

State-sum construction of two-dimensional functorial field theories

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Chapter 1

Introduction

The term quantum field theory refers to a wide range of models in physics which are expected to describe for example elementary particles in nature. A quantum field theory comes with a vector space called the state space, which is often an infinite-dimensional Hilbert space. A common feature of quantum field theories is locality that roughly says that the theory is only determined by what happens on a small scale and that spacelike separated fields are independent. Independent subsystems are assigned the tensor product of the individual state spaces, which reflects the quantum nature of the theory.

There are several proposals for axiomatising quantum field theories which implement the above features. One approach is called Algebraic or Axiomatic Quantum Field Theory [HK] which assigns to a region in spacetime an algebra of observables and is usually defined on manifolds with Lorentzian signature. The assignment of observables is compatible with inclusion and the observables on spacelike separated regions commute with each other, which implements locality.

The approach we take in this thesis is often called functorial quantum field theory. We will only consider theories on Riemannian manifolds with Euclidean signature, these are also referred to as statistical field theories. More precisely, a n -dimensional functorial quantum field theory is a symmetric monoidal functor from a geometric bordism category \mathcal{Bord}_n into a symmetric monoidal category \mathcal{S} , subject to suitable continuity conditions. The category \mathcal{Bord}_n has objects $(n - 1)$ -dimensional closed manifolds and morphisms are n -dimensional compact Riemannian manifolds with parametrised boundary modulo some equivalence relation. The target category \mathcal{S} in our examples will be the category of (super) vector spaces or the category of Hilbert spaces. The quantum nature of the theory is implemented in monoidality: disjoint union of manifolds are mapped to the tensor product of the vector spaces assigned to the connected components. Locality is reflected in functoriality, i.e. the assignment of morphisms between state spaces is compatible with cutting and glueing of manifolds. This definition is motivated by topological field theory [Ati] and 2-dimensional conformal field theory [Seg1, Seg2]. There are also several versions of functorial (topological) field theories, called extended field theories, which implement higher versions of locality, see e.g. [BD].

Functorial field theories come in different flavours, depending on the kind of bordism

category and the type of target category and there exist a vast variety of constructions of them. In this thesis we study two classes of 2-dimensional functorial quantum field theories and explain how one obtains examples of them via the state-sum construction. The first class is topological field theory on r -spin surfaces, where we equip surfaces with r -spin structures and consider them modulo diffeomorphisms. We give a combinatorial model of r -spin surfaces, which is convenient for the state-sum construction of r -spin topological field theories. We give an example of such a theory when r is even, which computes the Arf invariant of r -spin surfaces. As an application of this topological field theory and the combinatorial model we compute mapping class group orbits of r -spin surfaces, extending results of [Ran] and [GG]. This part of the thesis is available as a preprint in [RS1].

The second class of functorial field theories which we consider here is called area-dependent quantum field theory, where we consider surfaces up to area-preserving diffeomorphisms. Contrary to topological field theories, which have finite-dimensional state spaces, area-dependent quantum field theories allow for infinite-dimensional state spaces, which make them attractive to study. We classify these theories in terms of some algebraic data which we call commutative regularised Frobenius algebras. We consider defect lines, which are embedded 1-dimensional manifolds, and give a state-sum construction of area-dependent quantum field theories with defects where the input data is a set of regularised Frobenius algebras, which label surface components cut out by the defect lines, and a set of bimodules over these regularised algebras, which label the defect lines. We then show that under some assumption the fusion of defect lines corresponds to the tensor product of bimodules over these regularised algebras. The main example is 2-dimensional Yang-Mills theory with Wilson lines as defects, which we study in greater detail. This part of the thesis is available as a preprint in [RS2].

Chapter 2

Functorial quantum field theories

As we briefly explained in the introduction, functorial quantum field theory is a possible axiomatisation of quantum field theory. In Section 2.1 we give an overview of different classes of functorial field theories, like topological field theory and volume-dependent quantum field theory. Then we summarise our results on topological field theory on r -spin surfaces (Section 2.2) and on area-dependent quantum field theory with defects (Section 2.3). We assume familiarity with braided monoidal categories and refer the reader to [EGNO].

2.1 Functorial quantum field theories

In this section we will study some special classes of 2-dimensional functorial field theories. In order to put them into context we start discussing n -dimensional functorial field theories for arbitrary $n \in \mathbb{Z}_{\geq 1}$ and then specialise to dimensions 1 and 2. We stress that the purpose of this section is not to give precise definitions of functorial field theories, but to give a general picture. All manifolds in this section will be smooth and oriented unless specified otherwise.

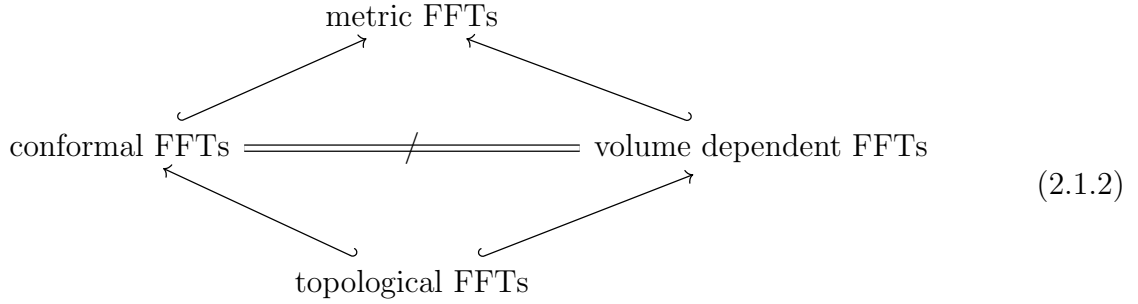
2.1.1 Topological and volume-dependent field theories

In order to formulate functorial quantum field theory in n dimensions, we need the notion of a bordism category $\mathcal{B}ord_n^{\text{metric}}$, where objects are closed $(n - 1)$ -dimensional manifolds and morphisms are equivalence classes of n -dimensional compact Riemannian manifolds with boundary parametrisation, which identifies the source and target objects with the boundary of the n -manifold. The equivalence relation is given by isometries compatible with the boundary parametrisation. Then a *metric functorial field theory* or *metric FFT* for short is a symmetric monoidal functor

$$\mathcal{Z} : \mathcal{B}ord_n^{\text{metric}} \rightarrow \mathcal{S} \tag{2.1.1}$$

to some symmetric monoidal category \mathcal{S} . Instead of trying to give a precise definition of $\mathcal{B}ord_n^{\text{metric}}$ we refer to [StTe] and we just note that finding examples of such theories in

dimensions higher than 1 is in general a hard task. One can however look at other bordism categories, where the equivalence relation is coarser than being isometric. In the following diagram we present three subclasses of metric FFTs and in the rest of this section we explain how these are defined.



One possibility to make the study of functorial field theories more tractable is to forget about the metric on the n -dimensional manifolds. This amounts to changing the source category in (2.1.1) to the category of topological bordisms \mathcal{Bord}_n , which is defined similarly as $\mathcal{Bord}_n^{\text{metric}}$, but without a metric for the n -dimensional manifolds, which are often referred to as *bordisms*, and where equivalence classes are taken with respect to diffeomorphisms. The disjoint union of manifolds endows this category with a symmetric monoidal structure. An n -dimensional topological functorial field theory or *TFT* is a symmetric monoidal functor

$$\mathcal{Z} : \mathcal{Bord}_n \rightarrow \mathcal{S} . \tag{2.1.3}$$

For a review on topological field theories see e.g. [Koc, Car, CR]. We write

$$\mathcal{Fun}^{\otimes, \text{sym}}(\mathcal{Bord}_n, \mathcal{S}) \tag{2.1.4}$$

for the category of n -dimensional TFTs and we note that it inherits the symmetric monoidal structure of \mathcal{S} . TFTs with values in $\mathcal{S} = \mathcal{Vect}$ have the property that their state spaces are finite-dimensional.

Lemma 2.1.1 (e.g. [CR, Sec.2.4]). *Let $n \in \mathbb{Z}_{\geq 1}$ and $\mathcal{Z} : \mathcal{Bord}_n \rightarrow \mathcal{Vect}$ be a TFT. Then for every object $O \in \mathcal{Bord}_n$ the vector space $\mathcal{Z}(O)$ is finite-dimensional.*

The key idea of the proof is that one can decompose a cylinder over any $(n - 1)$ dimensional closed manifold O as in Figure 2.1. Then evaluating the TFT yields duality morphisms for O .

A general feature of many quantum field theories is that their state spaces are infinite-dimensional. To recover this property, we could remember the conformal structure induced by the metric on bordisms. This way we arrive to the notion of the category of conformal bordisms and *conformal FFT* or *CFT*. In dimension $n = 2$, axioms for these theories have

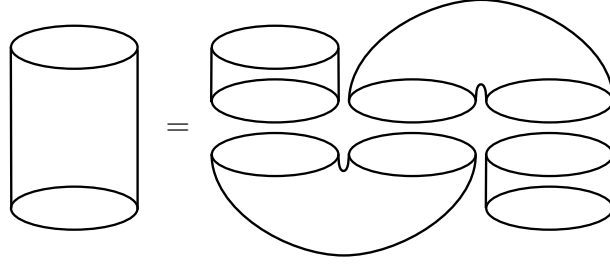


Figure 2.1: A decomposition of a cylinder over an $(n - 1)$ dimensional closed manifold, which we schematically draw as a circle.

been given in [Seg1, Seg2] and some existence results have been proven recently in [Ten]. Note that if a CFT is independent of the conformal structure, then it is necessarily a TFT.

Another way to allow for infinite-dimensional state spaces is to consider manifolds with volume. The category of n -dimensional bordisms with volume \mathcal{Bord}_n^{vol} has the same objects as \mathcal{Bord}_n and the morphisms are pairs (M, v) , where M is a morphism in \mathcal{Bord}_n and $v : \pi_0(M) \rightarrow \mathbb{R}_{>0}$ is a function representing the volume of each component of M . We furthermore allow zero volume cylinders in order to have identities in the category. One could alternatively define the morphisms to have Riemannian manifolds modulo volume form preserving diffeomorphism, which is the same information [Mos, Ban], so this way we would get an equivalent category. A *volume dependent FFT* is then a symmetric monoidal functor

$$\mathcal{Z} : \mathcal{Bord}_n^{vol} \rightarrow \mathcal{S} , \quad (2.1.5)$$

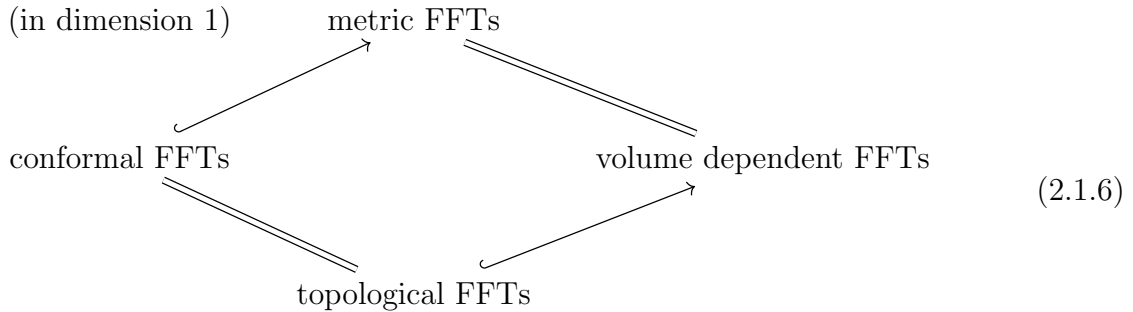
into a symmetric monoidal category with topological spaces as hom-sets, which is continuous on the hom-sets. The topology on the hom-sets of \mathcal{Bord}_n^{vol} is that of the disjoint union over $M \in \mathcal{Bord}_n(U, U')$ of $\mathbb{R}_{>0}^{|\pi_0(M)|}$ (or $\mathbb{R}_{\geq 0}$ for components of M that are cylinders). In other words, $\mathcal{Z}(U \xrightarrow{M} U', v)$ is jointly continuous as a function in the volume parameters assigned to the connected components of M with values in $\mathcal{S}(\mathcal{Z}(U), \mathcal{Z}(U'))$. For the precise definition in 2 dimensions we refer to Section 4.2. Volume dependent FFTs are essentially different from CFTs in the following sense. A diffeomorphism preserving the conformal class of the metric does not necessarily preserve the volume form and conversely a volume form preserving diffeomorphism does not necessarily preserve the conformal class.

Take a cylinder with volume $v \in \mathbb{R}_{>0}$ and decompose it as in Figure 2.1. Now each of the connected components has a positive volume and this cylinder is not the identity morphism in the category. In fact, in this category no object, except for the empty $(n - 1)$ -dimensional manifold, has a dual and the argument used in the proof of Lemma 2.1.1 does not apply. However this argument implies that if $\mathcal{S} = \mathcal{Hilb}$ then for $U \in \mathcal{Bord}_n^{vol}$ the Hilbert space $\mathcal{Z}(U)$ is separable (cf. Lemma 4.1.13 and Theorem 4.2.10). This also allows volume dependent FFTs with values in \mathcal{Hilb} to have infinite-dimensional state spaces. Note that if a volume dependent FFT is actually independent of the volume, then it is necessarily a TFT. Conversely one can show that if for all bordisms M the zero volume limit of $\mathcal{Z}(M)$ exists, then the zero volume limit of \mathcal{Z} is a TFT (Remark 4.2.11) and all state spaces $\mathcal{Z}(U)$

are necessarily finite-dimensional.

Remark 2.1.2. The category $\mathcal{B}ord_n^{vol}$, is enriched in \mathcal{Top} , the category of topological spaces, in particular the hom-sets are topological spaces. One could thus define volume-dependent theories to be \mathcal{Top} -enriched symmetric monoidal functors $\mathcal{B}ord_n^{vol} \rightarrow \mathcal{S}$ for some \mathcal{Top} -enriched symmetric monoidal target category \mathcal{S} . This would make the explicit mention of continuity in the volume parameters unnecessary. The reason we do not do this here is that it restricts the choice of target category. In particular, our main example – \mathcal{Hilb} with strong operator topology – is not \mathcal{Top} -enriched (Remark 4.1.11). On the other hand, \mathcal{Hilb} with norm topology is \mathcal{Top} -enriched, but this leads to another problem. Namely, the version of $\mathcal{B}ord_n^{vol}$ we use has identities in the form of zero-area cylinders (recall that only cylinder components are allowed to have zero area). This can be shown to imply that a volume-dependent QFT $\mathcal{B}ord_n^{vol} \rightarrow (\mathcal{Hilb} \text{ with norm-top.})$ must take values in finite-dimensional Hilbert spaces (Corollary 4.1.14). Hence, to have an interesting theory one has to remove the zero-volume cylinders. This can be done, but we do not pursue this further in the present thesis.

To further illustrate the relations of these subclasses of theories let us look at the case when $n = 1$. Now conformal invariance is equivalent to independence of the metric, therefore the notion of CFT and TFT is the same. On 1-dimensional manifolds the metric is given by the distance, hence metric FFTs and volume dependent theories coincide.



2.1.2 Topological field theories in two dimensions

In 2 dimensions, compact manifolds up to diffeomorphism are classified by non-negative integers: the number of connected components and the genus and the number of boundary components of each component. Due to this simple classification, topological field theories are well understood. They are given by commutative Frobenius algebras in \mathcal{S} , which are unital algebras and counital coalgebras such that the comultiplication is a module morphism. We write $c\mathcal{Frob}(\mathcal{S})$ for the symmetric monoidal category of commutative Frobenius algebras, see e.g. [Koc].

Theorem 2.1.3 ([Dij], [Abr], [Koc, Thm. 3.6.19]). *The functor*

$$\begin{aligned} \mathcal{F}un^{\otimes, \text{sym}}(\mathcal{B}ord_2, \mathcal{S}) &\xrightarrow{\sim} {}_c\mathcal{F}rob(\mathcal{S}) \\ \mathcal{Z} &\mapsto \mathcal{Z}(\mathbb{S}^1) \end{aligned} \tag{2.1.7}$$

is an equivalence of symmetric monoidal categories.

One can endow $\mathcal{B}ord_2$ with some extra structure in order to get more interesting TFTs. An example of such TFTs is called equivariant TFTs, where one considers surfaces with principal G -bundles for some finite group G . These can be classified similarly as ordinary TFTs and the corresponding algebraic structure is called crossed Frobenius G -algebra [Tur2]. In Chapter 3 we put r -spin structures on surfaces, which are certain principal \mathbb{Z}_r -bundles over the oriented orthonormal frame bundle of the surface, and study r -spin TFTs.

There is an extension of the bordism category where manifolds are endowed with a stratification, i.e. embedded lower dimensional manifolds, and a labeling of the strata with elements of some fixed sets. In the two dimensional case defects would correspond to embedded 0- and 1-dimensional submanifolds, but for simplicity here we only consider 1-dimensional defects. The connected components of these are labeled by a set D_1 (“defect conditions”) and the connected components of the complement of the defect lines are labeled by another set D_2 (“world sheet phases”). We write $\mathcal{B}ord_2^{def}(D_2, D_1)$ for this category, for details we refer to Section 4.2.3. The corresponding *topological field theories with defects* have been studied in 2 and higher dimensions e.g. in [DKR, Car, CRS].

The proof of Theorem 2.1.3 uses a generators and relations description of the bordism category and the existence of a normal form of compact surfaces. For bordisms with defects there is no such description currently available, which indicates that the corresponding TFTs are far more complex than TFTs without defects. Although there is no classification of defect TFTs similar to Theorem 2.1.3, there exists a systematic way of constructing examples of TFTs with defects in 2 dimensions from a simple set of algebraic data, which we briefly explain in the next section.

2.1.3 State-sum construction

The state-sum construction of TFTs is based on a cell decomposition of bordisms, which can be for example a triangulation or a PLCW decomposition, and some data assigned to the cells. This data is subject to conditions, which then ensure that the construction is invariant under local changes of the cell decomposition, such as the Pachner moves in 2 dimensions [Pac]. The first such models in 2 dimensions were proposed by [BP, FHK], this construction has been further generalised in [LP1] for so called open-closed TFTs and defect lines have been included in [DKR].

The algebraic data for the state-sum construction of 2-dimensional TFTs without defects with values in a symmetric monoidal category \mathcal{S} can be given in terms of a strongly separable symmetric Frobenius algebra in $A \in \mathcal{S}$. The state-sum TFT assigns $Z(A)$, the center of A , to a circle. The action of the functor on morphisms is more involved, the

rough idea is the following. One picks a triangulation of the surface and interprets the dual graph of this triangulation as a morphism in \mathcal{S} by assigning the multiplication and comultiplication of A to the trivalent vertices. Then one shows that this morphism is independent of the triangulation using the Frobenius algebra axioms.

For a TFT with defects the algebraic data is a set of strongly separable symmetric Frobenius algebras, which label components of the complement of the embedded 1-dimensional manifold. Defect lines are labeled by bimodules over these algebras. For the full description of 2-dimensional TFTs with defects we refer to the corresponding construction of area-dependent theories with defects in Section 4.3.

The 3-dimensional state-sum construction, called the Turaev-Viro model, was proposed in [TV]. The algebraic data assigned to the cells of a triangulation are given in terms of a spherical fusion category [BW]. An example of a 4-dimensional state-sum TFT is the Crane–Yetter model [CY], where the input data is a ribbon category, but no complete description of state-sum constructions in 4 dimensions exists.

2.2 Topological field theory on r -spin surfaces

The first main part of this thesis is about topological field theory on r -spin surfaces [RS1]. The r -spin group $Spin_2^r$ for r a positive integer is the r -fold cover of the rotation group SO_2 in 2 dimensions and $Spin_2^0 = \mathbb{R}$ is the universal cover. In higher dimensions the situation is essentially different, as the universal cover $Spin_d$ of SO_d is a 2-fold cover for every $d \geq 3$. One defines an r -spin structure over a surface Σ similarly as an ordinary spin structure. It is a principal $Spin_2^r$ -bundle over Σ which factorises through the oriented orthonormal frame bundle of Σ . With this definition a 1-spin surface is just an oriented surface, if $r = 2$ one obtains ordinary spin structures and the case $r = 0$ corresponds to framings of the surface.

We note here that for the above definition of an r -spin structure, more precisely for the definition of the oriented orthonormal frame bundle, one needs a Riemannian metric on the surface. We avoid this by considering instead the oriented frame bundle, which is principal bundle with structure group GL_2^+ , the group of 2×2 invertible matrices with positive determinant. One then replaces $Spin_2^r$ with the appropriate cover \widetilde{GL}_2^r of GL_2^+ . In the commutative diagram of Lie groups

$$\begin{array}{ccccc}
 & & Spin_2^r & \longrightarrow & SO_2 \\
 & \nearrow & \downarrow \tilde{\iota} & & \downarrow \iota \\
 \{0\} & \longrightarrow & \mathbb{Z}_r & & \{0\} \\
 & \searrow & \downarrow & & \nearrow \\
 & & \widetilde{GL}_2^r & \longrightarrow & GL_2^+
 \end{array} \tag{2.2.1}$$

the morphisms ι and $\tilde{\iota}$ are homotopy equivalences, hence the groupoids of r -spin structures defined in terms of $Spin_2^r$ - and SO_2 -bundles and in terms of \widetilde{GL}_2^r - and GL_2^+ -bundles are equivalent. For more details see e.g. [Nov, Sec. 3.2].

In order to be able to define the category of r -spin bordisms $\mathcal{B}ord_2^r$, we need compact r -spin surfaces whose boundary components are parametrised with annuli with r -spin structure. The isomorphism classes of r -spin structures on an annulus are in bijection with \mathbb{Z}_r . Therefore an object in $\mathcal{B}ord_2^r$ will be a finite set together with a map into \mathbb{Z}_r , which correspond to “circles with r -spin structures”. Morphisms are diffeomorphism classes of r -spin surfaces with boundary parametrisation. We define a 2-dimensional r -spin topological field theory with values in \mathcal{S} to be a symmetric monoidal functor

$$\mathcal{Z} : \mathcal{B}ord_2^r \rightarrow \mathcal{S} \quad (2.2.2)$$

into a symmetric monoidal category \mathcal{S} .

We give a combinatorial model of r -spin surfaces based on [Nov]. There the combinatorial model is given in terms of a triangulation, which for our purposes is cumbersome to work with, as a large number of triangles is needed even for the simplest surfaces. Our model uses a more convenient cell decomposition called PLCW-decompositions [Kir] (see Section 3.1.2). This allows one to describe a connected surface of genus g with b boundary components with $g + b \geq 1$ as a single 2-cell glued along the edges of a $(4g + 3b)$ -gon.

The combinatorial model consists of the following data:

- a PLCW decomposition of Σ such that each boundary component consists of a single edge and a single vertex,
- a choice of a marked edge for each face (before identification of the edges),
- an orientation of each edge,
- an edge index $s_e \in \mathbb{Z}_r$ for each edge e .

The edge indices need to satisfy an admissibility condition around each vertex, see Section 3.1.3. One obtains an r -spin structure from this combinatorial data by considering the trivial r -spin structure on each face and setting the transition functions between the faces using the edge indices and orientations. This r -spin structure extends uniquely to the vertices due to the admissibility condition. The above set of data encodes isomorphism classes of r -spin structures on a given surface redundantly. We determine the equivalence relation capturing this redundancy in Theorem 3.1.13.

The next application of the combinatorial model is the state-sum construction of r -spin TFTs (Section 3.2). This construction is again based on the one given in [Nov], but is considerably easier to evaluate on surfaces, as we work with PLCW decompositions instead of triangulations. Here we take \mathcal{S} to be an additive idempotent complete symmetric monoidal category with infinite direct sums in the $r = 0$ case. The input data is a Frobenius algebra $A \in \mathcal{S}$ with invertible window element $\mu \circ \Delta \circ \eta : \mathbb{I} \rightarrow A$, whose Nakayama automorphism N satisfies $N^r = \text{id}_A$. We write \mathcal{Z}_A for the r -spin TFT obtained from the construction. We show that the functor \mathcal{Z}_A equips the object

$$Z^r(A) := \bigoplus_{\lambda \in \mathbb{Z}_r} Z_\lambda, \quad (2.2.3)$$

where Z_λ is the value of \mathcal{Z}_A on the circle with r -spin structure corresponding to $\lambda \in \mathbb{Z}_r$, with a unital associative \mathbb{Z}_r -graded algebra structure which can be understood as a \mathbb{Z}_r -graded version of the centre of an algebra (Proposition 3.2.10). For $r = 2$, this algebraic structure on state spaces has also been found in [MS]. State-sum constructions in the $r = 2$ case were previously considered in [BT, NR, GK].

Frobenius algebras with $N^r = \text{id}$ appear in [DK] under the name of Λ_r -Frobenius algebras in relation to r -spin surfaces. In [Ster] Λ_r -Frobenius algebras have been used to describe r -spin TFTs defined on “open bordisms”, meaning that the objects in the bordism category are disjoint unions of intervals. Our r -spin TFTs are defined on “closed bordisms”, meaning that objects are disjoint unions of circles.

We give an example in the case when r is even. Let $\mathcal{S} := \mathcal{SVect}$ be the category of super vector spaces over some field k not of characteristic 2 and A the Clifford algebra $\mathcal{Cl}(1) = k \oplus k\theta$ in one odd generator θ . This algebra becomes a Frobenius algebra with the counit $\varepsilon(1) = 1/2$ and $\varepsilon(\theta) = 0$. One quickly checks that $\mathcal{Cl}(1)$ satisfies the conditions for the state-sum construction. Then $Z_\lambda = k\theta^\lambda$ for $\lambda \in \mathbb{Z}_r$ and the following holds (Section 3.4.1 and Theorem 3.4.8):

Theorem 2.2.1. *Let Σ be an r -spin surface of genus g with b ingoing boundary components of r -spin structures $\lambda_1, \dots, \lambda_b \in \mathbb{Z}_r$ and no outgoing boundary components. Then*

$$\mathcal{Z}_{\mathcal{Cl}(1)}(\Sigma)(\theta^{\lambda_1} \otimes \dots \otimes \theta^{\lambda_b}) = 2^{1-g} (-1)^{\text{Arf}(\Sigma)}, \quad (2.2.4)$$

where $\text{Arf}(\Sigma) \in \mathbb{Z}_2$ is the Arf-invariant of the r -spin structure of Σ as defined in [Ran, GG].

Since $\mathcal{Z}_{\mathcal{Cl}(1)}$ is an r -spin TFT, $\mathcal{Z}_{\mathcal{Cl}(1)}(\Sigma)$ is invariant under the action of the mapping class group of Σ . Therefore it follows that the r -spin Arf-invariant is constant on mapping class group orbits, a fact already shown in [Ran, GG] by different means. For usual spin structures, so $r = 2$, the fact that a spin-TFT can compute the Arf-invariant (incidentally, for the same algebra) was already noticed in [MS, Gun, BT, GK]. From this point of view Theorem 2.2.1 is not surprising as an r -spin structure for even r also defines a 2-spin structure, and this correspondence is compatible with the Arf-invariant.

In [Ran, Thm. 2.9] mapping class group orbits of r -spin structures on a connected surface $\Sigma_{g,b}$ of genus g with b boundary components have been calculated for $g, b \geq 1$ when $r = 2$, and for $g \geq 2, b \geq 1$ when $r > 0$; in [GG, Prop. 5] the orbits are given for $g \geq 0, b = 0$ in case $r > 0$. The r -spin Arf invariant has been used to distinguish two orbits for r even and $g \geq 2$. We extended these results for arbitrary g and b and give an alternative proof using the combinatorial formalism of Section 3.1. In [Sal, Prop. 3.13 and 3.15] a counting of orbits with a different treatment of boundary parametrisations is done.

In order to state our theorem we need to fix some conventions. We call an integer $d \in \mathbb{Z}_{\geq 0}$ a *divisor of r* if there exists an integer n such that $d \cdot n = r$. In particular, every non-negative integer, including 0, is a divisor of 0. Let us denote by $\gcd(a, b) \in \mathbb{Z}_{\geq 0}$ the non-negative generator of the ideal generated by a and b in \mathbb{Z} . Similarly one can define $\gcd(a, b, c) \in \mathbb{Z}_{\geq 0}$, etc. With this definition, $\gcd(a, 0) = a$ for all $a \in \mathbb{Z}_{\geq 0}$.

Let $\Sigma_{g,b}$ be a closed connected oriented surface of genus $g \geq 0$ with $b \geq 0$ ingoing boundary components and no outgoing boundary components. For $\lambda_1, \dots, \lambda_b \in \mathbb{Z}_r$ denote by $\mathcal{R}^r(\Sigma_{g,b})_{\lambda_1, \dots, \lambda_b}$ the set of isomorphism classes of r -spin structures on $\Sigma_{g,b}$ which near the boundary circles restrict to the annulus r -spin structures given by $\lambda_1, \dots, \lambda_b$ (see Section 3.1.1 for details).

We will also need the abelian group $O_0(r)$ defined as the quotient:

$$O_0(r) := (\mathbb{Z}_r)^b / \langle \hat{R}_i, \hat{H}_{ij}, G \mid i, j = 1 \dots b, i \neq j \rangle. \quad (2.2.5)$$

The generators $G, \hat{R}_i, \hat{H}_{ij} \in \prod_{i=1}^b \mathbb{Z}_r$ of the subgroup have components $(G)_i = 1$, $(\hat{R}_i)_k = \delta_{i,k}(\lambda_i - 1)$, $(\hat{H}_{ij})_i = (\hat{H}_{ij})_j = \lambda_i + \lambda_j - 1$ and $(\hat{H}_{ij})_k = 0$ for $k \neq i, j$.

Our second main result is:

Theorem 2.2.2. *Let $r \geq 0$ and let $\Sigma_{g,b}$ and $\lambda_1, \dots, \lambda_b$ be as above.*

1. *The set of isomorphism classes of r -spin structures $\mathcal{R}^r(\Sigma_{g,b})_{\lambda_1, \dots, \lambda_b}$ is non-empty if and only if*

$$2 - 2g \equiv \sum_{i=1}^b \lambda_i \pmod{r}. \quad (2.2.6)$$

2. *If the condition in Part 1 is satisfied, then the number of isomorphism classes is:*

r	b, g	$ \mathcal{R}^r(\Sigma_{g,b})_{\lambda_1, \dots, \lambda_b} $
0	$g = 0$ and $b \in \{0, 1\}$	1
	else	infinite
> 0	$b = 0$	r^{2g}
	$b \geq 1$	r^{2g+b-1}

3. *Suppose the condition in Part 1 is satisfied. Consider the action of the mapping class group of $\Sigma_{g,b}$ (which fixes the boundary pointwise) on $\mathcal{R}^r(\Sigma)_{\lambda_1, \dots, \lambda_b}$ by pullback. The number of orbits is*

g	conditions	number of orbits
0	(none)	$ O_0(r) $
1	r even and at least one λ_i odd	$2 \cdot \#(\text{divisors of } \gcd(r, \lambda_1, \dots, \lambda_b))$
	else	$\#(\text{divisors of } \gcd(r, \lambda_1, \dots, \lambda_b))$
≥ 2	r even	2
	r odd	1

Parts 1 and 2 of the theorem are proved in Proposition 3.1.19, Part 3 is proved in Section 3.5. The existence condition in Part 1 and the counting for $r > 0$ in Part 2 is well-known for closed surfaces from complex geometry, where it relates to roots of the canonical bundle. The counting in Parts 2 and 3 extends results obtained in [Ran, GG], as explained above, using different methods.

- Remark 2.2.3.** 1. We formulated Theorem 2.2.2 for ingoing boundary components to avoid notational complications. However, in the bordism category \mathcal{Bord}_2^r one naturally has ingoing and outgoing boundary components. To incorporate these, define $R_i = \lambda_i - 1$ for an ingoing boundary component and $R_i = 1 - \lambda_i$ for an outgoing boundary component. If one expresses Theorem 2.2.2 in terms of the R_i by replacing λ_i with $R_i + 1$ everywhere, the result applies to connected bordisms with both ingoing and outgoing boundary components. The proof in Proposition 3.1.19 and in Section 3.5 is given in terms of the R_i .
2. Let $X, Y \in \mathcal{Bord}_2^r$ and Σ a bordism in \mathcal{Bord}_2 with $|X|$ ingoing and $|Y|$ outgoing boundary components. Denote with $B \subset \mathcal{Bord}_2^r(X, Y)$ the subset of all morphisms which have the same underlying surface as Σ . Since morphisms in \mathcal{Bord}_2^r are diffeomorphism classes of r -spin bordisms, Part 3 of Theorem 2.2.2 precisely computes the number of elements in B .
3. We will see in Section 3.5 that $O_0(r)$ as defined in (2.2.5), and which appears in Part 3 of Theorem 2.2.2, is naturally in bijection with orbits of the mapping class group for $g = 0$ and $b \geq 0$. An explicit expression for the number of elements in $O_0(r)$ can be found in Lemma 3.1.18 ($b = 0$), Corollary 3.1.20 ($b = 1$), Equation 3.5.4 ($b = 2$) and Proposition 3.5.1 ($b \geq 2$), but the general result is somewhat cumbersome. Here we just list the answer for $b = 0, 1, 2$:

b	condition	$ O_0(r) $
0,1	(none)	1
2	$r = 0$ and $\lambda_1 = \lambda_2 = 1$	infinite
	else	$\gcd(r, \lambda_1 - 1)$

Recall that we assume the condition in Part 1 of Theorem 2.2.2 to hold. In particular, for $g = 0$, $b = 2$ we have $\lambda_1 + \lambda_2 \equiv 2 \pmod{r}$.

2.3 Area-dependent quantum field theory

In this section we consider 2-dimensional volume dependent FFTs in detail and we refer to such theories as *area-dependent QFTs*, or *aQFTs*¹ for short [RS2]. For simplicity we take

¹This should not be confused with Algebraic QFT or Axiomatic QFT, for which the abbreviation AQFT is used.

\mathcal{S} to be the category of Hilbert spaces with the strong operator topology on the hom-sets and we write for an aQFT $\mathcal{Z} : \mathcal{Bord}_2^{area} \rightarrow \mathcal{Hilb}$. The precise definition can be found in Section 4.2.1. Area-dependent theories in general have been considered in [Bru] and briefly in [Seg3, Sec. 1.4] (see also [Bar, Sec. 4.5]). A construction of area-dependent theories using triangulations with equal triangle area has been given in [CTS].

Recall that by Theorem 2.1.3 2d TFTs correspond to commutative Frobenius algebras and that by Lemma 2.1.1 the state spaces of TFTs are finite-dimensional. The state-sum construction of 2d TFTs in Section 2.1.3 takes a strongly separable (not necessarily commutative) Frobenius algebra A as an input (Section 2.1.3) and produces a TFT, which in turn corresponds to the centre $Z(A)$ of A by the above theorem.

The generalisation of these results to aQFTs is for the most part straightforward. We just add a positive real parameter to all structure morphisms, which we think of as “area parameters” and impose the condition that compositions of morphism depend on the sum of these areas.

For example, consider a unital associative algebra object A in \mathcal{Hilb} together with morphisms $\mu : A \otimes A \rightarrow A$, the multiplication, and $\eta : \mathbb{C} \rightarrow A$, the unit. These have to satisfy associativity and unitality:

$$\mu \circ (\text{id}_A \otimes \mu) = \mu \circ (\mu \otimes \text{id}_A) \quad , \quad \mu \circ (\text{id}_A \otimes \eta) = \mu \circ (\eta \otimes \text{id}_A) = \text{id}_A \quad . \quad (2.3.1)$$

A *regularised algebra* is then defined as follows (see Section 4.1.1). It is an object $A \in \mathcal{Hilb}$ together with two families of morphisms $\mu_a : A \otimes A \rightarrow A$ and $\eta_a : \mathbb{C} \rightarrow A$, for $a \in \mathbb{R}_{>0}$, such that, for all $a_1, a_2, b_1, b_2 \in \mathbb{R}_{>0}$ with $a_1 + a_2 = b_1 + b_2$,

$$\mu_{a_1} \circ (\text{id}_A \otimes \mu_{a_2}) = \mu_{b_1} \circ (\mu_{b_2} \otimes \text{id}_A) \quad , \quad \mu_{a_1} \circ (\text{id}_A \otimes \eta_{a_2}) = \mu_{b_1} \circ (\eta_{b_1} \otimes \text{id}_A) \quad . \quad (2.3.2)$$

The unit condition is one of the places where a little more thought is required: note that we do not demand that in the second equation in (2.3.2) we obtain id_A . Instead we *define* $P_a := \mu_{a_1} \circ (\text{id}_A \otimes \eta_{a_2}) : A \rightarrow A$, where $a = a_1 + a_2$. By the second condition in (2.3.2) this is indeed independent of the choice of a_1, a_2 in the decomposition $a = a_1 + a_2$. We now impose two conditions: P_a has to be continuous² in a and it has to satisfy $\lim_{a \rightarrow 0} P_a = \text{id}_A$. It is important for the formalism to *not* require μ_a and η_a to have zero-area limits on their own. Simple consequences of this definition are that μ_a and η_a are continuous in a , and that P_a is a semigroup, $P_a \circ P_b = P_{a+b}$.

The algebraic cornerstone of this part of the thesis is the notion of a *regularised Frobenius algebra* (RFA), which is a regularised algebra and coalgebra (with families Δ_a and ε_a for area-dependent coproduct and counit), subject to the usual compatibility condition, suitably decorated with area parameters (Definition 4.1.4). One difference between Frobenius algebras and RFAs is that the latter do not form a groupoid. Although RFA morphisms are mono and epi (Proposition 4.1.19), it may happen that the inverse of a homomorphism of Frobenius algebras is not bounded, hence not a morphism in \mathcal{Hilb} (Remark 4.1.20).

²For more general monoidal categories than \mathcal{Hilb} we need to add another continuity condition. We refer to Definition 4.1.1 for details. In \mathcal{Hilb} this condition is automatic – see (4.1.4) and Corollary 4.1.16.

In Section 4.1.5 we consider Hermitian RFAs, or \dagger -RFAs i.e. RFAs for which $\mu_a^\dagger = \Delta_a$ and $\eta_a^\dagger = \varepsilon_a$ for every $a \in \mathbb{R}_{>0}$, and classify them (Theorem 4.1.28):

Theorem 2.3.1. *Every Hermitian RFA is a Hilbert space direct sum of finite-dimensional Hermitian RFAs.*

Finite-dimensional RFAs in turn are very simple: they are just usual (by definition finite-dimensional) Frobenius algebras A together with an element H in the centre $Z(A)$ of A . The area-dependence is obtained by setting $P_a := \exp(aH)$ and defining $\mu_a := P_a \circ \mu$, etc., see Proposition 4.1.24. This makes RFAs sound not very interesting, but note that, conversely, for an infinite direct sum of finite-dimensional RFAs to again define an RFA one has to satisfy non-trivial bounds, as detailed in Proposition 4.1.18.

Our first main theorem in Chapter 4 generalises the classification of 2d TFTs in terms of commutative Frobenius algebras as given in Theorem 2.1.3. Let $\text{aQFT}(\mathcal{H}ilb)$ denote the category of aQFTs with values in $\mathcal{H}ilb$ and $c\mathcal{RFrob}(\mathcal{H}ilb)$ the category of commutative RFAs in $\mathcal{H}ilb$. In Theorem 4.2.10 we show:³

Theorem 2.3.2. *There is an equivalence of categories*

$$\begin{aligned} \text{aQFT}(\mathcal{H}ilb) &\xrightarrow{\sim} c\mathcal{RFrob}(\mathcal{H}ilb) \\ \mathcal{Z} &\mapsto \mathcal{Z}(\mathbb{S}^1) . \end{aligned} \tag{2.3.3}$$

In Sections 4.3.2 and 4.3.3 we furthermore generalise the state-sum construction of TFTs. We find that a strongly separable symmetric RFA A (as defined in Section 4.1.1) provides the data for the state-sum construction of an aQFT, and the resulting aQFT corresponds, via Theorem 2.3.2, to the commutative RFA given by the centre of A , see Theorem 4.3.11.

The main example of an aQFT is 2-dimensional Yang-Mills (2d YM) theory for a compact semisimple Lie group G [Mig, Rus, Wit1], in which case the Hilbert space assigned to a circle is $Cl^2(G)$, that is, square integrable class functions on G . We treat this example in detail in Section 4.4.

An important example of an RFA is $L^2(G)$, the square integrable functions on a compact semisimple Lie group G . Here, the structure maps μ_a and Δ_a do have zero area limits given by the convolution product and by $\Delta_0(f)(g, h) := f(gh)$. The unit and counit families η_a and ε_a on the other hand do not have $a \rightarrow 0$ limits, see Section 4.4.1 for details.

$L^2(G)$ is a \dagger -RFA and hence by Theorem 2.3.1 a direct sum of finite-dimensional RFAs (Proposition 4.4.2). Furthermore $L^2(G)$ is strongly separable and the 2d YM theory is defined via the state-sum construction from $L^2(G)$. The center of $L^2(G)$ is $Cl^2(G)$ which is the state space of the 2d YM for a circle.

We have seen that the first new feature one encounters when passing from 2d TFTs to aQFTs is the possibility of infinite-dimensional state spaces. When one develops the

³In [Seg3, Bar] the classification is instead in terms of algebras with a non-degenerate trace and an approximate unit. However, it is implicitly assumed there that the zero-area limit of the pair of pants with two in-going and one out-going boundary circles exists. This is not true for all examples as the commutative RFAs in Remark 4.1.32 illustrate.

theory in the presence of line defects one encounters a second new feature, namely that line defects can be transmissive to area or not. Let us explain this point in more detail.

The category of bordisms with area and defects $\mathcal{Bord}_2^{area, def}(D_2, D_1)$ is defined similarly as $\mathcal{Bord}_2^{def}(D_2, D_1)$ for defect labels D_1 and world sheet phases D_2 . It is natural to equip the connected components of the defect-submanifold with a length parameter $l \in \mathbb{R}_{>0}$. This is suggested by the motto: “If in an n -dimensional volume-dependent theory with defects the surrounding n -dimensional theory is trivial, one should end up with an $(n - 1)$ -dimensional volume-dependent theory.” We will attach an independent area parameter to each connected surface component of the complement of the defect-submanifold.

A defect aQFT is defined to be a symmetric monoidal functor

$$\mathcal{Z} : \mathcal{Bord}_2^{area, def}(D_2, D_1) \rightarrow \mathcal{Hilb} , \quad (2.3.4)$$

and where \mathcal{Z} is demanded to be continuous in the area and length parameters (Definition 4.2.17).

Let \mathcal{Z} be a defect aQFT and consider a surface Σ with defect lines where one such line (or circle) is labeled by $x \in D_1$. Suppose the area of the connected surface component to the right of that line is a and that to the left is b . The defect condition x is called *transmissive* if for all such surfaces Σ , $\mathcal{Z}(\Sigma)$ only depends on $a + b$, and not on a and b separately (i.e. not on $a - b$). We interpret this as area flowing through the defect line labeled x without affecting the value of \mathcal{Z} .

To construct examples of defect aQFTs in a systematic way, in Sections 4.3.5–4.3.7 we generalise the state-sum construction of defect TFTs given in [DKR] to accommodate area- and length-dependence. If defect aQFTs are evaluated on bordisms without defects, one just obtains an aQFT as before, though one which depends on the label from D_2 attached to the surface. Indeed, we will choose

$$D_2^{ss} = \{ \text{strongly separable symmetric RFAs} \} . \quad (2.3.5)$$

A defect line separating connected components of Σ labeled by A and B in D_2 is in turn labeled by an A - B -bimodule M , which is in addition *dualisable*. Bimodules over regularised algebras are defined in Section 4.1.6. They are objects $M \in \mathcal{Hilb}$ together with a bounded linear map $\rho_{a,l,b} : A \otimes M \otimes B \rightarrow M$, which now depends on three parameters $a, l, b \in \mathbb{R}_{>0}$, subject to some natural conditions, see Definition 4.1.37. In the state-sum construction a, b are interpreted as area and l as length in a rectangular plaquette bisected by the defect line. A bimodule is dualisable if it forms part of a dual pair of bimodules, we refer to Definition 4.1.43 for details. Altogether:

$$D_1^{ss} = \{ \text{dualisable bimodules over strongly separable symmetric RFAs} \} . \quad (2.3.6)$$

Our main result with regard to defect aQFTs is (Theorem 4.3.14 and Proposition 4.3.17):

Theorem 2.3.3. *The state-sum construction defines a defect aQFT*

$$\mathcal{Z}^{ss} : \mathcal{Bord}_2^{area, def}(D_2^{ss}, D_1^{ss}) \rightarrow \mathcal{Hilb} .$$

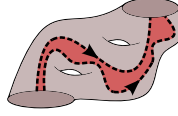


Figure 2.2: Surface with parallel defect lines. The defect lines are the dotted lines in the figure. In this figure they both start and end on a boundary component. The defect lines both have length l and the area of the surface component between them is a .

Crucially, one can define the tensor product $M \otimes_A N$ of bimodules. It satisfies a universal property (Definition 4.1.49), and it can be shown to exist in some natural cases⁴ at least in \mathcal{Hilb} (Proposition 4.1.60). The tensor product of bimodules is designed to model the “fusion of defect lines” in a defect aQFT in the following sense. Let $\Sigma(a)$ be a bordism with two parallel defect lines, one labeled by $M \in D_1$ and the other by $N \in D_1$, and assume the connected surface component separating them is labeled by $A \in D_2$ (Figure 2.2).⁵ Denote the area assigned to this component by a and assume that the two defect lines have the same length label l . Let Σ' be equal to $\Sigma(a)$, except that the component separating M and N has been collapsed, resulting in a single defect line which is now labeled by $M \otimes_A N$. Then (Theorem 4.3.20 and Remark 4.3.21):

Theorem 2.3.4. $\lim_{a \rightarrow 0} \mathcal{Z}^{ss}(\Sigma(a)) = \mathcal{Z}^{ss}(\Sigma')$.

An important example of a defect aQFT is again provided by 2d YM theory with G as above. In this case, the label set for two-dimensional connected components is just the strongly separable RFA $D_2 = \{L^2(G)\}$ (corresponding to the 2d YM theory without defects given by G), and a possible choice for D_1 is a collection of bimodules of the form $R \otimes L^2(G)$ where R denotes a finite-dimensional unitary representation of G . A defect line labeled by $R \otimes L^2(G) \in D_1$ corresponds to a Wilson line observable labeled by R . In the case G is connected, Wilson lines are transmissive, if and only if the G -representation R labeling it is a direct sum of trivial representations (Section 4.4.3). Examples of defects that are not Wilson lines can be obtained by twisting the action on the regular bimodule $L^2(G)$ by appropriate automorphisms of G (Lemma 4.4.16).

As expected, the fusion of Wilson lines labeled R and S is given by the G -representation $R \otimes S$, which in terms of Theorem 2.3.4 follows from the bimodule tensor product $(R \otimes L^2(G)) \otimes_{L^2(G)} (S \otimes L^2(G)) \cong (R \otimes S) \otimes L^2(G)$ (Proposition 4.4.10).

⁴These bimodules need to be left and right modules as well (which is not automatic). For details see Remark 4.1.38 and Lemma 4.1.59.

⁵These bimodules also need to be such that their tensor product has a dual, for more details see Lemma 4.1.55, the precise conditions of Theorem 4.3.20 and Remark 4.3.21.

Chapter 3

Topological field theory on r -spin surfaces and the Arf invariant

This chapter contains a detailed study of topological field theories on r -spin surfaces. This part of the thesis has appeared in [RS1]. In Section 3.1 we describe the combinatorial model for r -spin structures and state its main properties. In Section 3.2 we use this model to give a state-sum construction of r -spin TFTs, and we compute the value of these TFTs on several bordisms as an example. In Section 3.3, the action of a set of generators of the mapping class group on r -spin structures is expressed in terms of the data of the combinatorial model. In Section 3.4 we show that for r even, the r -spin state-sum TFT for the two-dimensional Clifford algebra computes the r -spin Arf-invariant. Section 3.5 contains the proof of Theorem 2.2.2 and also an explicit count of the mapping class group orbits in the genus 0 case. Finally, in Appendix 3.A we relate the description of r -spin structures in terms of PLCW-decompositions that we use here to the triangulation-based model of [Nov]. We furthermore give the proofs of those properties of the combinatorial model and of r -spin state-sum TFTs which require the triangulation-based model and have been omitted in the main text.

3.1 Combinatorial description of r -spin surfaces

In this section we present the combinatorial model for r -spin structures and state its properties. We start by reviewing the definition of an r -spin structure (Section 3.1.1) and of the decomposition of surfaces we will use (Section 3.1.2). The main results in this section are the bijection of the combinatorial data modulo an appropriate equivalence relation and isomorphism classes of r -spin structures (Theorem 3.1.13 in Section 3.1.3) and the counting of these isomorphism classes for compact connected surfaces (Proposition 3.1.19 in Section 3.1.5).

3.1.1 r -spin surfaces

Here we recall the definition of r -spin structures and of related notions, following [Nov]. Denote by $GL_2^+(\mathbb{R})$ the set of real 2×2 matrices of positive determinant, and let $p_{GL}^r : \widetilde{GL}_2^r \rightarrow GL_2^+(\mathbb{R})$ be the r -fold connected cover for $r \in \mathbb{Z}_{>0}$ and the universal cover for $r = 0$. Note that in both cases the fibres are isomorphic to $\mathbb{Z}_r = \mathbb{Z}/r\mathbb{Z}$. By a *surface* we mean an oriented two-dimensional smooth manifold. For a surface Σ we denote by $F_{GL^+}\Sigma \rightarrow \Sigma$ the oriented frame bundle over Σ (“oriented” means that orientation on the tangent space induced by the frame agrees with that of Σ).

- Definition 3.1.1.** 1. An r -spin structure on a surface Σ is a pair (η, p) , where $\eta : P_{\widetilde{GL}}\Sigma \rightarrow \Sigma$ is a principal \widetilde{GL}_2^r -bundle and $p : P_{\widetilde{GL}}\Sigma \rightarrow F_{GL^+}\Sigma$ is a bundle map intertwining the \widetilde{GL}_2^r - and GL_2^+ -actions on $P_{\widetilde{GL}}\Sigma$ and $F_{GL^+}\Sigma$ respectively.
2. An r -spin surface is a surface together with an r -spin structure.
3. A *morphism of r -spin surfaces* $\tilde{f} : \Sigma \rightarrow \Sigma'$ is a bundle map between the r -spin surfaces, such that the underlying map of surfaces f is a local diffeomorphism, and such that the diagram

$$\begin{array}{ccc}
 P_{\widetilde{GL}}\Sigma & \xrightarrow{\tilde{f}} & P_{\widetilde{GL}}\Sigma' \\
 p \downarrow & & \downarrow p' \\
 F_{GL^+}\Sigma & \xrightarrow{df_*} & F_{GL^+}\Sigma' \\
 \downarrow & & \downarrow \\
 \Sigma & \xrightarrow{f} & \Sigma'
 \end{array} \tag{3.1.1}$$

commutes, where df_* denotes the induced map from the derivative of f .

4. A *morphism of r -spin structures over Σ* is a morphism of r -spin surfaces whose underlying map of surfaces is the identity on Σ . We write

$$\mathcal{R}^r(\Sigma) \tag{3.1.2}$$

for the set of isomorphism classes of r -spin structures on Σ .

Note that $p : P_{\widetilde{GL}}\Sigma \rightarrow F_{GL^+}\Sigma$ is a \mathbb{Z}_r -principal bundle ($r \in \mathbb{Z}_{\geq 0}$). Also, morphisms of r -spin structures are always isomorphisms as they are maps of principal bundles. A *diffeomorphism of r -spin surfaces* is a morphism of r -spin surfaces with a diffeomorphism as underlying map of surfaces. Let us denote by

$$\mathcal{D}^r(\Sigma) \tag{3.1.3}$$

the diffeomorphism classes of r -spin surfaces with underlying surface Σ . Note that by construction we have a surjection

$$\mathcal{R}^r(\Sigma) \twoheadrightarrow \mathcal{D}^r(\Sigma) , \quad (3.1.4)$$

given by passing to orbits under the action of the mapping class group of Σ acting on $\mathcal{R}^r(\Sigma)$. As we shall see, this surjection is almost never injective.

Even though we do not need it in the rest of the paper, let us mention that a 0-spin structure is the same as a framing. A *framing* of Σ is a homotopy class of a *framing* of Σ is a homotopy class of trivialisations of the oriented frame bundle over Σ . of the oriented frame bundle over Σ . Let $T(\Sigma)$ denote the set of framings of Σ . We have:

Proposition 3.1.2. *There is a bijection $T(\Sigma) \xrightarrow{\sim} \mathcal{R}^0(\Sigma)$.*

Proof. Take a framing and pick a representative trivialisation, i.e. an isomorphism of GL_2^+ principal bundles $\varphi : F_{GL}\Sigma \xrightarrow{\sim} GL_2^+ \times \Sigma$. Define

$$\begin{aligned} p_\varphi &:= \left[\widetilde{GL}_2^0 \times \Sigma \xrightarrow{p_{GL}^0 \times \text{id}_\Sigma} GL_2^+ \times \Sigma \xrightarrow{\varphi^{-1}} F_{GL}\Sigma \right] , \\ \pi_\varphi &:= \left[\widetilde{GL}_2^0 \times \Sigma \xrightarrow{p_\varphi} F_{GL}\Sigma \rightarrow \Sigma \right] . \end{aligned} \quad (3.1.5)$$

Then $\rho_\varphi := (\pi_\varphi, p_\varphi)$ is a 0-spin structure. Changing φ by a homotopy gives an isomorphic 0-spin structure [Hus, Ch. 4, Thm. 9.9]. This defines a map $F : T(\Sigma) \rightarrow \mathcal{R}^0(\Sigma)$.

Next we define a map in the opposite direction. Since \widetilde{GL}_2^0 is contractible, for any 0-spin structure $\zeta = (\pi : P_{\widetilde{GL}}\Sigma \rightarrow \Sigma, p)$, π is a trivialisable \widetilde{GL}_2^0 principal bundle [Stee, Thm. 12.2]. Let $\tilde{\phi}_\zeta : P_{\widetilde{GL}}\Sigma \rightarrow \widetilde{GL}_2^0 \times \Sigma$ denote such a trivialisation. Then there exists a unique morphism of principal GL_2^+ bundles $\phi_\zeta : F_{GL}\Sigma \rightarrow GL_2^+ \times \Sigma$ such that

$$\begin{array}{ccc} P_{\widetilde{GL}}\Sigma & \xrightarrow{\tilde{\phi}_\zeta} & \widetilde{GL}_2^0 \times \Sigma \\ p \downarrow & & \downarrow p_{GL}^0 \times \text{id}_\Sigma \\ F_{GL}\Sigma & \xrightarrow{\phi_\zeta} & GL_2^+ \times \Sigma \end{array} \quad (3.1.6)$$

commutes. Again by contractability, any two choices of trivialisations $\tilde{\phi}_\zeta$ are homotopic and so the corresponding ϕ_ζ are homotopic, too. By the same argument, different choices of representatives of isomorphism classes of 0-spin structures give homotopic ϕ_ζ 's. This defines a map $G : \mathcal{R}^0(\Sigma) \xrightarrow{\sim} T(\Sigma)$.

The two maps F and G are inverse to each other. Indeed, for $[\zeta] \in \mathcal{R}^0(\Sigma)$, the 0-spin structure one obtains after constructing $F(G([\zeta]))$ is isomorphic to ζ via $\tilde{\phi}_\zeta$ as in (3.1.6), so that indeed $F(G([\zeta])) = [\zeta]$. Conversely, starting from a homotopy class of trivialisations $[\varphi] \in T(\Sigma)$, in computing $G(F([\varphi]))$ we see that in (3.1.6) we can take $\tilde{\phi}_\zeta = \text{id}$ and $\phi_\zeta = \varphi$, so that $G(F([\varphi])) = [\varphi]$. \square

After this aside on framings, let us return to r -spin surfaces and give a basic example which will later serve to parametrise the boundary components of r -spin bordisms.

Example 3.1.3. For $\kappa \in \mathbb{Z}$ let \mathbb{C}^κ denote the r -spin structure on \mathbb{C}^\times given by the trivial principal \widetilde{GL}_2^r -bundle $\widetilde{GL}_2^r \times \mathbb{C}^\times$ and the map

$$\begin{aligned} p^\kappa : \widetilde{GL}_2^r \times \mathbb{C}^\times &\rightarrow GL_2^+ \times \mathbb{C}^\times \\ (g, z) &\mapsto (z^\kappa \cdot p_{GL}^r(g), z) , \end{aligned} \quad (3.1.7)$$

where $z \in \mathbb{C}^\times$ acts on $M \in GL_2^+$ by

$$z.M = \begin{pmatrix} \operatorname{Re} z & -\operatorname{Im} z \\ \operatorname{Im} z & \operatorname{Re} z \end{pmatrix} M . \quad (3.1.8)$$

Since the \widetilde{GL}_2^r -action is from the right and p_{GL}^r is a group homomorphism, p^κ indeed intertwines the \widetilde{GL}_2^r - and GL_2^+ -actions.

Lemma 3.1.4 ([Nov, Sec.3.4]). \mathbb{C}^κ and $\mathbb{C}^{\kappa'}$ are isomorphic r -spin structures if and only if $\kappa \equiv \kappa' \pmod{r}$. The map $\mathbb{Z}_r \rightarrow \mathcal{R}^r(\mathbb{C}^\times)$, $\kappa \mapsto [\mathbb{C}^\kappa]$ is a bijection.

In the case that $r > 0$, it will be convenient to fix once and for all a set of representatives of \mathbb{Z}_r in \mathbb{Z} , say $\{0, 1, \dots, r-1\}$, and to agree that for $\lambda \in \mathbb{Z}_r$, \mathbb{C}^λ stands for \mathbb{C}^κ , with $\kappa \in \mathbb{Z}$ the chosen representative for λ .

Notations 3.1.5. For an r -spin surface Σ , by abuse of notation we will often use the same symbol Σ to denote its underlying surface. That is, Σ stands for the triple Σ, η, p from Definition 3.1.1 (1).

A *collar* is an open neighbourhood of \mathbb{S}^1 in \mathbb{C}^\times . An *ingoing* (resp. *outgoing*) *collar* is the intersection of a collar with the set $\{z \in \mathbb{C}^\times \mid |z| \geq 1\}$ (resp. $\{z \in \mathbb{C}^\times \mid |z| \leq 1\}$). A *boundary parametrisation* of a surface Σ is:

1. A disjoint decomposition $B_{\text{in}} \sqcup B_{\text{out}} = \pi_0(\partial\Sigma)$ (the in- and outgoing boundary components). B_{in} and/or B_{out} are allowed to be empty.
2. A collection of ingoing collars U_b , $b \in B_{\text{in}}$, and outgoing collars V_c , $c \in B_{\text{out}}$, together with a pair of orientation preserving embeddings

$$\phi_{\text{in}} : \bigsqcup_{b \in B_{\text{in}}} U_b \hookrightarrow \Sigma \hookleftarrow \bigsqcup_{c \in B_{\text{out}}} V_c : \phi_{\text{out}} . \quad (3.1.9)$$

We require that for each b , the restriction $\phi_{\text{in}}|_{U_b}$ maps \mathbb{S}^1 diffeomorphically to the connected component b of $\partial\Sigma$, and analogously for $\phi_{\text{out}}|_{V_c}$.

An r -spin *boundary parametrisation* of an r -spin surface Σ is:

1. A boundary parametrisation of the underlying surface Σ as above; we use the same notation as in (3.1.9).

2. A pair of maps fixing the restriction of the r -spin structure to the in- and outgoing boundary components

$$\begin{aligned} \lambda : B_{in} &\rightarrow \mathbb{Z}_r & \text{and} & & \mu : B_{out} &\rightarrow \mathbb{Z}_r. \\ b &\mapsto \lambda_b & & & c &\mapsto \mu_c \end{aligned} \quad (3.1.10)$$

3. A pair of morphisms of r -spin surfaces which parametrise the in- and outgoing boundary components by collars with r -spin structure,

$$\varphi_{in} : \bigsqcup_{b \in B_{in}} U_b^{\lambda_b} \hookrightarrow \Sigma \hookleftarrow \bigsqcup_{c \in B_{out}} V_c^{\mu_c} : \varphi_{out} . \quad (3.1.11)$$

Here, $U_b^{\lambda_b}$ is the restriction of \mathbb{C}^{λ_b} to the ingoing collar U_b , and analogously $V_c^{\mu_c} := \mathbb{C}^{\mu_c}|_{V_c}$. The maps of surfaces underlying $\varphi_{in/out}$ are required to be the maps $\phi_{in/out}$ in (3.1.9) from Part 1.

Note that by Lemma 3.1.4, the maps λ, μ in part 2 are not extra data, but are uniquely determined by the r -spin surface Σ and the boundary parametrisation.

For diffeomorphisms between r -spin surfaces with parametrised boundary we only require that they respect germs of the boundary parametrisation. In more detail, let Σ be as in (3.1.11) and let

$$\psi_{in} : \bigsqcup_{d \in B'_{in}} P_d^{\rho_d} \hookrightarrow \Xi \hookleftarrow \bigsqcup_{e \in B'_{out}} Q_e^{\sigma_e} : \psi_{out} \quad (3.1.12)$$

be another r -spin surface with boundary parametrisation. A *diffeomorphism of r -spin surfaces with boundary parametrisation* $\Sigma \rightarrow \Xi$ is an r -spin diffeomorphism $f : \Sigma \rightarrow \Xi$ subject to the following compatibility condition. Let $b \in B_{in}^\Sigma$ be an ingoing boundary component of Σ and let $f_*(b) \in \pi_0(\partial\Xi)$ be its image under f . We require that $f_*(b) \in B_{in}^\Xi$ and that $\lambda_b = \rho_{f_*(b)}$. Furthermore, there has to exist an ingoing collar C contained in both U_b and $P_{f_*(b)}^{\rho_{f_*(b)}}$ such that the diagram

$$\begin{array}{ccc} & U_b^{\lambda_b} & \xrightarrow{\varphi_{in}} \Sigma \\ C^{\lambda_b} & \swarrow & \downarrow f \\ & P_{f_*(b)}^{\rho_{f_*(b)}} & \xrightarrow{\psi_{in}} \Xi \end{array} \quad (3.1.13)$$

of r -spin morphisms commutes. An analogous condition has to hold for each outgoing boundary component $c \in B_{out}$.

By an *r -spin object* we mean a pair (X, ρ) consisting of a finite set X and a map $\rho : X \rightarrow \mathbb{Z}_r$, $x \mapsto \rho_x$. Below we will construct a category whose objects are r -spin objects, and whose morphisms are certain equivalence classes of r -spin surfaces, which we turn to now.

Definition 3.1.6. Let (X, ρ) and (Y, σ) be two r -spin objects. An r -spin bordism from (X, ρ) to (Y, σ) is a compact r -spin surface Σ with boundary parametrisation as in (3.1.11) together with bijections $\beta_{\text{in}} : X \xrightarrow{\sim} B_{\text{in}}$ and $\beta_{\text{out}} : Y \xrightarrow{\sim} B_{\text{out}}$ such that

$$\begin{array}{ccc} X & \xrightarrow{\beta_{\text{in}}} & B_{\text{in}} \\ & \searrow \rho & \swarrow \lambda \\ & \mathbb{Z}_r & \end{array} \quad \text{and} \quad \begin{array}{ccc} Y & \xrightarrow{\beta_{\text{out}}} & B_{\text{out}} \\ & \searrow \sigma & \swarrow \mu \\ & \mathbb{Z}_r & \end{array} \quad (3.1.14)$$

commute. We will often abbreviate an r -spin bordism Σ from (X, ρ) to (Y, σ) as $\Sigma : \rho \rightarrow \sigma$.

Given r -spin bordisms $\Sigma : \rho \rightarrow \sigma$ and $\Xi : \sigma \rightarrow \tau$, the *glued r -spin bordism* $\Xi \circ \Sigma : \rho \rightarrow \tau$ is defined as follows. Denote by Y the source of σ , i.e. $\sigma : Y \rightarrow \mathbb{Z}_r$. For every $y \in Y$, the boundary component $\beta_{\text{out}}^\Sigma(y)$ of Σ is glued to the boundary component $\beta_{\text{in}}^\Xi(y)$ of Ξ using the r -spin boundary parametrisations $\varphi_{\text{out}}^\Sigma$ and φ_{in}^Ξ . The diagrams in (3.1.14) ensure that the r -spin structures on the corresponding collars are restrictions of the same r -spin structure on \mathbb{C}^\times .

Two r -spin bordisms between the same r -spin objects, $\Sigma, \Sigma' : (X, \rho) \rightarrow (Y, \sigma)$ are called *equivalent* if there is a diffeomorphism $f : \Sigma \rightarrow \Sigma'$ of r -spin surfaces with boundary parametrisation such that with $f_* : \pi_0(\partial\Sigma) \rightarrow \pi_0(\partial\Sigma')$,

$$\begin{array}{ccc} & B_{\text{in}} & \\ \beta_{\text{in}} \nearrow & \downarrow f_* & \nwarrow \beta_{\text{out}} \\ X & & Y \\ \beta'_{\text{in}} \searrow & \downarrow f_* & \swarrow \beta'_{\text{out}} \\ & B'_{\text{in}} & \end{array} \quad \text{and} \quad \begin{array}{ccc} & B_{\text{out}} & \\ \beta_{\text{out}} \nwarrow & \downarrow f_* & \swarrow \beta'_{\text{out}} \\ & B'_{\text{out}} & \end{array} \quad (3.1.15)$$

commute. Let $[\Xi] : \sigma \rightarrow \tau$ and $[\Sigma] : \rho \rightarrow \sigma$ be equivalence classes of r -spin bordisms. The composition $[\Xi] \circ [\Sigma] := [\Xi \circ \Sigma] : \rho \rightarrow \tau$ is well defined, that is independent of the choice of representatives Ξ, Σ of the classes to be glued. In the following we will by abuse of notation write the same symbol Σ for an r -spin bordism Σ and its equivalence class $[\Sigma]$.

Definition 3.1.7. The *category of r -spin bordisms* \mathcal{Bord}_2^r has r -spin objects as objects and equivalence classes of r -spin bordisms as morphisms.

\mathcal{Bord}_2^r is a symmetric monoidal category with tensor product on objects and morphisms given by disjoint union. The identities and the symmetric structure are given by r -spin cylinders with appropriately parametrised boundary.

3.1.2 PLCW decompositions

In Section 3.1.3 we will use a cell decomposition to combinatorially encode r -spin structures on surfaces, and in Section 3.2.3 we will use this description to build an r -spin TFT. For explicit calculations it is helpful to keep the number of faces and edges to a minimum.

The notion of a PLCW decomposition from [Kir], and which we review in this section, is well suited for such calculations. For example, there is a PLCW decomposition of a torus consisting of 1 face, 2 edges and 1 vertex, see Figure 3.1. For comparison, using simplicial sets would require at least 2 faces, 3 edges and 1 vertex; using simplicial complexes (i.e. triangulations, as in [Nov]) would require at least 14 faces, 21 edges and 7 vertices (see e.g. [Lut]).

Now we turn to the definitions following [Kir]. Let $C \subset \mathbb{R}^N$ be a compact set, let \mathring{C} denote its interior and let $\dot{C} := C \setminus \mathring{C}$ denote its boundary. Let $B^N = [-1, 1]^N \in \mathbb{R}^N$ denote the closed N -ball, or rather a piece-wise linear (PL for short) version thereof. Then $\dot{B}^N = S^{N-1}$ is the (PL-version of the) $(N - 1)$ -sphere. A PL map $\varphi : C \rightarrow \mathbb{R}^M$ is called a *regular map* if $\varphi|_{\text{Int}(C)}$ is injective. A compact subset $C \subset \mathbb{R}^N$ is a *generalised n -cell* (or simply *cell*), if $\mathring{C} = \varphi(\mathring{B}^n)$ and $\dot{C} = \varphi(\dot{B}^n)$ for a regular map $\varphi : B^n \rightarrow C$, which we call a *characteristic map of C* . A *generalised cell decomposition* is a finite collection of cells such that the interiors of cells do not intersect and the boundary of any cell is a union of cells. Examples are shown in Figure 3.1 and in Figure 3.2. We denote the *n -skeleton* of K by K^n , which is the union of the set of k -cells K_k with $k \leq n$, and we define the *dimension* $\dim K$ of K to be the highest integer n for which the set of n -cells is nonempty. We denote the set of boundaries of an n -cell $C \in K_n$ by $\partial(C) \subset K_{n-1}$. A *regular cell map* $f : L \rightarrow K$ between generalised cell decompositions L and K is a piecewise linear map $f : \bigcup_{C \in L} C \rightarrow \bigcup_{D \in K} D$ such that for every $C \in L$ with characteristic map φ there is a cell $D = f(C) \in K$ for which $f \circ \varphi$ is a characteristic map. An example of a regular cell map is shown in Figure 3.1, a non-example is shown in Figure 3.2 b).

Definition 3.1.8. A *PLCW decomposition* K is a generalised cell decomposition of dimension n such that if $n > 0$

- K^{n-1} is a PLCW decomposition and
- for any n -cell $A \in K_n$ with characteristic map φ there is a PLCW decomposition L of S^{n-1} , such that $\varphi|_{S^{n-1}} : L \rightarrow K^{n-1}$ is a regular cell map.

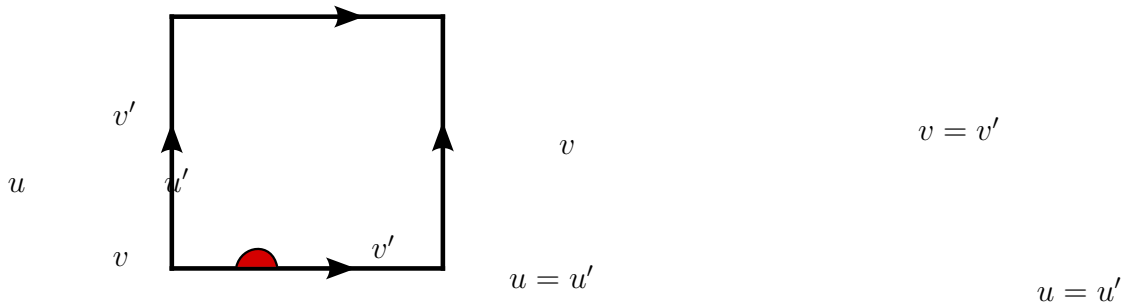


Figure 3.1: Glueing a torus from a rectangle. Each step is a regular cell map and each generalised cell decomposition is a PLCW decomposition.

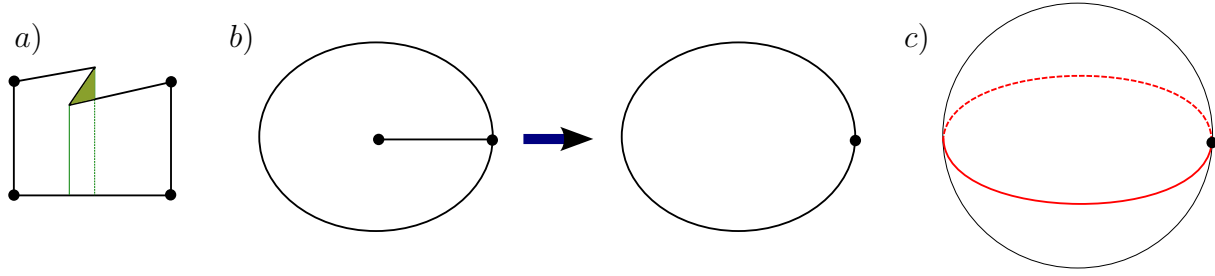


Figure 3.2: *a)* A generalised cell decomposition which is not a PLCW decomposition. There are one 2-cell, four 1-cells and four 0-cells. One can visualise it by folding a paper and glueing it only along the bottom edge. *b)* A triangle with two sides identified and a 1-gon, both PLCW decompositions. The map between them is not a regular cell map as the edge in the middle has no image. *c)* A PLCW decomposition of a sphere into two faces, one edge (red line) and one vertex.

Examples of PLCW decompositions are shown in Figure 3.1, Figure 3.2 *b)* and *c)*. A generalised cell decomposition which is not a PLCW decomposition is shown in Figure 3.2 *a)*. Each PLCW decomposition can be related by a series of local elementary moves (cf. Section 3.1.4 below), and each PLCW decomposition can be refined to a simplicial complex [Kir, Thm. 6.3]. For more details see [Kir, Sec. 6–8].

From now on we specialise to 2 dimensional PLCW decompositions. Let Σ be a compact surface with a PLCW decomposition $\Sigma_2, \Sigma_1, \Sigma_0$. We call these sets *faces*, *edges* and *vertices* respectively; one can think of faces as n -gons with $n \geq 1$. For $g + b \geq 1$, PLCW decompositions also allow for a decomposition of any compact connected surface $\Sigma_{g,b}$ of genus g and with b boundary components into a single face which is a $(4g + 3b)$ -gon, see Section 3.1.5.

To apply PLCW decompositions to smooth manifolds, we can use that a PLCW decomposition can be refined to a simplicial complex, and that PL cell maps for a simplicial complex can be approximated by smooth maps, giving smooth manifolds [Mun, Sec. 10].

3.1.3 Combinatorial description of r -spin structures

In this section we extend the combinatorial description of r -spin structures in [Nov], which uses a triangulation of the underlying surface, to PLCW decompositions. We will only consider PLCW decompositions where the boundary components consist of a single vertex and a single edge.

Let Σ be a surface with parametrised boundary, with a PLCW decomposition, with a marking of one edge of each face and an orientation of each edge. We do not require that the orientation of the boundary edges corresponds to the orientation of the boundary components, but we orient the faces according to the orientation of the surface. This induces an ordering of the edges of each face, the starting edge being the marked one, see Figure 3.3. By an *edge index assignment* we mean a map $s : \Sigma_1 \rightarrow \mathbb{Z}_r, e \mapsto s_e$.

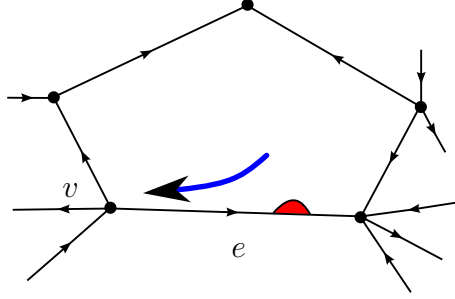


Figure 3.3: Figure of a face with adjacent edges and vertices in a marked PLCW decomposition. The orientation of the face is that of the paper plane, the orientation of the edges is indicated by an arrow on them. The half-dot indicates the marked edge of the face the half-dot lies in. The arrow in the middle shows the clockwise direction along the marked edge e and v is the vertex sitting on the boundary of e in clockwise direction. Note that the clockwise vertex v of the edge e is determined by the orientation of the face and not by the orientation of the edge e .

Definition 3.1.9. We call an assignment of edge markings, edge orientations and edge indices a *marking* of a PLCW decomposition and a PLCW decomposition together with a marking a *marked PLCW decomposition*.

For a vertex $v \in \Sigma_0$ let D_v be the number of faces whose marked edge has v as its boundary vertex in clockwise direction (with respect to the orientation of the face), as shown in Figure 3.3. Let $\partial^{-1}(v) \subset \Sigma_1$ denote the edges whose boundary contain v :

$$\partial^{-1}(v) := \{ e \in \Sigma_1 \mid v \in \partial(e) \} . \quad (3.1.16)$$

The orientation of an edge gives a starting and an ending vertex, which might be the same. Let N_v^{start} (resp. N_v^{end}) be the number of edges starting (resp. ending) at the vertex v and let

$$N_v = N_v^{\text{start}} + N_v^{\text{end}} . \quad (3.1.17)$$

We note that an edge which starts and ends at v contributes 1 to both N_v^{start} and to N_v^{end} . For every edge $e \in \partial^{-1}(v)$ let

$$\hat{s}_e = \begin{cases} -1 & \text{if } e \text{ starts and ends at } v, \\ s_e & \text{if } e \text{ is pointing out of } v, \\ -1 - s_e & \text{if } e \text{ is pointing into } v. \end{cases} \quad (3.1.18)$$

Recall the maps $\lambda : B_{\text{in}} \rightarrow \mathbb{Z}_r$ and $\mu : B_{\text{out}} \rightarrow \mathbb{Z}_r$ from (3.1.10), as well as our convention that we only consider PLCW decompositions with exactly one vertex and one edge on each boundary component. For a vertex u on a boundary component let us write by slight abuse of notation u for this boundary component and let

$$R_u := \begin{cases} \lambda_u - 1 & \text{if } u \in B_{\text{in}}, \\ 1 - \mu_u & \text{if } u \in B_{\text{out}}. \end{cases} \quad (3.1.19)$$

We call a marking *admissible* with given maps λ and μ , if for every vertex $v \in \Sigma_0$ which is not on the boundary and for every vertex $u \in \Sigma_0$ on the boundary vertex and one edge on each boundary component) the following conditions are satisfied:

$$\sum_{e \in \partial^{-1}(v)} \hat{s}_e \equiv D_v - N_v + 1 \pmod{r}, \quad (3.1.20)$$

$$\sum_{e \in \partial^{-1}(u)} \hat{s}_e \equiv D_u - N_u + 1 - R_u \pmod{r}. \quad (3.1.21)$$

For an arbitrary marking of a PLCW decomposition of Σ one can define an r -spin structure with r -spin boundary parametrisation on Σ minus its vertices by taking the trivial r -spin structure on faces and fixing the transition functions using the marking. The above r -spin structure extends uniquely to the vertices of Σ , if and only if the marking is admissible for λ and μ . The r -spin boundary parametrisations are given by the inclusion of r -spin collars (as prescribed by λ and μ) over the collars of the boundary parametrisation of Σ . For more details on this construction we refer the reader to Appendix 3.A.3.

Definition 3.1.10. Denote the r -spin structure with r -spin boundary parametrisation defined above by $\Sigma(s, \lambda, \mu)$.

There is some redundancy in the description of an r -spin structure via a marking. A one-to-one correspondence between certain equivalence classes of markings and isomorphism classes of r -spin structures will be given in Theorem 3.1.13 below. As preparation we first give a list of local modifications of the marking which lead to isomorphic r -spin structures.

Lemma 3.1.11. *The following changes of the marking of the PLCW decomposition of Σ (but keeping the PLCW decomposition fixed) give isomorphic r -spin structures:*

1. Flip the orientation of an edge e and change its edge index $s_e \mapsto -1 - s_e$ (see Figure 3.4(1)).
2. Move the marking on an edge e of a polygon to the following edge counterclockwise and change the edge index of the previously marked edge $s_e \mapsto s_e - 1$, if this edge is oriented counterclockwise, $s_e \mapsto s_e + 1$ otherwise (see Figure 3.4 (2a) and (2b)).
3. Let $k \in \mathbb{Z}$. Shift the edge index of each edge of a polygon by $+k$, if the edge is oriented counterclockwise with respect to the orientation of the polygon, and by $-k$ otherwise. If two edges of a polygon are identified (i.e. are given by the same $e \in \Sigma_1$), do not change its edge index. For an illustration, see Figure 3.4 Part 3. We call this a deck transformation.

These operations on the marking commute with each other in the sense that the final edge indices do not depend on the order in which a given set of operations 1–3 is applied.

Note that the operation in 3 is the same as moving around the marking of a face completely by applying operation 2. This lemma is proved in Appendix 3.A.4.

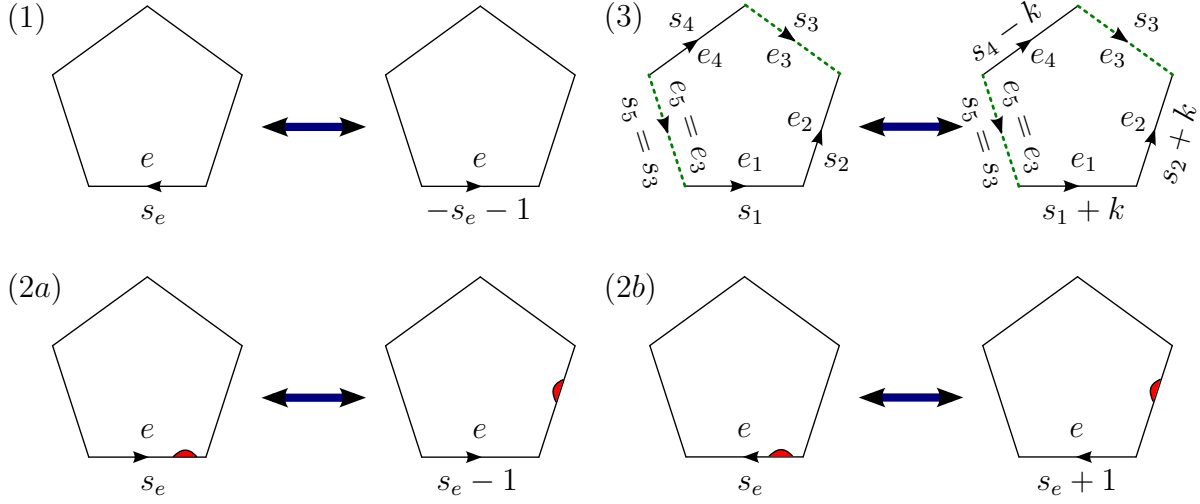


Figure 3.4: Moves of Lemma 3.1.11 for a face of Σ . All edge orientations and markings are arbitrary unless shown explicitly. (1) Flipping the edge orientation of e . (2a), (2b) Moving the edge marking for a face. (3) Shifting the edge indices for a face. The dotted edges e_3 and e_5 are identified, hence the edge index remains unchanged. The edges e_1 and e_2 are counterclockwise oriented, hence the $+k$ shift of the corresponding edge indices s_1 and s_2 , the edge e_4 is clockwise oriented, hence the $-k$ shift of s_4 .

Let Σ be a surface with a fixed PLCW decomposition. Write (m, o, s) for a given marking of Σ , where m denotes the edge markings of the faces, o the edge orientations and s the edge indices (cf. Definition 3.1.9). Let $\mathcal{M}(\Sigma)_{\lambda, \mu}^{PLCW}$ denote the set of all admissible markings for the maps λ and μ on Σ . The operations in Lemma 3.1.11 generate an equivalence relation \sim_{fix} on $\mathcal{M}(\Sigma)_{\lambda, \mu}^{PLCW}$. Let us denote equivalence classes by $[m, o, s]$. The following lemma gives a more concrete description of the equivalence classes.

Lemma 3.1.12. *Let $(m, o, s) \in \mathcal{M}(\Sigma)_{\lambda, \mu}^{PLCW}$. We have:*

1. *For every choice m', o' there is some s' such that $[m, o, s] \sim_{\text{fix}} [m', o', s']$.*
2. *For a given choice of edge indices \tilde{s} we have $[m, o, s] \sim_{\text{fix}} [m, o, \tilde{s}]$ if and only if s and \tilde{s} are related by a sequence of deck transformations (operation 3) in Lemma 3.1.11.*

Proof. The first statement is immediate from operations 1 and 2 in Lemma 3.1.11. For the second statement recall that operations 1–3 commute, and operation 3 is redundant. Any sequence of operations can thus be written as $M = \prod_e (\text{op. 1 for edge } e) \prod_f (\text{op. 2 for face } f)$. Since m and o do not change, operation 1 for an edge e must occur in pairs, leaving s_e unchanged, and operation 2 for a face f must occur in multiples of the number of edges of that face, so that the total change is expressible in terms of operation 3, $M = \prod_f (\text{op. 3 for face } f)$. \square

Let $\mathcal{R}^r(\Sigma)_{\lambda, \mu}$ denote the isomorphism classes of r -spin structures with r -spin boundary parametrisation for the maps λ and μ . The following theorem is proved in Appendix 3.A.4.

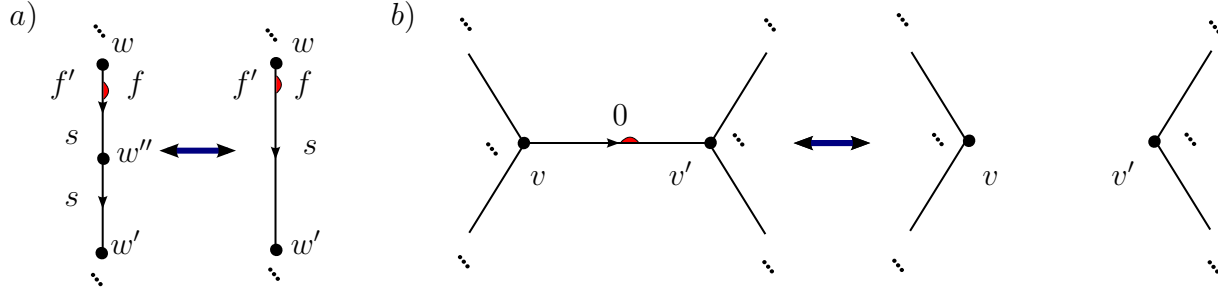


Figure 3.5: Elementary moves of a marked PLCW decomposition. Figure a) shows edges between faces f and f' (which are allowed to be the same). The edges are marked so that the vertex w is the clockwise vertex for the face f (cf. Figure 3.3). This convention is not restrictive as one can change the orientation of the edges and the markings using Lemma 3.1.11. In Figure b), on the left hand side the horizontal edge between the vertices v and v' (which are allowed to be the same) is marked for the top polygon, but not for the bottom polygon, and it has edge index 0. For the joint polygon on the right hand side, the marked edge is taken to be that from the bottom polygon on the left. Note that this latter convention for the markings is not restrictive, as using Lemma 3.1.11 one can move the markings around.

Theorem 3.1.13. *Let Σ be a surface with PLCW decomposition. The map*

$$\begin{aligned} \mathcal{M}(\Sigma)_{\lambda, \mu}^{PLCW} / \sim_{\text{fix}} &\longrightarrow \mathcal{R}^r(\Sigma)_{\lambda, \mu} \\ [m, o, s] &\longmapsto [\Sigma(s, \lambda, \mu)] \end{aligned} \quad (3.1.22)$$

is a bijection. On the right hand side it is understood that the edge markings and orientations of Σ are given by m, o .

Remark 3.1.14. When combined with Lemma 3.1.12, this shows that for a fixed edge marking and orientation the admissible edge index assignments up to deck transformations are in bijection with the isomorphism classes of r -spin structures with r -spin boundary parametrisation for the maps λ and μ .

3.1.4 Elementary moves on marked PLCW decompositions

In the previous section we defined the r -spin structure $\Sigma(s, \lambda, \mu)$ in terms of a marked PLCW decomposition, and we explained how to change the marking while staying within a given isomorphism class of r -spin structures. In this section we state how the marking needs to change when modifying the underlying PLCW decomposition by elementary moves in order to produce isomorphic r -spin structures.

Definition 3.1.15. An *elementary move* on a PLCW decomposition of a surface is either

- removing or adding a bivalent vertex as shown in Figure 3.5 a), or
- removing or adding an edge as shown in Figure 3.5 b).

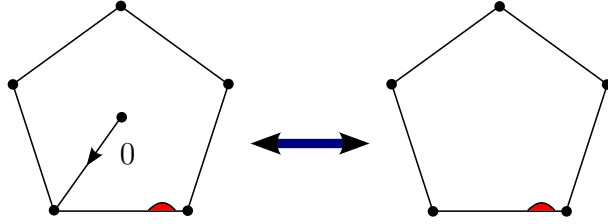


Figure 3.6: The edge index of a univalent vertex is fixed by only the marking and orientation of edges. Removing a univalent vertex induces an isomorphism of r -spin structures.

By [Kir, Thm. 7.4], any two PLCW decompositions can be related by elementary moves. We prove the following proposition in Appendix 3.A.4.

Proposition 3.1.16. *The elementary moves in Figure 3.5 induce isomorphisms of r -spin structures.*

The edge index of an edge with a univalent vertex is fixed by the orientation and the marking of the edge, in particular it is independent of the rest of the edge indices. In Lemma 3.A.6 we will show that removing univalent vertices induces an isomorphism of r -spin structures. For an illustration, see Figure 3.6.

3.1.5 Example: Connected r -spin surfaces

In this section we illustrate how one can use the combinatorial formalism to count isomorphism classes of r -spin structures. This recovers results obtained in [GG, Ran] using a different formalism.

Notations 3.1.17. Whenever it does not cause confusion we will use the same symbols for edge labels and for edge indices. For example for $e \in \Sigma_1$ we will simply write $e \in \mathbb{Z}_r$ instead of $s_e \in \mathbb{Z}_r$.

Lemma 3.1.18. *There exists r -spin structures on the sphere if and only if $r = 1$ or $r = 2$. If there exists an r -spin structure on the sphere then it is unique up to isomorphism.*

Proof. Let us consider the sphere decomposed into two 1-gons, one edge u and one vertex v as in Figure 3.2 c), with edge index u (cf. Notations 3.1.17). Let us collect the ingredients for the vertex condition (3.1.20). The edge u starts and ends at the vertex, therefore $\hat{u} = -1$ from (3.1.18).

The number N_v of in- and outgoing edges for v is $N_v = 1 + 1 = 2$, cf. (3.1.17). The number of faces with v in clockwise direction from their marked edge is $D_v = 2$, since the edge is marked for both faces. The vertex condition (3.1.20) then reads

$$-1 \equiv 2 - 2 + 1 \pmod{r},$$

which holds if and only if $r = 1$ or $r = 2$. The edge index u can be set arbitrarily by operation 3 in Lemma 3.1.11, and together with Remark 3.1.14 we see that for any two values of u the r -spin structures on the sphere are isomorphic. \square

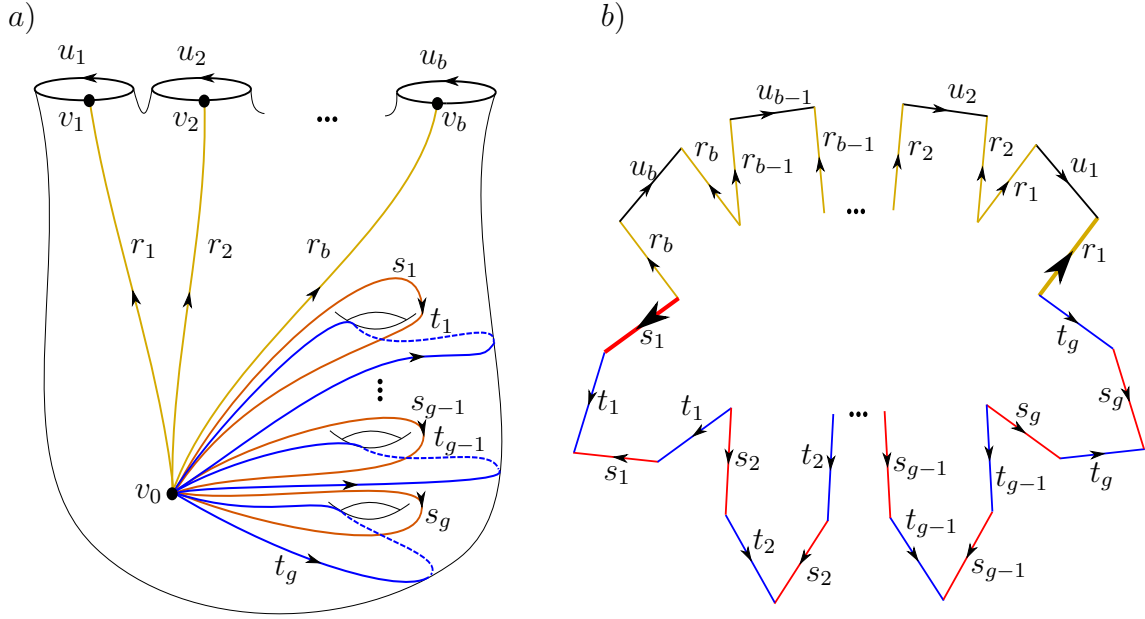


Figure 3.7: PLCW decomposition of $\Sigma_{g,b}$ for $g + b \geq 1$ using only one face, shown after glueing (Fig. a) and before glueing (Fig. b) – the edges labeled with the same symbols are identified. In Fig. b, the bigger arrows indicate the marked edge, namely edge r_1 in case $b > 0$ and edge s_1 in case $b = 0$.

Proposition 3.1.19. *Let $\Sigma_{g,b}$ be a connected surface of genus g with b boundary components and with maps λ and μ . There exists an r -spin structure on $\Sigma_{g,b}$ if and only if*

$$\chi(\Sigma_{g,b}) \equiv \sum_{u \in \pi_0(\partial\Sigma)} R_u \pmod{r}, \quad (3.1.23)$$

where $\chi(\Sigma_{g,b}) = 2 - 2g - b$ denotes the Euler characteristic and R_u was defined in (3.1.19). If (3.1.23) holds, the number $|\mathcal{R}^r(\Sigma_{g,b})_{\lambda,\mu}|$ of isomorphism classes of r -spin structures on $\Sigma_{g,b}$ is given by:

r	b, g	$ \mathcal{R}^r(\Sigma_{g,b})_{\lambda,\mu} $
0	$g = 0$ and $b \in \{0, 1\}$	1
	else	infinite
> 0	$b = 0$	r^{2g}
	$b \geq 1$	r^{2g+b-1}

A similar result has been obtained for the existence of r -spin structures on closed hyperbolic orbifolds for $r > 0$ in [GG, Thm. 3]. Note that in complex geometry, (3.1.23) (for $r > 0$ and $b = 0$) is just the condition for the existence of an r -th root of the canonical line bundle (see e.g. [Wit2]).

Proof. The case $g = b = 0$ has been discussed in Lemma 3.1.18, so we can assume $g + b \geq 1$. Decompose $\Sigma_{g,b}$ into a $(4g + 3b)$ -gon consisting of $2g + 2b$ inner edges, b boundary edges, one inner vertex v_0 and b boundary vertices v_j , $j = 1, \dots, b$, as shown in Figure 3.7 a) and b). Assign the edge indices s_i , t_i , r_j and u_j , where $i = 1, \dots, g$ and $j = 1, \dots, b$. Mark the edge s_1 if $g \neq 0$ or the edge r_1 if $g = 0$, see Figure 3.7 b).

We now evaluate the admissibility condition at each vertex. For the boundary vertex v_j there is the incoming inner edge r_j and the boundary edge u_j which starts and ends at the same vertex v_j . Therefore by (3.1.18), relative to v_j one has $\hat{r}_j = -r_j - 1$ and $\hat{u}_j = -1$. For either of the two markings (for $g \neq 0$ and for $g = 0$) $D_{v_j} = 0$ and $N_{v_j} = 3$, therefore we have

$$-r_j - 1 - 1 \equiv 0 - 3 + 1 - R_{u_j} \pmod{r} \quad \text{for } j = 1, \dots, b. \quad (3.1.24)$$

Thus the r_j are uniquely fixed by the boundary parametrisation λ, μ to be $r_j \equiv R_{u_j} \pmod{r}$ for all j .

For the inner vertex v_0 there are b edges leaving the vertex and $2g$ edges which start and end there. Therefore by (3.1.18), relative to v_0 one has $\hat{r}_j = r_j$ and $\hat{s}_i = \hat{t}_i = -1$. $D_{v_0} = 1$ and $N_{v_0} = 4g + b$, and so

$$\sum_{j=1}^b r_j - 2g \equiv 1 - (4g + b) + 1 \pmod{r}. \quad (3.1.25)$$

Combining (3.1.24) and (3.1.25) one obtains (3.1.23).

By Remark 3.1.14, for a fixed marking and orientation, edge index assignments up to deck transformations are in bijection with r -spin structures. From (3.1.24) and (3.1.25) every $(s_i, t_i, u_j) \in (\mathbb{Z}_r)^{2g+b}$ gives an admissible edge index assignment. A deck transformation on the face shifts the u_j parameters simultaneously and leaves the s_i and t_i parameters fixed. By a simple counting we get the number of isomorphism classes of r -spin structures. \square

Corollary 3.1.20. *There is a unique r -spin structure on the disk with boundary condition $\lambda = 2$ (ingoing boundary) or $\lambda = 0$ (outgoing boundary), and no r -spin structure else.*

Let $R_j := R_{u_j}$ from (3.1.19) and let us denote the r -spin structure on $\Sigma_{g,b}$ given by the parameters $s_i, t_i, u_j \in \mathbb{Z}_r$ for $i = 1, \dots, g$ and $j = 1, \dots, b$ from Figure 3.7 by

$$\Sigma_{g,b}(s_i, t_i, u_j, R_j) \quad (3.1.26)$$

(and recall from Notation 3.1.17 that the same symbols denote edges and the assigned edge indices).

3.2 State-sum construction of r -spin TFTs

Our first application of the combinatorial description of r -spin structures is a state-sum construction of r -spin TFTs, see [BT, NR, GK] for the 2-spin case and [Nov] for general

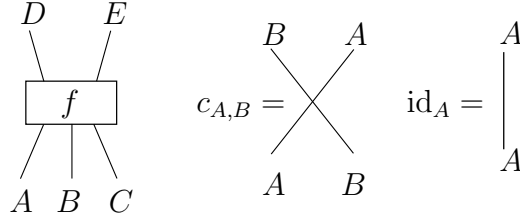


Figure 3.8: String diagram notation of a morphism $f \in \mathcal{S}(A \otimes B \otimes C, D \otimes E)$, the symmetric braiding $c_{A,B}$ and the identity id_A .

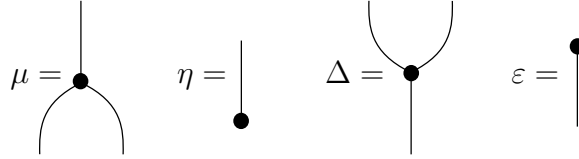


Figure 3.9: String diagrams we will use for the structure morphisms of a Frobenius algebra.

r . We generalise the construction in [Nov] from triangulations to PLCW-decompositions, which are much more convenient for explicit computations. In Sections 3.2.1 and 3.2.2 we present some algebraic preliminaries, and in Section 3.2.3 we explain how suitable Frobenius algebras produce an r -spin TFT via a state-sum construction (Theorem 3.2.8). In Section 3.2.4 we compute the value of the state-sum TFT on connected r -spin bordisms.

3.2.1 Algebraic notions

Let \mathcal{S} denote a strict symmetric monoidal category with tensor product \otimes , tensor unit \mathbb{I} and braiding c . We use the graphical calculus as shown in Figure 3.8, and we will omit the labels for objects if they are understood, as e.g. in Figure 3.9.

An object $A \in \mathcal{S}$ together with morphisms $\mu \in \mathcal{S}(A \otimes A, A)$ (multiplication), $\eta \in \mathcal{S}(\mathbb{I}, A)$ (unit), $\Delta \in \mathcal{S}(A, A \otimes A)$ (comultiplication) and $\varepsilon \in \mathcal{S}(A, \mathbb{I})$ (counit), see Figure 3.9, is a *Frobenius algebra* if the following relations hold:

$$\begin{array}{c} \text{Diagram 1} \end{array} = \begin{array}{c} \text{Diagram 2} \end{array} = \begin{array}{c} \text{Diagram 3} \end{array} = \begin{array}{c} \text{Diagram 4} \end{array} = \begin{array}{c} \text{Diagram 5} \end{array} \quad (3.2.1)$$

Diagram 1: A vertical line with a dot, and two lines from below merging into it. Diagram 2: A vertical line with a dot, and one line from below and one line from above. Diagram 3: A vertical line with a dot, and two lines from above merging into it. Diagram 4: A vertical line with a dot, and one line from below and one line from above. Diagram 5: A vertical line with a dot, and one line from below and one line from above.

$$\begin{array}{c} \text{Diagram 6} \end{array} = \begin{array}{c} \text{Diagram 7} \end{array} = \begin{array}{c} \text{Diagram 8} \end{array} \quad (3.2.2)$$

Diagram 6: A vertical line with a dot, and two lines from below merging into it. Diagram 7: A vertical line with a dot, and one line from below and one line from above. Diagram 8: A vertical line with a dot, and two lines from above merging into it.

These relations imply that a Frobenius algebra is in particular an associative algebra and

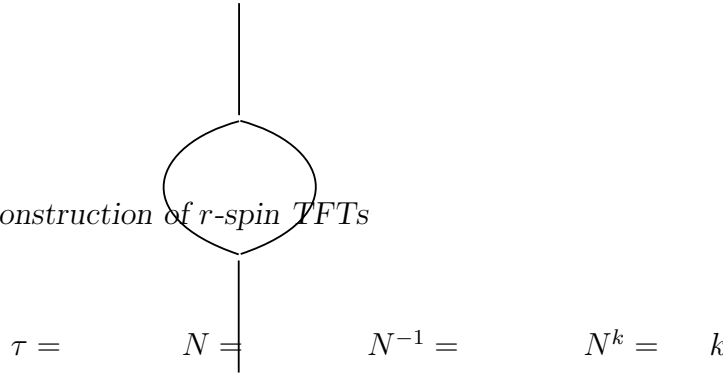


Figure 3.10: The window element τ , the Nakayama automorphism N , its inverse N^{-1} [Nov, Sec. 5.3] and our string diagram abbreviation for the k 'th power of N .

coassociative coalgebra, see [Koc, Prop. 2.3.24]. For more details on the definition of algebras, coalgebras and Frobenius algebras in monoidal categories we refer to e.g. [FS].

For a Frobenius algebra A we define the *window element* $\tau = \mu \circ \Delta \circ \eta$ [LP1] and the *Nakayama automorphism* $N = (\text{id}_A \otimes (\varepsilon \circ \mu)) \circ (c_{A,A} \otimes \text{id}_A) \circ (\text{id}_A \otimes (\Delta \circ \eta))$, see Figure 3.10. Then τ is central (as follows from an easy calculation) and N is a morphism of Frobenius algebras (see [FS] and [NR, Prop. 4.5]). A is called *symmetric* if $\varepsilon \circ \mu \circ c_{A,A} = \varepsilon \circ \mu$. It can be shown from a straightforward calculation that A is symmetric if and only if $N = \text{id}_A$.

Lemma 3.2.1. *Let $A \in \mathcal{S}$ Frobenius algebra with Nakayama automorphism N . Then for every $n \in \mathbb{Z}$*

The equation shows four string diagrams representing N^n for different values of n . The first diagram has two circles, each labeled n . The second diagram has one circle labeled n . The third diagram has one circle labeled $-n$. The fourth diagram has one circle labeled n . The diagrams are connected by equals signs.

Proof. The first and second equations are proven in [Nov, Lem. 5.12]. The third equation follows from a direct calculation. \square

A morphism $\kappa : \mathbb{I} \rightarrow A$ is called *invertible* if there is a morphism $\kappa' : \mathbb{I} \rightarrow A$ such that $\mu \circ (\kappa \otimes \kappa') = \eta = \mu \circ (\kappa' \otimes \kappa)$. In this case we write κ^{-1} instead of κ' for the unique inverse.

Let $r \in \mathbb{Z}_{\geq 0}$, $(A, \mu, \eta, \Delta, \varepsilon)$ be a Frobenius algebra in \mathcal{S} with invertible window element τ and with Nakayama automorphism N , such that $N^r = \text{id}_A$ (for $r = 0$ this last condition is empty). A Frobenius algebra with $N^r = \text{id}_A$ is called a Λ_r -Frobenius algebra in [DK, Prop. I.41]. Define

$$P_\lambda := (\tau^{-1} \cdot (-)) \circ \mu \circ c_{A,A} \circ (\text{id} \otimes N^{1-\lambda}) \circ \Delta \in \text{End}(A). \quad (3.2.4)$$

We collect some properties of P_λ in the following lemma.

Lemma 3.2.2. P_λ is an idempotent, and for any $\lambda_1, \lambda_2 \in \mathbb{Z}_r$ one has that:

$$N \circ P_{\lambda_1} = P_{\lambda_1} \circ N \quad (3.2.5)$$

$$\mu \circ (P_{\lambda_1} \otimes P_{\lambda_2}) = P_{\lambda_1 + \lambda_2} \circ \mu \circ (P_{\lambda_1} \otimes P_{\lambda_2}) , \quad \eta = P_0 \circ \eta , \quad (3.2.6)$$

$$(P_{\lambda_1} \otimes P_{\lambda_2}) \circ \Delta = (P_{\lambda_1} \otimes P_{\lambda_2}) \circ \Delta \circ P_{\lambda_1 + \lambda_2 - 2} , \quad \varepsilon = \varepsilon \circ P_2 . \quad (3.2.7)$$

Proof. That P_λ is an idempotent is a direct generalisation of [Nov, Lem. 5.12(1)]. The additional $(\tau^{-1} \cdot (-))$ removes the “bubble” $\mu \circ \Delta$. The identity (3.2.5) is immediate from the definition of P_λ in (3.2.4) and the fact that N is an automorphism of Frobenius algebras. The first identity in (3.2.6) is a more general version of [Nov, Lem. 6.8] and the proof works along the same lines. To show the second identity in each of (3.2.6) and (3.2.7) just write out the definition of P_λ , N and N^{-1} . For the first identity in (3.2.7) use (3.2.6) together with

$$((\varepsilon \circ \mu) \otimes \text{id}_A) \circ (\text{id}_A \otimes P_\lambda \otimes \text{id}_A) \circ (\text{id}_A \otimes (\Delta \circ \eta)) = P_{2-\lambda}, \quad (3.2.8)$$

which follows from a direct calculation. \square

3.2.2 The \mathbb{Z}_r -graded center

Let $A \in \mathcal{S}$ be a Frobenius algebra with invertible window element. Let \mathcal{S} be furthermore additive (in particular, finite direct sums distribute over tensor products) and assume that the idempotents P_λ split in \mathcal{S} , i.e.

$$P_\lambda = \left[A \xrightarrow{\pi_\lambda} Z_\lambda \xrightarrow{\iota_\lambda} A \right] , \quad \left[Z_\lambda \xrightarrow{\iota_\lambda} A \xrightarrow{\pi_\lambda} Z_\lambda \right] = \text{id}_{Z_\lambda} , \quad (3.2.9)$$

for some object $Z_\lambda \in \mathcal{S}$. For $r = 0$ assume furthermore that \mathcal{S} has countably infinite direct sums which distribute over the tensor product. We can now define:

Definition 3.2.3. Let $r \in \mathbb{Z}_{\geq 0}$. The \mathbb{Z}_r -graded center of a Frobenius algebra A with invertible window element and which satisfies $N^r = \text{id}$ is the direct sum

$$Z^r(A) := \bigoplus_{\lambda \in \mathbb{Z}_r} Z_\lambda . \quad (3.2.10)$$

This is a \mathbb{Z}_r -graded object and we call $\lambda \in \mathbb{Z}_r$ the degree of Z_λ .

Next we will endow $Z^r(A)$ with an algebra structure induced by A . Write

$$e_\lambda : Z_\lambda \rightarrow Z^r(A) \quad (3.2.11)$$

for the embeddings of the summands in (3.2.11) and p_λ for the induced projections which satisfy

$$\left[Z_{\lambda_1} \xrightarrow{e_{\lambda_1}} Z^r(A) \xrightarrow{p_{\lambda_2}} Z_{\lambda_2} \right] = \delta_{\lambda_1, \lambda_2} \text{id}_{Z_{\lambda_1}} . \quad (3.2.12)$$

Lemma 3.2.2 suggests to define, for $\lambda_1, \lambda_2 \in \mathbb{Z}_r$,

$$\mu_{\lambda_1, \lambda_2} := \left[Z_{\lambda_1} \otimes Z_{\lambda_2} \xrightarrow{\iota_{\lambda_1} \otimes \iota_{\lambda_2}} A \otimes A \xrightarrow{\mu} A \xrightarrow{\pi_{\lambda_1 + \lambda_2}} Z_{\lambda_1 + \lambda_2} \right] . \quad (3.2.13)$$

By the universal property of direct sums (which in the countably infinite case for $r = 0$ still distribute over \otimes by our assumptions) there is a unique map

$$\bar{\mu} : Z^r(A) \otimes Z^r(A) \longrightarrow Z^r(A) \quad (3.2.14)$$

which satisfies $\bar{\mu} \circ (e_{\lambda_1} \otimes e_{\lambda_2}) = e_{\lambda_1 + \lambda_2} \circ \mu_{\lambda_1, \lambda_2}$. Let us furthermore define

$$\bar{\eta} := \left[\mathbb{I} \xrightarrow{\eta} A \xrightarrow{\pi_0} Z_0 \xrightarrow{e_0} Z^r(A) \right] , \quad (3.2.15)$$

The morphisms $\bar{\mu}$ and $\bar{\eta}$ are degree preserving. It is straightforward to verify that $Z^r(A)$ together with $\bar{\mu}$ and $\bar{\eta}$ becomes an associative unital \mathbb{Z}_r -graded algebra in \mathcal{S} .

One can restrict the Nakayama automorphism of A on the Z_λ 's by

$$N_{Z_\lambda} := \left[Z_\lambda \xrightarrow{\iota_\lambda} A \xrightarrow{N} A \xrightarrow{\pi_\lambda} Z_\lambda \right] \quad (3.2.16)$$

As in [Nov, Lem. 5.12/3] one verifies that

$$N_{Z_\lambda}^{\gcd(1-\lambda, r)} = \text{id}_{Z_\lambda} . \quad (3.2.17)$$

Recall from the introduction that $\gcd(a, b)$ denotes the non-negative generator of the ideal $\langle a, b \rangle \subset \mathbb{Z}$. In particular, for $r = 0$ we have $\gcd(1 - \lambda, r) = |1 - \lambda|$. The product $\bar{\mu}$ is in general not commutative, but a simple computation shows that its components satisfy:

$$\begin{aligned} \mu_{\lambda_1, \lambda_2} \circ c_{Z_{\lambda_2}, Z_{\lambda_1}} &= \mu_{\lambda_2, \lambda_1} \circ \left(N_{\lambda_2}^{-\lambda_1} \otimes \text{id}_{Z_{\lambda_1}} \right) \\ &= \mu_{\lambda_2, \lambda_1} \circ \left(\text{id}_{Z_{\lambda_2}} \otimes N_{\lambda_1}^{+\lambda_2} \right) . \end{aligned} \quad (3.2.18)$$

Let $\bar{N} : Z^r(A) \rightarrow Z^r(A)$ be the unique morphism such that $\bar{N} \circ e_\lambda = e_\lambda \circ N_\lambda$ for all λ . Combining the fact that N is an automorphism of Frobenius algebras with the definition of $\bar{\mu}$ and $\bar{\eta}$ and using (3.2.5) shows that \bar{N} is an algebra automorphism. By (3.2.17) we have $\bar{N}^r = \text{id}$. We collect the above results in the following proposition.

Proposition 3.2.4. *Let A be as in Definition 3.2.3. The \mathbb{Z}_r -graded center $Z^r(A)$ of A is an associative unital algebra via $\bar{\mu}$, $\bar{\eta}$ and is equipped with the algebra automorphism \bar{N} satisfying $\bar{N}^r = \text{id}$. The algebra $Z^r(A)$ satisfies the commutativity conditions, for $\lambda \in \mathbb{Z}_r$,*

$$\begin{aligned} \bar{\mu} \circ c_{Z^r(A), Z^r(A)} \circ (\text{id} \otimes e_\lambda) &= \bar{\mu} \circ (\bar{N}^{-\lambda} \otimes e_\lambda) , \\ \bar{\mu} \circ c_{Z^r(A), Z^r(A)} \circ (e_\lambda \otimes \text{id}) &= \bar{\mu} \circ (e_\lambda \otimes \bar{N}^\lambda) . \end{aligned} \quad (3.2.19)$$

Corollary 3.2.5. *The component Z_0 of $Z^r(A)$ is a subalgebra and is the centre of A .*

Frobenius algebra structure for $r > 0$

For the rest of this section let us assume that $r > 0$. Since now \mathbb{Z}_r is finite, we can define the coproduct as the sum

$$\bar{\Delta} := \sum_{\lambda_1, \lambda_2 \in \mathbb{Z}_r} \left[Z^r(A) \xrightarrow{p_{\lambda_1 + \lambda_2 - 2}} Z_{\lambda_1 + \lambda_2 - 2} \xrightarrow{\Delta_{\lambda_1, \lambda_2}} Z_{\lambda_1} \otimes Z_{\lambda_2} \xrightarrow{e_{\lambda_1} \otimes e_{\lambda_2}} Z^r(A) \otimes Z^r(A) \right], \quad (3.2.20)$$

with component maps

$$\Delta_{\lambda_1, \lambda_2} := \left[Z_{\lambda_1 + \lambda_2 - 2} \xrightarrow{\iota_{\lambda_1 + \lambda_2 - 2}} A \xrightarrow{\Delta \circ (\tau \cdot (-))} A \otimes A \xrightarrow{\pi_{\lambda_1} \otimes \pi_{\lambda_2}} Z_{\lambda_2} \otimes Z_{\lambda_1} \right]. \quad (3.2.21)$$

We define the counit

$$\bar{\varepsilon} := \left[Z^r(A) \xrightarrow{p_2} Z_2 \xrightarrow{\iota_2} A \xrightarrow{(\tau^{-1} \cdot (-))} A \xrightarrow{\varepsilon} \mathbb{I} \right]. \quad (3.2.22)$$

The morphisms $\bar{\Delta}$ and $\bar{\varepsilon}$ have degree +2 and -2 respectively. Note that we inserted a multiplication with τ and its inverse in the definition of $\bar{\varepsilon}$ and $\bar{\Delta}$. The reason for this is that we want these maps to match the structure maps calculated in Section 3.2.4 from the state-sum construction.

It is straightforward to see that altogether $Z^r(A)$ becomes a Frobenius algebra, just verify (3.2.1) and (3.2.2) restricted to individual summands of $Z^r(A)$ by using Lemma 3.2.2 to move projectors past structure maps of A and by the properties of A itself. Altogether we have:

Proposition 3.2.6. *Let A be as in Definition 3.2.3. For $r > 0$, the \mathbb{Z}_r graded center of A together with $\bar{\mu}$, $\bar{\eta}$, $\bar{\Delta}$, $\bar{\varepsilon}$ is a \mathbb{Z}_r -graded Frobenius algebra. The morphisms $\bar{\mu}$ and $\bar{\eta}$ have degree 0, while $\bar{\Delta}$ has degree 2 and $\bar{\varepsilon}$ has degree -2.*

Remark 3.2.7. 1. The condition $N^r = \text{id}_A$ amounts to A being a representation of the group \mathbb{Z}_r . Instead of defining this in a general category, let k be a field and let us assume that $A \in \text{Rep}_k(\mathbb{Z}_r)$, the category of k -linear representations of \mathbb{Z}_r . Then the algebra $Z^r(A)$ is the full center of A as defined in [Dav], and is in particular a commutative algebra in $\mathcal{Z}(\text{Rep}_k(\mathbb{Z}_r))$, the monoidal center of $\text{Rep}_k(\mathbb{Z}_r)$. To see this one needs to check that $Z^r(A)$ has the form of the full center as given in [Dav, Prop. 9.6], which has been done in (3.2.18).

Note, however, that unless $r = 1$ or $r = 2$, the counit $\bar{\varepsilon}$ and the comultiplication $\bar{\Delta}$ are not degree preserving, i.e. $Z^r(A)$ is not a Frobenius algebra in $\mathcal{Z}(\mathbb{Z}_r)$ with these structure maps.

2. For $r = 0$ one still obtains for every $\lambda_1, \lambda_2 \in \mathbb{Z}$ a non-degeneracy condition, which we do not explain in detail.

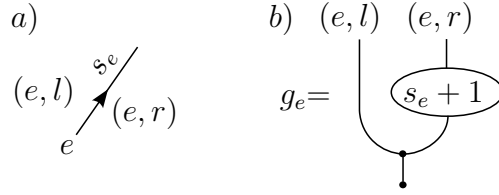


Figure 3.11: a) Left and right sides (e, l) and (e, r) of an inner edge e , determined by the orientation of Σ (paper orientation) and of e (arrow). The edge index of e is s_e . b) Convention for connecting tensor factors belonging to edge sides (e, l) and (e, r) of an inner edge e with the tensor factors belonging to the morphism g_e .

3.2.3 State-sum construction

Let again $r \geq 0$ and $A \in \mathcal{S}$ be a Frobenius algebra with invertible window element τ and with $N^r = \text{id}_A$. In this section we define a symmetric monoidal functor $\mathcal{Z}_A : \text{Bord}_2^r \rightarrow \mathcal{S}$, that is, a TFT on two-dimensional r -spin bordisms.

Recall the direct sum decomposition $Z^r(A) = \bigoplus_{\lambda \in \mathbb{Z}_r} Z_\lambda$ of the \mathbb{Z}_r -graded centre from Definition 3.2.3. We define the TFT \mathcal{Z}_A on objects as follows: Let $\rho : X \rightarrow \mathbb{Z}_r$ be an r -spin object. Then

$$\mathcal{Z}_A(\rho) := \bigotimes_{x \in X} Z_{\rho_x} . \quad (3.2.23)$$

To define \mathcal{Z}_A on morphisms is more involved and will take up the remainder of this section. Let (X, ρ) and (Y, σ) be two r -spin objects. Let $\Sigma : \rho \rightarrow \sigma$ be an r -spin bordism with maps $\lambda : B_{in} \rightarrow \mathbb{Z}_r$, $\mu : B_{out} \rightarrow \mathbb{Z}_r$. Choose a decorated PLCW decomposition $\Sigma_2, \Sigma_1, \Sigma_0$ of the surface Σ with admissible edge index assignment s such that the r -spin structure with parametrised boundary $\Sigma(s, \lambda, \mu)$ from Definition 3.1.10 is isomorphic to the r -spin structure of the r -spin bordism $\Sigma : \rho \rightarrow \sigma$. Recall that B_{in} and B_{out} denote the in- and outgoing boundary components respectively and that by our conventions they are in bijection with edges on the boundary.

For a face $f \in \Sigma_2$ which is an n_f -gon let us write (f, k) , $k = 1, \dots, n_f$ for the sides of f , where $(f, 1)$ denotes the marked edge of f , and the labeling proceeds counter-clockwise with respect to the orientation of f . We collect the sides of all faces into a set:

$$S := \{ (f, k) \mid f \in \Sigma_2, k = 1, \dots, n_f \} . \quad (3.2.24)$$

We double the set of edges by considering $\Sigma_1 \times \{l, r\}$, where “ l ” and “ r ” stand for left and right, respectively. Let $E \subset \Sigma_1 \times \{l, r\}$ be the subset of all (e, l) (resp. (e, r)), which have a face attached on the left (resp. right) side, cf. Figure 3.11 a). Thus for an inner edge $e \in \Sigma_1$ the set E contains both (e, l) and (e, r) , but for a boundary edge $e' \in \Sigma_1$ the set E contains either (e', l) or (e', r) . By construction of S and E we obtain a bijection

$$\Phi : E \xrightarrow{\sim} S \quad , \quad (e, x) \mapsto (f, k) , \quad (3.2.25)$$

where e is the k 'th edge on the boundary of the face f lying on the side x of e , counted counter-clockwise from the marked edge of f .

For every vertex $v \in \Sigma_0$ in the interior of Σ or on an ingoing boundary component of Σ choose a side of an edge $(e, x) \in E$ for which $v \in \partial(e)$. Let

$$V : \Sigma_0 \setminus B_{out} \rightarrow E \quad (3.2.26)$$

be the resulting function.

To define $\mathcal{Z}_A(\Sigma)$ we proceed with the following steps.

1. Let us introduce the tensor products

$$\begin{aligned} \mathcal{A}_S &:= \bigotimes_{(f,k) \in S} A^{(f,k)}, & \mathcal{A}_E &:= \bigotimes_{(e,x) \in E} A^{(e,x)}, \\ \mathcal{A}_{in} &:= \bigotimes_{b \in B_{in}} A^{(b,in)}, & \mathcal{A}_{out} &:= \bigotimes_{c \in B_{out}} A^{(c,out)}. \end{aligned} \quad (3.2.27)$$

Every tensor factor is equal to A , but the various superscripts will help us distinguish tensor factors in the source and target objects of the morphisms we define in the remaining steps.

2. For an edge $e \in \Sigma_1$ we set

$$g_e := \begin{cases} A^{(e,in)} \xrightarrow{N^{-s_e-1}} A^{(e,in)} & ; e \in B_{in} \\ \mathbb{I} \xrightarrow{\eta} A \xrightarrow{\Delta} A \otimes A \xrightarrow{\text{id}_A \otimes N^{s_e+1}} A^{(e,l)} \otimes A^{(e,out)} & ; e \in B_{out}, \text{ surface is left of } e \\ \mathbb{I} \xrightarrow{\eta} A \xrightarrow{\Delta} A \otimes A \xrightarrow{\text{id}_A \otimes N^{s_e+1}} A^{(e,out)} \otimes A^{(e,r)} & ; e \in B_{out}, \text{ surface is right of } e \\ \mathbb{I} \xrightarrow{\eta} A \xrightarrow{\Delta} A \otimes A \xrightarrow{\text{id}_A \otimes N^{s_e+1}} A^{(e,l)} \otimes A^{(e,r)} & ; e \text{ inner edge} \end{cases} \quad (3.2.28)$$

cf. Figure 3.11. Define the linear map

$$\mathcal{C} := \bigotimes_{e \in \Sigma_1} g_e : \mathcal{A}_{in} \rightarrow \mathcal{A}_E \otimes \mathcal{A}_{out}, \quad (3.2.29)$$

where it is understood that the tensor factors in $\mathcal{A}_E \otimes \mathcal{A}_{out}$ are assigned as indicated in (3.2.28).

3. Note that since all tensor factors in \mathcal{A}_E are algebras, so is \mathcal{A}_E itself. For $a : \mathbb{I} \rightarrow A$ and $(e, x) \in E$ write

$$a^{(e,x)} = \eta \otimes \cdots \otimes a \otimes \cdots \otimes \eta : \mathbb{I} \rightarrow \mathcal{A}_E, \quad (3.2.30)$$

where a maps to the tensor factor $A^{(e,x)}$. Define $z : \mathbb{I} \rightarrow \mathcal{A}_E$ as the following product in the k -algebra $\mathcal{S}(\mathbb{I}, \mathcal{A}_E)$:

$$z = \prod_{v \in \Sigma_0 \setminus B_{out}} (\tau^{-1})^{V(v)}. \quad (3.2.31)$$

Finally, we let \mathcal{Y} be the endomorphism of \mathcal{A}_E obtained by multiplying with z ,

$$\mathcal{Y} := \left[\mathcal{A}_E \xrightarrow{z \cdot (-)} \mathcal{A}_E \right]. \quad (3.2.32)$$

4. Let $\mu^{(1)} := \text{id}_A$ and let $\mu^{(n)}$ denote the n -fold product for $n \geq 2$. Assign to every face $f \in \Sigma_2$ obtained from an n_f -gon the morphism $\varepsilon \circ \mu^{(n_f)} : A_{(f,1)} \otimes \cdots \otimes A_{(f,n_f)} \rightarrow \mathbb{I}$ and take their tensor product:

$$\mathcal{F} := \bigotimes_{f \in \Sigma_2} (\varepsilon \circ \mu^{(n_f)}) : \mathcal{A}_S \rightarrow \mathbb{I} . \quad (3.2.33)$$

5. We will now put the above morphisms together to obtain a morphism $\mathcal{L} : \mathcal{A}_{in} \rightarrow \mathcal{A}_{out}$. Denote by Π_Φ the permutation of tensor factors induced by $\Phi : E \rightarrow S$,

$$\Pi_\Phi : \mathcal{A}_E \rightarrow \mathcal{A}_S . \quad (3.2.34)$$

Using this, we define

$$\mathcal{K} := \left[\mathcal{A}_E \xrightarrow{\gamma} \mathcal{A}_E \xrightarrow{\Pi_\Phi} \mathcal{A}_S \xrightarrow{\mathcal{F}} \mathbb{I} \right] , \quad (3.2.35)$$

$$\mathcal{L} := \left[\mathcal{A}_{in} \xrightarrow{\mathcal{C}} \mathcal{A}_E \otimes \mathcal{A}_{out} \xrightarrow{\mathcal{K} \otimes \text{id}_{\mathcal{A}_{out}}} \mathcal{A}_{out} \right] . \quad (3.2.36)$$

6. Let Π_{in} and Π_{out} denote the permutation of tensor factors induced by the maps β_{in} and β_{out} respectively:

$$\Pi_{in} : \mathcal{Z}_A(\rho) = \bigotimes_{x \in X} Z_{\rho_x} \rightarrow \bigotimes_{b \in B_{in}} Z_{\lambda_b} , \quad (3.2.37)$$

$$\Pi_{out} : \bigotimes_{c \in B_{out}} Z_{\mu_c} \rightarrow \bigotimes_{y \in Y} Z_{\sigma_y} = \mathcal{Z}_A(\sigma) . \quad (3.2.38)$$

Using these permutations and the embedding and projection maps $\iota_\lambda, \pi_\lambda$ from (3.2.9) we construct the morphisms linking the action of \mathcal{Z}_A on objects to the tensor products $\mathcal{A}_{in/out}$:

$$\mathcal{E}_{in} := \left[\mathcal{Z}_A(\rho) \xrightarrow{\Pi_{in}} \bigotimes_{b \in B_{in}} Z_{\lambda_b} \xrightarrow{\bigotimes_{b \in B_{in}} \iota_{\lambda_b}} \mathcal{A}_{in} \right] , \quad (3.2.39)$$

$$\mathcal{E}_{out} := \left[\mathcal{A}_{out} \xrightarrow{\bigotimes_{c \in B_{out}} \pi_{\mu_c}} \bigotimes_{c \in B_{out}} Z_{\mu_c} \xrightarrow{\Pi_{out}} \mathcal{Z}_A(\sigma) \right] . \quad (3.2.40)$$

We have now gathered all ingredients to define the action of \mathcal{Z}_A on morphisms:

$$\mathcal{Z}_A(\Sigma) := \left[\mathcal{Z}_A(\rho) \xrightarrow{\mathcal{E}_{in}} \mathcal{A}_{in} \xrightarrow{\mathcal{L}} \mathcal{A}_{out} \xrightarrow{\mathcal{E}_{out}} \mathcal{Z}_A(\sigma) \right] . \quad (3.2.41)$$

Theorem 3.2.8. *Let $A \in \mathcal{S}$ be a Frobenius algebra with invertible window element τ and with $N^r = \text{id}_A$.*

1. The morphism defined in (3.2.41) is independent of the choice of the marked PLCW decomposition and the assignment V .
2. The state-sum construction yields a symmetric monoidal functor $\mathcal{Z}_A : \mathcal{Bord}_2^r \rightarrow \mathcal{S}$ whose action on objects and morphisms is given by (3.2.23) and (3.2.41), respectively.

The proof of this theorem works by reducing to the corresponding statement for triangulations and is given in Appendix 3.A.5.

Remark 3.2.9. The above construction yields a TFT on the category of closed r -spin bordisms, where the complete boundary of the r -spin bordisms is parametrised, so the parametrised boundary is a closed manifold. One can define a different r -spin bordism category, called the open-closed r -spin bordism category, where only a one dimensional submanifold of the boundary of r -spin surfaces is parametrised. The subcategory of the latter generated by the open cup, the open pair of pants and their duals is called the open r -spin bordism category. In [Ster] a TFT on open r -spin bordisms was constructed using Λ_r -Frobenius algebras [DK, Prop. I.41] which are Frobenius algebras whose Nakayama automorphism N satisfies $N^r = \text{id}$.

3.2.4 Evaluation of state-sum TFTs on generating r -spin bordisms

In this section we apply the state-sum construction from Theorem 3.2.8 to pairs of pants and discs with r -spin structure. On the one hand, these bordisms generate \mathcal{Bord}_2^r , and on the other hand, we will recover the algebra structure of the \mathbb{Z}_r -graded center $Z^r(A)$ of A in this way. Finally, we evaluate \mathcal{Z}_A on a connected bordism of genus g with only ingoing boundary components.

Pair of pants as multiplication

Consider the r -spin 3-holed sphere parametrised as in Section 3.1.5 with 2 ingoing boundary components $B_{in} = \{u_1, u_2\}$ and 1 outgoing boundary component $B_{out} = \{u_3\}$ between r -spin objects $\rho : \{x_1, x_2\} \rightarrow \mathbb{Z}_r$ and $\sigma : \{y\} \rightarrow \mathbb{Z}_r$ with $\beta_{in}(x_i) = u_i$ ($i = 1, 2$) and $\beta_{out}(y) = u_3$. Let $\lambda_1 := \lambda_{u_1}$, $\lambda_2 := \lambda_{u_2}$ and $\lambda_3 := \mu_{u_3}$. Then by (3.1.19) $R_{u_1} = \lambda_1 - 1$, $R_{u_2} = \lambda_2 - 1$ and $R_{u_3} = 1 - \lambda_3$. Substituting these and $\chi(\Sigma_{0,3}) = 2 - 0 - 3 = -1$ in (3.1.23) gives

$$\lambda_1 + \lambda_2 \equiv \lambda_3 \pmod{r} . \quad (3.2.42)$$

Denote this r -spin bordism by

$$S_{1,2}(u_1, u_2, u_3, \lambda_1, \lambda_2) := \Sigma_{0,3}(u_1, u_2, u_3, \lambda_1 - 1, \lambda_2 - 1, 1 - \lambda_3) : \rho \rightarrow \sigma ,$$

(cf. (3.1.26)). The sets S , E are (see Figure 3.7)

$$S = \{ (f, k) \mid k = 1, \dots, 9 \} \simeq \{1, \dots, 9\} , \quad (3.2.43)$$

$$E = \{ (u_1, r), (u_2, r), (r_1, l), (r_1, r), (r_2, l), (r_2, r), (r_3, l), (r_3, r), (u_3, r) \} \\ \simeq \{1, \dots, 9\} , \quad (3.2.44)$$

where in (3.2.44) the isomorphism is given by the order of elements of E as listed. We have one inner vertex v_0 and 3 boundary vertices v_1 , v_2 and v_3 , with v_3 placed on the outgoing boundary component. We set

$$V(v_0) := (r_2, r), \quad V(v_1) := (u_1, r), \quad V(v_2) := (u_2, r) . \quad (3.2.45)$$

Following the steps of the state-sum construction we get:

2. For the various edge indices

$$\mathcal{C} = N^{-u_1-1} \otimes N^{-u_2-1} \otimes g_{r_1} \otimes g_{r_2} \otimes g_{r_3} \otimes g_{u_3}$$

from (3.2.29). Recall from Notation 3.1.17 that the same symbols denote edges and the assigned edge indices.

3. For the inner vertex and the ingoing vertices we set

$$z = (\tau^{-1})^{\otimes 2} \otimes \eta^{\otimes 3} \otimes (\tau^{-1}) \otimes \eta^{\otimes 3}$$

from (3.2.31) according to the map V in (3.2.45).

4. For the single 9-gon $\mathcal{F} = \varepsilon \circ \mu^{(9)}$ from (3.2.33).
5. The permutation is $\Pi_\Phi = (12543)(89)$ from (3.2.34) where we use the cycle notation for the permutation of tensor factors. After a calculation using associativity of the product and the last equation of (3.2.3), the morphism \mathcal{L} in (3.2.36) is

$$\left[A \otimes A \xrightarrow{P_{\lambda_1} \circ N^{-u_1-1} \otimes P_{\lambda_2} \circ N^{-u_2-1}} A \otimes A \xrightarrow{\mu} A \xrightarrow{P_{\lambda_1+\lambda_2} \circ N^{u_3+1}} A \right] . \quad (3.2.46)$$

6. For the in- and outgoing boundary components we get $\mathcal{E}_{in} = \iota_{\lambda_1} \otimes \iota_{\lambda_2}$ and $\mathcal{E}_{out} = \pi_{\lambda_1+\lambda_2}$ from (3.2.39) and (3.2.40), since the permutations induced by β_{in} and β_{out} from (3.2.37) and (3.2.38) are identities. Also note that $\rho_{x_1} = \lambda_1$, etc. Finally by composing \mathcal{L} with \mathcal{E}_{in} and \mathcal{E}_{out} as in (3.2.41) we obtain

$$\mathcal{Z}_A(S_{1,2}(u_1, u_2, u_3, \lambda_1, \lambda_2)) \\ = \left[Z_{\lambda_1} \otimes Z_{\lambda_2} \xrightarrow{N_{\lambda_1}^{-u_1} \otimes N_{\lambda_2}^{-u_2}} Z_{\lambda_1} \otimes Z_{\lambda_2} \xrightarrow{\mu_{\lambda_1, \lambda_2}} Z_{\lambda_1+\lambda_2} \xrightarrow{N_{\lambda_3}^{u_3}} Z_{\lambda_1+\lambda_2} \right] . \quad (3.2.47)$$

Observe that $\mathcal{Z}_A(S_{1,2}(0, 0, 0, \lambda_1, \lambda_2)) = \mu_{\lambda_1, \lambda_2}$ from (3.2.13).

Cup as unit

Consider a disk with outgoing boundary. By Corollary 3.1.20, we get a unique r -spin structure for boundary parametrisation, namely $\mu = 0$. Note that the map β_{out} is unique. Using the notation in (3.1.26) we write $S_{1,0} := \Sigma_{0,1}(u, 0) : \emptyset \rightarrow \rho$ with $\rho : \{*\} \rightarrow \mathbb{Z}_r$ $\rho_* = 0$. However, since the r -spin structure is actually independent of u we may as well set $u = 0$. We have

$$S = \{ (f, k) \mid k = 1, 2, 3 \} \simeq \{1, 2, 3\} , \quad (3.2.48)$$

$$E = \{ (r_1, l), (r_1, r), (u_1, r) \} \simeq \{1, 2, 3\} . \quad (3.2.49)$$

There is an inner vertex v_0 and an outgoing boundary vertex v_1 , and we set

$$V(v_0) := (u_1, r) . \quad (3.2.50)$$

By the state-sum construction one has

2. For the 2 edges $\mathcal{C} = g_{r_1} \otimes g_{u_1}$ from (3.2.29).
3. For the inner vertex $z = \eta^{\otimes 2} \otimes \tau^{-1}$ from (3.2.31).
4. For the single 3-gon $\mathcal{F} = \varepsilon \circ \mu^{(3)}$ from (3.2.33).
5. The permutation is $\Pi_\Phi = (23)$ from (3.2.34). Putting the above together according to (3.2.36) we get

$$\mathcal{L} = P_0 \circ \eta . \quad (3.2.51)$$

6. For the (empty) in- and outgoing boundary components we get $\mathcal{E}_{in} = \text{id}_{\mathbb{I}}$ and $\mathcal{E}_{out} = \pi_0$ from (3.2.39) and (3.2.40). From (3.2.41) we finally get

$$\mathcal{Z}_A(S_{1,0}) = \left[\mathbb{I} \xrightarrow{\eta} A \xrightarrow{\pi_0} Z_0 \right] . \quad (3.2.52)$$

Observe that $\left[\mathbb{I} \xrightarrow{\mathcal{Z}_A(S_{1,0})} Z_0 \xrightarrow{e_0} \oplus_{\lambda \in \mathbb{Z}_r} Z_\lambda \right] = \bar{\eta}$ from (3.2.15)).

Pair of pants as comultiplication

Consider a 3-holed sphere with the parametrisation as above, just with in- and outgoing boundary components exchanged, i.e. λ_1, λ_2 stand for outgoing boundary components, λ_3 for the ingoing etc. Then from (3.1.23) one has:

$$\lambda_1 + \lambda_2 - 2 \equiv \lambda_3 \pmod{r} . \quad (3.2.53)$$

Denote this r -spin surface with parametrised boundary by

$$S_{2,1}(u_1, u_2, u_3, \lambda_1, \lambda_2) := \Sigma_{0,3}(u_1, u_2, u_3, 1 - \lambda_1, 1 - \lambda_2, \lambda_3 - 1) : \sigma \rightarrow \rho ,$$

(cf. (3.1.26)). The morphism \mathcal{L} in (3.2.36) assigned to it by the state-sum construction is

$$\left[A \xrightarrow{P_{\lambda_1+\lambda_2-2} \circ N^{-u_3-1}} A \xrightarrow{\Delta \circ (\tau \cdot (-))} A \otimes A \xrightarrow{P_{\lambda_1} \circ N^{u_1+1} \otimes P_{\lambda_2} \circ N^{u_2+1}} A \otimes A \right]. \quad (3.2.54)$$

and from (3.2.41) one obtains

$$\mathcal{Z}_A(S_{2,1}(u_1, u_2, u_3, \lambda_1, \lambda_2)) = \left[Z_{\lambda_1+\lambda_2-2} \xrightarrow{N_{\lambda_1+\lambda_2-2}^{-u_3}} Z_{\lambda_1+\lambda_2-2} \xrightarrow{\Delta_{\lambda_1, \lambda_2}} Z_{\lambda_1} \otimes Z_{\lambda_2} \xrightarrow{N_{\lambda_1}^{u_1} \otimes N_{\lambda_2}^{u_2}} Z_{\lambda_1} \otimes Z_{\lambda_2} \right]. \quad (3.2.55)$$

Observe that $\mathcal{Z}_A(S_{2,1}(0, 0, 0, \lambda_1, \lambda_2)) = \Delta_{\lambda_1, \lambda_2}$ from (3.2.21). While the above morphism is defined also for $r = 0$, as was remarked in Section 3.2.2 one can sum these morphisms only in the case when $r \neq 0$, in which case one obtains (3.2.20).

Cap as counit

Consider an r -spin disk with ingoing boundary. By Corollary 3.1.20, the boundary parametrisation has $\lambda = 2$ and the r -spin structure is independent of the edge indices. Denote this r -spin surface with parametrised boundary with $S_{0,1} := \Sigma_{0,1}(0, 2) : \sigma \rightarrow \emptyset$, (cf. (3.1.26)), with $\sigma : \{*\} \rightarrow \mathbb{Z}_r$ $\sigma_* = 2$. By the state-sum construction one has

$$\mathcal{Z}_A(S_{0,1}) = \left[Z_2 \xrightarrow{\iota_2} A \xrightarrow{(\tau^{-1} \cdot (-))} A \xrightarrow{\varepsilon} \mathbb{I} \right]. \quad (3.2.56)$$

Observe that $\left[\bigoplus_{\lambda \in \mathbb{Z}_r} Z_\lambda \xrightarrow{p_2} Z_2 \xrightarrow{\mathcal{Z}_A(S_{1,0})} \mathbb{I} \right] = \bar{\varepsilon}$ from (3.2.22).

We collect the above computations for \mathcal{Z}_A evaluated on generators in the following proposition:

Proposition 3.2.10. *Let $A \in \mathcal{S}$ be a Frobenius algebra with invertible window element τ and with $N^r = \text{id}_A$, and let \mathcal{Z}_A be the r -spin TFT \mathcal{Z}_A defined in Theorem 3.2.8. The the \mathbb{Z}_r -graded center $Z^r(A)$ is equal to $\bigoplus_{\lambda \in \mathbb{Z}_r} \mathcal{Z}_A(\lambda)$ with product and unit (restricted to the corresponding graded components) given by $\mathcal{Z}_A(S_{1,2}(0, 0, 0, \lambda_1, \lambda_2))$ and $\mathcal{Z}_A(S_{1,0})$, respectively. For $r > 0$, we obtain an equality of Frobenius algebras.*

For $r = 2$, the above relation between state spaces and the \mathbb{Z}_r -graded center was already observed in [MS].

Connected r -spin bordisms

Finally, let us evaluate \mathcal{Z}_A on a general connected r -spin bordism with only ingoing boundary components, that is, on $\Sigma_{g,b}(s_i, t_i, u_j, \lambda_j - 1)$ in the notation of (3.1.26). Write

$$\varphi(s, t) := \text{Diagram} \quad (3.2.57)$$

Using the decomposition of $\Sigma_{g,b}$ from Figure 3.7 a), a straightforward computation along the same lines as above gives the following proposition.

Proposition 3.2.11. *Let $\Sigma_{g,b}(s_i, t_i, u_j, \lambda_j - 1)$ denote the r -spin surface of Definition 3.1.10 with only ingoing boundary components. Then*

$$\mathcal{Z}_A(\Sigma_{g,b}(s_i, t_i, u_j, \lambda_j - 1)) = \varepsilon \circ (\tau^{-1} \cdot (-)) \circ \prod_{i=1}^g \varphi(s_i, t_i) \circ \mu^{(b)} \circ \bigotimes_{j=1}^b (N^{-u_j-1} \circ \iota_{\lambda_j}). \quad (3.2.58)$$

3.3 Action of the mapping class group

Since the TFT is defined on diffeomorphism classes of r -spin surfaces with parametrised boundary, it is of natural interest to calculate these classes. Bundles related by homotopic underlying surface diffeomorphisms are isomorphic, therefore studying diffeomorphism classes of r -spin surfaces is the same as finding the orbits of the mapping class group acting on the set of isomorphism classes of r -spin structures on fixed surfaces. Here by *mapping class group* (MCG) we mean diffeomorphisms of the surface which restrict to the identity on the boundary, up to smooth homotopy which fix the boundary, see [FM, Sec. 2.1].

We consider $\Sigma_{g,b}$, a connected surface of genus g with b boundary components. Generators of the MCG of $\Sigma_{g,b}$ are given by Dehn twists along loops in $\Sigma_{g,b}$ as shown in Figure 3.12. For $g = 0$ this can be shown combining [FM, Thm. 4.9, Prop. 3.19 and Sect. 9.3]; for $g \geq 1$ this is shown in [FM, Sect. 4.4.4].

We will compute the action of these generators on the set $\mathcal{R}^r(\Sigma_{g,b})_{\lambda, \mu}$ of isomorphism classes of r -spin structures with parametrised boundary for given maps $\lambda : B_{in} \rightarrow \mathbb{Z}_r$ and $\mu : B_{out} \rightarrow \mathbb{Z}_r$ in terms of the parametrisation given in Section 3.1.5. We will then use these results in Section 3.5 in order to prove our main result, Theorem 2.2.2.

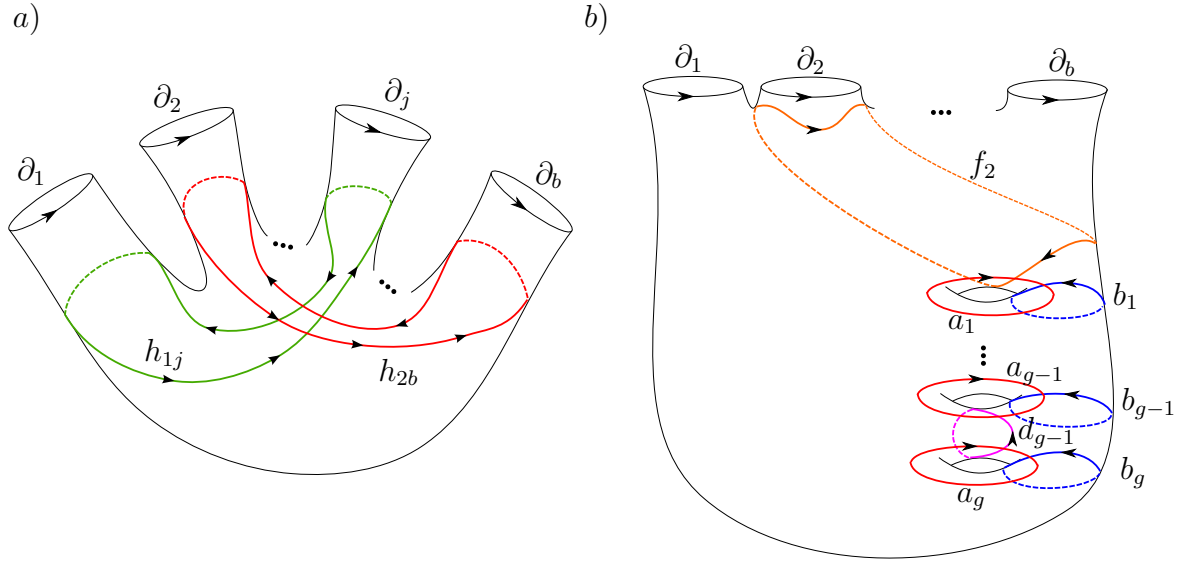


Figure 3.12: Dehn twists along the following loops provide a choice of generators of the MCG.
a) Loops in $\Sigma_{0,b}$: $\{\partial_i, h_{ij} \mid i, j = 1, \dots, b, i \neq j\}$. Here, ∂_j denotes the boundary component j and, for $b \geq 2$, h_{ij} denotes the connected sum of ∂_i and ∂_j . The connected sum is taken with respect to a choice of points on each loop and a path between these points, so that the result is as shown in the figure. Note that $h_{ij} = h_{ji}$.
b) Loops in $\Sigma_{g,b}$ for $g \geq 1$: $\{\partial_i, h_{ij}\}$ as before, and $\{f_i, a_l, b_l, d_x \mid i = 1, \dots, b, l = 1, \dots, g, x = 1, \dots, g-1\}$. Here, f_i denotes the connected sum of ∂_i and b_1 ; d_x denotes the connected sum of b_x and $-b_{x+1}$ and occurs only if $g \geq 2$.

Lemma 3.3.1. Consider a surface with a PLCW decomposition which has a cylinder inside decomposed into a square with identified opposite edges as in Figure 3.13. A Dehn twist around the edge labeled by t sends $t \mapsto t + s$ and does not change the other edge labels.

Proof. First refine the decomposition of Figure 3.13 a) as in Figure 3.14 a). This gives an isomorphic r -spin structure to the original one by Proposition 3.1.16. Pulling back the r -spin structure along the induced action of the Dehn twists yields the r -spin surface shown in Figure 3.14 b). Now apply Part 3 of Lemma 3.1.11 on the upper triangle to obtain Figure 3.14 c). Remove the middle edge by Proposition 3.1.16 to get the r -spin surface described by Figure 3.13 b). \square

Lemma 3.3.2. Recall the parametrisation of r -spin structures on $\Sigma_{g,b}$ from (3.1.26) and Figure 3.7. Let l denote a loop in $\Sigma_{g,b}$ and let D_l denote the isomorphism of r -spin surfaces induced by a Dehn twist around l . We write

$$D_l(\Sigma_{g,b}(s_i, t_i, u_j, R_j)) = \Sigma_{g,b}(s'_i, t'_i, u'_j, R_j). \quad (3.3.1)$$

Then the action of Dehn twists along the loops shown in Figure 3.12 is as listed in the following table (only the parameters that change are listed):

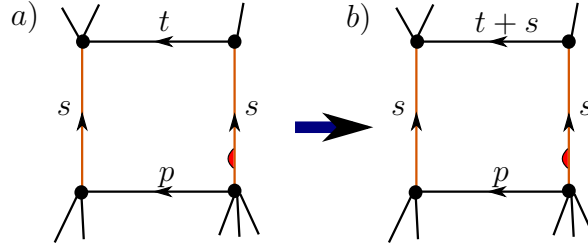


Figure 3.13: Action of a Dehn twist along the edge labeled by t . The two vertical edges labelled by s are identified.

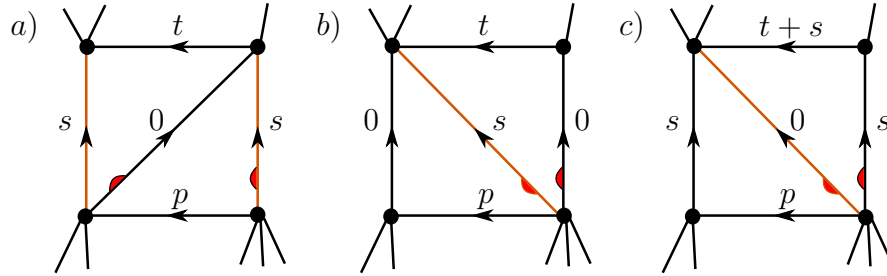


Figure 3.14: Pulling back the r -spin structure along an (inverse) Dehn twist along the edge labeled by t : a) insert the diagonal edge labelled 0; b) carry out a Dehn-twist along the upper horizontal edge labelled t ; c) apply a deck transformation to the top right triangle to change the label of the diagonal edge to 0.

loop	effect on parameters
∂_j	$u'_j = u_j - R_j$
h_{ij}	$u'_i = u_i + R_i + R_j + 1 \quad \text{and} \quad u'_j = u_j + R_i + R_j + 1$
a_i	$s'_i = s_i - t_i$
b_i	$t'_i = t_i - s_i$
f_j	$u'_j = u_j + s_1 + 1 + R_j \quad \text{and} \quad t'_1 = t_1 - s_1 - 1 - R_j$
d_i	$t'_i = t_i + s_{i+1} - s_i + 1 \quad \text{and} \quad t'_{i+1} = t_{i+1} - s_{i+1} + s_i - 1$

Proof. • a_i, b_i : For the loops a_i and b_i the statement is a direct consequence of Lemma 3.3.1. For example for a_1 split the edge t_1 in two by inserting a vertex, then insert an edge parallel to s_1 . Then apply the lemma and remove the previously added edge and vertex.

• ∂_j : For the loop ∂_j the statement follows along the same lines, together with (3.1.24).

• f_j : We prove the statement for f_j in the example $j = 2$. Let $s := s_1$ and $p := r_2$. First find the curve f_2 on the polygon decomposition, insert the dotted edges parallel to the loop f_2 using Proposition 3.1.16 and change orientations using Lemma 3.1.11 (1) as in Figure 3.15. We need to consider the part of the decomposition which is a cylinder glued together from two rectangles as shown in Figure 3.16 a). Then proceed with the sequence of steps shown in Figure 3.16. Finally apply a deck transformation by $-p - s - 1$ on the rectangle bounded by the edges with edge index $t_1, p + s + 1, u_2$ and $p + s$. The result

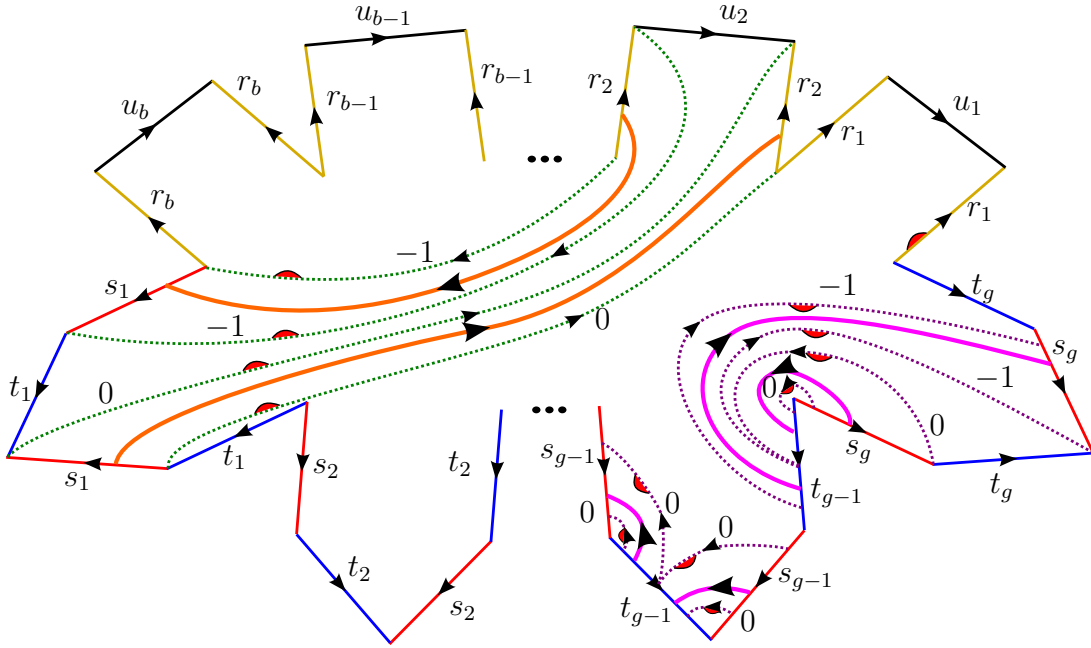


Figure 3.15: The loop f_2 (above, in 2 segments, between edges r_2 and s_1) and the loop d_{g-1} (below, in 4 segments, between edges t_{g-1} , s_{g-1} , t_g and s_g) on the PLCW decomposition.

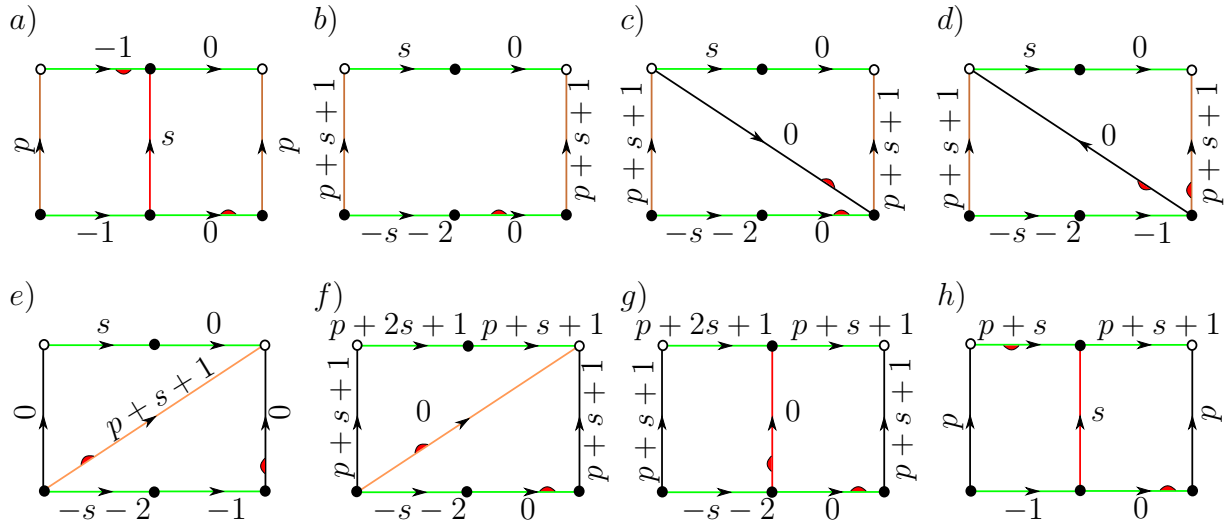


Figure 3.16: Calculation of the Dehn twist along the loop f_2 . a) The cylinder along the loop f_2 . The empty dot denotes the boundary vertex, the full dot the inner vertex. The vertical edges labeled by p are identified. b) Move the marking to the s edge (Lemma 3.1.11 (2)), shift the labels on the left square by $s+1$ (Lemma 3.1.11 (3)), and remove the middle edge (Proposition 3.1.16). c) Add an edge between two opposite corners with edge index 0. d) Move the markings and flip the middle edge orientation. e) Apply a Dehn twist along the top horizontal edges (marked s and 0). f) Apply a deck transformation to the top left triangle and move right marking. g) Remove the diagonal edge, insert new vertical edge. h) Shift edge indices on left square, move left marking.

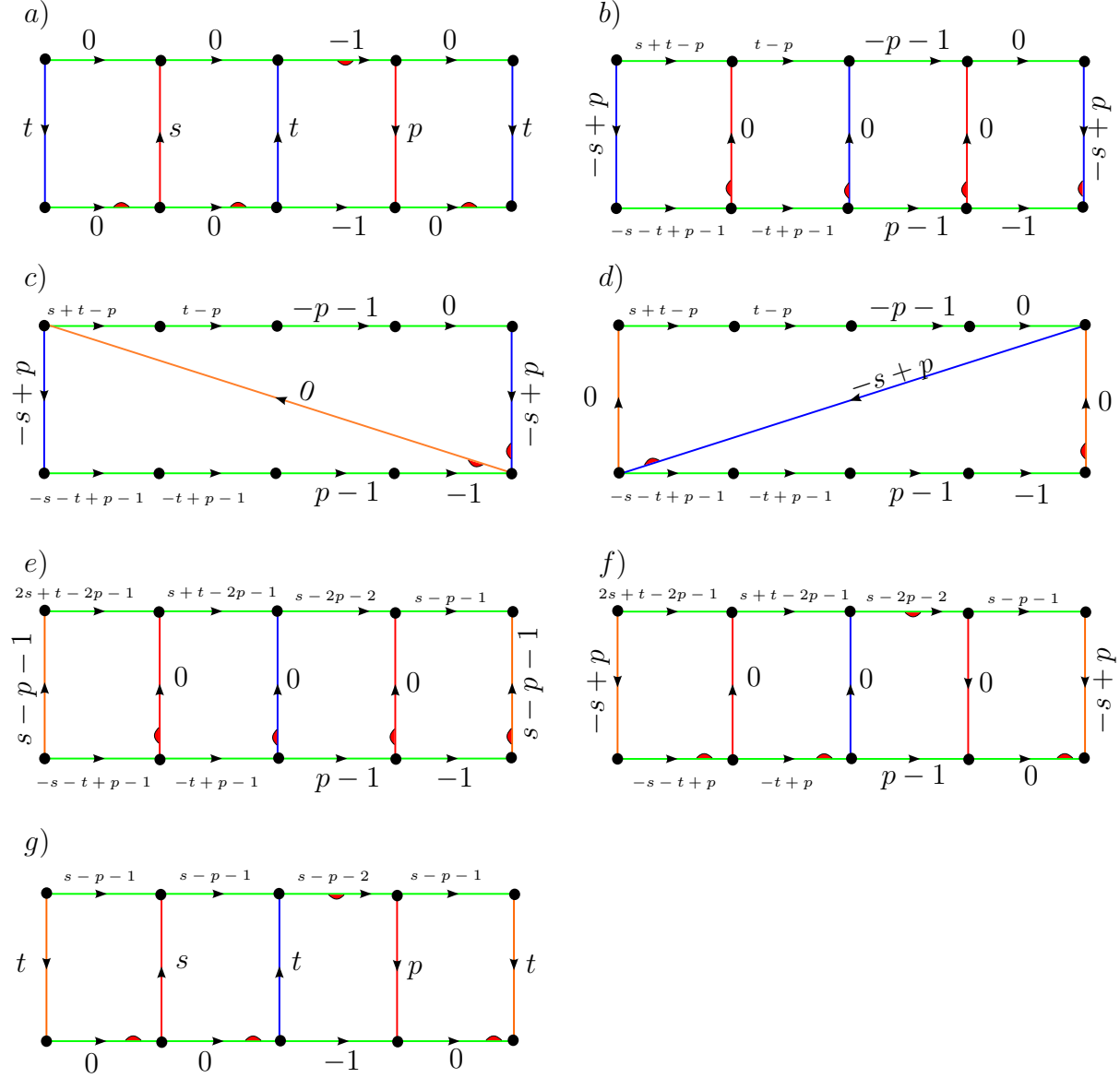


Figure 3.17: Calculation of a Dehn twist along the loop d_{g-1} . The left-most and right-most vertical edges are identified in all figures. a) The cylinder along the loop d_{g-1} . b) Move the markings, flip the “ p ” edge orientation and shift the edge indices on the 3 rectangles on the left. c) Remove the 3 inner edges and add a new edge. d) Do a Dehn twist along the d_{g-1} loop. e) Shift the edge indices on the upper triangle by $s - p - 1$; flip the orientation of the middle edge and then remove the middle edge; put back 3 edges. f) Move the markings, flip the first and fourth edge orientation. g) Shift the edge indices on the 3 rectangles on the left.

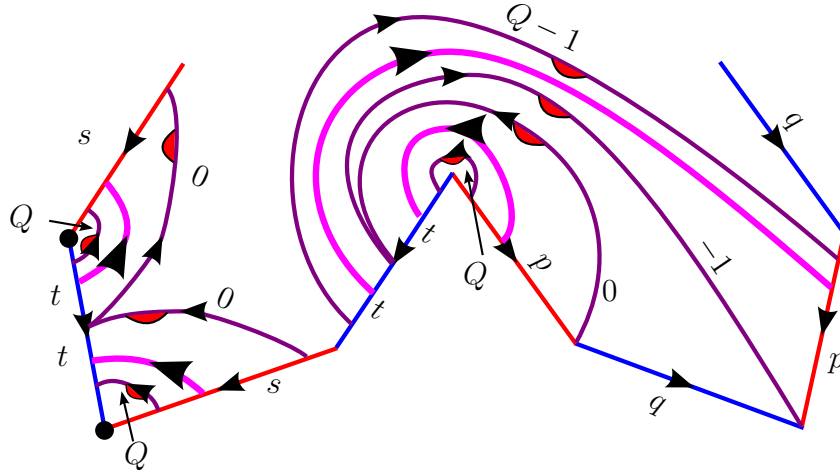


Figure 3.18: Part of the PLCW decomposition after a Dehn twist along the loop d_{g-1} . We shift the edge indices by $Q := s - p - 1$ on the following polygons: on the 2 triangles marked by dots; on the rectangle with edge labels $Q - 1$, p , -1 and t ; on the triangle below with edge labels -1 , q and 0 ; on the rectangle below with edge labels 0 , Q , -1 and t ; on the triangle below with edge labels Q , p and t .

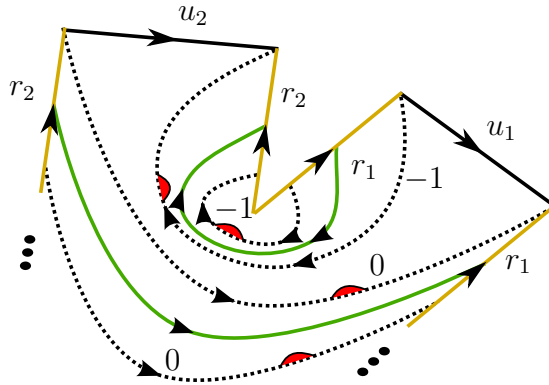


Figure 3.19: The loop h_{12} (in two segments between the edges r_1 and r_2) in the PLCW decomposition of $\Sigma_{g,b}$ and a cylinder around it.

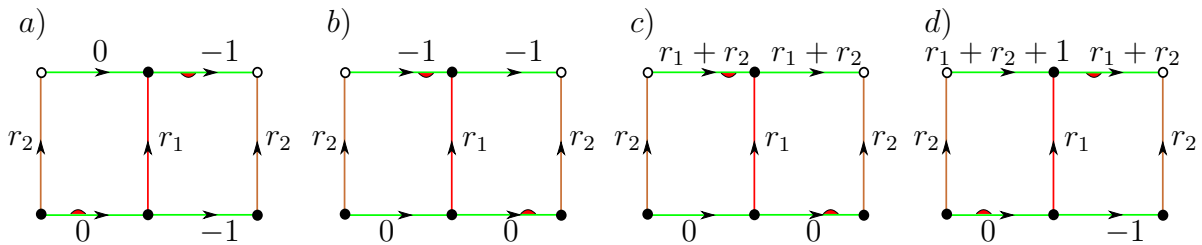


Figure 3.20: Dehn twist along the loop h_{12} . a) Take the cylinder from Figure 3.19. b) After changing the marking one obtains a similar cylinder as in Figure 3.16 a). c) Do the same steps as in Figure 3.16 to apply the Dehn twist. d) Change back the marking.

is a decomposition as on Figure 3.15 with t_1 replaced by $t_1 - p - s - 1$ and u_2 replaced by $u_2 + p + s + 1$. Now remove the newly added edges via Proposition 3.1.16 (flipping the edges labelled -1) to arrive to the statement.

- d_i : We treat the case $i = g - 1$ as an example by applying a similar argument as before. Let $s := s_{g-1}$, $p := s_g$, $t := t_{g-1}$ and $q := t_g$. First add vertices and dotted edges parallel to the loop d_{g-1} as shown in Figure 3.15. We will concentrate on the cylinder cut out by these edges, as shown in Figure 3.17 a). Proceed along the steps shown in Figure 3.17, after which one is left with the marked PLCW decomposition shown in Figure 3.18. Let $Q := s - p - 1$ and shift edge indices by Q according to the steps in Figure 3.18. This amounts to

$$t \mapsto t - Q \quad \text{and} \quad q \mapsto q + Q,$$

after removing the newly added edges and vertices.

- h_{ij} : We show the computation for $i = 1$, $j = 2$ as an example, for other values of i and j the argument is the same. Add vertices and dotted edges parallel to the loop h_{12} as shown in Figure 3.19. We then follow the steps in Figure 3.20. As the last step, one shifts the edge indices by $r_1 + r_2 + 1 = R_1 + R_2 + 1$ by a deck transformation on the square which has edges u_1 and u_2 on opposite sides. \square

3.4 r -spin TFT computing the Arf-invariant

In this section we give an example for the state-sum construction of r -spin TFTs, namely for the two-dimensional Clifford algebra in super vector spaces, and we compute its value on connected r -spin bordisms (Section 3.4.1). We then recall the definition of the Arf invariant for r -spin surfaces and observe that the TFT obtained from the Clifford algebra computes this invariant (Section 3.4.2).

3.4.1 r -spin TFT from a Clifford algebra

Let $r \in \mathbb{Z}_{\geq 0}$ be even and let k be a field not of characteristic 2. Let $\mathcal{Cl} \in \mathcal{SVect}$ be the Clifford algebra with one odd generator θ , i.e. $\mathcal{Cl} = k \oplus k\theta$ with $\theta^2 = 1$. We turn \mathcal{Cl} into a Frobenius algebra via

$$\varepsilon(1) = 2, \quad \varepsilon(\theta) = 0, \quad \Delta(1) = \frac{1}{2}(1 \otimes 1 + \theta \otimes \theta), \quad \Delta(\theta) = \frac{1}{2}(\theta \otimes 1 + 1 \otimes \theta). \quad (3.4.1)$$

Lemma 3.4.1. *For the Frobenius algebra \mathcal{Cl} the following hold.*

1. $\tau = \mu \circ \Delta \circ \eta = \eta$, hence \mathcal{Cl} has invertible window element.
2. The Nakayama automorphism is given by $N(\theta^m) = (-1)^m \theta^m$.
3. For $\lambda \in \mathbb{Z}_r$, $P_\lambda(\theta^m) = \frac{1}{2} [1 + (-1)^{\lambda-m}] \theta^m$, hence $Z_\lambda = k\theta^\lambda$.

4. The morphism $\varphi_{s,t}$ from (3.2.57) is given by $\varphi_{s,t} = \frac{1}{2}(-1)^{(s+1)(t+1)} \text{id}_{C\ell}$.

Proof. 1. $\tau(1) = \mu \circ \Delta \circ \eta(1) = \mu\left(\frac{1}{2}(\theta^n \otimes \theta^{-n})\right) = 1 = \eta(1)$. Its inverse is η .

2. $N(1) = 1$ in any Frobenius algebra. We calculate $N(\theta)$ in steps:

$$\theta \mapsto \theta \otimes (1 \otimes 1 + \theta \otimes \theta)/2 \mapsto (1 \otimes \theta \otimes 1 - \theta \otimes \theta \otimes \theta)/2 \mapsto -\theta.$$

3. We calculate $P_\lambda(\theta^m)$ in steps according to (3.2.4):

$$\begin{aligned} \theta^m &\mapsto \frac{1}{2}(\theta^m \otimes 1 + \theta^{m-1} \otimes \theta) \mapsto \frac{1}{2}(\theta^m \otimes 1 + (-1)^{1-\lambda}\theta^{m-1} \otimes \theta) \\ &\mapsto \frac{1}{2}(1 \otimes \theta^m + (-1)^{m-\lambda}\theta \otimes \theta^{m-1}) \mapsto \frac{1}{2}\theta^m(1 + (-1)^{m-\lambda}). \end{aligned}$$

We see that if λ and m have the same parity this is the identity, otherwise this is zero, i.e. P_λ is a projection onto $k \cdot \theta^\lambda$.

4. We calculate $\varphi_{s,t}(\theta^m)$ in steps according to (3.2.57):

$$\begin{aligned} \theta^m &\mapsto \frac{1}{2} \sum_{n=0}^1 \theta^{m-n} \otimes \theta^n \mapsto \frac{1}{4} \sum_{n,p=0}^1 \theta^{m-n} \otimes \theta^{n-p} \otimes \theta^p \\ &\mapsto \frac{1}{4} \sum_{n,p=0}^1 (-1)^{(s+1)(n-p)+(t+1)p} \theta^{m-n} \otimes \theta^{n-p} \otimes \theta^p \\ &\mapsto \frac{1}{4} \sum_{n,p=0}^1 (-1)^{(s+1)(n-p)+(t+1)p+(n-p)p} \theta^{m-n} \otimes \theta^p \otimes \theta^{n-p} \\ &\mapsto \frac{1}{4} \theta^m \sum_{n,p=0}^1 (-1)^{(s+1)(n-p)+(t+1)p+(n-p)p} \\ &= \frac{1}{4} \theta^m \sum_{n,p=0}^1 (-1)^{(s+1+p)(t+1+n-p)-(s+1)(t+1)} = \frac{1}{2} \theta^m (-1)^{(s+1)(t+1)}, \end{aligned}$$

where at the last step we execute first the summation over n for a fixed p and notice that we either get 0 or 2. □

Let $\mathcal{Z}_{C\ell}$ denote the TFT from Theorem 3.2.8 given by the Frobenius algebra $C\ell$ and recall from Section 3.1.5 the r -spin structure with parametrised boundary $\Sigma_{g,b}(s_i, t_i, u_j, \lambda_j - 1)$ with only ingoing boundary components and where $g + b \geq 1$. By calculating (3.2.58) in Proposition 3.2.11 and using (3.1.23) we get the following proposition.

Proposition 3.4.2. *The value of the TFT $\mathcal{Z}_{C\ell}$ is*

$$\mathcal{Z}_{C\ell}(\Sigma_{g,b}(s_i, t_i, u_j, \lambda_j - 1))(\theta^{\lambda_1} \otimes \cdots \otimes \theta^{\lambda_b}) = 2^{1-g} (-1)^{\sum_{n=1}^g (s_n+1)(t_n+1) + \sum_{j=1}^{b-1} (u_j - u_b) \lambda_j}. \quad (3.4.2)$$

The following corollary will be used to distinguish 2 MCG orbits on $\mathcal{R}^r(\Sigma_{g,b})_{\lambda,\mu}$ for $g \geq 2$ and r even.

Corollary 3.4.3. *Assume that $g \geq 1$ or that $b \geq 1$ and at least one of the λ_j 's is odd (by (3.1.23) in this case $b \geq 2$ and at least two λ_j 's are odd). Then the following map is surjective:*

$$\begin{aligned} \mathcal{R}^r(\Sigma_{g,b})_{\lambda} &\rightarrow \{+1, -1\} \\ [\Sigma_{g,b}(s_i, t_i, u_j, \lambda_j - 1)] &\mapsto 2^{g-1} \cdot \mathcal{Z}_{\mathcal{C}\ell}(\Sigma_{g,b}(s_i, t_i, u_j, \lambda_j - 1))(\theta^{\lambda_1} \otimes \cdots \otimes \theta^{\lambda_b}) \end{aligned} \quad (3.4.3)$$

Remark 3.4.4. 1. One can show, using a similar argument as in [Nov, Sec. 6.5], that for any choice of Frobenius algebra $A \in \mathcal{Vect}$ with invertible window element and with $N^r = \text{id}_A$ the TFT \mathcal{Z}_A of Section 3.2.3 is independent of the r -spin structure. The idea is that if there exists a symmetric Frobenius algebra structure on an algebra A , then \mathcal{Z}_A is independent of the r -spin structure for every other Frobenius algebra structure on A as well.

2. Let r be a positive integer and let us consider the category of \mathbb{Z}_r -graded k -vector spaces $\mathcal{Vect}_{\mathbb{Z}_r}$. By using the correspondence between braided monoidal structures on $\mathcal{Vect}_{\mathbb{Z}_r}$ and quadratic forms on \mathbb{Z}_r [JS] (see [FRS, App. A] for a review) one can check that for odd r there is only one symmetric monoidal structure on $\mathcal{Vect}_{\mathbb{Z}_r}$. For even r there are two: the trivial one inherited from \mathcal{Vect} and the non-trivial one given by the super grading.
3. One may wonder whether taking $\mathcal{Vect}_{\mathbb{Z}_r}$ with some choice of symmetric monoidal structure would yield more examples of r -spin TFTs than what one can find with target \mathcal{Vect} or \mathcal{SVect} . Part 2 shows that this is not so: All symmetric monoidal structures on $\mathcal{Vect}_{\mathbb{Z}_r}$ are inherited from \mathcal{Vect} or \mathcal{SVect} (and only from the former for r odd). Thus all algebras $A \in \mathcal{Vect}_{\mathbb{Z}_r}$ as in Theorem 3.2.8 are also algebras in \mathcal{Vect} , respectively \mathcal{SVect} , with the same properties, and produce the same results in the state-sum construction.

3.4.2 The r -spin Arf-invariant

Let Σ be a compact r -spin surface with parametrised boundary with maps $\lambda : B_{in} \rightarrow \mathbb{Z}_r$ and $\mu : B_{out} \rightarrow \mathbb{Z}_r$. By a *curve* in Σ we mean a smooth immersion $\gamma : [0, 1] \rightarrow \Sigma$ (i.e. γ has nowhere vanishing derivative), and which is either closed, or which starts and ends on the boundary of Σ . In the former case we require in addition that the tangent vectors at the start and end point agree: $\frac{d}{dt}\gamma(0) = \frac{d}{dt}\gamma(1)$. In the latter case we require that the start and end points are the images of $1 \in \mathbb{S}^1 \subset \mathbb{C}$ under the boundary parametrisation maps and that the tangent vector of the curve is the same as the tangent vector of the boundary curve. Two curves γ_0 and γ_1 with $\gamma_0(0) = \gamma_1(0)$ and $\gamma_0(1) = \gamma_1(1)$ are *homotopic* if there is a homotopy $(s, t) \mapsto \gamma_s(t)$ between them, such that for each s , γ_s is a curve in the above sense. In particular, since $\frac{d}{dt}\gamma$ must remain nonzero everywhere along the homotopy, one cannot “pull straight” a loop in the curve.

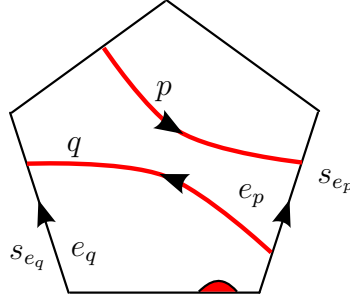


Figure 3.21: Two arcs $p, q \in A(\gamma)$ of a curve γ on a face f . Here $f_p = f_q = f$, $\hat{s}_{e_p} = -s_{e_p} - 1$, $\hat{s}_{e_q} = s_{e_q}$, $\hat{\delta}_{f_p}^p = +1$ and $\hat{\delta}_{f_q}^q = 0$.

Pick a lift $\gamma_F : [0, 1] \rightarrow F_{GL}\Sigma$ of γ to the oriented frame bundle by taking the tangent vector of γ (which is non-zero since γ is an immersion) and adding another non-zero and non-parallel vector such that the orientation induced by them agrees with the orientation of the surface. Such a lift of a curve in Σ to $F_{GL}\Sigma$ is unique up to homotopy, see e.g. [Nov, p. 26]. Also, if two curves in Σ are homotopic, then their lifts to $F_{GL}\Sigma$ are homotopic as well.

Consider a disc D around 1 in \mathbb{C}^\times with r -spin structure D^κ given by the restriction of \mathbb{C}^κ for $\kappa \in \mathbb{Z}_r$ as in Example 3.1.3. As on a contractible surface, all r -spin structures are isomorphic (see e.g. [Nov, Lem. 3.10]), there is an isomorphism of r -spin structures $D^0 \rightarrow D^\kappa$. In fact, there are exactly r such isomorphisms, and we pick the one which acts as the identity on the fibre over 1 (by Example 3.1.3, the fibre and projection over $1 \in \mathbb{C}^\times$ agree for all \mathbb{C}^κ). This construction will be needed to assign a holonomy to curves between different boundary components.

Recall that $P_{GL}\Sigma$ is a principal \mathbb{Z}_r bundle over $F_{GL}\Sigma$. Pick a lift $\tilde{\gamma} : [0, 1] \rightarrow P_{GL}\Sigma$ of γ_F to the r -spin bundle. Since the fibers of $P_{GL}\Sigma \xrightarrow{p} F_{GL}\Sigma$ are discrete, this lift is unique after fixing it at one point and homotopic curves in $F_{GL}\Sigma$ lift to homotopic curves in $P_{GL}\Sigma$. If γ is a closed curve let $\zeta(\gamma) \in \mathbb{Z}_r$ denote the holonomy of $\tilde{\gamma}$ at $\gamma(0) = \gamma(1)$. If γ is not closed, use the isomorphism $D^0 \rightarrow D^\kappa$ from above to identify the fibers \mathbb{Z}_r over the start- and end-point of γ_F , and let again $\zeta(\gamma) \in \mathbb{Z}_r$ denote the resulting holonomy of $\tilde{\gamma}$.

We now explain how to compute these holonomies in terms of the combinatorial description of r -spin structures. Take a decorated PLCW decomposition of Σ with edge index assignment s and consider the r -spin structure $\Sigma(s, \lambda, \mu)$ given by Definition 3.1.10. We may assume the PLCW decomposition to be fine enough so that its edges split γ into a set of arcs $A(\gamma)$ as in Figure 3.21. Then for every $a \in A(\gamma)$ there is a face $f_a \in \Sigma_2$ containing a and an edge e_a in the boundary of f_a where the arc a leaves the face f_a (see Figure 3.21). Let us assume that e_a is not a boundary edge. For s_{e_a} the edge index of the edge e_a let $\hat{s}_{e_a}^a = s_{e_a}$ if the edge e_a and a cross positively, and $\hat{s}_{e_a}^a = -s_{e_a} - 1$ otherwise (see again Figure 3.21 for conventions). Let $\hat{\delta}_{f_a}^a = +1$ if the clockwise vertex of the marked edge of the face f_a is on the right side of a (before glueing the edges) and $\hat{\delta}_{f_a}^a = 0$ otherwise. If γ is not a closed curve, let e_{start} (resp. e_{end}) denote the boundary edge where γ starts (resp. ends),

and let $s_{e_{\text{start}}}$ (resp. $s_{e_{\text{end}}}$) be its edge index. Recall that at the starting (ending) point of γ the tangent vector of γ is parallel to the boundary edge. Set $\hat{s}_{e_{\text{start}}} := -s_{e_{\text{start}}} - 1$ if the edge e_{start} and the tangent vector point in the same direction and $\hat{s}_{e_{\text{start}}} := s_{e_{\text{start}}}$ otherwise. Set $\hat{s}_{e_{\text{end}}} := s_{e_{\text{end}}}$ if the edge e_{end} and the tangent vector point in the same direction and $\hat{s}_{e_{\text{end}}} := -s_{e_{\text{end}}} - 1$ otherwise.

The proof of the following lemma relies on the relation to triangulations introduced in Appendix 3.A and is given in Appendix 3.A.6.

Lemma 3.4.5. *Let γ be a curve in Σ . Then:*

1. *If γ bounds a disc \mathbb{D} embedded in Σ , $\zeta(\gamma) = 1$ if γ is oriented counter-clockwise around the boundary of \mathbb{D} and $\zeta(\gamma) = -1$ otherwise.*
2. *If γ' is a curve homotopic to γ then $\zeta(\gamma') = \zeta(\gamma)$;*
3. *We have*

$$\zeta(\gamma) = \sum_{a \in A(\gamma)} (\hat{s}_{e_a}^a + \hat{\delta}_{f_a}^a) + \begin{cases} 0 & ; \gamma \text{ is closed} , \\ \hat{s}_{e_{\text{start}}} + 1 & ; \gamma \text{ is not closed} . \end{cases} \quad (3.4.4)$$

Note that in Part 3, in case the curve goes from boundary to boundary, the edge index of the boundary edge where the endpoint of the curve lies is included in the sum over $A(\gamma)$.

Let $g+b \geq 1$ and consider a compact connected surface $\Sigma_{g,b}$ of genus g with b boundary components with parametrised ingoing boundary and fix a set of curves in the surface $\Sigma_{g,b}$ as shown in Figure 3.22 a). Let us consider a marked PLCW decomposition of $\Sigma_{g,b}$ as in Section 3.1.5 and recall the corresponding r -spin structure $\Sigma_{g,b}(s_i, t_i, u_j, \lambda_j - 1)$ from (3.1.26).

Corollary 3.4.6. *The holonomies of the curves in Figure 3.22 are*

$$\zeta(a_i) = s_i, \quad \zeta(b_i) = t_i, \quad \zeta(c_j) = u_j - u_b + 1, \quad \zeta(\partial_j) = 1 - \lambda_j, \quad (3.4.5)$$

for $i = 1, \dots, g$ and $j = 1, \dots, b-1$.

Proof. We only show the calculation of the latter two holonomies. There is only one face, let us denote it by f . The tangent vectors of the edge u_b and the loop c_j point in the same direction and the loop starts at this edge ($u_b = e_{\text{start}}$), therefore $\hat{s}_{u_b}^f = \hat{s}_{e_{\text{start}}} = -u_b - 1$; the tangent vectors of the edge u_j and the loop c_j point in the same direction and the loop ends at this edge ($u_j = e_{\text{end}}$), therefore $\hat{s}_{u_j}^f = \hat{s}_{e_{\text{end}}} = u_j$; the clockwise vertex determined by the marked edge of the face f is on the right side of the curve c_j so $\hat{\delta}_f^{c_j} = 1$. Taking the sum of all these we get $\zeta(c_j) = u_j + 1 - u_b - 1 + 1 = u_j - u_b + 1$. The edge r_j and the loop ∂_j cross negatively and the clockwise vertex is on the right side of the loop, so we get $\zeta(\partial_j) = -r_j - 1 + 1 = 1 - \lambda_j$. \square

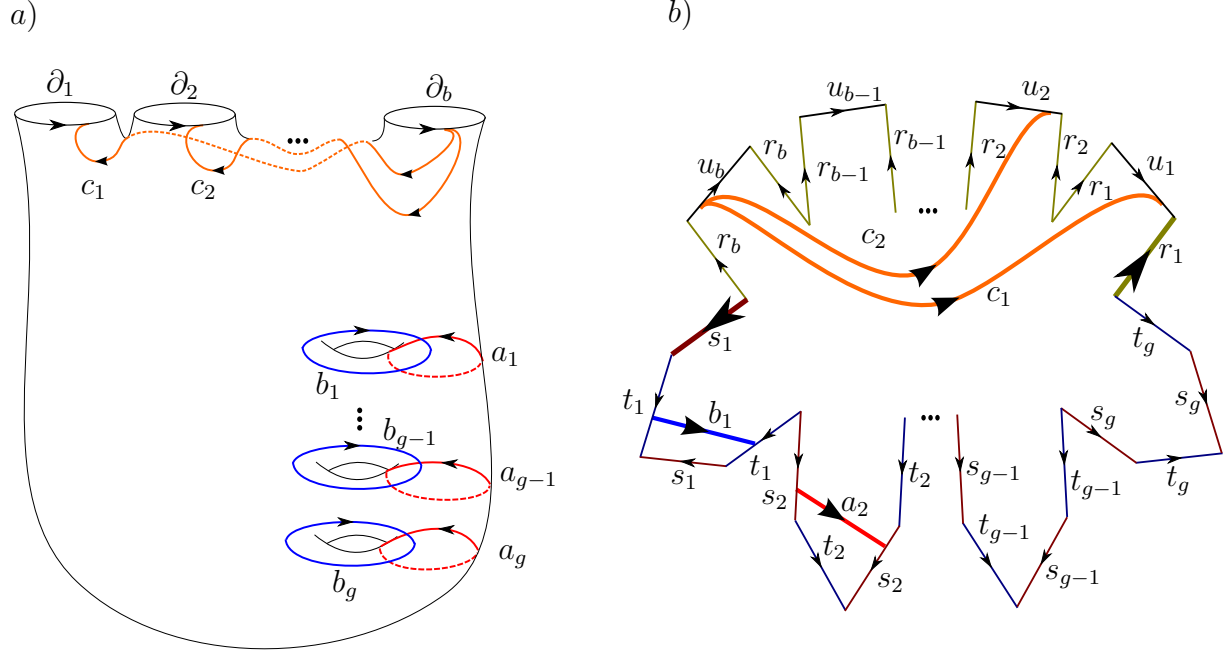


Figure 3.22: a) A set of curves in $\Sigma_{g,b}$: $\{a_i, b_i, c_j \mid i = 1, \dots, g; j = 1, \dots, b-1\}$. b) These curves in the PLCW decomposition of $\Sigma_{g,b}$ (cf. Figure 3.7). The bigger arrows on the edges show the marked edge: r_1 for $b > 0$ and s_1 for $b = 0$.

Definition 3.4.7 ([GG, Sec. 5] and [Ran, Sec. 2.4]). Let $r \geq 0$ be even. The r -spin Arf-invariant of the r -spin surface $\Sigma_{g,b}$ is

$$\text{Arf}(\Sigma_{g,b}) = \sum_{i=1}^g (\zeta(a_i) + 1) \cdot (\zeta(b_i) + 1) + \sum_{j=1}^{b-1} (\zeta(c_j) + 1) \cdot (\zeta(\partial_j) + 1) \pmod{2}. \quad (3.4.6)$$

Notice that for r even, r -spin structures naturally factorise through 2-spin structures. Therefore it makes sense to talk about the Arf-invariant of them, which was introduced for 2-spin structures [Joh]. $\text{Arf}(\Sigma_{g,b})$ is invariant under the action of the mapping class group of $\Sigma_{g,b}$, which has been proven in [GG, Lem. 7] and [Ran, Prop. 2.8]. We provide a different proof of this result in the corollary to the following theorem.

Theorem 3.4.8. *The TFT $\mathcal{Z}_{\mathcal{CL}}$ computes the r -spin Arf-invariant:*

$$\mathcal{Z}_{\mathcal{CL}}(\Sigma_{g,b}(s_i, t_i, u_j, \lambda_j - 1))(\theta^{\lambda_1} \otimes \dots \otimes \theta^{\lambda_b}) = 2^{1-g} \cdot (-1)^{\text{Arf}(\Sigma_{g,b}(s_i, t_i, u_j, \lambda_j - 1))}. \quad (3.4.7)$$

Proof. This is immediate from Proposition 3.4.2, Corollary 3.4.6 and Definition 3.4.7. \square

Since the morphisms in \mathcal{Bord}_2^r are diffeomorphism classes of r -spin bordisms (rel boundary), we get (cf. Remark 2.2.3(2)):

Corollary 3.4.9. *The r -spin Arf invariant is constant on mapping class group orbits.*

3.5 Counting mapping class group orbits

In this section we present the proof of Part 3 of Theorem 2.2.2. As we advertised it in Part 3 of Remark 2.2.3, we give an explicit expression for the number of mapping class group orbits of r -spin structures on $\Sigma_{0,b}$, i.e. $|O_0(r)|$, depending on the value of r and the R_i 's. Our proof follows the same ideas used in [GG] and in [Ran] to count orbits. Specifically, in [GG, Prop. 5] the number of orbits is given for general $r > 0$, $g \geq 0$ and $b = 0$, and in [Ran, Thm. 2.9] it is given for $r = 2$ in case $g = 1$, $b > 0$, and for general $r > 0$, in case $g \geq 2$, $b > 0$. Indeed, the authors of [GG, Ran] also calculate how the lifts of Dehn twists act on the isomorphism classes of r -spin structures in terms of a parametrisation and use these operations to reduce the parametrisation to a simpler form. Our proof uses the combinatorial model we introduced in Section 3.1.3, which is different from the parametrisation used in [GG, Ran], and we add the missing cases $g = 0$, $b > 0$ and $g = 1$, $b > 0$ for arbitrary $r \geq 0$.

Recall that a non-negative integer $d \in \mathbb{Z}_{\geq 0}$ is a divisor of r if there exists an integer n such that $d \cdot n = r$. In particular, with this definition every non-negative integer is a divisor of 0. As before we denote by $\gcd(a, b) \in \mathbb{Z}_{\geq 0}$ the non-negative integer that generates the ideal generated by a and b in \mathbb{Z} .

The $g = 0$ case

For $g = 0$ the MCG is generated by Dehn twists along loops ∂_j and h_{ij} shown in Figure 3.12. The cases $b = 0$ and $b = 1$ have been treated in Lemma 3.1.18 and in Corollary 3.1.20, so let us assume $b \geq 2$.

- Recall from Proposition 3.1.19 that the set of isomorphism classes of r -spin structures is given by $\prod_{i=1}^b \mathbb{Z}_r / \langle G \rangle$, where $G = (1, 1, \dots, 1)$.
- By applying Lemma 3.3.2 for the loop ∂_j , $\Sigma_{0,b}(u_j, R_j)$ and $\Sigma_{0,b}(u'_j, R_j)$ are in the same orbit if

$$u_j \equiv u'_j + R_j \pmod{r} . \quad (3.5.1)$$

- By applying Lemma 3.3.2 for the loop h_{ij} , $\Sigma_{0,b}(u_j, R_j)$ and $\Sigma_{0,b}(u'_j, R_j)$ are in the same orbit if

$$\begin{aligned} u_j &\equiv u'_j + R_i + R_j + 1 \pmod{r} , \\ u_i &\equiv u'_i + R_i + R_j + 1 \pmod{r} . \end{aligned} \quad (3.5.2)$$

Let $n \in \mathbb{Z}_{\geq 0}$ and let $\hat{R}_i, \hat{H}_{ij} \in \prod_{i=1}^b \mathbb{Z}_n$ for $i, j = 1, \dots, b$, $i \neq j$ have components $(\hat{R}_i)_k = \delta_{i,k} R_i$, $(\hat{H}_{ij})_i = (\hat{H}_{ij})_j = R_i + R_j + 1$ and $(\hat{H}_{ij})_k = 0$ for $k \neq i, j$. Let us define the quotient group

$$O_0(n) := (\mathbb{Z}_n)^b / \langle \hat{R}_i, \hat{H}_{ij}, G \rangle . \quad (3.5.3)$$

By construction, the set $O_0(r)$ is in bijection with orbits under the action of the MCG on isomorphism classes of r -spin structures. This proves the $g = 0$ case of Part 3 of Theorem 2.2.2. Notice that in case $b = 2$, $O_0(r)$ can be computed explicitly by hand:

$$O_0(r) = \mathbb{Z}_{\gcd(R_1, r)} \times \mathbb{Z}_{\gcd(R_2, r)} / \langle G \rangle \simeq \mathbb{Z}_{\gcd(R_1, r)} , \quad (3.5.4)$$

since by (3.1.23) $R_1 + R_2 \equiv 0 \pmod{r}$. We continue with computing the order of the set $O_0(r)$.

Explicit count of MCG orbits in the $g = 0$ case

Proposition 3.5.1. *The number of orbits $|O_0(r)|$ of the mapping class group on the set of isomorphism classes of r -spin structures on $\Sigma_{0,b}$ with $b \geq 2$ and with boundary parameters R_j , $j = 1, \dots, b$, is:*

- $r = 0$:
 - * $|O_0(0)| = \infty$ if $b = 2$ and $R_1 = R_2 = 0$,
 - * $|O_0(0)| = \gcd(2(R_j + 1)(R_k + 1), R_k(R_k + 1) | j, k \neq i, j \neq k)$ if $b \geq 3$ and $R_i = 0$ for an $i \in \{1, \dots, b\}$,
 - * $|O_0(0)| = |O_0(R_1 \cdot R_2 \cdots R_b)|$ else.
- $r > 0$: Let $r = p_1^{\alpha_1} \cdots p_L^{\alpha_L}$, $\alpha_i > 0$, be the prime decomposition of r . Then $|O_0(r)| = \prod_{i=1}^L |O_0(p_i^{\alpha_i})|$, with $|O_0(p_i^{\alpha_i})|$ as computed in Lemma 3.5.3.

In the following we give the proof of this proposition.¹ Let us first suppose that $r > 0$ and let $r = p_1^{\alpha_1} \cdots p_L^{\alpha_L}$ be the prime factorisation of r . The following lemma, whose proof is elementary, allows one to consider each $p_i^{\alpha_i}$ separately.

Lemma 3.5.2. *Let $\varphi : (\mathbb{Z}_n)^b \rightarrow (\mathbb{Z}_{p_1^{\alpha_1}})^b \times \cdots \times (\mathbb{Z}_{p_L^{\alpha_L}})^b$ be the isomorphism of abelian groups provided by the Chinese Remainder Theorem. Let $U = \langle u_1, \dots, u_N \rangle$ be the subgroup of $(\mathbb{Z}_n)^b$ generated by N elements $u_m \in (\mathbb{Z}_n)^b$, and let $V = \varphi(U)$ be its image. Then V is generated by the LN elements $u_m^{(l)}$, $l = 1, \dots, L$, $m = 1, \dots, N$, whose components in $(\mathbb{Z}_{p_i^{\alpha_i}})^b$ are*

$$(u_m^{(l)})_i = \begin{cases} u_m \pmod{p_i^{\alpha_i}} & i = l \\ 0 & i \neq l \end{cases} . \quad (3.5.5)$$

This lemma allows us to write $O_0(r) = \prod_{i=1}^L O_0(p_i^{\alpha_i})$. The key observation is the next lemma.

Lemma 3.5.3. *Let $\alpha \in \mathbb{Z}_{>0}$ and p a prime number. Then the order of $O_0(p^\alpha)$ is given as follows:*

¹We are indebted to Ehud Meir for showing us how to obtain an explicit expression for the order of $O_0(r)$ by first passing to the prime factorisation and then analysing each prime separately.

p	$\#\{R_k \text{ divisible by } p\}$	$ O_0(p^\alpha) $
2	0 or odd	1
	2	$\gcd(2^\alpha, \{R_k \text{ s.t. } 2 R_k\}, \{R_k + 1 \text{ s.t. } 2 \nmid R_k\})$
	> 2 and even	2
> 2	2	$\gcd(p^\alpha, \{R_k \text{ s.t. } p R_k\}, \{R_k + 1 \text{ s.t. } p \nmid R_k\})$
	$\neq 2$	1

Proof. We start by rewriting

$$O_0(p^\alpha) = \left(\prod_{i=1}^b \mathbb{Z}_{\gcd(R_i, p^\alpha)} \right) / \langle H_{ij}, G \rangle = \left(\prod_{i=1}^b \mathbb{Z}_{p^{\beta_i}} \right) / \langle H_{ij}, G \rangle, \quad (3.5.6)$$

where we defined $\beta_i \in \{0, 1, \dots, \alpha\}$ via $\gcd(R_i, p^\alpha) = p^{\beta_i}$. Let $H_{ij} \in \prod_{i=1}^b \mathbb{Z}_{p^{\beta_i}}$ for $i, j = 1, \dots, b$, $i \neq j$ have components $(H_{ij})_i = R_j + 1$, $(H_{ij})_j = R_i + 1$ and $(H_{ij})_k = 0$ for $k \neq i, j$. That is, $\hat{H}_{ij} = H_{ij}$ in $\prod_{i=1}^b \mathbb{Z}_{p^{\beta_i}}$.

Note that $R_i = S_i p^{\beta_i}$ for some integer S_i (which may still be divisible by p). Now if some R_i is not divisible by p , then $\beta_i = 0$ and so this factor can be omitted from the above product. Let

$$I \subset \{1, 2, \dots, b\} \quad (3.5.7)$$

consist of elements i for which $\beta_i > 0$. The generators H_{ij} now split into two sets, namely H_{ij} with $i, j \in I$, $i \neq j$, and $H_i^{(k)}$, with $i \in I$, $k \notin I$, whose only non-zero component is the i 'th one, which is equal to $R_k + 1$. We arrive at

$$O_0(p^\alpha) = \left(\prod_{i \in I} \mathbb{Z}_{p^{\beta_i}} \right) / \langle H_{ij}, H_i^{(k)}, G \rangle. \quad (3.5.8)$$

Pick a pair $i, j \in I$ with $i \neq j$ and such that $\beta_i \leq \beta_j$. Then in $A := \mathbb{Z}_{p^{\beta_i}} \times \mathbb{Z}_{p^{\beta_j}}$ we have $H_{ij} = (1, R_i + 1)$. Since R_i is divisible by p , $R_i + 1$ is not, and hence is invertible modulo p^{β_j} . Let $q \in \mathbb{Z}$ be such that $q(R_i + 1) \equiv 1 \pmod{p^{\beta_j}}$. Since $\beta_i \leq \beta_j$ this implies also that $q \equiv 1 \pmod{p^{\beta_i}}$ (as $R_i \equiv 0 \pmod{p^{\beta_i}}$). Altogether, in A we have $q(1, R_i + 1) = (1, 1)$. Conversely, $(R_i + 1)(1, 1) = (1, R_i + 1)$ in A , and so we can replace the generator H_{ij} by $\bar{H}_{ij} \in \prod_{k \in I} \mathbb{Z}_{p^{\beta_k}}$ which has entries 0 everywhere except for in positions $i, j \in I$, where it has entry 1. The group $O_0(p^\alpha)$ can thus be written as

$$O_0(p^\alpha) = \left(\prod_{i \in I} \mathbb{Z}_{p^{\beta_i}} \right) / \langle \bar{H}_{ij}, H_i^{(k)}, G \rangle = \left(\prod_{i \in I} \mathbb{Z}_{p^{\beta'_i}} \right) / \langle \bar{H}_{ij}, G \rangle, \quad (3.5.9)$$

where now, for $i \in I$,

$$p^{\beta'_i} = \gcd(p^\alpha, R_i, \{R_k + 1\}_{k \notin I}). \quad (3.5.10)$$

At this point we distinguish cases by the number of elements in I :

- $|I| = 0, 1$: In this case G already generates the group and so $O_0(p^\alpha) = \{0\}$.

- $|I| = 2$: Then $I = \{i, j\}$ for some $i \neq j$ and $\bar{H}_{ij} = G$. Thus $O_0(p^\alpha) \cong \mathbb{Z}_\gamma$ with $\gamma = \gcd(p^\alpha, \{R_k\}_{k \in I}, \{R_k + 1\}_{k \notin I})$.
- $|I| \geq 3$: Then $\bar{H}_{12} + \bar{H}_{13} - \bar{H}_{23}$ has 2 at the first component and 0 everywhere else. This means that one can take the first component (mod 2), and by a similar argument also for all the other components, that is

$$O_0(p^\alpha) = \{0\} \quad \text{if } p \neq 2. \quad (3.5.11)$$

If $p = 2$, first note that if $|I|$ is odd, then using G one can generate a 1 in any one component, with zeros in all other components, so furthermore

$$O_0(2^\alpha) = \{0\} \quad \text{if } |I| \text{ odd}. \quad (3.5.12)$$

If $|I|$ is even, by the above argument we can take every entry (mod 2), and it is then easy to see that there are exactly two orbits:

$$O_0(2^\alpha) \cong \mathbb{Z}_2 \quad \text{if } |I| \text{ even}. \quad (3.5.13)$$

□

Next we turn to the case $r = 0$.

Lemma 3.5.4. *If $R_i \neq 0$ for every $i = 1, \dots, b$ then $O_0(0) = O_0(R_1 \cdot R_2 \cdots R_b)$. If $R_i = 0$ for some i and $b = 2$ then the order of $O_0(0)$ is infinite. If $R_i = 0$ for some i and $b \geq 3$ then the order of $O_0(0)$ is $\gcd(2(R_j + 1)(R_k + 1), R_k(R_k + 1) | j, k \neq i, j \neq k)$.*

Proof. Let us assume that $R_i \neq 0$ for every $i = 1, \dots, b$. Then observe that

$$O_0(0) = \prod_{i=1}^b \mathbb{Z}_{R_i} / \langle H_{ij}, G \rangle = O_0(R_1 \cdot R_2 \cdots R_b), \quad (3.5.14)$$

by $\gcd(R_1 \cdot R_2 \cdots R_b, R_i) = R_i$ as in (3.5.6).

Assume that there is an i_0 such that $R_{i_0} = 0$. If $b = 2$ then using (3.1.23) we see that $R_1 = R_2 = 0$. Hence $\hat{R}_1 = \hat{R}_2 = 0$, $\hat{H}_{12} = G$ and so (3.5.3) reduces to $O_0(0) = \mathbb{Z}^2 / \langle G \rangle \cong \mathbb{Z}$.

Suppose now that $b \geq 3$. For simplicity we take $i_0 = 1$. Note that the element $\sum_{j>1} H_{1j} - G$ has $-1 + \sum_{j>1} (R_j + 1) = -2 + \sum_{j=1}^b (R_j + 1)$ as the first component and 0 everywhere else. But by (3.1.23) we have that $\sum_{i=1}^b (R_i + 1) = 2$, so that we get $G = \sum_{j>1} H_{1j}$, i.e. the generator G is redundant. Furthermore, the following elements are in the subgroup $\langle H_{ij} \rangle$ of $\mathbb{Z} \times \prod_{j=2}^b \mathbb{Z}_{R_j}$, for $i, j > 1$, $i \neq j$:

$$\begin{aligned} R_j H_{1j} &= (R_j(R_j + 1), 0, \dots, 0), \\ (R_i + 1)H_{1j} + (R_j + 1)H_{1i} - H_{ij} &= (2(R_i + 1)(R_j + 1), 0, \dots, 0). \end{aligned} \quad (3.5.15)$$

Write $g = \gcd(2(R_i + 1)(R_j + 1), R_j(R_j + 1) | i, j > 1, i \neq j)$ and consider the map

$$\phi: O_0(0) = \prod_{i=1}^b \mathbb{Z}_{R_i} / \langle H_{ij} \rangle \longrightarrow \mathbb{Z}_g \quad , \quad (a_1, \dots, a_b) \mapsto a_1 - \sum_{j=2}^b (R_j + 1)a_j . \quad (3.5.16)$$

Note that this map is indeed well-defined on the quotient and is a surjection. The map $\psi: \mathbb{Z}_g \rightarrow \prod_{i=1}^b \mathbb{Z}_{R_i} / \langle H_{ij} \rangle$, $m \mapsto (m, 0, \dots, 0)$ is equally well defined thanks to the elements in the subgroup listed in (3.5.15). By construction, $\phi \circ \psi = \text{id}$. We now show that $\psi \circ \phi = \text{id}$. The composition maps

$$(a_1, \dots, a_b) \mapsto \left(a_1 - \sum_{j=2}^b (R_j + 1)a_j , 0, \dots, 0 \right) . \quad (3.5.17)$$

By adding $\sum_{j=2}^b a_j H_{1,j}$ we get back (a_1, \dots, a_b) . Thus $|O_0(0)| = g$. \square

This completes the proof of Proposition 3.5.1, i.e. the explicit count of MCG orbits in the $g = 0$ case mentioned in Part 3 of Remark 2.2.3.

The $g = 1$ case

By Lemma 3.3.2, the set of MCG orbits is in bijection with

$$O_1(r) := \left(\mathbb{Z}_r^2 \times \prod_{i=1}^b \mathbb{Z}_{\gcd(R_i, r)} \right) / T , \quad (3.5.18)$$

the set of orbits under the action a group T generated by the following affine-linear transformations. Write an element of the above product as

$$\vec{x} = (s, t; u_1, \dots, u_b) . \quad (3.5.19)$$

Then T is generated by the transformations (recall that d_i in Lemma 3.3.2 only appears for $g > 1$)

$$\begin{aligned} T_G(\vec{x}) &= (s, t; u_1 + 1, \dots, u_n + 1) , \\ T_a(\vec{x}) &= (s - t, t; u_1, \dots, u_b) , \\ T_b(\vec{x}) &= (s, t - s; u_1, \dots, u_b) , \\ T_{f_j}(\vec{x}) &= (s, t - s - 1 - R_j; u_1, \dots, u_j + s + 1, \dots, u_b) \quad ; \quad 1 \leq j \leq b , \\ T_{h_{ij}}(\vec{x}) &= (s, t; u_1, \dots, u_i + R_j + 1, \dots, u_j + R_i + 1, \dots, u_b) \quad ; \quad 1 \leq i < j \leq b . \end{aligned} \quad (3.5.20)$$

It will be convenient to replace T_{f_j} by $T_j := T_b^{-1} T_{f_j}$ which acts as

$$T_j(\vec{x}) = (s, t - (R_j + 1); u_1, \dots, u_j + s + 1, \dots, u_b) . \quad (3.5.21)$$

Another convenient combination of generators is

$$T_S := T_a T_b^{-1} T_a(\vec{x}) = (-t, s; u_1, \dots, u_b) . \quad (3.5.22)$$

Note that T_a and T_b give an action of $SL(2, \mathbb{Z})$ on \mathbb{Z}_r^2 . The orbits of this action are parametrised by divisors of r :

Lemma 3.5.5. *Let D_r denote the set of divisors of r . The map $D_r \rightarrow \mathbb{Z}_r^2/SL(2, \mathbb{Z})$, $d \mapsto [(0, d)]$, is a bijection.*

Proof. Surjectivity: Let $(s, t) \in \mathbb{Z}^2$ be arbitrary and let $g := \gcd(s, t)$ and $d := \gcd(r, g)$, in particular $d \in D_r$. We can find $u, v \in \mathbb{Z}$ such that $us + vt = g$ and $x, y \in \mathbb{Z}$ such that $xr + yg = d$. Consider the elements

$$A = \begin{pmatrix} t/g & -s/g \\ u & v \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} g/d & -r/d \\ x & y \end{pmatrix} \quad (3.5.23)$$

in $SL(2, \mathbb{Z})$. They satisfy $A[(s, t)] = [(0, g)] = [(r, g)]$ and $B[(r, g)] = [(0, d)]$ in \mathbb{Z}_r^2 . So we have that $BA[(s, t)] = [(0, d)]$.

Injectivity: Let $d, d' \in D_r$ and assume that $(0, d)$ and $(0, d')$ lie on the same $SL(2, \mathbb{Z})$ -orbit. That is, there is an $A \in SL(2, \mathbb{Z})$ such that $A(0, d) = (0, d')$ holds in \mathbb{Z}_r^2 . It follows that there is an integer $a \in \mathbb{Z}$ such that $ad \equiv d' \pmod{r}$. Conversely, there is an integer a' such that $a'd' \equiv d \pmod{r}$. These relations, together with the fact that d and d' are divisors of r , show that the ideal $\langle r, d, d' \rangle$ in \mathbb{Z} generated by r, d, d' is equal to $\langle d \rangle$ and equal to $\langle d' \rangle$. But then $d = \pm d'$, and since both are non-negative, we have $d = d'$. \square

To analyse the set of orbits $O_1(r)$ we distinguish three cases by the number of boundary components.

- $b = 0$: In this case T is generated by T_a and T_b only and we can directly use Lemma 3.5.5 to conclude that $|O_1(r)| = |D_r|$.
- $b = 1$: In this case, $T_G(s, t; u_1) = (s, t; u_1 + 1)$, which removes the factor $\mathbb{Z}_{\gcd(R_1, r)}$ in (3.5.18). The remaining non-trivial generators acting now on \mathbb{Z}_r^2 are T_a, T_b and $T_1(s, t) = (s, t - (R_1 + 1))$. But by (3.1.23) we have $R_1 + 1 = 0$, and so T_1 also acts trivially. This reduces us to the $b = 0$ case and we again have $|O_1(r)| = |D_r|$.
- $b \geq 2$: The generators T_G and $T_{h_{ij}}$ commute with all generators. Let $U \subset T$ be the subgroup generated by T_G and $T_{h_{ij}}$ and write

$$A := \left(\mathbb{Z}_r^2 \times \prod_{i=1}^b \mathbb{Z}_{\gcd(R_i, r)} \right) / U \quad (3.5.24)$$

Note that the quotient by U amounts to dividing out a subgroup, and so A is still an abelian group. In the following, we will consider the action of T on A . By construction, we have $A/T = O_1(r)$.

For all $i \neq j$ we have

$$X_{ij}(\vec{x}) := T_j^{R_i+1} T_i^{R_j+1}(\vec{x}) = (s, t - 2(R_i + 1)(R_j + 1); u'_1, \dots, u'_b), \quad (3.5.25)$$

where $u'_i = u_i + (s + 1)(R_j + 1)$, $u'_j = u_j + (s + 1)(R_i + 1)$ and $u'_k = u_k$ for $k \neq i, j$. We can set u'_i and u'_j back to u_i and to u_j respectively by acting with $T_{h_{ij}}^{-s-1}$, and so in A we just have that

$$X_{ij}(\vec{x}) = (s, t - 2(R_i + 1)(R_j + 1); u_1, \dots, u_b). \quad (3.5.26)$$

Condition (3.1.23) now reads $\sum_{i=1}^b (R_i + 1) = 0$. Using this, we compute the iterated composition

$$\prod_{i=1, i \neq j}^b X_{ij}(\vec{x}) = (s, t + 2(R_j + 1)(R_j + 1); u_1, \dots, u_b) . \quad (3.5.27)$$

The R_j 's power of T_j acts as $T_j^{R_j}(\vec{x}) = (s, t - R_j(R_j + 1); u_1, \dots, u_b)$, so that altogether we find elements $Y_j \in T$ which act on A as

$$Y_j(\vec{x}) = (s, t - 2(R_j + 1); u_1, \dots, u_b) . \quad (3.5.28)$$

The cases R_j even and R_j odd behave differently:

- R_j even: the action of $T_j^{R_j}$ can be obtained as a power of Y_j ,
- R_j odd: an appropriate combination of $T_j^{R_j}$ and Y_j maps \vec{x} to $(s, t + R_j + 1; u_1, \dots, u_b)$.

We define

$$P_i := \begin{cases} 2(R_i + 1) & ; R_i \text{ even} \\ R_i + 1 & ; R_i \text{ odd} \end{cases} \quad (3.5.29)$$

and $g := \gcd(r, P_1, \dots, P_b)$. With this notation, T contains an element that maps $(s, t; \vec{u})$ to $(s, t + P_j; \vec{u})$, $j = 1, \dots, b$. By conjugating with T_S from (3.5.22) one furthermore obtains a group element that maps $(s, t; \vec{u})$ to $(s + P_j, t; \vec{u})$. We are therefore reduced to considering the T -orbits in

$$A' := \left(\mathbb{Z}_g^2 \times \prod_{i=1}^b \mathbb{Z}_{\gcd(R_i, r)} \right) / U . \quad (3.5.30)$$

As before, A' is an abelian group, and by construction we have $A'/T = O_1(r)$.

The above expression for g can be simplified. Indeed, the number of times a prime $p \geq 3$ divides P_i is equal to the number of times it divides $R_i + 1$, as the presence of a factor of 2 makes no difference. For the prime $p = 2$ note that 2 divides P_i exactly once if R_i is even, and at least once if R_i is odd. One easily checks that with

$$g' = \gcd(r, R_1 + 1, \dots, R_b + 1) \quad (3.5.31)$$

we have

$$g = \begin{cases} 2g' & ; r \text{ even and at least one } R_i \text{ even,} \\ g' & ; \text{else.} \end{cases} \quad (3.5.32)$$

At this point it is easy to give a lower bound on the number of orbits: Consider the projection $A' \rightarrow \mathbb{Z}_{g'}^2$, $(s, t, \vec{u}) \mapsto (s, t)$, with g' as in (3.5.31). On $\mathbb{Z}_{g'}^2$, the generators $T_G, T_j, T_{h_{ij}}$ all act trivially, so that we obtain a surjection

$$A'/T \longrightarrow \mathbb{Z}_{g'}^2 / SL(2, \mathbb{Z}) . \quad (3.5.33)$$

By Lemma 3.5.5, the right hand side consists of $D_{g'}$ many orbits. Altogether,

$$|O_1(r)| \geq |D_{g'}| . \quad (3.5.34)$$

We now give an upper bound for the number of orbits. From Lemma 3.5.5 we know that each orbit contains a representative

$$\vec{y} := (0, d; u_1, \dots, u_b), \quad (3.5.35)$$

where d now is a divisor of g . Since irrespective of the parity of R_j , adding $2(R_j + 1)$ to the second entry of A' acts trivially, we have $T_j^2(\vec{y}) = (0, d; u_1, \dots, u_j + 2, \dots, u_b)$.

We now go through various cases depending on the parity of r and the R_j :

- *r odd*: In this case $\gcd(r, R_j)$ is odd for all j , and the above shift by 2 can be replaced by a shift by 1, so that each orbit in A'/T contains an element $(0, d; 0, \dots, 0)$. Furthermore, by (3.5.32) we have $g = g'$ and so the lower bound (3.5.34) is strict.
- *r even*: Define $J \subset \{1, 2, \dots, b\}$ as

$$J = \{j \mid R_j \text{ even}\}. \quad (3.5.36)$$

Suppose that $j \notin J$, i.e. that R_j is odd. Then adding $R_j + 1$ acts trivially on the second component in A' , and so $T_j(\vec{y}) = (0, d; u_1, \dots, u_j + 1, \dots, u_b)$. In this way one can set all entries u_j of \vec{u} to zero for which $j \notin J$. Depending on the number of even R_j 's, we see different behaviour:

- ▷ $|J| = 0$: All R_j are odd and hence $g = g'$ and each orbit contains a representative $(0, d; 0, \dots, 0)$. Thus the lower bound (3.5.34) is strict.
- ▷ $|J|$ odd: This case cannot occur as $\sum_{j=1}^b (R_j + 1) = 0$ by (3.1.23). Indeed, taking this mod 2 and using that $R_j + 1$ is even for $j \notin J$ shows that $0 \equiv \sum_{j \in J} (R_j + 1) \equiv |J| \pmod{2}$.
- ▷ $|J| \geq 2$ even: We already know that every entry of \vec{u} can be reduced mod 2. Let $i, j \in J$ with $i \neq j$. Applying the generator $T_{h_{ij}}$ and reducing mod 2 shows that we can find an element of U that maps \vec{y} to $(0, d; u_1, \dots, u_i + 1, \dots, u_j + 1, \dots, u_b)$. Without loss of generality let us assume that $1 \in J$. Using the above shifts, and the mod 2 reduction we have anyway, we can transform $(0, d; \vec{u})$ to one of

$$(0, d; 0, 0, \dots, 0) \quad \text{or} \quad (0, d; 1, 0, \dots, 0). \quad (3.5.37)$$

Furthermore, acting with T_1 shows that, for $\varepsilon \in \{0, 1\}$,

$$(0, d; \varepsilon, 0, \dots, 0) \quad \text{and} \quad (0, d + (R_1 + 1); \varepsilon + 1 \pmod{2}, 0, \dots, 0) \quad (3.5.38)$$

lie on the same T -orbit.

By definition, d is a divisor of $g = 2g'$. But using (3.5.38) on a given orbit we can always find a representative of the form (3.5.37) where d is actually a divisor of g' . Indeed, in the present case g' is odd, and so if d divides g but not g' , it must be even, and $d + R_1 + 1$ is odd. As in the proof of Lemma 3.5.5 we can

use the $SL(2, \mathbb{Z})$ action to replace $d + R_1 + 1$ by $\gcd(0, d + R_1 + 1, 2g')$ which is odd and hence a divisor of g' .

From the surjection (3.5.33) we know that different divisors d of g' lie on different orbits of T .

It remains to show that for each $d \in D_{g'}$, the two elements in (3.5.37) lie on distinct orbits. We can assume without loss of generality that all boundary components are ingoing by changing the corresponding R_j to $-R_j$, since by this operation we do not change the parity. With this assumption we use Proposition 3.4.2 for $\Sigma_{1,b}(0, d, \varepsilon, 0, \dots, 0, R_1, \dots, R_b)$. One computes the RHS of (3.4.2) to be

$$(-1)^{d+1+\varepsilon \cdot (R_1+1)} . \quad (3.5.39)$$

Since $R_1 + 1$ is odd, different values of ε produce different signs.

Altogether, we have shown that in the present case, the number of orbits is

$$|O_1(r)| = 2|D_{g'}| . \quad (3.5.40)$$

This proves the $g = 1$ case of Part 3 of Theorem 2.2.2.

The $g \geq 2$ case

For $g \geq 2$ one can set $s_i = 0$ for every $i = 1, \dots, g$ as before and by using Lemma 3.3.2 for the loops ∂_j and f_j one can set $u_j = 0$ for every $j = 1, \dots, b$. Then using the lemma for the loops d_i one can set $t_i = 0$ for $i = 2, \dots, g$. Let us focus on (s_1, t_1, s_2, t_2) and apply the lemma for the following loops:

$$(0, t, 0, 0) \xrightarrow{\text{loop } d_1} (0, t-1, 0, +1) \xrightarrow{\text{loops } a_2, b_2} (0, t-1, 0, -1) \xrightarrow{\text{loop } d_1} (0, t-2, 0, 0).$$

This shows that there are at most 2 orbits for r even and 1 orbit for r odd. Again, as in the $g = 1$ case we assume that all boundary components are ingoing. By Corollary 3.4.3 there are at least 2 orbits for r even.

This completes the proof of the $g \geq 2$ case of Part 3 of Theorem 2.2.2 and thereby the proof of the entire theorem.

3.A Appendix: From triangulations to PLCW decompositions

By a *triangulation of a surface* we mean a smooth simplicial complex for the surface such that each boundary component consists of 3 edges and 3 vertices. In [Nov] a combinatorial description of r -spin surfaces was given using triangulations. The purpose of this appendix is to show how to obtain the combinatorial description of r -spin surfaces using PLCW decomposition of Section 3.1 from triangulations.

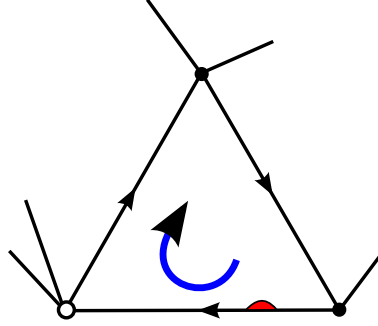


Figure 3.23: The 3 edges and 3 vertices of a boundary component together with the additional marking of one edge. The curly arrow shows the orientation of the boundary component and the empty vertex shows the ending vertex of the additionally marked edge.

3.A.1 r -spin surfaces with triangulations

Let us summarise the results of [Nov]. More precisely let us look at the differences between that formalism and the formalism developed in Section 3.1.

Let Σ be a marked triangulation of a surface with parametrised boundary, i.e. every edge has an orientation and an edge index and every face has a marked edge. Let us assume that all boundary components are ingoing and recall the notions of Section 3.1.3. Put an additional marking on one of the edges of each boundary component b . The induced orientation of the boundary component gives a starting and an ending vertex of this additionally marked edge, see Figure 3.23. For a boundary vertex u let $\alpha_u := +1$ if it is an ending vertex for the additionally marked edge and $\alpha_u := 0$ otherwise. We furthermore assume that the orientation of boundary edges agrees with the induced orientation of the boundary components. The marking is called *admissible* for a given map $\tilde{\lambda} : \pi_0(\partial\Sigma) \rightarrow \mathbb{Z}_r$ $b \mapsto \tilde{\lambda}_b$, if the following hold for every inner vertex v and every boundary vertex u on a boundary component b .

$$\sum_{e \in \partial^{-1}(v)} \hat{s}_e \equiv D_v - N_v + 1 \pmod{r}, \quad (3.A.1)$$

$$\sum_{e \in \partial^{-1}(u)} \hat{s}_e \equiv D_u - N_u + 1 + \alpha_u \cdot (1 - \tilde{\lambda}_b) \pmod{r}. \quad (3.A.2)$$

Here, $D_{v/u}$, $N_{v/u}$ and \hat{s}_e are defined as in Section 3.1.3. According to the construction in [Nov, Sec. 4.8] we proceed as follows:

- Define an r -spin structure on Σ minus edges and vertices by giving the interior of the faces the r -spin structure \mathbb{C}^0 .
- Define transition functions for every pair of faces fixed by the edge indices to extend the above to Σ minus vertices.
- There is a unique r -spin structure $\Sigma(\underline{s})$ extending to the vertices if and only if the edge index assignment is admissible. Extend the r -spin structure to the vertices.

- The r -spin boundary parametrisation map is the inclusion of the r -spin collars according to the map $\tilde{\lambda}$. The inclusions map $1 \in \mathbb{C}^{\tilde{\lambda}}$ to the boundary vertex determined by the extra marking of the given boundary component.

3.A.2 Distinguishing in- and outgoing boundary components

The glueing of r -spin surfaces with parametrised boundary is defined as follows. First for every $\kappa \in \mathbb{Z}_r$ we specify an r -spin lift $I_\varepsilon^\kappa : \mathbb{C}^\kappa \rightarrow \mathbb{C}^{2-\kappa}$ (\tilde{s}^ε in [Nov, Eqn. (3.35)]) of the map $z \mapsto z^{-1}$ given by an element $\varepsilon \in \mathbb{Z}_r$. Take two boundary components with r -spin structure on a neighbourhood of these components \mathbb{C}^κ and $\mathbb{C}^{2-\kappa}$. We can glue these boundary components along their r -spin boundary parametrisation composed with I_ε^κ .

To define outgoing boundary components we precompose the above boundary parametrisations with I_ε^κ for outgoing boundary components. (For convenience we will choose $\varepsilon = 0$, as different choices of ε can be seen as composition with different r -spin cylinders.) Then one can glue r -spin boundary components along in- and outgoing boundary parametrisations as described in Section 3.1.1. We now give more details on the construction.

Let Σ be an r -spin surface with ingoing r -spin boundary parametrisation

$$\tilde{\varphi} : \bigsqcup_{b \in \pi_0(\partial\Sigma)} U_b^{\tilde{\lambda}_b} \rightarrow \Sigma$$

for a map $\tilde{\lambda} : \pi_0(\partial\Sigma) \rightarrow \mathbb{Z}_r$ which maps $b \mapsto \tilde{\lambda}_b$. In order to distinguish in- and outgoing boundary components we first fix two sets $B_{in}, B_{out} \subset \pi_0(\partial\Sigma)$ as in Section 3.1.1. Let $\hat{I} : \mathbb{Z}_r \rightarrow \mathbb{Z}_r$ be the map $x \mapsto 2 - x$. We define maps $\lambda : B_{in} \rightarrow \mathbb{Z}_r$ and $\mu : B_{out} \rightarrow \mathbb{Z}_r$ by $\lambda := \tilde{\lambda}|_{B_{in}}$ and $\mu := \hat{I} \circ \tilde{\lambda}|_{B_{out}}$. For the in- and outgoing r -spin boundary parametrisations we set

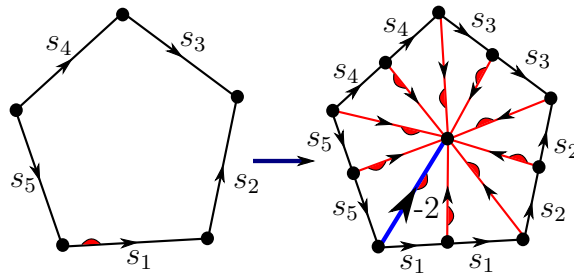
$$\varphi_{in} := \tilde{\varphi}|_{\bigsqcup_{b \in B_{in}} U_b^{\lambda_b}} \quad \text{and} \quad \varphi_{out} := \tilde{\varphi}|_{\bigsqcup_{c \in B_{out}} U_c^{\lambda_c}} \circ \left(\bigsqcup_{c \in B_{out}} I_0^{\lambda_c} \right)$$

respectively. The admissibility condition (3.A.2) needs to be changed since we are parametrising outgoing boundary components $c \in B_{out}$ with $\mathbb{C}^{2-\tilde{\lambda}_c}$ instead of with $\mathbb{C}^{\tilde{\lambda}_c}$. This means that for a vertex u on an outgoing boundary component c the factor α_u needs to be -1 instead of $+1$, since $1 - (2 - \tilde{\lambda}_c) = -(1 - \tilde{\lambda}_c)$.

3.A.3 Refining PLCW decompositions of r -spin surfaces

By a *series of radial subdivisions* we mean radially subdividing the 1-cells and then the 2-cells, see Figure 3.24. This means splitting each edge in two by adding a vertex, adding a vertex to the interior of each face and adding edges between this new vertex and all other vertices of this face. The following lemmas follow from straightforward calculations.

Lemma 3.A.1. *Let L be a PLCW decomposition obtained by a series of radial subdivisions from a PLCW decomposition K with admissible marking. Assign to new edges*



the markings, orientations and edge labels as shown in Figure 3.24. The vertex conditions (3.A.1) and (3.A.2) are satisfied at the old and new vertices.

Lemma 3.A.2. *Let L be a marked PLCW decomposition obtained by applying the steps in Lemma 3.A.1 twice on another PLCW decomposition K with admissible marking. Add 7 triangles at each boundary component and assign the marking to the new edges as shown in Figure 3.25 and put the extra markings on edges on boundary components so that the ending vertex is v_0 in Figure 3.25. Then the conditions (3.A.1) and (3.A.2) hold at old and new vertices.*

Sketch of proof. Let us assume that at each boundary component there are only two edges connecting to the single vertex: the boundary edge and another one coming from the interior of the surface. In such a situation the refinement is shown in Figure 3.25. The conditions (3.A.1) and (3.A.2) can be checked by hand at every vertex.

If there are boundary components where more edges connect to the boundary vertex from the interior in the original PLCW decomposition, checking the conditions (3.A.1) and (3.A.2) is similar, but we omit the figure here. \square

We now have all the ingredients needed to define an r -spin structure with r -spin boundary parametrisation using the tools developed by [Nov]. We proceed as follows.

- Take a surface with parametrised boundary and a marked PLCW decomposition with some edge indices s and maps $\lambda : B_{in} \rightarrow \mathbb{Z}_r$ and $\mu : B_{out} \rightarrow \mathbb{Z}_r$.
- Refine this marked PLCW decomposition as described in Lemma 3.A.2. This is a triangulation by [Kir, Thm. 6.3].

The new marking obtained this way is admissible in the sense of [Nov] (i.e. (3.A.1) and (3.A.2) hold) if and only if the marking of the original PLCW decomposition is admissible in the sense of Section 3.1.3 (i.e. (3.1.20) and (3.1.21) hold).

Definition 3.A.3. Let $\Sigma(s, \lambda, \mu)$ denote the r -spin structure on Σ obtained by the above steps.

3.A.4 Proofs for Section 3.1

Proof of Lemma 3.1.11. Operation 1 follows directly from part 2 of [Nov, Lem. 4.11].

For Operation 3 do a deck transformation [Nov, Part 1 of Lem. 4.11] on all triangles inside the polygon.

For Operation 2 first notice that moving the marking of a polygon to the next clockwise edge amounts to changing the edge indices as in Figure 3.26. This is done by a deck transformation on all filled triangles.

It is a straightforward calculation to show that these operations commute with each other. \square

Proof of Theorem 3.1.13. Let Σ be a surface with PLCW decomposition. Let Σ' the same surface, but now with a triangulation as obtained by a two-fold series of radial subdivisions as in Section 3.A.3. For clarity, in this proof we will write $\underline{\Sigma}$ for the surface without decomposition underlying both Σ and Σ' .

In [Nov, Sec. 4.8] $\mathcal{M}(\Sigma')_{\tilde{\lambda}}^{\text{triang}}$ the set of admissible markings for a fixed triangulation of $\underline{\Sigma}$ with only ingoing boundary components and fixed map $\tilde{\lambda}$ has been defined along with a similar equivalence relation as \sim_{fix} , which we denote by $\sim_{fix}^{\text{triang}}$. [Nov, Thm. 4.18] gives the isomorphism from the quotient of this set by $\sim_{fix}^{\text{triang}}$ to $\mathcal{R}^r(\underline{\Sigma})_{\tilde{\lambda}}$ the isomorphism classes of r -spin structures. By a simple reparametrisation as in Section 3.A.2 one obtains from this the set of admissible markings for in- and outgoing boundary components $\mathcal{M}(\Sigma')_{\lambda, \mu}^{\text{triang}}$

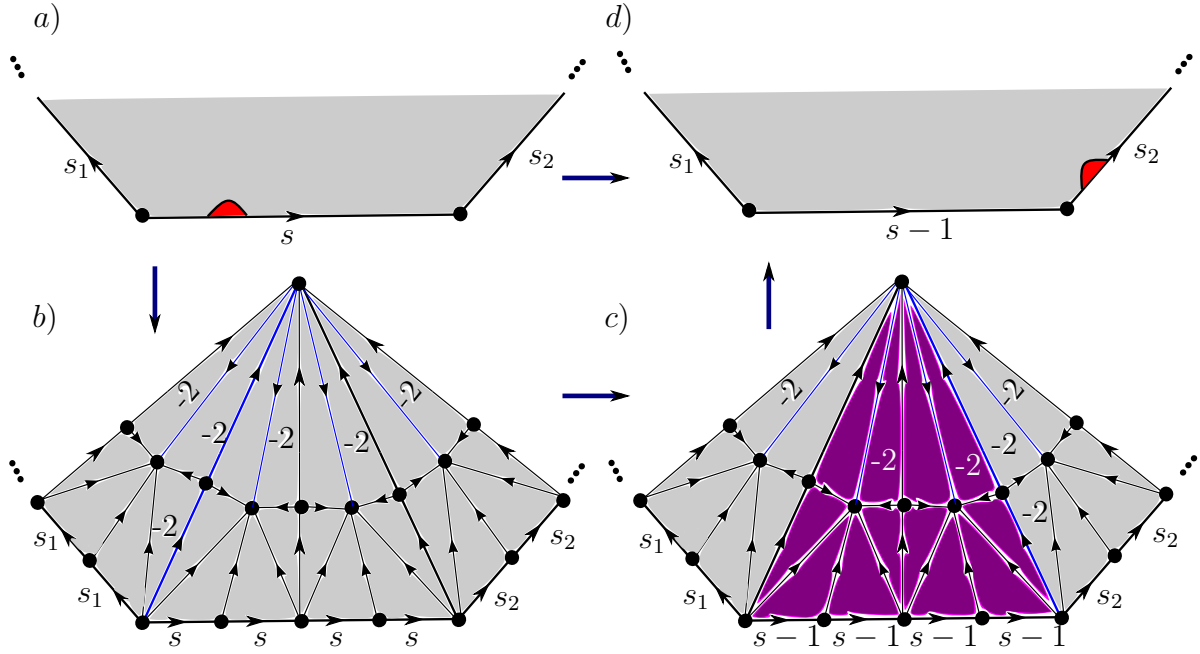


Figure 3.26: Shifting the marking on a face of a PLCW decomposition clockwise. All unlabeled edges have index -1. *a)* Part of a face of a marked PLCW decomposition showing the marked edge. *b)* The corresponding triangulation after two series of radial subdivisions. *c)* Execute a deck transformation on the 12 filled triangles. *d)* The PLCW decomposition with shifted marked edge which produces the triangulation shown in *c)*.

and the set of isomorphism classes of r -spin structures with in- and outgoing boundary components $\mathcal{R}^r(\underline{\Sigma})_{\lambda,\mu}$. Thus we get a bijection

$$\mathcal{M}(\Sigma')_{\lambda,\mu}^{\text{triang}} / \sim_{\text{fix}}^{\text{triang}} \xrightarrow{f} \mathcal{R}^r(\underline{\Sigma})_{\lambda,\mu} . \quad (3.A.3)$$

Let us denote by $\alpha : \mathcal{M}(\Sigma)_{\lambda,\mu}^{\text{PLCW}} \rightarrow \mathcal{M}(\Sigma')_{\lambda,\mu}^{\text{triang}}$ the map that sends a marked PLCW decomposition to its refinement according to Section 3.A.3. Since the generators of the equivalence relation \sim_{fix} are built up from generators of the equivalence relation $\sim_{\text{fix}}^{\text{triang}}$ (see the proof of Lemma 3.1.11 above), we get a well defined map

$$\mathcal{M}(\Sigma)_{\lambda,\mu}^{\text{PLCW}} / \sim_{\text{fix}} \xrightarrow{\bar{\alpha}} \mathcal{M}(\Sigma')_{\lambda,\mu}^{\text{triang}} / \sim_{\text{fix}}^{\text{triang}} . \quad (3.A.4)$$

By construction the composition of the maps (3.A.3) and (3.A.4) is the map (3.1.22) in the statement of the theorem. It therefore remains to show that $\bar{\alpha}$ is a bijection.

$\bar{\alpha}$ is surjective: Let (m', o', s') be an admissible marking of Σ' . As a first step, use the relation $\sim_{\text{fix}}^{\text{triang}}$ to change the edge markings m' and orientations o' to the form prescribed in Section 3.A.3, resulting in a marking (m'', o'', s'') . Next follow the algorithm described in Figure 3.27 to bring all edge indices of Σ' in the interior of faces of Σ to the form shown in Figure 3.25. Denote the resulting marking by (m'', o'', \tilde{s}) . Let e be an interior edge of

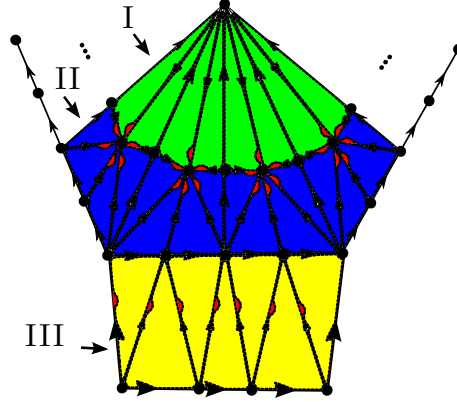


Figure 3.27: To convert the edge indices of all edges of the triangulation in the interior of some face of the PLCW decomposition to the form shown in Figure 3.25 apply the following algorithm to all faces: I) Pick a triangle in area I; proceeding clockwise around the vertex, use deck transformations on each triangle to bring the edge index of each edge radiating from the central vertex to the prescribed value (-1 or -2); note that the final edge in this procedure automatically has the correct index due to the admissibility condition around the central vertex. II) Pick a triangle t in region II which shares an edge with region I but whose neighbour t' in anti-clockwise direction of region II does not. Use a deck transformation on t to set the edge index of the edge on the boundary of region I to the value in Figure 3.25; proceed clockwise around region II setting the edge index between two triangles of region II to the correct value; the edge indices between II and I are determined by the admissibility condition (and so automatically as stated in Figure 3.25); finally, the edge between t' and t has the correct value by the admissibility condition around the vertex between region I and II shared by t and t' . III) If the face in question has a boundary component, then in region III one proceeds in the same way as in region II.

Σ and let e_1, \dots, e_4 be the edges of Σ' which cover e , and v_{12}, v_{23}, v_{34} the three additional vertices on e . The admissibility condition around v_{12}, v_{23}, v_{34} implies that the edge indices on e_1, \dots, e_4 must all be equal. The same argument shows that edge indices on boundary components are all equal. This shows that (m'', o'', \tilde{s}) lies in the image of α .

$\bar{\alpha}$ is injective: Let $(m, o, s), (m', o', s') \in \mathcal{M}(\Sigma)_{\lambda, \mu}^{\text{PLCW}}$ such that $\bar{\alpha}[(m, o, s)] = \bar{\alpha}[(m', o', s')]$, i.e. $\alpha(m, o, s) \sim_{\text{fix}}^{\text{triang}} \alpha(m', o', s')$. Notice that Lemma 3.1.12 and Remark 3.1.14 apply to marked triangulations as well. This means that we can assume that the marked edges and the edge orientations agree ($m = m'$ and $o = o'$) for the PLCW decomposition and the triangulation as well. Furthermore, $\alpha(m, o, s)$ and $\alpha(m, o, s')$ are related by a series D of deck transformation on the triangulation: $D(\alpha(m, o, s)) = \alpha(m, o, s')$.

Write $\delta_{\Delta}(k)$ for a deck transformation by k units on the triangle Δ of the triangulation of Σ' . Deck transformations on different triangles commute, so we can write the sequence of deck transformations as $D = \prod_{\Delta} \delta_{\Delta}(k_{\Delta})$. It is not hard to see that the identity $D(\alpha(m, o, s)) = \alpha(m, o, s')$ requires the k_{Δ} for all Δ belonging to a given face of the PLCW-decomposition of Σ to be equal. But this precisely means that D can be written as a product of deck transformations on the PLCW-decomposition of Σ , i.e.

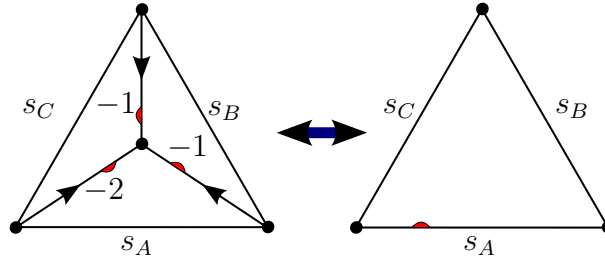


Figure 3.28: Pachner 3-1 move

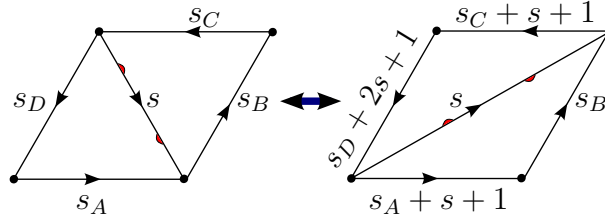


Figure 3.29: Pachner 2-2 move

$$(m, o, s) \sim_{\text{fix}} (m', o', s').$$

□

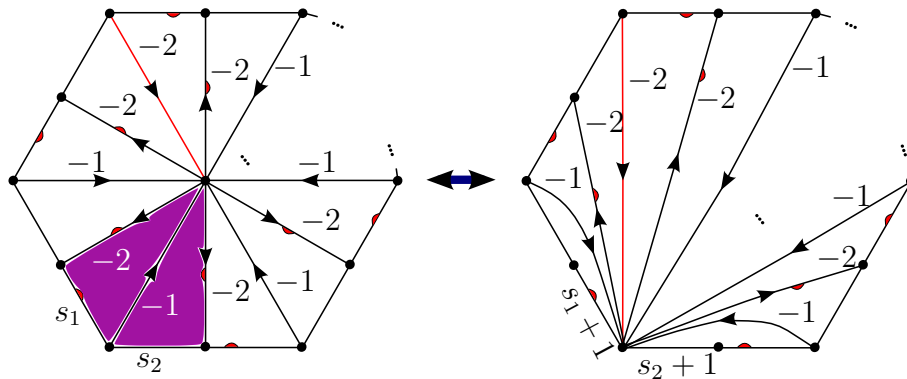
In the following we are going to give some tools that relate different marked triangulations and marked PLCW decompositions which parametrise isomorphic r -spin structures. First we recall [Nov, Prop. 4.19 and 4.20].

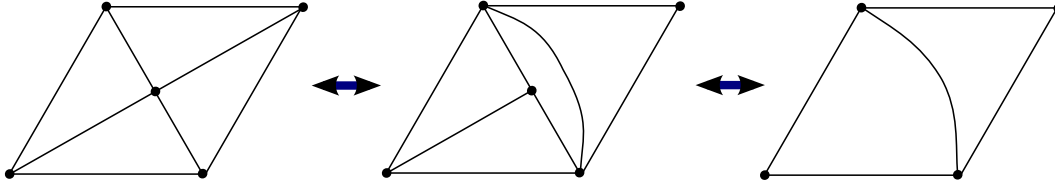
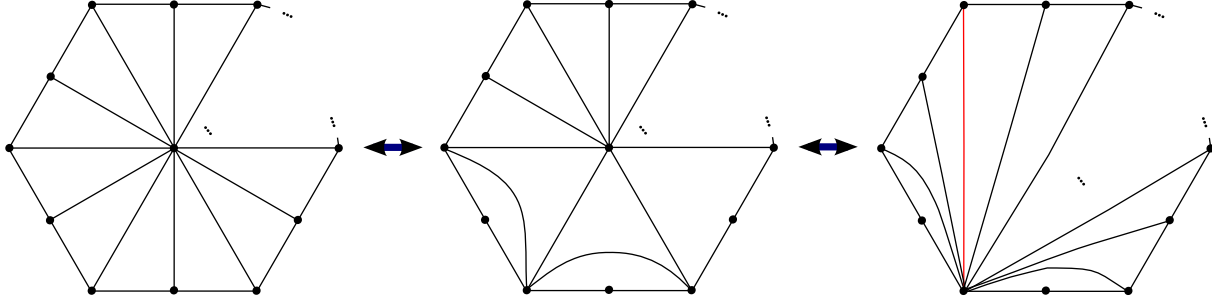
Lemma 3.A.4. *Let Σ and Σ' be two r -spin surfaces with triangulation and with the same underlying surface related by a Pachner 3-1 or 2-2 move as in Figure 3.28 and 3.29. Then these two r -spin structures are isomorphic.*

We define the T_n -moves for $n \geq 2$ as in Figure 3.30, which takes a $2n$ -gon glued together from $2n$ triangles to a $2n$ -gon glued together from $2(n-1)$ triangles.

Lemma 3.A.5. *The T_n move induces an isomorphism of r -spin structures.*

Proof. First we show that one can obtain the T_n move on a triangulation without any


 Figure 3.30: T_n move for $n \geq 2$. We remove or add the filled triangles.

**Figure 3.31:** T_2 move without marking**Figure 3.32:** Induction step for the T_{n+1} move

marking by a series of Pachner moves by induction on n . For $n = 2$ do a Pachner 2-2 move and then a Pachner 3-1 move as in Figure 3.31. Now assume that the statement holds for n and show for $n + 1$. First we do two Pachner 2-2 moves and then apply a T_n move as in Figure 3.32 to get exactly the T_{n+1} move.

Since the Pachner moves in Figures 3.28 and 3.29 only change the marking locally, it is enough to check how the marking can possibly change near the vertices that are touched by these moves. If one calculates (3.1.20) for these vertices before and after a T_n move one sees that the marking can only change according to Figure 3.30. \square

Lemma 3.A.6. *Removing a univalent vertex (whose edge was not marked) induces an isomorphism of r -spin structures.*

Proof. When we remove an edge from a PLCW decomposition we need to compare the associated triangulation with marking from Definition 3.A.3 and then use the above defined moves to go from one to the other. The part of the triangulations that need to be transformed into one another together with the transformation steps are shown in Figure 3.33. \square

Proof of Proposition 3.1.16.

Move b) in Figure 3.5 for $v \neq v'$: As in the proof of Lemma 3.A.6 we need to compare the marked triangulations associated to the marked PLCW decompositions. The part of the triangulations that need to be transformed into one another together with the transformation steps are shown in Figure 3.34.

Since we did local moves which induce isomorphisms of r -spin structures, it is enough to check how the edge indices will change at those vertices which have been touched by the above moves. These vertices are marked with a circle. Observe that at the vertices v_l ,

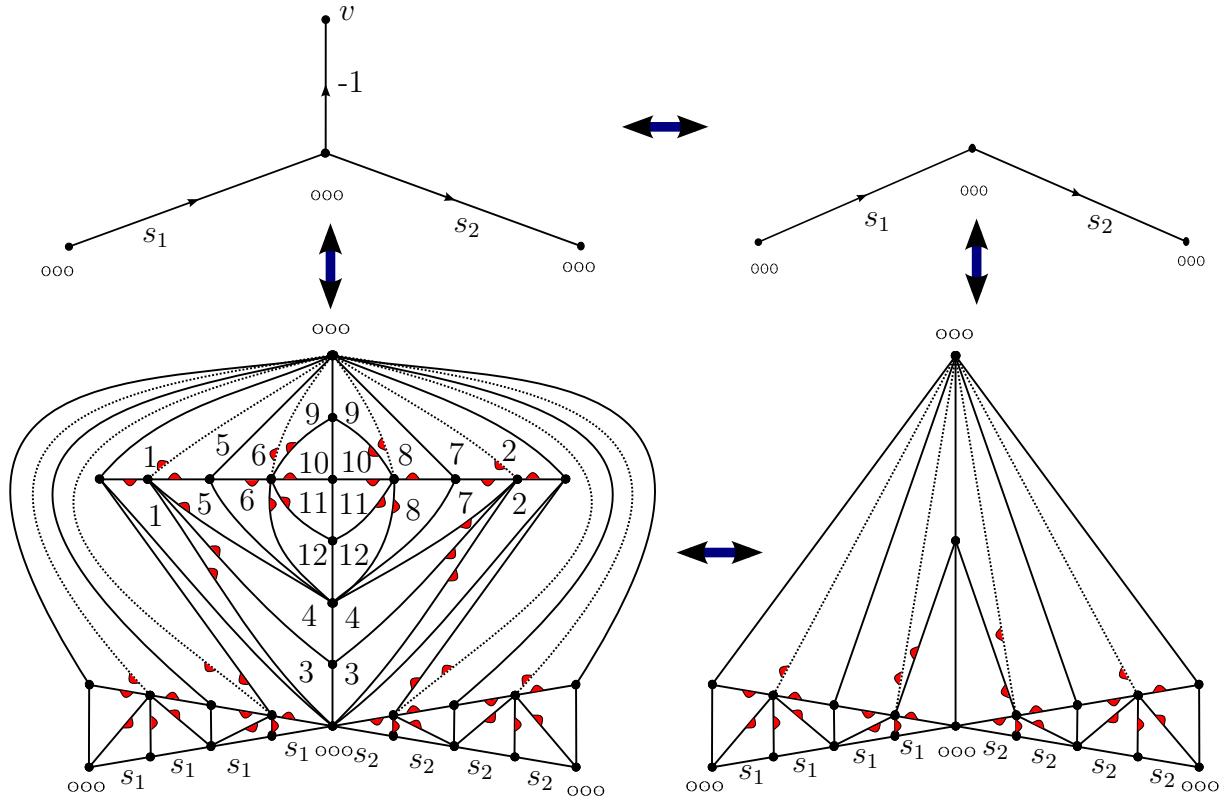


Figure 3.33: The part of the triangulations that need to be transformed into one another in case of removing or adding an univalent vertex v with its edge. The dotted edges have edge index -2, all other unlabeled edges have edge index -1. The orientation of the edges is left implicit, cf. Definition 3.A.3. We need to remove the 24 numbered triangles from the middle, we proceed by removing them in pairs. We use the T_n moves consecutively: first remove the two triangles marked by 1, then the two triangles marked by 2, etc until finally removing the two triangles marked by 12.

v_m and v_r one does not get any condition on s . The vertices v_l^{up} , v_l^{down} , v_r^{up} and v_r^{down} get identified with others.

Assume that the vertices v and v' are distinct and that $s'_i = s_i$ ($i = 1, \dots, 4$). At these two vertices one obtains $s \equiv 0 \pmod{r}$.

Move a) in Figure 3.5: When removing a bivalent vertex as in Figure 3.5 a), a similar argument applies.

Move b) in Figure 3.5 for $v = v'$: Indeed, look at the original PLCW decomposition and assume that the vertices v and v' are the same. Insert a bivalent vertex on the edge, remove one of the two new edges by the above and then the univalent vertex with its edge using Lemma 3.A.6. Again, one obtains $s \equiv 0 \pmod{r}$.

This completes the proof of the proposition. \square

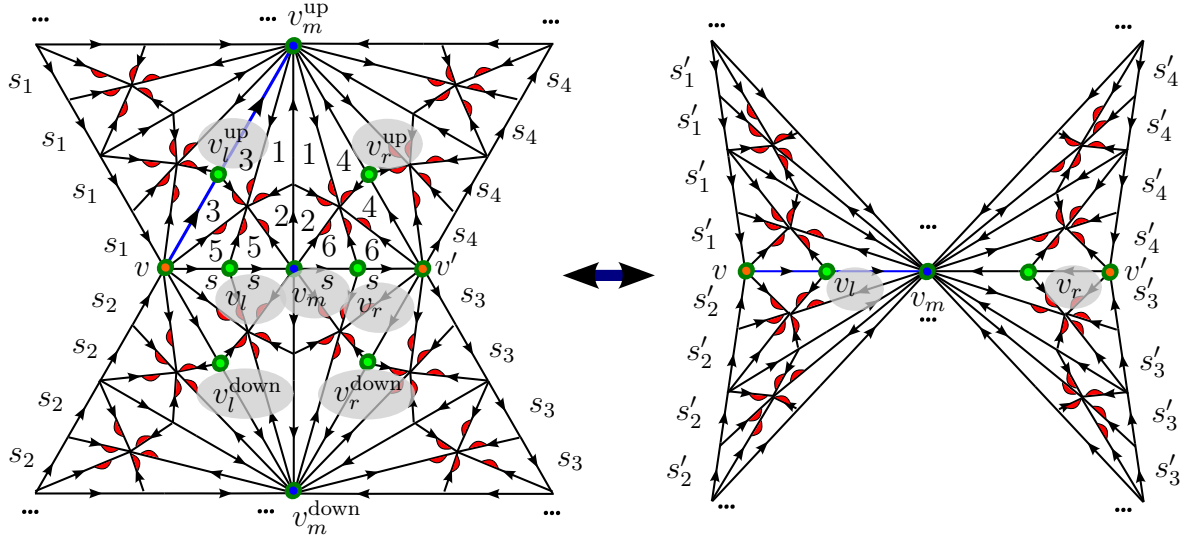


Figure 3.34: The part of the triangulations that need to be transformed into one another in case of removing or adding an edge between the vertices v and v' (cf. Figure 3.5 b)). The edges between v , v_l^{up} and v_m^{up} have edge index -2 , all other edges without edge index have edge index -1 . We need to remove the 24 triangles from the middle, of which 12 has been numbered in pairs. We use the T_n moves consecutively: first remove the two triangles marked by 1, then the two triangles marked by 2, etc until finally removing the two triangles marked by 6. Then do the same thing again for the mirror pairs.

3.A.5 Proof of Theorem 3.2.8

For Part 1 a direct computation shows that the morphism assigned to a PLCW decomposition and the morphism assigned to the triangulation obtained by the refinement of the PLCW decomposition are the same. One needs to use that multiplication with the τ^{-1} 's in the state-sum construction amount to canceling the “bubbles” $\mu \circ \Delta$. Independence of the choice of the function V follows from the fact that τ is a central element.

Next we check independence from the triangulation and from the choice of marking (for a given r -spin structure). Let us assume that Σ has b ingoing and no outgoing boundary components. Let $T_A(\Sigma)$ denote the morphism in \mathcal{S} assigned to Σ using a triangulation by the state sum construction of [Nov]. Note that we get three tensor factors of A for each boundary component, since each boundary component consists of three edges. Now we explain how to reduce $A^{\otimes 3}$ to A for each boundary component. Recall that we used the notation (13) for the cyclic permutation of the first and third tensor factors. Composing $T_A(\Sigma)$ with $\bigotimes_{i=1}^b (13) \circ (\Delta \otimes \text{id}_A) \circ \Delta \circ (\tau^{-2} \cdot (-)) \circ \iota_{\lambda_i}$, we obtain the morphism in (3.2.41). To show this we use that the factors of τ^{-1} remove the “bubbles” $\mu \circ \tau$. If Σ has outgoing boundary components, it is easy to see that composing with appropriate factors of $\Gamma_{i,j,\varepsilon}$ maps of [Nov, Sec. 5.4] and $\pi_{\lambda_i} \circ \mu^{(3)} \circ (13)$ again yields the morphism in (3.2.41). Independence of the details of the triangulation is shown in [Nov, Thm. 5.10]. This latter theorem also states that $T_A(\Sigma) = T_A(\Sigma')$ for isomorphic r -spin surfaces Σ and Σ' , so that the assignment $Z_A : \text{Bord}_2^r \rightarrow \mathcal{S}$ is well defined on morphisms.

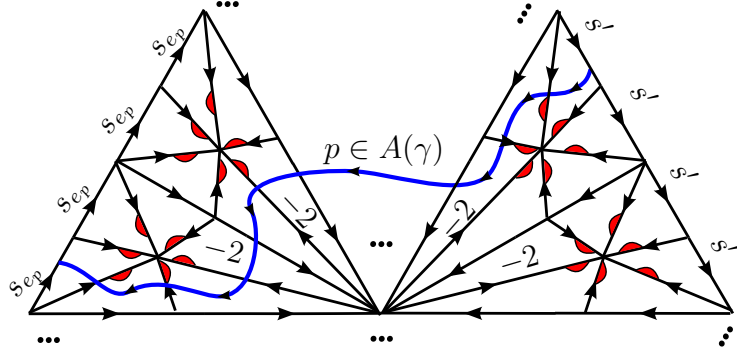


Figure 3.35: Detail of a face with interior edges of a refined PLCW decomposition with the segment $p \in A(\gamma)$ of the curve γ crossing it. Using Part 2, we can assume that the segment of the curve crosses as shown in the figure. All edge indices without edge labels are -1 . Notice that when crossing the dotted area, the lift of the curve does not pick up any of the ω_e contributions.

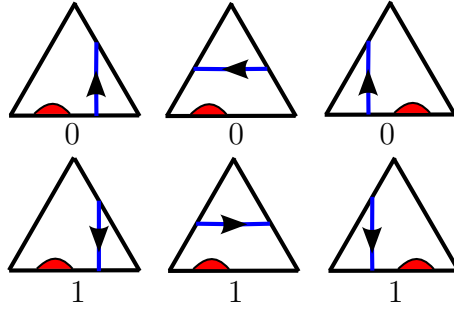


Figure 3.36: The different values of κ_e for different positions of the crossing curve segment. The edge e is where the line segment leaves the triangle.

For Part 2 functoriality can now be seen easily from the above discussion and by using [Nov, Prop. 5.11], since the embeddings and projectors ι_{λ_i} and π_{λ_i} compose to P_{λ_i} , which can be omitted due to [Nov, Prop. 5.13]. Monoidality and symmetry follow directly from the construction. This completes the proof of Theorem 3.2.8.

3.A.6 Proof of Lemma 3.4.5

Part 1 does not involve the marked PLCW decomposition and is shown in [Nov, Lem. 3.12].

Part 2 follows directly from the discussion in the main text: homotopic curves in Σ (in the sense described in the beginning of Section 3.4.2) have homotopic lifts in $F_{GL}\Sigma$ and homotopic curves in $F_{GL}\Sigma$ have the same lifts in $P_{GL}\Sigma$ after fixing them at the same starting point.

For Part 3, we are going to calculate the holonomy by summing up the contributions for all arcs $A(\gamma)$ as in (3.4.4).

The contribution for $p \in A(\gamma)$ can be computed as follows. Take the face f_p which p

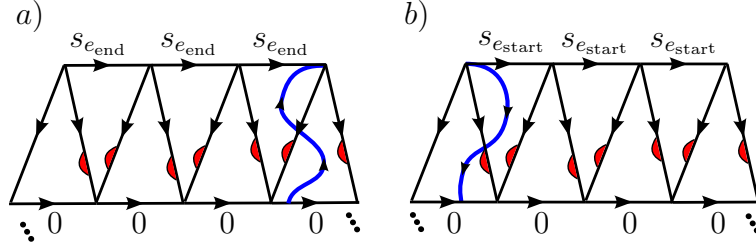


Figure 3.37: Detail of two (not necessarily different) faces with two boundary edges of a refined PLCW decomposition where the curve γ starts (b) and ends (a), i.e. at the image of $1 \in \mathbb{C}^\times$ under the boundary parametrisation. All edge indices are -1 unless otherwise noted.

crosses and take its refinement to a triangulation as in Section 3.A.3. Let us first assume that this face has only inner edges, as in Figure 3.35. Let e_{f_p} be the edge where p leaves the face f_p . The contribution of p can now be calculated by summing up for each triangle the “ ω_e ” contributions of [Nov, Section 4.7]. For a given triangle t and edge e , where the curve leaves t , the contribution is $\omega_e = \hat{s}_e + \kappa_e$ by [Nov, (4.33)], where κ_e is given in Figure 3.36.

First the curve crosses 3 triangles, which give a contribution of

$$(0 + 0) + (0 + 0) + (0 + 0) = 0 .$$

Notice that when afterwards crossing the dotted area, the lift of the curve does not pick up any of contributions: for every group of 4 triangles the contribution is

$$(-2 + 1) + (0 + 1) + (-2 + 1) + (0 + 1) = 0 .$$

If the marked edge of the face f_p is on the right side of p with respect to the orientation of f_p then the curve has crossed the corresponding edge with edge label -2 and the contribution is

$$\hat{\delta}_{f_p}^p = 1 .$$

Finally the curve crosses 6 triangles, which give a contribution of

$$(-2 + 1) + (0 + 1) + (-2 + 1) + (-1 + 1) + (-1 + 1) + (\hat{s}_{e_p} + 1) = \hat{s}_{e_p} .$$

This proves the formula (3.4.4) if γ is a closed curve.

If the curve γ starts and ends on the boundary of the surface then we take it into account as follows. The parts of the triangulation where γ starts and ends is shown in Figure 3.37. As described in the main text we have r -spin isomorphisms $D^\kappa \rightarrow D^0$ of some neighbourhoods of the starting and ending point of γ , both sending these two points to $1 \in D^0 \subset \mathbb{C}^0$. Under these isomorphisms the neighbourhood of 1, together with a part of γ and the boundary edges is shown in Figure 3.38. This way we can handle γ as a closed curve and by Part 3 we can modify the curve by a homotopy as in Figure 3.38, so that it crosses the edges e_{start} and e_{end} .

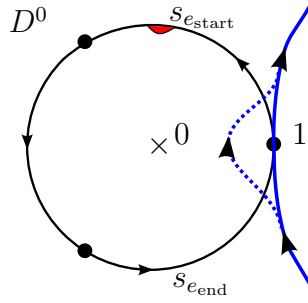


Figure 3.38: Detail of D^0 with the image of the identification of the neighbourhoods of the starting and ending point of γ . The circle denotes the two boundary components mapped onto each other. We obtain a closed curve which, by using Part 2, we are allowed to change by a homotopy to the dotted curve. This allows us to compute the holonomy in terms of the “ ω_e ” contributions as before.

We can now calculate the contribution of these crossed triangles as before. The curve first crosses the boundary triangle in Figure 3.38 picking up the contribution

$$\hat{s}_{e_{\text{start}}} + 1 .$$

Then it crosses the two triangles in Figure 3.37 b) picking up the contribution

$$(0 + 0) + (-1 + 0) .$$

After crossing inner edges finally it crosses the two triangles in Figure 3.37 a), using Figure 3.38, picking up the contribution

$$(0 + 1) + (0 + \hat{s}_{e_{\text{end}}}) .$$

Summing up the above contributions, we get formula (3.4.4).

This completes the proof of Lemma 3.4.5.

Chapter 4

Area-dependent quantum field theory with defects

In this chapter we present area-dependent quantum field theories in detail. This part of the thesis has appeared in [RS2]. In Section 4.1 we collect all the required algebraic preliminaries about regularised algebras and RFAs, as well as their modules. In Section 4.2 we state the definition of an aQFT without and with defects, and we show that aQFTs without defects correspond to commutative RFAs. Section 4.3 contains the state-sum constructions, first the one without defects and then the version with defects. It is shown how the data needed for the state-sum construction can be obtained from RFAs and from dualisable bimodules, and how the tensor product of bimodules and the fusion of defect lines are related. Finally, in Section 4.4 we give a detailed treatment of our main example, 2d YM theory with Wilson lines as defects.

4.1 Regularised Frobenius algebras

4.1.1 Definition of regularised algebras and Frobenius algebras

Let $(\mathcal{S}, \otimes, \mathbb{I})$ be a strict¹ monoidal category whose hom-sets are topological spaces such that composition is separately continuous.

We stress that we do not require the composition of \mathcal{S} to be jointly continuous, nor the tensor product of \mathcal{S} to be (jointly or separately) continuous. The reason is that our main example – the category of Hilbert spaces with bounded linear maps and strong operator topology – has none of these properties, see Remark 4.1.11 below.

Definition 4.1.1. A *regularised algebra* in \mathcal{S} is an object $A \in \mathcal{S}$ together with families of

¹Although our examples of such categories will not be strict, it can be show that one can always find an equivalent strict monoidal category with these properties such that the equivalence functor is a homeomorphism on hom-sets.

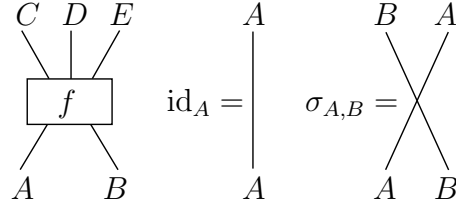


Figure 4.1: Graphical notation of morphisms in a strict (symmetric) monoidal category \mathcal{S} . Here a morphism $f \in \mathcal{S}(A \otimes B, C \otimes D \otimes E)$, the identity $\text{id}_A \in \mathcal{S}(A, A)$ and the symmetric braiding $\sigma_{A,B}$ are shown. The tensor product of morphisms is depicted by drawing the morphisms next to each other and composition of morphisms is stacking them on top of each other.

morphisms

$$\mu_a \in \mathcal{S}(A^{\otimes 2}, A) \quad \text{and} \quad \eta_a \in \mathcal{S}(\mathbb{I}, A) \quad (4.1.1)$$

for every $a \in \mathbb{R}_{>0}$, called *product* and *unit*, such that the following relations hold:

1. for every $a, a_1, a_2, b_1, b_2 \in \mathbb{R}_{>0}$, such that $a_1 + a_2 = b_1 + b_2 = a$,

$$\mu_{a_1} \circ (\text{id}_A \otimes \eta_{a_2}) = \mu_{b_1} \circ (\eta_{b_2} \otimes \text{id}_A) \quad , \quad (4.1.2)$$

$$\mu_{a_1} \circ (\text{id}_A \otimes \mu_{a_2}) = \mu_{b_1} \circ (\mu_{b_2} \otimes \text{id}_A) \quad , \quad (4.1.3)$$

2. Let $P_a \in \mathcal{S}(A, A)$ be given by (4.1.2), i.e. $P_a = \mu_{a_1} \circ (\text{id}_A \otimes \eta_{a_2})$. We require:

(a) $\lim_{a \rightarrow 0} P_a = \text{id}_A$.

(b) The assignments

$$\begin{aligned} \mathbb{R}_{\geq 0}^n &\rightarrow \mathcal{S}(A^{\otimes n}, A^{\otimes n}) \\ (a_1, \dots, a_n) &\mapsto P_{a_1} \otimes \dots \otimes P_{a_n} \end{aligned} \quad (4.1.4)$$

are jointly continuous for every $n \geq 1$.

Let $A, B \in \mathcal{S}$ be regularised algebras. A *morphism of regularised algebras* $A \xrightarrow{f} B$ is a morphism in \mathcal{S} such that for every $a \in \mathbb{R}_{>0}$

$$\eta_a^B = f \circ \eta_a^A, \quad \mu_a^B \circ (f \otimes f) = f \circ \mu_a^A.$$

Note that continuity is imposed only on P_a but not on μ_a or η_a . However, we will see shortly that continuity of μ_a and η_a is implied by the definition. On the other hand, it is important not to impose the existence of an $a \rightarrow 0$ limit on μ_a and η_a ; in Section 4.1.3 we will see examples where this limit does not exist, which would then have been excluded.

We will often use string diagram notation to represent morphisms in strict monoidal categories, our conventions are given in Figure 4.1. The morphisms in (4.1.1) are drawn as

$$\mu_a = \text{diagram} \quad \eta_a = \text{diagram} \quad (4.1.5)$$

and the relations in (4.1.2) and (4.1.3) are

$$\begin{array}{c} A \\ | \\ \bullet^{a_1} \\ / \quad \backslash \\ A \quad \mathbb{I} \end{array} = \begin{array}{c} A \\ | \\ \bullet^{b_1} \\ / \quad \backslash \\ \mathbb{I} \quad A \end{array} = \begin{array}{c} A \\ | \\ \bullet^a \\ | \\ A \end{array} = P_a \quad \text{and} \quad \begin{array}{c} A \\ | \\ \bullet^{a_1} \\ / \quad \backslash \\ A \quad A \end{array} = \begin{array}{c} A \\ | \\ \bullet^{b_1} \\ / \quad \backslash \\ A \quad A \end{array} \quad (4.1.6)$$

The next lemma gives some simple consequences of the above definition. In particular, part 4 shows that even though we imposed no continuity condition on the tensor product of \mathcal{S} , as far as morphisms built from a regularised algebra are concerned, everything is even jointly continuous.

Lemma 4.1.2. *Let A be a regularised algebra. Let $a_1, a_2, b_1, b_2, c_1, c_2 \in \mathbb{R}_{>0}$ such that $a_1 + a_2 = b_1 + b_2 = c_1 + c_2$.*

1. *Let $\eta'_a \in \mathcal{S}(\mathbb{I}, A)$ be a family of morphisms which satisfy (4.1.2). Then $\eta'_a = \eta_a$ for every $a \in \mathbb{R}_{>0}$.*
2. *$P_{a_1} \circ \eta_{a_2} = \eta_{a_1+a_2}$ and $P_{a_1} \circ P_{a_2} = P_{a_1+a_2}$.*
3. *$P_{a_1} \circ \mu_{a_2} = \mu_{b_1} \circ (P_{b_2} \otimes \text{id}) = \mu_{c_1} \circ (\text{id} \otimes P_{c_2}) = \mu_{a_1+a_2}$.*
4. *In the monoidal sub-category of \mathcal{S} tensor generated by A , μ_a and η_a , every morphism is jointly continuous in the parameters.*

Proof. Let $a, b, c \in \mathbb{R}_{>0}$.

Part 1: Let us write $P'_{a+b} := \mu_a \circ (\eta'_b \otimes \text{id}_A)$ for the morphism in (4.1.2). From (4.1.2) we have that

$$\mu_a \circ (\eta_b \otimes \eta'_c) = \mu_a \circ (\eta_c \otimes \eta'_b) \quad (4.1.7)$$

as both sides only depend on the sum of the parameters. We then have that

$$P_{a+b} \circ \eta'_c = P'_{a+b} \circ \eta_c, \quad (4.1.8)$$

and using that the composition is separately continuous together with $\lim_{a,b \rightarrow 0} P_{a+b} = \lim_{a,b \rightarrow 0} P'_{a+b} = \text{id}_A$ we get that $\eta'_c = \eta_c$ for every $c \in \mathbb{R}_{>0}$.

Part 2: The first equation follows from Part 1, because $P_b \circ \eta_a$ satisfies (4.1.2). The second equation follows from associativity (4.1.3) and from the first one:

$$\begin{aligned} P_a \circ P_b &= \mu_{a_1} \circ (\eta_{a_2} \otimes \mu_{b_1}) \circ (\eta_{b_2} \otimes \text{id}) = \mu_{b_1} \circ (P_a \circ \eta_{b_2} \otimes \text{id}) \\ &= \mu_{b_1} \circ (\eta_{a+b_2} \otimes \text{id}) = P_{a+b}, \end{aligned} \quad (4.1.9)$$

where $a = a_1 + a_2$ and $b = b_1 + b_2$.

Part 3: The first two equalities follow from the associativity of μ_a and the definition of P_a . For the last equality note that with $c = c_1 + c_2$ we have

$$P_c \circ P_b \circ \mu_a = P_{c+b} \circ \mu_a = \mu_c \circ (\eta_b \otimes \mu_a) = \mu_{c_1} \circ (\eta_{c_2} \otimes \mu_{a+b}) = P_c \circ \mu_{a+b} . \quad (4.1.10)$$

Finally we use separate continuity of the composition and $\lim_{c \rightarrow 0} P_c = \text{id}_A$.

Part 4: Let $\varphi_{a_1, \dots, a_N} : A^{\otimes n} \rightarrow A^{\otimes m}$ be a morphism in \mathcal{S} tensor generated by μ_a and η_a , involving a total of N copies of the latter two morphisms, with parameters a_1, \dots, a_N . One can write $\varphi_{a_1, \dots, a_N}$ in the form $\varphi_{\varepsilon_1}^{(1)} \circ \left(\bigotimes_{i=1}^N P_{a_i - \varepsilon_1 - \varepsilon_2} \right) \circ \varphi_{\varepsilon_2}^{(2)}$ for some $\varepsilon_1, \varepsilon_2 \in \mathbb{R}_{>0}$ and morphisms $\varphi_{\varepsilon_1}^{(1)}$ and $\varphi_{\varepsilon_2}^{(2)}$. Then by separate continuity of the composition of \mathcal{S} and joint continuity of $\bigotimes_{i=1}^N P_{a_i}$ in (4.1.4), the morphism $\varphi_{a_1, \dots, a_N}$ is jointly continuous in the parameters a_1, \dots, a_N . \square

As a special case of Part 4 of the above lemma we get:

Corollary 4.1.3. *In a regularised algebra the maps $a \mapsto \mu_a$ and $a \mapsto \eta_a$ are continuous.*

Next we introduce the dual concept to a regularised algebra. A *regularised coalgebra* in \mathcal{S} is an object $A \in \mathcal{S}$ together with families of morphisms

$$\Delta_a : A \rightarrow A^{\otimes 2} \quad \text{and} \quad \varepsilon_a : A \rightarrow \mathbb{I} \quad (4.1.11)$$

for $a \in \mathbb{R}_{>0}$, called *coproduct* and *counit*, such that the following relations hold: for all $a, a_1, a_2, b_1, b_2 > 0$, such that $a_1 + a_2 = b_1 + b_2 = a$,

$$(id_A \otimes \varepsilon_{a_2}) \circ \Delta_{a_1} = (\varepsilon_{b_2} \otimes id_A) \circ \Delta_{b_1} =: P'_a , \quad (4.1.12)$$

$$(id_A \otimes \Delta_{a_2}) \circ \Delta_{a_1} = (\Delta_{b_2} \otimes id_A) \circ \Delta_{b_1} , \quad (4.1.13)$$

$\lim_{a \rightarrow 0} P'_a = id_A$, and the assignments

$$\begin{aligned} \mathbb{R}_{\geq 0}^n &\rightarrow \mathcal{S}(A^{\otimes n}, A^{\otimes n}) \\ (a_1, \dots, a_n) &\mapsto P'_{a_1} \otimes \dots \otimes P'_{a_n} \end{aligned} \quad (4.1.14)$$

are jointly continuous for every $n \geq 1$. A *morphism of regularised coalgebras*, is a morphism of the objects which is compatible with Δ_a and ε_a in the obvious way. Note that for a regularised coalgebra the dual statements of Lemma 4.1.2 hold. For the morphisms in (4.1.11) we introduce the following graphical notation:

$$\Delta_a = \begin{array}{c} A \quad A \\ \diagdown \quad \diagup \\ \bullet \\ \mid \\ A \end{array} \quad \varepsilon_a = \begin{array}{c} \mathbb{I} \\ \bullet \\ \mid \\ A \end{array} \quad . \quad (4.1.15)$$

A key notion in this paper is the following:

Definition 4.1.4. A *regularised Frobenius algebra* (or RFA in short) in \mathcal{S} is a regularised algebra A , which is also a regularised coalgebra, such that

$$(4.1.16)$$

holds for all $a_1 + a_2 = b_1 + b_2 = c_1 + c_2$. A *morphism of RFAs*, is a morphism of regularised algebras and coalgebras.

In an RFA the semigroup homomorphism P_a from the algebra structure and P'_a from the coalgebra structure coincide:

Lemma 4.1.5. For an RFA we have $P_a = P'_a$ for all $a > 0$.

Proof. Let $a, b \in \mathbb{R}_{>0}$ be arbitrary. Choose $a_1, a_2, b_1, b_2 \in \mathbb{R}_{>0}$ such that $a = a_1 + a_2$, $b = b_1 + b_2$ and $a > b_1$ and $b > a_1$ (e.g. $b_1 = \frac{a}{2}$, $a_1 = \frac{b}{2}$). By relation (4.1.16) one has that

$$(\mu_{a_2} \otimes \text{id}_A) \circ (\text{id}_A \otimes \Delta_{b_2}) = \Delta_{a_1+a_2-b_1} \circ \mu_{b_1+b_2-a_1} \quad (4.1.17)$$

Composing (4.1.17) with $\text{id}_A \otimes \varepsilon_{b_1}$ from the left and with $\eta_{a_1} \otimes \text{id}_A$ from the right yields

$$P_a \circ P'_b = P'_a \circ P_b. \quad (4.1.18)$$

We can take the $b \rightarrow 0$ limit on both sides of (4.1.18) and use separate continuity of the composition in \mathcal{S} to get $P_a = P'_a$. \square

Remark 4.1.6. Requiring that $\lim_{a \rightarrow 0} P_a = \text{id}$ does not imply that P_a is mono or epi for every $a \in \mathbb{R}_{>0}$, as the following example in \mathcal{Hilb} illustrates.² Let $L \in \mathbb{R}_{>0}$ and $\mathcal{H} := L^2([0, L])$. Define $P_a \in \mathcal{B}(\mathcal{H})$ for $f \in \mathcal{H}$ to be right shift by a ,

$$(P_a(f))(x) = \begin{cases} 0 & \text{if } x < a \\ f(x - a) & \text{if } x \geq a \end{cases}. \quad (4.1.19)$$

This is neither mono nor epi for any choice of $a \in \mathbb{R}_{>0}$, and for $a \geq L$ we even have $P_a = 0$.

Usual (non-regularised) Frobenius algebras have an equivalent characterisation via a non-degenerate invariant pairing. The same is true in the regularised setting, as we now illustrate. Let A a regularised algebra A together with a family of morphisms $\varepsilon_a : A \rightarrow \mathbb{I}$ for $a \in \mathbb{R}_{>0}$ such that for all $a_1 + a_2 = b_1 + b_2$ we have $\varepsilon_{a_1} \circ \mu_{a_2} = \varepsilon_{b_1} \circ \mu_{b_2}$. We call the pairing $\beta_a := \varepsilon_{a_1} \circ \mu_{a_2}$ *non-degenerate* if there is a family of morphisms $\gamma_a : \mathbb{I} \rightarrow A^{\otimes 2}$ such that

$$(\text{id}_A \otimes \beta_{a_1}) \circ (\gamma_{a_2} \otimes \text{id}_A) = P_a = (\beta_{b_1} \otimes \text{id}_A) \circ (\text{id}_A \otimes \gamma_{b_2}) \quad (4.1.20)$$

for all $a_1 + a_2 = b_1 + b_2 = a$.

²We would like to thank Reiner Lauterbach for explaining this example to us.

Lemma 4.1.7. 1. For all $a, b > 0$,

$$(P_a \otimes \text{id}_A) \circ \gamma_b = \gamma_{a+b} = (\text{id}_A \otimes P_a) \circ \gamma_b, \quad (4.1.21)$$

2. The relation (4.1.20) defines γ_a uniquely.

3. The map $a \mapsto \gamma_a$ is continuous.

Proof. Part 1: From (4.1.20) one has that

$$(\text{id}_A \otimes \beta_b) \circ (\gamma_{a+x} \otimes \text{id}_A) = P_{a+b+x} = (\text{id}_A \otimes \beta_{b+x}) \circ (\gamma_a \otimes \text{id}_A). \quad (4.1.22)$$

Tensoring with id_A from the right and composing with γ_c from the right gives

$$(\text{id}_A \otimes P_{b+c}) \circ \gamma_{a+x} = (\text{id}_A \otimes P_{b+c+x}) \circ \gamma_a. \quad (4.1.23)$$

Taking the limit $b, c \rightarrow 0$ gives the second equation of (4.1.21). One obtains the first equation of (4.1.21) similarly.

Part 2: If Γ_a is a family of morphisms satisfying (4.1.20), then it also satisfies (4.1.21). Then

$$\begin{aligned} \Gamma_{a+b+c} &= (\text{id}_A \otimes P_{b+c}) \circ \Gamma_a = (\text{id}_A \otimes \beta_b \otimes \text{id}_A) \circ (\Gamma_a \otimes \gamma_c) = (P_{a+b} \otimes \text{id}_A) \circ \gamma_c \\ &= \gamma_{a+b+c}. \end{aligned} \quad (4.1.24)$$

Part 3: Continuity of γ_a is clear from Part 1 by continuity of P_a and separate continuity of the composition in \mathcal{S} . \square

We can now give the alternative characterisation of an RFA.

Proposition 4.1.8. Let A be a regularised algebra and let $\varepsilon_a : A \rightarrow \mathbb{I}$ be a family of morphisms such that the pairing $\varepsilon_{a_1} \circ \mu_{a_2}$ only depends on $a_1 + a_2$. If $\varepsilon_{a_1} \circ \mu_{a_2}$ is non-degenerate, then A is a regularised Frobenius algebra with counit ε_a and coproduct

$$\Delta_a := (\mu_{a_1} \otimes \text{id}_A) \circ (\text{id}_A \otimes \gamma_{a_2}), \quad (4.1.25)$$

for some $a_1 + a_2 = a$.

Proof. By (4.1.21) and Part 3 of Lemma 4.1.2, Δ_a indeed only depends on $a = a_1 + a_2$. Checking the algebraic relations (4.1.2), (4.1.3), (4.1.12), (4.1.13) and (4.1.16) of an RFA is analogous as for ordinary Frobenius algebras. From these follows that $P'_a = P_a$, so in particular $\lim_{a \rightarrow 0} P'_a = \text{id}_A$ and (4.1.14) holds. \square

Note that the converse of the proposition holds trivially: if A is a regularised Frobenius algebra then ε_a is non-degenerate in the above sense with $\gamma_a = \Delta_{a_1} \circ \eta_{a_2}$.

Let the category \mathcal{S} be in addition symmetric with braiding σ . Then we call a regularised algebra $A \in \mathcal{S}$ *commutative* if $\mu_a \circ \sigma = \mu_a$ for all $a \in \mathbb{R}_{>0}$. The *centre* of a regularised algebra A is an object $B \in \mathcal{S}$ and a morphism $i_B : B \rightarrow A$ such that

$$\mu_a \circ \sigma \circ (i_B \otimes \text{id}_A) = \mu_a \circ (i_B \otimes \text{id}_A) \quad (4.1.26)$$

for all $a \in \mathbb{R}_{>0}$, which is universal in the following sense. If there is an object C and a morphism $f : C \rightarrow A$ satisfying the above equation then there is a unique morphism $\tilde{f} : C \rightarrow B$ such that the diagram

$$\begin{array}{ccc} B & \xrightarrow{i_B} & A \\ \tilde{f} \uparrow & \nearrow f & \\ C & & \end{array} \quad (4.1.27)$$

commutes. This implies in particular that i_B is mono [Dav]. If the centre exists then one has the induced morphism $\tilde{P}_a \in \mathcal{S}(B, B)$ such that $P_a \circ i_B = i_B \circ \tilde{P}_a$.

Lemma 4.1.9. *If the centre of a regularised algebra exists, $\lim_{a \rightarrow 0} \tilde{P}_a$ exists and the maps*

$$(a_1, \dots, a_n) \mapsto \tilde{P}_{a_1} \otimes \dots \otimes \tilde{P}_{a_n} \quad (4.1.28)$$

are jointly continuous for every $n \geq 1$, then it is a commutative regularised algebra.

Proof. Similarly as one gets \tilde{P}_a , one has induced multiplication and unit $\tilde{\mu}$ and $\tilde{\eta}$. Checking associativity and unitality is now straightforward. The limit $\lim_{a \rightarrow 0} \tilde{P}_a = \text{id}_B$ follows from $P_a \circ i_B = i_B \circ \tilde{P}_a$ by separate continuity of composition in \mathcal{S} . \square

A regularised algebra is *separable* if there exists a family of morphisms $e_a \in \mathcal{S}(\mathbb{I}, A \otimes A)$ for every $a \in \mathbb{R}_{>0}$ such that

1. $(\mu_{a_1} \otimes \text{id}_A) \circ (\text{id}_A \otimes e_{a_2}) = (\text{id}_A \otimes \mu_{b_1}) \circ (e_{b_2} \otimes \text{id}_A)$ and
2. $\mu_{a_1} \circ e_{a_2} = \eta_a$.

The e_a are called *separability idempotents*. A regularised algebra A is *strongly separable* if it is separable and furthermore

3. $\sigma_{A,A} \circ e_a = e_a$.

These notions are direct generalisations of separability and strong separability for algebras, see e.g. [Kan, LP1].

For an RFA A , we call the family of morphisms $\tau_a := \mu_{a_1} \circ \Delta_{a_2} \circ \eta_{a_3}$ for $a_1, a_2, a_3 \in \mathbb{R}_{>0}$ with $a = a_1 + a_2 + a_3$ the *window element* of A , cf. [LP1, Def. 2.12]. We call the window element *invertible* if there exists a family of morphisms $z_a \in \mathcal{S}(\mathbb{I}, A)$ for $a \in \mathbb{R}_{>0}$ (the *inverse*) such that $\mu_{a_1} \circ (\tau_{a_2} \otimes z_{a_3}) = \eta_{a_1+a_2+a_3} = \mu_{a_1} \circ (z_{a_3} \otimes \tau_{a_3})$. From a direct computation one can verify that if there exists another family of morphisms z'_a which satisfies the above equation then $z'_a = z_a$ for every $a \in \mathbb{R}_{>0}$, that is the inverse of the window element is unique. In the following we write τ_a^{-1} for the inverse of τ_a . It is easy to check that the window element and its inverse satisfy (4.1.26), that is, they factorise through the centre if the centre exists.

An RFA is *symmetric* if $\varepsilon_{a_1} \circ \mu_{a_2} \circ \sigma = \varepsilon_{b_1} \circ \mu_{b_2}$. The following is a direct translation of [LP1, Thm. 2.14] for strong separability for symmetric Frobenius algebras.

Proposition 4.1.10. *A symmetric RFA is strongly separable if and only if its window element is invertible.*

Proof. Set $e_a := \Delta_{a_1} \circ \tau_{a_2}^{-1}$. Conversely set $\tau_a^{-1} := (\varepsilon_{a_1} \otimes \text{id}_A) \circ e_{a_2}$. \square

4.1.2 RFAs in the category of Hilbert spaces

Let \mathcal{Hilb} denote the symmetric monoidal category of Hilbert spaces and bounded linear maps with the strong operator topology on the hom-sets and the Hilbert space tensor product. We write $\mathcal{B}(\mathcal{H}, \mathcal{K}) := \mathcal{Hilb}(\mathcal{H}, \mathcal{K})$ and $\mathcal{B}(\mathcal{H}) := \mathcal{Hilb}(\mathcal{H}, \mathcal{H})$ for the hom-sets.

Remark 4.1.11. 1. In \mathcal{Hilb} the composition of morphisms is separately continuous, but not jointly continuous. The tensor product is not separately continuous, in fact even tensoring with the identity morphism of an infinite-dimensional Hilbert space is not continuous. Furthermore, taking adjoints is not continuous in \mathcal{Hilb} . For more details see [Hal, Prob. 211] and [KR, Sec. 2.6].

2. Instead of the strong operator topology one could use the so called ultrastrong-* operator topology in which taking adjoint is continuous and tensoring is separately continuous [Bla, Prop. I.8.6.4].³ However in this topology composition of morphisms is still not jointly continuous [RA, Prop. 46.1-2].

We give the following technical lemma which will be useful later.

Lemma 4.1.12. *Let $f : X \rightarrow X'$ and $g : Y \rightarrow Y'$ be morphisms in \mathcal{Hilb} , both mono (resp. epi). Then $f \otimes g : X \otimes Y \rightarrow X' \otimes Y'$ is mono (resp. epi).*

Proof. If f and g are both epi, then their image is dense. Then the algebraic tensor product of $\text{im}(f)$ and $\text{im}(g)$ is dense in $X \otimes Y$ and it is contained in $\text{im}(f \otimes g)$, which is hence dense. This means that $f \otimes g$ is epi.

If f and g are both mono, then f^\dagger and g^\dagger are both epi. But then $f^\dagger \otimes g^\dagger = (f \otimes g)^\dagger$ is epi and hence $f \otimes g$ is mono. \square

The next lemma shows in particular that an RFA in \mathcal{Hilb} has a Hilbert basis with at most countably many elements.

Lemma 4.1.13. *Let $A \in \mathcal{Hilb}$ be an RFA.*

1. *The Hilbert space underlying A is separable.*
2. *For all $a \in \mathbb{R}_{>0}$, P_a is a trace class operator (and hence compact).*

Proof. Part 1: Let $\{\phi_j \mid j \in I\}$ be a complete set of orthonormal vectors in A and write $\gamma_{a_2}(1) = \sum_{k,l \in I} \phi_k \otimes \phi_l \gamma_{a_2}^{kl}$. By [Kub, Cor. 5.28], independently of the countability of the indexing set I , there are at most countably many non-zero terms in the above sum. Thus for a given a_2 there is a countable set of pairs $(k, l) \in I \times I$ such that $\gamma_{a_2}^{kl} \neq 0$. Define $I(a_2) \subseteq I$ to be the countable set of all elements of I that appear in such a pair. Let

$$J := \bigcup_{n \in \mathbb{Z}_{>0}} I(1/n) \subseteq I \quad \text{and} \quad A_J := \overline{\text{span}\{\phi_j \mid j \in J\}} \subseteq A. \quad (4.1.29)$$

³We thank Yuki Arano for explaining this to us.

Note that J is countable and A_J is separable. By (4.1.20), for every $v \in A$ and $n \in \mathbb{Z}_{>0}$ we have that

$$P_{1/n}(v) \in A_J \quad \text{and} \quad \lim_{n \rightarrow \infty} P_{1/n}(v) = v, \quad (4.1.30)$$

since $\lim_{n \rightarrow \infty} P_{1/n} = \text{id}_A$ in the strong operator topology. Since A_J is closed, v is an element of A_J . We have shown that $A_J = A$, and hence that A is separable.

Part 2: First let us compute the following expression for some $a_1, a_2 \in \mathbb{R}_{>0}$:

$$\beta_{a_1} \circ \sigma \circ \gamma_{a_2}(1) = \sum_{j,k \in I} \beta_{a_1}(\phi_j \otimes \phi_k) \gamma_{a_2}^{kj}. \quad (4.1.31)$$

This is an absolutely convergent sum, since the lhs is a composition of bounded linear maps. We can rewrite this expression using (4.1.20) to get

$$\begin{aligned} \beta_{a_1} \circ \sigma \circ \gamma_{a_2}(1) &= \sum_{j,k,l \in I} \beta_{a_1}(\phi_j \otimes \phi_k) \gamma_{a_2}^{kl} \langle \phi_j | \phi_l \rangle \\ &= \sum_{j,k,l \in I} \langle \phi_j | (\beta_{a_1} \otimes \text{id}_A) \phi_j \otimes \phi_k \otimes \phi_l \gamma_{a_2}^{kl} \rangle \\ &= \sum_{j \in I} \langle \phi_j | (\beta_{a_1} \otimes \text{id}_A) \phi_j \otimes \sum_{k,l \in I} \phi_k \otimes \phi_l \gamma_{a_2}^{kl} \rangle \\ &= \sum_{j \in I} \langle \phi_j | (\beta_{a_1} \otimes \text{id}_A) \circ (\text{id}_A \otimes \gamma_{a_2}(1)) \phi_j \rangle \\ &= \sum_{j \in I} \langle \phi_j | P_a \phi_j \rangle, \end{aligned} \quad (4.1.32)$$

which is again an absolutely convergent sum. By [Con2, Ex. 18.2] P_a is a trace class operator if and only if $\sum_{j \in I} \langle \phi_j | P_a \phi_j \rangle$ is absolutely convergent for every choice of orthonormal basis $\{\phi_j\}$, which we just have shown. In this case we have that

$$\text{tr}(P_a) = \sum_{j \in I} \langle \phi_j | P_a \phi_j \rangle. \quad (4.1.33)$$

□

Let $A \in \mathcal{Hilb}$ be an RFA. By the Part 2 of Lemma 4.1.13 and [EN, Thm. II.4.29], $a \mapsto P_a$ (for $a > 0$) is norm continuous. The following corollary shows that if we had defined \mathcal{Hilb} to have the norm operator topology on hom-sets all examples of RFAs in \mathcal{Hilb} with self-adjoint P_a would be finite-dimensional.

Corollary 4.1.14. *Let $A \in \mathcal{Hilb}$ be an RFA such that $\lim_{a \rightarrow 0} P_a = \text{id}_A$ in the norm topology on $\mathcal{B}(A)$. Then A is finite-dimensional.*

Proof. From Lemma 4.1.13 (2) we know that P_a is compact for every $a \in \mathbb{R}_{>0}$. By [Con1, Prop. VI.3.4] the subspace of compact operators is closed in norm operator topology. These together with $\lim_{a \rightarrow 0} P_a = \text{id}_A$ imply that id_A is compact, which in turn implies that A is finite-dimensional. □

The following lemma will be instrumental in showing that various joint continuity conditions hold automatically in \mathcal{Hilb} . A similar statement can be found in [KMD, Sec. 2].

Lemma 4.1.15. *Let $\mathcal{H}_i \in \mathcal{Hilb}$ ($i = 1, 2$). Let X be a subset of a finite-dimensional normed vector space (e.g. $X = \mathbb{R}_{\geq 0}$). Equip X with the induced topology and let $a \mapsto S_a^{(i)}$ be two continuous maps $X \rightarrow \mathcal{B}(\mathcal{H}_i)$. Then the map $X^2 \rightarrow \mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$, $(a, b) \mapsto S_a^{(1)} \otimes S_b^{(2)}$ is jointly continuous.*

Proof. We will first show that the map $a \mapsto \|S_a^{(i)}\|$ is bounded on compact subsets of X . Let $K \subset X$ be compact. By strong continuity we have that for every $h \in \mathcal{H}_i$ the map $a \mapsto S_a^{(i)}(h)$ is continuous, so in particular the map $a \mapsto \|S_a^{(i)}(h)\|$ is continuous, hence bounded on K . By the Uniform Boundedness Principle [Con1, Ch. III.14] the map $a \mapsto \|S_a^{(i)}\|$ is bounded on K .

Now we turn to the claim in the lemma. Let $a_0, b_0 \in X$ and $\kappa, \varepsilon \in \mathbb{R}_{>0}$ be fixed. We will show that the map $(a, b) \mapsto S_a^{(1)} \otimes S_b^{(2)}$ is continuous at (a_0, b_0) .

For $T \in \mathcal{H}_1 \otimes \mathcal{H}_2$ take a sequence $\{T_n\}_n$ in the algebraic tensor product of \mathcal{H}_1 and \mathcal{H}_2 such that $T_n \xrightarrow{n \rightarrow \infty} T$. We have the estimate

$$\begin{aligned} & \left\| (S_a^{(1)} \otimes S_b^{(2)} - S_{a_0}^{(1)} \otimes S_{b_0}^{(2)})T \right\| \\ & \leq \left\| (S_a^{(1)} \otimes S_b^{(2)} - S_{a_0}^{(1)} \otimes S_{b_0}^{(2)}) \right\| \cdot \|T - T_n\| + \left\| (S_a^{(1)} \otimes S_b^{(2)} - S_{a_0}^{(1)} \otimes S_{b_0}^{(2)})T_n \right\|. \end{aligned} \quad (4.1.34)$$

We give an estimate for the first term on the rhs of (4.1.34). Fix some $\delta_1 > 0$. Then by the above boundedness result there is a $\kappa > 0$ such that for every $a, b \in X$ with $|a - a_0| + |b - b_0| < \delta_1$ we have

$$\|S_a^{(1)}\| < \kappa \quad \text{and} \quad \|S_b^{(2)}\| < \kappa. \quad (4.1.35)$$

So we have

$$\begin{aligned} \left\| S_a^{(1)} \otimes S_b^{(2)} - S_{a_0}^{(1)} \otimes S_{b_0}^{(2)} \right\| & \leq \|S_a^{(1)}\| \cdot \|S_b^{(2)}\| + \|S_{a_0}^{(1)}\| \cdot \|S_{b_0}^{(2)}\| \\ & \leq \kappa^2 + \|S_{a_0}^{(1)}\| \cdot \|S_{b_0}^{(2)}\| =: N_{a_0, b_0}^\kappa. \end{aligned} \quad (4.1.36)$$

Since $T_n \xrightarrow{n \rightarrow \infty} T$, we can choose n (which we keep fixed from now on) such that

$$\|T - T_n\| < \frac{\varepsilon}{2N_{a_0, b_0}^\kappa}. \quad (4.1.37)$$

Putting (4.1.36) and (4.1.37) together we get

$$\left\| (S_a^{(1)} \otimes S_b^{(2)} - S_{a_0}^{(1)} \otimes S_{b_0}^{(2)}) \right\| \cdot \|T - T_n\| \leq \frac{\varepsilon}{2}. \quad (4.1.38)$$

We give an estimate for the last term in (4.1.34). Recall that each T_n was chosen in the algebraic tensor product of \mathcal{H}_1 and \mathcal{H}_2 . Thus T_n is a finite sum of elementary tensors,

$$T_n = \sum_{j=1}^{t_n} x_n^j \otimes y_n^j \quad (4.1.39)$$

for $t_n \in \mathbb{Z}_{\geq 1}$, $x_n^j \in \mathcal{H}^{(1)}$ and $y_n^j \in \mathcal{H}^{(2)}$. Using this, we get:

$$\begin{aligned} & \left\| (S_a^{(1)} - S_{a_0}^{(1)}) \otimes S_b^{(2)} T_n + S_{a_0}^{(1)} \otimes (S_b^{(2)} - S_{b_0}^{(2)}) T_n \right\| \\ & \leq \sum_{j=1}^{t_n} \left(\left\| (S_a^{(1)} - S_{a_0}^{(1)}) x_n^j \right\| \cdot \left\| S_b^{(2)} \right\| \cdot \left\| y_n^j \right\| + \left\| S_{a_0}^{(1)} \right\| \cdot \left\| x_n^j \right\| \cdot \left\| (S_b^{(2)} - S_{b_0}^{(2)}) y_n^j \right\| \right). \end{aligned} \quad (4.1.40)$$

By strong continuity of $a \mapsto S_a^{(i)}$ we can chose $\delta_2 > 0$ such that for every $a, b \in X$ with $|a - a_0| + |b - b_0| < \delta_2$ we have

$$\left\| (S_a^{(1)} - S_{a_0}^{(1)}) x_n^j \right\| < \frac{\varepsilon}{4t_n \kappa \left\| y_n^j \right\|} \quad \text{and} \quad \left\| (S_b^{(2)} - S_{b_0}^{(2)}) y_n^j \right\| < \frac{\varepsilon}{4t_n \left\| S_{a_0}^{(1)} \right\| \cdot \left\| x_n^j \right\|}, \quad (4.1.41)$$

for every $j = 1, \dots, t_n$, since these are only finitely many conditions to satisfy. Let $\delta := \min \{\delta_1, \delta_2\}$. Then for every $a, b \in X$ with $|a - a_0| + |b - b_0| < \delta$ we have that

$$\left\| (S_a^{(1)} \otimes S_b^{(2)} - S_{a_0}^{(1)} \otimes S_{b_0}^{(2)}) T_n \right\| \leq \frac{\varepsilon}{2}. \quad (4.1.42)$$

Finally, using (4.1.38) and (4.1.42) we have that

$$\left\| (S_a^{(1)} \otimes S_b^{(2)} - S_{a_0}^{(1)} \otimes S_{b_0}^{(2)}) T \right\| < \varepsilon. \quad (4.1.43)$$

□

By iterating the previous lemma we see that the definition of a regularised algebra simplifies in \mathcal{Hilb} . Namely it is enough to check that $a \mapsto P_a$ is continuous, rather than having to consider multiple tensor products.

Corollary 4.1.16. *The continuity condition (4.1.4) in \mathcal{Hilb} is automatically satisfied for any $n \geq 2$ if it holds for $n = 1$.*

4.1.3 Examples of regularised algebras and RFAs in $\mathcal{Vect}^{\text{fd}}$ and \mathcal{Hilb}

Let $\mathcal{Vect}^{\text{fd}}$ denote the symmetric monoidal category of finite-dimensional complex vector spaces with the usual tensor product of vector spaces and the topology on the hom-sets induced by any norm on the vector spaces. In the following we list examples of regularised algebras and RFAs in $\mathcal{Vect}^{\text{fd}}$ and \mathcal{Hilb} .

1. Let A be an algebra in $\mathcal{Vect}^{\text{fd}}$ with multiplication μ and unit η and set $\mu_a := \mu \cdot e^{a\sigma}$, $\eta_a := \eta \cdot e^{a\sigma}$ for some $\sigma \in \mathbb{C}$. Then A is a regularised algebra. One can similarly obtain an RFA from a Frobenius algebra.

A Frobenius algebra in $\mathcal{Vect}^{\text{fd}}$ is always finite-dimensional. In Example 1 we equipped them with an RFA structure for which all $a \rightarrow 0$ limits exist. The converse also holds in the following sense.

Proposition 4.1.17. *Let $A \in \mathcal{RFrob}(\mathcal{Hilb})$. The following are equivalent.*

1. A is finite-dimensional.
2. All of the following limits exist:

$$\lim_{a \rightarrow 0} \eta_a, \quad \lim_{a \rightarrow 0} \mu_a, \quad \lim_{a \rightarrow 0} \varepsilon_a, \quad \lim_{a \rightarrow 0} \Delta_a.$$

Proof. ($1 \Rightarrow 2$): If A is finite-dimensional, then the map $a \mapsto P_a$ is norm continuous, hence $P_a = e^{aH}$ for some $H \in \mathcal{B}(A)$. Then $\eta_0 := e^{-aH} \eta_a$ is independent of a and $\eta_a = P_a \circ \eta_0$, hence $\lim_{a \rightarrow 0} \eta_a = \eta_0$ exists. One similarly proves that the other limits exist as well.

($2 \Leftarrow 1$): The morphisms given by these limits define a Frobenius algebra structure on A , hence A is finite-dimensional. \square

2. Consider the polynomial algebra $\mathbb{C}[x]$ and complete it with the Hilbert space structure given by $\langle x^n, x^m \rangle = \delta_{n,m} f(m)$ for some monotonously decreasing function $f : \mathbb{N} \rightarrow (0, 1] \subset \mathbb{R}$ and denote by $\overline{\mathbb{C}[x]}$ its Hilbert space completion. Let $P_a(x^n) := e^{a\sigma x} x^n$ for $\sigma \in \mathbb{R}$ (note the x in the exponent). We now show that this defines a bounded operator. Let $y \in \overline{\mathbb{C}[x]}$ with $y = \sum_{n \in \mathbb{N}} y_n x^n$. Then

$$\|P_a(y)\|^2 = \sum_{n,m \in \mathbb{N}} \left(\frac{a^m}{m!} \right)^2 f(n+m) |y_n|^2 \leq \sum_{n,m \in \mathbb{N}} \frac{a^{2m}}{(2m)!} f(n) |y_n|^2 \leq e^a \|y\|^2.$$

where we used that f is monotonously decreasing.

Let us assume that

$$\sup_{k \in \mathbb{N}} \left\{ \sum_{l=0}^k f(k) f(l) f(k-l) \right\} < \infty \quad (4.1.44)$$

holds, e.g. $f(m) = (1+m)^{-2}$ or $f(m) = e^{-m}$. Then the operator

$$\begin{aligned} M : \overline{\mathbb{C}[x]} &\rightarrow \overline{\mathbb{C}[x]} \otimes \overline{\mathbb{C}[x]} \\ x^k &\mapsto \sum_{l=0}^k f(k) x^{k-l} \otimes x^l \end{aligned} \quad (4.1.45)$$

is bounded. The adjoint of M is the standard multiplication

$$\begin{aligned} \mu : \overline{\mathbb{C}[x]} \otimes \overline{\mathbb{C}[x]} &\rightarrow \overline{\mathbb{C}[x]} \\ x^k \otimes x^l &\mapsto x^{k+l} , \end{aligned} \quad (4.1.46)$$

which is therefore also a bounded operator. Then defining $\mu_a := P_a \circ \mu$ and $\eta_a(1) := P_a(1)$ gives a regularised algebra in \mathcal{Hilb} . Note, however, that this regularised algebra cannot be turned into a regularised Frobenius algebra because P_a is not trace class, cf. Lemma 4.1.13.

3. Consider the Frobenius algebra $A := \mathbb{C}[x]/\langle x^d \rangle$ in $\mathcal{Vect}^{\text{fd}}$ with $\varepsilon(x^k) = \delta_{k,d-1}$. Let $h \in A$ and define $P_a(f) := e^{ah}f$, $\varepsilon_a := \varepsilon \circ P_a$, $\eta_a := P_a \circ \eta$ and $\mu_a := P_a \circ \mu$. Then $\mathbb{C}[x]/\langle x^d \rangle$ is an RFA, denoted A_h . Unless $d = 1$, this RFA is not separable.

Proposition 4.1.18. *Let I be a countable (possibly infinite) set. For $k \in I$ let $F_k \in \mathcal{Hilb}$ be a (possibly infinite-dimensional) RFA. Then $\bigoplus_{k \in I} F_k$ (the completed direct sum of Hilbert spaces) is an RFA in \mathcal{Hilb} if and only if, for every $a \in \mathbb{R}_{>0}$,*

$$\sup_{k \in I} \|\mu_a^k\| < \infty \quad \text{and} \quad \sup_{k \in I} \|\Delta_a^k\| < \infty , \quad (4.1.47)$$

$$\sum_{k \in I} \|\varepsilon_a^k\|^2 < \infty \quad \text{and} \quad \sum_{k \in I} \|\eta_a^k\|^2 < \infty , \quad (4.1.48)$$

where μ_a^k , Δ_a^k , ε_a^k and η_a^k denote the structure maps of F_k .

Proof. Let $F := \bigoplus_{k \in I} F_k$ and fix the value of a .

(\Rightarrow): Let us write x_k for the k 'th component of $x \in F = \bigoplus_{k \in I} F_k$. Then for every $k \in I$

$$\|\Delta_a^k\| = \sup_{\substack{x_k \in F_k \\ \|x_k\|=1}} \|\Delta_a^k(x_k)\| = \sup_{\substack{x_k \in F_k \\ \|x_k\|=1}} \|\Delta_a(x_k)\| \leq \sup_{\substack{x_k \in F_k \\ \|x_k\|=1}} \|\Delta_a\| \cdot \|x_k\| = \|\Delta_a\| < \infty ,$$

so in particular $\sup_k \|\Delta_a^k\| < \infty$. A similar proof applies to the case of μ_a . We calculate the norm of η_a :

$$\|\eta_a\|^2 = \|\eta_a(1)\|^2 = \sum_{k \in I} \|\eta_a^k(1)\|^2 = \sum_{k \in I} \|\eta_a^k\|^2 ,$$

which is finite if and only if η_a is a bounded operator. If ε_a is bounded, then by the Riesz Lemma there exists a unique $v \in F$ such that $\varepsilon_a(x) = \langle v, x \rangle$ and $\|\varepsilon_a\| = \|v\|$. Then $\langle v_k, x_k \rangle = \langle v, x_k \rangle = \varepsilon_a(x_k) = \varepsilon_a^k(x_k)$. So again by the Riesz Lemma $\|\varepsilon_a^k\| = \|v_k\|$. We have that

$$\|\varepsilon_a\|^2 = \|v\|^2 = \sum_{k \in I} \|v_k\|^2 = \sum_{k \in I} \|\varepsilon_a^k\|^2 .$$

(\Leftarrow): The operators η_a and ε_a are bounded by the previous discussion. For Δ_a one has that

$$\begin{aligned} \|\Delta_a\|^2 &= \sup_{\substack{x \in F \\ \|x\|=1}} \|\Delta_a(x)\|^2 = \sup_{\substack{x \in F \\ \|x\|=1}} \left\| \sum_{k \in I} \Delta_a(x_k) \right\|^2 = \sup_{\substack{x \in F \\ \|x\|=1}} \sum_{k \in I} \|\Delta_a^k(x_k)\|^2 \\ &\leq \sup_{\substack{x \in F \\ \|x\|=1}} \sum_{k \in I} \|\Delta_a^k\|^2 \|x_k\|^2 \leq \left(\sup_l \|\Delta_a^l\|^2 \right) \cdot \sup_{\substack{x \in F \\ \|x\|=1}} \sum_{k \in I} \|x_k\|^2 = \sup_l \|\Delta_a^l\|^2 < \infty , \end{aligned}$$

so Δ_a is bounded. For μ_a the proof is similar.

Then one needs to check that $a \mapsto P_a := \sum_{k \in I} P_a^k$ is continuous. Let $\varepsilon \in \mathbb{R}_{>0}$, $a_0 \in \mathbb{R}_{\geq 0}$ and $f \in \bigoplus_{k \in I} F_k$ with components f_k be fixed. Let $a' > a_0$ and $0 < E < \varepsilon$ be arbitrary. Since $P_a - P_{a_0}$ is a bounded operator, one can find $J_{a'} \subset I$ finite, such that for every $a < a'$

$$\sum_{j \in I \setminus J_{a'}} \|(P_a^j - P_{a_0}^j)f_j\|^2 < E .$$

Then let $\delta' > 0$ be such that for every $|a - a_0| < \delta'$

$$\sum_{j \in J_{a'}} \|(P_a^j - P_{a_0}^j)f_j\|^2 < \varepsilon - E ,$$

which can be chosen since the sum is finite and each P_a^j is continuous by assumption. Finally let $\delta := \min\{\delta', a' - a_0\}$. By construction we have that for every $|a - a_0| < \delta$,

$$\|(P_a - P_{a_0})f\|^2 = \sum_{j \in I} \|(P_a^j - P_{a_0}^j)f_j\|^2 < \varepsilon .$$

□

All examples of RFAs known to us are of the above form. For Hermitian RFAs, which we will introduce in Section 4.1.5, we can show that they are necessarily of the above form. Note that the same RFA F_k cannot appear infinitely many times, as otherwise the bounds (4.1.48) would be violated.

Now we continue our list of examples with some special cases.

4. Let $(\epsilon_k, \sigma_k)_{k \in I}$ be a countable family of pairs of complex numbers such that for all $a > 0$

$$\sup_{k \in I} |\epsilon_k e^{-a\sigma_k}| < \infty \quad \text{and} \quad \sum_{k \in I} \left| \frac{e^{-a\sigma_k}}{\epsilon_k} \right|^2 < \infty . \quad (4.1.49)$$

Then $A_{\epsilon, \sigma} := \bigoplus_{k \in I} \mathbb{C}f_k$, the Hilbert space generated by orthonormal vectors f_k , becomes an RFA by Proposition 4.1.18 via

$$\mu_a(f_k \otimes f_j) := \delta_{k,j} \epsilon_k f_k e^{-a\sigma_k} , \quad \eta_a(1) := \sum_{k \in I} \frac{f_k}{\epsilon_k} e^{-a\sigma_k} , \quad (4.1.50)$$

$$\Delta_a(f_k) := \frac{f_k \otimes f_k}{\epsilon_k} e^{-a\sigma_k} , \quad \varepsilon_a(f_k) := \epsilon_k e^{-a\sigma_k} . \quad (4.1.51)$$

This RFA is strongly separable (with $\tau_a = \eta_a$) and commutative.

5. Let $I := \mathbb{Z}_{>0}$ and consider the one-dimensional Hilbert spaces $\mathbb{C}f_k$ and $\mathbb{C}g_k$ with $\|f_k\|^2 = k^2$ and $\|g_k\|^2 = k^{-1}$. Let $F := \bigoplus_{k=1}^{\infty} \mathbb{C}f_k$ and $G := \bigoplus_{k=1}^{\infty} \mathbb{C}g_k$ be the Hilbert space direct sums, so that

$$\langle f_k, f_j \rangle_F = \delta_{k,j} k^2 \quad \text{and} \quad \langle g_k, g_j \rangle_G = \delta_{k,j} k^{-1} . \quad (4.1.52)$$

Define the maps

$$\begin{aligned} \mu_a^F(f_k \otimes f_j) &:= \delta_{k,j} e^{-ak^2} f_k , & \eta_a^F(1) &:= \sum_{k=1}^{\infty} e^{-ak^2} f_k , \\ \Delta_a^F(f_k) &:= e^{-ak^2} f_k \otimes f_k , & \varepsilon_a^F(f_k) &:= e^{-ak^2} , \end{aligned} \quad (4.1.53)$$

and similarly for G by changing f_k to g_k . These formulas define strongly separable (with $\tau_a = \eta_a$) commutative RFAs by the previous example with $(\epsilon_k, \sigma_k) = (k^{-1}, k^2)$ for F and with $(\epsilon_k, \sigma_k) = (k, k^2)$ for G . Note that $\lim_{a \rightarrow 0} \mu_a^F$ exists and has norm 1, but $\lim_{a \rightarrow 0} \mu_a^G$ does not: the set $\{ \|\mu_0^G(g_k \otimes g_k)\| / \|g_k \otimes g_k\| = k \mid k \in \mathbb{Z}_{>0} \}$ is not bounded.

Define the morphism of RFAs $\psi : F \rightarrow G$ as

$$\psi(f_k) = g_k \quad \text{for} \quad k = 1, 2, \dots .$$

It is an operator with $\|\psi\| = 1$ and is mono and epi, but it does not have a bounded inverse, as the set $\{ \|\psi^{-1}(g_k)\| / \|g_k\| = k^2 \mid k \in \mathbb{Z}_{>0} \}$ is not bounded. This is an example illustrating that the category \mathcal{Hilb} is not abelian: a morphism can be mono and epi without being invertible. The example also shows that RFA morphisms which are mono and epi need not preserve the existence of zero-area limits. Isomorphisms, on the other hand, being continuous with continuous inverse, do preserve the existence of limits.

6. Consider $L^2(G)$, the Hilbert space of square integrable functions on a compact semisimple Lie group G with the following morphisms:

$$\begin{aligned} \eta_a(1) &:= \sum_{V \in \hat{G}} e^{-a\sigma_V} \dim(V) \chi_V , & \mu(F)(x) &:= \int_G F(y, y^{-1}x) dy , \\ P_a(f) &:= \mu(\eta_a(1) \otimes f) , & \mu_a &:= P_a \circ \mu , \\ \varepsilon_a(f) &:= \int_G \eta_a(1)(x) f(x^{-1}) dx , & \Delta(f)(x, y) &:= f(xy) , & \Delta_a &:= \Delta \circ P_a , \end{aligned} \quad (4.1.54)$$

where $f \in L^2(G)$, $F \in L^2(G \times G) \cong L^2(G) \otimes L^2(G)$, \hat{G} is a set of representatives of isomorphism classes of finite-dimensional simple unitary G -modules, σ_V is the value of the Casimir operator of the Lie algebra of G in the simple module V , χ_V is the character of V , and \int_G denotes the Haar integral on G . These formulas define a strongly separable RFA in \mathcal{Hilb} (with $\tau_a = \eta_a$), for details see Section 4.4.1.

7. The centre of the previous RFA is $Cl^2(G)$, the Hilbert space of square integrable class functions on G , with multiplication, unit and counit given by the same formulas, but with the following coproduct:

$$\Delta_a(f) = \sum_{V \in \hat{G}} e^{-a\sigma_V} (\dim(V))^{-1} \chi_V \otimes \chi_V f_V, \quad (4.1.55)$$

where $f = \sum_{V \in \hat{G}} f_V \chi_V \in Cl^2(G)$. This is a strongly separable RFA in \mathcal{Hilb} (with $\tau_a(1) = \sum_{V \in \hat{G}} e^{-a\sigma_V} (\dim(V))^{-1} \chi_V$ and $\tau_a^{-1}(1) = \sum_{V \in \hat{G}} e^{-a\sigma_V} (\dim(V))^3 \chi_V$). For more details see Section 4.4.1.

4.1.4 Tensor products of RFAs and finite-dimensional RFAs

We denote the *category of regularised algebras in \mathcal{S}* by $\mathcal{RAlg}(\mathcal{S})$ and the *category of RFAs in \mathcal{S}* by $\mathcal{RFrob}(\mathcal{S})$. In this section we investigate under which conditions one can endow these categories with a monoidal structure. Then we describe the case $\mathcal{S} = \mathcal{Vect}^{\text{fd}}$ in detail.

Proposition 4.1.19. *Any morphism of RFAs is mono and epi.*

Proof. Let $\varphi : A \rightarrow B$ be a morphism of RFAs and let $\psi_{a,b} := (\text{id}_A \otimes \beta_b^B) \circ (\text{id}_A \otimes \varphi \otimes \text{id}_B) \circ (\gamma_a^A \otimes \text{id}_B)$. Then $\varphi \circ \psi_{a,b} = P_{a+b}^B$ and $\psi_{a,b} \circ \varphi = P_{a+b}^A$. We show that φ is epi, showing that it is mono is similar. Let $f, g \in \mathcal{S}(B, X)$ for an object X such that $f \circ \varphi = g \circ \varphi$. After composing with $\psi_{a,b}$ from the right for $a, b \in \mathbb{R}_{>0}$ we get $f \circ P_{a+b}^B = g \circ P_{a+b}^B$. This last equation holds for every $a, b \in \mathbb{R}_{>0}$, so we can take the limit $a, b \rightarrow 0$ to get $f = g$. \square

Remark 4.1.20. As we saw in Example 5, not every morphism of RFAs in \mathcal{Hilb} is invertible, hence $\mathcal{RFrob}(\mathcal{Hilb})$ is not a groupoid.

However we have the following:

Corollary 4.1.21. *The category $\mathcal{RFrob}(\mathcal{Vect}^{\text{fd}})$ is a groupoid.*

Proposition 4.1.22. *Assume that \mathcal{S} has a symmetric structure σ and that for $A, B \in \mathcal{RFrob}(\mathcal{S})$ the assignments*

$$(a_1, \dots, a_n) \mapsto P_{a_1}^A \otimes P_{a_1}^B \otimes \dots \otimes P_{a_n}^A \otimes P_{a_n}^B \quad (4.1.56)$$

are jointly continuous for every $n \geq 1$. Then $A \otimes B$ is an RFAs by

$$\begin{aligned} \mu_a^{A \otimes B} &:= (\mu_a^A \otimes \mu_a^B) \circ (\text{id} \otimes \sigma \otimes \text{id}), & \eta_a^{A \otimes B} &:= \eta_a^A \otimes \eta_a^B, \\ \Delta_a^{A \otimes B} &:= (\text{id} \otimes \sigma \otimes \text{id}) \circ (\Delta_a^A \otimes \Delta_a^B), & \varepsilon_a^{A \otimes B} &:= \varepsilon_a^A \otimes \varepsilon_a^B. \end{aligned} \quad (4.1.57)$$

Proof. Checking the algebraic relations is straightforward. The continuity of the maps in (4.1.56) assures that the continuity condition holds for the tensor product. \square

If condition (4.1.56) holds for every pair $A, B \in \mathcal{RFrob}(\mathcal{S})$ we can define a symmetric monoidal structure on $\mathcal{RFrob}(\mathcal{S})$, where the symmetric structure is inherited from \mathcal{S} . The tensor unit is the trivial RFA.

Proposition 4.1.23. $\mathcal{RFrob}(\mathcal{Hilb})$ and $\mathcal{RFrob}(\mathcal{Vect}^{\text{fd}})$ are symmetric monoidal categories with the above tensor product.

Proof. In $\mathcal{Vect}^{\text{fd}}$ the tensor product is continuous, so there the statement is trivial. In \mathcal{Hilb} Corollary 4.1.16 assures that the condition (4.1.56) holds for every pair $A, B \in \mathcal{RFrob}(\mathcal{Hilb})$. \square

Finite-dimensional regularised algebras and regularised Frobenius algebras

In the rest of this section we classify finite-dimensional regularised (Frobenius) algebras. The forgetful functor from finite-dimensional Hilbert spaces to $\mathcal{Vect}^{\text{fd}}$ is an equivalence of categories, therefore in this subsection we will only consider regularised (Frobenius) algebras in the latter.

Denote with $\mathcal{Alg}^Z(\mathcal{Vect}^{\text{fd}})$ the category with objects pairs (F, H) , where $F \in \mathcal{Vect}^{\text{fd}}$ is an algebra and $H \in Z(F)$ is an element in the centre of F , and morphisms $\phi : (F, H) \rightarrow (F', H')$ such that $\phi : F \rightarrow F'$ is a morphism of algebras and $\phi(H) = H'$. Analogously, denote with $\mathcal{Frob}^Z(\mathcal{Vect}^{\text{fd}})$ the category of pairs of Frobenius algebras and elements in their centre.

We define a functor $\mathcal{D} : \mathcal{RAlg}(\mathcal{Vect}^{\text{fd}}) \rightarrow \mathcal{Alg}^Z(\mathcal{Vect}^{\text{fd}})$ as follows: on objects as $\mathcal{D}(A) := (A, \frac{d}{da}\eta_a(1)|_{a=0})$ and on morphisms as identity. The same definition also gives a functor $\mathcal{D} : \mathcal{RFrob}(\mathcal{Vect}^{\text{fd}}) \rightarrow \mathcal{Frob}^Z(\mathcal{Vect}^{\text{fd}})$.

Proposition 4.1.24. *The functors*

$$\mathcal{D} : \mathcal{RAlg}(\mathcal{Vect}^{\text{fd}}) \rightarrow \mathcal{Alg}^Z(\mathcal{Vect}^{\text{fd}}) \quad \text{and} \quad \mathcal{D} : \mathcal{RFrob}(\mathcal{Vect}^{\text{fd}}) \rightarrow \mathcal{Frob}^Z(\mathcal{Vect}^{\text{fd}})$$

are equivalences of categories.

Proof. The inverse functor sends (A, H) to the regularised algebra A with $P_a := e^{aH}$, $\mu_a := P_a \circ \mu$ and $\eta_a := P_a \circ \eta$, where μ and η are the multiplication and unit of A . \square

Remark 4.1.25. Let $(A, H) \in \mathcal{Alg}^Z(\mathcal{Vect})$. Then

$$\mathcal{D}^{-1}(A, H) = \bigoplus_{\lambda \in \text{sp} H} \mathcal{D}^{-1}(A_\lambda, \text{Pr}_{A_\lambda}(H))$$

as regularised algebras, where A_λ denotes the generalised eigenspace of H corresponding to the eigenvalue λ and Pr_{A_λ} is the projection onto it. If $\mathcal{D}^{-1}(A, H)$ is furthermore an RFA then the above decomposition is valid as RFAs.

4.1.5 Hermitian RFAs in \mathcal{Hilb}

We start by recalling the notion of a dagger (or \dagger -) symmetric monoidal category \mathcal{S} , e.g. from [Sel]. A *dagger structure* on \mathcal{S} is a functor $(-)^{\dagger} : \mathcal{S} \rightarrow \mathcal{S}^{\text{opp}}$ which is identity on objects, $(-)^{\dagger\dagger} = \text{id}_{\mathcal{S}}$, $(f \otimes g)^{\dagger} = f^{\dagger} \otimes g^{\dagger}$ for any morphisms f, g and $\sigma_{U,V}^{\dagger} = \sigma_{V,U}$.

Let \mathcal{S} be as in the beginning of Section 4.1.1 and fix a \dagger -structure on \mathcal{S} . We do not require $(-)^{\dagger}$ to be continuous on hom-spaces, cf. Remark 4.1.11.

Definition 4.1.26. A *Hermitian regularised Frobenius algebra* (or \dagger -RFA for short) in \mathcal{S} is an RFA in \mathcal{S} for which $\mu_a^\dagger = \Delta_a$ and $\eta_a^\dagger = \varepsilon_a$ (and therefore $P_a = P_a^\dagger$). We denote by $\dagger\text{-}\mathcal{RFrob}(\mathcal{S})$ the full subcategory of $\mathcal{RFrob}(\mathcal{S})$ with objects given by \dagger -RFAs.

In the following we specialise to $\mathcal{S} = \mathcal{Hilb}$ with dagger structure given by the adjoint. Note that $\dagger\text{-}\mathcal{RFrob}(\mathcal{Hilb})$ is symmetric monoidal.

Example 4.1.27. Let us look at the examples from Section 4.1.3. In Example 1, if the Frobenius algebra $A \in \mathcal{Hilb}$ is a \dagger -Frobenius algebra (see e.g. [Vic, Def. 3.3]) and if $\sigma \in \mathbb{R}$ then P_a is self-adjoint and hence A is a \dagger -RFA. In Section 4.4.1 we will show that the RFAs in Examples 6 and 7 are \dagger -RFAs. The two RFAs in Example 5 are not \dagger -RFAs, as one can easily confirm that the summands $\mathbb{C}f_k$ and $\mathbb{C}g_k$ for $k > 1$ are not \dagger -RFAs. We compute e.g. for $\mathbb{C}f_k$ that

$$\langle f_k, \mu_a(f_k \otimes f_k) \rangle = e^{-ak^2} k^2 \quad \text{and} \quad \langle \Delta_a(f_k), f_k \otimes f_k \rangle = e^{-ak^2} k^4, \quad (4.1.58)$$

so clearly, if $k > 1$ then $\mu_a^\dagger \neq \Delta_a$.

Let $\dagger\text{-}\mathcal{Frob}^F(\mathcal{Hilb})$ denote the category which has objects countable families $\Phi = \{F_j, \sigma_j\}_{j \in I}$ of \dagger -Frobenius algebras F_j and real numbers σ_j , such that for every $a \in \mathbb{R}_{>0}$

$$\sup_{j \in I} \{e^{-a\sigma_j} \|\mu_j\|\} < \infty \quad \text{and} \quad \sum_{j \in I} e^{-2a\sigma_j} \|\eta_j\|^2 < \infty. \quad (4.1.59)$$

A morphism $\Psi : \Phi \rightarrow \Phi'$ consists of a bijection $f : I \xrightarrow{\sim} I'$ which satisfies $\sigma_j = \sigma_{f(j)}$ and a family of morphisms of Frobenius algebras $\psi_j : F_j \rightarrow F'_{f(j)}$ (which are automatically invertible [Koc, Lem. 2.4.5]). We will write $\Psi = (f, \{\psi_j\}_{j \in I})$.

Let $\Phi \in \dagger\text{-}\mathcal{Frob}^F(\mathcal{Hilb})$ with the notation from above. Then by Proposition 4.1.24, $\mathcal{D}^{-1}(F_j, \sigma_j \text{id}_{F_j})$ for $j \in I$ is an RFA. Using Proposition 4.1.18, we get an RFA structure on $\bigoplus_{j \in I} F_j$. The next theorem shows that the resulting functor is an equivalence.

Theorem 4.1.28. *There is an equivalence of categories $\dagger\text{-}\mathcal{Frob}^F(\mathcal{Hilb}) \rightarrow \dagger\text{-}\mathcal{RFrob}(\mathcal{Hilb})$ given by $\Phi \mapsto \bigoplus_{j \in I} F_j$.⁴*

Proof. We define the inverse functor. Let $F \in \dagger\text{-}\mathcal{RFrob}(\mathcal{Hilb})$ and fix $a \in \mathbb{R}_{>0}$. Then P_a is self-adjoint and therefore can be diagonalised. Let $\text{sp}_{\text{pt}}(P_a)$ denote the point spectrum⁵ of P_a . Furthermore, by Lemma 4.1.13 P_a is of trace class, and hence compact. Thus it has at most countably many eigenvalues and the eigenspaces with non-zero eigenvalues are finite-dimensional. Let

$$F = \bigoplus_{\alpha \in \text{sp}_{\text{pt}}(P_a)} F_\alpha \quad (4.1.60)$$

⁴We would like to thank André Henriques for explaining to us this decomposition of \dagger -RFAs, or rather the corresponding decomposition of Hermitian area-dependent QFTs via Corollary 4.2.14.

⁵The point spectrum of a bounded operator is the set of eigenvalues. Every compact operator on an infinite-dimensional Hilbert space has 0 in its spectrum, but it need not be an eigenvalue.

be the corresponding eigenspace decomposition of P_a .

Claim: The eigenvalue α of P_a on F_α is of the form $e^{-a\sigma_\alpha}$ for some $\sigma_\alpha \in \mathbb{R}$. In particular 0 is not an eigenvalue.

To show this, first assume that $c(a) := \alpha \neq 0$, so that F_α is finite-dimensional, and simultaneously diagonalise P_a , P_b and P_{a+b} on F_α . Then on a subspace where all three operators are constant with values $c(a)$, $c(b)$ and $c(a+b)$ one has that $c(a)c(b) = c(a+b)$. Furthermore $a \mapsto c(a)$ is a continuous function $\mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$ and $c(0) = 1$ since $a \mapsto P_a$ is strongly continuous at every $a \in \mathbb{R}_{\geq 0}$ and $\lim_{a \rightarrow 0} P_a = \text{id}_F$. So the unique solution to the above functional equation is $c(a) = e^{-a\sigma_\alpha}$ for some $\sigma_\alpha \in \mathbb{R}$.

Finally let us assume that $\alpha = 0$. Clearly, $\ker(P_a) \subseteq \ker(P_{a+b})$ for every $b \in \mathbb{R}_{\geq 0}$. Since P_a is self-adjoint, we have for $v \in F_0$ that $0 = P_a(v) = P_{a/2}^\dagger \circ P_{a/2}(v)$. But then $P_{a/2}(v) = 0$ and similarly, for every $n \in \mathbb{Z}_{\geq 0}$ we have that $P_{a/2^n}(v) = 0$. Altogether we have that $F_0 = \ker(P_a) = \ker(P_b)$ for every $b \in \mathbb{R}_{\geq 0}$. So $\lim_{a \rightarrow 0} P_a = \text{id}_F$ implies that $F_0 = \{0\}$.

Claim: The eigenspaces are \dagger -Frobenius algebras by restricting and projecting the structure maps of F .

To show this, first confirm that the structure maps do not mix eigenspaces of P_a , because P_a commutes with them. Then checking \dagger -RFA relations is straightforward and these are \dagger -Frobenius algebras, cf. Proposition 4.1.17.

Claim: The convergence conditions in (4.1.59) are satisfied by the above obtained family of \dagger -Frobenius algebras F_α and real numbers σ_α .

This can be shown directly by computing the norm of the structure maps.

Showing that the two functors give an equivalence of categories is now straightforward. \square

Corollary 4.1.29. *Let $A \in \mathcal{Hilb}$ be a \dagger -RFA. Then P_a is mono and epi.*

Proof. From the proof of Theorem 4.1.28 we see that P_a is mono. Since P_a is self-adjoint we get that P_a is epi. \square

Lemma 4.1.30. *Every \dagger -Frobenius algebra in \mathcal{Hilb} is semisimple.*

Proof. Let F denote a \dagger -Frobenius algebra in \mathcal{Hilb} and let $\zeta := \mu \circ \Delta = \Delta^* \circ \Delta$, which is an F - F -bimodule morphism and an F - F -bicomodule morphism. It is a self-adjoint operator, so it can be diagonalised and F decomposes into Hilbert spaces as

$$F = \bigoplus_{\alpha \in \text{sp}(\zeta)} F_\alpha, \quad (4.1.61)$$

where F_α is the eigenspace of ζ with eigenvalue α .

Now we show that (4.1.61) is a direct sum of Frobenius algebras. Let $\alpha \neq \beta$ and take $a \in F_\alpha$, $b \in F_\beta$. We have

$$\begin{aligned} \zeta(ab) &= a\zeta(b) = \beta ab \\ &= \zeta(a)b = \alpha ab \end{aligned} \quad (4.1.62)$$

since ζ is a bimodule morphism. Then (4.1.62) shows that $ab = 0$, so (4.1.61) is a decomposition as algebras.

Similarly one shows that (4.1.61) is a decomposition as coalgebras. We have for every $a \in F_\alpha$, using Sweedler notation:

$$\begin{aligned}\Delta(\zeta(a)) &= \zeta(a_{(1)}) \otimes a_{(2)} = a_{(1)} \otimes \zeta(a_{(2)}) \\ &= \alpha \Delta(a) = \alpha a_{(1)} \otimes a_{(2)} ,\end{aligned}\tag{4.1.63}$$

which shows that the comultiplication restricted to F_α lands in $F_\alpha \otimes F_\alpha$.

We now show that 0 is not in the spectrum. Let us assume otherwise. Then F_0 is a Frobenius algebra. We have $\zeta(x) = \Delta^* \circ \Delta(x) = 0$ for every $x \in F_0$, and so also $\Delta(x) = 0$, which is a contradiction to counitality. Therefore 0 is not in the spectrum of ζ , i.e. ζ is injective.

Now the only thing left to show is that each summand F_α is semisimple. Take $\Delta(1) \cdot \alpha^{-1}$ projected on $F_\alpha \otimes F_\alpha$. This is a separability idempotent for the algebra F_α , hence F_α is separable, hence semisimple. \square

Let $\epsilon \in \mathbb{C} \setminus \{0\}$, $\sigma \in \mathbb{R}$ and let $\mathbb{C}_{\epsilon, \sigma}$ denote the one-dimensional \dagger -RFA structure on \mathbb{C} given by

$$\begin{aligned}\varepsilon_a(1) &= e^{-a\sigma} \epsilon , & \Delta_a(1) &= \frac{e^{-a\sigma}}{\epsilon} 1 \otimes 1 , \\ \eta_a(1) &= e^{-a\sigma} \epsilon^* 1 , & \mu_a(1 \otimes 1) &= \frac{e^{-a\sigma}}{\epsilon^*} 1 .\end{aligned}\tag{4.1.64}$$

Let $C \in \mathcal{Hilb}$ be a one-dimensional \dagger -RFA and $c \in C$. Then by Proposition 4.1.24, $\varepsilon_a = \varepsilon_0 \circ P_a$. Set $\epsilon := \varepsilon_0(c) \in \mathbb{C}$ and $\sigma \in \mathbb{R}$ to be such that $P_a(c) = e^{-a\sigma} c$. Then

$$\begin{aligned}C &\rightarrow \mathbb{C}_{\epsilon, \sigma} \\ c &\mapsto 1\end{aligned}\tag{4.1.65}$$

is an isomorphism of RFAs.

Corollary 4.1.31. *Let C be a commutative \dagger -RFA in \mathcal{Hilb} . Then there is a family of numbers $\{\epsilon_j, \sigma_j\}_{j \in I}$, where $\epsilon_j \in \mathbb{C}$ and $\sigma_j \in \mathbb{R}$, satisfying*

$$\sup_{j \in I} \{e^{-a\sigma_j} |\epsilon_j|^{-1}\} < \infty \quad \text{and} \quad \sum_{j \in I} e^{-2a\sigma_j} |\epsilon_j|^2 < \infty\tag{4.1.66}$$

for every $a \in \mathbb{R}_{>0}$ such that $C \cong \bigoplus_{j \in I} \mathbb{C}_{\epsilon_j, \sigma_j}$ as RFAs.

Proof. By Theorem 4.1.28 and Lemma 4.1.30, C is a direct sum of semisimple algebras. By the Wedderburn-Artin theorem every semisimple commutative algebra is a direct sum of one-dimensional algebras. Using the isomorphism (4.1.65) we get the above family of numbers. The finiteness conditions come from (4.1.59). \square

Remark 4.1.32. In some cases none of the structure maps of a commutative Hermitian RFA admit an $a \rightarrow 0$ limit. A concrete example can be given as follows. Fix $1/2 > \delta > 0$. Then the family of numbers $\{n^{1/2+\delta}, n\}_{n \in \mathbb{Z}_{>0}}$ satisfies (4.1.66) and the structure maps μ_a , Δ_a , η_a , ε_a of the corresponding commutative \dagger -RFA from Corollary 4.1.31 do not have an $a \rightarrow 0$ limit.

Lemma 4.1.33. *Let $\varphi : \mathbb{C}_{\epsilon, \sigma} \rightarrow \mathbb{C}_{\epsilon', \sigma'}$ be a morphism of RFAs. Then $\varphi(1) = \epsilon/\epsilon' \in U(1)$ and $\sigma = \sigma'$.*

Proof. From $\varphi \circ \eta_a = \eta'_a$ one has that for every $a \in \mathbb{R}_{\geq 0}$, $\varphi(1)\epsilon^*e^{-a\sigma} = (\epsilon')^*e^{-a\sigma'}$. Since $\epsilon \neq 0$, $\epsilon' \neq 0$ and $\varphi(1) \neq 0$, one must have $\sigma = \sigma'$ and hence $\varphi(1)\epsilon^* = (\epsilon')^*$. One similarly obtains from $\varepsilon'_a \circ \varphi = \varepsilon_a$ that $\epsilon'\varphi(1) = \epsilon$. Combining these we get that $|\varphi(1)| = 1$ and that $\varphi(1) = \epsilon/\epsilon'$. \square

Proposition 4.1.34. *Every morphism of commutative \dagger -RFAs in \mathcal{Hilb} is unitary, in particular the category of commutative \dagger -RFAs in \mathcal{Hilb} is a groupoid.*

Proof. Let $\phi : C \rightarrow C'$ be a morphism of commutative \dagger -RFAs. By Corollary 4.1.31 we assume that $C = \bigoplus_{j \in I} \mathbb{C}_{\epsilon_j, \sigma_j}$ and $C' = \bigoplus_{j \in I'} \mathbb{C}_{\epsilon'_j, \sigma'_j}$. By a similar argument as in the proof of Lemma 4.1.33, we see that ϕ does not mix the $\mathbb{C}_{\epsilon_j, \sigma_j}$'s with different σ 's. Let $C_\sigma := \bigoplus_{\substack{j \in I \\ \sigma_j = \sigma}} \mathbb{C}_{\epsilon_j, \sigma_j}$ and define C'_σ similarly. These are both finite-dimensional, since these are eigenspaces of the P_a 's with eigenvalue $e^{-a\sigma}$. Let $\varphi := \phi|_{C_\sigma}$. Then φ is a morphism of finite-dimensional RFAs so it is a bijection by Corollary 4.1.21. Let us write $g_j = 1$ ($j = 1, n_\sigma$) for the generator of $\mathbb{C}_{\epsilon_j, \sigma_j}$ in C_σ and $g'_j = 1$ ($j = 1, n_\sigma$) for the generator of $\mathbb{C}_{\epsilon'_j, \sigma'_j}$ in C'_σ and write $\varphi(g_j) = \sum_{k=1}^{n_\sigma} \varphi^{jk} g'_k$.

From the equation $\varphi \circ \mu = \mu' \circ (\varphi \otimes \varphi)$ one has for every j, k, l that

$$\delta_{jk}(\epsilon_j^*)^{-1} \varphi^{jl} = \varphi^{jl} \varphi^{kl} ((\epsilon'_l)^*)^{-1}.$$

- If $j \neq k$ then $\varphi^{jl} \varphi^{kl} = 0$ for every such k and for every l . This means that in the matrix φ^{jl} in every row there might be at most one nonzero element. Since φ is bijective there is also at least one nonzero element in every row in the latter matrix and the same holds for every column. We conclude that the matrix of φ is the product of a permutation matrix π and a diagonal matrix D .
- If $j = k$ and if $\varphi^{jl} \neq 0$ then $\varphi^{jl} = (\epsilon'_l/\epsilon_j)^*$, which give the nonzero elements of the diagonal matrix.

Now $\pi^{-1} \circ \varphi$ restricts to RFA morphisms of the one-dimensional components, hence by Lemma 4.1.33 the diagonal matrix D is unitary. Therefore φ is unitary, ϕ is the direct sum of unitary matrices so ϕ is unitary and in particular invertible. \square

4.1.6 Modules over regularised algebras

We define modules over a regularised algebra in such a way that the action map now depends on two real parameters. This may seem odd at first sight but is motivated by the application to area-dependent field theory later on, see Section 4.2.3.

Definition 4.1.35. A *left module* over a regularised algebra A (or left A -module) in \mathcal{S} is an object $U \in \mathcal{S}$ together with a family of morphisms

$$\rho_{a,l} = \begin{array}{c} U \\ | \\ \bullet^{(a,l)} \\ / \quad | \\ A \quad U \end{array} \in \mathcal{S}(A \otimes U, U) \quad (4.1.67)$$

for every $a, l \in \mathbb{R}_{>0}$ called the *action*, such that they satisfy the following conditions.

1. For every $a = a_1 + a_2 = b_1 + b_2$ and $l = l_1 + l_2$

$$\rho_{a_1,l_1} \circ (\text{id}_A \otimes \rho_{a_2,l_2}) = \rho_{b_1,l} \circ (\mu_{b_2} \otimes \text{id}_U) . \quad (4.1.68)$$

and the morphisms

$$Q_{a,l}^U := \rho_{a,l} \circ (\eta_{a_2} \otimes \text{id}_U) \quad (4.1.69)$$

satisfy $\lim_{a,l \rightarrow 0} Q_{a,l}^U = \text{id}_U$.

2. The assignment

$$\begin{aligned} (\mathbb{R}_{>0}^2 \cup \{0\}) &\rightarrow \mathcal{S}(U, U) \\ (a, l) &\mapsto Q_{a,l}^U \end{aligned} \quad (4.1.70)$$

is jointly continuous.

One similarly defines right modules.

Note that the morphisms $Q_{a,l}^U$ form a semigroup,

$$Q_{a_1,l_1}^U \circ Q_{a_2,l_2}^U = Q_{a_1+a_2,l_1+l_2}^U, \quad (4.1.71)$$

and we have a continuous semigroup homomorphism $\mathbb{R}_{\geq 0}^2 \rightarrow \mathcal{S}(U, U)$, $(a, l) \mapsto Q_{a,l}^U$.

Remark 4.1.36. As in the case of regularised algebras, one would want to impose (4.1.70) for n -fold tensor products for every $n \geq 1$. However in Section 4.3.6 we will see that the natural condition would be to have this for a set of different modules, which would lead to the notion of “sets of mutually jointly continuous modules”, which is cumbersome to define. Instead, we will impose this condition later in Section 4.3.6. When considering regularised algebras and modules in \mathcal{Hilb} , this continuity condition will be automatic, see Lemma 4.1.15.

The definition of bimodules in terms of left and right modules requires an extra continuity assumption, so we spell it out in detail:

Definition 4.1.37. An A - B -bimodule over regularised algebras A and B is an object $U \in \mathcal{S}$ together with a family of morphisms

$$\rho_{a,l,b} = \begin{array}{c} U \\ | \\ \bullet^{(a,l,b)} \\ / \quad | \quad \backslash \\ A \quad U \quad B \end{array} \in \mathcal{S}(A \otimes U \otimes B, U) \quad (4.1.72)$$

for every $a, l, b \in \mathbb{R}_{>0}$ such that the following conditions hold.

1. For every $a = a_1 + a_2 = a'_1 + a'_2$, $b = b_1 + b_2 = b'_1 + b'_2$ and $l = l_1 + l_2$

$$\rho_{a_1, l_1, b_1} \circ (\text{id}_A \otimes \rho_{a_2, l_2, b_2} \otimes \text{id}_B) = \rho_{a'_1, l, b'_1} \circ \left(\mu_{a'_2}^A \otimes \text{id}_U \otimes \mu_{b'_2}^B \right), \quad (4.1.73)$$

the morphisms $Q_{a,l,b}^U := \rho_{a_1, l_1, b_1} \circ (\eta_{a_2}^A \otimes \text{id}_U \otimes \eta_{b_2}^B)$ satisfies that $\lim_{a,l,b \rightarrow 0} Q_{a,l,b}^U = \text{id}_U$ and

2. the map

$$\begin{aligned} (\mathbb{R}_{>0}^3 \cup \{0\}) &\rightarrow \mathcal{S}(U, U) \\ (a, l, b) &\mapsto Q_{a,l,b}^U \end{aligned} \quad (4.1.74)$$

is jointly continuous.

A bimodule is called *transmissive*, if $\rho_{a,l,b}$ depends only on $a + b$, or, in other words, if $\rho_{a+u, l, a-u}$ is independent of u . As with the inclusion of the extra parameter l , the notion of transmissivity is motivated by the application to area-dependent quantum field theory, see Section 4.2.3.

Remark 4.1.38. Let U be a left A -module and a right B -module such that the left and right actions $\rho_{a,l}^L$ and $\rho_{b,m}^R$ commute. That is for every $a, b \in \mathbb{R}_{>0}$ and $l_1 + l_2 = m_1 + m_2$

$$\rho_{a,l,b} := \rho_{a,l_1}^L \circ (\rho_{b,l_2}^R \otimes \text{id}_B) = \rho_{a,m_1}^R \circ (\text{id}_A \otimes \rho_{b,m_2}^L). \quad (4.1.75)$$

If $\rho_{a,l,b}$ is jointly continuous in the parameters, then U is an A - B -bimodule. Note that in contrast to the case of usual bimodules over associative algebras, which are defined to be left and right modules with commuting actions, here we have to impose the extra condition of joint continuity.

Conversely, let U be an A - B -bimodule with action $\rho_{a,l,b}$. If the limit

$$\rho_{a,l}^L := \lim_{b \rightarrow 0} \rho_{a,l,b_1} \circ (\text{id}_{A \otimes U} \otimes \eta_{b_2}^B) \quad (4.1.76)$$

with $b = b_1 + b_2$ exists for every $a, l \in \mathbb{R}_{>0}$ and remains jointly continuous in the limit, then U becomes a left A -module with action $\rho_{a,l}^L$. Similarly, if the analogous $a \rightarrow 0$ limit exists then U becomes a right B -module. In Appendix 4.A we give an example which illustrates that these limits do not always exist.

Example 4.1.39. Let $\text{Aut}_{\mathcal{RFrob}(\mathcal{S})}(A)$ denote the invertible morphisms in $\mathcal{RFrob}(\mathcal{S})(A, A)$. Then for $\alpha, \beta \in \text{Aut}_{\mathcal{RFrob}(\mathcal{S})}(A)$ we can define a transmissive bimodule structure ${}_{\alpha}A_{\beta}$ on A by twisting the multiplication from the two sides and letting the l -dependence be trivial. That is, for every $a, b, l \in \mathbb{R}_{>0}$ we define the action to be

$$\rho_{a,l,b} := \mu_a \circ (\text{id} \otimes \mu_b) \circ (\alpha \otimes \text{id}_A \otimes \beta), \quad (4.1.77)$$

which is jointly continuous in the parameters, since the composition in \mathcal{S} is separately continuous, and since we can rewrite $\rho_{a,l,b} = P_c \circ \rho_{a',l,b'}$ with $a' + b' + c = a + b$. Note that $\beta^{-1} : {}_{\alpha}A_{\beta} \rightarrow {}_{\beta^{-1} \circ \alpha}A_{\text{id}_A}$ is a bimodule isomorphism, so it is enough to consider twisting on one side.

The proof of the following proposition is similar to that of Proposition 4.1.18.

Proposition 4.1.40. Let $F = \bigoplus_{k \in I} F_k$ and $G = \bigoplus_{j \in J} G_j$ be RFAs in \mathcal{Hilb} as in Proposition 4.1.18. Let $M_{kj} \in \mathcal{Hilb}$ be an F_k - G_j -bimodule with action $\rho_{a,l,b}^{M_{kj}}$ for $k \in I$ and $j \in J$. Then $M := \bigoplus_{k \in I, j \in J} M_{kj}$ is an F - G -bimodule in \mathcal{Hilb} if and only if for every $a, l, b \in \mathbb{R}_{>0}$

$$\sup_{k \in I, j \in J} \left\{ \left\| \rho_{a,l,b}^{M_{kj}} \right\| \right\} < \infty. \quad (4.1.78)$$

A morphism $U \xrightarrow{\phi} V$ of left modules over a regularised algebra A is a morphism in \mathcal{S} which respects the action:

$$\phi \circ \rho_{a,l}^U = \rho_{a,l}^V \circ (\text{id}_A \otimes \phi), \quad (4.1.79)$$

for all $a, l \in \mathbb{R}_{>0}$. One similarly defines morphisms of right modules and bimodules. Denote with $A\text{-Mod}(\mathcal{S})$ the category of left modules over A in \mathcal{S} .

Recall from Proposition 4.1.24 that for a regularised algebra $A \in \mathcal{Vect}^{\text{fd}}$ the pair $\mathcal{D}(A) = (A, H)$ consists of the underlying algebra of A and an element H in its centre. Let $A\text{-Mod}^Z(\mathcal{Vect}^{\text{fd}})$ denote the following category. Its objects are pairs (U, H_U) , where U is a left A -module in $\mathcal{Vect}^{\text{fd}}$ and $H_U \in \text{End}_A(U)$. Its morphisms are left A -module morphisms $\phi : U \rightarrow V$, such that $H_V \circ \phi = \phi \circ H_U$.

As in the case of regularised algebras in $\mathcal{Vect}^{\text{fd}}$ (cf. Proposition 4.1.17), the semigroup $(a, l) \mapsto Q_{a,l}^U$ is norm continuous and hence $Q_{a,l}^U = e^{a \cdot H_A + l \cdot H_U}$ for $H_A, H_U \in \text{End}_A(U)$ such that $H_A \circ H_U = H_U \circ H_A$.

Let us define a functor $\mathcal{D} : A\text{-Mod}(\mathcal{Vect}^{\text{fd}}) \rightarrow A\text{-Mod}^Z(\mathcal{Vect}^{\text{fd}})$ as follows. The A -module structure on $\mathcal{D}(U)$ is given by $\rho^U = Q_{-a,-l}^U \circ \rho_{a,l}^U$ and H_U is defined as above. On morphisms \mathcal{D} is the identity.

Proposition 4.1.41. The functor $\mathcal{D} : A\text{-Mod}(\mathcal{Vect}^{\text{fd}}) \rightarrow A\text{-Mod}^Z(\mathcal{Vect}^{\text{fd}})$ is an equivalence of categories.

Proof. The A -module structure on $\mathcal{D}^{-1}(U, H_U)$ is given by $\rho_{a,l}^U := e^{a H_A + l H_U} \circ \rho^U$ with $H_A = \rho^U(H \otimes -) \in \text{End}_A(U)$, where ρ^U is the action on U . \square

Remark 4.1.42. Let $A, B \in \mathcal{Vect}^{\text{fd}}$ be regularised algebras and $U \in \mathcal{Vect}^{\text{fd}}$ an A - B -bimodule $U \in \mathcal{Vect}^{\text{fd}}$. As before we have $Q_{a,l,b}^U = e^{aH_A + lH_U + bH_B}$, where $H_A, H_U, H_B \in \text{End}_{A,B}(U)$ are bimodule homomorphisms. Then U is transmissive if and only if $H_A = H_B$.

Let us assume now that \mathcal{S} is symmetric. We now introduce a notion of duals for bimodules.

Definition 4.1.43. Let $A, B \in \mathcal{S}$ be regularised algebras. A *dual pair of bimodules* is an A - B -bimodule $U \in \mathcal{S}$ and a B - A -bimodule $V \in \mathcal{S}$ together with families of morphisms for every $a, l, b \in \mathbb{R}_{>0}$

$$\gamma_{a,l,b} \in \mathcal{S}(\mathbb{I}, V \otimes U), \quad \beta_{a,l,b} \in \mathcal{S}(U \otimes V, \mathbb{I}), \quad (4.1.80)$$

jointly continuous in the parameters, which we denote with

$$\gamma_{a,l,b} = \begin{array}{c} V \quad U \\ \searrow \quad \nearrow \\ \bullet \\ \mathbb{I} \end{array}, \quad \beta_{a,l,b} = \begin{array}{c} \mathbb{I} \\ \nearrow \quad \searrow \\ U \quad V \end{array}, \quad (4.1.81)$$

such that for $a_1 + a_2 = a$, $b_1 + b_2 = b$ and $l_1 + l_2 = l$ we have

$$Q_{a,l,b}^V = \begin{array}{c} V \quad \mathbb{I} \\ \searrow \quad \nearrow \\ \bullet \\ \mathbb{I} \end{array}, \quad Q_{a,l,b}^U = \begin{array}{c} \mathbb{I} \quad U \\ \nearrow \quad \searrow \\ U \quad \mathbb{I} \end{array}, \quad (4.1.82)$$

and for every $a_1 + a_2 = a_3 + a_4$, $b_1 + b_2 = b_3 + b_4$ and $l_1 + l_2 = l_3 + l_4$ we have

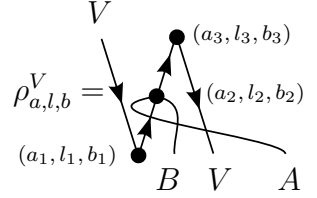
$$\begin{array}{c} V \quad U \\ \searrow \quad \nearrow \\ \bullet \\ (a_1, l_1, b_1) \end{array} = \begin{array}{c} V \quad U \\ \searrow \quad \nearrow \\ \bullet \\ (a_4, l_4, b_4) \end{array} \quad \text{and} \quad \begin{array}{c} \mathbb{I} \\ \nearrow \quad \searrow \\ U \quad V \end{array} = \begin{array}{c} \mathbb{I} \\ \nearrow \quad \searrow \\ U \quad V \end{array}. \quad (4.1.83)$$

Let us compare this situation to Lemma 4.1.7. There the continuity of γ_a in the parameter was automatic, but in Definition 4.1.43 we demanded continuity explicitly. The reason for this is that the argument in the proof of Lemma 4.1.7 does not apply, as we have not required that $\text{id}_V \otimes Q_{a,l,b}^U$ is continuous in the parameters, see Remark 4.1.36. However one can easily check that for every $a_1 + a_2 = a_3 + a_4$, $l_1 + l_2 = l_3 + l_4$ and $b_1 + b_2 = b_3 + b_4$

$$(\text{id}_V \otimes Q_{a_1, l_1, b_1}^U) \circ \gamma_{a_2, l_2, b_2} = (Q_{a_3, l_3, b_3}^V \otimes \text{id}_U) \circ \gamma_{a_4, l_4, b_4}. \quad (4.1.84)$$

Furthermore, in \mathcal{Hilb} this is equal to $\gamma_{a_1+a_2, l_1+l_2, b_1+b_2}$.

Note that (4.1.83) implies that the action on V is determined by the action on U :



$$\rho_{a,l,b}^V = \text{diagram} \quad (4.1.85)$$

We similarly define dual pairs of left and right modules and we omit the details here.

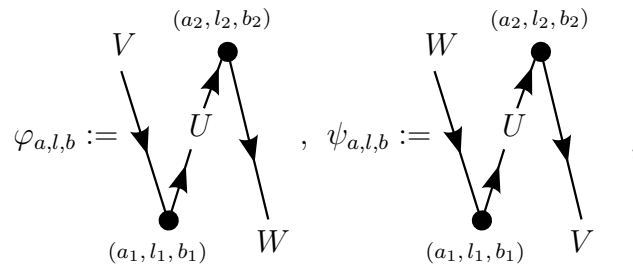
Example 4.1.44. Let $A \in \mathcal{S}$ be a symmetric RFA, $\alpha \in \text{Aut}_{\mathcal{R}\text{Frob}(\mathcal{S})}(A)$ and ${}_{\alpha}A_{\text{id}}$ be the twisted transmissive bimodule from Example 4.1.39. Then $({}_{\alpha}A_{\text{id}}, {}_{\alpha^{-1}}A_{\text{id}})$ is a dual pair of bimodules with duality morphisms

$$\beta_{a,l,b} = \varepsilon_a \circ \mu_b \circ (\text{id}_A \otimes \alpha) \quad \text{and} \quad \gamma_{a,l,b} = (\alpha^{-1} \otimes \text{id}_A) \circ \Delta_a \circ \eta_b \quad (4.1.86)$$

for $a, l, b \in \mathbb{R}_{>0}$. Note that these morphisms only depend on $a + b$.

Remark 4.1.45. If (U, V) is a dual pair of bimodules with duality morphisms $\gamma_{a,l,b}$ and $\beta_{a,l,b}$, then (V, U) is also a dual pair of bimodules with duality morphisms $\sigma_{V,U} \circ \gamma_{a,l,b}$ and $\beta_{a,l,b} \circ \sigma_{V,U}$.

Duals of bimodules over associative algebras are unique up to unique isomorphism. In the following we will see that under some assumptions this will be true for duals of bimodules over regularised algebras too. Let (U, V) and (U, W) be two dual pairs of bimodules and define



$$\varphi_{a,l,b} := \text{diagram} \quad , \quad \psi_{a,l,b} := \text{diagram} \quad , \quad (4.1.87)$$

which satisfy for $a = a_1 + a_2$, $b = b_1 + b_2$ and $l = l_1 + l_2$ that

$$\varphi_{a_1,l_1,b_1} \circ \psi_{a_2,l_2,b_2} = Q_{a,l,b}^V \quad \text{and} \quad \psi_{a_1,l_1,b_1} \circ \varphi_{a_2,l_2,b_2} = Q_{a,l,b}^W . \quad (4.1.88)$$

Using separate continuity of the composition and (4.1.88) one can show the following (we omit the details):

Lemma 4.1.46. *If the limits*

$$\lim_{a,l,b \rightarrow 0} \varphi_{a,l,b} \quad \text{and} \quad \lim_{a,l,b \rightarrow 0} \psi_{a,l,b} \quad (4.1.89)$$

exist, then $\varphi_{0,0,0}$ and $\psi_{0,0,0}$ are mutually inverse bimodule isomorphisms between V and W .

In general we do not know if $V \cong W$, not even in \mathcal{Hilb} .

Remark 4.1.47. A related concept of duals was introduced in [ABP] where duals are parametrised by Hilbert-Schmidt maps. The authors introduced the notion of a nuclear ideal in a symmetric monoidal category, which in \mathcal{Hilb} consists of Hilbert-Schmidt maps $\mathbf{HSO}(\mathcal{H}, \mathcal{K})$ for $\mathcal{H}, \mathcal{K} \in \mathcal{Hilb}$ [ABP, Thm. 5.9]. Part of the data is an isomorphism $\theta : \mathbf{HSO}(\mathcal{H}, \mathcal{K}) \xrightarrow{\cong} \mathcal{B}(\mathbb{C}, \overline{\mathcal{H}} \otimes \mathcal{K})$, where $\overline{\mathcal{H}}$ now denotes the conjugate Hilbert space. For $f \in \mathbf{HSO}(\mathcal{H}, \mathcal{K})$ and $g \in \mathbf{HSO}(\mathcal{K}, \mathcal{L})$ in a nuclear ideal the “compactness” relation holds:

$$(\text{id}_{\mathcal{L}} \otimes \theta(f^\dagger)^\dagger) \circ (\theta(g) \otimes \text{id}_{\mathcal{H}}) = g \circ f . \quad (4.1.90)$$

Our definition of duals fits into this framework as follows. Let A, B be a regularised algebras and U an A - B -bimodule in \mathcal{Hilb} with dual V . Then one can show that $Q_{a,l,b}^U$ is a trace class map, and hence Hilbert-Schmidt, cf. Lemma 4.1.13. Using the above notation let $\mathcal{H} = \mathcal{K} = \mathcal{L} := U$,

$$f := Q_{a,l,b}^U, \quad g := Q_{a',l',b'}^U, \quad \beta_{a,l,b}^U := \theta(f^\dagger)^\dagger, \quad \gamma_{a',l',b'}^U := \theta(g) . \quad (4.1.91)$$

Then (4.1.90) is exactly one half of the duality relation (4.1.82) and (U, \overline{U}) is a dual pair of bimodules in the sense of Definition 4.1.43.

4.1.7 Tensor product of modules over regularised algebras

Let $A \in \mathcal{S}$ be a regularised algebra, $M, N \in \mathcal{S}$ right and left A -modules respectively and $U \in \mathcal{S}$ an A - A -bimodule. Let $\rho_{a,l}^R := \rho_{a',l,a}^U \circ (\eta_{a'}^A \otimes \text{id}_{U \otimes A})$ and $\rho_{a,l}^L := \rho_{a,l,a'}^U \circ (\text{id}_{A \otimes U} \otimes \eta_{a'}^A)$ for $a' + a'' = a$.

Definition 4.1.48. The *tensor product of M and N over A* is an object $M \otimes_A N$ together with a morphism $\pi_{M \otimes_A N} : M \otimes N \rightarrow M \otimes_A N$ in \mathcal{S} , which is a coequaliser of $\rho_{a,l}^M \otimes Q_{a,l}^N$ and $Q_{a,l}^M \otimes \rho_{a,l}^N$ for every pair of parameters $(a, l) \in \mathbb{R}_{>0}^2$.

If \mathcal{S} is symmetric with braiding σ , one similarly defines the *cyclic tensor product* $(\pi_{\odot_A U} : U \rightarrow \odot_A U)$ to be a coequaliser of $\rho_{a,l}^L$ and $\rho_{a,l}^R \circ \sigma_{A,U}$ for every $(a, l) \in \mathbb{R}_{>0}^2$.

Let $A, B, C \in \mathcal{S}$ be regularised algebras, V a B - A -module and W an A - C -module. Let $\pi : V \otimes W \rightarrow V \otimes_A W$ denote a coequaliser of the morphisms

$$\rho_{a,b,c,l}^L := \begin{array}{c} V \quad W \\ | \quad | \\ (b_2, l, a) \quad (a_2, l, c_2) \\ | \quad | \quad | \quad | \\ b_1 \quad a_1 \quad c_1 \\ | \quad | \quad | \\ V \quad A \quad W \end{array}, \quad \rho_{a,b,c,l}^R := \begin{array}{c} V \quad W \\ | \quad | \\ (b_2, l, a_2) \quad (a, l, c_2) \\ | \quad | \quad | \quad | \\ b_1 \quad a_1 \quad c_1 \\ | \quad | \quad | \\ V \quad A \quad W \end{array}, \quad (4.1.92)$$

for every parameter $a_1, a_2, b_1, b_2, c_1, c_2, l \in \mathbb{R}_{>0}$ with $a = a_1 + a_2$, $b = b_1 + b_2$ and $c = c_1 + c_2$ (which of course may or may not exist). If the tensor product $B \otimes (-) \otimes C$ preserves

coequalisers of families of morphisms, then the universal property of the coequaliser induces a morphism $\bar{\rho}_{a,b,c,l} : B \otimes (V \otimes_A W) \otimes C \rightarrow V \otimes_A W$ from the morphism

$$\begin{array}{c}
 V \otimes_A W \\
 \downarrow \\
 \boxed{\pi} \\
 \downarrow \quad \downarrow \\
 \begin{array}{cc}
 (b, l, a_1) & (a_3, l, c) \\
 \swarrow \downarrow \searrow & \swarrow \downarrow \searrow \\
 B & V & W & C
 \end{array}
 \end{array} \quad , \quad (4.1.93)$$

where $a_1 + a_2 = a_3 + a_4$. If the limit $\bar{\rho}_{b,l,c} := \lim_{a \rightarrow 0} \bar{\rho}_{a,b,c,l}$ exists and is jointly continuous in all three parameters, then it gives a B - C -bimodule structure on $V \otimes_A W$.

Definition 4.1.49. The tensor product of V and W over A is the B - C -bimodule $V \otimes_A W$ with the action $\bar{\rho}_{b,c,l}$ together with the coequaliser $\pi : V \otimes W \rightarrow V \otimes_A W$.

The following proposition shows that in $\mathcal{Vect}^{\text{fd}}$ the tensor product of modules over regularised algebras reduces to the tensor product over ordinary associative algebras. The proof is straightforward and we omit it.

Proposition 4.1.50. Let A be a regularised algebra in $\mathcal{Vect}^{\text{fd}}$, M and N right and left A -modules, respectively. Let $\mathcal{D}(M) = (M, H_M)$ and $\mathcal{D}(N) = (N, H_N)$ be the corresponding underlying modules and module morphisms from Proposition 4.1.41. Then $\mathcal{D}(M \otimes_A N) = (M \otimes_A N, H_M \otimes_A H_N)$, where $M \otimes_A N$ is the tensor product of the underlying modules over the underlying algebra and $H_M \otimes_A H_N$ is the induced morphism on the tensor product.

For the rest of the section let \mathcal{S} be symmetric monoidal and idempotent complete, and $A \in \mathcal{S}$ a strongly separable regularised algebra with separability idempotents e_a . We define the following morphisms

$$\begin{array}{ccc}
 \begin{array}{c} M \quad N \\ | \quad | \\ \bullet \quad \bullet \\ \swarrow \quad \searrow \\ \boxed{e_{a_2}} \\ | \quad | \\ M \quad N \end{array} & , & \begin{array}{c} U \\ | \\ \bullet \\ \swarrow \quad \searrow \\ \boxed{e_{a_2}} \\ | \\ U \end{array} \\
 D_{a,l}^{M,N} & := & D_{a,l}^U
 \end{array} \quad , \quad \begin{array}{c} V \quad W \\ | \quad | \\ \bullet \quad \bullet \\ \swarrow \quad \searrow \\ \boxed{e_{a_2}} \\ | \quad | \\ V \quad W \end{array} \quad , \quad (4.1.94)$$

with $\sum_{i=1}^3 a_i = a$, $b = b_1 + b_2$, $c = c_1 + c_2$ and $l_1 + l_2 = l$. From a direct computation it follows that

$$D_{a_1,l_1}^{M,N} \circ D_{a_2,l_2}^{M,N} = D_{a_1+a_2,l_1+l_2}^{M,N} \quad (4.1.95)$$

$$D_{a_1,l_1}^U \circ D_{a_2,l_2}^U = D_{a_1+a_2,l_1+l_2}^U \quad (4.1.96)$$

$$D_{a_1,b_1,c_1,l_1}^{V,W} \circ D_{a_2,b_2,c_2,l_2}^{V,W} = D_{a_1+a_2,b_1+b_2,c_1+c_2,l_1+l_2}^{V,W} \quad (4.1.97)$$

for every $a_1, a_2, l_1, l_2, b_1, b_2 \in \mathbb{R}_{>0}$. So if $D_0^{M,N} := \lim_{a,l \rightarrow 0} D_{a,l}^{M,N}$ exists, then it is an idempotent. In this case we write

$$\begin{aligned} D_0^{M,N} &= \left[M \otimes N \xrightarrow{\pi} \text{im}(D_0^{M,N}) \xrightarrow{\iota} M \otimes N \right] \\ \text{id}_{\text{im}(D_0^{M,N})} &= \left[\text{im}(D_0^{M,N}) \xrightarrow{\iota} M \otimes N \xrightarrow{\pi} \text{im}(D_0^{M,N}) \right] \end{aligned} \quad (4.1.98)$$

for the projection π and embedding ι of its image $\text{im}(D_0^{M,N})$. Similarly, if $D_0^U := \lim_{a,l \rightarrow 0} D_{a,l}^U$ (resp. $D_0^{V,W} := \lim_{a,b,c,l \rightarrow 0} D_{a,b,c,l}^{V,W}$) exists, then it is also an idempotent and we similarly write π, ι and $\text{im}(D_0^U)$ (resp. $\text{im}(D_0^{V,W})$).

Let us assume that $\lim_{a,b,c,l \rightarrow 0} D_{a,b,c,l}^{V,W}$ exists and define

$$\begin{array}{c} \text{im}(D_0^{V,W}) \\ \downarrow \\ \boxed{\pi} \\ \downarrow \quad \downarrow \\ (b,l,a_1) \quad (a_3,l,c) \\ \swarrow \quad \searrow \\ \boxed{e_{a_2}} \\ \downarrow \\ \boxed{\iota} \\ \downarrow \\ B \quad \text{im}(D_0^{V,W}) \quad C \end{array} \quad (4.1.99)$$

$\tilde{\rho}_{a,b,c,l}^{V,W} :=$

for $a_1, a_2, a_3, b, c, l \in \mathbb{R}_{>0}$ with $a = a_1 + a_2 + a_3$.

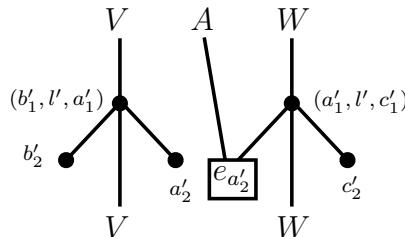
Proposition 4.1.51.

1. If $D_0^{M,N}$ exists then $(\pi, \text{im}(D_0^{M,N}))$ is the tensor product $M \otimes_A N$.
2. If D_0^U exists then $(\pi, \text{im}(D_0^U))$ is the cyclic tensor product $\odot_A U$.
3. If $D_0^{V,W}$ and $\tilde{\rho}_{b,l,c} := \lim_{a \rightarrow 0} \tilde{\rho}_{a,b,c,l}^{V,W}$ exists for every $b, c, l \in \mathbb{R}_{>0}$ and is jointly continuous in the parameters then $(\pi, \text{im}(D_0^{V,W}))$ with action $\tilde{\rho}_{b,l,c}$ is the tensor product $V \otimes_A W$.

Proof. We will only treat the third case, in the other two cases one proceeds analogously. We show that $(\pi, \text{im}(D_0^{V,W}))$ is a coequaliser of the morphisms in (4.1.92). Let $p := (a, b, c, l)$, $p' := (a', b', c', l')$ and $\varphi : V \otimes W \rightarrow Y$ be such that

$$\varphi \circ \rho_p^L = \varphi \circ \rho_{p'}^R. \quad (4.1.100)$$

Let $\tilde{\varphi} := \varphi \circ \iota$. We need to show that $\varphi = \tilde{\varphi} \circ \pi$ and that $\tilde{\varphi}$ is unique. Compose both sides of (4.1.100) with



with $a'_1 + a'_2 = a'$, $b'_1 + b'_2 = b'$ and $c'_1 + c'_2 = c'$ to get

$$\varphi \circ D_{p+p'}^{V,W} = \varphi \circ (Q_{p+p'}^V \otimes Q_{p+p'}^W).$$

Now taking the limit $p, p' \rightarrow 0$ gives $\varphi \circ D_0^{V,W} = \varphi$, which we needed to show. Uniqueness of $\tilde{\varphi}$ follows from $\pi \circ \iota = \text{id}_{\text{im}(D_0^{V,W})}$.

It is easy to see that the morphism $\bar{\rho}_{a,b,c,l}$ induced by (4.1.93) is the morphism in (4.1.99). \square

Corollary 4.1.52. *Consider A as a bimodule over itself. If D_0^A exists then $(\iota : \circlearrowleft_A A \rightarrow A)$ is the centre of A .*

Proof. Using the previous notation we show that $\iota : \circlearrowleft_A A \rightarrow A$ satisfies the universal property of the centre. So let $\varphi : Y \rightarrow A$ be such that

$$\mu_a \circ (\text{id}_A \otimes \varphi) = \mu_a \circ \sigma \circ (\text{id}_A \otimes \varphi). \quad (4.1.101)$$

Set $\tilde{\varphi} := \pi \circ \varphi$. We need to show that $\iota \circ \tilde{\varphi} = \varphi$. From (4.1.101) one obtains that $D_a \circ \varphi = P_a \circ \varphi$. Then taking the limit $a \rightarrow 0$ gives $D_0 \circ \varphi = \varphi$ which is what we needed to show. Uniqueness of $\tilde{\varphi}$ follows again from $\pi \circ \iota = \text{id}_{\circlearrowleft_A A}$. \square

Example 4.1.53. Let $A \in \mathcal{S}$ be a strongly separable symmetric RFA, $\alpha, \beta \in \text{Aut}_{\mathcal{RFrob}(\mathcal{S})}(A)$ ${}_{\alpha}A_{\text{id}}, {}_{\beta}A_{\text{id}}$ be the twisted transmissive bimodules from Example 4.1.39. Let us assume that $\lim_{a \rightarrow 0} D_a^{\alpha A_{\text{id}}, \beta A_{\text{id}}}$ from (4.1.94), $\lim_{a \rightarrow 0} \mu_a$ and $\lim_{a \rightarrow 0} \Delta_a$ exist. Then $({}_{\alpha}A_{\text{id}}) \otimes_A ({}_{\beta}A_{\text{id}}) = {}_{\alpha \circ \beta}A_{\text{id}}$ and the projection $\pi : {}_{\alpha}A_{\text{id}} \otimes {}_{\beta}A_{\text{id}} \rightarrow {}_{\alpha \circ \beta}A_{\text{id}}$ is given by $\pi = \mu_0 \circ (\beta \otimes \text{id}_A)$.

Remark 4.1.54. For A a strongly separable symmetric RFA, the tensor product over A actually automatically satisfies a stronger coequaliser condition. Let us illustrate this in the first case of Proposition 4.1.51: define $L_{a_1, a_2, l} := \rho_{a_1, l}^M \otimes Q_{a_2, l}^M$ and $R_{a_3, a_4, l} := Q_{a_3, l}^M \otimes \rho_{a_4, l}^N$ for $a_i > 0$. Then $\pi : M \otimes N \rightarrow M \otimes_A N$ is defined as the coequaliser of $L_{a, a, l}$ and $R_{a, a, l}$, but it is straightforward to verify that also $\pi \circ L_{a_1, a_2, l} = \pi \circ R_{a_3, a_4, l}$ holds for all $a_i > 0$ such that $a_1 + a_2 = a_3 + a_4$.

Using Proposition 4.1.51 and the dual action in (4.1.85) one can show the following.

Lemma 4.1.55. *Let (V, \bar{V}) be a dual pair of B - A -bimodules and (W, \bar{W}) a dual pair of A - C -bimodules. Let us assume that $D_0^{V,W}$ and $D_0^{\bar{W}, \bar{V}}$ exist and that $\lim_{a \rightarrow 0} \tilde{\rho}_{a,b,c,l}^{V,W}$ and $\lim_{a \rightarrow 0} \tilde{\rho}_{a,b,c,l}^{\bar{W}, \bar{V}}$ exist and are jointly continuous in their parameters. Let for $a, b, c, l \in \mathbb{R}_{>0}$*

$$\tilde{\gamma}_{a,b,c,l}^{V,W} := \begin{array}{c} \bar{W} \otimes_A \bar{V} \quad V \otimes_A W \\ \downarrow \quad \downarrow \\ \boxed{\pi} \quad \boxed{\pi} \\ \swarrow \quad \searrow \\ \bullet \quad \bullet \\ \swarrow \quad \searrow \\ \bullet \end{array} \quad (a, l, c; W) \quad (b, l, a; V) \quad \text{and} \quad \tilde{\beta}_{a,b,c,l}^{V,W} := \begin{array}{c} \bullet \quad (b, l, a; V) \\ \swarrow \quad \searrow \\ \bullet \quad (a, l, c; W) \\ \swarrow \quad \searrow \\ \boxed{\iota} \quad \boxed{\iota} \\ \downarrow \quad \downarrow \\ V \otimes_A W \quad \bar{W} \otimes_A \bar{V} \end{array}. \quad (4.1.102)$$

If $\lim_{a \rightarrow 0} \tilde{\gamma}_{a,b,c,l}^{V,W}$ and $\lim_{a \rightarrow 0} \tilde{\beta}_{a,b,c,l}^{V,W}$ exist for every $b, c, l \in \mathbb{R}_{>0}$ and are jointly continuous, then $(V \otimes_A W, \bar{W} \otimes_A \bar{V})$ is a dual pair of B - C -bimodules.

4.1.8 Tensor products in \mathcal{Hilb}

We now consider the case $\mathcal{S} = \mathcal{Hilb}$. Note that in \mathcal{Hilb} cokernels exist. If $f : X \rightarrow Y$ is a morphism in \mathcal{Hilb} then $\pi : Y \rightarrow Y/\overline{\text{im}(f)}$ is a cokernel of f where π is the canonical projection and $\overline{\text{im}(f)}$ is the closure of $\text{im}(f)$.

After some preparatory lemmas we will discuss tensor products of modules over regularised algebras.

Lemma 4.1.56. *Tensoring with identity in \mathcal{Hilb} preserves cokernels.*

Proof. Let $f : X \rightarrow Y$, $\pi_f = \text{coker}(f) : Y \rightarrow Y/\overline{\text{im}(f)}$, $Z \in \mathcal{Hilb}$ and $\pi_{f \otimes \text{id}_Z} := \text{coker}(f \otimes \text{id}_Z) : Y \otimes Z \rightarrow Y \otimes Z/\overline{\text{im}(f \otimes \text{id}_Z)}$. The claim of the lemma boils down to the observation that $\overline{\text{im}(f) \otimes Z} = \overline{\text{im}(f \otimes \text{id}_Z)}$, which in turn follows since both sides are closed and contain $\text{im}(f) \otimes Z$ as a dense subset. \square

Lemma 4.1.57. *Let A be a regularised algebra, and let M and N be right and left A -modules and U an A - A -bimodule. Let $p, q \in (\mathbb{R}_{>0})^2$ arbitrary and set $\varphi_p := \rho_p^M \otimes Q_p^N - Q_p^M \otimes \rho_p^N$. If Q_r^N and Q_r^M are epi for every $r \in (\mathbb{R}_{>0})^2$, then $\overline{\text{im}(\varphi_p)} = \overline{\text{im}(\varphi_q)}$.*

Proof. Let $p := (p_1, p_2)$ and $q := (q_1, q_2)$. It is enough to show that $\overline{\text{im}(\varphi_p)} = \overline{\text{im}(\varphi_q)}$ in the case when $p_1 > q_1$ and $p_2 > q_2$. Then we have that

$$\varphi_p = \varphi_q \circ (Q_{p-q}^M \otimes \text{id}_A \otimes Q_{p-q}^N) , \quad (4.1.103)$$

from which we directly get that $\overline{\text{im}(\varphi_p)} \subset \overline{\text{im}(\varphi_q)}$.

Now we show that $\overline{\text{im}(\varphi_q)} \subset \overline{\text{im}(\varphi_p)}$. We write $R := Q_{p-q}^M \otimes \text{id}_A \otimes Q_{p-q}^N$ and choose an arbitrary $y \in M \otimes A \otimes N$. Let $x = \varphi_q(y)$. By Lemma 4.1.12, R is epi, so we can choose a sequence $(z_n)_{n \in \mathbb{N}}$ in $M \otimes A \otimes N$, for which $\lim_{n \rightarrow \infty} R(z_n) = y$. Applying φ_q to both sides gives $\lim_{n \rightarrow \infty} \varphi_p(z_n) = x$, and thus $x \in \overline{\text{im}(\varphi_p)}$. \square

Proposition 4.1.58. *Let A be a regularised algebra, and let M and N be right and left A -modules and U an A - A -bimodule. If $Q_{l,b}^M$, $Q_{a,l}^N$ and $Q_{a,l,b}^U$ are epi for every $a, l, b \in \mathbb{R}_{>0}$ then the following tensor products exist:*

$$M \otimes_A N , \quad \circlearrowleft_A U . \quad (4.1.104)$$

Proof. Let us use the notation of Lemma 4.1.57. Let π be the projection

$$M \otimes N \rightarrow M \otimes N / \overline{\text{im}(\varphi_p)} \quad (4.1.105)$$

for some $p \in (\mathbb{R}_{>0})^2$, which is independent of p by Lemma 4.1.57. But this means exactly that π is the cokernel of φ_p for every $p \in (\mathbb{R}_{>0})^2$, and is hence a tensor product $M \otimes_A N$.

A similar argument shows that $\circlearrowleft_A U$ exists. \square

Lemma 4.1.59. *If $V \in \mathcal{Hilb}$ is a left B -module and a right A -module such that the two actions commute as in (4.1.75) then it is a B - A -bimodule via $\rho_{a,l,b}^V$ as in (4.1.75).*

Proof. The algebraic conditions are clear, and it remains to show that the two-sided action $\rho_{a,l,b}^V$ is jointly continuous in all three parameters. Since the composition is separately continuous and we have $Q_{a,l,b}^V \circ \rho_{a',l',b'}^V = \rho_{a+a',l+l',b+b'}^V$, it is enough to show that $Q_{a,l,b}^V$ is jointly continuous in all 3 parameters. Let $\rho_{a,l}^L$ be the left action and $\rho_{l,b}^R$ be the right action. Then we have $Q_{a,l,b}^V = Q_{a,l_1}^L \circ Q_{l_2,b}^R$ for any $a, b, l, l_1, l_2 \in \mathbb{R}_{>0}$ with $l_1 + l_2 = l$.

Let $\varepsilon > 0$ and $v \in V$. We show that $Q_{a,l,b}^V$ is continuous at $(a_0, l_0, b_0) \in (\mathbb{R}_{>0})^3$. Let us fix $0 < l'_0 < l_0$. For $l > l'_0$ we have the estimate

$$\begin{aligned} \|(Q_{a,l,b}^V - Q_{a_0,l_0,b_0}^V)v\| &= \left\| \left((Q_{a,l-l'_0}^L - Q_{a_0,l_0-l'_0}^L)Q_{l'_0,b_0}^R + Q_{a,l-l'_0}^L(Q_{l'_0,b}^R - Q_{l'_0,b_0}^R) \right) v \right\| \\ &\leq \left\| (Q_{a,l-l'_0}^L - Q_{a_0,l_0-l'_0}^L)Q_{l'_0,b_0}^R v \right\| + \left\| Q_{a,l-l'_0}^L \right\| \cdot \left\| (Q_{l'_0,b}^R - Q_{l'_0,b_0}^R)v \right\|. \end{aligned} \quad (4.1.106)$$

Using the joint continuity of $Q_{a,l}^L$ at the point $(a_0, l_0 - l'_0)$ we can find $\delta_1 > 0$ such that for every $a, l \in \mathbb{R}_{>0}$ with $|a - a_0| + |(l - l'_0) - (l_0 - l'_0)| < \delta_1$ the first term in the second line of (4.1.106) is smaller than $\varepsilon/2$. For $a, l \in \mathbb{R}_{>0}$ with $|a - a_0| + |l - l_0| \leq \delta_1$ and $l'_0 \leq l$ there exists a $K > 0$ such that $\|Q_{a,l-l'_0}^L\| < K$ since $(a, l) \rightarrow \|Q_{a,l}^L\|$ is continuous. Finally since $Q_{l'_0,b}^R$ is continuous in b we can choose $\delta_2 > 0$ such that $\|(Q_{l'_0,b}^R - Q_{l'_0,b_0}^R)v\| < \varepsilon/(2K)$ for every $b \in \mathbb{R}_{>0}$ with $|b - b_0| < \delta_2$. Altogether we have that $\|(Q_{a,l,b}^V - Q_{a_0,l_0,b_0}^V)v\| < \varepsilon$ for every $a, l, b \in \mathbb{R}_{>0}$ with $|a - a_0| + |l - l_0| + |b - b_0| < \min\{\delta_1, \delta_2, l_0 - l'_0\}$. \square

Recall that the converse statement of Lemma 4.1.59 is not true. In Appendix 4.A we give an example of a bimodule in \mathcal{Hilb} which is not a left module.

Proposition 4.1.60. *Let A, B, C be regularised algebras in \mathcal{Hilb} , V a B - A -bimodule, W an A - C -bimodule, both coming from left/right modules as in Lemma 4.1.59. If $Q_{b,l,a}^V$ and $Q_{a,l,c}^W$ are epi for every $a, b, c, l \in \mathbb{R}_{>0}$, then the tensor product of bimodules*

$$V \otimes_A W \quad (4.1.107)$$

exists.

Proof. By the assumption V is a right A -module and W is a left A -module. Since $Q_{b,l,a}^V$ and $Q_{a,l,c}^W$ are epi, the Q 's for the corresponding right and left module structures on V and W are epi as well. Let $\pi : V \otimes W \rightarrow V \otimes_A W$ be the tensor product of these right and left modules, which exists by Proposition 4.1.58. Since the left and right actions for V and W commute as in (4.1.75), π is a coequaliser for (4.1.92).

By Lemma 4.1.56, tensoring with identity preserves cokernels, so the universal property of the cokernel induces a morphism $\bar{\rho}_{a,b,c,l} : B \otimes (V \otimes_A W) \otimes C \rightarrow V \otimes_A W$ from (4.1.93). Since V is a left B -module and W is a right C module, the morphism in (4.1.93) with parameter $a = 0$ exists and induces the morphism $\bar{\rho}_{0,b,c,l}$, which is clearly the $a \rightarrow 0$ limit

of $\bar{\rho}_{a,b,c,l}$ and which is clearly jointly continuous in its parameters. So altogether we have shown that $V \otimes_A W$ as the tensor product of bimodules exists. \square

For the rest of this section we restrict our attention to tensor products over strongly separable regularised algebras.

Lemma 4.1.61. *Let A, M, N, U be as in Lemma 4.1.57 and suppose that A is strongly separable. If $Q_{a,l}^M, Q_{a,l}^N$ and $Q_{a,l}^U$ are epi for every $a, l, b \in \mathbb{R}_{>0}$, then the following limits of the maps in (4.1.94) exist:*

$$\lim_{a,l \rightarrow 0} D_{a,l}^{M,N}, \quad \lim_{a,l \rightarrow 0} D_{a,l}^U. \quad (4.1.108)$$

The above idempotents are projectors onto the respective tensor products.

Proof. We will only show that the first limit exists, the second can be shown similarly. Abbreviate $p = (a, l)$ and $D_p = D_{a,l}^{M,N}$. Recall the morphism φ_p from Lemma 4.1.57. Let us identify $M \otimes N / \overline{\text{im}(\varphi_p)}$ with the orthogonal complement of $\overline{\text{im}(\varphi_p)}$ in $M \otimes N$ and let D_0 be the projection onto this closed subspace. Since on this subspace the left and right actions are identified, one has that $D_p = D_0 \circ (Q_{p_1}^M \otimes Q_{p_2}^N)$, for appropriate $p, p_1, p_2 \in (\mathbb{R}_{>0})^2$, so $\lim_{p \rightarrow 0} D_p = D_0$. By Proposition 4.1.58 we know that the image of D_0 is the tensor product $M \otimes_A N$. \square

Proposition 4.1.62. *Let A be a strongly separable algebra and let V be a B - A -bimodule and W an A - C -bimodule, such that $Q_{b,l,a}^V$ and $Q_{a,l,c}^W$ are epi for every $a, b, c, l \in \mathbb{R}_{>0}$.*

1. *If the limit $\lim_{a \rightarrow 0} \tilde{\rho}_{a,b,c,l}$ of the morphism in (4.1.99) exists, then the tensor product*

$$V \otimes_A W \quad (4.1.109)$$

exists and is a B - C -bimodule via $\tilde{\rho}_{0,b,c,l}^{V,W}$ as in Proposition 4.1.51.

2. *If V and W are transmissive then $V \otimes_A W$ exists and is transmissive as well.*

Proof. Similarly as in Lemma 4.1.61, for V and W we again have that $V \otimes_A W$ exists as a cokernel. What is left to be shown is that we get an induced action on $V \otimes_A W$. By Lemma 4.1.56, tensoring with identity preserves cokernels, so we get an induced morphism from (4.1.93), which coincides with the morphism in (4.1.99). By our assumptions the $a \rightarrow 0$ limit of this morphism exists. It can be shown by iterating Lemma 4.1.15 that the action on $V \otimes_A W$ is jointly continuous in the parameters, so $V \otimes_A W$ is a bimodule.

If the bimodules were transmissive then the morphism $\tilde{\rho}_{a,b,c,l}^{V,W}$ from (4.1.99) depends only on $a + b + c$ and not on the differences of these parameters. In particular its $a \rightarrow 0$ limit and hence the tensor product $V \otimes_A W$ exists and the tensor product is transmissive. \square

We have seen two different conditions for the existence of tensor product of bimodules. In the state sum construction we will use both and our main examples will satisfy both of these conditions, too. Note that the existence of tensor product does not automatically mean that it closes on bimodules with duals. For the natural candidate for the dual of $V \otimes_A W$ to exist one would need to establish the existence of the limits in (4.1.102).

4.2 Area-dependent QFTs with and without defects as functors

In this section we define the symmetric monoidal categories of two-dimensional bordisms with area, with and without defects. Using these, area-dependent QFTs are defined as symmetric monoidal functors from such bordisms to a suitable target category \mathcal{S} . In the case without defects we classify such functors in terms of commutative regularised Frobenius algebras in \mathcal{S} , mirroring the result for two-dimensional topological field theories.

Below, by *manifold* we will always mean an oriented smooth manifold.

4.2.1 Bordisms with area and aQFTs

We first recall the definition of the category of 2-dimensional oriented bordisms [Koc, Car], and then extend this definition to include an assignment of an area to each connected component of a bordism. Using these notions we define area-dependent QFT as a symmetric monoidal functor with depends continuously on the area.

Let S be a compact closed 1-manifold. A *collar* of S is an open neighbourhood of S in $S \times \mathbb{R}$. An *incoming (outgoing) collar* of S is the intersection of $S \times [0, +\infty)$ (respectively $S \times (-\infty, 0]$) with a collar of S . Let S^{rev} denote S with the reversed orientation. A *surface* is a compact 2-dimensional manifold. A *boundary parametrisation* of a surface Σ is:

1. A pair of compact closed 1-manifolds S and T .
2. A choice of an incoming collar U of S and an outgoing collar V of T .
3. A pair of orientation preserving smooth embeddings

$$\phi_{\text{in}} : U \hookrightarrow \Sigma \hookleftarrow V : \phi_{\text{out}}, \quad (4.2.1)$$

such that $\phi_{\text{in}} \sqcup \phi_{\text{out}}$ maps $(S \times \{0\})^{\text{rev}} \sqcup (T \times \{0\})$ diffeomorphically to $\partial\Sigma$.

For two compact closed 1-manifolds S, T , a *bordism* $\Sigma : S \rightarrow T$ is a surface Σ together with a boundary parametrisation. The *in-out cylinder* over S is the bordism $S \times [0, 1] : S \rightarrow S$.

Let $\Sigma : S \rightarrow T$ be a bordism as in (4.2.1) and let $\Sigma' : S \rightarrow T$ with

$$\psi_{\text{in}} : X \hookrightarrow \Sigma' \hookleftarrow Y : \psi_{\text{out}} \quad (4.2.2)$$

be another bordism. The two bordisms $\Sigma, \Sigma' : S \rightarrow T$ are *equivalent* if there exist an orientation preserving diffeomorphism $f : \Sigma \rightarrow \Sigma'$, as well as incoming and outgoing collars C and D of S and T contained in the collars U, X and V, Y , respectively, such that the diagram

$$\begin{array}{ccccc}
 & & U & \xrightarrow{\varphi_{\text{in}}} & \Sigma & \xleftarrow{\varphi_{\text{out}}} & V & & \\
 & \nearrow & & & \downarrow f & & & \nwarrow & \\
 C & & & & & & & & D \\
 & \searrow & & & & & & \swarrow & \\
 & & X & \xrightarrow{\psi_{\text{in}}} & \Sigma' & \xleftarrow{\psi_{\text{out}}} & Y & &
 \end{array} \quad (4.2.3)$$

commutes.

Given two bordisms $\Sigma : S \rightarrow T$ and $\Xi : T \rightarrow W$, we define $\Xi \circ \Sigma : S \rightarrow W$ to be the surface glued using the boundary parametrisations ϕ_{out}^Σ and ϕ_{in}^Ξ and with ϕ_{in}^Σ and ϕ_{out}^Ξ parametrising the remaining boundary. The composition $[\Xi] \circ [\Sigma] := [\Xi \circ \Sigma] : S \rightarrow W$ is well defined, that is, it is independent of the choice of representatives Ξ, Σ of the classes to be glued.

The category of bordisms \mathcal{Bord}_2 has compact closed 1-manifolds as objects and equivalence classes of bordisms as morphisms.

The category \mathcal{Bord}_2 becomes a \dagger -category as follows. Let the functor $(-)^{\dagger} : \mathcal{Bord}_2 \rightarrow \mathcal{Bord}_2$ be identity on objects. Let $S \in \mathcal{Bord}_2$ and let us define the inversions

$$\begin{aligned} \iota_S : S \times \mathbb{R} &\rightarrow S \times \mathbb{R} \\ (s, t) &\mapsto (s, -t) . \end{aligned} \quad (4.2.4)$$

Let $\Sigma : S \rightarrow T$ be a bordism with boundary parametrisation maps as in (4.2.1). We define $(\Sigma)^{\dagger} : T \rightarrow S$ to be the bordism

$$\phi'_{\text{in}} := \phi_{\text{out}} \circ \iota_T : \iota_T(V) \hookrightarrow \Sigma^{\text{rev}} \hookleftarrow \iota_S(U) : \phi'_{\text{out}} := \phi_{\text{in}} \circ \iota_S , \quad (4.2.5)$$

with reversed orientation and new boundary parametrisation maps ϕ'_{in} and ϕ'_{out} .

After this quick review we can introduce the bordism category we are interested in:

Definition 4.2.1. A *bordism with area* $(\Sigma, \mathcal{A} : \pi_0(\Sigma) \rightarrow \mathbb{R}_{\geq 0}) : S \rightarrow T$ consists of a bordism $\Sigma : S \rightarrow T$ and an *area map* \mathcal{A} . The value of the area map is always strictly positive, except on connected components equivalent to in-out cylinders, where it is allowed to take value 0 as well. The value $\mathcal{A}(c)$ for $c \in \pi_0(\Sigma)$ is called the *area* of the component c .

Two bordisms with area $(\Sigma, \mathcal{A}), (\Sigma', \mathcal{A}') : S \rightarrow T$ are *equivalent* if the underlying bordisms are equivalent with diffeomorphism $f : \Sigma \rightarrow \Sigma'$ and if the following diagram commutes:

$$\begin{array}{ccc} \pi_0(\Sigma) & & \\ \downarrow f_* & \searrow \mathcal{A} & \\ & & \mathbb{R}_{\geq 0} , \\ & \nearrow \mathcal{A}' & \\ \pi_0(\Sigma') & & \end{array} \quad (4.2.6)$$

where $f_* : \pi_0(\Sigma) \rightarrow \pi_0(\Sigma')$ is the map induced by f .

Remark 4.2.2. Allowing zero area for connected components which are equivalent to in-out cylinders will ensure that the category of bordisms with area defined below has identities. Allowing zero area for all surfaces, in particular for “in-in” and “out-out” cylinders, would make state spaces of corresponding area-dependent quantum field theories finite-dimensional, see Remark 4.2.11 below. Requiring all surface components to have strictly positive area and adding identities to the category by hand would – at least in the example that the area-dependent theory takes values in \mathcal{Hilb} and under some natural assumptions – not give a richer theory. Hence we opted for the definition above.

Given two bordisms with area $(\Sigma, \mathcal{A}_\Sigma) : X \rightarrow Y$ and $(\Xi, \mathcal{A}_\Xi) : Y \rightarrow Z$, the *glued bordism with area* $(\Xi \circ \Sigma, \mathcal{A}_{\Xi \circ \Sigma}) : X \rightarrow Z$ is the glued bordism together with the new area map $\mathcal{A}_{\Xi \circ \Sigma}$ defined by assigning to each new connected component the sum of areas of the connected components which were glued together to build up the new connected component.

Let $[(\Xi, \mathcal{A}_\Xi)] : T \rightarrow W$ and $[(\Sigma, \mathcal{A}_\Sigma)] : S \rightarrow T$ be equivalence classes of bordisms with area. The composition $[(\Xi, \mathcal{A}_\Xi)] \circ [(\Sigma, \mathcal{A}_\Sigma)] := [(\Xi \circ \Sigma, \mathcal{A}_{\Xi \circ \Sigma})] : S \rightarrow W$ is again independent of the choice of representatives (Ξ, \mathcal{A}_Ξ) , $(\Sigma, \mathcal{A}_\Sigma)$ of the classes to be glued. In the following we will by abuse of notation write the same symbol (Σ, \mathcal{A}) for a bordism with area (Σ, \mathcal{A}) and its equivalence class $[(\Sigma, \mathcal{A})]$.

Definition 4.2.3. The *category of bordisms with area* \mathcal{Bord}_2^{area} has the same objects as \mathcal{Bord}_2 and equivalence classes of bordisms with area as morphisms.

Both \mathcal{Bord}_2 and \mathcal{Bord}_2^{area} are symmetric monoidal categories with tensor product on objects and morphisms given by disjoint union. The identities and the symmetric structure are given by equivalence classes of in-out cylinders (with zero area). There is a forgetful functor

$$F : \mathcal{Bord}_2^{area} \rightarrow \mathcal{Bord}_2, \quad (4.2.7)$$

which forgets the area map.

Next we introduce the following topology on hom-sets of \mathcal{Bord}_2^{area} . Fix a bordism $\Sigma : S \rightarrow T$ in \mathcal{Bord}_2 . Define the subset $U_\Sigma \subset \mathcal{Bord}_2^{area}(S, T)$ as

$$U_\Sigma := F^{-1}(\Sigma) = \{(\Sigma, \mathcal{A}) \mid \mathcal{A} : \pi_0(\Sigma) \rightarrow \mathbb{R}_{\geq 0}\} \cong (\mathbb{R}_{>0})^{N_n} \times (\mathbb{R}_{\geq 0})^{N_c}, \quad (4.2.8)$$

where N_c is the number of connected components of Σ equivalent to a cylinder over a connected 1-manifold and $N_n = |\pi_0(\Sigma)| - N_c$. The topology on U_Σ is that of $(\mathbb{R}_{>0})^{N_n} \times (\mathbb{R}_{\geq 0})^{N_c}$. We define the topology on $\mathcal{Bord}_2^{area}(S, T)$ to be the disjoint union topology of the sets U_Σ . One can quickly convince oneself of the following fact:

Lemma 4.2.4. *The composition and the tensor product of \mathcal{Bord}_2^{area} are jointly continuous.*

After these preparations we can finally define:

Definition 4.2.5. Let \mathcal{S} be a symmetric monoidal category whose hom-sets are topological spaces and whose composition is separately continuous. An *area-dependent quantum field theory with values in \mathcal{S}* or *aQFT* in short is a symmetric monoidal functor $\mathcal{Z} : \mathcal{Bord}_2^{area} \rightarrow \mathcal{S}$, such that for every $S, T \in \mathcal{Bord}_2^{area}$ the map

$$\begin{aligned} \mathcal{Z}_{S,T} : \mathcal{Bord}_2^{area}(S, T) &\rightarrow \mathcal{S}(\mathcal{Z}(S), \mathcal{Z}(T)) \\ (\Sigma, \mathcal{A}) &\mapsto \mathcal{Z}(\Sigma, \mathcal{A}) \end{aligned} \quad (4.2.9)$$

is continuous.

The continuity requirement can equivalently be stated as follows. For every bordism $\Sigma : S \rightarrow T$ in \mathcal{Bord}_2 , the map

$$\begin{aligned} U_\Sigma &\cong (\mathbb{R}_{>0})^{N_n} \times (\mathbb{R}_{\geq 0})^{N_c} \rightarrow \mathcal{S}(\mathcal{Z}(S), \mathcal{Z}(T)) \\ (\mathcal{A}(x))_{x \in \pi_0(\Sigma)} &\mapsto \mathcal{Z}(\Sigma, \mathcal{A}) \end{aligned} \quad (4.2.10)$$

is continuous.

The following lemma shows that it is enough to require this continuity condition to hold for cylinders with area. The proof is similar to the proof of Part 4 of Lemma 4.1.2 and we omit it.

Lemma 4.2.6. *Let $\mathcal{Z} : \mathcal{Bord}_2^{area} \rightarrow \mathcal{S}$ be a symmetric monoidal functor and for every $S \in \mathcal{Bord}_2^{area}$ let $(S \times [0, 1], \mathcal{A})$ denote a cylinder with area. If for every $S \in \mathcal{Bord}_2^{area}$ the assignment*

$$\begin{aligned} (\mathbb{R}_{\geq 0})^{|\pi_0(S)|} &\rightarrow \mathcal{S}(\mathcal{Z}(S), \mathcal{Z}(S)) \\ (\mathcal{A}(x))_{x \in \pi_0(S)} &\mapsto \mathcal{Z}(S \times [0, 1], \mathcal{A}), \end{aligned} \quad (4.2.11)$$

is continuous, then \mathcal{Z} is an aQFT.

aQFTs together with natural transformations form a category $\text{aQFT}(\mathcal{S})$. Assume that for $\mathcal{Z}_1, \mathcal{Z}_2 \in \text{aQFT}(\mathcal{S})$ and all $S, T \in \mathcal{Bord}_2^{area}$ the map

$$\begin{aligned} (\mathcal{Z}_1 \otimes \mathcal{Z}_2)_{S,T} : \mathcal{Bord}_2^{area}(S, T) &\rightarrow \mathcal{S}(\mathcal{Z}_1(S) \otimes \mathcal{Z}_2(S), \mathcal{Z}_1(T) \otimes \mathcal{Z}_2(T)) \\ (\Sigma, \mathcal{A}) &\mapsto \mathcal{Z}_1(\Sigma, \mathcal{A}) \otimes \mathcal{Z}_2(\Sigma, \mathcal{A}) \end{aligned} \quad (4.2.12)$$

is continuous. Then (4.2.12) defines an aQFT which we denote with $\mathcal{Z}_1 \otimes \mathcal{Z}_2$. If the continuity condition (4.2.12) holds for every $\mathcal{Z}_1, \mathcal{Z}_2 \in \text{aQFT}(\mathcal{S})$ then $\text{aQFT}(\mathcal{S})$ becomes a symmetric monoidal category. For example, combining Lemma 4.1.15 and Lemma 4.2.6 we see that $\text{aQFT}(\mathcal{Hilb})$ is symmetric monoidal.

The category \mathcal{Bord}_2^{area} becomes a \dagger -category via the functor which is the same as (4.2.5) on the bordisms and which does not change the area maps. Following the terminology of [Tur1, Sec. 5.2] we define:

Definition 4.2.7. Let us assume that \mathcal{S} is a \dagger -category. We call an aQFT $\mathcal{Z} : \mathcal{Bord}_2^{area} \rightarrow \mathcal{S}$ *Hermitian*, if the diagram

$$\begin{array}{ccc} \mathcal{Bord}_2^{area} & \xrightarrow{\mathcal{Z}} & \mathcal{S} \\ (-)^\dagger \downarrow & & \downarrow (-)^\dagger \\ \mathcal{Bord}_2^{area} & \xrightarrow{\mathcal{Z}} & \mathcal{S} \end{array} \quad (4.2.13)$$

commutes.

4.2.2 Equivalence of aQFTs and commutative RFAs

Let $\mathbb{S}_{n,m}^2 : (\mathbb{S}^1)^{\sqcup m} \rightarrow (\mathbb{S}^1)^{\sqcup n}$ denote the $(n + m)$ -holed sphere with m ingoing and n outgoing boundary components and let $(\mathbb{S}_{n,m}^2, a) : (\mathbb{S}^1)^{\sqcup m} \rightarrow (\mathbb{S}^1)^{\sqcup n}$ denote the corresponding

bordism with area a . Let us consider the following family of bordisms:

$$\bar{\eta}_a := (\mathbb{S}_{1,0}^2, a), \quad \bar{\varepsilon}_a := (\mathbb{S}_{0,1}^2, a), \quad \bar{\mu}_a := (\mathbb{S}_{1,2}^2, a), \quad \bar{\Delta}_a := (\mathbb{S}_{2,1}^2, a), \quad (4.2.14)$$

for $a \in \mathbb{R}_{>0}$. In addition, it is useful to set

$$\bar{P}_a := (\mathbb{S}_{1,1}^2, a). \quad (4.2.15)$$

Lemma 4.2.8. *The morphisms in (4.2.14) endow $\mathbb{S}^1 \in \mathcal{B}ord_2^{area}$ with the structure of a commutative regularised Frobenius algebra in $\mathcal{B}ord_2^{area}$.*

Proof. Checking the algebraic relations (4.1.2), (4.1.3), (4.1.12), (4.1.13) and (4.1.16) of an RFA and commutativity is analogous to the case of ordinary Frobenius algebras, see e.g. [Koc, Sec. 3.1]. The morphism P_a from Part 2 in Definition 4.1.1 is now given by \bar{P}_a in (4.2.15).

The limit $\lim_{a \rightarrow 0} \bar{P}_a = \text{id}_{\mathbb{S}^1}$ is immediate as the identities in $\mathcal{B}ord_2^{area}$ are cylinders with 0 area. The continuity condition in (4.1.4) follows equally directly from the definition of the topology on hom-sets in $\mathcal{B}ord_2^{area}$. \square

From the above lemma it is maybe not surprising that aQFTs with values in \mathcal{S} are in one-to-one correspondence with commutative RFAs in \mathcal{S} , in complete analogy to topological field theory. To give the precise statement and the equivalence functors, we need to introduce some notation.

Let $A \in \mathcal{S}$ be a commutative RFA. Let $a \in \mathbb{R}_{>0}$, $\mu_a^{(0)} := \eta_a$, $\mu_a^{(1)} := P_a$, $\mu_a^{(2)} := \mu_a$ and for $n \geq 3$

$$\mu_a^{(n)} := \mu_{a/2}^{(n-1)} \circ (\text{id}_{A \otimes (n-2)} \otimes \mu_{a/2}). \quad (4.2.16)$$

Let $\Delta_a^{(0)} := \varepsilon_a$, $\Delta_a^{(1)} := P_a$, $\Delta_a^{(2)} := \Delta_a$ and for $n \geq 3$

$$\Delta_a^{(n)} := (\text{id}_{A \otimes (n-2)} \otimes \Delta_{a/2}) \circ \Delta_{a/2}^{(n-1)}. \quad (4.2.17)$$

We will use the same notation for the structure maps (4.2.14) of the commutative RFA $\mathbb{S}^1 \in \mathcal{B}ord_2^{area}$.

For an object $S \in \mathcal{B}ord_2^{area}$ let

$$\mathcal{Z}^A(S) := \bigotimes_{x \in \pi_0(S)} A^{(x)}, \quad (4.2.18)$$

where $A^{(x)} = A$ for every $x \in \pi_0(S)$ and the superscript keeps track of tensor factors.

Let $S, T \in \mathcal{B}ord_2^{area}$, $(\Sigma_{g,b}, a) \in \mathcal{B}ord_2^{area}(S, T)$ a connected bordism with area a whose underlying surface is of genus g and has $b = |\pi_0(S)| + |\pi_0(T)|$ many boundary components. We say that $(\Sigma_{g,b}, a)$ is of *normal form*, if

$$(\Sigma_{g,b}, a) = \left[S \xrightarrow{\psi_S} (\mathbb{S}^1)^{\sqcup |\pi_0(S)|} \xrightarrow{\bar{\mu}_{a_1}^{(|\pi_0(S)|)}} \mathbb{S}^1 \xrightarrow{(\bar{\Delta}_{a_2/(2g)} \circ \bar{\mu}_{a_2/(2g)})^g} \mathbb{S}^1 \xrightarrow{\bar{\Delta}_{a_3}^{(|\pi_0(T)|)}} (\mathbb{S}^1)^{\sqcup |\pi_0(T)|} \xrightarrow{\psi_T} T \right] \quad (4.2.19)$$

for some $a_1, a_2, a_3 \in \mathbb{R}_{>0}$ such that $a_1 + a_2 + a_3 = a$ and orientation preserving diffeomorphisms ψ_S and ψ_T . Every connected bordism with area is equivalent to a bordism with area of normal form with the same area. By forgetting about the area, this is the normal form for ordinary bordisms, see e.g. [Koc, Sec. 1.4.16]. Let us pick a representative of $(\Sigma_{g,b}, a)$ which is of normal form and let

$$\mathcal{Z}^A(\Sigma_{g,b}, a) := \left[\begin{array}{c} \mathcal{Z}^A(S) \xrightarrow{\Psi_S} A^{\otimes |\pi_0(S)|} \xrightarrow{\mu_{a_1}^{(|\pi_0(S)|)}} A \\ A \xrightarrow{(\Delta_{a_2/(2g)} \circ \mu_{a_2/(2g)})^g} A \xrightarrow{\Delta_{a_3}^{(|\pi_0(T)|)}} A^{\otimes |\pi_0(T)|} \xrightarrow{\Psi_T} \mathcal{Z}^A(T) \end{array} \right], \quad (4.2.20)$$

where Ψ_S and Ψ_T denote the permutation of tensor factors induced by the bijections ψ_S and ψ_T respectively. For $g = 0$ the morphisms in the middle of the compositions in (4.2.19) and (4.2.20) are $\text{id}_{\mathbb{S}^1}$ and id_A respectively. For a bordism with area (Σ, \mathcal{A}) , where Σ is not necessarily connected we define

$$\mathcal{Z}^A(\Sigma, \mathcal{A}) := \otimes_{c \in \pi_0(\Sigma)} \mathcal{Z}^A(c, \mathcal{A}(c)). \quad (4.2.21)$$

Lemma 4.2.9.

1. Let $\mathcal{Z} \in \text{aQFT}(\mathcal{S})$. Then $\mathcal{Z}(\mathbb{S}^1)$, with structure maps given by the images of the bordisms (4.2.14) under \mathcal{Z} , is a commutative RFA.
2. Let $A \in \mathcal{S}$ be a commutative RFA. Then the assignments in (4.2.18) and (4.2.21) define an aQFT \mathcal{Z}^A .

Proof.

Part 1 follows directly from Lemma 4.2.8 and the continuity condition for \mathcal{Z} .

Part 2: Proving that \mathcal{Z}^A is a symmetric monoidal functor is similar to the case of topological field theories $\text{Bord}_2 \rightarrow \mathcal{S}$ [Abr, Thm. 3], and we omit it. The continuity condition (4.2.9) amounts to Lemma 4.1.2 Part 4. \square

Now consider the functor

$$\begin{aligned} G : \text{aQFT}(\mathcal{S}) &\rightarrow c\mathcal{RFrob}(\mathcal{S}) \\ \mathcal{Z} &\mapsto \mathcal{Z}(\mathbb{S}^1), \\ \left(\mathcal{Z} \xrightarrow{\theta} \mathcal{Z}' \right) &\mapsto \left(\mathcal{Z}(\mathbb{S}^1) \xrightarrow{\theta_{\mathcal{Z}(\mathbb{S}^1)}} \mathcal{Z}'(\mathbb{S}^1) \right). \end{aligned} \quad (4.2.22)$$

Theorem 4.2.10. *The functor G defined in (4.2.22) is an equivalence of categories.*

Proof. Let the inverse functor H be given by the assignments in Lemma 4.2.9 Part 2. Then it is easy to see that $G \circ H \cong \text{id}_{c\mathcal{RFrob}(\mathcal{S})}$. The rest of the proof is very similar to the proof of [Abr, Thm. 3], on the equivalence between 2-dimensional topological field theories and commutative Frobenius algebras, and we omit it. \square

Remark 4.2.11. If all zero area limits of $\mathcal{Z} \in \text{aQFT}(\mathcal{Hilb})$ exist, then the RFA $\mathcal{Z}(\mathbb{S}^1)$ is finite-dimensional. This follows from Theorem 4.2.10 and Proposition 4.1.17.

Proposition 4.2.12. Assume that \mathcal{S} is a symmetric monoidal category and that the conditions of Proposition 4.1.22 hold for every pair $A_1, A_2 \in c\mathcal{RFrob}(\mathcal{S})$. Then

1. the categories $c\mathcal{RFrob}(\mathcal{S})$ and $\text{aQFT}(\mathcal{S})$ are symmetric monoidal,
2. the functor G in (4.2.22) is an equivalence of symmetric monoidal categories.

Proof. As we already discussed after Proposition 4.1.22, $c\mathcal{RFrob}(\mathcal{S})$ is a symmetric monoidal category. The equivalence from Theorem 4.2.10 shows that the tensor product of aQFTs in (4.2.12) equally satisfies the continuity condition. Hence $\text{aQFT}(\mathcal{S})$ is monoidal (and clearly symmetric). It is easy to see that the equivalence G is symmetric monoidal. \square

Combining the above proposition with Proposition 4.1.23, we get:

Corollary 4.2.13. The categories $\text{aQFT}(\mathcal{Vect}^{\text{fd}})$ and $\text{aQFT}(\mathcal{Hilb})$ are symmetric monoidal.

Corollary 4.2.14. The restriction of the functor G in (4.2.22) to the category of Hermitian aQFTs with values in \mathcal{S} gives an equivalence to the category of \dagger -RFAs in \mathcal{S} .

Corollary 4.1.31 together with Corollary 4.2.14 shows that a Hermitian aQFT in \mathcal{Hilb} is determined by a countable family of numbers $\{\epsilon_i, \sigma_i\}_{i \in I}$ satisfying convergence conditions given in Corollary 4.1.31.

4.2.3 Bordisms and aQFTs with defects

We start by recalling some notions from field theories with defects [DKR, Car, CRS]. Let D_1 and D_2 denote sets, which we call *labels for defect lines* and *phases*, and $s, t : D_1 \rightarrow D_2$ maps of sets which we call *source* and *target* respectively. These maps describe the possible geometric configurations of defect lines and surface components, which we will explain in the following in more detail. We refer to this set of data as a *set of defect conditions* and write $\mathbb{D} := (D_1, D_2, s, t)$.

Using a fixed set of defect conditions \mathbb{D} we introduce some notions. Let $k \in \{1, 2\}$. A *k-manifold with defects* is a compact k -manifold X , together with (see Figure 4.2)

1. a finite decomposition into $(k-1)$ - and k -dimensional submanifolds $X = X_{[k-1]} \cup X_{[k]}$ and
2. maps $d_l : \pi_0(X_{[k+l-2]}) \rightarrow D_l$ for $l = 1, 2$,

such that the following hold.

- $X_{[k-1]} \cap X_{[k]} = \emptyset$,
- $X_{[k-1]}$ is an embedded oriented $(k-1)$ -dimensional submanifold, which is either closed or $\partial X_{[k-1]} \subset \partial X$

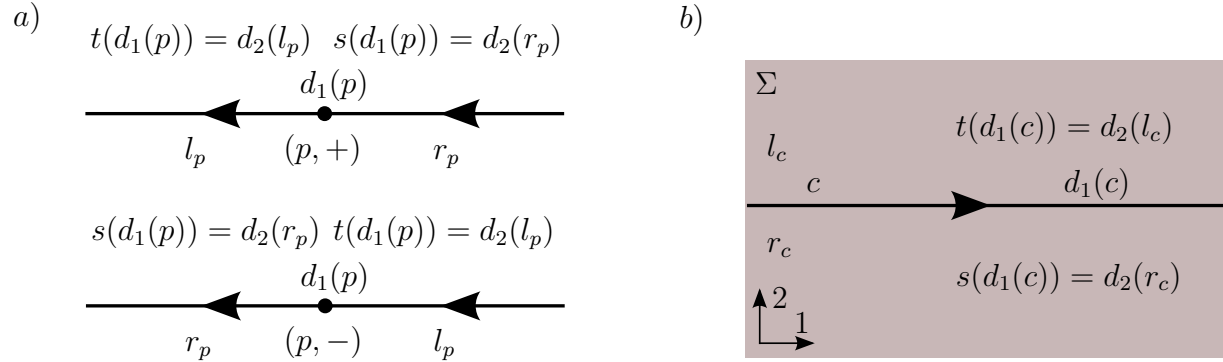


Figure 4.2: A neighbourhood of the submanifold $X_{[k-1]}$ in a k -manifold with defects.

a) case $k = 1$: The arrows show the orientation of the 1-manifold S , $(p, +)$ denotes a positively oriented point $p \in S_{[0]}$ and $(p, -)$ denotes a negatively oriented point. These orientations allow us to define a left and right side $l_p, r_p \in \pi_0(S_{[1]})$ of p . We require for $(p, +)$ that $t(d_1(p)) = d_2(l_p)$ and that $s(d_1(p)) = d_2(r_p)$ and for $(p, -)$ that the s and t are exchanged: $s(d_1(p)) = d_2(l_p)$ and $t(d_1(p)) = d_2(r_p)$.

b) case $k = 2$: The arrows marked with 1 and 2 show the orientation of the surface Σ , the arrow on the line shows the orientation of $c \in \pi_0(\Sigma_{[1]})$. The orientations of $\Sigma_{[1]}$ and $\Sigma_{[2]}$ allow us to define left and right side $l_c, r_c \in \pi_0(\Sigma_{[2]})$ of c . We require that for a defect line $d_1(c)$ the phase label on its right side is $s(d_1(c)) = d_2(r_c)$ and that the phase label on its left side is $t(d_1(c)) = d_2(l_c)$.

- $X_{[k]}$ is a k -dimensional submanifold with orientation induced from X and
- d_1 and d_2 are compatible with the maps s and t as shown in Figure 4.2.

We call a closed 1-manifold with defects a *defect object* and a 2-manifold with defects a *surface with defects*. In particular, for a defect object S the set $S_{[0]}$ is a finite set of distinct oriented points. For a surface with defects Σ , every connected component of $\Sigma_{[1]}$ is the image of a smooth embedding $[-1, 1] \rightarrow \Sigma$ or $\mathbb{S}^1 \rightarrow \Sigma$.

A *morphism of surfaces with defects* $f : \Sigma \rightarrow \Sigma'$ is an orientation preserving smooth map of surfaces such that the restrictions $f|_{\Sigma_{[k]}}$ map the submanifolds $\Sigma_{[k]}$ onto $\Sigma'_{[k]}$, they are diffeomorphisms onto their image, and they make the diagrams

$$\begin{array}{ccc}
 \pi_0(\Sigma_{[k]}) & \xrightarrow{f_*} & \pi_0(\Sigma'_{[k]}) \\
 & \searrow d_k \quad \swarrow d'_k & \\
 & D_k &
 \end{array} \tag{4.2.23}$$

commute for $k = 1, 2$.

Let S be a defect object. A *collar* of S is a surface with defects $C = C_{[1]} \cup C_{[2]}$ such that

- C is an open neighbourhood of $S \times \{0\}$ in $S \times \mathbb{R}$ and

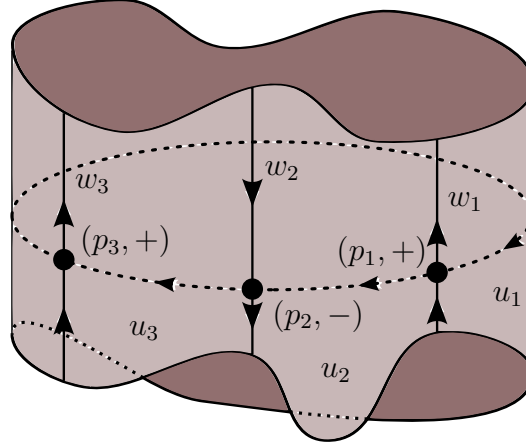


Figure 4.3: A collar $C = C_{[1]} \cup C_{[2]}$ of S . The dotted circle in the middle shows $S \times \{0\}$ with its orientation, the dots with labels (p_i, \pm) for $i = 1, \dots, 3$ show $S_{[1]}$ with orientations, the straight lines with the arrows show the submanifold $C_{[1]}$ with its orientation. In the figure both $C_{[1]}$ and $C_{[2]}$ have 3 connected components, the labels w_i and u_i for $i = 1, \dots, 3$ show the values of d_1 and d_2 respectively.

- $C_{[1]}$ is the intersection of $S_{[1]} \times \mathbb{R}$ with C with orientation induced from the orientation of $S_{[1]}$ as shown in Figure 4.3,
- $d_k(c) = d_k(c \cap (S \times \{0\}))$ for $c \in \pi_0(C_{[k]})$ and $k = 1, 2$.

An example of a collar is shown in Figure 4.3. An *ingoing (outgoing) collar with defects* is the intersection of a collar with defects and $S \times [0, +\infty)$ (respectively $S \times (-\infty, 0]$).

A *boundary parametrisation* of a surface with defects Σ consists of the following:

1. A pair of defect objects S and T .
2. An ingoing collar U of S and an outgoing collar V of T .
3. A pair of morphisms of surfaces with defects

$$\phi_{\text{in}} : U \hookrightarrow \Sigma \leftarrow V : \phi_{\text{out}} , \quad (4.2.24)$$

We require that $\phi_{\text{in}} \sqcup \phi_{\text{out}}$ maps $(S \times \{0\})^{\text{rev}} \sqcup T \times \{0\}$ diffeomorphically onto $\partial\Sigma$.

A *bordism with defects* $\Sigma : S \rightarrow T$ is a surface Σ together with a boundary parametrisation. The *in-out cylinder* over S is the bordism with defects $S \times [0, 1] : S \rightarrow S$. We define the equivalence of bordisms with defects similarly as in Section 4.2.1, now using diffeomorphisms of surfaces with defects that are compatible with the boundary parametrisation on common collars of defect objects. Given two bordisms with defects $\Sigma : S \rightarrow T$ and $\Xi : T \rightarrow W$, we can glue them along the boundary parametrisations to obtain a bordism with defects $\Xi \circ \Sigma : S \rightarrow W$. This glueing procedure is compatible with the above

notion of equivalence. The category of bordisms with defects $\mathcal{B}ord_{2,\mathbb{D}}^{def}$ has defect objects as objects and equivalence classes of bordisms with defects as morphisms.

After this preparation we turn to bordisms with area and defects.

Definition 4.2.15. A bordism with area and defects $(\Sigma, \mathcal{A}, \mathcal{L}) : S \rightarrow T$ consists of a bordism with defects $\Sigma : S \rightarrow T$, an area map $\mathcal{A} : \pi_0(\Sigma_{[2]}) \rightarrow \mathbb{R}_{\geq 0}$ and a length map $\mathcal{L} : \pi_0(\Sigma_{[1]}) \rightarrow \mathbb{R}_{\geq 0}$, which are only allowed to take value 0 on connected components of Σ equivalent to in-out cylinders with defects. The value $\mathcal{A}(c)$ for $c \in \pi_0(\Sigma_{[2]})$ is called the area of the component c and the value of $\mathcal{L}(x)$ for $x \in \pi_0(\Sigma_{[1]})$ is called the length of the defect line x .

Two bordisms with area and defects $(\Sigma, \mathcal{A}, \mathcal{L}), (\Sigma', \mathcal{A}', \mathcal{L}') : S \rightarrow T$ are equivalent if the underlying bordisms with defects are equivalent with diffeomorphism $f : \Sigma \rightarrow \Sigma'$ and if the following diagrams commute:

$$\begin{array}{ccc} \pi_0(\Sigma_{[2]}) & \xrightarrow{\mathcal{A}} & \mathbb{R}_{\geq 0} \\ f_* \downarrow & \nearrow \mathcal{A}' & \\ \pi_0(\Sigma'_{[2]}) & & \end{array} \quad \text{and} \quad \begin{array}{ccc} \pi_0(\Sigma_{[1]}) & \xrightarrow{\mathcal{L}} & \mathbb{R}_{\geq 0} \\ f_* \downarrow & \nearrow \mathcal{L}' & \\ \pi_0(\Sigma'_{[1]}) & & \end{array} . \quad (4.2.25)$$

Given two bordisms with area and defects $(\Sigma, \mathcal{A}_\Sigma, \mathcal{L}_\Sigma) : X \rightarrow Y$ and $(\Xi, \mathcal{A}_\Xi, \mathcal{L}_\Xi) : Y \rightarrow Z$, the glued bordism with area and defects $(\Xi \circ \Sigma, \mathcal{A}_{\Xi \circ \Sigma}, \mathcal{L}_{\Xi \circ \Sigma}) : X \rightarrow Z$ is the glued bordism with defects together with the new area map $\mathcal{A}_{\Xi \circ \Sigma}$ defined by assigning to each new connected component of $(\Xi \circ \Sigma)_{[2]}$ the sum of areas of the connected components which were glued together to build up the new connected component and with a similarly defined new length map $\mathcal{L}_{\Xi \circ \Sigma}$. As before, this glueing procedure is compatible with the above notion of equivalence.

Definition 4.2.16. The category of bordisms with area and defects $\mathcal{B}ord_{2,\mathbb{D}}^{area, def}$ has the same objects as $\mathcal{B}ord_{2,\mathbb{D}}^{def}$ and equivalence classes of bordisms with area and defects as morphisms.

Both $\mathcal{B}ord_{2,\mathbb{D}}^{def}$ and $\mathcal{B}ord_{2,\mathbb{D}}^{area, def}$ are symmetric monoidal categories with tensor product on objects and morphisms given by disjoint union. The identities and the symmetric structure are given by equivalence classes of in-out cylinders (with zero area and length).

We introduce a similar topology on hom-sets of $\mathcal{B}ord_{2,\mathbb{D}}^{area, def}$ as for $\mathcal{B}ord_{2,\mathbb{D}}^{area}$ only that we now need to take into account the topology related to the lengths.

Definition 4.2.17. Let \mathcal{S} be a symmetric monoidal category whose hom-sets are topological spaces and composition is separately continuous. A defect area-dependent quantum field theory with values in \mathcal{S} (or defect aQFT for short) is a symmetric monoidal functor $\mathcal{Z} : \mathcal{B}ord_{2,\mathbb{D}}^{area, def} \rightarrow \mathcal{S}$, such that for every $S, T \in \mathcal{B}ord_{2,\mathbb{D}}^{area, def}$ the map

$$\begin{aligned} \mathcal{Z}_{S,T} : \mathcal{B}ord_{2,\mathbb{D}}^{area, def}(S, T) &\rightarrow \mathcal{S}(\mathcal{Z}(S), \mathcal{Z}(T)) \\ (\Sigma, \mathcal{A}, \mathcal{L}) &\mapsto \mathcal{Z}(\Sigma, \mathcal{A}, \mathcal{L}) \end{aligned} \quad (4.2.26)$$

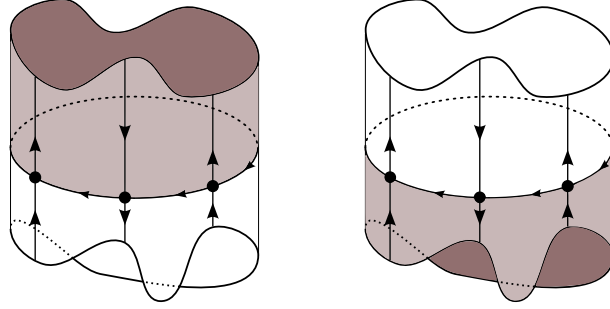


Figure 4.4: Collars on the opposite sides of $\mathbb{S}^1 \times \{0\}$.

is continuous.

Remark 4.2.18. Checking the continuity condition in (4.2.26) can be done by checking only for cylinders, similarly as in Lemma 4.2.6 for aQFTs without defects. To see this, one needs to cut surfaces with defects along circles which intersect with every defect line.

We turn the categories $\mathcal{Bord}_{2,\mathbb{D}}^{def}$ and $\mathcal{Bord}_{2,\mathbb{D}}^{area,def}$ into \dagger -categories in a similar way as \mathcal{Bord}_2 and \mathcal{Bord}_2^{area} in Section 4.2.1. That is, if $M : S \rightarrow T$ is a bordism with area and defects, then $M^\dagger : T \rightarrow S$ is a bordism with area and defects with $(M^\dagger)_{[k]} = M_{[k]}$ for $k = 1, 2$ with opposite orientation and with the same area maps and same defect labels. The boundary parametrisation is changed in the following way. The new collars are obtained from the old collars by extending the old ones and restricting them to the other side of $\mathbb{S}^1 \times \{0\}$ as illustrated in Figure 4.4. The boundary parametrisation maps are the old ones composed with the maps ι_S and ι_T from (4.2.4). We stress that in the definition of the dagger structure on $\mathcal{Bord}_{2,\mathbb{D}}^{area,def}$ we have not included an involution on the set of defect labels \mathbb{D} . This is important since we want the dagger to act as identity on objects. With these conventions it makes sense to consider $M^\dagger \circ M : T \rightarrow T$, which is relevant when considering reflection positivity, see e.g. [GJ, Ch. 6]. For a cylinder $C = S \times [0, 1]$ we have that $C^\dagger = C$.

Let us assume that \mathcal{S} is a dagger category. We call a defect aQFT $\mathcal{Z} : \mathcal{Bord}_{2,\mathbb{D}}^{area,def} \rightarrow \mathcal{S}$ *Hermitian* if it is compatible with the dagger structures.

In Section 4.3.5 we give a state-sum construction of defect aQFTs, and in Section 4.4.3 we discuss our main example, 2d YM theory with Wilson lines.

4.3 State-sum construction of aQFTs with defects

The state-sum construction of two-dimensional TFTs (see [BP, FHK] and e.g. [LP1, DKR]) has a straightforward generalisation to aQFTs which we investigate in this section. We start by giving the conditions on weights for plaquettes, edges and vertices in order to obtain state-sum aQFT without defects, and we explain the relation of these weights to RFAs, as well as the connection to the classification of aQFTs in terms of commutative RFAs assigned to \mathbb{S}^1 (Sections 4.3.1–4.3.3). Then we extend this state-sum construction

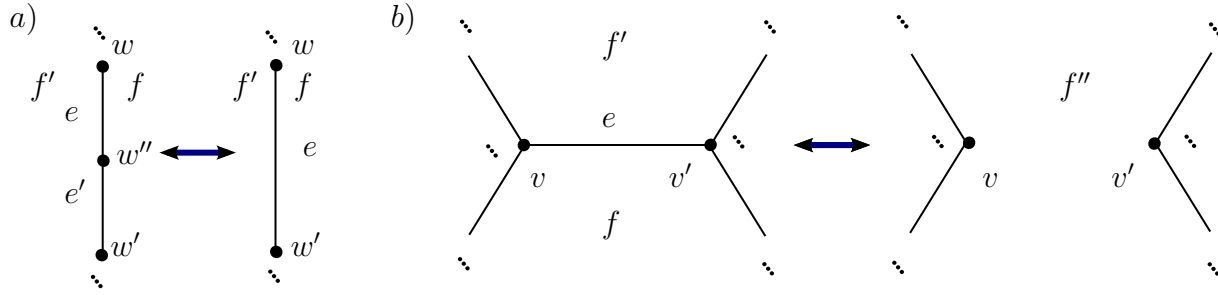


Figure 4.5: Elementary moves of PLCW decompositions with area. Figure a) shows edges e , e' and between faces f and f' . (The two faces are allowed to be the same.) When we remove the vertex w'' and the edge e' , the new area maps should be the same outside the shown region and such that the area of the connected component of the surface does not change. Figure b), shows an edge e between two faces f and f' . When we remove the edge e and merge the faces f and f' to f'' , the new area maps should again be the same outside the shown region and such that the area of the connected component of the surface does not change.

to aQFTs with defects and show that the weights for plaquettes traversed by defect lines are given by bimodules. We define the fusion of defect lines and show that it matches the tensor product of bimodules (Sections 4.3.4–4.3.7).

4.3.1 PLCW decompositions with area

In Section 4.3.2 we will use *PLCW decompositions* [Kir] to build aQFTs. For a compact surface Σ this consists of three sets Σ_0 , Σ_1 and Σ_2 whose elements are subsets of Σ . Their elements are called *vertices*, *edges* and *faces*. Faces are embeddings of polygons with $n \geq 1$ edges, edges are embeddings of intervals and vertices are just points in Σ . Faces are glued along edges so that vertices are glued to vertices. For example a PLCW decomposition of a cylinder $\mathbb{S}^1 \times [0, 1]$ could consist of a rectangle with two opposite edges glued together. From this one can obtain a PLCW decomposition of a torus $\mathbb{S}^1 \times \mathbb{S}^1$ by glueing together the other two opposite edges. For more details on PLCW decomposition we refer to [Kir] and for a short summary to Section 3.1.2.

We are going to need PLCW decompositions of surfaces with area, which we define now. Let (Σ, \mathcal{A}) be a surface with strictly positive area for each connected component and let Σ_0 , Σ_1 , Σ_2 be a PLCW decomposition of Σ . Let $\mathcal{A}_k : \Sigma_k \rightarrow \mathbb{R}_{>0}$ be maps for $k \in \{0, 1, 2\}$, which assign to vertices, edges and faces an *area*, such that for every connected component $x \in \pi_0(\Sigma)$ the sum of the areas of vertices, edges and faces of x is equal to its area $\mathcal{A}(x)$. A *PLCW decomposition of a surface with area* (Σ, \mathcal{A}) consists of a choice of Σ_k and \mathcal{A}_k for $k \in \{0, 1, 2\}$.

Definition 4.3.1. An *elementary move* on a PLCW decomposition of a surface is either

- removing or adding a bivalent vertex as shown in Figure 4.5 a), or
- removing or adding an edge as shown in Figure 4.5 b).

By [Kir, Thm. 7.4], any two PLCW decompositions can be related by these elementary moves. The elementary moves in Figure 4.5 map PLCW decompositions with area to PLCW decompositions with area.

4.3.2 State-sum construction without defects

Let us fix a symmetric monoidal idempotent complete category \mathcal{S} with symmetric structure σ which has topological spaces as hom-sets and separately continuous composition of morphisms.

Let $A \in \mathcal{S}$ be an object and consider the following families of morphisms

$$\zeta_a \in \mathcal{S}(A, A), \quad \beta_a \in \mathcal{S}(A^{\otimes 2}, \mathbb{I}) \quad \text{and} \quad W_a^n \in \mathcal{S}(\mathbb{I}, A^{\otimes n}) \quad (4.3.1)$$

for $a \in \mathbb{R}_{>0}$ and $n \in \mathbb{Z}_{\geq 1}$. We call β_a the *contraction* and W_a^n the *plaquette weights*. We will use the following graphical notation for these morphisms:

$$\zeta_a = \begin{array}{c} A \\ | \\ \diamond \\ | \\ A \end{array} \quad \beta_a = \begin{array}{c} \bullet \\ / \quad \backslash \\ A \quad A \end{array} \quad W_a^n = \begin{array}{c} A \quad A \quad \dots \quad A \\ | \quad | \quad \dots \quad | \\ \hline a; n \end{array} . \quad (4.3.2)$$

We introduce the morphisms $P_a, D_a : A \rightarrow A$ in order to be able to state the conditions these morphisms need to satisfy:

$$P_{a_1+a_2} := \begin{array}{c} \bullet \\ / \quad \backslash \\ \hline a_2; 2 \end{array} \quad \text{and} \quad D_{a_0+a_1+a_2+a_3} := \begin{array}{c} \bullet \quad \bullet \\ / \quad \backslash \quad / \quad \backslash \\ \hline a_0; 4 \end{array} \quad , \quad (4.3.3)$$

for every $a_0, a_1, a_2, a_3 \in \mathbb{R}_{>0}$ (it will follow from the axioms below that these compositions indeed depend on the sum of the parameters only).

Consider the following conditions on the morphisms in (4.3.1): for every $a, a_0, a_1, a_2, a_3 \in \mathbb{R}_{>0}$, and for every $n \in \mathbb{Z}_{\geq 1}$,

1. Cyclic symmetry:

$$\begin{array}{c} \dots \\ / \quad \backslash \\ \hline a; n \end{array} = \begin{array}{c} | \quad \dots \quad | \\ \hline a; n \end{array} \quad \text{and} \quad \begin{array}{c} \bullet \\ / \quad \backslash \\ \backslash \quad / \\ \bullet \end{array} = \begin{array}{c} \bullet \\ / \quad \backslash \\ A \quad A \end{array} . \quad (4.3.4)$$

2. Glueing plaquette weights:

$$\begin{array}{c} | \quad \dots \quad | \\ \hline a_1; n \end{array} \begin{array}{c} \bullet \\ / \quad \backslash \\ \hline a_2; m \end{array} = \begin{array}{c} | \quad \dots \quad | \\ \hline a_0 + a_1 + a_2; n + m - 2 \end{array} . \quad (4.3.5)$$

3. Removing a bubble:

$$\begin{array}{c} a_1 \\ \bullet \\ \diagup \quad \diagdown \\ \text{diamond} \\ \diagdown \quad \diagup \\ a_3; n+2 \end{array} = \begin{array}{c} \dots \\ a_1 + a_2 + a_3; n \end{array} . \quad (4.3.6)$$

4. “Moving ζ_a around”:

$$\begin{array}{c} a_1 \\ \text{diamond} \\ a_2; n \end{array} = \begin{array}{c} a_3 \\ \text{diamond} \\ a_1 + a_2 - a_3; n \end{array} \quad \text{and} \quad \begin{array}{c} a_2 \\ \bullet \\ \diagup \quad \diagdown \\ a_1 \quad a_3 \end{array} = \begin{array}{c} a_1 + a_2 - a_3 \\ \bullet \\ \diagup \quad \diagdown \\ a_1 \quad a_3 \end{array} . \quad (4.3.7)$$

5. $\lim_{a \rightarrow 0} P_a = \text{id}_A$ and the assignments

$$\begin{aligned} (\mathbb{R}_{\geq 0})^n &\rightarrow \mathcal{S}(A^{\otimes n}, A^{\otimes n}) \\ (a_1, \dots, a_n) &\mapsto P_{a_1} \otimes \dots \otimes P_{a_n} \end{aligned} \quad (4.3.8)$$

are jointly continuous for every $n \geq 1$.

6. The limit $\lim_{a \rightarrow 0} D_a$ exists.

Definition 4.3.2. We call the family of morphisms in (4.3.1) satisfying the above conditions *state-sum data* and denote it with

$$\mathbb{A} = (A, \zeta_a, \beta_a, W_a^n) . \quad (4.3.9)$$

Lemma 4.3.3. Let $\mathbb{A} = (A, \zeta_a, \beta_a, W_a^n)$ denote state-sum data. Then the assignments $a \mapsto \zeta_a$, $a \mapsto \beta_a$ and $a \mapsto W_a^n$ are continuous for every $n \geq 1$.

Proof. We only sketch that $a \mapsto W_a^n$ and $a \mapsto \zeta_a$ are continuous. By using Condition 2 we have that

$$\begin{array}{c} \dots \\ a; n \end{array} = \begin{array}{c} \dots \\ a - \varepsilon; n \end{array} \begin{array}{c} \bullet \\ \varepsilon_1 \\ \diagup \quad \diagdown \\ \varepsilon_2, 2 \end{array} = \begin{array}{c} \dots \\ a - \varepsilon; n \end{array} \begin{array}{c} \boxed{P_\varepsilon} \end{array} , \quad (4.3.10)$$

for every $a \geq \varepsilon \in \mathbb{R}_{>0}$ with $\varepsilon = \varepsilon_1 + \varepsilon_2$. So by separate continuity of the composition and Condition 5, the assignment $a \mapsto W_a^n$ is continuous. To see continuity of ζ_a we first use Conditions 2 and 4 and we get that

$$\zeta_a \circ P_{b+c} = \zeta_{a+b} \circ P_c \quad (4.3.11)$$

for every $a, b, c \in \mathbb{R}_{>0}$. Condition 5 now allows us to take the limit $c \rightarrow 0$, and continuity again follows from that of P_{b+c} . \square

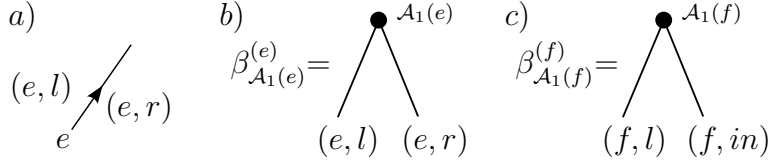


Figure 4.6: a) Left and right sides (e, l) and (e, r) of an inner edge e , determined by the orientation of Σ (paper orientation) and of e (arrow). b) Convention for connecting tensor factors belonging to edge sides (e, l) and (e, r) of an inner edge e with the tensor factors belonging to the morphism $\beta_{A_1(e)}^{(e)}$. c) Conventions for the labels of the tensor factors for an ingoing boundary edge f with $(f, l) \in E$.

Let us fix state-sum data \mathbb{A} using the notation of (4.3.9). In the rest of this section we define a symmetric monoidal functor $\mathcal{Z}_{\mathbb{A}} : \mathcal{Bord}_2^{area} \rightarrow \mathcal{S}$ using this data.

The next lemma is best proved after having established the relation between state-sum data and RFAs in Lemma 4.3.7 below, when it becomes a direct consequence of Lemma 4.3.10 and we omit the proof.

Lemma 4.3.4. *We have that*

$$D_a \circ D_b = D_{a+b} \quad (4.3.12)$$

for every $a, b \in \mathbb{R}_{\geq 0}$. In particular, the morphism $D_0 := \lim_{a \rightarrow 0} D_a \in \mathcal{S}(A, A)$ is idempotent.

Recall that we assumed that \mathcal{S} is idempotent complete, so the idempotent D_0 splits: let $Z(A) \in \mathcal{S}$ denote its image and let us write

$$D_0 = \left[A \xrightarrow{\pi_A} Z(A) \xrightarrow{\iota_A} A \right], \quad \left[Z(A) \xrightarrow{\iota_A} A \xrightarrow{\pi_A} Z(A) \right] = \text{id}_{Z(A)}, \quad (4.3.13)$$

We define the aQFT $\mathcal{Z}_{\mathbb{A}}$ on objects as follows: Let $S \in \mathcal{Bord}_2^{area}$. Then

$$\mathcal{Z}_{\mathbb{A}}(S) := \bigotimes_{x \in \pi_0(S)} Z(A)^{(x)}, \quad (4.3.14)$$

where $Z(A)^{(x)} = Z(A)$ and the superscript is used to label the tensor factors.

In the remainder of this section we give the definition of $\mathcal{Z}_{\mathbb{A}}$ on morphisms. Let $(\Sigma, \mathcal{A}) : S \rightarrow T$ be a bordism with area and let us assume that (Σ, \mathcal{A}) has no component with zero area. Choose a PLCW decomposition with area Σ_k, \mathcal{A}_k for $k \in \{0, 1, 2\}$ of the surface with area (Σ, \mathcal{A}) , such that the PLCW decomposition has exactly 1 edge on every boundary component. By this convention $\pi_0(S) \sqcup \pi_0(T)$ is in bijection with vertices on the boundary and with edges on the boundary.

Let us choose an edge for every face before glueing, which we call *marked edge*, and let us choose an orientation of every edge. For a face $f \in \Sigma_2$ which is an n_f -gon let us write (f, k) , $k = 1, \dots, n_f$ for the sides of f , where $(f, 1)$ denotes the marked edge of f , and the labeling proceeds counter-clockwise with respect to the orientation of f . We collect the sides of all faces into a set:

$$F := \{ (f, k) \mid f \in \Sigma_2, k = 1, \dots, n_f \}. \quad (4.3.15)$$

We double the set of edges by considering $\Sigma_1 \times \{l, r\}$, where “ l ” and “ r ” stand for left and right, respectively. Let $E \subset \Sigma_1 \times \{l, r\}$ be the subset of all (e, l) (resp. (e, r)), which have a face attached on the left (resp. right) side, cf. Figure 4.6 a). Thus for an inner edge $e \in \Sigma_1$ the set E contains both (e, l) and (e, r) , but for a boundary edge $e' \in \Sigma_1$ the set E contains either (e', l) or (e', r) . By construction of F and E we obtain a bijection

$$\Phi : F \xrightarrow{\sim} E, \quad (f, k) \mapsto (e, x), \quad (4.3.16)$$

where e is the k 'th edge on the boundary of the face f lying on the side x of e , counted counter-clockwise from the marked edge of f .

For every vertex $v \in \Sigma_0$ in the interior of Σ or on an ingoing boundary component of Σ choose a side of an edge $(e, x) \in E$ for which $v \in \partial(e)$. Let

$$V : \Sigma_0 \setminus \pi_0(T) \rightarrow E \quad (4.3.17)$$

be the resulting function.

To define $\mathcal{Z}_{\mathbb{A}}(\Sigma, \mathcal{A})$ we proceed with the following steps.

1. Let us introduce the tensor products

$$\begin{aligned} \mathcal{O}_F &:= \bigotimes_{(f,k) \in F} A^{(f,k)}, & \mathcal{O}_E &:= \bigotimes_{(e,x) \in E} A^{(e,x)}, \\ \mathcal{O}_{\text{in}} &:= \bigotimes_{b \in \pi_0(S)} A^{(b,\text{in})}, & \mathcal{O}_{\text{out}} &:= \bigotimes_{c \in \pi_0(T)} A^{(c,\text{out})}. \end{aligned} \quad (4.3.18)$$

Every tensor factor is equal to A , but the various superscripts will help us distinguish tensor factors in the source and target objects of the morphisms we define in the remaining steps.

2. Recall that by our conventions there is one edge in each boundary component and that we identified outgoing boundary edges with $\pi_0(T)$. Define the morphism

$$\mathcal{C} := \bigotimes_{e \in \Sigma_1 \setminus \pi_0(T)} \beta_{\mathcal{A}_1(e)}^{(e)} : \mathcal{O}_{\text{in}} \otimes \mathcal{O}_E \rightarrow \mathcal{O}_{\text{out}}, \quad (4.3.19)$$

where $\beta_{\mathcal{A}_1(e)}^{(e)} = \beta_{\mathcal{A}_1(e)}$, and where the tensor factors in $\mathcal{O}_{\text{in}} \otimes \mathcal{O}_E$ are assigned to those of $\beta_{\mathcal{A}_1(e)}$ according to Figure 4.6 b) and c).

3. Define the morphism

$$\mathcal{Y} := \prod_{v \in \Sigma_0 \setminus \pi_0(T)} \zeta_{\mathcal{A}_0(v)}^{(V(v))} \in \mathcal{S}(\mathcal{O}_E, \mathcal{O}_E), \quad (4.3.20)$$

where

$$\zeta_a^{(e,x)} = \text{id} \otimes \cdots \otimes \zeta_a \otimes \cdots \otimes \text{id} \in \mathcal{S}(\mathcal{O}_E, \mathcal{O}_E), \quad (4.3.21)$$

where ζ_a maps the tensor factor $A^{(e,x)}$ to itself.

4. Assign to every face $f \in \Sigma_2$ obtained from an n_f -gon the morphism

$$W_{\mathcal{A}_2(f)}^f = W_{\mathcal{A}_2(f)}^{(n_f)} : \mathbb{I} \rightarrow A_{(f,1)} \otimes \cdots \otimes A_{(f,n_f)} \quad (4.3.22)$$

and take their tensor product:

$$\mathcal{F} := \bigotimes_{f \in \Sigma_2} \left(W_{\mathcal{A}_2(f)}^f \right) : \mathbb{I} \rightarrow \mathcal{O}_F . \quad (4.3.23)$$

5. We will now put the above morphisms together to obtain a morphism $\mathcal{L} : \mathcal{A}_{\text{in}} \rightarrow \mathcal{A}_{\text{out}}$. Denote by Π_Φ the permutation of tensor factors induced by $\Phi : F \rightarrow E$,

$$\Pi_\Phi : \mathcal{O}_F \rightarrow \mathcal{O}_E . \quad (4.3.24)$$

Using this, we define

$$\mathcal{K} := \left[\mathbb{I} \xrightarrow{\mathcal{F}} \mathcal{O}_F \xrightarrow{\Pi_\Phi} \mathcal{O}_E \xrightarrow{\gamma} \mathcal{O}_E \right] , \quad (4.3.25)$$

$$\mathcal{L} := \left[\mathcal{O}_{\text{in}} \xrightarrow{\text{id}_{\mathcal{O}_{\text{in}}} \otimes \mathcal{K}} \mathcal{O}_{\text{in}} \otimes \mathcal{O}_E \xrightarrow{\mathcal{C}} \mathcal{O}_{\text{out}} \right] . \quad (4.3.26)$$

6. Using the embedding and projection maps ι_A, π_A from (4.3.13) we construct the morphisms:

$$\mathcal{E}_{\text{in}} := \bigotimes_{b \in \pi_0(S)} \iota_A^{(b)} : \mathcal{Z}_{\mathbb{A}}(S) \rightarrow \mathcal{O}_{\text{in}} , \quad \mathcal{E}_{\text{out}} := \bigotimes_{c \in \pi_0(T)} \pi_A^{(c)} : \mathcal{O}_{\text{out}} \rightarrow \mathcal{Z}_{\mathbb{A}}(T) , \quad (4.3.27)$$

where $\iota_A^{(b)} = \iota_A : Z(A)^{(b)} \rightarrow A^{(b)}$ and $\pi_A^{(b)} = \pi_A : A^{(b)} \rightarrow Z(A)^{(b)}$. We have all ingredients to define the action of $\mathcal{Z}_{\mathbb{A}}$ on morphisms:

$$\mathcal{Z}_{\mathbb{A}}(\Sigma, \mathcal{A}) := \left[\mathcal{Z}_{\mathbb{A}}(S) \xrightarrow{\mathcal{E}_{\text{in}}} \mathcal{O}_{\text{in}} \xrightarrow{\mathcal{L}} \mathcal{O}_{\text{out}} \xrightarrow{\mathcal{E}_{\text{out}}} \mathcal{Z}_{\mathbb{A}}(T) \right] . \quad (4.3.28)$$

Now that we defined $\mathcal{Z}_{\mathbb{A}(\mathbb{D})}$ on bordisms with strictly positive area, we turn to the general case. Let $(\Sigma, \mathcal{A}) : S \rightarrow T$ be a bordism with area and let $\Sigma_+ : S_+ \rightarrow T_+$ denote the connected component s of (Σ, \mathcal{A}) with strictly positive area. We have that in $\mathcal{Bord}_2^{\text{area}}$

$$(\Sigma, \mathcal{A}) = (\Sigma_+, \mathcal{A}_+) \sqcup (\Sigma \setminus \Sigma_+, 0) , \quad (4.3.29)$$

where \mathcal{A}_+ denotes the restriction of \mathcal{A} to $\pi_0(\Sigma_+)$. The bordism with zero area $(\Sigma \setminus \Sigma_+, 0)$ defines a permutation $\kappa : \pi_0(S \setminus S_+) \rightarrow \pi_0(T \setminus T_+)$. Let $\mathcal{Z}_{\mathbb{A}}(\Sigma \setminus \Sigma_+, 0) : \mathcal{Z}_{\mathbb{A}}(S \setminus S_+) \rightarrow \mathcal{Z}_{\mathbb{A}}(T \setminus T_+)$ be the induced permutation of tensor factors. We define

$$\mathcal{Z}_{\mathbb{A}}(\Sigma, \mathcal{A}) := \mathcal{Z}_{\mathbb{A}}(\Sigma \setminus \Sigma_+, 0) \otimes \mathcal{Z}_{\mathbb{A}}(\Sigma_+, \mathcal{A}_+) , \quad (4.3.30)$$

where $\mathcal{Z}_{\mathbb{A}}(\Sigma_+, \mathcal{A}_+)$ is defined in (4.3.28).

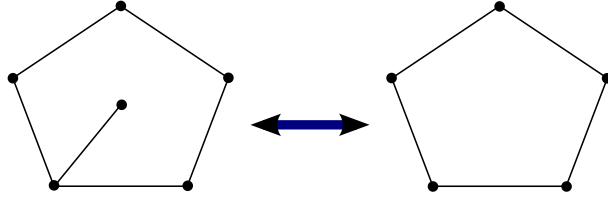


Figure 4.7: A face with a univalent vertex.

Theorem 4.3.5. *Let \mathbb{A} be state-sum data.*

1. *The morphism defined in (4.3.28) is independent of the choice of the PLCW decomposition with area, the choice of marked edges of faces, the choice of orientation of edges and the assignment V .*
2. *The state-sum construction yields an aQFT $\mathcal{Z}_{\mathbb{A}} : \mathcal{Bord}_2^{area} \rightarrow \mathcal{S}$ whose action on objects and morphisms is given by (4.3.14) and (4.3.30), respectively.*

Proof. Part 1:

First let us fix a PLCW decomposition with area. Independence of the choice of edges for faces and orientation of edges follows directly from Condition 1. Independence of the assignment V follows from iterating Conditions 1 and 4.

In order to show independence of the PLCW decomposition with area first notice that all conditions on \mathbb{A} depend on the sum of the parameters. This implies that the construction is independent of the distribution of area, i.e. the maps \mathcal{A}_k ($k \in \{0, 1, 2\}$). We need to check that the construction yields the same morphism for two different PLCW decompositions, but for this it is enough to check invariance under the elementary moves in Figure 4.5. Invariance under removing or adding an edge (Figure 4.5 b)) follows from Condition 2. To show invariance under splitting an edge by adding a vertex (Figure 4.5 a)) we use the trick used in the proof of [DKR, Lem. 3.5]. There the edge splitting is done inside a 2-gon (see [DKR, Fig. 14]), and that move in turn follows if one is allowed to add and remove univalent vertices as shown in Figure 4.7 (together with adding edges as in Figure 4.5 b)). But this follows from Condition 3.

Part 2:

We start by showing that if $(S \times [0, 1], \mathcal{A}) : S \rightarrow S$ is an in-out cylinder with positive area then the assignment

$$(\mathcal{A}(x))_{x \in \pi_0(S \times [0, 1])} \mapsto \mathcal{Z}_{\mathbb{A}}(S \times [0, 1], \mathcal{A}) \quad (4.3.31)$$

is continuous and the limit

$$\lim_{\mathcal{A} \rightarrow 0} \mathcal{Z}_{\mathbb{A}}(S \times [0, 1], \mathcal{A}) : \mathcal{Z}_{\mathbb{A}}(S) \rightarrow \mathcal{Z}_{\mathbb{A}}(S) \quad (4.3.32)$$

is a permutation of tensor factors.

Let us consider one connected component of $S \times [0, 1]$. By Part 1, we can pick a PLCW decomposition of this cylinder which consists of a square with two opposite edges identified,

and the other two edges being the in- and outgoing boundary components. The morphism \mathcal{L} from (4.3.26) is exactly D_a from (4.3.3), where a is the area of this component.

Now by looking at $\mathcal{Z}_{\mathbb{A}}(S \times [0, 1], \mathcal{A})$ with different area maps, we see that the difference is in the \mathcal{L} maps in (4.3.26), and is given by a factor of $\bigotimes_{x \in \pi_0(S)} P_{a_x}$ for some $a_x \in \mathbb{R}_{\geq 0}$. Therefore by separate continuity of the composition in \mathcal{S} and by Condition 5, the assignments in (4.3.31) are continuous for all positive parameters. By Condition 6 the limits in (4.3.32) exist, and we get the required permutations.

Next we show functoriality. We now assume that all components of the following bordisms have positive area. This is not a restriction since we can always take the areas of in-out cylinders to zero to get arbitrary bordisms with area. Let

$$\left[S \xrightarrow{(\Sigma, \mathcal{A}_{\Sigma})} T \xrightarrow{(\Xi, \mathcal{A}_{\Xi})} W \right]$$

be two bordisms with area. Pick PLCW decompositions with area so that at every outgoing boundary component of $(\Sigma, \mathcal{A}_{\Sigma})$ there is a square with two opposite edges identified and one edge on the boundary. Applying $\mathcal{Z}_{\mathbb{A}}$ on them we get

$$\begin{aligned} \mathcal{Z}_{\mathbb{A}}(\Sigma, \mathcal{A}_{\Sigma}) &:= \left[\mathcal{Z}_{\mathbb{A}}(S) \xrightarrow{\mathcal{E}_{\text{in}}^{\Sigma}} \mathcal{O}_{\text{in}}^{\Sigma} \xrightarrow{\psi \circ \mathcal{L}^{\Sigma}} \mathcal{O}_{\text{out}}^{\Sigma} \xrightarrow{\mathcal{E}_{\text{out}}^{\Sigma}} \mathcal{Z}_{\mathbb{A}}(T) \right], \\ \mathcal{Z}_{\mathbb{A}}(\Xi, \mathcal{A}_{\Xi}) &:= \left[\mathcal{Z}_{\mathbb{A}}(T) \xrightarrow{\mathcal{E}_{\text{in}}^{\Xi}} \mathcal{O}_{\text{in}}^{\Xi} \xrightarrow{\mathcal{L}^{\Xi}} \mathcal{O}_{\text{out}}^{\Xi} \xrightarrow{\mathcal{E}_{\text{out}}^{\Xi}} \mathcal{Z}_{\mathbb{A}}(W) \right], \end{aligned}$$

where $\psi = \bigotimes_{x \in \pi_0(T)} D_{a_x}$ for some $a_x \in \mathbb{R}_{>0}$. Note that $\mathcal{O}_{\text{out}}^{\Sigma} = \mathcal{O}_{\text{in}}^{\Xi}$. Composing these morphisms yields using (4.3.13) the morphism

$$\mathcal{Z}_{\mathbb{A}}(\Xi, \mathcal{A}_{\Xi}) \circ \mathcal{Z}_{\mathbb{A}}(\Sigma, \mathcal{A}_{\Sigma}) = \left[\mathcal{Z}_{\mathbb{A}}(S) \xrightarrow{\mathcal{E}_{\text{in}}^{\Sigma}} \mathcal{O}_{\text{in}}^{\Sigma} \xrightarrow{\psi \circ \mathcal{L}^{\Sigma}} \mathcal{O}_{\text{out}}^{\Sigma} \xrightarrow{\psi_0} \mathcal{O}_{\text{in}}^{\Xi} \xrightarrow{\mathcal{L}^{\Xi}} \mathcal{O}_{\text{out}}^{\Xi} \xrightarrow{\mathcal{E}_{\text{out}}^{\Xi}} \mathcal{Z}_{\mathbb{A}}(W) \right],$$

where $\psi_0 = \bigotimes_{x \in \pi_0(T)} D_0$. For the composition of these bordisms with area $(\Xi \circ \Sigma, \mathcal{A}_{\Xi \circ \Sigma})$ pick the PLCW decomposition with area obtained by glueing the two decompositions together at the boundary components corresponding to T . By construction, $\mathcal{L}^{\Xi \circ \Sigma}$ contains a copy of D_a for every connected component of T . Notice that when we compute $\mathcal{Z}_{\mathbb{A}}(\Xi, \mathcal{A}_{\Xi}) \circ \mathcal{Z}_{\mathbb{A}}(\Sigma, \mathcal{A}_{\Sigma})$, by (4.3.13), we also get a copy of D_0 for every connected component of T . Since $D_a \circ D_0 = D_a$ by Lemma 4.3.4, D_0 can be omitted and the above composition is equal to $\mathcal{Z}_{\mathbb{A}}(\Xi \circ \Sigma, \mathcal{A}_{\Xi \circ \Sigma})$.

The continuity conditions of Lemma 4.2.6 hold, as we have already checked them before; monoidality and symmetry follow from the construction, so altogether we have shown that $\mathcal{Z}_{\mathbb{A}}$ is indeed an aQFT. \square

Remark 4.3.6. By looking at this proof we see that Conditions 1-6 are not only sufficient, but also necessary, at least if one requires independence under the elementary moves of PLCW decompositions locally, that is, for the corresponding maps $\mathbb{I} \rightarrow A^{\otimes m}$.

4.3.3 State-sum data from RFAs

In this section we show that there is a one-to-one correspondence between state-sum data for a given object in \mathcal{S} and RFA structures on the same object (subject to certain conditions), see Theorem 4.3.9. Furthermore we show in Theorem 4.3.11 that the state-sum aQFT given in terms of such an RFA is classified by the centre of this RFA, cf. Theorem 4.2.10. We will keep using the notation of the previous section.

Lemma 4.3.7. *The state-sum data \mathbb{A} determines a strongly separable symmetric regularised Frobenius algebra structure on $A \in \mathcal{S}$ by setting*

$$\eta_{a_1} := W_{a_1}^1, \quad \mu_{a_1+a_2} := \left(id_A \otimes \tilde{B}_{a_1}^2 \right) \circ \left(W_{a_2}^3 \otimes id_{A^{\otimes 2}} \right), \quad (4.3.33)$$

$$\varepsilon_{a_1+a_2} := \beta_{a_1} \circ \left(W_{a_2}^1 \otimes id_A \right), \quad \Delta_{a_1+a_2} := \left(id_{A^{\otimes 2}} \otimes \beta_{a_1} \right) \circ \left(W_{a_2}^3 \otimes id_A \right), \quad (4.3.34)$$

for every $a_1, a_2 \in \mathbb{R}_{>0}$. In terms of the graphical calculus these morphisms are:

$$\begin{array}{c} \text{a vertical line with a dot at the bottom labeled } a \\ \hline \end{array} = \begin{array}{c} \text{a box labeled } [a; 1] \text{ with a line entering from the top} \\ \hline \end{array}, \quad \begin{array}{c} \text{a vertical line with a dot at the bottom labeled } a_1 + a_2 \\ \hline \end{array} = \begin{array}{c} \text{a box labeled } [a_2; 3] \text{ with two lines entering from the top, labeled } a_1/2 \text{ and } a_1/2 \\ \hline \end{array}, \quad (4.3.35)$$

$$\begin{array}{c} \text{a vertical line with a dot at the bottom labeled } a_1 + a_2 \\ \hline \end{array} = \begin{array}{c} \text{a box labeled } [a_1; 1] \text{ with a line entering from the top, labeled } a_2 \\ \hline \end{array}, \quad \begin{array}{c} \text{a vertical line with a dot at the bottom labeled } a_1 + a_2 \\ \hline \end{array} = \begin{array}{c} \text{a box labeled } [a_2; 3] \text{ with two lines entering from the top, labeled } a_1 \text{ and } a_1 \\ \hline \end{array}, \quad (4.3.36)$$

Let $\kappa(\mathbb{A})$ denote this RFA.

Proof. We are going to show that (4.1.2) holds. Checking the rest of the algebraic relations of an RFA is similar and it uses the algebraic relations listed in Conditions 1-4. The rhs of (4.1.2) is

$$\begin{array}{c} \text{a vertical line with a dot at the bottom labeled } a_1 \\ \hline \end{array} \begin{array}{c} \text{a vertical line with a dot at the bottom labeled } a_2 \\ \hline \end{array} \stackrel{\text{def.}}{=} \begin{array}{c} \text{a box labeled } [a_1''; 3] \text{ with a line entering from the top, labeled } a_1'/2 \\ \hline \end{array} \begin{array}{c} \text{a box labeled } [a_2; 1] \text{ with a line entering from the top, labeled } a_1'/2 \\ \hline \end{array} \stackrel{\text{Cond. 2}}{=} \begin{array}{c} \text{a box labeled } [a_1; 2] \text{ with a line entering from the top, labeled } a_2 \\ \hline \end{array} \stackrel{\text{def.}}{=} P_{a_1+a_2},$$

using Condition 2. The lhs is

$$\begin{array}{c} \text{a vertical line with a dot at the bottom labeled } a_1 \\ \hline \end{array} \begin{array}{c} \text{a vertical line with a dot at the bottom labeled } a_2 \\ \hline \end{array} \stackrel{\text{def.}}{=} \begin{array}{c} \text{a box labeled } [a_1'; 3] \text{ with a line entering from the top, labeled } a_1'/2 \\ \hline \end{array} \begin{array}{c} \text{a box labeled } [a_2; 1] \text{ with a line entering from the top, labeled } a_1'/2 \\ \hline \end{array} \stackrel{\text{Cond. 1}}{=} \begin{array}{c} \text{a box labeled } [a_1''; 3] \text{ with a line entering from the top, labeled } a_1'/2 \\ \hline \end{array} \begin{array}{c} \text{a box labeled } [a_2; 1] \text{ with a line entering from the top, labeled } a_1'/2 \\ \hline \end{array} \stackrel{\text{Cond. 2}}{=} \begin{array}{c} \text{a box labeled } [a_1; 2] \text{ with a line entering from the top, labeled } a_2 \\ \hline \end{array} \stackrel{\text{Cond. 1}}{=} \begin{array}{c} \text{a box labeled } [a_1; 2] \text{ with a line entering from the top, labeled } a_2 \\ \hline \end{array} \stackrel{\text{def.}}{=} P_{a_1+a_2}.$$

The continuity conditions for tensor products of P_a 's hold by Condition 5, which also states that $\lim_{a \rightarrow 0} P_a = id_A$. We have now shown that A is an RFA. It is symmetric by Condition 1. To show that A is strongly separable, by Proposition 4.1.10 we need to check that the window element of A is invertible. Similar to the calculation above, and using Condition 3, one checks that $\zeta_{a_1} \circ W_{a_2}^1$ is inverse to the window element. \square

Lemma 4.3.8. *Let $A \in \mathcal{S}$ be a strongly separable symmetric regularised Frobenius algebra with separability idempotent $e_a \in \mathcal{S}(\mathbb{I}, A^{\otimes 2})$. Define the following families of morphisms for $a_1, a_2, a_3 \in \mathbb{R}_{>0}$, $n \in \mathbb{Z}_{\geq 1}$,*

$$\zeta_{a_1+a_2+a_3} := (\varepsilon_{a_1} \otimes \mu_{a_2}) \circ (e_{a_3} \otimes \text{id}_A) , \quad \beta_{a_1+a_2} := \varepsilon_{a_1} \circ \mu_{a_2} , \quad (4.3.37)$$

$$W_{a_1+a_2}^n := \Delta_{a_1}^{(n)} \circ \eta_{a_2} , \quad \tilde{D}_{a_1+a_2+a_3} := \zeta_{a_1} \circ \mu_{a_2} \circ \sigma_{A,A} \circ \Delta_{a_3} , \quad (4.3.38)$$

where $\Delta_{a_1}^{(n)}$ is as in (4.2.17). Suppose that $\lim_{a \rightarrow 0} \tilde{D}_a$ exists. Then the families of morphisms ζ_a , β_a and W_a^n define state-sum data, which we denote with $\Omega(A)$. Also, the morphism D_a defined in (4.3.3) is the same as the morphism \tilde{D}_a defined in (4.3.38) for every $a \in \mathbb{R}_{\geq 0}$.

Proof. Conditions 5 and 6 are satisfied by our assumptions. The algebraic conditions can be checked by direct computation, here we only give the ideas how one can do this. Cyclicity in Condition 1 follows from the Frobenius relation (4.1.16) and A being symmetric. The glueing condition Condition 2 follows from the Frobenius relation (4.1.16) from counitality (4.1.12) and from coassociativity (4.1.13). Condition 3 follows from A being strongly separable, Condition 4 follows from the fact that the window element of A is commutative. \square

Let us fix an object $A \in \mathcal{S}$ and denote the sets

- $\mathbf{L} := \{\text{state-sum data on } A\}$,
- $\mathbf{F} := \{\text{strongly separable symmetric RFA structures on } A \text{ such that } \lim_{a \rightarrow 0} \tilde{D}_a \text{ exists}\}$.

From a direct calculation one can show the following theorem.

Theorem 4.3.9. *Let κ and Ω denote the maps of sets*

$$\begin{array}{ccc} & \xrightarrow{\kappa} & \\ \mathbf{L} & & \mathbf{F} \\ & \xleftarrow{\Omega} & \end{array} \quad (4.3.39)$$

defined by Lemmas 4.3.7 and 4.3.8 respectively. Then κ and Ω are inverse to each other.

In the following we make use of the notion of RFAs in order to prove some technical results used in the state-sum construction in Section 4.3.2. The following lemma is a direct generalisation of [LP1, Prop. 2.20] and was partially proved in Corollary 4.1.52.

Lemma 4.3.10. *Let \mathbb{A} be state-sum data and let A denote the corresponding RFA from Lemma 4.3.7. Then $D_a \circ D_b = D_{a+b}$ for every $a, b \in \mathbb{R}_{>0}$ and the image of the idempotent $D_0 := \lim_{a \rightarrow 0} D_a$ is the centre $Z(A)$ of the RFA A . It is an RFA with the restricted structure maps of A .*

Using Lemma 4.3.10 and Theorem 4.2.10, we have the direct translation of [LP1, Thm. 4.7].

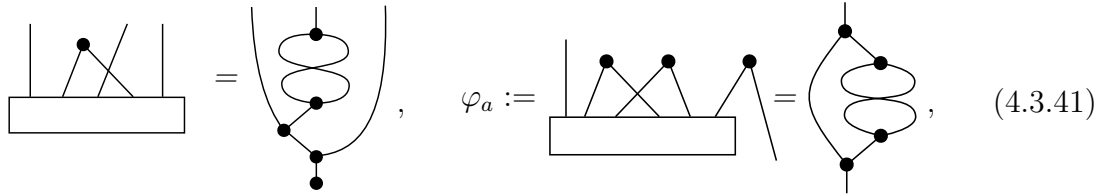
Theorem 4.3.11. *Let \mathbb{A} be state-sum data, let A denote the corresponding RFA and let $Z(A)$ denote its centre. Let $\mathcal{Z}_{\mathbb{A}}$ denote the state-sum aQFT of Theorem 4.3.5 and let G be the equivalence in (4.2.22). Then*

$$Z(A) = G(\mathcal{Z}_{\mathbb{A}}) . \quad (4.3.40)$$

The following lemma gives a concise expression for the value of a state-sum area-dependent QFT on a genus g surface with b outgoing boundary components.

Lemma 4.3.12. *Let \mathbb{A} be state-sum data and A the corresponding RFA.*

1. *We have the following identities:*



$$\varphi_a := \text{[diagram of box with three lines]} = \text{[diagram of box with loop and three dots]} , \quad (4.3.41)$$

2. *Let $(\Sigma_{g,b}, a) : \emptyset \rightarrow (\mathbb{S}^1)^{\sqcup b}$ be a connected bordism of genus g with $b \geq 1$ outgoing boundary components and area a . Then*

$$\mathcal{Z}_{\mathbb{A}}(\Sigma_{g,b}, a) = \pi^{\otimes b} \circ \Delta_{a_g}^{(b)} \circ \prod_{j=1}^g \varphi_{a_j} \circ (\zeta_{a'})^{1-b} \circ \eta_{a_0} , \quad (4.3.42)$$

with $a = b \cdot a' + \sum_{j=0}^{g+1} a_j$.

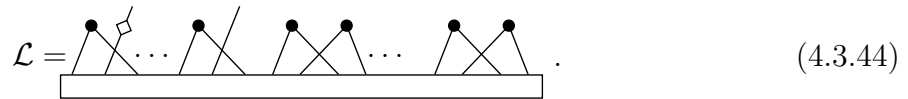
3. *Let $(C, a) : \mathbb{S}^1 \sqcup \mathbb{S}^1 \rightarrow \emptyset$ be a cylinder with two ingoing components and area a . Then*

$$\mathcal{Z}_{\mathbb{A}}(C, a) = \varepsilon_{a_0} \circ \zeta_{a_1} \circ \mu_{a_2} \circ (\iota \otimes \iota) , \quad (4.3.43)$$

with $a = a_0 + a_1 + a_2$.

Proof. Parts 1 and 3 follow from a simple calculation.

We will only sketch the proof for Part 2. Pick a PLCW decomposition of $\Sigma_{g,b}$ as shown in Figure 4.8 and apply the state-sum construction to get the morphism \mathcal{L} of (4.3.26), which is:



$$\mathcal{L} = \text{[diagram of box with triangles and dots]} . \quad (4.3.44)$$

The rest of the calculation is straightforward, but tedious, therefore we omit it here. Note that in order to get the D_0 's at the boundary components, which then cancel with the π 's, we need to insert $b - 1$ factors of $\zeta_{a'}$'s and their inverses. \square

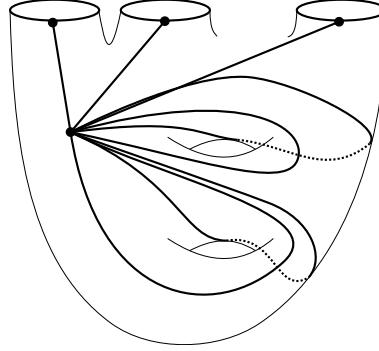


Figure 4.8: A convenient PLCW decomposition of $\Sigma_{g,b}$ using a single face, which is a $(4g + 3b)$ -gon. The dots show the vertices and the thick lines the edges.

In the rest of this section we discuss how one can build Hermitian aQFTs via the state-sum construction. Let us assume that \mathcal{S} is equipped with a \dagger -structure. The state-sum data \mathbb{A} is called *Hermitian* if it satisfies

$$\zeta_a^\dagger = \zeta_a, \quad \beta_a^\dagger = W_a^2 \quad \text{and} \quad (W_a^n)^\dagger = \begin{array}{c} \bullet \quad a_1/n \\ \bullet \quad a_1/n \\ \vdots \\ \bullet \quad a_1/n \\ \boxed{a - a_1; n} \end{array}, \quad (4.3.45)$$

for every $a, a_0 \in \mathbb{R}_{>0}$ and $n \in \mathbb{Z}_{\geq 1}$.

One can easily check the following statements. If \mathbb{A} is Hermitian, then the RFA $\kappa(\mathbb{A})$ from Lemma 4.3.7 is a \dagger -RFA. Conversely, let A be a \dagger -RFA. Then the state-sum data $\Omega(A)$ is Hermitian. Also, if \mathbb{A} is Hermitian, then the state-sum aQFT $\mathcal{Z}_{\mathbb{A}}$ is Hermitian.

4.3.4 PLCW decompositions with defects

In this section we introduce a cell decomposition of bordisms with defects, which is used in Section 4.3.5 to build defect aQFTs. We will use the notation of Sections 4.2.3 and 4.3.1 and fix a set of defect conditions $\mathbb{D} = (D_1, D_2, s, t)$.

Let $\Sigma = \Sigma_{[1]} \cup \Sigma_{[2]}$ be a surface with defects. A *PLCW decomposition with defects* of Σ is a PLCW decomposition $\Sigma_0, \Sigma_1, \Sigma_2$ of the surface Σ which satisfies the following conditions.

1. For every $p \in \Sigma_0$ the intersection $p \cap \Sigma_{[1]} = \emptyset$ is empty.
2. Every intersection of an element of Σ_1 and $\Sigma_{[1]}$ is transversal.
3. For every $e \in \Sigma_1$, $e \cap \Sigma_{[1]}$ is either empty or consists of precisely one point.
4. For every $f \in \Sigma_2$, if $f \cap \Sigma_{[1]} \neq \emptyset$, then it is diffeomorphic to an interval with the boundary points on edges of f .

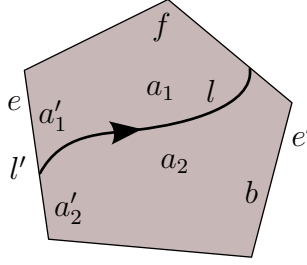


Figure 4.9: A face with defect $f \in \Sigma_2^{\text{defect}}$, the defect line is marked with a thick line with the arrow showing its orientation. For example, the edge e lies in Σ_1^{defect} and the edge e' in Σ_1^{empty} . The area maps have values $\mathcal{A}_2^{\text{defect}}(f) = (a_1, l, a_2)$, $\mathcal{A}_1^{\text{defect}}(e) = (a'_1, l', a'_2)$ and $\mathcal{A}_1^{\text{empty}}(e') = b$ for this particular face and two edges.

5. For every boundary component $b \in \pi_0(\partial\Sigma)$ for which $b \cap \Sigma_{[1]} = \emptyset$, there is 1 boundary edge.
6. For every boundary component $b \in \pi_0(\Sigma)$ for which $b \cap \Sigma_{[1]} \neq \emptyset$, every boundary edge contains exactly one point in $\Sigma_{[1]}$.

If the above conditions hold then the sets of faces and edges split in two disjoint sets. For $k \in \{1, 2\}$ let $\Sigma_k^{\text{empty}} \subseteq \Sigma_k$ be the subset of cells which do not intersect with $\Sigma_{[1]}$ (*empty cells*) and let $\Sigma_k^{\text{defect}} := \Sigma_k \setminus \Sigma_k^{\text{empty}}$ be the subset of cells which intersect with $\Sigma_{[1]}$ (*cells with a defect*). An example is shown in Figure 4.9. Let

$$\bar{V} : \Sigma_0 \rightarrow \Sigma_1 \quad (4.3.46)$$

be a map which assigns to a vertex v an edge e for which $v \in \partial(e)$, similarly as in (4.3.17). This map splits Σ_0 into two disjoint sets $\Sigma_0^{\text{empty}} := \bar{V}^{-1}(\Sigma_1^{\text{empty}})$ and $\Sigma_0^{\text{defect}} := \bar{V}^{-1}(\Sigma_1^{\text{defect}})$.

Let $(\Sigma, \mathcal{A}, \mathcal{L})$ be a surface with area and defects with strictly positive areas and lengths and let $\Sigma_0, \Sigma_1, \Sigma_2$ be a PLCW decomposition with defects of $\Sigma = \Sigma_{[1]} \cup \Sigma_{[2]}$. Let

$$\mathcal{A}_k^{\text{empty}} : \Sigma_k^{\text{empty}} \rightarrow \mathbb{R}_{>0}, \quad (4.3.47)$$

for $k \in \{0, 1, 2\}$ be maps which assign to empty vertices, edges and faces an *area*, and let

$$\mathcal{A}_k^{\text{defect}} : \Sigma_k^{\text{defect}} \rightarrow (\mathbb{R}_{>0})^3, \quad (4.3.48)$$

for $k \in \{0, 1, 2\}$ be maps which assign to vertices, edges and faces with defects an *area* on the half edges and half faces at the two sides of a defect line and a *length* of a defect line as explained in Figure 4.9. We require of the maps in (4.3.47) and (4.3.48) that for every connected component $x \in \pi_0(\Sigma_{[2]})$ the sum of the areas of corresponding vertices, (half) edges and (half) faces of x is equal to its area $\mathcal{A}(x)$. We require the analogous condition on lengths of defect lines. In the case $k = 0$ in (4.3.48) the three parameters $\mathcal{A}_k^{\text{defect}}(v)$ for a vertex v contribute to the same components as those of the edge $\bar{V}(v)$.

In the following we will write $\mathcal{A}_k(x)$ for both $\mathcal{A}_k^{\text{defect}}(x)$ and $\mathcal{A}_k^{\text{empty}}(x)$ and mean the latter depending on the type of $x \in \Sigma_k$. A *PLCW decomposition of a surface with area*

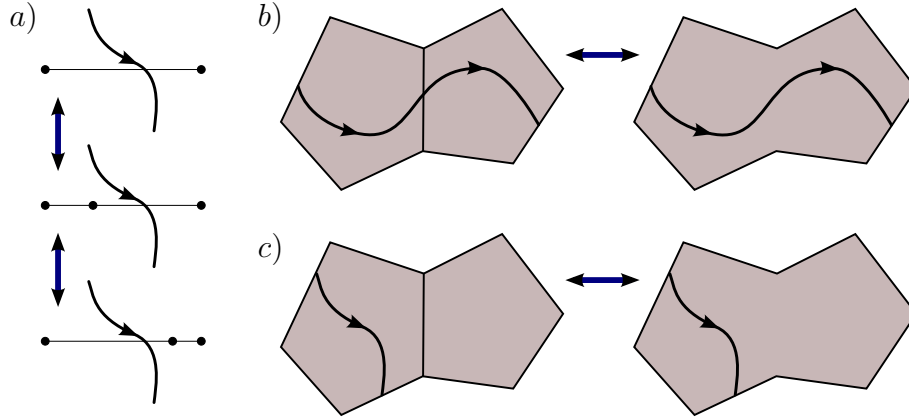


Figure 4.10: Additional elementary moves of PLCW decompositions with defects. In Figure a), an edge which is crossed by a defect line (curved line with arrow) is split in two by adding a new vertex (denoted with a dot). In Figure b), an edge which is crossed by a defect line is removed. In Figure c), an edge which is not crossed by any defect line is removed. Note that here only one of the faces can be crossed by defect lines.

and defects $(\Sigma, \mathcal{A}, \mathcal{L})$ consists of a choice of a PLCW decomposition with defects and a choice of maps as in (4.3.47) and (4.3.48).

Elementary moves on a PLCW decomposition with defects of a surface with defects (and area) are elementary moves of PLCW decompositions which respect the conditions listed above. The additional moves are shown in Figure 4.10. In [DKR, Lem. 3.6] it is argued that any two PLCW decompositions with defects can be related by these elementary moves.

4.3.5 State-sum construction with defects

After introducing PLCW decompositions with defects let us turn to the state-sum construction of defect aQFTs. We will again use the notation of Sections 4.2.3 and 4.3.4 and fix a set of defect conditions $\mathbb{D} = (D_1, D_2, s, t)$.

State sum data and some preparatory notions

As for the state-sum construction without defects in Section 4.3.2, we start with giving *state-sum data with defects* $\mathbb{A}(\mathbb{D})$. This consists of

1. state-sum data $\mathbb{A}_y = (A_y, \zeta_a^y, \beta_a^y, W_a^{y,n})$ for every $y \in D_2$ as in Definition 4.3.2,
2. a pair of objects $X_x, \bar{X}_x \in \mathcal{S}$ for every $x \in D_1$ together with the following families of morphisms:

$$\begin{aligned} \zeta_{a,l,b}^{x,\epsilon} &\in \mathcal{S}(X_x^\epsilon, X_x^\epsilon), \quad \beta_{a,l,b}^x \in \mathcal{S}(X_x \otimes \bar{X}_x, \mathbb{I}), \\ W_{a,l,b}^{x,n,m} &\in \mathcal{S}(\mathbb{I}, \bar{X}_x \otimes A_{t(x)}^{\otimes n} \otimes X_x \otimes A_{s(x)}^{\otimes m}) \end{aligned} \tag{4.3.49}$$

for every $a, l, b \in \mathbb{R}_{>0}$, every $n, m \geq 0$ and $\epsilon \in \{\pm\}$, where we used the following notation:

$$X_x^+ := X_x \quad \text{and} \quad X_x^- := \bar{X}_x . \quad (4.3.50)$$

We will use the following graphical notation for these morphisms. For $y \in D_2$,

$$\zeta_a^y = \begin{array}{c} A_y \\ | \\ \diamond \\ | \\ A_y \end{array} \quad (a; y) , \quad \beta_a^y = \begin{array}{c} \bullet \quad (a; y) \\ / \quad \backslash \\ A_y \quad A_y \end{array} \quad \text{and} \quad W_a^{y,n} = \begin{array}{c} A_y \quad A_y \quad A_y \\ | \quad | \quad | \\ \dots \\ \boxed{a; y, n} \end{array} , \quad (4.3.51)$$

$P_a^{(y)} \in \mathcal{S}(A_y, A_y)$ from (4.3.3) and for $x \in D_1$

$$\zeta_{a,l,b}^{x,\epsilon} = \begin{array}{c} X_x^\epsilon \\ | \\ \diamond \\ | \\ X_x^\epsilon \end{array} \quad (a, l, b; x, \epsilon) , \quad \beta_{a,l,b}^x = \begin{array}{c} \bullet \quad (a, l, b; x) \\ / \quad \backslash \\ X_x \quad \bar{X}_x \end{array} \quad \text{and} \quad W_{a,l,b}^{x,n,m} = \begin{array}{c} \bar{X}_x \quad A_{t(x)} \quad A_{t(x)} \quad X_x \quad A_{s(x)} \quad A_{s(x)} \\ \downarrow \quad | \quad | \quad \uparrow \quad | \quad | \\ \dots \\ \boxed{a, l, b; x, n, m} \end{array} . \quad (4.3.52)$$

Let us define

$$\begin{aligned} s(x, +) &:= s(x) , & t(x, +) &:= t(x) , \\ s(x, -) &:= t(x) , & t(x, -) &:= s(x) . \end{aligned} \quad (4.3.53)$$

By a *defect list of length n* we mean an equivalence class of ordered lists

$$\underline{x} := [(x_1, \epsilon_1, \dots, x_n, \epsilon_n)] , \quad (4.3.54)$$

where $x_i \in D_1$ and $\epsilon_i \in \{\pm\}$ ($i = 1, \dots, n$). The x_i, ϵ_i have to satisfy, for $i = 1, \dots, n$ and setting $x_{n+1} := x_1, \epsilon_{n+1} := \epsilon_1$,

$$s(x_i, \epsilon_i) = t(x_{i+1}, \epsilon_{i+1}) . \quad (4.3.55)$$

Two such lists $(x_1, \epsilon_1, \dots, x_n, \epsilon_n)$ and $(x'_1, \epsilon'_1, \dots, x'_n, \epsilon'_n)$ are equivalent if they are related by a cyclic permutation. Let us introduce the shorthand, for a chosen representative of \underline{x} ,

$$X_{\underline{x}} := X_{x_1}^{\epsilon_1} \otimes \dots \otimes X_{x_n}^{\epsilon_n} . \quad (4.3.56)$$

Different choices of representatives are related by cyclic permutations of tensor factors. Let us introduce the following morphisms:

$$Q_{a,l,b}^{(x,+)} := \begin{array}{c} \bullet \quad (a_1, l_1, b_1, x) \\ / \quad \backslash \\ X_x \quad \boxed{a_2, l_2, b_2; x, 0, 0} \end{array} , \quad Q_{a,l,b}^{(x,-)} := \begin{array}{c} \bar{X}_x \quad \bullet \quad (a_1, l_1, b_1, x) \\ \downarrow \quad / \quad \backslash \\ \boxed{a_2, l_2, b_2; x, 0, 0} \quad \bar{X}_x \end{array} , \quad (4.3.57)$$

where $a = a_1 + a_2$, $b = b_1 + b_2$ and $l = l_1 + l_2$,

$$F_{a,l,b}^{(x,+)} := \text{diagram}, \quad F_{a,l,b}^{(x,-)} := \text{diagram}, \quad (4.3.58)$$

where $a = a_0 + a_1 + a_2$, $b = b_1 + b_2$ and $l = l_1 + l_2$ and

$$E_{\underline{a}}^x := \text{diagram}, \quad (4.3.59)$$

where $\underline{a} = (a_1 + a'_1 + a''_1, \dots, a_n + a'_n + a''_n, l) \in (\mathbb{R}_{>0})^{n+1}$ and the values of y_i are determined by \underline{x} via (4.3.55), i.e. $y_i = s(x_i, \epsilon_i) = t(x_{i+1}, \epsilon_{i+1})$. Different choices of representatives of \underline{x} induce different morphisms via (4.3.59), which are related by conjugating with cyclic permutations of tensor factors.

With these preparations, we can now state the conditions state-sum data with defects $\mathbb{A}(\mathbb{D})$ have to satisfy. Namely, let $x \in D_1$, $\epsilon \in \{\pm\}$ and let \underline{x} be a defect list. Then:

1. Glueing plaquette weights with defects:

$$\text{diagram} = \text{diagram}, \quad (4.3.60)$$

for every $a = a_0 + a_1 + a_2$, $n = n_1 + n_2$, etc.

2. Glueing plaquette weights with and without defects:

$$\text{diagram} = \text{diagram}, \quad (4.3.61)$$

for every $a = a_0 + a_1 + a_2$, $b = b_0 + b_1 + b_2$, $l \in \mathbb{R}_{>0}$, $n = n_1 + n_2$, $m = m_1 + m_2$.

3. “Moving ζ ’s around”:

$$\begin{array}{c}
 \begin{array}{ccc}
 \begin{array}{c} (a_1, l_1, b_1; x, +) \\ \downarrow \quad \vdots \quad \downarrow \quad \downarrow \quad \downarrow \\ \boxed{a_2, l_2, b_2; x, n, m} \end{array} & = & \begin{array}{c} (b_1; s(x)) \\ \downarrow \quad \vdots \quad \downarrow \quad \downarrow \quad \downarrow \\ \boxed{a, l, b_2; x, n, m} \end{array} = \begin{array}{c} (b_1; s(x)) \\ \downarrow \quad \vdots \quad \downarrow \quad \downarrow \quad \downarrow \\ \boxed{a, l, b_2; x, n, m} \end{array}, \\
 \begin{array}{ccc}
 \begin{array}{c} (a_1, l_1, b_1; x, -) \\ \downarrow \quad \vdots \quad \downarrow \quad \downarrow \quad \downarrow \\ \boxed{a_2, l_2, b_2; x, n, m} \end{array} & = & \begin{array}{c} (a_1; t(x)) \\ \downarrow \quad \vdots \quad \downarrow \quad \downarrow \quad \downarrow \\ \boxed{a_2, l, b; x, n, m} \end{array} = \begin{array}{c} (a_1; t(x)) \\ \downarrow \quad \vdots \quad \downarrow \quad \downarrow \quad \downarrow \\ \boxed{a_2, l, b; x, n, m} \end{array}.
 \end{array}
 \end{array} \tag{4.3.62}$$

for $a_1 + a_2 = a$, $b_1 + b_2 = b$, $l_1 + l_2 = l$, $n, m \geq 0$.

4. The limit $\lim_{\underline{a} \rightarrow 0} E_{\underline{a}}^x$ exists, and $\lim_{a, b, l \rightarrow 0} Q_{a, l, b}^{(x, \epsilon)} = \text{id}_{X_x^\epsilon}$.
5. For every $n, m \geq 0$ with $n + m \geq 1$, $(x_i, \epsilon_i) \in D_1 \times \{\pm\}$ for $i = 1, \dots, n$, $p_j \in D_2$ for $j = 1, \dots, m$ the assignment

$$\begin{aligned}
 (\mathbb{R}_{\geq 0})^{3n+m} &\rightarrow \mathcal{S} \left(\bigotimes_{i=1}^n X_{x_i}^{\epsilon_i} \otimes \bigotimes_{j=1}^m A_{p_j}, \bigotimes_{i=1}^n X_{x_i}^{\epsilon_i} \otimes \bigotimes_{j=1}^m A_{p_j} \right) \\
 (a_1, l_1, b_1, \dots, a_n, l_n, b_n, c_1, \dots, c_m) &\mapsto \bigotimes_{i=1}^n Q_{a_i, l_i, b_i}^{(x_i, \epsilon_i)} \otimes \bigotimes_{j=1}^m P_{c_j}^{(p_j)}
 \end{aligned} \tag{4.3.63}$$

is jointly continuous.

We have the analogue of Lemma 4.3.4, which can be proven using Conditions 1, 2 and 4.

Lemma 4.3.13. For every defect list \underline{x} of length $n \in \mathbb{Z}_{\geq 1}$ and $\underline{a}, \underline{a}' \in (\mathbb{R}_{\geq 0})^{n+1}$,

$$E_{\underline{a}}^x \circ E_{\underline{a}'}^x = E_{\underline{a} + \underline{a}'}^x. \tag{4.3.64}$$

In particular, the morphism

$$E_0^x := \lim_{\underline{a} \rightarrow 0} E_{\underline{a}}^x \in \mathcal{S}(X_{\underline{x}}, X_{\underline{x}}) \tag{4.3.65}$$

is idempotent.

Let us fix state-sum data with defects $\mathbb{A}(\mathbb{D})$. In the rest of this section we define a symmetric monoidal functor $\mathcal{Z}_{\mathbb{A}(\mathbb{D})} : \text{Bord}_{2, \mathbb{D}}^{\text{area}, \text{def}} \rightarrow \mathcal{S}$ using this data.

By our assumptions, the idempotents in (4.3.65) split. Let $Z(X_{\underline{x}}) \in \mathcal{S}$ denote the image and write $\pi_{\underline{x}}$ and $\iota_{\underline{x}}$ for the projection and embedding, i.e.

$$E_0^x = \left[X_{\underline{x}} \xrightarrow{\pi_{\underline{x}}} Z(X_{\underline{x}}) \xrightarrow{\iota_{\underline{x}}} X_{\underline{x}} \right], \quad \text{id}_{Z(X_{\underline{x}})} = \left[Z(X_{\underline{x}}) \xrightarrow{\iota_{\underline{x}}} X_{\underline{x}} \xrightarrow{\pi_{\underline{x}}} Z(X_{\underline{x}}) \right]. \tag{4.3.66}$$

Note that different choices of representative in (4.3.56) give the same image $Z(X_{\underline{x}})$ since the idempotents E_0^x commute with cyclic permutations.

We will also write $X_{()}^{(b)} = A_{d_2(b)}$,

$$\iota_{()}^{(b)} = \iota_{A_{d_2(b)}} : Z(A_{d_2(b)}) \rightarrow A_{d_2(b)} \quad \text{and} \quad \pi_{()}^{(b)} = \pi_{A_{d_2(b)}} : A_{d_2(b)} \rightarrow Z(A_{d_2(b)}). \tag{4.3.67}$$

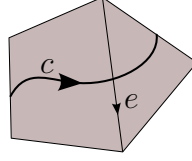


Figure 4.11: Positive crossing of an oriented edge e and a defect line c .

Defining $\mathcal{Z}_{\mathbb{A}(\mathbb{D})}$

We define the aQFT $\mathcal{Z}_{\mathbb{A}(\mathbb{D})}$ on objects as follows: Let $S \in \mathcal{Bord}_{2,\mathbb{D}}^{area,def}$ and $c \in \pi_0(S)$. If $c \cap S_{[0]} = \emptyset$ then let $\underline{x}(c) := ()$ be the empty list, and $Z(X_{()}^{(c)}) := Z(A_{d_2(c)})$. Otherwise, for every $c \in \pi_0(S)$ let

$$\underline{x}(c) := [(d_1(v), \epsilon(v))_{v \in c \cap S_{[0]}}] \quad (4.3.68)$$

be the defect list given by the defect labels $d_1(v)$ and orientations $\epsilon(v)$ of the defects in c in the cyclic order determined by the orientation of c . We define $\mathcal{Z}_{\mathbb{A}(\mathbb{D})}$ on objects as

$$\mathcal{Z}_{\mathbb{A}(\mathbb{D})}(S) := \bigotimes_{c \in \pi_0(S)} Z(X_{\underline{x}(c)}^{(c)}) , \quad (4.3.69)$$

where as in (4.3.14) the superscript is used to label the tensor factors.

The definition of $\mathcal{Z}_{\mathbb{A}(\mathbb{D})}$ on morphisms is again more involved. Let $(\Sigma, \mathcal{A}, \mathcal{L}) : S \rightarrow T$ be a bordism with area and defects and assume that it has no component with zero area or length. Choose a PLCW decomposition with area and defects (with the same notation as in Section 4.3.4) of the surface with area and defects $(\Sigma, \mathcal{A}, \mathcal{L})$.

Let us choose a marked edge for every face in Σ_2^{empty} and for every face in Σ_2^{defect} let the marked edge be the one where the defect line leaves. Also let us choose an orientation of every edge, requiring that the orientation of edges in Σ_1^{defect} are such that the edges and the defect lines cross positively as shown in Figure 4.11.

We introduce the sets of sides of faces F for faces and the set of sides of edges E and the bijection $\Phi : F \rightarrow E$ from (4.3.16) as in Section 4.3.2. We choose the map $V : \Sigma_0 \setminus \pi_0(T) \rightarrow E$ as in (4.3.17) so that the map \bar{V} from (4.3.46) satisfies

$$\bar{V}|_{\Sigma_0 \setminus \pi_0(T)} = \left[\Sigma_0 \setminus \pi_0(T) \xrightarrow{V} E \xrightarrow{\text{forget}} \Sigma_1 \right] , \quad (4.3.70)$$

where the map ‘forget’ is $(e, x) \mapsto e$. In addition, V has to satisfy that if v is on the left side of the defect line crossing the edge $\bar{V}(v)$ then $V(v) = (\bar{V}(v), r)$, otherwise $V(v) = (\bar{V}(v), l)$.

It will be convenient for the state-sum construction to know the phase labels of surface components in which faces and edges that are not intersected by defect lines lie. Similarly we will need to know the defect line labels of components intersected by faces and edges. Therefore we introduce the following for $k \in \{1, 2\}$:

- if $x \in \Sigma_k^{\text{empty}}$ we write $d_2(x) = d_1(x) = d_2(p)$ for the component $p \in \pi_0(\Sigma_{[2]})$ in which x lies,

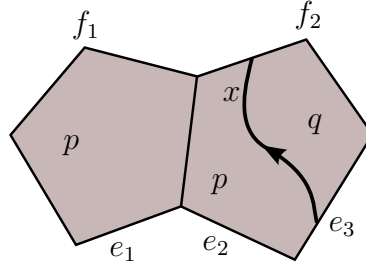


Figure 4.12: Notation for phase labels of empty cells and defect line labels of cells with defects. The phase label of the surface component left to the defect line is p , the phase label of the surface component to the right is q and the defect label is x , i.e. $t(x) = p$ and $s(x) = q$. The face f_1 and the edges e_1 and e_2 on the left are empty, i.e. $f_1 \in \Sigma_2^{\text{empty}}$ and $e_1, e_2 \in \Sigma_1^{\text{empty}}$. The corresponding phase labels are $d_2(f_1) = d_1(f_1) = d_2(e_1) = d_1(e_1) = d_2(e_2) = d_1(e_2) = p$. The face f_2 on the right and the edge e_3 on the right are intersected by a defect line, i.e. $f_2 \in \Sigma_2^{\text{defect}}$ and $e_3 \in \Sigma_1^{\text{defect}}$. The corresponding defect labels are $d_1(f_2) = d_1(e_3) = x$.

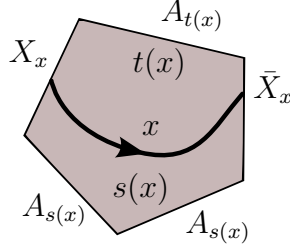


Figure 4.13: Objects from the state sum data with defects assigned to edges crossed by a defect line with defect line label $x \in D_1$.

- if $x \in \Sigma_k^{\text{defect}}$ we write $d_1(x) = d_1(q)$ for the defect line $q \in \pi_0(\Sigma_{[1]})$ intersected by x ,

which we illustrate in Figure 4.12.

After introducing these notations we are ready to define $\mathcal{Z}_{\mathbb{A}(\mathbb{D})}(\Sigma, \mathcal{A}, \mathcal{L})$. We proceed with the following steps.

1. Let $f \in \Sigma_2$ be a face with n_f sides. If $f \in \Sigma_2^{\text{empty}}$ then let $R^{(f,k)} := A_{d_2(f)}$. If $f \in \Sigma_2^{\text{defect}}$ then let n_f^o be the number of the edge where the defect with label x enters f . Then let

$$R^{(f,k)} := \begin{cases} \bar{X}_x & \text{if } k = 1, \\ A_{t(x)} & \text{if } 1 < k < n_f^o, \\ X_x & \text{if } k = n_f^o, \\ A_{s(x)} & \text{if } n_f^o < k, \end{cases} \quad (4.3.71)$$

and for a side of an edge $(e, y) \in E$

$$R^{(e,y)} := R^{\Phi^{-1}(e,y)}. \quad (4.3.72)$$

For these conventions see Figure 4.13.

Let us introduce the tensor products

$$\begin{aligned}\mathcal{O}_F &:= \bigotimes_{(f,k) \in F} R^{(f,k)} , & \mathcal{O}_E &:= \bigotimes_{(e,y) \in E} R^{(e,y)} , \\ \mathcal{O}_{\text{in}} &:= \bigotimes_{b \in \pi_0(S)} X_{\underline{x}(b)}^{(b,\text{in})} , & \mathcal{O}_{\text{out}} &:= \bigotimes_{c \in \pi_0(T)} X_{\underline{x}(c)}^{(c,\text{out})} ,\end{aligned}\tag{4.3.73}$$

using the notation from (4.3.66) and (4.3.71). The various superscripts will help us distinguish tensor factors in the source and target objects of the morphisms we define in the remaining steps.

2. We define the morphism

$$\mathcal{C} := \bigotimes_{e \in \Sigma_1 \setminus \pi_0(T)} \beta^{(e)} : \mathcal{O}_{\text{in}} \otimes \mathcal{O}_E \rightarrow \mathcal{O}_{\text{out}} ,\tag{4.3.74}$$

where $\beta^{(e)} = \beta_{\mathcal{A}_1(e)}^{d_1(e)}$ with the tensor factors given in Figure 4.6.

3. We define the morphism

$$\mathcal{Y} := \prod_{v \in \Sigma_0 \setminus \pi_0(T)} \zeta_{\mathcal{A}_0(v)}^{(V(v))} \in \mathcal{S}(\mathcal{O}_E, \mathcal{O}_E) ,\tag{4.3.75}$$

where

$$\zeta_a^{(e,y)} = \begin{cases} \text{id} \otimes \cdots \otimes \zeta_a^{d_2(e)} \otimes \cdots \otimes \text{id} & ; \text{ if } e \in \Sigma_1^{\text{empty}} \\ \text{id} \otimes \cdots \otimes \zeta_a^{d_1(e),+/-} \otimes \cdots \otimes \text{id} & ; \text{ if } e \in \Sigma_1^{\text{defect}} \end{cases} \in \mathcal{S}(\mathcal{O}_E, \mathcal{O}_E) ,\tag{4.3.76}$$

where ζ_a maps the tensor factor $R^{(e,y)}$ to itself, and $a \in \mathbb{R}_{>0}$ or $a \in \mathbb{R}_{>0}^3$.

4. For $f \in \Sigma_2^{\text{defect}}$ let n_f and n_f^o be as in step 1 and

$$W_{\mathcal{A}_2(f)}^f := W_{\mathcal{A}_2(f)}^{d_1(f), n_f - n_f^o, n_f^o - 2} ;\tag{4.3.77}$$

for $f \in \Sigma_2^{\text{empty}}$ let n_f be as before and

$$W_{\mathcal{A}_2(f)}^f := W_{\mathcal{A}_2(f)}^{d_1(f), n_f} .\tag{4.3.78}$$

In both cases the labeling of tensor factors is such that it matches (4.3.71). Define the morphism

$$\mathcal{F} := \bigotimes_{f \in \Sigma_2} \left(W_{\mathcal{A}_2(f)}^f \right) : \mathbb{I} \rightarrow \mathcal{O}_F .\tag{4.3.79}$$

5. We again put the above morphisms together as in Step 5 of Section 4.3.2:

$$\mathcal{K} := \left[\mathbb{I} \xrightarrow{\mathcal{F}} \mathcal{O}_F \xrightarrow{\Pi_\Phi} \mathcal{O}_E \xrightarrow{\mathcal{Y}} \mathcal{O}_E \right] , \quad (4.3.80)$$

$$\mathcal{L} := \left[\mathcal{O}_{\text{in}} \xrightarrow{\text{id}_{\mathcal{O}_{\text{in}}} \otimes \mathcal{K}} \mathcal{O}_{\text{in}} \otimes \mathcal{O}_E \xrightarrow{\mathcal{C}} \mathcal{O}_{\text{out}} \right] , \quad (4.3.81)$$

where Π_Φ is defined as in (4.3.24).

6. Using the embedding and projection maps from (4.3.66) we construct the following morphisms:

$$\mathcal{E}_{\text{in}} := \bigotimes_{b \in \pi_0(S)} \iota_{\underline{x}(b)}^{(b)} : \mathcal{Z}_{\mathbb{A}(\mathbb{D})}(S) \rightarrow \mathcal{O}_{\text{in}} , \quad \mathcal{E}_{\text{out}} := \bigotimes_{c \in \pi_0(T)} \pi_{\underline{x}(c)}^{(c)} : \mathcal{O}_{\text{out}} \rightarrow \mathcal{Z}_{\mathbb{A}(\mathbb{D})}(T) . \quad (4.3.82)$$

We finally define the action of $\mathcal{Z}_{\mathbb{A}(\mathbb{D})}$ on morphisms:

$$\mathcal{Z}_{\mathbb{A}(\mathbb{D})}(\Sigma, \mathcal{A}, \mathcal{L}) := \left[\mathcal{Z}_{\mathbb{A}(\mathbb{D})}(S) \xrightarrow{\mathcal{E}_{\text{in}}} \mathcal{O}_{\text{in}} \xrightarrow{\mathcal{L}} \mathcal{O}_{\text{out}} \xrightarrow{\mathcal{E}_{\text{out}}} \mathcal{Z}_{\mathbb{A}(\mathbb{D})}(T) \right] . \quad (4.3.83)$$

We defined $\mathcal{Z}_{\mathbb{A}(\mathbb{D})}$ on bordisms with defects with strictly positive area and length and now we give the definition in the general case. Let $(\Sigma, \mathcal{A}, \mathcal{L}) : S \rightarrow T$ be a bordism with area and defects and let $\Sigma_+ : S_+ \rightarrow T_+$ denote the connected component of (Σ, \mathcal{A}) with strictly positive area and length. The complement of Σ_+ again defines a permutation of tensor factors as in Section 4.3.2, so we define:

$$\mathcal{Z}_{\mathbb{A}(\mathbb{D})}(\Sigma, \mathcal{A}, \mathcal{L}) := \mathcal{Z}_{\mathbb{A}(\mathbb{D})}(\Sigma \setminus \Sigma_+, 0, 0) \otimes \mathcal{Z}_{\mathbb{A}(\mathbb{D})}(\Sigma_+, \mathcal{A}_+, \mathcal{L}_+) , \quad (4.3.84)$$

where \mathcal{A}_+ denotes the restriction of \mathcal{A} to $\pi_0((\Sigma_+)_k)$, $k = 1, 2$, and \mathcal{L}_+ is defined similarly and $\mathcal{Z}_{\mathbb{A}(\mathbb{D})}(\Sigma_+, \mathcal{A}_+, \mathcal{L}_+)$ is defined in (4.3.83).

We have the analogous theorem of Section 4.3.2.

Theorem 4.3.14. *Let $\mathbb{A}(\mathbb{D})$ be state-sum data with defects.*

1. *The morphism defined in (4.3.83) is independent of the choice of the PLCW decomposition with area and defects, the choice of marked edges of faces, the choice of orientation of edges and the assignment V .*
2. *The state-sum construction yields an aQFT $\mathcal{Z}_{\mathbb{A}(\mathbb{D})} : \mathcal{Bord}_{2,\mathbb{D}}^{\text{area}, \text{def}} \rightarrow \mathcal{S}$ given by (4.3.69) and (4.3.84), respectively.*

Sketch of proof. We only sketch some part of the proof of Part 1. We will check invariance under the additional elementary moves in Figure 4.10. Invariance under moves $b)$ and $c)$ directly follow from Condition 1 and 2 respectively. Invariance under move $a)$ can be shown using the same trick as in the proof of Theorem 4.3.5 Part 1 by combining the moves $b)$

and c) together with the move in Figure 4.7. We note that one needs to use Condition 3 to show independence of the choice of the map V .

Let $(C, \mathcal{A}, \mathcal{L})$ be a cylinder over a circle with defects with defect list \underline{x} and equal defect line lengths. The morphism in (4.3.81) associated to $(C, \mathcal{A}, \mathcal{L})$ is $E_{\underline{a}}^x$ from (4.3.59).

The proof of Part 2 goes along the same lines as the proof of Part 2 of Theorem 4.3.14. Joint continuity in the areas and lengths follows from Condition 5. \square

4.3.6 State-sum data with defects from bimodules

The purpose of this section is to give an algebraic characterisation of state-sum data with defects. We show that given state-sum data with defects for some objects we get a particular RFA and bimodule structure on the objects. This suggests that conversely given a particular RFA and bimodule structure on some objects we can get state-sum data on these objects. As before, we keep the notation from the previous sections.

Lemma 4.3.15. *For every $p \in D_2$ and $x \in D_1$ let us fix objects A_p and X_x in \mathcal{S} and state-sum data with defects $\mathbb{A}(\mathbb{D})$ for these objects. For $a = a_1 + a_2 + a_3$, $l = l_1 + l_2$ and $b = b_1 + b_2$ let*

$$\rho_{a,l,b}^x := \begin{array}{c} \begin{array}{c} (a_3, t(x)) \\ \nearrow \quad \searrow \\ \begin{array}{c} (a_1, l_1, b_1, x) \\ \nearrow \quad \searrow \\ A_{t(x)} \quad X_x \end{array} \end{array} \quad \begin{array}{c} X_x \\ \uparrow \\ \boxed{a_2, l_2, b_2; x, 1, 1} \end{array} \quad \begin{array}{c} (b_1, s(x)) \\ \nearrow \quad \searrow \\ A_{s(x)} \end{array} \end{array} . \quad (4.3.85)$$

Then the state-sum data $\mathbb{A}(\mathbb{D})$ determines

- a strongly separable symmetric RFA structure on A_p for every $p \in D_2$ as in Lemma 4.3.7 and
- a structure of a dual pair of $A_{t(x)}$ - $A_{s(x)}$ -bimodules on (X_x, \bar{X}_x) for every $x \in D_1$, where the actions on X_x and \bar{X}_x are given by $\rho_{a,l,b}^x$ and $\bar{\rho}_{a,l,b}^x$ from (4.3.85) respectively, and the pairing by $\beta_{a,l,b}^x$ and the copairing by $\gamma_{a,l,b}^x := W_{a,l,b}^{x,0,0}$.

Proof. Checking associativity (4.1.73) can be easily done using the graphical calculus and Conditions 1 and 2. The rest of the conditions on the action follow directly from the other conditions. Also checking that the duality morphisms satisfy (4.1.82) is straightforward. \square

Remark 4.3.16. We note that contrary to the state-sum construction of topological field theories, in general one cannot define left and right actions on the object X_x as the action (4.3.85) always comes with three strictly positive additive parameters. If for example the limit $\lim_{b \rightarrow 0} \rho_{a,l,b_1}^x \circ (\text{id}_{A_{t(x)} \otimes X_x} \otimes \eta_{b_2}^{A_{s(x)}})$ exists, then one can define a left action on X_x , cf. Remark 4.1.38. Note, however, that these limits need not exist, see Appendix 4.A for an example.

Proposition 4.3.17. *For every $p \in D_2$ and $x \in D_1$ let A_p be a strongly separable symmetric RFA and (X_x, \bar{X}_x) a dual pair of $A_{t(x)}\text{-}A_{s(x)}$ -bimodules with pairing $\beta_{a,l,b}^x$ and copairing $\gamma_{a,l,b}^x$. For $n, m \in \mathbb{Z}_{\geq 0}$ set*

with some distribution of the parameters on the rhs which sums up to a , b and l . Furthermore, let

where τ_a^{-1} denotes the inverse of the window element of $A_{s(x,\epsilon)}$. Suppose the following two conditions hold:

1. Let $(x_1, \epsilon_1; x_2, \epsilon_2) \in (D_1 \times \{\pm\})^2$ be such that $s(x_1, \epsilon_1) = t(x_2, \epsilon_2)$ from (4.3.55). Let $Y_i := X_{x_i}^{\epsilon_i}$ for $i = 1, 2$ and recall the morphisms $D_{a,b,c,l}^{Y_i, Y_{i+1}}$ and $D_{a,l}^{Y_i}$ from (4.1.94). We require that the limits

exist.

2. For every $n, m \in \mathbb{Z}_{>0}$ with $n + m \geq 1$, $(x_i, \epsilon_i) \in D_1 \times \{\pm\}$ for $i = 1, \dots, n$, $p_j \in D_2$

for $j = 1, \dots, m$ the assignment

$$\begin{aligned} (\mathbb{R}_{>0}^3 \cup \{0\})^n \times (\mathbb{R}_{\geq 0})^m &\rightarrow \mathcal{S} \left(\bigotimes_{i=1}^n X_{x_i}^{\epsilon_i} \otimes \bigotimes_{j=1}^m A_{p_j}, \bigotimes_{i=1}^n X_{x_i}^{\epsilon_i} \otimes \bigotimes_{j=1}^m A_{p_j} \right) \\ (a_1, l_1, b_1, \dots, a_n, l_n, b_n, c_1, \dots, c_m) &\mapsto \bigotimes_{i=1}^n Q_{a_i, l_i, b_i}^{X_{x_i}^{\epsilon_i}} \otimes \bigotimes_{j=1}^m P_{c_j}^{A_{p_j}} \end{aligned} \quad (4.3.89)$$

is jointly continuous.

Then (4.3.37), (4.3.38), (4.3.86), (4.3.87) and $\beta_{a,l,b}^x$ define state-sum data with defects.

Proof. From Lemma 4.3.8 we get the part of the state-sum data for elements of D_2 , so we turn directly to the part of the data for D_1 .

Checking the algebraic relations of Conditions 1, 2 and 3 can be done easily using the graphical calculus, and is similar to the case of RFAs and state-sum data without defects.

The last part of Condition 4 follows directly from X_x^ϵ being bimodules over RFAs. Condition 5 is just (4.3.89).

The only thing left to show is the first part of Condition 4, namely that for every defect list \underline{x} of length $n \geq 1$ the limit $\lim_{\underline{a} \rightarrow 0} E_{\underline{a}}^x$ exists. Let us introduce the shorthand $Y_i := X_{x_i}^{\epsilon_i}$. If $n = 1$ then $E_{\underline{a}}^x = D_{a+b,l}^{Y_i}$ from (4.1.94) and the limit $a, b, l \rightarrow 0$ exists by assumption. Now let $n \geq 2$. First rewrite $E_{\underline{a}}^x$ as

$$E_{\underline{a}}^x = \sigma_{Y_1 \otimes \dots \otimes Y_{n-1}, Y_n} \circ \left(D_{a_1, b_1, c_1, l_i}^{Y_n, Y_1} \otimes \text{id} \right) \circ \sigma_{Y_1 \otimes \dots \otimes Y_{n-1}, Y_n}^{-1} \circ \prod_{i=1}^{n-1} \text{id} \otimes D_{a_i, b_i, c_i, l_i}^{Y_i, Y_{i+1}} \otimes \text{id} \quad (4.3.90)$$

with the appropriate distribution of the parameters. Since the limits in (4.3.88) exist, we can rewrite (4.3.90) as

$$E_{\underline{a}}^x = \left(Q_{p_n}^{Y_n} \otimes \bigotimes_{i=1}^{n-1} Q_{p_i}^{Y_i} \right) \circ \tilde{E}_0^x, \quad (4.3.91)$$

with some distribution of the parameters, where \tilde{E}_0^x is the morphism obtained by taking the limits in the parameters of $D_{a_i, b_i, c_i, l_i}^{Y_i, Y_{i+1}}$ to 0 in (4.3.90) separately. The joint continuity condition in (4.3.89) together with (4.3.91) shows that the joint limit exists and is given by \tilde{E}_0^x . \square

Remark 4.3.18. For strongly separable symmetric RFAs and dual pairs of bimodules in \mathcal{Hilb} , conditions 1 and 2 in Proposition 4.3.17 are automatically satisfied, see Lemmas 4.1.15 and 4.1.61.

4.3.7 Defect fusion and tensor product of bimodules

In this section we are going to assume that the state-sum data is given in terms of strongly separable symmetric RFAs and dual pairs of bimodules for which the conditions of Proposition 4.3.17 hold. In Theorem 4.3.19 we show that the state spaces (4.3.69) can be explicitly computed in terms of tensor products of the bimodules over the intermediate RFAs, and in Theorem 4.3.20 we give the compatibility between the tensor product of bimodules and the fusion of defect lines.

Theorem 4.3.19. *Let $\mathbb{A}(\mathbb{D})$ be state-sum data given in terms of RFAs and bimodules as in Proposition 4.3.17, $\mathcal{Z}_{\mathbb{A}(\mathbb{D})}$ the state-sum aQFT from Section 4.3.5 and $S \in \text{Bord}_{2,\mathbb{D}}^{\text{area},\text{def}}$ be connected with corresponding defect list \underline{x} of length $n \geq 1$. Let us assume that for every $(x_i, \epsilon_i) \in D_1 \times \{\pm\}$ ($i = 1, 2$) satisfying $s(x_1, \epsilon_1) = t(x_2, \epsilon_2)$ of (4.3.55) the limit*

$$\lim_{a \rightarrow 0} \tilde{\rho}_{a,b,c,l}^{X_{x_1}^{\epsilon_1}, X_{x_2}^{\epsilon_2}} \quad (4.3.92)$$

of the morphism in (4.1.99) exists. Then with $B_i := A_{s(x_i, \epsilon_i)}$ we have

$$\mathcal{Z}_{\mathbb{A}(\mathbb{D})}(S) = Z(X_{\underline{x}}) = \circ_{B_n} X_{x_1}^{\epsilon_1} \otimes_{B_1} \cdots \otimes_{B_{n-1}} X_{x_n}^{\epsilon_n}. \quad (4.3.93)$$

Proof. We will prove the theorem for $n = 3$, for general n the proof is similar. Let $Y_i := X_{x_i}^{\epsilon_i}$ for $i = 1, 2, 3$. Let $D^{23} := \lim_{a,b,c,l \rightarrow 0} D_{a,b,c,l}^{Y_2, Y_3}$ from (4.1.94) and let π^{23} and ι^{23} denote the corresponding projection and embedding of its image $Y_2 \otimes_{B_2} Y_3$. By Proposition 4.1.51 $Y_2 \otimes_{B_2} Y_3$ is a B_1 - B_3 -bimodule. We show that $D^{123} := \lim_{a,b,c,l \rightarrow 0} D_{a,b,c,l}^{Y_1, Y_2 \otimes_{B_2} Y_3}$ exists:

$$\begin{aligned} \lim_{a,b,c,l \rightarrow 0} D_{a,b,c,l}^{Y_1, Y_2 \otimes_{B_2} Y_3} &= \lim_{a,b,c,l \rightarrow 0} \begin{array}{c} Y_1 \quad Y_2 \otimes_{B_2} Y_3 \\ | \quad | \\ (b_1, l, a_1) \quad (a_3, l, c_1) \\ \swarrow \quad \downarrow \quad \searrow \\ b_2 \quad \boxed{e_{a_2}} \quad c_2 \\ | \quad | \\ Y_1 \quad Y_2 \otimes_{B_2} Y_3 \end{array} \\ &\stackrel{\text{def.}}{=} \lim_{a,a',b,c,l \rightarrow 0} \begin{array}{c} Y_1 \quad Y_2 \otimes_{B_2} Y_3 \\ | \quad | \\ (b_1, l, a_1) \quad (a_3, l, a'_1) \quad (a'_3, l, c_1) \\ \swarrow \quad \downarrow \quad \searrow \\ b_2 \quad \boxed{e_{a_2}} \quad \boxed{e_{a'_2}} \quad c_2 \\ | \quad | \quad | \\ Y_1 \quad \boxed{\pi^{23}} \quad \boxed{\iota^{23}} \quad Y_2 \otimes_{B_2} Y_3 \end{array} = \begin{array}{c} Y_1 \quad Y_2 \otimes_{B_2} Y_3 \\ | \quad | \\ \boxed{\pi^{23}} \\ | \\ \boxed{D^{23}} \\ | \\ \boxed{D^{12}} \\ | \\ Y_1 \quad \boxed{\iota^{23}} \quad Y_2 \otimes_{B_2} Y_3 \end{array}, \quad (4.3.94) \end{aligned}$$

where in the last equation we used associativity of the action on Y_2 and took the limits separately (the joint limit exists for the same reason as in the proof of Proposition 4.3.17). Let π^{123} and ι^{123} denote the projection and embedding of the image of D^{123} which is $Y_1 \otimes_{B_1} Y_2 \otimes_{B_2} Y_3$. Note that the projectors for $(Y_1 \otimes_{B_1} Y_2) \otimes_{B_2} Y_3$ and $Y_1 \otimes_{B_1} (Y_2 \otimes_{B_2} Y_3)$ are the same, hence they have the same image and we can omit the brackets.

Similarly one shows using (4.3.90) that

$$D_0^{Y_1 \otimes_{B_1} Y_2 \otimes_{B_2} Y_3} = \lim_{a,l \rightarrow 0} \begin{array}{c} Y_1 \otimes_{B_1} Y_2 \otimes_{B_2} Y_3 \\ | \\ \bullet \quad (a_1, l, a_3) \\ | \\ Y_1 \otimes_{B_1} Y_2 \otimes_{B_2} Y_3 \\ | \\ \boxed{e_{a_2}} \end{array} = \begin{array}{c} Y_1 \otimes_{B_1} Y_2 \otimes_{B_2} Y_3 \\ | \\ \boxed{\pi^{123}} \\ | \\ \boxed{\pi^{23}} \\ | \\ \boxed{E_0^x} \\ | \\ \boxed{\iota^{23}} \\ | \\ \boxed{\iota^{123}} \\ | \\ Y_1 \otimes_{B_1} Y_2 \otimes_{B_2} Y_3 \end{array} . \quad (4.3.95)$$

Let π° and ι° denote the projection and embedding of the image of $D^{Y_1 \otimes_{B_1} Y_2 \otimes_{B_2} Y_3}$ which is $\circ_{B_3} Y_1 \otimes_{B_1} Y_2 \otimes_{B_2} Y_3$. Now a simple computation shows that

$$\begin{array}{c} Y_1 \otimes_{B_1} Y_2 \otimes_{B_2} Y_3 \\ | \\ \boxed{\pi^\circ} \\ | \\ \boxed{\pi^{123}} \\ | \\ Y_1 \quad Y_2 \quad Y_3 \\ | \quad | \quad | \\ \boxed{\pi^{23}} \end{array} \quad \text{and} \quad \begin{array}{c} Y_1 \quad Y_2 \quad Y_3 \\ | \quad | \quad | \\ \boxed{\iota^{23}} \\ | \\ \boxed{\iota^{123}} \\ | \\ \boxed{\iota^\circ} \\ | \\ Y_1 \otimes_{B_1} Y_2 \otimes_{B_2} Y_3 \end{array} \quad (4.3.96)$$

satisfy $\iota \circ \pi = E_0^x$ and $\pi \circ \iota = \text{id}_{\circ_{B_3} Y_1 \otimes_{B_1} Y_2 \otimes_{B_2} Y_3}$, that is the image of E_0^x is exactly $\circ_{B_3} Y_1 \otimes_{B_1} Y_2 \otimes_{B_2} Y_3$. \square

Let Σ be a bordism with defects. We say that two defect lines $x_0, x_1 \in \Sigma_{[1]}$ are *parallel* if there is an isotopy $t \mapsto x_t$ between them such that for every $t \in (0, 1)$ x_t does not intersect any defect line and in case x_0 and x_1 start and end on the boundary of Σ , x_t starts (resp. ends) on the same boundary component as x_0 and x_1 . Let us consider two bordisms with area and defects:

1. $(\Sigma, \mathcal{A}, \mathcal{L})$, which has two parallel defect lines with length l labeled with a B - A -bimodule V and an A - C -bimodule W , and with a surface component with area a between them and
2. $(\Sigma', \mathcal{A}', \mathcal{L}')$ which is the same as $(\Sigma, \mathcal{A}, \mathcal{L})$ except that the defect line x_0 is removed from $\Sigma_{[1]}$ and the surface component between x_0 and x_1 is collapsed. The remaining

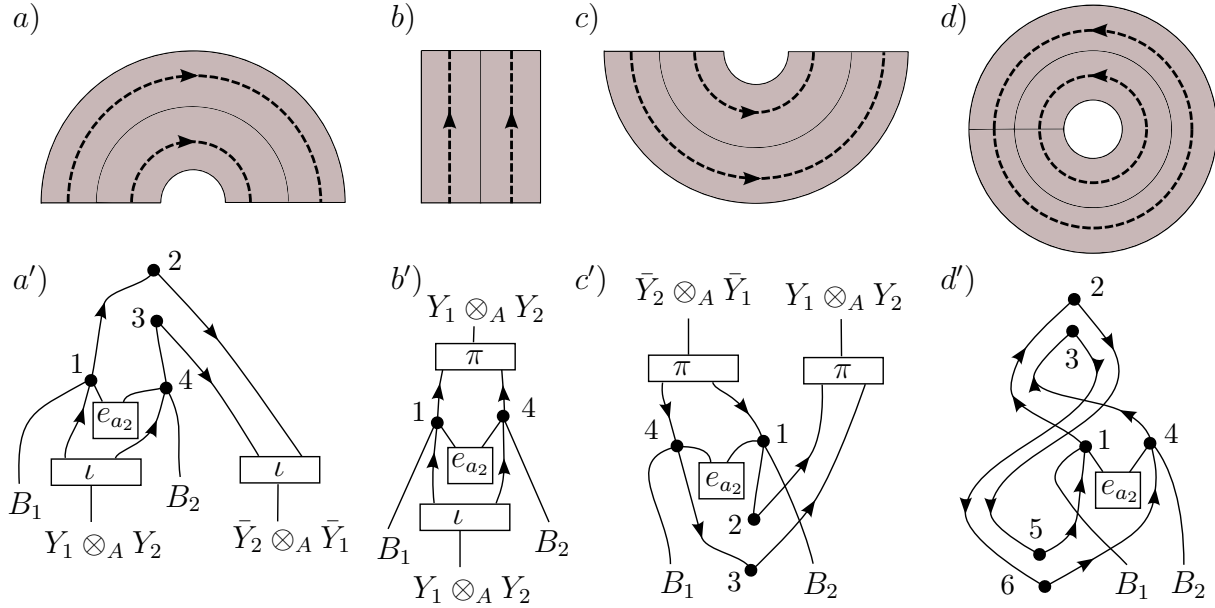


Figure 4.14: Detail of the chosen PLCW decomposition of Σ depending on the starting and ending points of the parallel defect lines (Parts a)-d)) and the part of the corresponding morphisms given by the state-sum construction (Parts a')-d')). In Part a) the defect lines start and end on an ingoing boundary component, in Part b) they start on an ingoing and end on an outgoing boundary component, in Part c) they start and end on an outgoing boundary component and in Part d) the defect lines are closed loops. In Parts a')-d') the numbers attached to the vertices for readability relate to the parameters as follows: $1 \hat{=} (b_1, l_1, a_1; Y_1)$, $2 \hat{=} (b_2, l_2, a_4; Y_1)$, $3 \hat{=} (a_3, l_2, c_2; Y_2)$, $4 \hat{=} (a_5, l_1, c_1; Y_2)$, $5 \hat{=} (b_3, l_3, a_6; Y_1)$, $6 \hat{=} (a_7, l_3, c_3; Y_2)$.

defect line x_1 is labeled by $V \otimes_A W$. The length of x_1 is l and the area and length of the other defect lines and surface components are unchanged.

Theorem 4.3.20. *Let $\mathbb{A}(\mathbb{D})$ be state-sum data satisfying the conditions of Theorem 4.3.19. Let us assume that, in addition, the limits*

$$\lim_{a \rightarrow 0} \tilde{\gamma}_{a,b,c,l}^{V,W} \quad \text{and} \quad \lim_{a \rightarrow 0} \tilde{\beta}_{a,b,c,l}^{V,W} \quad (4.3.97)$$

exist for every $b, c, l \in \mathbb{R}_{>0}$. Then we have

$$\mathcal{Z}_{\mathbb{A}(\mathbb{D})}(\Sigma', \mathcal{A}', \mathcal{L}') = \lim_{a \rightarrow 0} \mathcal{Z}_{\mathbb{A}(\mathbb{D})}(\Sigma, \mathcal{A}, \mathcal{L}). \quad (4.3.98)$$

Sketch of proof. Let us choose a PLCW decomposition of Σ which contains two rectangles, each of which is containing one of the parallel defect lines. For Σ' we choose a PLCW decomposition which contains a rectangle containing the defect line. Depending on where the defect lines start and end we have 4 essentially different cases that we need to consider and which we show in Figure 4.14 Parts a)-d).

The corresponding detail of the morphism $\mathcal{Z}_{\mathbb{A}(\mathbb{D})}(\Sigma, \mathcal{A}, \mathcal{L})$ is shown in Figure 4.14 Parts a' – d'). We see that after taking the limit $a \rightarrow 0$, and looking at the definition of the bimodule $Y_1 \otimes_A Y_2$ and its dual $\bar{Y}_2 \otimes_A \bar{Y}_1$ in Definition 4.1.49 and in Lemma 4.1.55, we obtain exactly $\mathcal{Z}_{\mathbb{A}(\mathbb{D})}(\Sigma', \mathcal{A}', \mathcal{L}')$. \square

Remark 4.3.21. For RFAs and bimodules in \mathcal{Hilb} which are left and right modules as well and for which $Q_{a,l,b}$ is epi for every $a, l, b \in \mathbb{R}_{>0}$ (cf. Remark 4.1.38 and Proposition 4.1.60), the conditions of Theorem 4.3.19 are automatically satisfied, since the limits in (4.3.88) exist by Lemma 4.1.61 and those in (4.3.92) by Proposition 4.1.60.

We stress here that one should not necessarily expect that the limits in (4.3.97) exist. It may happen that when one brings two defect lines near each other the correlators of the quantum field theory diverge, for an example in conformal field theory see e.g. [BB].

4.4 Example: 2d Yang-Mills theory

The state-sum construction of 2d Yang-Mills theory has been introduced by [Mig], was further developed for $G = U(N)$ in [Rus], and has been summarised in [Wit1]; for a review see [CMR]. There, partition functions and expectation values of Wilson loops were calculated. The proof of convergence of the (Boltzmann) plaquette weights has been shown in a different setting in [App]. In this section we will heavily rely on the representation theory of compact Lie groups, a standard reference is e.g. [Kna].

4.4.1 Two RFAs from a compact group G

Let G be a compact semisimple Lie group and $\int_G dx$ the Haar integral on G with the normalisation $\int_G 1 dx = 1$. We denote with $L^2(G)$ the Hilbert space of square integrable complex functions on G , where the scalar product of $f, g \in L^2(G)$ is given by $\langle f, g \rangle := \int f(x)^* g(x) dx$.

Let \hat{G} denote a set of representatives of isomorphism classes of finite-dimensional simple unitary G -modules. Then for $V \in \hat{G}$ with inner product $\langle -, - \rangle_V$ and an orthonormal basis $\{e_i^V\}_{i=1}^{\dim(V)}$ let

$$\begin{aligned} f_{ij}^V : G &\rightarrow \mathbb{C} \\ g &\mapsto (\dim(V))^{1/2} \langle e_i^V, g \cdot e_j^V \rangle_V \end{aligned} \quad (4.4.1)$$

denote a *matrix element function* and let M_V denote the linear span of these. The matrix element functions are orthonormal [Kna, Cor. 4.2]: for $V, W \in \hat{G}$, $i, j \in \{1, \dots, \dim(V)\}$ and $k, l \in \{1, \dots, \dim(W)\}$

$$\langle f_{ij}^V, f_{kl}^W \rangle = \delta_{ik} \delta_{jl} \delta_{V,W} \quad (4.4.2)$$

where $\delta_{V,W} = 1$ if $V = W$ and 0 otherwise. The *character* of V is defined as

$$\chi_V = (\dim(V))^{-1/2} \sum_{i=1}^{\dim(V)} f_{ii}^V. \quad (4.4.3)$$

The Peter-Weyl theorem provides a complete orthonormal basis of $L^2(G)$ in terms of matrix element functions and of the square integrable class functions $Cl^2(G)$ in terms of characters:

$$L^2(G) \cong \bigoplus_{V \in \hat{G}} M_V \quad \text{and} \quad Cl^2(G) \cong \bigoplus_{V \in \hat{G}} \mathbb{C} \cdot \chi_V \quad (4.4.4)$$

as Hilbert space direct sums. Note that $L^2(G) \otimes L^2(G) \cong L^2(G \times G)$ and $Cl^2(G) \otimes Cl^2(G) \cong Cl^2(G \times G)$ isometrically by mapping $f \otimes f'$ to the function $(g, g') \mapsto f(g)f'(g')$. We will often use these isomorphisms without further notice.

In the following we will define a \dagger -RFA structure on $L^2(G)$ and $Cl^2(G)$. Let us start with defining the operator

$$\begin{aligned} \Delta : L^2(G) &\rightarrow L^2(G) \otimes L^2(G) \\ f &\mapsto \Delta(f) = [(x, y) \mapsto f(xy)] \end{aligned} \quad (4.4.5)$$

which has norm 1. Let $\mu := \Delta^\dagger : L^2(G) \otimes L^2(G) \rightarrow L^2(G)$ be its adjoint, which is given by the convolution product. For $F \in L^2(G) \otimes L^2(G)$

$$\mu(F)(y) = \int_G F(x, x^{-1}y) dx . \quad (4.4.6)$$

Let $V \in \hat{G}$ and let us denote with $\sigma_V \in \mathbb{R}$ the value of the Casimir operator of G in the module V . We define for $a \in \mathbb{R}_{>0}$

$$\begin{aligned} \eta_a : \mathbb{C} &\rightarrow L^2(G) \\ 1 &\mapsto \eta_a(1) = \sum_{V \in \hat{G}} e^{-a\sigma_V} \dim(V) \chi_V . \end{aligned} \quad (4.4.7)$$

Lemma 4.4.1. *The sum in (4.4.7) is absolutely convergent for every $a \in \mathbb{R}_{>0}$.*

Proof. This follows from [App, Sec. 3], which we explain now. Let us fix a maximal torus of G and let T denote its Lie algebra, let $\Lambda^+ \subset T^*$ denote the set of dominant weights and let $(-, -)$ be the inner product on T^* induced by the Killing form and $|\cdot|$ the induced norm. We will use that, since G is semisimple, there is a bijection of sets [Kna, Thm. 5.5]

$$\begin{aligned} \hat{G} &\xrightarrow{\cong} \Lambda^+ \\ V &\mapsto \lambda_V , \\ V_\lambda &\leftarrow \lambda . \end{aligned} \quad (4.4.8)$$

From [Sug, (1.17)] and [App, (3.2)] we have that (by the Weyl dimension formula) for $V \in \hat{G}$ with dominant weight $\lambda_V \in \Lambda^+$

$$\dim(V) \leq N |\lambda_V|^m , \quad (4.4.9)$$

where $N \in \mathbb{R}_{>0}$ is a constant independent of V and $2m = \dim(G) - \text{rank}(G)$.

From [Sug, Lem. 1.1] we can express the value of the Casimir element in V using the highest weight λ_V of V and the half sum of simple roots ρ as

$$\sigma_V = (\lambda_V, \lambda_V + 2\rho) . \quad (4.4.10)$$

It follows directly [App, (3.5)] that

$$|\lambda_V|^2 \leq \sigma_V . \quad (4.4.11)$$

We can give an estimate for the norm of λ_V as follows. The choice of simple roots gives a bijection $\mathbb{Z}^{\text{rank}(G)} \rightarrow \Lambda^+$ which we write as $n \mapsto \lambda(n)$. Using the proof of [Sug, Lem. 1.3] there are $C_1, C_2 \in \mathbb{R}_{\geq 0}$ such that for every $n \in \mathbb{Z}^{\text{rank}(G)}$

$$C_1 \|n\| \leq |\lambda(n)| \leq C_2 \|n\| , \quad (4.4.12)$$

where $\|n\|^2 = \sum_{i=1}^{\text{rank}(G)} n_i^2$.

Let $b(j)$ denote the number of $n \in \mathbb{Z}^{\text{rank}(G)}$ with $\|n\|^2 = j$. We can easily give a (very rough) estimate of this by the volume of the $\text{rank}(G)$ -dimensional cube with edge length $2j^{1/2} + 1$:

$$b(j) \leq (2j^{1/2} + 1)^{\text{rank}(G)} . \quad (4.4.13)$$

We compute the squared norm of η_a following the computation in [App, Ex. 3.4.1].

$$\begin{aligned} \|\eta_a\|^2 &= \sum_{V \in \hat{G}} (\dim(V))^2 e^{-2a\sigma_V} \stackrel{(4.4.8)}{=} \sum_{\lambda \in \Lambda^+} (\dim(V_\lambda))^2 e^{-2a\sigma_{V_\lambda}} \\ &\stackrel{(4.4.9)}{\leq} N^2 \sum_{\lambda \in \Lambda^+} |\lambda|^{2m} e^{-2a\sigma_{V_\lambda}} \stackrel{(4.4.11)}{\leq} N^2 \sum_{\lambda \in \Lambda^+} |\lambda|^{2m} e^{-2a|\lambda|^2} \\ &\stackrel{(4.4.12)}{\leq} N^2 C_2^{2m} \sum_{n \in \mathbb{Z}^{\text{rank}(G)}} \|n\|^{2m} e^{-2aC_1 \|n\|^2} = N^2 C_2^{2m} \sum_{j=1}^{\infty} b(j) j^m e^{-2aC_1 j} \\ &\stackrel{(4.4.13)}{\leq} N^2 C_2^{2m} \sum_{j=1}^{\infty} (2j^{1/2} + 1)^{\text{rank}(G)} j^m e^{-2aC_1 j} , \end{aligned} \quad (4.4.14)$$

which converges. \square

Finally we define the counit as $\varepsilon_a := \eta_a^\dagger : L^2(G) \rightarrow \mathbb{C}$. Explicitly, for $f \in L^2(G)$,

$$\varepsilon_a(f) = \langle \eta_a, f \rangle = \sum_{V \in \hat{G}} e^{-a\sigma_V} \dim(V) \int_G \chi_V(x) f(x^{-1}) dx . \quad (4.4.15)$$

Again for $a \in \mathbb{R}_{>0}$ let

$$\begin{aligned} P_a : L^2(G) &\rightarrow L^2(G) \\ f &\mapsto \mu(\eta_a \otimes f) , \end{aligned} \quad (4.4.16)$$

$\mu_a := P_a \circ \mu$ and $\Delta_a := \Delta \circ P_a$.

Proposition 4.4.2. $L^2(G)$, together with the family of morphisms μ_a , η_a , Δ_a and ε_a for $a \in \mathbb{R}_{>0}$ defined above is a strongly separable symmetric \dagger -RFA in \mathcal{Hilb} .

Before proving this proposition let us state a lemma. Let $V \in \hat{G}$ and define

$$\begin{aligned} \mu_a^V &:= \mu_a|_{M_V \otimes M_V}, & \eta_a^V &:= e^{-a\sigma_V} \dim(V) \chi_V, \\ \Delta_a^V &:= \Delta|_{M_V}, & \varepsilon_a^V &:= \varepsilon_a|_{M_V}. \end{aligned} \quad (4.4.17)$$

From a computation using orthogonality of the f_{ij}^V we can obtain the following formulas:

$$P_a(f_{ij}^V) = e^{-a\sigma_V} f_{ij}^V \in M_V, \quad (4.4.18)$$

$$\mu_a(f_{ij}^V \otimes f_{kl}^V) = \delta_{jk} e^{-a\sigma_V} (\dim(V))^{-1/2} f_{il}^V \in M_V, \quad (4.4.19)$$

$$\Delta_a(f_{ij}^V) = e^{-a\sigma_V} (\dim(V))^{-1/2} \sum_{k=1}^{\dim(V)} f_{ik}^V \otimes f_{kj}^V \in M_V \otimes M_V, \quad (4.4.20)$$

$$\varepsilon_a(f_{ij}^V) = e^{-a\sigma_V} (\dim(V))^{1/2} \delta_{ij}. \quad (4.4.21)$$

Lemma 4.4.3. Let $V \in \hat{G}$. Then M_V is a strongly separable symmetric \dagger -RFA in \mathcal{Hilb} with the structure maps in (4.4.17).

Proof. Checking the algebraic relations is a straightforward calculation. As an example, we compute the window element of M_V .

$$\begin{aligned} \mu_{a_1}^V \circ \Delta_{a_2}^V \circ \eta_{a_3}^V &= \sum_{l=1}^{\dim(V)} \mu_{a_1}^V \circ \Delta_{a_2}^V(f_{ll}^V) e^{-a_3\sigma_V} (\dim(V))^{1/2} \\ &= \sum_{k,l=1}^{\dim(V)} \mu_{a_1}^V(f_{lk}^V \otimes f_{kl}^V) e^{-(a_2+a_3)\sigma_V} \\ &= \sum_{k,l=1}^{\dim(V)} f_{ll}^V e^{-(a_1+a_2+a_3)\sigma_V} (\dim(V))^{-1/2} \\ &= e^{-(a_1+a_2+a_3)\sigma_V} \dim(V) \chi_V = \eta_{a_1+a_2+a_3}^V, \end{aligned} \quad (4.4.22)$$

which is clearly invertible. □

Proof of Proposition 4.4.2. Let $V \in \hat{G}$ and let us compute the following norms.

$$\begin{aligned} \|\eta_a^V\|^2 &= e^{-2a\sigma_V} (\dim(V))^2 \langle \chi_V, \chi_V \rangle = e^{-2a\sigma_V} \dim(V) \sum_{k,l=1}^{\dim(V)} \langle f_{kk}^V, f_{ll}^V \rangle \\ &= e^{-2a\sigma_V} (\dim(V))^2. \end{aligned} \quad (4.4.23)$$

Let $\varphi = \sum_{i,j=1}^{\dim(V)} \varphi_{ij} f_{ij}^V \in M_V$ and compute

$$\|\Delta_a^V(\varphi)\|^2 = e^{-2a\sigma_V} (\dim(V))^{-1} \sum_{i,j,k=1}^{\dim(V)} |\varphi_{ij}|^2 \|f_{ik}^V \otimes f_{kj}^V\|^2 = e^{-2a\sigma_V} \|\varphi\|^2, \quad (4.4.24)$$

so $\|\Delta_a^V\| = e^{-a\sigma_V}$. Since M_V is a \dagger -RFA, $\|\varepsilon_a^V\| = \|\eta_a^V\|$ and $\|\mu_a^V\| = \|\Delta_a^V\|$.

We now would like to take the direct sum of the RFAs M_V for all $V \in \hat{G}$, so we check the conditions of Proposition 4.1.18: the sum is convergent since it is the squared norm of $\eta_a \in L^2(G)$ and the supremum is clearly bounded. Therefore $L^2(G)$ is an RFA.

Checking that $L^2(G)$ is strongly separable, symmetric and Hermitian is straightforward using Lemma 4.4.3. \square

Now we turn to define an RFA structure on $Cl^2(G)$.

Proposition 4.4.4. *The centre of $L^2(G)$ is $Cl^2(G)$ and it is a commutative \dagger -RFA.*

Proof. Let us compute the morphism D_a from (4.3.3), which is the same as \tilde{D}_a from (4.3.38) by Lemma 4.3.8. For $\varphi = \sum_{V \in \hat{G}} \sum_{i,j=1}^{\dim(V)} \varphi_{ij}^V f_{ij}^V \in L^2(G)$ we find:

$$\begin{aligned}
 D_a(\varphi) &= \mu_{a_2} \circ \sigma_{L^2(G), L^2(G)} \circ \Delta_{a_1}(\varphi) \\
 &= \mu_{a_2} \circ \sigma_{L^2(G), L^2(G)} \left(\sum_{V \in \hat{G}} \sum_{i,j,k=1}^{\dim(V)} \varphi_{ij}^V f_{ik}^V \otimes f_{kj}^V \right) e^{-a_1 \sigma_V (\dim(V))^{-1/2}} \\
 &= \sum_{V \in \hat{G}} \sum_{i,j,k=1}^{\dim(V)} \varphi_{ij}^V e^{-a \sigma_V (\dim(V))^{-1}} \delta_{ij} f_{kk}^V \\
 &= \sum_{V \in \hat{G}} \sum_{i=1}^{\dim(V)} \varphi_{ii}^V e^{-a \sigma_V (\dim(V))^{-1/2}} \chi_V .
 \end{aligned} \tag{4.4.25}$$

From this equation we immediately have that $D_a|_{Cl^2(G)} = P_a|_{Cl^2(G)}$. We now show that $D_a|_{(Cl^2(G))^\perp} = 0$. Using (4.4.4), we have that $\varphi \in (Cl^2(G))^\perp \subset L^2(G)$ if and only if for every $W \in \hat{G}$ $\langle \chi_W, \varphi \rangle = 0$. We can compute this using the orthogonality relation (4.4.2) to get the following: $\varphi \in (Cl^2(G))^\perp$ if and only if for every $W \in \hat{G}$ we have that $\sum_{k=1}^{\dim(W)} \varphi_{kk}^W = 0$. By (4.4.25) we get that $D_a(\varphi) = 0$.

Altogether, this shows that the limit $\lim_{a \rightarrow 0} D_a$ (in the strong operator topology) exists and D_0 is an orthogonal projection onto $Cl^2(G)$. Therefore by Lemma 4.3.10 the centre of $L^2(G)$ is $Cl^2(G)$. It is a \dagger -RFA, since $L^2(G)$ is a \dagger -RFA and D_0 is self-adjoint. \square

For completeness we give the comultiplication $\Delta_a^{Cl^2(G)}$ of $Cl^2(G)$. For $\varphi = \sum_{V \in \hat{G}} \varphi^V \chi_V \in Cl^2(G)$

$$\Delta_a^{Cl^2(G)}(\varphi) = \sum_{V \in \hat{G}} \varphi^V e^{-a \sigma_V (\dim(V))^{-2}} \chi_V \otimes \chi_V . \tag{4.4.26}$$

Remark 4.4.5. Note that for both $L^2(G)$ and $Cl^2(G)$, the $a \rightarrow 0$ limit of the multiplication and comultiplication exists (by definition), but the $a \rightarrow 0$ limit of the unit and counit does not.

4.4.2 State-sum construction of 2d Yang-Mills theory

In this section we give state-sum data for the 2d YM theory following [Wit1]. The plaquette weights $W_a^k : \mathbb{C} \rightarrow (L^2(G))^{\otimes k}$ for $k \in \mathbb{Z}_{\geq 0}$ and $a \in \mathbb{R} > 0$ are

$$W_a^k(1)(x_1, \dots, x_k) = \sum_{V \in \hat{G}} e^{-a\sigma_V \dim(V)} \chi_V(x_1 \cdots x_k) , \quad (4.4.27)$$

and the contraction and ζ_a are given by

$$\beta_a := (W_a^2)^\dagger , \quad \zeta_a := P_a , \quad (4.4.28)$$

where P_a is as in (4.4.16).

Proposition 4.4.6. *The morphisms (4.4.27) and (4.4.28) define state-sum data.*

Proof. We prove this by showing that the morphisms in (4.4.27) and (4.4.28) can be obtained from the RFA $L^2(G)$ via Lemma 4.3.8.

Since the inverse of the window element of $L^2(G)$ is simply η_a , we immediately get that $\zeta_a = P_a$. Let us look at the morphisms W_a^k . For $k = 1$ we have $W_a^1 = \eta_a$. Let us assume that for $k > 1$ we have $W_{a_1+a_2}^k = \Delta_{a_1}^{(k)} \circ \eta_{a_2}$. Then for $k + 1$ we see that

$$\begin{aligned} & (\text{id} \otimes \cdots \otimes \text{id} \otimes \Delta_{a_3}) \circ W_{a_1+a_2}^k(1) \\ &= (\text{id} \otimes \cdots \otimes \text{id} \otimes \Delta_{a_3}) \left(\sum_{V \in \hat{G}} e^{-(a_1+a_2)\sigma_V \dim(V)} \chi_V(x_1 \cdots x_k) \right) \\ &= \sum_{V \in \hat{G}} e^{-(a_1+a_2+a_3)\sigma_V \dim(V)} \chi_V(x_1 \cdots x_k x_{k+1}) . \end{aligned}$$

That $\beta_a = (W_a^2)^\dagger$ follows from $L^2(G)$ being a \dagger -RFA.

Finally we need to check that the limit $\lim_{a \rightarrow 0} D_a$ exists, which we have already checked in the proof of Proposition 4.4.4. \square

After this preparation we are ready to define 2d YM theory, which maps \mathbb{S}^1 to the centre of $L^2(G)$, see Proposition 4.4.4.

Definition 4.4.7. *The 2-dimensional Yang-Mills (2d YM) theory with gauge group G is the area-dependent QFT*

$$\begin{aligned} \mathcal{Z}_{\text{YM}}^G : \text{Bord}_2^{\text{area}} &\rightarrow \text{Hilb} \\ \mathbb{S}^1 &\mapsto C\ell^2(G) \end{aligned} \quad (4.4.29)$$

of Theorem 4.3.5 obtained from the state-sum data in (4.4.27) and (4.4.28).

Next we compute $\mathcal{Z}_{\text{YM}}^G$ on connected surfaces with area and $b \geq 0$ outgoing boundary components. For $b = 0$ the result agrees with [Wit1, Eqn. (2.51)] (see also [Rus, Eqn. (27)]).

Proposition 4.4.8. *Let $(\Sigma, a) : (\mathbb{S}^1)^{\sqcup b_{\text{in}}} \rightarrow (\mathbb{S}^1)^{\sqcup b_{\text{out}}}$ be a connected bordism of genus g with b_{in} ingoing and b_{out} outgoing boundary components and with area a . Then for $V_j \in \hat{G}$ for $j = 1, \dots, b_{\text{out}}$ we have*

$$\begin{aligned} & \mathcal{Z}_{\text{YM}}^G(\Sigma, a)(\chi_{V_1} \otimes \dots \otimes \chi_{V_{b_{\text{out}}}}) \\ &= \begin{cases} \sum_{V \in \hat{G}} e^{-a\sigma_V} (\dim(V))^{\chi(\Sigma)} \cdot (\chi_V)^{\otimes b_{\text{out}}} & \text{if } b_{\text{in}} = 0 \\ e^{-a\sigma_{V_1}} (\dim(V_1))^{\chi(\Sigma)} \cdot (\chi_{V_1})^{\otimes b_{\text{out}}} & \text{if } b_{\text{in}} \geq 1 \text{ and } V_1 = \dots = V_{b_{\text{in}}} \\ 0 & \text{else} \end{cases} \end{aligned} \quad (4.4.30)$$

where $\chi(\Sigma) = 2 - 2g - b_{\text{in}} - b_{\text{out}}$ is the Euler characteristic of Σ . For $b_{\text{in}} = 0$ ($b_{\text{out}} = 0$) the source (the target) is \mathbb{C} and the factors of χ_V or χ_{V_j} are absent.

Proof. We first consider the case that $b := b_{\text{out}} \geq 1$ and $b_{\text{in}} = 0$. The map φ_a from (4.3.41) is given by

$$\begin{aligned} \varphi_a(f_{ij}^V) &= \mu \circ (\text{id} \otimes (\mu \circ \sigma_{L^2(G), L^2(G)})) \left(\sum_{k,l=1}^{\dim(V)} e^{-a\sigma_V} (\dim(V))^{-1} f_{ik}^V \otimes f_{kl}^V \otimes f_{lj}^V \right) \\ &= \sum_{k,l=1}^{\dim(V)} e^{-a\sigma_V} (\dim(V))^{-2} \delta_{jk} \delta_{kl} f_{il}^V = e^{-a\sigma_V} (\dim(V))^{-2} f_{ij}^V. \end{aligned} \quad (4.4.31)$$

Using this, we compute for $a_0, \dots, a_{g+1} \in \mathbb{R}_{>0}$ with $a = \sum_{i=0}^{g+1} a_i$ that

$$\begin{aligned} \Delta_{a_{g+1}}^{(b)} \circ \prod_{i=1}^g \varphi_{a_i} \circ \eta_{a_0} &= \Delta_{a_{g+1}}^{(b)} \left(\sum_{V \in \hat{G}} e^{-(a-a_{g+1})\sigma_V} (\dim(V))^{1-2g} \chi_V \right) \\ &= \left[(x_1, \dots, x_b) \mapsto \left(\sum_{V \in \hat{G}} e^{-a\sigma_V} (\dim(V))^{1-2g} \chi_V(x_1 \dots x_b) \right) \right]. \end{aligned} \quad (4.4.32)$$

Finally, according to Part 2 of Lemma 4.3.12, we need to compose (4.4.32) with $\pi^{\otimes b}$ to get (4.4.30), where $\pi : L^2(G) \rightarrow Cl^2(G)$ is the projection onto the image of D_0 . To arrive at (4.4.30), we further compute

$$\begin{aligned} \pi^{\otimes b}(\chi_V(x_1 \dots x_b)) &= \pi^{\otimes b} \left(\sum_{k_1, \dots, k_b=1}^{\dim(V)} (\dim(V))^{-b/2} f_{k_1 k_2}^V(x_1) \dots f_{k_b k_1}^V(x_b) \right) \\ &= (\dim(V))^{1-b} \chi_V(x_1) \dots \chi_V(x_b). \end{aligned} \quad (4.4.33)$$

For the case $b_{\text{in}} = b_{\text{out}} = 0$ we use functoriality. Let Σ' the surface obtained by cutting out a disk from Σ . Compose $\mathcal{Z}_{\text{YM}}^G(\Sigma', a - a')$ with $\varepsilon_{a'}$ and use (4.4.21).

For the case $b_{\text{in}} \neq 0$ we need to turn back outgoing boundary components by composing with cylinders with two ingoing boundary components and with area a , which we denote with (C, a) . Using Part 3 of Lemma 4.3.12, for $U, W \in \hat{G}$ we have

$$\mathcal{Z}_{\text{YM}}^G(C, a)(\chi_U, \chi_W) = e^{-a\sigma_U} \delta_{U,V} . \quad (4.4.34)$$

Using the result for the $b_{\text{in}} = 0$ case and (4.4.34) we get the claimed expression. \square

Remark 4.4.9. As already noted in Remark 4.4.5, η_a and $\varepsilon_a|_{Cl^2(G)}$, i.e. the value of $\mathcal{Z}_{\text{YM}}^G$ on a disc with one outgoing (resp. one ingoing) boundary component, do not have zero area limits. On the other hand, the expression (4.4.30) has a zero area limit if $g + \frac{b_{\text{in}} + b_{\text{out}}}{2} \geq 2$. Indeed, the χ_V are orthogonal for different V and have norm $\|\chi_V\| = 1$, and for a given $\alpha \in \mathbb{Z}$ the sum $\sum_{V \in \hat{G}} (\dim(V))^\alpha$ converges if $\alpha \leq -2$. To see this, use the bijection from (4.4.8) and the estimate from (4.4.9) to get

$$\sum_{V \in \hat{G}} (\dim(V))^\alpha \leq \sum_{\lambda \in \Lambda^+} (\dim(V_\lambda))^\alpha \leq N \sum_{\lambda \in \Lambda^+} |\lambda|^{m\alpha} , \quad (4.4.35)$$

which converges for $-m\alpha > \text{rank}(G)$ by [Sug, Lem. 1.3]. Then use that $m = (\dim(G) - \text{rank}(G))/2$ and that $3\text{rank}(G) \leq \dim(G)$ to get $\alpha < -1$, and since α is an integer $\alpha \leq -2$. These limits are related in [Wit1] to volumes of moduli spaces of flat connections (see e.g. [KMT] for more results and references). For example for $G = SU(2)$ we have, for $g \geq 2$ and $b_{\text{in}} = b_{\text{out}} = 0$,

$$\lim_{a \rightarrow 0} \mathcal{Z}_{\text{YM}}^{SU(2)}(\Sigma, a)(1) = \sum_{n=1}^{\infty} n^{-2g+2} = \zeta(2g-2) , \quad (4.4.36)$$

where ζ is the Riemann zeta-function. For general G , these functions are also referred to as Witten zeta-functions, see e.g. [KMT].

4.4.3 Wilson lines and other defects

As we learned in Section 4.3.6, defect lines in the state-sum construction can be obtained from some bimodules over RFAs. In order to describe Wilson line observables in 2d YM theory, we are going to consider bimodules over $L^2(G)$ induced from finite-dimensional unitary G -modules.

Let $V \in \hat{G}$ and consider the Hilbert space $V \otimes L^2(G)$, which we identify with $L^2(G, V)$, the Hilbert space of square integrable functions with values in V . Let us define a map

$$\begin{aligned} \xi : V \otimes L^2(G) &\rightarrow L^2(G) \otimes V \otimes L^2(G) \\ v \otimes f &\mapsto [(x, y) \mapsto f(xy) y.v] \end{aligned} \quad (4.4.37)$$

One can easily check that $\|\xi\| = 1$. We define the left action of $L^2(G)$ on $V \otimes L^2(G)$ via the adjoint of ξ :

$$\begin{aligned} \rho_{0,0}^L &:= \xi^\dagger : L^2(G) \otimes V \otimes L^2(G) \rightarrow V \otimes L^2(G) \\ \varphi \otimes v \otimes f &\mapsto \left[x \mapsto \int_G \varphi(y) y.v f(y^{-1}x) dy \right] , \end{aligned} \quad (4.4.38)$$

and for $a, l \in \mathbb{R}_{>0}$

$$\rho_{a,l}^L := \rho_{0,0}^L(\eta_a \otimes -) \circ \rho_{0,0}^L, \quad (4.4.39)$$

with trivial l -dependence. In the rest of this section all length-dependence will be trivial, hence we drop the index l from the notation:

$$\rho_a^L := \rho_{a,l}^L, \quad Q_a^L := Q_{a,l}^L, \quad \text{etc.} \quad (4.4.40)$$

In Proposition 4.4.10 we prove that this is indeed an action, however one can also understand this from a different argument. If we consider $L^2(G)$ with pointwise multiplication and the same comultiplication Δ , then it is a unital and non-counital Hopf algebra. This Hopf algebra coacts on V via $v \mapsto [x \mapsto x.v]$, where, as above, we identified $L^2(G) \otimes V$ with $L^2(G, V)$. Then $L^2(G)$ coacts on $V \otimes L^2(G)$ as in (4.4.37) and taking the adjoint gives the action (4.4.39).

We define the right action of $L^2(G)$ on $V \otimes L^2(G)$ to be multiplication on the second factor:

$$\begin{aligned} \rho_b^R : V \otimes L^2(G) \otimes L^2(G) &\rightarrow V \otimes L^2(G) \\ v \otimes f \otimes \varphi &\mapsto v \otimes \mu_b(f \otimes \varphi). \end{aligned} \quad (4.4.41)$$

We will often write $\rho_0^L(\varphi \otimes v \otimes f) = \varphi.(v \otimes f)$, etc. By acting with η_a and η_b from the left and right, respectively, we get

$$Q_{a,b}^{V \otimes L^2(G)}(v \otimes f)(x) = \int_{G^2} \eta_a(y) y.v f(y^{-1}xz^{-1})\eta_b(z) dy dz. \quad (4.4.42)$$

Similarly as for $V \otimes L^2(G)$, we define the left action of $L^2(G)$ on $L^2(G) \otimes V$ to be multiplication on the first tensor factor:

$$\begin{aligned} \bar{\rho}_a^L : L^2(G) \otimes L^2(G) \otimes V &\rightarrow L^2(G) \otimes V \\ \varphi \otimes f \otimes v &\mapsto \mu_a(\varphi \otimes f) \otimes v, \end{aligned} \quad (4.4.43)$$

and we define the right action of $L^2(G)$ on $L^2(G) \otimes V$ as follows. First let

$$\begin{aligned} \bar{\rho}_0^R : L^2(G) \otimes V \otimes L^2(G) &\rightarrow L^2(G) \otimes V \\ f \otimes v \otimes \varphi &\mapsto \left[x \mapsto \int_G f(xy^{-1}) y^{-1}.v \varphi(y) dy \right], \end{aligned} \quad (4.4.44)$$

and finally

$$\bar{\rho}_b^R := \bar{\rho}_0^R(- \otimes \eta_b) \circ \bar{\rho}_0^R. \quad (4.4.45)$$

Next we define the duality morphisms for the pair $(V \otimes L^2(G), V^* \otimes L^2(G))$ of bimodules. Let $\{e_i^V\}_{i=1}^{\dim(V)}$ denote an orthonormal basis of V as in Section 4.4.1 and $\{\vartheta_i^V\}_{i=1}^{\dim(V)}$ the

dual basis. Let

$$\begin{aligned}\gamma_{0,b}(1) &:= \sum_{U \in \hat{G}} \sum_{k,l=1}^{\dim(U)} \sum_{j=1}^{\dim(V)} e^{-b\sigma_U} f_{kl}^U \otimes \vartheta_i^V \otimes e_i^V \otimes f_{lk}^U, \\ \gamma_{a,b}(1) &:= (\text{id}_{L^2(G) \otimes V^*} \otimes \rho_0^L(\eta_a \otimes -)) \circ \gamma_{0,b}(1),\end{aligned}\tag{4.4.46}$$

and

$$\begin{aligned}\beta_{0,b}(v \otimes f \otimes \vartheta \otimes g) &:= \vartheta(v) \int_{G^2} \eta_b(x) f(y) g(y^{-1}x^{-1}) dy dx, \\ \beta_{a,b} &:= \beta_{0,b} \circ (\rho_0^L(\eta_a \otimes -) \otimes \text{id}_{L^2(G) \otimes V^*}).\end{aligned}\tag{4.4.47}$$

Recall that we identified $V \otimes L^2(G)$ with square integrable functions on G with values in V , which we denote with $L^2(G, V)$. We will be particularly interested in a subspace of $L^2(G, V)$ consisting of G -invariant functions:

$$L^2(G, V)^G := \{ f \in L^2(G, V) \mid g.f(g^{-1}xg) = f(x) \text{ for every } g, x \in G \} .\tag{4.4.48}$$

Note that $L^2(G, \mathbb{C})^G = Cl^2(G)$.

Proposition 4.4.10. *Let $V, W \in \hat{G}$ and let V^* be the dual G -module of V . Then*

1. $V \otimes L^2(G)$ is a bimodule over $L^2(G)$ via (4.4.39) and (4.4.41), $L^2(G) \otimes V$ is a bimodule over $L^2(G)$ via (4.4.43) and (4.4.45),
2. $(V \otimes L^2(G)) \otimes_{L^2(G)} (W \otimes L^2(G)) = (V \otimes W) \otimes L^2(G)$,
3. $\circlearrowleft_{L^2(G)} (V \otimes L^2(G)) = L^2(G, V)^G$,
4. $(V \otimes L^2(G), L^2(G) \otimes V^*)$ is a dual pair of bimodules with duality morphisms given by (4.4.46) and (4.4.47).

If furthermore G is connected then

5. the bimodule $V \otimes L^2(G)$ is transmissive if and only if V is the trivial G -module $V = \mathbb{C}$.

Proof. Part 1:

We only treat the case of $V \otimes L^2(G)$, the proof for $L^2(G) \otimes V$ is similar. We start by showing associativity of the left action. Let $\varphi_1, \varphi_2 \in L^2(G)$ and $v \otimes \psi \in V \otimes L^2(G)$ and recall that we abbreviate $\rho_0^L(\varphi_1 \otimes v \otimes \psi) = \varphi_1.(v \otimes \psi)$. Then

$$\varphi_2.(\varphi_1.(v \otimes \psi))(x) = \int_{G^2} \varphi_2(z) \varphi_1(y) zy.v \psi(y^{-1}z^{-1}x) dy dz, \tag{4.4.49}$$

$$\mu(\varphi_2 \otimes \varphi_1).(v \otimes \psi)(x) = \int_{G^2} \varphi_2(z) \varphi_1(z^{-1}w) w.v \psi(w^{-1}x) dw dz. \tag{4.4.50}$$

Changing the integration variable $y = z^{-1}w$ in (4.4.49) we get (4.4.50). Using associativity of ρ_0^L and the unitality of μ , we get that $\lim_{a \rightarrow 0} Q_a^L = \text{id}_{V \otimes L^2(G)}$ for $Q_a^L = \rho_{a_1} \circ (\eta_{a_2} \otimes -)$. Clearly, the assignment $a \rightarrow \rho_a^L$ is continuous, and ρ_a^L satisfies the associativity (4.1.68). Therefore $V \otimes L^2(G)$ is a left $L^2(G)$ -module.

It is easy to see that $V \otimes L^2(G)$ is also a right $L^2(G)$ -module, so we are left to check two conditions. First, that the two actions commute as in (4.1.75), which can be shown similarly as associativity of ρ_a^L before. Second, that the two sided action is jointly continuous in the 3 parameters, which can be shown by a similar argument as in the proof of Lemma 4.1.15.

Part 2:

Let $\tilde{V} := V \otimes L^2(G)$, $\tilde{W} := W \otimes L^2(G)$, $v \otimes f \in \tilde{V}$ and $w \otimes g \in \tilde{W}$. We compute from (4.1.94) that

$$D_a^{\tilde{V}, \tilde{W}}(v \otimes f \otimes w \otimes g)(x, y) = \int_{G^2} v \otimes t.w f(s) \eta_a(s^{-1}xt) g(t^{-1}y) ds dt .$$

So using that $\lim_{a \rightarrow 0} P_a = \text{id}$, we get that

$$D_0^{\tilde{V}, \tilde{W}}(v \otimes f \otimes w \otimes g)(x, y) = \int_G v \otimes t.w f(xt) g(t^{-1}y) dt . \quad (4.4.51)$$

By Proposition 4.1.51, the image of the idempotent $D_0^{\tilde{V}, \tilde{W}}$ is the tensor product $\tilde{V} \otimes_{L^2(G)} \tilde{W}$. Let $\pi(v \otimes f \otimes w \otimes g) := v \otimes f.(w \otimes g)$ and $\iota(v \otimes w \otimes f)(x, y) := v \otimes x^{-1}.wf(xy)$. Then we have that $\pi \circ \iota = \text{id}_{V \otimes W \otimes L^2(G)}$ and

$$\iota \circ \pi(v \otimes f \otimes w \otimes g)(x, y) = v \otimes \int_G x^{-1}t.w f(t) g(t^{-1}xy) dt ,$$

which is equal to (4.4.51) after substituting $t' := x^{-1}t$. We have shown that π and ι is the projection and embedding of the image of $D_0^{\tilde{V}, \tilde{W}}$, so in particular the image is $\tilde{V} \otimes_{L^2(G)} \tilde{W} = V \otimes W \otimes L^2(G)$. The induced action on $V \otimes W \otimes L^2(G)$ from (4.1.99) is

$$\tilde{\rho}_{0,0}^{\tilde{V}, \tilde{W}} = \iota \circ (\rho_0^{\tilde{V}, L} \otimes \rho_0^{\tilde{W}, R}) \circ D_0^{\tilde{V}, \tilde{W}} \circ \pi , \quad (4.4.52)$$

which can be shown to agree with the action on $V \otimes W \otimes L^2(G)$ by a straightforward calculation.

Part 3:

Recall $(V \otimes L^2(G))^G$ from (4.4.48). Let $a \in \mathbb{R}_{>0}$, $v \in V$ and $f \in L^2(G)$. Then from (4.1.94) we have

$$\begin{aligned} D_a^{\tilde{V}}(v \otimes f)(x) &= \int_{G^2} \eta_a(yz^{-1}x) y.v f(y^{-1}z) dy dz \\ &\stackrel{w=yz^{-1}}{=} \int_{G^2} \eta_a(wx) y.v f(y^{-1}wy) dy dw . \end{aligned} \quad (4.4.53)$$

If $v \otimes f \in (V \otimes L^2(G))^G$ then $D_a^{\tilde{V}}(v \otimes f) = (\text{id}_V \otimes P_a)(v \otimes f)$ and hence $v \otimes f \in \text{im}(D_0^{\tilde{V}})$. Let $h \in G$ and compute

$$\begin{aligned}
h.D_a^{\tilde{V}}(v \otimes f)(h^{-1}xh) &= \int_{G^2} \eta_a(wh^{-1}xh) hy.v f(y^{-1}wy) dy dw \\
&\stackrel{\eta_a \in Cl^2(G)}{=} \int_{G^2} \eta_a(hwh^{-1}x) hy.v f(y^{-1}wy) dy dw \\
&\stackrel{z=hwh^{-1}}{=} \int_{G^2} \eta_a(zx) hy.v f(y^{-1}h^{-1}why) dy dz \\
&\stackrel{q=hy}{=} \int_{G^2} \eta_a(zx) q.v f(q^{-1}wq) dq dz \\
&= D_a^{\tilde{V}}(v \otimes f)(x) .
\end{aligned} \tag{4.4.54}$$

Since $h.(-) : V \otimes L^2(G) \rightarrow V \otimes L^2(G)$ is continuous, we can exchange it with $\lim_{a \rightarrow 0}(-)$, so $D_0^{\tilde{V}}(v \otimes f) \in (V \otimes L^2(G))^G$. Using the identification $V \otimes L^2(G) \cong L^2(G, V)$ we arrive at $\text{im}(D_0^{\tilde{V}}) = L^2(G, V)^G$, which is, by Proposition 4.1.51, $\circlearrowleft_{L^2(G)} (V \otimes L^2(G))$.

Part 4:

It is easy to see from the definition of $\beta_{a,b}$ and $\gamma_{a,b}$ that the zig-zag identities in (4.1.82) hold. So we only need to show that $\beta_{a,b}$ intertwines the actions as in (4.1.83). We compute

$$\begin{aligned}
\beta_{0,b}(\varphi.(v \otimes f) \otimes g \otimes \vartheta) &= \int_{G^3} \eta_b(z) \varphi(x) \vartheta(x.v) f(x^{-1}y) g(y^{-1}z^{-1}) dx dy dz , \\
\beta_{0,b}(v \otimes f \otimes (g \otimes \vartheta). \varphi) &= \int_{G^3} \eta_b(z) f(y) g(y^{-1}z^{-1}x^{-1}) (x^{-1}.\vartheta)(v) \varphi(x) dx dy dz \\
&\stackrel{G \text{ acts on } V^*}{=} \int_{G^3} \eta_b(z) \varphi(x) \vartheta(x.v) f(y) g(y^{-1}z^{-1}x^{-1}) dx dy dz \\
&\stackrel{\substack{y=x^{-1}u \\ z=x^{-1}wx}}{=} \int_{G^3} \eta_b(x^{-1}wx) \varphi(x) \vartheta(x.v) f(x^{-1}u) g(u^{-1}w^{-1}) dx du dw \\
&\stackrel{\eta_b \in Cl^2(G)}{=} \int_{G^3} \eta_b(w) \varphi(x) \vartheta(x.v) f(x^{-1}u) g(u^{-1}w^{-1}) dx du dw ,
\end{aligned} \tag{4.4.55}$$

which are equal. Composing with $Q_{a,0}^{V \otimes L^2(G)} \otimes Q_{a,0}^{V^* \otimes L^2(G)}$ shows that (4.1.83) holds for every $a, b \in \mathbb{R}_{>0}$ too.

Part 5:

Since $\rho_{a,b} = Q_{a,b} \circ \rho_{0,0}$, it is enough to consider $Q_{a,b}$. As we already noted in (4.1.71), $Q_{-,-} : (\mathbb{R}_{\geq 0})^2 \rightarrow \mathcal{B}(\tilde{V})$ is a two parameter strongly continuous semigroup. This defines two one parameter semigroups $Q_a^1 := Q_{a,0}$, $Q_b^2 := Q_{0,b}$ and $Q_{a,b}$ depends solely on $a + b$, if and only if these two one parameter semigroups are the same, see also the discussion before Definition 2.4 in [KS]. One parameter semigroups are completely determined by their generators, so we calculate these now.

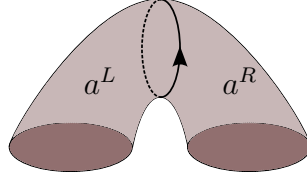


Figure 4.15: Cylinder (C, a^L, a^R, V) with ingoing boundaries and a Wilson line with label $V \in \hat{G}$. The area of the surface components left and right to the Wilson line is a^L and a^R respectively.

Let $v \in V$ and $W \in \hat{G}$. Then

$$\begin{aligned} Q_{a,b}(v \otimes f_{ij}^W)(x) &= e^{-b\sigma_W} Q_{a,0}(v \otimes f_{ij}^W)(x) \\ &= e^{-b\sigma_W} \int_G \sum_{U \in \hat{G}} e^{-a\sigma_U} \dim(U) \chi_U(s) s.v f_{ij}^W(s^{-1}x) ds . \end{aligned} \quad (4.4.56)$$

Using this, and writing H_i for the generator of Q^i for $i = 1, 2$, we have

$$\begin{aligned} H_1(v \otimes f_{ij}^W)(x) &= \frac{d}{da} Q_a^1(v \otimes f_{ij}^W)|_{a=0}(x) \\ &= \lim_{a \rightarrow 0} \sum_{U \in \hat{G}} -\sigma_U e^{-a\sigma_U} \dim(U) \int_G \chi_U(s) s.v f_{ij}^W(s^{-1}x) ds \end{aligned} \quad (4.4.57)$$

and $H_2(v \otimes f_{ij}^W) = -\sigma_W v \otimes f_{ij}^W$.

Let $v := e_k^V$ and $W := \mathbb{C}$. Note that $M_{\mathbb{C}}$ are constant functions. Then

$$H_1(e_k^V \otimes 1) = (-\sigma_V) \dim(V) \int_G \chi_V(s) s.e_k^V ds = -\sigma_V e_k^V \otimes 1 ,$$

which is nonzero if and only if $V \not\cong \mathbb{C}$. Furthermore, $H_2(e_k^V \otimes 1) = 0$. So if $V \not\cong \mathbb{C}$ then $V \otimes L^2(G)$ is not transmissive.

Clearly, if $V \cong \mathbb{C}$ then $\mathbb{C} \otimes L^2(G) = L^2(G)$ and by unitality of the product on $L^2(G)$ the bimodule $\mathbb{C} \otimes L^2(G)$ is transmissive. \square

In terms of Section 4.3.7, we can interpret these results as follows. Let $(\mathbb{S}^1, V, +)$ be a circle with a positively oriented marked point where a Wilson line with label $V \in \hat{G}$ crosses. Then the corresponding state space is

$$\mathcal{Z}_{\text{YM}}^G(\mathbb{S}^1, V, +) = L^2(G, V)^G . \quad (4.4.58)$$

Let $V, W \in \hat{G}$. Furthermore, the fusion of two Wilson lines with labels V and W is again a Wilson line with label $V \otimes W$.

In the following we show that the value of $\mathcal{Z}_{\text{YM}}^G$ on closed surfaces with Wilson lines agrees with the expression in [CMR, Sec. 3.5]. Let $(\Sigma, \mathcal{A}) = (\Sigma, \mathcal{A}, \mathcal{L})$ be a closed surface with area and defects with $\mathcal{L} = 0$. Since Σ is closed, the defect lines in Σ , denoted with

$\Sigma_{[1]}$, are closed curves. In order to compute $\mathcal{Z}_{\text{YM}}^G$ on (Σ, \mathcal{A}) we decompose it into convenient smaller pieces as follows. For every $x \in \Sigma_{[1]}$ with defect label $d_1(x) = V_x \otimes L^2(G)$ for some $V_x \in \hat{G}$ take a collar neighbourhood of x in Σ , which is a cylinder with x running around it. Denote the corresponding bordism with area and defects with both boundary components ingoing with (C_x, a_x^L, a_x^R, V_x) , where a_x^L and a_x^R are the area of the surface components to the left and right of x respectively. Denote with (Σ', \mathcal{A}') the bordism with area with all outgoing components, which is formed by removing $\bigsqcup_{x \in \Sigma_{[1]}} (C_x, a_x^L, a_x^R, V_x)$ from (Σ, \mathcal{A}) . We have

$$(\Sigma, \mathcal{A}) = \left(\bigsqcup_{x \in \Sigma_{[1]}} (C_x, a_x^L, a_x^R, V_x) \right) \circ (\Sigma', \mathcal{A}'). \quad (4.4.59)$$

Note that (Σ', \mathcal{A}') is a bordism with area but without defects, therefore using Proposition 4.4.8 and monoidality we can compute $\mathcal{Z}_{\text{YM}}^G$ on it. The final ingredient we need is:

Lemma 4.4.11. *Let (C, a, b, V) be a cylinder with a Wilson line with label $V \in \hat{G}$ as in Figure 4.15, $U \in \hat{G}$ and let $U \otimes V \cong \bigoplus_{W \in \hat{G}} W^{N_{U,V}^W}$ be the decomposition into simple G -modules, for some integers $N_{U,V}^W$. Then*

$$\mathcal{Z}_{\text{YM}}^G(C, a, b, V)(\chi_U \otimes \chi_W) = e^{-a\sigma_U - b\sigma_W} N_{U,V}^W. \quad (4.4.60)$$

Sketch of proof. The morphism $\mathcal{Z}_{\text{YM}}^G(C, a, b, V)$ is given by the diagram

$$\begin{array}{c}
 \bullet \quad (0, b_2; \tilde{V}) \\
 \swarrow \quad \searrow \\
 \bullet \quad (a, 0; \tilde{V}) \quad \bullet \quad (0, 0; \tilde{V}) \\
 \swarrow \quad \searrow \quad \swarrow \quad \searrow \\
 \bullet \quad (0, b_1; \tilde{V}) \quad \bullet \quad (0, b_1; \tilde{V}) \\
 \text{\scriptsize } Cl^2(G) \quad \text{\scriptsize } Cl^2(G)
 \end{array} \quad (4.4.61)$$

After a straightforward calculation and some manipulation of multiple integrals we get for $\varphi, \psi \in Cl^2(G)$ that

$$\mathcal{Z}_{\text{YM}}^G(C, a, b, V)(\varphi \otimes \psi) = \int_{G^3} \eta_a(z) \varphi(z^{-1}y) \chi_V(y) \psi(y^{-1}p^{-1}) \eta_b(p) dp dy dz. \quad (4.4.62)$$

Finally using

$$\int_G \chi_U(y) \chi_V(y) \chi_W(y^{-1}) = N_{U,V}^W, \quad (4.4.63)$$

which follows from basic properties of characters and character orthogonality, we get (4.4.60). \square

Remark 4.4.12. The computation of the defect cylinder in the above lemma allows one to interpret states of the 2d YM theory in terms of Wilson lines. Namely, let (D, a, b, V) be a disc with outgoing boundary and with embedded defect circle oriented anti-clockwise and labeled by $V \in \hat{G}$. The area inside the circle is a and the one outside is b . The corresponding amplitude is

$$\begin{aligned} \langle \chi_W, \mathcal{Z}_{\text{YM}}^G(D, a, b, V) \rangle &= \mathcal{Z}_{\text{YM}}^G(C, \frac{a}{2}, b, V)(\eta_{\frac{a}{2}} \otimes \chi_W) \\ &= \sum_{U \in \hat{G}} e^{-a\sigma_U - b\sigma_W} \dim(U) N_{U,V}^W. \end{aligned} \quad (4.4.64)$$

For a given W , the sum is finite. One checks from this expression that

$$\lim_{\substack{a \rightarrow \infty \\ b \rightarrow 0}} \mathcal{Z}_{\text{YM}}^G(D, a, b, V) = \chi_V. \quad (4.4.65)$$

Thus we can picture the state $\chi_V \in \mathcal{Z}_{\text{YM}}^G(\mathbb{S}^1)$ informally as the disc $(D, \infty, 0, V)$ with zero area outside of the circle and infinite area inside the circle (and which is hence not an allowed bordism with area). From this point of view the action (4.4.60) of the cylinder is no surprise as by Theorem 4.3.20 it amounts to the fusion of defect lines, which by Proposition 4.4.10 (2) is given by the tensor product of G -representations.

Proposition 4.4.13. For $x \in \Sigma_{[1]}$ let $\rho_R^x \in \pi_0(\Sigma')$ be the connected component which is glued to C_x on the right side of x in (4.4.59) and define $\rho_L^x \in \pi_0(\Sigma')$ similarly to be the connected component glued from the left. Using the notation from above we have

$$\mathcal{Z}_{\text{YM}}^G(\Sigma, \mathcal{A}) = \prod_{\rho \in \pi_0(\Sigma')} \prod_{x \in \Sigma_{[1]}} \sum_{U_\rho \in \hat{G}} e^{-a_\rho \sigma_{U_\rho}} (\dim(U_\rho)^{\chi(\rho)}) N_{U_{\rho_L^x}, V_x}^{U_{\rho_R^x}}, \quad (4.4.66)$$

where $a_\rho \in \mathbb{R}_{>0}$ is the area of ρ .

The expression in (4.4.66) matches the expression in [CMR, (3.28)] (see also [Rus, Sec. 5]).

Defects from automorphisms of G

Another way of obtaining bimodules is by twisting the actions on the trivial bimodule by an algebra automorphism as we saw in Example 4.1.39. In the rest of this section we will introduce automorphisms of $L^2(G)$ (seen as an RFA) using automorphisms of G .

Let $\alpha \in \text{Aut}(G)$, $V \in \hat{G}$ and denote with ${}_\alpha V$ the G -module obtained by precomposing the action on V with α . Let H denote the Haar measure on G and $\alpha^* H$ the induced measure. This is a left invariant normalised measure, hence by the uniqueness of such measures $\alpha^* H = H$. As a consequence, the Haar integral is invariant under $\text{Aut}(G)$.

Lemma 4.4.14. Let $\alpha \in \text{Aut}(G)$. Then precomposition with α is an automorphism of the RFA $L^2(G)$ and defines a group homomorphism $\text{Aut}(G) \rightarrow \text{Aut}_{\mathcal{R}\text{Frob}(\mathcal{H}\text{ilb})}(L^2(G))^{\text{op}}$.

Proof. Clearly, invariance of the Haar measure implies that $\alpha^* = (-) \circ \alpha$ is unitary. We first show that α^* commutes with the product. Let $\varphi_1, \varphi_2 \in L^2(G)$ and compute:

$$\begin{aligned} \mu((\varphi_1 \circ \alpha) \otimes (\varphi_2 \circ \alpha))(x) &= \int_G \varphi_1(\alpha(z)) \varphi_2(\alpha(z^{-1}x)) dz \\ &\stackrel{y=\alpha(z)}{=} \int_G \varphi_1(y) \varphi_2(y^{-1}\alpha(x)) dy = \mu(\varphi_1 \otimes \varphi_2)(\alpha(x)) , \end{aligned} \quad (4.4.67)$$

where we used that the Haar measure on G is invariant under α . Next, we show that $\eta_a \circ \alpha = \eta_a$.

$$\begin{aligned} \eta_a \circ \alpha &= \sum_{V \in \hat{G}} e^{-a\sigma_V} \dim(V) \chi_V \circ \alpha = \sum_{V \in \hat{G}} e^{-a\sigma_V} \dim(V) \chi_{\alpha V} \\ &= \sum_{V \in \hat{G}} e^{-a\sigma_{\alpha V}} \dim(\alpha V) \chi_{\alpha V} = \eta_a , \end{aligned} \quad (4.4.68)$$

where we used that αV has the same dimension as V and that the Casimir element is invariant under α . The latter can be understood as follows. The Lie group automorphism α induces an automorphism on the Lie algebra of G , and the Casimir element is defined in terms of an orthonormal basis of the Lie algebra with respect to an invariant non-degenerate pairing, for example the Killing form.

Since $L^2(G)$ is a \dagger -RFA and α^* is a unitary regularised algebra morphism, α^* is an RFA morphism. \square

Let $L_\alpha := {}_{\alpha^*}L^2(G)_{\text{id}}$ denote the transmissive twisted bimodule from Example 4.1.39. By Examples 4.1.44 and 4.1.53 these bimodules have duals and can be tensored together, i.e. we can label defect lines with them. For convenience we list these results here. Then

- $(L_\alpha, L_{\alpha^{-1}})$ is a dual pair of bimodules,
- $L_{\alpha_1} \otimes_{L^2(G)} L_{\alpha_2} \cong L_{\alpha_2 \circ \alpha_1}$, for $\alpha_1, \alpha_2 \in \text{Aut}(G)$,
- $\circ_{L^2(G)} L_\alpha \cong \{ f \in L^2(G) \mid f(gx\alpha(g^{-1})) = f(x) \text{ for every } g, x \in G \}$,

where the last equation can be computed from $D_0^{L_\alpha}$ of (4.1.94).

The following lemma can be proven similarly as Lemma 4.4.11.

Lemma 4.4.15. *Let $\alpha \in \text{Aut}(G)$ and (C, a, b, L_α) denote a cylinder as in Figure 4.15 with the defect line labeled with L_α . Then for $U, W \in \hat{G}$ we have*

$$\mathcal{Z}_{\text{YM}}^G(C, a, b, L_\alpha)(\chi_U \otimes \chi_W) = e^{-(a+b)\sigma_W} \delta_{\alpha U, W} . \quad (4.4.69)$$

The following lemma shows that for some particular choices of α , these bimodules could provide new examples.

Lemma 4.4.16. *Let $\alpha \in \text{Aut}(G)$ and $V \in \hat{G}$. Then*

1. $L_\alpha \cong L_{\text{id}} = L^2(G)$ as bimodules if and only if α is inner,

furthermore if G is connected,

2. $L_\alpha \cong V \otimes L^2(G)$ as bimodules if and only if α is inner and $V \cong \mathbb{C}$ as G -modules.

Proof. Part 1: Let us assume that $\alpha(x) = g^{-1}xg$ for some $g \in G$. We define $\varphi : L_\alpha \rightarrow L_{\text{id}}$ as $\varphi(f)(x) := f(gx)$, which is clearly bounded and invertible. To show that it is an intertwiner calculate for $\psi \in L^2(G)$ and $f \in L_\alpha$:

$$\varphi(\psi.f)(x) = \int_G \psi(g^{-1}yg)f(y^{-1}gx) = \int_G \psi(z)f(gz^{-1}x) = \psi.\varphi(f)(x) . \quad (4.4.70)$$

Conversely, let us assume that $L_\alpha \cong L_{\text{id}}$. Let $(\mathbb{S}^1 \times [0, 1], a, b, L_\alpha)$ be a cylinder as in Lemma 4.4.15, just with one of the boundary components being outgoing. Then we have that

$$\mathcal{Z}_{\text{YM}}^G(\mathbb{S}^1 \times [0, 1], a, b, L_\alpha)(\chi_V) = e^{-(a+b)\sigma_V} \chi_{\alpha V} . \quad (4.4.71)$$

But since $L_\alpha \cong L_{\text{id}}$, by a direct computation one can see that the operator in (4.4.71) is the same as the operator assigned to a cylinder without defect lines and with area $a + b$, so we have for every $V \in \hat{G}$ that

$$\chi_{\alpha V} = \chi_V , \quad (4.4.72)$$

which is equivalent to ${}_\alpha V \cong V$ for every $V \in \hat{G}$. This means that the highest weight of ${}_\alpha V$ and V are equal for every $V \in \hat{G}$, which holds if and only if α corresponds to the trivial automorphism of the Dynkin diagram of G . This is equivalent to α being an inner automorphism [Kna, Ch. VII].

Part 2 follows directly from the fact that L_α is transmissive, Part 1 of this lemma and Part 5 of Proposition 4.4.10. \square

Using Lemma 4.4.15 and Part 1 of Lemma 4.4.16 we can show the following proposition.

Proposition 4.4.17. *Let (Σ, \mathcal{A}) and (Σ', \mathcal{A}') be as in Proposition 4.4.13 with every defect line $x \in \Sigma_{[1]}$ labeled by L_{α_x} for $\alpha_x \in \text{Aut}(G)$. Then*

$$\mathcal{Z}_{\text{YM}}^G(\Sigma, \mathcal{A}) = \prod_{\rho \in \pi_0(\Sigma')} \prod_{x \in \Sigma_{[1]}} \sum_{U_\rho \in \hat{G}} e^{-a_\rho \sigma_{U_\rho}} (\dim(U_\rho)^{\chi(\rho)} \delta_{\alpha_x U_{\rho_L^x}, U_{\rho_R^x}} , \quad (4.4.73)$$

where $a_\rho \in \mathbb{R}_{>0}$ is the area of ρ . In particular, if α_x is inner for every $x \in \Sigma_{[1]}$ then (4.4.73) agrees with (4.4.30), the value of $\mathcal{Z}_{\text{YM}}^G$ on (Σ, \mathcal{A}) without defects.

The following is an example of a non-trivial twist-defect.

Example 4.4.18. Let us assume that G is furthermore simply connected. Then $\text{Out}(G)$, the group of outer automorphisms of G , is isomorphic to the group of automorphisms of the Dynkin diagram of G [Kna, Ch. VII].

Let $G := SU(N)$ for $N \geq 3$. Then $\text{Out}(G) \cong \mathbb{Z}_2$ and its generator, which we now denote with α , corresponds to complex conjugation. We have that ${}_{\alpha}V \cong V^*$ for every $V \in \hat{G}$. We can apply Proposition 4.4.17, so for example for a torus T^2 with one non-contractible defect line with defect label L_{α} we have

$$\mathcal{Z}_{\text{YM}}^{SU(N)}(T^2, a) = \sum_{U \in \hat{G}, U \cong U^*} e^{-a\sigma_U} . \quad (4.4.74)$$

4.A Appendix: A bimodule with singular limits

In this example we illustrate that not every bimodule over regularised algebras comes from a left and right module with commuting actions. Namely, we construct two regularised algebras A^L and A^R and an A^L - A^R -bimodule M , such that the two-sided action $\rho_{a,l,b}$ does not provide a left module structure as in Remark 4.1.38 since the limit in (4.1.76) does not exist.

Let A be $\mathbb{C}[x]/\langle x^2 \rangle$ as an algebra in \mathcal{Hilb} with orthonormal basis $\{1, x\}$. Let $n \in \mathbb{Z}_{\geq 1}$ and $M_n \in \mathcal{Hilb}$ be spanned by orthonormal vectors v_0 and v_1 . We define a left A -module structure on M_n by

$$x.v_0 = e^{n^2} v_1 \quad \text{and} \quad x.v_1 = 0 . \quad (4.A.1)$$

Since A is commutative, (4.A.1) defines a right A -module structure on M_n as well and together we have an A - A -bimodule structure.

Next we turn A into a regularised algebra in two ways. Let $h^L := x - n \in A$ and denote with $(A_n^L, \mu_a^{A_n^L}, \eta_a^{A_n^L})$ the regularised algebra structure on A defined as in Example 3 by setting

$$P_a^{A_n^L}(p) := e^{ah^L} p \quad (4.A.2)$$

for $p \in A_n^L$ and $a \in \mathbb{R}_{>0}$. Note that $\eta_a^{A_n^L} = e^{-an}(1 + ax)$. Similarly, define the regularised algebra A_n^R using $h^R := x - n^3$.

We turn the A - A -bimodule M_n from above into an A_n^L - A_n^R -bimodule over regularised algebras via Proposition 4.1.41 and by taking the l -dependence to be trivial. We denote the resulting action by $\rho_{a,b}^{M_n}$. The semigroup action is given by

$$Q_{a,b}^{M_n}(m) := e^{ah^L + bh^R}.m = e^{-an - bn^3}(1 + (a+b)x).m , \quad (4.A.3)$$

for $m \in M_n$.

Let us consider

$$A^L := \bigoplus_{n \in \mathbb{Z}_{\geq 1}} A_n^L , \quad A^R := \bigoplus_{n \in \mathbb{Z}_{\geq 1}} A_n^R \quad \text{and} \quad M := \bigoplus_{n \in \mathbb{Z}_{\geq 1}} M_n . \quad (4.A.4)$$

We claim that for every $a, b \in \mathbb{R}_{>0}$ we have

$$\sum_{n \in \mathbb{Z}_{\geq 1}} \left\| \eta_a^{A_n^L} \right\|^2 < \infty \quad \text{and} \quad \sum_{n \in \mathbb{Z}_{\geq 1}} \left\| \eta_b^{A_n^R} \right\|^2 < \infty \quad (4.A.5)$$

and furthermore

$$\sup_{n \in \mathbb{Z}_{\geq 1}} \left\{ \left\| \mu_a^{A_n^L} \right\| \right\} < \infty, \quad \sup_{n \in \mathbb{Z}_{\geq 1}} \left\{ \left\| \mu_b^{A_n^R} \right\| \right\} < \infty \quad \text{and} \quad \sup_{n \in \mathbb{Z}_{\geq 1}} \left\{ \left\| \rho_{a,b}^{M_n} \right\|^2 \right\} < \infty. \quad (4.A.6)$$

So by Proposition 4.1.18, A^L and A^R are regularised algebras and by Proposition 4.1.40 M is a A^L - A^R -bimodule. However the limit

$$\lim_{b \rightarrow 0} \rho_{a,b_1}^M \circ \left(\text{id}_{A^L \otimes M} \otimes \eta_{b_2}^{A^R} \right), \quad (4.A.7)$$

where $b = b_1 + b_2$, does not exist, i.e. M is not a left A^L -module.

Showing (4.A.5) is a direct calculation and we omit it. We now show that (4.A.6) holds. We compute for $p = p_0 + p_1 x \in A_n^L$, $m = m_0 v_0 + m_1 v_1 \in M_n$ and $q = q_0 + q_1 x \in A_n^R$ that

$$\begin{aligned} \rho_{a,b}^{M_n}(p \otimes m \otimes q) &= Q_{a,b}^{M_n}(pq.m) \\ &= Q_{a,b}^{M_n} \left(p_0 q_0 m_0 v_0 + \left((p_0 q_1 + p_1 q_0) m_0 e^{n^2} + p_0 q_0 m_1 \right) v_1 \right) \\ &= e^{-an-bn^3} \left(p_0 q_0 m_0 v_0 + \left(p_0 q_0 m_1 + [p_0 q_0(a+b) + (p_0 q_1 + p_1 q_0)] m_0 e^{n^2} \right) v_1 \right). \end{aligned} \quad (4.A.8)$$

Using this we compute the value of the adjoint of the action on $f = f_0 v_0 + f_1 v_1$ as

$$\begin{aligned} (\rho_{a,b}^{M_n})^\dagger(f) &= e^{-an-bn^3} \left(f_0 + (a+b)e^{n^2} f_1 \right) 1 \otimes v_0 \otimes 1 \\ &\quad + e^{-an-bn^3} f_1 \left(1 \otimes v_1 \otimes 1 + e^{n^2} (1 \otimes v_0 \otimes x + x \otimes v_0 \otimes 1) \right). \end{aligned} \quad (4.A.9)$$

Let f have $\|f\| = 1$ and compute the norm of (4.A.9)

$$\left\| (\rho_{a,b}^{M_n})^\dagger(f) \right\|^2 = e^{-2an-2bn^3} \left(\left| f_0 + (a+b)e^{n^2} f_1 \right|^2 + |f_1|^2 (1 + 2e^{2n^2}) \right), \quad (4.A.10)$$

from which we get by estimating $|f_0| \leq 1$ and $|f_1| \leq 1$ that

$$\left\| \rho_{a,b}^{M_n} \right\|^2 = \left\| (\rho_{a,b}^{M_n})^\dagger \right\|^2 \leq e^{-2an-2bn^3} \left(1 + 2(a+b)e^{n^2} + (2 + (a+b)^2) e^{2n^2} \right). \quad (4.A.11)$$

By a similar argument, without giving the details, we obtain the following estimates:

$$\left\| \mu_a^{A_n^L} \right\|^2 \leq e^{-2an} (2 + a + a^2) \quad \text{and} \quad \left\| \mu_b^{A_n^R} \right\|^2 \leq e^{-2bn^3} (2 + b + b^2). \quad (4.A.12)$$

From (4.A.11) and (4.A.12) it follows that (4.A.6) holds.

Finally we give a lower estimate of the norm of the morphism in (4.A.7) restricted to $A_n^L \otimes M_n$ without the $b \rightarrow 0$ limit:

$$\begin{aligned} \left\| \rho_{a,b_1}^M \circ \left(\text{id}_{A_n^L \otimes M_n} \otimes \eta_{b_2}^{A^R} \right) \right\|^2 &= \left\| \rho_{a,b}^{M_n} \circ \left(\text{id}_{A_n^L \otimes M_n} \otimes 1 \right) \right\|^2 \\ &\geq e^{-2an-2bn^3} \frac{1}{2} \left(\left(1 + (a+b)e^{n^2} \right)^2 + 1 + e^{2n^2} \right) . \end{aligned} \quad (4.A.13)$$

We arrived at this estimate by computing the norm of the adjoint of (4.A.7) before taking the limit $b \rightarrow 0$ evaluated at $f \in M_n$, as we did in (4.A.10), and then by choosing $f_0 = f_1 = \frac{1}{\sqrt{2}}$. Thus the $b \rightarrow 0$ limit in (4.A.7) cannot give a bounded operator.

Chapter 5

Outlook

The two projects in this thesis serve as good starting points for future projects. Here we give some of the directions in which one could continue working. We list some points for r -spin TFTs first:

- To our knowledge there are no essentially different examples of r -spin TFTs other than the ones presented in Chapter 3, and even those cannot distinguish all mapping class orbits of r -spin surfaces. In particular, when r is odd we only know examples of r -spin TFTs which are not sensitive to the r -spin structures on surfaces. A possible way of obtaining new examples, which may also be able to distinguish more mapping class group orbits could be using the orbifold or push construction of [SW].
- It is desirable to combine our closed r -spin TFTs with the open r -spin TFTs of [Ster]. Then we should find a classification in the spirit of Theorem 2.1.3, as it has been done in the oriented case in [LP1] and in the $r = 2$ case in [MS] and extend the state-sum construction of open-closed TFTs [LP2].
- This leads us to the next point, namely to consider fully extended r -spin TFTs. In the case of $r = 2$ in [Gun] it is shown that fully extended 2-spin TFTs correspond to Frobenius algebras with $N^2 = \text{id}$. This result should be possible to generalise to arbitrary values of r by looking at homotopy fixed points of the r -spin group action on the bordism bicategory using methods of [HV].
- Another direction is to include defect lines in r -spin TFTs. This is motivated by a larger project, where we want to understand the relation of (2-)spin TFTs to $\mathcal{N} = 1$ supersymmetric TFTs and then extend those notions to $\mathcal{N} = 2$ supersymmetric TFTs. This is important in the quest of understanding defect bicategories of Landau-Ginzburg models from a TFT point of view and compare results with [CM].

Now we give some ideas for aQFTs:

- The most natural direction to continue would be to consider 3-dimensional FFTs and state-sum constructions of them. In the topological case, the state-sum construction,

called the Turaev-Viro model [TV], takes as input a spherical fusion category. It would be interesting to understand the corresponding algebraic structure for the volume-dependent case. Furthermore one could study how different codimensional defects are related in 2- and 3-dimensional volume dependent theories.

- The tensor product of bimodules over RFAs should be studied further, especially to understand the condition better when dualisable bimodules close under tensor product. Then it would be interesting to see what kind of defect bicategories [DKR] one can obtain from the state-sum aQFTs from strongly separable RFAs and bimodules.
- A direct generalisation of 2d YM theory would be to replace $L^2(G)$ with an analytic quantum group. There are different versions of deformations of 2d YM theories (q - and (q, t) -deformed), e.g. in [SzTi]. We would like to understand these deformations using our framework for aQFTs with (analytic) quantum groups.
- A special case of 2d YM theory is when the group G is finite, which is a state-sum TFT from the group algebra $\mathbb{C}[G]$. The relation of this TFT to orbifolds, see e.g. [BCP, Ex. 1], is that state-sum models are orbifolds of the trivial theories. The present investigation suggests that including area-dependence may be useful to treat orbifolds by compact Lie groups, such as the one investigated in [GS].

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Summary

In this thesis we study two classes of 2-dimensional functorial field theories and give a state-sum construction of these theories. In the first part of this thesis we look at topological field theories on r -spin surfaces. We define a combinatorial model of r -spin surfaces, which is suitable for the state-sum construction. The latter takes a Frobenius algebra A , whose window element is invertible and whose Nakayama automorphism N satisfies $N^r = \text{id}$, as an input and produces an r -spin topological field theory \mathcal{Z}_A . For r even we give an example of such a state-sum topological field theory with values in super vector spaces, where $A = C\ell$ is the Clifford algebra with one odd generator and we show that $\mathcal{Z}_{C\ell}$ computes the Arf invariant of r -spin surfaces. As an application of the combinatorial model and this r -spin topological field theory we compute mapping class group orbits of r -spin structures extending results of Randal-Williams and Geiges, Gonzalo.

In the second part of the thesis we consider area-dependent quantum field theories. An important feature of these theories that, contrary to topological field theories, they allow for infinite-dimensional state-spaces. We classify these theories in terms of regularised Frobenius algebras and give a state-sum construction of them, for which the input data is now a strongly separable regularised Frobenius algebra. We then extend the state-sum construction to include defect lines, which we label with bimodules over strongly separable regularised Frobenius algebras. We show that the fusion of defect lines corresponds to the tensor product of bimodules over regularised algebras. The main example of area-dependent quantum field theories is 2-dimensional Yang-Mills theory with a compact semisimple Lie group G with Wilson lines as defects, which we treat in great detail. We finally introduce other defect lines by twisting by outer automorphisms of G .

Zusammenfassung

In dieser Doktorarbeit werden zwei Klassen von zweidimensionalen funktoriellen Quantenfeldtheorien untersucht und entsprechende Zustandssummenkonstruktionen werden gegeben. Im ersten Teil der Doktorarbeit werden topologische Feldtheorien auf r -Spin-Oberflächen betrachtet. Ein kombinatorisches Modell von r -Spin-Oberflächen wird definiert, die für die Zustandssummenkonstruktion geeignet ist. Diese Konstruktion nimmt eine Frobenius-Algebra A mit folgenden Eigenschaften als Input. Für den Nakayama-Automorphismus N von A gilt $N^r = \text{id}$ und das Fensterelement von A ist invertierbar. Die Konstruktion produziert eine r -Spin topologische Feldtheorie \mathcal{Z}_A . Wenn r eine gerade Zahl ist, wird ein Beispiel von einer solchen Zustandssummenfeldtheorie mit Werten in Supervektorräumen gegeben. Die Frobenius-Algebra $A = \mathcal{Cl}$ ist die Clifford-Algebra mit einem ungeraden Erzeuger. Es wird gezeigt, dass $\mathcal{Z}_{\mathcal{Cl}}$ die Arf-Invariante von r -Spin Oberflächen berechnet. Als Anwendung des kombinatorischen Modells und dieser r -Spin topologischen Feldtheorie werden Abbildungsklassenbahnen von r -Spin-Strukturen berechnet. Diese erweitern Resultaten von Randal-Williams und Geiges, Gonzalo.

Im zweiten Teil der Doktorarbeit werden flächenabhängige Quantenfeldtheorien betrachtet. Eine wichtige Eigenschaft dieser Theorien ist, im Gegensatz zu topologischen Feldtheorien, dass deren Zustandsräume unendlich dimensional sein dürfen. Diese Theorien werden durch regularisierte Frobenius-Algebren klassifiziert und eine Zustandssummenkonstruktion wird gegeben. Hier ist das Inputdatum eine stark separable regularisierte Frobenius-Algebra. Die Konstruktion wird dann um Defektlinien erweitert, die mit Bimoduln über stark separablen regularisierten Frobenius-Algebren bezeichnet sind. Es wird gezeigt, dass das Fusionprodukt von Defektlinien dem Tensorprodukt von Bimoduln über regularisierten Algebren entspricht. Das Hauptbeispiel flächenabhängiger Quantenfeldtheorien ist die zweidimensionale Yang-Mills Theorie mit einer kompakten halbeinfachen Lie-Gruppe G und mit Wilson-Linien als Defekten. Dieses Beispiel wird ausführlich betrachtet, und schließlich werden durch äußeren Automorphismen von G getwistete Defektlinien eingeführt und untersucht.