Logarithmic structures on commutative Hk-algebra spectra

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Introduction

This thesis is devoted to establish a theory of logarithmic structures on E_∞ differential graded k-algebras (E_∞ dgas) where k is a commutative ring with unit. The concept of logarithmic structures has its origin in algebraic geometry [Kat89]. The relevance of logarithmic structures for homotopy theory became apparent in the work of Hesselholt and Madsen who used logarithmic structures for the description of algebraic K-theory of local fields [HM03]. Motivated by the aim to extend structural results about the algebraic K-theory of commutative rings to the algebraic K-theory of commutative ring spectra, Rognes transferred this notion to homotopy theory [Rog09].

A *pre-log structure* on a commutative ring A is a commutative monoid M together with a map of commutative monoids $\alpha\colon M \to (A, \cdot)$ from M to the underlying multiplicative commutative monoid of A. The triple (A, M, α) is called a *pre-log ring*. The datum (M, α) is a *log structure* on A if the map $\alpha^{-1}(A^\times) \to A^\times$ from the sub commutative monoid $\alpha^{-1}(A^\times) \subseteq M$ of elements mapping to the units $A^\times$ of A is an isomorphism. A *log ring* is a commutative ring A equipped with a log structure (M, α). An easy example is the *trivial* log ring $(A, A^\times, A^\times \hookrightarrow A)$.

As a homotopical generalization of log rings, Rognes introduced *log ring spectra* where commutative symmetric ring spectra play the role of commutative rings. Exploiting the Quillen equivalence between commutative $\mathcal{I}$-spaces and E_∞ spaces [SS12], for a commutative symmetric ring spectrum A, there is a commutative $\mathcal{I}$-space $\Omega^{\mathcal{I}}(A)$ defined by $\mathbf{m} \mapsto \Omega^m(A(m))$, representing the underlying multiplicative E_∞ space of A. In this way, commutative $\mathcal{I}$-spaces may be viewed as a homotopical counterpart of commutative monoids, and Rognes related them to commutative symmetric ring spectra via a Quillen adjunction. Further, for a commutative symmetric ring spectrum A, there is a sub commutative $\mathcal{I}$-space $\mathrm{GL}_1^{\mathcal{I}}(A)$ of $\Omega^{\mathcal{I}}(A)$ that models the grouplike E_∞ space of units of A. One drawback is that both commutative $\mathcal{I}$-spaces $\Omega^{\mathcal{I}}(A)$ and $\mathrm{GL}_1^{\mathcal{I}}(A)$ do not carry any information about the negative dimensional homotopy groups of A. Consequently, the functors $\Omega^{\mathcal{I}}$ and $\mathrm{GL}_1^{\mathcal{I}}$ do not distinguish between a commutative symmetric ring spectrum and its connective cover so that they cannot detect periodicity phenomena in stable homotopy theory. For example, the connective cover map of complex topological K-theory $ku \to KU$ induces a weak equivalence $\mathrm{GL}_1^{\mathcal{I}}(ku) \xrightarrow{\sim} \mathrm{GL}_1^{\mathcal{I}}(KU)$.

Sagave and Schlichtkrull managed to overcome this problem in [SS12] by employing the more elaborate index category $\mathcal{J}$. The latter is given by Quillen's localization construction $\Sigma^{-1}\Sigma$ on the category of finite sets and bijections Σ. Hence, the classifying space $B\mathcal{J}$ is homotopy equivalent to $Q(S^0)$, which is the underlying additive E_∞ space of the sphere spectrum $\mathbb{S}$. The categories $\mathcal{I}$ and $\mathcal{J}$ are examples of *well-structured index categories* which is a suitable framework to obtain model structures on (structured) diagram spaces. Sagave and Schlichtkrull prove that for a well-structured index category $\mathcal{K}$ satisfying some assumptions, the model category of commutative $\mathcal{K}$-spaces is Quillen equivalent to E_∞ spaces over the classifying space $B\mathcal{K}$. So commutative $\mathcal{J}$-spaces are Quillen equivalent to

E_∞ spaces over $Q(S^0)$ that Sagave and Schlichtkrull describe as *graded* E_∞ spaces. For a commutative symmetric ring spectrum A, the commutative $\mathcal{J}$-space $\Omega^{\mathcal{J}}(A)$ is built from all spaces $\Omega^{m_2}(A(m_1))$. This makes it possible to specify a sub commutative $\mathcal{J}$-space $\mathrm{GL}_1^{\mathcal{J}}(A)$ of $\Omega^{\mathcal{J}}(A)$ from which we can recover all units in the graded ring $\pi_*(A)$.

A *pre-log structure* on a commutative symmetric ring spectrum A is a commutative $\mathcal{J}$-space M together with a map of commutative $\mathcal{J}$-spaces $\alpha\colon M \to \Omega^{\mathcal{J}}(A)$. The resulting *pre-log ring spectrum* (A, M, α) is a *log ring spectrum* if the base change map $\alpha^{-1}(\mathrm{GL}_1^{\mathcal{J}}(A)) \to \mathrm{GL}_1^{\mathcal{J}}(A)$ of the structure map α along the inclusion $\mathrm{GL}_1^{\mathcal{J}}(A) \to \Omega^{\mathcal{J}}(A)$ is a weak equivalence.

In joint work Rognes, Sagave and Schlichtkrull introduced *logarithmic topological Hochschild homology* which is an extension of ordinary topological Hochschild homology [RSS15]. The logarithmic topological Hochschild homology of appropriate pre-log ring spectra participates in interesting localization homotopy cofibre sequences that are similar to localization sequences for algebraic K-theory. This is significant for achieving results on algebraic K-theory of commutative ring spectra by means of localization techniques and trace maps from algebraic K-theory to topological Hochschild homology [RSS15]. For instance, topological K-theory spectra yield convenient logarithmic ring spectra which can be regarded as objects sitting in between the connective and the periodic versions of the respective topological K-theory spectra ([RSS15],[RSS18],[Sag14]). Moreover, the tamely ramified extension of the inclusion of the connective Adams summand ℓ into the p-local connective topological complex K-theory spectrum $ku_{(p)}$ is formally étale with respect to logarithmic topological Hochschild homology [RSS18].

The log ring spectra considered so far either come from log rings or involve topological K-theory spectra. The goal of this thesis is to provide a framework to gain new examples through algebraic objects. Richter and Shipley constructed a chain of Quillen equivalences connecting commutative Hk-algebra spectra to E_∞ dgas [RS17]. Using this, we develop a concept of log structures in the algebraic setting.

Pre-log structures on E_∞ dgas

An intermediate model category in Richter and Shipley's chain of Quillen equivalences between commutative Hk-algebra spectra and E_∞ dgas is the category of commutative symmetric ring spectra in simplicial k-modules $C(\mathrm{Sp}^\Sigma(s\mathrm{mod}))$ (see Subsection 2.3). We relate this model category to commutative $\bar{\mathcal{J}}$-spaces.

Proposition (Proposition 2.30)**.** *There is a Quillen adjunction*

$$C\mathcal{S}^{\bar{\mathcal{J}}} \underset{\Omega^{\bar{\mathcal{J}}}}{\overset{\Lambda^{\bar{\mathcal{J}}}}{\rightleftarrows}} C(\mathrm{Sp}^\Sigma(s\mathrm{mod})). \tag{0.1}$$

Here the category $\bar{\mathcal{J}}$ (see Definition 1.7) arises from the category $\mathcal{J}$ by defining an equivalence relation on the morphism sets of the latter. The idea for the category $\bar{\mathcal{J}}$

results from the fact that the action of the symmetric group Σ_n on the pointed space of the n-sphere S^n permutes coordinates, while the action of Σ_n on the n-sphere chain complex $\mathbb{S}^n(k)$ is just the sign action. The category $\bar{\mathcal{J}}$ is a well-structured index category, too (see Proposition 1.13), and the induced map of grouplike E_∞ spaces $B\mathcal{J} \to B\bar{\mathcal{J}}$ models the first Postnikov section of the sphere spectrum $\mathbb{S}$.

We employ the category $C(\mathrm{Sp}^\Sigma(s\mathrm{mod}))$ as a model for E_∞ dgas. Given the above Quillen adjunction (0.1), we define *pre-log structures* on E_∞ dgas as follows.

Definition (Definition 2.32)**.** Let A be an object in $C(\mathrm{Sp}^\Sigma(s\mathrm{mod}))$. A *pre-log structure* on A is a pair (M, α) consisting of a commutative $\bar{\mathcal{J}}$-space M and a map of commutative $\bar{\mathcal{J}}$-spaces $\alpha\colon M \to \Omega^{\bar{\mathcal{J}}}(A)$. If (M, α) is a pre-log structure on A, we call the triple (A, M, α) a *pre-log cdga*.

In consideration of Sagave and Schlichtkrull's definition of pre-log ring spectra, the following proposition confirms that the above definition of pre-log cdgas is reasonable.

Proposition (Proposition 2.36)**.** *For a positive fibrant object A in $C(\mathrm{Sp}^\Sigma(s\mathrm{mod}))$ and $(\mathbf{m}_1, \mathbf{m}_2)$ in $\bar{\mathcal{J}}$ such that $m_1 \geq 1$, the space $\Omega^{\bar{\mathcal{J}}}(A)(\mathbf{m}_1, \mathbf{m}_2)$ is weakly equivalent to the space $\Omega^{m_2}(U(A)(m_1))$ where U denotes the forgetful functor to commutative symmetric ring spectra in pointed simplicial sets.*

Making use of this result, we see that a homology class in the graded homology ring of an E_∞ dga gives rise to a pre-log cdga.

Example (Example 2.39)**.** Let A be a positive fibrant object in $C(\mathrm{Sp}^\Sigma(s\mathrm{mod}))$, and let $(\mathbf{m}_1, \mathbf{m}_2)$ be in $\bar{\mathcal{J}}$ such that $m_1 \geq 1$. Let $[x]$ be a homotopy class in $\pi_{m_2 - m_1}(U(A))$ represented by a map $x\colon S^{m_2} \to U(A)(m_1)$ in pointed spaces. The above proposition ensures that the latter corresponds to a point in the space $\Omega^{\bar{\mathcal{J}}}(A)(\mathbf{m}_1, \mathbf{m}_2)$. By adjunction there is a map of commutative $\bar{\mathcal{J}}$-spaces $\alpha\colon \mathbb{C}(F^{\bar{\mathcal{J}}}_{(\mathbf{m}_1,\mathbf{m}_2)}(*)) \to \Omega^{\bar{\mathcal{J}}}(A)$ where we write $\mathbb{C}$ for the monad associated to the commutativity operad in spaces and $F^{\bar{\mathcal{J}}}_{(\mathbf{m}_1,\mathbf{m}_2)}$ for the left adjoint of the evaluation functor with respect to $\bar{\mathcal{J}}$-level $(\mathbf{m}_1, \mathbf{m}_2)$. We obtain the pre-log cdga $(A, \mathbb{C}(F^{\bar{\mathcal{J}}}_{(\mathbf{m}_1,\mathbf{m}_2)}(*)), \alpha)$.

Apart from this, the previous proposition leads to the definition of *units* of A as a sub commutative $\bar{\mathcal{J}}$-space $\mathrm{GL}_1^{\bar{\mathcal{J}}}(A)$ of $\Omega^{\bar{\mathcal{J}}}(A)$ (see Definition 2.42), and with this a condition for a pre-log cdga to be a *log cdga*.

Definition (Definition 2.44)**.** Let A be a positive fibrant object in $C(\mathrm{Sp}^\Sigma(s\mathrm{mod}))$. A pre-log cdga (A, M, α) is a *log cdga* if the base change map $\alpha^{-1}(\mathrm{GL}_1^{\bar{\mathcal{J}}}(A)) \to \mathrm{GL}_1^{\bar{\mathcal{J}}}(A)$ of the structure map α along the inclusion $\mathrm{GL}_1^{\bar{\mathcal{J}}}(A) \to \Omega^{\bar{\mathcal{J}}}(A)$ is a weak equivalence.

An elementary example is the *trivial* log cdga $(A, \mathrm{GL}_1^{\bar{\mathcal{J}}}(A), \mathrm{GL}_1^{\bar{\mathcal{J}}}(A) \to \Omega^{\bar{\mathcal{J}}}(A))$. Furthermore, there is a construction called *logification* which turns a pre-log cdga into a log cdga (see Construction 2.47).

Group completion in commutative diagram spaces and logarithmic topological Hochschild homology of log cdgas

Commutative $\mathcal{K}$-spaces are Quillen equivalent to E_∞ spaces over $B\mathcal{K}$ for a well-structured index category $\mathcal{K}$ fulfilling a few assumptions. As a special case, commutative $\mathcal{I}$-spaces are Quillen equivalent to E_∞ spaces. Taking this into account, we prove the following theorem.

Theorem (Theorem 3.22)**.** *There is a chain of Quillen equivalences linking commutative $\mathcal{K}$-spaces to commutative $\mathcal{I}$-spaces over $B_{\mathcal{I}}\mathcal{K}$ where $B_{\mathcal{I}}\mathcal{K}$ is a commutative $\mathcal{I}$-space model of $B\mathcal{K}$.*

With the help of this and the additional assumption that the simplicial monoid $B\mathcal{K}$ is grouplike, we provide a notion of *group completion* in commutative $\mathcal{K}$-spaces. Our approach is model categorical which has the advantage that we get functorial group completions for all objects without extra conditions. We identify a left Bousfield localization on commutative $\mathcal{K}$-spaces as a *group completion model structure*. We do this by verifying that the latter is Quillen equivalent to a localized model structure on commutative $\mathcal{I}$-spaces over $B_{\mathcal{I}}\mathcal{K}$ in order to build on Sagave and Schlichtkrull's work on group completion in commutative $\mathcal{I}$-spaces [SS13].

Theorem (Theorem 3.37)**.** *Suppose that the simplicial monoid $B\mathcal{K}$ is grouplike. There is a group completion model structure on commutative $\mathcal{K}$-spaces in which fibrant replacements model group completions. A map of commutative $\mathcal{K}$-spaces is a group completion if the associated map of E_∞ spaces is a group completion in the usual sense.*

Specializing the index category to be $\bar{\mathcal{J}}$, the group completion functor on commutative $\bar{\mathcal{J}}$-spaces generates more examples of pre-log cdgas and is an essential foundation for the definition of *logarithmic topological Hochschild homology* of log cdgas.

Definition (Definition 4.23)**.** Let (A, M, α) be a cofibrant pre-log cdga. We define the *logarithmic topological Hochschild homology* $\mathrm{THH}(A, M, \alpha)$ via the pushout diagram

$$\begin{tikzcd}
\mathrm{THH}^{Hk}(\Lambda^{\bar{\mathcal{J}}}(M)) \arrow[r, "\cong"] \arrow[d] & \Lambda^{\bar{\mathcal{J}}}(B^{\mathrm{cy}}(M)) \arrow[r] & \Lambda^{\bar{\mathcal{J}}}((B^{\mathrm{cy}}(M))^{\mathrm{rep}}) \arrow[d] \\
\mathrm{THH}^{Hk}(A) \arrow[rr] & & \mathrm{THH}(A, M, \alpha).
\end{tikzcd} \tag{0.2}$$

The functor THH^{Hk} denotes ordinary topological Hochschild homology where the ground ring is given by the Eilenberg Mac Lane spectrum Hk. This can be identified with derived Hochschild homology which is also known as Shukla homology (see Remark 4.21). The left vertical map in the diagram (0.2) is determined by applying the functor THH^{Hk} to the adjoint $\Lambda^{\bar{\mathcal{J}}}(M) \to A$ of the structure map $\alpha\colon M \to \Omega^{\bar{\mathcal{J}}}(A)$. The top horizontal map in (0.2) is induced by the *repletion map* $B^{\mathrm{cy}}(M) \to (B^{\mathrm{cy}}(M))^{\mathrm{rep}}$ (see Definition 4.8) where the source $B^{\mathrm{cy}}(M)$ is the *cyclic bar construction* of the commutative $\bar{\mathcal{J}}$-space M (see Definition 4.1) and the repletion map can be viewed as a group completion relative to M. We show that the definition of logarithmic topological Hochschild homology is homotopy invariant under logification (see Proposition 4.25). More than that, we give a criterion for a morphism of log cdgas to be *formally étale* through the eyes of logarithmic topological Hochschild homology (see Definition 4.26) and present approaches towards examples.

In this work we provide several examples of pre-log cdgas and log cdgas. These mostly result from adapting the corresponding examples of (pre-) log ring spectra to the algebraic context. So far it is unclear how to construct interesting pre-log structures on E_∞ dgas so that e.g. tamely ramified extensions of E_∞ dgas give rise to formally log THH-étale morphisms of pre-log cdgas. As a guiding example, it would be crucial to establish suitable pre-log structures on the cochains of a space X with coefficients in k.

Organization

This thesis is organized as follows:
In the first section we introduce the category $\bar{\mathcal{J}}$ (see Definition 1.7). We discuss Sagave and Schlichtkrull's machinery of well-structured index categories and in doing so focus on the properties of the category $\bar{\mathcal{J}}$.
The second section is dedicated to the definition of log structures on E_∞ dgas. We first collect preliminary results about diagram spaces and symmetric spectra. Then we move on to Richter and Shipley's chain of Quillen equivalences between commutative Hk-algebra spectra and E_∞ dgas (see Theorem 2.16). Afterwards we relate symmetric spectra to $\bar{\mathcal{J}}$-spaces to derive from this the definition of pre-log structures on E_∞ dgas (see Definition 2.32). We specify units of E_∞ dgas (see Definition 2.42) and the logification process (see Construction 2.47). In addition, we give some examples of pre-log cdgas and log cdgas. Other than this, we discuss an alternative approach to set up log structures on E_∞ dgas via diagram chain complexes. Along with this, we provide a homotopy colimit formula on diagram chain complexes (see (2.47)) and argue that the latter does not have to admit a model structure in which the fibrant objects are precisely the objects that are homologically constant and the homotopy colimit functor detects the weak equivalences (see Example 2.57).
In the third section we start with analyzing the interaction of left Bousfield localizations with comma categories in a general context. We prove that in a sense, left Bousfield localization commutes with forming a comma category (see Proposition 3.4). We continue with stating Sagave and Schlichtkrull's chain of Quillen equivalences connecting commutative $\mathcal{K}$-spaces to E_∞ spaces over $B\mathcal{K}$ for a well-structured index category $\mathcal{K}$ satisfying a couple of assumptions (see Theorem 3.7). Motivated by this, we show that commutative $\mathcal{K}$-spaces are Quillen equivalent to commutative $\mathcal{I}$-spaces over $B_{\mathcal{I}}\mathcal{K}$ (see Theorem 3.22). This outcome together with the result on left Bousfield localizations and comma categories are substantial ingredients to characterize a localized model structure on commutative $\mathcal{K}$-spaces as a group completion model structure later (see Theorem 3.37).
Finally, in the last section we describe the cyclic and replete bar constructions as well as general repletion of commutative diagram spaces (see Definition 4.1, Construction 4.4,

Definition 4.8). After this, we set the index category to be $\bar{\mathcal{J}}$ and give sense to logarithmic topological Hochschild homology of log cdgas (see Definition 4.23).

Notation

Throughout this thesis, let k denote a discrete commutative ring with unit. We distinguish the different homs occuring in this work as follows. For a category $\mathcal{C}$ and X and Y objects in $\mathcal{C}$, we write $\mathcal{C}(X, Y)$ for the set of maps from X to Y and $\underline{\mathrm{Hom}}_{\mathcal{C}}(X, Y)$ for the internal hom object in $\mathcal{C}$. If the category $\mathcal{C}$ is enriched over a category $\mathcal{D}$, we denote the $\mathcal{D}$-enriched hom of X and Y in $\mathcal{C}$ by $\mathrm{Hom}^{\mathcal{C}}_{\mathcal{D}}(X, Y)$. More notation will be introduced as we need it.

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1 The category $\bar{\mathcal{J}}$

We introduce the category $\bar{\mathcal{J}}$ and discuss some properties. The category $\bar{\mathcal{J}}$ arises from Sagave and Schlichtkrull's category $\mathcal{J}$ by defining an equivalence relation on the morphism sets of the latter. With the category $\bar{\mathcal{J}}$ at hand, we develop a notion of *pre-log structures* on E_∞ differential graded k-algebras in Section 2. We explain Sagave and Schlichtkrull's concept of *well-structured index categories* and apply this to the category $\bar{\mathcal{J}}$. Well-structured index categories are a useful tool to obtain model structures on diagram spaces.

1.1 Definitions

In this subsection we recall Sagave and Schlichtkrull's definition of the category $\mathcal{J}$. We specify an equivalence relation on the morphism sets of this category to define the category $\bar{\mathcal{J}}$.

Let $\mathcal{I}$ be the category of finite sets with objects $\mathbf{m} = \{1, \dots, m\}$ for $m \geq 0$, with the convention that $\mathbf{0} = \emptyset$, and injective maps as morphisms. Every map in $\mathcal{I}(\mathbf{m}, \mathbf{n})$ can be factored into the standard inclusion $\iota_{\mathbf{m},\mathbf{n}} \colon \mathbf{m} \to \mathbf{n}$ followed by a permutation in Σ_n. For $n \geq m + 2$, this factorization is not unique. The morphism set $\mathcal{I}(\mathbf{m}, \mathbf{n})$ is isomorphic to Σ_n / Σ_{n-m}. The ordered concatenation $\sqcup$ makes $\mathcal{I}$ a symmetric strict monoidal category with unit $\mathbf{0}$ and non-trivial symmetry isomorphisms the shuffle maps $\chi_{\mathbf{m},\mathbf{n}} \colon \mathbf{m} \sqcup \mathbf{n} \to \mathbf{n} \sqcup \mathbf{m}$ (see [SS12, p. 2124]). The classifying space $B\mathcal{I}$ is contractible because the category $\mathcal{I}$ has the initial object $\mathbf{0}$.

We define an equivalence relation on the morphism set $\mathcal{I}(\mathbf{m}, \mathbf{n})$. A map $\alpha \colon \mathbf{m} \to \mathbf{n}$ is equivalent to a map $\alpha' \colon \mathbf{m} \to \mathbf{n}$ if and only if there exists a map σ in the alternating group A_n such that $\alpha = \sigma \circ \alpha'$. We denote the equivalence class of α by $[\alpha]$. Let $\mathcal{I}(\mathbf{m}, \mathbf{n})/\!\sim$ be the set of equivalence classes.

Lemma 1.1. *For a triple* $\mathbf{l}$, $\mathbf{m}$ *and* $\mathbf{n}$ *of objects in* $\mathcal{I}$, *there is a composition law* $\mathcal{I}(\mathbf{l}, \mathbf{m})/\!\sim \times \, \mathcal{I}(\mathbf{m}, \mathbf{n})/\!\sim \, \to \mathcal{I}(\mathbf{l}, \mathbf{n})/\!\sim$ *defined by* $([\alpha], [\beta]) \mapsto [\beta \circ \alpha]$.

Proof. We prove that this assignment is well-defined. Let $\alpha, \alpha' \colon \mathbf{l} \to \mathbf{m}$ and σ in A_m such that $\alpha = \sigma \circ \alpha'$, and let $\beta, \beta' \colon \mathbf{m} \to \mathbf{n}$ and τ in A_n such that $\beta = \tau \circ \beta'$. We have to show that $\beta \circ \alpha$ is equivalent to $\beta' \circ \alpha'$. We write $\beta' = \xi' \circ \iota_{\mathbf{m},\mathbf{n}}$ where ξ' is in Σ_n. Then we find that

$$\begin{aligned}
\beta \circ \alpha &= \tau \circ \beta' \circ \sigma \circ \alpha' \\
&= \tau \circ \xi' \circ \iota_{\mathbf{m},\mathbf{n}} \circ \sigma \circ \alpha' \\
&= \tau \circ \xi' \circ (\sigma \sqcup \mathrm{id}) \circ \iota_{\mathbf{m},\mathbf{n}} \circ \alpha' \\
&= \tau \circ \xi' \circ (\sigma \sqcup \mathrm{id}) \circ (\xi')^{-1} \circ \xi' \circ \iota_{\mathbf{m},\mathbf{n}} \circ \alpha' \\
&= \tau \circ \xi' \circ (\sigma \sqcup \mathrm{id}) \circ (\xi')^{-1} \circ \beta' \circ \alpha'
\end{aligned}$$

where $\omega = \tau \circ \xi' \circ (\sigma \sqcup \mathrm{id}) \circ (\xi')^{-1}$ is in Σ_n and

$$\begin{aligned}\mathrm{sgn}(\omega) &= \mathrm{sgn}(\tau) \cdot \mathrm{sgn}(\xi') \cdot \mathrm{sgn}(\sigma \sqcup \mathrm{id}) \cdot (\mathrm{sgn}(\xi'))^{-1} \\ &= \mathrm{sgn}(\tau) \cdot \mathrm{sgn}(\sigma) \\ &= 1.\end{aligned}$$

Beware that the sign $\mathrm{sgn}(\omega)$ does not depend on the choice of factorization of β' whereas the definition of the map ω does rely on the choice of factorization of β'. But we can proceed as above for any other choice of factorization of β' to obtain a suitable even permutation. □

Definition 1.2. We define $\bar{\mathcal{I}}$ to be the category with objects $\mathbf{m}$ in $\mathcal{I}$ and morphisms $\bar{\mathcal{I}}(\mathbf{m}, \mathbf{n}) = \mathcal{I}(\mathbf{m}, \mathbf{n})/\sim$.

Lemma 1.3. *The morphism set* $\bar{\mathcal{I}}(\mathbf{m}, \mathbf{n})$ *is determined by*

$$\bar{\mathcal{I}}(\mathbf{m}, \mathbf{n}) \cong \begin{cases} \emptyset, & m \geq n+1, \\ *, & m = 0, \\ *, & m = 1, n = 1, \\ *, & m \geq 1, n \geq m+2, \\ \Sigma_2, & m \geq 1, n = m+1, \\ \Sigma_2, & m \geq 2, n = m. \end{cases}$$

Proof. We show that $\bar{\mathcal{I}}(\mathbf{m}, \mathbf{n})$ is trivial for $m \geq 1$ and $n \geq m+2$. Let α be in $\bar{\mathcal{I}}(\mathbf{m}, \mathbf{n})$ such that $\alpha(i) = i$ for $1 \leq i \leq m$. We claim that every map α' in $\bar{\mathcal{I}}(\mathbf{m}, \mathbf{n})$ is equivalent to α. Let α' be in $\bar{\mathcal{I}}(\mathbf{m}, \mathbf{n})$. We define $\tilde{\sigma} \colon \mathrm{im}(\alpha') \to \mathbf{n}$ by $\tilde{\sigma}(i) = \alpha \circ (\alpha')^{-1}(i)$. We next want to define a permutation $\sigma \colon \mathbf{n} \to \mathbf{n}$ so that the restriction of σ to $\mathrm{im}(\alpha')$ is $\tilde{\sigma}$ and $\mathrm{sgn}(\sigma) = 1$. For this, we have to specify the restriction of σ to $\mathbf{n} \setminus \mathrm{im}(\alpha')$ in such a way that $\mathrm{sgn}(\sigma) = 1$. Without pinning the restriction of σ to $\mathbf{n} \setminus \mathrm{im}(\alpha')$ down yet, we write down all already visible inversions. If the number of these is even, we define the map σ to send $\mathbf{n} \setminus \mathrm{im}(\alpha')$ order-preservingly to $\{m+1, \ldots, n\}$. Let the number be odd. Let i be the smallest element in $\mathbf{n} \setminus \mathrm{im}(\alpha')$, and let $\hat{i}$ be the second smallest element in $\mathbf{n} \setminus \mathrm{im}(\alpha')$. We define $\sigma(i) = m+2$ and $\sigma(\hat{i}) = m+1$. We define the map σ to take the rest of the elements in $\mathbf{n} \setminus \mathrm{im}(\alpha')$ order-preservingly to $\{m+3, \ldots, n\}$. So we gain only one more inversion, namely $(i, \hat{i})$ so that $\mathrm{sgn}(\sigma) = 1$.
The remaining cases are clear. □

Note that there is a projection functor $\mathcal{I} \to \bar{\mathcal{I}}$. The category $\bar{\mathcal{I}}$ inherits a symmetric strict monoidal structure from $\mathcal{I}$. Let $\sqcup \colon \bar{\mathcal{I}} \times \bar{\mathcal{I}} \to \bar{\mathcal{I}}$ be the functor defined on objects by $(\mathbf{m}, \mathbf{n}) \mapsto \mathbf{m} \sqcup \mathbf{n}$ and on morphisms by $([\alpha], [\beta]) \mapsto [\alpha \sqcup \beta]$. This is well-defined: Let $\alpha, \alpha' \colon \mathbf{l} \to \mathbf{m}$ and σ in A_m such that $\alpha = \sigma \circ \alpha'$, and let $\beta, \beta' \colon \mathbf{n} \to \mathbf{p}$ and τ in A_p such that $\beta = \tau \circ \beta'$. Then $\sigma \sqcup \tau$ is in Σ_{m+p} with $\mathrm{sgn}(\sigma \sqcup \tau) = 1$, and

$$\begin{aligned}\alpha \sqcup \beta &= (\sigma \circ \alpha') \sqcup (\tau \circ \beta') \\ &= (\sigma \sqcup \tau) \circ (\alpha' \sqcup \beta').\end{aligned}$$

The unit is $\mathbf{0}$, and the symmetry isomorphisms are the equivalence classes of the shuffle maps $[\chi_{\mathbf{m},\mathbf{n}}]\colon \mathbf{m} \sqcup \mathbf{n} \to \mathbf{n} \sqcup \mathbf{m}$.

Definition 1.4. [SS12, Definition 4.2] The category $\mathcal{J}$ has as objects pairs $(\mathbf{m}_1, \mathbf{m}_2)$ of objects in $\mathcal{I}$, and a morphism $(\mathbf{m}_1, \mathbf{m}_2) \to (\mathbf{n}_1, \mathbf{n}_2)$ is a triple $(\alpha_1, \alpha_2, \rho)$ with $\alpha_1\colon \mathbf{m}_1 \to \mathbf{n_1}$ and $\alpha_2\colon \mathbf{m}_2 \to \mathbf{n}_2$ morphisms in $\mathcal{I}$ and $\rho\colon \mathbf{n}_1 \setminus \mathrm{im}(\alpha_1) \to \mathbf{n}_2 \setminus \mathrm{im}(\alpha_2)$ a bijection identifying the complement of α_1 in $\mathbf{n}_1$ with the complement of α_2 in $\mathbf{n}_2$. The composition of two morphisms

$$(\mathbf{l}_1, \mathbf{l}_2) \xrightarrow{(\alpha_1,\alpha_2,\rho)} (\mathbf{m}_1, \mathbf{m}_2) \xrightarrow{(\beta_1,\beta_2,\phi)} (\mathbf{n}_1, \mathbf{n}_2)$$

is defined by $(\beta_1 \circ \alpha_1, \beta_2 \circ \alpha_2, \psi)$, where ψ is the bijection

$$\mathbf{n}_1 \setminus \mathrm{im}(\beta_1 \circ \alpha_1) \xrightarrow{\psi} \mathbf{n}_2 \setminus \mathrm{im}(\beta_2 \circ \alpha_2)$$

specified by

$$\psi(s) = \begin{cases} \phi(s), & s \in \mathbf{n}_1 \setminus \mathrm{im}(\beta_1), \\ \beta_2 \circ \rho(t), & s = \beta_1(t) \in \mathrm{im}(\beta_1 \mid_{\mathbf{m}_1 \setminus \mathrm{im}(\alpha_1)}). \end{cases}$$

We also refer to ψ as $\phi \cup (\beta_2 \circ \rho \circ \beta_1^{-1})$. The ordered concatenation in both entries defines a symmetric strict monoidal structure on the category $\mathcal{J}$. Let $\sqcup\colon \mathcal{J} \times \mathcal{J} \to \mathcal{J}$ be the functor given on objects by $((\mathbf{m}_1, \mathbf{m}_2), (\mathbf{n}_1, \mathbf{n}_2)) \mapsto (\mathbf{m}_1 \sqcup \mathbf{n}_1, \mathbf{m}_2 \sqcup \mathbf{n}_2)$, and on morphisms by $((\alpha_1, \alpha_2, \rho), (\beta_1, \beta_2, \phi)) \mapsto (\alpha_1 \sqcup \beta_1, \alpha_2 \sqcup \beta_2, \rho \sqcup \phi)$, where $\rho \sqcup \phi$ is the bijection induced by the bijections ρ and ϕ. The unit is $(\mathbf{0}, \mathbf{0})$, and the symmetry isomorphisms are

$$(\mathbf{m}_1, \mathbf{m}_2) \sqcup (\mathbf{n}_1, \mathbf{n}_2) \xrightarrow{(\chi_{\mathbf{m}_1,\mathbf{n}_1}, \chi_{\mathbf{m}_2,\mathbf{n}_2}, \mathrm{id}_\emptyset)} (\mathbf{n}_1, \mathbf{n}_2) \sqcup (\mathbf{m}_1, \mathbf{m}_2)$$

[SS12, Proposition 4.3].

Remark 1.5. The category $\mathcal{J}$ is isomorphic to Quillen's localization construction $\Sigma^{-1}\Sigma$ on the category of finite sets and bijections Σ [SS12, Proposition 4.4]. Thus, the classifying space $B\mathcal{J}$ is homotopy equivalent to $Q(S^0)$ [SS12, Corollary 4.5]. The latter is the underlying additive E_∞ space of the sphere spectrum $\mathbb{S}$.

As for the category $\mathcal{I}$, we define an equivalence relation on the set $\mathcal{J}((\mathbf{m}_1, \mathbf{m}_2), (\mathbf{n}_1, \mathbf{n}_2))$. A map $(\alpha_1, \alpha_2, \rho)\colon (\mathbf{m}_1, \mathbf{m}_2) \to (\mathbf{n}_1, \mathbf{n}_2)$ is equivalent to a map $(\alpha'_1, \alpha'_2, \rho')\colon (\mathbf{m}_1, \mathbf{m}_2) \to (\mathbf{n}_1, \mathbf{n}_2)$ if and only if $\alpha_1 = \alpha'_1$ and there exists a map σ in A_{n_2} such that $\alpha_2 = \sigma \circ \alpha'_2$ and $\rho = \sigma \circ \rho'$, that is, $(\alpha_1, \alpha_2, \rho) = (\mathrm{id}_{\mathbf{n}_1}, \sigma, \mathrm{id}_\emptyset) \circ (\alpha'_1, \alpha'_2, \rho')$. We write $[\alpha_1, \alpha_2, \rho]$ for the equivalence class of $(\alpha_1, \alpha_2, \rho)$. Let $\mathcal{J}((\mathbf{m}_1, \mathbf{m}_2), (\mathbf{n}_1, \mathbf{n}_2))/\sim$ be the set of equivalence classes.

Lemma 1.6. *For a triple $(\mathbf{l}_1, \mathbf{l}_2)$, $(\mathbf{m}_1, \mathbf{m}_2)$ and $(\mathbf{n}_1, \mathbf{n}_2)$ of objects in $\mathcal{J}$, there is a composition law $\mathcal{J}((\mathbf{l}_1, \mathbf{l}_2), (\mathbf{m}_1, \mathbf{m}_2))/\sim \times \mathcal{J}((\mathbf{m}_1, \mathbf{m}_2), (\mathbf{n}_1, \mathbf{n}_2))/\sim \to \mathcal{J}((\mathbf{l}_1, \mathbf{l}_2), (\mathbf{n}_1, \mathbf{n}_2))/\sim$ defined by $([\alpha_1, \alpha_2, \rho], [\beta_1, \beta_2, \phi]) \mapsto [\beta_1 \circ \alpha_1, \beta_2 \circ \alpha_2, \phi \cup (\beta_2 \circ \rho \circ \beta_1^{-1})]$.*

Proof. We show that this assignment is well-defined. Let $(\alpha_1, \alpha_2, \rho)$ be equivalent to $(\alpha'_1, \alpha'_2, \rho')$ meaning that $\alpha_1 = \alpha'_1$ and that there is a σ in A_{m_2} such that $\alpha_2 = \sigma \circ \alpha'_2$ and

$\rho = \sigma \circ \rho'$, and let (β_1, β_2, ϕ) be equivalent to $(\beta_1', \beta_2', \phi')$ meaning that $\beta_1 = \beta_1'$ and that there is a τ in A_{n_2} such that $\beta_2 = \tau \circ \beta_2'$ and $\phi = \tau \circ \phi'$. We observe that

$$\begin{aligned}
&(\beta_1, \beta_2, \phi) \circ (\alpha_1, \alpha_2, \rho) \\
&= (\mathrm{id}_{\mathbf{n}_1}, \tau, \mathrm{id}_\emptyset) \circ (\beta_1', \beta_2', \phi') \circ (\mathrm{id}_{\mathbf{m}_1}, \sigma, \mathrm{id}_\emptyset) \circ (\alpha_1', \alpha_2', \rho') \\
&= (\mathrm{id}_{\mathbf{n}_1}, \tau, \mathrm{id}_\emptyset) \circ (\beta_1' \circ \mathrm{id}_{\mathbf{m}_1}, \beta_2' \circ \sigma, \phi') \circ (\alpha_1', \alpha_2', \rho') \\
&= (\mathrm{id}_{\mathbf{n}_1}, \tau, \mathrm{id}_\emptyset) \circ (\mathrm{id}_{\mathbf{n}_1}, (\beta_2' \circ \sigma \circ (\beta_2')^{-1}) \sqcup \mathrm{id}_{\mathbf{n}_2 \setminus \mathrm{im}(\beta_2')}, \mathrm{id}_\emptyset) \circ (\beta_1', \beta_2', \phi') \circ (\alpha_1', \alpha_2', \rho') \\
&= (\mathrm{id}_{\mathbf{n}_1}, \tau \circ ((\beta_2' \circ \sigma \circ (\beta_2')^{-1}) \sqcup \mathrm{id}_{\mathbf{n}_2 \setminus \mathrm{im}(\beta_2')}), \mathrm{id}_\emptyset) \circ (\beta_1', \beta_2', \phi') \circ (\alpha_1', \alpha_2', \rho'),
\end{aligned}$$

and that

$$\mathrm{sgn}(\tau \circ ((\beta_2' \circ \sigma \circ (\beta_2')^{-1}) \sqcup \mathrm{id}_{\mathbf{n}_2 \setminus \mathrm{im}(\beta_2')})) = 1.$$

Therefore, the composite $(\beta_1, \beta_2, \phi) \circ (\alpha_1, \alpha_2, \rho)$ is equivalent to $(\beta_1', \beta_2', \phi') \circ (\alpha_1', \alpha_2', \rho')$. □

Definition 1.7. Let $\bar{\mathcal{J}}$ be the category with objects $(\mathbf{m}_1, \mathbf{m}_2)$ in $\mathcal{J}$ and morphisms $\bar{\mathcal{J}}((\mathbf{m}_1, \mathbf{m}_2), (\mathbf{n}_1, \mathbf{n}_2)) = \mathcal{J}((\mathbf{m}_1, \mathbf{m}_2), (\mathbf{n}_1, \mathbf{n}_2))/\sim$.

Let $d \in \mathbb{Z}$. We write $\bar{\mathcal{J}}_d$ for the full subcategory of $\bar{\mathcal{J}}$ whose objects $(\mathbf{m}_1, \mathbf{m}_2)$ in $\bar{\mathcal{J}}$ satisfy $m_2 - m_1 = d$. The category $\bar{\mathcal{J}}$ is the disjoint union over $d \in \mathbb{Z}$ of the subcategories $\bar{\mathcal{J}}_d$,

$$\bar{\mathcal{J}} = \coprod_{d \in \mathbb{Z}} \bar{\mathcal{J}}_d.$$

There is a projection functor $\mathcal{J} \to \bar{\mathcal{J}}$. The category $\bar{\mathcal{J}}$ inherits a symmetric strict monoidal structure from $\mathcal{J}$. The functor $\sqcup \colon \bar{\mathcal{J}} \times \bar{\mathcal{J}} \to \bar{\mathcal{J}}$ sends $((\mathbf{m}_1, \mathbf{m}_2), (\mathbf{n}_1, \mathbf{n}_2))$ to $(\mathbf{m}_1 \sqcup \mathbf{n}_1, \mathbf{m}_2 \sqcup \mathbf{n}_2)$ and maps $([\alpha_1, \alpha_2, \rho], [\beta_1, \beta_2, \phi])$ to $[\alpha_1 \sqcup \beta_1, \alpha_2 \sqcup \beta_2, \rho \sqcup \phi]$. This is well-defined: Let $(\alpha_1, \alpha_2, \rho)$, $(\alpha_1', \alpha_2', \rho') \colon (\mathbf{l}_1, \mathbf{l}_2) \to (\mathbf{m}_1, \mathbf{m}_2)$ and σ in A_{m_2} such that $(\alpha_1, \alpha_2, \rho) = (\mathrm{id}_{\mathbf{m}_1}, \sigma, \mathrm{id}_\emptyset) \circ (\alpha_1', \alpha_2', \rho')$, and let (β_1, β_2, ϕ), $(\beta_1', \beta_2', \phi') \colon (\mathbf{n}_1, \mathbf{n}_2) \to (\mathbf{p}_1, \mathbf{p}_2)$ and τ in A_{p_2} such that $(\beta_1, \beta_2, \phi) = (\mathrm{id}_{\mathbf{p}_1}, \tau, \mathrm{id}_\emptyset) \circ (\beta_1', \beta_2', \phi')$. We remark that $\sigma \sqcup \tau$ is in $\Sigma_{m_2 + p_2}$ with $\mathrm{sgn}(\sigma \sqcup \tau) = 1$, and that

$$\begin{aligned}
(\alpha_1, \alpha_2, \rho) \sqcup (\beta_1, \beta_2, \phi) &= ((\mathrm{id}_{\mathbf{m}_1}, \sigma, \mathrm{id}_\emptyset) \circ (\alpha_1', \alpha_2', \rho')) \sqcup ((\mathrm{id}_{\mathbf{p}_1}, \tau, \mathrm{id}_\emptyset) \circ (\beta_1', \beta_2', \phi')) \\
&= (\mathrm{id}_{\mathbf{m}_1 \sqcup \mathbf{p}_1}, \sigma \sqcup \tau, \mathrm{id}_\emptyset) \circ ((\alpha_1', \alpha_2', \rho') \sqcup (\beta_1', \beta_2', \phi')).
\end{aligned}$$

The unit is $(\mathbf{0}, \mathbf{0})$, and the symmetry isomorphisms are

$$(\mathbf{m}_1, \mathbf{m}_2) \sqcup (\mathbf{n}_1, \mathbf{n}_2) \xrightarrow{[\chi_{\mathbf{m}_1, \mathbf{n}_1}, \chi_{\mathbf{m}_2, \mathbf{n}_2}, \mathrm{id}_\emptyset]} (\mathbf{n}_1, \mathbf{n}_2) \sqcup (\mathbf{m}_1, \mathbf{m}_2).$$

1.2 Well-structured index categories

We continue with introducing Sagave and Schlichtkrull's theory of *well-structured index categories* which is a convenient device to establish model structures on diagram spaces. For more background we refer to [SS12, §5]. As an application, we focus on the properties of the category $\bar{\mathcal{J}}$.

Let $(\mathcal{K}, \sqcup, \mathbf{0}_{\mathcal{K}})$ be a small symmetric monoidal category and let $\mathcal{A}$ be a subcategory of automorphisms. We assume that $\mathcal{A}$ is a *normal subcategory*, that is, for each isomorphism $\alpha\colon \mathbf{m} \to \mathbf{n}$ in $\mathcal{K}$ the object $\mathbf{m}$ belongs to $\mathcal{A}$ if and only if the object $\mathbf{n}$ does, and in this case conjugation by α specifies an isomorphism from the automorphism group $\mathcal{A}(\mathbf{m},\mathbf{m})$ to the automorphism group $\mathcal{A}(\mathbf{n},\mathbf{n})$ by sending γ to $\alpha \circ \gamma \circ \alpha^{-1}$. In addition, we require that the subcategory $\mathcal{A}$ is *multiplicative* meaning that the monoidal structure map $\sqcup\colon \mathcal{K} \times \mathcal{K} \to \mathcal{K}$ restricts to a functor $\mathcal{A} \times \mathcal{A} \to \mathcal{A}$. (But we do not demand that $\mathcal{A}$ contains the unit $\mathbf{0}_{\mathcal{K}}$ for the monoidal structure.) We think of $\mathbb{N}_0$, that is, the ordered set of natural numbers $0 \to 1 \to 2 \to \ldots$, as a symmetric monoidal category via the additive structure. For a small category $\mathcal{C}$, the set of connected components $\pi_0(\mathcal{NC})$ is defined to be the coequalizer of the maps

$$\mathcal{NC}[1] \underset{d_1}{\overset{d_0}{\rightrightarrows}} \mathcal{NC}[0].$$

Let $\mathcal{B}$ and $\mathcal{C}$ be categories, let $F\colon \mathcal{B} \to \mathcal{C}$ be a functor, and let c be an object of $\mathcal{C}$. Recall that the category $(F \downarrow c)$ of *objects of $\mathcal{B}$ over c* is the category in which an object is a pair (b,γ) where b is an object in $\mathcal{B}$ and γ is a morphism $\gamma\colon F(b) \to c$ in $\mathcal{C}$. A morphism $(b,\gamma) \to (b',\gamma')$ is a morphism $\beta\colon b \to b'$ in $\mathcal{B}$ such that the diagram

$$\begin{tikzcd} F(b) \arrow[rr, "F(\beta)"] \arrow[dr, "\gamma"'] & & F(b') \arrow[dl, "\gamma'"] \\ & c & \end{tikzcd}$$

commutes. We refer to this category as the comma category $(F \downarrow c)$. If $\mathcal{B} = \mathcal{C}$ and F is the identity functor, we write $(\mathcal{C} \downarrow c)$ for the comma category $(F \downarrow c)$. The comma category $(c \downarrow F)$ is defined dually (see [Mac98, II. §6]).

Definition 1.8. [SS12, Definition 5.2] A *well-structured relative index category* is a triple consisting of a small symmetric monoidal category $(\mathcal{K}, \sqcup, \mathbf{0}_{\mathcal{K}})$, a strong symmetric monoidal functor $\lambda\colon \mathcal{K} \to \mathbb{N}_0$, and a normal and multiplicative subcategory of automorphisms $\mathcal{A}$ in $\mathcal{K}$. These data are required to satisfy the following conditions.

(i) A morphism $\mathbf{m} \to \mathbf{n}$ in $\mathcal{K}$ is an isomorphism if and only if $\lambda(\mathbf{m}) = \lambda(\mathbf{n})$.

(ii) For each object $\mathbf{m}$ in $\mathcal{A}$ and each object $\mathbf{n}$ in $\mathcal{K}$, each connected component of the comma category $(\mathbf{m} \sqcup - \downarrow \mathbf{n})$ has a terminal object.

(iii) For each object $\mathbf{m}$ in $\mathcal{A}$ and each object $\mathbf{n}$ in $\mathcal{K}$, the canonical right action of the automorphism group $\mathcal{A}(\mathbf{m},\mathbf{m})$ on the comma category $(\mathbf{m} \sqcup - \downarrow \mathbf{n})$ induces a free action on the set of connected components of the comma category $(\mathbf{m} \sqcup - \downarrow \mathbf{n})$.

(iv) Let $\mathcal{K}_{\mathcal{A}}$ be the full subcategory of $\mathcal{K}$ generated by the objects in $\mathcal{A}$. The inclusion functor $\mathcal{K}_{\mathcal{A}} \to \mathcal{K}$ is homotopy right cofinal.

In condition (ii) and (iii) for each object $\mathbf{m}$ in $\mathcal{A}$ and each object $\mathbf{n}$ in $\mathcal{K}$, we employ the functor $\mathbf{m} \sqcup -\colon \mathcal{K} \to \mathcal{K}$ to form the comma category $(\mathbf{m} \sqcup - \downarrow \mathbf{n})$. We use the notation $(\mathcal{K}, \mathcal{A})$ to indicate a well-structured relative index category. Considering the case when $\mathcal{A}$

is the discrete category of identity morphisms in $\mathcal{K}$, denoted by $\mathcal{OK}$, the above definition breaks down to the notion of a *well-structured index category* which is the following definition.

Definition 1.9. [SS12, Definition 5.5] A *well-structured index category* $\mathcal{K}$ is a small symmetric monoidal category equipped with a strong symmetric monoidal functor $\lambda\colon \mathcal{K} \to \mathbb{N}_0$ such that

(i) a morphism $\mathbf{m} \to \mathbf{n}$ in $\mathcal{K}$ is an isomorphism if and only if $\lambda(\mathbf{m}) = \lambda(\mathbf{n})$, and

(ii) for each pair of objects $\mathbf{m}$ and $\mathbf{n}$ in $\mathcal{K}$, each connected component of the comma category $(\mathbf{m} \sqcup - \downarrow \mathbf{n})$ has a terminal object.

Example 1.10. (i) We endow the category $\mathcal{I}$ with the functor $\lambda\colon \mathcal{I} \to \mathbb{N}_0$ defined by $\lambda(\mathbf{m}) = m$. The category $\mathcal{I}$ is a well-structured index category by [SS12, Corollary 5.9]. Further, the category of finite sets and bijections Σ is the full automorphism subcategory of $\mathcal{I}$. The pair $(\mathcal{I}, \Sigma)$ specifies a well-structured relative index category by [SS12, Corollary 5.10].

(ii) Similarly, we enhance the category $\mathcal{J}$ with the functor $\lambda\colon \mathcal{J} \to \mathbb{N}_0$ given by $\lambda(\mathbf{m}_1, \mathbf{m}_2) = m_1$. Then the category $\mathcal{J}$ is a well-structured index category by [SS12, Corollary 5.9], and the pair $(\mathcal{J}, \Sigma \times \Sigma)$ defines a well-structured relative index category by [SS12, Corollary 5.10].

Sagave and Schlichtkrull show in [SS12] that a well-structured relative index category $(\mathcal{K}, \mathcal{A})$ gives rise to a certain model structure on $\mathcal{K}$-spaces which is proper, monoidal and lifts to the category of structured $\mathcal{K}$-spaces for any Σ-free operad. For a well-structured index category $\mathcal{K}$, the associated model structure on $\mathcal{K}$-spaces is called the *projective* $\mathcal{K}$-model structure (see [SS12, Definition 6.21], Proposition 2.7). If $\mathcal{K}$ is one of the categories $\mathcal{I}$ or $\mathcal{J}$ and $\mathcal{A}$ is given by the respective full automorphism subcategories, then $(\mathcal{K}, \mathcal{A})$ induces a *flat* model structure on $\mathcal{K}$-spaces (see [SS12, §3.8, §4.27]). At the beginning of Section 2 we collect some more results of [SS12].

Lemma 1.11. *Let* $(\mathbf{m}_1, \mathbf{m}_2)$ *and* $(\mathbf{n}_1, \mathbf{n}_2)$ *be objects in* $\bar{\mathcal{J}}$ *such that* $m_1 \leq n_1$ *and* $m_2 \leq n_2$.

(i) Every map $[\alpha_1, \alpha_2, \rho]\colon (\mathbf{m}_1, \mathbf{m}_2) \sqcup (\mathbf{k}_1, \mathbf{k}_2) \to (\mathbf{n}_1, \mathbf{n}_2)$ *in* $\bar{\mathcal{J}}$ *admits a factorization of the form*

$$\begin{tikzcd}
(\mathbf{m}_1, \mathbf{m}_2) \sqcup (\mathbf{k}_1, \mathbf{k}_2) \arrow[rr, "{[(\mathrm{id}_{\mathbf{m}_1}, \mathrm{id}_{\mathbf{m}_2}, \mathrm{id}_\emptyset) \sqcup (\gamma_1, \gamma_2, \omega)]}"] \arrow[dr, "{[\alpha_1, \alpha_2, \rho]}"'] & & (\mathbf{m}_1, \mathbf{m}_2) \sqcup (\mathbf{l}_1, \mathbf{l}_2) \arrow[dl, "{[\beta_1, \beta_2, \phi]}"] \\
& (\mathbf{n}_1, \mathbf{n}_2) &
\end{tikzcd}$$

with $[\beta_1, \beta_2, \phi]$ *an isomorphism in* $\bar{\mathcal{J}}$.

(ii) Suppose that the map $[\beta_1, \beta_2, \phi]\colon (\mathbf{m}_1, \mathbf{m}_2) \sqcup (\mathbf{l}_1, \mathbf{l}_2) \to (\mathbf{n}_1, \mathbf{n}_2)$ *is an isomorphism in* $\bar{\mathcal{J}}$*. Then a map* $[\alpha_1, \alpha_2, \rho]\colon (\mathbf{m}_1, \mathbf{m}_2) \sqcup (\mathbf{k}_1, \mathbf{k}_2) \to (\mathbf{n}_1, \mathbf{n}_2)$ *in* $\bar{\mathcal{J}}$ *factors as*

$$[\beta_1, \beta_2, \phi] \circ [(\mathrm{id}_{\mathbf{m}_1}, \mathrm{id}_{\mathbf{m}_2}, \mathrm{id}_\emptyset) \sqcup (\gamma_1, \gamma_2, \omega)]$$

if and only if $\beta_1|_{\mathbf{m}_1} = \alpha_1|_{\mathbf{m}_1}$ *in* $\mathcal{I}$ *and* $[\beta_2|_{\mathbf{m}_2}] = [\alpha_2|_{\mathbf{m}_2}]$ *in* $\bar{\mathcal{I}}$*. The map* $[\gamma_1, \gamma_2, \omega]$ *is unique if the factorization exists.*

Proof. (i) Let $(\alpha_1, \alpha_2, \rho)\colon (\mathbf{m}_1, \mathbf{m}_2) \sqcup (\mathbf{k}_1, \mathbf{k}_2) \to (\mathbf{n}_1, \mathbf{n}_2)$ be a representative of a map $[\alpha_1, \alpha_2, \rho]$ in $\bar{\mathcal{J}}$. We choose bijections $\beta_1\colon \mathbf{m}_1 \sqcup \mathbf{l}_1 \to \mathbf{n}_1$ and $\beta_2\colon \mathbf{m}_2 \sqcup \mathbf{l}_2 \to \mathbf{n}_2$ in $\mathcal{I}$ where the map β_1 is an extension of the map α_1 and the map β_2 is an extension of the map α_2. Let $\gamma_1 = \iota_{\mathbf{k}_1, \mathbf{l}_1}$ and $\gamma_2 = \iota_{\mathbf{k}_2, \mathbf{l}_2}$ be the standard inclusions. We get that $\beta_1 \circ (\mathrm{id}_{\mathbf{m}_1} \sqcup \iota_{\mathbf{k}_1, \mathbf{l}_1}) = \alpha_1$ and $\beta_2 \circ (\mathrm{id}_{\mathbf{m}_2} \sqcup \iota_{\mathbf{k}_2, \mathbf{l}_2}) = \alpha_2$ in $\mathcal{I}$. The bijection $\omega\colon \mathbf{l}_1 \setminus \mathrm{im}(\iota_{\mathbf{k}_1, \mathbf{l}_1}) \to \mathbf{l}_2 \setminus \mathrm{im}(\iota_{\mathbf{k}_2, \mathbf{l}_2})$ is specified by the following diagram of bijections

$$\begin{tikzcd}
\mathbf{n}_1 \setminus \mathrm{im}(\alpha_1) \arrow[r, "="] \arrow[dddd, "\rho"'] & \mathbf{n}_1 \setminus \mathrm{im}(\beta_1 \circ (\mathrm{id}_{\mathbf{m}_1} \sqcup \iota_{\mathbf{k}_1, \mathbf{l}_1})) \arrow[r, "="] & \mathrm{im}(\beta_1|_{(\mathbf{m}_1 \sqcup \mathbf{l}_1) \setminus \mathrm{im}(\mathrm{id}_{\mathbf{m}_1} \sqcup \iota_{\mathbf{k}_1, \mathbf{l}_1})}) \arrow[d, "\beta_1^{-1}"] \\
 & & (\mathbf{m}_1 \sqcup \mathbf{l}_1) \setminus \mathrm{im}(\mathrm{id}_{\mathbf{m}_1} \sqcup \iota_{\mathbf{k}_1, \mathbf{l}_1}) \arrow[d, "\omega"] \\
 & & (\mathbf{m}_2 \sqcup \mathbf{l}_2) \setminus \mathrm{im}(\mathrm{id}_{\mathbf{m}_2} \sqcup \iota_{\mathbf{k}_2, \mathbf{l}_2}) \arrow[d, "\beta_2"] \\
 & & \\
\mathbf{n}_2 \setminus \mathrm{im}(\alpha_2) \arrow[r, "="] & \mathbf{n}_2 \setminus \mathrm{im}(\beta_2 \circ (\mathrm{id}_{\mathbf{m}_2} \sqcup \iota_{\mathbf{k}_2, \mathbf{l}_2})) \arrow[r, "="] & \mathrm{im}(\beta_2|_{(\mathbf{m}_2 \sqcup \mathbf{l}_2) \setminus \mathrm{im}(\mathrm{id}_{\mathbf{m}_2} \sqcup \iota_{\mathbf{k}_2, \mathbf{l}_2})})
\end{tikzcd}$$

where we set $\omega = \beta_2^{-1} \circ \rho \circ \beta_1$. Hence, we obtain a factorization

$$(\alpha_1, \alpha_2, \rho) = (\beta_1, \beta_2, \mathrm{id}_\emptyset) \circ ((\mathrm{id}_{\mathbf{m}_1}, \mathrm{id}_{\mathbf{m}_2}, \mathrm{id}_\emptyset) \sqcup (\iota_{\mathbf{k}_1, \mathbf{l}_1}, \iota_{\mathbf{k}_2, \mathbf{l}_2}, \beta_2^{-1} \circ \rho \circ \beta_1))$$

in $\mathcal{J}$. Passing to equivalence classes this yields a factorization

$$[\alpha_1, \alpha_2, \rho] = [\beta_1, \beta_2, \mathrm{id}_\emptyset] \circ [(\mathrm{id}_{\mathbf{m}_1}, \mathrm{id}_{\mathbf{m}_2}, \mathrm{id}_\emptyset) \sqcup (\iota_{\mathbf{k}_1, \mathbf{l}_1}, \iota_{\mathbf{k}_2, \mathbf{l}_2}, \beta_2^{-1} \circ \rho \circ \beta_1)]$$

in $\bar{\mathcal{J}}$.

(ii) First assume that the map $[\alpha_1, \alpha_2, \rho]\colon (\mathbf{m}_1, \mathbf{m}_2) \sqcup (\mathbf{k}_1, \mathbf{k}_2) \to (\mathbf{n}_1, \mathbf{n}_2)$ in $\bar{\mathcal{J}}$ has a factorization

$$[\alpha_1, \alpha_2, \rho] = [\beta_1, \beta_2, \phi] \circ [(\mathrm{id}_{\mathbf{m}_1}, \mathrm{id}_{\mathbf{m}_2}, \mathrm{id}_\emptyset) \sqcup (\gamma_1, \gamma_2, \omega)].$$

Spelling out the composition this means that

$$[\alpha_1, \alpha_2, \rho] = [\beta_1 \circ (\mathrm{id}_{\mathbf{m}_1} \sqcup \gamma_1), \beta_2 \circ (\mathrm{id}_{\mathbf{m}_2} \sqcup \gamma_2), \beta_2 \circ \omega \circ \beta_1^{-1}].$$

From this we conclude that $\beta_1|_{\mathbf{m}_1} = \alpha_1|_{\mathbf{m}_1}$ in $\mathcal{I}$ and $[\beta_2|_{\mathbf{m}_2}] = [\alpha_2|_{\mathbf{m}_2}]$ in $\bar{\mathcal{I}}$.

Secondly, let $(\beta_1, \beta_2, \phi)\colon (\mathbf{m}_1, \mathbf{m}_2) \sqcup (\mathbf{l}_1, \mathbf{l}_2) \to (\mathbf{n}_1, \mathbf{n}_2)$ be a representative of a map $[\beta_1, \beta_2, \phi]$ in $\bar{\mathcal{J}}$, and let $(\alpha_1, \alpha_2, \rho)\colon (\mathbf{m}_1, \mathbf{m}_2) \sqcup (\mathbf{k}_1, \mathbf{k}_2) \to (\mathbf{n}_1, \mathbf{n}_2)$ be a representative of a map $[\alpha_1, \alpha_2, \rho]$ in $\bar{\mathcal{J}}$. We assume that $\beta_1|_{\mathbf{m}_1} = \alpha_1|_{\mathbf{m}_1}$ and $\beta_2|_{\mathbf{m}_2} = \alpha_2|_{\mathbf{m}_2}$ in $\mathcal{I}$. Let $\gamma_1\colon \mathbf{k}_1 \to \mathbf{l}_1$ be the map in $\mathcal{I}$ such that $\alpha_1|_{\mathbf{k}_1} = \beta_1|_{\mathbf{l}_1} \circ \gamma_1$, and let $\gamma_2\colon \mathbf{k}_2 \to \mathbf{l}_2$ be the map in $\mathcal{I}$ such that $\alpha_2|_{\mathbf{k}_2} = \beta_2|_{\mathbf{l}_2} \circ \gamma_2$. Let the bijection $\omega\colon \mathbf{l}_1 \setminus \mathrm{im}(\gamma_1) \to \mathbf{l}_2 \setminus \mathrm{im}(\gamma_2)$ be given by $\omega = \beta_2^{-1} \circ \rho \circ \beta_1$ (compare part (i)). Thus, the map $(\alpha_1, \alpha_2, \rho)$ can be factored as

$$(\beta_1, \beta_2, \phi) \circ ((\mathrm{id}_{\mathbf{m}_1}, \mathrm{id}_{\mathbf{m}_2}, \mathrm{id}_{\emptyset}) \sqcup (\gamma_1, \gamma_2, \omega))$$

in $\mathcal{J}$, and passing to equivalence classes the map $[\alpha_1, \alpha_2, \rho]$ can be factored as

$$[\beta_1, \beta_2, \phi] \circ [(\mathrm{id}_{\mathbf{m}_1}, \mathrm{id}_{\mathbf{m}_2}, \mathrm{id}_{\emptyset}) \sqcup (\gamma_1, \gamma_2, \omega)]$$

in $\bar{\mathcal{J}}$.

Note that if the factorization of the map $[\alpha_1, \alpha_2, \rho]$ exists, the map $[\gamma_1, \gamma_2, \omega]$ is uniquely determined by the maps $[\alpha_1, \alpha_2, \rho]$ and $[\beta_1, \beta_2, \phi]$ in $\bar{\mathcal{J}}$.

□

Corollary 1.12. *Let $(\mathbf{m}_1, \mathbf{m}_2)$ and $(\mathbf{n}_1, \mathbf{n}_2)$ be objects in $\bar{\mathcal{J}}$.*

(i) Suppose that $m_1 \leq n_1$ and $m_2 \leq n_2$. An object $((\mathbf{l}_1, \mathbf{l}_2), [\beta_1, \beta_2, \phi])$ is terminal in its connected component of the comma category $((\mathbf{m}_1, \mathbf{m}_2) \sqcup - \downarrow (\mathbf{n}_1, \mathbf{n}_2))$ if and only if the map $[\beta_1, \beta_2, \phi]$ is an isomorphism in $\bar{\mathcal{J}}$.

(ii) For the comma category $((\mathbf{m}_1, \mathbf{m}_2) \sqcup - \downarrow (\mathbf{n}_1, \mathbf{n}_2))$, the set of connected components $\pi_0(\mathcal{N}((\mathbf{m}_1, \mathbf{m}_2) \sqcup - \downarrow (\mathbf{n}_1, \mathbf{n}_2)))$ is isomorphic to the set $\mathcal{I}(\mathbf{m}_1, \mathbf{n}_1) \times \bar{\mathcal{I}}(\mathbf{m}_2, \mathbf{n}_2)$.

Proof. (i) Let the object $((\mathbf{l}_1, \mathbf{l}_2), [\beta_1, \beta_2, \phi])$ be terminal in its connected component of the comma category $((\mathbf{m}_1, \mathbf{m}_2) \sqcup - \downarrow (\mathbf{n}_1, \mathbf{n}_2))$. Assume that the map $[\beta_1, \beta_2, \phi]$ is not an isomorphism in $\bar{\mathcal{J}}$. From Lemma 1.11(i) we know that the map $[\beta_1, \beta_2, \phi]$ admits a factorization

$$[\beta_1, \beta_2, \phi] = [\tilde{\beta}_1, \tilde{\beta}_2, \tilde{\phi}] \circ [(\mathrm{id}_{\mathbf{m}_1}, \mathrm{id}_{\mathbf{m}_2}, \mathrm{id}_{\emptyset}) \sqcup (\gamma_1, \gamma_2, \omega)]$$

with $[\tilde{\beta}_1, \tilde{\beta}_2, \tilde{\phi}]$ an isomorphism in $\bar{\mathcal{J}}$. Since the map $[\beta_1, \beta_2, \phi]$ is not an isomorphism, the map $[\gamma_1, \gamma_2, \omega]$ cannot be an isomorphism in $\bar{\mathcal{J}}$. But this contradicts the assumption that the object $((\mathbf{l}_1, \mathbf{l}_2), [\beta_1, \beta_2, \phi])$ is terminal in its connected component. Therefore, the map $[\beta_1, \beta_2, \phi]$ is an isomorphism.

Reversely, let $((\mathbf{l}_1, \mathbf{l}_2), [\beta_1, \beta_2, \phi])$ be an object in $((\mathbf{m}_1, \mathbf{m}_2) \sqcup - \downarrow (\mathbf{n}_1, \mathbf{n}_2))$ with $[\beta_1, \beta_2, \phi]$ an isomorphism in $\bar{\mathcal{J}}$. Let $((\mathbf{k}_1, \mathbf{k}_2), [\alpha_1, \alpha_2, \rho])$ be an object in the connected component of $((\mathbf{l}_1, \mathbf{l}_2), [\beta_1, \beta_2, \phi])$. Then Lemma 1.11(ii) ensures that there is a unique morphism $[\gamma_1, \gamma_2, \omega]\colon (\mathbf{k}_1, \mathbf{k}_2) \to (\mathbf{l}_1, \mathbf{l}_2)$ such that

$$[\alpha_1, \alpha_2, \rho] = [\beta_1, \beta_2, \phi] \circ [(\mathrm{id}_{\mathbf{m}_1}, \mathrm{id}_{\mathbf{m}_2}, \mathrm{id}_{\emptyset}) \sqcup (\gamma_1, \gamma_2, \omega)]$$

in $\bar{\mathcal{J}}$. So the object $((\mathbf{l}_1, \mathbf{l}_2), [\beta_1, \beta_2, \phi])$ is terminal in its connected component.

(ii) If $m_1 \geq n_1 + 1$ or $m_2 \geq n_2 + 1$, the comma category $((\mathbf{m}_1, \mathbf{m}_2) \sqcup - \downarrow (\mathbf{n}_1, \mathbf{n}_2))$ is the empty category and the set $\mathcal{I}(\mathbf{m}_1, \mathbf{n}_1) \times \bar{\mathcal{I}}(\mathbf{m}_2, \mathbf{n}_2)$ is the empty set. Thus, the statement holds. Assume now that $m_1 \leq n_1$ and $m_2 \leq n_2$. The assignment $\pi_0(\mathcal{N}((\mathbf{m}_1, \mathbf{m}_2) \sqcup - \downarrow (\mathbf{n}_1, \mathbf{n}_2))) \to \mathcal{I}(\mathbf{m}_1, \mathbf{n}_1) \times \bar{\mathcal{I}}(\mathbf{m}_2, \mathbf{n}_2)$ that sends a class $[((\mathbf{l}_1, \mathbf{l}_2), [\beta_1, \beta_2, \phi])]$ to $(\beta_1|_{\mathbf{m}_1}, [\beta_2|_{\mathbf{m}_2}])$ defines a bijection.

□

Proposition 1.13. *The category $\bar{\mathcal{J}}$ together with the functor $\lambda\colon \bar{\mathcal{J}} \to \mathbb{N}_0$, given by $\lambda(\mathbf{m}_1, \mathbf{m}_2) = m_1$, determines a well-structured index category.*

Proof. (compare [SS12, proof of Proposition 5.8]) We notice that the functor λ is strong symmetric monoidal. Let $[\alpha_1, \alpha_2, \rho]\colon (\mathbf{m}_1, \mathbf{m}_2) \to (\mathbf{n}_1, \mathbf{n}_2)$ be a morphism in $\bar{\mathcal{J}}$. If the map $[\alpha_1, \alpha_2, \rho]$ is an isomorphism in $\bar{\mathcal{J}}$, it follows that $m_1 = n_1$. Conversely, if $\lambda(\mathbf{m}_1, \mathbf{m}_2) = \lambda(\mathbf{n}_1, \mathbf{n}_2)$, we observe that $0 = n_1 - m_1 = n_2 - m_2$ and so, $n_2 = m_2$. Hence, the map $[\alpha_1, \alpha_2, \rho]$ has to be an isomorphism. Furthermore, let $(\mathbf{m}_1, \mathbf{m}_2)$ and $(\mathbf{n}_1, \mathbf{n}_2)$ be a pair of objects in $\bar{\mathcal{J}}$. If $m_1 \geq n_1 + 1$ or $m_2 \geq n_2 + 1$, then the comma category $((\mathbf{m}_1, \mathbf{m}_2) \sqcup - \downarrow (\mathbf{n}_1, \mathbf{n}_2))$ is the empty category so that condition (ii) in Definition 1.9 is an empty statement. Suppose now that $m_1 \leq n_1$ and $m_2 \leq n_2$. From Corollary 1.12(i) we learn that an object $((\mathbf{k}_1, \mathbf{k}_2), [\alpha_1, \alpha_2, \rho])$ is terminal in its connected component of the comma category $((\mathbf{m}_1, \mathbf{m}_2) \sqcup - \downarrow (\mathbf{n}_1, \mathbf{n}_2))$ if and only if $[\alpha_1, \alpha_2, \rho]$ is an isomorphism in $\bar{\mathcal{J}}$. □

Remark 1.14. Let $\Sigma \times \bar{\Sigma}$ denote the full automorphism subcategory of $\bar{\mathcal{J}}$. We point out that the pair $(\bar{\mathcal{J}}, \Sigma \times \bar{\Sigma})$ does not specify a well-structured relative index category since condition (iii) in Definition 1.8 is not satisfied. Indeed, let $(\mathbf{m}_1, \mathbf{m}_2)$ be in $\Sigma \times \bar{\Sigma}$ and let $(\mathbf{n}_1, \mathbf{n}_2)$ be in $\bar{\mathcal{J}}$ such that $n_1 \geq m_1$, and $m_2 \geq 2$ and $n_2 \geq m_2 + 2$. By Corollary 1.12(ii) the set of connected components of the comma category $((\mathbf{m}_1, \mathbf{m}_2) \sqcup - \downarrow (\mathbf{n}_1, \mathbf{n}_2))$ is isomorphic to the set $\mathcal{I}(\mathbf{m}_1, \mathbf{n}_1) \times \bar{\mathcal{I}}(\mathbf{m}_2, \mathbf{n}_2)$. The latter is isomorphic to $\mathcal{I}(\mathbf{m}_1, \mathbf{n}_1)$ due to Lemma 1.3. Besides, the group $\bar{\Sigma}_{m_2} = \bar{\mathcal{I}}(\mathbf{m}_2, \mathbf{m}_2)$ is isomorphic to Σ_2 by Lemma 1.3. The action φ of $\Sigma_{m_1} \times \bar{\Sigma}_{m_2}$ on $\mathcal{I}(\mathbf{m}_1, \mathbf{n}_1) \times \bar{\mathcal{I}}(\mathbf{m}_2, \mathbf{n}_2)$,

$$(\Sigma_{m_1} \times \bar{\Sigma}_{m_2}) \times (\mathcal{I}(\mathbf{m}_1, \mathbf{n}_1) \times \bar{\mathcal{I}}(\mathbf{m}_2, \mathbf{n}_2)) \xrightarrow{\varphi} \mathcal{I}(\mathbf{m}_1, \mathbf{n}_1) \times \bar{\mathcal{I}}(\mathbf{m}_2, \mathbf{n}_2),$$

is given by

$$\varphi((\sigma, [\tau]), (\alpha, [*])) = (\alpha \circ \sigma, [*]).$$

If $[\tau] \neq [\mathrm{id}_{\mathbf{m}_2}]$, then $\varphi((\mathrm{id}_{\mathbf{m}_1}, [\tau]), (\alpha, [*])) = (\alpha, [*])$. Thus, the action φ is not free. In particular, we cannot employ Sagave and Schlichtkrull's machinery in [SS12] for obtaining a *flat* model structure on $\bar{\mathcal{J}}$-spaces.

To right induce the model structure on $\mathcal{K}$-spaces associated with a well-structured relative index category $(\mathcal{K}, \mathcal{A})$ on the category of commutative monoids in $\mathcal{K}$-spaces, the pair $(\mathcal{K}, \mathcal{A})$ has to fulfill the following property.

Definition 1.15. [SS12, Definition 5.3] A well-structured relative index category $(\mathcal{K}, \mathcal{A})$ is *very well-structured* if for each object $\mathbf{m}$ in $\mathcal{A}$, each object $\mathbf{n}$ in $\mathcal{K}$, and each $i \geq 1$, the canonical right action of the group $\Sigma_i \ltimes \mathcal{A}(\mathbf{m},\mathbf{m})^{\times i}$ on the comma category $(\mathbf{m}^{\sqcup i} \sqcup - \downarrow \mathbf{n})$ induces a free action on the set of connected components of the comma category $(\mathbf{m}^{\sqcup i} \sqcup - \downarrow \mathbf{n})$.

Remark 1.16. In the above Definition 1.15 the group $\Sigma_i \ltimes \mathcal{A}(\mathbf{m},\mathbf{m})^{\times i}$ is the semi-direct product of the symmetric group Σ_i acting from the right on the i-fold product of the group of automorphisms $\mathcal{A}(\mathbf{m},\mathbf{m})$. The action on the comma category $(\mathbf{m}^{\sqcup i} \sqcup - \downarrow \mathbf{n})$ is defined via the homomorphism $\Sigma_i \ltimes \mathcal{A}(\mathbf{m},\mathbf{m})^{\times i} \to \mathcal{K}(\mathbf{m}^{\sqcup i}, \mathbf{m}^{\sqcup i})$ which sends $(\sigma, (f_1, \ldots, f_i))$ to $\sigma_* \circ (f_1 \sqcup \ldots \sqcup f_i)$ where σ_* is the block permutation map.
Furthermore, we emphasize that $\mathcal{A}$ cannot contain the unit object $\mathbf{0}_{\mathcal{K}}$ if the pair $(\mathcal{K},\mathcal{A})$ is a very well-structured relative index category.

For a well-structured index category $\mathcal{K}$, we write $\mathcal{K}_+$ for the full subcategory of $\mathcal{K}$ whose objects $\mathbf{m}$ satisfy $\lambda(\mathbf{m}) \geq 1$. We denote the corresponding discrete subcategory of identity morphisms by $\mathcal{O}\mathcal{K}_+$.

Example 1.17. Let Σ_+ denote the full automorphism subcategory of $\mathcal{I}_+$, and let $\Sigma_+ \times \Sigma$ stand for the full automorphism subcategory of $\mathcal{J}_+$. The pairs $(\mathcal{I}, \mathcal{O}\mathcal{I}_+)$, $(\mathcal{I}, \Sigma_+)$, $(\mathcal{J}, \mathcal{O}\mathcal{J}_+)$ and $(\mathcal{J}, \Sigma_+ \times \Sigma)$ define very well-structured relative index categories by [SS12, Corollary 5.9, Corollary 5.10].

Let $e \in \mathbb{Z}_{\geq 0}$. We write $\bar{\mathcal{J}}_{(\geq e,-)}$ for the full subcategory of $\bar{\mathcal{J}}$ whose objects $(\mathbf{m}_1,\mathbf{m}_2)$ in $\bar{\mathcal{J}}$ satisfy $m_1 \geq e$. For $e = 0$, the category $\bar{\mathcal{J}}_{(\geq 0,-)}$ is the category $\bar{\mathcal{J}}$, and for $e = 1$, the category $\bar{\mathcal{J}}_{(\geq 1,-)}$ is the category $\bar{\mathcal{J}}_+$.

Lemma 1.18. *The inclusion functor $\iota_e \colon \bar{\mathcal{J}}_{(\geq e,-)} \to \bar{\mathcal{J}}$ is homotopy right cofinal.*

Proof. We proceed as in the proof of [SS12, Corollary 5.9]. Let $(\mathbf{m}_1,\mathbf{m}_2)$ be in $\bar{\mathcal{J}}$. We have to prove that the classifying space $B((\mathbf{m}_1,\mathbf{m}_2) \downarrow \iota_e)$ is contractible. We pick a morphism $[\alpha_1,\alpha_2,\rho] \colon (\mathbf{0},\mathbf{0}) \to (\mathbf{l}_1,\mathbf{l}_2)$ in $\bar{\mathcal{J}}$ such that $l_1 \geq e$. Let

$$- \sqcup [\alpha_1,\alpha_2,\rho] \colon ((\mathbf{m}_1,\mathbf{m}_2) \downarrow \bar{\mathcal{J}}) \to ((\mathbf{m}_1,\mathbf{m}_2) \downarrow \iota_e)$$

be the functor which sends an object $((\mathbf{a}_1,\mathbf{a}_2), [\eta_1,\eta_2,\psi] \colon (\mathbf{m}_1,\mathbf{m}_2) \to (\mathbf{a}_1,\mathbf{a}_2))$ to

$$((\mathbf{a}_1,\mathbf{a}_2) \sqcup (\mathbf{l}_1,\mathbf{l}_2), (\mathbf{m}_1,\mathbf{m}_2) = (\mathbf{m}_1,\mathbf{m}_2) \sqcup (\mathbf{0},\mathbf{0}) \xrightarrow{[(\eta_1,\eta_2,\psi) \sqcup (\alpha_1,\alpha_2,\rho)]} (\mathbf{a}_1,\mathbf{a}_2) \sqcup (\mathbf{l}_1,\mathbf{l}_2)),$$

and a morphism

$$\begin{tikzcd}
(\mathbf{m}_1,\mathbf{m}_2) \arrow[d, "{[\eta_1,\eta_2,\psi]}"'] \arrow[dr, bend left, "{[\theta_1,\theta_2,\omega]}"] & \\
(\mathbf{a}_1,\mathbf{a}_2) \arrow[r, "{[\beta_1,\beta_2,\phi]}"] & (\mathbf{b}_1,\mathbf{b}_2)
\end{tikzcd}$$

to

$$\begin{tikzcd}
(\mathbf{m}_1,\mathbf{m}_2) \arrow[d, "{[(\eta_1,\eta_2,\psi) \sqcup (\alpha_1,\alpha_2,\rho)]}"'] \arrow[dr, bend left, "{[(\theta_1,\theta_2,\omega) \sqcup (\alpha_1,\alpha_2,\rho)]}"] & \\
(\mathbf{a}_1,\mathbf{a}_2) \sqcup (\mathbf{l}_1,\mathbf{l}_2) \arrow[r, "{[(\beta_1,\beta_2,\phi) \sqcup (\mathrm{id}_{\mathbf{l}_1},\mathrm{id}_{\mathbf{l}_2},\mathrm{id}_{\emptyset})]}"] & (\mathbf{b}_1,\mathbf{b}_2) \sqcup (\mathbf{l}_1,\mathbf{l}_2).
\end{tikzcd}$$

In addition, let $(\iota_e)_*\colon ((\mathbf{m}_1,\mathbf{m}_2)\downarrow \iota_e)\to((\mathbf{m}_1,\mathbf{m}_2)\downarrow \bar{\mathcal{J}})$ be the functor induced by the inclusion functor $\iota_e\colon \bar{\mathcal{J}}_{(\geq e,-)}\to\bar{\mathcal{J}}$. There is a natural transformation

$$\mathrm{id}_{((\mathbf{m}_1,\mathbf{m}_2)\downarrow\iota_e)}\to(-\sqcup[\alpha_1,\alpha_2,\rho])\circ(\iota_e)_*.$$

For $((\mathbf{a}_1,\mathbf{a}_2),[\eta_1,\eta_2,\psi]\colon(\mathbf{m}_1,\mathbf{m}_2)\to(\mathbf{a}_1,\mathbf{a}_2))$, the object

$$((\mathbf{a}_1,\mathbf{a}_2),(\mathbf{m}_1,\mathbf{m}_2)\xrightarrow{[\eta_1,\eta_2,\psi]}(\mathbf{a}_1,\mathbf{a}_2))$$

is taken to

$$((\mathbf{a}_1,\mathbf{a}_2)\sqcup(\mathbf{l}_1,\mathbf{l}_2),(\mathbf{m}_1,\mathbf{m}_2)\xrightarrow{[(\eta_1,\eta_2,\psi)\sqcup(\alpha_1,\alpha_2,\rho)]}(\mathbf{a}_1,\mathbf{a}_2)\sqcup(\mathbf{l}_1,\mathbf{l}_2)).$$

Further, there is a natural transformation $\mathrm{id}_{((\mathbf{m}_1,\mathbf{m}_2)\downarrow\bar{\mathcal{J}})}\to(\iota_e)_*\circ(-\sqcup[\alpha_1,\alpha_2,\rho])$. For $((\mathbf{a}_1,\mathbf{a}_2),[\eta_1,\eta_2,\psi]\colon(\mathbf{m}_1,\mathbf{m}_2)\to(\mathbf{a}_1,\mathbf{a}_2))$, the object

$$((\mathbf{a}_1,\mathbf{a}_2),(\mathbf{m}_1,\mathbf{m}_2)\xrightarrow{[\eta_1,\eta_2,\psi]}(\mathbf{a}_1,\mathbf{a}_2))$$

is mapped to

$$((\mathbf{a}_1,\mathbf{a}_2)\sqcup(\mathbf{l}_1,\mathbf{l}_2),(\mathbf{m}_1,\mathbf{m}_2)\xrightarrow{[(\eta_1,\eta_2,\psi)\sqcup(\alpha_1,\alpha_2,\rho)]}(\mathbf{a}_1,\mathbf{a}_2)\sqcup(\mathbf{l}_1,\mathbf{l}_2)).$$

These natural transformations produce homotopies between the morphisms $\mathrm{id}_{B((\mathbf{m}_1,\mathbf{m}_2)\downarrow\iota_e)}$ and $B(-\sqcup[\alpha_1,\alpha_2,\rho])\circ B((\iota_e)_*)$, and between the morphisms $\mathrm{id}_{B((\mathbf{m}_1,\mathbf{m}_2)\downarrow\bar{\mathcal{J}})}$ and

$$B((\iota_e)_*)\circ B(-\sqcup[\alpha_1,\alpha_2,\rho]).$$

Hence, the space $B((\mathbf{m}_1,\mathbf{m}_2)\downarrow\iota_e)$ is homotopy equivalent to $B((\mathbf{m}_1,\mathbf{m}_2)\downarrow\bar{\mathcal{J}})$, which is contractible because the comma category $((\mathbf{m}_1,\mathbf{m}_2)\downarrow\bar{\mathcal{J}})$ has the initial object $((\mathbf{m}_1,\mathbf{m}_2),\mathrm{id}_{(\mathbf{m}_1,\mathbf{m}_2)})$. □

Proposition 1.19. *The pair $(\bar{\mathcal{J}},\mathcal{O}\bar{\mathcal{J}}_+)$ specifies a very well-structured relative index category.*

Proof. To understand that the pair $(\bar{\mathcal{J}},\mathcal{O}\bar{\mathcal{J}}_+)$ determines a well-structured relative index category, it remains to show (iv) in Definition 1.8, that is, that the inclusion functor $\bar{\mathcal{J}}_+\to\bar{\mathcal{J}}$ is homotopy right cofinal. But this follows from Lemma 1.18. The next step is to prove that the well-structured relative index category $(\bar{\mathcal{J}},\mathcal{O}\bar{\mathcal{J}}_+)$ is very well-structured. Let $(\mathbf{m}_1,\mathbf{m}_2)$ be in $\mathcal{O}\bar{\mathcal{J}}_+$, let $(\mathbf{n}_1,\mathbf{n}_2)$ be in $\bar{\mathcal{J}}$, and let $i\geq 1$. If $i\cdot m_1\geq n_1+1$ or $i\cdot m_2\geq n_2+1$, then the comma category $((\mathbf{m}_1,\mathbf{m}_2)^{\sqcup i}\sqcup-\downarrow(\mathbf{n}_1,\mathbf{n}_2))$ is empty so that the condition in Definition 1.15 automatically holds. Suppose that $i\cdot m_1\leq n_1$ and $i\cdot m_2\leq n_2$. We know from Corollary 1.12(ii) that the set of connected components of the comma category $((\mathbf{m}_1,\mathbf{m}_2)^{\sqcup i}\sqcup-\downarrow(\mathbf{n}_1,\mathbf{n}_2))$ is isomorphic to $\mathcal{I}(\mathbf{m}_1^{\sqcup i},\mathbf{n}_1)\times\bar{\mathcal{I}}(\mathbf{m}_2^{\sqcup i},\mathbf{n}_2)$. The action of the group $\Sigma_i\rtimes\{[\mathrm{id}_{\mathbf{m}_1},\mathrm{id}_{\mathbf{m}_2},\mathrm{id}_\emptyset]\}^{\times i}$ on the set $\mathcal{I}(\mathbf{m}_1^{\sqcup i},\mathbf{n}_1)\times\bar{\mathcal{I}}(\mathbf{m}_2^{\sqcup i},\mathbf{n}_2)$ is the map

$$(\Sigma_i\rtimes\{[\mathrm{id}_{\mathbf{m}_1},\mathrm{id}_{\mathbf{m}_2},\mathrm{id}_\emptyset]\}^{\times i})\times(\mathcal{I}(\mathbf{m}_1^{\sqcup i},\mathbf{n}_1)\times\bar{\mathcal{I}}(\mathbf{m}_2^{\sqcup i},\mathbf{n}_2))\to\mathcal{I}(\mathbf{m}_1^{\sqcup i},\mathbf{n}_1)\times\bar{\mathcal{I}}(\mathbf{m}_2^{\sqcup i},\mathbf{n}_2)$$

which sends the element $((\sigma,([\mathrm{id}_{\mathbf{m}_1},\mathrm{id}_{\mathbf{m}_2},\mathrm{id}_\emptyset],\ldots,[\mathrm{id}_{\mathbf{m}_1},\mathrm{id}_{\mathbf{m}_2},\mathrm{id}_\emptyset])),(\alpha,[\beta]))$ to the element $(\alpha\circ\sigma_*\circ(\mathrm{id}_{\mathbf{m}_1}\sqcup\ldots\sqcup\mathrm{id}_{\mathbf{m}_1}),[\beta\circ\sigma_*\circ(\mathrm{id}_{\mathbf{m}_2}\sqcup\ldots\sqcup\mathrm{id}_{\mathbf{m}_2})])$. This action is free because $m_1\geq 1$. □

The following lemma is a practical device for establishing model structures on diagram spaces where the indexing category is a product category.

Lemma 1.20. *Let $((\mathcal{K}, \sqcup, \mathbf{0}_{\mathcal{K}}), \mathcal{A}, \lambda_{\mathcal{K}})$ and $((\mathcal{L}, \tilde{\sqcup}, \mathbf{0}_{\mathcal{L}}), \mathcal{B}, \lambda_{\mathcal{L}})$ be well-structured relative index categories such that the morphisms in $\mathcal{K}$, and $\mathcal{L}$ respectively, preserve or raise the value of $\lambda_{\mathcal{K}}$, and $\lambda_{\mathcal{L}}$ respectively.*

(a) The pair $((\mathcal{K} \times \mathcal{L}, (\sqcup, \tilde{\sqcup}), (\mathbf{0}_{\mathcal{K}}, \mathbf{0}_{\mathcal{L}})), \mathcal{A} \times \mathcal{B})$, equipped with the functor $\lambda_{\mathcal{K}\times\mathcal{L}} \colon \mathcal{K} \times \mathcal{L} \to \mathbb{N}_0$ given by $\lambda_{\mathcal{K}\times\mathcal{L}}(\mathbf{k}, \mathbf{l}) = \lambda_{\mathcal{K}}(\mathbf{k}) + \lambda_{\mathcal{L}}(\mathbf{l})$, defines a well-structured relative index category.

(b) If the pairs $(\mathcal{K}, \mathcal{A})$ and $(\mathcal{L}, \mathcal{B})$ are very well-structured relative index categories, then so is the pair $(\mathcal{K} \times \mathcal{L}, \mathcal{A} \times \mathcal{B})$.

Proof. (a) The product of two small symmetric monoidal categories is again a small symmetric monoidal category. As the functors $\lambda_{\mathcal{K}}$ and $\lambda_{\mathcal{L}}$ are strong symmetric monoidal, so is the functor $\lambda_{\mathcal{K}\times\mathcal{L}}$. More than that, we notice that $\mathcal{A} \times \mathcal{B}$ is a normal and multiplicative subcategory of automorphisms in $\mathcal{K} \times \mathcal{L}$. We verify the conditions (i) to (iv) in Definition 1.8.

(i) A morphism $(\mathbf{k}, \mathbf{l}) \to (\mathbf{k}', \mathbf{l}')$ is an isomorphism in $\mathcal{K}\times\mathcal{L}$ if and only if $\mathbf{k} \to \mathbf{k}'$ is an isomorphism in $\mathcal{K}$ and $\mathbf{l} \to \mathbf{l}'$ is an isomorphism in $\mathcal{L}$. The latter is equivalent to $\lambda_{\mathcal{K}}(\mathbf{k}) = \lambda_{\mathcal{K}}(\mathbf{k}')$ and $\lambda_{\mathcal{L}}(\mathbf{l}) = \lambda_{\mathcal{L}}(\mathbf{l}')$. This implies that $\lambda_{\mathcal{K}\times\mathcal{L}}(\mathbf{k}, \mathbf{l}) = \lambda_{\mathcal{K}\times\mathcal{L}}(\mathbf{k}', \mathbf{l}')$. Taking into account the assumption that morphisms in $\mathcal{K}$, and $\mathcal{L}$ respectively, can only preserve or raise the value of $\lambda_{\mathcal{K}}$, and $\lambda_{\mathcal{L}}$ respectively, we can deduce from $\lambda_{\mathcal{K}\times\mathcal{L}}(\mathbf{k}, \mathbf{l}) = \lambda_{\mathcal{K}\times\mathcal{L}}(\mathbf{k}', \mathbf{l}')$ that $\lambda_{\mathcal{K}}(\mathbf{k}) = \lambda_{\mathcal{K}}(\mathbf{k}')$ and $\lambda_{\mathcal{L}}(\mathbf{l}) = \lambda_{\mathcal{L}}(\mathbf{l}')$.

(ii) Let $(\mathbf{a}, \mathbf{b})$ be in $\mathcal{A} \times \mathcal{B}$, and let $(\mathbf{k}, \mathbf{l})$ be in $\mathcal{K} \times \mathcal{L}$. The comma category $((\mathbf{a}, \mathbf{b})(\sqcup, \tilde{\sqcup}) - \downarrow (\mathbf{k}, \mathbf{l}))$ is isomorphic to the product of comma categories $(\mathbf{a} \sqcup - \downarrow \mathbf{k}) \times (\mathbf{b} \tilde{\sqcup} - \downarrow \mathbf{l})$. Hence, we obtain an isomorphism

$$\pi_0(\mathcal{N}((\mathbf{a}, \mathbf{b})(\sqcup, \tilde{\sqcup}) - \downarrow (\mathbf{k}, \mathbf{l}))) \cong \pi_0(\mathcal{N}(\mathbf{a} \sqcup - \downarrow \mathbf{k})) \times \pi_0(\mathcal{N}(\mathbf{b} \tilde{\sqcup} - \downarrow \mathbf{l})).$$

The terminal object in a connected component of the comma category

$$((\mathbf{a}, \mathbf{b})(\sqcup, \tilde{\sqcup}) - \downarrow (\mathbf{k}, \mathbf{l}))$$

is the product of the terminal objects of the corresponding connected components of the comma categories $(\mathbf{a} \sqcup - \downarrow \mathbf{k})$ and $(\mathbf{b} \tilde{\sqcup} - \downarrow \mathbf{l})$.

(iii) This condition follows by making use of the isomorphisms in (ii).

(iv) The category $(\mathcal{K} \times \mathcal{L})_{\mathcal{A}\times\mathcal{B}}$ is isomorphic to the product category $\mathcal{K}_{\mathcal{A}} \times \mathcal{L}_{\mathcal{B}}$. The inclusion functor $\mathcal{K}_{\mathcal{A}} \times \mathcal{L}_{\mathcal{B}} \to \mathcal{K} \times \mathcal{L}$ is homotopy right cofinal. To see this, let $(\mathbf{k}, \mathbf{l})$ be in $\mathcal{K} \times \mathcal{L}$. The space $\mathcal{N}((\mathbf{k}, \mathbf{l}) \downarrow (\mathcal{K}_{\mathcal{A}} \times \mathcal{L}_{\mathcal{B}} \to \mathcal{K} \times \mathcal{L}))$ is contractible as being isomorpic to the product of the contractible spaces $\mathcal{N}(\mathbf{k} \downarrow (\mathcal{K}_{\mathcal{A}} \to \mathcal{K}))$ and $\mathcal{N}(\mathbf{l} \downarrow (\mathcal{L}_{\mathcal{B}} \to \mathcal{L}))$.

(b) Let $(\mathbf{a}, \mathbf{b})$ be in $\mathcal{A} \times \mathcal{B}$, let $(\mathbf{k}, \mathbf{l})$ be in $\mathcal{K} \times \mathcal{L}$ and let $i \geq 1$. We again exploit the isomorphisms given in (a)(ii). So we get that the canonical right action of the group $\Sigma_i \rtimes (\mathcal{A}(\mathbf{a}, \mathbf{a})^{\times i} \times \mathcal{B}(\mathbf{b}, \mathbf{b})^{\times i})$ on the category $(\mathbf{a}^{\sqcup i} \sqcup - \downarrow \mathbf{k}) \times (\mathbf{b}^{\tilde{\sqcup} i} \tilde{\sqcup} - \downarrow \mathbf{l})$ induces a free action on $\pi_0(\mathcal{N}(\mathbf{a}^{\sqcup i} \sqcup - \downarrow \mathbf{k})) \times \pi_0(\mathcal{N}(\mathbf{b}^{\tilde{\sqcup} i} \tilde{\sqcup} - \downarrow \mathbf{l}))$ defined by the given free actions.

□

Remark 1.21. We point out that we only need the assumption about the morphisms in $\mathcal{K}$, and $\mathcal{L}$ respectively, preserving or raising the value of $\lambda_{\mathcal{K}}$, and $\lambda_{\mathcal{L}}$ respectively, to prove the reverse direction of condition (i) in Definition 1.8. Our main examples, the categories $\mathcal{I}$, $\mathcal{J}$ and $\bar{\mathcal{J}}$, satisfy this assumption.

1.3 The classifying space $B\bar{\mathcal{J}}$

In the sequel we determine the classifying space $B\bar{\mathcal{J}}$. We do this by identifying the non-negative components of $\bar{\mathcal{J}}$ with a Grothendieck construction (in the sense of Thomason [Tho79, Definition 1.1]). Our arguments are similar to [SS, 2.8-2.10].

For $n \geq 0$, we think of Σ_n as the category with a single object $*$ and morphisms $\Sigma_n(*, *) = \Sigma_n$. As for the category $\mathcal{I}$, we have an equivalence relation on the morphism set $\Sigma_n(*, *) = \Sigma_n \cong \mathcal{I}(\mathbf{n}, \mathbf{n})$. A map a in Σ_n is equivalent to a map a' in Σ_n if and only if there exists a map σ in A_n such that $a = \sigma \circ a'$. We remark that a map a in Σ_n is equivalent to a map a' in Σ_n if and only if $\operatorname{sgn}(a) = \operatorname{sgn}(a')$. Let $\bar{\Sigma}_n := \Sigma_n/\sim$ be the quotient set. We write $\bar{\Sigma}_n$ for the category with a single object $*$ and morphisms $\bar{\Sigma}_n(*, *) = \bar{\Sigma}_n$. For $n \leq 1$, the category $\bar{\Sigma}_n$ is trivial and for $n \geq 2$, the category $\bar{\Sigma}_n$ is isomorphic to $\Sigma_n/A_n \cong \{\pm 1\}$.

Let Cat denote the category of small categories. We consider the functor $\bar{\Sigma}$ from the category $\mathcal{I}$ to Cat, which maps an object $\mathbf{m}$ in $\mathcal{I}$ to the category $\bar{\Sigma}_m$ and takes a morphism $\alpha\colon \mathbf{m} \to \mathbf{n}$ in $\mathcal{I}$ to the functor $\bar{\Sigma}(\alpha)$. The latter sends a map $[a]$ in $\bar{\Sigma}_m$ to

$$\bar{\Sigma}(\alpha)[a] = [(\alpha, \mathrm{incl}) \circ (a \sqcup \mathrm{id}_{\mathbf{n}\setminus \mathrm{im}(\alpha)}) \circ (\alpha, \mathrm{incl})^{-1}]$$

in $\bar{\Sigma}_n$,

$$\begin{array}{ccc} \mathbf{n} & \xrightarrow{\quad\bar{\Sigma}(\alpha)[a]\quad} & \mathbf{n}. \\ {\scriptstyle [(\alpha,\mathrm{incl})^{-1}]}\Big\downarrow & & \Big\uparrow{\scriptstyle [(\alpha,\mathrm{incl})]} \\ \mathbf{m} \sqcup (\mathbf{n} \setminus \mathrm{im}(\alpha)) & \xrightarrow{[a \sqcup \mathrm{id}_{\mathbf{n}\setminus \mathrm{im}(\alpha)}]} & \mathbf{m} \sqcup (\mathbf{n} \setminus \mathrm{im}(\alpha)) \end{array}$$

This is well-defined: Suppose that a is equivalent to a', that is, there exists a map σ in A_m such that $a = \sigma \circ a'$. Then $(\alpha, \mathrm{incl}) \circ (\sigma \sqcup \mathrm{id}_{\mathbf{n}\setminus \mathrm{im}(\alpha)}) \circ (\alpha, \mathrm{incl})^{-1}$ is in Σ_n with $\operatorname{sgn}((\alpha, \mathrm{incl}) \circ (\sigma \sqcup \mathrm{id}_{\mathbf{n}\setminus \mathrm{im}(\alpha)}) \circ (\alpha, \mathrm{incl})^{-1}) = \operatorname{sgn}(\sigma) = 1$ and

$$\begin{aligned} &(\alpha, \mathrm{incl}) \circ (a \sqcup \mathrm{id}_{\mathbf{n}\setminus \mathrm{im}(\alpha)}) \circ (\alpha, \mathrm{incl})^{-1} \\ = &(\alpha, \mathrm{incl}) \circ (\sigma \sqcup \mathrm{id}_{\mathbf{n}\setminus \mathrm{im}(\alpha)}) \circ (\alpha, \mathrm{incl})^{-1} \circ (\alpha, \mathrm{incl}) \circ (a' \sqcup \mathrm{id}_{\mathbf{n}\setminus \mathrm{im}(\alpha)}) \circ (\alpha, \mathrm{incl})^{-1}. \end{aligned}$$

For $\mathbf{d}$ in $\mathcal{I}$, we write $\bar{\Sigma}(\mathbf{d} \sqcup -)$ for the functor obtained from $\bar{\Sigma}$ by precomposition with the endofunctor $\mathbf{d} \sqcup -$ on $\mathcal{I}$. For $\mathbf{d} = \mathbf{0}$, this is just the functor $\bar{\Sigma}$.

The Grothendieck construction on the $\mathcal{I}$-category $\bar{\Sigma}(\mathbf{d} \sqcup -)$, denoted by $\mathcal{I} \int \bar{\Sigma}(\mathbf{d} \sqcup -)$, is the category with objects $(\mathbf{m}, *)$, where $\mathbf{m}$ is in $\mathcal{I}$ and $*$ is in $\bar{\Sigma}_{d+m}$, and morphisms $(\alpha, [a])\colon (\mathbf{m}, *) \to (\mathbf{n}, *)$, where $\alpha\colon \mathbf{m} \to \mathbf{n}$ is in $\mathcal{I}$ and $[a]\colon * \to *$ is in $\bar{\Sigma}_{d+n}$. The composition of morphisms

$$(\mathbf{l}, *) \xrightarrow{(\alpha,[a])} (\mathbf{m}, *) \xrightarrow{(\beta,[b])} (\mathbf{n}, *)$$

is given by

$$(\beta, [b]) \circ (\alpha, [a]) = (\beta \circ \alpha, [b] \circ \bar{\Sigma}(\mathbf{d} \sqcup -)(\beta)[a]).$$

Lemma 1.22. *Let $\mathbf{d}$ be in $\mathcal{I}$. The category $\mathcal{I} \int \bar{\Sigma}(\mathbf{d} \sqcup -)$ is isomorphic to the category $\bar{\mathcal{J}}_d$.*

Proof. We prove the claim by determining two functors that are inverse to each other. On the one hand, let $F\colon \mathcal{I} \int \bar{\Sigma}(\mathbf{d} \sqcup -) \to \bar{\mathcal{J}}_d$ be the functor that sends $(\mathbf{m}, *)$ to $(\mathbf{m}, \mathbf{d} \sqcup \mathbf{m})$, and that takes a morphism $(\alpha, [a])\colon (\mathbf{m}, *) \to (\mathbf{n}, *)$ to the morphism

$$(\mathbf{m}, \mathbf{d} \sqcup \mathbf{m}) \xrightarrow{[\alpha, a\circ(\mathrm{id}_{\mathbf{d}} \sqcup \alpha), a\circ(\mathrm{id}_{\mathbf{d}} \sqcup (\alpha, \mathrm{incl}))|_{\mathbf{n}\setminus \mathrm{im}(\alpha)}]} (\mathbf{n}, \mathbf{d} \sqcup \mathbf{n}).$$

This is well-defined: Let a be equivalent to a' meaning that there is a σ in A_{d+n} such that $a = \sigma \circ a'$. Then $a \circ (\mathrm{id}_{\mathbf{d}} \sqcup \alpha) = \sigma \circ a' \circ (\mathrm{id}_{\mathbf{d}} \sqcup \alpha)$, and

$$a \circ (\mathrm{id}_{\mathbf{d}} \sqcup (\alpha, \mathrm{incl})) \mid_{\mathbf{n}\setminus \mathrm{im}(\alpha)} = \sigma \circ a' \circ (\mathrm{id}_{\mathbf{d}} \sqcup (\alpha, \mathrm{incl})) \mid_{\mathbf{n}\setminus \mathrm{im}(\alpha)} .$$

Hence, the morphism $(\alpha, a \circ (\mathrm{id}_{\mathbf{d}} \sqcup \alpha), a \circ (\mathrm{id}_{\mathbf{d}} \sqcup (\alpha, \mathrm{incl})) \mid_{\mathbf{n}\setminus \mathrm{im}(\alpha)})$ is equivalent to $(\alpha, a' \circ (\mathrm{id}_{\mathbf{d}} \sqcup \alpha), a' \circ (\mathrm{id}_{\mathbf{d}} \sqcup (\alpha, \mathrm{incl})) \mid_{\mathbf{n}\setminus \mathrm{im}(\alpha)})$.
On the other hand, let $G\colon \bar{\mathcal{J}}_d \to \mathcal{I} \int \bar{\Sigma}(\mathbf{d} \sqcup \_)$ be the functor that sends $(\mathbf{m}, \mathbf{d} \sqcup \mathbf{m})$ to $(\mathbf{m}, *)$, and that maps $[\alpha_1, \alpha_2, \rho]\colon (\mathbf{m}, \mathbf{d} \sqcup \mathbf{m}) \to (\mathbf{n}, \mathbf{d} \sqcup \mathbf{n})$ to

$$(\mathbf{m}, *) \xrightarrow{(\alpha_1, [(\alpha_2, \mathrm{incl}) \circ (\mathrm{id}_{\mathbf{d}\sqcup\mathbf{m}} \sqcup \rho) \circ (\mathrm{id}_{\mathbf{d}} \sqcup (\alpha_1, \mathrm{incl})^{-1})])} (\mathbf{n}, *).$$

Let $(\alpha_1, \alpha_2, \rho)$ be equivalent to $(\alpha_1', \alpha_2', \rho')$, that is, $\alpha_1 = \alpha_1'$ and there exists a σ in A_{d+n} such that $\alpha_2 = \sigma \circ \alpha_2'$ and $\rho = \sigma \circ \rho'$. We get that

$$\begin{aligned}
&(\alpha_2, \mathrm{incl}) \circ (\mathrm{id}_{\mathbf{d}\sqcup\mathbf{m}} \sqcup \rho) \circ (\mathrm{id}_{\mathbf{d}} \sqcup (\alpha_1, \mathrm{incl})^{-1}) \\
&= (\sigma \circ (\alpha_2', \mathrm{incl}) \circ (\mathrm{id}_{\mathbf{d}\sqcup\mathbf{m}} \sqcup \sigma)^{-1}) \circ ((\mathrm{id}_{\mathbf{d}\sqcup\mathbf{m}} \sqcup \sigma) \circ (\mathrm{id}_{\mathbf{d}\sqcup\mathbf{m}} \sqcup \rho')) \circ (\mathrm{id}_{\mathbf{d}} \sqcup (\alpha_1, \mathrm{incl})^{-1}) \\
&= \sigma \circ (\alpha_2', \mathrm{incl}) \circ (\mathrm{id}_{\mathbf{d}\sqcup\mathbf{m}} \sqcup \rho') \circ (\mathrm{id}_{\mathbf{d}} \sqcup (\alpha_1, \mathrm{incl})^{-1}).
\end{aligned}$$

Thus, G is well-defined, too.
We notice that $F \circ G(\mathbf{m}, \mathbf{d} \sqcup \mathbf{m}) = (\mathbf{m}, \mathbf{d} \sqcup \mathbf{m})$ and $G \circ F(\mathbf{m}, *) = (\mathbf{m}, *)$. More than that, we show that both compositions of functors are the identity on morphisms. First,

$$\begin{aligned}
&F \circ G([\alpha_1, \alpha_2, \rho]) \\
&= F(\alpha_1, [(\alpha_2, \mathrm{incl}) \circ (\mathrm{id}_{\mathbf{d}\sqcup\mathbf{m}} \sqcup \rho) \circ (\mathrm{id}_{\mathbf{d}} \sqcup (\alpha_1, \mathrm{incl})^{-1})]) \\
&= [\alpha_1, (\alpha_2, \mathrm{incl}) \circ (\mathrm{id}_{\mathbf{d}\sqcup\mathbf{m}} \sqcup \rho) \circ (\mathrm{id}_{\mathbf{d}} \sqcup (\alpha_1, \mathrm{incl})^{-1}) \circ (\mathrm{id}_{\mathbf{d}} \sqcup \alpha_1), \\
&\quad (\alpha_2, \mathrm{incl}) \circ (\mathrm{id}_{\mathbf{d}\sqcup\mathbf{m}} \sqcup \rho) \circ (\mathrm{id}_{\mathbf{d}} \sqcup (\alpha_1, \mathrm{incl})^{-1}) \circ (\mathrm{id}_{\mathbf{d}} \sqcup (\alpha_1, \mathrm{incl}))|_{\mathbf{n}\setminus \mathrm{im}(\alpha_1)}].
\end{aligned}$$

To realize that the second component $(\alpha_2, \mathrm{incl}) \circ (\mathrm{id}_{\mathbf{d}\sqcup\mathbf{m}} \sqcup \rho) \circ (\mathrm{id}_{\mathbf{d}} \sqcup (\alpha_1, \mathrm{incl})^{-1}) \circ (\mathrm{id}_{\mathbf{d}} \sqcup \alpha_1)$ is α_2, we spell out the composition in the diagram

$$\begin{tikzcd}
\mathbf{d} \sqcup \mathbf{m} \arrow[r, "\mathrm{id}_{\mathbf{d}}\sqcup\alpha_1"] \arrow[rrrd, bend right, "\alpha_2"'] & \mathbf{d} \sqcup \mathbf{n} \arrow[r, "\mathrm{id}_{\mathbf{d}}\sqcup(\alpha_1,\mathrm{incl})^{-1}"] & \mathbf{d} \sqcup \mathbf{m} \sqcup (\mathbf{n} \setminus \mathrm{im}(\alpha_1)) \arrow[r, "\mathrm{id}_{\mathbf{d}\sqcup\mathbf{m}}\sqcup\rho"] & \mathbf{d} \sqcup \mathbf{m} \sqcup ((\mathbf{d} \sqcup \mathbf{n}) \setminus \mathrm{im}(\alpha_2)) \arrow[d, "(\alpha_2,\mathrm{incl})"] \\
 & & & \mathbf{d} \sqcup \mathbf{n}.
\end{tikzcd}$$

In addition, the restriction of

$$(\alpha_2, \mathrm{incl}) \circ (\mathrm{id}_{\mathbf{d}\sqcup\mathbf{m}} \sqcup \rho) \circ (\mathrm{id}_{\mathbf{d}} \sqcup (\alpha_1, \mathrm{incl})^{-1}) \circ (\mathrm{id}_{\mathbf{d}} \sqcup (\alpha_1, \mathrm{incl})) = (\alpha_2, \mathrm{incl}) \circ (\mathrm{id}_{\mathbf{d}\sqcup\mathbf{m}} \sqcup \rho)$$

to $\mathbf{n} \setminus \mathrm{im}(\alpha_1)$ is ρ. Secondly,

$$\begin{aligned}
& G \circ F(\alpha, [a]) \\
= {} & G([\alpha, a \circ (\mathrm{id}_{\mathbf{d}} \sqcup \alpha), a \circ (\mathrm{id}_{\mathbf{d}} \sqcup (\alpha, \mathrm{incl}))|_{\mathbf{n}\setminus\mathrm{im}(\alpha)}]) \\
= {} & (\alpha, [(a \circ (\mathrm{id}_{\mathbf{d}} \sqcup \alpha), \mathrm{incl}) \circ (\mathrm{id}_{\mathbf{d}\sqcup\mathbf{m}} \sqcup (a \circ (\mathrm{id}_{\mathbf{d}} \sqcup (\alpha, \mathrm{incl}))|_{\mathbf{n}\setminus\mathrm{im}(\alpha)})) \circ (\mathrm{id}_{\mathbf{d}} \sqcup (\alpha, \mathrm{incl})^{-1})]).
\end{aligned}$$

The second component

$$[(a \circ (\mathrm{id}_{\mathbf{d}} \sqcup \alpha), \mathrm{incl}) \circ (\mathrm{id}_{\mathbf{d}\sqcup\mathbf{m}} \sqcup (a \circ (\mathrm{id}_{\mathbf{d}} \sqcup (\alpha, \mathrm{incl}))|_{\mathbf{n}\setminus\mathrm{im}(\alpha)})) \circ (\mathrm{id}_{\mathbf{d}} \sqcup (\alpha, \mathrm{incl})^{-1})]$$

can be identified with $[a]$ which we can read off from the diagram

$$\begin{tikzcd}
\mathbf{d} \sqcup \mathbf{n} \arrow[rdd, bend right, "a"'] & \mathbf{d} \sqcup \mathbf{m} \sqcup (\mathbf{n} \setminus \mathrm{im}(\alpha)) \arrow[l, "\mathrm{id}_{\mathbf{d}}\sqcup(\alpha,\mathrm{incl})"'] \arrow[d, "\mathrm{id}_{\mathbf{d}\sqcup\mathbf{m}}\sqcup(a\circ(\mathrm{id}_{\mathbf{d}}\sqcup(\alpha,\mathrm{incl}))|_{\mathbf{n}\setminus\mathrm{im}(\alpha)})"] \\
 & \mathbf{d} \sqcup \mathbf{m} \sqcup ((\mathbf{d} \sqcup \mathbf{n}) \setminus \mathrm{im}(a \circ (\mathrm{id}_{\mathbf{d}} \sqcup \alpha))) \arrow[d, "(a\circ(\mathrm{id}_{\mathbf{d}}\sqcup\alpha),\mathrm{incl})"] \\
 & \mathbf{d} \sqcup \mathbf{n}.
\end{tikzcd}$$

□

Let $e \in \mathbb{Z}_{\geq 0}$. We denote by $\mathcal{I}_{\geq e}$ the full subcategory of $\mathcal{I}$ with objects $\mathbf{m}$ in $\mathcal{I}$ such that $m \geq e$. For $e = 0$, the category $\mathcal{I}_{\geq 0}$ is the category $\mathcal{I}$, and for $e = 1$, the category $\mathcal{I}_{\geq 1}$ is the category $\mathcal{I}_+$.

Lemma 1.23. *Let $e, e' \in \mathbb{Z}_{\geq 0}$ such that $e' \geq e$. The inclusion functor $\iota_{e',e} \colon \mathcal{I}_{\geq e'} \to \mathcal{I}_{\geq e}$ is homotopy right cofinal.*

Proof. We argue as in the proofs of [SS12, Corollary 5.9.] and Proposition 1.19. Let $\mathbf{m}$ be in $\mathcal{I}_{\geq e}$. We have to show that the classifying space $B(\mathbf{m} \downarrow \iota_{e',e})$ is contractible. We choose a morphism $\alpha \colon \mathbf{0} \to \mathbf{l}$ in $\mathcal{I}$ such that $l \geq e'$. Let $- \sqcup \alpha \colon (\mathbf{m} \downarrow \mathcal{I}_{\geq e}) \to (\mathbf{m} \downarrow \iota_{e',e})$ be the functor given on objects by

$$(\mathbf{a}, \mathbf{m} \xrightarrow{\eta} \mathbf{a}) \mapsto (\mathbf{a} \sqcup \mathbf{l}, \mathbf{m} = \mathbf{m} \sqcup \mathbf{0} \xrightarrow{\eta\sqcup\alpha} \mathbf{a} \sqcup \mathbf{l})$$

and on morphisms by

$$\begin{tikzcd}[column sep=small] & \mathbf{m} \arrow[dl, "\eta"'] \arrow[dr, "\theta"] & \\ \mathbf{a} \arrow[rr, "\beta"] & & \mathbf{b} \end{tikzcd} \quad \mapsto \quad \begin{tikzcd}[column sep=small] & \mathbf{m} \arrow[dl, "\eta\sqcup\alpha"'] \arrow[dr, "\theta\sqcup\alpha"] & \\ \mathbf{a}\sqcup\mathbf{l} \arrow[rr, "\beta\sqcup\mathrm{id}_{\mathbf{l}}"] & & \mathbf{b}\sqcup\mathbf{l}. \end{tikzcd}$$

Furthermore, let $(\iota_{e',e})_*\colon (\mathbf{m}\downarrow\iota_{e',e}) \to (\mathbf{m}\downarrow\mathcal{I}_{\geq e})$ be the functor defined by the inclusion functor $\iota_{e',e}\colon \mathcal{I}_{\geq e'} \to \mathcal{I}_{\geq e}$. There are natural transformations between the identity functor $\mathrm{id}_{(\mathbf{m}\downarrow\iota_{e',e})}$ and $(-\sqcup\alpha)\circ(\iota_{e',e})_*$, and between the identity functor $\mathrm{id}_{(\mathbf{m}\downarrow\mathcal{I}_{\geq e})}$ and $(\iota_{e',e})_*\circ(-\sqcup\alpha)$. These give rise to homotopies between $\mathrm{id}_{B(\mathbf{m}\downarrow\iota_{e',e})}$ and $B(-\sqcup\alpha)\circ B((\iota_{e',e})_*)$, and between $\mathrm{id}_{B(\mathbf{m}\downarrow\mathcal{I}_{\geq e})}$ and $B((\iota_{e',e})_*)\circ B(-\sqcup\alpha)$. Thus, the space $B(\mathbf{m}\downarrow\iota_{e',e})$ is homotopy equivalent to $B(\mathbf{m}\downarrow\mathcal{I}_{\geq e})$. But the latter is contractible because the comma category $(\mathbf{m}\downarrow\mathcal{I}_{\geq e})$ has the initial object $(\mathbf{m},\mathrm{id}_{\mathbf{m}})$. □

Proposition 1.24. *The classifying space $B\bar{\mathcal{J}}$ is weakly equivalent to $\mathbb{Z}\times\mathbb{R}P^\infty$.*

Proof. The category $\bar{\mathcal{J}}$ is a permutative category. Hence, the classifying space $B\bar{\mathcal{J}}$ is an E_∞ space by [May74, Theorem 4.9]. Besides, since $B\bar{\mathcal{J}} = \coprod_{d\in\mathbb{Z}} B\bar{\mathcal{J}}_d$, we get that $\pi_0(B\bar{\mathcal{J}}) = \mathbb{Z}$. So $B\bar{\mathcal{J}}$ is a grouplike E_∞ space. From this we can conclude that all connected components $B\bar{\mathcal{J}}_d$ of $B\bar{\mathcal{J}}$ are homotopy equivalent. Let $d\in\mathbb{Z}_{\geq 0}$. Thomason's *homotopy colimit theorem* yields that there is a homotopy equivalence

$$\mathrm{hocolim}_{\mathcal{I}}\mathcal{N}(\bar{\Sigma}(\mathbf{d}\sqcup -)) \xrightarrow{\sim} \mathcal{N}(\mathcal{I}\int\bar{\Sigma}(\mathbf{d}\sqcup -))$$

[Tho79, Theorem 1.2]. Using that the inclusion functor $\mathcal{I}_{\geq 2}\to\mathcal{I}$ is homotopy right cofinal by Lemma 1.23, we obtain that the induced map of homotopy colimits

$$\mathrm{hocolim}_{\mathcal{I}_{\geq 2}}\mathcal{N}(\bar{\Sigma}(\mathbf{d}\sqcup -)) \to \mathrm{hocolim}_{\mathcal{I}}\mathcal{N}(\bar{\Sigma}(\mathbf{d}\sqcup -))$$

is a weak equivalence by [Hir03, Theorem 19.6.7.(1)]. Further, the functor $\bar{\Sigma}(\mathbf{d}\sqcup -)\colon \mathcal{I}_{\geq 2}\to \mathrm{Cat}$ is isomorphic to the constant functor $\mathrm{const}_{\mathcal{I}_{\geq 2}}\{\pm 1\}\colon \mathcal{I}_{\geq 2}\to\mathrm{Cat}$. This is because for $\mathbf{m}$ in $\mathcal{I}_{\geq 2}$, the category $\bar{\Sigma}_{d+m}$ is isomorphic to the category $\{\pm 1\}$, and for a morphism $\alpha\colon\mathbf{m}\to\mathbf{n}$ in $\mathcal{I}_{\geq 2}$, the functor $\bar{\Sigma}(\mathbf{d}\sqcup -)(\alpha)\colon \bar{\Sigma}_{d+m}\to\bar{\Sigma}_{d+n}$ is isomorphic to the identity functor on $\{\pm 1\}$. The latter holds as for a representative a of $[a]$ in $\bar{\Sigma}_{d+m}$, the sign of a is equal to the sign of $(\mathrm{id}_{\mathbf{d}}\sqcup\alpha,\mathrm{incl})\circ(a\sqcup\mathrm{id}_{(\mathbf{d}\sqcup\mathbf{m})\setminus\mathrm{im}(\mathrm{id}_{\mathbf{d}}\sqcup\alpha)})\circ(\mathrm{id}_{\mathbf{d}}\sqcup\alpha,\mathrm{incl})^{-1}$. Therefore, the functor $\mathcal{N}(\bar{\Sigma}(\mathbf{d}\sqcup -))$ can be identified with the functor $\mathcal{N}(\mathrm{const}_{\mathcal{I}_{\geq 2}}\{\pm 1\})$ which is isomorphic to $\mathrm{const}_{\mathcal{I}_{\geq 2}}\mathcal{N}\{\pm 1\}$. This implies that

$$\begin{aligned}\mathrm{hocolim}_{\mathcal{I}_{\geq 2}}\mathcal{N}(\bar{\Sigma}(\mathbf{d}\sqcup -)) &\cong \mathrm{hocolim}_{\mathcal{I}_{\geq 2}}\mathrm{const}_{\mathcal{I}_{\geq 2}}\mathcal{N}\{\pm 1\}\\ &\simeq \mathcal{N}\{\pm 1\}.\end{aligned}$$

By Lemma 1.22 the category $\mathcal{I}\int\bar{\Sigma}(\mathbf{d}\sqcup -)$ is isomorphic to the category $\bar{\mathcal{J}}_d$. Altogether, we find that

$$\begin{aligned} B\bar{\mathcal{J}}_d &= \mid \mathcal{N}\bar{\mathcal{J}}_d \mid \\ &\cong \mid \mathcal{N}(\mathcal{I} \int \bar{\Sigma}(\mathbf{d} \sqcup -)) \mid \\ &\simeq \mid \mathcal{N}\{\pm 1\} \mid \\ &\simeq \mathbb{R}P^\infty, \end{aligned}$$

and $B\bar{\mathcal{J}} \simeq \mathbb{Z} \times \mathbb{R}P^\infty$. □

Remark 1.25. Let $d \in \mathbb{Z}_{\geq 0}$. There is an alternative way to determine the classifying space $B\bar{\mathcal{J}}_d$. We write $\bar{\mathcal{J}}_{d,(\geq 2,-)}$ for the full subcategory of $\bar{\mathcal{J}}_d$ whose objects $(\mathbf{m}_1, \mathbf{m}_2)$ in $\bar{\mathcal{J}}_d$ satisfy $m_1 \geq 2$. It follows from Lemma 1.18 that the inclusion functor $\bar{\mathcal{J}}_{d,(\geq 2,-)} \to \bar{\mathcal{J}}_d$ is homotopy right cofinal. Thus, the induced map

$$\mathrm{hocolim}_{\bar{\mathcal{J}}_{d,(\geq 2,-)}} \mathrm{const}_{\bar{\mathcal{J}}_{d,(\geq 2,-)}} * \to \mathrm{hocolim}_{\bar{\mathcal{J}}_d} \mathrm{const}_{\bar{\mathcal{J}}_d} *$$

is a weak equivalence by [Hir03, Theorem 19.6.7.(1)]. Moreover, arguing as in the proof of Lemma 1.22, we see that the category $\bar{\mathcal{J}}_{d,(\geq 2,-)}$ is isomorphic to the category $\mathcal{I}_{\geq 2} \int \bar{\Sigma}(\mathbf{d} \sqcup -)$. As the functor $\bar{\Sigma}(\mathbf{d} \sqcup -) \colon \mathcal{I}_{\geq 2} \to \mathrm{Cat}$ is isomorphic to the constant functor $\mathrm{const}_{\mathcal{I}_{\geq 2}}\{\pm 1\} \colon \mathcal{I}_{\geq 2} \to \mathrm{Cat}$ (see proof of Proposition 1.24), we can identify the category $\mathcal{I}_{\geq 2} \int \bar{\Sigma}(\mathbf{d} \sqcup -)$ with the product category $\mathcal{I}_{\geq 2} \times \{\pm 1\}$. So we obtain that

$$\begin{aligned} \mathrm{hocolim}_{\bar{\mathcal{J}}_{d,(\geq 2,-)}} \mathrm{const}_{\bar{\mathcal{J}}_{d,(\geq 2,-)}} * &\cong \mathrm{hocolim}_{\mathcal{I}_{\geq 2} \times \{\pm 1\}} \mathrm{const}_{\mathcal{I}_{\geq 2} \times \{\pm 1\}} * \\ &= \mathcal{N}(\mathcal{I}_{\geq 2} \times \{\pm 1\}) \\ &\cong \mathcal{N}(\mathcal{I}_{\geq 2}) \times \mathcal{N}\{\pm 1\}. \end{aligned}$$

This is weakly equivalent to $\mathcal{N}\{\pm 1\}$, because the space $\mathcal{N}(\mathcal{I}_{\geq 2})$ is contractible.

Remark 1.26. The homotopy groups of the space $\mathbb{Z} \times \mathbb{R}P^\infty$ are given by

$$\pi_l(\mathbb{Z} \times \mathbb{R}P^\infty, *) \cong \begin{cases} \mathbb{Z}, & l = 0, \\ \mathbb{Z}/2\mathbb{Z}, & l = 1, \\ 0, & l \geq 2. \end{cases}$$

Considering the Quillen equivalence between grouplike E_∞ spaces and connective spectra [May09, Corollary 9.5], the induced map $B\mathcal{J} \to B\bar{\mathcal{J}}$ of grouplike E_∞ spaces models the first Postnikov section of the sphere spectrum $\mathbb{S}$ in connective spectra (see Remark 1.5). In Subsection 3.2 we recall Sagave and Schlichtkrull's chain of Quillen equivalences between commutative $\mathcal{K}$-spaces and E_∞ spaces over $B\mathcal{K}$ for a permutative well-structured index category $\mathcal{K}$ such that the inclusion functor $\mathcal{K}_+ \to \mathcal{K}$ is homotopy right cofinal and the pair $(\mathcal{K}, \mathcal{O}\mathcal{K}_+)$ is very well-structured (see Theorem 3.7). Applying this result to the category $\mathcal{J}$ yields that commutative $\mathcal{J}$-spaces are Quillen equivalent to E_∞ spaces over $Q(S^0)$ (see Remark 1.5). In analogy with algebra where $\mathbb{Z}$-graded monoids can be identified as monoids over the additive monoid $(\mathbb{Z},+)$ of the integers $\mathbb{Z}$, Sagave and Schlichtkrull view commutative $\mathcal{J}$-spaces as $Q(S^0)$-graded E_∞ spaces, where $Q(S^0)$ plays the role of $(\mathbb{Z},+)$ and the sphere spectrum $\mathbb{S}$ the role of the integers $\mathbb{Z}$ (see [SS12, p. 2120]).

The category $\bar{\mathcal{J}}$ is more algebraic than the category $\mathcal{J}$ because the grading is over $B\bar{\mathcal{J}} \simeq \mathbb{Z} \times \mathbb{R}P^\infty$ instead of $B\mathcal{J} \simeq Q(S^0)$. We think of commutative $\bar{\mathcal{J}}$-spaces as $(\mathbb{Z} \times \mathbb{R}P^\infty)$-graded E_∞ spaces. The fact that π_0 of the classifying space $B\bar{\mathcal{J}}$ is equal to $\mathbb{Z}$ can be interpreted as a $\mathbb{Z}$-grading, and the fact that the fundamental group of $B\bar{\mathcal{J}}$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$ corresponds to graded commutativity. Since the higher homotopy groups of $B\bar{\mathcal{J}}$ vanish, the category $\bar{\mathcal{J}}$ is in a sense minimal with these properties. For these reasons, the category $\bar{\mathcal{J}}$ is a suitable indexing category in the algebraic set-up. This also becomes more evident in the next section where we define pre-log structures on E_∞ dgas via commutative $\bar{\mathcal{J}}$-spaces.

2 Pre-log structures on E_∞ dgas

This section is devoted to establishing *pre-log structures* in the algebraic setting. We recall Richter and Shipley's chain of Quillen equivalences between commutative Hk-algebra spectra and E_∞ differential graded k-algebras (E_∞ dgas). Employing the intermediate model category of commutative symmetric ring spectra in simplicial k-modules in this chain, we construct the Quillen adjunction $(\Lambda^{\bar{\mathcal{J}}}, \Omega^{\bar{\mathcal{J}}})$ between the latter and commutative $\bar{\mathcal{J}}$-spaces, on which our definition of pre-log structures is based. We define *units* for E_∞ dgas to determine whether a pre-log structure is a *log structure*. We give some examples of *pre-log cdgas* and *log cdgas*. Further, we explain the drawbacks of another approach to specify pre-log structures via diagram chain complexes. For this, as minor results, we provide a homotopy colimit formula for diagram chain complexes and show that in contrast to diagram spaces indexed by a well-structured index category, diagram chain complexes indexed by a well-structured index category do not always carry a model structure in which the homotopy colimit functor detects the weak equivalences.

2.1 Preliminaries on diagram spaces

We start with collecting several results about diagram spaces from [SS12] which are relevant for our theory. For more details we refer to [SS12].

We write $\mathcal{S}$ for the category of spaces where spaces mean unpointed simplicial sets. Let $\mathcal{K}$ be a small category. A $\mathcal{K}$-space is a functor $M\colon \mathcal{K} \to \mathcal{S}$. The category of $\mathcal{K}$-spaces is the functor category $\mathcal{S}^{\mathcal{K}}$ [SS12, Definition 2.1]. The category $\mathcal{S}^{\mathcal{K}}$ is bicomplete with limits and colimits constructed $\mathcal{K}$-levelwise. Moreover, the category $\mathcal{S}^{\mathcal{K}}$ is enriched, tensored and cotensored over $\mathcal{S}$. For a $\mathcal{K}$-space M and a space T, the tensor $M \times T$ is the $\mathcal{K}$-space defined by $(M \times T)(\mathbf{k}) = M(\mathbf{k}) \times T$, and the cotensor M^T is the $\mathcal{K}$-space specified by $M^T(\mathbf{k}) = \underline{\mathrm{Hom}}_{\mathcal{S}}(T, M(\mathbf{k}))$ [SS12, Lemma 2.2].

For a $\mathcal{K}$-space M, the homotopy colimit of M over $\mathcal{K}$ can be modelled by the Bousfield-Kan homotopy colimit of M over $\mathcal{K}$, which is defined as the realization of the bisimplicial set

$$[s] \mapsto \coprod_{\mathbf{k}_0 \xleftarrow{\alpha_1} \ldots \xleftarrow{\alpha_s} \mathbf{k}_s \; \in \mathcal{N}\mathcal{K}[s]} M(\mathbf{k}_s) \tag{2.1}$$

(see [BK72, XII.§5]). A realization functor is provided by the diagonal functor ([BK72, XII.5.2], [Hir03, Theorem 15.11.6]).

Assume that $(\mathcal{K}, \sqcup, \mathbf{0}_{\mathcal{K}})$ is a small symmetric monoidal category. The symmetric monoidal structure of $\mathcal{K}$ and the cartesian product of $\mathcal{S}$ give rise to a closed symmetric monoidal structure of $\mathcal{S}^{\mathcal{K}}$, the Day convolution product of $\mathcal{S}^{\mathcal{K}}$ (see [Day70a, §3.2], [Day70b, §4]). For $\mathcal{K}$-spaces M and N, the monoidal product $M \boxtimes N$ is the left Kan extension of the

$\mathcal{K}$-levelwise cartesian product along $\sqcup\colon \mathcal{K}\times\mathcal{K}\to\mathcal{K}$,

$$
\begin{tikzcd}
\mathcal{K}\times\mathcal{K} \arrow[r, "\sqcup"] \arrow[d, "M\times N"'] & \mathcal{K} \arrow[ddl, "M\boxtimes N"] \\
\mathcal{S}\times\mathcal{S} \arrow[d, "\times"'] & \\
\mathcal{S}. &
\end{tikzcd}
$$

So the $\mathcal{K}$-space $M\boxtimes N$ is given by

$$(M\boxtimes N)(\mathbf{k}) = \mathrm{colim}_{\mathbf{m}\sqcup\mathbf{n}\to\mathbf{k}} M(\mathbf{m})\times N(\mathbf{n}) \tag{2.2}$$

where the colimit is taken over the comma category $(-\sqcup-\downarrow\mathbf{k})$. The monoidal unit is the $\mathcal{K}$-levelwise discrete $\mathcal{K}$-space $U^{\mathcal{K}} = \mathcal{K}(\mathbf{0}_{\mathcal{K}},-)$, and there are natural symmetry isomorphisms $\tau_{M,N}\colon M\boxtimes N\to N\boxtimes M$.

For an object $\mathbf{k}$ in the category $\mathcal{K}$, the evaluation functor $\mathrm{Ev}^{\mathcal{K}}_{\mathbf{k}}\colon \mathcal{S}^{\mathcal{K}}\to\mathcal{S}$ sends a $\mathcal{K}$-space M to the space $\mathrm{Ev}^{\mathcal{K}}_{\mathbf{k}}(M) = M(\mathbf{k})$. This functor possesses a left adjoint $F^{\mathcal{K}}_{\mathbf{k}}\colon \mathcal{S}\to\mathcal{S}^{\mathcal{K}}$, which maps a space T to the $\mathcal{K}$-space $F^{\mathcal{K}}_{\mathbf{k}}(T) = \mathcal{K}(\mathbf{k},-)\times T$. For $\mathbf{k}$ and $\mathbf{l}$ in $\mathcal{K}$, there is a natural isomorphism

$$F^{\mathcal{K}}_{\mathbf{k}}(S)\boxtimes F^{\mathcal{K}}_{\mathbf{l}}(T)\cong F^{\mathcal{K}}_{\mathbf{k}\sqcup\mathbf{l}}(S\times T), \tag{2.3}$$

for each pair S and T in $\mathcal{S}$.

In [SS12, §6] Sagave and Schlichtkrull describe various model structures on $\mathcal{K}$-spaces associated to a well-structured relative index category $(\mathcal{K},\mathcal{A})$. We restrict to the cases in which $\mathcal{A}$ is given by either $\mathcal{OK}$ or $\mathcal{OK}_+$. For the rest of this subsection suppose that $\mathcal{K}$ is a well-structured index category, that the inclusion functor $\mathcal{K}_+\to\mathcal{K}$ is homotopy right cofinal and that the pair $(\mathcal{K},\mathcal{OK}_+)$ is very well-structured. The category of spaces is equipped with the standard model structure. The latter is cofibrantly generated with generating cofibrations $I_{\mathcal{S}} = \{\partial\Delta_n\to\Delta_n, n\geq 0\}$, generating acyclic cofibrations $J_{\mathcal{S}} = \{\Lambda_{i,n}\to\Delta_n, n>0, 0\leq i\leq n\}$ and weak equivalences those maps which induce isomorphisms on homotopy groups (see [Hov99, §3, Theorem 3.6.5]).

Definition 2.1. (see [SS12, pp. 2148-2149])

(i) A map $f\colon M\to N$ of $\mathcal{K}$-spaces is a (positive) level equivalence/ (positive) level fibration if the map f is $\mathcal{K}_{(+)}$-levelwise a weak equivalence/ fibration of spaces.

(ii) A map $f\colon M\to N$ of $\mathcal{K}$-spaces is a (positive) $\mathcal{K}$-cofibration if the map f has the left lifting property with respect to (positive) level fibrations that are (positive) level equivalences.

(iii) We define the set $I^{(+)\mathrm{level}}_{\mathcal{S}^{\mathcal{K}}}$ by $I^{(+)\mathrm{level}}_{\mathcal{S}^{\mathcal{K}}} = \{F^{\mathcal{K}}_{\mathbf{k}}(i), \mathbf{k}\in\mathcal{OK}_{(+)}, i\in I_{\mathcal{S}}\}$, and the set $J^{(+)\mathrm{level}}_{\mathcal{S}^{\mathcal{K}}}$ by $J^{(+)\mathrm{level}}_{\mathcal{S}^{\mathcal{K}}} = \{F^{\mathcal{K}}_{\mathbf{k}}(j), \mathbf{k}\in\mathcal{OK}_{(+)}, j\in J_{\mathcal{S}}\}$.

Proposition 2.2. *([Hir03, Theorem 11.6.1], [SS12, Proposition 6.7]) The category of $\mathcal{K}$-spaces carries a cofibrantly generated (positive) projective level model structure with $I^{(+)\mathrm{level}}_{\mathcal{S}^{\mathcal{K}}}$ as its set of generating cofibrations and $J^{(+)\mathrm{level}}_{\mathcal{S}^{\mathcal{K}}}$ as its set of generating acyclic cofibrations. In this model structure the (positive) level equivalences/ (positive) level fibrations are the weak equivalences/ fibrations, and the (positive) $\mathcal{K}$-cofibrations are the cofibrations.*

Remark 2.3. The (positive) $\mathcal{K}$-cofibrations are characterized in [SS12, Proposition 6.8]. The (positive) projective level model structure on $\mathcal{S}^{\mathcal{K}}$ is an $\mathcal{S}$-model structure ([Hov99, Definition 4.2.18], [SS12, Proposition 6.10]) and proper [Hir03, Theorem 13.1.14].

Definition 2.4. (see [SS12, Definition 6.14, p. 2152])

(i) A map $f\colon M \to N$ of $\mathcal{K}$-spaces is a $\mathcal{K}$-equivalence if the induced map of homotopy colimits $\mathrm{hocolim}_{\mathcal{K}} f\colon \mathrm{hocolim}_{\mathcal{K}} M \to \mathrm{hocolim}_{\mathcal{K}} N$ is a weak equivalence of spaces.

(ii) A map $f\colon M \to N$ of $\mathcal{K}$-spaces is a (positive) $\mathcal{K}$-fibration if the map f is a (positive) level fibration and for every morphism $\alpha\colon \mathbf{k} \to \mathbf{l}$ in $\mathcal{K}_{(+)}$, the induced square

$$\begin{array}{ccc} M(\mathbf{k}) & \xrightarrow{M(\alpha)} & M(\mathbf{l}) \\ {\scriptstyle f(\mathbf{k})}\downarrow & & \downarrow{\scriptstyle f(\mathbf{l})} \\ N(\mathbf{k}) & \xrightarrow{N(\alpha)} & N(\mathbf{l}) \end{array}$$

is homotopy cartesian in spaces.

For a $\mathcal{K}$-space M and $\mathbf{k}$ in $\mathcal{K}$, there is a map of bisimplicial sets

$$\mathrm{const}_{\Delta^{\mathrm{op}}} M(\mathbf{k}) \to ([s] \mapsto \coprod_{\mathbf{k}_0 \xleftarrow{\alpha_1} \dots \xleftarrow{\alpha_s} \mathbf{k}_s \;\in \mathcal{N}\mathcal{K}[s]} M(\mathbf{k}_s))$$

which in simplicial degree $[s]$ is given by the inclusion, that is, the space $M(\mathbf{k})$ is sent by the identity to the summand $M(\mathbf{k})$ indexed by

$$\mathbf{k} \xleftarrow{\mathrm{id}} \dots \xleftarrow{\mathrm{id}} \mathbf{k} \in \mathcal{N}\mathcal{K}[s].$$

Applying the diagonal functor induces a map of spaces $M(\mathbf{k}) \to \mathrm{hocolim}_{\mathcal{K}} M$. The square

$$\begin{array}{ccc} M(\mathbf{k}) & \longrightarrow & \mathrm{hocolim}_{\mathcal{K}} M \\ \downarrow & & \downarrow \\ \{\mathbf{k}\} & \longrightarrow & B\mathcal{K} \end{array} \tag{2.4}$$

is a pullback square (see proof of [GJ09, Lemma IV.5.7]).

We say that a $\mathcal{K}$-space M is homotopy constant with respect to morphisms in $\mathcal{K}_{(+)}$ if for every morphism $\alpha\colon \mathbf{k} \to \mathbf{l}$ in $\mathcal{K}_{(+)}$, the induced map $M(\alpha)\colon M(\mathbf{k}) \to M(\mathbf{l})$ is a weak equivalence of spaces. (Positive) $\mathcal{K}$-fibrant $\mathcal{K}$-spaces are in particular homotopy constant with respect to morphisms in $\mathcal{K}_{(+)}$. The following proposition is a useful tool to determine the homotopy type of (positive) $\mathcal{K}$-fibrant $\mathcal{K}$-spaces.

Proposition 2.5. *Let M be a $\mathcal{K}$-space that is homotopy constant with respect to morphisms in $\mathcal{K}_{(+)}$. The pullback square* (2.4) *is homotopy cartesian for every object* $\mathbf{k}$ *in* $\mathcal{K}_{(+)}$.

Proof. First, we assume that the $\mathcal{K}$-space M is homotopy constant with respect to morphisms in $\mathcal{K}$. An application of [GJ09, Lemma IV.5.7] yields the claim. Let M be then homotopy constant with respect to morphisms in $\mathcal{K}_+$. Again, the result [GJ09, Lemma IV.5.7] implies that the pullback square

$$\begin{array}{ccc} M(\mathbf{k}) & \longrightarrow & \mathrm{hocolim}_{\mathcal{K}_+} M \\ \downarrow & & \downarrow \\ \{\mathbf{k}\} & \longrightarrow & B\mathcal{K}_+ \end{array}$$

is homotopy cartesian for every object $\mathbf{k}$ in $\mathcal{K}_+$. As the inclusion functor $\mathcal{K}_+ \to \mathcal{K}$ is homotopy right cofinal by assumption, the induced maps of homotopy colimits $\mathrm{hocolim}_{\mathcal{K}_+} M \to \mathrm{hocolim}_{\mathcal{K}} M$ and $\mathrm{hocolim}_{\mathcal{K}_+} \mathrm{const}_{\mathcal{K}_+} * \to \mathrm{hocolim}_{\mathcal{K}} \mathrm{const}_{\mathcal{K}} *$ are weak equivalences by [Hir03, Theorem 19.6.7.(1)]. So we can conclude that the pullback square (2.4) is homotopy cartesian. □

Remark 2.6. Let M be a $\mathcal{K}$-space that is homotopy constant with respect to morphisms in $\mathcal{K}_{(+)}$, and let $\mathbf{k}$ be in $\mathcal{K}_{(+)}$. We write z for the point $\{\mathbf{k}\}$. The homotopy cartesian square (2.4), which is a homotopy fibre square, induces a long exact sequence of homotopy groups

$$\begin{aligned} &\ldots \to \pi_3(B\mathcal{K}, z) \to \pi_2(M(\mathbf{k}), z) \to \pi_2(\mathrm{hocolim}_{\mathcal{K}} M, z) \to \pi_2(B\mathcal{K}, z) \to \pi_1(M(\mathbf{k}), z) \\ &\to \pi_1(\mathrm{hocolim}_{\mathcal{K}} M, z) \to \pi_1(B\mathcal{K}, z) \to \pi_0(M(\mathbf{k})) \to \pi_0(\mathrm{hocolim}_{\mathcal{K}} M) \to \pi_0(B\mathcal{K}). \end{aligned}$$

If the category $\mathcal{K}$ is $\mathcal{I}$, the classifying space $B\mathcal{I}$ is contractible. Consequently, the natural map $M(\mathbf{k}) \to \mathrm{hocolim}_{\mathcal{I}} M$ is a weak equivalence. As another example, assume that the category $\mathcal{K}$ is $\bar{\mathcal{J}}$. From Proposition 1.24 we know that the classifying space $B\bar{\mathcal{J}}$ is weakly equivalent to $\mathbb{Z} \times \mathbb{R}P^\infty$ and that hence the homotopy groups of the space $B\bar{\mathcal{J}}$ are given by

$$\pi_l(B\bar{\mathcal{J}}, z) \cong \pi_l(\mathbb{Z} \times \mathbb{R}P^\infty, *) \cong \begin{cases} \mathbb{Z}, & l = 0, \\ \mathbb{Z}/2\mathbb{Z}, & l = 1, \\ 0, & l \geq 2 \end{cases}$$

(see Remark 1.26). Therefore, we find that $\pi_l(M(\mathbf{k}_1, \mathbf{k}_2), z) \cong \pi_l(\mathrm{hocolim}_{\bar{\mathcal{J}}} M, z)$ for $l \geq 2$.

Moreover, let $\alpha\colon \mathbf{k} \to \mathbf{l}$ be a morphism in $\mathcal{K}_{(+)}$. The induced map $\alpha^*\colon F^{\mathcal{K}}_{\mathbf{l}}(*) \to F^{\mathcal{K}}_{\mathbf{k}}(*)$, defined by precomposition with the map α, is a $\mathcal{K}$-equivalence [SS12, Lemma 6.15]. We factor the map α^* through the mapping cylinder $\mathrm{Cyl}(\alpha^*)$ into a (positive) $\mathcal{K}$-cofibration j_{α^*} followed by a homotopy equivalence r_{α^*},

$$\begin{tikzcd} F^{\mathcal{K}}_{\mathbf{l}}(*) \arrow[r, tail, "j_{\alpha^*}"] \arrow[rr, bend left, "\alpha^*"] & \mathrm{Cyl}(\alpha^*) \arrow[r, "r_{\alpha^*}", "\sim"'] & F^{\mathcal{K}}_{\mathbf{k}}(*). \end{tikzcd} \tag{2.5}$$

Let $J^{(+)\prime}$ be the set of morphisms of the form $j_{\alpha^*}\Box i$ where $\Box$ stands for the pushout product and i is an element in $I_{\mathcal{S}}$. We specify the set $J^{(+)}_{\mathcal{S}^{\mathcal{K}}} = J^{(+)\mathrm{level}}_{\mathcal{S}^{\mathcal{K}}} \cup J^{(+)\prime}$ (see [SS12, p. 2152]).

Proposition 2.7. *[SS12, Proposition 6.16] The category of $\mathcal{K}$-spaces has a cofibrantly generated (positive) projective $\mathcal{K}$-model structure with $I^{(+)\mathrm{level}}_{\mathcal{S}^{\mathcal{K}}}$ as its set of generating cofibrations and $J^{(+)}_{\mathcal{S}^{\mathcal{K}}}$ as its set of generating acyclic cofibrations. In this model structure the $\mathcal{K}$-equivalences are the weak equivalences, the (positive) $\mathcal{K}$-fibrations are the fibrations and the (positive) $\mathcal{K}$-cofibrations are the cofibrations.*

Remark 2.8. The (positive) projective $\mathcal{K}$-model structure on $\mathcal{S}^{\mathcal{K}}$ is an $\mathcal{S}$-model structure [SS12, Proposition 6.19] and proper [SS12, Corollary 11.10.(i)].

Furthermore, Sagave and Schlichtkrull explain how to right induce model structures on $\mathcal{K}$-spaces to model structures on structured $\mathcal{K}$-spaces (see [SS12, §9]). Let $\mathcal{D}$ be an operad in spaces. We say that the operad $\mathcal{D}$ is Σ-free if the action of the symmetric group Σ_n on the space $\mathcal{D}(n)$ is free for all $n \geq 0$ [SS12, Definition 9.1]. For example, an E_∞ operad in spaces is a Σ-free operad which is contractible in all levels $n \geq 0$. The commutativity operad $\mathcal{C}$ in spaces, specified by $\mathcal{C}(n) = *$ for all levels $n \geq 0$, is not Σ-free. To the operad $\mathcal{D}$ we can associate a monad $\mathbb{D}$ by defining

$$\mathbb{D}(M) = \coprod_{n\geq 0} \mathcal{D}(n) \times_{\Sigma_n} M^{\boxtimes n}.$$

Here $M^{\boxtimes 0}$ indicates the monoidal unit $U^{\mathcal{K}}$ (see [SS12, p. 2161]). We write $D\mathcal{S}^{\mathcal{K}}$ for the category of $\mathbb{D}$-algebras in $\mathcal{K}$-spaces. The category $D\mathcal{S}^{\mathcal{K}}$ is bicomplete, and the forgetful functor from $D\mathcal{S}^{\mathcal{K}}$ to $\mathcal{S}^{\mathcal{K}}$ preserves limits and filtered colimits [SS12, Lemma 9.2]. Sagave and Schlichtkrull point out that in order to right induce model structures on $\mathcal{S}^{\mathcal{K}}$ to model structures on $D\mathcal{S}^{\mathcal{K}}$, the action of the symmetric group Σ_n on the $\mathcal{K}$-space $\mathcal{D}(n) \times M^{\boxtimes n}$ has to be *sufficiently* free for $n \geq 0$. This condition can be fulfilled by assuming that the operad $\mathcal{D}$ is Σ-free or by exploiting that the pair $(\mathcal{K}, \mathcal{O}\mathcal{K}_+)$ is very well-structured (see [SS12, p. 2162]).

Proposition 2.9. *[SS12, Proposition 9.3]*

(i) Suppose that the operad $\mathcal{D}$ is Σ-free. The projective $\mathcal{K}$-model structure on $\mathcal{S}^{\mathcal{K}}$ lifts to a right-induced model structure on $D\mathcal{S}^{\mathcal{K}}$. This (right-induced) projective $\mathcal{K}$-model structure on $D\mathcal{S}^{\mathcal{K}}$ is cofibrantly generated with $\mathbb{D}(I^{\mathrm{level}}_{\mathcal{S}^{\mathcal{K}}})$ as its set of generating cofibrations and $\mathbb{D}(J_{\mathcal{S}^{\mathcal{K}}})$ as its set of generating acyclic cofibrations.

(ii) The positive projective $\mathcal{K}$-model structure on $\mathcal{S}^{\mathcal{K}}$ lifts to a right-induced model structure on $D\mathcal{S}^{\mathcal{K}}$. This (right-induced) positive projective $\mathcal{K}$-model structure on $D\mathcal{S}^{\mathcal{K}}$ is cofibrantly generated with $\mathbb{D}(I^{+\mathrm{level}}_{\mathcal{S}^{\mathcal{K}}})$ as its set of generating cofibrations and $\mathbb{D}(J^{+}_{\mathcal{S}^{\mathcal{K}}})$ as its set of generating acyclic cofibrations.

Remark 2.10. We obtain the analogous result for the (positive) projective level model structure on $\mathcal{S}^{\mathcal{K}}$ [SS12, Proposition 9.3]. All these right-induced model structures on $D\mathcal{S}^{\mathcal{K}}$

are $\mathcal{S}$-model structures (see [SS12, p. 2163]) and right proper (see [SS12, p. 2170, Corollary 11.5]). Moreover, the positive projective $\mathcal{K}$-model structure on commutative $\mathcal{K}$-spaces $C\mathcal{S}^{\mathcal{K}}$ is proper [SS12, Corollary 11.10.(ii)].

Remark 2.11. Let $\mathcal{D}$ be an E_∞ operad. The adjunction

$$D(\mathcal{S}^{\mathcal{K}})^+ \underset{\text{id}}{\overset{\text{id}}{\rightleftarrows}} D\mathcal{S}^{\mathcal{K}},$$

which passes from the positive projective $\mathcal{K}$-model structure to the projective $\mathcal{K}$-model structure on $D\mathcal{S}^{\mathcal{K}}$, is a Quillen equivalence by [SS12, Proposition 9.8]. Further, the map ϵ of operads in spaces from $\mathcal{D}$ to the commutativity operad $\mathcal{C}$ induces a Quillen equivalence

$$D(\mathcal{S}^{\mathcal{K}})^+ \underset{\epsilon^*}{\overset{\epsilon_*}{\rightleftarrows}} C\mathcal{S}^{\mathcal{K}}$$

where the categories $D\mathcal{S}^{\mathcal{K}}$ and $C\mathcal{S}^{\mathcal{K}}$ are endowed with the respective positive projective $\mathcal{K}$-model structures (see [SS12, §9.11, Proposition 9.12]). We make use of both results in Section 3.

2.2 Preliminaries on symmetric spectra

In the following we briefly outline some facts about symmetric spectra. We use the general setting of symmetric spectra as introduced in [Hov01, §7-§9]. For a summary see also [RS17, §2].

Let $(\mathcal{C}, \otimes, \mathbf{1}_{\mathcal{C}})$ be a bicomplete closed symmetric monoidal category. The category Σ of finite sets and bijections is a subcategory of the category $\mathcal{I}$ and inherits a strict symmetric monoidal structure from the latter. A symmetric sequence in the category $\mathcal{C}$ is a functor $X \colon \Sigma \to \mathcal{C}$. The category of symmetric sequences is the functor category $\mathcal{C}^{\Sigma}$ [Hov01, Definition 7.1]. The category $\mathcal{C}^{\Sigma}$ inherits a closed symmetric monoidal structure from $\mathcal{C}$. For X and Y in $\mathcal{C}^{\Sigma}$, the monoidal product $X \odot Y$ is given by

$$(X \odot Y)(n) = \coprod_{p+q=n} \Sigma_n \times_{\Sigma_p \times \Sigma_q} X(p) \otimes Y(q).$$

The $\mathcal{C}$-enriched hom of X and Y in $\mathcal{C}^{\Sigma}$ is determined by

$$\mathrm{Hom}_{\mathcal{C}}^{\mathcal{C}^{\Sigma}}(X, Y) = \prod_{n \geq 0} \underline{\mathrm{Hom}}_{\mathcal{C}^{\Sigma_n}}(X(n), Y(n)).$$

Let L be an object in $\mathcal{C}$. The symmetric sequence $\mathrm{Sym}(L)$, specified in the nth level by $L^{\otimes n}$ equipped with the permutation action, is a commutative monoid in $\mathcal{C}^{\Sigma}$ (see [Hov01, p. 104]).

Definition 2.12. [Hov01, Definition 7.2] The category of *symmetric spectra* in the category $\mathcal{C}$ with respect to the object L, denoted by $\mathrm{Sp}^{\Sigma}(\mathcal{C}, L)$, is the category of right

$\mathrm{Sym}(L)$-modules in $\mathcal{C}^\Sigma$. That is, a symmetric spectrum X is a sequence of Σ_n-objects $X(n)$ in $\mathcal{C}$ together with Σ_n-equivariant maps $\sigma_{n,1}\colon X(n) \otimes L \to X(n+1)$ for all $n \geq 0$ such that the composites

$$\begin{tikzcd} X(n) \otimes L^{\otimes p} \arrow[r, "\sigma_{n,1} \otimes \mathrm{id}_{L^{\otimes p-1}}"] \arrow[rrr, bend left, "\sigma_{n,p}"] & X(n+1) \otimes L^{\otimes p-1} \arrow[r] & \dots \arrow[r] & X(n+p) \end{tikzcd}$$

are $\Sigma_n \times \Sigma_p$-equivariant for all $n, p \geq 0$. Morphisms in $\mathrm{Sp}^\Sigma(\mathcal{C}, L)$ are morphisms of symmetric sequences that are compatible with the right $\mathrm{Sym}(L)$-module structure.

For $m \geq 0$, the evaluation functor $\mathrm{Ev}_m\colon \mathrm{Sp}^\Sigma(\mathcal{C}, L) \to \mathcal{C}$ takes a symmetric spectrum X to its mth level $X(m)$ in $\mathcal{C}$. This functor has a left adjoint $F_m\colon \mathcal{C} \to \mathrm{Sp}^\Sigma(\mathcal{C}, L)$ such that $F_m(T)(n)$ is the initial object in $\mathcal{C}$ if $n \leq m-1$, and

$$F_m(T)(n) = \Sigma_n \times_{\Sigma_{n-m}} T \otimes L^{\otimes n-m}$$

if $n \geq m$ [Hov01, Definition 7.3]. Note that $F_0(\mathbf{1}_\mathcal{C}) = \mathrm{Sym}(L)$ (see [Hov01, p. 105]).

The category $\mathrm{Sp}^\Sigma(\mathcal{C}, L)$ is a closed symmetric monoidal category $(\mathrm{Sp}^\Sigma(\mathcal{C}, L), \wedge, \mathrm{Sym}(L))$, which is enriched, tensored and cotensored over the category $\mathcal{C}$. For X and Y in $\mathrm{Sp}^\Sigma(\mathcal{C}, L)$, the smash product $\wedge$ of X and Y is the symmetric spectrum $X \wedge Y$ defined as the coequalizer of

$$X \odot \mathrm{Sym}(L) \odot Y \rightrightarrows X \odot Y.$$

Here one map is induced by the right action of $\mathrm{Sym}(L)$ on X, and the other map is given by first applying the twist map in the symmetric monoidal structure on $\mathcal{C}^\Sigma$ and then employing the right action of $\mathrm{Sym}(L)$ on Y (see [Hov01, p. 105]). We write $C(\mathrm{Sp}^\Sigma(\mathcal{C}, L))$ for the category of commutative monoids in $(\mathrm{Sp}^\Sigma(\mathcal{C}, L), \wedge, \mathrm{Sym}(L))$. The $\mathcal{C}$-enriched hom of X and Y in $\mathrm{Sp}^\Sigma(\mathcal{C}, L)$ is the object $\mathrm{Hom}_\mathcal{C}^{\mathrm{Sp}^\Sigma(\mathcal{C},L)}(X, Y)$ in $\mathcal{C}$, described by the equalizer of

$$\mathrm{Hom}_\mathcal{C}^{\mathcal{C}^\Sigma}(X, Y) \rightrightarrows \mathrm{Hom}_\mathcal{C}^{\mathcal{C}^\Sigma}(X \odot \mathrm{Sym}(L), Y)$$

where one morphism executes the right action of $\mathrm{Sym}(L)$ on X and the other morphism implements the right action of $\mathrm{Sym}(L)$ on Y. For X in $\mathrm{Sp}^\Sigma(\mathcal{C}, L)$ and T in $\mathcal{C}$, the tensor of X over T, denoted by $X \wedge T$, is specified by $X \wedge F_0(T)$, which in spectrum degree n is given by $(X \wedge F_0(T))(n) = X(n) \otimes T$. The cotensor of X over T is defined by $\underline{\mathrm{Hom}}_{\mathrm{Sp}^\Sigma(\mathcal{C},L)}(F_0(T), X)$ that in spectrum degree n is given by $\underline{\mathrm{Hom}}_{\mathrm{Sp}^\Sigma(\mathcal{C},L)}(F_0(T), X)(n) = \underline{\mathrm{Hom}}_\mathcal{C}(T, X(n))$ (see [Hov01, p. 105]).

If the category $\mathcal{C}$ is left proper and cellular, and the object L is cofibrant in $\mathcal{C}$, then the category $\mathrm{Sp}^\Sigma(\mathcal{C}, L)$ has a projective level model structure. In this model structure the weak equivalences/ fibrations are levelwise weak equivalences/ fibrations in $\mathcal{C}$, and the cofibrations are determined by the left lifting property with respect to the class of acyclic fibrations [Hov01, Theorem 8.2]. The projective level model structure on $\mathrm{Sp}^{\Sigma}(\mathcal{C}, L)$ has a left Bousfield localization with respect to the set of maps

$$\{F_{n+1}(A^{\mathrm{cof}} \otimes L) \xrightarrow{\zeta_n^{A^{\mathrm{cof}}}} F_n(A^{\mathrm{cof}}), n \geq 0, A \in \{V, W : V \to W \in I_{\mathcal{C}}\}\}. \tag{2.6}$$

Here $(-)^{\mathrm{cof}}$ denotes a cofibrant replacement in $\mathcal{C}$, the object A runs through the domains and codomains of the generating cofibrations $I_{\mathcal{C}}$ of $\mathcal{C}$, and the map $\zeta_n^{A^{\mathrm{cof}}}$ is the adjoint of the map

$$A^{\mathrm{cof}} \otimes L \to F_n(A^{\mathrm{cof}})(n+1) = \Sigma_{n+1} \times A^{\mathrm{cof}} \otimes L$$

corresponding to the identity element of Σ_{n+1}. The localized model structure is called the projective stable model structure [Hov01, Definition 8.7].

Proposition 2.13. *[Hov01, Theorem 8.11] The projective stable model structure makes* $\mathrm{Sp}^{\Sigma}(\mathcal{C}, L)$ *into a symmetric monoidal model category.*

There are positive versions of both model structures which are necessary to right induce the respective model structures on $C(\mathrm{Sp}^{\Sigma}(\mathcal{C}, L))$. In the positive projective level model structure on $\mathrm{Sp}^{\Sigma}(\mathcal{C}, L)$ the weak equivalences/ fibrations are levelwise weak equivalences/ fibrations in $\mathcal{C}$ for positive levels, and the cofibrations are again specified by the left lifting property with respect to the class of acyclic fibrations. The positive cofibrations are precisely those cofibrations in the projective level model structure on $\mathrm{Sp}^{\Sigma}(\mathcal{C}, L)$ which are isomorphisms in spectrum level zero (see [RS17, p. 2018], compare [MMSS01, Theorem 14.1]). We adapt the localizing set (2.6) by taking only positive n into account, and form the left Bousfield localization of the positive projective level model structure to obtain the positive projective stable model structure on $\mathrm{Sp}^{\Sigma}(\mathcal{C}, L)$. We refer to [HSS00], [MMSS01], [Shi04], [Shi07], [RS17] and [PS18] in the case that $\mathcal{C}$ is the category of pointed simplicial sets, simplicial k-modules or (non-negative) chain complexes.

2.3 Commutative Hk-algebra spectra are Quillen equivalent to E_∞ dgas

In this subsection we give a short review of the chain of Quillen equivalences connecting commutative Hk-algebra spectra with E_∞ differential graded k-algebras. For more details see [Shi07] and [RS17].

In [SS03b] Schwede and Shipley prove the following theorem.

Theorem 2.14. *[SS03b, Theorem 5.1.6] There is a chain of Quillen equivalences connecting Hk-modules in* $\mathrm{Sp}^{\Sigma}(\mathcal{S}_*, S^1)$ *to unbounded chain complexes.*

In [Shi07] Shipley shows a structured version of this result.

Theorem 2.15. *[Shi07, Theorem 1.1] There is a chain of Quillen equivalences relating Hk-algebra spectra to unbounded differential graded k-algebras.*

To prove this, Shipley establishes the following chain of Quillen equivalences (2.7) between Hk-modules Hk-mod and unbounded chain complexes $\mathrm{Ch}(k)$, which is different to Schwede and Shipley's chain of Quillen equivalences in [SS03b, §B.1],

$$\begin{tikzcd}
Hk\text{-mod} \arrow[r, shift left, "\tilde{k}(-)\wedge_{\tilde{k}(Hk)}\mathrm{Sym}(\tilde{k}(S^1))"] & \mathrm{Sp}^\Sigma(s(k\text{-mod}), \tilde{k}(S^1)) \arrow[l, shift left, "U"] \arrow[r, shift right, "\Phi^*\circ N"'] & \mathrm{Sp}^\Sigma(\mathrm{ch}(k), \mathbb{S}^1(k)) \arrow[l, shift right, "L"'] \arrow[d, shift right, "i"'] \\
& \mathrm{Ch}(k) \arrow[r, shift left, "F_0"] & \mathrm{Sp}^\Sigma(\mathrm{Ch}(k), \mathbb{S}^1(k)) \arrow[l, shift left, "\mathrm{Ev}_0"] \arrow[u, shift right, "C_0"']
\end{tikzcd} \tag{2.7}$$

(see [Shi07, p. 357, Proposition 2.10, Proposition 4.9]). The category of Hk-modules in $\mathrm{Sp}^\Sigma(\mathcal{S}_*, S^1)$ carries the right-induced projective stable model structure, created by the forgetful functor to the projective stable model structure on $\mathrm{Sp}^\Sigma(\mathcal{S}_*, S^1)$ (see [SS00, Theorem 4.1.(2)]). The categories $\mathrm{Sp}^\Sigma(s(k\text{-mod}), \tilde{k}(S^1))$, $\mathrm{Sp}^\Sigma(\mathrm{ch}(k), \mathbb{S}^1(k))$ and $\mathrm{Sp}^\Sigma(\mathrm{Ch}(k), \mathbb{S}^1(k))$ are equipped with the respective projective stable model structures. The category of unbounded chain complexes $\mathrm{Ch}(k)$ has the projective model structure [Hov99, Theorem 2.3.11].
Let $\tilde{k}\colon \mathcal{S}_* \to s(k\text{-mod})$ be the functor such that $\tilde{k}(X)[n]$ is the free k-module on the non-basepoint simplices in $X[n]$. Applying the functor $\tilde{k}$ to each spectrum level of an Hk-module in $\mathrm{Sp}^\Sigma(\mathcal{S}_*, S^1)$ produces a $\tilde{k}(Hk)$-module in $\mathrm{Sp}^\Sigma(s(k\text{-mod}), \tilde{k}(S^1))$. In the Quillen equivalence $(\tilde{k}(-) \wedge_{\tilde{k}(Hk)} \mathrm{Sym}(\tilde{k}(S^1)), U)$ of (2.7), the functor U denotes the forgetful functor, and the underlying symmetric spectrum in pointed simplicial sets of $\mathrm{Sym}(\tilde{k}(S^1))$ is Hk (see [Shi07, pp. 357-358, p. 372]).
The subsequent Quillen equivalence $(L, \Phi^* \circ N)$ in (2.7) is an extension of the classical Dold-Kan correspondence between simplicial k-modules and non-negative chain complexes,

$$s(k\text{-mod}) \underset{N}{\overset{\Gamma}{\leftrightarrows}} \mathrm{ch}(k).$$

Since the category of symmetric sequences in k-modules $(k\text{-mod})^\Sigma$ is an abelian category, applying the normalization functor N levelwise, yields an equivalence of categories

$$(s(k\text{-mod}))^\Sigma \underset{N}{\overset{\Gamma}{\leftrightarrows}} (\mathrm{ch}(k))^\Sigma$$

[Ric15, Proposition 4.3]. The normalization functor N is lax symmetric monoidal (see [Mac63, Corollary VIII.8.9]). Hence, there is a map

$$\mathrm{Sym}(\mathbb{S}^1(k)) \xrightarrow{\Phi} N(\mathrm{Sym}(\tilde{k}(S^1)))$$

in $C(\mathrm{Sp}^\Sigma(\mathrm{ch}(k), \mathbb{S}^1(k)))$ which is induced by the shuffle transformation. We obtain a functor

$$\Phi^* \circ N\colon \mathrm{Sp}^\Sigma(s(k\text{-mod}), \tilde{k}(S^1)) \to \mathrm{Sp}^\Sigma(\mathrm{ch}(k), \mathbb{S}^1(k)),$$

where Φ^* is the associated change-of-rings morphism (see [Shi07, p. 358]). But the functor Γ is not lax *symmetric* monoidal because the Alexander-Whitney map is not

symmetric (see [SS03a, §2.3]). Schwede and Shipley explain in general how to produce a left adjoint on the categories of monoids and modules when given a Quillen adjunction between the underlying categories with a lax (symmetric) monoidal structure on the right adjoint plus some other assumptions (see [SS03a, §3.3]). In this way, the functor $\Phi^* \circ N$ possesses a left adjoint denoted by L.
The inclusion functor $i \colon \mathrm{ch}(k) \to \mathrm{Ch}(k)$ from non-negative chain complexes to unbounded chain complexes, whose right adjoint C_0 is the good truncation functor, induces the Quillen equivalence (i, C_0) in (2.7). The remaining Quillen equivalence in (2.7) is given by the adjoint pair (F_0, Ev_0).

Moreover, Richter and Shipley extend Shipley's result [Theorem 2.15] in the following sense.

Theorem 2.16. *[RS17, Corollary 8.3] There is a chain of Quillen equivalences between commutative Hk-algebra spectra and E_∞ differential graded k-algebras.*

The following diagram displays Richter and Shipley's chain of Quillen equivalences

$$
\begin{tikzcd}
C(Hk\text{-mod}) \arrow[r, shift left, "\tilde{k}(-)\wedge_{\tilde{k}(Hk)}\mathrm{Sym}(\tilde{k}(S^1))"] & C(\mathrm{Sp}^\Sigma(s(k\text{-mod}), \tilde{k}(S^1))) \arrow[l, shift left, "U"] \arrow[d, shift left, "\Phi^*\circ N"] \\
C(\mathrm{Sp}^\Sigma(\mathrm{Ch}(k), \mathbb{S}^1(k))) \arrow[r, shift left, "C_0"'] \arrow[d, shift left, "\epsilon^*"] & C(\mathrm{Sp}^\Sigma(\mathrm{ch}(k), \mathbb{S}^1(k))) \arrow[l, shift left, "i"'] \arrow[u, shift left, "L_N"] \\
E_\infty(\mathrm{Sp}^\Sigma(\mathrm{Ch}(k), \mathbb{S}^1(k)))^{s,+} \arrow[r, shift left, "\mathrm{id}"] \arrow[u, shift left, "\epsilon_*"] & E_\infty(\mathrm{Sp}^\Sigma(\mathrm{Ch}(k), \mathbb{S}^1(k)))^{s} \arrow[l, shift left, "\mathrm{id}"] \arrow[d, shift left, "\mathrm{Ev}_0"] \\
 & E_\infty \mathrm{Ch}(k) \arrow[u, shift left, "F_0"]
\end{tikzcd}
\tag{2.8}
$$

(see [RS17, Theorem 3.3, Theorem 4.1, Theorem 6.6, Corollary 7.3, Proposition 8.1, Theorem 8.2]). Here Richter and Shipley fix a cofibrant E_∞ operad in chain complexes (see [RS17, p. 2031]). The categories $C(Hk\text{-mod})$, $C(\mathrm{Sp}^\Sigma(s(k\text{-mod}), \tilde{k}(S^1)))$, $C(\mathrm{Sp}^\Sigma(\mathrm{ch}(k), \mathbb{S}^1(k)))$, $C(\mathrm{Sp}^\Sigma(\mathrm{Ch}(k), \mathbb{S}^1(k)))$ and $E_\infty(\mathrm{Sp}^\Sigma(\mathrm{Ch}(k), \mathbb{S}^1(k)))^{s,+}$ carry the right-induced positive projective stable model structures so that the forgetful functor to the positive projective stable model structures on the respective underlying categories of symmetric spectra determines the weak equivalences and fibrations [RS17, Theorem 3.1]. The category $E_\infty(\mathrm{Sp}^\Sigma(\mathrm{Ch}(k), \mathbb{S}^1(k)))^{s}$ is endowed with the right-induced projective stable model structure (see [Spi01, Theorem 4.3]), and the category $E_\infty\mathrm{Ch}(k)$ has the right-induced projective model structure created by the forgetful functor to the projective model structure on $\mathrm{Ch}(k)$ (see [Spi01, Theorem 4.3], [BM03, Theorem 3.1, Example 3.3.3]).
Note that the functor L_N in (2.8) is not equal to the functor L in (2.7). The functor L_N is constructed in [Ric15, Lemma 6.4], again using the general machinery of [SS03a, §3.3]. The map ϵ of operads in chain complexes from the fixed E_∞ operad to the commutativity operad gives rise to the Quillen equivalence (ϵ_*, ϵ^*) in (2.8). The adjacent Quillen equivalence in (2.8) passes from the right-induced positive projective stable model structure to

the right-induced projective stable model structure on $E_\infty(\mathrm{Sp}^\Sigma(\mathrm{Ch}(k), \mathbb{S}^1(k)))$.
A guiding example is the commutative Hk-algebra spectrum $F(X_+, Hk)$ (see [Scha, Example I.3.6, Example I.3.46],[Ric, §3.2], [RS17, p. 2013]) which under the chain of Quillen equivalences (2.8) corresponds to a chain model of the singular cochains on the space X with coefficients in k (see [RS17, p. 2013]). The E_∞ structure on the latter is parametrized by the Barratt-Eccles operad in chain complexes (see [BF04, §1.1, Theorem 2.1.1]). The homotopy groups of the function spectrum $F(X_+, Hk)$ are isomorphic to the cohomology groups of the space X with coefficients in k,

$$\pi_*(F(X_+, Hk)) \cong H^{-*}(X, k). \tag{2.9}$$

We employ later the intermediate category $C(\mathrm{Sp}^\Sigma(s(k\text{-mod}), \tilde{k}(S^1)))$ in (2.8) as well as the Quillen equivalence

$$\mathrm{Sp}^\Sigma(s(k\text{-mod}), \tilde{k}(S^1)) \underset{\Phi^* N}{\overset{L}{\rightleftarrows}} \mathrm{Sp}^\Sigma(\mathrm{ch}(k), \mathbb{S}^1(k))$$

in (2.7) to develop a notion of pre-log structures on E_∞ differential graded k-algebras.

From now on we abbreviate the category k-mod by mod, the category $s(k\text{-mod})$ by $s$mod, the category $\mathrm{ch}(k)$ by ch and the category $\mathrm{Ch}(k)$ by Ch. In addition, we denote the category $\mathrm{Sp}^\Sigma(\mathcal{S}_*, S^1)$ by Sp^Σ, the category $\mathrm{Sp}^\Sigma(s(k\text{-mod}), \tilde{k}(S^1))$ by $\mathrm{Sp}^\Sigma(s\mathrm{mod})$, and the category $\mathrm{Sp}^\Sigma(\mathrm{ch}(k), \mathbb{S}^1(k))$ by $\mathrm{Sp}^\Sigma(\mathrm{ch})$. Further, to ease notation we write $\mathbb{S}^m$ for the m-sphere chain complex $\mathbb{S}^m(k)$ for $m \in \mathbb{Z}$.

2.4 Symmetric spectra and $\bar{\mathcal{J}}$-spaces

Let $(\mathcal{C}, \otimes, \mathbf{1}_\mathcal{C})$ be a bicomplete closed symmetric monoidal category, and let L be an object in $\mathcal{C}$.

The category Sp^Σ can be viewed as diagram spectra with respect to the category of finite sets and bijections Σ [MMSS01, Example 4.2]. Recall that the category Σ is a subcategory of the category $\mathcal{I}$. The relation of the latter category and the category Sp^Σ is discussed in [SS12, §3.16], [Sch09, §3.1] and [Scha, I.§3.4]. Exploiting this, Sagave and Schlichtkrull define a strong symmetric monoidal functor $F_-(S^-)\colon \mathcal{J}^{\mathrm{op}} \to \mathrm{Sp}^\Sigma$ which sends $(\mathbf{m}_1, \mathbf{m}_2)$ to $F_{m_1}(S^{m_2})$ [SS12, Lemma 4.22]. Generalizing this definition, we can replace the category Sp^Σ by the category $\mathrm{Sp}^\Sigma(\mathcal{C}, L)$, so that we obtain the functor $F_-(L^{\otimes -})\colon \mathcal{J}^{\mathrm{op}} \to \mathrm{Sp}^\Sigma(\mathcal{C}, L)$ that takes $(\mathbf{m}_1, \mathbf{m}_2)$ to $F_{m_1}(L^{\otimes m_2})$. As an example, we investigate the functor $F_-(\mathbb{S}^-)\colon \mathcal{J}^{\mathrm{op}} \to \mathrm{Sp}^\Sigma(\mathrm{ch})$ and explain that the latter does factor through the projection $\mathcal{J}^{\mathrm{op}} \to \bar{\mathcal{J}}^{\mathrm{op}}$. For the functor $F_-(S^-)\colon \mathcal{J}^{\mathrm{op}} \to \mathrm{Sp}^\Sigma$, this is false. The reason for this is that while the action of the symmetric group Σ_n on the pointed simplicial set $\mathrm{Sym}(S^1)(n) = S^n$ permutes coordinates, the action of Σ_n on the chain complex $\mathrm{Sym}(\mathbb{S}^1)(n) = \mathbb{S}^n$ is only the sign action.

We begin with considering in the generalized setting the morphisms introduced by

Sagave and Schlichtkrull in [SS12, §3.16]. In the example where $\mathcal{C}$ is the category of chain complexes and L is the one-sphere chain complex $\mathbb{S}^1$, we point out that these morphisms are compatible with the equivalence relation imposed on the morphisms sets of the category $\mathcal{I}$ (see Subsection 1.1). Afterwards, we specify the functor $F_{-}(L^{\otimes -})\colon \mathcal{J}^{\mathrm{op}} \to \mathrm{Sp}^{\Sigma}(\mathcal{C}, L)$ and the induced functor $F_{-}(\mathbb{S}^{-})\colon \bar{\mathcal{J}}^{\mathrm{op}} \to \mathrm{Sp}^{\Sigma}(\mathrm{ch})$ and show that they are strong symmetric monoidal. With the help of the functor $F_{-}(\mathbb{S}^{-})\colon \bar{\mathcal{J}}^{\mathrm{op}} \to \mathrm{Sp}^{\Sigma}(\mathrm{ch})$, we build the Quillen adjunction $(\Lambda^{\bar{\mathcal{J}}}, \Omega^{\bar{\mathcal{J}}})$ between the model categories $C\mathcal{S}^{\bar{\mathcal{J}}}$ and $C(\mathrm{Sp}^{\Sigma}(s\mathrm{mod}))$, on which our definition of pre-log structures on E_∞ differential graded k-algebras is based.

For a finite set Z, we use the notation $L^{\otimes Z}$ for the $|Z|$-fold monoidal product of the object L in order to keep track of the different copies of L. If $Z = \emptyset$, we follow the convention that $L^{\otimes \emptyset} = \mathbf{1}_{\mathcal{C}}$.

Lemma 2.17. *(i) Let $\alpha\colon \mathbf{m} \to \mathbf{n}$ be a map in the category $\mathcal{I}$. The canonical extension of the map α to a bijection $(\alpha, \mathrm{incl})\colon \mathbf{m} \sqcup (\mathbf{n} \backslash \mathrm{im}(\alpha)) \to \mathbf{n}$ gives rise to an isomorphism*

$$L^{\otimes \mathbf{m}} \otimes L^{\otimes \mathbf{n}\backslash \mathrm{im}(\alpha)} \xrightarrow{(\alpha,\mathrm{incl})_*} L^{\otimes \mathbf{n}} \tag{2.10}$$

in $\mathcal{C}$. If α is an element in $\mathcal{I}(\mathbf{n}, \mathbf{n}) = \Sigma_n$, this specifies the usual Σ_n-action on $L^{\otimes \mathbf{n}}$. Plugging in the chain complex $\mathbb{S}^1$ for L, the equivalence class of α, which is $[\alpha]$ in the category $\bar{\mathcal{I}}$, induces an isomorphism

$$\mathbb{S}^{\mathbf{m}} \otimes \mathbb{S}^{\mathbf{n}\backslash \mathrm{im}(\alpha)} \xrightarrow{[(\alpha,\mathrm{incl})]_*} \mathbb{S}^{\mathbf{n}} \tag{2.11}$$

in chain complexes.

(ii) For a pair of morphisms $\alpha\colon \mathbf{l} \to \mathbf{m}$ and $\beta\colon \mathbf{m} \to \mathbf{n}$ in $\mathcal{I}$, there is a canonical bijection $(\beta, \mathrm{incl})\colon (\mathbf{m} \setminus \mathrm{im}(\alpha)) \sqcup (\mathbf{n} \setminus \mathrm{im}(\beta)) \to \mathbf{n} \setminus \mathrm{im}(\beta \circ \alpha)$. This leads to an isomorphism

$$L^{\otimes \mathbf{m}\backslash \mathrm{im}(\alpha)} \otimes L^{\otimes \mathbf{n}\backslash \mathrm{im}(\beta)} \xrightarrow{(\beta,\mathrm{incl})_*} L^{\otimes \mathbf{n}\backslash \mathrm{im}(\beta\circ\alpha)} \tag{2.12}$$

in $\mathcal{C}$. Inserting the chain complex $\mathbb{S}^1$ for L, and passing to the equivalence classes $[\alpha]$ and $[\beta]$ in $\bar{\mathcal{I}}$, we get an induced isomorphism

$$\mathbb{S}^{\mathbf{m}\backslash \mathrm{im}(\alpha)} \otimes \mathbb{S}^{\mathbf{n}\backslash \mathrm{im}(\beta)} \xrightarrow{[(\beta,\mathrm{incl})]_*} \mathbb{S}^{\mathbf{n}\backslash \mathrm{im}(\beta\circ\alpha)} \tag{2.13}$$

in chain complexes.

(iii) For morphisms $\alpha\colon \mathbf{m} \to \mathbf{n}$ and $\beta\colon \mathbf{p} \to \mathbf{q}$ in $\mathcal{I}$, there is a canonical identification of $(\mathbf{n} \sqcup \mathbf{q}) \setminus \mathrm{im}(\alpha \sqcup \beta)$ with $(\mathbf{n} \setminus \mathrm{im}(\alpha)) \sqcup (\mathbf{q} \setminus \mathrm{im}(\beta))$, and an associated isomorphism is

$$L^{\otimes (\mathbf{n}\sqcup\mathbf{q})\backslash \mathrm{im}(\alpha\sqcup\beta)} \cong L^{\otimes \mathbf{n}\backslash \mathrm{im}(\alpha)} \otimes L^{\otimes \mathbf{q}\backslash \mathrm{im}(\beta)} \tag{2.14}$$

in $\mathcal{C}$. Plugging in the chain complex $\mathbb{S}^1$ for L, we obtain for the corresponding equivalence classes $[\alpha]$ and $[\beta]$, an isomorphism

$$\mathbb{S}^{(\mathbf{n}\sqcup\mathbf{q})\backslash \mathrm{im}(\alpha\sqcup\beta)} \cong \mathbb{S}^{\mathbf{n}\backslash \mathrm{im}(\alpha)} \otimes \mathbb{S}^{\mathbf{q}\backslash \mathrm{im}(\beta)} \tag{2.15}$$

in chain complexes.

Proof. (i) For the isomorphism (2.10) compare [SS12, §3.16 (3.3)]. To see that the isomorphism (2.11) is well-defined, let α be equivalent to α', meaning that there is a σ in A_n such that $\alpha = \sigma \circ \alpha'$. As $\mathrm{sgn}(\sigma) = 1$, the induced map $\sigma_* \colon \mathbb{S}^{\mathbf{n}\setminus \mathrm{im}(\alpha')} \to \mathbb{S}^{\mathbf{n}\setminus \mathrm{im}(\alpha)}$ is the identity. Hence, we can identify the map $(\alpha, \mathrm{incl})_*$ with $(\alpha', \mathrm{incl})_*$.

(ii) For the isomorphism (2.12) compare [SS12, §3.16 (3.4)]. The isomorphism (2.13) is well-defined, because if α is equivalent to α' and β is equivalent to β', we can identify $\mathbb{S}^{\mathbf{m}\setminus \mathrm{im}(\alpha)}$ with $\mathbb{S}^{\mathbf{m}\setminus \mathrm{im}(\alpha')}$, $\mathbb{S}^{\mathbf{n}\setminus \mathrm{im}(\beta)}$ with $\mathbb{S}^{\mathbf{n}\setminus \mathrm{im}(\beta')}$ and $\mathbb{S}^{\mathbf{n}\setminus \mathrm{im}(\beta\circ\alpha)}$ with $\mathbb{S}^{\mathbf{n}\setminus \mathrm{im}(\beta'\circ\alpha')}$ as in part (i).

(iii) For the isomorphism (2.14) compare [SS12, §3.16 (3.5)]. The same argument as in part (i) ensures that the isomorphism (2.15) is independent of the choice of representatives of the respective equivalence classes.

□

Let X be an object in $\mathrm{Sp}^\Sigma(\mathcal{C}, L)$. For a map $\alpha \colon \mathbf{m} \to \mathbf{n}$ in $\mathcal{I}$, there is an induced structure map $\alpha_* \colon X(m) \otimes L^{\otimes \mathbf{n}\setminus \mathrm{im}(\alpha)} \to X(n)$ which is determined as follows. We choose a bijection $\beta \colon \mathbf{l} \to \mathbf{n} \setminus \mathrm{im}(\alpha)$ for an object $\mathbf{l}$ in $\mathcal{I}$, so that we get a bijection

$$\mathbf{m} \sqcup \mathbf{l} \xrightarrow{\mathrm{id}_{\mathbf{m}} \sqcup \beta} \mathbf{m} \sqcup (\mathbf{n} \setminus \mathrm{im}(\alpha)) \xrightarrow{(\alpha,\mathrm{incl})} \mathbf{n}.$$

This gives rise to the structure map α_*

$$\begin{tikzcd}
X(m) \otimes L^{\otimes \mathbf{n}\setminus \mathrm{im}(\alpha)} \arrow[r, "\mathrm{id}_{X(m)} \otimes (\beta_*)^{-1}"] \arrow[rrr, bend left, "\alpha_*"] & X(m) \otimes L^{\otimes \mathbf{l}} \arrow[r, "\sigma_{m,l}"] & X(m+l) \arrow[r, "((\alpha,\mathrm{incl})\circ(\mathrm{id}_{\mathbf{m}} \sqcup \beta))_*"] & X(n),
\end{tikzcd}$$

which does not depend on the choice of the map β. In this way, the standard inclusion $\iota_{\mathbf{m},\mathbf{m}\sqcup\mathbf{1}} \colon \mathbf{m} \to \mathbf{m} \sqcup \mathbf{1}$ induces the structure map $\sigma_{m,1} \colon X(m) \otimes L \to X(m+1)$ and the automorphisms of $\mathbf{m}$ yield the Σ_m-action on $X(m)$. For more details on this viewpoint of the category $\mathrm{Sp}^\Sigma(\mathcal{C}, L)$ compare [SS12, p. 2129] and [Sch09, §3.1].

Let $m \geq 0$. Recall the functor $F_m \colon \mathcal{C} \to \mathrm{Sp}^\Sigma(\mathcal{C}, L)$ from Subsection 2.2. Observing that the morphism set $\mathcal{I}(\mathbf{m}, \mathbf{n})$ is isomorphic to Σ_n/Σ_{n-m}, we notice that for an object C in $\mathcal{C}$, the object $F_m(C)(n)$ is isomorphic to $\coprod_{\alpha \in \mathcal{I}(\mathbf{m},\mathbf{n})} C \otimes L^{\otimes \mathbf{n}\setminus \mathrm{im}(\alpha)}$. A morphism $\beta \colon \mathbf{n} \to \mathbf{p}$ in $\mathcal{I}$ induces the structure map $\beta_* \colon F_m(C)(n) \otimes L^{\otimes \mathbf{p}\setminus \mathrm{im}(\beta)} \to F_m(C)(p)$. This sends a (coproduct) summand indexed by the map $\alpha \colon \mathbf{m} \to \mathbf{n}$ to the summand indexed by the composite map $\beta \circ \alpha$, by using the isomorphism (2.12) in Lemma 2.17 (ii),

$$C \otimes L^{\otimes \mathbf{n}\setminus \mathrm{im}(\alpha)} \otimes L^{\otimes \mathbf{p}\setminus \mathrm{im}(\beta)} \xrightarrow{\mathrm{id}_C \otimes (\beta,\mathrm{incl})_*} C \otimes L^{\otimes \mathbf{p}\setminus \mathrm{im}(\beta\circ\alpha)}.$$

Moreover, let m, $\tilde{m} \geq 0$, and let C and $\tilde{C}$ be objects in $\mathcal{C}$. There is a natural isomorphism

$$F_m(C) \wedge F_{\tilde{m}}(\tilde{C}) \xrightarrow{\cong} F_{m+\tilde{m}}(C \otimes \tilde{C}) \tag{2.16}$$

(see [Hov01, p. 105]). This isomorphism (2.16) can be made explicit by exploiting an alternative description of the smash product $\wedge$ and the isomorphisms (2.12) in Lemma 2.17(ii) and (2.14) in Lemma 2.17(iii). In general, for a pair of symmetric spectra X and Y in $\mathrm{Sp}^\Sigma(\mathcal{C}, L)$, the smash product $\wedge$ of X and Y in level n can be written as

$$(X \wedge Y)(n) = \operatorname{colim}_{\alpha\colon \mathbf{p}\sqcup\mathbf{q}\to\mathbf{n}} X(p) \otimes Y(q) \otimes L^{\otimes \mathbf{n}\setminus \mathrm{im}(\alpha)}$$

where the colimit is taken over the comma category $(- \sqcup - \downarrow \mathbf{n})$ (compare [SS12, p. 2130], [Sch09, p. 710]). Using this, the map (2.16) in level n is given by

$$\operatorname{colim}_{\alpha\colon \mathbf{p}\sqcup\mathbf{q}\to\mathbf{n}} \Big(\coprod_{\beta\in\mathcal{I}(\mathbf{m},\mathbf{p})} C \otimes L^{\otimes\mathbf{p}\setminus\mathrm{im}(\beta)}\Big) \otimes \Big(\coprod_{\tilde\beta\in\mathcal{I}(\tilde{\mathbf{m}},\mathbf{q})} \tilde C \otimes L^{\otimes\mathbf{q}\setminus\mathrm{im}(\tilde\beta)}\Big) \otimes L^{\otimes\mathbf{n}\setminus\mathrm{im}(\alpha)}$$
$$\longrightarrow \coprod_{\gamma\in\mathcal{I}(\mathbf{m}\sqcup\tilde{\mathbf{m}},\mathbf{n})} C \otimes \tilde C \otimes L^{\otimes\mathbf{n}\setminus\mathrm{im}(\gamma)}.$$

Here for each object $((\mathbf{p},\mathbf{q}), \alpha\colon \mathbf{p}\sqcup\mathbf{q}\to\mathbf{n})$ in the comma category $(- \sqcup - \downarrow \mathbf{n})$, the summand indexed by the maps $\beta\colon \mathbf{m}\to\mathbf{p}$ and $\tilde\beta\colon \tilde{\mathbf{m}}\to\mathbf{q}$ is sent to the summand indexed by the composite map

$$\mathbf{m}\sqcup\tilde{\mathbf{m}} \xrightarrow{\beta\sqcup\tilde\beta} \mathbf{p}\sqcup\mathbf{q} \xrightarrow{\alpha} \mathbf{n}$$

via the following composite of isomorphisms

$$C \otimes L^{\otimes\mathbf{p}\setminus\mathrm{im}(\beta)} \otimes \tilde C \otimes L^{\otimes\mathbf{q}\setminus\mathrm{im}(\tilde\beta)} \otimes L^{\otimes\mathbf{n}\setminus\mathrm{im}(\alpha)}$$
$$\xrightarrow{\mathrm{id}_C\otimes\tau_{L^{\otimes\mathbf{p}\setminus\mathrm{im}(\beta)},\tilde C}\otimes\mathrm{id}_{L^{\otimes\mathbf{q}\setminus\mathrm{im}(\tilde\beta)}\otimes L^{\otimes\mathbf{n}\setminus\mathrm{im}(\alpha)}}} C\otimes\tilde C\otimes L^{\otimes\mathbf{p}\setminus\mathrm{im}(\beta)} \otimes L^{\otimes\mathbf{q}\setminus\mathrm{im}(\tilde\beta)} \otimes L^{\otimes\mathbf{n}\setminus\mathrm{im}(\alpha)}$$
$$\longrightarrow C\otimes\tilde C\otimes L^{\otimes(\mathbf{p}\sqcup\mathbf{q})\setminus\mathrm{im}(\beta\sqcup\tilde\beta)} \otimes L^{\otimes\mathbf{n}\setminus\mathrm{im}(\alpha)}$$
$$\xrightarrow{\mathrm{id}_{C\otimes\tilde C}\otimes(\alpha,\mathrm{incl})_*} C\otimes\tilde C\otimes L^{\otimes\mathbf{n}\setminus\mathrm{im}(\alpha\circ(\beta\sqcup\tilde\beta))}$$

where the second isomorphism is induced by the isomorphism (2.14) in Lemma 2.17 (iii). Under the isomorphism (2.16), the symmetry isomorphism

$$F_m(C)\wedge F_{\tilde m}(\tilde C) \xrightarrow{\tau_{F_m(C),F_{\tilde m}(\tilde C)}} F_{\tilde m}(\tilde C)\wedge F_m(C)$$

matches with the map $F_{m+\tilde m}(C\otimes\tilde C) \to F_{\tilde m+m}(\tilde C\otimes C)$ which in spectrum degree n takes a summand indexed by the map $\alpha\colon \mathbf{m}\sqcup\tilde{\mathbf{m}}\to\mathbf{n}$ to the summand indexed by the composite map

$$\tilde{\mathbf{m}}\sqcup\mathbf{m} \xrightarrow{\chi_{\tilde{\mathbf{m}},\mathbf{m}}} \mathbf{m}\sqcup\tilde{\mathbf{m}} \xrightarrow{\alpha} \mathbf{n}$$

via the isomorphism

$$C\otimes\tilde C\otimes L^{\otimes\mathbf{n}\setminus\mathrm{im}(\alpha)} \xrightarrow{\tau_{C,\tilde C}\otimes\mathrm{id}_{L^{\otimes\mathbf{n}\setminus\mathrm{im}(\alpha)}}} \tilde C\otimes C\otimes L^{\otimes\mathbf{n}\setminus\mathrm{im}(\alpha\circ\chi_{\tilde{\mathbf{m}},\mathbf{m}})}$$

(compare [SS12, p. 2130]).

Construction 2.18. We define the functor $F_-(L^{\otimes -})\colon \mathcal{J}^{\mathrm{op}} \to \mathrm{Sp}^\Sigma(\mathcal{C}, L)$ that sends $(\mathbf{m}_1, \mathbf{m}_2)$ to $F_{m_1}(L^{\otimes m_2})$. A morphism $(\alpha_1, \alpha_2, \rho)\colon (\mathbf{m}_1, \mathbf{m}_2) \to (\mathbf{n}_1, \mathbf{n}_2)$ in $\mathcal{J}$ induces a morphism $(\alpha_1, \alpha_2, \rho)^*\colon F_{n_1}(L^{\otimes n_2}) \to F_{m_1}(L^{\otimes m_2})$ which in spectrum degree p,

$$\coprod_{\gamma \in \mathcal{I}(\mathbf{n}_1, \mathbf{p})} L^{\otimes \mathbf{n}_2} \otimes L^{\otimes \mathbf{p} \setminus \mathrm{im}(\gamma)} \xrightarrow{(\alpha_1,\alpha_2,\rho)^*(p)} \coprod_{\delta \in \mathcal{I}(\mathbf{m}_1, \mathbf{p})} L^{\otimes \mathbf{m}_2} \otimes L^{\otimes \mathbf{p} \setminus \mathrm{im}(\delta)},$$

takes a summand $L^{\otimes \mathbf{n}_2} \otimes L^{\otimes \mathbf{p} \setminus \mathrm{im}(\gamma)}$ indexed by the map $\gamma\colon \mathbf{n}_1 \to \mathbf{p}$ to the summand $L^{\otimes \mathbf{m}_2} \otimes L^{\otimes \mathbf{p} \setminus \mathrm{im}(\gamma \circ \alpha_1)}$ indexed by the composite map

$$\mathbf{m}_1 \xrightarrow{\alpha_1} \mathbf{n}_1 \xrightarrow{\gamma} \mathbf{p}.$$

The isomorphism $L^{\otimes \mathbf{n}_2} \otimes L^{\otimes \mathbf{p} \setminus \mathrm{im}(\gamma)} \to L^{\otimes \mathbf{m}_2} \otimes L^{\otimes \mathbf{p} \setminus \mathrm{im}(\gamma \circ \alpha_1)}$ is specified by the following chain of isomorphisms

$$\begin{tikzcd}
L^{\otimes \mathbf{n}_2} \otimes L^{\otimes \mathbf{p} \setminus \mathrm{im}(\gamma)} & L^{\otimes \mathbf{m}_2} \otimes L^{\otimes \mathbf{n}_2 \setminus \mathrm{im}(\alpha_2)} \otimes L^{\otimes \mathbf{p} \setminus \mathrm{im}(\gamma)} \arrow[l, "(\alpha_2, \mathrm{incl})_* \otimes \mathrm{id}_{L^{\otimes \mathbf{p} \setminus \mathrm{im}(\gamma)}}"'] \\
L^{\otimes \mathbf{m}_2} \otimes L^{\otimes \mathbf{p} \setminus \mathrm{im}(\gamma \circ \alpha_1)} & L^{\otimes \mathbf{m}_2} \otimes L^{\otimes \mathbf{n}_1 \setminus \mathrm{im}(\alpha_1)} \otimes L^{\otimes \mathbf{p} \setminus \mathrm{im}(\gamma)}. \arrow[l, "\mathrm{id}_{L^{\otimes \mathbf{m}_2}} \otimes (\gamma, \mathrm{incl})_*"'] \arrow[u, "\mathrm{id}_{L^{\otimes \mathbf{m}_2}} \otimes \rho_* \otimes \mathrm{id}_{L^{\otimes \mathbf{p} \setminus \mathrm{im}(\gamma)}}"']
\end{tikzcd}$$

Due to the adjunction of F_{n_1} to the evaluation functor Ev_{n_1}, the morphism $(\alpha_1, \alpha_2, \rho)^*$ is adjoint to a morphism $L^{\otimes n_2} \to F_{m_1}(L^{\otimes m_2})(n_1)$ that is determined by the composite

$$\begin{tikzcd}
L^{\otimes \mathbf{n}_2} \arrow[rrd, bend right] & L^{\otimes \mathbf{m}_2} \otimes L^{\otimes \mathbf{n}_2 \setminus \mathrm{im}(\alpha_2)} \arrow[l, "(\alpha_2, \mathrm{incl})_*"'] & L^{\otimes \mathbf{m}_2} \otimes L^{\otimes \mathbf{n}_1 \setminus \mathrm{im}(\alpha_1)} \arrow[l, "\mathrm{id}_{L^{\otimes \mathbf{m}_2}} \otimes \rho_*"'] \arrow[d, "\mathrm{incl}"] \\
 & & \coprod_{\delta \in \mathcal{I}(\mathbf{m}_1, \mathbf{n}_1)} L^{\otimes \mathbf{m}_2} \otimes L^{\otimes \mathbf{p} \setminus \mathrm{im}(\delta)}.
\end{tikzcd}$$

Lemma 2.19. *The assignment $F_-(L^{\otimes -})\colon \mathcal{J}^{\mathrm{op}} \to \mathrm{Sp}^\Sigma(\mathcal{C}, L)$ is a functor.*

Proof. (compare [SS12, proof of Lemma 4.22]) The above assignment makes use of the isomorphisms (2.10) in Lemma 2.17 (i) and (2.12) in Lemma 2.17 (ii). It holds that $F_-(L^{\otimes -})(\mathrm{id}_{\mathbf{m}_1}, \mathrm{id}_{\mathbf{m}_2}, \mathrm{id}_\emptyset) = \mathrm{id}_{F_{m_1}(L^{\otimes m_2})}$. Moreover, let

$$(\mathbf{l_1}, \mathbf{l}_2) \xrightarrow{(\alpha_1,\alpha_2,\rho)} (\mathbf{m}_1, \mathbf{m}_2) \xrightarrow{(\beta_1,\beta_2,\phi)} (\mathbf{n}_1, \mathbf{n}_2)$$

be a composable pair of morphisms in $\mathcal{J}$. We need to check that

$$(\alpha_1, \alpha_2, \rho)^* \circ (\beta_1, \beta_2, \phi)^* = (\beta_1 \circ \alpha_1, \beta_2 \circ \alpha_2, \phi \cup (\beta_2 \circ \rho \circ \beta_1^{-1}))^*. \tag{2.17}$$

For this, let p be a spectrum degree. We show that the diagram

$$\begin{tikzcd}
\coprod_{\gamma \in \mathcal{I}(\mathbf{n}_1, \mathbf{p})} L^{\otimes \mathbf{n}_2} \otimes L^{\otimes \mathbf{p} \setminus \mathrm{im}(\gamma)} \arrow[r, "(\beta_1,\beta_2,\phi)^*(p)"] \arrow[dr, "(\beta_1 \circ \alpha_1, \beta_2 \circ \alpha_2, \phi \cup (\beta_2 \circ \rho \circ \beta_1^{-1}))^*(p)"'] & \coprod_{\delta \in \mathcal{I}(\mathbf{m}_1, \mathbf{p})} L^{\otimes \mathbf{m}_2} \otimes L^{\otimes \mathbf{p} \setminus \mathrm{im}(\delta)} \arrow[d, "(\alpha_1,\alpha_2,\rho)^*(p)"] \\
 & \coprod_{\epsilon \in \mathcal{I}(\mathbf{l}_1, \mathbf{p})} L^{\otimes \mathbf{l}_2} \otimes L^{\otimes \mathbf{p} \setminus \mathrm{im}(\epsilon)}
\end{tikzcd}$$

commutes. The morphism $(\beta_1, \beta_2, \phi)^*(p)$ sends a summand $L^{\otimes \mathbf{n}_2} \otimes L^{\otimes \mathbf{p} \setminus \mathrm{im}(\gamma)}$ indexed by the map $\gamma\colon \mathbf{n}_1 \to \mathbf{p}$ to the summand $L^{\otimes \mathbf{m}_2} \otimes L^{\otimes \mathbf{p} \setminus \mathrm{im}(\gamma \circ \beta_1)}$ indexed by the composite map

$$\mathbf{m}_1 \xrightarrow{\beta_1} \mathbf{n}_1 \xrightarrow{\gamma} \mathbf{p}$$

via the following chain of isomorphisms

$$\begin{tikzcd}
L^{\otimes \mathbf{n}_2} \otimes L^{\otimes \mathbf{p} \setminus \mathrm{im}(\gamma)} & L^{\otimes \mathbf{m}_2} \otimes L^{\otimes \mathbf{n}_2 \setminus \mathrm{im}(\beta_2)} \otimes L^{\otimes \mathbf{p} \setminus \mathrm{im}(\gamma)} \arrow[l, "(\beta_2, \mathrm{incl})_* \otimes \mathrm{id}_{L^{\otimes \mathbf{p} \setminus \mathrm{im}(\gamma)}}"'] \\
L^{\otimes \mathbf{m}_2} \otimes L^{\otimes \mathbf{p} \setminus \mathrm{im}(\gamma \circ \beta_1)} & L^{\otimes \mathbf{m}_2} \otimes L^{\otimes \mathbf{n}_1 \setminus \mathrm{im}(\beta_1)} \otimes L^{\otimes \mathbf{p} \setminus \mathrm{im}(\gamma)}. \arrow[l, "\mathrm{id}_{L^{\otimes \mathbf{m}_2}} \otimes (\gamma, \mathrm{incl})_*"'] \arrow[u, "\mathrm{id}_{L^{\otimes \mathbf{m}_2}} \otimes \phi_* \otimes \mathrm{id}_{L^{\otimes \mathbf{p} \setminus \mathrm{im}(\gamma)}}"']
\end{tikzcd} \tag{2.18}$$

The morphism $(\alpha_1, \alpha_2, \rho)^*(p)$ then maps the summand $L^{\otimes \mathbf{m}_2} \otimes L^{\otimes \mathbf{p} \setminus \mathrm{im}(\gamma \circ \beta_1)}$ indexed by the composite map

$$\mathbf{m}_1 \xrightarrow{\beta_1} \mathbf{n}_1 \xrightarrow{\gamma} \mathbf{p}$$

to the summand $L^{\otimes \mathbf{l}_2} \otimes L^{\otimes \mathbf{p} \setminus \mathrm{im}(\gamma \circ \beta_1 \circ \alpha_1)}$ indexed by the composite map

$$\mathbf{l}_1 \xrightarrow{\alpha_1} \mathbf{m}_1 \xrightarrow{\beta_1} \mathbf{n}_1 \xrightarrow{\gamma} \mathbf{p}$$

via the following chain of isomorphisms

$$\begin{tikzcd}
L^{\otimes \mathbf{m}_2} \otimes L^{\otimes \mathbf{p} \setminus \mathrm{im}(\gamma \circ \beta_1)} & L^{\otimes \mathbf{l}_2} \otimes L^{\otimes \mathbf{m}_2 \setminus \mathrm{im}(\alpha_2)} \otimes L^{\otimes \mathbf{p} \setminus \mathrm{im}(\gamma \circ \beta_1)} \arrow[l, "(\alpha_2, \mathrm{incl})_* \otimes \mathrm{id}_{L^{\otimes \mathbf{p} \setminus \mathrm{im}(\gamma \circ \beta_1)}}"'] \\
L^{\otimes \mathbf{l}_2} \otimes L^{\otimes \mathbf{p} \setminus \mathrm{im}(\gamma \circ \beta_1 \circ \alpha_1)} & L^{\otimes \mathbf{l}_2} \otimes L^{\otimes \mathbf{m}_1 \setminus \mathrm{im}(\alpha_1)} \otimes L^{\otimes \mathbf{p} \setminus \mathrm{im}(\gamma \circ \beta_1)}. \arrow[l, "\mathrm{id}_{L^{\otimes \mathbf{l}_2}} \otimes (\gamma \circ \beta_1, \mathrm{incl})_*"'] \arrow[u, "\mathrm{id}_{L^{\otimes \mathbf{l}_2}} \otimes \rho_* \otimes \mathrm{id}_{L^{\otimes \mathbf{p} \setminus \mathrm{im}(\gamma \circ \beta_1)}}"']
\end{tikzcd} \tag{2.19}$$

On the other hand, the morphism $(\beta_1 \circ \alpha_1, \beta_2 \circ \alpha_2, \phi \cup (\beta_2 \circ \rho \circ \beta_1^{-1}))^*(p)$ takes the summand $L^{\otimes \mathbf{n}_2} \otimes L^{\otimes \mathbf{p} \setminus \mathrm{im}(\gamma)}$ indexed by the map $\gamma\colon \mathbf{n}_1 \to \mathbf{p}$ to the summand

$$L^{\otimes \mathbf{l}_2} \otimes L^{\otimes \mathbf{p} \setminus \mathrm{im}(\gamma \circ \beta_1 \circ \alpha_1)}$$

indexed by the composite map

$$\mathbf{l}_1 \xrightarrow{\alpha_1} \mathbf{m}_1 \xrightarrow{\beta_1} \mathbf{n}_1 \xrightarrow{\gamma} \mathbf{p}$$

via the following chain of isomorphisms

$$\begin{tikzcd}
L^{\otimes \mathbf{n}_2} \otimes L^{\otimes \mathbf{p} \setminus \mathrm{im}(\gamma)} & L^{\otimes \mathbf{l}_2} \otimes L^{\otimes \mathbf{n}_2 \setminus \mathrm{im}(\beta_2 \circ \alpha_2)} \otimes L^{\otimes \mathbf{p} \setminus \mathrm{im}(\gamma)} \arrow[l, "(\beta_2 \circ \alpha_2, \mathrm{incl})_* \otimes \mathrm{id}_{L^{\otimes \mathbf{p} \setminus \mathrm{im}(\gamma)}}"'] \\
L^{\otimes \mathbf{l}_2} \otimes L^{\otimes \mathbf{p} \setminus \mathrm{im}(\gamma \circ \beta_1 \circ \alpha_1)} & L^{\otimes \mathbf{l}_2} \otimes L^{\otimes \mathbf{n}_1 \setminus \mathrm{im}(\beta_1 \circ \alpha_1)} \otimes L^{\otimes \mathbf{p} \setminus \mathrm{im}(\gamma)}. \arrow[l, "\mathrm{id}_{L^{\otimes \mathbf{l}_2}} \otimes (\gamma, \mathrm{incl})_*"'] \arrow[u, "\mathrm{id}_{L^{\otimes \mathbf{l}_2}} \otimes (\phi \cup (\beta_2 \circ \rho \circ \beta_1^{-1}))_* \otimes \mathrm{id}_{L^{\otimes \mathbf{p} \setminus \mathrm{im}(\gamma)}}"']
\end{tikzcd} \tag{2.20}$$

It remains to argue that the chain of isomorphisms (2.18) composed with the chain of isomorphisms (2.19), which is

$$\begin{tikzcd}
L^{\otimes \mathbf{n}_2} \otimes L^{\otimes \mathbf{p} \setminus \mathrm{im}(\gamma)} & L^{\otimes \mathbf{m}_2} \otimes L^{\otimes \mathbf{n}_2 \setminus \mathrm{im}(\beta_2)} \otimes L^{\otimes \mathbf{p} \setminus \mathrm{im}(\gamma)} \arrow[l, "(\beta_2, \mathrm{incl})_* \otimes \mathrm{id}_{L^{\otimes \mathbf{p} \setminus \mathrm{im}(\gamma)}}"'] \\
L^{\otimes \mathbf{m}_2} \otimes L^{\otimes \mathbf{p} \setminus \mathrm{im}(\gamma \circ \beta_1)} & L^{\otimes \mathbf{m}_2} \otimes L^{\otimes \mathbf{n}_1 \setminus \mathrm{im}(\beta_1)} \otimes L^{\otimes \mathbf{p} \setminus \mathrm{im}(\gamma)} \arrow[l, "\mathrm{id}_{L^{\otimes \mathbf{m}_2}} \otimes (\gamma, \mathrm{incl})_*"'] \arrow[u, "\mathrm{id}_{L^{\otimes \mathbf{m}_2}} \otimes \phi_* \otimes \mathrm{id}_{L^{\otimes \mathbf{p} \setminus \mathrm{im}(\gamma)}}"'] \\
& L^{\otimes \mathbf{l}_2} \otimes L^{\otimes \mathbf{m}_2 \setminus \mathrm{im}(\alpha_2)} \otimes L^{\otimes \mathbf{p} \setminus \mathrm{im}(\gamma \circ \beta_1)} \arrow[ul, "(\alpha_2, \mathrm{incl})_* \otimes \mathrm{id}_{L^{\otimes \mathbf{p} \setminus \mathrm{im}(\gamma \circ \beta_1)}}"] \\
L^{\otimes \mathbf{l}_2} \otimes L^{\otimes \mathbf{p} \setminus \mathrm{im}(\gamma \circ \beta_1 \circ \alpha_1)} & L^{\otimes \mathbf{l}_2} \otimes L^{\otimes \mathbf{m}_1 \setminus \mathrm{im}(\alpha_1)} \otimes L^{\otimes \mathbf{p} \setminus \mathrm{im}(\gamma \circ \beta_1)}, \arrow[l, "\mathrm{id}_{L^{\otimes \mathbf{l}_2}} \otimes (\gamma \circ \beta_1, \mathrm{incl})_*"'] \arrow[u, "\mathrm{id}_{L^{\otimes \mathbf{l}_2}} \otimes \rho_* \otimes \mathrm{id}_{L^{\otimes \mathbf{p} \setminus \mathrm{im}(\gamma \circ \beta_1)}}"']
\end{tikzcd}$$

coincides with the chain of isomorphisms (2.20). This means we need to verify that the composite

$$\begin{aligned}
&(\mathrm{id}_{L^{\otimes \mathbf{l}_2}} \otimes (\gamma \circ \beta_1, \mathrm{incl})_*) \circ (\mathrm{id}_{L^{\otimes \mathbf{l}_2}} \otimes \rho_* \otimes \mathrm{id}_{L^{\otimes \mathbf{p} \setminus \mathrm{im}(\gamma \circ \beta_1)}})^{-1} \circ ((\alpha_2, \mathrm{incl})_* \otimes \mathrm{id}_{L^{\otimes \mathbf{p} \setminus \mathrm{im}(\gamma \circ \beta_1)}})^{-1} \\
&\circ (\mathrm{id}_{L^{\otimes \mathbf{m}_2}} \otimes (\gamma, \mathrm{incl})_*) \circ (\mathrm{id}_{L^{\otimes \mathbf{m}_2}} \otimes \phi_* \otimes \mathrm{id}_{L^{\otimes \mathbf{p} \setminus \mathrm{im}(\gamma)}})^{-1} \circ ((\beta_2, \mathrm{incl})_* \otimes \mathrm{id}_{L^{\otimes \mathbf{p} \setminus \mathrm{im}(\gamma)}})^{-1}
\end{aligned} \tag{2.21}$$

is equal to the composite

$$\begin{aligned}
&(\mathrm{id}_{L^{\otimes \mathbf{l}_2}} \otimes (\gamma, \mathrm{incl})_*) \circ (\mathrm{id}_{L^{\otimes \mathbf{l}_2}} \otimes (\phi \cup (\beta_2 \circ \rho \circ \beta_1^{-1}))_* \otimes \mathrm{id}_{L^{\otimes \mathbf{p} \setminus \mathrm{im}(\gamma)}})^{-1} \\
&\circ ((\beta_2 \circ \alpha_2, \mathrm{incl})_* \otimes \mathrm{id}_{L^{\otimes \mathbf{p} \setminus \mathrm{im}(\gamma)}})^{-1}.
\end{aligned} \tag{2.22}$$

For this, we analyze the following diagram of bijections in $\mathcal{I}$,

$$\begin{tikzcd}
\mathbf{n}_2 \sqcup (\mathbf{p} \setminus \mathrm{im}(\gamma)) & \mathbf{m}_2 \sqcup (\mathbf{n}_2 \setminus \mathrm{im}(\beta_2)) \sqcup (\mathbf{p} \setminus \mathrm{im}(\gamma)) \arrow[l, "(\beta_2, \mathrm{incl}) \sqcup \mathrm{id}_{\mathbf{p} \setminus \mathrm{im}(\gamma)}"'] \\
\mathbf{l}_2 \sqcup (\mathbf{n}_2 \setminus \mathrm{im}(\beta_2 \circ \alpha_2)) \sqcup (\mathbf{p} \setminus \mathrm{im}(\gamma)) \arrow[u, "(\beta_2 \circ \alpha_2, \mathrm{incl}) \sqcup \mathrm{id}_{\mathbf{p} \setminus \mathrm{im}(\gamma)}"] & \mathbf{m}_2 \sqcup (\mathbf{n}_1 \setminus \mathrm{im}(\beta_1)) \sqcup (\mathbf{p} \setminus \mathrm{im}(\gamma)) \arrow[u, "\mathrm{id}_{\mathbf{m}_2} \sqcup \phi \sqcup \mathrm{id}_{\mathbf{p} \setminus \mathrm{im}(\gamma)}"'] \arrow[d, "\mathrm{id}_{\mathbf{m}_2} \sqcup (\gamma, \mathrm{incl})"] \\
\mathbf{l}_2 \sqcup (\mathbf{n}_1 \setminus \mathrm{im}(\beta_1 \circ \alpha_1)) \sqcup (\mathbf{p} \setminus \mathrm{im}(\gamma)) \arrow[u, "\mathrm{id}_{\mathbf{l}_2} \sqcup (\phi \cup (\beta_2 \circ \rho \circ \beta_1^{-1})) \sqcup \mathrm{id}_{\mathbf{p} \setminus \mathrm{im}(\gamma)}"] \arrow[dddr, bend right, "\mathrm{id}_{\mathbf{l}_2} \sqcup (\gamma, \mathrm{incl})"'] & \mathbf{m}_2 \sqcup (\mathbf{p} \setminus \mathrm{im}(\gamma \circ \beta_1)) \\
& \mathbf{l}_2 \sqcup (\mathbf{m}_2 \setminus \mathrm{im}(\alpha_2)) \sqcup (\mathbf{p} \setminus \mathrm{im}(\gamma \circ \beta_1)) \arrow[u, "(\alpha_2, \mathrm{incl}) \sqcup \mathrm{id}_{\mathbf{p} \setminus \mathrm{im}(\gamma \circ \beta_1)}"'] \\
& \mathbf{l}_2 \sqcup (\mathbf{m}_1 \setminus \mathrm{im}(\alpha_1)) \sqcup (\mathbf{p} \setminus \mathrm{im}(\gamma \circ \beta_1)) \arrow[u, "\mathrm{id}_{\mathbf{l}_2} \sqcup \rho \sqcup \mathrm{id}_{\mathbf{p} \setminus \mathrm{im}(\gamma \circ \beta_1)}"'] \arrow[d, "\mathrm{id}_{\mathbf{l}_2} \sqcup (\gamma \circ \beta_1, \mathrm{incl})"] \\
& \mathbf{l}_2 \sqcup (\mathbf{p} \setminus \mathrm{im}(\gamma \circ \beta_1 \circ \alpha_1)).
\end{tikzcd} \tag{2.23}$$

We claim that composing the top horizontal map with the right vertical map

$$(\mathrm{id}_{\mathbf{l}_2} \sqcup (\gamma \circ \beta_1, \mathrm{incl})) \circ (\mathrm{id}_{\mathbf{l}_2} \sqcup \rho \sqcup \mathrm{id}_{\mathbf{p}\setminus \mathrm{im}(\gamma\circ\beta_1)})^{-1} \circ ((\alpha_2, \mathrm{incl}) \sqcup \mathrm{id}_{\mathbf{p}\setminus \mathrm{im}(\gamma\circ\beta_1)})^{-1}$$
$$\circ\, (\mathrm{id}_{\mathbf{m}_2} \sqcup (\gamma, \mathrm{incl})) \circ (\mathrm{id}_{\mathbf{m}_2} \sqcup \phi \sqcup \mathrm{id}_{\mathbf{p}\setminus \mathrm{im}(\gamma)})^{-1} \circ ((\beta_2, \mathrm{incl}) \sqcup \mathrm{id}_{\mathbf{p}\setminus \mathrm{im}(\gamma)})^{-1}$$

agrees with composing the left vertical map with the bottom diagonal map

$$(\mathrm{id}_{\mathbf{l}_2} \sqcup (\gamma, \mathrm{incl})) \circ (\mathrm{id}_{\mathbf{l}_2} \sqcup (\phi \cup (\beta_2 \circ \rho \circ \beta_1^{-1})) \sqcup \mathrm{id}_{\mathbf{p}\setminus \mathrm{im}(\gamma)})^{-1} \circ ((\beta_2 \circ \alpha_2, \mathrm{incl}) \sqcup \mathrm{id}_{\mathbf{p}\setminus \mathrm{im}(\gamma)})^{-1}.$$

To show this, we insert commutative diagrams in the diagram (2.23). For lack of space we use the following abbreviations

$$\begin{aligned}
A &= \mathbf{n}_2 \sqcup (\mathbf{p} \setminus \mathrm{im}(\gamma))\\
B &= \mathbf{m}_2 \sqcup (\mathbf{n}_2 \setminus \mathrm{im}(\beta_2)) \sqcup (\mathbf{p} \setminus \mathrm{im}(\gamma))\\
C &= \mathbf{m}_2 \sqcup (\mathbf{n}_1 \setminus \mathrm{im}(\beta_1)) \sqcup (\mathbf{p} \setminus \mathrm{im}(\gamma))\\
D &= \mathbf{m}_2 \sqcup (\mathbf{p} \setminus \mathrm{im}(\gamma \circ \beta_1))\\
E &= \mathbf{l}_2 \sqcup (\mathbf{m}_2 \setminus \mathrm{im}(\alpha_2)) \sqcup (\mathbf{p} \setminus \mathrm{im}(\gamma \circ \beta_1))\\
F &= \mathbf{l}_2 \sqcup (\mathbf{m}_1 \setminus \mathrm{im}(\alpha_1)) \sqcup (\mathbf{p} \setminus \mathrm{im}(\gamma \circ \beta_1))\\
G &= \mathbf{l}_2 \sqcup (\mathbf{p} \setminus \mathrm{im}(\gamma \circ \beta_1 \circ \alpha_1))\\
H &= \mathbf{l}_2 \sqcup (\mathbf{n}_2 \setminus \mathrm{im}(\beta_2 \circ \alpha_2)) \sqcup (\mathbf{p} \setminus \mathrm{im}(\gamma))\\
I &= \mathbf{l}_2 \sqcup (\mathbf{n}_1 \setminus \mathrm{im}(\beta_1 \circ \alpha_1)) \sqcup (\mathbf{p} \setminus \mathrm{im}(\gamma))\\
J &= \mathbf{l}_2 \sqcup (\mathbf{m}_2 \setminus \mathrm{im}(\alpha_2)) \sqcup (\mathbf{n}_2 \setminus \mathrm{im}(\beta_2)) \sqcup (\mathbf{p} \setminus \mathrm{im}(\gamma))\\
K &= \mathbf{l}_2 \sqcup (\mathbf{m}_2 \setminus \mathrm{im}(\alpha_2)) \sqcup (\mathbf{n}_1 \setminus \mathrm{im}(\beta_1)) \sqcup (\mathbf{p} \setminus \mathrm{im}(\gamma))\\
M &= \mathbf{l}_2 \sqcup (\mathbf{m}_1 \setminus \mathrm{im}(\alpha_1)) \sqcup (\mathbf{n}_1 \setminus \mathrm{im}(\beta_1)) \sqcup (\mathbf{p} \setminus \mathrm{im}(\gamma))
\end{aligned}$$

$$\begin{aligned}
a &= (\beta_2, \mathrm{incl}) \sqcup \mathrm{id}_{\mathbf{p}\setminus \mathrm{im}(\gamma)}\\
b &= \mathrm{id}_{\mathbf{m}_2} \sqcup \phi \sqcup \mathrm{id}_{\mathbf{p}\setminus \mathrm{im}(\gamma)}\\
c &= \mathrm{id}_{\mathbf{m}_2} \sqcup (\gamma, \mathrm{incl})\\
d &= (\alpha_2, \mathrm{incl}) \sqcup \mathrm{id}_{\mathbf{p}\setminus \mathrm{im}(\gamma\circ\beta_1)}\\
e &= \mathrm{id}_{\mathbf{l}_2} \sqcup \rho \sqcup \mathrm{id}_{\mathbf{p}\setminus \mathrm{im}(\gamma\circ\beta_1)}\\
f &= \mathrm{id}_{\mathbf{l}_2} \sqcup (\gamma \circ \beta_1, \mathrm{incl})\\
g &= (\beta_2 \circ \alpha_2, \mathrm{incl}) \sqcup \mathrm{id}_{\mathbf{p}\setminus \mathrm{im}(\gamma)}\\
h &= \mathrm{id}_{\mathbf{l}_2} \sqcup (\phi \cup (\beta_2 \circ \rho \circ \beta_1^{-1})) \sqcup \mathrm{id}_{\mathbf{p}\setminus \mathrm{im}(\gamma)}\\
i &= \mathrm{id}_{\mathbf{l}_2} \sqcup (\gamma, \mathrm{incl})\\
j &= (\alpha_2, \mathrm{incl}) \sqcup \mathrm{id}_{\mathbf{n}_2\setminus \mathrm{im}(\beta_2)} \sqcup \mathrm{id}_{\mathbf{p}\setminus \mathrm{im}(\gamma)}\\
q &= \mathrm{id}_{\mathbf{l}_2} \sqcup \mathrm{id}_{\mathbf{m}_2\setminus \mathrm{im}(\alpha_2)} \sqcup \phi \sqcup \mathrm{id}_{\mathbf{p}\setminus \mathrm{im}(\gamma)}\\
r &= \mathrm{id}_{\mathbf{l}_2} \sqcup \rho \sqcup \mathrm{id}_{\mathbf{n}_1\setminus \mathrm{im}(\beta_1)} \sqcup \mathrm{id}_{\mathbf{p}\setminus \mathrm{im}(\gamma)}\\
s &= \mathrm{id}_{\mathbf{l}_2} \sqcup \rho \sqcup \phi \sqcup \mathrm{id}_{\mathbf{p}\setminus \mathrm{im}(\gamma)}
\end{aligned}$$

$$
\begin{aligned}
t &= (\alpha_2, \mathrm{incl}) \sqcup \mathrm{id}_{\mathbf{n}_1 \setminus \mathrm{im}(\beta_1)} \sqcup \mathrm{id}_{\mathbf{p} \setminus \mathrm{im}(\gamma)} \\
u &= \mathrm{id}_{\mathbf{l}_2} \sqcup \mathrm{id}_{\mathbf{m}_2 \setminus \mathrm{im}(\alpha_2)} \sqcup (\gamma, \mathrm{incl}) \\
v &= \mathrm{id}_{\mathbf{l}_2} \sqcup \mathrm{id}_{\mathbf{m}_1 \setminus \mathrm{im}(\alpha_1)} \sqcup (\gamma, \mathrm{incl}) \\
w &= \mathrm{id}_{\mathbf{l}_2} \sqcup (\beta_1, \mathrm{incl}) \sqcup \mathrm{id}_{\mathbf{p} \setminus \mathrm{im}(\gamma)}.
\end{aligned}
$$

The diagram (2.23) is then the outer diagram of the following diagram of bijections in $\mathcal{I}$,

$$
\begin{tikzcd}
A & B \arrow[l, "a"'] & C \arrow[l, "b"'] \arrow[r, "c"] & D \\
H \arrow[u, "g"] & J \arrow[l] \arrow[u, "j"'] & K \arrow[l, "q"'] \arrow[u, "t"'] \arrow[r, "u"] & E \arrow[u, "d"'] \\
 & & M \arrow[u, "r"'] \arrow[ul, "s"] \arrow[r, "v"] \arrow[d, "w"'] & F \arrow[u, "e"'] \arrow[d, "f"'] \\
 & & I \arrow[uull, "h"] \arrow[r, "i"] & G
\end{tikzcd}
\tag{2.24}
$$

where the map $J \to H$ is induced by the bijection in Lemma 2.17(iii). We can read off that all diagrams being part of the diagram (2.24) commute. This implies the claim which in turn yields that the composite (2.21) is the same as the composite (2.22). Thus, we can conclude that the equation (2.17) is true. $\square$

Lemma 2.20. *The functor $F_{-}(\mathbb{S}^{-}) \colon \mathcal{J}^{\mathrm{op}} \to \mathrm{Sp}^{\Sigma}(\mathrm{ch})$ factors through the projection $\mathcal{J}^{\mathrm{op}} \to \bar{\mathcal{J}}^{\mathrm{op}}$.*

We refer to the induced functor $\bar{\mathcal{J}}^{\mathrm{op}} \to \mathrm{Sp}^{\Sigma}(\mathrm{ch})$ as $F_{-}(\mathbb{S}^{-})$, too.

Proof. Let $(\alpha_1, \alpha_2, \rho), (\alpha_1', \alpha_2', \rho') \colon (\mathbf{m}_1, \mathbf{m}_2) \to (\mathbf{n}_1, \mathbf{n}_2)$ be equivalent morphisms, that is, there exists a σ in A_{n_2} such that $(\alpha_1, \alpha_2, \rho) = (\mathrm{id}_{\mathbf{n}_1}, \sigma, \mathrm{id}_{\emptyset}) \circ (\alpha', \alpha_2', \rho')$. Hence, for the induced maps, we get that $(\alpha_1, \alpha_2, \rho)^* = (\alpha', \alpha_2', \rho')^* \circ (\mathrm{id}_{\mathbf{n}_1}, \sigma, \mathrm{id}_{\emptyset})^*$. Since $\mathrm{sgn}(\sigma) = 1$, the induced map $(\mathrm{id}_{\mathbf{n}_1}, \sigma, \mathrm{id}_{\emptyset})^*$ is the identity. Therefore, the map $[\alpha_1, \alpha_2, \rho]^*$ is well-defined. $\square$

Lemma 2.21. *The functor $F_{-}(L^{\otimes -}) \colon \mathcal{J}^{\mathrm{op}} \to \mathrm{Sp}^{\Sigma}(\mathcal{C}, L)$ is strong symmetric monoidal.*

Proof. (compare [SS12, proof of Lemma 4.22]) There is a morphism

$$
\mathrm{Sym}(L) \xrightarrow{\nu^{F_{-}(L^{\otimes -})}} F_0(\mathbf{1}_{\mathcal{C}})
$$

which is the identity. In addition, the isomorphism (2.16) yields that for $(\mathbf{m}_1, \mathbf{m}_2)$ and $(\mathbf{n}_1, \mathbf{n}_2)$ in $\mathcal{J}^{\mathrm{op}}$, there is an isomorphism

$$
F_{m_1}(L^{\otimes m_2}) \wedge F_{n_1}(L^{\otimes n_2}) \xrightarrow{\lambda^{F_{-}(L^{\otimes -})}_{(\mathbf{m}_1, \mathbf{m}_2), (\mathbf{n}_1, \mathbf{n}_2)}} F_{m_1 + n_1}(L^{\otimes m_2} \otimes L^{\otimes n_2}) = F_{m_1 + n_1}(L^{\otimes m_2 + n_2}).
\tag{2.25}
$$

One can check that this isomorphism (2.25) is natural in $((\mathbf{m}_1, \mathbf{m}_2), (\mathbf{n}_1, \mathbf{n}_2))$, so that we obtain a natural transformation $\lambda^{F_-(L^{\otimes -})}$, and that the latter together with $\nu^{F_-(L^{\otimes -})}$ is coherently associative, unital and commutative. We unravel commutativity. Let $(\mathbf{m}_1, \mathbf{m}_2)$ and $(\mathbf{n}_1, \mathbf{n}_2)$ be in $\mathcal{J}^{\mathrm{op}}$. We have to show that the diagram

$$\begin{tikzcd} F_{m_1}(L^{\otimes m_2}) \wedge F_{n_1}(L^{\otimes n_2}) \arrow[r, "\tau_{F_{m_1}(L^{\otimes m_2}), F_{n_1}(L^{\otimes n_2})}"] \arrow[d, "\lambda^{F_-(L^{\otimes -})}_{(\mathbf{m}_1,\mathbf{m}_2),(\mathbf{n}_1,\mathbf{n}_2)}"'] & F_{n_1}(L^{\otimes n_2}) \wedge F_{m_1}(L^{\otimes m_2}) \arrow[d, "\lambda^{F_-(L^{\otimes -})}_{(\mathbf{n}_1,\mathbf{n}_2),(\mathbf{m}_1,\mathbf{m}_2)}"] \\ F_{m_1+n_1}(L^{\otimes m_2+n_2}) \arrow[r, "(\chi_{\mathbf{n}_1,\mathbf{m}_1}, \chi_{\mathbf{n}_2,\mathbf{m}_2}, \mathrm{id}_\emptyset)^*"] & F_{n_1+m_1}(L^{\otimes n_2+m_2}) \end{tikzcd} \tag{2.26}$$

commutes. We investigate the above diagram (2.26) in spectrum degree n. For an object $((\mathbf{p}, \mathbf{q}), \alpha\colon \mathbf{p} \sqcup \mathbf{q} \to \mathbf{n})$ in the comma category $(- \sqcup - \downarrow \mathbf{n})$, the summand

$$L^{\otimes \mathbf{m}_2} \otimes L^{\otimes \mathbf{p} \setminus \mathrm{im}(\beta)} \otimes L^{\otimes \mathbf{n}_2} \otimes L^{\otimes \mathbf{q} \setminus \mathrm{im}(\tilde{\beta})} \otimes L^{\otimes \mathbf{n} \setminus \mathrm{im}(\alpha)} \tag{2.27}$$

indexed by the maps $\beta\colon \mathbf{m}_1 \to \mathbf{p}$ and $\tilde{\beta}\colon \mathbf{n}_1 \to \mathbf{q}$, is mapped to the summand

$$L^{\otimes \mathbf{n}_2} \otimes L^{\otimes \mathbf{q} \setminus \mathrm{im}(\tilde{\beta})} \otimes L^{\otimes \mathbf{m}_2} \otimes L^{\otimes \mathbf{p} \setminus \mathrm{im}(\beta)} \otimes L^{\otimes \mathbf{n} \setminus \mathrm{im}(\alpha \circ \chi_{\mathbf{q},\mathbf{p}})} \tag{2.28}$$

by the twist map $\tau_{F_{m_1}(L^{\otimes m_2}), F_{n_1}(L^{\otimes n_2})}(n)$. If we apply the morphism $\lambda^{F_-(L^{\otimes -})}_{(\mathbf{n}_1,\mathbf{n}_2),(\mathbf{m}_1,\mathbf{m}_2)}(n)$ to the summand (2.28), we get the summand

$$L^{\otimes \mathbf{n}_2 \sqcup \mathbf{m}_2} \otimes L^{\otimes \mathbf{n} \setminus \mathrm{im}(\alpha \circ \chi_{\mathbf{q},\mathbf{p}} \circ (\tilde{\beta} \sqcup \beta))} \tag{2.29}$$

indexed by

$$\mathbf{n}_1 \sqcup \mathbf{m}_1 \xrightarrow{\tilde{\beta} \sqcup \beta} \mathbf{q} \sqcup \mathbf{p} \xrightarrow{\chi_{\mathbf{q},\mathbf{p}}} \mathbf{p} \sqcup \mathbf{q} \xrightarrow{\alpha} \mathbf{n}.$$

The other way round, the morphism $\lambda^{F_-(L^{\otimes -})}_{(\mathbf{m}_1,\mathbf{m}_2),(\mathbf{n}_1,\mathbf{n}_2)}(n)$ sends the summand (2.27) to the summand

$$L^{\otimes \mathbf{m}_2 \sqcup \mathbf{n}_2} \otimes L^{\otimes \mathbf{n} \setminus \mathrm{im}(\alpha \circ (\beta \sqcup \tilde{\beta}))} \tag{2.30}$$

indexed by

$$\mathbf{m}_1 \sqcup \mathbf{n}_1 \xrightarrow{\beta \sqcup \tilde{\beta}} \mathbf{p} \sqcup \mathbf{q} \xrightarrow{\alpha} \mathbf{n}.$$

The morphism $(\chi_{\mathbf{n}_1,\mathbf{m}_1}, \chi_{\mathbf{n}_2,\mathbf{m}_2}, \mathrm{id}_\emptyset)^*(n)$ takes the summand (2.30) to the summand

$$L^{\otimes \mathbf{n}_2 \sqcup \mathbf{m}_2} \otimes L^{\otimes \mathbf{n} \setminus \mathrm{im}(\alpha \circ (\beta \sqcup \tilde{\beta}) \circ \chi_{\mathbf{n}_1,\mathbf{m}_1})} \tag{2.31}$$

indexed by

$$\mathbf{n}_1 \sqcup \mathbf{m}_1 \xrightarrow{\chi_{\mathbf{n}_1,\mathbf{m}_1}} \mathbf{m}_1 \sqcup \mathbf{n}_1 \xrightarrow{\beta \sqcup \tilde{\beta}} \mathbf{p} \sqcup \mathbf{q} \xrightarrow{\alpha} \mathbf{n}.$$

We notice that

$$\alpha \circ \chi_{\mathbf{q},\mathbf{p}} \circ (\tilde{\beta} \sqcup \beta) = \alpha \circ (\beta \sqcup \tilde{\beta}) \circ \chi_{\mathbf{n}_1,\mathbf{m}_1}$$

so that the summand (2.29) agrees with the summand (2.31). Hence, the diagram (2.26) commutes. $\square$

Corollary 2.22. *The functor $F_-(\mathbb{S}^-)\colon \bar{\mathcal{J}}^{\mathrm{op}} \to \mathrm{Sp}^\Sigma(\mathrm{ch})$ is strong symmetric monoidal.*

Proof. This follows from Lemma 2.20 and Lemma 2.21. □

We recall the Dold-Kan correspondence on the level of symmetric spectra, that is, the Quillen equivalence

$$\mathrm{Sp}^\Sigma(s\mathrm{mod}) \underset{\Phi^*\circ N}{\overset{L}{\leftrightarrows}} \mathrm{Sp}^\Sigma(\mathrm{ch})$$

(see [Shi07, Proposition 2.10.(2)], (2.7)).

Definition 2.23. Let $\Omega^{\bar{\mathcal{J}}}\colon \mathrm{Sp}^\Sigma(s\mathrm{mod}) \to (s\mathrm{mod})^{\bar{\mathcal{J}}}$ be the functor that sends a symmetric spectrum A to the $\bar{\mathcal{J}}$-simplicial k-module $\Omega^{\bar{\mathcal{J}}}(A) = \mathrm{Hom}_{s\mathrm{mod}}^{\mathrm{Sp}^\Sigma(s\mathrm{mod})}(L(F_-(\mathbb{S}^-)), A)$, and a morphism of symmetric spectra $f\colon A \to B$ to the induced map of $\bar{\mathcal{J}}$-simplicial k-modules

$$\mathrm{Hom}_{s\mathrm{mod}}^{\mathrm{Sp}^\Sigma(s\mathrm{mod})}(L(F_-(\mathbb{S}^-)), A) \xrightarrow{\mathrm{Hom}_{s\mathrm{mod}}^{\mathrm{Sp}^\Sigma(s\mathrm{mod})}(\mathrm{id}, f)} \mathrm{Hom}_{s\mathrm{mod}}^{\mathrm{Sp}^\Sigma(s\mathrm{mod})}(L(F_-(\mathbb{S}^-)), B).$$

Remark 2.24. Let A be in $\mathrm{Sp}^\Sigma(s\mathrm{mod})$, and let $(\mathbf{m}_1, \mathbf{m}_2)$ be in $\bar{\mathcal{J}}$. Using adjunctions, we can break down the definition of $\Omega^{\bar{\mathcal{J}}}(A)(\mathbf{m}_1, \mathbf{m}_2)$ as follows

$$\begin{aligned}
\Omega^{\bar{\mathcal{J}}}(A)(\mathbf{m}_1, \mathbf{m}_2) &= \mathrm{Hom}_{s\mathrm{mod}}^{\mathrm{Sp}^\Sigma(s\mathrm{mod})}(L(F_{m_1}(\mathbb{S}^{m_2})), A) \\
&= \mathrm{Hom}_{\mathrm{mod}}^{\mathrm{Sp}^\Sigma(s\mathrm{mod})}(L(F_{m_1}(\mathbb{S}^{m_2})) \wedge \tilde{k}(\Delta_{(-)_+}), A) \\
&\cong \mathrm{Hom}_{\mathrm{mod}}^{\mathrm{Sp}^\Sigma(s\mathrm{mod})}(L(F_{m_1}(\mathbb{S}^{m_2})), F_{s\mathrm{mod}}(\tilde{k}(\Delta_{(-)_+}), A)) \\
&\cong \mathrm{Hom}_{\mathrm{mod}}^{\mathrm{Sp}^\Sigma(\mathrm{ch})}(F_{m_1}(\mathbb{S}^{m_2}), \phi^* \circ N(F_{s\mathrm{mod}}(\tilde{k}(\Delta_{(-)_+}), A))) \\
&\cong \mathrm{Hom}_{\mathrm{mod}}^{\mathrm{ch}}(\mathbb{S}^{m_2}, N(\underline{\mathrm{Hom}}_{s\mathrm{mod}}(\tilde{k}(\Delta_{(-)_+}), A(m_1)))).
\end{aligned}$$

This is in simplicial degree $[q]$ isomorphic to the m_2-cycles of the chain complex $N(\underline{\mathrm{Hom}}_{s\mathrm{mod}}(\tilde{k}(\Delta_{q_+}), A(m_1)))$ denoted by $Z_{m_2}(N(\underline{\mathrm{Hom}}_{s\mathrm{mod}}(\tilde{k}(\Delta_{q_+}), A(m_1))))$.

In the following we write $\hat{\otimes}$ for the symmetric monoidal product in simplicial k-modules which is for M and $\tilde{M}$ in $s\mathrm{mod}$ in simplicial degree $[q]$ given by $(M \hat{\otimes} \tilde{M})[q] = M[q] \otimes \tilde{M}[q]$ with a diagonal action of face and degeneracy operators.

Example 2.25. A model in $C(\mathrm{Sp}^\Sigma(s\mathrm{mod}))$ for the function spectrum $F(X_+, Hk)$ in $C(Hk\text{-mod})$ is the object $\underline{\mathrm{Hom}}_{\mathrm{Sp}^\Sigma(s\mathrm{mod})}(F_0(\tilde{k}(X_+)), \mathrm{Sym}(\tilde{k}(S^1)))$ which we denote by $F_{s\mathrm{mod}}(\tilde{k}(X_+), \mathrm{Sym}(\tilde{k}(S^1)))$. To see this, we apply the forgetful functor

$$U\colon C(\mathrm{Sp}^\Sigma(s\mathrm{mod})) \to C(Hk\text{-mod})$$

to $F_{\mathrm{smod}}(\tilde{k}(X_+), \mathrm{Sym}(\tilde{k}(S^1)))$. Let n be a spectrum degree. We obtain that

$$\begin{aligned}
U(F_{\mathrm{smod}}(\tilde{k}(X_+), \mathrm{Sym}(\tilde{k}(S^1))))(n) &= U(\underline{\mathrm{Hom}}_{\mathrm{smod}}(\tilde{k}(X_+), \mathrm{Sym}(\tilde{k}(S^1))(n))) \\
&= U(\mathrm{Hom}^{\mathrm{smod}}_{\mathrm{mod}}(\tilde{k}(X_+)\hat{\otimes}\tilde{k}(\Delta_{(-)_+}), \mathrm{Sym}(\tilde{k}(S^1))(n))) \\
&\cong U(\mathrm{Hom}^{\mathrm{smod}}_{\mathrm{mod}}(\tilde{k}(X_+ \wedge \Delta_{(-)_+}), \mathrm{Sym}(\tilde{k}(S^1))(n))) \\
&\cong \mathcal{S}_*(X_+ \wedge \Delta_{(-)_+}, Hk(n)) \\
&= \underline{\mathrm{Hom}}_{\mathcal{S}_*}(X_+, Hk(n)) \\
&= F(X_+, Hk)(n).
\end{aligned}$$

If we apply the functor $\Omega^{\bar{\mathcal{J}}}$ to $F_{\mathrm{smod}}(\tilde{k}(X_+), \mathrm{Sym}(\tilde{k}(S^1)))$, we get that for $(\mathbf{m}_1, \mathbf{m}_2)$ in $\bar{\mathcal{J}}$, the simplicial k-module $\Omega^{\bar{\mathcal{J}}}(F_{\mathrm{smod}}(\tilde{k}(X_+), \mathrm{Sym}(\tilde{k}(S^1))))(\mathbf{m}_1, \mathbf{m}_2)$ is isomorphic to

$$\begin{aligned}
&Z_{m_2}(N(\underline{\mathrm{Hom}}_{\mathrm{smod}}(\tilde{k}(\Delta_{(-)_+}), \underline{\mathrm{Hom}}_{\mathrm{smod}}(\tilde{k}(X_+), \mathrm{Sym}(\tilde{k}(S^1))(m_1))))) \\
\cong\ &Z_{m_2}(N(\underline{\mathrm{Hom}}_{\mathrm{smod}}(\tilde{k}(\Delta_{(-)_+} \wedge X_+), \tilde{k}(S^{m_1}))))
\end{aligned}$$

(see Remark 2.24).

Lemma 2.26. *The functor* $\Omega^{\bar{\mathcal{J}}} \colon \mathrm{Sp}^{\Sigma}(s\mathrm{mod}) \to (s\mathrm{mod})^{\bar{\mathcal{J}}}$ *possesses a left adjoint functor* $L_{\Omega^{\bar{\mathcal{J}}}} \colon (s\mathrm{mod})^{\bar{\mathcal{J}}} \to \mathrm{Sp}^{\Sigma}(s\mathrm{mod})$.

Proof. First, we observe that for $(\mathbf{m}_1, \mathbf{m}_2)$ in $\bar{\mathcal{J}}$, a simplicial k-module M and a symmetric spectrum A, there are isomorphisms

$$\begin{aligned}
(s\mathrm{mod})^{\bar{\mathcal{J}}}(F^{\bar{\mathcal{J}}}_{(\mathbf{m}_1,\mathbf{m}_2)}(M), \Omega^{\bar{\mathcal{J}}}(A)) &\cong s\mathrm{mod}(M, \mathrm{Ev}^{\bar{\mathcal{J}}}_{(\mathbf{m}_1,\mathbf{m}_2)} \circ \Omega^{\bar{\mathcal{J}}}(A)) \\
&= s\mathrm{mod}(M, \mathrm{Hom}^{\mathrm{Sp}^{\Sigma}(s\mathrm{mod})}_{s\mathrm{mod}}(L(F_{m_1}(\mathbb{S}^{m_2})), A)) \\
&\cong \mathrm{Sp}^{\Sigma}(s\mathrm{mod})(F_0(M) \wedge L(F_{m_1}(\mathbb{S}^{m_2})), A).
\end{aligned}$$

Thus, the functor $\mathrm{Ev}^{\bar{\mathcal{J}}}_{(\mathbf{m}_1,\mathbf{m}_2)} \circ \Omega^{\bar{\mathcal{J}}}$ is right adjoint to the functor $F_0(-) \wedge L(F_{m_1}(\mathbb{S}^{m_2}))$. Considering this, for $(\mathbf{m}_1, \mathbf{m}_2)$ in $\bar{\mathcal{J}}$ and M in $s\mathrm{mod}$, we set

$$L_{\Omega^{\bar{\mathcal{J}}}}(F^{\bar{\mathcal{J}}}_{(\mathbf{m}_1,\mathbf{m}_2)}(M)) = F_0(M) \wedge L(F_{m_1}(\mathbb{S}^{m_2})).$$

Every object X in $(s\mathrm{mod})^{\bar{\mathcal{J}}}$ can be written as a coequalizer of

$$\textstyle\bigoplus_{[\alpha_1,\alpha_2,\rho] \in \bar{\mathcal{J}}((\mathbf{m}_1,\mathbf{m}_2),(\mathbf{n}_1,\mathbf{n}_2))} F^{\bar{\mathcal{J}}}_{(\mathbf{n}_1,\mathbf{n}_2)}(X(\mathbf{m}_1,\mathbf{m}_2)) \rightrightarrows \bigoplus_{(\mathbf{p}_1,\mathbf{p}_2) \in \bar{\mathcal{J}}} F^{\bar{\mathcal{J}}}_{(\mathbf{p}_1,\mathbf{p}_2)}(X(\mathbf{p}_1,\mathbf{p}_2))$$

where on a summand indexed by $[\alpha_1, \alpha_2, \rho]$, one map is determined by the morphism

$$\begin{aligned}
&k(\bar{\mathcal{J}}((\mathbf{n}_1,\mathbf{n}_2),-))\hat{\otimes}X(\mathbf{m}_1,\mathbf{m}_2) \\
&\xrightarrow{\mathrm{id}_{k(\bar{\mathcal{J}}((\mathbf{n}_1,\mathbf{n}_2),-))}\hat{\otimes}X([\alpha_1,\alpha_2,\rho])} k(\bar{\mathcal{J}}((\mathbf{n}_1,\mathbf{n}_2),-))\hat{\otimes}X(\mathbf{n}_1,\mathbf{n}_2),
\end{aligned}$$

and the other map is given by the morphism

$$\begin{aligned}&k(\bar{\mathcal{J}}((\mathbf{n}_1,\mathbf{n}_2),-))\hat{\otimes}X(\mathbf{m}_1,\mathbf{m}_2)\\&\xrightarrow{k([\alpha_1,\alpha_2,\rho]^*)\hat{\otimes}\mathrm{id}_{X(\mathbf{m}_1,\mathbf{m}_2)}} k(\bar{\mathcal{J}}((\mathbf{m}_1,\mathbf{m}_2),-))\hat{\otimes}X(\mathbf{m}_1,\mathbf{m}_2).\end{aligned}$$

As a left adjoint, $L_{\Omega^{\bar{\mathcal{J}}}}$ has to preserve colimits and thus for X in $(s\mathrm{mod})^{\bar{\mathcal{J}}}$, we define $L_{\Omega^{\bar{\mathcal{J}}}}(X)$ as the coequalizer of

$$\begin{aligned}&\textstyle\bigoplus_{[\alpha_1,\alpha_2,\rho]\in\bar{\mathcal{J}}((\mathbf{m}_1,\mathbf{m}_2),(\mathbf{n}_1,\mathbf{n}_2))} L_{\Omega^{\bar{\mathcal{J}}}}(F^{\bar{\mathcal{J}}}_{(\mathbf{n}_1,\mathbf{n}_2)}(X(\mathbf{m}_1,\mathbf{m}_2)))\\&\rightrightarrows \textstyle\bigoplus_{(\mathbf{p}_1,\mathbf{p}_2)\in\bar{\mathcal{J}}} L_{\Omega^{\bar{\mathcal{J}}}}(F^{\bar{\mathcal{J}}}_{(\mathbf{p}_1,\mathbf{p}_2)}(X(\mathbf{p}_1,\mathbf{p}_2))).\end{aligned}\tag{2.32}$$

We check that the functor $L_{\Omega^{\bar{\mathcal{J}}}}$ is indeed left adjoint to the functor $\Omega^{\bar{\mathcal{J}}}$. Let X be in $(s\mathrm{mod})^{\bar{\mathcal{J}}}$, and let A be in $\mathrm{Sp}^\Sigma(s\mathrm{mod})$. Let $L_{\Omega^{\bar{\mathcal{J}}}}(X)\to A$ be a map in $\mathrm{Sp}^\Sigma(s\mathrm{mod})$. By definition this is a map from the coequalizer of (2.32) to A. Exploiting that for $(\mathbf{p}_1,\mathbf{p}_2)$ in $\bar{\mathcal{J}}$, the functor $L_{\Omega^{\bar{\mathcal{J}}}}\circ F^{\bar{\mathcal{J}}}_{(\mathbf{p}_1,\mathbf{p}_2)}$ is left adjoint to the functor $\mathrm{Ev}^{\bar{\mathcal{J}}}_{(\mathbf{p}_1,\mathbf{p}_2)}\circ\Omega^{\bar{\mathcal{J}}}$, the latter corresponds to morphisms $X(\mathbf{p}_1,\mathbf{p}_2)\to\Omega^{\bar{\mathcal{J}}}(A)(\mathbf{p}_1,\mathbf{p}_2)$ in $s\mathrm{mod}$ for $(\mathbf{p}_1,\mathbf{p}_2)$ in $\bar{\mathcal{J}}$, such that for every morphism $(\mathbf{m}_1,\mathbf{m}_2)\to(\mathbf{n}_1,\mathbf{n}_2)$ in $\bar{\mathcal{J}}$, the induced square

$$\begin{array}{ccc}X(\mathbf{m}_1,\mathbf{m}_2) & \longrightarrow & X(\mathbf{n}_1,\mathbf{n}_2)\\ \downarrow & & \downarrow\\ \Omega^{\bar{\mathcal{J}}}(A)(\mathbf{m}_1,\mathbf{m}_2) & \longrightarrow & \Omega^{\bar{\mathcal{J}}}(A)(\mathbf{n}_1,\mathbf{n}_2)\end{array}$$

commutes. This specifies a morphism $X\to\Omega^{\bar{\mathcal{J}}}(A)$ in $(s\mathrm{mod})^{\bar{\mathcal{J}}}$. □

In addition, there is a free-forgetful adjunction (k,U) between $\bar{\mathcal{J}}$-spaces and $\bar{\mathcal{J}}$-simplicial k-modules. Composing the latter with the adjunction $(L_{\Omega^{\bar{\mathcal{J}}}},\Omega^{\bar{\mathcal{J}}})$, we obtain an adjunction between $\mathcal{S}^{\bar{\mathcal{J}}}$ and $\mathrm{Sp}^\Sigma(s\mathrm{mod})$,

$$\mathcal{S}^{\bar{\mathcal{J}}} \underset{U}{\overset{k}{\rightleftarrows}} (s\mathrm{mod})^{\bar{\mathcal{J}}} \underset{\Omega^{\bar{\mathcal{J}}}}{\overset{L_{\Omega^{\bar{\mathcal{J}}}}}{\rightleftarrows}} \mathrm{Sp}^\Sigma(s\mathrm{mod}).\tag{2.33}$$

Recall that in general, given an adjunction

$$\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D}$$

between symmetric monoidal categories $(\mathcal{C},\otimes,\mathbf{1}_\mathcal{C})$ and $(\mathcal{D},\odot,\mathbf{1}_\mathcal{D})$ with a lax symmetric monoidal structure (ν^G,λ^G) on the right adjoint G, the left adjoint F inherits a lax symmetric comonoidal structure $(\tilde{\nu}^F,\tilde{\lambda}^F)$ as follows. The counit map $\tilde{\nu}^F\colon F(\mathbf{1}_\mathcal{C})\to\mathbf{1}_\mathcal{D}$

is the adjoint of the unit map $\nu^G\colon \mathbf{1}_{\mathcal{C}} \to G(\mathbf{1}_{\mathcal{D}})$. For X and Y in $\mathcal{C}$, the natural map $\tilde{\lambda}^F_{X,Y}\colon F(X \otimes Y) \to FX \odot FY$ is defined as the composite

$$F(X \otimes Y) \xrightarrow{F(\eta_X \otimes \eta_Y)} F(GFX \otimes GFY) \xrightarrow{F(\lambda^G_{FX,FY})} FG(FX \odot FY) \xrightarrow{\epsilon_{FX \odot FY}} FX \odot FY$$

where η denotes the adjunction unit and ϵ the adjunction counit. If on the contrary, the left adjoint F has a strong symmetric monoidal structure (ν^F, λ^F), this gives rise to a lax symmetric monoidal structure (ν^G, λ^G) on the right adjoint G. Explicitly, the unit map $\nu^G\colon \mathbf{1}_{\mathcal{C}} \to G(\mathbf{1}_{\mathcal{D}})$ is specified by the composite

$$\mathbf{1}_{\mathcal{C}} \xrightarrow{\eta_{\mathbf{1}_{\mathcal{C}}}} GF(\mathbf{1}_{\mathcal{C}}) \xrightarrow{G((\nu^F)^{-1})} G(\mathbf{1}_{\mathcal{D}}).$$

For V and W in $\mathcal{D}$, the natural map $\lambda^G_{V,W}\colon GV \otimes GW \to G(V \odot W)$ is determined by the composite

$$GV \otimes GW \xrightarrow{\eta_{GV \otimes GW}} GF(GV \otimes GW) \xrightarrow{G((\lambda^F_{GV,GW})^{-1})} G(FGV \odot FGW) \xrightarrow{G(\epsilon_V \odot \epsilon_W)} G(V \odot W)$$

(see [SS03a, §3.2]).

Lemma 2.27. *The functor $\Omega^{\bar{\mathcal{J}}}\colon \mathrm{Sp}^\Sigma(s\mathrm{mod}) \to (s\mathrm{mod})^{\bar{\mathcal{J}}}$ is lax symmetric monoidal.*

Proof. To prove the statement, we use that on the one hand the functor

$$L\colon \mathrm{Sp}^\Sigma(s\mathrm{mod}) \to \mathrm{Sp}^\Sigma(\mathrm{ch})$$

is lax symmetric comonoidal because its right adjoint $\Phi^* \circ N$ is lax symmetric monoidal (see [Shi07, Proposition 2.10.(2)]), and on the other hand that the functor

$$F_-(\mathbb{S}^-)\colon \bar{\mathcal{J}}^{\mathrm{op}} \to \mathrm{Sp}^\Sigma(\mathrm{ch})$$

is strong symmetric monoidal by Corollary 2.22. As the functor L is lax symmetric comonoidal, L comes with a morphism $\tilde{\nu}^L\colon L(\mathrm{Sym}(\mathbb{S}^1)) = L(F_0(\mathbb{S}^0)) \to \mathrm{Sym}(\tilde{k}(S^1))$ and natural morphisms $\tilde{\lambda}^L_{X,Y}\colon L(X \wedge Y) \to L(X) \wedge L(Y)$ for X and Y in $\mathrm{Sp}^\Sigma(\mathrm{ch})$. The morphism $\tilde{\nu}^L$ induces a map $\mathrm{const}_{\Delta^{\mathrm{op}}} k \to \mathrm{Hom}^{\mathrm{Sp}^\Sigma(s\mathrm{mod})}_{s\mathrm{mod}}(L(F_0(\mathbb{S}^0)), \mathrm{Sym}(\tilde{k}(S^1)))$ which is adjoint to the required map $\nu^{\Omega^{\bar{\mathcal{J}}}}\colon k(U^{\bar{\mathcal{J}}}) \to \Omega^{\bar{\mathcal{J}}}(\mathrm{Sym}(\tilde{k}(S^1)))$. Let A and B be in $\mathrm{Sp}^\Sigma(s\mathrm{mod})$, and let $(\mathbf{m}_1, \mathbf{m}_2)$ and $(\mathbf{n}_1, \mathbf{n}_2)$ be in $\bar{\mathcal{J}}$. We get a natural morphism

$$\Omega^{\bar{\mathcal{J}}}(A)(\mathbf{m}_1, \mathbf{m}_2) \hat{\otimes} \Omega^{\bar{\mathcal{J}}}(B)(\mathbf{n}_1, \mathbf{n}_2) \to \Omega^{\bar{\mathcal{J}}}(A \wedge B)(\mathbf{m}_1 \sqcup \mathbf{n}_1, \mathbf{m}_2 \sqcup \mathbf{n}_2)$$

via the composition

$$
\begin{aligned}
&\Omega^{\bar{\mathcal{J}}}(A)(\mathbf{m}_1,\mathbf{m}_2)\hat{\otimes}\Omega^{\bar{\mathcal{J}}}(B)(\mathbf{n}_1,\mathbf{n}_2)\\
&= \mathrm{Hom}^{\mathrm{Sp}^\Sigma(s\mathrm{mod})}_{s\mathrm{mod}}(L(F_{m_1}(\mathbb{S}^{m_2})),A)\hat{\otimes}\mathrm{Hom}^{\mathrm{Sp}^\Sigma(s\mathrm{mod})}_{s\mathrm{mod}}(L(F_{n_1}(\mathbb{S}^{n_2})),B)\\
&\to \mathrm{Hom}^{\mathrm{Sp}^\Sigma(s\mathrm{mod})}_{s\mathrm{mod}}(L(F_{m_1}(\mathbb{S}^{m_2}))\wedge L(F_{n_1}(\mathbb{S}^{n_2})),A\wedge B)\\
&\xrightarrow{\mathrm{Hom}^{\mathrm{Sp}^\Sigma(s\mathrm{mod})}_{s\mathrm{mod}}(\tilde{\lambda}^L_{F_{m_1}(\mathbb{S}^{m_2}),F_{n_1}(\mathbb{S}^{n_2})},\mathrm{id})} \mathrm{Hom}^{\mathrm{Sp}^\Sigma(s\mathrm{mod})}_{s\mathrm{mod}}(L(F_{m_1}(\mathbb{S}^{m_2})\wedge F_{n_1}(\mathbb{S}^{n_2})),A\wedge B)\\
&\xrightarrow{\mathrm{Hom}^{\mathrm{Sp}^\Sigma(s\mathrm{mod})}_{s\mathrm{mod}}(L((\lambda^{F_-(\mathbb{S}^-)}_{(\mathbf{m}_1,\mathbf{m}_2),(\mathbf{n}_1,\mathbf{n}_2)})^{-1}),\mathrm{id})} \mathrm{Hom}^{\mathrm{Sp}^\Sigma(s\mathrm{mod})}_{s\mathrm{mod}}(L(F_{m_1+n_1}(\mathbb{S}^{m_2+n_2})),A\wedge B)\\
&= \Omega^{\bar{\mathcal{J}}}(A\wedge B)(\mathbf{m}_1\sqcup\mathbf{n}_1,\mathbf{m}_2\sqcup\mathbf{n}_2).
\end{aligned}
$$

This map gives rise to a natural morphism $\lambda^{\Omega^{\bar{\mathcal{J}}}}_{A,B}\colon \Omega^{\bar{\mathcal{J}}}(A)\boxtimes\Omega^{\bar{\mathcal{J}}}(B)\to\Omega^{\bar{\mathcal{J}}}(A\wedge B)$. Since the map $\nu^{\Omega^{\bar{\mathcal{J}}}}$ as well as the natural transformation $\lambda^{\Omega^{\bar{\mathcal{J}}}}$ are defined by using the morphisms making the functor L lax symmetric comonoidal and the functor $F_-(\mathbb{S}^-)$ strong symmetric monoidal, they are coherently associative, unital and commutative. We spell out unitality. Let A be in $\mathrm{Sp}^\Sigma(s\mathrm{mod})$. We have to show that the following diagrams commute

$$
\begin{tikzcd}
k(U^{\bar{\mathcal{J}}})\boxtimes\Omega^{\bar{\mathcal{J}}}(A) \arrow[r, "\nu^{\Omega^{\bar{\mathcal{J}}}}\boxtimes\mathrm{id}_{\Omega^{\bar{\mathcal{J}}}(A)}"] \arrow[d, "\text{left unitor}"'] & \Omega^{\bar{\mathcal{J}}}(\mathrm{Sym}(\tilde{k}(S^1)))\boxtimes\Omega^{\bar{\mathcal{J}}}(A) \arrow[d, "\lambda^{\Omega^{\bar{\mathcal{J}}}}_{\mathrm{Sym}(\tilde{k}(S^1)),A}"] \\
\Omega^{\bar{\mathcal{J}}}(A) & \Omega^{\bar{\mathcal{J}}}(\mathrm{Sym}(\tilde{k}(S^1))\wedge A) \arrow[l, "\Omega^{\bar{\mathcal{J}}}(\text{left unitor})"']
\end{tikzcd}
$$

$$
\begin{tikzcd}
\Omega^{\bar{\mathcal{J}}}(A)\boxtimes k(U^{\bar{\mathcal{J}}}) \arrow[r, "\mathrm{id}_{\Omega^{\bar{\mathcal{J}}}(A)}\boxtimes\nu^{\Omega^{\bar{\mathcal{J}}}}"] \arrow[d, "\text{right unitor}"'] & \Omega^{\bar{\mathcal{J}}}(A)\boxtimes\Omega^{\bar{\mathcal{J}}}(\mathrm{Sym}(\tilde{k}(S^1))) \arrow[d, "\lambda^{\Omega^{\bar{\mathcal{J}}}}_{A,\mathrm{Sym}(\tilde{k}(S^1))}"] \\
\Omega^{\bar{\mathcal{J}}}(A) & \Omega^{\bar{\mathcal{J}}}(A\wedge\mathrm{Sym}(\tilde{k}(S^1))). \arrow[l, "\Omega^{\bar{\mathcal{J}}}(\text{right unitor})"']
\end{tikzcd}
$$

We argue that the second diagram commutes. To prove that the first diagram commutes, we can proceed analogously. Let $(\mathbf{m}_1,\mathbf{m}_2)$ and $(\mathbf{n},\mathbf{n})$ be in $\bar{\mathcal{J}}$. It suffices to show that the map

$$
\begin{aligned}
&\mathrm{Hom}^{\mathrm{Sp}^\Sigma(s\mathrm{mod})}_{s\mathrm{mod}}(L(F_{m_1}(\mathbb{S}^{m_2})),A)\hat{\otimes}k(\bar{\mathcal{J}}((\mathbf{0},\mathbf{0}),(\mathbf{n},\mathbf{n})))\\
&\xrightarrow{\text{right unitor}} \mathrm{Hom}^{\mathrm{Sp}^\Sigma(s\mathrm{mod})}_{s\mathrm{mod}}(L(F_{m_1+n}(\mathbb{S}^{m_2+n})),A)
\end{aligned}
$$

is equal to the composite

$$\begin{aligned}
&\mathrm{Hom}_{s\mathrm{mod}}^{\mathrm{Sp}^\Sigma(s\mathrm{mod})}(L(F_{m_1}(\mathbb{S}^{m_2})), A)\hat{\otimes} k(\bar{\mathcal{J}}((\mathbf{0},\mathbf{0}),(\mathbf{n},\mathbf{n})))\\
&\xrightarrow{\mathrm{id}_{\Omega^{\bar{\mathcal{J}}}(A)(\mathbf{m}_1,\mathbf{m}_2)}\hat{\otimes}\nu^{\Omega^{\bar{\mathcal{J}}}}(\mathbf{n},\mathbf{n})}\\
&\mathrm{Hom}_{s\mathrm{mod}}^{\mathrm{Sp}^\Sigma(s\mathrm{mod})}(L(F_{m_1}(\mathbb{S}^{m_2})), A)\hat{\otimes}\mathrm{Hom}_{s\mathrm{mod}}^{\mathrm{Sp}^\Sigma(s\mathrm{mod})}(L(F_n(\mathbb{S}^n)), \mathrm{Sym}(\tilde{k}(S^1)))\\
&\xrightarrow{\lambda^{\Omega^{\bar{\mathcal{J}}}}_{A,\mathrm{Sym}(\tilde{k}(S^1))}} \mathrm{Hom}_{s\mathrm{mod}}^{\mathrm{Sp}^\Sigma(s\mathrm{mod})}(L(F_{m_1+n}(\mathbb{S}^{m_2+n})), A\wedge \mathrm{Sym}(\tilde{k}(S^1)))\\
&\xrightarrow{\Omega^{\bar{\mathcal{J}}}(\text{right unitor})(\mathbf{m}_1\sqcup\mathbf{n},\mathbf{m}_2\sqcup\mathbf{n})} \mathrm{Hom}_{s\mathrm{mod}}^{\mathrm{Sp}^\Sigma(s\mathrm{mod})}(L(F_{m_1+n}(\mathbb{S}^{m_2+n})), A).
\end{aligned}$$

Let $[\alpha_1,\alpha_2,\rho]\colon (\mathbf{0},\mathbf{0})\to(\mathbf{n},\mathbf{n})$ be a morphism in $\bar{\mathcal{J}}$. The (right unitor)-morphism sends a map $f\colon L(F_{m_1}(\mathbb{S}^{m_2}))\to A$ to

$$L(F_{m_1+n}(\mathbb{S}^{m_2+n})) \xrightarrow{L([\mathrm{id}_{\mathbf{m}_1}\sqcup\alpha_1,\mathrm{id}_{\mathbf{m}_2}\sqcup\alpha_2,\rho]^*)} L(F_{m_1}(\mathbb{S}^{m_2})) \xrightarrow{f} A. \tag{2.34}$$

The map $\mathrm{id}_{\Omega^{\bar{\mathcal{J}}}(A)(\mathbf{m}_1,\mathbf{m}_2)}\hat{\otimes}\nu^{\Omega^{\bar{\mathcal{J}}}}(\mathbf{n},\mathbf{n})$ takes $f\colon L(F_{m_1}(\mathbb{S}^{m_2}))\to A$ to

$$(L(F_{m_1}(\mathbb{S}^{m_2}))\xrightarrow{f} A, L(F_n(\mathbb{S}^n))\xrightarrow{L([\alpha_1,\alpha_2,\rho]^*)} L(F_0(\mathbb{S}^0))\xrightarrow{\tilde{\nu}^L}\mathrm{Sym}(\tilde{k}(S^1))),$$

which is then mapped to

$$\begin{aligned}
&L(F_{m_1+n}(\mathbb{S}^{m_2+n}))\xrightarrow{L((\lambda^{F_-(\mathbb{S}^-)}_{(\mathbf{m}_1,\mathbf{m}_2),(\mathbf{n},\mathbf{n})})^{-1})} L(F_{m_1}(\mathbb{S}^{m_2})\wedge F_n(\mathbb{S}^n))\\
&\xrightarrow{\tilde{\lambda}^L_{F_{m_1}(\mathbb{S}^{m_2}),F_n(\mathbb{S}^n)}} L(F_{m_1}(\mathbb{S}^{m_2}))\wedge L(F_n(\mathbb{S}^n))\\
&\xrightarrow{\mathrm{id}_{L(F_{m_1}(\mathbb{S}^{m_2}))}\wedge L([\alpha_1,\alpha_2,\rho]^*)} L(F_{m_1}(\mathbb{S}^{m_2}))\wedge L(F_0(\mathbb{S}^0))\\
&\xrightarrow{f\wedge\tilde{\nu}^L} A\wedge\mathrm{Sym}(\tilde{k}(S^1))\xrightarrow{\text{right unitor}} A
\end{aligned}\tag{2.35}$$

by the composite $\Omega^{\bar{\mathcal{J}}}(\text{right unitor})(\mathbf{m}_1\sqcup\mathbf{n},\mathbf{m}_2\sqcup\mathbf{n})\circ\lambda^{\Omega^{\bar{\mathcal{J}}}}_{A,\mathrm{Sym}(\tilde{k}(S^1))}$. Using that the morphisms $\tilde{\nu}^L$ and $\tilde{\lambda}^L$ are counital and that $\tilde{\lambda}^L$ is a natural transformation, we see that the composite (2.34) coincides with the composite (2.35). $\square$

The free-forgetful adjunction (k,U) in (2.33) lifts to the level of commutative monoids as the left adjoint k is strong symmetric monoidal. On the grounds that the functor $\Omega^{\bar{\mathcal{J}}}\colon \mathrm{Sp}^\Sigma(s\mathrm{mod})\to(s\mathrm{mod})^{\bar{\mathcal{J}}}$ is lax symmetric monoidal by Lemma 2.27, it defines a functor on the level of commutative monoids, $\Omega^{\bar{\mathcal{J}}}\colon C(\mathrm{Sp}^\Sigma(s\mathrm{mod}))\to C((s\mathrm{mod})^{\bar{\mathcal{J}}})$. Although the lax symmetric monoidal structure on the functor $\Omega^{\bar{\mathcal{J}}}$ gives rise to a lax symmetric comonoidal structure on the functor $L_{\Omega^{\bar{\mathcal{J}}}}$ (see remarks before Lemma 2.27), these comonoidal structure morphisms are not isomorphisms in general so that the functor $L_{\Omega^{\bar{\mathcal{J}}}}$ is not lax symmetric monoidal. In particular, the functor $L_{\Omega^{\bar{\mathcal{J}}}}$ does not pass to a functor on the monoid categories (see [SS03a, p. 303]). Applying the usual machinery of [SS03a, §3.3], we figure out a left adjoint to the functor $\Omega^{\bar{\mathcal{J}}}\colon C(\mathrm{Sp}^\Sigma(s\mathrm{mod}))\to C((s\mathrm{mod})^{\bar{\mathcal{J}}})$ as follows.

Lemma 2.28. *The functor* $\Omega^{\bar{\mathcal{J}}}\colon C(\mathrm{Sp}^\Sigma(s\mathrm{mod})) \to C((s\mathrm{mod})^{\bar{\mathcal{J}}})$ *has a left adjoint functor* $L^{\mathrm{mon}}_{\Omega^{\bar{\mathcal{J}}}}\colon C((s\mathrm{mod})^{\bar{\mathcal{J}}}) \to C(\mathrm{Sp}^\Sigma(s\mathrm{mod}))$.

Proof. Taking into account that the functor $\mathbb{C} \circ L_{\Omega^{\bar{\mathcal{J}}}}\colon (s\mathrm{mod})^{\bar{\mathcal{J}}} \to C(\mathrm{Sp}^\Sigma(s\mathrm{mod}))$ is left adjoint to the functor $\Omega^{\bar{\mathcal{J}}} \circ U\colon C(\mathrm{Sp}^\Sigma(s\mathrm{mod})) \to (s\mathrm{mod})^{\bar{\mathcal{J}}}$, we set

$$L^{\mathrm{mon}}_{\Omega^{\bar{\mathcal{J}}}}(\mathbb{C}(M)) = \mathbb{C}(L_{\Omega^{\bar{\mathcal{J}}}}(M))$$

for M in $(s\mathrm{mod})^{\bar{\mathcal{J}}}$. Every object X in $C((s\mathrm{mod})^{\bar{\mathcal{J}}})$ can be expressed as a coequalizer of

$$\mathbb{C}\mathbb{C}(X) \rightrightarrows \mathbb{C}(X)$$

where one map is induced by the structure map $\xi\colon \mathbb{C}(X) \to X$ and the other map is the multiplication of the monad $\mathbb{C}$ applied to X which is $\mu^{\mathbb{C}}_X\colon \mathbb{C}\mathbb{C}(X) \to \mathbb{C}(X)$. So for X in $C((s\mathrm{mod})^{\bar{\mathcal{J}}})$, we define $L^{\mathrm{mon}}_{\Omega^{\bar{\mathcal{J}}}}(X)$ as the coequalizer of

$$L^{\mathrm{mon}}_{\Omega^{\bar{\mathcal{J}}}}(\mathbb{C}\mathbb{C}(X)) \underset{L^{\mathrm{mon}}_{\Omega^{\bar{\mathcal{J}}}}(\mu^{\mathbb{C}}_X)}{\overset{\mathbb{C}(L_{\Omega^{\bar{\mathcal{J}}}}(\xi))}{\rightrightarrows}} L^{\mathrm{mon}}_{\Omega^{\bar{\mathcal{J}}}}(\mathbb{C}(X)). \tag{2.36}$$

Let X be in $C((s\mathrm{mod})^{\bar{\mathcal{J}}})$, and let A be in $C(\mathrm{Sp}^\Sigma(s\mathrm{mod}))$. Let $L^{\mathrm{mon}}_{\Omega^{\bar{\mathcal{J}}}}(X) \to A$ be a morphism in $C(\mathrm{Sp}^\Sigma(s\mathrm{mod}))$. By definition this is a morphism from the coequalizer of (2.36) to A. Considering that the functor $L^{\mathrm{mon}}_{\Omega^{\bar{\mathcal{J}}}} \circ \mathbb{C}$ is defined as $\mathbb{C} \circ L_{\Omega^{\bar{\mathcal{J}}}}$, that the functor $\Omega^{\bar{\mathcal{J}}}$ commutes with the forgetful functor U, and that there are adjoint pairs $(L_{\Omega^{\bar{\mathcal{J}}}}, \Omega^{\bar{\mathcal{J}}})$ and $(\mathbb{C}, U)$, the latter corresponds to a map $X \to \Omega^{\bar{\mathcal{J}}}(A)$ in $C((s\mathrm{mod})^{\bar{\mathcal{J}}})$. □

Altogether, we get a commutative diagram of adjunctions

$$\begin{tikzcd} \mathcal{S}^{\bar{\mathcal{J}}} \arrow[r, shift left, "k"] \arrow[d, shift right, "\mathbb{C}"'] & (s\mathrm{mod})^{\bar{\mathcal{J}}} \arrow[l, shift left, "U"] \arrow[r, shift left, "L_{\Omega^{\bar{\mathcal{J}}}}"] \arrow[d, shift right, "\mathbb{C}"'] & \mathrm{Sp}^\Sigma(s\mathrm{mod}) \arrow[l, shift left, "\Omega^{\bar{\mathcal{J}}}"] \arrow[d, shift right, "\mathbb{C}"'] \\ C\mathcal{S}^{\bar{\mathcal{J}}} \arrow[u, shift right, "U"'] \arrow[r, shift left, "k"] & C((s\mathrm{mod})^{\bar{\mathcal{J}}}) \arrow[u, shift right, "U"'] \arrow[l, shift left, "U"] \arrow[r, shift left, "L^{\mathrm{mon}}_{\Omega^{\bar{\mathcal{J}}}}"] & C(\mathrm{Sp}^\Sigma(s\mathrm{mod})). \arrow[u, shift right, "U"'] \arrow[l, shift left, "\Omega^{\bar{\mathcal{J}}}"] \end{tikzcd} \tag{2.37}$$

We continue with showing that the adjunctions in the above diagram (2.37) are homotopically well-behaved.

Lemma 2.29. *For every morphism* $[\alpha_1, \alpha_2, \rho]\colon (\mathbf{m}_1, \mathbf{m}_2) \to (\mathbf{n}_1, \mathbf{n}_2)$ *in* $\bar{\mathcal{J}}$*, the induced map* $[\alpha_1, \alpha_2, \rho]^*\colon F_{n_1}(\mathbb{S}^{n_2}) \to F_{m_1}(\mathbb{S}^{m_2})$ *is a stable equivalence in* $\mathrm{Sp}^\Sigma(\mathrm{ch})$.

Proof. We observe that we can write a morphism $[\alpha_1, \alpha_2, \rho]\colon (\mathbf{m}_1, \mathbf{m}_2) \to (\mathbf{n}_1, \mathbf{n}_2)$ in $\bar{\mathcal{J}}$ as a composition of an isomorphism and a morphism which is the equivalence class of a morphism where the first and second entry are the standard inclusions. More precisely, we choose factorizations $\alpha_1 = \xi_1 \circ \iota_{\mathbf{m}_1,\mathbf{n}_1}$ where ξ_1 is in Σ_{n_1} and $\alpha_2 = \xi_2 \circ \iota_{\mathbf{m}_2,\mathbf{n}_2}$ where ξ_2 is in Σ_{n_2}, so that

$$[\alpha_1, \alpha_2, \rho] = [\xi_1, \xi_2, \mathrm{id}_\emptyset] \circ [\iota_{\mathbf{m}_1,\mathbf{n}_1}, \iota_{\mathbf{m}_2,\mathbf{n}_2}, \xi_2^{-1} \circ \rho \circ \xi_1].$$

The induced map $[\xi_1, \xi_2, \mathrm{id}_\emptyset]^*$ is a level equivalence in $\mathrm{Sp}^\Sigma(\mathrm{ch})$. Considering the localizing set (2.6), we notice that the composition

$$\begin{aligned}
&F_{m_1+n_1-m_1}(\mathbb{S}^{m_2+n_2-m_2}) = F_{m_1+n_1-m_1}(\mathbb{S}^{m_2+n_2-m_2-1} \otimes \mathbb{S}^1) \\
&\xrightarrow{\zeta^{\mathbb{S}^{m_2+n_2-m_2-1}}_{m_1+n_1-m_1-1}} F_{m_1+n_1-m_1-1}(\mathbb{S}^{m_2+n_2-m_2-1}) = F_{m_1+n_1-m_1-1}(\mathbb{S}^{m_2+n_2-m_2-2} \otimes \mathbb{S}^1) \\
&\xrightarrow{\zeta^{\mathbb{S}^{m_2+n_2-m_2-2}}_{m_1+n_1-m_1-2}} \dots \\
&\dots \\
&\to F_{m_1+1}(\mathbb{S}^{m_2+1}) = F_{m_1+1}(\mathbb{S}^{m_2} \otimes \mathbb{S}^1) \xrightarrow{\zeta^{\mathbb{S}^{m_2}}_{m_1}} F_{m_1}(\mathbb{S}^{m_2})
\end{aligned} \tag{2.38}$$

is a stable equivalence in $\mathrm{Sp}^\Sigma(\mathrm{ch})$. The above composite (2.38) is isomorphic to the induced map $[\iota_{\mathbf{m}_1,\mathbf{n}_1}, \iota_{\mathbf{m}_2,\mathbf{n}_2}, \xi_2^{-1} \circ \rho \circ \xi_1]^*$ so that the latter is a stable equivalence in $\mathrm{Sp}^\Sigma(\mathrm{ch})$, too. We conclude that the morphism $[\alpha_1, \alpha_2, \rho]^*$ is a stable equivalence in $\mathrm{Sp}^\Sigma(\mathrm{ch})$. □

Proposition 2.30. *(i) The adjunction* $(L_{\Omega\bar{\mathcal{J}}} \circ k, U \circ \Omega^{\bar{\mathcal{J}}})$ *is a Quillen adjunction with respect to the (positive) projective* $\bar{\mathcal{J}}$*-model structure on* $\mathcal{S}^{\bar{\mathcal{J}}}$ *and the (positive) projective stable model structure on* $\mathrm{Sp}^\Sigma(s\mathrm{mod})$.

(ii) The adjunction $(L^{\mathrm{mon}}_{\Omega\bar{\mathcal{J}}} \circ k, U \circ \Omega^{\bar{\mathcal{J}}})$ *is a Quillen adjunction with respect to the positive projective* $\bar{\mathcal{J}}$*-model structure on* $C\mathcal{S}^{\bar{\mathcal{J}}}$ *and the positive projective stable model structure on* $C(\mathrm{Sp}^\Sigma(s\mathrm{mod}))$.

Proof. (i) We prove that the functor $L_{\Omega\bar{\mathcal{J}}} \circ k$ preserves (positive) cofibrations and (positive) acyclic cofibrations. According to [Hov99, Lemma 2.1.20], it suffices to show that the functor $L_{\Omega\bar{\mathcal{J}}} \circ k$ maps the generating (positive) cofibrations in $\mathcal{S}^{\bar{\mathcal{J}}}$ to (positive) cofibrations in $\mathrm{Sp}^\Sigma(s\mathrm{mod})$ and the generating (positive) acyclic cofibrations to (positive) acyclic cofibrations. Let $F^{\bar{\mathcal{J}}}_{(\mathbf{m}_1,\mathbf{m}_2)}(i)\colon F^{\bar{\mathcal{J}}}_{(\mathbf{m}_1,\mathbf{m}_2)}(\partial\Delta_l) \to F^{\bar{\mathcal{J}}}_{(\mathbf{m}_1,\mathbf{m}_2)}(\Delta_l)$ be an element in $I^{(+)\mathrm{level}}_{\mathcal{S}^{\bar{\mathcal{J}}}}$. We find that the map $L_{\Omega\bar{\mathcal{J}}} \circ k(F^{\bar{\mathcal{J}}}_{(\mathbf{m}_1,\mathbf{m}_2)}(i))$ can be identified with the map

$$F_0(k(\partial\Delta_l)) \wedge L(F_{m_1}(\mathbb{S}^{m_2})) \xrightarrow{F_0(k(i)) \wedge \mathrm{id}_{L(F_{m_1}(\mathbb{S}^{m_2}))}} F_0(k(\Delta_l)) \wedge L(F_{m_1}(\mathbb{S}^{m_2})). \tag{2.39}$$

The map $F_0(k(i))$ is a cofibration in $\mathrm{Sp}^\Sigma(s\mathrm{mod})$, and the object $L(F_{m_1}(\mathbb{S}^{m_2}))$ is cofibrant in $\mathrm{Sp}^\Sigma(s\mathrm{mod})$, because the object $F_{m_1}(\mathbb{S}^{m_2})$ is cofibrant in $\mathrm{Sp}^\Sigma(\mathrm{ch})$ and L is a left Quillen functor. Applying the *pushout product axiom* to the two cofibrations $F_0(k(i))\colon F_0(k(\partial\Delta_l)) \to F_0(k(\Delta_l))$ and $0 \to L(F_{m_1}(\mathbb{S}^{m_2}))$ yields that the map (2.39) is a cofibration in $\mathrm{Sp}^\Sigma(s\mathrm{mod})$ (see [Shi07, Proposition 2.9]). Assume that $(\mathbf{m}_1, \mathbf{m}_2)$ is in $\bar{\mathcal{J}}_+$. The map $L_{\Omega\bar{\mathcal{J}}} \circ k(F^{\bar{\mathcal{J}}}_{(\mathbf{m}_1,\mathbf{m}_2)}(i))$ is isomorphic to the map

$$F_{m_1}(k(\partial\Delta_l)\hat{\otimes}\Gamma(\mathbb{S}^{m_2})) \xrightarrow{F_{m_1}(k(i)\hat{\otimes}\mathrm{id}_{\Gamma(\mathbb{S}^{m_2})})} F_{m_1}(k(\Delta_l)\hat{\otimes}\Gamma(\mathbb{S}^{m_2}))$$

which is the zero map $\mathrm{const}_{\Delta^{\mathrm{op}}} 0 \to \mathrm{const}_{\Delta^{\mathrm{op}}} 0$ in spectrum level zero. Hence, the map $L_{\Omega^{\bar{\mathcal{J}}}} \circ k(F^{\bar{\mathcal{J}}}_{(\mathbf{m}_1,\mathbf{m}_2)}(i))$ is a positive cofibration in $\mathrm{Sp}^\Sigma(s\mathrm{mod})$.

Let $F^{\bar{\mathcal{J}}}_{(\mathbf{m}_1,\mathbf{m}_2)}(j)$ be an element in $J^{(+)\mathrm{level}}_{\mathcal{S}^{\bar{\mathcal{J}}}}$. The *pushout product axiom* with respect to the acyclic cofibration $F_0(k(j))$ and the cofibration $0 \to L(F_{m_1}(\mathbb{S}^{m_2}))$ ensures that the morphism $L_{\Omega^{\bar{\mathcal{J}}}} \circ k(F^{\bar{\mathcal{J}}}_{(\mathbf{m}_1,\mathbf{m}_2)}(j))$ is an acyclic cofibration in $\mathrm{Sp}^\Sigma(s\mathrm{mod})$. If $(\mathbf{m}_1, \mathbf{m}_2)$ is in $\bar{\mathcal{J}}_+$, the morphism $L_{\Omega^{\bar{\mathcal{J}}}} \circ k(F^{\bar{\mathcal{J}}}_{(\mathbf{m}_1,\mathbf{m}_2)}(j))$ being isomorphic to $F_{m_1}(k(j)\hat{\otimes}\mathrm{id}_{\Gamma(\mathbb{S}^{m_2})})$ is the zero map $\mathrm{const}_{\Delta^{\mathrm{op}}} 0 \to \mathrm{const}_{\Delta^{\mathrm{op}}} 0$ in spectrum level zero, and thus a positive acyclic cofibration in $\mathrm{Sp}^\Sigma(s\mathrm{mod})$.

Let $[\alpha_1, \alpha_2, \rho]\colon (\mathbf{m}_1, \mathbf{m}_2) \to (\mathbf{n}_1, \mathbf{n}_2)$ be a morphism in $\bar{\mathcal{J}}_{(+)}$, and let i be a generating cofibration in spaces so that $j_{[\alpha_1,\alpha_2,\rho]^*}\square i$ is an element in $J^{(+)\prime}$. We claim that the morphism $L_{\Omega^{\bar{\mathcal{J}}}} \circ k(j_{[\alpha_1,\alpha_2,\rho]^*}\square i)$ is an acyclic cofibration in $\mathrm{Sp}^\Sigma(s\mathrm{mod})$. As the functor $L_{\Omega^{\bar{\mathcal{J}}}} \circ k$ respects tensors and colimits, we obtain that the morphism $L_{\Omega^{\bar{\mathcal{J}}}} \circ\ k(j_{[\alpha_1,\alpha_2,\rho]^*}\square i)$ is isomorphic to $(L_{\Omega^{\bar{\mathcal{J}}}} \circ k(j_{[\alpha_1,\alpha_2,\rho]^*}))\square k(i)$. Recall the factorization of the map $[\alpha_1, \alpha_2, \rho]^*$ through the mapping cylinder $\mathrm{Cyl}([\alpha_1, \alpha_2, \rho]^*)$ (see (2.5)),

$$
\begin{tikzcd}
F^{\bar{\mathcal{J}}}_{(\mathbf{n}_1,\mathbf{n}_2)}(*) \arrow[r, tail, "j_{[\alpha_1,\alpha_2,\rho]^*}"] \arrow[rr, bend left, "{[\alpha_1,\alpha_2,\rho]^*}"] & \mathrm{Cyl}([\alpha_1, \alpha_2, \rho]^*) \arrow[r, "r_{[\alpha_1,\alpha_2,\rho]^*}", "\sim"'] & F^{\bar{\mathcal{J}}}_{(\mathbf{m}_1,\mathbf{m}_2)}(*).
\end{tikzcd}
$$

From the first part of the proof we deduce that the morphism $L_{\Omega^{\bar{\mathcal{J}}}} \circ k(j_{[\alpha_1,\alpha_2,\rho]^*})$ is a cofibration in $\mathrm{Sp}^\Sigma(s\mathrm{mod})$. The functor $L_{\Omega^{\bar{\mathcal{J}}}} \circ k$ preserves mapping cylinders so that we get a factorization of the morphism $L_{\Omega^{\bar{\mathcal{J}}}} \circ k([\alpha_1, \alpha_2, \rho]^*)$, being isomorphic to $\mathrm{id}_{F_0(\mathrm{const}_{\Delta^{\mathrm{op}}} k)} \wedge L([\alpha_1, \alpha_2, \rho]^*)$, through the respective mapping cylinder (compare [HSS00, Construction 3.1.7]). The map $[\alpha_1, \alpha_2, \rho]^*\colon F_{n_1}(\mathbb{S}^{n_2}) \to F_{m_1}(\mathbb{S}^{m_2})$ is a stable equivalence by Lemma 2.29, between cofibrant objects in $\mathrm{Sp}^\Sigma(\mathrm{ch})$. As L is a left Quillen functor and the object $F_0(\mathrm{const}_{\Delta^{\mathrm{op}}} k)$ is cofibrant in $\mathrm{Sp}^\Sigma(s\mathrm{mod})$, the map $\mathrm{id}_{F_0(\mathrm{const}_{\Delta^{\mathrm{op}}} k)} \wedge L([\alpha_1, \alpha_2, \rho]^*)$ is a stable equivalence (see [Shi07, Proposition 2.9]). Further, the map $L_{\Omega^{\bar{\mathcal{J}}}} \circ k(r_{[\alpha_1,\alpha_2,\rho]^*})$ is a stable equivalence (compare [HSS00, Construction 3.1.7]). By two out of three the map $L_{\Omega^{\bar{\mathcal{J}}}} \circ k(j_{[\alpha_1,\alpha_2,\rho]^*})$ is a stable equivalence. Moreover, the map $k(i)$ is a cofibration in simplicial k-modules. The *pushout product axiom* with respect to the acyclic cofibration $L_{\Omega^{\bar{\mathcal{J}}}} \circ k(j_{[\alpha_1,\alpha_2,\rho]^*})$ and the cofibration $k(i)$ implies that the map $(L_{\Omega^{\bar{\mathcal{J}}}} \circ k(j_{[\alpha_1,\alpha_2,\rho]^*}))\square k(i)$ is an acyclic cofibration in $\mathrm{Sp}^\Sigma(s\mathrm{mod})$. If $[\alpha_1, \alpha_2, \rho]$ is a morphism in $\bar{\mathcal{J}}_+$, the morphism $(L_{\Omega^{\bar{\mathcal{J}}}} \circ k(j_{[\alpha_1,\alpha_2,\rho]^*}))\square k(i)$ is the zero map $\mathrm{const}_{\Delta^{\mathrm{op}}} 0 \to \mathrm{const}_{\Delta^{\mathrm{op}}} 0$ in spectrum level zero, and so a positive acyclic cofibration in $\mathrm{Sp}^\Sigma(s\mathrm{mod})$.

(ii) Since the model structure on $C\mathcal{S}^{\bar{\mathcal{J}}}$ as well as on $C(\mathrm{Sp}^\Sigma(s\mathrm{mod}))$ is created by the respective forgetful functors, the functor $U \circ \Omega^{\bar{\mathcal{J}}} \colon C(\mathrm{Sp}^\Sigma(s\mathrm{mod})) \to C\mathcal{S}^{\bar{\mathcal{J}}}$ is a right Quillen functor.

$\square$

Definition 2.31. We define $\Lambda^{\bar{\mathcal{J}}} = L^{\mathrm{mon}}_{\Omega^{\bar{\mathcal{J}}}} \circ k$.

From now on we mostly omit the forgetful functor U in the notation of the functor $U \circ \Omega^{\bar{\mathcal{J}}}$.

2.5 Definition of pre-log structures

We now have the ingredients to define *pre-log stuctures* on E_∞ differential graded k-algebras. As a model for the latter, we employ the category $C(\mathrm{Sp}^\Sigma(s\mathrm{mod}))$, which is an intermediate category in the chain of Quillen equivalences between $C(Hk\text{-mod})$ and $E_\infty\mathrm{Ch}$ (see (2.8)). We provide several examples of *pre-log cdgas*.

Definition 2.32. (compare [SS12, Definition 4.31]) Let A be in $C(\mathrm{Sp}^\Sigma(s\mathrm{mod}))$. A *pre-log structure* on A is a pair (M, α) consisting of a commutative $\bar{\mathcal{J}}$-space M and a map of commutative $\bar{\mathcal{J}}$-spaces $\alpha\colon M \to \Omega^{\bar{\mathcal{J}}}(A)$. If (M, α) is a pre-log structure on A, we call the triple (A, M, α) a *pre-log cdga*. A map of pre-log cdgas $(f, f^b)\colon (A, M, \alpha) \to (B, N, \beta)$ is a map $f\colon A \to B$ in $C(\mathrm{Sp}^\Sigma(s\mathrm{mod}))$ and a map $f^b\colon M \to N$ in $C\mathcal{S}^{\bar{\mathcal{J}}}$ such that $\Omega^{\bar{\mathcal{J}}}(f^b) \circ \alpha = \beta \circ f^b$. We call the resulting category the *category of pre-log cdgas*.

Remark 2.33. (compare [RSS15, Definition 4.5 and the corresponding remark]) The category of pre-log cdgas carries a cofibrantly generated projective model structure, in which a map (f, f^b) is a weak equivalence/ fibration if and only if both the map f and the map f^b are weak equivalences/ fibrations. In this model structure a pre-log cdga (A, M, α) is cofibrant if the commutative $\bar{\mathcal{J}}$-space M is positive cofibrant and the adjoint structure map $\mathrm{ad}(\alpha)\colon \Lambda^{\bar{\mathcal{J}}}(M) \to A$ is a positive cofibration of commutative symmetric ring spectra in simplicial k-modules.

Example 2.34. (compare [Sag14, Example 3.5]) Let M be a commutative $\bar{\mathcal{J}}$-space. The adjunction unit of $(\Lambda^{\bar{\mathcal{J}}}, U \circ \Omega^{\bar{\mathcal{J}}})$ gives rise to the *canonical pre-log structure* $(M, M \to \Omega^{\bar{\mathcal{J}}}(\Lambda^{\bar{\mathcal{J}}}(M)))$ on $\Lambda^{\bar{\mathcal{J}}}(M)$.

Example 2.35. (compare [Sag14, Example 3.6]) Let (B, N, β) be a pre-log cdga, and let $f\colon A \to B$ be a morphism in $C(\mathrm{Sp}^\Sigma(s\mathrm{mod}))$. The pullback diagram

$$\begin{tikzcd}
(\Omega^{\bar{\mathcal{J}}}(f))_*(N) \arrow[r, "(\Omega^{\bar{\mathcal{J}}}(f))_*(\beta)"] \arrow[d] & \Omega^{\bar{\mathcal{J}}}(A) \arrow[d, "\Omega^{\bar{\mathcal{J}}}(f)"] \\
N \arrow[r, "\beta"] & \Omega^{\bar{\mathcal{J}}}(B)
\end{tikzcd} \tag{2.40}$$

induces a pre-log structure $((\Omega^{\bar{\mathcal{J}}}(f))_*(N), (\Omega^{\bar{\mathcal{J}}}(f))_*(\beta))$ on A. We call this pre-log structure the *direct image pre-log structure* on A with respect to the map f and the pre-log structure (N, β). Because of the commutativity of the pullback diagram (2.40), there is a morphism of pre-log cdgas $(A, (\Omega^{\bar{\mathcal{J}}}(f))_*(N), (\Omega^{\bar{\mathcal{J}}}(f))_*(\beta)) \to (B, N, \beta)$.

For a (positive) fibrant object A in $\mathrm{Sp}^\Sigma(s\mathrm{mod})$, the following proposition is crucial to understanding the homotopy type of the $\bar{\mathcal{J}}$-space $U(\Omega^{\bar{\mathcal{J}}}(A))$.

Proposition 2.36. *Let A be (positive) fibrant in* $\mathrm{Sp}^\Sigma(s\mathrm{mod})$*, and let* $(\mathbf{m}_1, \mathbf{m}_2)$ *be in* $\bar{\mathcal{J}}_{(+)}$*. The space* $U(\Omega^{\bar{\mathcal{J}}}(A))(\mathbf{m}_1, \mathbf{m}_2)$ *is weakly equivalent to the space* $\Omega^{m_2}(U(A)(m_1))$*.*

Proof. We prove that the symmetric spectrum $L(F_{m_1}(\mathbb{S}^{m_2}))$ is weakly equivalent to $F_{m_1}(\tilde{k}(S^{m_2}))$. Then we exploit this to show the claim.
The symmetric spectrum $L(F_{m_1}(\mathbb{S}^{m_2}))$ is isomorphic to $F_{m_1}(\Gamma(\mathbb{S}^{m_2}))$. Using that

$$\mathbb{S}^{m_2} = \mathbb{S}^{1^{\otimes m_2}} \cong (N(\tilde{k}(S^1)))^{\otimes m_2}$$

(see [Shi07, p. 358]), we get a map

$$\Gamma(\mathbb{S}^{m_2}) \cong \Gamma((N(\tilde{k}(S^1)))^{\otimes m_2}) \xrightarrow{\Gamma(\lambda^N_{\tilde{k}(S^1),\dots,\tilde{k}(S^1)})} \Gamma(N(\tilde{k}(S^1)^{\hat{\otimes} m_2})) \cong \tilde{k}(S^{m_2}).$$

Here the lax symmetric monoidal structure map of the normalization functor $\lambda^N_{\tilde{k}(S^1),\dots,\tilde{k}(S^1)}$ is given by the shuffle map (see [Mac63, Corollary VIII.8.9]) which is a chain homotopy equivalence. As the functor Γ preserves weak equivalences, the induced map $\Gamma(\lambda^N_{\tilde{k}(S^1),\dots,\tilde{k}(S^1)})$ is a weak equivalence. Since the left Quillen functor F_{m_1} preserves weak equivalences between cofibrant objects, the induced map

$$L(F_{m_1}(\mathbb{S}^{m_2})) \cong F_{m_1}(\Gamma(\mathbb{S}^{m_2})) \xrightarrow{F_{m_1}(\Gamma(\lambda^N_{\tilde{k}(S^1),\dots,\tilde{k}(S^1)}))} F_{m_1}(\tilde{k}(S^{m_2}))$$

is a weak equivalence.
The category $\mathrm{Sp}^\Sigma(s\mathrm{mod})$ with respect to the (positive) projective stable model structure is a monoidal model category [Shi07, Proposition 2.9], and an analogon of [Hir03, Corollary 9.3.3.(2)] in the case that the enrichment is over simplicial k-modules holds. So given the weak equivalence $F_{m_1}(\Gamma(\lambda^N_{\tilde{k}(S^1),\dots,\tilde{k}(S^1)}))\colon L(F_{m_1}(\mathbb{S}^{m_2})) \to F_{m_1}(\tilde{k}(S^{m_2}))$ between (positive) cofibrant objects and a (positive) fibrant object A in $\mathrm{Sp}^\Sigma(s\mathrm{mod})$, the induced map

$$\mathrm{Hom}^{\mathrm{Sp}^\Sigma(s\mathrm{mod})}_{s\mathrm{mod}}(F_{m_1}(\tilde{k}(S^{m_2})), A) \to \mathrm{Hom}^{\mathrm{Sp}^\Sigma(s\mathrm{mod})}_{s\mathrm{mod}}(L(F_{m_1}(\mathbb{S}^{m_2})), A) \tag{2.41}$$

is a weak equivalence. We apply the forgetful functor $U\colon s\mathrm{mod} \to \mathcal{S}_*$ to the simplicial k-module $\mathrm{Hom}^{\mathrm{Sp}^\Sigma(s\mathrm{mod})}_{s\mathrm{mod}}(F_{m_1}(\tilde{k}(S^{m_2})), A)$,

$$\begin{aligned} U(\mathrm{Hom}^{\mathrm{Sp}^\Sigma(s\mathrm{mod})}_{s\mathrm{mod}}(F_{m_1}(\tilde{k}(S^{m_2})), A)) &\cong U(\mathrm{Hom}^{\mathrm{Sp}^\Sigma(s\mathrm{mod})}_{\mathrm{mod}}(F_{m_1}(\tilde{k}(S^{m_2} \wedge \Delta_{(-)_+})), A)) \\ &\cong U(\mathrm{Hom}^{s\mathrm{mod}}_{\mathrm{mod}}(\tilde{k}(S^{m_2} \wedge \Delta_{(-)_+}), A(m_1))) \\ &\cong \underline{\mathrm{Hom}}_{\mathcal{S}_*}(S^{m_2}, U(A)(m_1)) \\ &= \Omega^{m_2}(U(A)(m_1)). \end{aligned} \tag{2.42}$$

□

Example 2.37. Recall that a pre-log structure on a discrete commutative ring A is a pair (M, α) consisting of a commutative monoid M and a map α from M to the underlying multiplicative commutative monoid of A denoted by $(A,\cdot)$ [Rog09, Definition 2.1]. For instance, let $\langle x\rangle = \{x^j, j\geq 0\}$ be the free commutative monoid on a generator x. There is a pre-log structure on the polynomial ring $k[x]$ given by the pair (M,ξ) where the map $\xi\colon \langle x\rangle \to (k[x],\cdot)$ sends x^j to $1_k\cdot x^j$ (compare [Rog09, Definition 2.12]). The pre-log ring $(k[x], \langle x\rangle, \xi)$ gives rise to a pre-log cdga as follows. Applying the Eilenberg-Mac Lane functor H to $k[x]$ yields the commutative Hk-algebra spectrum $H(k[x])$. The underlying multiplicative commutative monoid of zero simplices of the space $(H(k[x]))(0)$ is equal to $(k[x],\cdot)$. We can view the commutative monoid $\langle x\rangle$ as a discrete simplicial commutative monoid. There is a composite map of spaces

$$\begin{aligned}
&\langle x\rangle \xrightarrow{\xi} (H(k[x]))(0) \to U(\tilde{k}(H(k[x])) \wedge_{\tilde{k}(Hk)} \mathrm{Sym}(\tilde{k}(S^1)))(0)\\
&\xrightarrow{\cong} U(\mathrm{Hom}_{smod}^{\mathrm{Sp}^\Sigma(smod)}(F_0(\tilde{k}(S^0)), \tilde{k}(H(k[x])) \wedge_{\tilde{k}(Hk)} \mathrm{Sym}(\tilde{k}(S^1))))\\
&\xrightarrow{\cong} U(\mathrm{Hom}_{smod}^{\mathrm{Sp}^\Sigma(smod)}(L(F_0(\mathbb{S}^0)), \tilde{k}(H(k[x])) \wedge_{\tilde{k}(Hk)} \mathrm{Sym}(\tilde{k}(S^1))))\\
&= U(\Omega^{\bar{\mathcal{J}}}(\tilde{k}(H(k[x])) \wedge_{\tilde{k}(Hk)} \mathrm{Sym}(\tilde{k}(S^1))))(\mathbf{0},\mathbf{0}).
\end{aligned} \tag{2.43}$$

Here the second map is specified by the adjunction unit of the Quillen equivalence $(\tilde{k}(-) \wedge_{\tilde{k}(Hk)} \mathrm{Sym}(\tilde{k}(S^1)), U)$ (see Subsection 2.3). For the third and fourth map see the proof of Proposition 2.36, (2.42) and (2.41), and note the isomorphisms

$$L(F_0(\mathbb{S}^0)) \cong F_0(\Gamma(\mathbb{S}^0)) \cong F_0(\tilde{k}(S^0)).$$

The composite morphism (2.43) in $\mathcal{S}$ is adjoint to a morphism

$$\alpha_x\colon F^{\bar{\mathcal{J}}}_{(\mathbf{0},\mathbf{0})}(\langle x\rangle) \to U(\Omega^{\bar{\mathcal{J}}}(\tilde{k}(H(k[x])) \wedge_{\tilde{k}(Hk)} \mathrm{Sym}(\tilde{k}(S^1))))$$

in $C\mathcal{S}^{\bar{\mathcal{J}}}$ so that we get the pre-log cdga $(\tilde{k}(H(k[x]))\wedge_{\tilde{k}(Hk)}\mathrm{Sym}(\tilde{k}(S^1)), F^{\bar{\mathcal{J}}}_{(\mathbf{0},\mathbf{0})}(\langle x\rangle), \alpha_x)$.

Remark 2.38. Let A be (positive) fibrant in $\mathrm{Sp}^\Sigma(s\mathrm{mod})$, let $(\mathbf{m}_1,\mathbf{m}_2)$ be in $\bar{\mathcal{J}}_{(+)}$ and let $l\in\mathbb{Z}_{\geq 0}$. We note that

$$\begin{aligned}
\pi_{l+m_2-m_1}(U(A)) &= \mathrm{colim}_u \pi_{l+m_2-m_1+u}(U(A)(u))\\
&\cong \mathrm{colim}_u \pi_{l+u}(\Omega^{m_2-m_1}(U(A)(u))).
\end{aligned}$$

Exploiting that $U(A)$ is (positive) fibrant in $\mathrm{Sp}^\Sigma(\mathcal{S}_*, S^1)$, this is isomorphic to

$$\begin{aligned}
\mathrm{colim}_u \pi_{l+u}(\Omega^{m_2}(U(A)(m_1+u))) &\cong \mathrm{colim}_u \pi_l(\Omega^{m_2+u}(U(A)(m_1+u)))\\
&\cong \pi_l(\Omega^{m_2}(U(A)(m_1))).
\end{aligned}$$

By Proposition 2.36 the latter is isomorphic to $\pi_l(U(\Omega^{\bar{\mathcal{J}}}(A))(\mathbf{m}_1,\mathbf{m}_2),*)$.

Example 2.39. (compare [Sag14, Example 3.4]) Let A be a positive fibrant object in $C(\mathrm{Sp}^\Sigma(s\mathrm{mod}))$, and let $(\mathbf{m}_1,\mathbf{m}_2)$ be an object in $\bar{\mathcal{J}}_+$. Let $[x]$ be a homotopy class in $\pi_{m_2-m_1}(U(A))$ represented by a map $x\colon S^{m_2}\to U(A)(m_1)$ in $\mathcal{S}_*$. The latter corresponds to a point in $U(\Omega^{\bar{\mathcal{J}}}(A))(\mathbf{m}_1, \mathbf{m}_2)$ (see Remark 2.38). By adjunction we obtain a map of commutative $\bar{\mathcal{J}}$-spaces $\alpha\colon \mathbb{C}(F^{\bar{\mathcal{J}}}_{(\mathbf{m}_1,\mathbf{m}_2)}(*)) \to U(\Omega^{\bar{\mathcal{J}}}(A))$. We set $C(x) = \mathbb{C}(F^{\bar{\mathcal{J}}}_{(\mathbf{m}_1,\mathbf{m}_2)}(*))$ and call the pre-log structure $(C(x), \alpha)$ the *free pre-log structure* on A. This yields the pre-log cdga $(A, C(x), \alpha)$. Analogously to [SS13, Example 3.7] we compute that

$$\begin{aligned}
\mathrm{hocolim}_{\bar{\mathcal{J}}} C(x) &\cong \mathrm{hocolim}_{\bar{\mathcal{J}}} \coprod_{n\geq 0} \bar{\mathcal{J}}((\mathbf{m}_1^{\sqcup n}, \mathbf{m}_2^{\sqcup n}), -)/\Sigma_n \\
&\cong \coprod_{n\geq 0} (\mathrm{hocolim}_{\bar{\mathcal{J}}} \bar{\mathcal{J}}((\mathbf{m}_1^{\sqcup n}, \mathbf{m}_2^{\sqcup n}), -))/\Sigma_n \\
&\cong \coprod_{n\geq 0} B((\mathbf{m}_1^{\sqcup n}, \mathbf{m}_2^{\sqcup n}) \downarrow \bar{\mathcal{J}})/\Sigma_n.
\end{aligned}$$

Taking into account that $m_1 \geq 1$, the space $B((\mathbf{m}_1^{\sqcup n}, \mathbf{m}_2^{\sqcup n}) \downarrow \bar{\mathcal{J}})$ carries a free Σ_n-action. Further, since $((\mathbf{m}_1^{\sqcup n}, \mathbf{m}_2^{\sqcup n}), \mathrm{id}_{(\mathbf{m}_1^{\sqcup n}, \mathbf{m}_2^{\sqcup n})})$ is the initial object in the comma category $((\mathbf{m}_1^{\sqcup n}, \mathbf{m}_2^{\sqcup n}) \downarrow \bar{\mathcal{J}})$, the space $B((\mathbf{m}_1^{\sqcup n}, \mathbf{m}_2^{\sqcup n}) \downarrow \bar{\mathcal{J}})$ is contractible. Thus, the space $\coprod_{n\geq 0} B((\mathbf{m}_1^{\sqcup n}, \mathbf{m}_2^{\sqcup n}) \downarrow \bar{\mathcal{J}})/\Sigma_n$ is weakly equivalent to $\coprod_{n\geq 0} B\Sigma_n$.

2.6 Units, log structures and logification

In this subsection we define *units* of commutative $\bar{\mathcal{J}}$-spaces and of commutative symmetric ring spectra in simplicial k-modules. With this notion at hand, we can formulate a condition for a pre-log cdga to be a *log cdga*. Moreover, we explain a construction called *logification*, which turns a pre-log cdga into a log cdga.

Let $(\mathcal{K}, \sqcup, \mathbf{0}_{\mathcal{K}})$ be a small symmetric monoidal category. The homotopy colimit functor $\mathrm{hocolim}_{\mathcal{K}}$ (see (2.1)) is lax monoidal. For $\mathcal{K}$-spaces M and N, there is a natural composite

$$\begin{aligned}
&\mathrm{hocolim}_{\mathcal{K}} M \times \mathrm{hocolim}_{\mathcal{K}} N \xrightarrow{\cong} \mathrm{hocolim}_{(\mathbf{k},\mathbf{l})\in\mathcal{K}\times\mathcal{K}} M(\mathbf{k}) \times N(\mathbf{l}) \\
&\to \mathrm{hocolim}_{(\mathbf{k},\mathbf{l})\in\mathcal{K}\times\mathcal{K}} (M \boxtimes N)(\mathbf{k} \sqcup \mathbf{l}) \to \mathrm{hocolim}_{\mathcal{K}} (M \boxtimes N)
\end{aligned} \tag{2.44}$$

where the second map is determined by the universal natural transformation of $(\mathcal{K} \times \mathcal{K})$-diagrams $M(\mathbf{k}) \times N(\mathbf{l}) \to (M \boxtimes N)(\mathbf{k} \sqcup \mathbf{l})$ and the third map is induced by the monoidal structure of $\mathcal{K}$ (compare [SS13, p. 641]). So for a commutative $\mathcal{K}$-space M, the space $\mathrm{hocolim}_{\mathcal{K}} M$ is a simplicial monoid with product

$$\begin{aligned}
&\mathrm{hocolim}_{\mathcal{K}} M \times \mathrm{hocolim}_{\mathcal{K}} M \xrightarrow{\cong} \mathrm{hocolim}_{(\mathbf{k},\mathbf{l})\in\mathcal{K}\times\mathcal{K}} M(\mathbf{k}) \times M(\mathbf{l}) \\
&\to \mathrm{hocolim}_{(\mathbf{k},\mathbf{l})\in\mathcal{K}\times\mathcal{K}} M(\mathbf{k} \sqcup \mathbf{l}) \to \mathrm{hocolim}_{\mathcal{K}} M.
\end{aligned}$$

As the symmetry isomorphism $\chi_{\mathbf{k},\mathbf{l}}\colon \mathbf{k} \sqcup \mathbf{l} \to \mathbf{l} \sqcup \mathbf{k}$ in $\mathcal{K}$ can differ from the identity, the simplicial monoid $\mathrm{hocolim}_{\mathcal{K}} M$ is not commutative in general (see [Sag16, pp. 1209-1210]). Arguing as in [Sch09, §6.1], one can show that the space $\mathrm{hocolim}_{\mathcal{K}} M$ is an E_∞ space. In particular, the monoid of connected components $\pi_0(\mathrm{hocolim}_{\mathcal{K}} M)$ is commutative since the induced morphism $M(\chi_{\mathbf{k},\mathbf{l}})\colon M(\mathbf{k} \sqcup \mathbf{l}) \to M(\mathbf{l} \sqcup \mathbf{k})$ provides a path between the products xy and yx of elements x in $M(\mathbf{k})$ and y in $M(\mathbf{l})$. We recall the following definition from [Sag16].

Definition 2.40. [Sag16, Definition 2.5] Let M be a commutative $\mathcal{K}$-space.

(i) If the commutative monoid $\pi_0(\mathrm{hocolim}_{\mathcal{K}} M)$ is a group, we say that M is *grouplike*.

(ii) The *units* $M^\times$ of M is the grouplike sub commutative $\mathcal{K}$-space of M with $M^\times(\mathbf{k})$ consisting of those path components of $M(\mathbf{k})$ which map to a unit in the commutative monoid $\pi_0(\mathrm{hocolim}_{\mathcal{K}} M)$.

Remark 2.41. Let $\mathcal{K}$ be a permutative well-structured index category. Suppose that the inclusion functor $\mathcal{K}_+ \to \mathcal{K}$ is homotopy right cofinal, and that the pair $(\mathcal{K}, \mathcal{OK}_+)$ is very well-structured. In Subsection 3.2 we state Sagave and Schlichtkrull's chain of Quillen equivalences between commutative $\mathcal{K}$-spaces and E_∞ spaces over the classifying space $B\mathcal{K}$ under which a commutative $\mathcal{K}$-space M corresponds to the E_∞ space $\mathrm{hocolim}_{\mathcal{K}} M$ over $B\mathcal{K}$. Exploiting this, we can justify the definition of a grouplike commutative $\mathcal{K}$-space (see the beginning of Subsection 3.7).
The inclusion of the units $M^\times \to M$ realizes the inclusion $(\pi_0(\mathrm{hocolim}_{\mathcal{K}} M))^\times \to \pi_0(\mathrm{hocolim}_{\mathcal{K}} M)$ (see [Sag16, p. 1210]).

Furthermore, Remark 2.38 motivates the following definition.

Definition 2.42. (compare [SS12, Definition 4.25]) Let A be a positive fibrant object in $C(\mathrm{Sp}^\Sigma(s\mathrm{mod}))$. The *units* of A, denoted by $\mathrm{GL}_1^{\bar{\mathcal{J}}}(A)$, is the commutative $\bar{\mathcal{J}}$-space $(\Omega^{\bar{\mathcal{J}}}(A))^\times$.

Remark 2.43. (compare [SS12, p. 2137], [Sag14, p. 460]) Let A be positive fibrant in $C(\mathrm{Sp}^\Sigma(s\mathrm{mod}))$. The inclusion of the units $i_A\colon \mathrm{GL}_1^{\bar{\mathcal{J}}}(A) \to \Omega^{\bar{\mathcal{J}}}(A)$ is an inclusion of path components and hence a positive $\bar{\mathcal{J}}$-fibration in $C\mathcal{S}^{\bar{\mathcal{J}}}$.

Definition 2.44. (compare [SS12, Definition 4.33]) Let A be a positive fibrant object in $C(\mathrm{Sp}^\Sigma(s\mathrm{mod}))$, and let (M, α) be a pre-log structure on A. We consider the pullback diagram

$$\begin{tikzcd} \alpha^{-1}(\mathrm{GL}_1^{\bar{\mathcal{J}}}(A)) \arrow[r] \arrow[d] & \mathrm{GL}_1^{\bar{\mathcal{J}}}(A) \arrow[d, "i_A"] \\ M \arrow[r, "\alpha"] & \Omega^{\bar{\mathcal{J}}}(A). \end{tikzcd} \tag{2.45}$$

If the base change map $\alpha^{-1}(\mathrm{GL}_1^{\bar{\mathcal{J}}}(A)) \to \mathrm{GL}_1^{\bar{\mathcal{J}}}(A)$ in (2.45) is a weak equivalence, the pair (M, α) is a *log structure* on A. We then call the triple (A, M, α) a *log cdga*.

As the positive projective $\bar{\mathcal{J}}$-model structure on commutative $\bar{\mathcal{J}}$-spaces is right proper (see Remark 2.10), the above condition for a pre-log cdga to be a log cdga is homotopy invariant.

Example 2.45. (compare [SS12, p. 2142]) The inclusion of the units $i_A\colon \mathrm{GL}_1^{\bar{\mathcal{J}}}(A) \to \Omega^{\bar{\mathcal{J}}}(A)$ induces the *trivial* log cdga $(A, \mathrm{GL}_1^{\bar{\mathcal{J}}}(A), i_A)$.

Example 2.46. (compare [Sag14, Example 3.9]) Let (B, N, β) be a log cdga, and let $f\colon A \to B$ be a morphism between positive fibrant objects in $C(\mathrm{Sp}^{\Sigma}(s\mathrm{mod}))$. Assume that either the morphism β is a positive $\bar{\mathcal{J}}$-fibration or that the morphism f is a positive fibration. Without loss of generality suppose that the former holds. We consider the commutative cube

$$\begin{tikzcd}
 & (\mathrm{GL}_1^{\bar{\mathcal{J}}}(f))_*(\beta^{-1}(\mathrm{GL}_1^{\bar{\mathcal{J}}}(B))) \arrow[rr, two heads] \arrow[dl] \arrow[dd] & & \mathrm{GL}_1^{\bar{\mathcal{J}}}(A) \arrow[dl, "\mathrm{GL}_1^{\bar{\mathcal{J}}}(f)"] \arrow[dd, "i_A"] \\
\beta^{-1}(\mathrm{GL}_1^{\bar{\mathcal{J}}}(B)) \arrow[rr, two heads, "\sim"] \arrow[dd] & & \mathrm{GL}_1^{\bar{\mathcal{J}}}(B) \arrow[dd, "i_B"] & \\
 & (\Omega^{\bar{\mathcal{J}}}(f))_*(N) \arrow[rr, two heads, "(\Omega^{\bar{\mathcal{J}}}(f))_*(\beta)"] \arrow[dl] & & \Omega^{\bar{\mathcal{J}}}(A) \arrow[dl, "\Omega^{\bar{\mathcal{J}}}(f)"] \\
N \arrow[rr, two heads, "\beta"] & & \Omega^{\bar{\mathcal{J}}}(B). &
\end{tikzcd}$$

The front, top and bottom square are pullback squares so that the back square is a pullback square, too. Thus, the commutative $\bar{\mathcal{J}}$-space $(\mathrm{GL}_1^{\bar{\mathcal{J}}}(f))_*(\beta^{-1}(\mathrm{GL}_1^{\bar{\mathcal{J}}}(B)))$ is isomorphic to the pullback $((\Omega^{\bar{\mathcal{J}}}(f))_*(\beta))^{-1}(\mathrm{GL}_1^{\bar{\mathcal{J}}}(A))$. Due to (B, N, β) being a log cdga, the base change map $\beta^{-1}(\mathrm{GL}_1^{\bar{\mathcal{J}}}(B)) \to \mathrm{GL}_1^{\bar{\mathcal{J}}}(B)$ is a $\bar{\mathcal{J}}$-equivalence. Therefore, the base change map $((\Omega^{\bar{\mathcal{J}}}(f))_*(\beta))^{-1}(\mathrm{GL}_1^{\bar{\mathcal{J}}}(A)) \to \mathrm{GL}_1^{\bar{\mathcal{J}}}(A)$ is a $\bar{\mathcal{J}}$-equivalence so that the triple $(A, (\Omega^{\bar{\mathcal{J}}}(f))_*(N), (\Omega^{\bar{\mathcal{J}}}(f))_*(\beta))$ is a log cdga, and

$$(A, (\Omega^{\bar{\mathcal{J}}}(f))_*(N), (\Omega^{\bar{\mathcal{J}}}(f))_*(\beta)) \to (B, N, \beta)$$

is a morphism of log cdgas.

The following construction associates a log cdga to a pre-log cdga.

Construction 2.47. (compare [RSS15, Construction 4.22], [Sag14, Construction 3.11]) Let (A, M, α) be a pre-log cdga where A is positive fibrant. We factor the base change map $\alpha^{-1}(\mathrm{GL}_1^{\bar{\mathcal{J}}}(A)) \to \mathrm{GL}_1^{\bar{\mathcal{J}}}(A)$ in (2.45) into a positive cofibration followed by a positive acyclic $\bar{\mathcal{J}}$-fibration,

$$\begin{tikzcd}
\alpha^{-1}(\mathrm{GL}_1^{\bar{\mathcal{J}}}(A)) \arrow[r, tail] \arrow[rr, bend left] & G \arrow[r, two heads, "\sim"] & \mathrm{GL}_1^{\bar{\mathcal{J}}}(A).
\end{tikzcd}$$

We define the commutative $\bar{\mathcal{J}}$-space M^a by the pushout square

$$\begin{tikzcd}
\alpha^{-1}(\mathrm{GL}_1^{\bar{\mathcal{J}}}(A)) \arrow[r, tail] \arrow[d] & G \arrow[d] \\
M \arrow[r, tail, "f^b"] & M^a.
\end{tikzcd} \tag{2.46}$$

Since the positive projective $\bar{\mathcal{J}}$-model structure on $C\mathcal{S}^{\bar{\mathcal{J}}}$ is left proper (see Remark 2.10), the square (2.46) is actually a homotopy pushout square. The maps $\alpha\colon M \to \Omega^{\bar{\mathcal{J}}}(A)$ and

$$G \xrightarrow{\sim} \mathrm{GL}_1^{\bar{\mathcal{J}}}(A) \xrightarrow{i_A} \Omega^{\bar{\mathcal{J}}}(A)$$

give rise to a map $\tilde{\alpha}\colon M^a \to \Omega^{\bar{\mathcal{J}}}(A)$ by the universal property of the pushout. Let $\mathrm{ad}(\tilde{\alpha})\colon \Lambda^{\bar{\mathcal{J}}}(M^a) \to A$ be the adjoint map in $C(\mathrm{Sp}^{\Sigma}(s\mathrm{mod}))$. We factor the map

$$A \wedge_{\Lambda^{\bar{\mathcal{J}}}(M)} \Lambda^{\bar{\mathcal{J}}}(M^a) \xrightarrow{\mathrm{id}_A \wedge_{\Lambda^{\bar{\mathcal{J}}}(M)} \mathrm{ad}(\tilde{\alpha})} A \wedge_{\Lambda^{\bar{\mathcal{J}}}(M)} A \to A,$$

into a positive cofibration followed by a positive acyclic fibration in $C(\mathrm{Sp}^{\Sigma}(s\mathrm{mod}))$,

$$\begin{tikzcd} A \wedge_{\Lambda^{\bar{\mathcal{J}}}(M)} \Lambda^{\bar{\mathcal{J}}}(M^a) \arrow[r, tail, "i"] \arrow[rr, bend left] & A^a \arrow[r, two heads, "\sim"] & A. \end{tikzcd}$$

Let the map $\alpha^a\colon M^a \to \Omega^{\bar{\mathcal{J}}}(A^a)$ in $C\mathcal{S}^{\bar{\mathcal{J}}}$ be the adjoint of the map

$$\Lambda^{\bar{\mathcal{J}}}(M^a) \xrightarrow{g} A \wedge_{\Lambda^{\bar{\mathcal{J}}}(M)} \Lambda^{\bar{\mathcal{J}}}(M^a) \overset{i}{\rightarrowtail} A^a$$

in $C(\mathrm{Sp}^{\Sigma}(s\mathrm{mod}))$ where the map g is the cobase change map. The diagram

$$\begin{tikzcd} \Lambda^{\bar{\mathcal{J}}}(M) \arrow[r, tail, "\Lambda^{\bar{\mathcal{J}}}(f^b)"] \arrow[d, "\mathrm{ad}(\alpha)"'] & \Lambda^{\bar{\mathcal{J}}}(M^a) \arrow[d, "g"] \arrow[dr, "\mathrm{ad}(\alpha^a)"] & \\ A \arrow[r, tail, "h"] \arrow[rr, tail, bend right, "f"'] & A \wedge_{\Lambda^{\bar{\mathcal{J}}}(M)} \Lambda^{\bar{\mathcal{J}}}(M^a) \arrow[r, tail, "i"] & A^a, \end{tikzcd}$$

where the map h is the cobase change map and the map f is defined by composition, commutes. Hence, we obtain a map of pre-log cdgas $(f, f^b)\colon (A, M, \alpha) \to (A^a, M^a, \alpha^a)$. If the pre-log cdga (A, M, α) is cofibrant, then so is (A^a, M^a, α^a).

Lemma 2.48. *Let (A, M, α) be a pre-log cdga where A is positive fibrant. The pre-log cdga (A^a, M^a, α^a) is a log cdga. If the pre-log cdga (A, M, α) is a log cdga, the map $(f, f^b)\colon (A, M, \alpha) \to (A^a, M^a, \alpha^a)$ is a weak equivalence.*

We call the map $(f, f^b)\colon (A, M, \alpha) \to (A^a, M^a, \alpha^a)$ the *logification* of (A, M, α).

Proof. The proof is analogous to the proof of [Sag14, Lemma 3.12]. Let $\widehat{\mathrm{GL}_1^{\bar{\mathcal{J}}}(A)}$ be the sub $\bar{\mathcal{J}}$-space of $\Omega^{\bar{\mathcal{J}}}(A)$ with $\widehat{\mathrm{GL}_1^{\bar{\mathcal{J}}}(A)}(\mathbf{m}_1, \mathbf{m}_2)$ consisting of those path components of $\Omega^{\bar{\mathcal{J}}}(A)(\mathbf{m}_1, \mathbf{m}_2)$ which do not map to a unit in the commutative monoid $\pi_0(\mathrm{hocolim}_{\bar{\mathcal{J}}} \Omega^{\bar{\mathcal{J}}}(A))$. In other words, $\widehat{\mathrm{GL}_1^{\bar{\mathcal{J}}}(A)}$ denotes the complement of $\mathrm{GL}_1^{\bar{\mathcal{J}}}(A)$

in $\Omega^{\bar{\mathcal{J}}}(A)$. Further, let $M = \widetilde{M} \coprod \widehat{M}$ be the decomposition of the underlying $\bar{\mathcal{J}}$-space of M into the part $\widetilde{M} = M \times_{\Omega^{\bar{\mathcal{J}}}(A)} \mathrm{GL}_1^{\bar{\mathcal{J}}}(A)$ which maps to the units and the part $\widehat{M} = M \times_{\Omega^{\bar{\mathcal{J}}}(A)} \widehat{\mathrm{GL}_1^{\bar{\mathcal{J}}}(A)}$ which maps to the nonunits. There are isomorphisms

$$\begin{aligned} M^a &= M \boxtimes_{\alpha^{-1}(\mathrm{GL}_1^{\bar{\mathcal{J}}}(A))} G \\ &= M \boxtimes_{\widetilde{M}} G \\ &\cong (\widetilde{M} \coprod \widehat{M}) \boxtimes_{\widetilde{M}} G \\ &\cong G \coprod (\widehat{M} \boxtimes_{\widetilde{M}} G). \end{aligned}$$

Note that G maps to the units because the map $G \to \mathrm{GL}_1^{\bar{\mathcal{J}}}(A)$ is a positive acyclic $\bar{\mathcal{J}}$-fibration, and that $\widehat{M} \boxtimes_{\widetilde{M}} G$ maps to the nonunits. Therefore, we get that

$$\begin{aligned} (\alpha^a)^{-1}(\mathrm{GL}_1^{\bar{\mathcal{J}}}(A^a)) &= M^a \times_{\Omega^{\bar{\mathcal{J}}}(A^a)} \mathrm{GL}_1^{\bar{\mathcal{J}}}(A^a) \\ &\cong G \end{aligned}$$

which is $\bar{\mathcal{J}}$-equivalent to $\mathrm{GL}_1^{\bar{\mathcal{J}}}(A^a)$. We conclude that the triple (A^a, M^a, α^a) is a log cdga.

Suppose that the triple (A, M, α) is a log cdga. Since the map $\alpha^{-1}(\mathrm{GL}_1^{\bar{\mathcal{J}}}(A)) \to G$ is then a positive acyclic cofibration, the cobase change map $f^b \colon M \to M^a$ is a positive acyclic cofibration. This together with the map $f \colon A \to A^a$ being a stable equivalence, yields that the map $(f, f^b) \colon (A, M, \alpha) \to (A^a, M^a, \alpha^a)$ is a weak equivalence. $\square$

2.7 An approach via diagram chain complexes

We begin with collecting a few results about diagram chain complexes. We present the idea of an approach to define pre-log structures on E_∞ dgas via diagram chain complexes and explain the reasons why we have refrained from this. In connection to this, we provide a homotopy colimit formula for diagram chain complexes, and argue that diagram chain complexes do not have to admit a model structure in which the fibrant objects are precisely the objects that are homologically constant and the homotopy colimit functor detects the weak equivalences.

Let $(\mathcal{K}, \sqcup, \mathbf{0}_\mathcal{K})$ be a small symmetric monoidal category. A $\mathcal{K}$-chain complex is a functor $X \colon \mathcal{K} \to \mathrm{Ch}$. The category of $\mathcal{K}$-chain complexes is the functor category $\mathrm{Ch}^\mathcal{K}$. The symmetric monoidal structures of $\mathcal{K}$ and Ch give rise to the Day convolution product of $\mathrm{Ch}^\mathcal{K}$ (see [Day70a, §3.2], [Day70b, §4], compare Subsection 2.1). For an object $\mathbf{k}$ in the category $\mathcal{K}$, the evaluation functor $\mathrm{Ev}_\mathbf{k}^\mathcal{K} \colon \mathrm{Ch}^\mathcal{K} \to \mathrm{Ch}$ sends a $\mathcal{K}$-chain complex X to the chain complex $\mathrm{Ev}_\mathbf{k}^\mathcal{K}(X) = X(\mathbf{k})$. Its left adjoint $F_\mathbf{k}^\mathcal{K} \colon \mathrm{Ch} \to \mathrm{Ch}^\mathcal{K}$ takes a chain complex T to the $\mathcal{K}$-chain complex $F_\mathbf{k}^\mathcal{K}(T) = \mathbb{S}^0(k(\mathcal{K}(\mathbf{k}, -))) \otimes T$.

Recall that the category of chain complexes has a projective model structure [Hov99, Definition 2.3.3] that is cofibrantly generated with $I_{\mathrm{Ch}} = \{i_l \colon \mathbb{S}^l \to \mathbb{D}^{l+1}, l \in \mathbb{Z}\}$ as its set of generating cofibrations, $J_{\mathrm{Ch}} = \{0 \to \mathbb{D}^l, l \in \mathbb{Z}\}$ as its set of generating acyclic cofibrations and homology isomorphisms as its weak equivalences [Hov99, Theorem 2.3.11]. The category of $\mathcal{K}$-chain complexes then carries a cofibrantly generated projective level model structure with $I^{\mathrm{level}}_{\mathrm{Ch}^{\mathcal{K}}} = \{F^{\mathcal{K}}_{\mathbf{k}}(i), \mathbf{k} \in \mathcal{K}, i \in I_{\mathrm{Ch}}\}$ as its set of generating cofibrations and $J^{\mathrm{level}}_{\mathrm{Ch}^{\mathcal{K}}} = \{F^{\mathcal{K}}_{\mathbf{k}}(j), \mathbf{k} \in \mathcal{K}, j \in J_{\mathrm{Ch}}\}$ as its set of generating acyclic cofibrations. In this model structure a map is a weak equivalence/ fibration if it is so $\mathcal{K}$-levelwise in chain complexes. The cofibrations are determined by the left lifting property with respect to acyclic fibrations [Hir03, Theorem 11.6.1]. A cofibration in the projective level model structure is $\mathcal{K}$-levelwise a cofibration in chain complexes [Hir03, Proposition 11.6.3]. In particular, this is $\mathcal{K}$-levelwise in every chain degree a monomorphism, that is, this is $\mathcal{K}$-levelwise a cofibration in the injective model structure on chain complexes [Hov99, Theorem 2.3.13].

A model for the homotopy colimit on diagram chain complexes is provided by an algebraic analogon of the Bousfield-Kan homotopy colimit on diagram spaces (see [BK72, XII. §5], (2.1)). It is the composition of a *simplicial replacement functor* and a suitable substitute for the diagonal functor. We explain these functors and show that their composition defines a model for the homotopy colimit functor.

Definition 2.49. (see [Joa11, Definition 4.3], [RG14, Definition 2.5]) For a $\mathcal{K}$-chain complex X, we define the *simplicial replacement* of X as the simplicial chain complex $\mathrm{sr}(X)$ which in simplicial degree $[p]$ is given by

$$\mathrm{sr}(X)[p] = \bigoplus_{\mathbf{k}_0 \xleftarrow{\alpha_1} \dots \xleftarrow{\alpha_p} \mathbf{k}_p \in \mathcal{N}\mathcal{K}[p]} X(\mathbf{k}_p).$$

The face maps $d_i \colon \mathrm{sr}(X)[p] \to \mathrm{sr}(X)[p-1]$ for $0 \le i \le p$, are specified as follows. A summand $X(\mathbf{k}_p)$ indexed by

$$\mathbf{k}_0 \xleftarrow{\alpha_1} \dots \xleftarrow{\alpha_p} \mathbf{k}_p$$

is mapped by

- the identity to $X(\mathbf{k}_p)$ indexed by

$$\mathbf{k}_1 \xleftarrow{\alpha_2} \dots \xleftarrow{\alpha_p} \mathbf{k}_p$$

  for $i = 0$,

- the identity to $X(\mathbf{k}_p)$ indexed by

$$\mathbf{k}_0 \xleftarrow{\alpha_1} \dots \xleftarrow{\alpha_{i-1}} \mathbf{k}_{i-1} \xleftarrow{\alpha_i \circ \alpha_{i+1}} \mathbf{k}_{i+1} \xleftarrow{\alpha_{i+2}} \dots \xleftarrow{\alpha_p} \mathbf{k}_p$$

  for $1 \le i \le p-1$,

- $X(\alpha_p)$ to $X(\mathbf{k}_{p-1})$ indexed by

$$\mathbf{k}_0 \xleftarrow{\alpha_1} \dots \xleftarrow{\alpha_{p-1}} \mathbf{k}_{p-1}$$

  for $i = p$.

The degeneracy maps $s_j\colon \mathrm{sr}(X)[p] \to \mathrm{sr}(X)[p+1]$ for $0 \le j \le p$, send a summand $X(\mathbf{k}_p)$ indexed by

$$\mathbf{k}_0 \xleftarrow{\alpha_1} \dots \xleftarrow{\alpha_p} \mathbf{k}_p$$

by the identity to $X(\mathbf{k}_p)$ indexed by

$$\mathbf{k}_0 \xleftarrow{\alpha_1} \dots \xleftarrow{\alpha_j} \mathbf{k}_j \xleftarrow{\mathrm{id}} \mathbf{k}_j \xleftarrow{\alpha_{j+1}} \mathbf{k}_{j+1} \xleftarrow{\alpha_{j+2}} \dots \xleftarrow{\alpha_p} \mathbf{k}_p.$$

This definition is functorial. We call the functor $\mathrm{Ch}^{\mathcal{K}} \to \mathrm{Ch}^{\Delta^{\mathrm{op}}}$ the *simplicial replacement functor* (sr).

Recall that a *double complex* Z is a family $\{Z_{p,q}, (p,q) \in \mathbb{Z} \times \mathbb{Z}\}$ of k-modules together with k-linear *horizontal differentials* $d^h\colon Z_{p,q} \to Z_{p-1,q}$ and k-linear *vertical differentials* $d^v\colon Z_{p,q} \to Z_{p,q-1}$ such that $d^h \circ d^h = 0 = d^v \circ d^v$ and $d^h \circ d^v = -d^v \circ d^h$.
The *Moore functor* C associates to a simplicial chain complex Y a double complex CY concentrated in the first and fourth quadrant. The double complex CY in bidegree (p,q) is given by $CY_{p,q} = (Y[p])_q$. The horizontal differential $d^h\colon (Y[p])_q \to (Y[p-1])_q$ is defined as $d^h = \Sigma_{i=0}^{p}(-1)^i d_i$ where the d_i are the simplicial face maps of Y, and the vertical differential $d^v\colon (Y[p])_q \to (Y[p])_{q-1}$ is the differential of the chain complex $Y[p]$.
The *total complex functor* with respect to the direct sum, denoted by $\mathrm{Tot}^{\oplus}$, from double complexes to chain complexes takes a double complex Z with horizontal differentials d^h and vertical differentials d^v to $\mathrm{Tot}^{\oplus}(Z)$, which in chain degree $l \in \mathbb{Z}$ is defined by $(\mathrm{Tot}^{\oplus}(Z))_l = \bigoplus_{p+q=l} Z_{p,q}$. The differential $d^{\mathrm{Tot}^{\oplus}(Z)}\colon (\mathrm{Tot}^{\oplus}(Z))_l \to (\mathrm{Tot}^{\oplus}(Z))_{l-1}$ is given by $d^{\mathrm{Tot}^{\oplus}(Z)}(x) = (-1)^q d^h(x) + d^v(x)$ for every homogeneous element $x \in Z_{p,q}$. (Concerning the signs, we follow the convention of Rodríguez González (see [RG, pp. 153-154, pp. 157-158], [RG12, Example (2.4)]).)
The composition of the Moore functor and the total complex functor serves as a substitute for the diagonal functor in the topological setting. Next we prove that the following composition of functors

$$\mathrm{Ch}^{\mathcal{K}} \xrightarrow{\mathrm{sr}} \mathrm{Ch}^{\Delta^{\mathrm{op}}} \xrightarrow{C} \mathrm{ch}(\mathrm{Ch}) \xrightarrow{\mathrm{Tot}^{\oplus}} \mathrm{Ch} \tag{2.47}$$

defines a model for the homotopy colimit functor. For this, we first show that the functor $\mathrm{Tot}^{\oplus} \circ C \circ \mathrm{sr}$ preserves level equivalences. Secondly, we define a natural transformation from the functor $\mathrm{Tot}^{\oplus} \circ C \circ \mathrm{sr}$ to the colimit functor $\mathrm{colim}_{\mathcal{K}}$, and notice that for a cofibrant $\mathcal{K}$-chain complex X, the map $\mathrm{Tot}^{\oplus}(C\mathrm{sr}(X)) \to \mathrm{colim}_{\mathcal{K}} X$ is a homology isomorphism.

Remark 2.50. We could replace the Moore functor C by the normalization functor N. The latter takes a simplicial chain complex Y to the double complex CY/DY where DY is the degenerate sub double chain complex of CY, that is, for horizontal degree $p \ge 0$, the chain complex $DY_{p,*}$ is generated by the images of the degeneracies s_j, so that $DY_{p,q} = \Sigma_{j=0}^{p-1} s_j(CY_{p-1,q})$ (see [Wei94, p. 266]).

Let Z be a double complex. To compute the homology of $\mathrm{Tot}^{\oplus}(Z)$ we can use the spectral sequence $\{E^r_{p,q}(Z), d^r\}$ that arises from filtering $\mathrm{Tot}^{\oplus}(Z)$ by the columns of Z.

This spectral sequence starts with $E^0_{p,q}(Z) = Z_{p,q}$ and the zeroth differentials d^0 are the vertical differentials d^v of Z so that $E^1_{p,q}(Z) = H^v_q(Z_{p,*})$. The first differentials $d^1 \colon H^v_q(Z_{p,*}) \to H^v_q(Z_{p-1,*})$ are induced on homology from the horizontal differentials d^h of Z. Since for a $\mathcal{K}$-chain complex X, the double complex $C\mathrm{sr}(X)$ is sitting in the first and fourth quadrant, the spectral sequence $E^r_{p,q}(C\mathrm{sr}(X))$ converges to $H_{p+q}(\mathrm{Tot}^{\oplus}(C\mathrm{sr}(X)))$ (see [Wei94, pp. 141-142]).

Proposition 2.51. *If $f \colon X \to Y$ is a level equivalence in $\mathcal{K}$-chain complexes, the induced map $\mathrm{Tot}^{\oplus}(C\mathrm{sr}(f))$ is a homology isomorphism.*

Proof. As homology commutes with direct sums, in every simplicial degree $[p]$ the map $\mathrm{sr}(f)[p] \colon \mathrm{sr}(X)[p] \to \mathrm{sr}(Y)[p]$ is a homology isomorphism. We consider the spectral sequences $\{E^r_{p,q}(C\mathrm{sr}(X)), d^r\}$ and $\{E^r_{p,q}(C\mathrm{sr}(Y)), d^r\}$ obtained by filtering $\mathrm{Tot}^{\oplus}(C\mathrm{sr}(X))$ by the colums of $C\mathrm{sr}(X)$, and $\mathrm{Tot}^{\oplus}(C\mathrm{sr}(Y))$ by the colums of $C\mathrm{sr}(Y)$ respectively. Because the induced map

$$E^1_{p,q}(C\mathrm{sr}(X)) = H^v_q(C\mathrm{sr}(X)_{p,*}) \xrightarrow{H^v_q(C\mathrm{sr}(f)_{p,*})} H^v_q(C\mathrm{sr}(Y)_{p,*}) = E^1_{p,q}(C\mathrm{sr}(Y))$$

is an isomorphism for all p and q, the map

$$H_*(\mathrm{Tot}^{\oplus}(C\mathrm{sr}(X))) \xrightarrow{H_*(\mathrm{Tot}^{\oplus}(C\mathrm{sr}(f)))} H_*(\mathrm{Tot}^{\oplus}(C\mathrm{sr}(Y)))$$

is an isomorphism by [Wei94, Comparison theorem 5.2.12]. $\square$

Let X be a $\mathcal{K}$-chain complex. We define a map

$$\partial_1 \colon \mathrm{sr}(X)[1] = \bigoplus_{\mathbf{k}_0 \xleftarrow{\alpha_1} \mathbf{k}_1 \in \mathcal{NK}[1]} X(\mathbf{k}_1) \to \mathrm{sr}(X)[0] = \bigoplus_{\mathbf{k}_0 \in \mathcal{K}} X(\mathbf{k}_0)$$

that in chain degree $l \in \mathbb{Z}$ sends an element x in $(X(\mathbf{k}_1))_l$ where $X(\mathbf{k}_1)$ is indexed by $\alpha_1 \colon \mathbf{k}_1 \to \mathbf{k}_0$, to the element $x - (X(\alpha_1))_l(x)$. The map ∂_1 is a chain map because $X(\alpha_1)$ is so. The cokernel of ∂_1, denoted by $\mathrm{coker}(\partial_1)$, and determined by the pushout diagram

$$\begin{tikzcd}
\bigoplus_{\mathbf{k}_0 \xleftarrow{\alpha_1} \mathbf{k}_1 \in \mathcal{NK}[1]} X(\mathbf{k}_1) \arrow[r, "\partial_1"] \arrow[d] & \bigoplus_{\mathbf{k}_0 \in \mathcal{K}} X(\mathbf{k}_0) \arrow[d, "\pi"] \\
0 \arrow[r] & \mathrm{coker}(\partial_1),
\end{tikzcd}$$

can be identified with $\mathrm{colim}_{\mathcal{K}} X$ (see [Joa11, Remark 4.5]). We specify the required map from $\mathrm{Tot}^{\oplus}(C\mathrm{sr}(X))$ to $\mathrm{colim}_{\mathcal{K}} X$ with the help of $\mathrm{coker}(\partial_1)$.

Definition 2.52. (see [Joa11, Remark 4.5]) Let $\Psi_X \colon \mathrm{Tot}^{\oplus}(C\mathrm{sr}(X)) \to \mathrm{colim}_{\mathcal{K}} X$ be the map that in chain degree $l \in \mathbb{Z}$ is given by

$$\bigoplus_{p+q=l} \bigoplus_{\mathbf{k}_0 \xleftarrow{\alpha_1} \cdots \xleftarrow{\alpha_p} \mathbf{k}_p \in \mathcal{NK}[p]} (X(\mathbf{k}_p))_q \xrightarrow{(\Psi_X)_l} (\mathrm{colim}_{\mathcal{K}} X)_l,$$

$$x \longmapsto \begin{cases} 0, & p \geq 1, \\ \pi_l(x), & p = 0. \end{cases}$$

Lemma 2.53. *The map Ψ_X is a chain map.*

Proof. Let $l \in \mathbb{Z}$. We have to show that the diagram

$$\begin{tikzcd} \bigoplus_{p+q=l} \bigoplus_{\mathbf{k}_0 \xleftarrow{\alpha_1} \dots \xleftarrow{\alpha_p} \mathbf{k}_p \in \mathcal{N}\mathcal{K}[p]} (X(\mathbf{k}_p))_q \arrow[r, "(\Psi_X)_l"] \arrow[d, "d^{\mathrm{Tot}^{\oplus}(C\mathrm{sr}(X))}"'] & (\mathrm{colim}_{\mathcal{K}} X)_l \arrow[d, "d^{\mathrm{colim}_{\mathcal{K}} X}"] \\ \bigoplus_{p+q=l-1} \bigoplus_{\mathbf{k}_0 \xleftarrow{\alpha_1} \dots \xleftarrow{\alpha_p} \mathbf{k}_p \in \mathcal{N}\mathcal{K}[p]} (X(\mathbf{k}_p))_q \arrow[r, "(\Psi_X)_{l-1}"] & (\mathrm{colim}_{\mathcal{K}} X)_{l-1} \end{tikzcd} \tag{2.48}$$

commutes. Let x be in $(X(\mathbf{k}_p))_q$ where $X(\mathbf{k}_p)$ is indexed by

$$\mathbf{k}_0 \xleftarrow{\alpha_1} \dots \xleftarrow{\alpha_p} \mathbf{k}_p.$$

We need to consider three different cases:

- If $p = 0$ and $q = l$, the composite $(\Psi_X)_{l-1} \circ d^{\mathrm{Tot}^{\oplus}(C\mathrm{sr}(X))}$ sends x to $\pi_{l-1} d^{X(\mathbf{k}_0)}(x)$ which as π is a chain map is equal to $d^{X(\mathbf{k}_0)}\pi_l(x)$. The latter is the value of x under the composite $d^{\mathrm{colim}_{\mathcal{K}} X} \circ (\Psi_X)_l$.
- If $p = 1$ and $q = l - 1$, the differential $d^{\mathrm{Tot}^{\oplus}(C\mathrm{sr}(X))}$ maps x to
$$d^{\mathrm{Tot}^{\oplus}(C\mathrm{sr}(X))}(x) = (-1)^{l-1}(x - (X(\alpha_1))_{l-1}(x)) + d^{X(\mathbf{k}_1)}(x).$$
The map $(\Psi_X)_{l-1}$ then takes the latter to $\pi_{l-1}((-1)^{l-1}(x - (X(\alpha_1))_{l-1}(x))) + 0$ which is zero. Applying $(\Psi_X)_l$ to x is zero, too.
- If $p \geq 2$ and $q = l - p$, the differential $d^{\mathrm{Tot}^{\oplus}(C\mathrm{sr}(X))}$ sends x to
$$d^{\mathrm{Tot}^{\oplus}(C\mathrm{sr}(X))}(x) = (-1)^{l-p}\Sigma_{i=0}^{p}(-1)^i d_i(x) + d^{X(\mathbf{k}_p)}(x)$$
which is then taken to zero by the map $(\Psi_X)_{l-1}$. If we apply the map $(\Psi_X)_l$ to x, this is also zero.

In all three cases the diagram (2.48) commutes. Therefore, the map Ψ_X is a chain map. $\square$

The map Ψ_X is natural in X so that we obtain a natural transformation

$$\Psi \colon \mathrm{Tot}^{\oplus} \circ C \circ \mathrm{sr} \to \mathrm{colim}_{\mathcal{K}}.$$

Proposition 2.54. *Let X be a cofibrant $\mathcal{K}$-chain complex. The map $\Psi_X \colon \mathrm{Tot}^{\oplus}(C\mathrm{sr}(X)) \to \mathrm{colim}_{\mathcal{K}} X$ is a homology isomorphism.*

Proof. We start with the case that X is of the form $F_{\mathbf{k}}^{\mathcal{K}}(L)$ where L is a cofibrant chain complex. The double complex $C\mathrm{sr}(F_{\mathbf{k}}^{\mathcal{K}}(L))$ in bidegree (p, q) is given by

$$\begin{aligned} C\mathrm{sr}(F_{\mathbf{k}}^{\mathcal{K}}(L))_{p,q} &= \bigoplus_{\mathbf{k}_0 \xleftarrow{\alpha_1} \dots \xleftarrow{\alpha_p} \mathbf{k}_p \in \mathcal{N}\mathcal{K}[p]} (\mathbb{S}^0(k(\mathcal{K}(\mathbf{k}, \mathbf{k}_p))) \otimes L)_q \\ &\cong \bigoplus_{\mathbf{k}_0 \xleftarrow{\alpha_1} \dots \xleftarrow{\alpha_p} \mathbf{k}_p \leftarrow \mathbf{k} \in \mathcal{N}(\mathbf{k} \downarrow \mathcal{K})[p]} L_q. \end{aligned}$$

We see that $C\mathrm{sr}(F^{\mathcal{K}}_{\mathbf{k}}(L)) \cong CL[\mathcal{N}(\mathbf{k} \downarrow \mathcal{K})]$. As the comma category $(\mathbf{k} \downarrow \mathcal{K})$ has the initial object $(\mathbf{k}, \mathrm{id}_{\mathbf{k}})$, its nerve $\mathcal{N}(\mathbf{k} \downarrow \mathcal{K})$ is contractible. Hence, we deduce that

$$H_*(\mathrm{Tot}^{\oplus}(CL[\mathcal{N}(\mathbf{k} \downarrow \mathcal{K})])) \cong H_*(\mathrm{Tot}^{\oplus}(C\mathrm{const}_{\Delta^{\mathrm{op}}} L)).$$

The normalization of $\mathrm{const}_{\Delta^{\mathrm{op}}} L$ is concentrated in the zeroth column where it is L. Thus, we obtain that

$$\begin{aligned} H_*(\mathrm{Tot}^{\oplus}(C\mathrm{const}_{\Delta^{\mathrm{op}}} L)) &\cong H_*(\mathrm{Tot}^{\oplus}(N\mathrm{const}_{\Delta^{\mathrm{op}}} L)) \\ &= H_*(L). \end{aligned}$$

Moreover, it holds that $\mathrm{colim}_{\mathcal{K}} F^{\mathcal{K}}_{\mathbf{k}}(L) \cong L$. The map

$$\Psi_{F^{\mathcal{K}}_{\mathbf{k}}(L)} \colon \mathrm{Tot}^{\oplus}(C\mathrm{sr}(F^{\mathcal{K}}_{\mathbf{k}}(L))) \to \mathrm{colim}_{\mathcal{K}} F^{\mathcal{K}}_{\mathbf{k}}(L) \cong L$$

is induced by projecting $C\mathrm{sr}(F^{\mathcal{K}}_{\mathbf{k}}(L))$ onto the zeroth column and in chain degree $l \in \mathbb{Z}$ can be identified with the composite

$$\bigoplus_{p+q=l} \bigoplus_{\mathbf{k}_0 \xleftarrow{\alpha_1} \dots \xleftarrow{\alpha_p} \mathbf{k}_p \leftarrow \mathbf{k} \in \mathcal{N}(\mathbf{k}\downarrow\mathcal{K})[p]} L_q \to \bigoplus_{\mathbf{k}_0 \leftarrow \mathbf{k} \in (\mathbf{k}\downarrow\mathcal{K})} L_l \to L_l.$$

We conclude that the map $\Psi_{F^{\mathcal{K}}_{\mathbf{k}}(L)}$ induces an isomorphism in homology.
For the next step we assume that the map $\Psi_{X_0} \colon \mathrm{Tot}^{\oplus}(C\mathrm{sr}(X_0)) \to \mathrm{colim}_{\mathcal{K}} X_0$ is a homology isomorphism, and that X_1 is the pushout obtained by attaching an element of $I^{\mathrm{level}}_{\mathrm{Ch}^{\mathcal{K}}}$ to X_0. The functors $\mathrm{Tot}^{\oplus}$, C (respectively N), sr and $\mathrm{colim}_{\mathcal{K}}$ commute with colimits, so in particular with pushouts. Let $F^{\mathcal{K}}_{\mathbf{k}}(i_l) \colon F^{\mathcal{K}}_{\mathbf{k}}(\mathbb{S}^l) \to F^{\mathcal{K}}_{\mathbf{k}}(\mathbb{D}^{l+1})$ be an element in $I^{\mathrm{level}}_{\mathrm{Ch}^{\mathcal{K}}}$. The map $F^{\mathcal{K}}_{\mathbf{k}}(i_l)$ is $\mathcal{K}$-levelwise a cofibration in the injective model structure on chain complexes. The induced morphism $\mathrm{Tot}^{\oplus}(C\mathrm{sr}(F^{\mathcal{K}}_{\mathbf{k}}(i_l)))$ is still a cofibration in the injective model structure on chain complexes. We investigate the diagram

$$\begin{tikzcd}
\mathrm{Tot}^{\oplus}(C\mathrm{sr}(F^{\mathcal{K}}_{\mathbf{k}}(\mathbb{D}^{l+1}))) \arrow[d, "\Psi_{F^{\mathcal{K}}_{\mathbf{k}}(\mathbb{D}^{l+1})}"'] & \mathrm{Tot}^{\oplus}(C\mathrm{sr}(F^{\mathcal{K}}_{\mathbf{k}}(\mathbb{S}^l))) \arrow[l, "\mathrm{Tot}^{\oplus}(C\mathrm{sr}(F^{\mathcal{K}}_{\mathbf{k}}(i_l)))"'] \arrow[r] \arrow[d, "\Psi_{F^{\mathcal{K}}_{\mathbf{k}}(\mathbb{S}^l)}"] & \mathrm{Tot}^{\oplus}(C\mathrm{sr}(X_0)) \arrow[d, "\Psi_{X_0}"] \\
\mathrm{colim}_{\mathcal{K}} F^{\mathcal{K}}_{\mathbf{k}}(\mathbb{D}^{l+1}) & \mathrm{colim}_{\mathcal{K}} F^{\mathcal{K}}_{\mathbf{k}}(\mathbb{S}^l) \arrow[l, "\mathrm{colim}_{\mathcal{K}}(F^{\mathcal{K}}_{\mathbf{k}}(i_l))"'] \arrow[r] & \mathrm{colim}_{\mathcal{K}} X_0.
\end{tikzcd}$$

By assumption the vertical maps are homology isomorphisms. The gluing lemma with respect to the injective model structure on chain complexes (see [Hir03, Proposition 13.5.3]) yields that the induced map of pushouts $\Psi_{X_1} \colon \mathrm{Tot}^{\oplus}(C\mathrm{sr}(X_1)) \to \mathrm{colim}_{\mathcal{K}} X_1$ is a homology isomorphism.
In general, for a cofibrant $\mathcal{K}$-chain complex X the map $\mathrm{const}_{\mathcal{K}} 0 \to X$ is a retract of a transfinite composition of pushouts of elements in $I^{\mathrm{level}}_{\mathrm{Ch}^{\mathcal{K}}}$. We can assume that X is itself a $I^{\mathrm{level}}_{\mathrm{Ch}^{\mathcal{K}}}$-cell complex. So there is an ordinal λ and a λ-sequence $\{X_\alpha, \alpha < \lambda\}$ such that $X_0 = \mathrm{const}_{\mathcal{K}} 0$, $X = \mathrm{colim}_{\alpha<\lambda} X_\alpha$, and each of the maps $X_\alpha \to X_{\alpha+1}$ is the pushout of an element in $I^{\mathrm{level}}_{\mathrm{Ch}^{\mathcal{K}}}$. By an inductive argument we find that the map

$\Psi_{X_\alpha}\colon \mathrm{Tot}^{\oplus}(C\mathrm{sr}(X_\alpha)) \to \mathrm{colim}_{\mathcal{K}} X_\alpha$ is a homology isomorphism. It remains to show that the map

$$\mathrm{colim}_{\alpha<\lambda}\mathrm{Tot}^{\oplus}(C\mathrm{sr}(X_\alpha)) \xrightarrow{\mathrm{colim}_{\alpha<\lambda}\Psi_{X_\alpha}} \mathrm{colim}_{\alpha<\lambda}\mathrm{colim}_{\mathcal{K}} X_\alpha \tag{2.49}$$

is a homology isomorphism. For this, we consider the diagram

$$\begin{tikzcd} \mathrm{Tot}^{\oplus}(C\mathrm{sr}(X_0)) \arrow[r] \arrow[d, "\Psi_{X_0}"'] & \mathrm{Tot}^{\oplus}(C\mathrm{sr}(X_1)) \arrow[r] \arrow[d, "\Psi_{X_1}"] & \mathrm{Tot}^{\oplus}(C\mathrm{sr}(X_2)) \arrow[r] \arrow[d, "\Psi_{X_2}"] & \dots \\ \mathrm{colim}_{\mathcal{K}} X_0 \arrow[r] & \mathrm{colim}_{\mathcal{K}} X_1 \arrow[r] & \mathrm{colim}_{\mathcal{K}} X_2 \arrow[r] & \dots . \end{tikzcd} \tag{2.50}$$

In the above diagram (2.50) all objects are cofibrant in the injective model structure on chain complexes because every object is so. Since $X_\alpha \to X_{\alpha+1}$ is a cofibration in $\mathcal{K}$-chain complexes, the induced maps $\mathrm{Tot}^{\oplus}(C\mathrm{sr}(X_\alpha)) \to \mathrm{Tot}^{\oplus}(C\mathrm{sr}(X_{\alpha+1}))$ and $\mathrm{colim}_{\mathcal{K}} X_\alpha \to \mathrm{colim}_{\mathcal{K}} X_{\alpha+1}$ are cofibrations in the injective model structure on chain complexes. It then follows from [Hir03, Proposition 15.10.12] that the map $\mathrm{colim}_{\alpha<\lambda}\Psi_{X_\alpha}$ (2.49) is a homology isomorphism. □

We conclude that $\mathrm{Tot}^{\oplus} \circ C \circ \mathrm{sr}$ defines a model for the homotopy colimit functor on $\mathcal{K}$-chain complexes.

Remark 2.55. This model for a homotopy colimit functor has been studied by Rodríguez González. She introduces so-called *simplicial descent categories* ([RG, Definition 2.1.6], [RG14, Definition 3.5]) and argues that in this framework a model for the homotopy colimit functor is the composition of a simplicial replacement functor [RG14, Definition 2.5] and a simple functor, which is part of the datum of a simplicial descent category [RG14, Theorem 3.1]. The category of chain complexes is a simplicial descent category [RG, Proposition 5.2.1]. The simple functor is defined by $\mathrm{Tot}^{\oplus} \circ C$ (see [RG, p. 164]), or $\mathrm{Tot}^{\oplus} \circ N$ respectively (see [RG, Remark 5.2.3]). In [Joa11] Joachimi uses this model for a homotopy colimit functor on $\mathcal{I}$-chain complexes. Our argumentation above provides an independent proof of Rodríguez González' result.

We employ this model in the sequel. We note that we can adjust all arguments made so far in this Subsection 2.7 to restrict to $\mathcal{K}$-non-negative chain complexes $\mathrm{ch}^{\mathcal{K}}$.

Originally, we thought of using the intermediate category $C(\mathrm{Sp}^{\Sigma}(\mathrm{ch}))$ in the chain of Quillen equivalences between $C(Hk\text{-mod})$ and $E_\infty\mathrm{Ch}$ (see (2.8)) to define pre-log structures in the algebraic setting. There is an adjunction

$$\mathrm{ch}^{\bar{\mathcal{J}}} \underset{\tilde{\Omega}^{\bar{\mathcal{J}}}}{\overset{\tilde{\mathbb{S}}^{\bar{\mathcal{J}}}}{\rightleftarrows}} \mathrm{Sp}^{\Sigma}(\mathrm{ch}) \tag{2.51}$$

where for A in $\mathrm{Sp}^{\Sigma}(\mathrm{ch})$, the $\bar{\mathcal{J}}$-chain complex $\tilde{\Omega}^{\bar{\mathcal{J}}}(A)$ is defined by

$$\tilde{\Omega}^{\bar{\mathcal{J}}}(A) = \mathrm{Hom}_{\mathrm{ch}}^{\mathrm{Sp}^{\Sigma}(\mathrm{ch})}(F_{-}(\mathbb{S}^{-}), A).$$

From the latter, we deduce that for $(\mathbf{m}_1, \mathbf{m}_2)$ in $\bar{\mathcal{J}}$ and M in ch, the symmetric spectrum $\tilde{\mathbb{S}}^{\bar{\mathcal{J}}}(F^{\bar{\mathcal{J}}}_{(\mathbf{m}_1,\mathbf{m}_2)}(M))$ is specified by $\tilde{\mathbb{S}}^{\bar{\mathcal{J}}}(F^{\bar{\mathcal{J}}}_{(\mathbf{m}_1,\mathbf{m}_2)}(M)) = F_0(M) \wedge F_{m_1}(\mathbb{S}^{m_2})$, so that for X in $\mathrm{ch}^{\bar{\mathcal{J}}}$, the symmetric spectrum $\tilde{\mathbb{S}}^{\bar{\mathcal{J}}}(X)$ is determined by a suitable coequalizer. The functor $\tilde{\mathbb{S}}^{\bar{\mathcal{J}}}$ has an explicit description, namely for X in $\mathrm{ch}^{\bar{\mathcal{J}}}$, the symmetric spectrum $\tilde{\mathbb{S}}^{\bar{\mathcal{J}}}(X)$ in spectrum degree n is given by

$$\tilde{\mathbb{S}}^{\bar{\mathcal{J}}}(X)(n) = \bigoplus_{l \geq 0} \mathbb{S}^l \otimes_{k\Sigma_l} X(\mathbf{n}, \mathbf{l}). \tag{2.52}$$

As a result of the functor $F_-(\mathbb{S}^-)$ being strong symmetric monoidal by Corollary 2.22, the functor $\tilde{\mathbb{S}}^{\bar{\mathcal{J}}}$ is strong symmetric monoidal and the functor $\tilde{\Omega}^{\bar{\mathcal{J}}}$ is lax symmetric monoidal. Thus, the adjunction (2.51) lifts to the level of commutative monoids,

$$C(\mathrm{ch}^{\bar{\mathcal{J}}}) \underset{\tilde{\Omega}^{\bar{\mathcal{J}}}}{\overset{\tilde{\mathbb{S}}^{\bar{\mathcal{J}}}}{\rightleftarrows}} C(\mathrm{Sp}^{\Sigma}(\mathrm{ch})). \tag{2.53}$$

Apart from taking the index category $\bar{\mathcal{J}}$ instead of $\mathcal{J}$, this is analogous to Sagave and Schlichtkrull's adjunction $(\mathbb{S}^{\mathcal{J}}, \Omega^{\mathcal{J}})$ between (commutative) $\mathcal{J}$-spaces and (commutative) symmetric ring spectra in pointed spaces (see [SS12, §4.21]), on which the definition of pre-log structures in the topological setting is based (see [SS12, Definition 4.31]). The advantage of this approach is that the left adjoint $\tilde{\mathbb{S}}^{\bar{\mathcal{J}}}$ is easier to understand (see (2.52)) and has better properties than the functor $\Lambda^{\bar{\mathcal{J}}}$, for example in contrast to the latter, the functor $\tilde{\mathbb{S}}^{\bar{\mathcal{J}}}$ is strong symmetric monoidal. Nevertheless, one drawback of working with this adjunction is that so far, we have not figured out a reasonable model structure on $\mathrm{ch}^{\bar{\mathcal{J}}}$, beside the projective level model structure [Hir03, Theorem 11.6.1]. We would aim to specify a model structure on $\mathrm{ch}^{\bar{\mathcal{J}}}$ such that the fibrant objects are precisely the objects that are homologically constant with respect to morphisms in $\bar{\mathcal{J}}$ and such that the adjunction (2.51) is a Quillen adjunction where $\mathrm{Sp}^{\Sigma}(\mathrm{ch})$ is equipped with the projective stable model structure. For this, we proceed as in [Dug01, §5].

Let $\mathcal{K}$ be a small category. We consider the left Bousfield localization of the projective level model structure on $\mathrm{ch}^{\mathcal{K}}$ [Hir03, Theorem 11.6.1] with respect to the set

$$S = \{F^{\mathcal{K}}_{\mathbf{l}}(V) \xrightarrow{\alpha^*} F^{\mathcal{K}}_{\mathbf{k}}(V), \alpha \in \mathcal{K}(\mathbf{k}, \mathbf{l}), V \in \{\mathbb{S}^l, \mathbb{D}^{l+1}, l \in \mathbb{Z}_{\geq -1}\}\}$$

where V runs through the domains and codomains of the generating cofibrations in the projective model structure on ch. The left Bousfield localization exists, because $\mathrm{ch}^{\mathcal{K}}$ is cofibrantly generated, left proper and locally presentable, as ch is so.

Lemma 2.56. *An object X is fibrant in the localized model structure on $\mathrm{ch}^{\mathcal{K}}$ if and only if X is homologically constant with respect to morphisms in $\mathcal{K}$.*

Proof. We argue as in the proof of [Dug01, Theorem 5.2.(c)]. A $\mathcal{K}$-chain complex X is fibrant in the localized model structure if and only if X is S-local, that is, for every element $\alpha^* \colon F^{\mathcal{K}}_{\mathbf{l}}(V) \to F^{\mathcal{K}}_{\mathbf{k}}(V)$ in S, the induced map of homotopy function complexes

$$\underline{\mathrm{ch}^{\mathcal{K}}}(F^{\mathcal{K}}_{\mathbf{k}}(V), X) \xrightarrow{\underline{\mathrm{ch}^{\mathcal{K}}}(\alpha^*, \mathrm{id})} \underline{\mathrm{ch}^{\mathcal{K}}}(F^{\mathcal{K}}_{\mathbf{l}}(V), X) \tag{2.54}$$

is a weak equivalence. Using adjointness, the map (2.54) is a weak equivalence if and only if the map of homotopy function complexes

$$\underline{\mathrm{ch}}(V, X(\mathbf{k})) \xrightarrow{\underline{\mathrm{ch}}(\mathrm{id}, X(\alpha))} \underline{\mathrm{ch}}(V, X(\mathbf{l}))$$

is a weak equivalence. This holds precisely when X is homologically constant by [Dug01, Proposition A.5]. □

Dugger shows that if the nerve of the indexing category $\mathcal{K}$ is contractible, the S-local equivalences are the maps for which the induced maps of homotopy colimits are homology isomorphisms [Dug01, Theorem 5.2.(a)]. The classifying space of the category $\bar{\mathcal{J}}$ is weakly equivalent to $\mathbb{Z} \times \mathbb{R}P^\infty$ by Proposition 1.24, so in particular not contractible. However, we know that the well-structured index category $\bar{\mathcal{J}}$ (see Proposition 1.13) gives rise to the projective $\bar{\mathcal{J}}$-model structure on $\bar{\mathcal{J}}$-spaces in which the homotopy colimit functor $\mathrm{hocolim}_{\bar{\mathcal{J}}}$ detects the weak equivalences (see [SS12, Proposition 6.16], Proposition 2.7). But the following example illustrates that Dugger's result does not generalize to $\bar{\mathcal{J}}$-chain complexes.

Example 2.57. Let k be $\mathbb{Z}$. We define a $\bar{\mathcal{J}}$-chain complex X by

$$X(\mathbf{m}_1, \mathbf{m}_2) = \begin{cases} \mathbb{S}^0(\mathbb{Q}), & m_1 = m_2, \\ 0, & m_1 \neq m_2. \end{cases}$$

The category $\bar{\mathcal{J}}_0$ is isomorphic to the category $\mathcal{I} \int \bar{\Sigma}$ due to Lemma 1.22. For a morphism $(\alpha, [a])\colon (\mathbf{m}, *) \to (\mathbf{n}, *)$, let the induced map

$$X(\mathbf{m}, \mathbf{m}) \xrightarrow{X([\alpha, a\circ\alpha, a\circ(\alpha, \mathrm{incl})|_{\mathbf{n}\setminus \mathrm{im}(\alpha)}])} X(\mathbf{n}, \mathbf{n})$$

be multiplication by $\mathrm{sgn}(a)$. The $\bar{\mathcal{J}}$-chain complex X is homologically constant, and hence by Lemma 2.56 fibrant in the localized model structure on $\mathrm{ch}^{\bar{\mathcal{J}}}$.
We compute the chain complex $\mathrm{hocolim}_{\bar{\mathcal{J}}} X$. Since the inclusion functor $\bar{\mathcal{J}}_{0,(\geq 2,-)} \to \bar{\mathcal{J}}_0$ is homotopy right cofinal by Lemma 1.18, the induced map of homotopy colimits

$$\mathrm{hocolim}_{\bar{\mathcal{J}}_{0,(\geq 2,-)}} X \to \mathrm{hocolim}_{\bar{\mathcal{J}}_0} X \cong \mathrm{hocolim}_{\bar{\mathcal{J}}} X$$

is a weak equivalence by [Hir03, Theorem 19.6.7.(1)]. Recall that the category $\bar{\mathcal{J}}_{0,(\geq 2,-)}$ is isomorphic to the product category $\mathcal{I}_{\geq 2} \times \Sigma_2$ (see Remark 1.25). Using this, we obtain that

$$\begin{aligned} \mathrm{hocolim}_{\bar{\mathcal{J}}_{0,(\geq 2,-)}} X &\cong \mathrm{hocolim}_{\mathcal{I}_{\geq 2} \times \Sigma_2} X \\ &\simeq \mathrm{hocolim}_{\mathcal{I}_{\geq 2}} \mathrm{hocolim}_{\Sigma_2} X. \end{aligned}$$

Let $\mathbf{m}$ be in $\mathcal{I}_{\geq 2}$. The double complex $N\mathrm{sr}(X(\mathbf{m}, \mathbf{m}))$ is $\mathbb{Q}$ in bidegrees $(0,0)$ and $(1,0)$, and zero otherwise. The only non-trivial (horizontal) differential $\mathbb{Q} \to \mathbb{Q}$ is multiplication by 2. Thus, the homology of $\mathrm{hocolim}_{\Sigma_2} X(\mathbf{m}, \mathbf{m}) = \mathrm{Tot}^{\oplus}(N\mathrm{sr}(X(\mathbf{m}, \mathbf{m})))$

is trivial. Alternatively, a result by Quillen (see [Qui73, p. 91]) yields that the homology of $\mathrm{hocolim}_{\Sigma_2} X(\mathbf{m},\mathbf{m})$ is isomorphic to the group homology of Σ_2 with coefficients in $\mathbb{Q}$ where Σ_2 acts on $\mathbb{Q}$ by the sign operation, which is trivial. So the chain complex $\mathrm{hocolim}_{\Sigma_2} X(\mathbf{m},\mathbf{m})$ is quasi-isomorphic to the zero chain complex. As the functor $\mathrm{hocolim}_{\mathcal{I}_{\geq 2}}$ preserves level equivalences by Proposition 2.51, the chain complex $\mathrm{hocolim}_{\bar{\mathcal{J}}_{0,(\geq 2,-)}} X$ is quasi-isomorphic to the zero chain complex.
We conclude that the map $X \to \mathrm{const}_{\bar{\mathcal{J}}} 0$ is a map between homologically constant $\bar{\mathcal{J}}$-chain complexes whose induced map of homotopy colimits is a homology isomorphism. Further, the map $X \to \mathrm{const}_{\bar{\mathcal{J}}} 0$ is not a level equivalence. But [Hir03, Proposition 3.3.4.(1)] and *Ken Brown's lemma* imply that a weak equivalence between fibrant objects in the localized model structure on $\mathrm{ch}^{\bar{\mathcal{J}}}$ is a level equivalence. Therefore, the functor $\mathrm{hocolim}_{\bar{\mathcal{J}}}$ does not detect the weak equivalences in the localized model structure on $\mathrm{ch}^{\bar{\mathcal{J}}}$.

So far, we cannot characterize the S-local equivalences in $\mathrm{ch}^{\bar{\mathcal{J}}}$. Another disadvantage of employing the adjunction (2.51), or (2.53) respectively, is that morally the category of (commutative) $\bar{\mathcal{J}}$-chain complexes does not seem to be an appropriate category to set up pre-log structures. To understand this, recall the notion of pre-log structures on discrete commutative rings (see [Rog09, Definition 2.1], Example 2.37). In this sense, the object $\tilde{\Omega}^{\bar{\mathcal{J}}}(A)$ should model the underlying multiplicative commutative monoid of an object A in $C(\mathrm{Sp}^{\Sigma}(\mathrm{ch}))$. But $\bar{\mathcal{J}}$-chain complexes have additive structure. The differentials are k-linear maps that are responsible for the homotopy theoretical information. We do not know yet how to get rid of the additive structure on a $\bar{\mathcal{J}}$-chain complex without loosing homotopy theory.
Furthermore, we can show an analogon of [SS12, Proposition 4.24], namely that for an object A in $C(\mathrm{Sp}^{\Sigma}(\mathrm{ch}))$, there is an isomorphism of graded commutative k-algebras $H_*(A) \cong H_0(\tilde{\Omega}^{\bar{\mathcal{J}}}(A))$. This should motivate a definition of units. But again we face the above problem. We are uncertain in which category the units should live such that their definition is homotopy invariant. Besides, hitherto, we have not come up with an analogon of the restriction of path components like in the topological setting (see [SS12, p. 2137, Definition 4.25]). For these reasons we have constructed another adjunction (2.37) introduced in Subsection 2.4 to make sense of pre-log structures on E_∞ dgas.

3 Group completion in commutative diagram spaces

In this section we develop a notion of group completion in commutative diagram spaces shaped by a permutative well-structured index category $\mathcal{K}$ whose classifying space $B\mathcal{K}$ is grouplike. Our approach is model categorical, which has the advantage that it provides functorial group completions for all objects without further assumptions. We form the left Bousfield localization of the positive projective $\mathcal{K}$-model structure on commutative $\mathcal{K}$-spaces with respect to a set of maps that corepresents shear maps. We characterize this localized model structure and argue that it is indeed a *group completion model structure*. Sagave and Schlichtkrull describe group completion in commutative $\mathcal{I}$-spaces in [SS13]. We construct a chain of Quillen equivalences between commutative $\mathcal{K}$-spaces and commutative $\mathcal{I}$-spaces over a commutative $\mathcal{I}$-space model of $B\mathcal{K}$, and exploit this to build on Sagave and Schlichtkrull's work. Having a concept of group completion in commutative $\bar{\mathcal{J}}$-spaces at our disposal, we present other examples of pre-log cdgas.

3.1 Useful results about comma categories and left Bousfield localizations

We start with collecting a few general results about the interaction of left Bousfield localizations with comma categories. The main outcome is that in a sense, left Bousfield localization commutes with forming a comma category (see Proposition 3.4). This is a crucial ingredient in the upcoming Subsection 3.7 when specifying a *group completion model structure* on commutative diagram spaces.

For background on left Bousfield localizations we refer to [Hir03]. Recall that left Bousfield localizations exist if the model category is for example left proper and cellular [Hir03, Theorem 4.1.1] or left proper and combinatorial meaning locally presentable and cofibrantly generated [Bar10, Theorem 4.7].

Let $\mathcal{C}$ be a cofibrantly generated simplicial model category which is proper and cellular. Let Z be an object in $\mathcal{C}$. The comma category $(\mathcal{C} \downarrow Z)$ inherits a cofibrantly generated simplicial model structure from the category $\mathcal{C}$ which is again proper and cellular ([Hir03, Theorem 7.6.5.(1)], [Hir, Theorem 1.5, Theorem 1.7]). In this overcategory model structure on $(\mathcal{C} \downarrow Z)$ a map

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "\rho_X"'] & & Y \arrow[dl, "\rho_Y"] \\ & Z & \end{tikzcd}$$

is a weak equivalence/ fibration/ cofibration if and only if the underlying map f is so in $\mathcal{C}$. The space of maps from (X, ρ_X) to (Y, ρ_Y) is defined by the pullback diagram

$$\begin{tikzcd} \mathrm{Hom}_{\mathcal{S}}^{(\mathcal{C}\downarrow Z)}((X,\rho_X),(Y,\rho_Y)) \arrow[r] \arrow[d] & \mathrm{Hom}_{\mathcal{S}}^{\mathcal{C}}(X,Y) \arrow[d, "\mathrm{Hom}_{\mathcal{S}}^{\mathcal{C}}(\mathrm{id},\rho_Y)"] \\ \{\rho_X\} \arrow[r] & \mathrm{Hom}_{\mathcal{S}}^{\mathcal{C}}(X,Z). \end{tikzcd}$$

For (X, ρ_X) in $(\mathcal{C} \downarrow Z)$ and T in $\mathcal{S}$, the tensor of (X, ρ_X) and T in $(\mathcal{C} \downarrow Z)$ is determined by

$$X \otimes T \xrightarrow{\mathrm{id}_X \otimes t_T} X \otimes \Delta_0 \cong X \xrightarrow{\rho_X} Z$$

where $t_T \colon T \to \Delta_0$ is the unique map from T to the terminal object Δ_0 in $\mathcal{S}$. The cotensor of (X, ρ_X) and T in $(\mathcal{C} \downarrow Z)$, denoted by $(X, \rho_X)^T$, is specified by the pullback diagram

$$\begin{tikzcd}
(X, \rho_X)^T \arrow[r] \arrow[d] & X^T \arrow[d, "(\rho_X)^T"] \\
Z^{\Delta_0} \arrow[r, "Z^{t_T}"] & Z^T.
\end{tikzcd}$$

Let S be a set of maps with cofibrant domains and codomains in $\mathcal{C}$. An object X in $\mathcal{C}$ is called S-local if X is fibrant in $\mathcal{C}$ and for every element $f \colon A \to B$ of S, the induced morphism of simplicial mapping spaces $\mathrm{Hom}^{\mathcal{C}}_{\mathcal{S}}(f, \mathrm{id}) \colon \mathrm{Hom}^{\mathcal{C}}_{\mathcal{S}}(B, X) \to \mathrm{Hom}^{\mathcal{C}}_{\mathcal{S}}(A, X)$ is a weak equivalence (see [Hir03, Definition 3.1.4.(1)(a)]). Let $\mathcal{C}_{\mathrm{loc}(S)}$ be the left Bousfield localization of $\mathcal{C}$ with respect to S (see [Hir03, Theorem 4.1.1]). The fibrant objects of $\mathcal{C}_{\mathrm{loc}(S)}$ are the S-local objects of $\mathcal{C}$ [Hir03, Theorem 4.1.1.(2)]. From now on we assume that Z is an S-local object in $\mathcal{C}$. We write S_Z for the set of morphisms in $(\mathcal{C} \downarrow Z)$ of the form

$$\begin{tikzcd}
A \arrow[rr, "f"] \arrow[dr, "\rho_A"'] & & B \arrow[dl, "\rho_B"] \\
& Z &
\end{tikzcd}$$

where f is an element of S. We form the left Bousfield localization of the overcategory model structure on $(\mathcal{C} \downarrow Z)$ with respect to S_Z and denote the localized model structure by $(\mathcal{C} \downarrow Z)_{\mathrm{loc}(S_Z)}$. We compare the latter to $(\mathcal{C}_{\mathrm{loc}(S)} \downarrow Z)$ which is the overcategory model structure with respect to $\mathcal{C}_{\mathrm{loc}(S)}$. Our next goal is to show that these two model structures agree.

The following definition is similar to [Hir03, Definition 4.2.2]. Let $\Lambda(S)$ be the set of maps obtained by choosing for every element $f \colon A \to B$ of S a factorization into a cofibration followed by an acyclic fibration

$$\begin{tikzcd}
A \arrow[r, tail, "f^{\mathrm{cof}}"] \arrow[rr, bend left, "f"] & B' \arrow[r, two heads, "p", "\sim"'] & B
\end{tikzcd} \tag{3.1}$$

in $\mathcal{C}$, and considering the pushout product of the map f^{cof} and a generating cofibration i in spaces, that is,

$$\Lambda(S) = \{A \otimes \Delta_n \coprod_{A \otimes \partial \Delta_n} B' \otimes \partial \Delta_n \xrightarrow{f^{\mathrm{cof}} \square i} B' \otimes \Delta_n, f \in S, i \in I_{\mathcal{S}}\}.$$

For the reason that the model category $\mathcal{C}_{\mathrm{loc}(S)}$ is simplicial [Hir03, Theorem 4.1.1.(4)], the maps in $\Lambda(S)$ are acyclic cofibrations in $\mathcal{C}_{\mathrm{loc}(S)}$. We define $\overline{\Lambda(S)} = \Lambda(S) \cup J_{\mathcal{C}}$ where $J_{\mathcal{C}}$

is the set of generating acyclic cofibrations in $\mathcal{C}$. The domains of the maps in $\overline{\Lambda(S)}$ are small with respect to $\overline{\Lambda(S)}$-cell by [Hir03, Theorem 12.4.3, Theorem 12.4.4].

Lemma 3.1. *A map $g\colon X \to Z$ is a fibration in $\mathcal{C}_{\mathrm{loc}(S)}$ if and only if the map g has the right lifting property with respect to $\overline{\Lambda(S)}$.*

Proof. Let the map $g\colon X \to Z$ be a fibration in $\mathcal{C}_{\mathrm{loc}(S)}$. Then the map g is a fibration in $\mathcal{C}$ by [Hir03, Proposition 3.3.3.(1)(c)] so that g has the right lifting property with respect to $J_{\mathcal{C}}$. Let $f^{\mathrm{cof}}\square i\colon A \otimes \Delta_n \coprod_{A\otimes\partial\Delta_n} B' \otimes \partial\Delta_n \to B' \otimes \Delta_n$ be an element in $\Lambda(S)$. Taking into account that $\mathcal{C}$ is a simplicial model category, given the cofibration $f^{\mathrm{cof}}\colon A \to B'$ and the fibration $g\colon X \to Z$ in $\mathcal{C}$, we get that the induced map

$$\mathrm{Hom}^{\mathcal{C}}_{\mathcal{S}}(B', X) \xrightarrow{\mathrm{Hom}^{\mathcal{C}}_{\mathcal{S}\square}(f^{\mathrm{cof}},g)} \mathrm{Hom}^{\mathcal{C}}_{\mathcal{S}}(A, X) \times_{\mathrm{Hom}^{\mathcal{C}}_{\mathcal{S}}(A,Z)} \mathrm{Hom}^{\mathcal{C}}_{\mathcal{S}}(B', Z) \tag{3.2}$$

is a fibration. Since the object Z is S-local, the object X is S-local by [Hir03, Proposition 3.3.14.(1)]. This implies that the map (3.2) is a weak equivalence (see [Hir03, Corollary 9.3.3.(2)]). Due to adjointness, the map (3.2) being an acyclic fibration is equivalent to the map g having the right lifting property with respect to $f^{\mathrm{cof}}\square i$.

On the other hand, we assume that the map $g\colon X \to Z$ has the right lifting property with respect to $\overline{\Lambda(S)}$. As the map g has the right lifting property with respect to $J_{\mathcal{C}}$, the map g is a fibration in $\mathcal{C}$. Using that the map g has the right lifting property with respect to $\Lambda(S)$ and adjointness, we can conclude that the induced map

$$\mathrm{Hom}^{\mathcal{C}}_{\mathcal{S}}(B', X) \xrightarrow{\mathrm{Hom}^{\mathcal{C}}_{\mathcal{S}\square}(f^{\mathrm{cof}},g)} \mathrm{Hom}^{\mathcal{C}}_{\mathcal{S}}(A, X) \times_{\mathrm{Hom}^{\mathcal{C}}_{\mathcal{S}}(A,Z)} \mathrm{Hom}^{\mathcal{C}}_{\mathcal{S}}(B', Z)$$

is an acyclic fibration for every f in S. Because the object Z is S-local, we obtain that the map $\mathrm{Hom}^{\mathcal{C}}_{\mathcal{S}}(f^{\mathrm{cof}}, \mathrm{id})\colon \mathrm{Hom}^{\mathcal{C}}_{\mathcal{S}}(B', X) \to \mathrm{Hom}^{\mathcal{C}}_{\mathcal{S}}(A, X)$ is a weak equivalence for every f in S. Thus, the object X is S-local by [Hir03, Corollary 9.3.3.(2)]. So the map g is a fibration in $\mathcal{C}$ between S-local objects which is equivalent to g being a fibration in $\mathcal{C}_{\mathrm{loc}(S)}$ by [Hir03, Proposition 3.3.16.(1)]. □

Remark 3.2. Let $g\colon X \to Z$ be a map in $\mathcal{C}$. As a result of the previous lemma, we can apply the *small object argument* with respect to $\overline{\Lambda(S)}$ to get a factorization of the map g into an acyclic cofibration followed by a fibration

$$\begin{tikzcd} X \arrow[r, tail, "\sim"] \arrow[rr, bend left, "g"] & \tilde{X} \arrow[r, two heads] & Z \end{tikzcd} \tag{3.3}$$

in $\mathcal{C}_{\mathrm{loc}(S)}$. Moreover, [Hir, Lemma 1.4] yields that the map $X \to \tilde{X}$ is a $\overline{\Lambda(S)}$-cell complex in $\mathcal{C}_{\mathrm{loc}(S)}$ if an only if the map

$$\begin{tikzcd} X \arrow[rr] \arrow[dr, "g"'] & & \tilde{X} \arrow[dl] \\ & Z & \end{tikzcd}$$

is a $(\overline{\Lambda(S)})_Z$-cell complex in $(\mathcal{C}_{\mathrm{loc}(S)} \downarrow Z)$. Therefore, the factorization (3.3) provides a fibrant replacement of the object (X, g) in $(\mathcal{C}_{\mathrm{loc}(S)} \downarrow Z)$.

Lemma 3.3. *Let*

$$\begin{tikzcd} V \arrow[rr, "h"] \arrow[dr] & & W \arrow[dl, "q_0"] \\ & Z & \end{tikzcd}$$

be a map between cofibrant objects in the category $(\mathcal{C} \downarrow Z)$ *such that the map* q_0 *is homotopic to a map* $q_1 \colon W \to Z$ *in* $\mathcal{C}$*. The map*

$$\begin{tikzcd} V \arrow[rr, "h"] \arrow[dr] & & W \arrow[dl, "q_0"] \\ & Z & \end{tikzcd}$$

is a weak equivalence in $(\mathcal{C} \downarrow Z)_{\mathrm{loc}(S_Z)}$ *if and only if the map*

$$\begin{tikzcd} V \arrow[rr, "h"] \arrow[dr] & & W \arrow[dl, "q_1"] \\ & Z & \end{tikzcd}$$

is a weak equivalence in $(\mathcal{C} \downarrow Z)_{\mathrm{loc}(S_Z)}$.

Proof. Let $j = 0, 1$. Let $(X, \rho_X \colon X \to Z)$ be an S_Z-local object in the category $(\mathcal{C} \downarrow Z)$. The space $\mathrm{Hom}_{\mathcal{S}}^{(\mathcal{C}\downarrow Z)}((W, q_j), (X, \rho_X))$ is defined by the pullback square

$$\begin{tikzcd} \mathrm{Hom}_{\mathcal{S}}^{(\mathcal{C}\downarrow Z)}((W, q_j), (X, \rho_X)) \arrow[r] \arrow[d] & \mathrm{Hom}_{\mathcal{S}}^{\mathcal{C}}(W, X) \arrow[d, two heads, "\mathrm{Hom}_{\mathcal{S}}^{\mathcal{C}}(\mathrm{id}, \rho_X)"] \\ \{q_j\} \arrow[r] & \mathrm{Hom}_{\mathcal{S}}^{\mathcal{C}}(W, Z), \end{tikzcd}$$

which is a homotopy pullback square because $\mathcal{S}$ is right proper and the map $\mathrm{Hom}_{\mathcal{S}}^{\mathcal{C}}(\mathrm{id}, \rho_X)$ is a fibration by [Hir03, Proposition 9.3.1.(2)]. A homotopy from the map q_0 to the map q_1 corresponds to a path between the points $\{q_0\}$ and $\{q_1\}$ in the space $\mathrm{Hom}_{\mathcal{S}}^{\mathcal{C}}(W, Z)$. This implies that the homotopy fibre $\mathrm{Hom}_{\mathcal{S}}^{(\mathcal{C}\downarrow Z)}((W, q_0), (X, \rho_X))$ is weakly equivalent to the homotopy fibre $\mathrm{Hom}_{\mathcal{S}}^{(\mathcal{C}\downarrow Z)}((W, q_1), (X, \rho_X))$ (see also [Hir03, Proposition 13.4.7]). By the same arguments the homotopy fibre $\mathrm{Hom}_{\mathcal{S}}^{(\mathcal{C}\downarrow Z)}((V, q_0 \circ h), (X, \rho_X))$ is weakly equivalent to the homotopy fibre $\mathrm{Hom}_{\mathcal{S}}^{(\mathcal{C}\downarrow Z)}((V, q_1 \circ h), (X, \rho_X))$. Thus, the map

$$\mathrm{Hom}_{\mathcal{S}}^{(\mathcal{C}\downarrow Z)}((W, q_0), (X, \rho_X)) \xrightarrow{\mathrm{Hom}_{\mathcal{S}}^{(\mathcal{C}\downarrow Z)}(h, \mathrm{id})} \mathrm{Hom}_{\mathcal{S}}^{(\mathcal{C}\downarrow Z)}((V, q_0 \circ h), (X, \rho_X))$$

is a weak equivalence if and only if the map

$$\mathrm{Hom}_{\mathcal{S}}^{(\mathcal{C}\downarrow Z)}((W, q_1), (X, \rho_X)) \xrightarrow{\mathrm{Hom}_{\mathcal{S}}^{(\mathcal{C}\downarrow Z)}(h, \mathrm{id})} \mathrm{Hom}_{\mathcal{S}}^{(\mathcal{C}\downarrow Z)}((V, q_1 \circ h), (X, \rho_X))$$

is a weak equivalence. This finishes the proof. □

Proposition 3.4. *The model structures* $(\mathcal{C} \downarrow Z)_{\mathrm{loc}(S_Z)}$ *and* $(\mathcal{C}_{\mathrm{loc}(S)} \downarrow Z)$ *coincide.*

Proof. Due to an argument of Joyal [Joy, Proposition E.1.10], it suffices to show that the cofibrations and the fibrant objects in both model structures agree. The cofibrations in both model structures are the same, as these are the cofibrations in $\mathcal{C}$.
Let

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr] & & B \arrow[dl] \\ & Z & \end{tikzcd}$$

be in S_Z, with a factorization into a cofibration followed by an acyclic fibration

$$\begin{tikzcd} A \arrow[r, tail, "f^{\mathrm{cof}}"] \arrow[rr, bend left, "f"] \arrow[dr] & B' \arrow[r, two heads, "\sim"] & B \arrow[dl] \\ & Z & \end{tikzcd}$$

in $(\mathcal{C} \downarrow Z)$. Forgetting the augmentation to Z, the map $f^{\mathrm{cof}} \colon A \to B'$ is a weak equivalence in $\mathcal{C}_{\mathrm{loc}(S)}$. Therefore, it follows from [Hir03, Proposition 3.3.18.(1)] that

$$(\mathcal{C} \downarrow Z)_{\mathrm{loc}(S_Z)} \underset{\mathrm{id}}{\overset{\mathrm{id}}{\rightleftarrows}} (\mathcal{C}_{\mathrm{loc}(S)} \downarrow Z)$$

is a Quillen adjunction. It remains to prove that the fibrant objects in $(\mathcal{C} \downarrow Z)_{\mathrm{loc}(S_Z)}$ are fibrant in $(\mathcal{C}_{\mathrm{loc}(S)} \downarrow Z)$. We argue that the functor

$$\mathrm{id} \colon (\mathcal{C}_{\mathrm{loc}(S)} \downarrow Z) \to (\mathcal{C} \downarrow Z)_{\mathrm{loc}(S_Z)} \tag{3.4}$$

is also a left Quillen functor. For this, we check that this functor (3.4) preserves weak equivalences. As the functor (3.4) is already a right Quillen functor, it is enough to show that the functor (3.4) preserves fibrant replacements of objects in $(\mathcal{C}_{\mathrm{loc}(S)} \downarrow Z)$.
Let $(X, \rho_X \colon X \to Z)$ be an object in $(\mathcal{C}_{\mathrm{loc}(S)} \downarrow Z)$. From Remark 3.2 we know that we obtain a fibrant replacement of (X, ρ_X) in $(\mathcal{C}_{\mathrm{loc}(S)} \downarrow Z)$ by applying the *small object argument* with respect to $(\overline{\Lambda(S)})_Z$,

$$\begin{tikzcd} X \arrow[rr, tail, "\sim"] \arrow[dr, "\rho_X"'] & & \tilde{X} \arrow[dl, two heads, "\rho_{\tilde{X}}"] \\ & Z. & \end{tikzcd} \tag{3.5}$$

We need to verify that the $(\overline{\Lambda(S)})_Z$-cell complex (3.5) is a weak equivalence in the model category $(\mathcal{C} \downarrow Z)_{\mathrm{loc}(S_Z)}$.
We start with explaining that the maps in $(\overline{\Lambda(S)})_Z$ are weak equivalences in $(\mathcal{C} \downarrow Z)_{\mathrm{loc}(S_Z)}$. The maps in $(J_{\mathcal{C}})_Z$ are weak equivalences in $(\mathcal{C} \downarrow Z)_{\mathrm{loc}(S_Z)}$, as they are weak equivalences in $(\mathcal{C} \downarrow Z)$ by [Hir03, Proposition 3.3.3.(1)(a)]. Let

$$\begin{tikzcd} A \otimes \Delta_n \coprod_{A \otimes \partial\Delta_n} B' \otimes \partial\Delta_n \arrow[r, "f^{\mathrm{cof}} \square i"] \arrow[dr] & B' \otimes \Delta_n \arrow[d, "\rho_{(B' \otimes \Delta_n)}"] \\ & Z \end{tikzcd} \tag{3.6}$$

be a map in $(\Lambda(S))_Z$. The map $\rho_{(B'\otimes\Delta_n)}\colon B'\otimes\Delta_n \to Z$ is homotopic to the map

$$B'\otimes\Delta_n \xrightarrow[\sim]{\mathrm{id}_{B'}\otimes t_{\Delta_n}} B'\otimes\Delta_0 \cong B' \xrightarrow{\rho_{B'}} Z$$

in $\mathcal{C}$, because the space Δ_n is contractible. We claim that the map

$$\begin{tikzcd} A \arrow[rr, "f^{\mathrm{cof}}"] \arrow[dr] & & B' \arrow[dl, "\rho_{B'}"] \\ & Z & \end{tikzcd}$$

is an acyclic cofibration in $(\mathcal{C}\downarrow Z)_{\mathrm{loc}(S_Z)}$. The map f^{cof} is a cofibration in $\mathcal{C}$. We consider the diagram

$$\begin{tikzcd} A \arrow[rr, bend left, "f"] \arrow[r, tail, "f^{\mathrm{cof}}"] \arrow[d] & B' \arrow[r, two heads, "p", "\sim"'] \arrow[dl, "\rho_{B'}"] & B \\ Z & & \end{tikzcd}$$

in $\mathcal{C}$ (see (3.1)). The map p is an isomorphism in the homotopy category $\mathrm{Ho}(\mathcal{C})$, and so the map $\rho_{B'}\circ p^{-1}\colon B\to Z$ defines a map in $\mathrm{Ho}(\mathcal{C})$. Because B is cofibrant and Z is fibrant in $\mathcal{C}$, the map $\rho_{B'}\circ p^{-1}$ can be represented by a map $w\colon B\to Z$ in $\mathcal{C}$ such that the composite $w\circ p$ is homotopic to $\rho_{B'}$. The composite map

$$\begin{tikzcd} A \arrow[rr, bend left, "f"] \arrow[r, tail, "f^{\mathrm{cof}}"] \arrow[d] & B' \arrow[r, two heads, "p", "\sim"'] \arrow[dl, "w\circ p"] & B \arrow[dll, bend left, "w"] \\ Z & & \end{tikzcd}$$

lies in S_Z and hence is a weak equivalence in $(\mathcal{C}\downarrow Z)_{\mathrm{loc}(S_Z)}$. Two out of three ensures that the map $f^{\mathrm{cof}}\colon (A, w\circ p\circ f^{\mathrm{cof}}) \to (B', w\circ p)$ is a weak equivalence in $(\mathcal{C}\downarrow Z)_{\mathrm{loc}(S_Z)}$. As the composite $w\circ p$ is homotopic to the map $\rho_{B'}$, Lemma 3.3 yields that the map $f^{\mathrm{cof}}\colon (A, \rho_{B'}\circ f^{\mathrm{cof}}) \to (B', \rho_{B'})$ is a weak equivalence in $(\mathcal{C}\downarrow Z)_{\mathrm{loc}(S_Z)}$. Since $(\mathcal{C}\downarrow Z)_{\mathrm{loc}(S_Z)}$ is a simplicial model category by [Hir03, Theorem 4.1.1.(4)] and the map $f^{\mathrm{cof}}\colon (A, \rho_{B'}\circ f^{\mathrm{cof}}) \to (B', \rho_{B'})$ is an acyclic cofibration in $(\mathcal{C}\downarrow Z)_{\mathrm{loc}(S_Z)}$, the map

$$\begin{tikzcd} A\otimes\Delta_n \coprod_{A\otimes\partial\Delta_n} B'\otimes\partial\Delta_n \arrow[rr, "f^{\mathrm{cof}}\square i"] \arrow[dr] & & B'\otimes\Delta_n \arrow[d, "\mathrm{id}_{B'}\otimes t_{\Delta_n}", "\sim"'] \\ & Z & B'\cong B'\otimes\Delta_0 \arrow[l, "\rho_{B'}"'] \end{tikzcd}$$

is an acyclic cofibration in $(\mathcal{C}\downarrow Z)_{\mathrm{loc}(S_Z)}$. This together with the fact that the composite $\rho_{B'}\circ(\mathrm{id}_{B'}\otimes t_{\Delta_n})$ is homotopic to the map $\rho_{(B'\otimes\Delta_n)}$ implies that the map

$$(A\otimes\Delta_n \coprod_{A\otimes\partial\Delta_n} B'\otimes\partial\Delta_n, \rho_{(B'\otimes\Delta_n)}\circ(f^{\mathrm{cof}}\square i)) \to (B'\otimes\Delta_n, \rho_{(B'\otimes\Delta_n)})$$

(see (3.6)) is a weak equivalence in $(\mathcal{C} \downarrow Z)_{\mathrm{loc}(S_Z)}$ by Lemma 3.3.
The map (3.5) being a $(\overline{\Lambda(S)})_Z$-cell complex means that there is an ordinal λ and a λ-sequence $\{(X_\alpha, \rho_{X_\alpha}), \alpha < \lambda\}$ such that $(X_0, \rho_{X_0}) = (X, \rho_X)$ and

$$\mathrm{colim}_{\alpha<\lambda}(X_\alpha, \rho_{X_\alpha}) = (\tilde{X}, \rho_{\tilde{X}}),$$

and each of the maps

$$\begin{tikzcd} X_\alpha \arrow[rr] \arrow[dr] & & X_{\alpha+1} \arrow[dl] \\ & Z & \end{tikzcd}$$

is a pushout of an element in $(\overline{\Lambda(S)})_Z$. We have shown above that every map in $(\overline{\Lambda(S)})_Z$ is an acyclic cofibration in $(\mathcal{C} \downarrow Z)_{\mathrm{loc}(S_Z)}$. Hence, each of the maps

$$\begin{tikzcd} X_\alpha \arrow[rr] \arrow[dr] & & X_{\alpha+1} \arrow[dl] \\ & Z & \end{tikzcd}$$

is an acyclic cofibration in $(\mathcal{C} \downarrow Z)_{\mathrm{loc}(S_Z)}$. So the transfinite composition (3.5) is an acyclic cofibration in $(\mathcal{C} \downarrow Z)_{\mathrm{loc}(S_Z)}$. In this way, we can conclude that the functor (3.4) respects fibrant replacements. $\square$

Let $g\colon V \to W$ be a weak equivalence in $\mathcal{C}$. Since $\mathcal{C}$ is right proper, the adjunction

$$(\mathcal{C} \downarrow V) \underset{g^*}{\overset{g_*}{\rightleftarrows}} (\mathcal{C} \downarrow W),$$

induced by composition with and pullback along the map g, is a Quillen equivalence. The following lemma addresses this question in the localized setting.

Lemma 3.5. *Let $g\colon V \to W$ be a weak equivalence in $\mathcal{C}$. The induced adjunction*

$$(\mathcal{C}_{\mathrm{loc}(S)} \downarrow V) \underset{g^*}{\overset{g_*}{\rightleftarrows}} (\mathcal{C}_{\mathrm{loc}(S)} \downarrow W)$$

is a Quillen equivalence.

Proof. By definition of the model structures, the functor g_* preserves and reflects cofibrations and weak equivalences. Let $(X, \rho_X\colon X \to W)$ be a fibrant object in the model category $(\mathcal{C}_{\mathrm{loc}(S)} \downarrow W)$. Let

$$\begin{tikzcd} (V \times_W X)^{\mathrm{cof}} \arrow[rr] \arrow[dr] & & V \times_W X \arrow[dl] \\ & V & \end{tikzcd}$$

be a cofibrant replacement of $(V \times_W X, V \times_W X \to V)$ in $(\mathcal{C}_{\mathrm{loc}(S)} \downarrow V)$. We consider the diagram

$$\begin{tikzcd}
(V \times_W X)^{\mathrm{cof}} \arrow[dr, "\sim"] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& V \times_W X \arrow[r] \arrow[d] & X \arrow[d, two heads, "\rho_X"] \\
& V \arrow[r, "g", "\sim"'] & W
\end{tikzcd}$$

in $(\mathcal{C}_{\mathrm{loc}(S)} \downarrow W)$. Since the map ρ_X is a fibration by [Hir03, Proposition 3.3.3.(1)(c)] and the map g is a weak equivalence in the right proper model category $\mathcal{C}$, the base change map $V \times_W X \to X$ is a weak equivalence in $\mathcal{C}$ and hence in $\mathcal{C}_{\mathrm{loc}(S)}$ by [Hir03, Proposition 3.3.3.(1)(a)]. Therefore, the composite $(V \times_W X)^{\mathrm{cof}} \to V \times_W X \to X$ is a weak equivalence in $\mathcal{C}_{\mathrm{loc}(S)}$. The claim follows by [Hov99, Corollary 1.3.16]. $\square$

3.2 Diagram spaces are Quillen equivalent to spaces over the classifying space of the indexing category

Let $\mathcal{K}$ be a well-structured index category with classifying space $B\mathcal{K}$. Consider the Barratt-Eccles operad in spaces which has as its nth space the classifying space of the translation category of the symmetric group Σ_n and hence is an E_∞ operad in spaces. Here we define an E_∞ (diagram) space to be a (diagram) space with an action of the Barratt-Eccles operad in spaces. In this subsection we briefly recall Sagave and Schlichtkrull's chain of Quillen equivalences connecting commutative $\mathcal{K}$-spaces with E_∞ spaces over $B\mathcal{K}$. For more details we refer to [SS12, §13].

Let $E\mathcal{K}$ be the $\mathcal{K}$-space specified by $\mathbf{k} \mapsto B(\mathcal{K} \downarrow \mathbf{k})$. Forgetting the augmentation $\mathcal{K}$-levelwise gives rise to a map of $\mathcal{K}$-spaces $u\colon E\mathcal{K} \to \mathrm{const}_{\mathcal{K}} B\mathcal{K}$. The adjoint map $\mathrm{colim}_{\mathcal{K}} E\mathcal{K} \to B\mathcal{K}$ is an isomorphism (see [SS12, p. 2178]).

Theorem 3.6. *[SS12, Theorem 13.2] There is a chain of Quillen equivalences connecting $\mathcal{K}$-spaces equipped with the projective $\mathcal{K}$-model structure to spaces over $B\mathcal{K}$ carrying the overcategory model structure with respect to the standard model structure on spaces.*

Sagave and Schlichtkrull establish in [SS12, §13.1] the following chain of Quillen equivalences,

$$\mathcal{S}^{\mathcal{K}} \underset{q}{\overset{t_*}{\leftrightarrows}} (\mathcal{S}^{\mathcal{K}} \downarrow E\mathcal{K}) \underset{p}{\overset{u_*}{\rightleftarrows}} (\mathcal{S}^{\mathcal{K}} \downarrow \mathrm{const}_{\mathcal{K}} B\mathcal{K}) \underset{\mathrm{const}_{\mathcal{K}}}{\overset{\mathrm{colim}_{\mathcal{K}}}{\rightleftarrows}} (\mathcal{S} \downarrow B\mathcal{K}) \tag{3.7}$$

(see [SS12, Lemma 13.3, Lemma 13.4]). The comma categories $(\mathcal{S}^{\mathcal{K}} \downarrow E\mathcal{K})$ and $(\mathcal{S}^{\mathcal{K}} \downarrow \mathrm{const}_{\mathcal{K}} B\mathcal{K})$ come with the overcategory model structures with respect to $\mathcal{S}^{\mathcal{K}}$. The first Quillen equivalence (t_*, q) in (3.7) is induced by composition with and pullback along the map $t\colon E\mathcal{K} \to \mathrm{const}_{\mathcal{K}} *$. The adjacent Quillen equivalence (u_*, p) in (3.7) is

determined by composition with and pullback along the map u.
In the special case that the nerve of the category $\mathcal{K}$ is contractible, the adjunction

$$\mathcal{S}^{\mathcal{K}} \underset{\mathrm{const}_{\mathcal{K}}}{\overset{\mathrm{colim}_{\mathcal{K}}}{\rightleftarrows}} \mathcal{S}$$

is a Quillen equivalence [SS12, Proposition 6.23].

If the category $\mathcal{K}$ is permutative, the map $u\colon E\mathcal{K} \to \mathrm{const}_{\mathcal{K}} B\mathcal{K}$ is a map in $E_\infty \mathcal{S}^{\mathcal{K}}$ [SS12, Lemma 13.8]. Under this assumption, Sagave and Schlichtkrull provide a structured version of (3.7).

Theorem 3.7. *[SS12, Theorem 13.12] Let $\mathcal{K}$ be a permutative well-structured index category. Suppose that the inclusion functor $\mathcal{K}_+ \to \mathcal{K}$ is homotopy right cofinal, and that the pair $(\mathcal{K}, \mathcal{O}\mathcal{K}_+)$ is very well-structured. There is a chain of Quillen equivalences relating commutative $\mathcal{K}$-spaces endowed with the positive projective $\mathcal{K}$-model structure to E_∞ spaces over $B\mathcal{K}$ given the overcategory model structure with respect to the (right-induced) standard model structure on E_∞ spaces.*

The following diagram displays the extended chain of Quillen equivalences,

$$\begin{tikzcd} C\mathcal{S}^{\mathcal{K}} \arrow[r, shift left, "\epsilon_*"] & E_\infty(\mathcal{S}^{\mathcal{K}})^+ \arrow[l, shift left, "\epsilon^*"] \arrow[r, shift left, "\mathrm{id}"] & E_\infty \mathcal{S}^{\mathcal{K}} \arrow[l, shift left, "\mathrm{id}"] \arrow[r, shift right, leftarrow, "t_*"'] \arrow[r, shift left, leftarrow] & (E_\infty \mathcal{S}^{\mathcal{K}} \downarrow E\mathcal{K}) \arrow[d, shift right, "u_*"'] \\ & (E_\infty \mathcal{S} \downarrow B\mathcal{K}) \arrow[r, shift left, leftarrow, "\mathrm{colim}_{\mathcal{K}}"] \arrow[r, shift right, "\mathrm{const}_{\mathcal{K}}"'] & & (E_\infty \mathcal{S}^{\mathcal{K}} \downarrow \mathrm{const}_{\mathcal{K}} B\mathcal{K}) \arrow[u, shift right, "p"'] \end{tikzcd} \tag{3.8}$$

(see [SS12, Lemma 13.9, Lemma 13.10, Theorem 13.11]). The category $E_\infty(\mathcal{S}^{\mathcal{K}})^+$ has the positive projective $\mathcal{K}$-model structure. The category $E_\infty \mathcal{S}^{\mathcal{K}}$ carries the projective $\mathcal{K}$-model structure. The comma categories in (3.8) are equipped with the overcategory model structures. The map of operads in spaces from the Barratt-Eccles operad to the commutativity operad gives rise to the Quillen equivalence (ϵ_*, ϵ^*) in (3.8). The subsequent Quillen equivalence in (3.8) passes from the positive projective $\mathcal{K}$-model structure to the projective $\mathcal{K}$-model structure on $E_\infty \mathcal{S}^{\mathcal{K}}$. The composite derived functor from $C\mathcal{S}^{\mathcal{K}}$ to $(E_\infty \mathcal{S} \downarrow B\mathcal{K})$ sends M to the induced map of E_∞ spaces $\mathrm{hocolim}_{\mathcal{K}} M \to B\mathcal{K}$. If the nerve of the category $\mathcal{K}$ is contractible, the chain (3.8) boils down to the following chain of Quillen equivalences between commutative $\mathcal{K}$-spaces and E_∞ spaces,

$$C\mathcal{S}^{\mathcal{K}} \underset{\epsilon^*}{\overset{\epsilon_*}{\leftrightarrows}} E_\infty(\mathcal{S}^{\mathcal{K}})^+ \underset{\mathrm{id}}{\overset{\mathrm{id}}{\rightleftarrows}} E_\infty \mathcal{S}^{\mathcal{K}} \underset{\mathrm{const}_{\mathcal{K}}}{\overset{\mathrm{colim}_{\mathcal{K}}}{\rightleftarrows}} E_\infty \mathcal{S} \tag{3.9}$$

(see [SS12, Proposition 13.6, Theorem 13.7]). For example, the category of commutative $\mathcal{I}$-spaces is Quillen equivalent to E_∞ spaces.

Motivated by Sagave and Schlichtkrull's result that on the one hand, commutative $\mathcal{K}$-spaces are Quillen equivalent to E_∞ spaces over $B\mathcal{K}$ and on the other hand, E_∞ spaces are Quillen equivalent to commutative $\mathcal{I}$-spaces, we aim to construct a similar chain of Quillen equivalences between commutative $\mathcal{K}$-spaces and commutative $\mathcal{I}$-spaces over a commutative $\mathcal{I}$-space model of $B\mathcal{K}$. An explicit commutative $\mathcal{I}$-space model of $B\mathcal{K}$ is presented in [Sol11] and [SS16]. In the next subsection we recall its definition and some relevant properties. In addition, we introduce a $(\mathcal{K} \times \mathcal{I})$-space which will play the role of $E\mathcal{K}$ in (3.7) and (3.8).

3.3 The diagram spaces $E_{\mathcal{I}}\mathcal{K}$ and $B_{\mathcal{I}}\mathcal{K}$

In this subsection let $(\mathcal{K}, \sqcup, \mathbf{0}_{\mathcal{K}})$ be a small permutative category. Schlichtkrull and Solberg specify a commutative $\mathcal{I}$-space model of the classifying space $B\mathcal{K}$ (see [SS16, §4.14, §7], [Sol11, Example 3.1.12]). This is defined by applying the nerve functor $\mathcal{I}$-levelwise to the following functor. Let $\Phi_B(\mathcal{K})\colon \mathcal{I} \to \mathrm{Cat}$ be the functor that takes $\mathbf{m}$ to the category $\Phi_B(\mathcal{K})(\mathbf{m})$ with objects m-tupels $(\mathbf{k}_1, \dots, \mathbf{k}_m)$ in $\mathcal{K}$ and morphisms

$$\Phi_B(\mathcal{K})(\mathbf{m})((\mathbf{k}_1, \dots, \mathbf{k}_m), (\mathbf{k}'_1, \dots, \mathbf{k}'_m)) = \mathcal{K}(\mathbf{k}_1 \sqcup \dots \sqcup \mathbf{k}_m, \mathbf{k}'_1 \sqcup \dots \sqcup \mathbf{k}'_m).$$

The convention is that $\Phi_B(\mathcal{K})(\mathbf{0})$ is the category with the empty string $\emptyset$ as its only object and morphisms $\Phi_B(\mathcal{K})(\mathbf{0})(\emptyset, \emptyset) = \mathcal{K}(\mathbf{0}_{\mathcal{K}}, \mathbf{0}_{\mathcal{K}})$. A map $\alpha\colon \mathbf{m} \to \mathbf{n}$ gives rise to a functor $\Phi_B(\mathcal{K})(\alpha)\colon \Phi_B(\mathcal{K})(\mathbf{m}) \to \Phi_B(\mathcal{K})(\mathbf{n})$, which maps an object $(\mathbf{k}_1, \dots, \mathbf{k}_m)$ to the n-tupel $(\mathbf{k}_{\overline{\alpha}^{-1}(1)}, \dots, \mathbf{k}_{\overline{\alpha}^{-1}(n)})$ where

$$\mathbf{k}_{\overline{\alpha}^{-1}(j)} = \begin{cases} \mathbf{k}_i, & \alpha(i) = j, \\ \mathbf{0}_{\mathcal{K}}, & j \notin \mathrm{im}(\alpha), \end{cases} \tag{3.10}$$

for $j = 1, \dots, n$. The functor $\Phi_B(\mathcal{K})(\alpha)$ sends a morphism $\gamma\colon \mathbf{k}_1 \sqcup \dots \sqcup \mathbf{k}_m \to \mathbf{k}'_1 \sqcup \dots \sqcup \mathbf{k}'_m$ to a morphism $\mathbf{k}_{\overline{\alpha}^{-1}(1)} \sqcup \dots \sqcup \mathbf{k}_{\overline{\alpha}^{-1}(n)} \to \mathbf{k}'_{\overline{\alpha}^{-1}(1)} \sqcup \dots \sqcup \mathbf{k}'_{\overline{\alpha}^{-1}(n)}$ that is determined by the commutative diagram

$$\begin{tikzcd} \mathbf{k}_{\overline{\alpha}^{-1}(1)} \sqcup \dots \sqcup \mathbf{k}_{\overline{\alpha}^{-1}(n)} \arrow[r] \arrow[d] & \mathbf{k}'_{\overline{\alpha}^{-1}(1)} \sqcup \dots \sqcup \mathbf{k}'_{\overline{\alpha}^{-1}(n)} \arrow[d] \\ \mathbf{k}_1 \sqcup \dots \sqcup \mathbf{k}_m \arrow[r, "\gamma"] & \mathbf{k}'_1 \sqcup \dots \sqcup \mathbf{k}'_m \end{tikzcd}$$

in $\mathcal{K}$. The vertical maps are given by unique bijections as a result of the category $\mathcal{K}$ being permutative. We skip showing that $\Phi_B(\mathcal{K})$ is functorial in $\mathcal{I}$ because this is similar to our argumentation in Construction 3.8. We define the $\mathcal{I}$-space $B_{\mathcal{I}}\mathcal{K}$ as $\mathcal{N}(\Phi_B(\mathcal{K}))$. The $\mathcal{I}$-space $B_{\mathcal{I}}\mathcal{K}$ is homotopy constant with respect to morphisms in $\mathcal{I}_+$. In particular, the map $B\mathcal{K} = B_{\mathcal{I}}\mathcal{K}(\mathbf{1}) \to \mathrm{hocolim}_{\mathcal{I}} B_{\mathcal{I}}\mathcal{K}$ is a weak equivalence (see [SS16, Lemma 4.15, Proposition 4.18, Theorem 4.19, Theorem 7.1], [Sol11, Lemma 5.2.3, Theorem 5.2.9]). Furthermore, the $\mathcal{I}$-category $\Phi_B(\mathcal{K})$ is a commutative monoid in $\mathrm{Cat}^{\mathcal{I}}$. There is a functor $\lambda^{\Phi_B(\mathcal{K})}_{\mathbf{m},\mathbf{n}}\colon \Phi_B(\mathcal{K})(\mathbf{m}) \times \Phi_B(\mathcal{K})(\mathbf{n}) \to \Phi_B(\mathcal{K})(\mathbf{m} \sqcup \mathbf{n})$, specified on objects by

$$((\mathbf{k}_1, \dots, \mathbf{k}_m), (\mathbf{l}_1, \dots, \mathbf{l}_n)) \mapsto (\mathbf{k}_1, \dots, \mathbf{k}_m, \mathbf{l}_1, \dots, \mathbf{l}_n)$$

and on morphisms by $(\gamma, \delta) \mapsto \gamma \sqcup \delta$, which is natural in $(\mathbf{m}, \mathbf{n})$. A unit for this multiplication is given by $\emptyset$ in $\Phi_B(\mathcal{K})(\mathbf{0})$. This together with the functor $\lambda^{\Phi_B(\mathcal{K})}_{\mathbf{m},\mathbf{n}}$ is coherently associative, unital and commutative. Since the nerve functor is a right adjoint, we can deduce that the $\mathcal{I}$-space $B_{\mathcal{I}}\mathcal{K}$ is a commutative monoid in the category $\mathcal{S}^{\mathcal{I}}$ (see [SS16, Proposition 4.16, p. 7334], [Sol11, Proposition 5.1.2]).

Next we introduce a $(\mathcal{K} \times \mathcal{I})$-space which we employ in the upcoming Subsections 3.4 and 3.5 to build a chain of Quillen equivalences between (commutative) $\mathcal{K}$-spaces and (commutative) $\mathcal{I}$-spaces over $B_{\mathcal{I}}\mathcal{K}$.

Construction 3.8. Let $\Phi_E(\mathcal{K})\colon \mathcal{K} \times \mathcal{I} \to \mathrm{Cat}$ be the functor that sends $(\mathbf{k}, \mathbf{m})$ to the category $\Phi_E(\mathcal{K})(\mathbf{k}, \mathbf{m})$ with objects $((\mathbf{k}_1, \ldots, \mathbf{k}_m), \rho\colon \mathbf{k}_1 \sqcup \ldots \sqcup \mathbf{k}_m \to \mathbf{k})$, where $(\mathbf{k}_1, \ldots, \mathbf{k}_m)$ is an m-tupel in $\mathcal{K}$, and morphisms

$$\begin{tikzcd} \mathbf{k}_1 \sqcup \ldots \sqcup \mathbf{k}_m \arrow[rr, "\gamma"] \arrow[dr, "\rho"'] & & \mathbf{k}'_1 \sqcup \ldots \sqcup \mathbf{k}'_m \arrow[dl, "\rho'"] \\ & \mathbf{k}. & \end{tikzcd}$$

For $\mathbf{k}$ in $\mathcal{K}$, we define $\Phi_E(\mathcal{K})(\mathbf{k}, \mathbf{0})$ to be the category with objects $(\emptyset, \rho\colon \mathbf{0}_{\mathcal{K}} \to \mathbf{k})$ and morphisms

$$\begin{tikzcd} \mathbf{0}_{\mathcal{K}} \arrow[rr, "\gamma"] \arrow[dr, "\rho"'] & & \mathbf{0}_{\mathcal{K}} \arrow[dl, "\rho'"] \\ & \mathbf{k}. & \end{tikzcd}$$

A morphism $\psi\colon \mathbf{k} \to \mathbf{l}$ in $\mathcal{K}$ induces a functor $\Phi_E(\mathcal{K})(\psi, \mathrm{id})\colon \Phi_E(\mathcal{K})(\mathbf{k}, \mathbf{m}) \to \Phi_E(\mathcal{K})(\mathbf{l}, \mathbf{m})$, defined by postcomposing with the map ψ, that is, an object

$$((\mathbf{k}_1, \ldots, \mathbf{k}_m), \mathbf{k}_1 \sqcup \ldots \sqcup \mathbf{k}_m \xrightarrow{\rho} \mathbf{k})$$

is mapped to

$$((\mathbf{k}_1, \ldots, \mathbf{k}_m), \mathbf{k}_1 \sqcup \ldots \sqcup \mathbf{k}_m \xrightarrow{\rho} \mathbf{k} \xrightarrow{\psi} \mathbf{l}),$$

and a morphism

$$\begin{tikzcd} \mathbf{k}_1 \sqcup \ldots \sqcup \mathbf{k}_m \arrow[rr, "\gamma"] \arrow[dr, "\rho"'] & & \mathbf{k}'_1 \sqcup \ldots \sqcup \mathbf{k}'_m \arrow[dl, "\rho'"] \\ & \mathbf{k} & \end{tikzcd}$$

to

$$\begin{tikzcd} \mathbf{k}_1 \sqcup \ldots \sqcup \mathbf{k}_m \arrow[rr, "\gamma"] \arrow[dr, "\rho"'] & & \mathbf{k}'_1 \sqcup \ldots \sqcup \mathbf{k}'_m \arrow[dl, "\rho'"] \\ & \mathbf{k} \arrow[d, "\psi"] & \\ & \mathbf{l}. & \end{tikzcd}$$

We see that $\Phi_E(\mathcal{K})(\mathbf{k}, \mathbf{m})$ is functorial in $\mathbf{k}$. Let $\alpha\colon \mathbf{m} \to \mathbf{n}$ be a morphism in $\mathcal{I}$. The map α produces a functor $\Phi_E(\mathcal{K})(\mathrm{id}, \alpha)\colon \Phi_E(\mathcal{K})(\mathbf{k}, \mathbf{m}) \to \Phi_E(\mathcal{K})(\mathbf{k}, \mathbf{n})$, which takes an object $((\mathbf{k}_1, \ldots, \mathbf{k}_m), \rho\colon \mathbf{k}_1 \sqcup \ldots \sqcup \mathbf{k}_m \to \mathbf{k})$ to

$$((\mathbf{k}_{\overline{\alpha}^{-1}(1)}, \ldots, \mathbf{k}_{\overline{\alpha}^{-1}(n)}), \mathbf{k}_{\overline{\alpha}^{-1}(1)} \sqcup \ldots \sqcup \mathbf{k}_{\overline{\alpha}^{-1}(n)} \to \mathbf{k}_1 \sqcup \ldots \sqcup \mathbf{k}_m \xrightarrow{\rho} \mathbf{k}).$$

Here $\mathbf{k}_{\overline{\alpha}^{-1}(j)}$ is defined as in (3.10) for $j = 1, \ldots, n$, and the bijection

$$\mathbf{k}_{\overline{\alpha}^{-1}(1)} \sqcup \ldots \sqcup \mathbf{k}_{\overline{\alpha}^{-1}(n)} \to \mathbf{k}_1 \sqcup \ldots \sqcup \mathbf{k}_m$$

is unique because the category $\mathcal{K}$ is permutative. A morphism

$$\begin{tikzcd} \mathbf{k}_1 \sqcup \ldots \sqcup \mathbf{k}_m \arrow[rr, "\gamma"] \arrow[dr, "\rho"'] & & \mathbf{k}'_1 \sqcup \ldots \sqcup \mathbf{k}'_m \arrow[dl, "\rho'"] \\ & \mathbf{k} & \end{tikzcd}$$

is sent to

$$\begin{tikzcd} \mathbf{k}_{\overline{\alpha}^{-1}(1)} \sqcup \ldots \sqcup \mathbf{k}_{\overline{\alpha}^{-1}(n)} \arrow[rr] \arrow[d] & & \mathbf{k}'_{\overline{\alpha}^{-1}(1)} \sqcup \ldots \sqcup \mathbf{k}'_{\overline{\alpha}^{-1}(n)} \arrow[d] \\ \mathbf{k}_1 \sqcup \ldots \sqcup \mathbf{k}_m \arrow[rr, "\gamma"] \arrow[dr, "\rho"'] & & \mathbf{k}'_1 \sqcup \ldots \sqcup \mathbf{k}'_m \arrow[dl, "\rho'"] \\ & \mathbf{k}. & \end{tikzcd}$$

We argue that $\Phi_E(\mathcal{K})$ is functorial in $\mathcal{I}$. Let

$$\mathbf{m} \xrightarrow{\alpha} \mathbf{n} \xrightarrow{\beta} \mathbf{p}$$

be a composite of morphisms in $\mathcal{I}$. We need to verify that

$$\Phi_E(\mathcal{K})(\mathrm{id}, \beta \circ \alpha) = \Phi_E(\mathcal{K})(\mathrm{id}, \beta) \circ \Phi_E(\mathcal{K})(\mathrm{id}, \alpha). \tag{3.11}$$

Let $((\mathbf{k}_1, \ldots, \mathbf{k}_m), \rho\colon \mathbf{k}_1 \sqcup \ldots \sqcup \mathbf{k}_m \to \mathbf{k})$ be an object in $\Phi_E(\mathcal{K})(\mathbf{k}, \mathbf{m})$. The composite $\Phi_E(\mathcal{K})(\mathrm{id}, \beta) \circ \Phi_E(\mathcal{K})(\mathrm{id}, \alpha)$ sends $((\mathbf{k}_1, \ldots, \mathbf{k}_m), \rho\colon \mathbf{k}_1 \sqcup \ldots \sqcup \mathbf{k}_m \to \mathbf{k})$ to the tuple consisting of the p-tuple $(\mathbf{k}_{\overline{\alpha}^{-1}\overline{\beta}^{-1}(1)}, \ldots, \mathbf{k}_{\overline{\alpha}^{-1}\overline{\beta}^{-1}(p)})$ and the augmentation map

$$\mathbf{k}_{\overline{\alpha}^{-1}\overline{\beta}^{-1}(1)} \sqcup \ldots \sqcup \mathbf{k}_{\overline{\alpha}^{-1}\overline{\beta}^{-1}(p)} \to \mathbf{k}_{\overline{\alpha}^{-1}(1)} \sqcup \ldots \sqcup \mathbf{k}_{\overline{\alpha}^{-1}(n)} \to \mathbf{k}_1 \sqcup \ldots \sqcup \mathbf{k}_m \xrightarrow{\rho} \mathbf{k}$$

where

$$\mathbf{k}_{\overline{\alpha}^{-1}\overline{\beta}^{-1}(j)} = \begin{cases} \mathbf{k}_{\overline{\alpha}^{-1}(i)}, & \beta(i) = j, \\ \mathbf{0}_{\mathcal{K}}, & j \notin \mathrm{im}(\beta), \end{cases}$$

for $j = 1, \ldots, p$. On the other hand, the functor $\Phi_E(\mathcal{K})(\mathrm{id}, \beta \circ \alpha)$ maps the object $((\mathbf{k}_1, \ldots, \mathbf{k}_m), \rho\colon \mathbf{k}_1 \sqcup \ldots \sqcup \mathbf{k}_m \to \mathbf{k})$ to

$$((\mathbf{k}_{\overline{(\beta\circ\alpha)}^{-1}(1)}, \ldots, \mathbf{k}_{\overline{(\beta\circ\alpha)}^{-1}(p)}), \mathbf{k}_{\overline{(\beta\circ\alpha)}^{-1}(1)} \sqcup \ldots \sqcup \mathbf{k}_{\overline{(\beta\circ\alpha)}^{-1}(p)} \to \mathbf{k}_1 \sqcup \ldots \sqcup \mathbf{k}_m \xrightarrow{\rho} \mathbf{k}).$$

Taking into account that the p-tuple $(\mathbf{k}_{\overline{\alpha}^{-1}\overline{\beta}^{-1}(1)}, \ldots, \mathbf{k}_{\overline{\alpha}^{-1}\overline{\beta}^{-1}(p)})$ is equal to the p-tuple $(\mathbf{k}_{\overline{(\beta\circ\alpha)}^{-1}(1)}, \ldots, \mathbf{k}_{\overline{(\beta\circ\alpha)}^{-1}(p)})$ and that the composite bijection

$$\mathbf{k}_{\overline{\alpha}^{-1}\overline{\beta}^{-1}(1)} \sqcup \ldots \sqcup \mathbf{k}_{\overline{\alpha}^{-1}\overline{\beta}^{-1}(p)} \to \mathbf{k}_{\overline{\alpha}^{-1}(1)} \sqcup \ldots \sqcup \mathbf{k}_{\overline{\alpha}^{-1}(n)} \to \mathbf{k}_1 \sqcup \ldots \sqcup \mathbf{k}_m$$

coincides with the bijection

$$\mathbf{k}_{\overline{(\beta\circ\alpha)}^{-1}(1)} \sqcup \ldots \sqcup \mathbf{k}_{\overline{(\beta\circ\alpha)}^{-1}(p)} \to \mathbf{k}_1 \sqcup \ldots \sqcup \mathbf{k}_m,$$

it follows that the equation (3.11) is true on objects. Similarly, we understand that the equation (3.11) holds on morphisms. Therefore, $\Phi_E(\mathcal{K})$ is a $(\mathcal{K} \times \mathcal{I})$-category.

Definition 3.9. We define $E_{\mathcal{I}}\mathcal{K}$ to be the $(\mathcal{K}\times\mathcal{I})$-space resulting from applying the nerve functor $(\mathcal{K}\times\mathcal{I})$-levelwise to $\Phi_E(\mathcal{K})$,

$$E_{\mathcal{I}}\mathcal{K} = \mathcal{N}(\Phi_E(\mathcal{K})).$$

Remark 3.10. The $\mathcal{K}$-space $E_{\mathcal{I}}\mathcal{K}(-,\mathbf{1})$ can be identified with $E\mathcal{K}$. More than that, for $(\mathbf{k},\mathbf{m})$ in $\mathcal{K}\times\mathcal{I}_+$, the category $\Phi_E(\mathcal{K})(\mathbf{k},\mathbf{m})$ has the terminal object

$$((\mathbf{k},\mathbf{0}_{\mathcal{K}},\dots,\mathbf{0}_{\mathcal{K}}),\mathbf{k}\sqcup\mathbf{0}_{\mathcal{K}}\sqcup\dots\sqcup\mathbf{0}_{\mathcal{K}}=\mathbf{k}\xrightarrow{\mathrm{id}}\mathbf{k}).$$

Thus, the map $t\colon E_{\mathcal{I}}\mathcal{K}\to \mathrm{const}_{\mathcal{K}\times\mathcal{I}}*$ is a weak equivalence in all levels $(\mathbf{k},\mathbf{m})$ in $\mathcal{K}\times\mathcal{I}_+$. Since the inclusion functor $\mathcal{K}\times\mathcal{I}_+\to\mathcal{K}\times\mathcal{I}$ is homotopy right cofinal (see Lemma 1.23), the induced map

$$\mathrm{hocolim}_{\mathcal{K}\times\mathcal{I}}E_{\mathcal{I}}\mathcal{K}\xrightarrow{\mathrm{hocolim}_{\mathcal{K}\times\mathcal{I}}t}\mathrm{hocolim}_{\mathcal{K}\times\mathcal{I}}\mathrm{const}_{\mathcal{K}\times\mathcal{I}}*$$

is a weak equivalence by two out of three (see [Hir03, Theorem 19.6.7.(1)]).

Proposition 3.11. *The $(\mathcal{K}\times\mathcal{I})$-category $\Phi_E(\mathcal{K})$ is a commutative monoid in* $\mathrm{Cat}^{\mathcal{K}\times\mathcal{I}}$. *In particular, the $(\mathcal{K}\times\mathcal{I})$-space $E_{\mathcal{I}}\mathcal{K}$ is a commutative monoid in $\mathcal{S}^{\mathcal{K}\times\mathcal{I}}$.*

Proof. Let $(\mathbf{k},\mathbf{m})$ and $(\mathbf{l},\mathbf{n})$ be in $\mathcal{K}\times\mathcal{I}$. We define a functor

$$\lambda^{\Phi_E(\mathcal{K})}_{(\mathbf{k},\mathbf{m}),(\mathbf{l},\mathbf{n})}\colon \Phi_E(\mathcal{K})(\mathbf{k},\mathbf{m})\times\Phi_E(\mathcal{K})(\mathbf{l},\mathbf{n})\to\Phi_E(\mathcal{K})(\mathbf{k}\sqcup\mathbf{l},\mathbf{m}\sqcup\mathbf{n})$$

as follows. An object $(((\mathbf{k}_1,\dots,\mathbf{k}_m),\rho\colon\mathbf{k}_1\sqcup\dots\sqcup\mathbf{k}_m\to\mathbf{k}),((\mathbf{l}_1,\dots,\mathbf{l}_n),\nu\colon\mathbf{l}_1\sqcup\dots\sqcup\mathbf{l}_n\to\mathbf{l}))$ is mapped to $((\mathbf{k}_1,\dots,\mathbf{k}_m,\mathbf{l}_1,\dots,\mathbf{l}_n),\rho\sqcup\nu\colon\mathbf{k}_1\sqcup\dots\sqcup\mathbf{k}_m\sqcup\mathbf{l}_1\sqcup\dots\sqcup\mathbf{l}_n\to\mathbf{k}\sqcup\mathbf{l})$, and a morphism

$$\left(\begin{tikzcd}[ampersand replacement=\&] \mathbf{k}_1\sqcup\dots\sqcup\mathbf{k}_m \arrow[rr,"\gamma"] \arrow[dr,"\rho"'] \& \& \mathbf{k}'_1\sqcup\dots\sqcup\mathbf{k}'_m, \arrow[dl,"\rho'"] \& \mathbf{l}_1\sqcup\dots\sqcup\mathbf{l}_n \arrow[rr,"\delta"] \arrow[dr,"\nu"'] \& \& \mathbf{l}'_1\sqcup\dots\sqcup\mathbf{l}'_n \arrow[dl,"\nu'"] \\ \& \mathbf{k} \& \& \& \mathbf{k} \& \end{tikzcd}\right)$$

to

$$\begin{tikzcd} \mathbf{k}_1\sqcup\dots\sqcup\mathbf{k}_m\sqcup\mathbf{l}_1\sqcup\dots\sqcup\mathbf{l}_n \arrow[rr,"\gamma\sqcup\delta"] \arrow[dr,"\rho\sqcup\nu"'] & & \mathbf{k}'_1\sqcup\dots\sqcup\mathbf{k}'_m\sqcup\mathbf{l}'_1\sqcup\dots\sqcup\mathbf{l}'_n \arrow[dl,"\rho'\sqcup\nu'"] \\ & \mathbf{k}\sqcup\mathbf{l}. & \end{tikzcd}$$

The functor $\lambda^{\Phi_E(\mathcal{K})}_{(\mathbf{k},\mathbf{m}),(\mathbf{l},\mathbf{n})}$ is natural in $((\mathbf{k},\mathbf{m}),(\mathbf{l},\mathbf{n}))$. A unit for this multiplication is specified by $(\emptyset,\mathrm{id}\colon\mathbf{0}_{\mathcal{K}}\to\mathbf{0}_{\mathcal{K}})$ in $\Phi_E(\mathcal{K})(\mathbf{0}_{\mathcal{K}},\mathbf{0})$. The latter together with the functor $\lambda^{\Phi_E(\mathcal{K})}_{(\mathbf{k},\mathbf{m}),(\mathbf{l},\mathbf{n})}$ is coherently associative, unital and commutative. We spell out commutativity. For this, we need to show that the diagram

$$\begin{tikzcd} \Phi_E(\mathcal{K})(\mathbf{k},\mathbf{m})\times\Phi_E(\mathcal{K})(\mathbf{l},\mathbf{n}) \arrow[r,"\lambda^{\Phi_E(\mathcal{K})}_{(\mathbf{k},\mathbf{m}),(\mathbf{l},\mathbf{n})}"] \arrow[d,"\tau_{\Phi_E(\mathcal{K})(\mathbf{k},\mathbf{m}),\Phi_E(\mathcal{K})(\mathbf{l},\mathbf{n})}"'] & \Phi_E(\mathcal{K})(\mathbf{k}\sqcup\mathbf{l},\mathbf{m}\sqcup\mathbf{n}) \arrow[d,"\Phi_E(\mathcal{K})(\chi_{\mathbf{k},\mathbf{l}},\chi_{\mathbf{m},\mathbf{n}})"] \\ \Phi_E(\mathcal{K})(\mathbf{l},\mathbf{n})\times\Phi_E(\mathcal{K})(\mathbf{k},\mathbf{m}) \arrow[r,"\lambda^{\Phi_E(\mathcal{K})}_{(\mathbf{l},\mathbf{n}),(\mathbf{k},\mathbf{m})}"] & \Phi_E(\mathcal{K})(\mathbf{l}\sqcup\mathbf{k},\mathbf{n}\sqcup\mathbf{m}), \end{tikzcd} \tag{3.12}$$

commutes where $\chi_{\mathbf{k},\mathbf{l}}$ denotes the symmetry isomorphism in $\mathcal{K}$. Let

$$(((\mathbf{k}_1,\ldots,\mathbf{k}_m),\mathbf{k}_1\sqcup\ldots\sqcup\mathbf{k}_m\xrightarrow{\rho}\mathbf{k}),((\mathbf{l}_1,\ldots,\mathbf{l}_n),\mathbf{l}_1\sqcup\ldots\sqcup\mathbf{l}_n\xrightarrow{\nu}\mathbf{l})) \tag{3.13}$$

be an object in $\Phi_E(\mathcal{K})(\mathbf{k},\mathbf{m})\times\Phi_E(\mathcal{K})(\mathbf{l},\mathbf{n})$. The composite $\Phi_E(\mathcal{K})(\chi_{\mathbf{k},\mathbf{l}},\chi_{\mathbf{m},\mathbf{n}})\circ\lambda^{\Phi_E(\mathcal{K})}_{(\mathbf{k},\mathbf{m}),(\mathbf{l},\mathbf{n})}$ sends (3.13) to the tuple consisting of the $(n+m)$-tuple $(\mathbf{l}_1,\ldots,\mathbf{l}_n,\mathbf{k}_1,\ldots,\mathbf{k}_m)$ and the augmentation map

$$\mathbf{l}_1\sqcup\ldots\sqcup\mathbf{l}_n\sqcup\mathbf{k}_1\sqcup\ldots\sqcup\mathbf{k}_m\xrightarrow{(\chi_{\mathbf{m},\mathbf{n}})_*}\mathbf{k}_1\sqcup\ldots\sqcup\mathbf{k}_m\sqcup\mathbf{l}_1\sqcup\ldots\sqcup\mathbf{l}_n\xrightarrow{\rho\sqcup\nu}\mathbf{k}\sqcup\mathbf{l}\xrightarrow{\chi_{\mathbf{k},\mathbf{l}}}\mathbf{l}\sqcup\mathbf{k}$$

where $(\chi_{\mathbf{m},\mathbf{n}})_*$ denotes the bijection induced by the shuffle map $\chi_{\mathbf{m},\mathbf{n}}$. The other way round, the composite $\lambda^{\Phi_E(\mathcal{K})}_{(\mathbf{l},\mathbf{n}),(\mathbf{k},\mathbf{m})}\circ\tau_{\Phi_E(\mathcal{K})(\mathbf{k},\mathbf{m}),\Phi_E(\mathcal{K})(\mathbf{l},\mathbf{n})}$ sends (3.13) to the tuple $((\mathbf{l}_1,\ldots,\mathbf{l}_n,\mathbf{k}_1,\ldots,\mathbf{k}_m),\nu\sqcup\rho\colon\mathbf{l}_1\sqcup\ldots\sqcup\mathbf{l}_n\sqcup\mathbf{k}_1\sqcup\ldots\sqcup\mathbf{k}_m\to\mathbf{l}\sqcup\mathbf{k})$. For the reason that $\chi_{\mathbf{k},\mathbf{l}}\circ(\rho\sqcup\nu)\circ(\chi_{\mathbf{m},\mathbf{n}})_*=\nu\sqcup\rho$, the diagram (3.12) commutes on objects. Likewise, one can verify that the diagram (3.12) commutes on morphisms. $\square$

Forgetting the augmentation $(\mathcal{K}\times\mathcal{I})$-levelwise induces a morphism

$$\Phi_E(\mathcal{K})\to\mathrm{const}_{\mathcal{K}}\Phi_B(\mathcal{K}),$$

which is compatible with the respective commutative monoid structures and hence in $C(\mathrm{Cat}^{\mathcal{K}\times\mathcal{I}})$. Applying the nerve functor yields a map $u\colon E_{\mathcal{I}}\mathcal{K}\to\mathrm{const}_{\mathcal{K}}B_{\mathcal{I}}\mathcal{K}$ in $C\mathcal{S}^{\mathcal{K}\times\mathcal{I}}$. The adjoint of the map u is $\mathrm{colim}_{\mathcal{K}}E_{\mathcal{I}}\mathcal{K}\to B_{\mathcal{I}}\mathcal{K}$, which is an isomorphism.

3.4 $\mathcal{K}$-spaces are Quillen equivalent to $\mathcal{I}$-spaces over $B_{\mathcal{I}}\mathcal{K}$

Let $\mathcal{K}$ be a well-structured index category which is permutative. We prove that there is a chain of Quillen equivalences between $\mathcal{K}$-spaces endowed with the projective $\mathcal{K}$-model structure, and $\mathcal{I}$-spaces over $B_{\mathcal{I}}\mathcal{K}$ carrying the overcategory model structure with respect to the projective $\mathcal{I}$-model structure on $\mathcal{I}$-spaces.

We make use of the following lemma frequently in this subsection.

Lemma 3.12. *Let $\mathcal{K}$ and $\mathcal{L}$ be well-structured index categories. Let M be $(\mathcal{K}\times\mathcal{L})$-cofibrant in $\mathcal{S}^{\mathcal{K}\times\mathcal{L}}$. The map induced by the canonical map from the homotopy colimit to the colimit*

$$\mathrm{hocolim}_{\mathcal{K}}\mathrm{hocolim}_{\mathcal{L}}M\to\mathrm{hocolim}_{\mathcal{K}}\mathrm{colim}_{\mathcal{L}}M \tag{3.14}$$

is a weak equivalence.

Proof. As the object M is $(\mathcal{K}\times\mathcal{L})$-cofibrant, the object $\mathrm{colim}_{\mathcal{L}}M$ is $\mathcal{K}$-cofibrant by [Hir03, Theorem 11.6.8.(1)]. The map (3.14) fits into the commutative diagram

$$\begin{tikzcd}
\mathrm{hocolim}_{\mathcal{K}}\mathrm{hocolim}_{\mathcal{L}}M \arrow[r, "\sim", no head] \arrow[drrr, bend right] & \mathrm{hocolim}_{\mathcal{K}\times\mathcal{L}}M \arrow[r] & \mathrm{colim}_{\mathcal{K}\times\mathcal{L}}M\cong & \mathrm{colim}_{\mathcal{K}}\mathrm{colim}_{\mathcal{L}}M. \\
 & & & \mathrm{hocolim}_{\mathcal{K}}\mathrm{colim}_{\mathcal{L}}M \arrow[u]
\end{tikzcd}$$

Here the horizontal and the vertical map are weak equivalences by [SS12, Lemma 6.22]. It follows from two out of three that the map (3.14) is a weak equivalence. $\square$

Lemma 3.13. *Let the category $\mathcal{S}^{\mathcal{K}\times\mathcal{I}}$ be equipped with the projective $(\mathcal{K}\times\mathcal{I})$-model structure and the category $\mathcal{S}^{\mathcal{K}}$ with the projective $\mathcal{K}$-model structure. The adjunction*

$$\mathcal{S}^{\mathcal{K}\times\mathcal{I}} \underset{\mathrm{const}_{\mathcal{I}}}{\overset{\mathrm{colim}_{\mathcal{I}}}{\rightleftarrows}} \mathcal{S}^{\mathcal{K}} \tag{3.15}$$

is a Quillen equivalence.

Proof. (compare [SS12, proof of Proposition 6.23]) We start with showing that the functor $\mathrm{const}_{\mathcal{I}}$ is a right Quillen functor. Let $f\colon M \to N$ be a $\mathcal{K}$-fibration in $\mathcal{S}^{\mathcal{K}}$. The induced map $\mathrm{const}_{\mathcal{I}}f\colon \mathrm{const}_{\mathcal{I}}M \to \mathrm{const}_{\mathcal{I}}N$ is a level fibration in $\mathcal{S}^{\mathcal{K}\times\mathcal{I}}$. Let $(\mathbf{k},\mathbf{m}) \to (\mathbf{l},\mathbf{n})$ be a morphism in $\mathcal{K}\times\mathcal{I}$. The induced square

$$\begin{array}{ccc} (\mathrm{const}_{\mathcal{I}}M)(\mathbf{k},\mathbf{m}) = M(\mathbf{k}) & \longrightarrow & (\mathrm{const}_{\mathcal{I}}M)(\mathbf{l},\mathbf{n}) = M(\mathbf{l}) \\ \downarrow & & \downarrow \\ (\mathrm{const}_{\mathcal{I}}N)(\mathbf{k},\mathbf{m}) = N(\mathbf{k}) & \longrightarrow & (\mathrm{const}_{\mathcal{I}}N)(\mathbf{l},\mathbf{n}) = N(\mathbf{l}) \end{array}$$

is homotopy cartesian. Hence, the map $\mathrm{const}_{\mathcal{I}}f$ is a $(\mathcal{K}\times\mathcal{I})$-fibration in $\mathcal{S}^{\mathcal{K}\times\mathcal{I}}$. In addition, for a $\mathcal{K}$-space M, we have the following weak equivalences

$$\begin{aligned} \mathrm{hocolim}_{\mathcal{K}\times\mathcal{I}}\mathrm{const}_{\mathcal{I}}M &\simeq \mathrm{hocolim}_{\mathcal{K}}\mathrm{hocolim}_{\mathcal{I}}\mathrm{const}_{\mathcal{I}}M \\ &\simeq \mathrm{hocolim}_{\mathcal{K}}(\mathcal{N}\mathcal{I}\times M) \\ &\simeq \mathrm{hocolim}_{\mathcal{K}}M. \end{aligned}$$

This implies that the functor $\mathrm{const}_{\mathcal{I}}$ preserves weak equivalences. Thus, the adjunction (3.15) is a Quillen adjunction.
Let M be a $(\mathcal{K}\times\mathcal{I})$-cofibrant $(\mathcal{K}\times\mathcal{I})$-space, and let N be a $\mathcal{K}$-fibrant $\mathcal{K}$-space. Assume that there is a map $f\colon \mathrm{colim}_{\mathcal{I}}M \to N$ in $\mathcal{S}^{\mathcal{K}}$ with adjoint $\mathrm{ad}(f)\colon M \to \mathrm{const}_{\mathcal{I}}N$ in $\mathcal{S}^{\mathcal{K}\times\mathcal{I}}$. We consider the diagram

$$\begin{array}{ccc} \mathrm{hocolim}_{\mathcal{K}}\mathrm{colim}_{\mathcal{I}}M & \xrightarrow{\mathrm{hocolim}_{\mathcal{K}}f} & \mathrm{hocolim}_{\mathcal{K}}N. \\ \sim\uparrow & & \uparrow\sim \\ \mathrm{hocolim}_{\mathcal{K}}\mathrm{hocolim}_{\mathcal{I}}M & \xrightarrow{\mathrm{hocolim}_{\mathcal{K}}\mathrm{hocolim}_{\mathcal{I}}\mathrm{ad}(f)} & \mathrm{hocolim}_{\mathcal{K}}\mathrm{hocolim}_{\mathcal{I}}\mathrm{const}_{\mathcal{I}}N \end{array}$$

Here the vertical maps come from the canonical map from the homotopy colimit to the colimit, using in the case of the right vertical map that the functor $\mathrm{colim}_{\mathcal{I}}\circ\mathrm{const}_{\mathcal{I}}$ is the identity functor. The left vertical map is a weak equivalence by Lemma 3.12. By two out of three we can conclude that the map f is a weak equivalence if and only if its adjoint $\mathrm{ad}(f)$ is so. □

Lemma 3.14. *Let the category $\mathcal{S}^{\mathcal{K}\times\mathcal{I}}$ carry the projective $(\mathcal{K}\times\mathcal{I})$-model structure, and the comma category $(\mathcal{S}^{\mathcal{K}\times\mathcal{I}}\downarrow E_{\mathcal{I}}\mathcal{K})$ the overcategory model structure. The adjunction*

$$(\mathcal{S}^{\mathcal{K}\times\mathcal{I}}\downarrow E_{\mathcal{I}}\mathcal{K}) \underset{q}{\overset{t_*}{\rightleftarrows}} \mathcal{S}^{\mathcal{K}\times\mathcal{I}},$$

induced by composition with and pullback along the map $t\colon E_{\mathcal{I}}\mathcal{K} \to \mathrm{const}_{\mathcal{K}\times\mathcal{I}}*$, defines a Quillen equivalence.

Proof. This holds because the model category $\mathcal{S}^{\mathcal{K}\times\mathcal{I}}$ is right proper (see Remark 2.8) and the map t is a $(\mathcal{K}\times\mathcal{I})$-equivalence (see Remark 3.10). $\square$

Lemma 3.15. *Let $\mathcal{S}^{\mathcal{K}\times\mathcal{I}}$ have the projective $(\mathcal{K}\times\mathcal{I})$-model structure and $\mathcal{S}^{\mathcal{I}}$ the projective $\mathcal{I}$-model structure. Let the comma categories $(\mathcal{S}^{\mathcal{K}\times\mathcal{I}}\downarrow E_{\mathcal{I}}\mathcal{K})$, $(\mathcal{S}^{\mathcal{K}\times\mathcal{I}}\downarrow \mathrm{const}_{\mathcal{K}}B_{\mathcal{I}}\mathcal{K})$ and $(\mathcal{S}^{\mathcal{I}}\downarrow B_{\mathcal{I}}\mathcal{K})$ be endowed with the overcategory model structures. The composite adjunction*

$$(\mathcal{S}^{\mathcal{K}\times\mathcal{I}}\downarrow E_{\mathcal{I}}\mathcal{K}) \underset{p}{\overset{u_*}{\rightleftarrows}} (\mathcal{S}^{\mathcal{K}\times\mathcal{I}}\downarrow \mathrm{const}_{\mathcal{K}}B_{\mathcal{I}}\mathcal{K}) \underset{\mathrm{const}_{\mathcal{K}}}{\overset{\mathrm{colim}_{\mathcal{K}}}{\rightleftarrows}} (\mathcal{S}^{\mathcal{I}}\downarrow B_{\mathcal{I}}\mathcal{K}),$$

where the first adjunction is defined by composition with and pullback along the map $u\colon E_{\mathcal{I}}\mathcal{K}\to \mathrm{const}_{\mathcal{K}}B_{\mathcal{I}}\mathcal{K}$, is a Quillen equivalence.

Proof. (compare [SS12, proof of Lemma 13.4]) By definition of the model structures the functor u_* respects cofibrations and weak equivalences. Analogous to the proof of Lemma 3.13 we can see that the functor $\mathrm{const}_{\mathcal{K}}$ preserves fibrations and weak equivalences. To show that the Quillen adjunction $(\mathrm{colim}_{\mathcal{K}}\circ u_*, p\circ \mathrm{const}_{\mathcal{K}})$ is a Quillen equivalence, we make use of [Hov99, Corollary 1.3.16]. First, we prove that the functor $\mathrm{colim}_{\mathcal{K}}\circ u_*$ reflects weak equivalences between cofibrant objects. Let

$$\begin{tikzcd} M \arrow[rr, "f"] \arrow[dr] & & N \arrow[dl] \\ & E_{\mathcal{I}}\mathcal{K} & \end{tikzcd}$$

be a map between cofibrant objects in $(\mathcal{S}^{\mathcal{K}\times\mathcal{I}}\downarrow E_{\mathcal{I}}\mathcal{K})$ such that the induced map $\mathrm{hocolim}_{\mathcal{I}}\mathrm{colim}_{\mathcal{K}}f$ is a weak equivalence. From Lemma 3.12 and two out of three we deduce that the map $\mathrm{hocolim}_{\mathcal{K}\times\mathcal{I}}f$ is a weak equivalence. Secondly, we check that for a fibrant object $(Y,\ Y\twoheadrightarrow B_{\mathcal{I}}\mathcal{K})$ in $(\mathcal{S}^{\mathcal{I}}\downarrow B_{\mathcal{I}}\mathcal{K})$, the derived counit

$$\begin{tikzcd} \mathrm{colim}_{\mathcal{K}}(E_{\mathcal{I}}\mathcal{K}\times_{\mathrm{const}_{\mathcal{K}}B_{\mathcal{I}}\mathcal{K}}\mathrm{const}_{\mathcal{K}}Y)^{\mathrm{cof}} \arrow[r] \arrow[dr] & \mathrm{colim}_{\mathcal{K}}(E_{\mathcal{I}}\mathcal{K}\times_{\mathrm{const}_{\mathcal{K}}B_{\mathcal{I}}\mathcal{K}}\mathrm{const}_{\mathcal{K}}Y) \arrow[r] & Y \arrow[dl, bend left] \\ & B_{\mathcal{I}}\mathcal{K} & \end{tikzcd}$$

is a weak equivalence. We investigate the following diagram where we abbreviate the functor $\mathrm{hocolim}_{\mathcal{K}}$ by $(-)_{h\mathcal{K}}$, and the functor $\mathrm{hocolim}_{\mathcal{I}}$ by $(-)_{h\mathcal{I}}$ respectively,

$$\begin{tikzcd} (((E_{\mathcal{I}}\mathcal{K}\times_{\mathrm{const}_{\mathcal{K}}B_{\mathcal{I}}\mathcal{K}}\mathrm{const}_{\mathcal{K}}Y)^{\mathrm{cof}})_{h\mathcal{K}})_{h\mathcal{I}} \arrow[r, "\sim"] \arrow[d, "\sim"'] & (\mathrm{colim}_{\mathcal{K}}(E_{\mathcal{I}}\mathcal{K}\times_{\mathrm{const}_{\mathcal{K}}B_{\mathcal{I}}\mathcal{K}}\mathrm{const}_{\mathcal{K}}Y)^{\mathrm{cof}})_{h\mathcal{I}} \arrow[d] \\ ((E_{\mathcal{I}}\mathcal{K}\times_{\mathrm{const}_{\mathcal{K}}B_{\mathcal{I}}\mathcal{K}}\mathrm{const}_{\mathcal{K}}Y)_{h\mathcal{K}})_{h\mathcal{I}} \arrow[r] \arrow[d] & (\mathrm{colim}_{\mathcal{K}}(E_{\mathcal{I}}\mathcal{K}\times_{\mathrm{const}_{\mathcal{K}}B_{\mathcal{I}}\mathcal{K}}\mathrm{const}_{\mathcal{K}}Y))_{h\mathcal{I}} \arrow[d] \\ ((\mathrm{const}_{\mathcal{K}}Y)_{h\mathcal{K}})_{h\mathcal{I}} \arrow[r] & Y_{h\mathcal{I}}. \end{tikzcd}$$

The first horizontal map is a weak equivalence by Lemma 3.12. To see that the composite map in the second column is a weak equivalence, it suffices to argue that the composite

$$((E_{\mathcal{I}}\mathcal{K}\times_{\mathrm{const}_{\mathcal{K}}B_{\mathcal{I}}\mathcal{K}}\mathrm{const}_{\mathcal{K}}Y)_{h\mathcal{K}})_{h\mathcal{I}} \to ((\mathrm{const}_{\mathcal{K}}Y)_{h\mathcal{K}})_{h\mathcal{I}} \to Y_{h\mathcal{I}}$$

is a weak equivalence. For this, we analyze the diagram

$$\begin{array}{ccccc} \mathrm{hocolim}_{\mathcal{K}}(E_{\mathcal{I}}\mathcal{K} \times_{\mathrm{const}_{\mathcal{K}} B_{\mathcal{I}}\mathcal{K}} \mathrm{const}_{\mathcal{K}} Y) & \longrightarrow & \mathrm{hocolim}_{\mathcal{K}}\mathrm{const}_{\mathcal{K}} Y & \xrightarrow{\mathrm{pr}} & Y \\ \downarrow & & \downarrow & & \twoheaddownarrow \\ \mathrm{hocolim}_{\mathcal{K}} E_{\mathcal{I}}\mathcal{K} & \longrightarrow & \mathrm{hocolim}_{\mathcal{K}}\mathrm{const}_{\mathcal{K}} B_{\mathcal{I}}\mathcal{K} & \xrightarrow{\mathrm{pr}} & B_{\mathcal{I}}\mathcal{K} \end{array} \tag{3.16}$$

in $\mathcal{S}^{\mathcal{I}}$. Here the second top and bottom horizontal map pr are given by the projection maps, using that for an $\mathcal{I}$-space Z, we have

$$\mathrm{hocolim}_{\mathcal{K}}\mathrm{const}_{\mathcal{K}} Z \xrightarrow{\sim} B\mathcal{K} \times Z \xrightarrow{\mathrm{pr}} Z.$$

The pullback $E_{\mathcal{I}}\mathcal{K} \times_{\mathrm{const}_{\mathcal{K}} B_{\mathcal{I}}\mathcal{K}} \mathrm{const}_{\mathcal{K}} Y$ in $\mathcal{S}^{\mathcal{K}\times\mathcal{I}}$ is $\mathcal{I}$-levelwise a pullback in $\mathcal{S}^{\mathcal{K}}$. Therefore, [SS12, Lemma 11.2] yields that the left square in (3.16) is a pullback square in $\mathcal{S}^{\mathcal{I}}$. Besides, the right square in (3.16) is a pullback square, too. Consequently, the outer square in (3.16) is a pullback square, which is a homotopy pullback square as the map $Y \to B_{\mathcal{I}}\mathcal{K}$ is an $\mathcal{I}$-fibration by assumption. To obtain that the upper horizontal composite map in (3.16) is an $\mathcal{I}$-equivalence, it is enough to show that the lower horizontal composite map in (3.16) is an $\mathcal{I}$-equivalence, for the reason that $\mathcal{S}^{\mathcal{I}}$ is right proper (see Remark 2.8). Exploiting that the $(\mathcal{K} \times \mathcal{I})$-space $E_{\mathcal{I}}\mathcal{K}$ is homotopy constant with respect to morphisms in $\mathcal{K} \times \mathcal{I}_+$, and that the $\mathcal{I}$-space $B_{\mathcal{I}}\mathcal{K}$ is homotopy constant with respect to morphisms in $\mathcal{I}_+$, it remains to prove that the composite

$$\mathrm{hocolim}_{\mathcal{K}} E_{\mathcal{I}}\mathcal{K}(-, \mathbf{1}) \to \mathrm{hocolim}_{\mathcal{K}}\mathrm{const}_{\mathcal{K}} B_{\mathcal{I}}\mathcal{K}(\mathbf{1}) \to B_{\mathcal{I}}\mathcal{K}(\mathbf{1}) \tag{3.17}$$

is a weak equivalence. But the map (3.17) can be identified with the map

$$\mathrm{hocolim}_{\mathcal{K}} E\mathcal{K} \to \mathrm{colim}_{\mathcal{K}} E\mathcal{K} \cong B\mathcal{K},$$

which is a weak equivalence because the $\mathcal{K}$-space $E\mathcal{K}$ is $\mathcal{K}$-cofibrant (see [Hir03, Proposition 14.8.9], [SS12, Lemma 6.22]). □

Theorem 3.16. *The category of $\mathcal{K}$-spaces is Quillen equivalent to the category of $\mathcal{I}$-spaces over $B_{\mathcal{I}}\mathcal{K}$.*

Proof. Putting together the results of Lemma 3.13, Lemma 3.14 and Lemma 3.15, we get the following chain of Quillen equivalences between $\mathcal{S}^{\mathcal{K}}$ and $(\mathcal{S}^{\mathcal{I}} \downarrow B_{\mathcal{I}}\mathcal{K})$,

$$\mathcal{S}^{\mathcal{K}} \underset{\mathrm{const}_{\mathcal{I}}}{\overset{\mathrm{colim}_{\mathcal{I}}}{\rightleftarrows}} \mathcal{S}^{\mathcal{K}\times\mathcal{I}} \underset{q}{\overset{t_*}{\rightleftarrows}} (\mathcal{S}^{\mathcal{K}\times\mathcal{I}} \downarrow E_{\mathcal{I}}\mathcal{K}) \underset{p}{\overset{u_*}{\rightleftarrows}} (\mathcal{S}^{\mathcal{K}\times\mathcal{I}} \downarrow \mathrm{const}_{\mathcal{K}} B_{\mathcal{I}}\mathcal{K}) \underset{\mathrm{const}_{\mathcal{K}}}{\overset{\mathrm{colim}_{\mathcal{K}}}{\rightleftarrows}} (\mathcal{S}^{\mathcal{I}} \downarrow B_{\mathcal{I}}\mathcal{K}). \tag{3.18}$$

□

3.5 Commutative $\mathcal{K}$-spaces are Quillen equivalent to commutative $\mathcal{I}$-spaces over $B_{\mathcal{I}}\mathcal{K}$

Let $\mathcal{K}$ be a well-structured index category which is permutative. Suppose that the inclusion functor $\mathcal{K}_+ \to \mathcal{K}$ is homotopy right cofinal, and that the pair $(\mathcal{K}, \mathcal{O}\mathcal{K}_+)$ is very well-structured. The left adjoints involved in the above chain of Quillen equivalences (3.18) are strong symmetric monoidal. Hence, the adjunctions in (3.18) lift to the level of commutative monoids,

$$\begin{tikzcd} C\mathcal{S}^{\mathcal{K}} \arrow[r, shift right, "\mathrm{const}_{\mathcal{I}}"'] & C\mathcal{S}^{\mathcal{K}\times\mathcal{I}} \arrow[l, shift right, "\mathrm{colim}_{\mathcal{I}}"'] \arrow[r, shift right, "q"'] & (C\mathcal{S}^{\mathcal{K}\times\mathcal{I}} \downarrow E_{\mathcal{I}}\mathcal{K}) \arrow[l, shift right, "t_*"'] \arrow[r, shift left, "u_*"] & (C\mathcal{S}^{\mathcal{K}\times\mathcal{I}} \downarrow \mathrm{const}_{\mathcal{K}} B_{\mathcal{I}}\mathcal{K}) \arrow[l, shift left, "p"] \arrow[d, shift right, "\mathrm{colim}_{\mathcal{K}}"'] \\ & & & (C\mathcal{S}^{\mathcal{I}} \downarrow B_{\mathcal{I}}\mathcal{K}). \arrow[u, shift right, "\mathrm{const}_{\mathcal{K}}"'] \end{tikzcd} \tag{3.19}$$

Our aim in this subsection is to prove that the adjunctions in the chain (3.19) are Quillen equivalences. As in Subsection 3.2 we consider the Barratt-Eccles operad in spaces and define an E_∞ monoid in diagram spaces to be a diagram space with an action of the Barratt-Eccles operad in spaces. In the case of the adjunctions $(\mathrm{colim}_{\mathcal{I}}, \mathrm{const}_{\mathcal{I}})$ and $(\mathrm{colim}_{\mathcal{K}} \circ \pi_*, p \circ \mathrm{const}_{\mathcal{K}})$ we employ the respective adjunctions on the level of E_∞ monoids for exploiting that the underlying diagram space of a cofibrant E_∞ diagram space is cofibrant which is not true for a cofibrant commutative diagram space. The proofs in this subsection are mainly based on the corresponding proofs in the previous subsection.

Lemma 3.17. *Let $E_\infty(\mathcal{S}^{\mathcal{K}\times\mathcal{I}})^+$ denote the positive projective $(\mathcal{K}\times\mathcal{I})$-model structure, and $E_\infty(\mathcal{S}^{\mathcal{I}})^+$ the positive projective $\mathcal{I}$-model structure. The adjunction*

$$E_\infty(\mathcal{S}^{\mathcal{K}\times\mathcal{I}})^+ \underset{\mathrm{const}_{\mathcal{I}}}{\overset{\mathrm{colim}_{\mathcal{I}}}{\rightleftarrows}} E_\infty(\mathcal{S}^{\mathcal{K}})^+$$

is a Quillen equivalence.

Proof. Since both model structures are right-induced, the functor $\mathrm{const}_{\mathcal{I}}$ is a right Quillen functor (see Lemma 3.13). Let M be positive cofibrant in $E_\infty(\mathcal{S}^{\mathcal{K}\times\mathcal{I}})^+$. The identity functor $\mathrm{id}\colon E_\infty(\mathcal{S}^{\mathcal{K}\times\mathcal{I}})^+ \to E_\infty\mathcal{S}^{\mathcal{K}\times\mathcal{I}}$ passing from the positive projective $(\mathcal{K}\times\mathcal{I})$-model structure to the projective $(\mathcal{K}\times\mathcal{I})$-model structure, defines the left adjoint in a Quillen equivalence [SS12, Proposition 9.8]. Thus, the object M is cofibrant in $E_\infty\mathcal{S}^{\mathcal{K}\times\mathcal{I}}$, and it follows from [SS12, Corollary 12.3] that the underlying $(\mathcal{K}\times\mathcal{I})$-space of M is cofibrant in the projective $(\mathcal{K}\times\mathcal{I})$-model structure on $\mathcal{S}^{\mathcal{K}\times\mathcal{I}}$. By the same arguments the underlying $\mathcal{K}$-space of $\mathrm{colim}_{\mathcal{I}}M$ is cofibrant in the projective $\mathcal{K}$-model structure on $\mathcal{S}^{\mathcal{K}}$. Given this, we can continue as in the proof of Lemma 3.13. □

Corollary 3.18. *Let the category $C\mathcal{S}^{\mathcal{K}\times\mathcal{I}}$ be endowed with the positive projective $(\mathcal{K}\times\mathcal{I})$-model structure and $C\mathcal{S}^{\mathcal{K}}$ with the positive projective $\mathcal{K}$-model structure. The adjunction*

$$C\mathcal{S}^{\mathcal{K}\times\mathcal{I}} \underset{\mathrm{const}_{\mathcal{I}}}{\overset{\mathrm{colim}_{\mathcal{I}}}{\rightleftarrows}} C\mathcal{S}^{\mathcal{K}} \tag{3.20}$$

is a Quillen equivalence.

Proof. The functor $\mathrm{const}_{\mathcal{I}}$ is a right Quillen functor. We consider the following square of Quillen adjunctions,

$$\begin{tikzcd} C\mathcal{S}^{\mathcal{K}\times\mathcal{I}} \arrow[r, shift left, "\mathrm{colim}_{\mathcal{I}}"] \arrow[d, shift left, "\epsilon^*"] & C\mathcal{S}^{\mathcal{K}} \arrow[l, shift left, "\mathrm{const}_{\mathcal{I}}"] \arrow[d, shift left, "\epsilon^*"] \\ E_\infty(\mathcal{S}^{\mathcal{K}\times\mathcal{I}})^+ \arrow[u, shift left, "\epsilon_*"] \arrow[r, shift left, "\mathrm{colim}_{\mathcal{I}}"] & E_\infty(\mathcal{S}^{\mathcal{K}})^+. \arrow[u, shift left, "\epsilon_*"] \arrow[l, shift left, "\mathrm{const}_{\mathcal{I}}"] \end{tikzcd}$$

Here the map of operads in spaces from the Barratt-Eccles operad to the commutativity operad induces the vertical adjunctions which are Quillen equivalences by [SS12, Proposition 9.12]. The last lemma and two out of three for Quillen equivalences [Hov99, Corollary 1.3.15] ensure that (3.20) is a Quillen equivalence. □

Lemma 3.19. *Let the category $C\mathcal{S}^{\mathcal{K}\times\mathcal{I}}$ be equipped with the positive projective $(\mathcal{K}\times\mathcal{I})$-model structure and the comma category $(C\mathcal{S}^{\mathcal{K}\times\mathcal{I}} \downarrow E_{\mathcal{I}}\mathcal{K})$ with the overcategory model structure. The adjunction*

$$(C\mathcal{S}^{\mathcal{K}\times\mathcal{I}} \downarrow E_{\mathcal{I}}\mathcal{K}) \underset{q}{\overset{t_*}{\rightleftarrows}} C\mathcal{S}^{\mathcal{K}\times\mathcal{I}}$$

defines a Quillen equivalence.

Proof. This follows from the model category $C\mathcal{S}^{\mathcal{K}\times\mathcal{I}}$ being right proper (see Remark 2.10) and the morphism $t\colon E_{\mathcal{I}}\mathcal{K} \to \mathrm{const}_{\mathcal{K}\times\mathcal{I}}*$ being a $(\mathcal{K}\times\mathcal{I})$-equivalence (see Remark 3.10). □

Lemma 3.20. *Let $E_\infty(\mathcal{S}^{\mathcal{K}\times\mathcal{I}})^+$ denote the positive projective $(\mathcal{K}\times\mathcal{I})$-model structure, and $E_\infty(\mathcal{S}^{\mathcal{I}})^+$ the positive projective $\mathcal{I}$-model structure. Let the comma categories $(E_\infty(\mathcal{S}^{\mathcal{K}\times\mathcal{I}})^+ \downarrow \epsilon^*(E_{\mathcal{I}}\mathcal{K}))$, $(E_\infty(\mathcal{S}^{\mathcal{K}\times\mathcal{I}})^+ \downarrow \epsilon^*(\mathrm{const}_{\mathcal{K}}B_{\mathcal{I}}\mathcal{K}))$ and $(E_\infty(\mathcal{S}^{\mathcal{I}})^+ \downarrow \epsilon^*(B_{\mathcal{I}}\mathcal{K}))$ carry the overcategory model structures. The composite adjunction*

$$(E_\infty(\mathcal{S}^{\mathcal{K}\times\mathcal{I}})^+\downarrow\epsilon^*(E_{\mathcal{I}}\mathcal{K})) \underset{p}{\overset{u_*}{\rightleftarrows}} (E_\infty(\mathcal{S}^{\mathcal{K}\times\mathcal{I}})^+\downarrow\epsilon^*(\mathrm{const}_{\mathcal{K}}B_{\mathcal{I}}\mathcal{K})) \underset{\mathrm{const}_{\mathcal{K}}}{\overset{\mathrm{colim}_{\mathcal{K}}}{\rightleftarrows}} (E_\infty(\mathcal{S}^{\mathcal{I}})^+\downarrow\epsilon^*(B_{\mathcal{I}}\mathcal{K}))$$

is a Quillen equivalence.

Proof. By definition of the model structures and Lemma 3.15 the functor $p\circ\mathrm{const}_{\mathcal{K}}$ is a right Quillen functor. Like in the proof of Lemma 3.17 we make use of [SS12, Proposition 9.8] and [SS12, Corollary 12.3] so that we can argue as in the proof of Lemma 3.15. □

Corollary 3.21. *Let the category $C\mathcal{S}^{\mathcal{K}\times\mathcal{I}}$ come with the positive projective $(\mathcal{K}\times\mathcal{I})$-model structure, and $C\mathcal{S}^{\mathcal{I}}$ with the positive projective $\mathcal{I}$-model structure. Let the comma categories $(C\mathcal{S}^{\mathcal{K}\times\mathcal{I}} \downarrow E_{\mathcal{I}}\mathcal{K})$, $(C\mathcal{S}^{\mathcal{K}\times\mathcal{I}} \downarrow \mathrm{const}_{\mathcal{K}}B_{\mathcal{I}}\mathcal{K})$ and $(C\mathcal{S}^{\mathcal{I}} \downarrow B_{\mathcal{I}}\mathcal{K})$ possess the overcategory model structures. The composite adjunction*

$$(C\mathcal{S}^{\mathcal{K}\times\mathcal{I}} \downarrow E_{\mathcal{I}}\mathcal{K}) \underset{p}{\overset{u_*}{\rightleftarrows}} (C\mathcal{S}^{\mathcal{K}\times\mathcal{I}} \downarrow \mathrm{const}_{\mathcal{K}}B_{\mathcal{I}}\mathcal{K}) \underset{\mathrm{const}_{\mathcal{K}}}{\overset{\mathrm{colim}_{\mathcal{K}}}{\rightleftarrows}} (C\mathcal{S}^{\mathcal{I}} \downarrow B_{\mathcal{I}}\mathcal{K}) \tag{3.21}$$

is a Quillen equivalence.

Proof. The composite right adjoint $p \circ \mathrm{const}_{\mathcal{K}}$ is right Quillen by the definition of the model structures and Lemma 3.15. We consider the following diagram of Quillen adjunctions,

$$\begin{tikzcd}
(C\mathcal{S}^{\mathcal{K}\times\mathcal{I}} \downarrow E_{\mathcal{I}}\mathcal{K}) \arrow[r, shift left, "u_*"] \arrow[d, shift left, "\epsilon^*"] & (C\mathcal{S}^{\mathcal{K}\times\mathcal{I}} \downarrow \mathrm{const}_{\mathcal{K}} B_{\mathcal{I}}\mathcal{K}) \arrow[l, shift left, "p"] \arrow[r, shift left, "\mathrm{colim}_{\mathcal{K}}"] \arrow[d, shift left, "\epsilon^*"] & (C\mathcal{S}^{\mathcal{I}} \downarrow B_{\mathcal{I}}\mathcal{K}) \arrow[l, shift left, "\mathrm{const}_{\mathcal{K}}"] \arrow[d, shift left, "\epsilon^*"] \\
(E_\infty(\mathcal{S}^{\mathcal{K}\times\mathcal{I}})^+ \downarrow \epsilon^*(E_{\mathcal{I}}\mathcal{K})) \arrow[u, shift left, "\epsilon_*"] \arrow[r, shift left, "u_*"] & (E_\infty(\mathcal{S}^{\mathcal{K}\times\mathcal{I}})^+ \downarrow \epsilon^*(\mathrm{const}_{\mathcal{K}} B_{\mathcal{I}}\mathcal{K})) \arrow[u, shift left, "\epsilon_*"] \arrow[l, shift left, "p"] \arrow[r, shift left, "\mathrm{colim}_{\mathcal{K}}"] & (E_\infty(\mathcal{S}^{\mathcal{I}})^+ \downarrow \epsilon^*(B_{\mathcal{I}}\mathcal{K})). \arrow[u, shift left, "\epsilon_*"] \arrow[l, shift left, "\mathrm{const}_{\mathcal{K}}"]
\end{tikzcd}$$

Again the map of operads in spaces from the Barratt-Eccles operad to the commutativity operad gives rise to the vertical Quillen equivalences by [SS12, Proposition 9.12]. It follows from Lemma 3.20 and two out of three for Quillen equivalences [Hov99, Corollary 1.3.15] that (3.21) is a Quillen equivalence. □

Theorem 3.22. *The category of commutative $\mathcal{K}$-spaces is Quillen equivalent to the category of commutative $\mathcal{I}$-spaces over $B_{\mathcal{I}}\mathcal{K}$.*

Proof. From Corollary 3.18, Lemma 3.19 and Corollary 3.21 we know that the adjunctions in (3.19) are Quillen equivalences. □

Remark 3.23. The above theorem allows to think of commutative $\mathcal{K}$-spaces as commutative $\mathcal{I}$-spaces over $B_{\mathcal{I}}\mathcal{K}$. In this sense, the category $\mathcal{I}$ is universal among the indexing categories satisfying the assumptions of $\mathcal{K}$.

Let R denote a fibrant replacement functor in $C\mathcal{S}^{\mathcal{K}}$, and Q a cofibrant replacement functor in $(C\mathcal{S}^{\mathcal{K}\times\mathcal{I}} \downarrow \mathrm{const}_{\mathcal{K}} B_{\mathcal{I}}\mathcal{K})$. We write $I_{\mathcal{K}}^{\mathcal{I}}$ for the composite derived functor $\mathrm{colim}_{\mathcal{K}} Q(u_*(E_{\mathcal{I}}\mathcal{K} \times \mathrm{const}_{\mathcal{I}} R(-)))$ from $C\mathcal{S}^{\mathcal{K}}$ to $(C\mathcal{S}^{\mathcal{I}} \downarrow B_{\mathcal{I}}\mathcal{K})$ (see (3.19)). Let R' stand for a fibrant replacement functor in $E_\infty\mathcal{S}$, and Q' for a cofibrant replacement functor in $E_\infty(\mathcal{S}^{\mathcal{I}})^+$. We write $I^{\mathcal{I}}$ for the composite derived functor $\epsilon_*(Q'(\mathrm{const}_{\mathcal{I}} R(-)))$ from $E_\infty\mathcal{S}$ to $C\mathcal{S}^{\mathcal{I}}$ (see (3.9)).

Proposition 3.24. *Let M be a commutative $\mathcal{K}$-space. The image $I_{\mathcal{K}}^{\mathcal{I}}(M)$, where we forget the augmentation to $B_{\mathcal{I}}\mathcal{K}$, is weakly equivalent to $I^{\mathcal{I}}(\mathrm{hocolim}_{\mathcal{K}} M)$.*

Proof. We take advantage of the commutativity of the following diagram of Quillen equivalences,

$$\begin{tikzcd}
E_\infty(\mathcal{S}^{\mathcal{K}})^+ \arrow[r, shift left, "\epsilon_*"] \arrow[d, shift left, "\mathrm{const}_{\mathcal{I}}"] & C\mathcal{S}^{\mathcal{K}} \arrow[l, shift left, "\epsilon^*"] \arrow[d, shift left, "\mathrm{const}_{\mathcal{I}}"] \\
E_\infty(\mathcal{S}^{\mathcal{K}\times\mathcal{I}})^+ \arrow[u, shift left, "\mathrm{colim}_{\mathcal{I}}"] \arrow[r, shift left, "\epsilon_*"] \arrow[d, shift left, "q"] & C\mathcal{S}^{\mathcal{K}\times\mathcal{I}} \arrow[u, shift left, "\mathrm{colim}_{\mathcal{I}}"] \arrow[l, shift left, "\epsilon^*"] \arrow[d, shift left, "q"] \\
(E_\infty(\mathcal{S}^{\mathcal{K}\times\mathcal{I}})^+ \downarrow \epsilon^*(E_{\mathcal{I}}\mathcal{K})) \arrow[u, shift left, "t_*"] \arrow[r, shift left, "\epsilon_*"] \arrow[d, shift right, "u_*"'] & (C\mathcal{S}^{\mathcal{K}\times\mathcal{I}} \downarrow E_{\mathcal{I}}\mathcal{K}) \arrow[u, shift left, "t_*"] \arrow[l, shift left, "\epsilon^*"] \arrow[d, shift right, "u_*"'] \\
(E_\infty(\mathcal{S}^{\mathcal{K}\times\mathcal{I}})^+ \downarrow \epsilon^*(\mathrm{const}_{\mathcal{K}}B_{\mathcal{I}}\mathcal{K})) \arrow[u, shift right, "p"'] \arrow[r, shift left, "\epsilon_*"] \arrow[d, shift right, "\mathrm{colim}_{\mathcal{K}}"'] & (C\mathcal{S}^{\mathcal{K}\times\mathcal{I}} \downarrow \mathrm{const}_{\mathcal{K}}B_{\mathcal{I}}\mathcal{K}) \arrow[u, shift right, "p"'] \arrow[l, shift left, "\epsilon^*"] \arrow[d, shift right, "\mathrm{colim}_{\mathcal{K}}"'] \\
(E_\infty(\mathcal{S}^{\mathcal{I}})^+ \downarrow \epsilon^*(B_{\mathcal{I}}\mathcal{K})) \arrow[u, shift right, "\mathrm{const}_{\mathcal{K}}"'] \arrow[r, shift left, "\epsilon_*"] & (C\mathcal{S}^{\mathcal{I}} \downarrow B_{\mathcal{I}}\mathcal{K}). \arrow[u, shift right, "\mathrm{const}_{\mathcal{K}}"'] \arrow[l, shift left, "\epsilon^*"]
\end{tikzcd} \tag{3.22}$$

We can assume that M is positive $\mathcal{K}$-fibrant in $C\mathcal{S}^{\mathcal{K}}$. The image $I^{\mathcal{I}}_{\mathcal{K}}(M)$ is equal to $(\mathrm{colim}_{\mathcal{K}}Q(u_*(E_{\mathcal{I}}\mathcal{K} \times \mathrm{const}_{\mathcal{I}}M)), \mathrm{colim}_{\mathcal{K}}Q(u_*(E_{\mathcal{I}}\mathcal{K} \times \mathrm{const}_{\mathcal{I}}M)) \to B_{\mathcal{I}}\mathcal{K})$. The space $\mathrm{hocolim}_{\mathcal{I}}I^{\mathcal{I}}(\mathrm{hocolim}_{\mathcal{K}}M)$ is weakly equivalent to $\mathrm{hocolim}_{\mathcal{K}}M$. So we have to show that the latter is weakly equivalent to $\mathrm{hocolim}_{\mathcal{I}}\mathrm{colim}_{\mathcal{K}}Q(u_*(E_{\mathcal{I}}\mathcal{K} \times \mathrm{const}_{\mathcal{I}}M))$. Exploiting the Quillen equivalences in (3.22), we observe that M is $\mathcal{K}$-equivalent to the image of a cofibrant object $(N, N \to \epsilon^*(E_{\mathcal{I}}\mathcal{K}))$ in $(E_\infty(\mathcal{S}^{\mathcal{K}\times\mathcal{I}})^+ \downarrow \epsilon^*(E_{\mathcal{I}}\mathcal{K}))$ under the functor $\epsilon_* \circ \mathrm{colim}_{\mathcal{I}} \circ t_*$, that is,

$$\mathrm{hocolim}_{\mathcal{K}}M \simeq \mathrm{hocolim}_{\mathcal{K}}\epsilon_*(\mathrm{colim}_{\mathcal{I}}t_*(N)). \tag{3.23}$$

Since $\mathrm{colim}_{\mathcal{I}}t_*(N)$ is positive cofibrant in $E_\infty(\mathcal{S}^{\mathcal{K}})^+$, its underlying $\mathcal{K}$-space is cofibrant in the projective $\mathcal{K}$-model structure on $\mathcal{S}^{\mathcal{K}}$. Hence, the map

$$\mathrm{hocolim}_{\mathcal{K}}\mathrm{colim}_{\mathcal{I}}t_*(N) \to \mathrm{colim}_{\mathcal{K}}\mathrm{colim}_{\mathcal{I}}t_*(N)$$

is a weak equivalence by [SS12, Lemma 6.22]. Using that the derived unit of the adjunction (ϵ_*, ϵ^*) is a weak equivalence, we get that the map

$$\mathrm{hocolim}_{\mathcal{K}}\mathrm{colim}_{\mathcal{I}}t_*(N) \to \mathrm{hocolim}_{\mathcal{K}}\epsilon^*(\epsilon_*(\mathrm{colim}_{\mathcal{I}}t_*(N))) \tag{3.24}$$

is a weak equivalence. But in view of (3.23), the target in (3.24) is weakly equivalent to $\mathrm{hocolim}_{\mathcal{K}}\epsilon^*(M)$ which is the space $\mathrm{hocolim}_{\mathcal{K}}M$. Likewise, we obtain the following chain of weak equivalences

$$\begin{aligned}
\mathrm{colim}_{\mathcal{I}}\mathrm{colim}_{\mathcal{K}}u_*(N) &\xleftarrow{\sim} \mathrm{hocolim}_{\mathcal{I}}\mathrm{colim}_{\mathcal{K}}u_*(N) \\
&\xrightarrow{\sim} \mathrm{hocolim}_{\mathcal{I}}\epsilon^*(\epsilon_*(\mathrm{colim}_{\mathcal{K}}u_*(N))) \\
&\simeq \mathrm{hocolim}_{\mathcal{I}}\epsilon^*(\mathrm{colim}_{\mathcal{K}}Q(u_*(E_{\mathcal{I}}\mathcal{K} \times \mathrm{const}_{\mathcal{I}}M))).
\end{aligned}$$

Altogether, we find that

$$\begin{aligned}\mathrm{hocolim}_{\mathcal{K}}\epsilon^*(M) &\simeq \mathrm{colim}_{\mathcal{K}}\mathrm{colim}_{\mathcal{I}}t_*(N)\\ &\cong \mathrm{colim}_{\mathcal{I}}\mathrm{colim}_{\mathcal{K}}u_*(N)\\ &\simeq \mathrm{hocolim}_{\mathcal{I}}\epsilon^*(\mathrm{colim}_{\mathcal{K}}Q(u_*(E_{\mathcal{I}}\mathcal{K}\times \mathrm{const}_{\mathcal{I}}M)))\end{aligned}$$

which finishes the proof. □

3.6 Localized model structures on commutative diagram spaces

Let $\mathcal{K}$ be a well-structured index category which is permutative. Suppose that the inclusion functor $\mathcal{K}_+ \to \mathcal{K}$ is homotopy right cofinal, and that the pair $(\mathcal{K}, \mathcal{O}\mathcal{K}_+)$ is very well-structured. In this subsection we left Bousfield localize the positive projective $\mathcal{K}$-model structure on commutative $\mathcal{K}$-spaces with respect to a set of maps which corepresent shear maps. To better understand the localized model structure, we use the Quillen equivalence between commutative $\mathcal{K}$-spaces and commutative $\mathcal{I}$-spaces over $B_{\mathcal{I}}\mathcal{K}$ established in the last subsection. The goal is to show that if the simplicial monoid $B\mathcal{K}$ is grouplike, the localized model structure on commutative $\mathcal{K}$-spaces is Quillen equivalent to the overcategory model structure on commutative $\mathcal{I}$-spaces over $B_{\mathcal{I}}\mathcal{K}$ with respect to a localized model structure on commutative $\mathcal{I}$-spaces. The results in this subsection play an important role in the next subsection where we prove the latter and argue that the localized model structure on commutative $\mathcal{K}$-spaces is indeed a *group completion model structure*.

Left Bousfield localizations exist on all model categories involved in the chain of Quillen equivalences (3.19) between commutative $\mathcal{K}$-spaces and commutative $\mathcal{I}$-spaces over $B_{\mathcal{I}}\mathcal{K}$. For this, we can argue as in the proof of [Sag16, Lemma 7.16]: The positive projective $\mathcal{K}$-model structure on commutative $\mathcal{K}$-spaces is cofibrantly generated and left proper (see Proposition 2.9, Remark 2.10). The forgetful functor $C\mathcal{S}^{\mathcal{K}} \to \mathcal{S}^{\mathcal{K}}$ preserves filtered colimits [SS12, Lemma 9.2] so that the category $C\mathcal{S}^{\mathcal{K}}$ is locally presentable by [Bor94, Example 5.2.2.b, Theorem 5.5.9]. Alternatively, in the case that the category $\mathcal{K}$ is $\mathcal{I}$, Sagave and Schlichtkrull verify in [SS13, §A] that the positive projective $\mathcal{I}$-model structure on $C\mathcal{S}^{\mathcal{I}}$ is cellular.

To transfer the Quillen equivalences from the previous subsection to the localized setting, we rely on a criterion by Hirschhorn [Hir03, Theorem 3.3.20.(1)(b)]. The latter says that given a Quillen equivalence

$$\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D}$$

and a class S of maps in $\mathcal{C}$, the left Bousfield localization of $\mathcal{C}$ with respect to S is Quillen equivalent to the left Bousfield localization of $\mathcal{D}$ with respect to the image of S under the derived functor of F.

Let $\mathbf{k}$ and $\mathbf{l}$ be in $\mathcal{K}_+$. We define a map

$$\mathbb{C}(F^{\mathcal{K}}_{\mathbf{k}\sqcup\mathbf{l}}(*)) \boxtimes \mathbb{C}(F^{\mathcal{K}}_{\mathbf{l}}(*)) \xrightarrow{s^{\mathcal{K}}_{\mathbf{k},\mathbf{l}}} \mathbb{C}(F^{\mathcal{K}}_{\mathbf{k}}(*)) \boxtimes \mathbb{C}(F^{\mathcal{K}}_{\mathbf{l}}(*))$$

in $C\mathcal{S}^{\mathcal{K}}$ as follows. Because the monoidal product $\boxtimes$ is the coproduct in $C\mathcal{S}^{\mathcal{K}}$, it suffices to determine morphisms

$$\mathbb{C}(F^{\mathcal{K}}_{\mathbf{k}\sqcup\mathbf{l}}(*)) \to \mathbb{C}(F^{\mathcal{K}}_{\mathbf{k}}(*)) \boxtimes \mathbb{C}(F^{\mathcal{K}}_{\mathbf{l}}(*))$$

and

$$\mathbb{C}(F^{\mathcal{K}}_{\mathbf{l}}(*)) \to \mathbb{C}(F^{\mathcal{K}}_{\mathbf{k}}(*)) \boxtimes \mathbb{C}(F^{\mathcal{K}}_{\mathbf{l}}(*)).$$

The latter is given by the inclusion of $\mathbb{C}(F^{\mathcal{K}}_{\mathbf{l}}(*))$ into the coproduct $\mathbb{C}(F^{\mathcal{K}}_{\mathbf{k}}(*)) \boxtimes \mathbb{C}(F^{\mathcal{K}}_{\mathbf{l}}(*))$, denoted by $i^{\mathcal{K}}_{\mathbf{l}}$. By adjunction a morphism $\mathbb{C}(F^{\mathcal{K}}_{\mathbf{k}\sqcup\mathbf{l}}(*)) \to \mathbb{C}(F^{\mathcal{K}}_{\mathbf{k}}(*)) \boxtimes \mathbb{C}(F^{\mathcal{K}}_{\mathbf{l}}(*))$ in $C\mathcal{S}^{\mathcal{K}}$ corresponds to a point in $(\mathbb{C}(F^{\mathcal{K}}_{\mathbf{k}}(*)) \boxtimes \mathbb{C}(F^{\mathcal{K}}_{\mathbf{l}}(*)))(\mathbf{k} \sqcup \mathbf{l})$ in $\mathcal{S}$. We know that

$$(\mathbb{C}(F^{\mathcal{K}}_{\mathbf{k}}(*)) \boxtimes \mathbb{C}(F^{\mathcal{K}}_{\mathbf{l}}(*)))(\mathbf{k} \sqcup \mathbf{l}) \cong \operatorname{colim}_{\mathbf{p}\sqcup\mathbf{q}\to\mathbf{k}\sqcup\mathbf{l}}(\coprod_{i\geq 0} \mathcal{K}(\mathbf{k}^{\sqcup i}, \mathbf{p})/\Sigma_i) \times (\coprod_{j\geq 0} \mathcal{K}(\mathbf{l}^{\sqcup j}, \mathbf{q})/\Sigma_j) \tag{3.25}$$

(see (2.2), (2.3)). We send a point $*$ to $(\mathrm{id}_{\mathbf{k}}, \mathrm{id}_{\mathbf{l}})$ in $\mathcal{K}(\mathbf{k}, \mathbf{k}) \times \mathcal{K}(\mathbf{l}, \mathbf{l})$, which is the $(i = 1, j = 1)$-summand in

$$(\coprod_{i\geq 0} \mathcal{K}(\mathbf{k}^{\sqcup i}, \mathbf{k})/\Sigma_i) \times (\coprod_{j\geq 0} \mathcal{K}(\mathbf{l}^{\sqcup j}, \mathbf{l})/\Sigma_j)$$

where the latter is indexed by $((\mathbf{k}, \mathbf{l}), \mathrm{id}\colon \mathbf{k}\sqcup\mathbf{l} \to \mathbf{k}\sqcup\mathbf{l})$ in the colimit system. Postcomposing with the canonical map to the colimit (3.25) specifies a point in

$$(\mathbb{C}(F^{\mathcal{K}}_{\mathbf{k}}(*)) \boxtimes \mathbb{C}(F^{\mathcal{K}}_{\mathbf{l}}(*)))(\mathbf{k} \sqcup \mathbf{l}).$$

Let $a^{\mathcal{K}}_{\mathbf{k},\mathbf{l}}\colon \mathbb{C}(F^{\mathcal{K}}_{\mathbf{k}\sqcup\mathbf{l}}(*)) \to \mathbb{C}(F^{\mathcal{K}}_{\mathbf{k}}(*)) \boxtimes \mathbb{C}(F^{\mathcal{K}}_{\mathbf{l}}(*))$ be the adjoint map in $C\mathcal{S}^{\mathcal{K}}$. So we set $s^{\mathcal{K}}_{\mathbf{k},\mathbf{l}} = i^{\mathcal{K}}_{\mathbf{l}} + a^{\mathcal{K}}_{\mathbf{k},\mathbf{l}}$. Let $S^{\mathcal{K}}$ be then the following set of maps

$$S^{\mathcal{K}} = \{\mathbb{C}(F^{\mathcal{K}}_{\mathbf{k}\sqcup\mathbf{l}}(*)) \boxtimes \mathbb{C}(F^{\mathcal{K}}_{\mathbf{l}}(*)) \xrightarrow{s^{\mathcal{K}}_{\mathbf{k},\mathbf{l}}} \mathbb{C}(F^{\mathcal{K}}_{\mathbf{k}}(*)) \boxtimes \mathbb{C}(F^{\mathcal{K}}_{\mathbf{l}}(*)), \mathbf{k}, \mathbf{l} \in \mathcal{K}_+\}.$$

The domains and codomains of the maps in $S^{\mathcal{K}}$ are cofibrant in the positive projective $\mathcal{K}$-model structure on $C\mathcal{S}^{\mathcal{K}}$. We denote the left Bousfield localization of the positive projective $\mathcal{K}$-model structure on $C\mathcal{S}^{\mathcal{K}}$ with respect to $S^{\mathcal{K}}$ by $(C\mathcal{S}^{\mathcal{K}})_{\mathrm{loc}(S^{\mathcal{K}})}$.

Lemma 3.25. *The induced adjunction*

$$(C\mathcal{S}^{\mathcal{K}\times\mathcal{I}})_{\mathrm{loc}(S^{\mathcal{K}\times\mathcal{I}})} \underset{\mathrm{const}_{\mathcal{I}}}{\overset{\mathrm{colim}_{\mathcal{I}}}{\rightleftarrows}} (C\mathcal{S}^{\mathcal{K}})_{\mathrm{loc}(S^{\mathcal{K}})}$$

is a Quillen equivalence.

Proof. Let $(\mathbf{k},\mathbf{m})$ and $(\mathbf{l},\mathbf{n})$ be in $\mathcal{K}_+ \times \mathcal{I}_+$. We observe that

$$\begin{aligned}
\mathrm{colim}_{\mathcal{I}}(\mathbb{C}(F^{\mathcal{K}\times\mathcal{I}}_{(\mathbf{k}\sqcup\mathbf{l},\mathbf{m}\sqcup\mathbf{n})}(*)) \boxtimes \mathbb{C}(F^{\mathcal{K}\times\mathcal{I}}_{(\mathbf{l},\mathbf{n})}(*))) &\cong \mathrm{colim}_{\mathcal{I}}\mathbb{C}(F^{\mathcal{K}\times\mathcal{I}}_{(\mathbf{k}\sqcup\mathbf{l},\mathbf{m}\sqcup\mathbf{n})}(*)) \boxtimes \mathrm{colim}_{\mathcal{I}}\mathbb{C}(F^{\mathcal{K}\times\mathcal{I}}_{(\mathbf{l},\mathbf{n})}(*)) \\
&\cong \mathbb{C}(\mathrm{colim}_{\mathcal{I}}F^{\mathcal{K}\times\mathcal{I}}_{(\mathbf{k}\sqcup\mathbf{l},\mathbf{m}\sqcup\mathbf{n})}(*)) \boxtimes \mathbb{C}(\mathrm{colim}_{\mathcal{I}}F^{\mathcal{K}\times\mathcal{I}}_{(\mathbf{l},\mathbf{n})}(*)) \\
&\cong \mathbb{C}(F^{\mathcal{K}}_{\mathbf{k}\sqcup\mathbf{l}}(*)) \boxtimes \mathbb{C}(F^{\mathcal{K}}_{\mathbf{l}}(*)).
\end{aligned}$$

In the same way, we figure out that

$$\mathrm{colim}_{\mathcal{I}}(\mathbb{C}(F^{\mathcal{K}\times\mathcal{I}}_{(\mathbf{k},\mathbf{m})}(*)) \boxtimes \mathbb{C}(F^{\mathcal{K}\times\mathcal{I}}_{(\mathbf{l},\mathbf{n})}(*))) \cong \mathbb{C}(F^{\mathcal{K}}_{\mathbf{k}}(*)) \boxtimes \mathbb{C}(F^{\mathcal{K}}_{\mathbf{l}}(*)).$$

We need to argue that the diagram

$$\begin{tikzcd}
\mathrm{colim}_{\mathcal{I}}(\mathbb{C}(F^{\mathcal{K}\times\mathcal{I}}_{(\mathbf{k}\sqcup\mathbf{l},\mathbf{m}\sqcup\mathbf{n})}(*)) \boxtimes \mathbb{C}(F^{\mathcal{K}\times\mathcal{I}}_{(\mathbf{l},\mathbf{n})}(*))) \arrow[r, "\cong"] \arrow[d, "\mathrm{colim}_{\mathcal{I}} s^{\mathcal{K}}_{(\mathbf{k},\mathbf{m}),(\mathbf{l},\mathbf{n})}"'] & \mathbb{C}(F^{\mathcal{K}}_{\mathbf{k}\sqcup\mathbf{l}}(*)) \boxtimes \mathbb{C}(F^{\mathcal{K}}_{\mathbf{l}}(*)) \arrow[d, "s^{\mathcal{K}}_{\mathbf{k},\mathbf{l}}"] \\
\mathrm{colim}_{\mathcal{I}}(\mathbb{C}(F^{\mathcal{K}\times\mathcal{I}}_{(\mathbf{k},\mathbf{m})}(*)) \boxtimes \mathbb{C}(F^{\mathcal{K}\times\mathcal{I}}_{(\mathbf{l},\mathbf{n})}(*))) \arrow[r, "\cong"] & \mathbb{C}(F^{\mathcal{K}}_{\mathbf{k}}(*)) \boxtimes \mathbb{C}(F^{\mathcal{K}}_{\mathbf{l}}(*))
\end{tikzcd} \tag{3.26}$$

commutes. The inclusion into the coproduct $i^{\mathcal{K}}_{\mathbf{l}} \colon \mathbb{C}(F^{\mathcal{K}}_{\mathbf{l}}(*)) \to \mathbb{C}(F^{\mathcal{K}}_{\mathbf{k}}(*)) \boxtimes \mathbb{C}(F^{\mathcal{K}}_{\mathbf{l}}(*))$ can be identified with the map

$$\begin{aligned}
&\mathbb{C}(F^{\mathcal{K}}_{\mathbf{l}}(*)) \cong \mathrm{colim}_{\mathcal{I}}\mathbb{C}(F^{\mathcal{K}\times\mathcal{I}}_{(\mathbf{l},\mathbf{n})}(*)) \\
&\xrightarrow{\mathrm{colim}_{\mathcal{I}} i^{\mathcal{K}\times\mathcal{I}}_{(\mathbf{l},\mathbf{n})}} \mathrm{colim}_{\mathcal{I}}(\mathbb{C}(F^{\mathcal{K}\times\mathcal{I}}_{(\mathbf{k},\mathbf{m})}(*)) \boxtimes \mathbb{C}(F^{\mathcal{K}\times\mathcal{I}}_{(\mathbf{l},\mathbf{n})}(*))) \cong \mathbb{C}(F^{\mathcal{K}}_{\mathbf{k}}(*)) \boxtimes \mathbb{C}(F^{\mathcal{K}}_{\mathbf{l}}(*)).
\end{aligned}$$

In addition, the map $a^{\mathcal{K}}_{\mathbf{k},\mathbf{l}} \colon \mathbb{C}(F^{\mathcal{K}}_{\mathbf{k}\sqcup\mathbf{l}}(*)) \to \mathbb{C}(F^{\mathcal{K}}_{\mathbf{k}}(*)) \boxtimes \mathbb{C}(F^{\mathcal{K}}_{\mathbf{l}}(*))$ agrees with the map

$$\begin{aligned}
&\mathbb{C}(F^{\mathcal{K}}_{\mathbf{k}\sqcup\mathbf{l}}(*)) \cong \mathrm{colim}_{\mathcal{I}}\mathbb{C}(F^{\mathcal{K}\times\mathcal{I}}_{(\mathbf{k}\sqcup\mathbf{l},\mathbf{m}\sqcup\mathbf{n})}(*)) \\
&\xrightarrow{\mathrm{colim}_{\mathcal{I}} a^{\mathcal{K}\times\mathcal{I}}_{(\mathbf{k},\mathbf{m}),(\mathbf{l},\mathbf{n})}} \mathrm{colim}_{\mathcal{I}}(\mathbb{C}(F^{\mathcal{K}\times\mathcal{I}}_{(\mathbf{k},\mathbf{m})}(*)) \boxtimes \mathbb{C}(F^{\mathcal{K}\times\mathcal{I}}_{(\mathbf{l},\mathbf{n})}(*))) \cong \mathbb{C}(F^{\mathcal{K}}_{\mathbf{k}}(*)) \boxtimes \mathbb{C}(F^{\mathcal{K}}_{\mathbf{l}}(*)).
\end{aligned}$$

Therefore, the diagram (3.26) commutes so that the map $\mathrm{colim}_{\mathcal{I}} s^{\mathcal{K}\times\mathcal{I}}_{(\mathbf{k},\mathbf{m}),(\mathbf{l},\mathbf{n})}$ is isomorphic to the map $s^{\mathcal{K}}_{\mathbf{k},\mathbf{l}}$. Thus, it holds that $\mathrm{colim}_{\mathcal{I}} S^{\mathcal{K}\times\mathcal{I}} \cong S^{\mathcal{I}}$. The claim then follows from Corollary 3.18 and [Hir03, Theorem 3.3.20.(1)(b)]. □

Before we go on, we again fix some notation. Let M be a commutative $\mathcal{K}$-space. Let $S^{\mathcal{K}}_M$ be the set of morphisms in $(\mathcal{CS}^{\mathcal{K}} \downarrow M)$ whose projection to $\mathcal{CS}^{\mathcal{K}}$ is an element in $S^{\mathcal{K}}$. We form the left Bousfield localization of the overcategory model structure on $(\mathcal{CS}^{\mathcal{K}} \downarrow M)$ with respect to $S^{\mathcal{K}}_M$ and write $(\mathcal{CS}^{\mathcal{K}} \downarrow M)_{\mathrm{loc}(S^{\mathcal{K}}_M)}$ for the localized model structure.

Lemma 3.26. *The induced adjunction*

$$(\mathcal{CS}^{\mathcal{K}\times\mathcal{I}} \downarrow E_{\mathcal{I}}\mathcal{K})_{\mathrm{loc}(S^{\mathcal{K}\times\mathcal{I}}_{E_{\mathcal{I}}\mathcal{K}})} \underset{q}{\overset{t_*}{\rightleftarrows}} (\mathcal{CS}^{\mathcal{K}\times\mathcal{I}})_{\mathrm{loc}(S^{\mathcal{K}\times\mathcal{I}})}$$

defines a Quillen equivalence.

Proof. As before we want to deduce the claim from Lemma 3.19 and [Hir03, Theorem 3.3.20.(1)(b)]. So we have to check that the image of $S^{\mathcal{K}\times\mathcal{I}}_{E_{\mathcal{I}}\mathcal{K}}$ under the functor t_* is isomorphic to $S^{\mathcal{K}\times\mathcal{I}}$. For this, it is enough to show that for every $(\mathbf{k},\mathbf{m})$ and $(\mathbf{l},\mathbf{n})$ in $\mathcal{K}_+\times\mathcal{I}_+$, there exists a map $\mathbb{C}(F^{\mathcal{K}\times\mathcal{I}}_{(\mathbf{k},\mathbf{m})}(*))\boxtimes\mathbb{C}(F^{\mathcal{K}\times\mathcal{I}}_{(\mathbf{l},\mathbf{n})}(*))\to E_{\mathcal{I}}\mathcal{K}$ in $\mathcal{CS}^{\mathcal{K}\times\mathcal{I}}$. Let $(\mathbf{k},\mathbf{m})$ and $(\mathbf{l},\mathbf{n})$ be in $\mathcal{K}_+\times\mathcal{I}_+$. The spaces $E_{\mathcal{I}}\mathcal{K}(\mathbf{k},\mathbf{m})$ and $E_{\mathcal{I}}\mathcal{K}(\mathbf{l},\mathbf{n})$ are non-empty, so that by adjunction there are maps $\mathbb{C}(F^{\mathcal{K}\times\mathcal{I}}_{(\mathbf{k},\mathbf{m})}(*))\to E_{\mathcal{I}}\mathcal{K}$ and $\mathbb{C}(F^{\mathcal{K}\times\mathcal{I}}_{(\mathbf{l},\mathbf{n})}(*))\to E_{\mathcal{I}}\mathcal{K}$. Note that at this point we need the assumption that $m,n\geq 1$. Since the monoidal product $\boxtimes$ is the coproduct in $\mathcal{CS}^{\mathcal{K}\times\mathcal{I}}$, these maps produce a map $\mathbb{C}(F^{\mathcal{K}\times\mathcal{I}}_{(\mathbf{k},\mathbf{m})}(*))\boxtimes\mathbb{C}(F^{\mathcal{K}\times\mathcal{I}}_{(\mathbf{l},\mathbf{n})}(*))\to E_{\mathcal{I}}\mathcal{K}$ in $\mathcal{CS}^{\mathcal{K}\times\mathcal{I}}$. □

Lemma 3.27. *The induced adjunction*

$$(\mathcal{CS}^{\mathcal{K}\times\mathcal{I}}\downarrow E_{\mathcal{I}}\mathcal{K})_{\mathrm{loc}(S^{\mathcal{K}\times\mathcal{I}}_{E_{\mathcal{I}}\mathcal{K}})} \underset{p\circ\mathrm{const}_{\mathcal{K}}}{\overset{\mathrm{colim}_{\mathcal{K}}\circ u_*}{\rightleftarrows}} (\mathcal{CS}^{\mathcal{I}}\downarrow B_{\mathcal{I}}\mathcal{K})_{\mathrm{loc}(S^{\mathcal{I}}_{B_{\mathcal{I}}\mathcal{K}})}$$

is a Quillen equivalence.

Proof. As in the proof of Lemma 3.25, we get that the image of $S^{\mathcal{K}\times\mathcal{I}}$ under the functor $\mathrm{colim}_{\mathcal{K}}$ is isomorphic to $S^{\mathcal{I}}$. Besides, we know that

$$\mathrm{colim}_{\mathcal{K}}E_{\mathcal{I}}\mathcal{K}\cong B_{\mathcal{I}}\mathcal{K}=\mathrm{colim}_{\mathcal{K}}\mathrm{const}_{\mathcal{K}}B_{\mathcal{I}}\mathcal{K}.$$

Therefore, Corollary 3.21 and [Hir03, Theorem 3.3.20.(1)(b)] imply the claim. □

Our next aim is to show that if the simplicial monoid $B\mathcal{K}$ is grouplike, the model category $(\mathcal{CS}^{\mathcal{I}}\downarrow B_{\mathcal{I}}\mathcal{K})_{\mathrm{loc}(S^{\mathcal{I}}_{B_{\mathcal{I}}\mathcal{K}})}$ is Quillen equivalent to $((\mathcal{CS}^{\mathcal{I}})_{\mathrm{loc}(S^{\mathcal{I}})}\downarrow B_{\mathcal{I}}\mathcal{K})$. We prove this in Subsection 3.7. We end this subsection with yet another auxiliary result which is needed in Subsection 3.7.

Lemma 3.28. *Let $g\colon V\to W$ be an $\mathcal{I}$-equivalence in $\mathcal{CS}^{\mathcal{I}}$, and let V and W be homotopy constant with respect to morphisms in $\mathcal{I}_+$. The adjunction*

$$(\mathcal{CS}^{\mathcal{I}}\downarrow V)_{\mathrm{loc}(S^{\mathcal{I}}_V)} \underset{g^*}{\overset{g_*}{\rightleftarrows}} (\mathcal{CS}^{\mathcal{I}}\downarrow W)_{\mathrm{loc}(S^{\mathcal{I}}_W)}, \qquad (3.27)$$

induced by composition with and pullback along the map g, determines a Quillen equivalence.

Proof. To prove the claim, we apply a criterion of Hovey [Hov01, Proposition 2.3]. First, we notice that for an element in $S^{\mathcal{I}}_V$, the image under the functor g_* lies in $S^{\mathcal{I}}_W$ and thus is a weak equivalence in $(\mathcal{CS}^{\mathcal{I}}\downarrow W)_{\mathrm{loc}(S^{\mathcal{I}}_W)}$.
Secondly, let $(X,\rho_X\colon X\to V)$ be an $S^{\mathcal{I}}_V$-local object in $(\mathcal{CS}^{\mathcal{I}}\downarrow V)$. Let the diagram

$$\begin{tikzcd} X \arrow[r, tail, "j", "\sim"'] \arrow[d, two heads, "\rho_X"'] & \tilde{X} \arrow[ddl, two heads, bend left, "\rho_{\tilde{X}}"] \\ V \arrow[d, "g"', "\sim"] & \\ W & \end{tikzcd}$$

display a factorization of the map $g \circ \rho_X$ into an acyclic cofibration j followed by a fibration $\rho_{\tilde{X}}$ in the positive projective $\mathcal{I}$-model structure on $C\mathcal{S}^{\mathcal{I}}$. The object X is homotopy constant with respect to morphisms in $\mathcal{I}_+$, because the object V is so and the map ρ_X is a positive $\mathcal{I}$-fibration. Likewise, the object $\tilde{X}$ is homotopy constant with respect to morphisms in $\mathcal{I}_+$, as the object W is so and the map $\rho_{\tilde{X}}$ is a positive $\mathcal{I}$-fibration. It follows by Proposition 2.5 and [Hir03, Proposition 13.3.14] that the maps j and g are positive level equivalences. We claim that $(\tilde{X}, \rho_{\tilde{X}})$ is an $S^{\mathcal{I}}_W$-local object in $(C\mathcal{S}^{\mathcal{I}} \downarrow W)$. For this, it remains to check that for an element

$$\begin{tikzcd} \mathbb{C}(F^{\mathcal{I}}_{\mathbf{m}\sqcup\mathbf{n}}(*)) \boxtimes \mathbb{C}(F^{\mathcal{I}}_{\mathbf{n}}(*)) \arrow[rr, "s^{\mathcal{I}}_{\mathbf{m},\mathbf{n}}"] \arrow[dr] & & \mathbb{C}(F^{\mathcal{I}}_{\mathbf{m}}(*)) \boxtimes \mathbb{C}(F^{\mathcal{I}}_{\mathbf{n}}(*)) \arrow[dl, "\rho_{\mathbb{C}(F^{\mathcal{I}}_{\mathbf{m}}(*))\boxtimes\mathbb{C}(F^{\mathcal{I}}_{\mathbf{n}}(*))}"] \\ & W & \end{tikzcd}$$

in $S^{\mathcal{I}}_W$, the induced map

$$\begin{tikzcd} \mathrm{Hom}^{(C\mathcal{S}^{\mathcal{I}}\downarrow W)}_{\mathcal{S}}((\mathbb{C}(F^{\mathcal{I}}_{\mathbf{m}}(*)) \boxtimes \mathbb{C}(F^{\mathcal{I}}_{\mathbf{n}}(*)), \rho_{\mathbb{C}(F^{\mathcal{I}}_{\mathbf{m}}(*))\boxtimes\mathbb{C}(F^{\mathcal{I}}_{\mathbf{n}}(*))}), (\tilde{X}, \rho_{\tilde{X}})) \arrow[d, "\mathrm{Hom}^{(C\mathcal{S}^{\mathcal{I}}\downarrow W)}_{\mathcal{S}}(s^{\mathcal{I}}_{\mathbf{m},\mathbf{n}}, \mathrm{id})"] \\ \mathrm{Hom}^{(C\mathcal{S}^{\mathcal{I}}\downarrow W)}_{\mathcal{S}}((\mathbb{C}(F^{\mathcal{I}}_{\mathbf{m}\sqcup\mathbf{n}}(*)) \boxtimes \mathbb{C}(F^{\mathcal{I}}_{\mathbf{n}}(*)), \rho_{\mathbb{C}(F^{\mathcal{I}}_{\mathbf{m}}(*))\boxtimes\mathbb{C}(F^{\mathcal{I}}_{\mathbf{n}}(*))} \circ s^{\mathcal{I}}_{\mathbf{m},\mathbf{n}}), (\tilde{X}, \rho_{\tilde{X}})) \end{tikzcd} \tag{3.28}$$

is a weak equivalence. To ease notation we set

$$A = \mathbb{C}(F^{\mathcal{I}}_{\mathbf{m}\sqcup\mathbf{n}}(*)) \boxtimes \mathbb{C}(F^{\mathcal{I}}_{\mathbf{n}}(*))$$
$$B = \mathbb{C}(F^{\mathcal{I}}_{\mathbf{m}}(*)) \boxtimes \mathbb{C}(F^{\mathcal{I}}_{\mathbf{n}}(*)).$$

By adjunction the space $\mathrm{Hom}^{C\mathcal{S}^{\mathcal{I}}}_{\mathcal{S}}(B, \tilde{X})$ is weakly equivalent to $\tilde{X}(\mathbf{m}) \times \tilde{X}(\mathbf{n})$. Since the map j is a positive level equivalence, the latter is weakly equivalent to $X(\mathbf{m}) \times X(\mathbf{n})$, which again by adjunction is weakly equivalent to $\mathrm{Hom}^{C\mathcal{S}^{\mathcal{I}}}_{\mathcal{S}}(B, X)$. Hence, the map $\mathrm{Hom}^{C\mathcal{S}^{\mathcal{I}}}_{\mathcal{S}}(\mathrm{id}, j) \colon \mathrm{Hom}^{C\mathcal{S}^{\mathcal{I}}}_{\mathcal{S}}(B, X) \to \mathrm{Hom}^{C\mathcal{S}^{\mathcal{I}}}_{\mathcal{S}}(B, \tilde{X})$ is a weak equivalence. By the same arguments, we find that the map $\mathrm{Hom}^{C\mathcal{S}^{\mathcal{I}}}_{\mathcal{S}}(\mathrm{id}, g) \colon \mathrm{Hom}^{C\mathcal{S}^{\mathcal{I}}}_{\mathcal{S}}(B, V) \to \mathrm{Hom}^{C\mathcal{S}^{\mathcal{I}}}_{\mathcal{S}}(B, W)$ is a weak equivalence. In particular, the map $\mathrm{Hom}^{C\mathcal{S}^{\mathcal{I}}}_{\mathcal{S}}(\mathrm{id}, g)$ induces an isomorphism on π_0. We consider the diagram

$$\begin{tikzcd} * \arrow[r] \arrow[d] & \mathrm{Hom}^{C\mathcal{S}^{\mathcal{I}}}_{\mathcal{S}}(B, V) \arrow[d, "\sim"', "\mathrm{Hom}^{C\mathcal{S}^{\mathcal{I}}}_{\mathcal{S}}(\mathrm{id}, g)"] & \mathrm{Hom}^{C\mathcal{S}^{\mathcal{I}}}_{\mathcal{S}}(B, X) \arrow[l, two heads, "\mathrm{Hom}^{C\mathcal{S}^{\mathcal{I}}}_{\mathcal{S}}(\mathrm{id}, \rho_X)"'] \arrow[d, "\sim"', "\mathrm{Hom}^{C\mathcal{S}^{\mathcal{I}}}_{\mathcal{S}}(\mathrm{id}, j)"] \\ * \arrow[r] & \mathrm{Hom}^{C\mathcal{S}^{\mathcal{I}}}_{\mathcal{S}}(B, W) & \mathrm{Hom}^{C\mathcal{S}^{\mathcal{I}}}_{\mathcal{S}}(B, \tilde{X}). \arrow[l, two heads, "\mathrm{Hom}^{C\mathcal{S}^{\mathcal{I}}}_{\mathcal{S}}(\mathrm{id}, \rho_{\tilde{X}})"'] \end{tikzcd}$$

The induced map of homotopy pullbacks

$$\mathrm{Hom}^{(C\mathcal{S}^{\mathcal{I}}\downarrow V)}_{\mathcal{S}}((B, B \to V), (X, \rho_X)) \to \mathrm{Hom}^{(C\mathcal{S}^{\mathcal{I}}\downarrow W)}_{\mathcal{S}}((B, B \to W), (\tilde{X}, \rho_{\tilde{X}}))$$

is a weak equivalence for every given augmentation $B \to W$ (see [Hir03, Proposition 13.4.7]). Analogously, we obtain that the induced map

$$\mathrm{Hom}_{\mathcal{S}}^{(C\mathcal{S}^{\mathcal{I}}\downarrow V)}((A, A \to V), (X, \rho_X)) \to \mathrm{Hom}_{\mathcal{S}}^{(C\mathcal{S}^{\mathcal{I}}\downarrow W)}((A, A \to W), (\tilde{X}, \rho_{\tilde{X}}))$$

is a weak equivalence. By assumption (X, ρ_X) is an $S^{\mathcal{I}}_V$-local object in $(C\mathcal{S}^{\mathcal{I}} \downarrow V)$. Two out of three then implies that the map (3.28) is a weak equivalence. It is left to show that (X, ρ_X) is weakly equivalent to $(V \times_W \tilde{X}, V \times_W \tilde{X} \to V)$ in $(C\mathcal{S}^{\mathcal{I}} \downarrow V)$. For this, we investigate the commutative diagram

$$\begin{tikzcd}
X \arrow[dr] \arrow[drr, bend left, "j", "\sim"'] \arrow[ddr, bend right, "\rho_X"'] & & \\
& V \times_W \tilde{X} \arrow[r] \arrow[d] & \tilde{X} \arrow[d, two heads, "\rho_{\tilde{X}}"] \\
& V \arrow[r, "g", "\sim"'] & W.
\end{tikzcd}$$

Since the model category $C\mathcal{S}^{\mathcal{I}}$ is right proper (see Remark 2.10) and two out of three holds, the map $X \to V \times_W \tilde{X}$ is an $\mathcal{I}$-equivalence. Taken together, the criterion of Hovey [Hov01, Proposition 2.3] ensures that (3.27) is a Quillen equivalence. □

3.7 The group completion model structure on commutative diagram spaces

Let $\mathcal{K}$ be a well-structured index category which is permutative. Suppose that the inclusion functor $\mathcal{K}_+ \to \mathcal{K}$ is homotopy right cofinal, and that the pair $(\mathcal{K}, \mathcal{OK}_+)$ is very well-structured. We finish the proof that if the simplicial monoid $B\mathcal{K}$ is grouplike, the model category $(C\mathcal{S}^{\mathcal{K}})_{\mathrm{loc}(S^{\mathcal{K}})}$ is Quillen equivalent to $((C\mathcal{S}^{\mathcal{I}})_{\mathrm{loc}(S^{\mathcal{I}})} \downarrow B_{\mathcal{I}}\mathcal{K})$. With this at hand, we justify that $(C\mathcal{S}^{\mathcal{K}})_{\mathrm{loc}(S^{\mathcal{K}})}$ defines a *group completion model structure*. We build on work of Sagave and Schlichtkrull in [SS13] where they describe group completion in commutative $\mathcal{I}$-spaces.

Recall from e.g. [May74] that a (simplicial or topological) monoid M is *grouplike* if the monoid of connected components $\pi_0(M)$ is a group. For an associative (simplicial or topological) monoid M, we write $B(M) = B(*, M, *)$ for the bar construction of M with respect to the cartesian product. A map of homotopy commutative (simplicial or topological) monoids $M \to N$ is a *group completion* if N is grouplike and the induced map of bar constructions $B(M) \to B(N)$ is a weak equivalence in spaces. Taking into account that in the simplicial setting the map of simplicial monoids $M \to \Omega((B(M))^{\mathrm{fib}})$ is a group completion where $(-)^{\mathrm{fib}}$ denotes a fibrant replacement functor, this implies that the simplicial monoid N is weakly equivalent to $\Omega((B(M))^{\mathrm{fib}})$. In the topological setting we assume that the topological monoids M and N are well-based, to conclude from $M \to N$ being a group completion that N is weakly equivalent to $\Omega(B(M))$. Spaces with an action of the Barratt-Eccles operad in spaces are simplicial monoids because the associativity operad in spaces is a sub operad of the Barratt-Eccles operad in spaces.

In view of these notions and on the grounds that commutative $\mathcal{K}$-spaces are Quillen equivalent to E_∞ spaces over $B\mathcal{K}$ (see Subsection 3.2), the definition of a grouplike commutative $\mathcal{K}$-space [Definition 2.40(i)] is sensible. In addition, we make the following definition.

Definition 3.29. (compare [Sag16, Definition 5.4]) A map $M \to N$ of commutative $\mathcal{K}$-spaces is a *group completion* if N is grouplike and the induced map of bar constructions $B(\mathrm{hocolim}_{\mathcal{K}} M) \to B(\mathrm{hocolim}_{\mathcal{K}} N)$ is a weak equivalence in spaces.

Example 3.30. The commutative $\mathcal{I}$-space $B_{\mathcal{I}}\mathcal{K}$ is grouplike if the monoid of connected components $\pi_0(\mathrm{hocolim}_{\mathcal{I}} B_{\mathcal{I}}\mathcal{K}) \cong \pi_0(B\mathcal{K})$ is a group, that is, if the simplicial monoid $B\mathcal{K}$ is grouplike. For example, if $\mathcal{K}$ is given by $\mathcal{I}$, $\mathcal{J}$ or $\bar{\mathcal{J}}$, the commutative $\mathcal{I}$-space $B_{\mathcal{I}}\mathcal{K}$ is grouplike.

Lemma 3.31. *A commutative $\mathcal{I}$-space M is fibrant in $(C\mathcal{S}^{\mathcal{I}})_{\mathrm{loc}(S^{\mathcal{I}})}$ if and only if M is positive $\mathcal{I}$-fibrant and grouplike.*

Proof. Assume that the commutative $\mathcal{I}$-space M is fibrant in $(C\mathcal{S}^{\mathcal{I}})_{\mathrm{loc}(S^{\mathcal{I}})}$ which means that M is $S^{\mathcal{I}}$-local in $C\mathcal{S}^{\mathcal{I}}$ by [Hir03, Theorem 4.1.1.(2)]. So in particular, the commutative $\mathcal{I}$-space M is positive $\mathcal{I}$-fibrant. To show that M is grouplike, let $\mathbf{m}$ and $\mathbf{n}$ be in $\mathcal{I}_+$. The space $\mathrm{Hom}_{\mathcal{S}}^{C\mathcal{S}^{\mathcal{I}}}(\mathbb{C}(F_{\mathbf{m}}^{\mathcal{I}}(*)) \boxtimes \mathbb{C}(F_{\mathbf{n}}^{\mathcal{I}}(*)), M)$ is weakly equivalent to

$$\mathrm{Hom}_{\mathcal{S}}^{C\mathcal{S}^{\mathcal{I}}}(\mathbb{C}(F_{\mathbf{m}}^{\mathcal{I}}(*)), M) \times \mathrm{Hom}_{\mathcal{S}}^{C\mathcal{S}^{\mathcal{I}}}(\mathbb{C}(F_{\mathbf{n}}^{\mathcal{I}}(*)), M),$$

which by adjointness is weakly equivalent to $M(\mathbf{m}) \times M(\mathbf{n})$. Likewise, the space $\mathrm{Hom}_{\mathcal{S}}^{C\mathcal{S}^{\mathcal{I}}}(\mathbb{C}(F_{\mathbf{m}\sqcup\mathbf{n}}^{\mathcal{I}}(*)) \boxtimes \mathbb{C}(F_{\mathbf{n}}^{\mathcal{I}}(*)), M)$ is weakly equivalent to $M(\mathbf{m} \sqcup \mathbf{n}) \times M(\mathbf{n})$. The map

$$\mathrm{Hom}_{\mathcal{S}}^{C\mathcal{S}^{\mathcal{I}}}(\mathbb{C}(F_{\mathbf{m}}^{\mathcal{I}}(*)) \boxtimes \mathbb{C}(F_{\mathbf{n}}^{\mathcal{I}}(*)), M) \xrightarrow{\mathrm{Hom}_{\mathcal{S}}^{C\mathcal{S}^{\mathcal{I}}}(s_{\mathbf{m},\mathbf{n}}^{\mathcal{I}}, \mathrm{id})} \mathrm{Hom}_{\mathcal{S}}^{C\mathcal{S}^{\mathcal{I}}}(\mathbb{C}(F_{\mathbf{m}\sqcup\mathbf{n}}^{\mathcal{I}}(*)) \boxtimes \mathbb{C}(F_{\mathbf{n}}^{\mathcal{I}}(*)), M)$$

is a weak equivalence if and only if the map $M(\mathbf{m}) \times M(\mathbf{n}) \to M(\mathbf{m} \sqcup \mathbf{n}) \times M(\mathbf{n})$ given by the shear map $(x, y) \mapsto (xy, y)$ is a weak equivalence. Recall that for $\mathbf{p}$ in $\mathcal{I}$, there is a pullback square

$$\begin{array}{ccc} M(\mathbf{p}) & \longrightarrow & \mathrm{hocolim}_{\mathcal{I}} M \\ \downarrow & & \downarrow \\ \{\mathbf{p}\} & \longrightarrow & B\mathcal{I} \end{array} \tag{3.29}$$

(see [GJ09, proof of Lemma IV.5.7], remarks before Proposition 2.5). If $\mathbf{p}$ is in $\mathcal{I}_+$, the above square (3.29) is homotopy cartesian by Proposition 2.5. Since the classifying space $B\mathcal{I}$ is contractible, the base change map $M(\mathbf{p}) \to \mathrm{hocolim}_{\mathcal{I}} M$ is a weak equivalence in this case. We consider the induced diagram

$$\begin{array}{ccc} M(\mathbf{m}) \times M(\mathbf{n}) & \longrightarrow & M(\mathbf{m} \sqcup \mathbf{n}) \times M(\mathbf{n}) \\ \downarrow\sim & & \downarrow\sim \\ \mathrm{hocolim}_{\mathcal{I}} M \times \mathrm{hocolim}_{\mathcal{I}} M & \longrightarrow & \mathrm{hocolim}_{\mathcal{I}} M \times \mathrm{hocolim}_{\mathcal{I}} M. \end{array} \tag{3.30}$$

The horizontal maps in (3.30) are the shear maps. For the bottom horizontal morphism in (3.30) this means that in simplicial degree $[s]$ an element (x, y) in $M(\mathbf{m}_s)[s] \times M(\mathbf{n}_s)[s]$ indexed by

$$(\mathbf{m}_0 \xleftarrow{\alpha_1} \ldots \xleftarrow{\alpha_s} \mathbf{m}_s \in \mathcal{N}\mathcal{I}[s], \mathbf{n}_0 \xleftarrow{\beta_1} \ldots \xleftarrow{\beta_s} \mathbf{n}_s \in \mathcal{N}\mathcal{I}[s])$$

is mapped to (xy, y) in $M(\mathbf{m}_s \sqcup \mathbf{n}_s)[s] \times M(\mathbf{n}_s)[s]$ indexed by

$$(\mathbf{m}_0 \sqcup \mathbf{n}_0 \xleftarrow{\alpha_1 \sqcup \beta_1} \ldots \xleftarrow{\alpha_s \sqcup \beta_s} \mathbf{m}_s \sqcup \mathbf{n}_s \in \mathcal{N}\mathcal{I}[s], \mathbf{n}_0 \xleftarrow{\beta_1} \ldots \xleftarrow{\beta_s} \mathbf{n}_s \in \mathcal{N}\mathcal{I}[s]).$$

We see that the diagram (3.30) is commutative as follows. Let $[s]$ be a simplicial degree, and let (x, y) be in $M(\mathbf{m})[s] \times M(\mathbf{n})[s]$. The left vertical map in (3.30) sends (x, y) to (x, y) in $M(\mathbf{m})[s] \times M(\mathbf{n})[s]$ indexed by

$$(\mathbf{m} \xleftarrow{\mathrm{id}} \ldots \xleftarrow{\mathrm{id}} \mathbf{m} \in \mathcal{N}\mathcal{I}[s], \mathbf{n} \xleftarrow{\mathrm{id}} \ldots \xleftarrow{\mathrm{id}} \mathbf{n}_s \in \mathcal{N}\mathcal{I}[s]).$$

The bottom horizontal map in (3.30) takes the latter to (xy, y) in $M(\mathbf{m} \sqcup \mathbf{n})[s] \times M(\mathbf{n})[s]$ indexed by

$$(\mathbf{m} \sqcup \mathbf{n} \xleftarrow{\mathrm{id}} \ldots \xleftarrow{\mathrm{id}} \mathbf{m} \sqcup \mathbf{n} \in \mathcal{N}\mathcal{I}[s], \mathbf{n} \xleftarrow{\mathrm{id}} \ldots \xleftarrow{\mathrm{id}} \mathbf{n} \in \mathcal{N}\mathcal{I}[s]).$$

The other way round, the top horizontal morphism in (3.30) maps (x, y) to (xy, y) in $M(\mathbf{m} \sqcup \mathbf{n})[s] \times M(\mathbf{n})[s]$ which is then sent to (xy, y) in $M(\mathbf{m} \sqcup \mathbf{n})[s] \times M(\mathbf{n})[s]$ indexed by

$$(\mathbf{m} \sqcup \mathbf{n} \xleftarrow{\mathrm{id}} \ldots \xleftarrow{\mathrm{id}} \mathbf{m} \sqcup \mathbf{n} \in \mathcal{N}\mathcal{I}[s], \mathbf{n} \xleftarrow{\mathrm{id}} \ldots \xleftarrow{\mathrm{id}} \mathbf{n} \in \mathcal{N}\mathcal{I}[s])$$

by the right vertical morphism. Because the map $\mathrm{Hom}_{\mathcal{S}}^{\mathcal{C}\mathcal{S}^{\mathcal{I}}}(s^{\mathcal{I}}_{\mathbf{m},\mathbf{n}}, \mathrm{id})$ is a weak equivalence by assumption, two out of three yields that the bottom horizontal map in (3.30) is a weak equivalence. If we apply the realization functor to the latter, we obtain that this is a weak equivalence between cofibrant objects in topological spaces and hence a homotopy equivalence. It follows from [Whi78, III.(4.17)] that the E_∞ space $\mathrm{hocolim}_{\mathcal{I}} M$ is grouplike.

Reversely, let M be positive $\mathcal{I}$-fibrant and grouplike. Exploiting that M is grouplike, [Whi78, III.(4.17)] ensures that the bottom map in (3.30) is a weak equivalence. Two out of three implies that the top map in (3.30) is a weak equivalence for all $\mathbf{m}$ and $\mathbf{n}$ in $\mathcal{I}_+$, which is equivalent to the map $\mathrm{Hom}_{\mathcal{S}}^{\mathcal{C}\mathcal{S}^{\mathcal{I}}}(s^{\mathcal{I}}_{\mathbf{m},\mathbf{n}}, \mathrm{id})$ being a weak equivalence for all $\mathbf{m}$ and $\mathbf{n}$ in $\mathcal{I}_+$. Hence, the commutative $\mathcal{I}$-space M is $S^{\mathcal{I}}$-local. □

Remark 3.32. Sagave and Schlichtkrull establish in [SS13] a *group completion model structure* on commutative $\mathcal{I}$-spaces as the left Bousfield localization of the positive projective $\mathcal{I}$-model structure with respect to a certain universal group completion map (see [SS13, §5]). The cofibrations and the fibrant objects in their group completion model structure agree with the ones in $(\mathcal{C}\mathcal{S}^{\mathcal{I}})_{\mathrm{loc}(S^{\mathcal{I}})}$ (see [SS13, Lemma 5.6], Lemma 3.31). According to [Joy, Proposition E.1.10], we can conclude that both model structures agree. In particular, we get that a map $M \to N$ is a weak equivalence in $(\mathcal{C}\mathcal{S}^{\mathcal{I}})_{\mathrm{loc}(S^{\mathcal{I}})}$ if and only if the induced map of bar constructions $B(\mathrm{hocolim}_{\mathcal{I}} M) \to B(\mathrm{hocolim}_{\mathcal{I}} N)$ is a weak equivalence in spaces (see [SS13, Theorem 1.3, §5]).

Remark 3.33. Suppose that $\pi_0(B\mathcal{K})$ is a group. Let $j\colon B_{\mathcal{I}}\mathcal{K} \overset{\sim}{\rightarrowtail} (B_{\mathcal{I}}\mathcal{K})^{\mathrm{fib}}$ be a fibrant replacement of $B_{\mathcal{I}}\mathcal{K}$ in the positive projective $\mathcal{I}$-model structure on $C\mathcal{S}^{\mathcal{I}}$. The commutative $\mathcal{I}$-space $(B_{\mathcal{I}}\mathcal{K})^{\mathrm{fib}}$ is positive $\mathcal{I}$-fibrant and grouplike and consequently $S^{\mathcal{I}}$-local by Lemma 3.31.

Proposition 3.34. *Suppose that $\pi_0(B\mathcal{K})$ is a group. The identity functor*

$$\mathrm{id}\colon (C\mathcal{S}^{\mathcal{I}} \downarrow B_{\mathcal{I}}\mathcal{K})_{\mathrm{loc}(S^{\mathcal{I}}_{B_{\mathcal{I}}\mathcal{K}})} \to ((C\mathcal{S}^{\mathcal{I}})_{\mathrm{loc}(S^{\mathcal{I}})} \downarrow B_{\mathcal{I}}\mathcal{K})$$

is the left adjoint in a Quillen equivalence.

Proof. The identity functor $\mathrm{id}\colon (C\mathcal{S}^{\mathcal{I}} \downarrow B_{\mathcal{I}}\mathcal{K}) \to ((C\mathcal{S}^{\mathcal{I}})_{\mathrm{loc}(S^{\mathcal{I}})} \downarrow B_{\mathcal{I}}\mathcal{K})$ is a left Quillen functor (see [Hir03, Proposition 3.3.4.(1)]). If we factor an element in $S^{\mathcal{I}}_{B_{\mathcal{I}}\mathcal{K}}$ into a cofibration followed by an acyclic fibration in $(C\mathcal{S}^{\mathcal{I}} \downarrow B_{\mathcal{I}}\mathcal{K})$, the identity functor $\mathrm{id}\colon (C\mathcal{S}^{\mathcal{I}} \downarrow B_{\mathcal{I}}\mathcal{K}) \to ((C\mathcal{S}^{\mathcal{I}})_{\mathrm{loc}(S^{\mathcal{I}})} \downarrow B_{\mathcal{I}}\mathcal{K})$ takes the cofibration from the factorization to a weak equivalence in $((C\mathcal{S}^{\mathcal{I}})_{\mathrm{loc}(S^{\mathcal{I}})} \downarrow B_{\mathcal{I}}\mathcal{K})$. Thus, it follows by [Hir03, Proposition 3.3.18.(1)] that

$$(C\mathcal{S}^{\mathcal{I}} \downarrow B_{\mathcal{I}}\mathcal{K})_{\mathrm{loc}(S^{\mathcal{I}}_{B_{\mathcal{I}}\mathcal{K}})} \underset{\mathrm{id}}{\overset{\mathrm{id}}{\rightleftarrows}} ((C\mathcal{S}^{\mathcal{I}})_{\mathrm{loc}(S^{\mathcal{I}})} \downarrow B_{\mathcal{I}}\mathcal{K})$$

is a Quillen adjunction. Note that this is the same argument as in the beginning of the proof of Proposition 3.4. Further, let $j\colon B_{\mathcal{I}}\mathcal{K} \overset{\sim}{\rightarrowtail} (B_{\mathcal{I}}\mathcal{K})^{\mathrm{fib}}$ be a fibrant replacement of $B_{\mathcal{I}}\mathcal{K}$ in the positive projective $\mathcal{I}$-model structure on $C\mathcal{S}^{\mathcal{I}}$. We investigate the following diagram of adjunctions

$$\begin{tikzcd}
(C\mathcal{S}^{\mathcal{I}} \downarrow B_{\mathcal{I}}\mathcal{K})_{\mathrm{loc}(S^{\mathcal{I}}_{B_{\mathcal{I}}\mathcal{K}})} \arrow[r, shift left, "j_*"] \arrow[d, shift right, "\mathrm{id}"'] & (C\mathcal{S}^{\mathcal{I}} \downarrow (B_{\mathcal{I}}\mathcal{K})^{\mathrm{fib}})_{\mathrm{loc}(S^{\mathcal{I}}_{(B_{\mathcal{I}}\mathcal{K})^{\mathrm{fib}}})} \arrow[l, shift left, "j^*"] \arrow[d, shift right, "\mathrm{id}"'] \\
((C\mathcal{S}^{\mathcal{I}})_{\mathrm{loc}(S^{\mathcal{I}})} \downarrow B_{\mathcal{I}}\mathcal{K}) \arrow[r, shift left, "j_*"] \arrow[u, shift right, "\mathrm{id}"'] & ((C\mathcal{S}^{\mathcal{I}})_{\mathrm{loc}(S^{\mathcal{I}})} \downarrow (B_{\mathcal{I}}\mathcal{K})^{\mathrm{fib}}). \arrow[l, shift left, "j^*"] \arrow[u, shift right, "\mathrm{id}"']
\end{tikzcd}$$

Here the horizontal adjunctions come from composition with and pullback along the map j. As the map j is an $\mathcal{I}$-equivalence, Lemma 3.5 implies that the bottom adjunction is a Quillen equivalence. Since the objects $B_{\mathcal{I}}\mathcal{K}$ and $(B_{\mathcal{I}}\mathcal{K})^{\mathrm{fib}}$ are homotopy constant with respect to morphisms in $\mathcal{I}_+$, Lemma 3.28 ensures that the top adjunction is a Quillen equivalence. Because the object $(B_{\mathcal{I}}\mathcal{K})^{\mathrm{fib}}$ is $S^{\mathcal{I}}$-local (see Remark 3.33), we gain from Proposition 3.4 that the upper right and the lower right model structure coincide. Applying two out of three for Quillen equivalences [Hov99, Corollary 1.3.15] finishes the proof. □

Theorem 3.35. *Suppose that $\pi_0(B\mathcal{K})$ is a group. There is a chain of Quillen equivalences between $(C\mathcal{S}^{\mathcal{K}})_{\mathrm{loc}(S^{\mathcal{K}})}$ and $((C\mathcal{S}^{\mathcal{I}})_{\mathrm{loc}(S^{\mathcal{I}})} \downarrow B_{\mathcal{I}}\mathcal{K})$.*

Proof. From Lemma 3.25, Lemma 3.26, Lemma 3.27 and Proposition 3.34, we obtain the following chain of Quillen equivalences connecting $(\mathcal{CS}^{\mathcal{K}})_{\mathrm{loc}(S^{\mathcal{K}})}$ with $((\mathcal{CS}^{\mathcal{I}})_{\mathrm{loc}(S^{\mathcal{I}})} \downarrow B_{\mathcal{I}}\mathcal{K})$

$$\begin{tikzcd}
(\mathcal{CS}^{\mathcal{K}})_{\mathrm{loc}(S^{\mathcal{K}})} \arrow[r, shift right, "\mathrm{const}_{\mathcal{I}}"'] & (\mathcal{CS}^{\mathcal{K}\times\mathcal{I}})_{\mathrm{loc}(S^{\mathcal{K}\times\mathcal{I}})} \arrow[l, shift right, "\mathrm{colim}_{\mathcal{I}}"'] \arrow[r, shift right, "q"'] & (\mathcal{CS}^{\mathcal{K}\times\mathcal{I}} \downarrow E_{\mathcal{I}}\mathcal{K})_{\mathrm{loc}(S^{\mathcal{K}\times\mathcal{I}}_{E_{\mathcal{I}}\mathcal{K}})} \arrow[l, shift right, "t_*"'] \arrow[d, shift right, "\mathrm{colim}_{\mathcal{K}}\circ u_*"'] \\
& ((\mathcal{CS}^{\mathcal{I}})_{\mathrm{loc}(S^{\mathcal{I}})} \downarrow B_{\mathcal{I}}\mathcal{K}) \arrow[r, shift right, "\mathrm{id}"'] & (\mathcal{CS}^{\mathcal{I}} \downarrow B_{\mathcal{I}}\mathcal{K})_{\mathrm{loc}(S^{\mathcal{I}}_{B_{\mathcal{I}}\mathcal{K}})}. \arrow[l, shift right, "\mathrm{id}"'] \arrow[u, shift right, "p\mathrm{const}_{\mathcal{K}}"']
\end{tikzcd}$$

□

With the Quillen equivalence between $(\mathcal{CS}^{\mathcal{K}})_{\mathrm{loc}(S^{\mathcal{K}})}$ and $((\mathcal{CS}^{\mathcal{I}})_{\mathrm{loc}(S^{\mathcal{I}})} \downarrow B_{\mathcal{I}}\mathcal{K})$ at hand, we are able to describe the fibrant objects and the weak equivalences in $(\mathcal{CS}^{\mathcal{K}})_{\mathrm{loc}(S^{\mathcal{K}})}$.

Lemma 3.36. *Suppose that $\pi_0(B\mathcal{K})$ is a group. A commutative $\mathcal{K}$-space M is fibrant in $(\mathcal{CS}^{\mathcal{K}})_{\mathrm{loc}(S^{\mathcal{K}})}$ if and only if M is positive $\mathcal{K}$-fibrant and grouplike.*

Proof. First, let M be fibrant in $(\mathcal{CS}^{\mathcal{K}})_{\mathrm{loc}(S^{\mathcal{K}})}$, that is, M is $S^{\mathcal{K}}$-local in $\mathcal{CS}^{\mathcal{K}}$ by [Hir03, Theorem 4.1.1.(2)]. By definition we get that M is positive $\mathcal{K}$-fibrant. Let the map $j\colon B_{\mathcal{I}}\mathcal{K} \xrightarrow{\sim} (B_{\mathcal{I}}\mathcal{K})^{\mathrm{fib}}$ be a fibrant replacement of $B_{\mathcal{I}}\mathcal{K}$ in the positive projective $\mathcal{I}$-model structure on $\mathcal{CS}^{\mathcal{I}}$. Exploiting that the model category $(\mathcal{CS}^{\mathcal{K}})_{\mathrm{loc}(S^{\mathcal{K}})}$ is Quillen equivalent to $((\mathcal{CS}^{\mathcal{I}})_{\mathrm{loc}(S^{\mathcal{I}})} \downarrow (B_{\mathcal{I}}\mathcal{K})^{\mathrm{fib}})$ (see proof of Proposition 3.34, Theorem 3.35), we can assume that M is weakly equivalent to the image of a fibrant object $(N,\ \rho_N\colon\ N \to\ (B_{\mathcal{I}}\mathcal{K})^{\mathrm{fib}})$ in $((\mathcal{CS}^{\mathcal{I}})_{\mathrm{loc}(S^{\mathcal{I}})} \downarrow (B_{\mathcal{I}}\mathcal{K})^{\mathrm{fib}})$ under the composite derived functors from $((\mathcal{CS}^{\mathcal{I}})_{\mathrm{loc}(S^{\mathcal{I}})} \downarrow (B_{\mathcal{I}}\mathcal{K})^{\mathrm{fib}})$ to $(\mathcal{CS}^{\mathcal{K}})_{\mathrm{loc}(S^{\mathcal{K}})}$. Since the map ρ_N is a fibration in $(\mathcal{CS}^{\mathcal{I}})_{\mathrm{loc}(S^{\mathcal{I}})}$ and the commutative $\mathcal{I}$-space $(B_{\mathcal{I}}\mathcal{K})^{\mathrm{fib}}$ is $S^{\mathcal{I}}$-local (see Remark 3.33), the commutative $\mathcal{I}$-space N is $S^{\mathcal{I}}$-local as well by [Hir03, Proposition 3.3.14.(1)]. It follows from Lemma 3.31 that N is grouplike. But from Proposition 3.24 we know that the simplicial monoid $\mathrm{hocolim}_{\mathcal{I}}N$ is weakly equivalent to $\mathrm{hocolim}_{\mathcal{K}}M$, in particular that the commutative monoid $\pi_0(\mathrm{hocolim}_{\mathcal{I}}N)$ is isomorphic to $\pi_0(\mathrm{hocolim}_{\mathcal{K}}M)$. As the former is a group, we can conclude that the commutative $\mathcal{K}$-space M is grouplike.

Secondly, let the commutative $\mathcal{K}$-space M be positive $\mathcal{K}$-fibrant and grouplike. Using the chain of Quillen equivalences between $\mathcal{CS}^{\mathcal{K}}$ and $(\mathcal{CS}^{\mathcal{I}} \downarrow B_{\mathcal{I}}\mathcal{K})$ (see Theorem 3.22), we can assume that M is $\mathcal{K}$-equivalent to the image of a fibrant object $(N, \rho_N\colon N \to B_{\mathcal{I}}\mathcal{K})$ in $(\mathcal{CS}^{\mathcal{I}} \downarrow B_{\mathcal{I}}\mathcal{K})$ under the composite derived functors from $(\mathcal{CS}^{\mathcal{I}} \downarrow B_{\mathcal{I}}\mathcal{K})$ to $\mathcal{CS}^{\mathcal{K}}$. Because M is grouplike, the shear map

$$\mathrm{hocolim}_{\mathcal{K}}M \times \mathrm{hocolim}_{\mathcal{K}}M \to \mathrm{hocolim}_{\mathcal{K}}M \times \mathrm{hocolim}_{\mathcal{K}}M \tag{3.31}$$

is a weak equivalence by [Whi78, III.(4.17)]. As the simplicial monoid $\mathrm{hocolim}_{\mathcal{K}}M$ is weakly equivalent to $\mathrm{hocolim}_{\mathcal{I}}N$ by Proposition 3.24, the map (3.31) is a weak equivalence if and only if the shear map

$$\mathrm{hocolim}_{\mathcal{I}}N \times \mathrm{hocolim}_{\mathcal{I}}N \to \mathrm{hocolim}_{\mathcal{I}}N \times \mathrm{hocolim}_{\mathcal{I}}N \tag{3.32}$$

is a weak equivalence. Since the map ρ_N is a positive $\mathcal{I}$-fibration and the commutative $\mathcal{I}$-space $B_{\mathcal{I}}\mathcal{K}$ is homotopy constant with respect to morphisms in $\mathcal{I}_+$, the commutative $\mathcal{I}$-space N is homotopy constant with respect to morphisms in $\mathcal{I}_+$, too. Thus, Proposition 2.5 yields that for every $\mathbf{m}$ and $\mathbf{n}$ in $\mathcal{I}_+$, the map (3.32) is a weak equivalence if and only if the shear map

$$N(\mathbf{m}) \times N(\mathbf{n}) \to N(\mathbf{m} \sqcup \mathbf{n}) \times N(\mathbf{n}) \tag{3.33}$$

is a weak equivalence. Using adjointness, the shear map (3.33) is a weak equivalence if and only if the map

$$\mathrm{Hom}_{\mathcal{S}}^{C\mathcal{S}^{\mathcal{I}}}(\mathbb{C}(F_{\mathbf{m}}^{\mathcal{I}}(*))\boxtimes\mathbb{C}(F_{\mathbf{n}}^{\mathcal{I}}(*)), N) \xrightarrow{\mathrm{Hom}_{\mathcal{S}}^{C\mathcal{S}^{\mathcal{I}}}(s_{\mathbf{m},\mathbf{n}}^{\mathcal{I}},\mathrm{id})} \mathrm{Hom}_{\mathcal{S}}^{C\mathcal{S}^{\mathcal{I}}}(\mathbb{C}(F_{\mathbf{m}\sqcup\mathbf{n}}^{\mathcal{I}}(*))\boxtimes\mathbb{C}(F_{\mathbf{n}}^{\mathcal{I}}(*)), N)$$

is a weak equivalence. Because the commutative $\mathcal{I}$-space $B_{\mathcal{I}}\mathcal{K}$ is grouplike, we get that for every $\mathbf{m}$ and $\mathbf{n}$ in $\mathcal{I}_+$, the map

$$\begin{aligned}&\mathrm{Hom}_{\mathcal{S}}^{C\mathcal{S}^{\mathcal{I}}}(\mathbb{C}(F_{\mathbf{m}}^{\mathcal{I}}(*)) \boxtimes \mathbb{C}(F_{\mathbf{n}}^{\mathcal{I}}(*)), B_{\mathcal{I}}\mathcal{K})\\ &\xrightarrow{\mathrm{Hom}_{\mathcal{S}}^{C\mathcal{S}^{\mathcal{I}}}(s_{\mathbf{m},\mathbf{n}}^{\mathcal{I}},\mathrm{id})} \mathrm{Hom}_{\mathcal{S}}^{C\mathcal{S}^{\mathcal{I}}}(\mathbb{C}(F_{\mathbf{m}\sqcup\mathbf{n}}^{\mathcal{I}}(*)) \boxtimes \mathbb{C}(F_{\mathbf{n}}^{\mathcal{I}}(*)), B_{\mathcal{I}}\mathcal{K})\end{aligned}$$

is a weak equivalence (see proof of Lemma 3.31). Applying [Hir03, Proposition 13.3.14] ensures that the map

$$\begin{gathered}\mathrm{Hom}_{\mathcal{S}}^{(C\mathcal{S}^{\mathcal{I}}\downarrow B_{\mathcal{I}}\mathcal{K})}((\mathbb{C}(F_{\mathbf{m}}^{\mathcal{I}}(*)) \boxtimes \mathbb{C}(F_{\mathbf{n}}^{\mathcal{I}}(*)), \mathbb{C}(F_{\mathbf{m}}^{\mathcal{I}}(*)) \boxtimes \mathbb{C}(F_{\mathbf{n}}^{\mathcal{I}}(*)) \to B_{\mathcal{I}}\mathcal{K}), (N, \rho_N))\\ \Big\downarrow \mathrm{Hom}_{\mathcal{S}}^{(C\mathcal{S}^{\mathcal{I}}\downarrow B_{\mathcal{I}}\mathcal{K})}(s_{\mathbf{m},\mathbf{n}}^{\mathcal{I}},\mathrm{id})\\ \mathrm{Hom}_{\mathcal{S}}^{(C\mathcal{S}^{\mathcal{I}}\downarrow B_{\mathcal{I}}\mathcal{K})}((\mathbb{C}(F_{\mathbf{m}\sqcup\mathbf{n}}^{\mathcal{I}}(*)) \boxtimes \mathbb{C}(F_{\mathbf{n}}^{\mathcal{I}}(*)), \mathbb{C}(F_{\mathbf{m}\sqcup\mathbf{n}}^{\mathcal{I}}(*)) \boxtimes \mathbb{C}(F_{\mathbf{n}}^{\mathcal{I}}(*)) \to B_{\mathcal{I}}\mathcal{K}), (N, \rho_N))\end{gathered}$$

is a weak equivalence for every possible augmentation $\mathbb{C}(F_{\mathbf{m}}^{\mathcal{I}}(*)) \boxtimes \mathbb{C}(F_{\mathbf{n}}^{\mathcal{I}}(*)) \to B_{\mathcal{I}}\mathcal{K}$. Taking into account that Quillen equivalences induce weak equivalences between the homotopy types of mapping spaces, we obtain that for every $\mathbf{k}$ and $\mathbf{l}$ in $\mathcal{K}_+$, the map

$$\mathrm{Hom}_{\mathcal{S}}^{C\mathcal{S}^{\mathcal{K}}}(\mathbb{C}(F_{\mathbf{k}}^{\mathcal{K}}(*))\boxtimes\mathbb{C}(F_{\mathbf{l}}^{\mathcal{K}}(*)), M) \xrightarrow{\mathrm{Hom}_{\mathcal{S}}^{C\mathcal{S}^{\mathcal{K}}}(s_{\mathbf{k},\mathbf{l}}^{\mathcal{K}},\mathrm{id})} \mathrm{Hom}_{\mathcal{S}}^{C\mathcal{S}^{\mathcal{K}}}(\mathbb{C}(F_{\mathbf{k}\sqcup\mathbf{l}}^{\mathcal{K}}(*))\boxtimes\mathbb{C}(F_{\mathbf{l}}^{\mathcal{K}}(*)), M)$$

is a weak equivalence. Therefore, the commutative $\mathcal{K}$-space M is $S^{\mathcal{K}}$-local. □

Theorem 3.37. *Suppose that the simplicial monoid $B\mathcal{K}$ is grouplike. The localized model structure $(C\mathcal{S}^{\mathcal{K}})_{\mathrm{loc}(S^{\mathcal{K}})}$ can be characterized as follows.*

- *A map $M \to N$ is a weak equivalence if and only if the induced map of bar constructions $B(\mathrm{hocolim}_{\mathcal{K}}M) \to B(\mathrm{hocolim}_{\mathcal{K}}N)$ is a weak equivalence of spaces.*
- *The cofibrations are the cofibrations in the positive projective $\mathcal{K}$-model structure on $C\mathcal{S}^{\mathcal{K}}$.*
- *A commutative $\mathcal{K}$-space M is fibrant if and only if M is positive $\mathcal{K}$-fibrant and grouplike. Fibrant replacements model group completions. Fibrations are determined by the right lifting property with respect to the class of acyclic cofibrations.*

We call the model structure $(C\mathcal{S}^{\mathcal{K}})_{\mathrm{loc}(S^{\mathcal{K}})}$ the *group completion model structure*. We denote a fibrant replacement functor in this model structure by $(-)^{\mathrm{gp}}$.

Proof. Let $f\colon M \to N$ be a map of commutative $\mathcal{K}$-spaces. Taking advantage of the Quillen equivalence between $(C\mathcal{S}^{\mathcal{K}})_{\mathrm{loc}(S^{\mathcal{K}})}$ and $((C\mathcal{S}^{\mathcal{I}})_{\mathrm{loc}(S^{\mathcal{I}})} \downarrow B_{\mathcal{I}}\mathcal{K})$ (see Theorem 3.35) and Proposition 3.24, the map f is a weak equivalence if and only if the map

$$I^{\mathcal{I}}(\mathrm{hocolim}_{\mathcal{K}} M) \xrightarrow{I^{\mathcal{I}}(\mathrm{hocolim}_{\mathcal{K}} f)} I^{\mathcal{I}}(\mathrm{hocolim}_{\mathcal{K}} N)$$

is a weak equivalence in $(C\mathcal{S}^{\mathcal{I}})_{\mathrm{loc}(S^{\mathcal{I}})}$. This holds if and only if the induced map of bar constructions

$$B(\mathrm{hocolim}_{\mathcal{I}} I^{\mathcal{I}}(\mathrm{hocolim}_{\mathcal{K}} M)) \xrightarrow{B(\mathrm{hocolim}_{\mathcal{I}} I^{\mathcal{I}}(\mathrm{hocolim}_{\mathcal{K}} f))} B(\mathrm{hocolim}_{\mathcal{I}} I^{\mathcal{I}}(\mathrm{hocolim}_{\mathcal{K}} N)) \tag{3.34}$$

is a weak equivalence in spaces (see Remark 3.32). But the map (3.34) is a weak equivalence if and only if the map

$$B(\mathrm{hocolim}_{\mathcal{K}} M) \xrightarrow{B(\mathrm{hocolim}_{\mathcal{K}} f)} B(\mathrm{hocolim}_{\mathcal{K}} N)$$

is a weak equivalence. The statement about the fibrant objects in $(C\mathcal{S}^{\mathcal{K}})_{\mathrm{loc}(S^{\mathcal{K}})}$ is precisely Lemma 3.36. □

Remark 3.38. Sagave discusses group completion in commutative $\mathcal{J}$-spaces in [Sag16]. His approach is model categorical as well. As we do, Sagave defines *the group completion model structure* on commutative $\mathcal{J}$-spaces as the left Bousfield localization of the positive projective $\mathcal{J}$-model structure with respect to the set $S^{\mathcal{J}}$ (see [Sag16, Theorem 5.5, pp. 1242-1243]). In contrast to our work, in order to describe $(C\mathcal{S}^{\mathcal{J}})_{\mathrm{loc}(S^{\mathcal{J}})}$, Sagave constructs a chain of Quillen equivalences between the localized model structure on commutative $\mathcal{J}$-spaces and the stable model structure on Γ-spaces over a certain explicit Γ-space defined through the permutative category $\mathcal{J}$ (see [Sag16, Definition 3.5, Theorem 5.10, Corollary 7.17]). It is unclear whether Sagave's approach also works for the category $\bar{\mathcal{J}}$ instead of the category $\mathcal{J}$. The reason for this is that in the proof of [Sag16, Lemma 7.22] we need that the monoidal structure map of the functor $\mathrm{hocolim}_{\bar{\mathcal{J}}}$ (see (2.44)) is a weak equivalence for positive cofibrant commutative $\bar{\mathcal{J}}$-spaces. To prove that the monoidal structure map of the functor $\mathrm{hocolim}_{\mathcal{J}}$ is a weak equivalence, one makes use of the flat model structure on $\mathcal{J}$-spaces (see [Sag14, Lemma 2.11]). But we do not have a flat model structure on $\bar{\mathcal{J}}$-spaces (see Remark 1.14), and so far we do not know how to prove the statement.

3.8 More examples of pre-log cdgas

Having a suitable notion of group completion in commutative $\bar{\mathcal{J}}$-spaces motivates other examples of pre-log cdgas. We construct a certain direct image pre-log structure on a given commutative symmetric ring spectrum in simplicial k-modules associated to a homotopy class in the homotopy groups of the latter. This pre-log structure is an analogon of a quite useful pre-log structure in the topological setting (see [RSS18, §4, §6-§8], [Sag14, §4, §6]).

In the sequel let A be a positive fibrant object in $C(\mathrm{Sp}^{\Sigma}(s\mathrm{mod}))$. Let $(\mathbf{m}_1, \mathbf{m}_2)$ be in $\bar{\mathcal{J}}_+$, and let $[x]$ be a homotopy class in $\pi_{m_2-m_1}(U(A))$ represented by a map $x\colon S^{m_2} \to U(A)(m_1)$ in $\mathcal{S}_*$.

Lemma 3.39. *There is a localization map $j\colon A \to A[1/x]$ of positive fibrant objects in $C(\mathrm{Sp}^{\Sigma}(s\mathrm{mod}))$ such that the induced map $\pi_*(U(j))\colon \pi_*(U(A)) \to \pi_*(U(A[1/x]))$ takes the homotopy class $[x]$ in $\pi_{m_2-m_1}(U(A))$ to a unit in $\pi_{m_2-m_1}(U(A[1/x]))$, and there is an isomorphism of graded commutative rings $\pi_*(U(A[1/x])) \cong (\pi_*(U(A)))[1/[x]]$.*

Proof. (compare [Schb, Example I.4.65]) Let $p \geq 0$. We define the simplicial k-module $\widetilde{A[1/x]}(p)$ as

$$\widetilde{A[1/x]}(p) = \underline{\mathrm{Hom}}_{s\mathrm{mod}}(\tilde{k}(S^{m_2 p}), A((1+m_1)p)).$$

The symmetric group Σ_p acts on the simplicial k-modules $\tilde{k}(S^{m_2 p})$ and $A((1+m_1)p)$ by permuting the p blocks of m_2 respectively $1+m_1$, that is, by restriction along the diagonal embedding $\Sigma_p \to \Sigma_{m_2 p}$ respectively $\Sigma_p \to \Sigma_{(1+m_1)p}$. The action of the symmetric group Σ_p on the simplicial k-module $\underline{\mathrm{Hom}}_{s\mathrm{mod}}(\tilde{k}(S^{m_2 p}), A((1+m_1)p))$ is then given by conjugation. For $p, q \geq 0$, there are $\Sigma_p \times \Sigma_q$-equivariant multiplication maps

$$\begin{aligned} &\underline{\mathrm{Hom}}_{s\mathrm{mod}}(\tilde{k}(S^{m_2 p}), A((1+m_1)p)) \hat{\otimes} \underline{\mathrm{Hom}}_{s\mathrm{mod}}(\tilde{k}(S^{m_2 q}), A((1+m_1)q)) \\ &\to \underline{\mathrm{Hom}}_{s\mathrm{mod}}(\tilde{k}(S^{m_2(p+q)}), A((1+m_1)(p+q))) \end{aligned}$$

which send a pair (f, g) to the composite map

$$\begin{aligned} &\tilde{k}(S^{m_2(p+q)}) \cong \tilde{k}(S^{m_2 p}) \hat{\otimes} \tilde{k}(S^{m_2 q}) \xrightarrow{f \hat{\otimes} g} A((1+m_1)p) \hat{\otimes} A((1+m_1)q) \\ &\xrightarrow{\mu^A_{(1+m_1)p,(1+m_1)q}} A((1+m_1)(p+q)) \end{aligned}$$

where μ^A denotes the multiplication map of A. Moreover, for $p \geq 0$, let

$$A(p) \xrightarrow{\tilde{j}(p)} \widetilde{A[1/x]}(p)$$

be the Σ_p-equivariant morphism that is adjoint to the composite morphism

$$\begin{aligned} &A(p) \hat{\otimes} \tilde{k}(S^{m_2 p}) \cong A(p) \hat{\otimes} \tilde{k}(S^{m_2})^{\hat{\otimes} p} \xrightarrow{\mathrm{id}_{A(p)} \hat{\otimes} \mathrm{ad}(x)^{\hat{\otimes} p}} A(p) \hat{\otimes} A(m_1)^{\hat{\otimes} p} \\ &\xrightarrow{\mu^A_{p,m_1,\ldots,m_1}} A((1+m_1)p) \end{aligned}$$

where $\mathrm{ad}(x)$ stands for the adjoint map of the map x. For $p \geq 0$, we define unit maps $\eta^{\widetilde{A[1/x]}}(p)\colon \tilde{k}(S^1)^{\hat{\otimes} p} \to \widetilde{A[1/x]}(p)$ as the composite

$$\tilde{k}(S^1)^{\hat{\otimes} p} \xrightarrow{\eta^{A}(p)} A(p) \xrightarrow{\tilde{j}(p)} \widetilde{A[1/x]}(p)$$

where we write η^A for the unit map of A. These data assemble to an object $\widetilde{A[1/x]}$ in $C(\mathrm{Sp}^\Sigma(s\mathrm{mod}))$ and a morphism $\tilde{j}\colon A \to \widetilde{A[1/x]}$ in $C(\mathrm{Sp}^\Sigma(s\mathrm{mod}))$. Applying the forgetful functor $U\colon C(\mathrm{Sp}^\Sigma(s\mathrm{mod})) \to C(\mathrm{Sp}^\Sigma)$ to $\widetilde{A[1/x]}$ is isomorphic to $(U(A))[1/x]$ as specified in [Schb, Example I.4.65]. In addition, the underlying morphism $U(\tilde{j})\colon U(A) \to U(\widetilde{A[1/x]})$ in $C(\mathrm{Sp}^\Sigma)$ can be identified with the morphism $U(A) \to (U(A))[1/x]$ provided by Schwede in [Schb, Example I.4.65]. Employing a fibrant replacement $\widetilde{A[1/x]} \overset{\sim}{\rightarrowtail} A[1/x]$ of $\widetilde{A[1/x]}$ in $C(\mathrm{Sp}^\Sigma(s\mathrm{mod}))$ yields a morphism j

$$\begin{tikzcd} A \arrow[r, "\tilde{j}"] \arrow[rr, bend left, "j"] & \widetilde{A[1/x]} \arrow[r, tail, "\sim"] & A[1/x] \end{tikzcd}$$

in $C(\mathrm{Sp}^\Sigma(s\mathrm{mod}))$ that has the desired properties by [Schb, Corollary I.4.69]. For this, we remark that the homotopy groups of $U(\widetilde{A[1/x]})$ and $U(A[1/x])$ are isomorphic because the symmetric spectrum $(U(A))[1/x]$ is semistable by [Schb, Proposition 4.67]. $\square$

Construction 3.40. (compare [Sag14, Construction 4.2]) We consider the free pre-log structure $(C(x), \alpha)$ on A (see Example 2.39). The homotopy class $[x]$ is a unit in $\pi_*(U(A[1/x]))$ so that the composite map

$$C(x) \xrightarrow{\alpha} \Omega^{\bar{\mathcal{J}}}(A) \xrightarrow{\Omega^{\bar{\mathcal{J}}}(j)} \Omega^{\bar{\mathcal{J}}}(A[1/x])$$

factors through the map $i_{A[1/x]}\colon \mathrm{GL}_1^{\bar{\mathcal{J}}}(A[1/x]) \to \Omega^{\bar{\mathcal{J}}}(A[1/x])$. We factor the resulting map $C(x) \to \mathrm{GL}_1^{\bar{\mathcal{J}}}(A[1/x])$ into an acyclic cofibration followed by a fibration in the group completion model structure $(C\mathcal{S}^{\bar{\mathcal{J}}})_{\mathrm{loc}(S^{\bar{\mathcal{J}}})}$ (see Theorem 3.37),

$$\begin{tikzcd} C(x) \arrow[r, tail, "\sim"] \arrow[rr, bend left] & (C(x))^{\mathrm{gp}} \arrow[r, two heads] & \mathrm{GL}_1^{\bar{\mathcal{J}}}(A[1/x]). \end{tikzcd}$$

Since the commutative $\bar{\mathcal{J}}$-space $\mathrm{GL}_1^{\bar{\mathcal{J}}}(A[1/x])$ is positive $\bar{\mathcal{J}}$-fibrant and grouplike, it is fibrant in $(C\mathcal{S}^{\bar{\mathcal{J}}})_{\mathrm{loc}(S^{\bar{\mathcal{J}}})}$ by Lemma 3.36. Hence, the commutative $\bar{\mathcal{J}}$-space $(C(x))^{\mathrm{gp}}$ is fibrant in $(C\mathcal{S}^{\bar{\mathcal{J}}})_{\mathrm{loc}(S^{\bar{\mathcal{J}}})}$ so that the map of commutative $\bar{\mathcal{J}}$-spaces $C(x) \to (C(x))^{\mathrm{gp}}$ specifies a group completion indeed. Let $D'(x)$ be the pullback of the diagram

$$(C(x))^{\mathrm{gp}} \twoheadrightarrow \Omega^{\bar{\mathcal{J}}}(A[1/x]) \xleftarrow{\Omega^{\bar{\mathcal{J}}}(j)} \Omega^{\bar{\mathcal{J}}}(A)$$

in $C\mathcal{S}^{\bar{\mathcal{J}}}$. By the universal property of the pullback we obtain a map $C(x) \to D'(x)$ which we factor into a positive cofibration followed by a positive acyclic $\bar{\mathcal{J}}$-fibration in $C\mathcal{S}^{\bar{\mathcal{J}}}$,

$$\begin{tikzcd} C(x) \arrow[r, tail] \arrow[rr, bend left] & D(x) \arrow[r, two heads, "\sim"] & D'(x). \end{tikzcd}$$

The single steps in this construction yield the commutative diagram

$$
\begin{tikzcd}
C(x) \arrow[dr, tail] \arrow[rrrrdd, bend left, "\alpha"] \arrow[rrddd, tail, bend right] & & & & \\
& D(x) \arrow[dr, two heads, "\sim"] & & & \\
& & D'(x) \arrow[rr, two heads] \arrow[d] & & \Omega^{\bar{\mathcal{J}}}(A) \arrow[d, "\Omega^{\bar{\mathcal{J}}}(j)"] \\
& & (C(x))^{\mathrm{gp}} \arrow[r, two heads] & \mathrm{GL}_1^{\bar{\mathcal{J}}}(A[1/x]) \arrow[r, two heads, "i_{A[1/x]}"] & \Omega^{\bar{\mathcal{J}}}(A[1/x])
\end{tikzcd}
$$

in $\mathcal{CS}^{\bar{\mathcal{J}}}$. Note that the pre-log structure $(D'(x), D'(x) \to \Omega^{\bar{\mathcal{J}}}(A))$ is the direct image pre-log structure on A with respect to the map j and the pre-log structure

$$((C(x))^{\mathrm{gp}}, (C(x))^{\mathrm{gp}} \to \mathrm{GL}_1^{\bar{\mathcal{J}}}(A[1/x]) \xrightarrow{i_{A[1/x]}} \Omega^{\bar{\mathcal{J}}}(A[1/x]))$$

(see Example 2.35). We call the pre-log structure

$$(D(x), D(x) \to D'(x) \to \Omega^{\bar{\mathcal{J}}}(A))$$

the *direct image pre-log structure* on A associated with x. There is a sequence of morphisms of pre-log cdgas

$$
\begin{array}{c}
(A, C(x), \alpha) \\
\downarrow \\
(A, D(x), D(x) \to D'(x) \to \Omega^{\bar{\mathcal{J}}}(A)) \\
\downarrow\sim \\
(A, D'(x), D'(x) \to \Omega^{\bar{\mathcal{J}}}(A)) \\
\downarrow \\
(A, (\Omega^{\bar{\mathcal{J}}}(j))_*(\mathrm{GL}_1^{\bar{\mathcal{J}}}(A[1/x])), (\Omega^{\bar{\mathcal{J}}}(j))_*(i_{A[1/x]})) \\
\downarrow \\
(A[1/x], \mathrm{GL}_1^{\bar{\mathcal{J}}}(A[1/x]), i_{A[1/x]})
\end{array}
\tag{3.35}
$$

where the fourth pre-log cdga is A together with the direct image log structure with respect to the map j and the trivial log structure $(\mathrm{GL}_1^{\bar{\mathcal{J}}}(A[1/x]), i_{A[1/x]})$ (see Example 2.35, Example 2.46). The last two pre-log cdgas in the above sequence (3.35) are log cdgas in fact.

From Example 2.39 we know that the space $\mathrm{hocolim}_{\bar{\mathcal{J}}} C(x)$ is weakly equivalent to $\coprod_{n \geq 0} B\Sigma_n$. The Barratt-Priddy-Quillen theorem implies that the group completion $\mathrm{hocolim}_{\bar{\mathcal{J}}} C(x) \to \mathrm{hocolim}_{\bar{\mathcal{J}}} (C(x))^{\mathrm{gp}}$ is weakly equivalent to $\coprod_{n \geq 0} B\Sigma_n \to Q(S^0)$. The next lemma determines the homotopy type of the space $\mathrm{hocolim}_{\bar{\mathcal{J}}} D(x)$ in a special case.

Lemma 3.41. *Let A be a positive fibrant object in $C(\mathrm{Sp}^\Sigma(s\mathrm{mod}))$ such that $\pi_q(U(A)) = 0$ for $q \leq -1$. Let $(\mathbf{m}_1, \mathbf{m}_2)$ be in $\bar{\mathcal{J}}_+$, and let $[x]$ be a homotopy class in $\pi_*(U(A))$ of even positive degree $m_2 - m_1$ represented by a map $x\colon S^{m_2} \to U(A)(m_1)$ in $\mathcal{S}_*$. The space $\mathrm{hocolim}_{\bar{\mathcal{J}}} D(x)$ is weakly equivalent to the non-negative path components of $Q(S^0)$ denoted by $(Q(S^0))_{\geq 0}$. The composite map*

$$\mathrm{hocolim}_{\bar{\mathcal{J}}} C(x) \to \mathrm{hocolim}_{\bar{\mathcal{J}}} D(x) \to \mathrm{hocolim}_{\bar{\mathcal{J}}} (C(x))^{\mathrm{gp}}$$

is weakly equivalent to

$$\coprod_{n\geq 0} B\Sigma_n \to (Q(S^0))_{\geq 0} \to Q(S^0)$$

where the latter is the canonical factorization of the group completion map through the inclusion of the non-negative path components of $Q(S^0)$.

Proof. (compare [Sag14, proof of Lemma 4.6]) As the commutative $\bar{\mathcal{J}}$-space $D(x)$ is $\bar{\mathcal{J}}$-equivalent to $D'(x)$, it suffices to prove the statement for the latter. The pullback $D'(x)$ can be computed $\bar{\mathcal{J}}$-levelwise. Let $(\mathbf{n}_1, \mathbf{n}_2)$ be in $\bar{\mathcal{J}}_+$. The space $D'(x)(\mathbf{n}_1, \mathbf{n}_2)$ is the pullback of the diagram

$$(C(x))^{\mathrm{gp}}(\mathbf{n}_1, \mathbf{n}_2) \twoheadrightarrow \Omega^{\bar{\mathcal{J}}}(A[1/x])(\mathbf{n}_1, \mathbf{n}_2) \xleftarrow{\Omega^{\bar{\mathcal{J}}}(j)(\mathbf{n}_1,\mathbf{n}_2)} \Omega^{\bar{\mathcal{J}}}(A)(\mathbf{n}_1, \mathbf{n}_2).$$

Recall from Remark 2.38 that for a positive fibrant object B in $C(\mathrm{Sp}^\Sigma(s\mathrm{mod}))$ and $l \in \mathbb{Z}_{\geq 0}$, there is an isomorphism

$$\pi_{l+n_2-n_1}(U(B)) \cong \pi_l(U(\Omega^{\bar{\mathcal{J}}}(B))(\mathbf{n}_1, \mathbf{n}_2)). \tag{3.36}$$

First, suppose that $n_2 - n_1 \leq -1$. Taking into account (3.36) (for $l = 0$), under the base change map $D'(x)(\mathbf{n}_1, \mathbf{n}_2) \to \Omega^{\bar{\mathcal{J}}}(A)(\mathbf{n}_1, \mathbf{n}_2)$ a point in $D'(x)(\mathbf{n}_1, \mathbf{n}_2)$ is sent to a point in $\Omega^{\bar{\mathcal{J}}}(A)(\mathbf{n}_1, \mathbf{n}_2)$ which would represent a power of an inverse of the homotopy class $[x]$ in $\pi_*(U(A))$. But it follows from the assumption that $\pi_q(U(A)) = 0$ for $q \leq -1$, that the space $D'(x)(\mathbf{n}_1, \mathbf{n}_2)$ is empty.

Secondly, suppose that $n_2 - n_1 \geq 0$. Again in view of (3.36), the fact that the morphism $\pi_*(U(j))\colon \pi_*(U(A)) \to \pi_*(U(A[1/x]))$ is an isomorphism in non-negative degrees ensures that the morphism $\Omega^{\bar{\mathcal{J}}}(j)(\mathbf{n}_1, \mathbf{n}_2)\colon \Omega^{\bar{\mathcal{J}}}(A)(\mathbf{n}_1, \mathbf{n}_2) \to \Omega^{\bar{\mathcal{J}}}(A[1/x])(\mathbf{n}_1, \mathbf{n}_2)$ is a weak equivalence. Hence, due to the model category $\mathcal{S}$ being right proper, the base change map $D'(x)(\mathbf{n}_1, \mathbf{n}_2) \to (C(x))^{\mathrm{gp}}(\mathbf{n}_1, \mathbf{n}_2)$ is a weak equivalence. Observing that the inclusion functor $\bar{\mathcal{J}}_+ \to \bar{\mathcal{J}}$ is homotopy right cofinal by Lemma 1.18 and that the space $\mathrm{hocolim}_{\bar{\mathcal{J}}} (C(x))^{\mathrm{gp}}$ is weakly equivalent to $Q(S^0)$, finishes the proof. □

Remark 3.42. In the topological setting forming pre-log structures involving the commutative $\mathcal{J}$-space $D(x)$ [Sag14, Construction 4.2] is convenient, e.g. when identifying examples of log THH-étale morphisms of pre-log ring spectra (see [RSS18, §6]) or calculating logarithmic topological Hochschild homology of pre-log ring spectra in examples. The latter in turn helps to determine the ordinary topological Hochschild homology of the underlying commutative symmetric ring spectra (see [RSS18, §7-§8]). Our main example, the commutative Hk-algebra spectrum $F(X_+, Hk)$ for a space X, usually has non-trivial negative homotopy groups (recall the isomorphism $\pi_*(F(X_+, Hk)) \cong H^{-*}(X, k)$ (2.9)). We point out that we do not have an analogon of the previous lemma in the case that A is a positive fibrant object in $C(\mathrm{Sp}^\Sigma(s\mathrm{mod}))$ such that $\pi_q(U(A)) = 0$ for $q \geq 1$, that $(\mathbf{m}_1, \mathbf{m}_2)$ is in $\bar{\mathcal{J}}_+$, and that $[x]$ is a homotopy class in $\pi_*(U(A))$ of even negative degree $m_2 - m_1$. The reason for this is that from the isomorphism (3.36) and the fact that the morphism $\pi_*(U(j))\colon \pi_*(U(A)) \to \pi_*(U(A[1/x]))$ is an isomorphism in non-positive degrees, we can only deduce that for $(\mathbf{n}_1, \mathbf{n}_2)$ in $\bar{\mathcal{J}}_+$ with $n_2 - n_1 \leq 0$, the map

$$\pi_l(U(\Omega^{\bar{\mathcal{J}}}(A))(\mathbf{n}_1, \mathbf{n}_2)) \xrightarrow{\pi_l(U(\Omega^{\bar{\mathcal{J}}}(j))(\mathbf{n}_1, \mathbf{n}_2))} \pi_l(U(\Omega^{\bar{\mathcal{J}}}(A[1/x]))(\mathbf{n}_1, \mathbf{n}_2))$$

is an isomorphism for $l + n_2 - n_1 \leq 0$, that is, $l \leq -(n_2 - n_1)$. But from this we cannot conclude that the map $\Omega^{\bar{\mathcal{J}}}(j)(\mathbf{n}_1, \mathbf{n}_2)\colon \Omega^{\bar{\mathcal{J}}}(A)(\mathbf{n}_1, \mathbf{n}_2) \to \Omega^{\bar{\mathcal{J}}}(A[1/x])(\mathbf{n}_1, \mathbf{n}_2)$ is a weak equivalence.

Another problem of working with the commutative $\bar{\mathcal{J}}$-space $D(x)$ is that so far we do not have an explicit description of (the homotopy type or the homology groups of) $\Lambda^{\bar{\mathcal{J}}}(D(x))$.

4 Logarithmic topological Hochschild homology of log cdgas

In the last section we introduce *logarithmic topological Hochschild homology* of log cdgas. Our approach resembles Rognes, Sagave and Schlichtkrull's work in [RSS15]. We make use of the results in Section 3. First we discuss cyclic and replete bar constructions as well as general repletion of commutative diagram spaces. After this, we set the index category to be $\bar{\mathcal{J}}$ and define logarithmic topological Hochschild homology of pre-log cdgas. We show that this definition is homotopy invariant under logification. Furthermore, we specify *formally log* THH-*étale* morphisms of pre-log cdgas and present two approaches towards examples.

In the upcoming first two subsections, let $\mathcal{K}$ denote a well-structured index category which is permutative and whose classifying space $B\mathcal{K}$ is grouplike. Suppose that the inclusion functor $\mathcal{K}_+ \to \mathcal{K}$ is homotopy right cofinal and that the pair $(\mathcal{K}, \mathcal{O}\mathcal{K}_+)$ is a very well-structured relative index category.

4.1 The cyclic and replete bar constructions

We present the cyclic and replete bar constructions of commutative $\mathcal{K}$-spaces. The concepts in [RSS15, §3.1, §3.3] directly generalize from the category $\mathcal{J}$ to a category $\mathcal{K}$ which satisfies the above assumptions.

Definition 4.1. (compare [RSS15, Definition 3.1, Definition 3.2]) Let M be a commutative $\mathcal{K}$-space. Let $B^{\mathrm{cy}}_\bullet(M)$ be the cyclic commutative $\mathcal{K}$-space given by $[n] \mapsto M^{\boxtimes n+1}$. The face maps $d_i\colon M^{\boxtimes n+1} \to M^{\boxtimes n}$ for $0 \le i \le n-1$, multiply adjacent copies of M by using the multiplication map $\mu^M\colon M \boxtimes M \to M$. The face map $d_n\colon M^{\boxtimes n+1} \to M^{\boxtimes n}$ employs the symmetry isomorphism for the symmetric monoidal product $\boxtimes$ and the multiplication map μ^M,

$$\begin{tikzcd} M^{\boxtimes n} \boxtimes M \arrow[r, "\tau_{M^{\boxtimes n}, M}"] \arrow[rr, bend left, "d_n"] & M \boxtimes M^{\boxtimes n} \arrow[r, "\mu^M \boxtimes \mathrm{id}_{M^{\boxtimes n-1}}"] & M^{\boxtimes n}. \end{tikzcd}$$

The degeneracy maps $s_j\colon M^{\boxtimes n+1} \to M^{\boxtimes n+2}$ for $0 \le j \le n$, insert copies of M along the unit map $U^{\mathcal{K}} \to M$. The cyclic operator $t_n\colon M^{\boxtimes n+1} \to M^{\boxtimes n+1}$ is specified by the symmetry isomorphism for the symmetric monoidal product $\boxtimes$. The *cyclic bar construction* $B^{\mathrm{cy}}(M)$ is defined as the realization of $B^{\mathrm{cy}}_\bullet(M)$. The iterated multiplication maps of M give rise to a natural augmentation map $(\epsilon_M)_\bullet\colon B^{\mathrm{cy}}_\bullet(M) \to \mathrm{const}_\Lambda M$ whose realization is $\epsilon_M\colon B^{\mathrm{cy}}(M) \to M$.

A realization functor from simplicial objects in $C\mathcal{S}^{\mathcal{K}}$ to $C\mathcal{S}^{\mathcal{K}}$ is given by applying the diagonal functor from bisimplicial sets to simplicial sets $\mathcal{K}$-levelwise.

Remark 4.2. To define the cyclic bar construction $B^{\mathrm{cy}}(M)$ it actually suffices to assume that M is an associative $\mathcal{K}$-space.

Remark 4.3. The category of commutative $\mathcal{K}$-spaces is tensored over spaces (see [SS12, pp. 2163-2164]). In this way, the cyclic bar construction of M admits a different description, namely $B^{\mathrm{cy}}(M) \cong M \otimes S^1$ in $\mathcal{CS}^{\mathcal{K}}$. The augmentation map ϵ_M is determined by the map $S^1 \to *$ (compare [RSS15, Lemma 3.3]). As the functor $(-) \otimes S^1$ is a left Quillen functor, the commutative $\mathcal{K}$-space $B^{\mathrm{cy}}(M)$ has a well-defined homotopy type if the commutative $\mathcal{K}$-space M is cofibrant in the positive projective $\mathcal{K}$-model structure on $\mathcal{CS}^{\mathcal{K}}$.

Let M be a commutative $\mathcal{K}$-space, and let

$$M \overset{\eta_M}{\underset{\sim}{\rightarrowtail}} M^{\mathrm{gp}} \twoheadrightarrow \mathrm{const}_{\mathcal{K}} * \tag{4.1}$$

be a functorial fibrant replacement of M in the group completion model structure $(\mathcal{CS}^{\mathcal{K}})_{\mathrm{loc}(S^{\mathcal{K}})}$ (see Theorem 3.37).

Construction 4.4. (compare [RSS15, Construction 3.11, Definition 3.12]) We factor the map η_M in (4.1) into an acyclic cofibration followed by a fibration in the positive projective $\mathcal{K}$-model structure on $\mathcal{CS}^{\mathcal{K}}$

$$\begin{tikzcd} M \arrow[r, tail, "\sim"] \arrow[rr, bend left, "\eta_M"] & \tilde{M} \arrow[r, two heads, "q"] & M^{\mathrm{gp}}. \end{tikzcd}$$

Let $B^{\mathrm{rep}}_\bullet(M)$ be the cyclic commutative $\mathcal{K}$-space that is defined by the pullback diagram

$$\begin{tikzcd} B^{\mathrm{rep}}_\bullet(M) \arrow[r] \arrow[d] & B^{\mathrm{cy}}_\bullet(M^{\mathrm{gp}}) \arrow[d, "(\epsilon_{M^{\mathrm{gp}}})_\bullet"] \\ \mathrm{const}_\Lambda \tilde{M} \arrow[r, "\mathrm{const}_\Lambda q"] & \mathrm{const}_\Lambda M^{\mathrm{gp}} \end{tikzcd} \tag{4.2}$$

in cyclic commutative $\mathcal{K}$-spaces. By the universal property of the pullback we obtain a natural map $(\nu_M)_\bullet \colon B^{\mathrm{cy}}_\bullet(M) \to B^{\mathrm{rep}}_\bullet(M)$,

$$\begin{tikzcd} B^{\mathrm{cy}}_\bullet(M) \arrow[r, "(\nu_M)_\bullet"] \arrow[rr, bend left, "B^{\mathrm{cy}}_\bullet(\eta_M)"] \arrow[d, "(\epsilon_M)_\bullet"'] & B^{\mathrm{rep}}_\bullet(M) \arrow[r] \arrow[d] & B^{\mathrm{cy}}_\bullet(M^{\mathrm{gp}}) \arrow[d, "(\epsilon_{M^{\mathrm{gp}}})_\bullet"] \\ \mathrm{const}_\Lambda M \arrow[r] & \mathrm{const}_\Lambda \tilde{M} \arrow[r, "\mathrm{const}_\Lambda q"] & \mathrm{const}_\Lambda M^{\mathrm{gp}}. \end{tikzcd} \tag{4.3}$$

The *replete bar construction* $B^{\mathrm{rep}}(M)$ is defined as the realization of $B^{\mathrm{rep}}_\bullet(M)$. The induced map $\nu_M \colon B^{\mathrm{cy}}(M) \to B^{\mathrm{rep}}(M)$ is called the *repletion map*.

Remark 4.5. Since the model category $\mathcal{CS}^{\mathcal{K}}$ is right proper (see Remark 2.10) and the map $q\colon \tilde{M} \to M^{\mathrm{gp}}$ is a positive $\mathcal{K}$-fibration, the realization of the pullback diagram (4.2) is a homotopy pullback diagram.

Lemma 4.6. *If a map $f\colon M \to N$ is a weak equivalence in the group completion model structure $(\mathcal{CS}^{\mathcal{K}})_{\mathrm{loc}(S^{\mathcal{K}})}$ between grouplike objects, then the map f is a $\mathcal{K}$-equivalence.*

Proof. Let $j\colon\ M \overset{\sim}{\rightarrowtail\!\!\!\rightarrow} M^{\mathrm{fib}}$ and $j'\colon\ N \overset{\sim}{\rightarrowtail\!\!\!\rightarrow} N^{\mathrm{fib}}$ be functorial fibrant replacements in the positive projective $\mathcal{K}$-model structure on $\mathcal{CS}^{\mathcal{K}}$. Since the maps j and j' are weak equivalences in $(\mathcal{CS}^{\mathcal{K}})_{\mathrm{loc}(S^{\mathcal{K}})}$ by [Hir03, Proposition 3.3.3.(1)(a)], it follows from two out of three that the induced map $f^{\mathrm{fib}}\colon M^{\mathrm{fib}} \to N^{\mathrm{fib}}$ is a weak equivalence in $(\mathcal{CS}^{\mathcal{K}})_{\mathrm{loc}(S^{\mathcal{K}})}$. The commutative $\mathcal{K}$-space M^{fib} as well as the commutative $\mathcal{K}$-space N^{fib} is positive $\mathcal{K}$-fibrant and grouplike and hence fibrant in $(\mathcal{CS}^{\mathcal{K}})_{\mathrm{loc}(S^{\mathcal{K}})}$ by Lemma 3.36. Therefore, the map f^{fib} is a $\mathcal{K}$-equivalence. Two out of three implies that the map f is a $\mathcal{K}$-equivalence. $\square$

Corollary 4.7. *(compare [RSS15, Lemma 3.13]) If the commutative $\mathcal{K}$-space M is positive cofibrant and grouplike, the repletion map $\nu_M\colon B^{\mathrm{cy}}(M) \to B^{\mathrm{rep}}(M)$ is a $\mathcal{K}$-equivalence.*

Proof. As the map $\eta_M\colon M \to M^{\mathrm{gp}}$ is a weak equivalence in $(\mathcal{CS}^{\mathcal{K}})_{\mathrm{loc}(S^{\mathcal{K}})}$ between grouplike objects, Lemma 4.6 ensures that the map η_M is a $\mathcal{K}$-equivalence. By two out of three the map $q\colon \tilde{M} \to M^{\mathrm{gp}}$ is a positive acyclic $\mathcal{K}$-fibration. Consequently, the base change map $B^{\mathrm{rep}}(M) \to B^{\mathrm{cy}}(M^{\mathrm{gp}})$ is a positive acyclic $\mathcal{K}$-fibration (see (4.3)). Because the map η_M is a $\mathcal{K}$-equivalence between positive cofibrant objects, the induced map $B^{\mathrm{cy}}(\eta_M)$ is a $\mathcal{K}$-equivalence. Thus, two out of three yields that the repletion map ν_M is a $\mathcal{K}$-equivalence. $\square$

4.2 General repletion

In this subsection we deal with a more general concept of repletion which can be considered as a relative version of group completion. We transfer Rognes, Sagave and Schlichtkrull's results in [RSS15, §3.4] concerning the index category $\mathcal{J}$ to the index category $\mathcal{K}$. We leave out those proofs which translate straightaway and refer to [RSS15] for more details.

Definition 4.8. (compare [RSS15, Definition 3.14]) Let $f\colon N \to M$ be a map of commutative $\mathcal{K}$-spaces. The *repletion* $N^{\mathrm{rep}} \to M$ of N over M is given by factoring the map f in the group completion model structure $(\mathcal{CS}^{\mathcal{K}})_{\mathrm{loc}(S^{\mathcal{K}})}$ as an acyclic cofibration followed by a fibration

$$\begin{tikzcd} N \arrow[r, tail, "\nu_N", "\sim"'] \arrow[rr, bend left, "f"] & N^{\mathrm{rep}} \arrow[r, two heads] & M. \end{tikzcd} \tag{4.4}$$

We call the map $\nu_N\colon N \to N^{\mathrm{rep}}$ the *repletion map.*

Definition 4.9. (compare [RSS15, Definition 3.16]) A map $f\colon N \to M$ of commutative $\mathcal{K}$-spaces is *virtually surjective* if the map f gives rise to a surjective homomorphism of abelian groups

$$\pi_0(\mathrm{hocolim}_{\mathcal{K}} N^{\mathrm{gp}}) \xrightarrow{\pi_0(\mathrm{hocolim}_{\mathcal{K}} f^{\mathrm{gp}})} \pi_0(\mathrm{hocolim}_{\mathcal{K}} M^{\mathrm{gp}}).$$

Example 4.10. Let M be a commutative $\mathcal{K}$-space. The augmentation map

$$B^{\mathrm{cy}}(M) \xrightarrow{\epsilon_M} M$$

has a multiplicative section $M \to B^{\mathrm{cy}}(M)$ which is the realization of the inclusion of the constant cyclic object $\mathrm{const}_\Lambda M$ to $B^{\mathrm{cy}}_\bullet(M)$. Therefore, the map ϵ_M is virtually surjective (see [RSS15, proof of Proposition 3.15]).

Lemma 4.11. *(compare [RSS15, Corollary 3.18]) Let $f\colon N \to M$ be a virtually surjective map of commutative $\mathcal{K}$-spaces. The repletion N^{rep} is $\mathcal{K}$-equivalent to the homotopy pullback of the diagram*

$$N^{\mathrm{gp}} \xrightarrow{f^{\mathrm{gp}}} M^{\mathrm{gp}} \xleftarrow{\eta_M} M$$

in the positive projective $\mathcal{K}$-model structure on $\mathcal{CS}^{\mathcal{K}}$. The repletion map $\nu_N\colon N \to N^{\mathrm{rep}}$ is given as above in (4.4)*.*

Remark 4.12. Assume that the category $\mathcal{K}$ is $\mathcal{I}$, and let $f\colon N \to M$ be a virtually surjective map of commutative $\mathcal{I}$-spaces. In [SS13] the repletion N^{rep} of N over M is actually defined as the homotopy pullback of the diagram

$$N^{\mathrm{gp}} \xrightarrow{f^{\mathrm{gp}}} M^{\mathrm{gp}} \xleftarrow{\eta_M} M$$

in the positive projective $\mathcal{I}$-model structure on $\mathcal{CS}^{\mathcal{I}}$. The maps $\eta_N\colon N \to N^{\mathrm{gp}}$ and $f\colon N \to M$ give rise to the repletion map $\nu_N\colon N \to N^{\mathrm{rep}}$ by the universal property of the pullback (see [SS13, Remark 5.15], compare [Rog09, Definition 8.2]).

Lemma 4.13. *(compare [RSS15, Lemma 3.19]) Let M be a commutative $\mathcal{K}$-space. The commutative $\mathcal{K}$-spaces $B^{\mathrm{cy}}(M^{\mathrm{gp}})$ and $(B^{\mathrm{cy}}(M))^{\mathrm{gp}}$ are $\mathcal{K}$-equivalent as commutative $\mathcal{K}$-spaces under $B^{\mathrm{cy}}(M)$ and over M^{gp}.*

The last two lemmas yield the following proposition, which relates the replete bar construction introduced in the previous subsection to the general repletion defined above.

Proposition 4.14. *(compare [RSS15, Proposition 3.15]) Let M be a commutative $\mathcal{K}$-space. There is a natural chain of $\mathcal{K}$-equivalences under $B^{\mathrm{cy}}(M)$ and over $\tilde{M}$ connecting the replete bar construction $B^{\mathrm{rep}}(M)$ to the repletion $(B^{\mathrm{cy}}(M))^{\mathrm{rep}}$ of the augmentation map $\epsilon_M\colon B^{\mathrm{cy}}(M) \to M$.*

Lemma 4.15. *The homotopy pushout (with respect to the positive projective $\mathcal{K}$-model structure on commutative $\mathcal{K}$-spaces) of grouplike objects is grouplike.*

Proof. Let

$$C \longleftarrow A \longrightarrow B$$

be a diagram of commutative $\mathcal{K}$-spaces where C and B are grouplike. Assume without loss of generality that one of the maps is already a positive cofibration so that there is a pushout diagram

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ C & \longrightarrow & C \boxtimes_A B \end{array}$$

in commutative $\mathcal{K}$-spaces. Recall that in general for a $\mathcal{K}$-space M, every point in $\pi_0(\mathrm{hocolim}_{\mathcal{K}} M)$ is represented by a zero simplex x in $M(\mathbf{k}_0)[0]$ with $\mathbf{k}_0$ in $\mathcal{K}$. There is a composite of simplicial monoids

$$\mathrm{hocolim}_{\mathcal{K}} C \times \mathrm{hocolim}_{\mathcal{K}} B \to \mathrm{hocolim}_{\mathcal{K}}(C \boxtimes B) \to \mathrm{hocolim}_{\mathcal{K}}(C \boxtimes_A B) \tag{4.5}$$

where the first map is the monoidal structure map of the functor $\mathrm{hocolim}_{\mathcal{K}}$ (see (2.44)) and the second map is induced by the quotient map $C \boxtimes B \to C \boxtimes_A B$. The composite (4.5) in simplicial degree zero is given by the composite

$$\begin{aligned}
&\coprod_{\mathbf{k}_0 \in \mathcal{K}} C(\mathbf{k}_0)[0] \times \coprod_{\mathbf{l}_0 \in \mathcal{K}} B(\mathbf{l}_0)[0] \xrightarrow{\cong} \coprod_{(\mathbf{k}_0, \mathbf{l}_0) \in \mathcal{K} \times \mathcal{K}} C(\mathbf{k}_0)[0] \times B(\mathbf{l}_0)[0] \\
&\to \coprod_{(\mathbf{k}_0, \mathbf{l}_0) \in \mathcal{K} \times \mathcal{K}} \mathrm{colim}_{\mathbf{m} \sqcup \mathbf{n} \to \mathbf{k}_0 \sqcup \mathbf{l}_0} C(\mathbf{m})[0] \times B(\mathbf{n})[0] \\
&\to \coprod_{\mathbf{p}_0 \in \mathcal{K}} \mathrm{colim}_{\mathbf{m} \sqcup \mathbf{n} \to \mathbf{p}_0} C(\mathbf{m})[0] \times B(\mathbf{n})[0] \\
&\to \coprod_{\mathbf{p}_0 \in \mathcal{K}} (C \boxtimes_A B)(\mathbf{p}_0)[0].
\end{aligned} \tag{4.6}$$

The last map in the composite (4.6) is the quotient map and hence surjective. Applying the functor π_0 to the composite (4.5) yields the following composite of commutative monoids

$$\pi_0(\mathrm{hocolim}_{\mathcal{K}} C \times \mathrm{hocolim}_{\mathcal{K}} B) \to \pi_0(\mathrm{hocolim}_{\mathcal{K}}(C \boxtimes B)) \to \pi_0(\mathrm{hocolim}_{\mathcal{K}}(C \boxtimes_A B)). \tag{4.7}$$

Assume that there is a point in $\pi_0(\mathrm{hocolim}_{\mathcal{K}}(C \boxtimes_A B))$. This is represented by a pair of zero simplices (x, y) in $C(\mathbf{m})[0] \times B(\mathbf{n})[0]$ indexed by a map $\mathbf{m} \sqcup \mathbf{n} \to \mathbf{p}_0$ in $\mathcal{K}$ with $\mathbf{p}_0$ in $\mathcal{K}$. The preimage in $\pi_0(\mathrm{hocolim}_{\mathcal{K}} C \times \mathrm{hocolim}_{\mathcal{K}} B)$ under the composite (4.7) is just represented by the pair of zero simplices (x, y) in $C(\mathbf{m})[0] \times B(\mathbf{n})[0]$. Thus, the composite (4.7) is surjective. The surjectivity of a map of commutative monoids where the source is a group implies that the target is a group, too. The commutative monoid $\pi_0(\mathrm{hocolim}_{\mathcal{K}} C \times \mathrm{hocolim}_{\mathcal{K}} B)$ being isomorphic to $\pi_0(\mathrm{hocolim}_{\mathcal{K}} C) \times \pi_0(\mathrm{hocolim}_{\mathcal{K}} B)$ is a group. Consequently, the commutative monoid $\pi_0(\mathrm{hocolim}_{\mathcal{K}}(C \boxtimes_A B))$ is a group. $\square$

The remaining results in this subsection are mainly consequences of the theory in Section 3, but play an important role in the upcoming subsection where we prove that *logarithmic topological Hochschild homology* is homotopy invariant under logification. Recall the composite derived functor $I_{\mathcal{K}}^{\mathcal{I}}$ from commutative $\mathcal{K}$-spaces to commutative $\mathcal{I}$-spaces over $B_{\mathcal{I}}\mathcal{K}$ (see Theorem 3.22 and remarks below Remark 3.23). We note that in general, a left adjoint in a Quillen equivalence preserves pointwise cofibrant homotopy cocartesian and pointwise cofibrant homotopy cartesian squares, whereas a right adjoint in a Quillen equivalence respects pointwise fibrant homotopy cocartesian and pointwise fibrant homotopy cartesian squares. In the following we frequently use that the derived composite functor $I_{\mathcal{K}}^{\mathcal{I}}$ preserves homotopy cocartesian as well as homotopy cartesian squares.

Lemma 4.16. *(i) Let the map $f\colon M \to N$ be a group completion in commutative $\mathcal{K}$-spaces. The induced map $I^{\mathcal{I}}_{\mathcal{K}}(f)\colon I^{\mathcal{I}}_{\mathcal{K}}(M) \to I^{\mathcal{I}}_{\mathcal{K}}(N)$ is a group completion in commutative $\mathcal{I}$-spaces, when forgetting the augmentation to $B_{\mathcal{I}}\mathcal{K}$.*

(ii) Let $f\colon N \to M$ be a virtually surjective map of commutative $\mathcal{K}$-spaces. The induced map $I^{\mathcal{I}}_{\mathcal{K}}(f)\colon I^{\mathcal{I}}_{\mathcal{K}}(N) \to I^{\mathcal{I}}_{\mathcal{K}}(M)$ is a virtually surjective map in commutative $\mathcal{I}$-spaces, when forgetting the augmentation to $B_{\mathcal{I}}\mathcal{K}$.

Proof. (i) From Proposition 3.24 we know that the induced map $I^{\mathcal{I}}_{\mathcal{K}}(f)$ is a weak equivalence if and only if the map $I^{\mathcal{I}}(\mathrm{hocolim}_{\mathcal{K}} f)\colon I^{\mathcal{I}}(\mathrm{hocolim}_{\mathcal{K}} M) \to I^{\mathcal{I}}(\mathrm{hocolim}_{\mathcal{K}} N)$ is an $\mathcal{I}$-equivalence. Due to $\pi_0(\mathrm{hocolim}_{\mathcal{I}} I^{\mathcal{I}}(\mathrm{hocolim}_{\mathcal{K}} N)) \cong \pi_0(\mathrm{hocolim}_{\mathcal{K}} N)$ being a group, the target of the map $I^{\mathcal{I}}_{\mathcal{K}}(f)$ is grouplike. In addition, the induced map of bar constructions

$$B(\mathrm{hocolim}_{\mathcal{I}} I^{\mathcal{I}}(\mathrm{hocolim}_{\mathcal{K}} M)) \xrightarrow{B(\mathrm{hocolim}_{\mathcal{I}} I^{\mathcal{I}}(\mathrm{hocolim}_{\mathcal{K}} f))} B(\mathrm{hocolim}_{\mathcal{I}} I^{\mathcal{I}}(\mathrm{hocolim}_{\mathcal{K}} N))$$

is a weak equivalence in spaces, because the map

$$B(\mathrm{hocolim}_{\mathcal{K}} M) \xrightarrow{B(\mathrm{hocolim}_{\mathcal{K}} f)} B(\mathrm{hocolim}_{\mathcal{K}} N)$$

is so.

(ii) From part (i) we get that the map $I^{\mathcal{I}}_{\mathcal{K}}(f)$ is virtually surjective if the induced map

$$\begin{aligned}&\pi_0(\mathrm{hocolim}_{\mathcal{I}} I^{\mathcal{I}}(\mathrm{hocolim}_{\mathcal{K}} N^{\mathrm{gp}}))\\ &\xrightarrow{\pi_0(\mathrm{hocolim}_{\mathcal{I}} I^{\mathcal{I}}(\mathrm{hocolim}_{\mathcal{K}} f^{\mathrm{gp}}))} \pi_0(\mathrm{hocolim}_{\mathcal{I}} I^{\mathcal{I}}(\mathrm{hocolim}_{\mathcal{K}} M^{\mathrm{gp}}))\end{aligned} \tag{4.8}$$

is a surjective homomorphism of abelian groups. But this holds because the map (4.8) is isomorphic to the map $\pi_0(\mathrm{hocolim}_{\mathcal{K}} f^{\mathrm{gp}})$.

□

In the following lemma we use the characterization of repletion from Lemma 4.11, and Remark 4.12 respectively.

Lemma 4.17. *Let $f\colon N \to M$ be a virtually surjective map of commutative $\mathcal{K}$-spaces. Let N^{rep} be the repletion of N over M with repletion map $\nu_N\colon N \to N^{\mathrm{rep}}$. The commutative $\mathcal{I}$-space $I^{\mathcal{I}}_{\mathcal{K}}(N^{\mathrm{rep}})$ provides a repletion of $I^{\mathcal{I}}_{\mathcal{K}}(N)$ over $I^{\mathcal{I}}_{\mathcal{K}}(M)$ with repletion map $I^{\mathcal{I}}_{\mathcal{K}}(\nu_N)\colon I^{\mathcal{I}}_{\mathcal{K}}(N) \to I^{\mathcal{I}}_{\mathcal{K}}(N^{\mathrm{rep}})$, when forgetting the augmentation to $B_{\mathcal{I}}\mathcal{K}$.*

Proof. By Lemma 4.11 the repletion N^{rep} is $\mathcal{K}$-equivalent to the homotopy pullback of the diagram

$$N^{\mathrm{gp}} \xrightarrow{f^{\mathrm{gp}}} M^{\mathrm{gp}} \xleftarrow{\eta_M} M$$

in the positive projective $\mathcal{K}$-model structure on $\mathcal{CS}^{\mathcal{K}}$. As the derived composite functor $I^{\mathcal{I}}_{\mathcal{K}}$ preserves homotopy cartesian squares, the commutative $\mathcal{I}$-space $I^{\mathcal{I}}_{\mathcal{K}}(N^{\mathrm{rep}})$ is $\mathcal{I}$-equivalent to the homotopy pullback of the diagram

$$I^{\mathcal{I}}_{\mathcal{K}}(N^{\mathrm{gp}}) \xrightarrow{I^{\mathcal{I}}_{\mathcal{K}}(f^{\mathrm{gp}})} I^{\mathcal{I}}_{\mathcal{K}}(M^{\mathrm{gp}}) \xleftarrow{I^{\mathcal{I}}_{\mathcal{K}}(\eta_M)} I^{\mathcal{I}}_{\mathcal{K}}(M)$$

in the positive projective $\mathcal{I}$-model structure on $\mathcal{CS}^{\mathcal{I}}$. It follows from Lemma 4.16(i) that the induced maps $I_{\mathcal{K}}^{\mathcal{I}}(\eta_M)\colon I_{\mathcal{K}}^{\mathcal{I}}(M) \to I_{\mathcal{K}}^{\mathcal{I}}(M^{\mathrm{gp}})$ and $I_{\mathcal{K}}^{\mathcal{I}}(\eta_N)\colon I_{\mathcal{K}}^{\mathcal{I}}(N) \to I_{\mathcal{K}}^{\mathcal{I}}(N^{\mathrm{gp}})$ are group completions, and from Lemma 4.16(ii) that the map $I_{\mathcal{K}}^{\mathcal{I}}(f)\colon I_{\mathcal{K}}^{\mathcal{I}}(N) \to I_{\mathcal{K}}^{\mathcal{I}}(M)$ is virtually surjective. Therefore, the repletion of $I_{\mathcal{K}}^{\mathcal{I}}(N)$ over $I_{\mathcal{K}}^{\mathcal{I}}(M)$ is $\mathcal{I}$-equivalent to $I_{\mathcal{K}}^{\mathcal{I}}(N^{\mathrm{rep}})$, and the repletion map is specified by $I_{\mathcal{K}}^{\mathcal{I}}(\nu_N)\colon I_{\mathcal{K}}^{\mathcal{I}}(N) \to I_{\mathcal{K}}^{\mathcal{I}}(N^{\mathrm{rep}})$. □

Proposition 4.18. *Let*

$$\begin{tikzcd}
& N_2 \arrow[rr] \arrow[dd] & & N_2^{\mathrm{rep}} \arrow[rr] \arrow[dd] & & M_2 \arrow[dd] \\
N_1 \arrow[ur] \arrow[rr] \arrow[dd] & & N_1^{\mathrm{rep}} \arrow[ur] \arrow[rr] \arrow[dd] & & M_1 \arrow[ur] \arrow[dd] & \\
& N_4 \arrow[rr] & & N_4^{\mathrm{rep}} \arrow[rr] & & M_4 \\
N_3 \arrow[ur] \arrow[rr] & & N_3^{\mathrm{rep}} \arrow[ur] \arrow[rr] & & M_3 \arrow[ur] &
\end{tikzcd} \tag{4.9}$$

be a diagram of commutative $\mathcal{K}$-spaces in which N_i^{rep} is the repletion of the horizontal composite map $f_i\colon N_i \to M_i$ for $i = 1, \dots, 4$. If the left and right hand faces are homotopy cocartesian and the map f_i is virtually surjective for $i = 1, \dots, 4$, the middle square of repletions is homotopy cocartesian.

Proof. We apply the derived composite functor $I_{\mathcal{K}}^{\mathcal{I}}$ to the diagram (4.9) to obtain the following diagram of positive cofibrant commutative $\mathcal{I}$-spaces where we forget the augmentation to $B_{\mathcal{I}}\mathcal{K}$,

$$\begin{tikzcd}
& I_{\mathcal{K}}^{\mathcal{I}}(N_2) \arrow[rr] \arrow[dd] & & I_{\mathcal{K}}^{\mathcal{I}}(N_2^{\mathrm{rep}}) \arrow[rr] \arrow[dd] & & I_{\mathcal{K}}^{\mathcal{I}}(M_2) \arrow[dd] \\
I_{\mathcal{K}}^{\mathcal{I}}(N_1) \arrow[ur] \arrow[rr] \arrow[dd] & & I_{\mathcal{K}}^{\mathcal{I}}(N_1^{\mathrm{rep}}) \arrow[ur] \arrow[rr] \arrow[dd] & & I_{\mathcal{K}}^{\mathcal{I}}(M_1) \arrow[ur] \arrow[dd] & \\
& I_{\mathcal{K}}^{\mathcal{I}}(N_4) \arrow[rr] & & I_{\mathcal{K}}^{\mathcal{I}}(N_4^{\mathrm{rep}}) \arrow[rr] & & I_{\mathcal{K}}^{\mathcal{I}}(M_4). \\
I_{\mathcal{K}}^{\mathcal{I}}(N_3) \arrow[ur] \arrow[rr] & & I_{\mathcal{K}}^{\mathcal{I}}(N_3^{\mathrm{rep}}) \arrow[ur] \arrow[rr] & & I_{\mathcal{K}}^{\mathcal{I}}(M_3) \arrow[ur] &
\end{tikzcd} \tag{4.10}$$

From Lemma 4.16(ii) we get that the induced map $I_{\mathcal{K}}^{\mathcal{I}}(f_i)$ is virtually surjective for $i = 1, \dots, 4$. Besides Lemma 4.17 implies that the commutative $\mathcal{I}$-space $I_{\mathcal{K}}^{\mathcal{I}}(N_i^{\mathrm{rep}})$ is $\mathcal{I}$-equivalent to the repletion of the map $I_{\mathcal{K}}^{\mathcal{I}}(f_i)$ for $i = 1, \dots, 4$. Considering that the derived composite functor $I_{\mathcal{K}}^{\mathcal{I}}$ respects homotopy cocartesian squares, the left and right hand faces of (4.10) are homotopy cocartesian. Given this, we can proceed precisely as in the proof of [RSS15, Lemma 4.26] replacing the category $\mathcal{J}$ by the category $\mathcal{I}$. Note that the statement made at the beginning of the proof of [RSS15, Lemma 4.26] is proved by Lemma 4.15. □

Remark 4.19. In the proof of [RSS15, Lemma 4.26] Rognes, Sagave and Schlichtkrull exploit the fact that the monoidal structure map of the functor $\mathrm{hocolim}_{\mathcal{J}}$ is a weak equivalence for positive cofibrant $\mathcal{J}$-spaces. As we do not know whether this holds in general (e.g. for the category $\bar{\mathcal{J}}$ see Remark 3.38), we make a detour by using the Quillen equivalence between commutative $\mathcal{K}$-spaces and commutative $\mathcal{I}$-spaces over $B_{\mathcal{I}}\mathcal{K}$ (see Theorem 3.22). The monoidal structure map of the functor $\mathrm{hocolim}_{\mathcal{I}}$ is a weak equivalence for positive cofibrant commutative $\mathcal{I}$-spaces by [SS13, Lemma 2.25, Proposition 3.2].

4.3 Logarithmic topological Hochschild homology

We define *logarithmic topological Hochschild homology* of pre-log cdgas and show that this is homotopy invariant under logification. Our approach is similar to the one in the topological context by Rognes, Sagave and Schlichtkrull in [RSS15, §4].

Let A be an object in $C(\mathrm{Sp}^\Sigma(s\mathrm{mod}))$. We specify a cyclic object $B^{\mathrm{cy}}_\bullet(A)$ in $C(\mathrm{Sp}^\Sigma(s\mathrm{mod}))$ by $B^{\mathrm{cy}}_n(A) = A^{\wedge n+1}$, with structure maps induced by the multiplication and unit map of A and the symmetry isomorphism for the symmetric monoidal smash product $\wedge$.

Definition 4.20. (compare [Shi00, Definition 4.1.2], [RSS15, Definition 3.5]) Let A be a positive cofibrant object in $C(\mathrm{Sp}^\Sigma(s\mathrm{mod}))$. We define the *topological Hochschild homology* $\mathrm{THH}^{\mathrm{Sym}(\tilde{k}(S^1))}(A)$ as the realization of the cyclic object $\mathrm{THH}^{\mathrm{Sym}(\tilde{k}(S^1))}_\bullet(A) = B^{\mathrm{cy}}_\bullet(A)$.

A realization functor from simplicial objects in $C(\mathrm{Sp}^\Sigma(s\mathrm{mod}))$ to $C(\mathrm{Sp}^\Sigma(s\mathrm{mod}))$ is given by applying the diagonal functor from bisimplicial k-modules to simplicial k-modules in each spectrum level. Because of the cofibrancy assumption, the definition of topological Hochschild homology is homotopy invariant.

Remark 4.21. Recall the Quillen equivalence

$$C(Hk\text{-mod}) \underset{U}{\overset{\tilde{k}(-)\wedge_{\tilde{k}(Hk)}\mathrm{Sym}(\tilde{k}(S^1))}{\rightleftarrows}} C(\mathrm{Sp}^\Sigma(s\mathrm{mod}))$$

(see (2.8)) and that the left adjoint $\tilde{k}(-) \wedge_{\tilde{k}(Hk)} \mathrm{Sym}(\tilde{k}(S^1))$ is strong symmetric monoidal. For a positive cofibrant object A in $C(Hk\text{-mod})$, the topological Hochschild homology $\mathrm{THH}^{Hk}(A)$ in $C(Hk\text{-mod})$ corresponds to $\mathrm{THH}^{\mathrm{Sym}(\tilde{k}(S^1))}(\tilde{k}(A) \wedge_{\tilde{k}(Hk)} \mathrm{Sym}(\tilde{k}(S^1)))$ in $C(\mathrm{Sp}^\Sigma(s\mathrm{mod}))$. Moreover, considering that (commutative) Hk-algebra spectra are Quillen equivalent to (E_∞) dgas (see Theorem 2.15, Theorem 2.16), the functor THH^{Hk} can be identified with derived Hochschild homology HH^k which is known as Shukla homology (see [DS07, §4.4]). For instance, let A be a commutative k-algebra and let $(HA)^{\mathrm{cof}} \overset{\sim}{\twoheadrightarrow} HA$ be a cofibrant replacement of HA in $C(Hk\text{-mod})$. If A is a commutative k-flat k-algebra, the underlying Hk-module spectrum of HA is flat and there are the following isomorphisms of commutative k-algebras

$$\begin{aligned} \mathrm{HH}^k_*(A) &\cong \mathrm{THH}^{Hk}_*(HA) \\ &\cong \mathrm{THH}^{Hk}_*((HA)^{\mathrm{cof}}) \\ &\cong \mathrm{THH}^{\mathrm{Sym}(\tilde{k}(S^1))}_*(\tilde{k}((HA)^{\mathrm{cof}}) \wedge_{\tilde{k}(Hk)} \mathrm{Sym}(\tilde{k}(S^1))). \end{aligned} \tag{4.11}$$

For the first isomorphism in (4.11) see also [EKMM97, Theorem IX.1.7].

Recall the functor $\Lambda^{\bar{\mathcal{J}}}$ from Definition 2.31.

Lemma 4.22. *Let M be a cofibrant object in the positive projective $\bar{\mathcal{J}}$-model structure on $C\mathcal{S}^{\bar{\mathcal{J}}}$. There is a natural isomorphism*

$$\mathrm{THH}^{\mathrm{Sym}(\tilde{k}(S^1))}(\Lambda^{\bar{\mathcal{J}}}(M)) \cong \Lambda^{\bar{\mathcal{J}}}(B^{\mathrm{cy}}(M)).$$

Proof. We denote the realization functor by $|-|$. As a left adjoint, the functor $\Lambda^{\bar{\mathcal{J}}}$ commutes with coproducts. Thus, we obtain that

$$\begin{aligned}\mathrm{THH}^{\mathrm{Sym}(\tilde{k}(S^1))}(\Lambda^{\bar{\mathcal{J}}}(M)) &= |\{[n] \mapsto (\Lambda^{\bar{\mathcal{J}}}(M))^{\wedge n+1}\}| \\ &\cong |\{[n] \mapsto \Lambda^{\bar{\mathcal{J}}}(M^{\boxtimes n+1})\}|,\end{aligned}$$

which is isomorphic to $\Lambda^{\bar{\mathcal{J}}}(B^{\mathrm{cy}}(M))$. □

Definition 4.23. (compare [RSS15, Definition 4.6]) Let (A, M, α) be a cofibrant pre-log cdga. We define the *logarithmic topological Hochschild homology* $\mathrm{THH}(A, M, \alpha)$ via the pushout diagram

$$\begin{tikzcd}
\mathrm{THH}^{\mathrm{Sym}(\tilde{k}(S^1))}(\Lambda^{\bar{\mathcal{J}}}(M)) \arrow[r, "\cong"] \arrow[d, tail, "\mathrm{THH}^{\mathrm{Sym}(\tilde{k}(S^1))}(\mathrm{ad}(\alpha))"'] & \Lambda^{\bar{\mathcal{J}}}(B^{\mathrm{cy}}(M)) \arrow[r, "\Lambda^{\bar{\mathcal{J}}}(\nu_{B^{\mathrm{cy}}(M)})"] & \Lambda^{\bar{\mathcal{J}}}((B^{\mathrm{cy}}(M))^{\mathrm{rep}}) \arrow[d] \\
\mathrm{THH}^{\mathrm{Sym}(\tilde{k}(S^1))}(A) \arrow[rr] & & \mathrm{THH}(A, M, \alpha).
\end{tikzcd} \tag{4.12}$$

Remark 4.24. The first upper horizontal map in the pushout diagram (4.12) is given by the isomorphism from Lemma 4.22. The cofibrancy assumption on (A, M, α) requires the commutative $\bar{\mathcal{J}}$-space M to be positive cofibrant and the adjoint structure map $\mathrm{ad}(\alpha)\colon \Lambda^{\bar{\mathcal{J}}}(M) \to A$ to be a positive cofibration (see Remark 2.33). Therefore, the pushout square (4.12) is in fact a homotopy pushout square, and the definition of logTHH is homotopy invariant.
Rognes, Sagave and Schlichtkrull use the replete bar construction $B^{\mathrm{rep}}(M)$ instead of the repletion $(B^{\mathrm{cy}}(M))^{\mathrm{rep}}$ to define logarithmic topological Hochschild homology of a pre-log ring spectrum [RSS15, Definition 4.6]. But the object $B^{\mathrm{rep}}(M)$ is in general not positive cofibrant. Nevertheless, Rognes, Sagave and Schlichtkrull show that their definition is homotopy invariant [RSS15, Proposition 4.9] by introducing the criterion of $\mathbb{S}^{\mathcal{J}}$*-goodness* (see [RSS15, §8]). To check $\mathbb{S}^{\mathcal{J}}$-goodness, Rognes, Sagave and Schlichtkrull employ the explicit decription of the functor $\mathbb{S}^{\mathcal{J}}$ ([SS12, (4.5)], see proof of [RSS15, Lemma 8.4]). Contrary to this, although we can define $\Lambda^{\bar{\mathcal{J}}}$*-goodness* analogously, we do not know how to verify the latter due to the abstract definition of the functor $\Lambda^{\bar{\mathcal{J}}}$. The advantage of Rognes, Sagave and Schlichtkrull's definition is that the logarithmic topological Hochschild homology of a pre-log ring spectrum is isomorphic to the realization of a cyclic commutative symmetric ring spectrum (see [RSS15, remarks after Definition 4.6]).

Next we show that logarithmic topological Hochschild homology is homotopy invariant under logification. The proof resembles the proof of [RSS15, Theorem 4.24]. Besides, it is an application of the results from Section 3 and the previous two subsections.

Proposition 4.25. *Let A be a positive fibrant commutative symmetric ring spectrum in simplicial k-modules, and let (A, M, α) be a cofibrant pre-log cdga. The logification map $(A, M, \alpha) \to (A^a, M^a, \alpha^a)$ gives rise to a stable equivalence*

$$\mathrm{THH}(A, M, \alpha) \to \mathrm{THH}(A^a, M^a, \alpha^a).$$

Proof. (compare [RSS15, proof of Theorem 4.24]) We need to verify that the map of pushouts

$$\begin{aligned} &\mathrm{THH}^{\mathrm{Sym}(\tilde{k}(S^1))}(A) \wedge_{\Lambda^{\bar{\mathcal{J}}}(B^{\mathrm{cy}}(M))} \Lambda^{\bar{\mathcal{J}}}((B^{\mathrm{cy}}(M))^{\mathrm{rep}}) \\ \to\ &\mathrm{THH}^{\mathrm{Sym}(\tilde{k}(S^1))}(A^a) \wedge_{\Lambda^{\bar{\mathcal{J}}}(B^{\mathrm{cy}}(M^a))} \Lambda^{\bar{\mathcal{J}}}((B^{\mathrm{cy}}(M^a))^{\mathrm{rep}}) \end{aligned}$$

is a stable equivalence.

Let $P \overset{\sim}{\twoheadrightarrow} \alpha^{-1}(\mathrm{GL}_1^{\bar{\mathcal{J}}}(A))$ be a cofibrant replacement of $\alpha^{-1}(\mathrm{GL}_1^{\bar{\mathcal{J}}}(A))$ in the positive projective $\bar{\mathcal{J}}$-model structure on commutative $\bar{\mathcal{J}}$-spaces. We investigate the following pushout square

$$\begin{tikzcd} B^{\mathrm{cy}}(P) \arrow[rr, "B^{\mathrm{cy}}(\eta_P)"] \arrow[d, tail, "\nu_{B^{\mathrm{cy}}(P)}"'] & & B^{\mathrm{cy}}(P^{\mathrm{gp}}) \arrow[d, "\tau"] \\ (B^{\mathrm{cy}}(P))^{\mathrm{rep}} \arrow[r] & (B^{\mathrm{cy}}(P))^{\mathrm{rep}} \boxtimes_{B^{\mathrm{cy}}(P)} B^{\mathrm{cy}}(P^{\mathrm{gp}}). & \end{tikzcd}$$

As the map $\nu_{B^{\mathrm{cy}}(P)}$ is an acyclic cofibration in the group completion model structure $(\mathcal{CS}^{\bar{\mathcal{J}}})_{\mathrm{loc}(S^{\bar{\mathcal{J}}})}$, so is the cobase change map τ. The source $B^{\mathrm{cy}}(P^{\mathrm{gp}})$ of the map τ is grouplike because of Lemma 4.13. Due to the induced map $\pi_0(\mathrm{hocolim}_{\bar{\mathcal{J}}} \nu_{B^{\mathrm{cy}}(P)})$ being surjective, the target of the map τ is grouplike as well. By Lemma 4.6 the map τ is a $\bar{\mathcal{J}}$-equivalence. There is a composite map

$$P \overset{\sim}{\twoheadrightarrow} \alpha^{-1}(\mathrm{GL}_1^{\bar{\mathcal{J}}}(A)) \longrightarrow \mathrm{GL}_1^{\bar{\mathcal{J}}}(A) \xrightarrow{i_A} \Omega^{\bar{\mathcal{J}}}(A), \tag{4.13}$$

which factors through $\mathrm{GL}_1^{\bar{\mathcal{J}}}(A)$. Hence, the composite map (4.13) extends over the map $\eta_P \colon P \to P^{\mathrm{gp}}$. Let the map $\Lambda^{\bar{\mathcal{J}}}(P) \to A$ be the adjoint of the composite map (4.13). The induced map

$$\Lambda^{\bar{\mathcal{J}}}(B^{\mathrm{cy}}(P)) \xrightarrow{\cong} \mathrm{THH}^{\mathrm{Sym}(\tilde{k}(S^1))}(\Lambda^{\bar{\mathcal{J}}}(P)) \to \mathrm{THH}^{\mathrm{Sym}(\tilde{k}(S^1))}(A)$$

factors as a composite

$$\Lambda^{\bar{\mathcal{J}}}(B^{\mathrm{cy}}(P)) \xrightarrow{\Lambda^{\bar{\mathcal{J}}}(B^{\mathrm{cy}}(\eta_P))} \Lambda^{\bar{\mathcal{J}}}(B^{\mathrm{cy}}(P^{\mathrm{gp}})) \to \mathrm{THH}^{\mathrm{Sym}(\tilde{k}(S^1))}(A).$$

We observe that the pushout of the diagram

$$\mathrm{THH}^{\mathrm{Sym}(\tilde{k}(S^1))}(A) \longleftarrow \Lambda^{\bar{\mathcal{J}}}(B^{\mathrm{cy}}(P)) \overset{\Lambda^{\bar{\mathcal{J}}}(\nu_{B^{\mathrm{cy}}(P)})}{\rightarrowtail} \Lambda^{\bar{\mathcal{J}}}((B^{\mathrm{cy}}(P))^{\mathrm{rep}})$$

is isomorphic to the pushout of the diagram

$$\mathrm{THH}^{\mathrm{Sym}(\tilde{k}(S^1))}(A) \longleftarrow \Lambda^{\bar{\mathcal{J}}}(B^{\mathrm{cy}}(P^{\mathrm{gp}})) \overset{\Lambda^{\bar{\mathcal{J}}}(\tau)}{\rightarrowtail} \Lambda^{\bar{\mathcal{J}}}((B^{\mathrm{cy}}(P))^{\mathrm{rep}} \boxtimes_{B^{\mathrm{cy}}(P)} B^{\mathrm{cy}}(P^{\mathrm{gp}})).$$

Since the map τ is a positive acyclic cofibration, the induced map $\Lambda^{\bar{\mathcal{J}}}(\tau)$ is a positive acyclic cofibration. Therefore, the cobase change map

$$\mathrm{THH}^{\mathrm{Sym}(\tilde{k}(S^1))}(A) \to \mathrm{THH}^{\mathrm{Sym}(\tilde{k}(S^1))}(A) \wedge_{\Lambda^{\bar{\mathcal{J}}}(B^{\mathrm{cy}}(P))} \Lambda^{\bar{\mathcal{J}}}((B^{\mathrm{cy}}(P))^{\mathrm{rep}}) \tag{4.14}$$

is a positive acyclic cofibration.
Further, we factor the composite

$$P \overset{\sim}{\twoheadrightarrow} \alpha^{-1}(\mathrm{GL}_1^{\bar{\mathcal{J}}}(A)) \longrightarrow \mathrm{GL}_1^{\bar{\mathcal{J}}}(A)$$

into a cofibration followed by an acyclic fibration in the positive projective $\bar{\mathcal{J}}$-model structure on commutative $\bar{\mathcal{J}}$-spaces

$$P \rightarrowtail G \overset{\sim}{\twoheadrightarrow} \mathrm{GL}_1^{\bar{\mathcal{J}}}(A).$$

Because of [Hir03, Proposition 13.5.3], the induced map of pushouts $M \boxtimes_P G \to M^a$ is a $\bar{\mathcal{J}}$-equivalence. We analyze the diagram

$$\begin{tikzcd}
 & \Lambda^{\bar{\mathcal{J}}}(B^{\mathrm{cy}}(G)) \arrow[rr] \arrow[dd, tail] & & \Lambda^{\bar{\mathcal{J}}}(B^{\mathrm{cy}}(M^a)) \arrow[dd, tail] \\
\Lambda^{\bar{\mathcal{J}}}(B^{\mathrm{cy}}(P)) \arrow[ur, tail] \arrow[rr] \arrow[dd, tail] & & \Lambda^{\bar{\mathcal{J}}}(B^{\mathrm{cy}}(M)) \arrow[ur, tail] \arrow[dd, tail] & \\
 & \Lambda^{\bar{\mathcal{J}}}((B^{\mathrm{cy}}(G))^{\mathrm{rep}}) \arrow[rr] \arrow[dd] & & \Lambda^{\bar{\mathcal{J}}}((B^{\mathrm{cy}}(M^a))^{\mathrm{rep}}) \arrow[dd] \\
\Lambda^{\bar{\mathcal{J}}}((B^{\mathrm{cy}}(P))^{\mathrm{rep}}) \arrow[ur] \arrow[rr] \arrow[dd] & & \Lambda^{\bar{\mathcal{J}}}((B^{\mathrm{cy}}(M))^{\mathrm{rep}}) \arrow[ur] \arrow[dd] & \\
 & \Lambda^{\bar{\mathcal{J}}}(G) \arrow[rr] & & \Lambda^{\bar{\mathcal{J}}}(M^a). \\
\Lambda^{\bar{\mathcal{J}}}(P) \arrow[ur, tail] \arrow[rr] & & \Lambda^{\bar{\mathcal{J}}}(M) \arrow[ur, tail] &
\end{tikzcd} \tag{4.15}$$

As the commutative $\bar{\mathcal{J}}$-space G is $S^{\bar{\mathcal{J}}}$-local by Lemma 3.36 and the map $(B^{\mathrm{cy}}(G))^{\mathrm{rep}} \to G$ is a fibration in $(C\mathcal{S}^{\bar{\mathcal{J}}})_{\mathrm{loc}(S^{\bar{\mathcal{J}}})}$, the commutative $\bar{\mathcal{J}}$-space $(B^{\mathrm{cy}}(G))^{\mathrm{rep}}$ is $S^{\bar{\mathcal{J}}}$-local by [Hir03, Proposition 3.3.14.(1)]. In particular, the latter is grouplike due to Lemma 3.36 so that the map $\nu_{B^{\mathrm{cy}}(G)} \colon B^{\mathrm{cy}}(G) \to (B^{\mathrm{cy}}(G))^{\mathrm{rep}}$ is an acyclic cofibration in the positive projective $\bar{\mathcal{J}}$-model structure by Lemma 4.6. Thus, the induced map

$$\Lambda^{\bar{\mathcal{J}}}(B^{\mathrm{cy}}(G)) \xrightarrow{\Lambda^{\bar{\mathcal{J}}}(\nu_{B^{\mathrm{cy}}(G)})} \Lambda^{\bar{\mathcal{J}}}((B^{\mathrm{cy}}(G))^{\mathrm{rep}}) \tag{4.16}$$

is a positive acyclic cofibration. Let the map $\Lambda^{\bar{\mathcal{J}}}(G) \to A$ be the adjoint of the composite

$$G \xrightarrow{\sim} \mathrm{GL}_1^{\bar{\mathcal{J}}}(A) \xrightarrow{i_A} \Omega^{\bar{\mathcal{J}}}(A),$$

and let $\Lambda^{\bar{\mathcal{J}}}(B^{\mathrm{cy}}(G)) \to \mathrm{THH}^{\mathrm{Sym}(\tilde{k}(S^1))}(A)$ be the induced morphism. We then apply the functor $(-) \wedge_{\Lambda^{\bar{\mathcal{J}}}(B^{\mathrm{cy}}(G))} \mathrm{THH}^{\mathrm{Sym}(\tilde{k}(S^1))}(A)$ to the upper left hand square in (4.15), and consider the map from the pushout

$$\begin{aligned} &(\Lambda^{\bar{\mathcal{J}}}((B^{\mathrm{cy}}(P))^{\mathrm{rep}}) \wedge_{\Lambda^{\bar{\mathcal{J}}}(B^{\mathrm{cy}}(P))} \Lambda^{\bar{\mathcal{J}}}(B^{\mathrm{cy}}(G))) \wedge_{\Lambda^{\bar{\mathcal{J}}}(B^{\mathrm{cy}}(G))} \mathrm{THH}^{\mathrm{Sym}(\tilde{k}(S^1))}(A) \\ \cong\ & \Lambda^{\bar{\mathcal{J}}}((B^{\mathrm{cy}}(P))^{\mathrm{rep}}) \wedge_{\Lambda^{\bar{\mathcal{J}}}(B^{\mathrm{cy}}(P))} \mathrm{THH}^{\mathrm{Sym}(\tilde{k}(S^1))}(A) \end{aligned}$$

to $\Lambda^{\bar{\mathcal{J}}}((B^{\mathrm{cy}}(G))^{\mathrm{rep}}) \wedge_{\Lambda^{\bar{\mathcal{J}}}(B^{\mathrm{cy}}(G))} \mathrm{THH}^{\mathrm{Sym}(\tilde{k}(S^1))}(A)$. This is a stable equivalence, because the map (4.14) is a stable equivalence and the map (4.16) is a positive acyclic cofibration. Moreover, in view of the functors $\Lambda^{\bar{\mathcal{J}}}$ and $B^{\mathrm{cy}} \cong (-) \otimes S^1$ being left Quillen, the top and the bottom face of the diagram (4.15) are homotopy cocartesian. The middle square in (4.15) is homotopy cocartesian because of Proposition 4.18 and the functor $\Lambda^{\bar{\mathcal{J}}}$ being left Quillen. Consequently, the square

$$\begin{array}{ccc} \mathrm{THH}(A) \wedge_{\Lambda^{\bar{\mathcal{J}}}(B^{\mathrm{cy}}(P))} \Lambda^{\bar{\mathcal{J}}}((B^{\mathrm{cy}}(P))^{\mathrm{rep}}) & \longrightarrow & \mathrm{THH}(A) \wedge_{\Lambda^{\bar{\mathcal{J}}}(B^{\mathrm{cy}}(M))} \Lambda^{\bar{\mathcal{J}}}((B^{\mathrm{cy}}(M))^{\mathrm{rep}}) \\ \downarrow & & \downarrow \\ \mathrm{THH}(A) \wedge_{\Lambda^{\bar{\mathcal{J}}}(B^{\mathrm{cy}}(G))} \Lambda^{\bar{\mathcal{J}}}((B^{\mathrm{cy}}(G))^{\mathrm{rep}}) & \longrightarrow & \mathrm{THH}(A^a) \wedge_{\Lambda^{\bar{\mathcal{J}}}(B^{\mathrm{cy}}(M^a))} \Lambda^{\bar{\mathcal{J}}}((B^{\mathrm{cy}}(M^a))^{\mathrm{rep}}) \end{array} \tag{4.17}$$

where we abbreviate the functor $\mathrm{THH}^{\mathrm{Sym}(\tilde{k}(S^1))}$ by THH is homotopy cocartesian. Taking into account that the left vertical map in (4.17) is a stable equivalence, so is the right vertical map, which finishes the proof. □

4.4 Formally log THH-étale morphisms

In the sequel we specify *formally log* THH-*étale* morphisms of pre-log cdgas. We focus on two approaches to find examples. On the one hand, tamely ramified extensions of commutative rings, and on the other hand, formally THH-étale morphisms of commutative Hk-algebra spectra should give rise to formally log THH-étale morphisms of pre-log cdgas.

Definition 4.26. (compare [RSS18, p. 510]) A map of cofibrant pre-log cdgas $(A, M, \alpha) \to (B, N, \beta)$ is *formally log* THH-*étale* if the induced square

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ \mathrm{THH}(A, M, \alpha) & \longrightarrow & \mathrm{THH}(B, N, \beta) \end{array}$$

is homotopy cocartesian in $C(\mathrm{Sp}^{\Sigma}(s\mathrm{mod}))$.

Remark 4.27. Let $(f, f^b)\colon (A, M, \alpha) \to (B, N, \beta)$ be a map of cofibrant pre-log cdgas. We consider the induced diagram

$$\begin{tikzcd}
 & & A \arrow[rr] \arrow[dd] & & B \arrow[dd] \\
\Lambda^{\bar{\mathcal{J}}}(M) \arrow[urr] \arrow[rrr] \arrow[dd] & & & \Lambda^{\bar{\mathcal{J}}}(N) \arrow[ur] \arrow[dd] & \\
 & & \mathrm{THH}^{\mathrm{Sym}(\tilde{k}(S^1))}(A) \arrow[rr] \arrow[dd] & & \mathrm{THH}^{\mathrm{Sym}(\tilde{k}(S^1))}(B) \arrow[dd] \\
\Lambda^{\bar{\mathcal{J}}}(B^{\mathrm{cy}}(M)) \arrow[urr] \arrow[rrr] \arrow[dd] & & & \Lambda^{\bar{\mathcal{J}}}(B^{\mathrm{cy}}(N)) \arrow[ur] \arrow[dd] & \\
 & & \mathrm{THH}(A, M, \alpha) \arrow[rr] & & \mathrm{THH}(B, N, \beta). \\
\Lambda^{\bar{\mathcal{J}}}((B^{\mathrm{cy}}(M))^{\mathrm{rep}}) \arrow[urr] \arrow[rrr] & & & \Lambda^{\bar{\mathcal{J}}}((B^{\mathrm{cy}}(N))^{\mathrm{rep}}) \arrow[ur] &
\end{tikzcd} \tag{4.18}$$

To show that the map (f, f^b) is formally log THH-étale, we need to prove that the back face of the diagram (4.18) is homotopy cocartesian. For this, it suffices to show that the top face and the front face of (4.18) are homotopy cocartesian. To see this, we observe that the lower left hand square and the lower right hand square of (4.18) are homotopy cocartesian by definition. The inner square results from applying the functor $\mathrm{THH}^{\mathrm{Sym}(\tilde{k}(S^1))}$ to the top face and is consequently homotopy cocartesian. Thus, the bottom face of (4.18) is homotopy cocartesian. As the top, front and bottom face of (4.18) are homotopy cocartesian, we can conclude that the back face of (4.18) is homotopy cocartesian (compare [RSS18, proof of Theorem 6.3]).
In general it is not sufficient that the upper back square of the diagram (4.18) is homotopy cocartesian to deduce that the map (f, f^b) is formally log THH-étale.

In the upcoming Example 4.29, we investigate a map of pre-log cdgas induced by a tamely ramified extension of polynomial rings. We make use of the notions cyclic bar construction, group completion, virtually surjective morphism and repletion of commutative monoids as defined in [Rog09, §3]. Moreover, the following lemma is helpful.

Lemma 4.28. *Let $\langle x \rangle$ denote the (discrete simplicial) free commutative monoid on a generator x.*

(i) The cyclic bar construction $B^{\mathrm{cy}}(\langle x \rangle)$ decomposes as a disjoint union of cyclic sets

$$B^{\mathrm{cy}}(\langle x \rangle) = \coprod_{j \geq 0} B^{\mathrm{cy}}(\langle x \rangle, j) \simeq * \sqcup \coprod_{j \geq 1} S^1(j)$$

where the cyclic subset $B^{\mathrm{cy}}(\langle x \rangle, j) = \epsilon^{-1}_{\langle x \rangle}(x^j)$ consists of the simplices $(m_0, \ldots, m_q)$ with $m_0 \cdots m_q = x^j$. The cyclic subset $B^{\mathrm{cy}}(\langle x \rangle, 0)$ is a point, while the cyclic subset $B^{\mathrm{cy}}(\langle x \rangle, j)$ for $j \geq 1$, is S^1-equivariantly homotopy equivalent to S^1 with the degree j action, which we write as $S^1(j)$.

(ii) The group completion $\langle x\rangle^{\mathrm{gp}} = \langle x, x^{-1}\rangle = \{x^j, j \in \mathbb{Z}\}$ *has the cyclic bar construction* $B^{\mathrm{cy}}(\langle x\rangle^{\mathrm{gp}})$*, which decomposes as a disjoint union of cyclic sets*

$$B^{\mathrm{cy}}(\langle x\rangle^{\mathrm{gp}}) = \coprod_{j\in\mathbb{Z}} B^{\mathrm{cy}}(\langle x\rangle^{\mathrm{gp}}, j) \simeq \coprod_{j\in\mathbb{Z}} S^1(j)$$

where $B^{\mathrm{cy}}(\langle x\rangle^{\mathrm{gp}}, j) = \epsilon^{-1}_{\langle x\rangle^{\mathrm{gp}}}(x^j)$*. The cyclic bar construction* $B^{\mathrm{cy}}(\langle x\rangle^{\mathrm{gp}})$ *contains the replete bar construction*

$$B^{\mathrm{rep}}(\langle x\rangle) = \coprod_{j\geq 0} B^{\mathrm{cy}}(\langle x\rangle^{\mathrm{gp}}, j) \simeq \coprod_{j\geq 0} S^1(j)$$

as the non-negatively indexed summands. The repletion map

$$B^{\mathrm{cy}}(\langle x\rangle) \xrightarrow{\nu_{\langle x\rangle}} B^{\mathrm{rep}}(\langle x\rangle)$$

decomposes as the disjoint union of the inclusions $B^{\mathrm{cy}}(\langle x\rangle, j) \to B^{\mathrm{cy}}(\langle x\rangle^{\mathrm{gp}}, j)$ *for* $j \geq 0$*. For* $j \geq 1$*, this inclusion is an* S^1*-equivariant homotopy equivalence, and for* $j = 0$*, this inclusion identifies the source with the* S^1*-fixed points of the target.*

(iii) The commutative symmetric ring spectrum $\Lambda^{\bar{\mathcal{J}}}(F^{\bar{\mathcal{J}}}_{(\mathbf{0},\mathbf{0})}(\langle x\rangle))$ *is naturally isomorphic to* $F_0(k[x])$*.*

(iv) The commutative symmetric ring spectrum $\Lambda^{\bar{\mathcal{J}}}(B^{\mathrm{cy}}(F^{\bar{\mathcal{J}}}_{(\mathbf{0},\mathbf{0})}(\langle x\rangle)))$ *is naturally isomorphic to* $\Lambda^{\bar{\mathcal{J}}}(F^{\bar{\mathcal{J}}}_{(\mathbf{0},\mathbf{0})}(B^{\mathrm{cy}}(\langle x\rangle)))$*.*

(v) The group completion $\eta_{\langle x\rangle}\colon \langle x\rangle \to \langle x\rangle^{\mathrm{gp}}$ *induces the group completions*

$$F^{\bar{\mathcal{J}}}_{(\mathbf{0},\mathbf{0})}(\langle x\rangle) \xrightarrow{F^{\bar{\mathcal{J}}}_{(\mathbf{0},\mathbf{0})}(\eta_{\langle x\rangle})} F^{\bar{\mathcal{J}}}_{(\mathbf{0},\mathbf{0})}(\langle x\rangle^{\mathrm{gp}}) \tag{4.19}$$

and

$$F^{\bar{\mathcal{J}}}_{(\mathbf{0},\mathbf{0})}(B^{\mathrm{cy}}(\langle x\rangle)) \xrightarrow{F^{\bar{\mathcal{J}}}_{(\mathbf{0},\mathbf{0})}(B^{\mathrm{cy}}(\eta_{\langle x\rangle}))} F^{\bar{\mathcal{J}}}_{(\mathbf{0},\mathbf{0})}B^{\mathrm{cy}}(\langle x\rangle^{\mathrm{gp}}) \tag{4.20}$$

in $C\mathcal{S}^{\bar{\mathcal{J}}}$*.*

(vi) The repletion $(B^{\mathrm{cy}}(F^{\bar{\mathcal{J}}}_{(\mathbf{0},\mathbf{0})}(\langle x\rangle)))^{\mathrm{rep}}$ *is* $\bar{\mathcal{J}}$*-equivalent to* $F^{\bar{\mathcal{J}}}_{(\mathbf{0},\mathbf{0})}(B^{\mathrm{rep}}(\langle x\rangle))$*. The repletion map*

$$B^{\mathrm{cy}}(F^{\bar{\mathcal{J}}}_{(\mathbf{0},\mathbf{0})}(\langle x\rangle)) \xrightarrow{\nu_{B^{\mathrm{cy}}(F^{\bar{\mathcal{J}}}_{(\mathbf{0},\mathbf{0})}(\langle x\rangle))}} (B^{\mathrm{cy}}(F^{\bar{\mathcal{J}}}_{(\mathbf{0},\mathbf{0})}(\langle x\rangle)))^{\mathrm{rep}}$$

can be identified with the induced map

$$F^{\bar{\mathcal{J}}}_{(\mathbf{0},\mathbf{0})}(B^{\mathrm{cy}}(\langle x\rangle)) \xrightarrow{F^{\bar{\mathcal{J}}}_{(\mathbf{0},\mathbf{0})}(\nu_{\langle x\rangle})} F^{\bar{\mathcal{J}}}_{(\mathbf{0},\mathbf{0})}(B^{\mathrm{rep}}(\langle x\rangle)).$$

Proof. (i) This statement is [Rog09, Proposition 3.20] which follows from the proof of [Hes96, Lemma 2.2.3].

(ii) This statement is [Rog09, Proposition 3.21].

(iii) Let A be an object in $C(\mathrm{Sp}^{\Sigma}(s\mathrm{mod}))$. Let $C\mathcal{S}$ denote the category of commutative simplicial monoids. Exploiting adjunctions, we find the following isomorphisms

$$\begin{aligned}
C(\mathrm{Sp}^{\Sigma}(s\mathrm{mod}))(\Lambda^{\bar{\mathcal{J}}}(F^{\bar{\mathcal{J}}}_{(\mathbf{0},\mathbf{0})}(\langle x\rangle)), A) &\cong C\mathcal{S}^{\bar{\mathcal{J}}}(F^{\bar{\mathcal{J}}}_{(\mathbf{0},\mathbf{0})}(\langle x\rangle), U(\Omega^{\bar{\mathcal{J}}}(A))) \\
&\cong C\mathcal{S}(\langle x\rangle, U(\Omega^{\bar{\mathcal{J}}}(A))(\mathbf{0},\mathbf{0})).
\end{aligned}$$

Using the isomorphisms $L(F_0(\mathbb{S}^0)) \cong F_0(\Gamma(\mathbb{S}^0)) \cong F_0(\mathrm{const}_{\Delta^{\mathrm{op}}} k)$ and adjunctions, we obtain that

$$\begin{aligned}
U(\Omega^{\bar{\mathcal{J}}}(A))(\mathbf{0},\mathbf{0}) &= U(\mathrm{Hom}^{\mathrm{Sp}^{\Sigma}(s\mathrm{mod})}_{s\mathrm{mod}}(L(F_0(\mathbb{S}^0)), A)) \\
&\cong U(\mathrm{Hom}^{\mathrm{Sp}^{\Sigma}(s\mathrm{mod})}_{s\mathrm{mod}}(F_0(\mathrm{const}_{\Delta^{\mathrm{op}}} k), A)) \\
&\cong U(\underline{\mathrm{Hom}}_{s\mathrm{mod}}(k(\Delta_0), A(0))) \\
&\cong \underline{\mathrm{Hom}}_{\mathcal{S}}(\Delta_0, U(A(0))) \\
&\cong U(A(0)).
\end{aligned}$$

Moreover, again by adjunction we get that

$$C(\mathrm{Sp}^{\Sigma}(s\mathrm{mod}))(F_0(k[x]), A) \cong C\mathcal{S}(\langle x\rangle, U(A(0))).$$

Thus, the object $\Lambda^{\bar{\mathcal{J}}}(F^{\bar{\mathcal{J}}}_{(\mathbf{0},\mathbf{0})}(\langle x\rangle))$ can be identified with the object $F_0(k[x])$ in $C(\mathrm{Sp}^{\Sigma}(s\mathrm{mod}))$.

(iv) We obtain that

$$\begin{aligned}
\Lambda^{\bar{\mathcal{J}}}(B^{\mathrm{cy}}(F^{\bar{\mathcal{J}}}_{(\mathbf{0},\mathbf{0})}(\langle x\rangle))) &\cong \Lambda^{\bar{\mathcal{J}}}(F^{\bar{\mathcal{J}}}_{(\mathbf{0},\mathbf{0})}(\langle x\rangle) \otimes S^1) \\
&\cong \Lambda^{\bar{\mathcal{J}}}(F^{\bar{\mathcal{J}}}_{(\mathbf{0},\mathbf{0})}(\langle x\rangle \otimes S^1)) \\
&= \Lambda^{\bar{\mathcal{J}}}(F^{\bar{\mathcal{J}}}_{(\mathbf{0},\mathbf{0})}(B^{\mathrm{cy}}(\langle x\rangle)))
\end{aligned}$$

where we make use of the fact that the left adjoint functor $F^{\bar{\mathcal{J}}}_{(\mathbf{0},\mathbf{0})}$ commutes with tensors. For more details on the definition of $B^{\mathrm{cy}}(\langle x\rangle)$ see [Rog09, Definition 3.16, Proposition 3.20].

(v) The space $\mathrm{hocolim}_{\bar{\mathcal{J}}} F^{\bar{\mathcal{J}}}_{(\mathbf{0},\mathbf{0})}(*)$ is contractible and hence, the commutative monoid

$$\begin{aligned}
\pi_0(\mathrm{hocolim}_{\bar{\mathcal{J}}} F^{\bar{\mathcal{J}}}_{(\mathbf{0},\mathbf{0})}(\langle x\rangle^{\mathrm{gp}})) &\cong \pi_0(\langle x\rangle^{\mathrm{gp}} \times \mathrm{hocolim}_{\bar{\mathcal{J}}} F^{\bar{\mathcal{J}}}_{(\mathbf{0},\mathbf{0})}(*)) \\
&\cong \pi_0(\langle x\rangle^{\mathrm{gp}}) \times \pi_0(\mathrm{hocolim}_{\bar{\mathcal{J}}} F^{\bar{\mathcal{J}}}_{(\mathbf{0},\mathbf{0})}(*)) \\
&\cong \pi_0(\langle x\rangle^{\mathrm{gp}}) \\
&= \langle x\rangle^{\mathrm{gp}}
\end{aligned}$$

is a group. The induced map

$$B(\mathrm{hocolim}_{\bar{\mathcal{J}}} F^{\bar{\mathcal{J}}}_{(\mathbf{0},\mathbf{0})}(\langle x\rangle)) \to B(\mathrm{hocolim}_{\bar{\mathcal{J}}} F^{\bar{\mathcal{J}}}_{(\mathbf{0},\mathbf{0})}(\langle x\rangle^{\mathrm{gp}}))$$

is a weak equivalence, because the map $B(\langle x\rangle) \to B(\langle x\rangle^{\mathrm{gp}})$ is a weak equivalence. Similarly, considering that the simplicial commutative monoid $(B^{\mathrm{cy}}(\langle x\rangle))^{\mathrm{gp}}$ is weakly equivalent to $B^{\mathrm{cy}}(\langle x\rangle^{\mathrm{gp}})$ (see [Rog09, Proposition 3.20, Proposition 3.21], part (i), part (ii)), the map (4.20) specifies a group completion.

(vi) As the augmentation map $\epsilon_{F^{\bar{\mathcal{J}}}_{(\mathbf{0},\mathbf{0})}(\langle x\rangle)} \colon B^{\mathrm{cy}}(F^{\bar{\mathcal{J}}}_{(\mathbf{0},\mathbf{0})}(\langle x\rangle)) \to F^{\bar{\mathcal{J}}}_{(\mathbf{0},\mathbf{0})}(\langle x\rangle)$ is virtually surjective in $\mathcal{CS}^{\bar{\mathcal{J}}}$, Lemma 4.11 implies that the repletion $(B^{\mathrm{cy}}(F^{\bar{\mathcal{J}}}_{(\mathbf{0},\mathbf{0})}(\langle x\rangle)))^{\mathrm{rep}}$ is $\bar{\mathcal{J}}$-equivalent to the homotopy pullback of the diagram

$$(B^{\mathrm{cy}}(F^{\bar{\mathcal{J}}}_{(\mathbf{0},\mathbf{0})}(\langle x\rangle)))^{\mathrm{gp}} \xrightarrow{\epsilon_{F^{\bar{\mathcal{J}}}_{(\mathbf{0},\mathbf{0})}(\langle x\rangle)}} (F^{\bar{\mathcal{J}}}_{(\mathbf{0},\mathbf{0})}(\langle x\rangle))^{\mathrm{gp}} \xleftarrow{\eta_{F^{\bar{\mathcal{J}}}_{(\mathbf{0},\mathbf{0})}(\langle x\rangle)}} F^{\bar{\mathcal{J}}}_{(\mathbf{0},\mathbf{0})}(\langle x\rangle).$$

We employ the above group completions (4.19) and (4.20). The map (4.19) is an inclusion of path components and hence a positive $\bar{\mathcal{J}}$-fibration. Exploiting that the left adjoint functor $F^{\bar{\mathcal{J}}}_{(\mathbf{0},\mathbf{0})}$ commutes with tensors and that the model category $\mathcal{CS}^{\bar{\mathcal{J}}}$ is right proper (see Remark 2.10), the repletion $(B^{\mathrm{cy}}(F^{\bar{\mathcal{J}}}_{(\mathbf{0},\mathbf{0})}(\langle x\rangle)))^{\mathrm{rep}}$ is $\bar{\mathcal{J}}$-equivalent to the pullback of the diagram

$$F^{\bar{\mathcal{J}}}_{(\mathbf{0},\mathbf{0})}(B^{\mathrm{cy}}(\langle x\rangle^{\mathrm{gp}})) \xrightarrow{F^{\bar{\mathcal{J}}}_{(\mathbf{0},\mathbf{0})}(\epsilon_{\langle x\rangle})} F^{\bar{\mathcal{J}}}_{(\mathbf{0},\mathbf{0})}(\langle x\rangle^{\mathrm{gp}}) \xleftarrow{F^{\bar{\mathcal{J}}}_{(\mathbf{0},\mathbf{0})}(\eta_{\langle x\rangle})} F^{\bar{\mathcal{J}}}_{(\mathbf{0},\mathbf{0})}(\langle x\rangle).$$

The pullback is computed $\bar{\mathcal{J}}$-levelwise, and the functor $F^{\bar{\mathcal{J}}}_{(\mathbf{0},\mathbf{0})}$ preserves pullbacks. Therefore, the repletion $(B^{\mathrm{cy}}(F^{\bar{\mathcal{J}}}_{(\mathbf{0},\mathbf{0})}(\langle x\rangle)))^{\mathrm{rep}}$ is $\bar{\mathcal{J}}$-equivalent to $F^{\bar{\mathcal{J}}}_{(\mathbf{0},\mathbf{0})}(B^{\mathrm{rep}}(\langle x\rangle))$ (see [Rog09, p. 427]).

□

Example 4.29. Let p be an odd prime, and let k be the ring of p-local integers $\mathbb{Z}_{(p)}$. Let $\langle v_0\rangle \to \langle u_0\rangle$ be the map of free commutative monoids that sends v_0 to u_0^{p-1}. Since the greatest common divisor of p and $p-1$ is 1, the induced map of polynomial rings $k[v_0] \to k[u_0]$ is a tamely ramified extension. Applying the Eilenberg-Mac Lane functor H yields a map of commutative Hk-algebra spectra $H(k[v_0]) \to H(k[u_0])$. Recall from Example 2.37 that there is a pre-log structure $(F^{\bar{\mathcal{J}}}_{(\mathbf{0},\mathbf{0})}(\langle v_0\rangle), \alpha_{v_0})$ on $\tilde{k}(H(k[v_0])) \wedge_{\tilde{k}(Hk)} \mathrm{Sym}(\tilde{k}(S^1))$, and $(F^{\bar{\mathcal{J}}}_{(\mathbf{0},\mathbf{0})}(\langle u_0\rangle), \alpha_{u_0})$ on $\tilde{k}(H(k[u_0])) \wedge_{\tilde{k}(Hk)} \mathrm{Sym}(\tilde{k}(S^1))$ respectively. The induced diagram

$$\begin{array}{ccc}
F^{\bar{\mathcal{J}}}_{(\mathbf{0},\mathbf{0})}(\langle v_0\rangle) & \longrightarrow & \Omega^{\bar{\mathcal{J}}}(\tilde{k}(H(k[v_0])) \wedge_{\tilde{k}(Hk)} \mathrm{Sym}(\tilde{k}(S^1))) \\
\downarrow & & \downarrow \\
F^{\bar{\mathcal{J}}}_{(\mathbf{0},\mathbf{0})}(\langle u_0\rangle) & \longrightarrow & \Omega^{\bar{\mathcal{J}}}(\tilde{k}(H(k[u_0])) \wedge_{\tilde{k}(Hk)} \mathrm{Sym}(\tilde{k}(S^1)))
\end{array}$$

commutes so that we obtain a map of pre-log cdgas

$$\begin{aligned}&(\tilde{k}(H(k[v_0])) \wedge_{\tilde{k}(Hk)} \mathrm{Sym}(\tilde{k}(S^1)), F^{\bar{\mathcal{J}}}_{(\mathbf{0},\mathbf{0})}(\langle v_0\rangle), \alpha_{v_0})\\ &\to (\tilde{k}(H(k[u_0])) \wedge_{\tilde{k}(Hk)} \mathrm{Sym}(\tilde{k}(S^1)), F^{\bar{\mathcal{J}}}_{(\mathbf{0},\mathbf{0})}(\langle u_0\rangle), \alpha_{u_0}).\end{aligned} \tag{4.21}$$

For a discrete simplicial free commutative monoid $\langle x\rangle$, the commutative $\bar{\mathcal{J}}$-space $F^{\bar{\mathcal{J}}}_{(\mathbf{0},\mathbf{0})}(\langle x\rangle)$ is usually not cofibrant in the positive projective $\bar{\mathcal{J}}$-model structure on $C\mathcal{S}^{\bar{\mathcal{J}}}$ (compare [RSS15, remarks before Lemma 5.1]). To determine whether the map (4.21) is formally log THH-étale, we actually need to pass to the induced map of cofibrant replacements. But the abstract definition of the functor $\Lambda^{\bar{\mathcal{J}}}$ makes it hard to check whether the latter is formally log THH-étale. Unfortunately, we are only able to show that the map (4.21) is formally log THH-étale in a *naive* sense where we mean by naive that we do not care whether the objects have well-defined homotopy types. For this, we make use of Remark 4.27, that is, we argue that with respect to the map (4.21) the top face and the front face of the diagram (4.18) are pushout squares in $C(\mathrm{Sp}^\Sigma(s\mathrm{mod}))$.

From Lemma 4.28(iii) we know that for a discrete simplicial free commutative monoid $\langle x\rangle$, the commutative symmetric ring spectrum $\Lambda^{\bar{\mathcal{J}}}(F^{\bar{\mathcal{J}}}_{(\mathbf{0},\mathbf{0})}(\langle x\rangle))$ is naturally isomorphic to $F_0(k[x])$. The latter is isomorphic to the commutative symmetric ring spectrum $\tilde{k}(F_0(\langle x\rangle_+))$ which again is isomorphic to $\tilde{k}(F_0(\langle x\rangle_+) \wedge Hk) \wedge_{\tilde{k}(Hk)} \mathrm{Sym}(\tilde{k}(S^1))$ as the functor $\tilde{k}$ is strong symmetric monoidal. In addition, using that $F_0(\langle x\rangle_+) \wedge Hk$ is weakly equivalent to $H(k[x])$, the induced square

$$\begin{tikzcd} F_0(\langle v_0\rangle_+) \wedge Hk \arrow[r] \arrow[d] & H(k[v_0]) \arrow[d] \\ F_0(\langle u_0\rangle_+) \wedge Hk \arrow[r] & H(k[u_0]) \end{tikzcd} \tag{4.22}$$

is a pushout square in the category $C(Hk\text{-mod})$. From applying the left adjoint functor $\tilde{k}(-) \wedge_{\tilde{k}(Hk)} \mathrm{Sym}(\tilde{k}(S^1))$ to the square (4.22), we deduce that the top face of the diagram (4.18) is a pushout square in the category $C(\mathrm{Sp}^\Sigma(s\mathrm{mod}))$.

We move on to analyze with respect to the map (4.21) the front face of the diagram (4.18). We take advantage of Lemma 4.28(iv)-(vi). Further, we analyze the diagram

$$\begin{tikzcd} \langle v_0\rangle \arrow[r] \arrow[d] & \langle u_0\rangle \arrow[d] \\ B^{\mathrm{cy}}(\langle v_0\rangle) \arrow[r] \arrow[d] & B^{\mathrm{cy}}(\langle u_0\rangle) \arrow[d] \\ B^{\mathrm{rep}}(\langle v_0\rangle) \arrow[r] & B^{\mathrm{rep}}(\langle u_0\rangle). \end{tikzcd} \tag{4.23}$$

Plugging in Hesselholt's and Rognes' calculations (see Lemma 4.28(i)-(ii)), we can identify the diagram (4.23) with the diagram

$$\begin{array}{ccc} \coprod_{j\geq 0} * & \longrightarrow & \coprod_{j\geq 0} * \\ \downarrow & & \downarrow \\ * \sqcup \coprod_{j\geq 1} S^1(j) & \longrightarrow & * \sqcup \coprod_{j\geq 1} S^1(j) \\ \downarrow & & \downarrow \\ \coprod_{j\geq 0} S^1(j) & \longrightarrow & \coprod_{j\geq 0} S^1(j). \end{array} \tag{4.24}$$

The top horizontal morphism sends a point $*$ indexed by j to the point $*$ indexed by $j(p-1)$. The middle horizontal morphism maps the point $*$ to the point $*$, and by the identity a circle $S^1(j)$ indexed by j to $S^1(j(p-1))$ indexed by $j(p-1)$. The bottom horizontal morphism takes by the identity a circle $S^1(j)$ indexed by j to $S^1(j(p-1))$ indexed by $j(p-1)$. Computing the pushout of the composite left vertical map and the top horizontal map, we find that the outer square of (4.24) is a pushout square. Consequently, the outer square of the induced diagram

$$\begin{array}{ccc} \Lambda^{\bar{\mathcal{J}}}(F^{\bar{\mathcal{J}}}_{(\mathbf{0},\mathbf{0})}(\langle v_0\rangle)) & \longrightarrow & \Lambda^{\bar{\mathcal{J}}}(F^{\bar{\mathcal{J}}}_{(\mathbf{0},\mathbf{0})}(\langle u_0\rangle)) \\ \downarrow & & \downarrow \\ \Lambda^{\bar{\mathcal{J}}}(F^{\bar{\mathcal{J}}}_{(\mathbf{0},\mathbf{0})}(B^{\mathrm{cy}}(\langle v_0\rangle))) & \longrightarrow & \Lambda^{\bar{\mathcal{J}}}(F^{\bar{\mathcal{J}}}_{(\mathbf{0},\mathbf{0})}(B^{\mathrm{cy}}(\langle u_0\rangle))) \\ \downarrow & & \downarrow \\ \Lambda^{\bar{\mathcal{J}}}(F^{\bar{\mathcal{J}}}_{(\mathbf{0},\mathbf{0})}(B^{\mathrm{rep}}(\langle v_0\rangle))) & \longrightarrow & \Lambda^{\bar{\mathcal{J}}}(F^{\bar{\mathcal{J}}}_{(\mathbf{0},\mathbf{0})}(B^{\mathrm{rep}}(\langle u_0\rangle))) \end{array}$$

is a pushout square in $C(\mathrm{Sp}^{\Sigma}(s\mathrm{mod}))$. But as for a discrete simplicial free commutative monoid $\langle x\rangle$, the commutative $\bar{\mathcal{J}}$-space $F^{\bar{\mathcal{J}}}_{(\mathbf{0},\mathbf{0})}(B^{\mathrm{rep}}(\langle x\rangle))$ is not positive cofibrant, we do not know whether the commutative symmetric ring spectrum $\Lambda^{\bar{\mathcal{J}}}((B^{\mathrm{cy}}(F^{\bar{\mathcal{J}}}_{(\mathbf{0},\mathbf{0})}(\langle x\rangle)))^{\mathrm{rep}})$ is stably equivalent to $\Lambda^{\bar{\mathcal{J}}}(F^{\bar{\mathcal{J}}}_{(\mathbf{0},\mathbf{0})}(B^{\mathrm{rep}}(\langle x\rangle)))$. Altogether, we conclude, that the map (4.21) is log THH-étale in a naive sense. We hope to improve this ansatz to get a homotopy meaningful statement.

Another source of examples may come from the general assumption that tamely ramified extensions of commutative ring spectra correspond to formally log THH-étale morphisms of pre-log ring spectra (see [Rog14, Remark 7.3], [RSS18, p. 510]). An illustrating example for this is the inclusion of the connective Adams summand into the p-local connective topological complex K-theory $\ell \to ku_{(p)}$ which is tamely ramified (see [Aus05, §10.4], [DLR, Theorem 4.1]) and induces a formally log THH-étale morphism of pre-log ring spectra [RSS18, Theorem 6.2]. Our approch is to find tamely ramified extensions of commutative Hk-algebra spectra and to construct suitable pre-log structures such that the induced morphism of pre-log cdgas is formally log THH-étale.

Definition 4.30. [Rog08, Definition 9.2.1] A map $f\colon A \to B$ in $C(\mathrm{Sp}^{\Sigma})$ is *formally* THH-*étale* if the map $B \to \mathrm{THH}^{A}(B)$ is a weak equivalence.

Remark 4.31. The failure of the map $B \to \mathrm{THH}^A(B)$ to be a weak equivalence detects ramification. Unramified maps of commutative ring spectra are formally THH-étale (see [DLR, p. 2], [Ric, §8]). In [DLR] Dundas, Lindenstrauss and Richter provide several examples which propose that (relative) topological Hochschild homology is a suitable tool for measuring ramification. It is work in progress to develop a conceptional notion of tame and wild ramification of maps between commutative ring spectra (see [DLR, p. 2]). Tame ramification might be visible if $\mathrm{THH}^A_*(B)$ resembles $\mathrm{HH}^C_*(D)$ for $C \to D$ a tamely ramified extension of number rings (see [DLR, §4]).

In the sense of [DLR] we present an example of a tamely ramified extension of commutative Hk-algebra spectra. Recall the cohomology rings of the infinite complex projective space $\mathbb{C}P^\infty$ and the infinite quaternionic projective space $\mathbb{H}P^\infty$,

$$H^{-*}(\mathbb{H}P^\infty, k) \cong \pi_*(F((\mathbb{H}P^\infty)_+, Hk)) \cong k[y_{-4}]$$
$$H^{-*}(\mathbb{C}P^\infty, k) \cong \pi_*(F((\mathbb{C}P^\infty)_+, Hk)) \cong k[x_{-2}]$$

(see e.g. [Hat02, Theorem 3.19, p. 222]). There is a quotient map from $\mathbb{C}P^\infty$ to $\mathbb{H}P^\infty$ arising from writing both spaces as quotients of the infinite sphere S^∞ and the inclusion from S^1 into S^3,

$$\mathbb{C}P^\infty = S^\infty/S^1 \to S^\infty/S^3 = \mathbb{H}P^\infty. \tag{4.25}$$

To better understand this map, we investigate the skeleton filtrations of both spaces. For $\mathbb{C}P^\infty$, we find that

$$* = \mathbb{C}P^0 \subseteq \mathbb{C}P^1 \subseteq \mathbb{C}P^2 \subseteq \dots$$

so that the space $\mathbb{C}P^n$ is the $2n$-skeleton of $\mathbb{C}P^\infty$. Inductively, the space $\mathbb{C}P^{n+1}$ arises from $\mathbb{C}P^n$ by attaching a single $2(n+1)$-cell via the projection map

$$S^{2n+1} \to S^{2n+1}/S^1 = \mathbb{C}P^n$$

so that there is a pushout diagram

$$\begin{array}{ccc} S^{2n+1} & \longrightarrow & S^{2n+1}/S^1 = \mathbb{C}P^n \\ \downarrow & & \downarrow \\ D^{2n+2} & \longrightarrow & \mathbb{C}P^{n+1}. \end{array}$$

Similarly, for $\mathbb{H}P^\infty$, we consider the skeleton filtration

$$* = \mathbb{H}P^0 \subseteq \mathbb{H}P^1 \subseteq \mathbb{H}P^2 \subseteq \dots$$

so that the space $\mathbb{H}P^{\tilde{n}}$ is the $4\tilde{n}$-skeleton of $\mathbb{H}P^\infty$. Inductively, the space $\mathbb{H}P^{\tilde{n}+1}$ is defined by attaching a single $4(\tilde{n}+1)$-cell to $\mathbb{H}P^{\tilde{n}}$ via the projection map

$$S^{4\tilde{n}+3} \to S^{4\tilde{n}+3}/S^3 = \mathbb{H}P^{\tilde{n}}$$

so that there is a pushout diagram

$$\begin{array}{ccc} S^{4\tilde{n}+3} & \longrightarrow & S^{4\tilde{n}+3}/S^3 = \mathbb{H}P^{\tilde{n}} \\ \downarrow & & \downarrow \\ D^{4\tilde{n}+4} & \longrightarrow & \mathbb{H}P^{\tilde{n}+1}. \end{array}$$

For $n = 2\tilde{n}+1$, the map $\mathbb{C}P^{2\tilde{n}+1} = S^{4\tilde{n}+3}/S^1 \to S^{4\tilde{n}+3}/S^3 = \mathbb{H}P^{\tilde{n}}$ yields a map $\mathbb{C}P^{2\tilde{n}+2} \to \mathbb{H}P^{\tilde{n}+1}$ by the universal property of the pushout. In particular, for $\tilde{n} = 0$, we obtain a map $\mathbb{C}P^2 \to \mathbb{H}P^1 \cong S^4$ which ensures that the above map (4.25) takes the 4-cell of $\mathbb{C}P^\infty$ homeomorphically onto the 4-cell of $\mathbb{H}P^\infty$. From this, we can deduce that the induced map on cohomology sends the generator y_{-4} in $H^4(\mathbb{H}P^\infty, k)$ to x_{-2}^2 in $H^4(\mathbb{C}P^\infty, k)$, and thus, y_{-4}^l to x_{-2}^{2l} for $l \geq 0$ (see [Hat02, Example 4.L.4]). Assume that 2 is a unit in k, for example for p an odd prime, $k = \mathbb{Z}_{(p)}$ or $k = \mathbb{F}_p$. The following proposition indicates that the induced map of function spectra

$$F((\mathbb{H}P^\infty)_+, Hk) \to F((\mathbb{C}P^\infty)_+, Hk)$$

is a tamely ramified extension.

Proposition 4.32. *Assume that 2 is a unit in k. There is an isomorphism of augmented graded commutative $k[x_{-2}]$-algebras*

$$\mathrm{THH}_*^{F((\mathbb{H}P^\infty)_+, Hk)}(F((\mathbb{C}P^\infty)_+, Hk)) \cong k[x_{-2}] \rtimes k\langle z_0, z_1, \dots\rangle$$

where $k[x_{-2}] \rtimes M$ denotes a square-zero extension of $k[x_{-2}]$ by a $k[x_{-2}]$-module M. The degree of z_j is $-(2j+1)$, for $j \geq 0$.

Proof. We proceed similarly to the proof of [DLR, Theorem 4.1]. We observe that

$$\begin{aligned}\pi_*(F((\mathbb{C}P^\infty)_+, Hk)) &\cong k[x_{-2}] \\ &\cong k[y_{-4}][x_{-2}]/(x_{-2}^2 - y_{-4}) \\ &\cong \pi_*(F((\mathbb{H}P^\infty)_+, Hk))[x_{-2}]/(x_{-2}^2 - y_{-4})\end{aligned}$$

is projective over $\pi_*(F((\mathbb{H}P^\infty)_+, Hk))$. Thus, we can apply the Bökstedt spectral sequence with π_* as the homology theory,

$$\begin{aligned}E^2_{s,t} &= \mathrm{HH}_{s,t}^{\pi_*(F((\mathbb{H}P^\infty)_+, Hk))}(\pi_*(F((\mathbb{C}P^\infty)_+, Hk)), \pi_*(F((\mathbb{C}P^\infty)_+, Hk))) \\ &\Rightarrow \pi_{s+t}(\mathrm{THH}^{F((\mathbb{H}P^\infty)_+, Hk)}(F((\mathbb{C}P^\infty)_+, Hk))).\end{aligned}$$

Exploiting [LL92, (1.6)], we obtain that $\mathrm{HH}_*^{k[y_{-4}]}(k[x_{-2}], k[x_{-2}])$ is isomorphic to the homology of the complex

$$\dots \to \Sigma^{-6}k[x_{-2}] \xrightarrow{0} \Sigma^{-4}k[x_{-2}] \xrightarrow{2x_{-2}} \Sigma^{-2}k[x_{-2}] \xrightarrow{0} k[x_{-2}]. \tag{4.26}$$

As 2 is a unit in $k[x_{-2}]$ we get that

$$\mathrm{HH}_s^{k[y_{-4}]}(k[x_{-2}], k[x_{-2}]) \cong \begin{cases} k[x_{-2}], & s = 0, \\ \Sigma^{-2(2j+1)}k, & s = 2j+1, j \geq 0, \\ 0, & s = 2j, j \geq 1. \end{cases}$$

Therefore, the E^2-page consists of $k[x_{-2}]$ in the zeroth column and k in bidegrees $(2j+1, -2(2j+1))$ for $j \geq 0$. Due to $\mathrm{THH}^{F((\mathbb{H}P^\infty)_+, Hk)}(F((\mathbb{C}P^\infty)_+, Hk))$ being an augmented commutative $F((\mathbb{C}P^\infty)_+, Hk)$-algebra, it follows that $F((\mathbb{C}P^\infty)_+, Hk)$ splits off $\mathrm{THH}^{F((\mathbb{H}P^\infty)_+,Hk)}(F((\mathbb{C}P^\infty)_+, Hk))$. Consequently, the zeroth column cannot be hit by any differentials. Next we argue that for degree reasons there are no non-trivial differentials d^r, for $r \geq 2$, so that the spectral sequence collapses at the E^2-page. Let $r \geq 2$. First, we consider $d^r\colon E^r_{2j+1,-2(2j+1)} \to E^r_{2j+1-r,-2(2j+1)+r-1}$.

- If $2j+1 \leq r-1$, then $E^r_{2j+1-r,-2(2j+1)+r-1} = 0$.
- If $2j+1 = r$, then $E_{0,-2(j+1)} = k(x_{-2}^{j+1})$ cannot be hit by d^r.
- Let $2j+1 \geq r+1$. If r is odd, then $E^r_{2j+1-r,-2(2j+1)+r-1} = 0$ because $2j+1-r$ is even. If r is even, then $E^r_{2j+1-r,-2(2j+1)+r-1} = 0$ as $-2(2j+1)+r-1$ is odd.

Secondly, we investigate $d^r\colon E^r_{2j+1+r,-2(2j+1)-r+1} \to E^r_{2j+1,-2(2j+1)}$.

- If r is odd, then $E^r_{2j+1+r,-2(2j+1)-r+1} = 0$ since $2j+1+r$ is even.
- If r is even, then $E^r_{2j+1+r,-2(2j+1)-r+1} = 0$ on account of $-2(2j+1)-r+1$ being odd.

Moreover, in every fixed total degree there is only one term on the E^2-page. Hence, there are no additive extensions, so that additively we can conclude the desired result. Since $\mathrm{THH}_*^{F((\mathbb{H}P^\infty)_+,Hk)}(F((\mathbb{C}P^\infty)_+, Hk))$ is an augmented graded commutative $\pi_*(F((\mathbb{C}P^\infty)_+, Hk))$-algebra and that everything in the augmentation ideal is sitting in odd degrees, there can only be the trivial multiplication between any two elements in the augmentation ideal. The spectral sequence is a spectral sequence of $\pi_*(F((\mathbb{C}P^\infty)_+, Hk))$-modules. The zeroth column $k[x_{-2}]$ acts on $k(z_j)$ trivially. $\square$

Remark 4.33. If we drop the assumption that 2 is a unit in k in Proposition 4.32, we do not know whether the spectral sequence collapses at the E^2-page or whether we can exclude additive extensions (see the proof of Proposition 4.32). For example, let $k = \mathbb{Z}_{(2)}$. As in the proof of Proposition 4.32, we figure out that $\mathrm{HH}_*^{\mathbb{Z}_{(2)}[y_{-4}]}(\mathbb{Z}_{(2)}[x_{-2}], \mathbb{Z}_{(2)}[x_{-2}])$ is isomorphic to the homology of the complex (4.26). We compute that

$$\mathrm{HH}_s^{\mathbb{Z}_{(2)}[y_{-4}]}(\mathbb{Z}_{(2)}[x_{-2}], \mathbb{Z}_{(2)}[x_{-2}]) \cong \begin{cases} \mathbb{Z}_{(2)}[x_{-2}], & s = 0, \\ \Sigma^{-2(2j+1)}\mathbb{Z}_{(2)}[x_{-2}]/(2x_{-2}), & s = 2j+1, j \geq 0, \\ 0, & s = 2j, j \geq 1. \end{cases}$$

For another example $k = \mathbb{F}_2$, using again the complex (4.26), we calculate that

$$\mathrm{HH}_s^{\mathbb{F}_2[y_{-4}]}(\mathbb{F}_2[x_{-2}], \mathbb{F}_2[x_{-2}]) \cong \begin{cases} \mathbb{F}_2[x_{-2}], & s = 0, \\ \Sigma^{-2s}\mathbb{F}_2[x_{-2}], & s \geq 1. \end{cases}$$

Assume that 2 is a unit in k. In future work we aim to find appropriate pre-log structures on $F_{\mathrm{smod}}(\tilde{k}((\mathbb{H}P^\infty)_+), \mathrm{Sym}(\tilde{k}(S^1)))$, and on $F_{\mathrm{smod}}(\tilde{k}((\mathbb{C}P^\infty)_+), \mathrm{Sym}(\tilde{k}(S^1)))$ respectively, to prove the following conjecture.

Conjecture 4.34. *The induced map of commutative symmetric ring spectra in simplicial k-modules*

$$F_{s\mathrm{mod}}(\tilde{k}((\mathbb{H}P^{\infty})_+), \mathrm{Sym}(\tilde{k}(S^1))) \to F_{s\mathrm{mod}}(\tilde{k}((\mathbb{C}P^{\infty})_+), \mathrm{Sym}(\tilde{k}(S^1)))$$

gives rise to a map of pre-log cdgas which is formally log THH*-étale.*

Outlook

In this thesis we introduced log cdgas and as a corresponding homotopical invariant logarithmic topological Hochschild homology whose definition is based on a suitable notion of group completion in commutative $\bar{\mathcal{J}}$-spaces. We could translate several examples of pre-log ring spectra to examples of pre-log cdgas. For instance, the trivial, the canonical, the direct image or the free pre-log structure on an E_∞ dga is defined similarly as the respective pre-log structure on a commutative symmetric ring spectrum. Further, the direct image pre-log structure $D(x)$ on an E_∞ dga associated to a homology class $[x]$ in its graded homology ring was specified in the same way as in the topological set-up, once we had at hand a feasible concept of group completion in commutative $\bar{\mathcal{J}}$-spaces. But as already pointed out, we are only able to determine the homotopy type of $D(x)$ if the homology class $[x]$ has positive degree. In particular, this does not apply in general for our guiding example, the cochains on a space X with coefficients in k. So far, we have not figured out other convenient examples of pre-log structures of algebraic nature. This is surprising considering that studying dgas in contrast to ring spectra, it is often easier to give a complete description of their homotopy type. Therefore, an extension of this thesis clearly should be the construction of additional examples of pre-log structures on E_∞ dgas. These should be useful to create concrete examples of formally log THH-étale morphisms of pre-log cdgas. One first step could be to work out appropriate pre-log structures so that the tamely ramified extension of commutative Hk-algebra spectra

$$F((\mathbb{H}P^\infty)_+, Hk) \to F((\mathbb{C}P^\infty)_+, Hk)$$

induces a formally log THH-étale morphism of pre-log cdgas indeed. Hitherto, one difficulty to compute logarithmic topological Hochschild homology of a pre-log cdga or to identify formally log THH-étale morphisms of pre-log cdgas has been the abstract definition of the functor $\Lambda^{\bar{\mathcal{J}}}$. Up to now, we can explicitly express the latter on free commutative $\bar{\mathcal{J}}$-spaces only. But as remarked in [RSS15, Remark 5.8], the free pre-log structure does not seem to be interesting unless the E_∞ dga is concentrated in chain degree zero. We hope to overcome this problem in the future.

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Summary

In the last decade Rognes, Sagave and Schlichtkrull have established a theory of logarithmic structures on commutative symmetric ring spectra ([Rog09], [SS12], [Sag14]). This is relevant for obtaining results on the algebraic K-theory of commutative ring spectra through localization techniques or trace maps from algebraic K-theory to topological Hochschild homology ([RSS15], [RSS18]). Recently, Richter and Shipley showed that for a commutative ring k with unit, there is a chain of Quillen equivalences between commutative Hk-algebra spectra and E_∞ differential graded k-algebras (E_∞ dgas) [RS17]. Motivated by the aim to gain new examples of log ring spectra via algebraic objects, we develop a concept of logarithmic structures on E_∞ dgas.

Considering the intermediate model category of commutative symmetric ring spectra in simplicial k-modules $C(\mathrm{Sp}^\Sigma(s\mathrm{mod}))$ in Richter and Shipley's chain of Quillen equivalences, we relate this to commutative $\bar{\mathcal{J}}$-spaces via the Quillen adjunction $(\Lambda^{\bar{\mathcal{J}}}, \Omega^{\bar{\mathcal{J}}})$. Here a crucial step has been to figure out an index category which is suitable in this algebraic context and fulfils the axioms of a well-structured index category. It turns out that the category $\bar{\mathcal{J}}$ that arises from Sagave and Schlichtkrull's category $\mathcal{J}$ by determining an equivalence relation on the morphism sets, is a reasonable choice. The induced map of classifying spaces $B\mathcal{J} \to B\bar{\mathcal{J}}$ models the first Postnikov section of the sphere spectrum $\mathbb{S}$. Given the Quillen adjunction $(\Lambda^{\bar{\mathcal{J}}}, \Omega^{\bar{\mathcal{J}}})$, we specify pre-log structures on E_∞ dgas. For a positive fibrant object A in $C(\mathrm{Sp}^\Sigma(s\mathrm{mod}))$ and $(\mathbf{m}_1, \mathbf{m}_2)$ in $\bar{\mathcal{J}}$ such that $m_1 \geq 1$, the space $\Omega^{\bar{\mathcal{J}}}(A)(\mathbf{m}_1, \mathbf{m}_2)$ is weakly equivalent to the space $\Omega^{m_2}(U(A))(m_1)$ where U denotes the forgetful functor to commutative symmetric ring spectra. This result motivates the definition of units of an E_∞ dga and with it a condition for a pre-log structure to be a log structure. Moreover, we explain a construction called logification which assigns a log cdga to a pre-log cdga. We provide several examples of pre-log cdgas and log cdgas. Apart from this, we discuss the defects of an alternative approach to set up pre-log structures via diagram chain complexes. In connection to this, we present a homotopy colimit formula on diagram chain complexes. With the latter, we argue that diagram chain complexes do not have to possess a model structure in which the fibrant objects are precisely the objects that are homologically constant and the homotopy colimit functor detects the weak equivalences.

As an important tool, we study group completion in commutative diagram spaces. We recall Sagave and Schlichtkrull's chain of Quillen equivalences linking commutative $\mathcal{K}$-spaces to E_∞ spaces over the classifying space $B\mathcal{K}$ where $\mathcal{K}$ is a well-structured index category satisfying a few assumptions. We then show that there is a chain of Quillen equivalences between commutative $\mathcal{K}$-spaces and commutative $\mathcal{I}$-spaces over a commutative $\mathcal{I}$-space model of $B\mathcal{K}$. Assuming that the simplicial monoid $B\mathcal{K}$ is grouplike and building on Sagave and Schlichtkrull's work on group completion in commutative $\mathcal{I}$-spaces [SS13], we identify a localized model structure on commutative $\mathcal{K}$-spaces as a group completion model structure. Here a map of commutative $\mathcal{K}$-spaces is a group completion if the associated map of E_∞ spaces is so in the usual sense.

Having a feasible concept of group completion in commutative $\bar{\mathcal{J}}$-spaces yields more examples of pre-log cdgas and is a substantial foundation for the definition of logarithmic

topological Hochschild homology of pre-log cdgas. We verify that the latter is homotopy invariant under logification. More than that, we give a criterion for a morphism of log cdgas to be formally étale with respect to logarithmic topological Hochschild homology and present approaches towards examples.

Zusammenfassung

Im vergangenen Jahrzehnt haben Rognes, Sagave und Schlichtkrull eine Theorie von Logstrukturen auf kommutativen symmetrischen Ringspektren entwickelt ([Rog09], [SS12], [Sag14]). Diese ist zum Beispiel relevant, um mittels Lokalisierungstechniken und Spurabbildungen von algebraischer K-Theorie zur topologischen Hochschild-Homologie Resultate über algebraische K-Theorie von kommutativen Ringspektren zu erhalten ([RSS15], [RSS18]). Kürzlich haben Richter und Shipley gezeigt, dass es für einen kommutativen Ring k mit Eins eine Kette von Quillenäquivalenzen zwischen kommutativen Hk-Algebraspektren und E_∞ differentiell graduierten k-Algebren (E_∞ dgas) gibt [RS17]. Die Hoffnung ist, neue Beispiele von Logringspektren zu finden, die von algebraischen Objekten stammen. Aus diesem Grund schaffen wir ein Konzept von Logstrukturen auf E_∞ dgas.

Wir betrachten die Zwischenmodellkategorie der kommutativen symmetrischen Ringspektren in simplizialen k-Moduln $C(\mathrm{Sp}^\Sigma(s\mathrm{mod}))$ in Richter und Shipleys Kette von Quillenäquivalenzen und setzen diese durch die Quillenadjuntion $(\Lambda^{\bar{\mathcal{J}}}, \Omega^{\bar{\mathcal{J}}})$ in Bezug zu kommutativen $\bar{\mathcal{J}}$-Räumen. Hierbei ist es entscheidend gewesen, eine für den algebraischen Kontext geeignete Indexkategorie zu ermitteln. Diese sollte die Axiome einer gut-strukturierten Indexkategorie erfüllen, damit wir Sagave und Schlichtkrulls Methodik anwenden können, um Modellstrukturen auf zugehörigen (strukturierten) Diagrammräumen zu etablieren. Es stellt sich heraus, dass die Kategorie $\bar{\mathcal{J}}$, welche induziert ist von Sagave und Schlichtkrulls Kategorie $\mathcal{J}$, indem wir eine Äquivalenzrelation auf den Morphismenmengen spezifizieren, eine passende Wahl ist. Die induzierte Abbildung von klassifizierenden Räumen $B\mathcal{J} \to B\bar{\mathcal{J}}$ modelliert den ersten Postnikovschnitt des Sphärenspektrums $\mathbb{S}$.

Mit Hilfe der Quillenadjunktion $(\Lambda^{\bar{\mathcal{J}}}, \Omega^{\bar{\mathcal{J}}})$ definieren wir Prälogstrukturen auf E_∞ dgas. Für ein positiv faserndes Objekt A in $C(\mathrm{Sp}^\Sigma(s\mathrm{mod}))$ und $(\mathbf{m}_1, \mathbf{m}_2)$ in $\bar{\mathcal{J}}$ mit $m_1 \geq 1$, ist der Raum $\Omega^{\bar{\mathcal{J}}}(A)(\mathbf{m}_1, \mathbf{m}_2)$ schwach äquivalent zum Raum $\Omega^{m_2}(U(A))(m_1)$, wobei U für den Vergissfunktor zu kommutativen symmetrischen Ringspektren steht. Dies motiviert die Definition von Einheiten einer E_∞ dga, und damit eine Bedingung für eine Prälog cdga eine Log cdga zu sein. Außerdem erläutern wir eine Logifizierung genannte Konstruktion, die einer Prälog cdga eine Log cdga zuordnet. Wir führen mehrere Beispiele von Prälog cdgas und Log cdgas auf. Abgesehen davon stellen wir einen ursprünglichen Ansatz, Prälogstrukturen durch Diagrammkettenkomplexe zu erklären, vor, sowie die Gründe, die uns davon abgehalten haben, diesen weiter zu verfolgen. Damit im Zusammenhang geben wir eine Homotopiekolimesformel auf Diagrammkettenkomplexen an. Mit dieser zeigen wir, dass Diagrammkettenkomplexe keine Modellstruktur besitzen müssen, in der die fasernden Objekte genau die Objekte sind, die homologie-konstant sind, und der Homotopiekolimesfunktor die schwachen Äquivalenzen detektiert.

Des Weiteren beschäftigen wir uns mit Gruppenvervollständigung in kommutativen Diagrammräumen. Wir wiederholen Sagave und Schlichtkrulls Kette von Quillenäquivalenzen, die kommutative $\mathcal{K}$-Räume mit E_∞ Räumen über dem klassifizierenden Raum $B\mathcal{K}$ verbindet, wobei $\mathcal{K}$ eine gut-strukturierte Indexkategorie ist, die noch einige Voraussetzungen erfüllt. Wir beweisen dann, dass es eine Kette von Quillenäquivalenzen

zwischen kommutativen $\mathcal{K}$-Räumen und kommutativen $\mathcal{I}$-Räumen über einem kommutativen $\mathcal{I}$-Raum-Modell von $B\mathcal{K}$ gibt. Unter der Annahme, dass das simpliziale Monoid $B\mathcal{K}$ gruppenähnlich ist, und der Benutzung von Sagave und Schlichtkrulls Arbeit zur Gruppenvervollständigung in kommutativen $\mathcal{I}$-Räumen [SS13], identifizieren wir eine lokalisierte Modellstruktur auf kommutativen $\mathcal{K}$-Räumen als eine Gruppenvervollständigungsmodellstruktur. Hierbei ist ein Morphismus von kommutativen $\mathcal{K}$-Räumen eine Gruppenvervollständigung, falls dies der assoziierte Morphismus von E_∞ Räumen im gewöhnlichen Sinne ist. Mit diesem Konzept von Gruppenvervollständigung in kommutativen $\bar{\mathcal{J}}$-Räumen geben wir weitere Beispiele von Prälog cdgas an. Ferner ist dieses essentiell für die Definition von logarithmischer topologischer Hochschild-Homologie von Prälog cdgas. Wir verifizieren, dass letzteres homotopieinvariant unter Logifizierung ist. Weiterhin präsentieren wir ein Kriterium für einen Morphismus von Log cdgas formal étale bezüglich logarithmisch topologischer Hochschild-Homologie zu sein und diskutieren Herangehensweisen, um Beispiele dafür zu finden.

Eidesstattliche Erklärung

Hiermit erkläre ich an Eides statt, dass ich die vorliegende Dissertationsschrift selbst verfasst und keine anderen als die angegebenen Quellen und Hilfsmittel benutzt habe.

Ort, Datum

Unterschrift