirac's theorem to k-graphs. Among other notions of cycles in k-graphs (e.g., Berge cycles), the following 'uniform' cycles have attracted much attention. For integers $1 \leqslant \ell \leqslant k-1$ and $m \geqslant 3$, a k-graph F with $m(k-\ell)$ vertices and m edges is called an *ℓ-cycle* if its vertices can be ordered cyclically so that each of its edges consists of k consecutive vertices and every two consecutive edges (in the natural order of the edges) share exactly ℓ vertices. Usually $(k-1)$-cycles are also referred to as *tight* cycles. We say that a k-graph contains a *Hamiltonian ℓ-cycle* if it contains an ℓ-cycle as a spanning subgraph. In view of Dirac's theorem, minimum d-degree conditions that force Hamiltonian ℓ-cycles (for $1 \leqslant d,\ \ell \leqslant k-1$) have been studied intensively [3, 4, 14, 16, 30, 35–37, 42, 52, 53, 67, 69–72].

Let $\mathbb{G}^{(k)}(n,p)$ denote the binomial random k-graph on n vertices, where each k-tuple forms an edge independently with probability p. The threshold for the existence of Hamiltonian ℓ-cycles has been studied by Dudek and Frieze [19, 20], who proved that for $\ell = 1$ the threshold is $(\log n)/n^{k-1}$, and for $\ell \geqslant 2$ the threshold is $1/n^{k-\ell}$ (they also determined sharp thresholds for every $k \geqslant 4$ and $\ell = k-1$).

Krivelevich, Kwan, and Sudakov [50] considered randomly perturbed k-graphs, which are k-graphs obtained by adding random edges to a fixed k-graph. They proved the following theorem, which mirrors the result of Bohman, Frieze, and Martin [8] for randomly perturbed graphs mentioned earlier.

Theorem 11. *For any $k \geqslant 2$ and $\alpha > 0$, there is $c_k = c_k(\alpha)$ for which the following holds. Let H be a k-graph on $n \in (k-1)\mathbb{N}$ vertices with $\delta_{k-1}(H) \geqslant \alpha n$.*

If $p = c_k n^{-(k-1)}$, then the union $H \cup \mathbb{G}^{(k)}(n,p)$ asymptotically almost surely contains a Hamiltonian 1-cycle.

The authors of [50] also obtained a similar result for perfect matchings. These results are tight up to the value of c_k, as shown by a simple 'bipartite' construction. McDowell and Mycroft [58] and, subsequently, Han and Zhao [38] extended Theorem 11 to Hamiltonian ℓ-cycles and other degree conditions.

1.4.2 Powers of Hamiltonian cycles

Powers of cycles are natural generalizations of cycles. Given $k \geqslant 2$ and $r \geqslant 1$, we say that a k-graph with m vertices is an *r^{th} power of a tight cycle* if its vertices can be ordered cyclically so that each consecutive $k+r-1$ vertices span a copy of $K^{(k)}_{k+r-1}$, the complete k-graph on $k+r-1$ vertices, and there are no other edges than the ones forced by this condition. This extends the notion of (tight) cycles in hypergraphs, which corresponds to the case $r = 1$.

The existence of powers of paths and cycles has also been intensively studied. For example, the famous Pósa–Seymour conjecture, which was proved by Komlós, Sárközy, and Szemerédi [46,47] for sufficiently large graphs, states that every n-vertex graph with minimum degree at least $rn/(r+1)$ contains the r^{th} power of a Hamiltonian cycle. A general result of Riordan [68] implies that, for $r \geqslant 3$, the threshold for the existence of the r^{th} power of a Hamiltonian cycle in $\mathbb{G}(n,p)$ is $n^{-1/r}$. The case $r = 2$ was investigated by Kühn and Osthus [54], who proved that $p \geqslant n^{-1/2+\varepsilon}$ suffices for the existence of the square of a Hamiltonian cycle in $\mathbb{G}(n,p)$, which is sharp up to the n^{ε} factor. This was further sharpened by Nenadov and Škorić [62] to $p \geqslant C(\log n)^4/\sqrt{n}$. Moreover, Bennett, Dudek, and Frieze [5] proved a result for the square of a Hamiltonian cycle in randomly perturbed graphs, extending the result of Bohman, Frieze, and Martin [8].

Theorem 12. *For any $\alpha > 0$ there is $K > 0$ such that the following holds. Let G be a n-vertex graph with $\delta(G) \geqslant (1/2+\alpha)n$ and suppose $p = p(n) \geqslant Kn^{-2/3}\log^{1/3} n$. Then the union $H \cup \mathbb{G}(n,p)$ a.a.s. contains the square of a Hamiltonian cycle.*

Very recently Dudek, Reiher, Ruciński, and Schacht [21] obtained the following result.

Theorem 13. *For every $\alpha > 0$ and $k \geqslant 1$ there exists $C > 0$ such that if G is an n-vertex graph with $\delta(G) \geqslant (\frac{k-1}{k} + \alpha)n$, then $G \cup G(n, \frac{C}{n})$ a.a.s. contains the k^{th} power of a Hamiltonian cycle.*

Note that in Theorem 12 the randomness that is required is much weaker than the one needed in the result for the pure random model (which is essentially $n^{-1/2}$). The authors of [5] also asked for similar results for higher powers of Hamilton cycles in randomly perturbed graphs.

Parczyk and Person [64, Theorem 3.7] proved that, for $k \geqslant 3$ and $r \geqslant 2$, the threshold for the existence of an r^{th} power of a tight Hamilton cycle in the random k-graph $\mathbb{G}^{(k)}(n,p)$ is $n^{-\binom{k+r-2}{k-1}^{-1}}$. Our main result, Theorem 14 below, shows that if we consider randomly perturbed k-graphs $H \cup \mathbb{G}^{(k)}(n,p)$ with $\delta_{k-1}(H)$ reasonably large, then $p = p(n) \geqslant n^{-\binom{k+r-2}{k-1}^{-1}-\varepsilon}$ is enough to guarantee the existence of an r^{th} power of a tight Hamilton cycle with high probability.

Theorem 14 (Main result)**.** *For all integers $k \geqslant 2$ and $r \geqslant 1$ such that $k + r \geqslant 4$ and $\alpha > 0$, there is $\varepsilon > 0$ such that the following holds. Suppose H is a k-graph on n vertices with*

$$\delta_{k-1}(H) \geqslant \left(1 - \binom{k+r-2}{k-1}^{-1} + \alpha\right) n \tag{1.5}$$

and $p = p(n) \geqslant n^{-\binom{k+r-2}{k-1}^{-1}-\varepsilon}$. Then a.a.s. the union $H \cup \mathbb{G}^{(k)}(n,p)$ contains the r^{th} power of a tight Hamiltonian cycle.

We remark that our proof only gives a small ε, and it would be interesting to know if one can get a larger gap in comparison with the result in the purely random model, as in Theorem 12. We remark that the case $k \geqslant 3$ and $r = 1$ of Theorem 14 was first proved by McDowell and Mycroft [58]. Other results in randomly perturbed graphs can be found in [2, 12, 13, 38, 51].

The core of the proof of Theorem 14 follows the *Absorbing Method* introduced by Rödl, Ruciński, and Szemerédi in [70], combined with results concerning binomial random hypergraphs.

2 On the local density problem for graphs of given odd-girth

In this chapter we will prove the following result.

Theorem 15. *If a graph G is homomorphic to a generalised Andrásfai graph F_d^k for some integers $k \geqslant 2$ and $d \geqslant 1$, then G is not $(\frac{1}{2}, \frac{1}{2(2k+1)^2})$-dense.*

In Section 2.1 we show an alternative geometric characterisation of generalised Andrásfai graph and investigate some properties of this representation. The proof of Theorem 15 will be based on this geometric characterisation and is part of Section 2.2.

2.1 A geometric characterisation of generalised Andrásfai graphs

We consider graphs G that are homomorphic to some generalised Andrásfai graph F_d^k. For the proof of Theorem 15 it will be convenient to work with a geometric representation of such graphs G. In that representation we will arrange the vertices of G on the unit circle $\mathbb{R}/\mathbb{Z}$ and edges between two vertices x and y may only appear depending on their angle with respect to the centre of the circle (see Lemma 16). For the proof of Theorem 15 it suffices to consider edge maximal graphs G that are homomorphic to F_d^k for some integers $k \geqslant 2$ and $d \geqslant 1$. In other words, we may assume G is a blow-up of F_d^k.

For example, let G be a blow-up of $F_2 = C_5$. One can distribute the vertices of F_2 equally spaced on the unit circle (see Figure 2.1). Then we place all vertices of G that correspond to the blow-up class of v_i into a small ε-ball around v_i on the unit circle (cf. green arcs in Figure 2.1). For a sufficiently small ε, all vertices in an ε-ball around v_i have the same neighbours and they can be characterised by having their smaller angle with respect to the centre bigger than 120° (cf. red

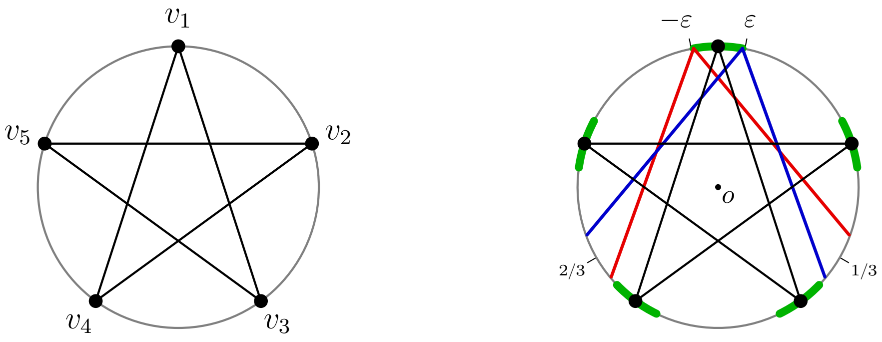

Figure 2.1: A copy of $F_2 = C_5$ and a representation of a blow-up on the unit circle.

and blue lines in Figure 2.1). The following lemma states this fact for blow-ups of generalised Andrásfai graphs.

Lemma 16. *If G is a blow-up of a generalised Andrásfai graph F_d^k for some integers $k \geqslant 2$ and $d \geqslant 1$, then the vertices of G can be arranged on the unit circle $\mathbb{R}/\mathbb{Z}$ with centre o such that*

$$\{x, y\} \in E(G) \qquad \Longleftrightarrow \qquad \sphericalangle xoy > \frac{k-1}{2k-1} \cdot 360^\circ\,, \tag{2.1}$$

where $\sphericalangle xoy$ denotes the smaller angle between x and y with respect to o.

We remark that conversely every graph $G = (V, E)$ with $V \subseteq \mathbb{R}/\mathbb{Z}$ satisfying (2.1) is a blow-up of F_d^k for some appropriate $d \geqslant 1$. However, since this direction is not utilised here, we omit the formal proof of this observation.

Proof of Lemma 16. For integers $k \geqslant 2$ and $d \geqslant 1$ let G be a blow-up of the generalised Andrásfai graph F_d^k (defined in (1.4)) signified by some graph homomorphism $\varphi\colon G \to F_d^k$ and let $m = (2k-1)(d-1) + 2$ be the number of vertices of F_d^k. Set

$$\varepsilon = \frac{1}{2(2k-1)m}\,.$$

For $i \in [m]$ we arrange the vertices of G that are contained in $\varphi^{-1}(v_i)$ in the ε-ball around the point $\frac{i-1}{m}$. Owing to the symmetry it suffices to check that (2.1) holds for an arbitrary vertex $x \in \varphi^{-1}(v_1) \subseteq V(G)$.

By definition of F_d^k the neighbourhood of v_1 is

$$N(v_1) = \{v_{(k-1)(d-1)+2}, \dots, v_{k(d-1)+2}\}\,.$$

Note that the choice of ε gives

$$\left(\tfrac{(k-1)(d-1)+1}{m}-\varepsilon,\tfrac{k(d-1)+1}{m}+\varepsilon\right)=\left(\tfrac{k-1}{2k-1}+\varepsilon,\tfrac{k}{2k-1}-\varepsilon\right)$$

and, consequently, all neighbours y of x are placed in the interval $(\frac{k-1}{2k-1}+\varepsilon,\frac{k}{2k-1}-\varepsilon)$. Since $x\in\varphi^{-1}(v_1)$ itself is placed in $(-\varepsilon,\varepsilon)$, this implies the forward direction of (2.1). The converse direction follows from the observation

$$\left(\tfrac{i-1}{m}-\varepsilon,\tfrac{i-1}{m}+\varepsilon\right)\cap\left(\tfrac{k-1}{2k-1}-\varepsilon,\tfrac{k}{2k-1}+\varepsilon\right)=\varnothing$$

for every $i\in[m]\smallsetminus\{(k-1)(d-1)+2,\dots,k(d-1)+2\}$. $\square$

We close this section with a few useful estimates on the number of vertices contained in intervals of $\mathbb{R}/\mathbb{Z}$ for geometric representations of blow-ups G of generalised Andrásfai graphs. Let V be the set of points of the unit circle that are identified with the vertices of G. For an interval $I\subseteq\mathbb{R}/\mathbb{Z}$, we write $\lambda(I)$ for the number of vertices of G contained in I, i.e.,

$$\lambda(I)=|V\cap I|\,. \tag{2.2}$$

This defines expressions such as $\lambda\big([a,b]\big)$, $\lambda\big([a,b)\big)$, etc.

Since subsets of $\lfloor n/2\rfloor$ vertices are of special interest, we denote for every $\xi\in\mathbb{R}/\mathbb{Z}$ by z_ξ the vertex from V with the property

$$\lambda\big([\xi,z_\xi]\big)=\lfloor n/2\rfloor\,. \tag{2.3}$$

In the proof of Theorem 15 we shall use the following lemma and, since the proof will be carried out by contradiction, the graphs G that we shall consider also satisfy the density assumption for parts *(iv)* and *(v)*.

Lemma 17. *For integers $k\geqslant 2$ and $d\geqslant 1$ let $G=(V,E)$ be a blow-up of the generalised Andrásfai graph F_d^k having a geometric representation with $V\subseteq\mathbb{R}/\mathbb{Z}$ satisfying (2.1) and $|V|=n$. Then the following holds for every interval $I\subseteq\mathbb{R}/\mathbb{Z}$:*

(i) If I has length at most $\frac{k-1}{2k-1}$, then $V\cap I$ is an independent set in G and $\lambda(I)\leqslant\alpha(G)$.

(ii) If I has length at most $\frac{1}{2k-1}$, then $\lambda(I)\leqslant(2k-3)\alpha(G)-(k-2)n$.

(iii) If I has length at least $\frac{1}{2k-1}$, then $\lambda(I)\geqslant n-2\alpha(G)$.

If in addition G is $(\frac{1}{2}, \frac{1}{2(2k+1)^2})$-dense and $2(2k+1) \mid n$, then the following holds for $\xi \in \mathbb{R}/\mathbb{Z}$:

(iv) If $\lambda\big([\xi, \xi + \frac{k-1}{2k-1}]\big) = \alpha(G)$, then $\lambda\big([\xi, z_\xi - \frac{k-1}{2k-1})\big) > 2\alpha(G) - \frac{2k-1}{2k+1}n$.

(v) We have $\lambda\big((\xi - \frac{1}{2k-1}, \xi + \frac{1}{2k-1})\big) > \frac{4}{2k+1}n - 2\lambda\big((\xi + \frac{k-1}{2k-1}, \xi - \frac{k-1}{2k-1})\big)$.

Proof. Part *(i)* follows directly from the definition of the geometric representation in (2.1). For part *(ii)* we note that

$$(2k-3)\frac{k-1}{2k-1} = (k-2) + \frac{1}{2k-1}\,.$$

Consequently, there exist $2k-3$ consecutive intervals of length $\frac{k-1}{2k-1}$ that wrap $k-2$ times around $\mathbb{R}/\mathbb{Z}$ in such a way that only I is covered $k-1$ times. Therefore, *(i)* yields

$$(2k-3)\alpha(G) \geqslant (k-2)n + \lambda(I)$$

and the desired estimate follows.

Part *(iii)* is also a consequence of *(i)* and the observation that there are two intervals of length at most $\frac{k-1}{2k-1}$ that together with I cover $\mathbb{R}/\mathbb{Z}$ once.

In the proofs of parts *(iv)* and *(v)* we make use of the inequality

$$\lambda\big([\xi, z_\xi - \tfrac{k-1}{2k-1})\big) > \frac{2n}{2k+1} - 2\lambda\big((\xi + \tfrac{k-1}{2k-1}, z_\xi]\big)\,, \tag{2.4}$$

which we show first. For that we note that (2.1) implies

$$e_G\big([\xi, z_\xi] \cap V\big) \leqslant \lambda\big([\xi, z_\xi - \tfrac{k-1}{2k-1})\big) \cdot \lambda\big((\xi + \tfrac{k-1}{2k-1}, z_\xi]\big)\,.$$

Hence, the additional assumption that G is $(\frac{1}{2}, \frac{1}{2(2k+1)^2})$-dense combined with the simplest case of the inequality between the arithmetic and geometric mean yields

$$\Big(\frac{n}{2k+1}\Big)^2 < 2e_G\big([\xi, z_\xi] \cap V\big) \leqslant \frac{1}{4}\Big(\lambda\big([\xi, z_\xi - \tfrac{k-1}{2k-1})\big) + 2\lambda\big((\xi + \tfrac{k-1}{2k-1}, z_\xi]\big)\Big)^2,$$

which establishes (2.4).

The remaining parts *(iv)* and *(v)* follow from (2.4). In fact, for *(iv)* the additional assumption $\lambda\big([\xi, \xi + \frac{k-1}{2k-1}]\big) = \alpha(G)$ yields $\lambda\big((\xi + \frac{k-1}{2k-1}, z_\xi]\big) = n/2 - \alpha(G)$ and, hence, *(iv)* follows from (2.4).

For the proof of *(v)* we will apply (2.4) twice. First we apply it for the given $\xi \in \mathbb{R}/\mathbb{Z}$ and, since by *(i)* we also have $z_\xi \in (\xi + \frac{k-1}{2k-1}, \xi + \frac{k}{2k-1})$, we obtain

$$\lambda\big([\xi, \xi + \tfrac{1}{2k-1})\big) \geqslant \lambda\big([\xi, z_\xi - \tfrac{k-1}{2k-1})\big) \overset{(2.4)}{>} \frac{2n}{2k+1} - 2\lambda\big((\xi + \tfrac{k-1}{2k-1}, z_\xi]\big)\,. \tag{2.5}$$

The second symmetric application of (2.4) in $-\mathbb{R}/\mathbb{Z}$ to $-\xi$ yields

$$\lambda\big((\xi - \tfrac{1}{2k-1}, \xi]\big) \overset{(2.4)}{>} \frac{2n}{2k+1} - 2\lambda\big([z'_\xi, \xi - \tfrac{k-1}{2k-1})\big)\,, \tag{2.6}$$

for $z'_\xi \in (z_\xi, \xi)$ with $\lambda\big([z'_\xi, \xi]\big) = n/2$. Consequently, if $\xi \notin V$ then summing the inequalities (2.5) and (2.6) yields part (v). However, if $\xi \in V$ then still the same conclusion follows, since $(2k+1) \mid n$ implies that the right-hand sides of (2.5) and (2.6) are integers and both inequalities are strict. □

2.2 Blow-ups of generalised Andrásfai graphs

In this section we establish Theorem 15. For that it suffices to show that blow-ups G of generalised Andrásfai graphs F^k_d are not $(\frac{1}{2}, \frac{1}{2(2k+1)^2})$-dense and we will appeal to the geometric representation from Lemma 16 of such graphs. The strategy of our proofs is that we try to find an interval of consecutive vertices spanning few edges. To this end we distinguish two cases depending on the independence number $\alpha(G)$ and start with the case that $\alpha(G)$ is not too large.

Proposition 18. *If G is a blow-up of a generalised Andrásfai graph F^k_d for some integers $k \geqslant 2$ and $d \geqslant 1$ with $|V(G)| = n$ and $\alpha(G) < \frac{k}{2k+1}n$, then G is not $(\frac{1}{2}, \frac{1}{2(2k+1)^2})$-dense.*

Proof. Let G be a blow-up of F^k_d with $|V(G)| = n$ and $\alpha(G) < \frac{k}{2k+1}n$. Without loss of generality we may assume that n is divisible by $2(2k+1)$. This follows from the observation, that a graph G is $(\frac{1}{2}, \frac{1}{2(2k+1)^2})$-dense if and only if the balanced blow-up of G obtained by replacing each vertex by $2(2k+1)$ vertices is $(\frac{1}{2}, \frac{1}{2(2k+1)^2})$-dense.

Suppose for the sake of contradiction that G is $(\frac{1}{2}, \frac{1}{2(2k+1)^2})$-dense. From now on consider the geometric representation of G given by Lemma 16. Let V be the set of points of the unit circle that are identified with the vertices of G. Recall that in (2.2) we defined $\lambda(I)$ as the number of vertices contained in an interval $I \subseteq \mathbb{R}/\mathbb{Z}$. It will sometimes be convenient to count vertices on the boundary of an interval only with weight $1/2$. For that we write terms like $\lambda(\langle a, b\rangle)$, $\lambda(\langle a, b))$, where the brackets "$\langle$" or "$\rangle$" mark that the left or right end-point of the respective interval is only counted $1/2$ if it is a vertex. Also recall that for $\xi \in \mathbb{R}/\mathbb{Z}$ we defined $z_\xi \in V$ in (2.3). Since by our assumption $\alpha(G) < n/2$, we infer from part (i) of Lemma 17 that

$$z_\xi \in \big(\xi + \tfrac{k-1}{2k-1}, \xi + \tfrac{k}{2k-1}\big)\,, \tag{2.7}$$

which yields together with Lemma 17(i) that

$$\sum_{x \in V \cap [\xi, z_\xi - \frac{k-1}{2k-1})} \left|N_G(x) \cap (x, z_\xi]\right| = e_G\big([\xi, z_\xi] \cap V\big)\,. \tag{2.8}$$

Moreover, part (ii) of Lemma 17 applied to intervals $[x+\frac{k-1}{2k-1}, x+\frac{k}{2k-1}]$ combined with the assumption $\alpha(G) < \frac{k}{2k+1}n$ leads to

$$\begin{aligned}\lambda\big(\langle x + \tfrac{k-1}{2k-1}, x + \tfrac{k}{2k-1}\rangle\big) &\leqslant \lambda\big([x + \tfrac{k-1}{2k-1}, x + \tfrac{k}{2k-1}]\big)\\ &\leqslant (2k-3)\alpha(G) - (k-2)n < (2k-3)\frac{k}{2k+1}n - (k-2)n = \frac{2}{2k+1}n\end{aligned}$$

for every vertex $x \in V$. Consequently,

$$\begin{aligned}&\sum_{x \in V} \Big(\lambda\big(\langle x - \tfrac{k-1}{2k-1}, x\rangle\big) + \lambda\big(\langle x, x + \tfrac{k-1}{2k-1}\rangle\big)\Big)\\ &= \sum_{x \in V} \Big(\lambda\big(\langle x, x+1\rangle\big) - \lambda\big(\langle x + \tfrac{k-1}{2k-1}, x + \tfrac{k}{2k-1}\rangle\big)\Big) > n^2 - \frac{2}{2k+1}n^2 = \frac{2k-1}{2k+1}n^2\end{aligned}$$

and by symmetry we may assume that

$$\sum_{x \in V} \lambda\big(\langle x, x + \tfrac{k-1}{2k-1}\rangle\big) > \frac{1}{2} \cdot \frac{2k-1}{2k+1}n^2\,. \tag{2.9}$$

In view of (2.9) the following claim seems a bit surprising and, in fact, it will lead to the desired contradiction. For a simpler notation we set

$$V_\xi = V \cap [\xi, z_\xi - \tfrac{k-1}{2k-1}) \tag{2.10}$$

for $\xi \in \mathbb{R}/\mathbb{Z}$.

Claim 19. *For every $\xi \in \mathbb{R}/\mathbb{Z}$ we have*

$$\sum_{x \in V_\xi} \Big(\lambda\big(\langle x, x + \tfrac{k-1}{2k-1}\rangle\big) - \frac{1}{2} \cdot \frac{2k-1}{2k+1}n\Big) < 0\,.$$

Proof of Claim 19. Fix some $\xi \in \mathbb{R}/\mathbb{Z}$. Since we assume that G is $(\frac{1}{2}, \frac{1}{2(2k+1)^2})$-dense, we have

$$\sum_{x \in V_\xi} \lambda\big((x + \tfrac{k-1}{2k-1}, z_\xi]\big) = \sum_{x \in V_\xi} \left|N_G(x) \cap (x, z_\xi]\right| \overset{(2.8)}{=} e_G\big([\xi, z_\xi] \cap V\big) > \frac{n^2}{2(2k+1)^2}\,.$$

Therefore,

$$\begin{aligned}
\sum_{x\in V_\xi} \lambda\big(\langle x, x+\tfrac{k-1}{2k-1}]\big) &= \sum_{x\in V_\xi}\Big(\lambda\big(\langle x, z_\xi]\big) - \lambda\big((x+\tfrac{k-1}{2k-1}, z_\xi]\big)\Big) \\
&< \sum_{x\in V_\xi} \lambda\big(\langle x, z_\xi]\big) - \frac{n^2}{2(2k+1)^2} \\
&= \sum_{x\in V_\xi}\Big(\lambda\big([\xi, z_\xi]\big) - \lambda\big([\xi, x\rangle\big)\Big) - \frac{n^2}{2(2k+1)^2} \\
&\overset{(2.3)}{=} |V_\xi|\cdot\frac{n}{2} - \frac{n^2}{2(2k+1)^2} - \sum_{x\in V_\xi}\lambda\big([\xi, x\rangle\big)\,. \qquad (2.11)
\end{aligned}$$

We observe

$$\sum_{x\in V_\xi}\lambda\big([\xi, x\rangle\big) = \sum_{i=1}^{|V_\xi|}(i-\tfrac{1}{2}) = \frac{|V_\xi|^2}{2} \qquad (2.12)$$

and combining (2.11) and (2.12) yields

$$\begin{aligned}
&\sum_{x\in V_\xi}\Big(\lambda\big(\langle x, x+\tfrac{k-1}{2k-1}]\big) - \frac{1}{2}\cdot\frac{2k-1}{2k+1}n\Big) \\
<&|V_\xi|\cdot\Big(\frac{n}{2} - \frac{1}{2}\cdot\frac{2k-1}{2k+1}n\Big) - \frac{n^2}{2(2k+1)^2} - \frac{|V_\xi|^2}{2} = -\frac{1}{2}\Big(|V_\xi| - \frac{n}{2k+1}\Big)^2 \leqslant 0\,,
\end{aligned}$$

which establishes the claim. □

Now set $V^* = \big\{\xi \in \mathbb{R}/\mathbb{Z} \colon \xi + \tfrac{k-1}{2k-1} \in V\big\}$. Starting with an arbitrary $x(0) \in V^*$ we define recursively a sequence of members of V^* by putting

$$x(i+1) = z_{x(i)} - \tfrac{k-1}{2k-1}$$

for every $i \in \mathbb{N}$. Since V^* is finite, this sequence is eventually periodic and thus we could have chosen $x(0)$ such that $x(m) = x(0)$ holds for some $m \geqslant 2$. Let $w \in \mathbb{N}$ denote the number of times we wind around the circle when reaching $x(m)$ from $x(0)$ by this construction. By Claim 19 we know that

$$\sum_{i=0}^{m-1}\sum_{x\in V_{x(i)}}\Big(\lambda\big(\langle x, x+\tfrac{k-1}{2k-1}\rangle\big) - \frac{1}{2}\cdot\frac{2k-1}{2k+1}n\Big) < 0\,.$$

On the other hand, (2.9) yields

$$\begin{aligned}
&\sum_{i=0}^{m-1}\sum_{x\in V_{x(i)}}\Big(\lambda\big(\langle x, x+\tfrac{k-1}{2k-1}\rangle\big) - \frac{1}{2}\cdot\frac{2k-1}{2k+1}n\Big) \\
\overset{(2.10)}{=}& w\cdot\sum_{x\in V}\Big(\lambda\big(\langle x, x+\tfrac{k-1}{2k-1}\rangle\big) - \frac{1}{2}\cdot\frac{2k-1}{2k+1}n\Big) \\
\overset{(2.9)}{>}& w\cdot\Big(\frac{1}{2}\cdot\frac{2k-1}{2k+1}n^2 - \frac{1}{2}\cdot\frac{2k-1}{2k+1}n^2\Big) = 0\,,
\end{aligned}$$

which is a contradiction and concludes the proof of Proposition 18. □

It is left to consider the case when G contains a large independent set.

Proposition 20. *If G is a blow-up of a generalised Andrásfai graph F_d^k for some integers $k \geqslant 2$ and $d \geqslant 1$ with $|V(G)| = n$ and $\alpha(G) \geqslant \frac{k}{2k+1}n$, then G is not $(\frac{1}{2}, \frac{1}{2(2k+1)^2})$-dense.*

Proof. Similarly as in the proof of Proposition 18 we consider the geometric representation of an n-vertex graph G that is a blow-up of a generalised Andrásfai graph F_d^k and identify the vertex set of G with some set $V \subseteq \mathbb{R}/\mathbb{Z}$ so that (2.1) holds. Again we may assume without loss of generality that n is divisible by $2(2k+1)$ and we suppose for a contradiction that G is $(\frac{1}{2}, \frac{1}{2(2k+1)^2})$-dense. In particular, $\alpha(G) < n/2$ and the additional assumptions for parts *(iv)* and *(v)* of Lemma 17 are satisfied.

Observe that every independent set of G is contained in some interval of $\mathbb{R}/\mathbb{Z}$ of length $\frac{k-1}{2k-1}$. Therefore, without loss of generality we may assume that $[0, \frac{k-1}{2k-1}]$ contains a maximum independent set, i.e.,

$$\lambda\big([0, \tfrac{k-1}{2k-1}]\big) = \alpha(G) \geqslant \frac{k}{2k+1}n\,.$$

Recall that in (2.3) we defined a point z_0 with $\lambda\big([0, z_0]\big) = n/2$. Let the vertex z' be defined similarly by $\lambda\big([z', \frac{k-1}{2k-1}]\big) = n/2$. Then we have

$$\begin{aligned}
\lambda\big((z_0, z')\big) &= n - \lambda\big([z', 0)\big) - \lambda\big([0, \tfrac{k-1}{2k-1}]\big) - \lambda\big((\tfrac{k-1}{2k-1}, z_0]\big) \\
&= n - (n/2 - \alpha(G)) - \alpha(G) - (n/2 - \alpha(G)) \\
&= \alpha(G)
\end{aligned}$$

and since z_0, $z' \in V$ the maximality of $\alpha(G)$ discloses that the interval $[z_0, z']$ has at least the length $\frac{k-1}{2k-1}$. Hence there is a closed subinterval $[b_k, b_0]$ of $[z_0, z']$ whose length is exactly $\frac{k-1}{2k-1}$. We complete b_0 and b_k to the vertices of a regular $(2k-1)$-gon, i.e., we consider the points $b_i \in \mathbb{R}/\mathbb{Z}$ for $i \in \{0, \dots, 2k-2\}$ such that the intervals $[b_i, b_{i+1}]$ have length $\frac{1}{2k-1}$ (see Figure 2.2). Notice that $\alpha(G) < n/2$ entails

$$z_0 \in (b_{k-1}, b_k]\,. \tag{2.13}$$

Below we apply Lemma 17 to obtain several bounds on the numbers $\lambda\big([b_1, b_{k-1}]\big)$ and $\lambda\big([b_{k+1}, b_{2k-2}]\big)$ that eventually lead to the desired contradiction. Applying

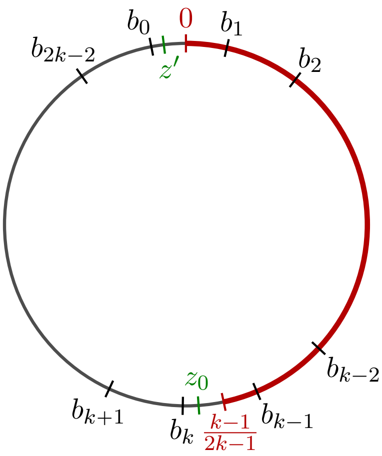

Figure 2.2: Largest independent set of G is contained in the interval $[0, \frac{k-1}{2k-1}]$ and the intervals $[0, z_0]$, $[z', \frac{k-1}{2k-1}]$ contain $n/2$ vertices each. The b_i form a regular $(2k-1)$-gon.

Lemma 17 *(iv)* with $\xi = 0$ gives

$$\lambda\big([0, b_1)\big) = \lambda\big([0, b_k - \tfrac{k-1}{2k-1})\big) \overset{(2.13)}{\geqslant} \lambda\big([0, z_0 - \tfrac{k-1}{2k-1})\big) > 2\alpha(G) - \frac{2k-1}{2k+1}n$$

and, by symmetry, we also have

$$\lambda\big((b_{k-1}, \tfrac{k-1}{2k-1}]\big) > 2\alpha(G) - \frac{2k-1}{2k+1}n\,.$$

Consequently, we arrive at

$$\begin{aligned}\lambda\big([b_1, b_{k-1}]\big) &= \lambda\big([0, \tfrac{k-1}{2k-1}]\big) - \lambda\big([0, b_1)\big) - \lambda\big((b_{k-1}, \tfrac{k-1}{2k-1}]\big) \\ &< \alpha(G) - 2\left(2\alpha(G) - \frac{2k-1}{2k+1}n\right) = \frac{4k-2}{2k+1}n - 3\alpha(G)\,. \end{aligned} \tag{2.14}$$

In particular, for the case $k = 2$ this implies

$$0 \leqslant \lambda\big([b_1, b_1]\big) < \frac{6}{5}n - 3\alpha(G)\,,$$

which contradicts our assumption $\alpha(G) \geqslant 2n/5$. Similarly, for $k = 3$ inequality (2.14) combined with Lemma 17 *(iii)* gives

$$n - 2\alpha(G) \leqslant \lambda\big([b_1, b_2]\big) < \frac{10}{7}n - 3\alpha(G)\,,$$

which again contradicts the assumption $\alpha(G) \geqslant 3n/7$ of this case. Consequently, for the rest of the proof we can assume that $k \geqslant 4$.

Next we note that both intervals (b_{k-1}, b_{2k-2}) and (b_{k+1}, b_1) have length $\frac{k-1}{2k-1}$ and, hence, Lemma 17 *(i)* implies

$$\lambda\big((b_{k-1}, b_{2k-2})\big) + \lambda\big((b_{k+1}, b_1)\big) \leqslant 2\alpha(G)$$

and, therefore,

$$\lambda\big((b_{k+1}, b_{2k-2})\big) \leqslant 2\alpha(G) - \lambda\big((b_{k-1}, b_1)\big) = 2\alpha(G) - \Big(n - \lambda\big([b_1, b_{k-1}]\big)\Big)\,. \quad (2.15)$$

Finally, below we will verify

$$\frac{4k-5}{2k+1}n - 2\alpha(G) - \lambda\big([b_1, b_{k-1}]\big) < \lambda\big((b_{k+1}, b_{2k-2})\big)\,. \quad (2.16)$$

Before we prove (2.16), we note that using (2.15) as an upper bound for the right-hand side of (2.16) leads to

$$\frac{6k-4}{2k+1}n - 4\alpha(G) < 2\lambda\big([b_1, b_{k-1}]\big) \overset{(2.14)}{<} \frac{8k-4}{2k+1}n - 6\alpha(G)\,.$$

This inequality contradicts the assumption $\alpha(G) \geqslant \frac{k}{2k+1}$ of the proposition and, hence, we conclude the proof by establishing (2.16).

For the proof of inequality (2.16) we appeal to Lemma 17 *(v)* with $\xi = b_i$ for every $i = 2, \ldots, k-2$. We set

$$I_i = (b_i - \tfrac{1}{2k-1}, b_i + \tfrac{1}{2k-1}) = (b_{i-1}, b_{i+1})$$

and then in view of

$$(b_i + \tfrac{k-1}{2k-1}, b_i - \tfrac{k-1}{2k-1}) = (b_{i+k-1}, b_{i+k})$$

part *(v)* translates to

$$\lambda(I_i) > \frac{4}{2k+1}n - 2\lambda\big((b_{i+k-1}, b_{i+k})\big)\,. \quad (2.17)$$

Furthermore, we note that for every $i \in \{2, \ldots, k-2\}$ we have $I_i \subseteq [b_1, b_{k-1}]$ and each of the two families

$$\mathcal{I}_0 = \{I_i \colon i \text{ even and } 2 \leqslant i \leqslant k-2\} \quad \text{and} \quad \mathcal{I}_1 = \{I_i \colon i \text{ odd and } 2 \leqslant i \leqslant k-2\}$$

consists of mutually disjoint intervals. Moreover, we can add the interval $[b_1, b_2)$ to $\mathcal{I}_1$ and $(b_{k-2}, b_{k-1}]$ either to $\mathcal{I}_1$ (when k is even) or to $\mathcal{I}_0$ (when k is odd) and still each family consists of mutually disjoint intervals all contained in $[b_1, b_{k-1}]$. As a result we get

$$2\lambda\big([b_1, b_{k-1}]\big) \geqslant \lambda\big([b_1, b_2)\big) + \sum_{i=2}^{k-2} \lambda(I_i) + \lambda\big((b_{k-2}, b_{k-1}]\big)\,.$$

Moreover, using the estimate from Lemma 17 *(iii)* for $\lambda\big([b_1, b_2)\big)$ and $\lambda\big((b_{k-2}, b_{k-1}]\big)$ and (2.17) for every term in the middle sum, we arrive at

$$\begin{aligned}
2\lambda\big([b_1, b_{k-1}]\big) &> \big(n - 2\alpha(G)\big) + \sum_{i=2}^{k-2} \left(\frac{4n}{2k+1} - 2\lambda\big((b_{i+k-1}, b_{i+k})\big) \right) + \big(n - 2\alpha(G)\big) \\
&\geqslant 2n - 4\alpha(G) + (k-3) \cdot \frac{4n}{2k+1} - 2\lambda\big((b_{k+1}, b_{2k-2})\big) \\
&= \frac{8k-10}{2k+1} n - 4\alpha(G) - 2\lambda\big((b_{k+1}, b_{2k-2})\big)\,.
\end{aligned}$$

Rearranging the last inequality gives (2.16) and this concludes the proof. $\square$

3 Squares of Hamiltonian cycles in 3-uniform hypergraphs

In this chapter we will prove the following result.

Theorem 21. *For every $\alpha > 0$ there exists an integer n_0 such that every 3-uniform hypergraph H with $n \geqslant n_0$ vertices and with minimum pair-degree $\delta_2(H) \geqslant (\frac{4}{5}+\alpha)n$ contains a squared Hamiltonian cycle.*

3.1 Building squared Hamiltonian Cycles in Hypergraphs

In this section we will show the outline of the proof of Theorem 21. We start by presenting the dependencies of the auxiliary constants we use in the propositions required for the proof of Thereom 21. We write $a \gg b$ to indicate that b will be chosen sufficiently small depending on a and all other constants appearing on the left of b. In Theorem 21 some α with $1 \gg \alpha > 0$ is given. We fix the auxiliary constants ϑ_* and an integer $M \in \mathbb{N}$, such that

$$1 \gg \alpha \gg 1/M \gg \vartheta_* \gg 1/n\,.$$

The connecting lemma stated below plays a crucial rôle in the proof of Theorem 21. It asserts that any two disjoint triples of vertices can be connected by many "short" squared paths.

Proposition 22. *(Connecting Lemma) There are an integer M and $\vartheta_* > 0$, such that for all sufficiently large hypergraphs $H = (V, E)$ with $\delta_2(H) \geqslant (4/5 + \alpha)|V|$ and all disjoint triples (a, b, c) and (x, y, z) with $abc, xyz \in E$ there exists $m < M$ for which there are at least $\vartheta_* n^m$ squared paths from abc to xyz with m internal vertices.*

The proof of the connecting lemma forms the content of Section 3.2. We can connect any two squared paths by the connecting lemma using their start or endtriples, but for our constructions it will be important that we do not interfere with any already constructed subpath. Therefore we put a small *reservoir* of vertices aside, such that if we do not connect too many times it is possible to use vertices of the reservoir set only. The following lemma, which we prove in Section 3.3, shows the existence of such a set.

Proposition 23. *(Reservoir Lemma) Suppose that for a given $\alpha > 0$ the constants $1/M \gg \vartheta_*$ are as provided by the connecting lemma and that $H = (V, E)$ is a sufficiently large hypergraph with $|V| = n$ and $\delta_2(H) \geqslant (4/5 + \alpha)n$. Then there exists a reservoir set $\mathcal{R} \subseteq V$ of size $|\mathcal{R}| \leqslant \vartheta_*^2 n$ such that for all $\mathcal{R}' \subseteq \mathcal{R}$ with $|\mathcal{R}'| \leqslant \vartheta_*^4 n$ and for all disjoint triples (a, b, c) and (x, y, z) with $abc, xyz \in E$ there exists a connecting squared path in H with less than M internal vertices all of which belong to $\mathcal{R} \setminus \mathcal{R}'$.*

Moreover, we put aside an absorbing path P_A, which will absorb an arbitrary but not too large set X of leftover vertices at the end of the proof, such that we get a squared Hamiltonian cycle.

Proposition 24 (Absorbing path)**.** *Let $\alpha \gg 1/M \gg \vartheta_*$ be as usual and let $H = (V, E)$ be a sufficiently large hypergraph with $|V| = n$ and minimum pair-degree $\delta_2(H) \geqslant (4/5 + \alpha)n$. There exists an (absorbing) squared path $P_A \subseteq H - \mathcal{R}$ such that*

(1) $|V(P_A)| \leqslant \vartheta_* n$,

(2) for every set $X \subseteq V \setminus V(P_A)$ with $|X| \leqslant 2\vartheta_*^2 n$ there is a squared path in H whose set of vertices is $V(P_A) \cup X$ and whose end-triples are the same as those of P_A.

In Section 3.4 we prove Proposition 24 and in Section 3.5 we will show the following theorem.

Theorem 25. *Given $\alpha, \mu > 0$, and $Q \in \mathbb{N}$ there exists $n_0 \in \mathbb{N}$ such that in every hypergraph H with $v(H) = n \geqslant n_0$ and $\delta_2(H) \geqslant (3/4 + \alpha)n$ all but at most μn vertices of H can be covered by vertex-disjoint squared paths with Q vertices.*

Also in Section 3.5 we use this theorem to prove the existence of an almost spanning squared cycle that covers all but at most $2\vartheta_*^2 n$ vertices.

Proposition 26. *Given $\alpha > 0$, let $\vartheta_* > 0$ and $M \in \mathbb{N}$ be the constants from the connecting lemma. There exists $n_0 \in \mathbb{N}$ such that in every hypergraph H with $v(H) = n \geqslant n_0$ and $\delta_2(H) \geqslant (4/5 + \alpha)n$ all but at most $2\vartheta_*^2 n$ vertices of H can be covered by a squared cycle such that some absorbing squared path P_A is an induced subgraph of this cycle.*

Combining Proposition 24 and Proposition 26 implies the existence of a squared Hamiltonian cycle and therefore proves Theorem 21.

3.2 Connecting Lemma

We will show some of our results with the constant $\frac{3}{4}$ and others for $\frac{4}{5}$. Moreover we fix the auxiliary constants $\beta, \gamma, \vartheta_*$ and integers $K, \ell, M \in \mathbb{N}$ obeying the hierarchy

$$1 \gg \alpha \gg \beta, \gamma, 1/\ell \gg 1/K \gg 1/M \gg \vartheta_* \gg 1/n\,.$$

3.2.1 Connecting properties

We prove that the graph properties stated in the following lemma imply a connecting property and use this lemma later to show that some auxiliary graphs G_3 and G_v have this connecting property.

Lemma 27. *Let $\gamma \leqslant 1/16$ and let $G = (V, E)$ with $|V| = n$ be a graph with $\delta(G) \geqslant \sqrt{\gamma}n$ such that for every partition $X \cup Y = V$ of the vertex set with $|X|, |Y| \geqslant \sqrt{\gamma}n$ we have $e_G(X, Y) \geqslant \gamma n^2$.*

Then for every pair of distinct vertices $x, y \in V(G)$ there exists some $s = s(x, y) \leqslant 4/\gamma$ for which there are at least $\Omega(n^{s-1})$ many x-y-walks of length s.

Proof. For an arbitrary vertex $x \in V$ and an integer $i \geqslant 1$ we define

$$\begin{aligned} Z_x^i = \{z \in V\colon\ & \text{there are at least } (\gamma^2/4)^s n^{s-1} \ x\text{-}z\text{-walks of length } s \text{ in } G \\ & \text{for some } s \leqslant i\}\,. \end{aligned}$$

For $i \geqslant 2$ we have $Z_x^i \supseteq Z_x^{i-1}$ and therefore

$$|Z_x^i| \geqslant |Z_x^1| = |N_G(x)| \geqslant \delta(G) \geqslant \sqrt{\gamma} n \,.$$

Now we show that for every integer i with $1 \leqslant i \leqslant 2/\gamma$ at least one of the following holds:

$$|V \smallsetminus Z_x^i| < \sqrt{\gamma} n \qquad \text{or} \qquad |Z_x^{i+1} \smallsetminus Z_x^i| \geqslant \frac{\gamma n}{2} \,. \tag{3.1}$$

If $|V \smallsetminus Z_x^i| \geqslant \sqrt{\gamma} n$, then the assumption yields that

$$e_G(Z_x^i, V \smallsetminus Z_x^i) \geqslant \gamma n^2 \,.$$

This implies that at least $\gamma n/2$ vertices in $V \smallsetminus Z_x^i$ have at least $\gamma n/2$ neighbours in Z_x^i. For such a vertex $u \in V \smallsetminus Z_x^i$ at least a proportion of $1/i \geqslant \gamma/2$ of its neighbours in Z_x^i is connected to x by walks of the same length, which implies $u \in Z_x^{i+1}$. As this argument applies to $\gamma n/2$ vertices outside Z_x^i we thus obtain $|Z_x^{i+1} \smallsetminus Z_x^i| \geqslant \gamma n/2$, which concludes the proof of (3.1).

It is not possible that the right outcome of (3.1) holds for each positive $i \leqslant 2/\gamma$. Therefore we have $|V \smallsetminus Z_x^j| < \sqrt{\gamma} n$ for $j = \lfloor 2/\gamma \rfloor$. So for $x, y \in V$ at least $n - 2\sqrt{\gamma} n \geqslant n/2$ vertices z are contained in the intersection $Z_x^j \cap Z_y^j$. For each $z \in Z_x^j \cap Z_y^j$ we get constants $s_1, s_2 \leqslant j \leqslant 2/\gamma$ such that there are at least $(\gamma^2/4)^{s_1} n^{s_1 - 1}$ x-z-walks of length s_1 and there are at least $(\gamma^2/4)^{s_2} n^{s_2 - 1}$ z-y-walks of length s_2. Therefore, for $s_z = s_1 + s_2 \geqslant 2$ there are at least $(\gamma^2/4)^{s_z} n^{s_z - 2}$ x-y-walks of length s_z passing through z.

There are at least $n/2$ vertices this argument applies to and by the box principle at least $\frac{n}{2}/\frac{4}{\gamma^2}$ of them give rise to the same pair (s_1, s_2) and, consequently, the same value of s_z. Moreover, the walks obtained for those vertices are distinct and hence for some $s(x,y) \in [2, 4/\gamma]$ there are at least

$$(\gamma^2 n/8) \cdot (\gamma^2/4)^{s(x,y)} n^{s(x,y)-2} \geqslant \tfrac{1}{2} (\gamma^2/4)^{4/\gamma + 1} n^{s(x,y)-1}$$

x-y-walks of length $s(x,y)$. $\square$

3.2.2 The auxiliary graph G_3

The first auxiliary graph we will study is the following.

Definition 28. *For a 3-uniform hypergraph $H = (V, E)$ we define the auxiliary graph G_3 (see Fig. 3.1) as the graph with vertex set $V(G_3) = V$ and*

$$xy \in E(G_3) \iff x \neq y \text{ and } \#\{(a,b,c) \in V^3 \colon abcx \text{ and } abcy \text{ are } K_4^{(3)}\} \geqslant \beta n^3 \,.$$

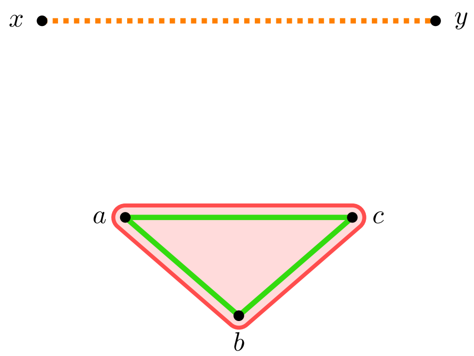

Figure 3.1: We have an edge $xy \in E(G_3)$ iff there are "many" edges $abc \in E(H)$ for which $ab, ac, bc \in L(x) \cap L(y)$.

The main result of this subsection is the following proposition.

Proposition 29. *Given $\alpha > 0$ there exist $n_0, \ell \in \mathbb{N}$ such that in every hypergraph H with $v(H) = n \geqslant n_0$ and $\delta_2(H) \geqslant (3/4 + \alpha)n$ for every pair of distinct vertices $x, y \in V(G)$ there exists some $t = t(x, y) \leqslant \ell$ for which there are at least $\Omega(n^{t-1})$ x-y-walks of length t in G_3.*

The next lemma gives us a lower bound on the minimum degree of G_3.

Lemma 30. *If $n \gg \alpha^{-1}$ and H is a hypergraph on n vertices with minimum pair-degree $\delta_2(H) \geqslant (3/4 + \alpha)n$, then $\delta(G_3) \geqslant (1/4 + \alpha)n$.*

Proof. Let $x \in V$ and $\beta < \alpha/8$. We count the ordered quadruples $(a, b, c, y) \in V^4$, such that $\{a, b, c, y\}$ and $\{x, a, b, c\}$ induce distinct tetrahedra in H. That is, we estimate the size of the set

$$A_x = \{(a,b,c,y) \in V^4 \colon x \neq y \text{ and } xabc \text{ and } abcy \text{ are } K_4^{(3)}\} \,.$$

Due to our assumption about $\delta_2(H)$ the number A of triples $(a, b, c) \in V^3$, which form a $K_4^{(3)}$ with x, can be estimated by

$$\begin{aligned} A &= \#\{(a,b,c) \in V^3 \colon abcx \text{ is a } K_4^{(3)}\} \\ &\geqslant (n-1)\Big(\frac{3n}{4} + \alpha n\Big)\Big(\frac{n}{4} + 3\alpha n\Big) \\ &\geqslant \frac{n^3}{8} \end{aligned} \tag{3.2}$$

for n sufficiently large. Using the minimum pair-degree condition again we obtain

$$|A_x| \geqslant A\Big(\frac{n}{4} + 3\alpha n - 1\Big) \geqslant \Big(\frac{1}{4} + 2\alpha\Big)An\,. \tag{3.3}$$

On the other hand, the assumption $d_{G_3}(x) \leqslant n/4 + \alpha n$ would imply that

$$\begin{aligned}|A_x| &= \sum_{y \in V \smallsetminus \{x\}} \#\{(a,b,c) \in V^3 \colon abcy \text{ and } abcx \text{ are } K_4^{(3)}\} \\ &\leqslant n \cdot \beta n^3 + (n/4 + \alpha n)A\,.\end{aligned}$$

Together with (3.3) this yields that

$$\Big(\frac{1}{4} + 2\alpha\Big)An \leqslant \beta n^4 + \Big(\frac{1}{4} + \alpha\Big)An\,,$$

i.e., $\beta n^3 \geqslant \alpha A \overset{(3.2)}{\geqslant} \alpha n^3/8$. Since $\beta < \alpha/8$ this is a contradiction and shows that the minimum degree of G_3 is at least $(1/4 + \alpha)n$. $\square$

Lemma 31. *If $\beta, \gamma \ll \alpha$ and H is a hypergraph on n vertices with minimum pair-degree $\delta_2(H) \geqslant (3/4 + \alpha)n$, then for every partition $X \cupdot Y = V$ of the vertex set with $|X|, |Y| \geqslant (1/4 + \alpha/2)n$ we have $e_{G_3}(X, Y) \geqslant \gamma n^2$.*

Proof. W.l.o.g. we can assume that $|X| \leqslant |Y|$. Since $|X| \geqslant (1/4 + \alpha/2)n$, we know that $|Y| \leqslant (3/4 - \alpha/2)n$. Counting the ordered triples with two vertices in X and one in Y which induce an edge in H, we get

$$\begin{aligned}&\#\{(x,x',y) \in X^2 \times Y \colon xx'y \in E(H)\} \\ &= \sum_{(x,y) \in X \times Y} |N(x,y) \cap X| \\ &\geqslant |X||Y| \cdot (\delta_2(H) - |Y|) \\ &\geqslant \frac{3}{16}n^2 \cdot \frac{3\alpha n}{2} = \frac{9\alpha}{32}n^3\,.\end{aligned}$$

The number of $K_4^{(3)}$ including such a triple (x,x',y) can thus be estimated by

$$\begin{aligned}&\big|\{(x,x',y,y') \in X^2 \times Y^2 \colon xx'yy' \text{ is a } K_4^{(3)}\}\big| \\ +&\big|\{(x,x',x'',y) \in X^3 \times Y \colon xx'x''y \text{ is a } K_4^{(3)}\}\big| \\ \geqslant&\frac{9\alpha n^3}{32} \cdot \frac{n}{4} = \frac{9\alpha}{128}n^4\,.\end{aligned}$$

Now we will distinguish two cases depending on whether the number of $K_4^{(3)}$ with exactly two or exactly three vertices in X is bigger than $\frac{9\alpha}{256}n^4$.

Case 1. $\#\{(x,x',y,y') \in X^2 \times Y^2 \colon xx'yy' \text{ is a } K_4^{(3)}\} \geqslant \dfrac{9\alpha}{256}n^4$

Define $A \subseteq X^2 \times Y^2 \times V$ to be the set of all quintuples (x,x',y,y',z) satisfying

(i) $xx'yy'$ is a $K_4^{(3)}$;

(ii) $zxx', zyy' \in E(H)$;

(iii) and at least three of $zxy, zx'y', zxy', zx'y$ are edges in H.

We claim that the size of A can be bounded from below by

$$|A| \geqslant \frac{9\alpha^2}{64} n^5 \,. \tag{3.4}$$

As we are in Case 1, it suffices to prove that every tetrahedron $(x, x', y, y') \in X^2 \times Y^2$ extends to at least $4\alpha n$ members of A.

Writing

$$f(z) = |\{xy, xy', x'y, x'y'\} \cap E(L_z)| + 2|\{xx', yy'\} \cap E(L_z)|$$

for every $z \in V$ we get

$$\begin{aligned}\sum_{z \in V} f(z) &= d_H(x, y) + d_H(x, y') + d_H(x', y) + d_H(x', y') + 2d_H(x, x') + 2d_H(y, y') \\ &\geqslant 8\delta_2(H) \geqslant (6 + 8\alpha)n \,.\end{aligned}$$

As $f(z) \leqslant 8$ holds for each $z \in V$ it follows that there are at least $4\alpha n$ vertices with $f(z) \geqslant 7$. For each of them we have $(x, x', y, y', z) \in A$. Thereby (3.4) is proved.

To derive an upper bound on $|A|$, we break the symmetry in (iii). Denoting by A' the set of quintuples $(x, x', y, y', z) \in X^2 \times Y^2 \times V$ satisfying (i), (ii), and

(iv) $xy'z, x'yz, x'y'z \in E(H)$

we have

$$|A| \leqslant 4|A'| \,. \tag{3.5}$$

Moreover

$$\begin{aligned}|A'| &\leqslant \sum_{(x,y) \in X \times Y} \#\{(x', y', z) \in X \times Y \times V : xx'y'z \text{ and } x'yy'z \text{ are } K_4^{(3)}\} \\ &\leqslant e_{G_3}(X, Y) \cdot |X||Y||V| + |X||Y| \cdot \beta n^3 \\ &\leqslant \frac{1}{4} e_{G_3}(X, Y) n^3 + \frac{1}{4}\beta n^5 \,.\end{aligned}$$

Therefore with (3.4) and (3.5) it follows that

$$e_{G_3}(X,Y) \geqslant \Big(\frac{9\alpha^2}{64} - \beta\Big)n^2\,.$$

Case 2. $\#\{(x,x',x'',y) \in X^3 \times Y : xx'x''y \text{ is a } K_4^{(3)}\} \geqslant \dfrac{9\alpha}{256}n^4$

Define $A \subseteq X^3 \times Y \times V$ to be the set of all quintuples (x,x',x'',y,z) satisfying

(*i*) $xx'x''y$ is a $K_4^{(3)}$;

(*ii*) if $z \in Y$ at least one of the vertex sets $\{x,x'',y\}, \{x,x',y\}, \{x',x'',y\}$ induces a triangle in L_z;

(*iii*) if $z \in X$ the vertex set $\{x,x',x''\}$ induces a triangle in L_z .

We claim that the size of A can be bounded from below by

$$|A| \geqslant \frac{27\alpha^2}{256}n^5\,. \tag{3.6}$$

As we are in Case 2, it suffices to prove that every tetrahedron $(x,x',x'',y) \in X^3 \times Y$ extends to at least $3\alpha n$ members of A.

Writing

$$f(z) = |\{xy, xx', xx'', x'x'', x'y, x''y\} \cap E(L_z)|$$

for every $z \in V$ we get

$$\begin{aligned}\sum_{z\in V} f(z) &= d_H(x,y) + d_H(x,x') + d_H(x,x'') + d_H(x',x'') + d_H(x',y) + d_H(x'',y)\\ &\geqslant 6\delta_2(H) \geqslant (9/2 + 6\alpha)n\,.\end{aligned}$$

If $z \in Y$ is a vertex with $(x,x',x'',y,z) \notin A$ then $f(z) \leqslant 4$ and if $z \in X$ is a vertex with $(x,x',x'',y,z) \notin A$ then $f(z) \leqslant 5$. Hence we have

$$\begin{aligned}&(9/2+6\alpha)n\\ &\leqslant 5|X| + 4|Y| + \big|\{z \in X : (x,x',x'',y,z) \in A\}\big| + 2\big|\{z \in Y : (x,x',x'',y,z) \in A\}\big|\,.\end{aligned}$$

Since $5|X| + 4|Y| = 4n + |X| \leqslant 9/2n$, it follows that

$$3\alpha n \leqslant \big|\{z \in X : (x,x',x'',y,z) \in A\}\big| + \big|\{z \in Y : (x,x',x'',y,z) \in A\}\big|\,,$$

as claimed.

Like before in Case 1 we obtain the upper bound

$$|A| \leqslant \beta n^5 + e_{G_3}(X,Y)n^3 \,.$$

Therefore with (3.6) it follows that

$$e_{G_3}(X,Y) \geqslant \Big(\frac{27\alpha^2}{256} - \beta\Big)n^2 \,. \qquad \square$$

Proof of Proposition 29. Because of Lemma 27, Lemma 30, and Lemma 31 it remains to check that for every partition $V = X \cup Y$ with $\sqrt{\gamma}n \leqslant |X| \leqslant (1/4 + \alpha/2)n$ we have $e_{G_3}(X,Y) \geqslant \gamma n^2$. This follows easily from

$$e_{G_3}(X,Y) = \sum_{x \in X} d_Y^{G_3}(x) \geqslant \delta(G_3) \cdot |X| - |X|^2$$

and Lemma 30. $\square$

3.2.3 The auxiliary graphs G_v

The second kind of auxiliary graphs we will study is the following.

Definition 32. *For a 3-uniform hypergraph $H = (V,E)$ and a vertex $v \in V$ we define the auxiliary graph G_v (see Fig. 3.2) as the graph with vertex set $V(G_v) = V \smallsetminus \{v\}$ and*

$$xy \in E(G_v) \Longleftrightarrow x \neq y \text{ and } \#\{(a,b) \in V^2 \colon xabv \text{ and } yabv \text{ are } K_4^{(3)}\} \geqslant \beta n^2 \,.$$

The main result of this subsection is the following proposition.

Proposition 33. *Given $\alpha > 0$ there exist $n_0, \ell \in \mathbb{N}$ such that in every hypergraph H with $v(H) = n \geqslant n_0$ and $\delta_2(H) \geqslant (3/4 + \alpha)n$ for every pair of distinct vertices $x, y \in V(G)$ there exists some $t = t(x,y) \leqslant \ell$ for which there are at least $\Omega(n^{t-1})$ x-y-walks of length t in G_v.*

The next lemma gives us a lower bound on the minimum degree of G_v.

Lemma 34. *If $n \gg \alpha^{-1}$ and H is a hypergraph on n vertices with minimum pair-degree $\delta_2(H) \geqslant (3/4 + \alpha)n$, then $\delta(G_v) \geqslant (1/4 + \alpha)n$.*

Proof. Let $x \in V \smallsetminus \{v\}$. We count the triples $(a,b,y) \in V^3$, such that $\{y,a,b,v\}$ and $\{x,a,b,v\}$ induce distinct tetrahedra in H. That is, we estimate the size of the set

$$A_x = \{(a,b,y) \in V^3 \colon x \neq y \neq v \text{ and } xabv \text{ and } yabv \text{ are } K_4^{(3)}\} \,.$$

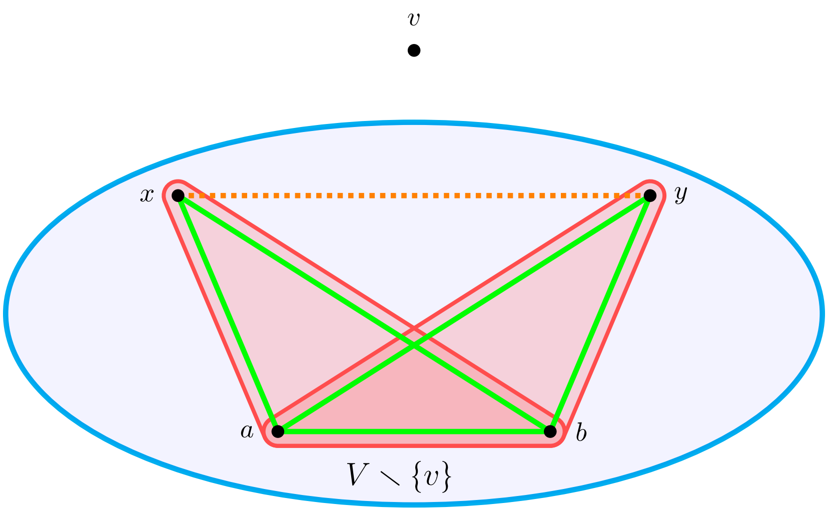

Figure 3.2: We have $xy \in E(G_v)$ iff there are "many" pairs $(a, b) \in V^2$ for which $abx, aby \in E(H)$ and abx, aby span triangles in L_v.

Due to our assumption about $\delta_2(H)$ the number A of pairs $(a, b) \in V^2$, which form a $K_4^{(3)}$ with x and v, can be estimated by

$$\begin{aligned} A &= \#\{(a,b) \in V^2 \colon abxv \text{ is a } K_4^{(3)}\} \\ &\geqslant \Big(\frac{3n}{4} + \alpha n\Big)\Big(\frac{n}{4} + 3\alpha n\Big) \\ &\geqslant \frac{n^2}{8}\,. \end{aligned} \tag{3.7}$$

Moreover we have

$$|A_x| \geqslant A\Big(\frac{n}{4} + 3\alpha n - 1\Big) \geqslant \Big(\frac{1}{4} + 2\alpha\Big) An\,. \tag{3.8}$$

On the other hand, the assumption $d_{G_v}(x) \leqslant n/4 + \alpha n$ would imply that

$$|A_x| = \sum_{y \in V \smallsetminus \{v,x\}} \#\{(a,b) \in V^2 \colon abvy \text{ and } abvx \text{ are } K_4^{(3)}\} \leqslant n \cdot \beta n^2 + (n/4 + \alpha n)A\,.$$

Together with (3.8) this yields that

$$\Big(\frac{1}{4} + 2\alpha\Big) An \leqslant \beta n^3 + \Big(\frac{1}{4} + \alpha\Big) An\,,$$

i.e., $\beta n^2 \geqslant \alpha A \overset{(3.7)}{\geqslant} \alpha n^2/8$. Since $\beta < \alpha/8$ this is a contradiction and shows that the minimum degree of G_v is at least $(1/4 + \alpha)n$. $\square$

Lemma 35. *If $n \gg \beta^{-1}, \gamma^{-1} \gg \alpha^{-1}$ and H is a hypergraph on n vertices with minimum pair-degree $\delta_2(H) \geqslant (3/4+\alpha)n$, then for every partition $X \cup Y = V \smallsetminus \{v\}$ of the vertex set with $|X|, |Y| \geqslant (1/4 + \alpha/2)n$ we have $e_{G_v}(X, Y) \geqslant \gamma n^2$.*

Proof. We begin by showing that the set

$$A_\star = \{(x,y,z) \in X \times Y \times (V \smallsetminus \{v\}) \colon vxyz \text{ is a } K_4^{(3)} \text{ in } H\},$$

satisfies

$$|A_\star| \geqslant \frac{n^3}{32}\,. \tag{3.9}$$

For the proof of this fact we may assume that $|X| \leqslant |Y|$. Thus $|X| \in [\frac{n}{4}, \frac{n}{2}]$ and hence

$$\begin{aligned}|A_\star| &\geqslant |X| \cdot (\delta_2(H) - |X|) \cdot (3\delta_2(H) - 2n)\\ &\geqslant |X| \cdot \Big(\frac{3}{4}n - |X|\Big) \cdot \frac{n}{4}\\ &\geqslant \frac{n^2}{8} \cdot \frac{n}{4} = \frac{n^3}{32}\,,\end{aligned}$$

as desired.

It follows that

$$|A_\star \cap (X \times Y \times X)| + |A_\star \cap (X \times Y^2)| = |A_\star| \geqslant \frac{n^3}{32}$$

and w.l.o.g. we can assume that $|A_\star \cap (X \times Y \times X)| \geqslant n^3/64$. Now we study the set

$$A_{\star\star} = \{(a,b,y,z) \in X^2 \times Y \times (V \smallsetminus \{v\}) \colon abvy, abvz \text{ are } K_4^{(3)} \text{ and } yz \in E(L_v)\}\,.$$

Given any triple $(a,y,b) \in A_\star \cap (X \times Y \times X)$ the quadruple $abvy$ forms a tetrahedron, there are at least $3\delta_2(H) - 2n$ vertices z for which $abvz$ forms a tetrahedron as well, and for at most $n - \delta_2(H)$ of those the condition $yz \in E(L_v)$ fails. Hence

$$\begin{aligned}|A_{\star\star}| &\geqslant |A_\star \cap (X \times Y \times X)| \cdot [(3\delta_2(H) - 2n) - (n - \delta_2(H))]\\ &\geqslant 4\alpha n \cdot |A_\star \cap (X \times Y \times X)| \geqslant \frac{\alpha}{16} n^4\,.\end{aligned}$$

Case 1. $|A_{\star\star} \cap (X^2 \times Y \times X)| \geqslant \alpha n^4/32$.

Owing to

$$\begin{aligned}\frac{\alpha n^4}{32} &\leqslant |A_{\star\star} \cap (X^2 \times Y \times X)|\\ &\leqslant \sum_{(z,y) \in X \times Y} \#\{(a,b) \in X^2 \colon abzv \text{ and } abvy \text{ are } K_4^{(3)}\}\\ &\leqslant \beta n^2 |X||Y| + e_{G_v}(X,Y) \cdot n^2\\ &\leqslant \beta n^2 \cdot n^2/4 + e_{G_v}(X,Y) \cdot n^2\end{aligned}$$

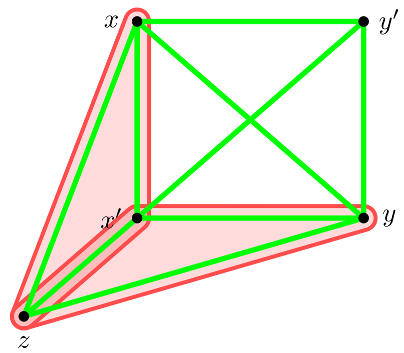

Figure 3.3: Example of a quintuple in A, where the link graph of v is indicated in green and hyperedges of H in red.

we have

$$e_{G_v}(X,Y) \geqslant \Big(\frac{\alpha}{32} - \frac{\beta}{4}\Big)n^2\,,$$

as desired.

Case 2. $|A_{\star\star} \cap (X^2 \times Y^2)| \geqslant \alpha n^4/32$

Define $A \subseteq X^2 \times Y^2 \times (V \smallsetminus \{v\})$ to be the set of all quintuples (x,x',y,y',z) satisfying

(i) $xx'yy'$ is a K_4 in L_v

(ii) at least one of xx', yy' forms a $K_4^{(3)}$ with v and z

(iii) at least one of $xy, xy', x'y, x'y'$ forms a $K_4^{(3)}$ with v and z.

Notice that condition (i) holds for every $(x,x',y,y') \in A_{\star\star} \cap (X^2 \times Y^2)$. Let us now fix some such quadruple (x,x',y,y'). Due to our assumption about $\delta_2(H)$ we have

$$\begin{aligned} d_H(x,y) + d_H(x,y') + d_H(x',y) + d_H(x',y') + 2\big(d_H(x,x') + d_H(y,y')\big)\\ +2\big(d_{L_v}(x) + d_{L_v}(x') + d_{L_v}(y) + d_{L_v}(y')\big) \geqslant 16\delta_2(H) \geqslant (12+16\alpha)n\,. \end{aligned}$$

So writing

$$\begin{aligned} f(z) = |\{xy, xy', x'y, x'y'\} \cap E(L_z)| + 2|\{xx', yy'\} \cap E(L_z)|\\ + 2|\{vx, vx', vy, vy'\} \cap E(L_z)| \end{aligned}$$

for every $z \in V$ we get

$$\sum_{z \in V} f(z) \geqslant (12+16\alpha)n\,.$$

If z is a vertex with $(x, x', y, y', z) \notin A$, then $f(z) \leqslant 12$, and hence we have

$$\#\{z \in V\colon (x, x', y, y', z) \in A\} \geqslant 16\alpha n/4 = 4\alpha n\,.$$

Applying this argument to every $(x, x', y, y') \in A_{\star\star} \cap (X^2 \times Y^2)$ we obtain, since we are in Case 2, that

$$|A| \geqslant \frac{\alpha}{32} n^4 \cdot 4\alpha n = \frac{\alpha^2}{8} n^5\,. \tag{3.10}$$

Now let us denote by A_x (resp. A_y) the number of quintuples (x, x', y, y', z) in $X^2 \times Y^2 \times (V \smallsetminus \{v\})$ such that

- $xx'vz$ (resp. $yy'vz$) and $x'yvz$ are $K_4^{(3)}$.

By symmetry we have

$$A_x + A_y \geqslant \frac{1}{4}|A| \overset{(3.10)}{\geqslant} \frac{\alpha^2}{32} n^5\,.$$

Consequently at least one of A_x, A_y is at least $\frac{\alpha^2}{64} n^5$. In either case one can prove that $e_{G_v}(X, Y) \geqslant \gamma n^2$ and below we display the argument assuming $A_x \geqslant \frac{\alpha^2}{64} n^5$. In this case

$$\begin{aligned} A_x &\leqslant \sum_{(x,y)\in X\times Y} \#\{(x', y', z) \in V^3\colon xx'zv \text{ and } yx'zv \text{ are } K_4^{(3)}\} \\ &\leqslant n \sum_{(x,y)\in X\times Y} \#\{(x', z) \in V^2\colon xx'zv \text{ and } yx'zv \text{ are } K_4^{(3)}\} \\ &\leqslant |X||Y|\beta n^3 + e_{G_v}(X, Y) n^3 \end{aligned}$$

yields

$$e_{G_v}(X, Y) \geqslant \Big(\frac{\alpha^2}{64} - \frac{\beta}{4}\Big) n^2\,,$$

as desired. The case $A_y \geqslant \frac{\alpha^2}{64} n^5$ is similar. □

Proof of Proposition 33. Because of Lemma 34 and the fact that

$$e_{G_v}(X, Y) = \sum_{x\in X} d_Y^{G_v}(x) \geqslant \delta(G_v) \cdot |X| - |X|^2\,,$$

Lemma 35 is already true if $|X|, |Y| \geqslant \sqrt{\gamma} n$. Therefore the assumptions of Lemma 27 hold for the graph G_v, which implies Proposition 33. □

3.2.4 Connecting Lemma

For the rest of this section we will use the constant $\frac{4}{5}$, i.e., the minimum pair-degree hypothesis $\delta_2(H) \geqslant (4/5 + \alpha)n$.

Definition 36. *For a 3-uniform hypergraph $H = (V, E)$ and vertices $v, r, s \in V$ we write*

$$N_v(r, s) = N(r, s, v) = N(r, v) \cap N(s, v) \cap N(r, s)\,.$$

Notice that our minimum pair-degree condition entails

$$|N_v(r, s)| \geqslant n/4 \tag{3.11}$$

for all $v, r, s \in V$.

Definition 37. *Given $n \gg \alpha^{-1}$, a hypergraph H on n vertices with minimum pair-degree $\delta_2(H) \geqslant (4/5 + \alpha)n$ and two distinct vertices $v, w \in V(H)$ we define the auxiliary graph G_{vw} by $V(G_{vw}) = N(v, w)$ and*

$$uu' \in E(G_{vw}) \Longleftrightarrow uu'vw \text{ is a } K_4^{(3)}\,.$$

Due to our assumption about the minimum pair-degree we know that the size n' of the vertex set satisfies $n' = |V(G_{vw})| \geqslant (4/5 + \alpha)n$.

Lemma 38. *Let $v, w \in V$ and $b, x \in V(G_{vw})$. There are at least $\Omega(n^2)$ walks of length 3 from b to x in G_{vw}.*

Proof. For a vertex $r \in V(G_{vw})$ we have

$$\begin{aligned} d_{G_{vw}}(r) &\geqslant |V(G_{vw})| - 2(n - \delta_2(H)) \\ &\geqslant \frac{|V(G_{vw})|}{2} + \frac{\delta_2(H)}{2} - 2(n - \delta_2(H)) \\ &= \frac{|V(G_{vw})|}{2} + \frac{5\delta_2(H)}{2} - 2n \geqslant \frac{n'}{2} + \frac{5\alpha n}{2} \geqslant \Big(\frac{1}{2} + \alpha\Big)n'\,. \end{aligned}$$

Thus the minimum degree of the auxiliary graph G_{vw} can be bounded from below by $\delta(G_{vw}) \geqslant (1/2 + \alpha)n'$ and any two vertices of G_{vw} have at least $2\alpha n'$ common neighbours in G_{vw}. Due to this and the minimum vertex degree condition in G_{vw} we can therefore find at least

$$\frac{n'}{2} \cdot 2\alpha n' = \alpha(n')^2 \geqslant \frac{\alpha}{2} n^2$$

walks of length 3 from b to x in G_{vw}. This shows Lemma 38. $\square$

Lemma 39. *If $vbc, vxy \in E$ and $|N_v(b,c) \cap N_v(x,y)| = m$, then there are at least $\Omega(m^2n^2)$ quadruples (w_0, b_1, c_1, w_1) such that $bcw_0b_1c_1w_1xy$ is*

- *a walk in H and*
- *a squared walk in L_v.*

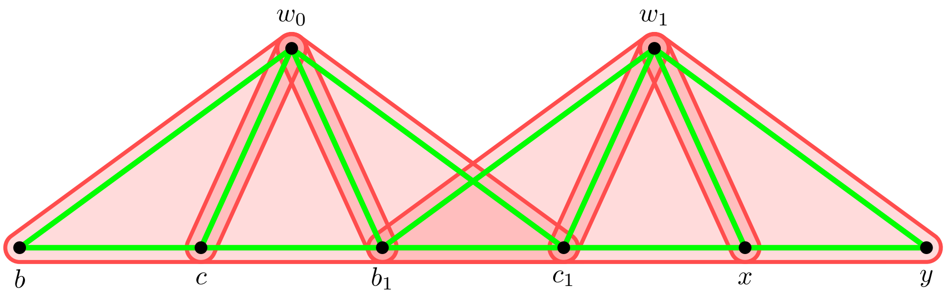

Figure 3.4: Quadruple (w_0, b_1, c_1, w_1) that fulfills the conditions of Lemma 39, where the link graph of v is indicated in green and hyperedges of H in red.

Proof. For every $w \in N_v(b,c) \cap N_v(x,y)$ Lemma 38 states that there are at least $\Omega(n^2)$ walks in G_{vw} from c to x of length 3. Let

$$X_{b_1c_1} = \{w \in N_v(b,c) \cap N_v(x,y) \colon cb_1c_1x \text{ is a walk in } G_{vw}\}$$

for $b_1, c_1 \in V$. Thus

$$\sum_{(b_1,c_1)\in V^2} |X_{b_1c_1}| \geqslant \Omega(mn^2)$$

and therefore the Cauchy-Schwarz inequality yields that

$$\sum_{(b_1,c_1)\in V^2} |X_{b_1c_1}|^2 \geqslant \Omega(m^2n^2)\,.$$

If $b_1, c_1 \in V$ and $w_0, w_1 \in X_{b_1c_1}$, then $bcw_0b_1c_1w_1xy$ has the desired properties. $\square$

Proposition 40. *There is an integer K, such that for all edges $abc, xyz \in E$ and vertices $v \in N(a,b,c) \cap N(x,y,z)$ there are for some $k = k(abc, xyz) \leqslant K$ with $k \equiv 1 \pmod 3$ at least $\Omega(n^k)$ many $(u_1, \ldots, u_k) \in V^k$ for which $abcu_1 \ldots u_kxyz$ is*

- *a walk in H*

- *a squared walk in L_v .*

Proof. Recall that in Proposition 33 we found an integer ℓ and a function $t\colon V^{(2)} \to [\ell]$ such that for all distinct $r, s \in V$ there are $\Omega(n^{t(r,s)-1})$ walks of length $t(r,s)$ from r to s in G_v. By the box principle there exists an integer $t \leqslant \ell$ such that the set $\mathcal{Q} \subseteq N_v(b,c) \times N_v(x,y)$ of all pairs $(u,u') \in N_v(b,c) \times N_v(x,y)$ with $t(u,u') = t$ satisfies

$$|\mathcal{Q}| \geqslant \frac{|N_v(b,c)| \cdot |N_v(x,y)|}{\ell} \overset{(3.11)}{\geqslant} \frac{n^2}{16\ell} .$$

For each walk $v_0 v_1 \dots v_t$ in G_v there are by Definition 32 at least $(\beta n^2)^t$ many $(2t)$-tuples $(b_1, c_1, \dots, b_t, c_t)$ such that

(i) $b_i c_i v \in E$ for $i = 1, \dots, t$,

(ii) $v_0 \in N_v(b_1, c_1)$ and $v_t \in N_v(b_t, c_t)$,

(iii) $v_i \in N_v(b_i, c_i) \cap N_v(b_{i+1}, c_{i+1})$ for $i = 1, \dots, t-1$.

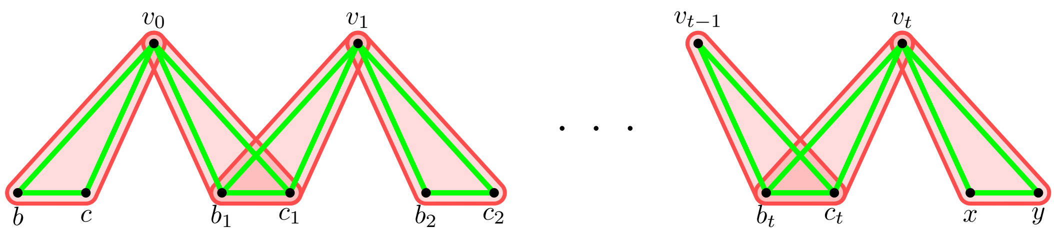

Figure 3.5: A $(3t+1)$-tuple $(v_0, v_1, \dots, v_t, b_1, c_1, \dots, b_t, c_t) \in V^{3t+1}$ satisfying (i), (ii), (iii), and (iv), where the link graph of v is indicated in green and hyperedges of H in red.

Consequently, there are at least

$$\frac{n^2}{16\ell} \cdot \Omega(n^{t-1}) \cdot (\beta n^2)^t = \Omega(n^{3t+1})$$

$(3t+1)$-tuples $(v_0, v_1, \dots, v_t, b_1, c_1, \dots, b_t, c_t) \in V^{3t+1}$ satisfying (i), (ii), (iii) as well as

(iv) $v_0 \in N_v(b,c)$ and $v_t \in N_v(x,y)$.

On the other hand, we can also write the number of these $(3t+1)$-tuples as

$$\sum_{\vec{v}\in\Psi} |I_0(\vec{v})| \cdot |I_1(\vec{v})| \cdot \ldots \cdot |I_t(\vec{v})| \,,$$

where

$$\Psi = \{(b_1, c_1, \ldots, b_t, c_t) \in V^{2t} \colon b_i c_i v \in E \text{ for } i = 1, \ldots, t\}$$

and for fixed $\vec{v} = (b_1, c_1, \ldots, b_t, c_t) \in \Psi$

- $I_0(\vec{v}) = N_v(b, c) \cap N_v(b_1, c_1)$
- $I_i(\vec{v}) = N_v(b_i, c_i) \cap N_v(b_{i+1}, c_{i+1})$ for $i = 1, \ldots, t-1$
- $I_t(\vec{v}) = N_v(b_t, c_t) \cap N_v(x, y)$.

Altogether we have thereby shown that

$$\sum_{\vec{v}\in\Psi} |I_0(\vec{v})| \cdot |I_1(\vec{v})| \cdot \ldots \cdot |I_t(\vec{v})| \geqslant \Omega(n^{3t+1}) \,. \tag{3.12}$$

Due to (3.12) and Lemma 39 there are at least

$$\begin{aligned}
&\sum_{\vec{v}\in\Psi} \Omega(|I_0(\vec{v})|^2 n^2) \cdot \ldots \cdot \Omega(|I_t(\vec{v})|^2 n^2) \\
&\geqslant \Omega(n^{2t+2}) \sum_{\vec{v}\in\Psi} (|I_0(\vec{v})| \cdot \ldots \cdot |I_t(\vec{v})|)^2 \\
&\geqslant \Omega(n^{2t+2}) \frac{\Big(\sum_{\vec{v}\in\Psi} |I_0(\vec{v})| \cdot \ldots \cdot |I_t(\vec{v})|\Big)^2}{|\Psi|} \\
&\geqslant \Omega(n^{2t+2}) \Big(\frac{\Omega(n^{3t+1})}{n^t}\Big)^2 = \Omega(n^{6t+4})
\end{aligned}$$

$(6t+4)$-tuples, which fulfill the conditions of Proposition 40.

Since $6t + 4 \equiv 1 \pmod 3$ this concludes the proof. □

Definition 41. *We call a sequence of vertices $v_1 \ldots v_h$ a squared v-walk from abc to xyz with h interior vertices if $abcv_1 \ldots v_h xyz$ is a walk in H and a squared walk in L_v.*

Proposition 42. *For all $abc, xyz \in E$ and $v \in N(a, b, c) \cap N(x, y, z)$ there are for some $k' = k'(abc, xyz, v) \leqslant K + 2$ with $k' \equiv 0 \pmod 3$ at least $\Omega(n^{k'})$ many squared v-walks with k' interior vertices from abc to xyz.*

Proof. We choose vertices $d \in N_v(b, c)$ and $e \in N_v(c, d)$, and with Proposition 40 we find at least $\Omega(n^k)$ many squared v-walks from cde to xyz, where $k = k(cde, xyz) \leqslant K$ and $k \equiv 1 \pmod 3$. Notice that if $u_1 \dots u_k$ is such a walk, then $deu_1 \dots u_k$ is a squared v-walk from abc to xyz.

Since $|N_v(b, c)|, |N_v(c, d)| \geqslant n/4$ holds by (3.11), there are for some $k \leqslant K$ with $k \equiv 1 \pmod 3$ at least $\frac{n^2/16}{K} = \Omega(n^2)$ pairs (d, e) with $k(cde, xyz) = k$. Now altogether there are $\Omega(n^{k+2})$ squared v-walks from abc to xyz with $k + 2$ interior vertices. This implies Proposition 42, since $k + 2 \equiv 0 \pmod 3$. □

Lemma 43. *If $abc, xyz \in E$ and $|N(a, b, c) \cap N(x, y, z)| = m$, then there is an integer $t = t(abc, xyz) \leqslant (K + 2)/3$ such that at least $\Omega(m^{t+1}n^{3t})$ squared walks from abc to xyz with $4t + 1$ interior vertices exist.*

Proof. For every $w \in N(a, b, c) \cap N(x, y, z)$ Proposition 42 states that for some integer $k' = k'(w) \leqslant K + 2$ with $k' \equiv 0 \pmod 3$ there are at least $\Omega(n^{k'})$ many squared w-walks from abc to xyz with k' interior vertices. By the box principle there exists an integer $k'' \leqslant K + 2$ with $k'' \equiv 0 \pmod 3$ such that the set $\mathcal{Q} \subseteq N(a, b, c) \cap N(x, y, z)$ of all vertices $w' \in N(a, b, c) \cap N(x, y, z)$ with $k'(w') = k''$ satisfies

$$|\mathcal{Q}| \geqslant \frac{|N(a, b, c) \cap N(x, y, z)|}{K + 2} = \frac{m}{K + 2}\,.$$

For $P = (u_1, \dots, u_{k''}) \in V^{k''}$ let $X_P \subseteq \mathcal{Q}$ be the set of vertices $u \in \mathcal{Q}$ such that P is a squared u-walk from abc to xyz. Since $|\mathcal{Q}| \geqslant m/(K + 2)$, the average size of X_P is at least $\Omega(m/(K + 2)) = \Omega(m)$ by Proposition 42 and double counting. Since

$$\frac{\sum_{P \in V^{k''}} X_P^{k''/3+1}}{n^{k''}} \geqslant \left(\frac{\sum_{P \in V^{k''}} X_P}{n^{k''}}\right)^{k''/3+1} \geqslant \Omega(m^{k''/3+1})\,,$$

we get

$$\sum_{P \in V^{k''}} X_P^{k''/3+1} \geqslant \Omega(m^{k''/3+1} n^{k''})\,.$$

Since $k'' \equiv 0 \pmod 3$ and every ordered k''-tuple P of vertices gives rise to $X_P^{k''/3+1}$ squared walks from abc to xyz with $4k''/3+1$ interior vertices, this implies Lemma 43 with $t = k''/3$. □

Finally we come to the main result of this section stated earlier as Proposition 22.

Proposition 44 (Connecting Lemma). *There are an integer M and $\vartheta_* > 0$, such that for all disjoint triples (a,b,c) and (x,y,z) with $abc, xyz \in E$ there exists $m < M$ for which there are at least $\vartheta_* n^m$ squared paths from abc to xyz with m internal vertices.*

Proof. Recall that in Proposition 29 we found an integer ℓ and a function $t\colon V^{(2)} \to [\ell]$ such that for all distinct $r, s \in V$ there are $\Omega(n^{t(r,s)-1})$ walks of length $t(r,s)$ from r to s in G_3. By the box principle there exists an integer $t \leqslant \ell$ such that the set $\mathcal{Q} \subseteq N(a,b,c) \times N(x,y,z)$ of pairs $(u,u') \in N(a,b,c) \times N(x,y,z)$ with $t(u,u') = t$ satisfies

$$|\mathcal{Q}| \geqslant \frac{|N(a,b,c)| \cdot |N(x,y,z)|}{\ell} \geqslant \frac{n^2}{16\ell}\,.$$

For each path $v_0 v_1 \dots v_t$ in G_3 there are by Definition 28 at least $(\beta n^3)^t$ many $(3t)$-tuples $(a_1, b_1, c_1, \dots, a_t, b_t, c_t)$ such that

(i) $a_i b_i c_i \in E$ for $i = 1, \dots, t$

(ii) $v_0 \in N(a_1, b_1, c_1)$ and $v_t \in N(a_t, b_t, c_t)$

(iii) $v_i \in N(a_i, b_i, c_i) \cap N(a_{i+1}, b_{i+1}, c_{i+1})$ for $i = 1, \dots, t-1$.

Consequently, there are at least

$$\frac{n^2}{16\ell} \cdot \Omega(n^{t-1}) \cdot (\beta n^3)^t = \Omega(n^{4t+1})$$

$(4t+1)$-tuples $(v_0, \dots, v_t, a_1, b_1, c_1, \dots, a_t, b_t, c_t) \in V^{4t+1}$ satisfying (i), (ii), (iii) as well as

(iv) $v_0 \in N(a,b,c)$ and $v_t \in N(x,y,z)$.

On the other hand, we can also write the number of these $(4t+1)$-tuples as

$$\sum_{\vec{v} \in \Psi} |I_0(\vec{v})| \cdot |I_1(\vec{v})| \cdot \ldots \cdot |I_t(\vec{v})|\,,$$

where

$$\Psi = \{(a_1, b_1, c_1, \dots, a_t, b_t, c_t) \in V^{3t} \colon a_i b_i c_i \in E \text{ for } i = 1, \dots, t\}$$

and for fixed $\vec{v} = (a_1, b_1, c_1, \dots, a_t, b_t, c_t) \in \Psi$

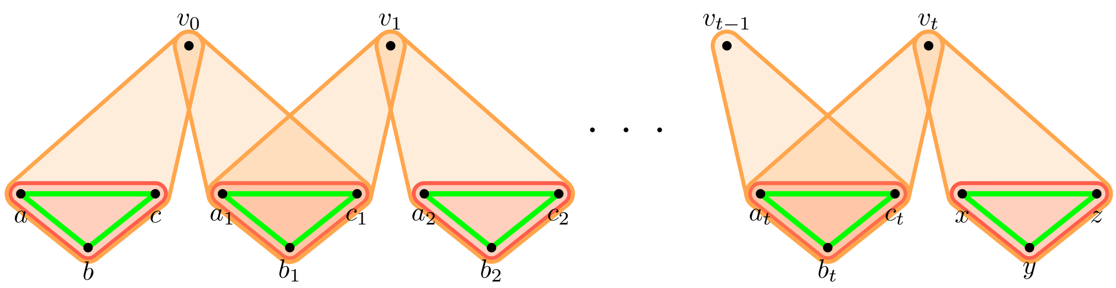

Figure 3.6: A $(4t+1)$-tuple $(v_0, \ldots, v_t, a_1, b_1, c_1, \ldots, a_t, b_t, c_t) \in V^{4t+1}$ satisfying *(i)*, *(ii)*, *(iii)*, and *(iv)*, where orange quadruples indicate a copy of $K_4^{(3)}$, hyperedges of H are indicated in red, and green pairs are in the link graph of the corresponding v_i.

- $I_0(\vec{v}) = N(a, b, c) \cap N(a_1, b_1, c_1)$
- $I_i(\vec{v}) = N(a_i, b_i, c_i) \cap N(a_{i+1}, b_{i+1}, c_{i+1})$ for $i = 1, \ldots, t-1$
- $I_t(\vec{v}) = N(a_t, b_t, c_t) \cap N(x, y, z)$

Altogether we have thereby shown that

$$\sum_{\vec{v} \in \Psi} |I_0(\vec{v})| \cdot |I_1(\vec{v})| \cdot \ldots \cdot |I_t(\vec{v})| \geqslant \Omega(n^{4t+1}) .$$

Lemma 43 gives us for every $\vec{v} \in \Psi$ some integers

- $t_0(\vec{v}) = t(abc, a_1 b_1 c_1)$
- $t_i(\vec{v}) = t(a_i b_i c_i, a_{i+1} b_{i+1} c_{i+1})$ for $i = 1, 2, \ldots, t-1$
- and $t_t(\vec{v}) = t(a_t b_t c_t, xyz)$.

By the box principle there are $\Psi_\star \subseteq \Psi$ and a $(t+1)$-tuple $(t_0, \ldots, t_t)$ in $[1, (K+2)/3]^{t+1}$ such that

$$\sum_{\vec{v} \in \Psi_\star} |I_0(\vec{v})| \cdot |I_1(\vec{v})| \cdot \ldots \cdot |I_t(\vec{v})| \geqslant \Omega(n^{4t+1}) \tag{3.13}$$

and $t_i(\vec{v}) = t_i$ for all $i \in \{0, \ldots, t\}$ and $\vec{v} \in \Psi_*$. Set $m = 4t + 4\sum_{i=0}^{t} t_i + 1$. Due to Lemma 43 there are at least

$$\begin{aligned}
&\sum_{\vec{v} \in \Psi_\star} \Omega(|I_0(\vec{v})|^{t_0+1} n^{3t_0}) \cdot \ldots \cdot \Omega(|I_t(\vec{v})|^{t_t+1} n^{3t_t}) \\
&= \Omega(n^{3\sum_{i=0}^{t} t_i}) \sum_{\vec{v} \in \Psi_\star} |I_0(\vec{v})|^{t_0+1} \cdot \ldots \cdot |I_t(\vec{v})|^{t_t+1}
\end{aligned}$$

m-tuples, which up to repeated vertices fulfill the conditions of Proposition 44. Let $T = \max(t_0, \ldots, t_t)$. Since

$$|I_i(\vec{v})|^{T+1} = |I_i(\vec{v})|^{t_i+1} \cdot |I_i(\vec{v})|^{T-t_i} \leqslant |I_i(\vec{v})|^{t_i+1} \cdot n^{T-t_i},$$

we get

$$\begin{aligned}
n^{T(t+1)-\sum_{i=0}^{t} t_i} \sum_{\vec{v}\in\Psi_\star} \prod_{i=0}^{t} |I_i(\vec{v})|^{t_i+1} &= \sum_{\vec{v}\in\Psi_\star} \prod_{i=0}^{t} n^{T-t_i} |I_i(\vec{v})|^{t_i+1} \\
&\geqslant \sum_{\vec{v}\in\Psi_\star} |I_0(\vec{v})|^{T+1} \cdot \ldots \cdot |I_t(\vec{v})|^{T+1} = \sum_{\vec{v}\in\Psi_\star} (|I_0(\vec{v})| \cdot \ldots \cdot |I_t(\vec{v})|)^{T+1} \\
&\geqslant \left(\frac{\sum_{\vec{v}\in\Psi_\star} |I_0(\vec{v})| \cdot \ldots \cdot |I_t(\vec{v})|}{|\Psi_\star|} \right)^{T+1} \cdot |\Psi_\star| \overset{(3.13)}{\geqslant} \left(\frac{\Omega(n^{4t+1})}{n^{3t}} \right)^{T+1} \cdot n^{3t} \\
&\geqslant \Omega(n^{3t+(t+1)(T+1)}),
\end{aligned}$$

which implies that

$$\begin{aligned}
\Omega(n^{3\sum_{i=0}^{t} t_i}) &\sum_{\vec{v}\in\Psi_\star} |I_0(\vec{v})|^{t_0+1} \cdot \ldots \cdot |I_t(\vec{v})|^{t_t+1} \\
&\geqslant \Omega(n^{3t+(t+1)+\sum_{i=0}^{t} t_i + 3\sum_{i=0}^{t} t_i}) = \Omega(n^m).
\end{aligned}$$

At most $O(n^{m-1})$ tuples can fail being paths due to repeated vertices, thus there are $\Omega(n^m)$ squared paths from abc to xyz. This proves Proposition 44 with $M = \lceil 4\ell + 4(\ell+1) \cdot \frac{K+2}{3} + 2 \rceil$, since

$$m = 4t + 4\sum_{i=0}^{t} t_i + 1 \leqslant 4\ell + 4(\ell+1) \cdot \frac{K+2}{3} + 1. \qquad \square$$

3.3 Reservoir Set

In all proofs using a reservoir lemma the reservoir set $\mathcal{R}$ is obtained by taking a random subset of V. On a technical level there are several possibilities which properties of $\mathcal{R}$ one actually requires and below we follow closely the approach of [67].

Proposition 45. *Let ϑ_* and M be the constants given by the Connecting Lemma. Then there exists a reservoir set $\mathcal{R} \subseteq V$ with $\frac{\vartheta_*^2 n}{2} \leqslant |\mathcal{R}| \leqslant \vartheta_*^2 n$, such that for all disjoint triples (a, b, c) and (x, y, z) with $abc, xyz \in E$ there are at least $\vartheta_* |\mathcal{R}|^{m(abc,xyz)}/2$ connecting squared paths in H all of whose $m(abc, xyz) < M$ internal vertices belong to $\mathcal{R}$.*

Proof. Consider a random subset $\mathcal{R} \subseteq V$ with elements included independently with probability

$$p = \Big(1 - \frac{3}{10M}\Big)\vartheta_*^2 .$$

Therefore $|\mathcal{R}|$ is binomially distributed and Chernoff's inequality yields

$$\mathbb{P}(|\mathcal{R}| < \vartheta_*^2 n/2) = o(1) . \tag{3.14}$$

Since

$$\vartheta_*^2 n \geqslant (4/3)^{1/M} pn \geqslant (1+c)\mathbb{E}[|\mathcal{R}|]$$

for some sufficiently small $c = c(M) > 0$, we have

$$\mathbb{P}(|\mathcal{R}| > \vartheta_*^2 n) \leqslant \mathbb{P}\big(|\mathcal{R}| > (4/3)^{1/M} pn\big) = o(1) . \tag{3.15}$$

The Connecting Lemma ensures that for all triples (a, b, c) and (x, y, z) there are at least $\vartheta_* n^m$ squared paths connecting them with $m = m(abc, xyz) < M$ internal vertices.

Let $X = X((a, b, c), (x, y, z))$ be the random variable counting the number of squared paths from (a, b, c) to (x, y, z) with m internal vertices in $\mathcal{R}$. We get

$$\mathbb{E}[X] \geqslant p^m \vartheta_* n^m . \tag{3.16}$$

Including or not including a particular vertex into $\mathcal{R}$ affects the random variable X by at most mn^{m-1}, wherefore the Azuma-Hoeffding inequality (see, e.g., [39, Corollary 2.27]) implies

$$\begin{aligned}\mathbb{P}\big(X \leqslant \tfrac{2}{3}\vartheta_*(pn)^m\big) &\overset{(3.16)}{\leqslant} \mathbb{P}\big(X \leqslant \tfrac{2}{3}\mathbb{E}[X]\big)\\ &\leqslant \exp\left(-\frac{2\mathbb{E}[X]^2}{9n(mn^{m-1})^2}\right) = \exp(-\Omega(n)) .\end{aligned} \tag{3.17}$$

Since there are at most n^6 pairs of triples that we have to consider, the union bound and (3.14), (3.15) tell us that asymptotically almost surely the reservoir $\mathcal{R}$ satisfies

$$\frac{\vartheta_*^2 n}{2} \leqslant |\mathcal{R}| \leqslant (4/3)^{1/M} pn \leqslant \vartheta_*^2 n \tag{3.18}$$

and

$$X((a, b, c), (x, y, z)) \geqslant \frac{2}{3}\vartheta_*(pn)^m \tag{3.19}$$

for all pairs of disjoint edges $abc, xyz \in E$. In particular, there is some $\mathcal{R} \subseteq V$ satisfying (3.18) and (3.19). Now it follows that

$$X((a, b, c), (x, y, z)) \geqslant \vartheta_* |\mathcal{R}|^m / 2$$

holds for all $abc, xyz \in E$ as well, meaning that $\mathcal{R}$ has the desired properties. $\square$

Lemma 46. *Let $\mathcal{R} \subseteq V$ be a reservoir set, ϑ_* the constant given by the Connecting Lemma and $\mathcal{R}' \subseteq \mathcal{R}$ an arbitrary subset of size at most $\vartheta_*^4 n$. Then for all triples (a, b, c) and (x, y, z) there exist a connecting squared path with $m(abc, xyz) < M$ internal vertices in H whose internal vertices belong to the set $\mathcal{R} \smallsetminus \mathcal{R}'$.*

Proof. Let $m = m(abc, xyz)$. Since $|\mathcal{R}| \geqslant \frac{\vartheta_*^2 n}{2}$ and $\vartheta_* \ll M^{-1}$, we can arrange that

$$|\mathcal{R}'| \leqslant \vartheta_*^4 n \leqslant \frac{\vartheta_*}{4m} |\mathcal{R}| \,.$$

Every vertex in $\mathcal{R}'$ is a member of at most $m|\mathcal{R}|^{m-1}$ squared paths with internal vertices in $\mathcal{R}$. Consequently, there are at least

$$\frac{\vartheta_*}{2} |\mathcal{R}|^m - |\mathcal{R}'| m |\mathcal{R}|^{m-1} \geqslant \frac{\vartheta_*}{2} |\mathcal{R}|^m - \frac{\vartheta_*}{4m} m |\mathcal{R}|^m > 0$$

such paths with all internal vertices in $\mathcal{R} \smallsetminus \mathcal{R}'$. $\square$

To conclude this section we remark that taken together Proposition 45 and Lemma 46 entail Proposition 23.

3.4 Absorbing Path

The goal of this section is to establish Proposition 24 which, let us recall, requires the minimum degree condition $\delta_2(H) \geqslant (4/5 + \alpha)|V(H)|$. The common assumptions of all statements of this section are that we have

- $1 \gg \alpha \gg M^{-1} \gg \vartheta_* \gg n^{-1}$ such that the connecting lemma holds,
- a hypergraph $H = (V, E)$ with $|V| = n$ and $\delta_2(H) \geqslant (4/5 + \alpha)n$,
- and a reservoir set $\mathcal{R} \subseteq V$ satisfying, in particular, that $|\mathcal{R}| \leqslant \vartheta_*^2 n$.

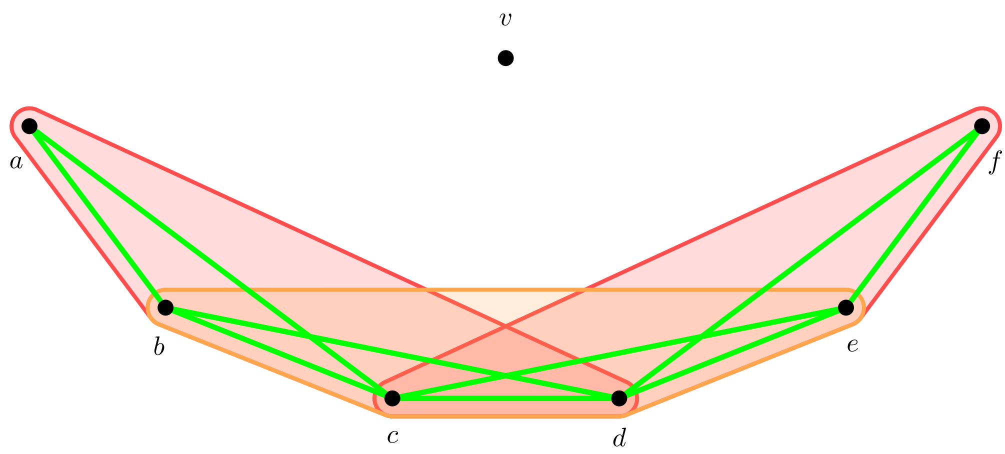

Figure 3.7: Example of a v-absorber, where the link graph of v is indicated in green and orange or red 4-edges indicate a copy of $K_4^{(3)}$.

Definition 47. *Given a vertex $v \in V$ and a 6-tuple $(a, b, c, d, e, f) \in (V \setminus \{v\})^6$ of distinct vertices, we call such a 6-tuple v-absorber if $abcdef$ and $abcvdef$ are squared paths in H.*

Lemma 48. *For every $v \in V$ there are at least $\alpha^3 n^6$ many v-absorbers in $(V \setminus \mathcal{R})^6$.*

Proof. Given $v \in V$ we choose the vertices of the 6-tuple in alphabetic order. For the first vertex we have n possible choices and for the second we still have more than $4n/5$ possibilities, since we only have the condition that $vab \in E$. For the third vertex we already have 3 conditions, namely $abc, vbc, vac \in E$. Consequently, we have more than $2n/5$ choices for c. For the vertices d, e, f we always have 5 conditions, so we have for each of them at least $5\alpha n$ possible choices. This implies that for given $v \in V$ we find more than

$$n \cdot 4n/5 \cdot 2n/5 \cdot (5\alpha n)^3 = 40\alpha^3 n^6$$

6-tuples meeting all the requirements from the v-absorber definition except that some of the 7 vertices $v, a, \ldots, f$ might coincide. There are at most $\binom{7}{2} n^5 = 21n^5$ such bad 6-tuples and at most $6\vartheta_*^2 n^6$ members of V^6 can use a vertex from the reservoir. Consequently, the number of v-absorbers in $(V \setminus \mathcal{R})^6$ is at least $\left(40\alpha^3 - \frac{21}{n} - 6\vartheta_*^2\right) n^6 \geqslant \alpha^3 n^6$. □

Lemma 49. *There is a set $\mathcal{F} \subseteq (V \setminus \mathcal{R})^6$ with the following properties:*

(1) $|\mathcal{F}| \leqslant 8\alpha^{-3}\vartheta_*^2 n$,

(2) all vertices of every 6-tuple in $\mathcal{F}$ are distinct and the 6-tuples in $\mathcal{F}$ are pairwise disjoint,

(3) if $(a,b,c,d,e,f) \in \mathcal{F}$, then $abcdef$ is a squared path in H

(4) and for every $v \in V$ there are at least $2\vartheta_*^2 n$ many v-absorbers in $\mathcal{F}$.

Proof. Consider a random selection $\mathcal{X} \subseteq (V \smallsetminus \mathcal{R})^6$ containing each 6-tuple independently with probability $p = \gamma n^{-5}$, where $\gamma = 4\vartheta_*^2/\alpha^3$. Since $\mathbb{E}[|\mathcal{X}|] \leqslant pn^6 = \gamma n$, Markov's inequality yields

$$\mathbb{P}(|\mathcal{X}| > 2\gamma n) \leqslant 1/2\,. \tag{3.20}$$

We call two distinct 6-tuples from V^6 *overlapping* if there is a vertex occurring in both. There are at most $36n^{11}$ ordered pairs of overlapping 6-tuples. Let P be the random variable giving the number of such pairs both of whose components are in $\mathcal{X}$. Since $\mathbb{E}[P] \leqslant 36n^{11}p^2 = 36\gamma^2 n$ and $12\gamma \leqslant \vartheta_*$, Markov's inequality yields

$$\mathbb{P}(P > \vartheta_*^2 n) \leqslant \mathbb{P}(P > 144\gamma^2 n) \leqslant \frac{1}{4}\,. \tag{3.21}$$

In view of Lemma 48 for each vertex $v \in V$ the set A_v containing all v-absorbers in $(V \smallsetminus \mathcal{R})^6$ has the property $\mathbb{E}[|A_v \cap \mathcal{X}|] \geqslant \alpha^3 n^6 p = \alpha^3 \gamma n = 4\vartheta_*^2 n$. Since $|A_v \cap \mathcal{X}|$ is binomially distributed, Chernoff's inequality gives for every $v \in V$

$$\mathbb{P}(|A_v \cap \mathcal{X}| \leqslant 3\vartheta_*^2 n) \leqslant \exp(-\Omega(n)) < \frac{1}{5n}\,. \tag{3.22}$$

Owing to (3.20), (3.21), and (3.22) there is an "instance" $\mathcal{F}_\star$ of $\mathcal{X}$ satisfying the following:

- $|\mathcal{F}_\star| \leqslant 2\gamma n$,
- $\mathcal{F}_\star$ contains at most $\vartheta_*^2 n$ ordered pairs of overlapping 6-tuples,
- and for every $v \in V$ the number of v-absorbers in $\mathcal{F}_\star$ is at least $3\vartheta_*^2 n$.

If we delete from $\mathcal{F}_\star$ all the 6-tuples containing some vertex more than once, all that belong to an overlapping pair, and all violating (3), we get a set $\mathcal{F}$ which fulfills (1), since $|\mathcal{F}| \leqslant |\mathcal{F}_\star|$. The properties (2) and (3) hold by construction and for (4) we recall that v-absorbers satisfy (3) by definition. Therefore the set $\mathcal{F}$ has all the desired properties. $\square$

We are now ready to prove Proposition 24, which we restate for the reader's convenience.

Proposition 50 (Absorbing path)**.** *There exists an (absorbing) squared path $P_A \subseteq H - \mathcal{R}$ such that*

(1) $|V(P_A)| \leqslant \vartheta_* n$,

(2) for every set $X \subseteq V \smallsetminus V(P_A)$ with $|X| \leqslant 2\vartheta_*^2 n$ there is a squared path in H whose set of vertices is $V(P_A) \cup X$ and whose end-triples are the same as those of P_A.

Proof. Let $\mathcal{F} \subseteq (V \smallsetminus \mathcal{R})^6$ be as obtained in Lemma 49. Recall that $\mathcal{F}$ is a family of at most $8\alpha^{-3}\vartheta_*^2 n$ vertex-disjoint squared paths with six vertices.

We will prove that there is a path $P_A \subseteq H - \mathcal{R}$ with the following properties:

(a) P_A contains all members of $\mathcal{F}$ as subpaths,

(b) $|V(P_A)| \leqslant (M+6)|\mathcal{F}|$.

Basically we will construct such a path P_A starting with any member of $\mathcal{F}$ by applying the connecting lemma $|\mathcal{F}| - 1$ times, attaching on further part from $\mathcal{F}$ each time.

Let $\mathcal{F}_* \subseteq \mathcal{F}$ be a maximal subset such that some path $P_A^* \subseteq H - \mathcal{R}$ has the properties (a) and (b) with $\mathcal{F}$ replaced by $\mathcal{F}_*$. Obviously $P_A^* \neq \varnothing$. From (b) and $1 \gg \alpha, M^{-1} \gg \vartheta_*$ we infer

$$|V(P_A^*)| \leqslant (M+6)|\mathcal{F}_*| \leqslant 2M|\mathcal{F}| \leqslant 16M\alpha^{-3}\vartheta_*^2 n \leqslant \vartheta_*^{3/2} n \tag{3.23}$$

and thus our upper bound on the size of the reservoir leads to

$$|V(P_A^*)| + |\mathcal{R}| \leqslant 2\vartheta_*^{3/2} n \leqslant \frac{\vartheta_* n}{2M}\,. \tag{3.24}$$

Assume for the sake of contradiction that $\mathcal{F}_* \neq \mathcal{F}$. Let (x, y, z) be the ending triple of P_A^* and let P be an arbitrary path in $\mathcal{F} \smallsetminus \mathcal{F}_*$ with starting triple (u, v, w). Then the connecting lemma tells us that there are at least $\vartheta_* n^m$ connecting squared paths with m interior vertices, where $m = m(xyz, uvw) < M$. By (3.24) at least half of them are disjoint to $V(P_A^*) \cup \mathcal{R}$. At least one such connection gives us a path $P_A^{**} \subseteq H - \mathcal{R}$ starting with P_A^*, ending with P and satisfying

$$|V(P_A^{**})| = |V(P_A^*)| + m + |V(P)| \leqslant |V(P_A^*)| + m + 6 \leqslant (M+6)(|\mathcal{F}_*| + 1)\,.$$

So $\mathcal{F}_* \cup \{P\}$ contradicts the maximality of $\mathcal{F}_*$ and proves that we have indeed $\mathcal{F}_* = \mathcal{F}$. Therefore there exists a path P_A with the properties (a) and (b).

As proved in (3.23) this path satisfies condition (1) of Proposition 50. To establish (2) one absorbs the up to at most $2\vartheta_*^2 n$ vertices in X one by one into P_A. This is possible due to (a) combined with (4) from Lemma 49. $\square$

3.5 Almost spanning cycle

The main work of this section goes into the proof of Theorem 25, which will occupy the Subsections 3.5.1–3.5.4. Having obtained this result we will deduce Proposition 26 in Subsection 3.5.5.

The proof of Theorem 25 itself is structured as follows. In Subsection 3.5.1 we derive an "approximate version" of Pikhurko's $K_4^{(3)}$-factor theorem (see Lemma 51) by imitating his proof from [65]. This lemma leads to Theorem 25 in the light of the hypergraph regularity method, which we recall in Subsection 3.5.2. In Subsection 3.5.3 we explain why "tetrahedra in the reduced hypergraph" correspond to regular "tetrads" large fractions of which can be covered by long squared paths. Finally in Subsection 3.5.4 we put everything together and complete the proof of Theorem 25.

3.5.1 $K_4^{(3)}$-tilings

The subsequent lemma will later be applied to a hypergraph obtained by means of the regularity lemma.

Lemma 51. *Let $t \geqslant 36$, $0 < \alpha < 1/4$ and $\tau \ll \alpha$. Given a hypergraph G on t vertices such that all but at most τt^2 unordered pairs $xy \in V^{(2)}$ of distinct vertices satisfy $d(x, y) \geqslant (3/4 + \alpha)t$, it is possible to delete at most $2\sqrt{\tau}t + 13$ vertices and find a $K_4^{(3)}$-factor afterwards.*

The following proof is similar to Pikhurko's argument establishing [65, Theorem 1].

Proof. Let us call a pair of vertices *bad* if its pair-degree is smaller than $(3/4 + \alpha)t$. Moreover we will call a subhypergraph of G *good* if it does not contain any bad pair of vertices.

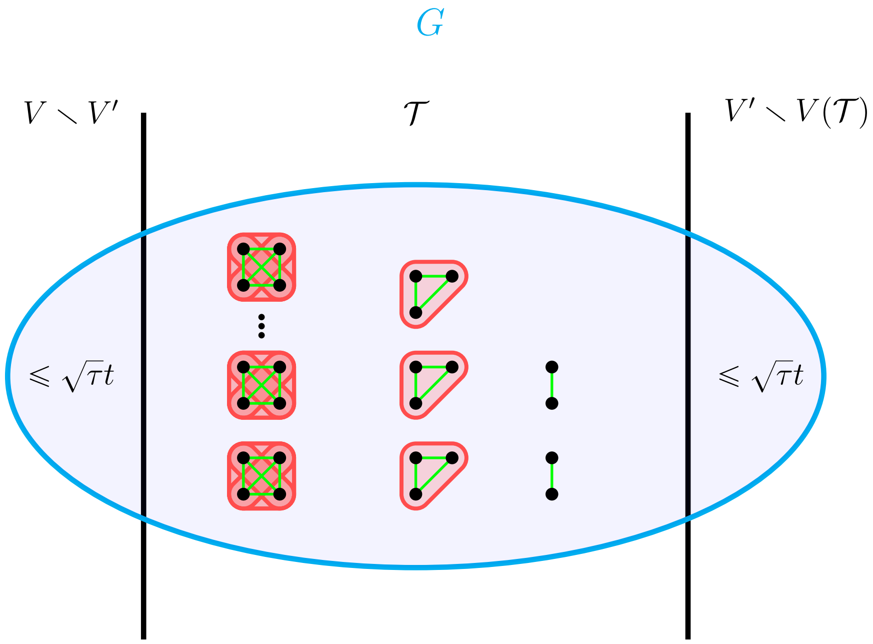

Figure 3.8: Example of a tiling $\mathcal{T}$ with maximal weight, where good pairs are indicated by green edges.

First of all we will delete vertices which are in many bad pairs. More precisely we will successively delete vertices if such a vertex is in at least $\sqrt{\tau}t$ bad pairs. Since there are at most τt^2 bad pairs, we are deleting at most $\sqrt{\tau}t$ vertices and in the remaining hypergraph $G' = (V', E')$ every vertex is in at most $\sqrt{\tau}t$ bad pairs.

Let $\mathcal{F}$ be a set of hypergraphs. By an *$\mathcal{F}$-tiling* in G we mean a collection of vertex-disjoint good subgraphs, each of which is isomorphic to some member of $\mathcal{F}$. Moreover let $w_2 = 2$, $w_3 = 6$, and $w_4 = 11$ be weight factors.

In the following we will consider a $\{K_2^{(3)}, K_3^{(3)}, K_4^{(3)}\}$-tiling $\mathcal{T}$ in G' that maximises the weight function $w(\mathcal{T}) = w_2\ell_2 + w_3\ell_3 + w_4\ell_4$, where ℓ_i denote the number of copies of $K_i^{(3)}$ in $\mathcal{T}$.

At most $\sqrt{\tau}t$ vertices of V' are missed by the tiling $\mathcal{T}$. Indeed, otherwise we find a good subgraph isomorphic to $K_2^{(3)}$ not in the tiling, since every vertex in V' is in at most $\sqrt{\tau}t$ bad pairs. Because $w_2 > 0$ this is a contradiction to the maximality of $\mathcal{T}$.

We say a hypergraph $F \in \mathcal{T}$ makes a *connection* with the vertex $x \in V' \setminus V(F)$ (denoted by $(F, x) \in \mathcal{C}$) if $|V(F)| \leqslant 3$ and $V(F) \cup \{x\}$ spans a complete good hypergraph. Examining the properties of connections, we get the following results.

- A $K_i^{(3)}$-subgraph $F \in \mathcal{T}$ with $i \leqslant 3$ can only make a connection to a vertex x that belongs to a $K_j^{(3)}$-subgraph of $\mathcal{T}$ with $j > i$.

Otherwise moving x to F would increase the weight of $\mathcal{T}$, since $w_4+w_2-2w_3 = 1$, $w_4 - w_2 - w_3 = 3$, $w_3 - 2w_2 = 2$, and all other possible weight changes are positive as well.

- Each $K_2^{(3)}$-subgraph F in $\mathcal{T}$ makes at least $(\frac{3}{4} + \frac{\alpha}{2})t$ connections.

Let $\{a, b\}$ be the vertex set of $K_2^{(3)}$-subgraph F of $\mathcal{T}$. The subgraph F makes a connection with a vertex $x \in V' \setminus V(F)$ if $abx \in E(G)$ and ab, ax, bx are good pairs. Recalling that ab is a good pair due to the definition of tiling, we can relax the second condition to ax, bx being good pairs. There are at least $(\frac{3}{4} + \alpha - \sqrt{\tau})t$ vertices in $V' \setminus V(F)$ that form an edge with ab in G. Since every vertex in V' is in at most $\sqrt{\tau}t$ bad pairs, at most $2\sqrt{\tau}t$ vertices, which form an edge with ab in G, can fail the second condition. Thus, every $K_2^{(3)}$-subgraph F of $\mathcal{T}$ makes at least $(\frac{3}{4} + \alpha - 3\sqrt{\tau})t$ connections, which due to $\tau < \frac{\alpha^2}{36}$ is more than $(\frac{3}{4} + \frac{\alpha}{2})t$.

- Every $K_3^{(3)}$-subgraph F in $\mathcal{T}$ makes at least $(\frac{1}{4} + \alpha)t$ connections.

For each $K_3^{(3)}$-subgraph F of $\mathcal{T}$ there are at least $(\frac{9}{4} + \alpha)t$ edges that intersect it in exactly two vertices and consists of no bad pairs. Let c denote the number of connections made by a $K_3^{(3)}$-subgraph of $\mathcal{T}$. Thus, we get

$$\Big(\frac{9}{4} + \alpha\Big)t \leqslant 3c + 2(t - 3 - c)\,,$$

i.e.,

$$\Big(\frac{9}{4} + \alpha\Big)t - 2t + 6 \leqslant c\,.$$

- $\ell_3 \leqslant 3$.

Otherwise let F_1, F_2, F_3, F_4 be $K_3^{(3)}$-subgraphs in $\mathcal{T}$. All connections made by a F_i belong to a $K_4^{(3)}$-subgraph of $\mathcal{T}$ by the first bullet above. An upper bound for the number of $K_4^{(3)}$ in $\mathcal{T}$ is $\lfloor t/4 \rfloor$. Since

$$4\Big(\frac{1}{4} + \alpha\Big)t > 4\lfloor t/4 \rfloor\,,$$

the vertices of some $K_4^{(3)}$-subgraph F of $\mathcal{T}$ make at least 5 connections with F_1, F_2, F_3, F_4. Therefore we find two distinct vertices $x, y \in V(F)$ and $i, j \in [4]$

with $i \neq j$, such that $(F_i, x), (F_j, y) \in \mathcal{C}$. Moving x to F_i and y to F_j and thereby reducing F to a $K_2^{(3)}$ would increase the weight of $\mathcal{T}$, since $2(w_4 - w_3) + (w_2 - w_4) = 1$. Thus, we get a contradiction to the maximality of $\mathcal{T}$.

Case 1. $\ell_2 \geqslant 3$

Let F_1, F_2, F_3 be $K_2^{(3)}$-subgraphs in $\mathcal{T}$.

- There is no $K_3^{(3)}$-subgraph $F \in \mathcal{T}$ with the property that F_1, F_2, F_3 make more than 3 connections to F.

Otherwise we could find distinct vertices $x, y \in V(F)$ and $i, j \in [3]$ with $i \neq j$, such that $(F_i, x), (F_j, y) \in \mathcal{C}$. Moving x to F_i and y to F_j and thereby eliminating F would increase the weight of $\mathcal{T}$, since $2(w_3 - w_2) - w_3 = 2$. Thus, we get a contradiction to the maximality of $\mathcal{T}$.

- There is no $K_4^{(3)}$-subgraph $F \in \mathcal{T}$ with the property that F_1, F_2, F_3 make more than 8 connections to F.

Otherwise we could find distinct vertices $x_1, x_2, x_3 \in V(F)$, such that $(F_i, x_i) \in \mathcal{C}$ for every $i \in [3]$. This is because every bipartite graph with nine edges and partition classes of size 3 and 4 contains a matching of size 3. Moving each x_i to F_i and thereby eliminating F would increase the weight of $\mathcal{T}$, since $3(w_3 - w_2) - w_4 = 1$. Thus, we get a contradiction to the maximality of $\mathcal{T}$.

Finally, by estimating the number of connections created by F_1, F_2, F_3 we obtain

$$3\Big(\frac{3}{4} + \frac{\alpha}{2}\Big)t \leqslant 3\ell_3 + 8\ell_4 \,.$$

Since $\ell_3 \leqslant 3$ and $\ell_4 \leqslant \lfloor t/4 \rfloor$, we have

$$\Big(\frac{9}{4} + \frac{3}{2}\alpha\Big)t \leqslant 9 + 8\lfloor t/4 \rfloor \,,$$

which contradicts $t \geqslant 36$.

Case 2. $\ell_2 \leqslant 2$

We have deleted $\sqrt{\tau}t$ vertices from G to obtain the graph G', another $\sqrt{\tau}t$ vertices can be missed by the tiling $\mathcal{T}$, and at most $2\ell_2 + 3\ell_3 \leqslant 13$ vertices of $V(\mathcal{T})$ are not covered by $K_4^{(3)}$ subgraphs. Therefore it is possible to delete at most $2\sqrt{\tau}t + 13$ vertices and find a $K_4^{(3)}$-factor afterwards. □

3.5.2 Hypergraph regularity method

We denote by $K(X, Y)$ the complete bipartite graph with vertex partition $X \dot\cup Y$. For a bipartite graph $P = (X \dot\cup Y, E)$ we say it is (δ_2, d_2)-*quasirandom* if

$$\big|e(X', Y') - d_2|X'||Y'|\big| \leqslant \delta_2|X||Y|$$

holds for all subsets $X' \subseteq X$ and $Y' \subseteq Y$, where $e(X', Y')$ denotes the number of edges in P with one vertex in X' and one in Y'. Given a k-partite graph $P = (X_1 \dot\cup \ldots \dot\cup X_k, E)$ with $k \geqslant 2$ we say P is (δ_2, d_2)-*quasirandom*, if all naturally induced bipartite subgraphs $P[X_i, X_j]$ are (δ_2, d_2)-quasirandom. Moreover, for a tripartite graph $P = (X \dot\cup Y \dot\cup Z, E)$ we denote by

$$\mathcal{K}_3(P) = \big\{\{x, y, z\} \subseteq X \cup Y \cup Z : xy, xz, yz \in E\big\}$$

the triples of vertices in P spanning a triangle. For a (δ_2, d_2)-quasirandom tripartite graph $P = (X \dot\cup Y \dot\cup Z, E)$ the so-called *triangle counting lemma* implies that

$$d_2^3|X||Y||Z| - 3\delta_2|X||Y||Z| \leqslant |\mathcal{K}_3(P)| \leqslant d_2^3|X||Y||Z| + 3\delta_2|X||Y||Z| . \tag{3.25}$$

Definition 52. *Given a 3-uniform hypergraph $H = (V, E_H)$ and a tripartite graph $P = (X \dot\cup Y \dot\cup Z, E)$ with $X \cup Y \cup Z \subseteq V$ we say H is (δ_3, d_3)-quasirandom with respect to P if for every tripartite subgraph $Q \subseteq P$ we have*

$$\big||E_H \cap \mathcal{K}_3(Q)| - d_3|\mathcal{K}_3(Q)|\big| \leqslant \delta_3|\mathcal{K}_3(P)| .$$

Furthermore, we say H is δ_3-quasirandom with respect to P, if it is (δ_3, d_3)-quasirandom for some $d_3 \geqslant 0$.

We define the *relative density* of H with respect to P by

$$d(H|P) = \frac{|E_H \cap \mathcal{K}_3(P)|}{|\mathcal{K}_3(P)|} ,$$

where $d(H|P) = 0$ if $\mathcal{K}_3(P) = \varnothing$.

A refined version of the regularity lemma (see [73, Theorem 2.3]) states the following.

Lemma 53 (Regularity Lemma)**.** *For every $\delta_3 > 0$, every $\delta_2 \colon \mathbb{N} \to (0, 1]$, and every $t_0 \in \mathbb{N}$ there exists an integer T_0 such that for every $n \geqslant t_0$ and every n-vertex 3-uniform hypergraph $H = (V, E_H)$ the following holds.*

There are integers t and ℓ with $t_0 \leqslant t \leqslant T_0$ and $\ell \leqslant T_0$ and there exists a vertex partition $V_0 \cupdot V_1 \cupdot \ldots \cupdot V_t = V$ and for all $1 \leqslant i < j \leqslant t$ there exists a partition

$$\mathcal{P}^{ij} = \{P_\alpha^{ij} = (V_i \cupdot V_j, E_\alpha^{ij}) \colon 1 \leqslant \alpha \leqslant \ell\}$$

of the edge set of the complete bipartite graph $K(V_i, V_j)$ satisfying the following properties

(1) $|V_0| \leqslant \delta_3 n$ and $|V_1| = \ldots = |V_t|$,

(2) for every $1 \leqslant i < j \leqslant t$ and $\alpha \in [\ell]$ the bipartite graph P_α^{ij} is $(\delta_2(\ell), 1/\ell)$-quasirandom, and

(3) H is δ_3-quasirandom w.r.t $P_{\alpha\beta\gamma}^{ijk}$ for all but at most $\delta_3 t^3 \ell^3$ tripartite graphs

$$P_{\alpha\beta\gamma}^{ijk} = P_\alpha^{ij} \cupdot P_\beta^{ik} \cupdot P_\gamma^{jk} = (V_i \cupdot V_j \cupdot V_k, E_\alpha^{ij} \cupdot E_\beta^{ik} \cupdot E_\gamma^{jk}),$$

with $1 \leqslant i < j < k \leqslant t$ and $\alpha, \beta, \gamma \in [\ell]$.

The tripartite graphs $P_{\alpha\beta\gamma}^{ijk}$ appearing in (3) are usually called *triads*. Furthermore we will use the following version of the embedding lemma from [60].

Lemma 54. *For every $p \in \mathbb{N}$ and $\xi, d_3 > 0$ there exist $\delta_3 > 0$ and functions $\delta_2 \colon \mathbb{N} \to (0,1)$, $N \colon \mathbb{N} \to \mathbb{N}$ such that the following holds.*

Let $\ell \in \mathbb{N}$ and let $G = \bigcupdot_{1 \leqslant i < j \leqslant p} G^{ij}$ be a p-partite graph with vertex partition $V_1 \cupdot \ldots \cupdot V_p$, where $|V_1| = \ldots = |V_p| = n \geqslant N(\ell)$, such that each $G^{ij} = G[V_i, V_j]$ is $(\delta_2(\ell), 1/\ell)$-quasirandom. Moreover, let H be a 3-uniform hypergraph that is (δ_3, d_{ijk})-quasirandom with respect to G^{ijk} for all $1 \leqslant i < j < k \leqslant p$, where $G^{ijk} = G[V_i, V_j, V_k]$ and $d_{ijk} \geqslant d_3$. Then the number $|\mathcal{K}_p(H)|$ of complete, 3-uniform hypergraphs on p vertices in H with one vertex from each V_i satisfies

$$|\mathcal{K}_p(H)| \geqslant (1 - \xi) d_3^{\binom{p}{3}} (1/\ell)^{\binom{p}{2}} n^p .$$

3.5.3 Squared paths in quasirandom tetrads

The Embedding Lemma 54 can be utilised to find a squared path in appropriate 4-partite environments.

Lemma 55. *Given $Q \in \mathbb{N}$ and $d_3 > 0$, there exist $\delta_3 > 0$, and functions $\delta_2 \colon \mathbb{N} \to (0,1]$ and $N \colon \mathbb{N} \to \mathbb{N}$, such that that the following holds for every $\ell \in \mathbb{N}$.*

Let $P = (V_1 \cup V_2 \cup V_3 \cup V_4, E_P)$ be a 4-partite graph with $|V_1| = \ldots = |V_4| = n$ and $n \geqslant N(\ell)$ such that $P^{ij} = (V_i \cup V_j, E^{ij})$ is $(\delta_2(\ell), 1/\ell)$-quasirandom for every pair $ij \in [4]^{(2)}$. Suppose H is a 4-partite, 3-uniform hypergraph with vertex classes $V_1, \ldots, V_4$, which satisfies for every $ijk \in [4]^{(3)}$ that H is (δ_3, d_{ijk})-quasirandom w.r.t. the tripartite graphs $P^{ijk} = P^{ij} \cup P^{ik} \cup P^{jk}$ for some $d_{ijk} \geqslant d_3$. Then there exists a squared path with Q vertices in H.

Proof. For $p = Q$, $\xi = 1/2$ and the current d_3 let $\delta_3 > 0$ and functions $\delta_2 \colon \mathbb{N} \to (0,1)$, $N \colon \mathbb{N} \to \mathbb{N}$ be given by Lemma 54. Moreover, let $W_1, \ldots, W_Q$ be disjoint vertex sets of size n. Choose for every $j \in [Q]$ and $i \in [4]$ with $i \equiv j \pmod 4$ a bijective function $\varphi_j \colon V_i \to W_j$. We copy E_P and $E(H)$ onto $W_1 \cup \ldots \cup W_Q$ in the following way.

- If for $1 \leqslant i < j \leqslant Q$ the integers $i', j' \in [4]$ satisfying $i \equiv i' \pmod 4$ and $j \equiv j' \pmod 4$ are distinct, let E_W^{ij} be the bipartite graph on $W_i \cup W_j$ defined by
$$xy \in E^{i'j'} \iff \varphi_i(x)\varphi_j(y) \in E_W^{ij}$$
for all $x \in V_{i'}$ and $y \in V_{j'}$.

- If for $1 \leqslant i < j < k \leqslant Q$ the integers $i', j', k' \in [4]$ satisfying $i \equiv i' \pmod 4$, $j \equiv j' \pmod 4$, and $k \equiv k' \pmod 4$ are distinct, let H_W^{ijk} be the tripartite hypergraph on $W_i \cup W_j \cup W_k$ defined by
$$xyz \in E(H) \iff \varphi_i(x)\varphi_j(y)\varphi_k(z) \in H_W^{ijk}$$
for all $x \in V_{i'}$, $y \in V_{j'}$, and $z \in V_{k'}$.

For technical reasons we also need to specify bipartite graphs E_W^{ij} for distinct $i, j \in [Q]$ that are congruent modulo 4 in order to make Lemma 54 applicable. The precise choice of these graphs is immaterial in the following and we just take arbitrary $(\delta_2(\ell), 1/\ell)$-quasirandom bipartite graphs. E.g., we could declare all theses graphs to be isomorphic to P^{12}. Similarly, we need to define 3-partite hypergraphs H_W^{ijk} for distinct $i, j, k \in [Q]$ at least two of which are congruent modulo 4. This time we may just take the complete 3-partite hypergraphs between W_i, W_j, W_k, which are certainly $(\delta_3, 1)$-quasirandom with respect to $(W_i \cup W_j \cup W_k, E_W^{ij} \cup E_W^{ik} \cup E_W^{jk})$.

By Lemma 54 applied to $G_W = (W_1 \dot\cup \dots \dot\cup W_Q, E_W)$, where $E_W = \dot\bigcup_{1 \leqslant i < j \leqslant Q} E_W^{ij}$ and the hypergraph $H_W = \dot\bigcup_{1 \leqslant i < j < k \leqslant Q} H_W^{ijk}$ we find at least $(1/2)n^Q(1/\ell)^{\binom{Q}{2}} d_3^{\binom{Q}{3}}$ squared paths $v_1 \dots v_Q$ in H_W with $v_i \in W_i$ for every $i \in [Q]$. Notice that every squared such path in H_W corresponds to a squared walk in H via the inverses of the maps φ_i. It may happen that vertices get identified under this correspondence and therefore there might be squared paths in H_W not yielding squared paths in H. However $\binom{Q}{2} n^{Q-1}$ is a straightforward upper bound on the number of times this can occur and since for n sufficiently large we have

$$\frac{1}{2} n^Q (1/\ell)^{\binom{Q}{2}} d_3^{\binom{Q}{3}} > \binom{Q}{2} n^{Q-1},$$

we find at least one squared path in H. □

Lemma 56. *Given $Q \in \mathbb{N}$ with $Q \equiv 0 \pmod 4$, $d_3 > 0$, and $\nu > 0$. There exist $\delta_3 > 0$, $\delta_2 \colon \mathbb{N} \to (0,1)$, and $N \colon \mathbb{N} \to \mathbb{N}$, such that the following holds for every $\ell \in \mathbb{N}$. Let $P = (V_1 \dot\cup V_2 \dot\cup V_3 \dot\cup V_4, E_P)$ be a 4-partite graph with $|V_1| = \dots = |V_4| = n \geqslant N(\ell)$ and let $P^{ij} = (V_i \dot\cup V_j, E^{ij})$ be $(\delta_2(\ell), 1/\ell)$-quasirandom for every $ij \in [4]^{(2)}$. Suppose that H is a 3-uniform hypergraph, which satisfies for every $ijk \in [4]^{(3)}$ that H is (δ_3, d_{ijk})-quasirandom with respect to the tripartite graph $P^{ijk} = P^{ij} \dot\cup P^{ik} \dot\cup P^{jk}$ for some $d_{ijk} \geqslant d_3$. Then all but at most νn vertices of $V_1 \dot\cup \dots \dot\cup V_4$ can be covered by vertex-disjoint squared paths with Q vertices each.*

Proof. Let $\delta_3^* > 0$, $\delta_2^* \colon \mathbb{N} \to (0,1]$, $N^* \colon \mathbb{N} \to \mathbb{N}$ be the number and functions obtained by applying Lemma 55 to Q and $d_3/2$. Define

$$\delta_3 = \frac{\delta_3^* \nu^3}{128}, \quad \delta_2(\ell) = \min\left(\frac{\delta_2^*(\ell)\nu^2}{16}, \frac{\nu^2}{144\ell^3}\right), \quad N(\ell) = \left\lceil \frac{4N^*(\ell)}{\nu} \right\rceil$$

for each $\ell \in \mathbb{N}$. Let $P = (V_1 \dot\cup V_2 \dot\cup V_3 \dot\cup V_4, E_P)$ and H be as described above for some $\ell \in \mathbb{N}$. Consider a maximal collection $S_1, \dots, S_m$ of vertex-disjoint squared paths on Q vertices in H. For $i \in [4]$ let $V_i' \subseteq V_i$ denote the set of vertices not belonging to any of these paths. Due to $4 \mid Q$ the sets $V_1', \dots, V_4'$ have the same size, say n^*. If $n^* < \nu n/4$ we are done, so assume from now on that $n^* \geqslant \nu n/4$. Then our choice of $\delta_2(\ell)$ implies that the bipartite graphs $P^{ij}[V_i' \dot\cup V_j']$ are $(\delta_2^{**}(\ell), 1/\ell)$-quasirandom, where $\delta_2^{**}(\ell) = \min(\delta_2^*(\ell), \frac{1}{9\ell^3})$. So by Lemma 55 we get a contradiction to the maximality of m provided we can show that H is (δ_3^*, d_{ijk})-quasirandom w.r.t. the subtriads P_*^{ijk} of P^{ijk} induced by $V_i' \cup V_j' \cup V_k'$. This is indeed the case, since the triangle counting lemma yields that

$$
\begin{aligned}
|\mathcal{K}_3(P^{123})| &\leqslant |V_1||V_2||V_3|\big(1/\ell^3 + 3\delta_2(\ell)\big) \\
&\overset{n^* \geqslant \nu n/4}{\leqslant} \frac{4^3|V_1'||V_2'||V_3'|}{\nu^3}(1/\ell^3 + 3\delta_2(\ell)) \\
&\leqslant \frac{64 \cdot \mathcal{K}_3(P[V_1', V_2', V_3'])}{\nu^3} \cdot \frac{(1/\ell^3 + 3\delta_2(\ell))}{(1/\ell^3 - 3\delta_2^{**}(\ell))} \\
&\leqslant 128 \cdot \frac{\mathcal{K}_3(P_*^{123})}{\nu^3}\,,
\end{aligned}
$$

i.e.,

$$
\delta_3|\mathcal{K}_3(P^{123})| \leqslant \delta_3^*|\mathcal{K}_3(P_*^{123})|\,,
$$

and the same argument applies to every other triple $ijk \in [4]^{(3)}$. □

3.5.4 Vertex-disjoint squared paths with Q vertices

Next we restate and prove Theorem 25.

Theorem 57. *Given $\alpha, \mu > 0$ and $Q \in \mathbb{N}$ there exists $n_0 \in \mathbb{N}$ such that in every hypergraph H with $v(H) = n \geqslant n_0$ and $\delta_2(H) \geqslant (3/4 + \alpha)n$ all but at most μn vertices of H can be covered by vertex-disjoint squared paths with Q vertices.*

Proof. As we could replace Q by $4Q$ if necessary we may suppose that Q is a multiple of 4. Pick sufficiently small $d_3, \nu, \tau \ll \alpha, \mu$ and let $\delta_3 > 0$, $\delta_2 \colon \mathbb{N} \to (0,1)$, $N \colon \mathbb{N} \to \mathbb{N}$ be the number and functions obtained by applying Lemma 56 to Q, ν, and d_3. W.l.o.g. δ_3, $\delta_2(\cdot)$ are sufficiently small, such that $\delta_3 \ll \alpha, \tau$, and $\delta_2(\ell) \ll \alpha, \ell^{-1}, \tau$. For t_0 sufficiently large we can use Lemma 53 with δ_3, δ_2, t_0 and get an integer T_0. Finally we let n_0 be sufficiently large.

Now let H be a 3-uniform hypergraph with $v(H) = n \geqslant n_0$ and $\delta_2(H) \geqslant (\frac{3}{4} + \alpha)n$. Due to Lemma 53 there exists a vertex partition $V_0 \,\dot\cup\, V_1 \,\dot\cup\, \dots \,\dot\cup\, V_t = V$ and pair partitions

$$
\mathcal{P}^{ij} = \{P_\alpha^{ij} = (V_i \,\dot\cup\, V_j, E_\alpha^{ij}) \colon 1 \leqslant \alpha \leqslant \ell\}
$$

of the complete bipartite graphs $K(V_i, V_j)$ for $1 \leqslant i < j \leqslant t$ satisfying (1)-(3).

We call a triad $P_{\alpha\beta\gamma}^{ijk}$ *dense* if $d(H|P_{\alpha\beta\gamma}^{ijk}) \geqslant \alpha/10$. For every pair $i_*j_* \in [t]^{(2)}$ and every $\lambda \in [\ell]$ we denote the set of dense triads involving V_{i_*}, V_{j_*}, and $P_\lambda^{i_*j_*}$ by $\mathcal{D}_\lambda(i_*, j_*)$.

Claim 58. *For every $i_* j_* \in [t]^{(2)}$ we have $|\mathcal{D}_\lambda(i_*, j_*)| \geqslant (\frac{3}{4} + \frac{\alpha}{2})\ell^2 t$.*

Proof. Notice that Lemma 53(1) yields

$$\frac{n(1-\delta_3)}{t} \leqslant |V_k| \leqslant \frac{n}{t} \tag{3.26}$$

for every $k \in [t]$. Appealing to the $(\delta_2(\ell), 1/\ell)$-quasirandomness of $P_\lambda^{i_* j_*}$ we infer

$$\begin{aligned} |E_\lambda^{i_* j_*}| &\geqslant \Big(\frac{1}{\ell} - \delta_2(\ell)\Big)|V_{i_*}||V_{j_*}| \\ &\geqslant \Big(\frac{1}{\ell} - \delta_2(\ell)\Big)\Big(\frac{(1-\delta_3)n}{t}\Big)^2 . \end{aligned}$$

Together with the lower bound on $\delta_2(H)$ and $|V_0| \leqslant \delta_3 n$ it follows that

$$\Big(\frac{1}{\ell} - \delta_2(\ell)\Big)\Big(\frac{(1-\delta_3)n}{t}\Big)^2\Big(\frac{3}{4} + \alpha - \delta_3\Big)n \leqslant \sum_{xy \in E_\lambda^{i_* j_*}} |N(x,y) \smallsetminus V_0| . \tag{3.27}$$

On the other hand we can derive an upper bound on the right side by counting the edges in each triad using $E_\lambda^{i_* j_*}$ separately. Due to the triangle counting lemma and (3.26) each such triad contains at most

$$\Big(\frac{1}{\ell^3} + 3\delta_2(\ell)\Big)\Big(\frac{n}{t}\Big)^3$$

triangles. Therefore we have

$$\sum_{xy \in E_\lambda^{i_* j_*}} |N(x,y) \smallsetminus V_0| \leqslant t\ell^2 \frac{\alpha}{10}\Big(\frac{n}{t}\Big)^3\Big(\frac{1}{\ell^3} + 3\delta_2(\ell)\Big) + |\mathcal{D}_\lambda(i_*, j_*)|\Big(\frac{n}{t}\Big)^3\Big(\frac{1}{\ell^3} + 3\delta_2(\ell)\Big) .$$

Combined with (3.27) this leads because of $\delta_3 \ll \alpha$ and $\delta_2 \ll \alpha/\ell^3$ to

$$|\mathcal{D}_\lambda(i_*, j_*)| \geqslant (3/4 + \alpha/2)\ell^2 t . \qquad \square$$

For every $f\colon [t]^2 \to [\ell]$ we define a hypergraph J_f on the vertex set $[t]$ such that a 3-element set $\{i, j, k\}$ is an edge of J_f if the triad $P^{ijk}_{f(ij)f(ik)f(jk)}$ is dense and H is δ_3-quasirandom w.r.t. this triad.

Claim 59. *There is $f\colon [t]^{(2)} \to [\ell]$ such that all but at most τt^2 pairs $ij \in [t]^{(2)}$ have at least pair-degree $(\frac{3}{4} + \frac{\alpha}{8})t$ in J_f.*

Proof. Let D_f be the hypergraph on $[t]$ whose edges are the triples ijk such that the triad $P^{ijk}_{f(ij)f(ik)f(jk)}$ is dense, and let R_f be the hypergraph consisting of all sets $\{i, j, k\}$ such that H is δ_3-quasirandom with respect to the triad $P^{ijk}_{f(ij)f(ik)f(jk)}$.

Clearly, $J_f = D_f \cap R_f$. We will show that if we choose f uniformly at random, then with positive probability $E(\overline{R_f}) \leqslant 2\delta_3 t^3$ and $\delta_2(D_f) \geqslant (3/4 + \alpha/4)t$ hold.

The expected value of the number of missing edges in R_f is

$$\mathbb{E}(E(\overline{R_f})) \leqslant \frac{1}{\ell^3} \cdot \delta_3 t^3 \ell^3 = \delta_3 t^3 ,$$

since by Lemma 53(3) there are at most $\delta_3 t^3 \ell^3$ irregular triads. Thus, due to Markov's inequality

$$\mathbb{P}(E(\overline{R_f}) > 2\delta_3 t^3) < \frac{\delta_3 t^3}{2\delta_3 t^3} = \frac{1}{2} . \tag{3.28}$$

Now fix a pair $i_* j_* \in [t]^{(2)}$. Estimating the expected value of $d_{D_f}(i_*, j_*)$, we get for every $\lambda \in [\ell]$ that

$$\begin{aligned} &\mathbb{E}\big(d_{D_f}(i_*, j_*) | f(i_*, j_*) = \lambda\big) \\ &= \frac{1}{\ell^{\binom{t}{2}-1}} \sum_{f\colon [t]^2 \to [\ell], f(i_*,j_*)=\lambda} d_{D_f}(i_*, j_*) \\ &= \frac{|\mathcal{D}_\lambda(i_*, j_*)|}{\ell^2} . \end{aligned}$$

By Claim 58 it follows that

$$\mathbb{E}\big(d_{D_f}(i_*, j_*) | f(i_*, j_*) = \lambda\big) \geqslant (3/4 + \alpha/2)t .$$

Moreover, for $f\colon [t]^2 \to [\ell]$ with $f(i_*, j_*) = \lambda$ the value of $d_{D_f}(i_*, j_*)$ is completely determined by the $2(t-2)$ numbers $f(i,j)$ with $|\{i,j\} \cap \{i_*, j_*\}| = 1$ and if one changes one of these $2(t-2)$ values of f, then $d_{D_f}(i_*, j_*)$ can change by at most 1. Thus, the Azuma-Hoeffding inequality (see, e.g., [39, Corollary 2.27]) leads to

$$\mathbb{P}\big(d_{D_f}(i_*, j_*) < (3/4 + \alpha/4)t \big| f(i_*, j_*) = \lambda\big) < \exp\Big(-\frac{2(\alpha t/4)^2}{2(t-2)}\Big) .$$

Therefore,

$$\mathbb{P}\big(d_{D_f}(i_*, j_*) < (3/4 + \alpha/4)t \big| f(i_*, j_*) = \lambda\big) < e^{-\Omega(t)}$$

for each $\lambda \in [\ell]$ and hence

$$\mathbb{P}(d_{D_f}(i_*, j_*) < (3/4 + \alpha/4)t) < e^{-\Omega(t)} . \tag{3.29}$$

Therefore the probability that some pair has a pair-degree less than $(3/4 + \alpha/4)t$ is less than $t^2/e^{\Omega(t)}$, which proves that with probability greater then $1/2$ the minimum pair-degree of D_f is at least $(3/4 + \alpha/4)t$. Together with (3.28) this shows that the probability that a function f fulfills $E(\overline{R_f}) \leqslant 2\delta_3 t^3$ and $\delta_2(D_f) \geqslant (3/4+\alpha/4)t$ is greater than zero.

From now on let $f\colon [t]^2 \to [\ell]$ be a fixed function having these two properties. Notice that $D_f \cap R_f$ arise from D_f by deleting at most $2\delta_3 t^3$ edges. We can estimate the number $\overline{\tau} t^2$ of pairs, which have afterwards a pair-degree smaller than $(3/4+\alpha/8)t$, by

$$\overline{\tau} t^2 \alpha t/8 \leqslant 6\delta_3 t^3 \,.$$

Thus $\overline{\tau} \leqslant \frac{48\delta_3}{\alpha}$ and by our choice of $\delta_3 \ll \alpha, \tau$ it follows that $\overline{\tau} \leqslant \tau$. In other words, there are indeed at most τt^2 pairs $ij \in [t]^{(2)}$ whose pair-degree in J_f is smaller than $(\frac{3}{4}+\frac{\alpha}{8})t$. □

From now on we will denote the bipartite graph $P^{ij}_{f(i,j)}$ simply by P^{ij}, where f is the function obtained in Claim 59. Due to Claim 59 we can apply Lemma 51 to J_f with $\alpha' = \alpha/8$ instead of α and find a $K_4^{(3)}$-factor missing at most $2\sqrt{\tau}t + 13$ vertices with $\tau \ll \alpha'$. Since $Q \equiv 0 \pmod 4$, we can apply Lemma 56 to the "tetrads" corresponding to these $K_4^{(3)}$ in the reduced hypergraph. Therefore all but at most

$$\frac{n}{t}(2\sqrt{\tau}t + 13) + \frac{t}{4} \cdot \nu \cdot \frac{n}{t} + \delta_3 n \leqslant \mu n$$

vertices can be covered by vertex-disjoint squared paths with Q vertices. □

3.5.5 Almost squared cycle

Finally we establish Proposition 26 by connecting the absorbing path and a collection of many long squared paths provided by the foregoing theorem, which yields an almost spanning squared cycle.

Proposition 60. *Given $\alpha > 0$, let $\vartheta_* > 0$, $M \in \mathbb{N}$ be the constants from the connecting lemma and let P_A be an absorbing squared path. There exists $n_0 \in \mathbb{N}$ such that in every hypergraph H with $v(H) = n \geqslant n_0$ and $\delta_2(H) \geqslant (4/5+\alpha)n$ all but at most $2\vartheta_*^2 n$ vertices of H can be covered by a squared cycle and P_A is an induced subhypergraph of this cycle.*

Proof. Applying Theorem 57 to the hypergraph $H \smallsetminus (P_A \cup \mathcal{R})$, where $\mathcal{R}$ is the reservoir set, with $\alpha' = \alpha/2$ instead of α, with some $Q \geqslant 2M\vartheta_*^{-4}$ divisible by 4, and $\mu = \vartheta_*^2$. We get less than n/Q squared paths with Q vertices and miss at most μn vertices. We will connect these paths and the absorbing path P_A to a squared cycle by using Lemma 46, which is applicable each time, since $M(\frac{n}{Q}+1) \leqslant \vartheta_*^4 n$ for $Q \geqslant 2M\vartheta_*^{-4}$ and n sufficiently large. Therefore we just used vertices of the reservoir set. Because $\mu \leqslant \vartheta_*^2$ and $|R| \leqslant \vartheta_*^2 n$ we miss at most $\mu n + |R| \leqslant 2\vartheta_*^2 n$ vertices. $\square$

4 Powers of tight Hamilton cycles in randomly perturbed hypergraphs

In this chapter we will prove the following result.

Theorem 61 (Main result)**.** *For all integers $k \geqslant 2$ and $r \geqslant 1$ such that $k+r \geqslant 4$ and $\alpha > 0$, there is $\varepsilon > 0$ such that the following holds. Suppose H is a k-graph on n vertices with*

$$\delta_{k-1}(H) \geqslant \left(1 - \binom{k+r-2}{k-1}^{-1} + \alpha\right) n \tag{4.1}$$

and $p = p(n) \geqslant n^{-\binom{k+r-2}{k-1}^{-1} - \varepsilon}$. Then a.a.s. the union $H \cup \mathbb{G}^{(k)}(n,p)$ contains the r^{th} power of a tight Hamiltonian cycle.

This chapter is organized as follows. In Section 4.1 we prove some results concerning random hypergraphs. Section 4.2 contains two essential lemmas in our approach, namely, Lemma 65 (Connecting Lemma) and Lemma 66 (Absorbing Lemma). In Section 4.3 we prove our main result, Theorem 61. Some remarks concerning the hypotheses in Theorem 61 are given in Section 4.4. Throughout this chapter, we omit floor and ceiling functions.

4.1 Subgraphs of random hypergraphs

In this section we prove some results related to binomial random k-graphs. We will apply Chebyshev's inequality and Janson's inequality to prove some concentration results that we shall need. For convenience, we state these two inequalities in the form we need (inequalities (4.2) and (4.3) below follow, respectively, from Janson's and Chebyshev's inequalities; see, e.g., [39, Theorem 2.14, Equation (1.2)]).

We first recall Janson's inequality. Let Γ be a finite set and let Γ_p be a random subset of Γ such that each element of Γ is included in Γ_p independently with probability p. Let $\mathcal{S}$ be a family of non-empty subsets of Γ and for each $S \in \mathcal{S}$, let I_S be the indicator random variable for the event $S \subseteq \Gamma_p$. Thus each I_S is a Bernoulli random variable $\mathrm{Be}(p^{|S|})$. Let $X := \sum_{S\in\mathcal{S}} I_S$ and $\lambda := \mathbb{E}(X)$. Let $\Delta_X := \sum_{S\cap T\neq\varnothing} \mathbb{E}(I_S I_T)$, where the sum is over all ordered pairs $S, T \in \mathcal{S}$ (note that the sum includes the pairs (S, S) with $S \in \mathcal{S}$). Then Janson's inequality says that, for any $0 \leqslant t \leqslant \lambda$,

$$\mathbb{P}(X \leqslant \lambda - t) \leqslant \exp\left(-\frac{t^2}{2\Delta_X}\right). \tag{4.2}$$

Next note that $\mathrm{Var}(X) = \mathbb{E}(X^2) - \mathbb{E}(X)^2 \leqslant \Delta_X$. Then, by Chebyshev's inequality,

$$\mathbb{P}(X \geqslant 2\lambda) \leqslant \frac{\mathrm{Var}(X)}{\lambda^2} \leqslant \frac{\Delta_X}{\lambda^2}. \tag{4.3}$$

Consider the random k-graph $\mathbb{G}^{(k)}(n, p)$ on an n-vertex set V. Note that we can view $\mathbb{G}^{(k)}(n, p)$ as Γ_p with $\Gamma = \binom{V}{k}$. For two k-graphs G and H, let $G \cap H$ (or $G \cup H$) denote the k-graph with vertex set $V(G) \cap V(H)$ (or $V(G) \cup V(H)$) and edge set $E(G) \cap E(H)$ (or $E(G) \cup E(H)$). Finally, let

$$\Phi_F = \Phi_F(n, p) = \min\{n^{v_H} p^{e_H} : H \subseteq F \text{ and } e_H > 0\}.$$

The following simple proposition is useful.

Proposition 62. *Let F be a k-graph with s vertices and f edges and let $G := \mathbb{G}^{(k)}(n, p)$. Let $\mathcal{A}$ be a family of ordered s-subsets of $V = V(G)$. For each $A \in \mathcal{A}$, let I_A be the indicator random variable of the event that A spans a labelled copy of F in G. Let $X = \sum_{A\in\mathcal{A}} I_A$. Then $\Delta_X \leqslant s!2^{2s}n^{2s}p^{2f}/\Phi_F$.*

Proof. Order the vertices of F arbitrarily. For each ordered s-subset A of V, let α_A be the bijection from $V(F)$ to A following the orders of $V(F)$ and A. Let F_A be the labelled copy of F spanned on A. For any $T \subseteq V(F)$ with $e_F(T) > 0$, denote by W_T the set of all pairs A, $B \in \mathcal{A}$ such that $A \cap B = \alpha_A(T)$. If T has s' vertices and $F[T]$ has f' edges, then for every $\{A, B\} \in W_T$, $F_A \cup F_B$ has exactly $2s - s'$ vertices and at least $2f - f'$ edges. Therefore, we can bound Δ_X by

$$\Delta_X \leqslant \sum_{T\subseteq V(F)} |W_T| p^{2f-f'}.$$

Given integers n and b, let $(n)_b := n(n-1)(n-2)\cdots(n-b+1) = n!/(n-b)!$. Note that there are at most $\binom{n}{2s-s'}$ choices for the vertex set of $F_A \cup F_B$, and there are at most

$$(2s-s')_s \cdot \binom{s}{s'} s! \leqslant (2s-s')!s!2^s$$

ways to label each $(2s-s')$-set to get $\{A,B\}$. Thus we have $|W_T| \leqslant s!2^s n^{2s-s'}$ and

$$\Delta_X \leqslant \sum_{T \subseteq V(F)} s!2^s n^{2s-s'} p^{2f-f'} \leqslant \sum_{T \subseteq V(F)} s!2^s n^{2s} p^{2f}/\Phi_F \leqslant s!2^{2s} n^{2s} p^{2f}/\Phi_F,$$

because there are at most 2^s choices for T. $\square$

The following lemma gives the properties of $\mathbb{G}^{(k)}(n,p)$ that we will use. Throughout the rest of the paper, we write $\alpha \ll \beta \ll \gamma$ to mean that 'we can choose the positive constants α, β and γ from right to left'. More precisely, there are functions f and g such that, given γ, whenever $\beta \leqslant f(\gamma)$ and $\alpha \leqslant g(\beta)$, the subsequent statement holds. Hierarchies of other lengths are defined similarly.

Lemma 63. *Let F be a labelled k-graph with b vertices and a edges. Suppose $1/n \ll 1/C \ll \gamma, 1/a, 1/b, 1/s$. Let V be an n-vertex set, and let $\mathcal{F}_1, \dots, \mathcal{F}_t$ be $t \leqslant n^s$ families of γn^b ordered b-sets on V. If $p = p(n)$ is such that $\Phi_F(n,p) \geqslant Cn$, then the following properties hold for the binomial random k-graph $G = \mathbb{G}^{(k)}(n,p)$ on V.*

(*i*) *With probability at least $1-\exp(-n)$, every induced subgraph of G of order γn contains a copy of F.*

(*ii*) *With probability at least $1-\exp(-n)$, for every $i \in [t]$, there are at least $(\gamma/2)n^b p^a$ ordered b-sets in $\mathcal{F}_i$ that span labelled copies of F.*

(*iii*) *With probability at least $1-1/\sqrt{n}$, there are at most $2n^b p^a$ ordered b-sets of vertices of G that span labelled copies of F.*

(*iv*) *With probability at least $1-1/\sqrt{n}$, the number of overlapping (i.e., not vertex-disjoint) pairs of copies of F in G is at most $4b^2 n^{2b-1} p^{2a}$.*

Proof. Let $\mathcal{A}$ be a family of ordered b-sets of vertices in V. For each $A \in \mathcal{A}$, let I_A be the indicator random variable of the event that A spans a labelled copy of F in G. Let $X_{\mathcal{A}} = \sum_{A \in \mathcal{A}} I_A$. From the hypothesis that $\Phi_F \geqslant Cn$ and Proposition 62, we have

$$\Delta_X \leqslant b!2^{2b} n^{2b} p^{2a}/\Phi_F \leqslant b!2^{2b} n^{2b} p^{2a}/(Cn). \tag{4.4}$$

Furthermore, let $\mathcal{S}$ consist of the edge sets of the labelled copies of F spanned on A in the complete k-graph on V for all $A \in \mathcal{A}$. Since we can write $X_{\mathcal{A}} = \sum_{S \in \mathcal{S}} I_S$, where I_S is the indicator variable for the event $S \subseteq E(G)$, we can apply (4.2) to $X_{\mathcal{A}}$.

For *(i)*, fix a vertex set W of G with $|W| = \gamma n$. Let $\mathcal{A}$ be the family of all labelled b-sets in W. Let $X_{\mathcal{A}}$ be the random variable that counts the number of members of $\mathcal{A}$ that span a labelled copy of F and thus $\mathbb{E}[X_{\mathcal{A}}] = (\gamma n)_b p^a$. By (4.4) and (4.2) and the fact that $1/C \ll \gamma, 1/b$, we have $\mathbb{P}(X_{\mathcal{A}} = 0) \leqslant \exp(-2n)$. By the union bound, the probability that there exists a vertex set W of size γn such that $X_{\mathcal{A}} = 0$ is at most $2^n \exp(-2n) \leqslant \exp(-n)$, which proves *(i)*.

For *(ii)*, fix $i \in [t]$ and let $X_{\mathcal{F}_i}$ be the random variable that counts the members of $\mathcal{F}_i$ that span F. Note that $\mathbb{E}[X_{\mathcal{F}_i}] = \gamma n^b p^a$. Thus (4.2) implies that $\mathbb{P}\big(X_{\mathcal{F}_i} \leqslant (\gamma/2) n^b p^a\big) \leqslant \exp(-2n)$. By the union bound and the fact that $n^s \exp(-2n) \leqslant \exp(-n)$, we see that *(ii)* holds.

For *(iii)*, let X_3 be the random variable that counts the number of labelled copies of F in G. Since $\mathbb{E}(X_3) = (n)_b p^a$, by (4.4) and (4.3), we obtain

$$\mathbb{P}(X_3 \geqslant 2p^a n^b) \leqslant \mathbb{P}(X_3 \geqslant 2\mathbb{E}[X_3]) \leqslant \frac{\Delta_{X_3}}{\mathbb{E}[X_3]^2} \leqslant \frac{b! 2^{2b} n^{2b} p^{2a}/(Cn)}{((n)_b p^a)^2} \leqslant \frac{1}{\sqrt{n}}.$$

For *(iv)*, let Y be the random variable that denotes the number of overlapping pairs of copies of F in G. We first estimate $\mathbb{E}[Y]$. We write $Y = \sum_{A \in \mathcal{Q}} I_A$, where $\mathcal{Q}$ is the collection of the edge sets of overlapping pairs of labelled copies of F in the complete k-graph on n vertices. Note that if two overlapping copies of F do not share any edge, then they induce at most $2b-1$ vertices and exactly $2a$ edges. Note that for $1 \leqslant i \leqslant b$, there are

$$\binom{n}{2b-i}(2b-i)_b \binom{b}{i} b! = (n)_{2b-i} \binom{b}{i} (b)_i \leqslant (n)_{2b-i} (b)_i^2$$

members of $\mathcal{Q}$ whose two copies of F share exactly i vertices. Thus, the number of choices for the vertex sets of pairs of copies which induce at most $2b-2$ vertices is at most $\sum_{2 \leqslant i \leqslant b} (n)_{2b-i}(b)_i^2 \leqslant n^{2b-1}$. By the definition of Δ_{X_3} and (4.4) we have

$$n^{2b-1} b^2 p^{2a}/2 \leqslant \mathbb{E}[Y] \leqslant (n)_{2b-1} b^2 \cdot p^{2a} + n^{2b-1} \cdot p^{2a} + \Delta_{X_3} \leqslant 2b^2 n^{2b-1} p^{2a}.$$

We next compute Δ_Y. For each $A \in \mathcal{Q}$, let S_A denote the k-graph induced by A (thus S_A is the union of two overlapping copies of F). For each $A, B \in \mathcal{Q}$, write $S_A := F_1 \cup F_2$ and $S_B := F_3 \cup F_4$, where each F_i is a copy of F for $i \in [4]$ such that $E(F_1) \cap E(F_3) \neq \varnothing$. Define $H_1 := F_1 \cap F_2$, $H_2 := (F_1 \cup F_2) \cap F_3$

and $H_3 := (F_1 \cup F_2 \cup F_3) \cap F_4$. Since $V(F_1) \cap V(F_2) \neq \varnothing$, $V(F_3) \cap V(F_4) \neq \varnothing$, and $E(F_1) \cap E(F_3) \neq \varnothing$, we know that $v_{H_i} \geqslant 1$ for $i = 1, 2, 3$. We claim that $n^{v_{H_i}} p^{e_{H_i}} \geqslant n$ for $i = 1, 2, 3$. Indeed, since each H_i is a subgraph of F, if $e_{H_i} \geqslant 1$, then $n^{v_{H_i}} p^{e_{H_i}} \geqslant \Phi_F \geqslant Cn$; otherwise $e_{H_i} = 0$ and then we have $n^{v_{H_i}} p^{e_{H_i}} = n^{v_{H_i}} \geqslant n^1 = n$. So we have

$$n^{v_{H_1}} p^{e_{H_1}} \cdot n^{v_{H_2}} p^{e_{H_2}} \cdot n^{v_{H_3}} p^{e_{H_3}} \geqslant n^3. \tag{4.5}$$

Now we define $\Delta_{H_1,H_2,H_3} = \sum_{A,B} \mathbb{E}[I_A I_B]$, where the sum is over the pairs $\{A, B\}$ with $A \cap B \neq \varnothing$ that generate H_1, H_2, H_3. Observe that the sum contains at most

$$\binom{n}{4b - v_{H_1} - v_{H_2} - v_{H_3}} (4b - v_{H_1} - v_{H_2} - v_{H_3})_b^4 < n^{4b-(v_{H_1}+v_{H_2}+v_{H_3})} (4b)^{3b}$$

terms. Thus, from (4.5), we obtain

$$\begin{aligned} \Delta_{H_1,H_2,H_3} = \sum_{A,B} \mathbb{E}[I_A I_B] &\leqslant (4b)^{3b} n^{4b-(v_{H_1}+v_{H_2}+v_{H_3})} p^{4a-(e_{H_1}+e_{H_2}+e_{H_3})} \\ &\leqslant (4b)^{3b} n^{4b-3} p^{4a}. \end{aligned}$$

Let $D = D(b, k, r)$ be the number of choices for H_1, H_2, H_3, thus

$$\Delta_Y = \sum_{H_1,H_2,H_3} \Delta_{H_1,H_2,H_3} \leqslant D(4b)^{3b} n^{4b-3} p^{4a}.$$

Therefore, by (4.3) and the fact that n is large enough, we get

$$\mathbb{P}\big(Y \geqslant 4b^2 n^{2b-1} p^{2a}\big) \leqslant \mathbb{P}\big(Y \geqslant 2\mathbb{E}[Y]\big) \leqslant \frac{\Delta_Y}{\mathbb{E}[Y]^2} \leqslant \frac{D(4b)^{3b} n^{4b-3} p^{4a}}{(n^{2b-1} p^{2a}/2)^2} \leqslant \frac{1}{\sqrt{n}}.$$

This verifies *(iv)*. □

For $m \geqslant k + r - 1$, denote by $P_m^{k,r}$ the r^{th} power of a k-uniform tight path on m vertices. Similarly, write $C_m^{k,r}$ for the r^{th} power of a k-uniform tight cycle on m vertices. For simplicity we say that $P_m^{k,r}$ is an (r, k)-*path* and $C_m^{k,r}$ is an (r, k)-*cycle*. We write P_m^r for $P_m^{k,r}$ whenever k is clear from the context. Moreover, the ends of P_m^r are its first and last $k + r - 1$ vertices (with the order in the (r, k)-path). We end this section by computing $\Phi_{P_b^r}$ for $p = p(n) \geqslant n^{-\binom{k+r-2}{k-1}^{-1} - \varepsilon}$ as in Theorem 61. For $b \geqslant k + r - 1$, let

$$g(b) := \left(b - \frac{(k-1)(k+r-1)}{k}\right) \binom{k+r-2}{k-1}.$$

Clearly g is an increasing function. Note that the number of edges in $P_m^{k,r}$ is given by

$$\begin{aligned}\left|E\left(P_m^{k,r}\right)\right| &= \binom{k+r-1}{k} + (m-(k+r-1))\binom{k+r-2}{k-1} \\ &= \left(m - \frac{(k-1)(k+r-1)}{k}\right)\binom{k+r-2}{k-1} = g(m).\end{aligned}$$

Proposition 64. *Suppose $k \geqslant 2$, $r \geqslant 1$, $b \geqslant k+r-1$, $k+r \geqslant 4$ and $C > 0$. Let ε be such that $0 < \varepsilon < \min\left\{(2g(b))^{-1}, \left(3\binom{k+r-1}{k}\right)^{-1}\right\}$.*

Suppose $1/n \ll 1/C,\ 1/k,\ 1/r,\ 1/b$. If $p = p(n) \geqslant n^{-\binom{k+r-2}{k-1}^{-1}-\varepsilon}$, then $\Phi_{P_b^r} \geqslant Cn$.

Proof. Let H be a subgraph of P_b^r. Since for any integer $k+r-1 \leqslant b' \leqslant b$, any subgraph of $P_{b'}^r$ has at most $g(b')$ edges, we have the following observations.

(a) If $e_H > g(b')$ for some $b' \geqslant k+r-1$, then $v_H \geqslant b'+1$;

(b) if $e_H > \binom{i}{k}$ for some $k-1 \leqslant i < k+r-1$, then $v_H \geqslant i+1$.

By (a), we have

$$\begin{aligned}\min_{g(k+r-1)<e_H\leqslant g(b)} n^{v_H}p^{e_H} &= \min_{k+r-1\leqslant b'<b}\left(\min_{g(b')<e_H\leqslant g(b'+1)} n^{v_H}p^{e_H}\right) \\ &\geqslant \min_{k+r-1\leqslant b'<b} n^{b'+1}p^{g(b'+1)}.\end{aligned}$$

Since $p \geqslant n^{-1/\binom{k+r-2}{k-1}-\varepsilon}$, and $g(b'+1) > 0$, the following holds for any $b' < b$:

$$\begin{aligned}n^{b'+1}p^{g(b'+1)} &\geqslant n^{b'+1}\left(n^{-1/\binom{k+r-2}{k-1}-\varepsilon}\right)^{g(b'+1)} \\ &= n^{-g(b'+1)\varepsilon}n^{(k-1)(k+r-1)/k} \geqslant n^{-g(b)\varepsilon}n^{(k-1)(k+r-1)/k} \geqslant Cn,\end{aligned}$$

where we used $(k-1)(k+r-1)/k \geqslant 3/2$ and $g(b)\varepsilon < 1/2$. Therefore,

$$\min_{g(k+r-1)<e_H\leqslant g(b)} n^{v_H}p^{e_H} \geqslant Cn. \tag{4.6}$$

On the other hand, noting that $g(k+r-1) = \binom{k+r-1}{k}$, by (b) we have

$$\begin{aligned}\min_{0<e_H\leqslant g(k+r-1)} n^{v_H}p^{e_H} &= \min_{k-1\leqslant i<k+r-1}\left(\min_{\binom{i}{k}<e_H\leqslant\binom{i+1}{k}} n^{v_H}p^{e_H}\right) \\ &\geqslant \min_{k-1\leqslant i<k+r-1} n^{i+1}p^{\binom{i+1}{k}}.\end{aligned}$$

Since $p \geqslant n^{-1/\binom{k+r-2}{k-1}-\varepsilon}$, and $\binom{i+1}{k}\varepsilon \leqslant 1/3$ for any $k-1 \leqslant i \leqslant k+r-2$, if $i \geqslant 2$, then

$$n^{i+1}p^{\binom{i+1}{k}} \geqslant n^{i+1}n^{-(1/\binom{k+r-2}{k-1}+\varepsilon)\frac{i+1}{k}\binom{i}{k-1}} \geqslant n^{i+1-\frac{i+1}{k}-\binom{i+1}{k}\varepsilon} \geqslant Cn.$$

Otherwise $i = 1$ and thus $k = 2$, in which case we have $n^{i+1}p^{\binom{i+1}{k}} = n^2 p \geqslant Cn$. Therefore,

$$\min_{0<e_H\leqslant g(k+r-1)} n^{v_H}p^{e_H} \geqslant Cn. \tag{4.7}$$

From (4.6) and (4.7), we have $\Phi_{P_b^r} \geqslant Cn$, as desired. $\square$

4.2 The Connecting and Absorbing Lemmas

For brevity, throughout the rest of this paper, we write

$$h := k+r-1, \quad t := g(2h), \quad c := \binom{k+r-2}{k-1}^{-1}.$$

Recall that the ends of an (r,k)-path are ordered h-sets that span a copy of $K_h^{(k)}$ in H.

4.2.1 The Connecting Lemma

Given a k-graph H and two ordered h-sets of vertices A and B each spanning a copy of $K_h^{(k)}$ in H, we say that an ordered $2h$-set of vertices C *connects* A and B if $C \cap A = C \cap B = \varnothing$ and the concatenation ACB spans a labelled copy of P_{4h}^r. We are now ready to state our connecting lemma.

Lemma 65 (Connecting Lemma)**.** *Suppose $1/n \ll \varepsilon \ll \beta \ll \alpha' \ll 1/k, 1/r$. Let H be an n-vertex k-graph with $\delta_{k-1}(H) \geqslant (1-c+\alpha')\,n$ and suppose $p = p(n) \geqslant n^{-c-\varepsilon}$. Then a.a.s. $H \cup \mathbb{G}^{(k)}(n,p)$ contains a set $\mathcal{C}$ of vertex-disjoint copies of P_{2h}^r with $|\mathcal{C}| \leqslant \beta n$ such that, for every pair of disjoint ordered h-sets spanning a copy of $K_h^{(k)}$ in H, there are at least $\beta^2 n/(2h)^2$ ordered copies of P_{2h}^r in $\mathcal{C}$ that connect them.*

Proof. Let $\mathcal{S}$ be the set of pairs of disjoint ordered h-sets that each span a copy of $K_h^{(k)}$ in H. Fix $\{S, S'\} \in \mathcal{S}$ and write $S := (v_1, \ldots, v_h)$ and $S' := (w_h, \ldots, w_1)$. Since $\delta_{k-1}(H) \geqslant (1-c+\alpha')\,n$, we can extend S to an (r,k)-path with vertices $(v_1, \ldots, v_{2h})$ such that the vertices of this (r,k)-path are disjoint with $\{w_h, \ldots, w_1\}$ and there are at least $(\alpha' n/2)^h$ choices for the ordered set $(v_{h+1}, \ldots, v_{2h})$. Similarly, we can extend S' to an (r,k)-path $(w_{2h}, \ldots, w_1)$ such that the vertices of this (r,k)-path are disjoint with $\{v_1, \ldots, v_{2h}\}$ and there are at least $(\alpha' n/2)^h$ choices for the ordered set $(w_{2h}, \ldots, w_{h+1})$.

Therefore there are at least $(\alpha' n/2)^{2h} \geqslant 24\beta n^{2h}$ possible choices for the ordered $2h$-sets $(v_{h+1}, \ldots, v_{2h}, w_{2h}, \ldots, w_{h+1})$. Let $\mathcal{C}_{S,S'}$ be a collection of exactly $24\beta n^{2h}$ such ordered $2h$-sets of vertices. Clearly if an ordered set C in $\mathcal{C}_{S,S'}$ spans a copy of P^r_{2h}, then C connects S and S'.

Now we will use the edges of $G = \mathbb{G}^{(k)}(n,p)$ to obtain the desired copies of P^r_{2h} that connect the pairs in $\mathcal{S}$. Let $\mathcal{T}$ be the set of all labelled copies of P^r_{2h} in G. We claim that the following properties hold with probability at least $1 - 3/\sqrt{n}$:

(a) $|\mathcal{T}| \leqslant 2p^t n^{2h}$;

(b) for every $\{S, S'\} \in \mathcal{S}$, at least $12\beta p^t n^{2h}$ members of $\mathcal{T}$ connect S and S';

(c) the number of overlapping pairs of members of $\mathcal{T}$ is at most $4(2h)^2 p^{2t} n^{4h-1}$.

To see that the claim above holds, note that by Proposition 64, we can apply Lemma 63 with $F = P^r_{2h}$, $\gamma = 24\beta$ and $\mathcal{C}_{S,S'}$ in place of $\mathcal{F}_i$. Items (a), (b) and (c) follow, respectively, from Lemma 63 *(iii)*, *(ii)* and *(iv)*.

Next we select a random collection $\mathcal{C}'$ by including each member of $\mathcal{T}$ independently with probability $q := \beta/(2(2h)^2 n^{2h-1} p^t)$. We remark that $q < 1$, since $n^{2h-1}p^t \geqslant C$ due to Proposition 64. By using Chernoff's inequality (for (i) and (ii) below) and Markov's inequality (for (iii) below), we know that there is a choice of $\mathcal{C}'$ that satisfies the following properties:

(i) $|\mathcal{C}'| \leqslant 2q|\mathcal{T}| \leqslant \beta n$;

(ii) for every $\{S, S'\} \in \mathcal{S}$, there are at least $12\beta(q/2)n^{2h}p^t = 3\beta^2 n/(2h)^2$ members of $\mathcal{C}'$ that connect S and S';

(iii) the number of overlapping pairs of members of $\mathcal{C}'$ is at most $8(2h)^2 q^2 n^{4h-1} p^{2t} = 2\beta^2 n/(2h)^2$.

Deleting one member from each overlapping pair, we obtain a collection $\mathcal{C}$ of vertex disjoint copies of P^r_{2h} with $|\mathcal{C}| \leqslant \beta n$, and such that, for every pair of disjoint ordered h-sets each spanning a $K^{(k)}_h$ in H, there are at least $3\beta^2 n/(2h)^2 - 2\beta^2 n/(2h)^2 = \beta^2 n/(2h)^2$ sets of $2h$ vertices connecting them. □

4.2.2 The Absorbing Lemma

In this subsection we prove our absorbing lemma.

Lemma 66 (Absorbing Lemma)**.** *Suppose $1/n \ll \varepsilon \ll \zeta \ll \alpha \ll 1/k, 1/r$. Let H be an n-vertex k-graph with $\delta_{k-1}(H) \geqslant (1-c+\alpha)n$ and suppose $p = p(n) \geqslant n^{-c-\varepsilon}$. Then a.a.s. $H \cup \mathbb{G}^{(k)}(n,p)$ contains an (r,k)-path P_{abs} of order at most $6h\zeta n$ such that, for every set $X \subseteq V(H) \setminus V(P_{\mathrm{abs}})$ with $|X| \leqslant \zeta^2 n/(2h)^2$, there is an (r,k)-path in H on $V(P_{\mathrm{abs}}) \cup X$ that has the same ends as P_{abs}.*

We call the (r,k)-path P_{abs} in Lemma 66 an *absorbing path*. We now define *absorbers*.

Definition 67. *Let v be a vertex of a k-graph. An ordered $2h$-set of vertices $(w_1, \dots, w_{2h})$ is a v-*absorber* if $(w_1, \dots, w_{2h})$ spans a labelled copy of P^r_{2h} and $(w_1, \dots, w_h, v, w_{h+1}, \dots, w_{2h})$ spans a labelled copy of P^r_{2h+1}.*

Proof of Lemma 66. Suppose $1/n \ll \varepsilon \ll \zeta \ll \beta \ll \alpha \ll 1/k,\ 1/r$. We split the proof into two parts. We first find a set $\mathcal{F}$ of absorbers and then connect them to an (r,k)-path by using Lemma 65 (Connecting Lemma). We will expose $G = \mathbb{G}^{(k)}(n,p)$ in two rounds: $G = G_1 \cup G_2$ with G_1 and G_2 independent copies of $\mathbb{G}^{(k)}(n,p')$, where $(1-p')^2 = 1-p$.

Fix a vertex v. By the codegree condition of H, we can extend v to a labelled copy of P^r_{2h+1} in the form $(w_1, \dots, w_h, v, w_{h+1}, \dots, w_{2h})$ such that there are at least $(\alpha n/2)^{2h} \geqslant 24\zeta n^{2h}$ choices for the ordered $2h$-set $(w_1, \dots, w_{2h})$. Let $\mathcal{A}_v$ be a collection of exactly $24\zeta n^{2h}$ such ordered $2h$-sets. By definition, if an ordered set A in $\mathcal{A}_v$ spans a labelled copy of P^r_{2h}, then A is a v-absorber.

Now consider $G_1 = \mathbb{G}^{(k)}(n,p')$ and let $\mathcal{T}$ be the set of all labelled copies of P^r_{2h} in G_1. By Proposition 64, we can apply Lemma 63 with $F = P^r_{2h}$ and $\mathcal{A}_v$ in place of $\mathcal{F}_i$. Using the union bound we conclude that the following properties hold with probability at least $1 - 3/\sqrt{n}$:

(a) $|\mathcal{T}| \leqslant 2p^t n^{2h}$;

(b) for every vertex v in H, at least $12\zeta p^t n^{2h}$ members of $\mathcal{T}$ are v-absorbers;

(c) the number of overlapping pairs of members of $\mathcal{T}$ is at most $4(2h)^2 p^{2t} n^{4h-1}$.

Next we select a random collection $\mathcal{F}'$ by including each member of $\mathcal{T}$ independently with probability $q = \zeta/(2(2h)^2 p^t n^{2h-1}) < 1$. In view of the properties above, by using Chernoff's inequality (for (i) and (ii) below) and Markov's inequality (for (iii) below), we know that there is a choice of $\mathcal{F}'$ that satisfies the following properties:

(i) $|\mathcal{F}'| \leqslant \zeta n$;

(ii) for every vertex v, at least $12\zeta(q/2)p^t n^{2h} = 3\zeta^2 n/(2h)^2$ members of $\mathcal{F}'$ are v-absorbers;

(iii) there are at most $8(2h)^2 q^2 n^{4h-1} p^{2t} = 2\zeta^2 n/(2h)^2$ overlapping pairs of members of $\mathcal{F}'$.

By deleting from $\mathcal{F}'$ one member from each overlapping pair and all members that are not in $\mathcal{T}$, we obtain a collection $\mathcal{F}$ of vertex-disjoint copies of P^r_{2h} such that $|\mathcal{F}| \leqslant \zeta n$, and for every vertex v, there are at least

$$3\zeta^2 n/(2h)^2 - 2\zeta^2 n/(2h)^2 = \zeta^2 n/(2h)^2$$

v-absorbers.

Now we connect these absorbers using Lemma 65. Let $V' = V(H) \smallsetminus V(\mathcal{F})$ and $n' = |V'|$. In particular, $n' \geqslant n/2$ is sufficiently large. Now consider $H' = H[V']$ and $G' = G_2[V'] = \mathbb{G}^{(k)}(n', p')$. Since $|V(\mathcal{F})| \leqslant 2h \cdot \zeta n \leqslant \alpha^2 n$, we have $\delta_{k-1}(H') \geqslant (1 - c + \alpha/2)\, n$. We apply Lemma 65 to H' and G' with $\alpha' = \alpha/2$ and β, and conclude that a.a.s. $H' \cup G'$ contains a set $\mathcal{C}$ of vertex-disjoint copies of P^r_{2h} such that $|\mathcal{C}| \leqslant \beta n$ and for every pair of ordered h-sets in V', there are at least $\beta^2 n$ members of $\mathcal{C}$ connecting them.

For each copy of P^r_{2h} in $\mathcal{F}$, we greedily extend its two ends by h vertices such that all new paths are pairwise vertex disjoint and also vertex disjoint from $V(\mathcal{C})$. This is possible because of the codegree condition of H_0 and the fact that $|V(\mathcal{F})| + 2h|\mathcal{F}| + |V(\mathcal{C})| \leqslant 2h\zeta n + 2h\zeta n + 2h \cdot \beta n < \alpha n/4$. Note that both ends of these (r,k)-paths P^r_{4h} are in $V' \smallsetminus V(\mathcal{C})$. Since $\zeta n \leqslant \beta^2 n'/(2h)^2$, we can greedily connect these P^r_{4h}. Let P_{abs} be the resulting (r,k)-path. By construction, $|V(P_{\mathrm{abs}})| \leqslant (4h+2h)\cdot\zeta n = 6h\zeta n$. Moreover, for any $X \subseteq V \smallsetminus V(P_{\mathrm{abs}})$ such that $|X| \leqslant \zeta^2 n/(2h)^2$, since each vertex v has at least $\zeta^2 n/(2h)^2$ v-absorbers in $\mathcal{F}$, we can absorb them greedily and conclude that there is an (r,k)-path on $V(P_{\mathrm{abs}}) \cup X$ that has the same ends as P_{abs}. $\square$

4.3 Proof of Theorem 61

We now combine Lemmas 65 and 66 to prove Theorem 61.

Proof of Theorem 61. Suppose $1/n \ll \varepsilon \ll \beta \ll \zeta \ll \alpha, 1/k, 1/r$. Furthermore, recall that $c := \binom{k+r-2}{k-1}^{-1}$ and suppose that $H \cup \mathbb{G}^{(k)}(n,p)$ is an n-vertex k-graph with $\delta_{k-1}(H) \geqslant (1-c+\alpha)\,n$ and $p = p(n) \geqslant n^{-c-\varepsilon}$. We will expose $G := \mathbb{G}^{(k)}(n,p)$ in three rounds: $G = G_1 \cup G_2 \cup G_3$ with G_1, G_2 and G_3 three independent copies of $\mathbb{G}^{(k)}(n,p')$, where $(1-p')^3 = 1-p$. Note that $p' > p/3 > n^{-c-2\varepsilon}$.

By Lemma 66 with 2ε in place of ε, a.a.s. the k-graph $H \cup G_1$ contains an absorbing (r,k)-path P_{abs} of order at most $6h\zeta n$, that is, for every set $X \subseteq V(H) \setminus V(P_{\text{abs}})$ such that $|X| \leqslant \zeta^2 n/(2h)^2$, there is an (r,k)-path in H on $V(P_{\text{abs}}) \cup X$ which has the same ends as P_{abs}. Let $V' = V(H) \setminus V(P_{\text{abs}})$ and $n' = |V'|$. In particular, $n' \geqslant (1-6h\zeta)n$ and, since ζ is small enough, we have $(n')^{c+\varepsilon} \geqslant n^{c+\varepsilon}/2$. Thus $p' > p/2 \geqslant n^{-c-\varepsilon}/2 \geqslant (n')^{-c-\varepsilon}/4 \geqslant (n')^{-c-2\varepsilon}$.

Now consider $H' = H[V']$ and let $G_2' := \mathbb{G}^{(k)}(n',p')$ be the subgraph of G_2 induced by V'. Note that $\delta_{k-1}(H') \geqslant \delta_{k-1}(H) - |V(P_{\text{abs}})| \geqslant (1-c+\alpha/2)\,n'$. By Lemma 65, a.a.s. the k-graph $H' \cup G_2'$ contains a set $\mathcal{C}$ of vertex-disjoint copies of P_{2h}^r such that $|\mathcal{C}| \leqslant \beta n$ and for every pair of disjoint ordered h-sets in V' that each spans a copy of $K_h^{(k)}$, there are at least $\beta^2 n'/(2h)^2$ members of $\mathcal{C}$ connecting them. Since $|V(\mathcal{C})| + |V(P_{\text{abs}})| \leqslant 2h \cdot \beta n + 6h\zeta n \leqslant \alpha n/2$, we can greedily extend the two ends of P_{abs} by h vertices so that the two new ends E_1, E_2 are in $V' \setminus V(\mathcal{C})$.

Let $m := g^{-1}(1/(2\varepsilon))$. Note that $m \geqslant 1/\sqrt{\varepsilon}$ because ε is small enough and g is linear. By Proposition 64, we can apply Lemma 63 *(i)* with $b = m$ to G_3 and conclude that a.a.s. every induced subgraph of G_3 of order βn contains a copy of P_m^r. Thus we can greedily find at most $\sqrt{\varepsilon} n$ vertex-disjoint copies of P_m^r in $V' \setminus (V(\mathcal{C}) \cup E_1 \cup E_2)$, which together covers all but at most βn vertices of $V' \setminus V(\mathcal{C})$. Since $\sqrt{\varepsilon} n + 1 \leqslant \beta^2 n'/(2h)^2$, we can greedily connect these (r,k)-paths P_m^r and P_{abs} to an (r,k)-cycle Q^r. Let $R := V(H) \setminus V(Q^r)$ and note that $|R| \leqslant |V(\mathcal{C})| + \beta n \leqslant (2h+1)2\beta n \leqslant \zeta^2 n/(2h)^2$. Since P_{abs} is an absorber, there is an (r,k)-path on $V(P_{\text{abs}}) \cup R$ which has the same ends as P_{abs}. So we can replace P_{abs} by this (r,k)-path in Q^r and obtain the r^{th} power of a tight Hamiltonian cycle.

Moreover, since all previous steps can be achieved a.a.s., by the union bound, $H \cup G$ a.a.s. contains the desired r^{th} power of a tight Hamiltonian cycle. □

4.4 Concluding remarks

Let us briefly discuss the hypotheses in Theorem 61. Note that, for $r = 1$, the condition in (4.1) is simply $\delta_{k-1}(H) \geqslant \alpha n$, with α any arbitrary positive constant. Thus, in this case, our theorem is in the spirit of the original Bohman, Frieze, and Martin [8] set-up, in the sense that we have a similar minimum degree condition on the deterministic graph H. However, if $r > 1$, then our minimum condition (4.1) is of the form $\delta_{k-1}(H) \geqslant (\sigma + \alpha)n$ for some $\sigma = \sigma(k, r) > 0$ (and arbitrarily small $\alpha > 0$). Thus, for $r > 1$, our result is more in line with Theorem 12 of Bennett, Dudek, and Frieze [5] (in fact, we have $\sigma(2, 2) = 1/2$ in our result, which matches the minimum degree condition in Theorem 12). It is natural to ask whether one can weaken the condition in (4.1) to $\delta_{k-1}(H) \geqslant \alpha n$, that is, whether one can have $\sigma = 0$. This problem was settled positively by Böttcher, Montgomery, Parczyk, and Person for graphs [13]. They showed that for each $k \geqslant 2$ and $\alpha > 0$, there is some $\eta > 0$, such that if G_α is an n-vertex graph with minimum degree at least αn, then $G_\alpha \cup \mathbb{G}(n, n^{-1/k-\eta})$ a.a.s. contains the k^{th} power of a Hamiltonian cycle. However, the problem remains open for k-graphs ($k \geqslant 3$).

Question 4.4.1. *Let integers $k \geqslant 3$ and $r \geqslant 2$ and $\alpha > 0$ be given. Is there $\varepsilon > 0$ such that, if H is a k-graph on n vertices with $\delta_{k-1}(H) \geqslant \alpha n$ and $p = p(n) \geqslant n^{-\binom{k+r-2}{k-1}^{-1}-\varepsilon}$, then a.a.s. $H \cup \mathbb{G}^{(k)}(n, p)$ contains the r^{th} power of a tight Hamiltonian cycle?*

Some remarks on the value of $\sigma = \sigma(k, r)$ in our degree condition (4.1) follow. These remarks show that, even though $\sigma > 0$ if $r > 1$, the value of σ is (in the cases considered) below the value that guarantees that H on its own contains the r^{th} power of a tight Hamilton cycle.

Let us first consider the case $k = 2$, that is, the case of graphs. In this case, $\sigma = 1 - 1/r$ and condition (4.1) is $\delta(H) \geqslant (1 - 1/r + \alpha)n$. We observe that this condition does *not* guarantee that H contains the r^{th} power of a Hamilton cycle; the minimum degree condition that does is $\delta(H) \geqslant (1 - 1/(r + 1))n = rn/(r + 1)$, and this value is optimal.

Let us now consider the case $k = 3$ and $4 \mid n$. In this case, a construction of Pikhurko [65] shows that the condition $\delta_2(H) \geqslant 3n/4$ does not guarantee the existence of the square of a tight Hamilton cycle in H (in fact, his construction is stronger and shows that this condition does not guarantee a $K_4^{(3)}$-factor in H). Our minimum degree condition for $k = 3$ and $r = 2$ is $\delta_2(H) \geqslant (2/3 + \alpha)n$.

Moreover, Lo and Zhao [56] showed that in an r-graph H the minimum codegree $\delta_{r-1}(H)$ has to be at least $\Big(1 - \Theta\big(\frac{\ln t}{t^{r-1}}\big)\Big)n$ to ensure the existence of a $K_t^{(r)}$.

Finally, a simple calculation shows that the expected number of P_n^r in $\mathbb{G}^{(k)}(n,p)$ is $o(1)$ if $p \leqslant n^{-\binom{k+r-2}{k-1}^{-1}}$ and $\varepsilon > 0$. Thus, for such a p, a.a.s. $\mathbb{G}^{(k)}(n,p)$ does *not* contain the r^{th} power of a tight Hamiltonian cycle.

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Appendix

Summary/Zusammenfassung

We present three results concerning different aspects of extremal and probabilistic combinatorics and their proofs. In the first part we study the local density conditions of graphs homomorphic to a generalised Andrásfai graph. This is motivated by the conjecture of Erdős that every n-vertex graph with the property that any $\lfloor n/2 \rfloor$ vertices span more than $n^2/50$ edges contains a triangle.

The second part of this thesis is dedicated to a Hamiltonian cycle problem in 3-uniform hypergraphs. We study which minimum pair-degree condition suffices to ensure the existence of a squared Hamiltonian cycle in a 3-uniform hypergraph. This is motivated by Pósa's conjecture which asked for a minimum degree condition that implies the existence of a second power of a Hamiltonian cycle in a graph.

In the third part we continue the study of Hamiltonian cycle problems, but this time in randomly perturbed k-uniform hypergraphs $H \cup \mathbb{G}^{(k)}(n,p)$. We investigate which conditions on the parameters $\delta_{k-1}(H)$ and p ensure the existence of an r^{th} power of a tight Hamiltonian cycle.

Wir stellen drei Resultate, die verschiedene Aspekte der extremalen und probabilistischen Kombinatorik betreffen, und deren Beweise vor. Im ersten Teil untersuchen wir lokale Dichtebedingungen von Graphen, die homomorph zu einem generalisierten Andrásfai-Graphen sind. Diese Arbeit ist durch eine Vermutung von Erdős motiviert, welche besagt, dass jeder Graph auf n Ecken, in dem jede Eckenmenge der Größe $\lfloor n/2 \rfloor$ mindestens $n^2/50$ Kanten aufspannt, ein Dreieck enthält.

Der zweite Teil dieser Arbeit widmet sich Hamiltonkreisproblemen in 3-uniformen Hypergraphen. Wir untersuchen, welche minimale Paargradbedingung ausreichend ist, um die Existenz eines Quadrathamiltonkreises in 3-uniformen Hypergraphen

zu gewährleisten. Dies ist motiviert durch Pósa's Vermutung, welche nach einer Minimalgradbedingungen fragt, die die Existenz eines Quadrathamiltonkreises in Graphen sicherstellt.

Im dritten Teil werden ebenfalls Hamiltonkreisprobleme untersucht. Dieses Mal jedoch in k-uniformen Hypergraphen der Form $H \cup \mathbb{G}^{(k)}(n, p)$, wobei H für einen vorgegebenen (deterministischen) k-uniformen Hypergraphen steht und $\mathbb{G}^{(k)}(n, p)$ für das binomiale Modell eines zufälligen k-uniformen Hypergraphen mit Kantenwahrscheinlichkeit p. Wir untersuchen, welche Bedingungen an $\delta_{k-1}(H)$ und p die Existenz einer r-ten Potenz eines Hamiltonkreises gewährleisten.

Publications related to this thesis

Articles

[84] W. Bedenknecht, G. O. Mota, Chr. Reiher, and M. Schacht, *On the local density problem for graphs of given odd-girth*, available at arXiv:1609.05712. To appear in Journal of Graph Theory.

[85] W. Bedenknecht and Chr. Reiher, *Squares of Hamiltonian cycles in 3-uniform hypergraphs*, available at arXiv:1712.08231. Submitted to Random Structures & Algorithms.

[86] W. Bedenknecht, J. Han, Y. Kohayakawa, and G. O. Mota, *Powers of tight Hamilton cycles in randomly perturbed hypergraphs*, available at arXiv:1802.08900. Submitted to Random Structures & Algorithms.

Extended Abstracts

[87] W. Bedenknecht, G. O. Mota, C. Reiher, and M. Schacht, *On the local density problem for graphs of given odd-girth*, LAGOS'17—IX Latin and American Algorithms, Graphs and Optimization, Electron. Notes Discrete Math., vol. 62, Elsevier Sci. B. V., Amsterdam, 2017, pp. 39–44. MR3746696

Declaration on my contributions

Section 1.2 and Chapter 2 are based on the paper *On the local density problem for graphs of given odd-girth* [84], which is joint work with Guilherme Oliveira Mota, Christian Reiher, and Mathias Schacht. We started working on this problem in 2015 when Guilherme Oliveira Mota was in Hamburg. During this time we came up with an initial proof strategy for Andrásfai graphs, I drafted a first version of the proof, which we jointly proofread and also changed to a version for generalised Andrásfai graphs.

Section 1.3 and Chapter 3 are based on the paper *Squares of Hamiltonian cycles in 3-uniform hypergraphs* [85], on which I worked together with Christian Reiher. He introduced me to this problem and together we discussed possible strategies for the proof. I then drafted the first version of the paper, which we jointly proofread.

Section 1.4 and Chapter 4 are based on the paper *Powers of tight Hamilton cycles in randomly perturbed hypergraphs* [86], which is joint work with Jie Han, Yoshiharu Kohayakawa, and Guilherme Oliveira Mota. I was introduced to this problem by Mathias Schacht, who suggested it because I already worked on a similar problem with my supervisor. The work on this topic started during my DAAD-funded research visit to São Paulo. We jointly figured out the details of the proof, which follows a similar strategy as [85], and drafted a first version of the paper during my time there. After the research visit we jointly proofread the proof .

Eidesstattliche Erklärung

Hiermit erkläre ich an Eides statt, dass ich die vorliegende Dissertationsschrift selbst verfasst und keine anderen als die angegebenen Quellen als Hilfsmittel benutzt habe.

Ort, Datum	Wiebke Bedenknecht