Non-semisimple modular tensor categories from small quantum groups

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Part I

Introduction

Introduction

The representation theory of finite groups plays an important role in many different areas not only in mathematics but also in physics and even chemistry. Different aspects of this theory are relevant for different fields. The classification of finite simple groups, for instance, wouldn't have been accomplished without the character theory of their representations and the process led to completely new branches of modern mathematics, such as the theory of vertex algebras. An application in physics, more specifically quantum mechanics, is for example the study of the symmetry group of the Hamiltonian or of the permutation group in many-body problems.

It is well known that the finite-dimensional representations over an algebraically closed field k of a finite group G form a category $\mathsf{Rep}(G)$, which comes with several additional structures. First of all, for any two G-representations V, W we can build the direct sum of k-vector spaces $V \oplus W$, which carries again the structure of a G-representation. In characteristic zero, every G-representation V decomposes into a direct sum of irreducible G-representations, of which only finitely many equivalence classes exist. In this case, $\mathsf{Rep}(G)$ is what is called a *finite semisimple category.* For $\mathrm{char} k \neq 0$ this is in general not the case and the category $\mathsf{Rep}(G)$ is called non-semisimple. Moreover, we can build the algebraic tensor product $V \otimes W$ of two G-representations, which is endowed with the diagonal G-action. Together with the trivial G-representation $\mathbb{I}$, which plays the role of the so-called *tensor unit* for this tensor product, this turns $\mathsf{Rep}(G)$ into what is called a *monoidal category.* In the semisimple case, character theory is used in order to determine the *fusion rules* of $\mathsf{Rep}(G)$, i.e. the decomposition of the tensor product $U_i \otimes U_j$ of two irreducible G-representations into a finite direct sum of powers of irreducible representations:

$$U_i \otimes U_j \cong \bigoplus_{k \in I} U_k^{\oplus m_{ij}^k}$$

The number $m_{ij}^k \in \mathbb{Z}_{\geq 0}$ is called the *multiplicity* of U_k in $U_i \otimes U_j$. Furthermore, for every G-representation V we can endow the dual space V^* with a G-action given by precomposition with the inverse. The canonical evaluation and coevaluation maps on V and V^* are G-intertwiners satisfying certain compatibility conditions. This *duality* together with the above tensor structure turns $\mathsf{Rep}(G)$ into a so-called *fusion category.* An additional structure inherited from the category of finite-dimensional vector space is the action of the symmetric group S_n on n-fold tensor products

$V \otimes \cdots \otimes V$ of a G-representation V. This action appears for example in Schur-Weyl duality and gives rise to what is called a *braiding* on $\mathsf{Rep}(G)$, i.e. a natural isomorphism $\sigma_{V,W} : V \otimes W \to W \otimes V$ satisfying certain coherence conditions. In addition, this braiding is *symmetric*, i.e. it satisfies the symmetry condition $\sigma_{W,V}\sigma_{V,W} = \mathrm{id}_{V \otimes W}$, which is due to the fact that we have an action of S_n instead of the braid group $B_n \twoheadrightarrow S_n$. The category $\mathsf{Rep}(G)$ can be seen as the prototypical example of a *symmetric braided* fusion category. In fact, it is an important result by Deligne [Del90] that for $\mathrm{char} k = 0$ every *positive ribbon* symmetric braided fusion category $\mathcal{C}$ is equivalent to $\mathsf{Rep}(G)$ for some group G. Without going into detail here, we simply recall that ribbon categories allow for the notion of *dimension* of an object and positivity requires these dimensions to take non-negative integral values. In this thesis, we construct braided categories which are exactly of the opposite type, in the sense that the braiding is maximally *non-degenerate*, i.e. $c_{V,W} \circ c_{W,V} = \mathrm{id}$ for all W if and only if V is isomorphic to direct sum of finitely many copies of the tensor unit $\mathbb{I}$. Braided fusion categories with non-degenerate braiding are called *modular*, and they are fairly well-understood in the finite semisimple case [EGNO15], with important examples coming for instance from the previously mentioned vertex algebra theory. Much less is known about non-semisimple modular tensor categories, although they play an important role in logarithmic conformal field theory and low-dimensional topology (for a motivation, see Section 0.2). In particular one is interested in a good stock of examples. In the semisimple case, given a braided fusion category that is not modular, under some conditions it is possible to turn this into a modular category and the idea is to "mod out" the largest subcategory of the form $\mathsf{Rep}(G)$ [Bru00]. This procedure goes under the name *modularization* and we are interested in a non-semisimple analogue.

One of the main goals of this thesis is to produce examples of non-semisimple modular tensor categories as representation categories of factorizable quasi-Hopf algebras (see Section 0.3). Before we give a motivation (see Section 0.2), we are going to introduce the relevant objects and concepts, starting with the notion of a non-semisimple modular tensor category. We assume the reader to be familiar with the material presented in [Kas95].

0.1 Non-semisimple modular tensor categories

To begin with, we fix the notion of a premodular category. Throughout this thesis, k denotes an algebraically closed field of characteristic zero. A monoidal k-linear abelian category $\mathcal{C}$ is called premodular if it has a braiding $c_{X,Y} : X \otimes Y \to Y \otimes X$, duals $X^\vee$ and a self-dual twist $\theta_X : X \to X$. Moreover, it should satisfy several finiteness conditions which can be summarized by the requirement that $\mathcal{C}$ is equivalent to the category Rep_A of finite-dimensional representations of a finite-dimensional k-algebra A as a k-linear abelian category. In particular, $\mathcal{C}$ possesses only finitely many isomorphism classes of simple objects. In other words, a premodular category is a finite k-linear abelian *ribbon* category. Note that in contrast to [EGNO15], we did not assume semisimplicity, meaning that objects in $\mathcal{C}$ do not have to be isomorphic to direct sums of simple objects. One of the main features of ribbon categories is that they admit a consistent theory of traces of morphisms $Tr : End(X) \to End(\mathbb{I}) \cong k$. We can thus define the so-called S-matrix

$$S := (s_{XY})_{X,Y \in \mathcal{O}(\mathcal{C})},$$

with entries $s_{XY} := Tr(c_{Y,X}c_{X,Y})$. These can be interpreted as invariants of the Hopf link, i.e. the link consisting of two circles passing through each other, with circles colored by elements X, Y in the set $\mathcal{O}(\mathcal{C})$ of equivalence classes of simple objects. In the semisimple case, the premodular category $\mathcal{C}$ is called modular, if this S-matrix is non-degenerate. As we will see in the motivational part, it is desirable to have a notion of modularity in the non-semisimple setting. This has been defined by Lyubashenko [KL01]: in a premodular category, the coend

$$\mathbb{K}_{\mathcal{C}} := \int^{X \in \mathcal{C}} X^{\vee} \otimes X$$

has a canonical structure of a Hopf algebra (see next section) in $\mathcal{C}$. For example, if $\mathcal{C} = \mathsf{Rep}_H$ is the representation category of a finite-dimensional Hopf algebra H, then $\mathbb{K}_{\mathcal{C}}$ is isomorphic to the dual Hopf algebra H^* endowed with the coadjoint action. Moreover, the monodromies $c_{Y^{\vee},X}c_{X,Y^{\vee}}$ induce a symmetric Hopf pairing $\omega_{\mathcal{C}} : \mathbb{K}_{\mathcal{C}} \otimes \mathbb{K}_{\mathcal{C}} \to \mathbb{I}$ via the universal property of the coend. The premodular category $\mathcal{C}$ is called modular if $\omega_{\mathcal{C}}$ is non-degenerate. In the finite semisimple case, the coend is given by $\mathbb{K}_{\mathcal{C}} = \bigoplus_{i \in I} U_i^{\vee} \otimes U_i$, where the U_i's are representatives of the finitely many equivalence classes of simple objects. In general it is not easy to compute the coend and hence to check whether a premodular category is modular. Fortunately, Shimizu [Shi16] gave several equivalent conditions for $\mathcal{C}$ to be modular of which the following two turned out to be very useful for us: An object $X \in \mathcal{C}$ is called transparent, if $c_{Y,X}c_{X,Y} = \mathrm{id}_{X \otimes Y}$ holds for every other object $Y \in \mathcal{C}$. The transparent objects form a full subcategory $\mathcal{C}' \subseteq \mathcal{C}$, which is called the Müger center of $\mathcal{C}$. Shimizu showed that $\mathcal{C}$ is modular if and only if $\mathcal{C}' \cong \mathsf{Vect}_k$. On the other hand, Etingof, Nykshych and Ostrik [ENO04] introduced a braided tensor functor

$$F : \mathcal{C} \boxtimes \mathcal{C}^{rev} \to \mathcal{Z}(\mathcal{C}) \tag{1}$$

from the Deligne product $\mathcal{C} \boxtimes \mathcal{C}^{rev}$ to the center $\mathcal{Z}(\mathcal{C})$ of $\mathcal{C}$, which is defined in terms of the braiding and the reverse braiding in $\mathcal{C}$. The category $\mathcal{C}$ is called factorizable if this functor is an equivalence. Again, Shimizu showed that for a premodular category factorizability is equivalent to modularity.
If $\mathcal{C}$ is a premodular category, it is very natural to ask the question whether we can associate to $\mathcal{C}$ a modular category $\mathcal{D}$, which should be minimal in some sense. In the semisimple setting, the relevant notions have been introduced by Bruguierès [Bru00]: a linear ribbon functor $F : \mathcal{C} \to \mathcal{D}$ between a semisimple premodular category $\mathcal{C}$ and a semisimple modular category $\mathcal{D}$ is called a modularization if it is dominant, i.e. for every object $D \in \mathcal{D}$ we have $\mathrm{id}_D = p \circ i$ for some $i : D \to F(C)$, $p : F(C) \to D$, $C \in \mathcal{C}$. It has been shown in [Bru00] and [Müg00] that a premodular category $\mathcal{C}$

admits a modularization if and only if the twist θ_X is trivial for every transparent object $X \in \mathcal{C}$. For the non-semisimple case, the following definition is used in the thesis:

Definition 0.1.1. *A linear ribbon functor between premodular categories $F : \mathcal{C} \to \mathcal{D}$ is a modularization of $\mathcal{C}$, if $\mathcal{D}$ is modular and*

$$F\left(\mathbb{K}_{\mathcal{C}}/Rad(\omega_{\mathcal{C}})\right) \cong \mathbb{K}_{\mathcal{D}},$$

where $Rad(\omega_{\mathcal{C}})$ denotes the radical of the Hopf pairing $\omega_{\mathcal{C}}$.

We should point out here, that there are other approaches in order to define a non-semisimple modularization. For example, the notion of a dominant functor still makes sense in this case. It is therefore tempting to define a modularization of $\mathcal{C}$ as an exact sequence $\mathcal{C}' \to \mathcal{C} \xrightarrow{F} \mathcal{D}$ of tensor categories in the sense of [BN11], where $\mathcal{C}'$ is the Müger center of $\mathcal{C}$ and $\mathcal{D}$ is modular. At least in the setting of Thm. 0.6, both definitions coincide. It would also be interesting to generalize the actual construction of a modularization in [Bru00] to the non-semisimple case. A step in this direction has been made in [BN11], where the authors show that a dominant functor $F : \mathcal{C} \to \mathcal{D}$ with exact right adjoint is equivalent to the free module functor $\mathcal{C} \to \mathrm{mod}_{\mathcal{C}}(A)$ for some commutative algebra A in the center of $\mathcal{C}$. One of the reasons for choosing the given definition is its closeness to one of the equivalent definitions of a modular tensor category. We conjecture that it reduces to the definition of Bruguierès in the semisimple case.

Before we describe the other important objects in this thesis in more detail, we give a motivation for studying non-semisimple modular tensor categories.

0.2 Motivation

One of the main reasons to study modular tensor categories is the role they play in the description of 2d conformal field theories obeying suitable finiteness conditions. More precisely, it is believed that the chiral part of a 2d conformal field theory, more precisely the monodromies of chiral blocks, is encoded in the underlying modular tensor category. This has been made precise in the finite semisimple case [Zhu96][Hua05]. For certain classes of two-dimensional conformal field theories, the associated modular tensor categories arise as representation categories of certain vertex operator algebras (VOA). For a detailed introduction to VOA's, we refer to [FBZ04].

Recall that a VOA is an infinite-dimensional graded vector space V, together with a product $V \otimes V \to V((z, z^{-1}))$ (state-field correspondence), where $V((z, z^{-1}))$ denotes the space of formal Laurent series with cofficients in V, a unit $|0\rangle \in V$ (vacuum vector) and an operator $T : V \to V$ (translation operator) subject to several axioms. Moreover, a VOA is required to carry an action of the Virasoro algebra, which should be compatible with the other structure. Without going too much into detail, we just mention that a VOA allows for the definition of so-called chiral n-point functions (even though they are multivalued) on the sphere, which are also called conformal blocks. Conformal blocks with three marked points define a tensor product on the representation category $V-\mathsf{Mod}$ and the structure of conformal blocks with four marked points determines an associativity constraint for this tensor product, turning $V-\mathsf{Mod}$ into a monoidal category. Also, moving the marked points around each other and considering the corresponding monodromy leads naturally to a braiding on this category, which should be non-degenerate by construction. If a VOA moreover allows for duals and a ribbon structure, its representation category is expected to form a modular tensor category, even though this has not been made precise in the general case.

The physically interesting examples of chiral 2d conformal field theories are often neither finite nor semisimple. It is therefore desirable to build a good stock of examples in both cases, with a mathematically precise description in terms of modular tensor categories. As we have mentioned, in this thesis we are interested in the finite but non-semisimple case, giving rise to so-called logarithmic conformal field theories. One of the best understood examples of a logarithmic conformal field comes from so-called $\mathcal{W}$-algebras:
Without going into detail here, we just remark that there is a general approach (see [FT10]) to construct a non-semisimple VOA from a simple complex simply-laced Lie algebra $\mathfrak{g}$ and a $2p$th root of unity generalizing the so-called triplet VOA $\mathcal{W}(p)$ (see [FGST06] for a definition), which is the case $\mathfrak{g}=\mathfrak{sl}_2$. The class of VOA's coming from this approach is referred to as $\mathcal{W}$-algebras. It is believed that the representation categories of this class of VOA's are ribbon equivalent to representation categories of certain finite-dimensional quasi-Hopf algebras. In the case of the so-called triplet VOA $\mathcal{W}(p)$, an equivalence of $\mathbb{C}$-linear categories between $\mathcal{W}(p)-\mathsf{Mod}$ and the representation category of the small quantum group $u_q(\mathfrak{sl}_2)$ at a primitive $2p$th root of unity q was proven in [NT09]. However, it was soon observed that this particular example of a small quantum group (see Sec. 0.4) does not allow for an R-matrix, meaning that the above equivalence can not be extended to an equivalence of braided categories. Recently, it was shown that the coproduct of $u_q(\mathfrak{sl}_2)$ can be modified in a way, such that one obtains a factorizable ribbon quasi-Hopf algebra $\bar{u}_q^{\Phi}(\mathfrak{sl}_2)$, coinciding with $u_q(\mathfrak{sl}_2)$ as an algebra [CGR17]. At least in the case $p=2$, this quasi-Hopf algebra was known before and it has been shown that in this case we indeed obtain a ribbon equivalence [GR17].

Since the premodular and modular categories we are interested in are representation categories of (quasi-)Hopf algebras with several additional structures, we will now describe these objects in more detail.

0.3 Factorizable ribbon (quasi-)Hopf algebras

Before we discuss the case of quasi-Hopf algebras, we briefly recall the relevant notions for an ordinary Hopf algebra. A bialgebra is an associative unital k-algebra (H,m,η) together with algebra maps $\Delta : H \to H \otimes H$ and $\epsilon : H \to k$ turning H into a coassociative counital coalgebra. A convolution inverse of the identity $\mathrm{id} : H \to H$ is called an antipode and denoted by S. If it exists, it is unique and H is then called a Hopf algebra. The coproduct Δ allows to define an H-module structure on the tensor product $V \otimes W$ of two H-modules V, W. This turns the category Rep_H of finite-dimensional left H-modules into a k-linear abelian monoidal category with trivial associator, which is finite if H is finite-dimensional. Moreover, pre-composition with the antipode (or its inverse) defines a left H-module structure on the dual space V^* of an H-module V. Together with the ordinary evaluation and coevaluation from finite-dimensional vector spaces, this turns Rep_H into a category with left and right duals, more precisely, into a rigid category. A braiding on Rep_H can be achieved by an additional structure on the Hopf algebra H, namely a universal R-matrix $R \in H \otimes H$. In the following, we will often use the suggestive notation $R = R^1 \otimes R^2$. The axioms for an R-matrix are modelled in a way, s.t. the action $v \otimes w \mapsto R^2.w \otimes R^1.v$ is an H-intertwiner and satisfies the hexagon axioms of a braided category. A Hopf algebra with R-matrix is called quasi-triangular. Finally, a ribbon element $\nu \in H$ is defined in a way, s.t. the action $v \mapsto \nu.v$ defines a ribbon twist in Rep_H. In sum, a finite-dimensional ribbon Hopf algebra has a premodular representation category.

Inspired by the previous section it is now natural to ask, whether the premodular category Rep_H is modular. To answer this, we first introduce the notion of factorizability for a Hopf algebra. A finite-dimensional quasi-triangular Hopf algebra (H, R) is called factorizable if its monodromy matrix $M := R_{21}R \in H \otimes H$, where $R_{21} = R^2 \otimes R^1$ is non-degenerate, i.e. we can write $M = \sum_i E^i \otimes F^i$ for two bases $E^i, F^j \in H$. In [Sch01], Schneider proves that factorizability of Rep_H is equivalent to H being factorizable. If H is a finite-dimensional ribbon Hopf algebra, this means that Rep_H is modular if and only if H is factorizable.

For the class of Hopf algebras we considered in this thesis, namely small quantum groups (see section 0.4), our scan showed that only a few of them are factorizable. As ribbon Hopf algebras, the remaining still define premodular categories, which one could ask to be modularizable. In the cases when this is possible, the resulting category is very rarely again the representation category of a Hopf algebra (see 0.6), but rather of a quasi-Hopf algebra, which we want to introduce now:

As we pointed out above, the representation category of a bialgebra H has a monoidal structure with trivial associator in the sense that the forgetful functor $\mathsf{Rep}_H \to \mathsf{Vect}_k$ is monoidal. This is due to the fact that, as a coalgebra, H is a coassociative. In order to allow for non-trivial associators, we have to weaken the coassociativity axiom by picking an invertible element $\Phi \in H \otimes H \otimes H$, s.t. $(H \otimes \Delta)\Delta(h) \cdot \Phi = \Phi \cdot (\Delta \otimes H)\Delta(h)$. Moreover, Φ should satisfy a 3-cycle condition, so that the action $u \otimes v \otimes w \in U \otimes V \otimes W \mapsto \Phi^1.u \otimes \Phi^2.v \otimes \Phi.w \in U \otimes V \otimes W$ satisfies the pentagon axioms of a monoidal category. An algebra H together with a coassociator Φ and a quasi-coassociative coproduct satisfying all the other axioms of a bialgebra is called a quasi-bialgebra. A direct consequence of this change of axioms is that the antipode $S : H \to H$ is not unique anymore. It comes with two additional elements $\alpha, \beta \in H$, s.t. they define evaluation and coevaluation for the dual space

V^*, endowed with the same module structure as before. Such a collection $(H, \Phi, S, \alpha, \beta)$ is called a quasi-Hopf algebra. Again, an R-matrix $R \in H \otimes H$ for a quasi-Hopf algebra is defined in a way, s.t. the action $v \otimes w \in V \otimes W \mapsto R^2.w \otimes R^1.v \in W \otimes V$ satisfies the hexagon axioms, but now with non-trivial associator coming from Φ. The notions of ribbon quasi-Hopf algebras and factorizable Hopf algebras are adjusted similarly (see [BN03] and [BT04]). Again, for a finite-dimensional ribbon quasi-Hopf algebra, the representation category Rep_H will be modular if H is factorizable.
In order to construct the above mentioned factorizable quasi-Hopf algebras, we furthermore need quasi-analogues of several well-known constructions from the Hopf-case, such as Yetter-Drinfeld modules, Nichols algebras, Radford biproducts and Drinfeld doubles. All these notions exist also in the case of a quasi-Hopf algebra and are carefully introduced in the second part of this thesis.

0.4 Example: Small quantum groups and their representations

In this thesis, we provide modular tensor categories as representation categories of a particular family of finite-dimensional Hopf algebras and quasi-Hopf algebras. The former we want to describe now. We first note that for convenience, we set $k = \mathbb{C}$ in the first part of the thesis although we only need k to be algebraically closed and of characteristic zero. even though we could have worked with It is known that the universal enveloping algebra of a semisimple finite-dimensional complex Lie algebra $\mathfrak{g}$ can be naturally deformed to a Hopf algebra $U_q(\mathfrak{g})$ over the field of rational functions over $\mathbb{Q}$. With some care, it is possible to specialize the indeterminate q to any specific value in $\mathbb{C}^\times$. From now on, q will be a primitive ℓth root of unity. This case is particularly interesting, since in contrast to the generic case the representation theory of $U_q(\mathfrak{g})$ is not semisimple. In [Lus90], Lusztig constructs a surjetive homomorphism from $U_q(\mathfrak{g})$ to the ordinary enveloping algebra $U(\mathfrak{g})$. He realizes that the kernel of this homomorphism is a finite-dimensional Hopf algebra, which is called the Lusztig kernel or small quantum group and denoted by $u_q(\mathfrak{g})$. The small quantum group is generated by skew-primitive elements $E_{\alpha_i}, F_{\alpha_j}$ and grouplikes $K^{\pm}_{\alpha_k}$, where the α_i's are choices of simple roots in the root lattice of $\mathfrak{g}$. Amongst other, these generators satisfy the relations

$$K_{\alpha_i} E_{\alpha_j} K_{\alpha_i}^{-1} = q^{(\alpha_i, \alpha_j)} E_{\alpha_j} \qquad\qquad K_{\alpha_i} F_{\alpha_j} K_{\alpha_i}^{-1} = q^{-(\alpha_i, \alpha_j)} F_{\alpha_j}.$$

They can be interpreted as an action of the root lattice Λ_R on $u_q(\mathfrak{g})$. This action allows us to extend the small quantum group by an arbitrary intermediate lattice $\Lambda_R \subseteq \Lambda \subseteq \Lambda_W$, which will serve as an additional parameter for our search for factorizable Hopf algebras. In [Lus90], Lusztig also constructs an "almost R-matrix" $\bar{\Theta} \in u_q(\mathfrak{g})^- \otimes u_q(\mathfrak{g})^+$ in the Borel part of $u_q(\mathfrak{g})$, from which we construct actual factorizable R-matrices in the first part of this thesis.
We should point out that part of our construction of the quasi-Hopf algebras arising as modularizations of small quantum groups is based on [AS02], where the authors give a more abstract characterization of the Borel part of small quantum groups in terms of Nichols algebras of Cartan

type.

This thesis is divided into two parts: in the first part we determine those (extended) small quantum groups, which have a (pre-)modular representation category. In the second part, we describe the modularization of the premodular representation categories. To this end, we systematically construct a whole class of finite-dimensional quasi-triangular quasi-Hopf algebras generalizing small quantum groups and the above mentioned quasi-Hopf algebra $\bar{u}^{\Phi}(\mathfrak{sl}_2)$.
We now describe the two parts of the thesis in more detail:

0.5 Factorizable R-matrices for small quantum groups

The central objects of the first part of the thesis are extensions of small quantum groups, as introduced in section 0.4. They are parameterized by the following data:

- a finite-dimensional simple complex Lie algebra $\mathfrak{g}$
- a natural number ℓ, determining a primitive root of unity $q = \exp(2\pi i/\ell)$
- an intermediate lattice $\Lambda_R \subseteq \Lambda \subseteq \Lambda_W$ between the root lattice Λ_R and the weight lattice Λ_W
- a sublattice $\Lambda' \subseteq \mathrm{Cent}_{\Lambda_R}(\Lambda_R)$ of the centralizer $\mathrm{Cent}_{\Lambda_R}(\Lambda_R) \subseteq \Lambda_R$.

As we have mentioned, Lusztig showed that there is a so-called quasi-R-matrix $\bar{\Theta} \in u_q^-(\mathfrak{g}) \otimes u_q^+(\mathfrak{g})$ for the small quantum group $u_q(\mathfrak{g})$, behaving similar to a proper R-matrix. In [Mül98], Müller showed that the ansatz $R = R_0\bar{\Theta}$ gives rise to an actual R-matrix of the extended small quantum group $u := u_q(\mathfrak{g}, \Lambda, \Lambda')$ if certain conditions are fulfilled. Here, $R_0 \in u^0 \otimes u^0$ entirely lives in the coradical $u^0 = \mathbb{C}[\Lambda/\Lambda']$ of u. In [LN15], it was checked explicitly, for which input parameters (with fixed Λ') and for which choices of R_0 these conditions are fulfilled. In [LO17], we reinterpreted the conditions in [Mül98] as the non-degeneracy of a certain bihomomorphism $\hat{f} : G_1 \times G_2 \to \mathbb{C}^\times$ of finite abelian groups. From this, we were able to extend the result in [LN15] to an arbitrary sublattice $\Lambda' \subseteq \mathrm{Cent}_{\Lambda_R}(\Lambda_R)$ and give a single closed condition for the existence of an R-matrix of the above form, involving all relevant parameters.
In Theorem 5.1.6 we give a necessary and sufficient condition for an R-matrix of the form $R = R_0\bar{\Theta}$ to be factorizable, which corresponds to Rep_u being factorizable. To this end, we show that the invertibility of the monodromy matrix of R is equivalent to the invertibility of the monodromy matrix of R_0. It turned out that it is invertible if and only if the radical Rad_0 of the so-called symmetrization $\mathrm{Sym}(\hat{f})$ (see Def. 5.1.2) of the above mentioned bihomomorphism $\hat{f}$ is trivial. For further use, we compute the radical of $\mathrm{Sym}(\hat{f})$ for all possible choices of R_0.
We prove that the simple transparent objects in Rep_u are all 1-dimensional and that the corresponding group of all transparent objects is in fact isomorphic to Rad_0. Finally, we show that for

every R-matrix of the form $R = R_0\bar{\Theta}$ there exists a ribbon element $\nu \in u$, turning Rep_u into a pre-modular category. We summarize our results in the following table, containing all quasi-triangular extensions of small quantum groups together with their group of transparent objects. The columns of the table are labeled by

1. the finite dimensional simple complex Lie algebra $\mathfrak{g}$
2. the natural number ℓ, determining the primitive root of unity $q = \exp\left(\frac{2\pi i}{\ell}\right)$
3. the number of possible R-matrices for the Lusztig ansatz
4. the subgroups $H_i \subseteq H = \Lambda/\Lambda_R$ introduced in Theorem 3.1.3
5. the subgroups H_i in terms of generators given by multiples of fundamental dominant weights $\lambda_i \in \Lambda_W$
6. the group pairing $g : H_1 \times H_2 \to \mathbb{C}^\times$ determined by its values on generators
7. the group of transparent objects $T \subseteq \Lambda/\Lambda'$ introduced in Lemma 5.4.5.

$\mathfrak{g}$	ℓ	$\#$	$H_i \cong$	$H_{i\ (i=1,2)}$	g	$T \subseteq \Lambda/\Lambda'$
all	ℓ odd	1	$\mathbb{Z}_1$	$\langle 0 \rangle$	$g = 1$	$\mathbf{0}$
	$\ell \equiv 0 \bmod 4$	1				$\mathbb{Z}_2^n$
$A_{n\geq 1}$ $\pi_1 = \mathbb{Z}_{n+1}$		∞	$\mathbb{Z}_d$ $d \mid n+1$	$\langle \hat{d}\lambda_n \rangle$ $\hat{d} = \frac{n+1}{d}$	$g(\hat{d}\lambda_n, \hat{d}\lambda_n) = \exp\left(\frac{2\pi i k}{d}\right)$ $\gcd\left(n+1, d\ell, k\ell - \frac{n+1}{d}n\right) = 1$	$\mathbb{Z}_2^{n-1}$, $2 \nmid x$ $\mathbb{Z}_2^n$, $2 \mid x$ $x = \frac{d\ell}{\gcd(\ell, \hat{d})}$
$B_{n\geq 2}$ $\pi_1 = \mathbb{Z}_2$	$\ell \equiv 2 \bmod 4$	1	$\mathbb{Z}_1$	$\langle 0 \rangle$	$g = 1$	$\mathbf{0}$
	$\ell \equiv 2 \bmod 4$	2	$\mathbb{Z}_2$	$\langle \lambda_n \rangle$	$g(\lambda_n, \lambda_n) = \pm 1$	$\mathbb{Z}_2$
	$\ell \equiv 0 \bmod 4$	2	$\mathbb{Z}_2$	$\langle \lambda_n \rangle$	$g(\lambda_n, \lambda_n) = \pm 1$	$\mathbb{Z}_2^n$
	ℓ odd	1	$\mathbb{Z}_2$	$\langle \lambda_n \rangle$	$g(\lambda_n, \lambda_n) = (-1)^{n+1}$	$\mathbb{Z}_2$
	$\ell \equiv 2 \bmod 4$	1	$\mathbb{Z}_1$	$\langle 0 \rangle$	$g = 1$	$\mathbb{Z}_2^{n-2}$

$C_{n\geq 3}$ $\pi_1=\mathbb{Z}_2$	$\ell\equiv 2 \bmod 4$	1	$\mathbb{Z}_2$	$\langle\lambda_n\rangle$	$g(\lambda_n,\lambda_n)=1$	$\mathbb{Z}_2^{n-1}$
	$\ell\equiv 0 \bmod 4$	2			$g(\lambda_n,\lambda_n)=\pm 1$	$\mathbb{Z}_2^{n}$
	ℓ odd	1			$g(\lambda_n,\lambda_n)=-1$	$\mathbb{Z}_2$
$D_{2n\geq 4}$	$\ell\equiv 2 \bmod 4$	1	$\mathbb{Z}_1$	$\langle 0\rangle$	$g=1$	$\mathbb{Z}_2^{2(n-1)}$
	$\ell\equiv 2 \bmod 4$	1	$\mathbb{Z}_2$	$H_1\cong\langle\lambda_{2n-1}\rangle$ $H_2\cong\langle\lambda_{2n}\rangle$	$g(\lambda_{2n-1},\lambda_{2n})=(-1)^n$	$\mathbb{Z}_2^{2n}$
	$\ell\equiv 0 \bmod 4$	$2\delta_{2\vert n}$			$g(\lambda_{2n-1},\lambda_{2n})=\pm 1$, n even	
	ℓ odd	1			$g(\lambda_{2n-1},\lambda_{2n})=-1$	$\mathbf{0}$

$\pi_1 = \mathbb{Z}_2 \times \mathbb{Z}_2$	$\ell \equiv 2 \bmod 4$	1	$\mathbb{Z}_2$	$\langle \lambda_{2n} \rangle$	$g(\lambda_{2n}, \lambda_{2n}) = (-1)^{n+1}$	$\mathbb{Z}_2^{2n-1}$
	$\ell \equiv 0 \bmod 4$	$2\delta_{2\nmid n}$			$g(\lambda_{2n}, \lambda_{2n}) = \pm 1$, n odd	$\mathbb{Z}_2^{2n}$
	ℓ odd	1			$g(\lambda_{2n}, \lambda_{2n}) = -1$	$\mathbb{Z}_2$
	ℓ even	2	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\langle \lambda_{2n}, \lambda_{2n+1} \rangle$	$g(\lambda_{2(n-1)+i}, \lambda_{2(n-1)+j}) = \pm 1$	$\mathbb{Z}_2^{2n}$
	ℓ odd				$det(K) = K_{12} + K_{21} = 0 \bmod 2$	$\mathbb{Z}_2$
					$det(K) = K_{12} + K_{21} = 1 \bmod 2$	$\mathbb{Z}_2^2$
$D_{2n+1 \geq 5}$ $\pi_1 = \mathbb{Z}_4$	$\ell \equiv 2 \bmod 4$	1	$\mathbb{Z}_1$	$\langle 0 \rangle$	$g = 1$	$\mathbb{Z}_2^{2n}$
	$\ell \equiv 2 \bmod 4$	1	$\mathbb{Z}_2$	$\langle 2\lambda_{2n+1} \rangle$	$g(2\lambda_{2n+1}, 2\lambda_{2n+1}) = 1$	$\mathbb{Z}_2^{2n+1}$
	$\ell \equiv 0 \bmod 4$	2			$g(2\lambda_{2n+1}, 2\lambda_{2n+1}) = \pm 1$	
	ℓ odd	1			$g(2\lambda_{2n+1}, 2\lambda_{2n+1}) = -1$	$\mathbb{Z}_2$
	ℓ even	4	$\mathbb{Z}_4$	$\langle \lambda_{2n+1} \rangle$	$g(\lambda_{2n+1}, \lambda_{2n+1}) = c$, $c^4 = 1$	$\mathbb{Z}_2^{2n+1}$
	ℓ odd	2			$g(\lambda_{2n+1}, \lambda_{2n+1}) = \pm 1$	$\mathbb{Z}_2$
E_6 $\pi_1 = \mathbb{Z}_3$	$\ell \equiv 2 \bmod 4$	1	$\mathbb{Z}_1$	$\langle 0 \rangle$	$g = 1$	$\mathbb{Z}_2^6$
	$\ell \equiv 0 \bmod 3$	3	$\mathbb{Z}_3$	$\langle \lambda_n \rangle$	$g(\lambda_n, \lambda_n) = c$, $c^3 = 1$	$\mathbb{Z}_2^6$, $2 \mid \ell$
	$\ell \equiv 1 \bmod 3$	2			$g(\lambda_n, \lambda_n) = 1, \exp\left(\frac{2\pi i 2}{3}\right)$	
	$\ell \equiv 2 \bmod 3$	2			$g(\lambda_n, \lambda_n) = 1, \exp\left(\frac{2\pi i}{3}\right)$	$\mathbf{0}$, $2 \nmid \ell$

E_7 $\pi_1 = \mathbb{Z}_2$	$\ell \equiv 2 \bmod 4$	1	$\mathbb{Z}_1$	$\langle 0 \rangle$	$g = 1$	$\mathbb{Z}_2^6$
	ℓ even	2	$\mathbb{Z}_2$	$\langle \lambda_n \rangle$	$g(\lambda_n, \lambda_n) = \pm 1$	$\mathbb{Z}_2^n$
	ℓ odd	1			$g(\lambda_n, \lambda_n) = 1$	$\mathbb{Z}_2$
E_8	$\ell \equiv 2 \bmod 4$	1	$\mathbb{Z}_1$	$\langle 0 \rangle$	$g = 1$	$\mathbb{Z}_2^8$
F_4	$\ell \equiv 2 \bmod 4$	1	$\mathbb{Z}_1$	$\langle 0 \rangle$	$g = 1$	$\mathbb{Z}_2^2$
G_2	$\ell \equiv 2 \bmod 4$	1	$\mathbb{Z}_1$	$\langle 0 \rangle$	$g = 1$	$\mathbb{Z}_2^2$

Table 1: Solutions for R-matrices

From the last column, one can already see that the premodular category Rep_u is almost never modular, which is equivalent to T being trivial, for an even root of unity q. The only exception is when $\mathfrak{g} = B_n$ and $\ell \equiv 2 \bmod 4$. The canonical next step is to use the method of modularization, as described in Section 0.1.

0.6 Modularization of small quantum groups

We now turn to the second part of this thesis, the modularization of the premodular category Rep_u from the previous section. We start with the representation category Rep_{u^0} of the Cartan part $u^0 = \mathbb{C}[G]$, where $G = \Lambda/\Lambda'$, (see Sec. 0.5) which we from now on identify with the category of finite dimensional $\widehat{G}$-graded vector spaces $\mathsf{Vect}_{\widehat{G}}$. The braiding on Rep_{u^0} induced by the element $R_0 \in \mathbb{C}[G] \otimes \mathbb{C}[G]$ is transported to a braiding on $\mathsf{Vect}_{\widehat{G}}$. It is induced by a bihomomorphism $\sigma : \widehat{G} \times \widehat{G} \to \mathbb{C}^\times$, which is simply the Fourier transform of R_0. The non-modularity of this category is in a precise sense measured by the radical $T := Rad(\mathrm{B})$ of the associated bihomomorphism $\mathrm{B} := \sigma\sigma^T$, so the idea is to simply take the quotient $\widehat{G}/T$. The problem is that σ might not be well-defined on $\widehat{G}/T$ and so we wouldn't have a braiding on $\mathsf{Vect}_{\widehat{G}/T}$. The solution to this problem is to allow for so-called abelian 3-cocycles $(\omega, \sigma) \in Z^3_{ab}(\widehat{G}/T)$ (see [Mac52]) instead of ordinary bihomomorphisms $\sigma \in Bihom(\widehat{G}/T)$. Categorically speaking, this corresponds to allowing for a non-trivial associator on $\mathsf{Vect}_{\widehat{G}/T}$, induced by the 3-cocycle $\omega \in Z^3(\widehat{G}/T)$. This 3-cocycle fulfills two additional compatibility conditions with the 2-cochain $\sigma \in C^2(\widehat{G}/T)$, corresponing to the hexagon equations of a braided monoidal category. In Section 5 in the second part of the thesis, we start with an abelian group G, an abelian 3-cocycle $(\omega, \sigma) \in Z^3_{ab}(\widehat{G}/T)$ and a finite set of elements $\chi_i \in \widehat{G}$. From this, we can define a Nichols algebra $B(V)$ in the braided monoidal category $\mathsf{Vect}^{(\omega,\sigma)}_{\widehat{G}/T}$.

Surprisingly, this Nichols algebra satisfies the same relations as the Nichols algebra associated to the braided vector space V with diagonal braiding given by $q_{ij} = \sigma(\bar{\chi}_i, \bar{\chi}_j)$. If this Nichols algebra is finite-dimensional, we can build the Drinfeld double $D(u(\omega,\sigma)^{\leq 0})$ of the Radford biproduct $u(\omega,\sigma)^{\leq 0} := B(V)\# k^{\widehat{G}}_\omega$ over the function algebra $k^{\widehat{G}}_\omega$, considered as a quasi-Hopf algebra with coassociator induced by ω. Both notions have been generalized to the quasi-Hopf case (see [BN02] and [HN99b]). After this, we define an both algebra and coalgebra embedding $j : k\widehat{G} \to D(\omega,\sigma)$, which takes values in the center of $D(\omega,\sigma)$. After modding out a certain biideal associated to this embedding, we end up with a finite-dimensional quasi-Hopf algebra $u(\omega,\sigma)$, which we refer to as small quasi-quantum group. We show that there exist a canonical choice of R-matrix for this quasi-Hopf algebra, induced by the 2-cochain σ. Moreover, we show that $u(\omega,\sigma)$ is factorizable if and only if $T = 0$. We give a list of important relations of the small quasi-quantum group $u(\omega,\sigma)$:

$$\Delta(F_i) = K_{\bar{\chi}_i} \otimes F_i \left(\sum_{\chi,\psi} \theta(\chi|\bar{\chi}_i,\psi)^{-1}\omega(\bar{\chi}_i,\psi,\chi)^{-1}\, \delta_\chi \otimes \delta_\psi \right) + F_i \otimes 1 \left(\sum_{\chi,\psi} \omega(\bar{\chi}_i,\chi,\psi)^{-1}\, \delta_\chi \otimes \delta_\psi \right)$$

$$\Delta(E_i) = \left(\sum_{\chi,\psi} \theta(\psi|\chi\bar{\chi}_i,\chi_i)^{-1}\omega(\psi,\chi,\bar{\chi}_i)^{-1}\, \delta_\chi \otimes \delta_\psi \right) E_i \otimes \bar{K}_{\chi_i} + \left(\sum_{\chi,\psi} \omega(\chi,\psi,\bar{\chi}_i)^{-1}\, \delta_\chi \otimes \delta_\psi \right) 1 \otimes E_i$$

$$\Delta(K_\chi) = (K_\chi \otimes K_\chi)P_\chi^{-1} \qquad \Delta(\bar{K}_\chi) = (\bar{K}_\chi \otimes \bar{K}_\chi)P_\chi, \qquad P_\chi := \sum_{\psi,\xi} \theta(\chi|\psi,\xi)\, \delta_\psi \otimes \delta_\xi$$

$$[E_i^a K_{\chi_i}, F_j^b]_\sigma = \delta_{ij}\sigma(\chi_i,\bar{\chi}_i)\left(1 - K_{\chi_i}\bar{K}_{\chi_i}\right) \left(\sum_\xi \frac{a_i(\xi)b_i(\xi\chi_i)}{\omega(\bar{\chi}_i,\chi_i,\xi)}\, \delta_\xi \right), \text{ where}$$

$$E_i^a := E_i \left(\sum_\xi a_i(\xi)\, \delta_\xi \right) \qquad F_j^b := F_j \left(\sum_\xi b_j(\xi)\, \delta_\xi \right), \quad \text{with } a_i, b_j \text{ solutions to Eq. 5.8.}$$

$$K_\chi E_i = \sigma(\chi,\chi_i)E_i K_\chi Q_{\chi,\chi_i}^{-1}, \qquad \bar{K}_\chi E_i = \sigma(\chi_i,\chi)E_i\bar{K}_\chi Q_{\chi,\chi_i}, \qquad Q_{\chi,\psi} := \sum_\xi \theta(\chi|\xi,\psi)\, \delta_\xi$$

$$K_\chi F_i = \sigma(\chi,\bar{\chi}_i)F_i K_\chi Q_{\chi,\bar{\chi}_i}^{-1}, \qquad \bar{K}_\chi F_i = \sigma(\bar{\chi}_i,\chi)F_i\bar{K}_\chi Q_{\chi,\bar{\chi}_i}$$

$$S(F_i) = -\left(\sum_\psi \omega(\bar{\psi},\bar{\chi}_i,\chi_i\psi)d\sigma(\chi_i,\psi,\bar{\psi})\theta(\bar{\psi}|\psi\chi_i,\bar{\chi}_i)^{-1}\delta_\psi \right) K_{\chi_i}F_i$$

$$S(E_i) = -E_i\bar{K}_{\chi_i}^{-1} \left(\sum_\psi \frac{\omega(\bar{\chi}_i\bar{\xi},\bar{\chi}_i,\xi)}{\omega(\bar{\xi},\bar{\chi}_i,\chi_i)}\, \delta_\xi \right)$$

$$\epsilon(K_\chi) = \epsilon(\bar{K}_\chi) = 1, \qquad \epsilon(E_i) = \epsilon(F_i) = 0, \qquad 1_{u(\omega,\sigma)} = K_1$$

In the next section we interpret the initial small quantum group u as a small quasi-quantum group $u(1,\sigma)$ associated to the data $G = \Lambda/\Lambda'$ and $\chi_i = q^{(\alpha_i,\_)}$. If the associated quadratic form $Q(\chi) := \sigma(\chi,\chi)$ of the abelian 3-cocycle $(1,\sigma)$ vanishes on T, then for every set-theoretic section

$s : \widehat{G}/T \to \widehat{G}$, we find an explicit abelian 3-cocycle $(\bar{\omega}, \bar{\sigma}) \in Z^3_{ab}(\widehat{G})$, s.t. $\pi^*(\bar{\omega}, \bar{\sigma})$ is cohomologous to $(1, \sigma)$. We refer to the condition $Q|_T = 1$ as $u(1, \sigma)$ being modularizable. Moreover, we find an algebra embedding $M : u(\omega, \sigma) \to u(1, \sigma)$. Our modularization functor is going to be a restriction functor along this algebra homomorphism. The main result of this part of the thesis is Thm. 6.0.6:

Theorem (Lentner-O-Gainutdinov, 2018)**.** *Let u be an extended small quantum group with R-matrix $R = R_0\bar{\Theta}$. If u is modularizable in the above sense, then there exists a factorizable small quasi-quantum group $u(\bar{\omega}, \bar{\sigma})$ and a modularization in the sense of Def. 0.1.1*

$$F : \mathsf{Rep}_u \longrightarrow \mathsf{Rep}_{u(\bar{\omega},\bar{\sigma})}.$$

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Part II

Factorizability of small quantum groups

Chapter 1. Introduction

The aim of the first part of this thesis is to provide modular tensor categories from extended *small quantum groups* $u_q(\mathfrak{g})$ at a primitive ℓth root of unity q for a finite-dimensional simple complex Lie algebra $\mathfrak{g}$. Lusztig [Lus90] has constructed these finite-dimensional Hopf algebras and provided an ansatz for an R-matrix $R_0\bar{\Theta}$, where the fixed element $\bar{\Theta} \in u_q(\mathfrak{g})^- \otimes u_q(\mathfrak{g})^+$ is constructed from a dual basis of PBW-generators, while $R_0 \in u_q(\mathfrak{g})^0 \otimes u_q(\mathfrak{g})^0$ is a free parameter subject to certain constraints given in [Mül98]. Lusztig gives a canonical solution for R_0 whenever ℓ has no common divisors with root lengths, otherwise there are cases where no R-matrix exists [KS11]. Of particular interest in conformal field theory [FGST06, FT10, GR17] is the most extreme case where all root lengths (α, α) divide ℓ. In particular, this thesis adresses the question which modular tensor category appear in these cases. In Lemma 4.5.1, we observe that these extremal cases give especially nice R-matrices. However, in general they are not factorizable and will require modularization in order to match the CFT side. This will be done in the second part of the thesis.

But even if there are no common divisors with the root length, the resulting braided tensor categories may not fulfill the non-degeneracy condition and hence provides no modular tensor category. Both obstacles for being factorizable, existence and non-degeneracy of an R-matrix, can be be resolved by extending the Cartan part of the small quantum group by a choice of a lattice $\Lambda_R \subseteq \Lambda \subseteq \Lambda_W$ between root- and weight-lattice, respectively a choice of a subgroup of the fundamental group $\pi_1 := \Lambda_W/\Lambda_R$, corresponding to a choice of a Lie group between adjoint and simply-connected form. These extensions are already present in [Lus90] as the choice of two lattices X, Y with pairing $X \times Y \to \mathbb{C}^\times$ (root datum). In this way the number of possible R-matrices increases and the purpose of our paper is to study them all.

In [LN15], the authors have already constructed some solutions R_0 in this spirit (under some additional assumptions). In this thesis, we conclude this effort: First we introduce more systematical techniques that allow us to compute a list of all quasitriangular structures (without additional assumptions, so we find more solutions). Then our new techniques allow us to determine, which of these choices fulfill the non-degeneracy condition. We also determine which cases have a ribbon structure. A main role in the first part is played by a natural pairing a_ℓ on the fundamental group

π_1 which depends only on the common divisors of ℓ with the fundamental group and encapsulates the essential ℓ-dependence. In the generic case, the non-degeneracy of the braiding turns out to depend only on the 2-torsion of the abelian group in question.

Our result produces a list of modular tensor categories for representations of small quantum groups. Moreover, we use our methods to explicitly describe the so-called group of transparent objects in the non-factorizable cases. The main output of this part of the thesis is Table 1, where we list all solutions for quasi-triangular small quantum groups together with their group of transparent objects.

Chapter 2. Preliminaries

2.1 Lie-Theory

Throughout this part of the thesis, $\mathfrak{g}$ denotes a finite-dimensional simple complex Lie algebra. We fix a choice of simple roots $\Delta = \{\alpha_i \,|\, i \in I\}$, so that the Cartan matrix C is given by $C_{ij} = 2\frac{(\alpha_i,\alpha_j)}{(\alpha_i,\alpha_i)}$, where $(\ ,\ )$ denotes the normalized Killing form. For a root α, we define $d_\alpha := \frac{(\alpha,\alpha)}{2}$ and set $d_i = d_{\alpha_i}$.
By $\Lambda_R := \mathbb{Z}[\Delta]$ and $\Lambda_R^\vee := \mathbb{Z}[\Delta^\vee]$ we denote the (co)root lattice of $\mathfrak{g}$.
By Λ_W, we denote the *weight lattice* spanned by fundamental dominant weights λ_i, which are defined by the equation $(\lambda_i, \alpha_j) = \delta_{i,j} d_i$. Finally, we define the *co-weight lattice* $\Lambda_W^\vee$ as the $\mathbb{Z}$-span of the elements $\lambda_i^\vee := \frac{\lambda_i}{d_i}$. The quotient $\pi_1 := \Lambda_W/\Lambda_R$ is called the *fundamental group* of $\mathfrak{g}$.
One can easily see that the Killing form restricts to a perfect pairing $(\ ,\ ) : \Lambda_W^\vee \times \Lambda_R \to \mathbb{Z}$ and that we get a string of inclusions $\Lambda_R \subseteq \Lambda_R^\vee \subseteq \Lambda_W \subseteq \Lambda_W^\vee$.

2.2 Lusztig's Ansatz for R-matrices

The starting point for the works [Mül98] and [LN15] was Lusztig's ansatz in [Lus93], Sec. 32.1, for a universal R-matrix of $U_q(\mathfrak{g})$. Namely, for a specific element $\bar{\Theta} \in U_q^{\geq 0} \otimes U_q^{\leq 0}$ from a dual basis and a suitable (not further specified) element in the coradical $R_0 \in U_q^0 \otimes U_q^0$ we are looking for R-matrices of the form

$$R = R_0 \bar{\Theta}.$$

We remark that there is no claim that all possible R-matrices are of this form. However they are an interesting source of examples, motivated by the interpretation of $u_q(\mathfrak{g})$ as a quotient of a Drinfeld double and thus well-behaved with respect to the triangular decomposition. This ansatz has been successfully generalized to general diagonal Nichols algebras in [AY15]. In our more general setting $U_q(\mathfrak{g}, \Lambda, \Lambda')$, we have

$$R_0 \in \mathbb{C}[\Lambda/\Lambda'] \otimes \mathbb{C}[\Lambda/\Lambda'].$$

This ansatz has been worked out by Müller in his Dissertation [Mül98] for small quantum groups $u_q(\mathfrak{g})$ which we will use in the following, leading to a system of quadratic equation on R_0 that are equivalent to R being an R-matrix:

Theorem 2.2.1 (cf. [Mül98], Thm. 8.2). *Let $u := u_q(\mathfrak{g})$. (a) There is a unique family of elements $\Theta_\beta \in u_\beta^- \otimes u_\beta^+$, $\beta \in \Lambda_R$, such that $\Theta_0 = 1 \otimes 1$ and $\Theta = \sum_\beta \Theta_\beta \in u \otimes u$ satisfies $\Delta(x)\Theta = \Theta\bar{\Delta}(x)$ for all $x \in\ u$.*
(b) Let B be a vector space-basis of u^-, such that $B_\beta = B \cap u_\beta^-$ is a basis of u_β^- for all β. Here, u_β^- refers to the natural Λ_R-grading of u^-. Let $\{b^* \mid b \in B_\beta\}$ be the basis of u_β^+ dual to B_β under the non-degenerate bilinear form $(\cdot\,,\cdot)\colon u^- \otimes u^+ \to \mathbb{C}$. We have

$$\Theta_\beta = (-1)^{\mathrm{tr}\beta} q_\beta \sum_{b \in B_\beta} b^- \otimes b^{*+} \in u_\beta^- \otimes u_\beta^+.$$

Theorem 2.2.2 (cf. [Mül98], Theorem 8.11). *Let $\Lambda' \subset \{\mu \in \Lambda \mid K_\mu$ central in $u_q(\mathfrak{g}, \Lambda)\}$ a subgroup of Λ, and G_1, G_2 subgroups of $G := \Lambda/\Lambda'$, containing Λ_R/Λ'. In the following, $\mu, \mu_1, \mu_2 \in G_1$ and $\nu, \nu_1, \nu_2 \in G_2$.*
The element $R = R_0\bar{\Theta}$ with an arbitrary $R_0 = \sum_{\mu,\nu} f(\mu,\nu) K_\mu \otimes K_\nu$ is a R-matrix for $u_q(\mathfrak{g}, \Lambda, \Lambda')$, if and only if for all $\alpha \in \Lambda_R$ and μ, ν the following holds:

$$f(\mu+\alpha,\nu) = q^{-(\nu,\alpha)} f(\mu,\nu), \quad f(\mu,\nu+\alpha) = q^{-(\mu,\alpha)} f(\mu,\nu), \tag{2.1}$$

$$\sum_{\nu_1+\nu_2=\nu} f(\mu_1,\nu_1) f(\mu_2,\nu_2) = \delta_{\mu_1,\mu_2} f(\mu_1,\nu), \qquad \sum_{\mu_1+\mu_2=\mu} f(\mu_1,\nu_1) f(\mu_2,\nu_2) = \delta_{\nu_1,\nu_2} f(\mu,\nu_1), \tag{2.2}$$

$$\sum_\mu f(\mu,\nu) = \delta_{\nu,0}, \quad \sum_\nu f(\mu,\nu) = \delta_{\mu,0}. \tag{2.3}$$

Chapter 3. Conditions for the Existence of R-Matrices

3.1 A first set of conditions on the group Λ/Λ'

The target of our efforts is a Hopf algebra called small quantum group $u_q(\mathfrak{g},\Lambda,\Lambda')$ with Cartan part $u_q^0=\mathbb{C}[\Lambda/\Lambda']$. It is defined e.g. in [LN15] and depends on lattices Λ,Λ' defined below. For $\Lambda=\Lambda_R$ the root lattice and this is the usual small quantum group; the choice of Λ' differs in literature.

In the previous section we have discussed an $R=R_0\bar{\Theta}$-matrix for the quantum group $u_q(\mathfrak{g},\Lambda,\Lambda')$ can be obtained from an R_0-matrix of the form

$$R_0=\sum_{\mu,\nu\in\Lambda} f(\mu,\nu)K_\mu\otimes K_\nu\in\mathbb{C}[\Lambda/\Lambda']\otimes\mathbb{C}[\Lambda/\Lambda'].$$

In the following we collect necessary and sufficient conditions for $R=R_0\bar{\Theta}$ to be an R-matrix.

Definition 3.1.1. *We fix once-and-for-all a finite-dimensional simple complex Lie algebra $\mathfrak{g}$ and a lattice Λ between root- and weight-lattice*

$$\Lambda_R\subseteq\Lambda\subseteq\Lambda_W.$$

These choices have a nice geometric interpretation as quantum groups associated to different Lie groups associated to the Lie algebra $\mathfrak{g}$.

Another interesting choice is $\Lambda_R\subseteq\Lambda\subseteq\Lambda_W^\vee\cong\Lambda_R^*$, which would below pose no additional complications and may produce further interesting factorizable R-matrices.

Definition 3.1.2. *We fix once-and-for-all a primitive ℓ-th root of unity q.*
For $\Lambda_1,\Lambda_2\subseteq\Lambda_W^\vee$ we define the sublattice

$$Cent_{\Lambda_1}(\Lambda_2):=\{\,\nu\in\Lambda_1\,|\,(\nu,\mu)\in\ell\cdot\mathbb{Z}\quad\forall\mu\in\Lambda_2\}.$$

Informally, this is the centralizer with respect to the braiding $q^{-(\nu,\mu)}$.

Contrary to [LN15] we do not fix Λ' but we prove later 3.1.6 that there is a necessary choice for Λ'. In this way, we get more solutions than in [LN15]. The only condition necessary to ensure that the Hopf algebra $u_q(\mathfrak{g},\Lambda,\Lambda')$ is well-defined is $\Lambda'\subseteq\mathrm{Cent}_{\Lambda_R}(\Lambda_R)$.

Theorem 3.1.3. *(c.f. [LN15] Thm. 3.4) The R_0-matrix is necessarily of the form*

$$f(\mu,\nu) = \frac{1}{d|\Lambda_R/\Lambda'|} \cdot q^{-(\mu,\nu)} g(\bar{\mu},\bar{\nu}) \delta_{\bar{\mu}\in H_1} \delta_{\bar{\nu}\in H_2} \tag{3.1}$$

where H_1, H_2 are subgroups of $H := \Lambda/\Lambda_R \subseteq \pi_1$ with equal cardinality $|H_1| = |H_2| =: d$ (not necessarily isomorphic!) and $g \colon H_1 \times H_2 \to \mathbb{C}^\times$ is a pairing of groups.

The necessity of this form (in particular that the support of f is indeed a subgroup!) amounts to a combinatorial problem of its own interest, which we solved for π_1 cyclic in [LN15] and for $\mathbb{Z}_2 \times \mathbb{Z}_2$ by hand; a closed proof for all abelian groups would be interesting.

Definition 3.1.4. *Let $g : G \times H \to \mathbb{C}^\times$ be a finite group pairing, then the* left radical *is defined as*

$$Rad_L(g) := \{\lambda \in G \,|\, g(\lambda,\eta) = 1 \,\forall\, \eta \in H\}.$$

Similarly, the right radical is defined as

$$Rad_R(g) := \{\eta \in H \,|\, g(\lambda,\eta) = 1 \,\forall\, \lambda \in G\}.$$

The pairing g is called non-degenerate *if $Rad_L(g) = 0$. If in addition $Rad_R(g) = 0$, g is called* perfect.

For an R_0-matrix of this form, a sufficient condition is that they fulfill the so-called *diamond-equations* (see [LN15] Def. 2.7) for each element $0 \neq \zeta \in (\mathrm{Cent}(\Lambda_R) \cap \Lambda)/\Lambda'$.
However, we will now go into a different, more systematic direction that makes use of the following observation:

Lemma 3.1.5. *An R_0-matrix of the form given in Theorem 3.1.3 is a solution to the equations in Theorem 2.2.2, and hence produces an R-matrix $R_0\bar{\Theta}$ iff the restriction to the support*

$$\hat{f} := d|\Lambda_R/\Lambda'| \cdot f : \; G_1 \times G_2 \to \mathbb{C}^\times$$

is a perfect *group pairing, where $G_i := \Lambda_i/\Lambda' \subseteq \Lambda/\Lambda' =: G$.*

Proof. We first show that a solution with restriction to the support a nondegenerate pairing solves the equation:
The first equations are obviousely fulfilled for the form assumed.

$$f(\mu+\alpha,\nu) = q^{-(\nu,\alpha)} f(\mu,\nu), \quad f(\mu,\nu+\alpha) = q^{-(\mu,\alpha)} f(\mu,\nu),$$

For the other equations the sums get only contributions in the support $\Lambda_1/\Lambda' \times \Lambda_2/\Lambda'$. The quantities $f(\mu,\nu) \cdot d|\Lambda_R/\Lambda'|$ for fixed ν (or μ) are characters on the respective support, and by the assumed non-degeneracy all $\nu \neq 0$ give rise to different nontrivial characters. Then the second and third relations follows from orthogonality of characters. Note that since $d|\Lambda_R/\Lambda'| = |G_1| = |G_2|$ (equality of the latter was an assumption!) we were able to chose the right normalization.

For the other direction assume a solution of the given form to the equations. Then already the third equation shows that no $f(-,\nu)$ may be the trivial character and hence the form on the support is nondegenerate and hence perfect by $|G_1| = |G_2|$. $\square$

Corollary 3.1.6. *A first consequence of the perfectness of $\hat{f}$ (i.e. a necessary condition for quasi-triangularity) is:*

$$Cent_{\Lambda_R}(\Lambda_1) = Cent_{\Lambda_R}(\Lambda_2) = \Lambda'.$$

This fixes Λ' uniquely. Morover in cases $\Lambda_1 \neq \Lambda_2$, which can only happen for $\mathfrak{g} = D_{2n}$ where π_1 is noncyclic, we get an additional constraint relating Λ_1, Λ_2.

In our case, the only possibility for $\Lambda_1 \neq \Lambda_2$, s.t. $G_1 \cong G_2$ is $\mathfrak{g} = D_{2n}$. In this case, we have $\mathrm{Cent}_{\Lambda_R}(\Lambda_W) = \mathrm{Cent}_{\Lambda_R}(\Lambda_R)$ and thus the above condition is always fulfilled.
Our main goal for the new approach on quasitriangularity as well as the later modularity is to reduce this non-degeneracy condition for $\hat{f}$ to a non-degeneracy condition for g on $H_1, H_2 \subset \pi_1$ that can be checked explicitly.

3.2 A natural form on the fundamental group

We now define for each triple $(\Lambda, \Lambda_1, \Lambda_2)$ and each ℓth root of unity q a natural pairing a_ℓ on the subgroups $H_i := \Lambda_i/\Lambda_R$ of the fundamental group $\pi_1 := \Lambda_W/\Lambda_R$. The simplest example is $a_\ell = e^{-2\pi i(\mu,\nu)}$. In general it is a transportation of the natural form $q^{-(\mu,\nu)}$ (which does not factorize over Λ_R) to H_i by a suitable isomorphism A_ℓ.
This isomorphism A_ℓ will encapsulate the crucial dependence on the common divisors of ℓ, $|H|$ and the root lengths d_i; moreover, for different H these forms are *not* simply restrictions of one another. Then, we can moreover transport any given pairing g together with $q^{-(\mu,\nu)}$ along the isomorphism A_ℓ to the H_i and thus define forms a_ℓ^g on H. The main result of this section is in Theorem 3.2.7 that the non-degeneracy condition in Lemma 3.1.5 for $R_0(f)$ depending on H_i, g is equivalent to a_ℓ^g being non-degenerate.

Definition 3.2.1. *Let $\Lambda \subseteq \Lambda_W^\vee$ be a sublattice, s.t. $\Lambda_R \subseteq \Lambda$. By $\hat{\Lambda} \subset \Lambda_W^\vee$ we denote the unique sublattice, s.t. the symmetric bilinear form $(\,.\,,\,.\,) : \Lambda_W^\vee \times \Lambda_W^\vee \to \mathbb{Q}$ induces a commuting diagram*

$$\begin{tikzcd}
\Lambda_R \arrow[r, hook] \arrow[d, "\cong"] & \hat{\Lambda} \arrow[r, hook] \arrow[d, "\cong"] & \Lambda_W^\vee \arrow[d, "\cong"] \\
{\Lambda_W^\vee}^* \arrow[r, hook] & \Lambda^* \arrow[r, hook] & \Lambda_R^*
\end{tikzcd} \quad , \tag{3.2}$$

where $\Lambda^* := Hom_{\mathbb{Z}}(\Lambda, \mathbb{Z})$. In particular, we have $\hat{\Lambda}_R = \Lambda_W^\vee$ and $\hat{\Lambda}_W^\vee = \Lambda_R$.

Definition 3.2.2. *A* centralizer transfer map *is an group endomorphism $A_\ell \in End_{\mathbb{Z}}(\Lambda)$, s.t.*

1. $A_\ell(\Lambda) \stackrel{!}{=} \Lambda \cap \ell \cdot \hat{\Lambda}_R = Cent_\Lambda^\ell(\Lambda_R)$
2. $A_\ell(\Lambda_R) \stackrel{!}{=} \Lambda_R \cap \ell \cdot \hat{\Lambda} = Cent_{\Lambda_R}^\ell(\Lambda)$.

Such a A_ℓ induces a group isomorphism

$$\Lambda/\Lambda_R \xrightarrow{\sim} Cent_\Lambda^\ell(\Lambda_R)/Cent_{\Lambda_R}^\ell(\Lambda).$$

Of course A_ℓ is not unique.

Question 3.2.3. *Are there abstract arguments for the existence of these isomorphism and for its explicit form?*

We will calculate explicit expressions for A_ℓ depending on the cases in the next section. At this point we give the generic answers:

Example 3.2.4. *For $\Lambda = \Lambda_W^\vee$ we have $A_\ell = \ell \cdot id$.*
For $\Lambda = \Lambda_R$ the two conditions are equivalent, so existence is trivial (resp. obviously the two trivial groups are isomorphic) and we may simply take for A_ℓ any base change between left and right side. The expression may however be nontrivial.

Lemma 3.2.5. *Assume* $\gcd(\ell,\ |\Lambda_W^\vee/\Lambda|) = 1$, *then $A_\ell = \ell \cdot id$. In particular this is the case if ℓ is prime to all root lengths and all divisors of the Cartan matrix.*

Moreover if $\ell = \ell_1\ell_2$ with $\gcd(\ell_1,\ |\Lambda_W^\vee/\Lambda|) = 1$, *then $A_\ell = \ell_1 \cdot A_{\ell_2}$.*

This means we only have to calculate A_ℓ for all divisors ℓ of $|\Lambda_W^\vee/\Lambda|$, which is a subset of all divisors of root lengths times divisors of the Cartan matrix.

Proof. For the first condition we need to show for any $\lambda \in \Lambda_W^\vee$ that $\ell\lambda \in \Lambda$ already implies $\lambda \in \Lambda$. But if by assumption the order of the quotient group $\Lambda_W^\vee/\Lambda$ is prime to ℓ, then $\ell\cdot$ is an isomorphism on this abelian group, hence follows the assertion. For the second condition applies the same argument noting that $|\hat{\Lambda}/\Lambda_R| = |\Lambda_W^\vee/\Lambda|$.

For the second claim we simply consider the inclusion chains

$$A_\ell(\Lambda) \subset \Lambda \cap \ell_2 \cdot \hat{\Lambda}_R \subset \Lambda \cap \ell \cdot \hat{\Lambda}_R$$

$$A_\ell(\Lambda_R) \subset \Lambda \cap \ell_2 \cdot \hat{\Lambda} \subset \Lambda_R \cap \ell \cdot \hat{\Lambda},$$

where a first isomorphism is given by A_{ℓ_2} and again $\ell_1\cdot$ is a second isomorphism because it is prime to the index. ☐

Our main result of this chapter is the following:

Theorem 3.2.6. *Let $\Lambda_R \subseteq \Lambda_1, \Lambda_2 \subseteq \Lambda_W$ be intermediate lattices, s.t. the condition in Corollary 3.1.6 is fulfilled, i.e. $Cent_{\Lambda_R}(\Lambda_1) = Cent_{\Lambda_R}(\Lambda_2) = \Lambda'$. Assume we have a centralizer transfer map A_ℓ.*

1. *The following form is well defined on the quotients:*

$$\begin{aligned} a_g^\ell :\ \Lambda_1/\Lambda_R \times \Lambda_2/\Lambda_R &\longrightarrow \mathbb{C}^\times \\ (\bar{\lambda}, \bar{\mu}) &\longmapsto q^{-(\lambda, A_\ell(\mu))} \cdot g(\lambda, A_\ell(\mu)). \end{aligned}$$

2. *Let*

$$Cent^g_{\Lambda_1}(\Lambda_2) := \{\lambda \in \Lambda_1 \mid q^{(\lambda,\mu)} = g(\lambda,\mu)\ \forall \mu \in \Lambda_2\}.$$

   Then the inclusion $Cent^g_{\Lambda_1}(\Lambda_2) \hookrightarrow \Lambda_1$ induces an isomorphism

$$Cent^g_{\Lambda_1}(\Lambda_2)/\Lambda' \cong Rad(a_g^\ell). \tag{3.3}$$

Corollary 3.2.7. *The quasi-triangularity conditions for a choice R_0 are by Lemma 3.1.5 equivalent to the non-degeneracy of the group pairing on $\Lambda_1/\Lambda' \times \Lambda_2/\Lambda'$:*

$$\hat{f}(\lambda, \mu) = q^{-(\lambda,\mu)} g(\lambda, \mu).$$

By the previous theorem this condition is now equivalent to the nondegeneracy of a_g^ℓ.

This condition on the fundamental group, which is a finite abelian group and mostly cyclic, can be checked explicitly once a_g^ℓ has been calculated.

Proof of Thm. 3.2.6. The first part of the theorem is a direct consequence of the definition of the centralizer transfer matrix A_ℓ. For the second part, we first notice that by assumption we have a commutative diagram of finite abelian groups

$$\begin{tikzcd} \Lambda_R/\Lambda' \arrow[r, hook] \arrow[d, "q^{-(\cdot,\cdot)}"] & \Lambda_1/\Lambda' \arrow[r, two heads] \arrow[d, "\hat{f}"] & \Lambda_1/\Lambda_R \arrow[d, "\hat{f}'"] \\ (\Lambda_2/\mathrm{Cent}_{\Lambda_2}(\Lambda_R))^\wedge \arrow[r, hook] & (\Lambda_2/\Lambda')^\wedge \arrow[r, two heads] & (\mathrm{Cent}_{\Lambda_2}(\Lambda_R)/\Lambda')^\wedge, \end{tikzcd}$$

where $G^\wedge$ denotes the dual group of a group G.
Now, by the five lemma we know that $\hat{f}$ is an isomorphism if and only if the induced map $\hat{f}'$ is an isomorphism. Post-composing this map with the dualized centralizer transfer matrix $A_\ell^\wedge$: $(\mathrm{Cent}_{\Lambda_2}(\Lambda_R)/\Lambda')^\wedge \cong (\Lambda_2/\Lambda_R)^\wedge$ gives a_g^ℓ. ☐

Chapter 4. Explicit calculation for every simple Lie algebra

In the following, we want to compute the endomorphism $A_\ell \in \mathrm{End}_{\mathbb{Z}}(\Lambda)$ and the pairing a_ℓ on the fundamental group explicitly in terms of the Cartan matrices and the common divisors of ℓ with root lengths and divisors of the Cartan matrix. We will finally give a list for all $\mathfrak{g}$.

4.1 Technical Tools

We choose the basis of simple roots α_i for Λ_R and the dual basis of fundamental coweights $\lambda_i^\vee$ for the dual lattice $\Lambda_W^\vee$ with $(\alpha_i, \lambda_j^\vee) = \delta_{i,j}$.

For any choice $\Lambda \subset \Lambda_W \subset \Lambda_W^\vee$, let A_Λ be a *basis matrix* i.e. any $\mathbb{Z}$-linear isomorphism $\Lambda_W^\vee \to \Lambda$ sending the basis $\lambda_i^\vee$ of $\Lambda_W^\vee$ to some basis μ_i of Λ. It is unique up to pre-composition of a unimodular matrix $U \in \mathrm{SL}_n(\mathbb{Z})$.
The dual basis $A_{\hat{\Lambda}}$ of $\hat{\Lambda}$ is defined by

$$(A_{\hat{\Lambda}}(\lambda_i^\vee), A_\Lambda(\lambda_j^\vee)) = \delta_{ij}.$$

Explicitly, $A_{\hat{\Lambda}}$ is given by $A_{\hat{\Lambda}} = (A_\Lambda^{-1} A_R)^T$, where $(A_R)_{ij} = (\alpha_i, \alpha_j)$. Now, let $A_\Lambda = P_\Lambda S_\Lambda Q_\Lambda$ be the unique Smith decomposition of A_Λ, which means: P_Λ, Q_Λ are unimodular and S_Λ is diagonal with diagonal entries $(S_\Lambda)_{ii} =: d_i^\Lambda$, such that $d_i^\Lambda \mid d_j^\Lambda$ for $i < j$.

Example 4.1.1. *For the root lattice the $d_i^{\Lambda_R}$ are the divisors of scalar product matrix (α_i, α_j). Their product is*

$$\prod_i d_i^{\Lambda_R} = |\Lambda_W^\vee / \Lambda_R| = \left(\prod_i d_i\right) \cdot |\pi_1|, \qquad d_i = \frac{(\alpha_i, \alpha_i)}{2}.$$

For the coweight lattice all $d_i^{\Lambda_W^\vee} = 1$. For the weight lattice we recover the familiar $d_i^{\Lambda_W} = d_i$.

Without loss of generality, we will assume the basis matrices A_Λ to be symmetric, i.e. $Q_\Lambda = P_\Lambda^T$. We then have the following Lemma:

Lemma 4.1.2. *Let $\Lambda_R \subseteq \Lambda \subseteq \Lambda_W^\vee$ be a lattice. We define lattices*

$$A_{Cent} := (P_\Lambda^T)^{-1} D_\ell P_\Lambda^{-1} \qquad D_\ell := Diag\left(\frac{\ell}{gcd(\ell, d_i^\Lambda)}\right).$$

Then,

$$Cent_{\Lambda_R}(\Lambda) = A_R A_{Cent} \Lambda_W^\vee \qquad Cent_\Lambda(\Lambda_R) = A_\Lambda A_{Cent} \Lambda_W^\vee.$$

Proof. We compute explicitly,

$$\begin{aligned}
\mathrm{Cent}_{\Lambda_R}(\Lambda) &= \Lambda_R \cap \ell \cdot \hat{\Lambda} \\
&= A_R \Lambda_W^\vee \cap (A_\Lambda^{-1} A_R)^T \ell \Lambda_W^\vee \\
&= (A_\Lambda^{-1} A_R)^T (((A_\Lambda^{-1} A_R)^T)^{-1} A_R \Lambda_W^\vee \cap \ell \Lambda_W^\vee) \\
&= A_R A_\Lambda^{-1} (A_\Lambda \cap \ell \Lambda_W^\vee) \\
&= A_R (P_\Lambda S_\Lambda P_\Lambda^T)^{-1} (P_\Lambda S_\Lambda P_\Lambda^T \Lambda_W^\vee \cap \ell \Lambda_W^\vee) \\
&= A_R (P_\Lambda^T)^{-1} S_\Lambda^{-1} (S_\Lambda \Lambda_W^\vee \cap \ell \Lambda_W^\vee) \\
&= A_R (P_\Lambda^T)^{-1} S_\Lambda^{-1} \mathrm{Diag}(\mathrm{lcm}(S_{\Lambda_{ii}}, \ell)) \Lambda_W^\vee \\
&= A_R (P_\Lambda^T)^{-1} D_\ell \Lambda_W^\vee = A_R A_{\mathrm{Cent}} \Lambda_W^\vee.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\mathrm{Cent}_\Lambda(\Lambda_R) &= \Lambda \cup \ell \hat{\Lambda}_R \\
&= \Lambda \cup \ell \Lambda_W^\vee \\
&= A_\Lambda \Lambda_W^\vee \cup \ell \Lambda_W^\vee \\
&= P_\Lambda S_\Lambda P_\Lambda^T \Lambda_W^\vee \cup \ell \Lambda_W^\vee \\
&= P_\Lambda (S_\Lambda \Lambda_W^\vee \cup \ell \Lambda_W^\vee) \\
&= P_\Lambda S_\Lambda D_\ell \Lambda_W^\vee \\
&= A_\Lambda (P_\Lambda^T)^{-1} D_\ell \Lambda_W^\vee = A_\Lambda A_{\mathrm{Cent}} \Lambda_W^\vee.
\end{aligned}$$

In particular, this means $A_{\hat{\Lambda}} \mathrm{Cent}_\Lambda(\Lambda_R) = \mathrm{Cent}_{\Lambda_R}(\Lambda)$. □

4.2 Case $\Lambda = \Lambda_W$

In order to exhaust all cases that appear in our setting, we continue with $\Lambda = \Lambda_W$:

Lemma 4.2.1. *In the case $\Lambda = \Lambda_W$, the centralizer transfer matrix A_ℓ is of the following form:*

$$A_\ell = \begin{cases} A_{\Lambda_W} A_{Cent} Q_C^T P_C^{-1} A_{\Lambda_W}^{-1}, & gcd(\ell, |\pi_1|) \neq 1 \\ \ell \cdot id, & else. \end{cases}$$

Here, $C = P_C S_C Q_C$ denotes the Smith decomposition of the Cartan matrix of $\mathfrak{g}$.

Proof. As we noted in Example 4.1.1, we have $A_{\Lambda_W} = \mathrm{Diag}(d_i)$, for d_i being the ith root length. Since $d_i \in \{1, p\}$ for some prime number p, up to a permutation A_{Λ_W} is already in Smith normal form: this means that P_{Λ_W} is a permutation matrix of the form $(P_{\Lambda_W})_{ij} = \delta_{j,\sigma(i)}$ for some $\sigma \in S_n$, s.t. $d_{\sigma(1)} \leq \cdots \leq d_{\sigma(n)}$. It follows that $A_{\mathrm{Cent}} = \mathrm{Diag}\left(\frac{\ell}{\gcd(\ell, d_i)}\right)$.
Using the definition $C_{ij} = \frac{(\alpha_i, \alpha_j)}{d_i}$, in the case $\gcd(\ell, |\pi_1|) \neq 1$ we obtain

$$A_{\mathrm{Cent}} C^T = C A_{\mathrm{Cent}}.$$

Thus,

$$\begin{aligned} A_\ell A_R &= A_{\Lambda_W} A_{\mathrm{Cent}} Q_C^T P_C^{-1} A_{\Lambda_W}^{-1} A_R \\ &= A_R C^{-1} A_{\mathrm{Cent}} Q_C^T P_C^{-1} C \\ &= A_R A_{\mathrm{Cent}} (C^T)^{-1} Q_C^T P_C^{-1} C \\ &= A_R A_{\mathrm{Cent}}. \end{aligned}$$

By the previous Lemma, this proves the first condition for A_ℓ. The second condition follows immediately from the previous Lemma.
The case $\gcd(\ell, |\pi_1|) = 1$ follows from Lemma 3.2.5 and the fact that $|\pi_1| = |\Lambda_W^\vee / \Lambda_R^\vee|$. □

4.3 Case A_n

In the following example, we treat the case $\mathfrak{g} = A_n$ with fundamental group $\Lambda_W/\Lambda_R = \mathbb{Z}_{n+1}$ for general intermediate lattices $\Lambda_R \subseteq \Lambda \subseteq \Lambda_W$.

Example 4.3.1. *In order to compute the centralizer transfer map A_ℓ, we first compute the Smith decomposition of A_R:*

$$\begin{aligned} A_R &= \begin{pmatrix} 2 & -1 & 0 & \dots & & 0 \\ -1 & 2 & -1 & 0 & & 0 \\ 0 & -1 & 2 & \ddots & \ddots & \vdots \\ 0 & 0 & \ddots & \ddots & -1 & 0 \\ \vdots & & \ddots & -1 & 2 & -1 \\ 0 & \dots & & 0 & -1 & 2 \end{pmatrix} \\ &= \begin{pmatrix} -1 & 0 & 0 & \dots & & 0 \\ 2 & -1 & 0 & & & 0 \\ 0 & 2 & -1 & \ddots & & \vdots \\ 0 & 0 & \ddots & \ddots & & 0 \\ \vdots & & \ddots & 2 & -1 & 0 \\ 0 & \dots & & 0 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & \dots & & 0 \\ 0 & 1 & 0 & & & 0 \\ 0 & 0 & 1 & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & & \\ & & & \ddots & 1 & 0 \\ 0 & \dots & & & 0 & n+1 \end{pmatrix} \begin{pmatrix} -2 & 1 & 0 & \dots & & 0 \\ -3 & 0 & 1 & \ddots & & 0 \\ -4 & 0 & 0 & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & 1 & 0 \\ -n & & & & 0 & 1 \\ 1 & 0 & \dots & & 0 & 0 \end{pmatrix} \end{aligned}$$

A sublattice $\Lambda_R \subsetneq \Lambda \subsetneq \Lambda_W$ is uniquely determined by a divisor $d \mid n+1$, so that $\Lambda/\Lambda_R \cong \mathbb{Z}_d$ and is generated by the multiple $\hat{d}\lambda_n$, where $\hat{d} := \frac{n+1}{d}$. Then

$$d_i^{\Lambda} = \begin{cases} 1, \; i < n \\ d, \; i = n \end{cases} .$$

Since A_n is simply laced with cyclic fundamental group, the formula $A_\Lambda = P_R S_\Lambda P_R^T$ gives us symmetric basis matrices of sublattices $\Lambda_R \subseteq \Lambda \subseteq \Lambda_W$. We also substitute the above basis matrix of the root lattice A_R by $A_R(Q_R)^{-1}P_R^T$. It is then easy to see that the definition $A_\ell := P_R D_\ell P_R^T$ gives a centralizer transfer matrix. We calculate it explicitly:

$$(A_\ell)_{ij} = (P_R D_\ell P_R^{-1})_{ij} = \begin{cases} \delta_{ij}, & i < n \\ (n+1-j)\left(\frac{\ell}{gcd(\ell,d)} - 1\right), & i = n \text{ and } j < n \\ \frac{\ell}{gcd(\ell,d)}, & i = j = n. \end{cases}$$

Now a form g is uniquely determined by a dth root of unity $g(\chi,\chi) = exp(\frac{2\pi i \cdot k}{d}) = \zeta_d^k$ with some k. Then we calculate the form a_g^ℓ on the generator:

$$\begin{aligned} a_g^\ell(\chi,\chi) &= q^{-(\chi, A_\ell(\chi))} g(\chi, A_\ell(\chi)) \\ &= q^{-\frac{(n+1)^2 \cdot \ell}{d^2 gcd(\ell,\hat{d})}(\lambda_n^\vee, \lambda_n^\vee)} \cdot g(\chi,\chi)^{\frac{\ell}{gcd(\ell,\hat{d})}} \\ &= exp\left(\frac{2\pi i \cdot (k\ell - \hat{d}n)}{d \cdot gcd(\ell,\hat{d})}\right). \end{aligned}$$

For example the trivial g (i.e. $k=0$) gives an R-matrix for all lattices Λ (defined by $\hat{d}d = n+1$) iff $\frac{\hat{d}}{gcd(\ell,\hat{d})}$ is coprime to d. For ℓ coprime to the divisor $n+1$ this amounts to all lattices associated to decompositions of $n+1$ into two coprime factors.

4.4 Case D_n

Finally, we consider the root lattice D_n. Since we have $\pi_1(D_{2n\geq 4}) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ and $\pi_1(D_{2n+1\geq 5}) \cong \mathbb{Z}_4$, it is appropriate to split this investigation in two steps. We start with $D_{2n\geq 4}$. In order to compute the respective Smith decompositions, we used the software *Wolfram Mathematica*.

Example 4.4.1. *In the case $D_{2n\geq 4}$, we have three different possibilities for the lattices $\Lambda_R \subseteq \Lambda_1, \Lambda_2 \subseteq \Lambda_W$:*

1. *$\Lambda_1 \neq \Lambda_2$, $H_1 \cong H_2 \cong \mathbb{Z}_2$: In this case, the subgroups $\Lambda_i/\Lambda_R \subseteq \Lambda_R$ are spanned by the fundamental weights $\lambda_{2(n-1)+i}$. As in the case A_n, we define the centralizer transfer map $A_\ell := P_R D_\ell P_R^{-1}$ on H_2. This is possible since the symmetric basis matrix $A_{\Lambda_2} = P_R S_{\Lambda_2} P_R^T$*

of Λ_2 is already in Smith normal form. Using the software Wolfram Mathematica in order to compute P_R, we obtain $A_\ell(\lambda_{2n}) = \frac{\ell}{gcd(2,\ell)}$. Combining this with $(\lambda_{2n-1}, \lambda_{2n}) = \frac{n-1}{2}$, we get

$$a_g^\ell(\lambda_{2n-1}, \lambda_{2n}) = exp\left(\frac{2\pi i \cdot (kl - 2(n-1))}{2 \cdot gcd(2,\ell)}\right)$$

for $g(\lambda_{2n-1}, \lambda_{2n}) = exp\left(\frac{2\pi i k}{2}\right)$.

2. *$\Lambda_1 = \Lambda_2$, $H_i \cong \mathbb{Z}_2$: Without restrictions and in order to use the same definition for A_ℓ as above, we choose Λ_i, s.t. the group Λ_i/Λ_R is spanned by λ_{2n}. Combining the above result $A_\ell(\lambda_{2n}) = \frac{\ell}{gcd(2,\ell)}$ with $(\lambda_{2n}, \lambda_{2n}) = \frac{n}{2}$, we obtain*

$$a_g^\ell(\lambda_{2n}, \lambda_{2n}) = exp\left(\frac{2\pi i \cdot (kl - 2n)}{2 \cdot gcd(2,\ell)}\right)$$

for $g(\lambda_{2n}, \lambda_{2n}) = exp\left(\frac{2\pi i k}{2}\right)$.

3. *$\Lambda_1 = \Lambda_2 = \Lambda_W$, $H \cong \mathbb{Z}_2 \times \mathbb{Z}_2$: A group pairing $g : (\mathbb{Z}_2 \times \mathbb{Z}_2) \times (\mathbb{Z}_2 \times \mathbb{Z}_2) \to \mathbb{C}^\times$ is uniquely defined by a matrix $K \in \mathfrak{gl}(2, \mathbb{F}_2)$, so that*

$$g(\lambda_{2(n-1)+i}, \lambda_{2(n-1)+j}) = exp\left(\frac{2\pi i K_{ij}}{2}\right).$$

Since D_n is simply-laced, we have $A_\ell = \ell \cdot Id$. Using $(\lambda_{2(n-1)+i}, \lambda_{2(n-1)+j})$ mod $2 = \delta_{i+j odd}$, we obtain

$$a_\ell^g(\lambda_{2(n-1)+i}, \lambda_{2(n-1)+j}) = exp\left(\frac{2\pi i \cdot K_{ij}\ell}{2}\right)(-1)^{i+j}.$$

The last step is the case $D_{2n+1\geq 5}$:

Example 4.4.2. *Since it it is simply-laced and its fundamental group is cyclic, the case $D_{2n+1\geq 5}$ can be treated very similar to A_n. We distinguish two cases:*

1. *$\Lambda_1 = \Lambda_2$, $H_i = \langle 2\lambda_{2n+1}\rangle \cong \mathbb{Z}_2$. As in the case A_n, we define the centralizer transfer map $A_\ell := P_R D_\ell P_R^{-1}$ on H_2. Using $(\lambda_{2n+1}, \lambda_{2n+1}) = \frac{2n+1}{4}$, we obtain*

$$a_g^\ell(2\lambda_{2n+1}, 2\lambda_{2n+1}) = exp\left(\frac{2\pi i \cdot (k\ell - 2(2n+1))}{2 \cdot gcd(2,\ell)}\right).$$

for $g(2\lambda_{2n+1}, 2\lambda_{2n+1}) = exp\left(\frac{2\pi i k}{2}\right)$.

2. *$\Lambda_1 = \Lambda_2 = \Lambda_W$, $H = \langle \lambda_{2n+1}\rangle \cong \mathbb{Z}_4$. By an analagous argument as above, we obtain*

$$a_g^\ell(\lambda_{2n+1}, \lambda_{2n+1}) = exp\left(\frac{2\pi i \cdot (k\ell - (2n+1))}{4}\right).$$

for $g(\lambda_{2n+1}, \lambda_{2n+1}) = exp\left(\frac{2\pi i k}{4}\right)$.

4.5 Table of all quasitriangular quantum groups

In the following table, we list all simple Lie algebras and check for which non-trivial choices of Λ, Λ_i, ℓ and g the element $R_0\bar{\Theta}$ is an R-matrix. As before, we define $H_i := \Lambda_i/\Lambda_R$ and $H := \Lambda/\Lambda_R$. In the cyclic case, if x_i are generators of the H_i, then the pairing is uniquely defined by an element $1 \leq k \leq |H_i|$, s.t. $g(x_1, x_2) = \exp\left(\frac{2\pi i k}{|H_i|}\right)$. In the case D_{2n}, $\Lambda = \Lambda_W$, g is uniquely defined by a 2×2-matrix $K \in \mathfrak{gl}(2, \mathbb{F}_2)$, s.t. $g(\lambda_{2(n-1)+i}, \lambda_{2(n-1)+j}) = \exp\left(\frac{2\pi i K_{ij}^g}{2}\right)$ for $i, j \in \{1, 2\}$.
The columns of the following table are labeled by

1. the finite dimensional simple complex Lie algebra $\mathfrak{g}$
2. the natural number ℓ, determining the root of unity $q = \exp\left(\frac{2\pi i}{\ell}\right)$
3. the number of possible R-matrices for the Lusztig ansatz
4. the subgroups $H_i \subseteq H = \Lambda/\Lambda_R$ introduced in Theorem 3.1.3
5. the subgroups H_i in terms of generators given by multiples of fundamental dominant weights $\lambda_i \in \Lambda_W$
6. the group pairing $g : H_1 \times H_2 \to \mathbb{C}^\times$ determined by its values on generators
7. the group pairing $a_g^\ell \subseteq \Lambda/\Lambda'$ introduced in Theorem 3.2.6 determined by its values on generators.

$\mathfrak{g}$	ℓ	$\#$	$H_i \cong$	$H_{i\ (i=1,2)}$	g	a_g^ℓ
all		1	$\mathbb{Z}_1$	$\langle 0 \rangle$	$g = 1$	1
$A_{n\geq 1}$ $\pi_1 = \mathbb{Z}_{n+1}$		∞	$\mathbb{Z}_d$ $d \mid n+1$	$\langle \hat{d}\lambda_n \rangle$ $\hat{d} = \frac{n+1}{d}$	$g(\hat{d}\lambda_n, \hat{d}\lambda_n) = \exp\left(\frac{2\pi i k}{d}\right)$ $\gcd\left(n+1, d\ell, k\ell - \frac{n+1}{d}n\right) = 1$	$\exp\left(\frac{2\pi i\cdot(k\ell - \hat{d}n)}{d\cdot\gcd(\ell,\hat{d})}\right)$
$B_{n\geq 2}$ $\pi_1 = \mathbb{Z}_2$	ℓ even	2	$\mathbb{Z}_2$	$\langle \lambda_n \rangle$	$g(\lambda_n, \lambda_n) = \pm 1$	-1
	ℓ odd	1			$g(\lambda_n, \lambda_n) = (-1)^{n+1}$	$\exp\left(\frac{2\pi i\cdot(k\ell - n)}{2}\right)$
$C_{n\geq 3}$ $\pi_1 = \mathbb{Z}_2$	$\ell \equiv 2 \bmod 4$	1	$\mathbb{Z}_2$	$\langle \lambda_n \rangle$	$g(\lambda_n, \lambda_n) = 1$	$\exp\left(\frac{2\pi i\cdot(k\frac{\ell}{2}+1)}{2}\right)$
	$\ell \equiv 0 \bmod 4$	2			$g(\lambda_n, \lambda_n) = \pm 1$	-1
	ℓ odd	1			$g(\lambda_n, \lambda_n) = -1$	$\exp\left(\frac{2\pi i\cdot(k\ell - 2n)}{2}\right)$
$D_{2n\geq 4}$ $\pi_1 = \mathbb{Z}_2 \times \mathbb{Z}_2$	$\ell \equiv 2 \bmod 4$	1	$\mathbb{Z}_2$	$H_1 \cong \langle \lambda_{2n-1} \rangle$ $H_2 \cong \langle \lambda_{2n} \rangle$	$g(\lambda_{2n-1}, \lambda_{2n}) = (-1)^n$	$\exp\left(\frac{2\pi i\cdot(k\frac{\ell}{2} - n+1))}{2}\right)$
	$\ell \equiv 0 \bmod 4$	$2\delta_{2\mid n}$			$g(\lambda_{2n-1}, \lambda_{2n}) = \pm 1$, n even	
	ℓ odd	1			$g(\lambda_{2n-1}, \lambda_{2n}) = -1$	$\exp\left(\frac{2\pi i\cdot(k\ell - 2(n-1))}{2}\right)$
	$\ell \equiv 2 \bmod 4$	1	$\mathbb{Z}_2$	$\langle \lambda_{2n} \rangle$	$g(\lambda_{2n}, \lambda_{2n}) = (-1)^{n+1}$	$\exp\left(\frac{2\pi i(k\frac{\ell}{2} - n)}{2}\right)$
	$\ell \equiv 0 \bmod 4$	$2\delta_{2\nmid n}$			$g(\lambda_{2n}, \lambda_{2n}) = \pm 1$, n odd	
	ℓ odd	1			$g(\lambda_{2n}, \lambda_{2n}) = -1$	$\exp\left(\frac{2\pi i(k\ell - 2n)}{2}\right)$
	ℓ even	2	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\langle \lambda_{2n}, \lambda_{2n+1} \rangle$	$g(\lambda_{2(n-1)+i}, \lambda_{2(n-1)+j}) = \pm 1$	$\exp\left(\frac{2\pi i\cdot K_{ij}\ell}{2}\right)(-1)^{i+j}$
	ℓ odd				$det(K) = K_{12} + K_{12} \bmod 2$	
$D_{2n+1\geq 5}$	$\ell \equiv 2 \bmod 4$	1	$\mathbb{Z}_2$	$\langle 2\lambda_{2n+1} \rangle$	$g(2\lambda_{2n+1}, 2\lambda_{2n+1}) = 1$	$\exp\left(\frac{2\pi i\cdot(k\frac{\ell}{2} - 2n-1)}{2}\right)$
	$\ell \equiv 0 \bmod 4$	2			$g(2\lambda_{2n+1}, 2\lambda_{2n+1}) = \pm 1$	
	ℓ odd	1			$g(2\lambda_{2n+1}, 2\lambda_{2n+1}) = -1$	$\exp\left(\frac{2\pi i\cdot(k\ell - 2(2n+1))}{2}\right)$

$\pi_1 = \mathbb{Z}_4$						
	ℓ even	4	$\mathbb{Z}_4$	$\langle \lambda_{2n+1} \rangle$	$g(\lambda_{2n+1}, \lambda_{2n+1}) = c,\ c^4 = 1$	$\exp\left(\frac{2\pi i \cdot (k\ell - (2n+1))}{4}\right)$
	ℓ odd	2			$g(\lambda_{2n+1}, \lambda_{2n+1}) = \pm 1$	
E_6 $\pi_1 = \mathbb{Z}_3$	$\ell \equiv 0 \bmod 3$	3	$\mathbb{Z}_3$	$\langle \lambda_n \rangle$	$g(\lambda_n, \lambda_n) = c,\ c^3 = 1$	$\exp\left(\frac{2\pi i \cdot (k\ell - 1)}{3}\right)$
	$\ell \equiv 1 \bmod 3$	2			$g(\lambda_n, \lambda_n) = 1, \exp\left(\frac{2\pi i 2}{3}\right)$	
	$\ell \equiv 2 \bmod 3$	2			$g(\lambda_n, \lambda_n) = 1, \exp\left(\frac{2\pi i}{3}\right)$	
E_7 $\pi_1 = \mathbb{Z}_2$	ℓ even	2	$\mathbb{Z}_2$	$\langle \lambda_n \rangle$	$g(\lambda_n, \lambda_n) = \pm 1$	$\exp\left(\frac{2\pi i \cdot (k\ell - 1)}{2}\right)$
	ℓ odd	1			$g(\lambda_n, \lambda_n) = 1$	

Table 4.1: Solutions for R_0-matrices

The Lie algebras E_8, F_4 and G_2 have trivial fundamental groups and thus have no non-trivial solution. We want to emphasize once more that the choice $\Lambda_i = \Lambda_R$ always leads to a quasitriangular quantum group.

The following Lemma connects our results with Lusztig's original result:

Lemma 4.5.1. *In Lusztig's definition of a quantum group he uses the quotient $\Lambda'_{Lusz} = 2\,Cent_{\Lambda_R}(2\Lambda_W)$. This coincide with our choice $\Lambda' = Cent_{\Lambda_R}(\Lambda_1 + \Lambda_2)$, if and only if*

$$2\gcd(\ell, d_i^\Lambda) = \gcd(\ell, 2d_i^W), \tag{4.1}$$

where the d_i^Λ denote the invariant factors of $\Lambda_W^\vee/\Lambda$ and the d_i^W denote the invariant factors of $\Lambda_W^\vee/\Lambda_W$ (i.e. ordered root lengths).

In particular, for ℓ odd these choices never coincide. For $\Lambda = \Lambda_W$, $\Lambda' = \Lambda'_{Lusz}$ holds if and only if $2d_i \mid \ell$. This is the most extreme case of divisibility and it is precisely the case appearing in logarithmic conformal field theories.

Proof. We first note that in our cases, $\Lambda' = \mathrm{Cent}_{\Lambda_R}(\Lambda_1 + \Lambda_2) = \mathrm{Cent}_{\Lambda_R}(\Lambda)$. We have,

$$\begin{aligned} 2\mathrm{Cent}_{\Lambda_R}(2\Lambda_W) &= 2(\Lambda_R \cap \widehat{2\Lambda_W}) \\ &= A_R 2(\Lambda_W^\vee \cap A_W^{-1} \frac{\ell}{2} \Lambda_W^\vee) \\ &= A_R \mathrm{Diag}\left(\frac{2\ell}{\gcd(\ell, 2d_i^W)}\right) \Lambda_W^\vee. \end{aligned}$$

By Lemma 4.1.2, this coincides with Λ' if and only if equation (4.1) holds. □

Chapter 5. Factorizability of Quantum Group $\boldsymbol{R}$-matrices

We first recall the definition of factorizable braided tensor categories and factorizable Hopf algebras, respectively.

Definition 5.0.1. *[EGNO15] A braided tensor category $\mathcal{C}$ is* factorizable *if the canonical braided tensor functor $G : \mathcal{C} \boxtimes \mathcal{C}^{op} \to \mathcal{Z}(\mathcal{C})$ is an equivalence of categories.*

In [Sch01], Schneider gave a different characterization of factorizable Hopf algebras in terms of its Drinfeld double, leading to the following theorem:

Definition 5.0.2. *A finite-dimensional quasitriangular Hopf algebra (H, R) is called* factorizable *if its* monodromy matrix *$M := R_{21} \cdot R \in H \otimes H$ is non-degenerate, i.e. the following linear map is bijective*

$$H^* \to H \quad \phi \mapsto (id \otimes \phi)(M).$$

Equivalenty, this means we can write $M = \sum_i R_1^i \otimes R_2^i$ for two basis' $R_1^i, R_2^j \in H$.

Theorem 5.0.3. *Let (H, R) be a finite-dimensional quasitriangular Hopf algebra. Then the category of finite-dimensional H-modules $H - \mathsf{mod}_{fd}$ is factorizable if and only if (H, R) is a factorizable Hopf algebra.*

Shimizu [Shi16] has recently proven a number of equivalent characterizations of factorizability for arbitrary (in particular non-semisimple) braided tensor categories. Besides the two previous characterizations (equivalence to Drinfeld center and nondegeneracy of the monodromy matrix), factorizability is equivalent to the fact that the so-called transparent objects are all trivial, see Theorem 5.4.2 below, which will become visible during our analysis later.

5.1 Monodromy matrix in terms of R_0

In order to obtain conditions for the factorizability of the quasitriangular small quantum groups $(u_q(\mathfrak{g}, \Lambda, \Lambda'), R_0(f)\bar{\Theta})$ as in Theorem 2.2.2 in terms of $\mathfrak{g}, q, \Lambda$ and f, we start by calculating the monodromy matrix $M := R_{21} \cdot R \in u_q(\mathfrak{g}, \Lambda, \Lambda') \otimes u_q(\mathfrak{g}, \Lambda, \Lambda')$ in general as far as possible:

Lemma 5.1.1. *For $R = R_0(f)\bar{\Theta}$ as in Theorem 2.2.2, the factorizability of R is equivalent to the invertibility of the following complex-valued matrix m with entries indexed by elements in $\mu, \nu \in \Lambda/\Lambda'$:*

$$m_{\mu,\nu} := \sum_{\mu',\nu' \in \Lambda/\Lambda'} f(\mu - \mu', \nu - \nu') f(\nu', \mu').$$

Proof. We first plug in the expressions for R_0 from Theorem 3.1.3 and $\bar{\Theta}$ from Theorem 2.2.2 and simplify:

$$\begin{aligned}
M &:= R_{21} \cdot R \\
&= (R_0)_{21} \cdot \bar{\Theta}_{21} \cdot R_0 \cdot \bar{\Theta} \\
&= \left(\sum_{\mu_1,\nu_1 \in \Lambda} f(\mu_1, \nu_1) K_{\nu_1} \otimes K_{\mu_1} \right) \cdot \left(\sum_{\beta_1 \in \Lambda_R^+} (-1)^{\mathrm{tr}\beta_1} q_{\beta_1} \sum_{b_1 \in B_{\beta_2}} b_1^{*+} \otimes b_1^- \right) \\
&\cdot \left(\sum_{\mu_2,\nu_2 \in \Lambda} f(\mu_2, \nu_2) K_{\mu_2} \otimes K_{\nu_2} \right) \cdot \left(\sum_{\beta_2 \in \Lambda_R^+} (-1)^{\mathrm{tr}\beta_2} q_{\beta_2} \sum_{b_2 \in B_{\beta_2}} b_2^- \otimes b_2^{*+} \right) \\
&= \sum_{\beta_1,\beta_2 \in \Lambda_R^+} (-1)^{\mathrm{tr}\beta_1+\beta_2} q_{\beta_1} q_{\beta_2} \left(\sum_{\mu_1,\mu_2,\nu_1,\nu_2 \in \Lambda} f(\mu_1, \nu_1) f(\mu_2, \nu_2) q^{\beta_1(\nu_2-\mu_2)} K_{\nu_1+\mu_2} \otimes K_{\mu_1+\nu_2} \right) \\
&\cdot \left(\sum_{b_1 \in B_{\beta_1}, b_2 \in B_{\beta_2}} b_1^{*+} b_2^- \otimes b_1^- b_2^{*+} \right),
\end{aligned}$$

where $\Lambda_R^+ = \mathbb{N}_0[\Delta]$. The last equation holds since $b_1^- \in u_{\beta_1}^-$ and hence fulfills $K_{\nu_2} b_1^- = q^{-\beta_1\nu_2} b_1^- K_{\nu_2}$ and similarly for b_1^{*+}. We have two triangular decompositions

$$u_q = u_q^0 u_q^- u_q^+ \qquad u_q = u_q^0 u_q^+ u_q^-,$$

and the Λ_R^+-gradation on $u_q^\pm$ induces a gradation

$$u_q \otimes u_q \cong \bigoplus_{\beta_1,\beta_2} (u^0 \otimes u^0)(u_q^+{}_{\beta_1} u_q^-{}_{\beta_2} \otimes u_q^-{}_{\beta_1} u_q^+{}_{\beta_2}).$$

The factorizability of R is equivalent to the invertibility of M interpreted as a metrix indexed by the PBW basis. The grading implies a block matrix form of M, so the invertibility M is equivalent to the invertibility of $M^{\beta_1,\beta_2} \in (u_q \otimes u_q)_{(\beta_1,\beta_2)}$ for every $\beta_1, \beta_2 \in \Lambda_R^+$ as follows

$$M^{\beta_1,\beta_2} := \left(\sum_{\mu_1,\mu_2,\nu_1,\nu_2 \in \Lambda} f(\mu_1, \nu_1) f(\mu_2, \nu_2) q^{\beta_1(\nu_2-\mu_2)} K_{\nu_1+\mu_2} \otimes K_{\mu_1+\nu_2} \right) \left(\sum_{b_1 \in B_{\beta_1}, b_2 \in B_{\beta_2}} b_1^{*+} b_2^- \otimes b_1^- b_2^{*+} \right).$$

Since the second sum in M^{β_1,β_2} runs over a basis in $u_q^+{}_{\beta_1}u_q^-{}_{\beta_2}\otimes u_q^-{}_{\beta_1}u_q^+{}_{\beta_2}$, the invertibility of M is equivalent to the invertibility for all $\beta_1\in\Lambda_R^+$ the following element:

$$\begin{aligned}
M_0^{\beta_1} &:= \sum_{\mu_1,\mu_2,\nu_1,\nu_2\in\Lambda/\Lambda'} q^{\beta_1(\nu_2-\mu_2)}f(\mu_1,\nu_1)f(\mu_2,\nu_2)K_{\nu_1+\mu_2}\otimes K_{\mu_1+\nu_2}\\
&= \sum_{\mu,\nu\in\Lambda/\Lambda'} K_\nu\otimes K_\mu\cdot\left(\sum_{\mu',\nu'\in\Lambda/\Lambda'} q^{\beta_1(\mu'-\nu')}f(\mu-\mu',\nu-\nu')f(\nu',\mu')\right).
\end{aligned}$$

Since $K_\nu\otimes K_\mu$ is a vector space basis of $u_q^0\otimes u_q^0=\mathbb{C}[\Lambda/\Lambda']\otimes\mathbb{C}[\Lambda/\Lambda']$, this in turn is equivalent to the invertibility of the following family of matrices m^{β_1} for all $\beta_1\in\Lambda_R^+$ with rows/columns indexed by elements in $\mu,\nu\in\Lambda/\Lambda'$:

$$m_{\mu,\nu}^{\beta_1} := \sum_{\mu',\nu'\in\Lambda/\Lambda'} f(\mu-\mu',\nu-\nu')f(\nu',\mu')q^{\beta_1(\mu'-\nu')}.$$

We now use the fact that R was indeed an R-matrix: By property (2.1) in Theorem 2.2.2 we have

$$m_{\mu,\nu}^{\beta_1} = \sum_{\mu',\nu'\in\Lambda/\Lambda'} f(\mu-\mu',\nu-\nu')f(\nu'+\beta_1,\mu')q^{-\beta_1\nu'}.$$

Since the invertibility of a matrix $m_{\mu,\nu}$ is equivalent to the invertibility of any matrix $m_{\mu,\nu+\beta_1}$, we may substitute $\nu'\mapsto\nu'+\beta_1,\nu\mapsto\nu+\beta_1$, pull the constant factor $q^{-\beta_1^2}$ in front (which also does not affect invertibility) and hence eliminate the first β_1 from the condition. Hence the invertibility of R is equivalent to the invertibility of the following family of matrices $\tilde{m}^{\beta_1}$ for all $\beta_1\in\Lambda_R^+$:

$$\tilde{m}_{\mu,\nu}^{\beta_1} := \sum_{\mu',\nu'\in\Lambda/\Lambda'} f(\mu-\mu',\nu-\nu')f(\nu',\mu')q^{-\beta_1\nu'}.$$

We may now use the same procedure to eliminate the second β_1, hence the invertibility of R is equivalent to the invertibility of the following matrix with rows/columns indiced by elements in $\mu,\nu\in\Lambda/\Lambda'$:

$$m_{\mu,\nu} := \sum_{\mu',\nu'\in\Lambda/\Lambda'} f(\mu-\mu',\nu-\nu')f(\nu',\mu').$$

This was the assertion we wanted to prove. □

Definition 5.1.2. *Let $g:G_1\times G_2\to\mathbb{C}^\times$ be a group pairing. It induces a symmetric form on the product $G_1\times G_2$ we denote by Sym(g):*

$$\begin{aligned}
Sym(g):\quad & (G_1\times G_2)^{\times 2}\longrightarrow\mathbb{C}^\times\\
& ((\mu_1,\mu_2),(\nu_1,\nu_2))\longmapsto g(\mu_1,\nu_2)g(\nu_1,\mu_2).
\end{aligned}$$

Lemma 5.1.3. *If $g:G_1\times G_2\to\mathbb{C}^\times$ is a perfect pairing of abelian groups, then the symmetric form $Sym(g)$ is perfect.*

Proof. By assumption, $g \times g$ defines an isomorphism between $G_1 \times G_2$ to $\widehat{G_2} \times \widehat{G_1}$. The symmetric form $Sym(g)$ is given by the composition of this isomorphism with the canonical isomorphism $\widehat{G_2} \times \widehat{G_1} \cong \widehat{G_1 \times G_2}$. This proves the claim. $\square$

Consider for a finite abelian group G and subgroups $G_1, G_2 \leq G$ the canonical exact sequence

$$0 \to G_1 \cap G_2 \to G_1 \times G_2 \to G_1 + G_2 \to 0 \tag{5.1}$$

For $\mu \in G_1 + G_2$, we denote its fiber by

$$(G_1 \times G_2)_\mu := \{(\mu_1, \mu_2) \in G_1 \times G_2 \mid \mu_1 + \mu_2 = \mu\}.$$

Moreover, we define

$$\begin{aligned}
\mathrm{Rad} :&= \{ \ (\mu_1, \mu_2) \in G_1 \times G_2 \mid \mathrm{Sym}(\hat{f})((\mu_1, \mu_2), x) = 1 \ \forall x \in (G_1 \times G_2)_0 \ \} \\
\mathrm{Rad}_\mu :&= \mathrm{Rad} \cap (G_1 \times G_2)_\mu \\
\mathrm{Rad}_0^\perp :&= \{ \ \mu_1 + \mu_2 \in G \mid (\mu_1, \mu_2) \in \mathrm{Rad} \ \}
\end{aligned}$$

Lemma 5.1.4. *We have two split exact sequences:*

$$0 \to Rad_0 \to Rad \to Rad_0^\perp \to 0$$
$$0 \to Rad_0^\perp \to G \to Rad_0 \to 0.$$

Proof. The first sequence is exact by definition of the three groups. Moreover, we know

$$\mathrm{Rad} = \ker(\hat{\imath} \circ \mathrm{Sym}(\hat{f})) \cong \ker(\hat{\imath}) = \mathrm{im}(\hat{\pi}) \cong \hat{G} \cong G,$$

where $\hat{\imath}, \hat{\pi}$ denote the duals of the inclusion and projection in (5.1). In example 5.1.8 we will see that in the case $G_1 = G_2 = G$, $\hat{f}$ symmetric, Rad_0 is the 2-torsion subgroup of G, and the second map in the second exact sequence is just the projection, hence both diagrams split in this case. If $\hat{f}$ is asymmetric, we will see in section 5.3 that Rad_0 is isomorphic to $\mathbb{Z}_2^k$ for some $k \geq 2$, thus

$$\begin{aligned}
\mathrm{Rad}_0^\perp &\longrightarrow \mathrm{Rad} \\
x &\longmapsto \sum_{\tilde{x} \in \mathrm{Rad}_x} \tilde{x}
\end{aligned}$$

is a section of the first exact sequence. Here we used that the sum over all elements in $\mathbb{Z}_2^k$ vanishes. Again, it follows that both diagrams split. Finally, if $G_1 \neq G_2$ (i.e. in the case D_{2n}), then $\hat{f} = q^{-(\cdot,\cdot)}$ on $G_1 \cap G_2$. By the same argument as in example 5.1.8, Rad_0 is the 2-torsion subgroup of $G_1 \cap G_2$. But we have $G \cong G_1 \cap G_2 \times \pi_1$ in this case, hence both sequences split. $\square$

Corollary 5.1.5. *Using the projection $\alpha : G \to Rad_0^\perp$ and the inclusion $\beta : Rad_0^\perp \to Rad$ from the above lemma, we can define a symmetric form on G:*

$$Sym_G(\hat{f}) : G \times G \longrightarrow \mathbb{C}^\times \tag{5.2}$$
$$(\mu, \nu) \longmapsto Sym(\hat{f})(\beta \circ \alpha(\mu), \beta \circ \alpha(\nu)). \tag{5.3}$$

Moreover, we have $Rad(Sym_G(\hat{f})) \cong Rad_0$.

□

Theorem 5.1.6. *We have shown in Theorem 2.2.2 and Lemma 3.1.5 that the assumption that $R = R_0(f)\bar{\Theta}$ is an R-matrix is equivalent to the existence of subgroups $G_1, G_2 \subset \Lambda/\Lambda'$ of same order some $d|\Lambda_R/\Lambda'|$ and f restricting up to a scalar to a non-degenerate pairing $\hat{f} : G_1 \times G_2 \to \mathbb{C}^\times$ and f vanishes otherwise.*
In this notation the matrix m as defined in the previous lemma can be rewritten as:

$$m_{\mu,\nu} = \frac{1}{d^2|\Lambda_R/\Lambda'|^2} \sum_{\substack{\tilde{\mu}\in(G_1\times G_2)_\mu \\ \tilde{\nu}\in(G_1\times G_2)_\nu}} Sym(\hat{f})(\tilde{\mu}, \tilde{\nu}).$$

It is invertible if and only if $Rad_0 = 0$. In this case,

$$m_{\mu,\nu} = \frac{|G_1 \cap G_2|}{d^2|\Lambda_R/\Lambda'|^2} Sym_G(\hat{f}).$$

We first note that $\mathrm{Rad}_0 = 0$ implies $\mathrm{Rad}_0^\perp = G$ and thus $G = G_1 + G_2$. Together with Corollary 3.1.6 this implies

Corollary 5.1.7.

$$\Lambda' = Cent_{\Lambda_R}(\Lambda).$$

Before we proof the theorem, we first give a simple example:

Example 5.1.8. *Let $G_1 = G_2 = G$ (correspondingly $\Lambda_1 = \Lambda_2 = \Lambda$) and assume $\hat{f}$ is symmetric non-degenerate, then the radical measures 2-torsion:*

$$Rad(Sym_G(\hat{f})) \cong Rad_0 = \{\mu \in G \mid 2\mu = 0\}.$$

Again, this is the only case appearing for cyclic fundamental groups. Hence in all cases except $\mathfrak{g} = D_{2n}$ factorizability is equivalent to $|\Lambda/\Lambda'|$ being odd.

Proof of Thm. 5.1.6. The first part of the theorem follows by applying lemma 3.1.5 to the matrix m as given in the previous lemma. Now, assume that m is invertible. We must have $G = G_1 + G_2$, otherwise the matrix has zero-columns and -rows, differently formulated: the fibers $(G_1 \times G_2)_\mu$ in the short exact sequence must be non-empty for all $\mu \in G$. If on the other hand, $\mathrm{Rad}_0 = 0$, then $\mathrm{Rad}_0^\perp = G$ and thus $G_1 + G_2 = G$ must also hold, thus we assume this from now on. By the short exact sequence the fiber $(G_1 \times G_2)_0 \cong G_1 \cap G_2$, other fibers are of the explicit form $\tilde{\mu} + (G_1 \times G_2)_0$

for some choice of representative $\tilde{\mu}$. Therefore,

$$\begin{aligned}
m_{\mu,\nu} &= \frac{1}{d^2|\Lambda_R/\Lambda'|^2} \sum_{\substack{\tilde{\mu}\in(G_1\times G_2)_\mu \\ \tilde{\nu}\in(G_1\times G_2)_\nu}} \mathrm{Sym}(\hat{f})(\tilde{\mu},\tilde{\nu}) \\
&= \frac{1}{d^2|\Lambda_R/\Lambda'|^2} \sum_{\tilde{\nu}\in(G_1\times G_2)_\nu} \mathrm{Sym}(\hat{f})(\tilde{\mu},\tilde{\nu}) \sum_{\tilde{\eta}\in(G_1\times G_2)_0} \mathrm{Sym}(\hat{f})(\tilde{\eta},\tilde{\nu}) \\
&= \frac{|G_1\cap G_2|}{d^2|\Lambda_R/\Lambda'|^2} \sum_{\tilde{\nu}\in(G_1\times G_2)_\nu} \mathrm{Sym}(\hat{f})(\tilde{\mu},\tilde{\nu}) \cdot \delta_{\mathrm{Sym}(f)(\tilde{\nu},_)|_{G_1\cap G_2}=1} = (*).
\end{aligned}$$

Fix as above a representative $\tilde{\nu}$ of the fiber of ν, i.e. $\tilde{\nu}\in(G_1\times G_2)_\nu$ such that $\mathrm{Sym}(f)(\tilde{\nu},\_)|_{G_1\cap G_2}=1$ holds. Two elements fulfilling this property differ by an element in the subgroup $\mathrm{Rad}_0 \leq G_1\cap G_2$, thus

$$\begin{aligned}
(*) &= \frac{|G_1\cap G_2|}{d^2|\Lambda_R/\Lambda'|^2} \mathrm{Sym}(\hat{f})(\tilde{\mu},\tilde{\nu}) \sum_{\tilde{\xi}\in\mathrm{Rad}_0} \mathrm{Sym}(\hat{f})(\tilde{\xi},\tilde{\nu}) \cdot \delta_{\mathrm{Sym}(f)(\tilde{\nu},_)|_{G_1\cap G_2}=1} \\
&= \frac{|G_1\cap G_2||\mathrm{Rad}_0|}{d^2|\Lambda_R/\Lambda'|^2} \mathrm{Sym}(\hat{f})(\tilde{\mu},\tilde{\nu}) \cdot \delta_{\mathrm{Sym}(\hat{f})(\tilde{\nu},_)|_{G_1\cap G_2}=1}\, \delta_{\mathrm{Sym}(\hat{f})(\tilde{\mu},_)|_{\mathrm{Rad}_0}=1}.
\end{aligned}$$

Since m is symmetric, we have

$$\begin{aligned}
m_{\mu,\nu} &= \frac{|G_1\cap G_2||\mathrm{Rad}_0|}{d^2|\Lambda_R/\Lambda'|^2} \mathrm{Sym}(\hat{f})(\tilde{\mu},\tilde{\nu}) \cdot \delta_{\mathrm{Sym}(\hat{f})(\tilde{\nu},_)|_{G_1\cap G_2}=1}\, \delta_{\mathrm{Sym}(\hat{f})(\tilde{\mu},_)|_{G_1\cap G_2}=1} \\
&= \frac{|G_1\cap G_2||\mathrm{Rad}_0|}{d^2|\Lambda_R/\Lambda'|^2} \mathrm{Sym}_G(\hat{f})(\mu,\nu)\delta_{\mathrm{Rad}_\mu\neq\emptyset}\delta_{\mathrm{Rad}_\nu\neq\emptyset}.
\end{aligned}$$

and this is invertible if an only if $\mathrm{Rad}_0 \cong \mathrm{Rad}(\mathrm{Sym}_G(\hat{f})) = 0$.

□

5.2 Factorizability for symmetric $R_0(f)$

For $R_0 = \sum_{\mu,\nu} f(\mu,\nu) K_\mu \otimes K_\nu$ being the Cartan part of an R-matrix, assume that $\hat{f} = |G|f$ on G is symmetric. We have shown in Example 5.1.8 that factorizability is equivalent to $|G|$ being odd. We now want to give a necessary and sufficient condition for this:

Lemma 5.2.1. *Let $\Lambda_R \subseteq \Lambda \subseteq \Lambda_W$ be an arbitrary intermediate lattice for a certain irreducible root system. Then the order of the group $G = \Lambda/Cent_{\Lambda_R}(\Lambda)$ is odd if and only if both of the following conditions are satisfied:*

1. *$|\Lambda/\Lambda_R|$ is odd*
2. *ℓ is either odd or ($\ell \equiv 2$ mod 4, $\mathfrak{g} = B_n$, $\Lambda = \Lambda_R$) including A_1.*

Proof. We saw that in all our cases, there exists an isomorphism $\Lambda/\Lambda_R \cong \mathrm{Cent}_\Lambda(\Lambda_R)/\mathrm{Cent}_{\Lambda_R}(\Lambda)$. Moreover, from Lemma 4.1.2 we know that $|\Lambda/\mathrm{Cent}_\Lambda(\Lambda_R)| = \det(D_\ell)$, where D_ℓ was the diagonal matrix $\mathrm{Diag}\left(\frac{\ell}{\gcd(\ell,d_i^\Lambda)}\right)$ with d_i^Λ being the invariant factors of the lattice Λ (i.e. the diagonal entries of the Smith normal form of a basis matrix of Λ). Thus,

$$\begin{aligned}
|G| &= |\Lambda/\mathrm{Cent}_{\Lambda_R}(\Lambda)| \\
&= |\Lambda/\mathrm{Cent}_\Lambda(\Lambda_R)||\mathrm{Cent}_\Lambda(\Lambda_R)/\mathrm{Cent}_{\Lambda_R}(\Lambda)| \\
&= |\Lambda/\mathrm{Cent}_\Lambda(\Lambda_R)||\Lambda/\Lambda_R| \\
&= \det(D_\ell)|\Lambda/\Lambda_R| \\
&= \prod_{i=1}^n \frac{\ell}{\gcd(\ell,d_i^\Lambda)}\,|\Lambda/\Lambda_R|.
\end{aligned}$$

Clearly, this term is odd if ℓ and $|\Lambda/\Lambda_R|$ are odd. In the case ($\ell \equiv 2 \bmod 4$, $\mathfrak{g} = B_n$, $\Lambda = \Lambda_R$), the Smith normal form S_R of the basis matrix A_R is given by $2 \cdot \mathrm{id}$. Thus, $|G|$ is odd in this case. On the other hand, let $|G|$ be odd:
We first consider the case ℓ **even**. A necessary condition for $|\Lambda/\Lambda'|$ odd is that the multiplicity m_ℓ of the prime 2 in $\prod_{i=1}^n \frac{\ell}{\gcd(\ell,d_i^\Lambda)}$ is at most the multiplicity m_{π_1} of the prime 2 in $|\pi_1|$. We check this condition for rank $n > 1$:

- For $\mathfrak{g}$ simply-laced (or triply-laced $\mathfrak{g} = G_2$) we have all $d_i = 1$, hence $n \mid m_\ell$ (equality for $\ell = 2 \bmod 4$). The cases D_n with $m_{\pi_1} = 2$ have rank $n \geq 4$, all others except A_n have $m_{\pi_1} = 0, 1$, so the necessary condition $m_\ell \leq m_{\pi_1}$ is never fulfilled. The cases A_n have $2^{m_{\pi_1}}|(n+1) \leq (m_\ell + 1) \overset{!}{\leq} (m_{\pi_1} + 1)$ which can only be true in rank $n = 1$ treated above.

- For $\mathfrak{g}$ doubly-laced of rank $n > 1$, we always have always $m_{\pi_1} = 0, 1$ but m_ℓ can be considerably smaller than above, namely for $\ell = 2 \bmod 4$ equal to the number of short simple roots $d_{\alpha_i} = 1$ (otherwise m_ℓ again increases by n for every factor 2 in ℓ), hence the necessary condition $m_\ell \leq m_{\pi_1}$ can be fulfilled only for B_n (which would also include A_1 above for $n = 1$). More precisely, since $m_\ell = m_{\pi_1}$ and the decomposition for Λ/Λ' has an additional factor $|\Lambda/\Lambda_R|$, it can only be odd for $\Lambda = \Lambda_R$.

On the other hand, if ℓ is **odd**, then the whole product term is odd. But since $|G|$ was assumed to be odd, also $|\Lambda/\Lambda'|$ must be odd. $\square$

Corollary 5.2.2. *Let $\Lambda = \Lambda_R$. In the previous section we have seen that $\hat{f} = q^{-(\cdot,\cdot)}$ gives always an R-matrix in this case. By the proof of the previous Lemma, we have*

$$Rad_0 \cong \prod_{i=1}^n \mathbb{Z}_{\gcd\left(2,\frac{\ell}{\gcd(\ell,d_i^R)}\right)},$$

where the d_i^R denote the inveriant factors of $\Lambda_W^\vee/\Lambda_R$.

5.3 Factorizability for D_{2n}, R_0 antisymmetric

The split case $\mathfrak{g} = D_{2n}$, $G = G_1 \times G_2$ is clearly factorizable, so the only remaining case for which we have to check factorizabilty is $\mathfrak{g} = D_{2n}, \Lambda = \Lambda_W$ for $\hat{f}$ being not symmetric. We know that in this case, the corresponding form g on Λ/Λ_R is uniquely defined by a 2×2-matrix $K \in \mathfrak{gl}(2, \mathbb{F}_2)$, s.t. $g(\lambda_{2(n-1)+i}, \lambda_{2(n-1)+j}) = \exp\left(\frac{2\pi i K_{ij}}{2}\right)$ for $i, j \in \{1, 2\}$. From this we see that if g is not symmetric, it must be antisymmetric, i.e. $g(\mu, \nu) = g(\nu, \mu)^{-1}$. Thus, the following lemma applies in this case, and hence there are no factorizable R-matrices for D_{2n}, $\Lambda = \Lambda_W$.

Lemma 5.3.1. *For $\mathfrak{g}$ simply-laced and $\Lambda = \Lambda_W$, let $\hat{f} = q^{-(\cdot,\cdot)}g : G \times G \to \mathbb{C}^\times$ be a non-degenerate form as in Thm. 3.1.3 and Lemma 3.1.5, s.t. the form $g : \pi_1 \times \pi_1 \to \mathbb{C}^\times$ is asymmetric. Then,*

$$Rad_0 \cong \bigoplus_{i=1}^{n} \mathbb{Z}_{\gcd(2, \ell d_i^R)},$$

where the d_i^R denote the invariant factors of π_1. In particular, $Rad_0 = 0$ holds if and only if $\gcd(2, \ell|\pi_1|) = 1$.

Proof. We recall the definition of $\mathrm{Rad}_0(\mathrm{Sym}_G(\hat{f}))$ in this case:

$$\begin{aligned}
\mathrm{Rad}_0(\mathrm{Sym}_G(\hat{f})) &= \{\mu \in G \mid f(\nu, \mu)^{-1} = f(\mu, \nu) \quad \forall \nu \in G\ \} \\
&= \{\mu \in G \mid q^{(\nu,\mu)} g(\nu, \mu)^{-1} = q^{-(\mu,\nu)} g(\mu, \nu) \quad \forall \nu \in G\ \} \\
&= \{\mu \in G \mid q^{(\nu,\mu)} = q^{-(\mu,\nu)} \quad \forall \nu \in G\ \} \\
&= \{\mu \in G \mid q^{(2\mu,\nu)} = 1 \quad \forall \nu \in G\ \} \\
&= \{\mu \in G \mid 2\mu \in \mathrm{Cent}_{2\Lambda_W}(\Lambda_W)/2\mathrm{Cent}_{\Lambda_R}(\Lambda_W)\ \} = (*)
\end{aligned}$$

For $\mathfrak{g}$ is simply-laced, we have $\Lambda_W = \Lambda_W^\vee$, thus

$$\begin{aligned}
(*) &\cong \mathrm{Cent}_{2\Lambda_W}(\Lambda_W)/2\mathrm{Cent}_{\Lambda_R}(\Lambda_W) \\
&= (2\Lambda_W \cap \ell A_R \Lambda_W)/2\ell A_R \Lambda_W \\
&= P_R \mathrm{Diag}(\mathrm{lcm}(2, \ell d_i^R))\Lambda_W / P_R 2\ell S_R \Lambda_W \\
&= \Lambda_W / \mathrm{Diag}(\gcd(2, \ell d_i^R))\Lambda_W.
\end{aligned}$$

This proves the claim. □

5.4 Transparent objects in non-factorizable cases

In this section, we determine the transparent objects in the representation category of $u_q(\mathfrak{g}, \Lambda)$ with our R-matrix given by $R_0\bar{\Theta}$ and $R_0 = \frac{1}{|\Lambda/\Lambda'|} \sum_{\mu,\nu \in \Lambda/\Lambda'} \hat{f}$ with $\hat{f}$ a group pairing $\Lambda_1/\Lambda' \times \Lambda_2/\Lambda' \to \mathbb{C}^\times$.

Definition 5.4.1. *Let $\mathcal{C}$ be a braided monoidal category with braiding c. An object $V \in \mathcal{C}$ is called* transparent *if the double braiding $c_{W,V} \circ c_{V,W}$ is the identity on $V \otimes W$ for all $W \in \mathcal{C}$.*

The following theorem by Shimizu gives a very important characterization of factorizable categories:

Theorem 5.4.2 ([Shi16], Thm. 1.1)**.** *A braided finite tensor category is factorizable if and only if the transparent objects are direct sums of finitely many copies of the unit object.*

Corollary 5.4.3. *In particular, for a Hopf algebra H the representation category $H - \mathsf{mod}_{fd}$ is factorizable if and only if the transparent objects are multiples of the trivial representation and vice versa.*

Since in our cases $\Lambda_1 \neq \Lambda_2$ can only appear in D_{2n}, and we know those are factorizable, we shall in the following restrict ourselves to the case $\Lambda_1 = \Lambda_2 = \Lambda$. The proof below works also in the more general case, but requires more notation. As usual we first reduce the Hopf algebra question to the group ring and then solve the group theoretical problem.

Lemma 5.4.4. *If a $u_q(\mathfrak{g})$-module V, with a highest-weight vector v and $K_\mu v = \chi(K_\mu)v$, is a transparent object, then necessarily the 1-dimensional Λ/Λ'-module $\mathbb{C}_\chi$ is a transparent object over the Hopf algebra $\mathbb{C}[\Lambda/\Lambda']$ with R-matrix R_0. If V is 1-dimensional, then V is transparent if and only if $\mathbb{C}_\chi$ is.*

Proof. Let V be transparent. For every $\psi : \Lambda/\Lambda' \to \mathbb{C}^\times$ we have another finite-dimensional module $W := u_q(\mathfrak{g}) \otimes_{u_q(\mathfrak{g})^+} \mathbb{C}_\psi$ with highest weight vector $w = 1 \otimes 1_\psi$ which we can test this assumption against

$$c^2 : V \otimes W \to W \otimes V \to V \otimes W.$$

We calculate the effect of c^2 on the highest-weight vectors $v \otimes w$:

$$c^2(v \otimes w) = \tau_{W \otimes V} R_0 \bar{\Theta} \tau_{V \otimes W} R_0 \bar{\Theta}(v \otimes w)$$

Because v, w were assumed highest-weight vectors, the $\bar{\Theta}$ act trivially. Hence follows that $\mathbb{C}_\chi, \mathbb{C}_\psi$ have a trivial double braiding over the Hopf algebra $\mathbb{C}[\Lambda/\Lambda']$ with R-matrix R_0. Because we could achieve this result for any ψ this means that $\mathbb{C}_\chi$ is transparent as asserted.

Now, let $V = \mathbb{C}_\chi$ be 1-dimensional over $u_q(\mathfrak{g})$ and transparent over $\mathbb{C}[\Lambda/\Lambda']$, and let w be any element in any module W, then again the two Θ act trivially, one time because $v = 1_\chi$ is a highest weight vector, and one time because it is also a lowest weight vector. But if the double-braiding of $v = 1_\chi$ with any element w is trivial, then $V = \mathbb{C}_\chi$ is already tranparent over $u_q(\mathfrak{g})$. $\square$

Lemma 5.4.5. *$\mathbb{C}_\chi$ is a transparent object over the Hopf algebra $\mathbb{C}[\Lambda/\Lambda']$ with R-matrix R_0 iff it is an f-transformed of the radical of $Sym_G(\hat{f})$, i.e.*

$$\chi(\mu) = f(\mu, \xi) \qquad \xi \in Rad_0.$$

Proof. Since f is nondegenerate, we can assume $\chi(\mu) = f(\mu,\xi)$ and wish to prove $\mathbb{C}_\chi$ is transparent iff $\xi \in \mathrm{Rad}_0$. We test transparency against any module $\mathbb{C}_\psi$ and also write $\psi(\mu) = f(\lambda,\mu)$ (note the order of the argument). We evaluate the double-braiding on $1_\chi \otimes 1_\psi$ and get the following scalar factor, which needs to be $= 1$ for all ψ in order to make $\mathbb{C}_\chi$ transparent:

$$
\begin{aligned}
&\frac{1}{|G|^2}\sum_{\mu,\nu}\chi(\mu)\psi(\nu)\sum_{\substack{\mu_1+\mu_2=\mu\\ \nu_1+\nu_2=\mu}}\mathrm{Sym}(\hat{f})((\mu_1,\mu_2),(\nu_1,\nu_2))\\
&=\frac{1}{|G|^2}\sum_{\mu,\nu}f(\mu,\xi)f(\lambda,\nu)\sum_{\substack{\mu_1+\mu_2=\mu\\ \nu_1+\nu_2=\mu}}f(\mu_1,\nu_1)f(\nu_2,\mu_2)\\
&=\frac{1}{|G|^2}\sum_{\mu,\nu}f(\mu,\xi)f(\lambda,\nu)\sum_{\nu_1,\mu_1}f(\mu_1,\nu_1)\;f(\nu,\mu)f^{-1}(\nu_1,\mu)f^{-1}(\nu,\mu_1)f(\nu_1,\mu_1)\\
&=\frac{1}{|G|}\sum_{\nu}f(\lambda,\nu)\sum_{\nu_1,\mu_1}f(\mu_1,\nu_1)\;\delta_{\xi=-\nu+\nu_1}f^{-1}(\nu,\mu_1)f(\nu_1,\mu_1)\\
&=\frac{1}{|G|}\sum_{\nu}f(\lambda,\nu)\sum_{\mu_1}f(\mu_1,\xi+\nu)\;f(\xi,\mu_1)\\
&=f^{-1}(\lambda,\xi)\;f^{-1}(\xi,\lambda)=\mathrm{Sym}_G(\hat{f})(\lambda,\xi)
\end{aligned}
$$

This scalar factor of the double braiding is equal $+1$ for all λ (and hence all $\mathbb{C}_\psi$) iff $\xi \in \mathrm{Rad}_0$ as asserted. □

The previous two lemmas combined imply that any irreducible transparent $u_q(\mathfrak{g})$-module has necessarily the characters $\chi(\mu) = f(\mu,\xi), \xi \in \mathrm{Rad}_0$ as highest-weights, and conversely if such a character χ gives rise to 1-dimensional $u_q(\mathfrak{g})$-modules (i.e. $\chi|_{2\Lambda_R} = 1$), then these are guaranteed transparent objects. Hence the final step is to give more closed expressions for the f-transformed characters χ of the radical depending on the case and check the 1-dimensionality condition.

In all cases where f is symmetric we have seen in 5.1.8 that $\mathrm{Rad}_0(\mathrm{Sym}_G(\hat{f}))$ is the 2-torsion subgroup of Λ/Λ', so in these cases χ gives rise to a 1-dimensional object.

Corollary 5.4.6. *If f is symmetric (true for all cases except D_{2n}) then the irreducible transparent objects are all 1-dimensional $\mathbb{C}_\chi$ where the characters χ are the f-transformed of the elements in the radical of the bimultiplicative form Sym$(\hat{f})|_G$ on $G = \Lambda/\Lambda'$. In particular, the group of transparent objects is isomorphic to this radical as an abelian group.*

Corollary 5.4.7. *In the case of symmetric f (all cases except D_{2n}) the fact that Rad$_0$ is the 2-torsion of Λ/Λ' and f-transformation is a group isomorphism shows:*

The group T of transparent objects consists of $\mathbb{C}_\chi$ where $\chi|_{2\Lambda} = 1$ i.e. the two-torsion of the character group.

The remaining case in D_{2n} with f nonsymmetric and has been done by hand in Lemma 5.3.1.
In Table 1, we gave a list of all quasi-triangular small quantum groups as in Table 4.1, where we replaced the entries in the last column by the respective subgroups of transparent objects $T \subseteq \Lambda/\Lambda'$. If the quantum group is factorizable, this is indicated by a bold $\mathbf{0}$. Since ($\Lambda = \Lambda_R$, ℓ odd) is always a solution, we omitted this from the table.

Chapter 6. Quantum groups with a ribbon structure

In Thm. 8.23 in [Mül98], the existence of ribbon structures for $u_q(\mathfrak{g},\Lambda)$ is proven. In this section we construct a ribbon structure for all cases. In the proof, we use several auxiliary results from [Mül98].

Theorem 6.0.1. *Let $u_q(\mathfrak{g},\Lambda)$ be quasitriangular Hopf algebra, with an R-matrix satisfying the conditions in Theorem 2.2.2 and let $u := S(R_{(2)})R_{(1)}$. Then $v := K_{\nu_0}^{-1}u$ is a ribbon element in $u_q(\mathfrak{g},\Lambda)$.*

Proof. We consider the natural $\mathbb{N}_0[\alpha_i \,|\, i \in I]$-grading on the Borel parts $u^{\pm} := u_q(\mathfrak{g},\Lambda)^{\pm}$ [Lus93]. Since $u^{\pm}$ is finite-dimensional, there exists a maximal $\nu_0 \in \mathbb{N}_0[\alpha_i \,|\, i \in I]$, s.t. the homogeneous component $u^{\pm}_{\nu_0}$ is non-trivial. More explicitly ν_0 is of the form:

$$\nu_0 = \sum_{\alpha\in\Phi^+} (\ell_\alpha - 1)\alpha,$$

where $\ell_\alpha := \frac{\ell}{\gcd(\ell,2d_\alpha)}$.
Using the formulas $u = (\sum f(\mu,\nu)K_{\mu+\nu})^{-1}\vartheta$ and $S(u) = (\sum f(\mu,\nu)K_{\mu+\nu})^{-1}S(\vartheta)$, where $\vartheta = \sum \bar{\Theta}^{(2)}S^{-1}(\bar{\Theta}^{(2)})$, Mueller proves the formula $K^2_{-\nu_0} = u^{-1}S(u)$. Using the fact that u commutes with all grouplike elements, this implies $v^2 = uS(u)$. In order to show that v is central, we first show that $K^{-1}_{\nu_0+2\rho}$ is a central element. By the K,E-relations, this is equivalent to

$$\nu_0 + 2\rho \in \mathrm{Cent}_\Lambda(\Lambda_R), \tag{6.1}$$

where $\rho = \frac{1}{2}\sum_{\alpha\in\Phi^+}\alpha$ is the Weyl vector.

We calculate directly that this is always the case:

$$\begin{aligned}(\nu_0 + 2\rho, \beta) &= q^{\sum_{\alpha\in\Phi^+}(\ell_\alpha - 1 + 1)(\alpha,\beta)} \\ &= q^{\ell\sum_{\alpha\in\Phi^+}\frac{1}{\gcd(\ell,2d_\alpha)}\cdot 2d_\alpha(\alpha^\vee,\beta)} = 1.\end{aligned}$$

Since $K_{2\rho}ux = xK_{2\rho}u$ holds for all $x \in u_q(\mathfrak{g}, \Lambda)$ (see [Mül98], Lemma 8.22 and 8.19), we have

$$\begin{aligned} vx &= K_{\nu_0}^{-1}ux = K_{\nu_0+2\rho}^{-1}K_{2\rho}ux \\ &= K_{\nu_0+2\rho}^{-1}xK_{2\rho}u = xK_{\nu_0+2\rho}^{-1}K_{2\rho}u = xv, \end{aligned}$$

hence v is central.

$\square$

Part III

Small quasi-quantum groups and modularization

Chapter 1. Introduction

In this part of the thesis, we modularize the representation categories of those small quantum groups, which turned out to be quasi-triangular but not factorizable in the first part. To this end, we construct a whole class of finite-dimensional quasi-Hopf algebras, generalizing extended small quantum groups. Part of this construction is based on the approach in [AS02], where the Borel part of a small quantum group is chararcterized as a Radford biproduct $B(V)\#kG$ of the finite dimensional Nichols algebra $B(V)$ of some diagonally braided vector space V with the group algebra kG of a finite abelian group G together with a bihomomorphism on G. Using the notation of the first part of this thesis, G is simply the quotient Λ/Λ'. Moreover, we have $V = \oplus_i F_i\mathbb{C}$ with brading matrix q_{ij} induced by a bihomomorphism σ on the dual group $\widehat{G}$, which is simply the Fourier transform of the R-matrix R_0 on $\mathbb{C}[G]$. Since Nichols algebras live in general abelian braided monoidal categories and the Radford biproduct is defined also for quasi-Hopf algebras [BN02], we can perform the analogue construction for a group algebra $\mathbb{C}[G]_\omega$, now considered as quasi-Hopf algebra, whith non-trivial coassociator induced by a 3-cocycle ω on the dual group $\widehat{G}$. The bihomomorphism σ is then replaced by a 2-cochain σ on $\widehat{G}$, satisfying a so-called abelian 3-cocycle condition together with ω. Given an abelian 3-cocycle (ω, σ), we construct in Sec. 5 a quasi-triangular quasi-Hopf algebra $u(\sigma, \omega)$. We are interested in (σ, ω) such that the associated Nichols algebra $B(V)$ is finite-dimensional; the corresponding braiding matrices $q_{ij} = \sigma(\chi_i, \chi_j)$ are classified in [Hec09] and include those corresponding to the Borel parts of small quantum groups. After finding relations for the Nichols algebra, we build the Drinfeld double of the Radford biproduct $u(\omega, \sigma)^{\leq 0} := B(V)\#\mathbb{C}[G]_\omega$, which has also been defined in the quasi-Hopf setting [HN99b]. After modding out a certain biideal, we end up with a finite-dimensional quasi-Hopf algebra $u(\omega, \sigma)$, which has a canonical quasi-triangular structure by construction. In Chapter 5, we describe generators, relations, grouplikes, etc. for this quasi-Hopf algebra.

In Chapter 6 we turn to our previous (extended) small quantum groups $u_q(\mathfrak{g}, \Lambda) = u(1, \sigma)$ with $\sigma(\chi_i, \chi_j) = q^{(\alpha_i, \alpha_j)}$. In the first part of this thesis we have calculated the transparent objects of the braided tensor category of representations $\mathsf{Rep}_{u_q(\mathfrak{g},\Lambda)}$. In Thm. 6.0.6 we now give for any suitable datum $(\mathfrak{g}, q, \Lambda, R_0)$ an abelian 3-cocycle $(\bar{\sigma}, \bar{\omega})$ on the dual of a certain subgroup $\bar{G} \subseteq \Lambda/\Lambda'$, such that the corresponding quasi-Hopf algebra $u(\bar{\sigma}, \bar{\omega})$ constructed in the last chapter is a subalgebra of $u_q(\mathfrak{g},\Lambda)$ and restriction along this algebra inclusion defines a modularization of $\mathsf{Rep}_{u_q(\mathfrak{g},\Lambda)}$. A key ingredient here is Cor. 5.4.6, which guarantees that the transparent objects in $\mathsf{Rep}_{u_q(\mathfrak{g},\Lambda)}$ are 1-dimensional. This allows us to trace back the modularization of $\mathsf{Rep}_{u_q(\mathfrak{g},\Lambda)}$ to the modularization of $\mathsf{Rep}_{\mathbb{C}[\Lambda/\Lambda']}$.

Chapter 2. Preliminaries

2.1 Quasi-Hopf algebras

We start with the definition of a quasi-Hopf algebra as introduced in [Dri89].

Definition 2.1.1. *A quasi-bialgebra is an algebra H with algebra homomorphisms $\Delta : H \to H \otimes H$ and $\epsilon : H \to k$, together with an invertible* coassociator *$\phi \in H \otimes H \otimes H$, such that the following conditions hold:*

$$(id \otimes \Delta)(\Delta(a)) \cdot \phi = \phi \cdot (\Delta \otimes id)(\Delta(a))$$
$$(id \otimes id \otimes \Delta)(\phi) \cdot (\Delta \otimes id \otimes id)(\phi) = (1 \otimes \phi) \cdot (id \otimes \Delta \otimes id)(\phi) \cdot (\phi \otimes id)$$
$$(\epsilon \otimes id)\Delta(h) = h = (id \otimes \epsilon)\Delta(h)$$
$$(id \otimes \epsilon \otimes id)(\phi) = 1$$

A quasi-antipode (S, α, β) for H consists of algebra anti-automorphism $S : H \to H$, together with elements $\alpha, \beta \in H$, s.t.

$$S(h_{(1)})\alpha h_{(2)} = \epsilon(h)\alpha, \qquad h_{(1)}\beta S(h_{(2)}) = \epsilon(h)\beta,$$
$$X^1\beta S(X^2)\alpha X^3 = 1, \qquad S(x^1)\alpha x^2 \beta x^3 = 1.$$

Similar to Sweedlers' notation, we used the short-hand notation $\phi = X^1 \otimes X^2 \otimes X^3$. For the inverse, we use small letters $\phi^{-1} = x^1 \otimes x^2 \otimes x^3$. If more than one associator appears in an equation we use letters X, Y, Z and x, y, z, respectively.

In the first condition, ϕ can be understood as a coassociativity constraint for the coproduct Δ. In particular, $\phi = 1 \otimes 1 \otimes 1$ implies coassociativity and H becomes an ordinary Hopf algebra. The second condition guarantees that the representation category Rep_H of H has a canonical monoidal structure by left-action of $\phi \in H \otimes H \otimes H$:

Theorem 2.1.2. *The category of representations of a quasi-Hopf algebra is a monoidal category if endowed with a tensor product $V \otimes W$ given by Δ, unit object given by k_ϵ and a nontrivial associator $(U \otimes V) \otimes W \to U \otimes (V \otimes W)$ given by:*

$$\omega_{U,V,W} : u \otimes v \otimes w \mapsto X^1 u \otimes X^2 v \otimes X^3 w. \tag{2.1}$$

In fact, every finite tensor category with quasi-fiber functor is equivalent to the representation category of a quasi-Hopf algebra (see [EGNO15], Thm. 5.13.7). A canonical source of equivalences between representation categories of quasi-Hopf algebras are twists:

Definition 2.1.3. *[HN99a] Let H be a quasi-bialgebra, and $J \in H \otimes H$ an invertible element, s.t. $(\epsilon \otimes id)(J) = (id \otimes \epsilon)(J) = 1$. Given a twist, we can define a new quasi-bialgebra H^J which is H as an algebra, with the same counit, the coproduct is given by*

$$\Delta^J(x) = J\Delta(x)J^{-1},$$

and the associator given by

$$\phi^J = U_R \phi U_L^{-1},$$

where $U_L := (J \otimes 1)(\Delta \otimes id)(J)$ and $U_R := (1 \otimes J)(id \otimes \Delta)(J)$. Moreover, we have

$$\alpha^J = S(J^{(-1)})\alpha J^{(-2)} \qquad \beta^J = J^1 \beta S(J^2).$$

The quasi-bialgebra H^J is called twist equivalent to H, by the twist $J = J^1 \otimes J^2$. If a twist appears more then once in an equation, we use letters J,K,L,... for them.

Example 2.1.4. *Let G be a finite group, $\omega \in Z^3(G, \mathbb{C}^\times)$ be a group 3-cocycle. The dual algebra k^G equipped with the coassociator*

$$\phi = \sum_{g_1,g_2,g_3 \in G} \omega(g_1, g_2, g_3)\, \delta_{g_1} \otimes \delta_{g_2} \otimes \delta_{g_3} \qquad (\text{where } \delta_g(h) := \delta_{g,h})$$

is a quasi-Hopf algebra which we denote by k^G_ω.
The category of representations of k^G_ω is identified with the category Vect^ω_G of G-graded vector spaces with associator $\omega_{g_1,g_2,g_3} : (\mathbb{C}_{g_1} \otimes \mathbb{C}_{g_2}) \otimes \mathbb{C}_{g_3} \to \mathbb{C}_{g_1} \otimes (\mathbb{C}_{g_2} \otimes \mathbb{C}_{g_3})$ for simple objects $\mathbb{C}_{g_i}$ given by

$$\omega_{g_1,g_2,g_3} : 1_{g_1} \otimes 1_{g_2} \otimes 1_{g_3} \mapsto \omega(g_1, g_2, g_3) \cdot 1_{g_1} \otimes 1_{g_2} \otimes 1_{g_3}.$$

The following example is due to [DPR92]:

Example 2.1.5. *Again, let G be a finite group, $\omega \in Z^3(G, \mathbb{C}^\times)$ be a group 3-cocycle. Then there is a quasi-Hopf algebra $D^\omega(G)$ with a basis $g \otimes \delta_h$. The coassociator of this quasi-Hopf algebra is given by*

$$\phi = \sum_{g_1,g_2,g_3 \in G} \omega(g_1, g_2, g_3)\, (e \otimes \delta_{g_1}) \otimes (e \otimes \delta_{g_2}) \otimes (e \otimes \delta_{g_3}).$$

Product and coproduct are given by

$$(h_1 \otimes \delta_{g_1}) \cdot (h_2 \otimes \delta_{g_2}) = \delta_{h_1 g_1 h_1^{-1}, g_2} \frac{\omega(h_2, h_1, h_1^{-1} h_2^{-1} g_2 h_1 h_2)\omega(g_2, h_2, h_1)}{\omega(h_2, h_2^{-1} g_2 h_2, h_1)} (h_2 h_1 \otimes \delta_{g_2})$$
$$\Delta(h \otimes \delta_g) = \sum_{g_1 g_2 = g} \frac{\omega(h, h^{-1} g_1 h, h^{-1} g_2 h)\omega(g_1, g_2, h)}{\omega(g_1, h, h^{-1} g_2 h)} (h \otimes \delta_{g_1}) \otimes (h \otimes \delta_{g_2})$$

Moreover, there is a compatible R-matrix

$$R = \sum_{g \in G} (e \otimes \delta_g) \otimes (g \otimes 1_{k^G}),$$

such that the category of representations is a braided monoidal category.

Modules over $D^\omega(G)$ can be seen as Yetter-Drinfeld modules with a projective coaction instead of an ordinary coaction (see [Maj98], Prop. 2.2 or [Voc10], Def. 6.1). More generally the following universal construction has been estabilshed in [Sch02]:

Example 2.1.6. *Let H be a quasi-Hopf algebra, then there is a Drinfeld double DH which is again a quasi-Hopf algebra with R-matrix. For example in our first case $Dk^G_\omega = D^\omega(G)$.*
It has the universal property that the braided category of DH-representation is equivalent to the Drinfeld center of the monoidal category Rep_H of H-representations. Similarly, one can introduce Yetter-Drinfeld modules over a quasi-Hopf algebra, and this braided monoidal category is equivalent to the previous categories.

For later use, we introduce several important elements which were first defined in [Dri89]: He showed that for an arbitrary quasi-Hopf algebra there is an invertible element $f = f^1 \otimes f^2 \in H \otimes H$, which we refer to as Drinfeld twist, satisfying

$$f\Delta(S(h))f^{-1} = (S \otimes S)(\Delta^{cop}(h)). \tag{2.2}$$

Before we give f and f^{-1} explicitly, we follow [BN02] by defining elements

$$p_R = p^1 \otimes p^2 = x^1 \otimes x^2 \beta S(x^3) \qquad p_L = \tilde{p}^1 \otimes \tilde{p}^2 := X^2 S^{-1}(X^1\beta) \otimes X^3$$
$$q_R = q^1 \otimes q^2 = X^1 \otimes S^{-1}(\alpha X^3)X^2 \qquad q_L = \tilde{q}^1 \otimes \tilde{q}^2 := S(x^1)\alpha x^2 \otimes x^3,$$

satisfying the following useful equalities:

$$\Delta(h_{(1)})p_R(1 \otimes S(h_{(2)})) = p_R(h \otimes 1) \qquad \Delta(h_{(2)})p_L(S^{-1}(h_{(1)}) \otimes 1) = p_L(1 \otimes h) \tag{2.3}$$
$$(1 \otimes S^{-1}(h_{(2)}))q_R\Delta(h_{(1)}) = (h \otimes 1)q_R \qquad (S(h_{(1)}) \otimes 1)q_L\Delta(h_{(2)}) = (1 \otimes h)q_L. \tag{2.4}$$

Furthermore, we define elements

$$\delta = \delta^1 \otimes \delta^2 = x^1 \beta S(x^3_{(2)}\tilde{p}^2) \otimes x^2 S(x^3_{(1)}\tilde{p}^1) \tag{2.5}$$
$$\gamma = \gamma^1 \otimes \gamma^2 = S(q^2 x^1_{(2)})x^2 \otimes S(q^1 x^1_{(1)})\alpha x^3 \tag{2.6}$$

The Drinfeld twist f and its inverse f^{-1} are then explicitly given by:

$$\begin{aligned} f &= (S \otimes S)(\Delta^{cop}(p^1))\gamma\Delta(p^2) \\ f^{-1} &= \Delta(\tilde{q}^1)\delta(S \otimes S)(\Delta^{cop}(\tilde{q}^2)). \end{aligned} \tag{2.7}$$

In addition to Eq. 2.2, the Drinfeld twist satisfies

$$f\Delta(\alpha) = \gamma \qquad \Delta(\beta)f^{-1} = \delta.$$

Let $J \in H \otimes H$ be a twist. The respective elements on the twisted quasi-Hopf algebra H^J are denoted by $p_J, q_J, \delta_J, \ldots$. Using the above identities, one can show:

$$\begin{aligned}
p_{RJ} &= U_L^1 p^1 \otimes U_L^2 p_R^2 S(U_L^3) & p_{LJ} &= U_R^2 \tilde{p}^1 S^{-1}(U_R^1) \otimes U_R^3 \tilde{p}^2 \\
q_{RJ} &= q^1 U_L^{(-1)} \otimes S^{-1}(U_L^{(-3)}) q^2 U_L^{(-2)} & q_{LJ} &= S(U_R^{-1}) \tilde{q}^1 U_R^{(-2)} \otimes \tilde{q}^2 U_R^{(-3)} \\
f_J &= S(J^{(-2)}) f^1 K^{(-1)} \otimes S(J^{(-1)}) f^2 K^{(-2)} & f_J^{-1} &= J^1 f^{(-1)} S(K^2) \otimes J^2 f^{(-2)} S(K^1).
\end{aligned}$$

Finally, we introduce the notion of a quasitriangular ribbon quasi-Hopf algebra. We start with the notion of quasi-triangularity:

Definition 2.1.7. *A quasi-triangular quasi-Hopf algebra is a quasi-Hopf algebra H together with an invertible element $R \in H \otimes H$, the so-called R-matrix, s.t. the following conditions are fulfilled:*

$$\begin{aligned}
R\Delta(h) &= \Delta^{op}(h)R \qquad \forall\, h \in H \\
(\Delta \otimes id) &= \phi_{321} R_{13} \phi_{132}^{-1} R_{23} \phi \\
(id \otimes \Delta) &= \phi_{231}^{-1} R_{13} \phi_{213} R_{12} \phi^{-1}
\end{aligned}$$

The definition of a quasi-triangular quasi-Hopf algebra has an important symmetry: If $R = R^1 \otimes R^2 \in H \otimes H$ is an R-matrix for H, then so is $R_{21}^{-1} \in H \otimes H$.
The following lemma is proven in [BN03]:

Lemma 2.1.8. *Let (H, R) be a quasi-triangular quasi-Hopf algebra. We define the Drinfeld element $u \in H$ as follows:*

$$u := S(R^2 p^2) \alpha R^1 p^1.$$

This element is invertible and satisfies $S^2(h) = uhu^{-1}$.

We will need the following Lemma in Section 5.6:

Lemma 2.1.9. *Let H be a quasi-Hopf algebra. Assume that $\delta(S \otimes S)(\gamma_{21}) = \beta S(\alpha) \otimes \beta S(\alpha)$. Then we have*

$$\Delta(\beta S(\alpha)) = (\beta S(\alpha) \otimes \beta S(\alpha))\, (S \otimes S)(f_{21}^{-1}) f.$$

Proof. We have

$$\begin{aligned}
\Delta(\beta S(\alpha)) &= \Delta(\beta)\Delta(S(\alpha)) = \delta f \Delta(S(\alpha)) = \delta(S \otimes S)(\Delta^{cop}(\alpha)) f \\
&= \delta(S \otimes S)(f_{21}^{-1} \gamma_{21}) f = \delta(S \otimes S)(\gamma_{21})(S \otimes S)(f_{21}^{-1}) f \\
&= (\beta S(\alpha) \otimes \beta S(\alpha))\, (S \otimes S)(f_{21}^{-1}) f.
\end{aligned}$$

□

We now recall the definition of a ribbon quasi-Hopf algebra. It has been shown in [BN03] that the following definition of a quasi-triangular ribbon quasi-Hopf algebra is equivalent to the original one given in [AC92].

Definition 2.1.10. *A quasi-triangular quasi-Hopf algebra (H, R) is called ribbon if there exists a central element $\nu \in H$, s.t.*

$$\nu^2 = uS(u), \qquad S(\nu) = \nu,$$
$$\epsilon(\nu) = 1, \qquad \Delta(\nu) = (\nu \otimes \nu)(R_{21}R)^{-1}.$$

We will need the following Lemma in Section 5.6:

Lemma 2.1.11. *Let (H, R) be a quasi-triangular quasi-Hopf algebra with Drinfeld element $u \in H$. As in Lemma 2.1.9, we assume that $\delta(S \otimes S)(\gamma_{21}) = \beta S(\alpha) \otimes \beta S(\alpha)$ holds. Then $\nu := \beta S(\alpha)u$ satisfies the condition $\Delta(\nu) = (\nu \otimes \nu)(R_{21}R)^{-1}$.*

Proof. In [BN03], the authors prove the following identity:

$$\Delta(u) = f^{-1}(S \otimes S)(f_{21})(u \otimes u)(R_{21}R)^{-1}.$$

Using this and Lemma 2.1.9, we obtain

$$\begin{aligned}\Delta(\nu) &= \Delta(\beta S(\alpha)u) = \Delta(\beta S(\alpha))\Delta(u)\\ &= (\beta S(\alpha) \otimes \beta S(\alpha))\,(S \otimes S)(f_{21}^{-1})ff^{-1}(S \otimes S)(f_{21})(u \otimes u)(R_{21}R)^{-1} = (\nu \otimes \nu)(R_{21}R)^{-1}.\end{aligned}$$

□

2.2 Abelian 3-cocycles on G

From now on, let G be a finite abelian group.

Definition 2.2.1. *[Mac52][JS93][1] An abelian 3-cocycle $(\sigma, \omega) \in Z^3_{ab}(G)$ is a pair consisting of a ordinary 3-cocycle $\omega \in Z^3(G, \mathbb{C}^\times)$ and an ordinary 2-cochain $\sigma \in C^2(G, \mathbb{C}^\times)$, s.t. the following two equations hold:*

$$\frac{\omega(y,x,z)}{\omega(x,y,z)\omega(y,z,x)} = \frac{\sigma(x,y+z)}{\sigma(x,y)\sigma(x,z)} \tag{2.8}$$

$$\frac{\omega(z,x,y)\omega(x,y,z)}{\omega(x,z,y)} = \frac{\sigma(x+y,z)}{\sigma(x,z)\sigma(y,z)} \tag{2.9}$$

An abelian 3-coboundary is of the form $d_{ab}\kappa := (\kappa/\kappa^T, d\kappa^{-1})$ for any ordinary 2-cochain $\kappa \in C^2_{ab}(G) := C^2(G, \mathbb{C}^\times)$. Here, $\kappa^T(x,y) := \kappa(y,x)$. The quotient group of abelian cohomology classes is the abelian cohomology group $H^3_{ab}(G)$.

[1] Note that we follow [JS93] and thus have a different convention than [Mac52], this amounts to having ω^{-1} everywhere.

Theorem 2.2.2. *[Mac52] To any abelian 3-cocycle (ω, σ) there is an associated quadratic form $Q(g) := \sigma(g, g)$ on the group G and the associated symmetric bihomomorphism $\mathrm{B}(g, h) := \sigma(g, h)\sigma(h, g)$. We have an identity*

$$\mathrm{B}(g, h) = \frac{Q(x + y)}{Q(x)Q(y)}. \tag{2.10}$$

This implies, that the symmetric bihomomorphism B characterizes the quadratic form up to a homomorphism $\eta \in Hom(G, \{\pm 1\})$.
The assignment $\Phi : (\omega, \sigma) \mapsto Q$ yields a group isomorphism between abelian 3-cohomology classes $H^3_{ab}(G)$ and quadratic forms $QF(G)$ on G.

As we shall see in the next section, abelian cohomology classes classify different braiding/tensor structures on the category of G-graded vector spaces.

Example 2.2.3. *For $G = \mathbb{Z}_n$ we have two cases*

- *For odd n we have $H^3_{ab}(\mathbb{Z}_n) = \mathbb{Z}_n$ with representatives (ω, σ) for $k = 0, \ldots n - 1$ given by $\sigma(g^i, g^j) = \zeta_n^{kij}, \omega = 1$ and the respective quadratic form is given by*

$$Q(g^i) = \zeta_n^{ki^2}, \qquad \mathrm{B}(g^i, g^j) = \zeta_n^{2kij}$$

- *For even n we have $H^3_{ab}(\mathbb{Z}_n) = \mathbb{Z}_{2n}$ with quadratic forms for $k = 0, \ldots 2n - 1$*

$$Q(g^i) = \zeta_{2n}^{ki^2}, \qquad \mathrm{B}(g^i, g^j) = \zeta_{2n}^{2kij}$$

  For even k we have again representatives given by $\sigma(g^i, g^j) = \zeta_n^{(k/2)ij}, \omega = 1$, but for odd k we have only representatives with ω in the nontrivial cohomology class of $H^3(\mathbb{Z}_n, \mathbb{Z}_2)$.

In particular $G = \mathbb{Z}_4$ has four abelian cohomology classes, two of which have trivial ω, and two of which have nontrivial ω and nondegenerate B.

Proposition 2.2.4. *Let $G = \bigoplus_{i=1}^n \mathbb{Z}_{m_i}$ be a finite abelian group with generators g_i, $i = 1, \ldots n$.*

1. *A quadratic form $Q \in QF(G)$ is uniquely determined by elements $0 \leq r_i \leq \gcd(2, m_i)m_i - 1$ and $0 \leq r_{kl} \leq \gcd(m_k, m_l) - 1$ (for $k < l$), so that*

$$Q(g_i) = \exp\left(\frac{2\pi i \cdot r_i}{\gcd(2, m_i)m_i}\right) \qquad \mathrm{B}(g_k, g_l) = \exp\left(\frac{2\pi i \cdot r_{kl}}{\gcd(m_k, m_l)}\right) \quad (k < l).$$

2. *For a quadratic form $Q \in QF(G)$, the abelian 3-cocycle $(\omega, \sigma) \in Z^3_{ab}(G)$ given by*

$$\sigma(a, b) := \prod_{i=1}^n Q(g_i)^{a_i b_i} \prod_{1 \leq k < l \leq n} \mathrm{B}(g_k, g_l)^{a_k b_l}$$
$$\omega(a, b, c) := \prod_{i=1}^n Q(g_i)^{m_i \delta_{a_i + b_i \geq m_i} c_i}$$

   satisfies $Q(g) = \sigma(g, g)$.

Proof. From Theorem 2.2.2 it is easy to see that for an element $g = g_1^{x_1} \dots g_n^{x_n} \in G$, $Q(g)$ decomposes as follows:

$$Q(g) = \prod_{i=1}^{n} Q(g_i)^{x_i^2} \cdot \prod_{i<j} \mathrm{B}(g_k, g_l)^{x_k x_l}.$$

As B is a bihomomorphism on the finite abelian group G, we have

$$\mathrm{B}(g_k, g_l) = \exp\left(\frac{2\pi i \cdot r_{kl}}{\gcd(m_k, m_l)}\right)$$

for some $0 \le r_{kl} \le \gcd(m_k, m_l) - 1$. From this formula also follows

$$Q(g_i) = \exp\left(\frac{2\pi i \cdot \tilde{r}_i}{m_i^2}\right)$$

for some $0 \le \tilde{r}_i \le m_i^2 - 1$. Combining this with the axiom $Q(g_i) = Q(-g_i)$ leads to

$$Q(g_i) = \exp\left(\frac{2\pi i \cdot r_i}{\gcd(2, m_i) m_i}\right)$$

for some $0 \le r_i \le \gcd(2, m_i) m_i - 1$. It is a straightforward computation to check that the pair (ω, σ) defined in the second part of the proposition satisfies the axioms of an abelian 3-cocycle. Finally, using the above decomposition of $Q(g)$, it is easy to see that $\sigma(g, g) = Q(g)$. $\square$

2.3 Modular structures on Vect_G

Theorem 2.3.1. *Let $(\sigma, \omega) \in Z^3_{ab}(G)$ be an abelian 3-cocycle on the finite abelian group G. This induces a canonical braided monoidal structure on the category Vect_G of G-graded vector spaces. On simple objects $\mathbb{C}_{g_i}$, associator, unitors and braiding are given by:*

$$\begin{aligned}
\omega_{g_1,g_2,g_3} : (\mathbb{C}_{g_1} \otimes \mathbb{C}_{g_2}) \otimes \mathbb{C}_{g_3} \to \mathbb{C}_{g_1} \otimes (\mathbb{C}_{g_2} \otimes \mathbb{C}_{g_3}), &\qquad 1_{g_1} \otimes 1_{g_2} \otimes 1_{g_3} \mapsto \omega(g_1, g_2, g_3) \cdot 1_{g_1} \otimes 1_{g_2} \otimes 1_{g_3} \\
l_g : \mathbb{C}_0 \otimes \mathbb{C}_g \to \mathbb{C}_g, &\qquad 1_0 \otimes 1_g \mapsto \omega(0, 0, g)^{-1} \cdot 1_g \\
r_g : \mathbb{C}_g \otimes \mathbb{C}_0 \to \mathbb{C}_g, &\qquad 1_g \otimes 1_0 \mapsto \omega(g, 0, 0) \cdot 1_g \\
\sigma_{g_1,g_2} : \mathbb{C}_{g_1} \otimes \mathbb{C}_{g_2} \to \mathbb{C}_{g_2} \otimes \mathbb{C}_{g_1}, &\qquad 1_{g_1} \otimes 1_{g_2} \mapsto \sigma(g_1, g_2) \cdot 1_{g_1} \otimes 1_{g_2}.
\end{aligned}$$

The resulting braided monoidal category is denoted by $\mathsf{Vect}_G^{(\sigma,\omega)}$. All braided monoidal structures on Vect_G are classified up to braided monoidal equivalence by the third abelian cohomology group $H^3_{ab}(G)$ modulo automorphisms on G.

Proof. This is Exercise 8.4.8 in [EGNO15]. $\square$

Remark 2.3.2. *Since every 3-cocycle is equivalent to a normalized one, we can choose the unitors to be trivial.*

We recall the definition of a pre-modular category and modularization in the semisimple case:

Definition 2.3.3. *A fusion category is a rigid semisimple k-linear monoidal category $\mathcal{C}$ with only finitely many isomorphism classes of simple objects, such that $End(\mathbb{I}) \cong k$. A braided fusion category is called pre-modular if it has a ribbon structure, i.e. an element $\theta \in Aut(id_{\mathcal{C}})$, s.t.*

$$\begin{aligned}
&\theta_{X \otimes Y} = (\theta_X \otimes \theta_Y) \circ c_{Y,X} \circ c_{X,Y} \\
&(\theta_X)^* = \theta_{X^*},
\end{aligned}$$

where c denotes the braiding in $\mathcal{C}$.

Definition 2.3.4. *In a pre-modular category, we have categorical traces and categorical dimensions:*

$$\begin{aligned}
tr(f) &:= d_X \circ c_{X,X^*} \circ ((\theta_X \circ f) \otimes id_{X^*}) \circ b_X : \mathbb{I} \to \mathbb{I} \qquad f \in End(X) \\
dim(X) &:= tr(id_X) \qquad X \in \mathcal{C},
\end{aligned}$$

where d and b denote evaluation and coevaluation in the rigid category $\mathcal{C}$.

Definition 2.3.5. *Let $\mathcal{C}$ be a pre-modular category. The so-called S-matrix of $\mathcal{C}$, $\mathcal{S} = (\mathcal{S}_{XY})_{X,Y \in \mathcal{O}(\mathcal{C})}$, is indexed by the set $\mathcal{O}(\mathcal{C})$ of isomorphism classes of simple objects in $\mathcal{C}$ with entries defined by*

$$\mathcal{S}_{XY} := tr(c_{Y,X} c_{X,Y}).$$

A pre-modular category is said to be modular if its S-matrix is non-degenerate. A linear ribbon functor $F : \mathcal{C} \to \mathcal{D}$ between pre-modular categories is said to be a modularization if

1. *it is dominant, i.e. for every object $D \in \mathcal{D}$ we have $id_D = p \circ i$ for some $i : D \to F(C)$, $p : F(C) \to D$, $C \in \mathcal{C}$.*
2. *$\mathcal{D}$ is modular.*

If such a functor exist, then $\mathcal{C}$ is called modularizable.

Lemma 2.3.6. *For an abelian 3-cocycle $(\sigma, \omega) \in Z^3_{ab}(G)$, let $\mathsf{Vect}_G^{(\sigma,\omega)}$ be the corresponding braided monoidal category from Thm. 2.3.1. For a character $\eta : G \to \{\pm 1\}$, the twist $\Theta_g : \mathbb{C}_g \to \mathbb{C}_g$ given by multiplication with $Q(g) \cdot \eta(g)$ defines a ribbon structure on $\mathsf{Vect}_G^{(\sigma,\omega)}$. We denote the resulting ribbon catgegory by $\mathsf{Vect}_G^{(\sigma,\omega,\eta)}$. All ribbon structures on Vect_G up to ribbon category equivalence are classified by elements $(Q, \eta) \in QF(G) \oplus Hom(G, \{\pm 1\})$ modulo autmorphisms on G.*

Proof. We have already seen that braided monoidal structures on Vect_G are classified by $H^3_{ab}(G)$ modulo automorphisms on G. By Theorem 2.2.2 for every such class there is a unique quadratic form Q, which satisfies equation 2.10. This means that the quadratic form defines a ribbon structure. On the other hand, the theorem says that two quadratic forms have the same associated symmetric bihomomorphism $\frac{Q(x+y)}{Q(x)Q(y)}$ if and only if they differ by a homomorphism $\eta : G \to \{\pm 1\}$. From exercise 8.4.6 in [EGNO15] we know that a braided monoidal equivalence between $\mathsf{Vect}_G^{(\sigma_1,\omega_1)}$ and $\mathsf{Vect}_G^{(\sigma_2,\omega_2)}$ is uniquely determined by an automorphism $f : G \to G$ s.t. $Q_1 = Q_2 \circ f$ together with some choice of $\kappa : G \times G \to \mathbb{C}$, s.t. $d_{ab}\kappa = (\omega_1, \sigma_2)^{-1} f^*(\omega_2, \sigma_2)$. Given a ribbon structure $\eta_1 \in \mathrm{Hom}(G, \{\pm 1\})$ on $\mathsf{Vect}_G^{(\sigma_1,\omega_1)}$ and $\eta_2 = \eta_1 \circ f^{-1}$ on $\mathsf{Vect}_G^{(\sigma_2,\omega_2)}$ it is easy to see that this functor is a ribbon equivalence. $\square$

Remark 2.3.7. *By definition of the associated symmetric bilinear form* B *of* $(\sigma, \omega) \in Z^3_{ab}(G)$, *the pre-modular category* $\mathsf{Vect}_G^{(\sigma,\omega,\eta)}$ *is modular if and only if* B *is non-degenerate.*

Chapter 3. Modularization of $\mathsf{Vect}_G^{(\omega,\sigma)}$

Proposition 3.0.1. *For a given ribbon structure (σ,ω,η) on Vect_G, the pre-modular category $\mathsf{Vect}_G^{(\sigma,\omega,\eta)}$ is modularizable if and only if both Q and η are trivial on the radical $T := Rad(\mathrm{B})$ of the associated symmetric bilinear form $\mathrm{B} : G \times G \to \mathbb{C}^\times$ of $(\sigma,\omega) \in Z^3_{ab}(G)$. Explicitly, we construct a functor*

$$F : \mathsf{Vect}_G^{(\sigma,\omega,\eta)} \to \mathsf{Vect}_{G/T}^{(\bar\sigma,\bar\omega,\bar\eta)}$$

to a modular category $\mathsf{Vect}_{G/T}^{(\bar\sigma,\bar\omega,\bar\eta)}$, where the triple (σ,ω,η) on G factors to a triple $(\bar\sigma,\bar\omega,\bar\eta)$ on G/T.

Proof. By Paragraph 2.11.6 in [DGNO10] we have $dim(\mathbb{C}_g) = \eta(g)$. Then, the first part of the proposition is an easy application of theorem 3.1 in [Bru00]. We now construct the modularization functor explicitly:
Let $\mathsf{Vect}_G^{(\sigma,\omega,\eta)}$ be a pre-modular category satisfying the conditions in proposition 3.0.1.

We first want to find the modular target category of the modularization functor:
The condition $Q|_T = 1$ implies that Q factors to a well-defined quadratic form $\bar Q : G/T \to \mathbb{C}^\times$. Let $(\bar\sigma,\bar\omega) \in Z^3_{ab}(G/T)$ denote a representative of $\Phi^{-1}(\bar Q)$, where the isomorphism $\Phi : H^3_{ab}(G/T) \to QF(G/T)$ was introduced in theorem 2.2.2. Furthermore, since $\eta : G \to \mathbb{C}^\times$ was a character, $\eta|_T = 1$ implies that it factors through a character $\bar\eta : G/T \to \mathbb{C}^\times$. Hence, we obtain a pre-modular category $\mathsf{Vect}_{G/T}^{(\bar\sigma,\bar\omega,\bar\eta)}$. Since T was defined as the radical of B, the new associated symmetric form $\bar{\mathrm{B}} : G/T \times G/T \to \mathbb{C}^\times$ is non-degenerate and by remark 2.3.7, $\mathsf{Vect}_{G/T}^{(\bar\sigma,\bar\omega,\bar\eta)}$ is even modular.

Now, we need to construct a linear ribbon functor $F : \mathsf{Vect}_G^{(\sigma,\omega,\eta)} \to \mathsf{Vect}_{G/T}^{(\bar\sigma,\bar\omega,\bar\eta)}$:
Clearly, the projection $\pi : G \to G/T$ induces a functor $\mathsf{Vect}_G \to \mathsf{Vect}_{G/T}$. We want to endow this functor with a monoidal structure that is compatible with braiding and twist. This amounts to finding a 2-cochain $\kappa : G \times G \to \mathbb{C}$, s.t.

$$\pi^*(\bar\sigma,\bar\omega) = d_{ab}\kappa \cdot (\sigma,\omega).$$

Since the associated quadratic form of the abelian 3-cocycle $(\tilde\sigma,\tilde\omega) = (\sigma,\omega)^{-1}\pi^*(\bar\sigma,\bar\omega)$ vanishes by assumption and since Φ is an isomorphism, $(\tilde\sigma,\tilde\omega)$ must be an abelian coboundary, hence κ exists.

Compatibility with the twist is due to $\bar{\eta} \circ \pi = \eta$ and $\bar{Q} \circ \pi = Q$. This functor is clearly dominant, since it sends simple objects to simple objects and all simple are in the image. $\square$

Remark 3.0.2. *In [Bru00], the modularised category is constructed as the category of modules of a commutative algebra $\mathcal{T}$ inside the non-modular category $\mathcal{C}$. As an object, $\mathcal{T}$ is the direct sum of all transparent objects. We remark that in our case $\mathcal{C} = \mathsf{Vect}_G^{(\sigma,\omega,\eta)}$ our explicit modularization functor and our modularised category is equivalent to Brugieres' construction for $\mathcal{T} := \oplus_{t \in T} \mathbb{C}_t$.*

Chapter 4. Quantum groups $u_q(\mathfrak{g}, \Lambda)$ and R-matrices

In this section, we recall the some results from the first part. To begin with, we collect some data in order to define the small quantum group $u_q(\mathfrak{g}, \Lambda)$:
In the following, let

- $\mathfrak{g}$ be a simple complex finite-dimensional Lie algebra with simple roots $\alpha_1, \ldots, \alpha_n$ and Killing form (α_i, α_j),
- q be a primitive ℓth root of unity, where $\ell \in \mathbb{N}$,
- $\Lambda_R \subseteq \Lambda \subseteq \Lambda_W$ be an intermediate lattice between the root lattice Λ_R and weight lattice Λ_W of $\mathfrak{g}$, equivalently a subgroup of the fundamental group $H \subseteq \pi_1 := \Lambda_W/\Lambda_R$,
- Λ' be the centralizer of the root lattice with respect to Λ,

  $$\Lambda' = \mathrm{Cent}_{\Lambda_R}(\Lambda) := \{\alpha \in \Lambda_R \mid q^{(\alpha,\nu)} = 1 \quad \forall \nu \in \Lambda\}$$

  and the quotient group $G := \Lambda/\Lambda'$,
- $\Lambda_R \subseteq \Lambda_1, \Lambda_2 \subseteq \Lambda$ sublattices, equivalently subgroups $G_i := \Lambda_i/\Lambda' \subseteq G$ of common index $d := |G_i|$, s.t. $G_1 + G_2 = G$. Note that in all cases except $\mathfrak{g} = D_{2n}$, we have a cyclic fundamental group and thus $\Lambda_1 = \Lambda_2 = \Lambda$ and $G_1 = G_2 = G$,
- $f : G_1 \times G_2 \to \mathbb{C}^\times$ be a non-degenerate bilinear form,
- $H_i := \Lambda_i/\Lambda \subseteq H$.

Theorem 4.0.1. *[LO17] For the above data, let $u_q(\mathfrak{g}, \Lambda)$ be the so-called small quantum group with coradical $u_q^0(\mathfrak{g}, \Lambda) \cong \mathbb{C}[G]$, as defined for example in [Len16], Def. 5.3. In most cases, this will be isomorphic to the Frobenius-Lusztig kernel.*
Then, an R-matrix for this quantum group is given by

$$R = R_0(f)\bar{\Theta},$$

where $\Theta \in u_q^-(\mathfrak{g},\Lambda) \otimes u_q^+(\mathfrak{g},\Lambda)$ is the universal quasi-R-matrix constructed by Lusztig (see [Lus93], Thm. 4.1.2.) and $R_0(f)$ is given by

$$R_0(f) := \frac{1}{d} \sum_{\mu \in G_1 \nu \in G_2} f(\mu,\nu) K_\mu \otimes K_\nu \in u_q^0(\mathfrak{g},\Lambda) \otimes u_q^0(\mathfrak{g},\Lambda).$$

The non-degenerate bilinear form $f : G_1 \times G_2 \to \mathbb{C}^\times$ is of the explicit form

$$f(\mu,\nu) = q^{-(\mu,\nu)} \cdot g(\bar{\mu},\bar{\nu}), \tag{4.1}$$

where $g : H_1 \times H_2 \to \mathbb{C}^\times$ is another bilinear form.
Moreover, every small quantum group $u_q(\mathfrak{g},\Lambda)$ with R-matrix of the form $R_0(f)\bar{\Theta}$ admits a ribbon element of the form $\nu = \gamma^{-1}u$, where u denotes the Drinfeld element in $u_q(\mathfrak{g},\Lambda)$ and γ is a spherical pivotal element in $u_q(\mathfrak{g},\Lambda)$.

In [LO17], we listed all possible bilinear forms g, s.t. the corresponding bilinear form f is non-degenerate. Furthermore, we gave necessary and sufficient conditions on f for the corresponding R-matrix $R = R_0(f)\bar{\Theta}$ to be factorizable and checked again explicitly when this will be the case.

Remark 4.0.2. *The element $R_0(f) \in \mathbb{C}[G] \otimes \mathbb{C}[G]$ itself is an R-matrix of the group algebra $\mathbb{C}[G]$, leading to a braiding σ in the category $\mathsf{Rep}_{\mathbb{C}[G]} \cong \mathsf{Rep}G \cong \mathsf{Vect}_{\widehat{G}}$ which is defined on simple objects (i.e. characters) $\mathbb{C}_\chi, \mathbb{C}_\psi \in \mathsf{Vect}_{\widehat{G}}$ by*

$$\begin{aligned}\sigma_{\chi,\psi}(1_\chi \otimes 1_\psi) &= \left(\frac{1}{d} \sum_{\mu \in G_1 \nu \in G_2} f(\mu,\nu)\chi(\mu)\psi(\nu)\right) \cdot 1_\psi \otimes 1_\chi \\ &= \chi\left(f^{-1}(\bar{\psi}|_{G_2})\right) \cdot 1_\psi \otimes 1_\chi \\ &= \sigma(\chi,\psi) \cdot 1_\psi \otimes 1_\chi,\end{aligned}$$

where in the second line the bicharacter $f : G_1 \times G_2 \to \mathbb{C}^\times$ is interpreted as a homomorphism $G_1 \to \widehat{G_2}$. From this, it is clear that σ is a bilinear form on $\widehat{G}$.

So far, we introduced the small quantum group $u_q(\mathfrak{g},\Lambda)$ associated to a simple Lie algebra $\mathfrak{g}$, an intermediate lattice $\Lambda_R \subseteq \Lambda \subseteq \Lambda_W$ and an ℓth root of unity q with R-matrix induced by a bilinear form f. We now look at the explicit case $\mathfrak{g} = \mathfrak{sl}_2$, $\Lambda_1 = \Lambda_2 = \Lambda_W$.

Example 4.0.3.
- *The Cartan part $u_q(\mathfrak{sl}_2,\Lambda_W)_0$ of this quantum group is given by $\mathbb{C}[G] = \mathbb{C}[\Lambda_W/Cent_{\Lambda_R}(\Lambda_W)]$. Since $\Lambda_R = 2\Lambda_W = \mathbb{Z}\cdot\alpha$, we have $G \cong \mathbb{Z}_{2\ell}$.*
- *For the defining bilinear form $f : G \times G \to \mathbb{C}^\times$ of the R_0-matrix we have two possibilities, namely $f_\pm(\lambda,\lambda) = \pm q^{-(\lambda,\lambda)} = \pm\exp(\frac{-\pi i}{\ell})$, where $\lambda = [\frac{\alpha}{2}]$ is a generator of G. From table 1 in [LO17], we see that in the case $2 \nmid \ell$ only f_+ is non-degenerate. In the even case, both choices are allowed.*

- *The radical $Rad(f \cdot f^T) \subseteq G$ is given by $\ell\Lambda_W/\ell\Lambda_R \cong \mathbb{Z}_2$. Dualizing the representation category of $\mathbb{C}[G]$ with braiding induced by $f_\pm$ leads to the braided monoidal category $\mathsf{Vect}_{\widehat{G}}^{(\sigma_\pm,1)}$, where $\sigma_\pm(\chi,\chi) := f_\pm(\lambda,\lambda)^{-1}$, where $\chi := f(\lambda,\_)$ is a generator of $\widehat{G}$. We always use the non-degenerate form f to identify G and $\widehat{G}$. In particular, the radical $T := Rad(\mathrm{B}) \subseteq \widehat{G}$ is isomorphic to, but not equal to the dual of $Rad(f \cdot f^T)$. It is generated by $\tau := \chi^\ell$.*

- *We now want to check, when the conditions for modularizability given in proposition 3.0.1 are satisfied. It is easy to see that the corresponding quadratic form $Q_\pm(\chi) = \sigma_\pm(\chi,\chi)$ is trivial on T if and only if $Q_\pm = Q_{(-1)^\ell}$. Combined with the non-degeneracy condition from above, this excludes the case $2 \nmid \ell$. From now on, we restrict to the case $2 \mid \ell$ and $f = f_+$. Here, both possibilities $\eta_\pm(\chi) = \pm 1$ are allowed. We are now looking for an explicit abelian 3-cocycle $(\bar{\sigma},\bar{\omega}) \in Z^3_{ab}(\widehat{G}/T)$ corresponding to the pushed down quadratic form $\bar{Q}_+(\bar{\chi}) = Q_+(\chi)$ on $\widehat{G}/T$. It turns out that the following definition does the job:*

$$\bar{\omega}(\bar{\chi}^i,\bar{\chi}^j,\bar{\chi}^k) := q^{\frac{i(j+k-[j+k])}{2}} \qquad 0 \le i,j,k \le \ell-1$$
$$\bar{\sigma}(\bar{\chi}^i,\bar{\chi}^j) := q^{\frac{ij}{2}} \qquad 0 \le i,j \le \ell-1.$$

  It is immediately clear, that $\bar{\sigma}$ won't be bilinear anymore. For further use, we introduce a 2-cochain

$$\zeta_t(\bar{\chi}^i,\bar{\chi}^j) := \begin{cases} q^{\frac{-tj}{2}} & \text{if } i \text{ odd} \\ 1 & \text{else} \end{cases} \qquad 0 \le i,j \le \ell-1, \quad 2 \nmid t,$$

  leading to an equivalent abelian 3-cocycle $(\bar{\omega}_t,\bar{\sigma}_t) = (d\zeta_t\bar{\omega}, \frac{\zeta_t}{\zeta_t^T}\bar{\sigma})$:

$$\bar{\omega}_t(\bar{\chi}^i,\bar{\chi}^j,\bar{\chi}^k) := \begin{cases} q^{-tk} & \text{if } i,j \text{ odd} \\ 1 & \text{else} \end{cases} \qquad 0 \le i,j,k \le \ell-1$$
$$\bar{\sigma}_t(\bar{\chi}^i,\bar{\chi}^j) := q^{\frac{(i-\delta_{2\nmid i}t)(j+\delta_{2\nmid j}t)+t^2}{2}} \qquad 0 \le i,j \le \ell-1.$$

- ***Summarizing:** For $\Lambda = \Lambda_W$, we have for ℓ odd a single R-matrix, which is not modularizable and for ℓ even, we have two R-matrices, one of which is not modularizable and one of which modularizes to a modular tensor category with $|\mathbb{Z}_\ell| = \ell$ many simple objects. We have two choices for the ribbon structure in this category.*

Chapter 5. Definition of a quasi-Hopf algebra $u(\omega, \sigma)$

Let k be an algebraically closed field of characteristic zero. In this section we construct a quasi-Hopf algebra $u(\omega, \sigma)$ from the following data:

- a finite abelian group G
- an abelian 3-cocycle $(\omega, \sigma) \in Z^3_{ab}(\widehat{G})$ on its dual
- a subset $\{\, \chi_i \in \widehat{G}\,\}_{1 \leq i \leq n} \subseteq \widehat{G}$

The following theorem summarizes the results of this chapter:

Theorem 5.0.1. *Given the above data, there is a quasi-Hopf algebra $u(\omega, \sigma)$ with the following properties:*

1. *The quasi-Hopf algebra $u(\omega, \sigma)$ contains $k^{\widehat{G}}_\omega$ (see Ex. 2.1.4) as a quasi-Hopf subalgebra. We introduce the elements*

$$K_\chi := \sum_{\psi \in \widehat{G}} \sigma(\chi, \psi)\, \delta_\psi \in k^{\widehat{G}}$$
$$\bar{K}_\chi := \sum_{\psi \in \widehat{G}} \sigma(\psi, \chi)\, \delta_\psi \in k^{\widehat{G}}$$

   They are grouplike if and only if the 2-cocycle $\theta(\chi) \in Z^2(\widehat{G})$ defined in 5.1.5 is trivial.

2. *Let V be the k-vector space spanned by basis elements $\{F_i\}_{1 \leq i \leq n}$ and endowed with the Yetter-Drinfeld module structure over $k^{\widehat{G}}_\omega$ from Cor. 5.1.4. In particular, action, coaction and braiding are given by:*

$$L.F_i := \bar{\chi}_i(L) \qquad L \in G \cong \widehat{\widehat{G}} \subseteq k^{\widehat{G}},$$
$$\delta(F_i) := L_i \otimes F_i, \qquad L_i := K_{\bar{\chi}_i}$$
$$c_{V,V}(F_i \otimes F_j) := q_{ij} F_j \otimes F_i, \qquad q_{ij} := \sigma(\bar{\chi}_i, \bar{\chi}_j).$$

   Let $B(V)$ denote the corresponding Nichols algebra (see Section 5.2). We assume $B(V)$ to be finite-dimensional. The quasi-Hopf algebra $u(\omega, \sigma)$ contains the Radford biproduct $u(\omega, \sigma)^{\leq} :=$

$B(V)\#k^{\widehat{G}}_\omega$ (see Section 5.3) as a quasi-Hopf subalgebra. Explicit relations are given in Cor. 5.2.4 and Prop. 5.3.1. Under the assumption from Def. 5.0.3, they simplify considerably (see Ex. 5.0.5).

3. *The quasi-Hopf algebra $u(\omega,\sigma)$ is a quotient of the Drinfeld double $D(u(\omega,\sigma)^{\leq})$ (see Section 5.4.2) of the Radford biproduct $u^{\leq 0}$* [1]. *After defining certain elements $E_i \in \left(u^{\leq 0}\right)^*$ the quasi-quantum group $u(\omega,\sigma)$ is generated by E_i's, F_j's and δ_χ's. Note that the elements K_χ do not necessarily form a basis of $k^{\widehat{G}}$. A full list of relations is given at the end of Section 5.5.*

4. *The finite-dimensional quasi-Hopf algebra $u(\omega,\sigma)$ has a canonical quasi-triangular structure defined in Section 5.6.*

5. *As a vector space, we have $u(\omega,\sigma) \cong B(V) \otimes kG \otimes B(V^*)$.*

Example 5.0.2. *The reader familiar with ordinary quantum groups at roots of unity would certainly expect such a construction of a "quasi-quantum group", since up to the up to the technicalities of quasi-Hopf algebras it is based on the construction of quantum groups as Drinfeld doubles of Nichols algebras [AS02].*

Definition 5.0.3. *For a given datum $(\omega,\sigma,\chi_i \in \widehat{G})$ as above, the abelian 3-cocycle $(\omega,\sigma) \in Z^3_{ab}(\widehat{G})$ is called nice if the following two conditions are fulfilled:*

$$\omega(\chi_i,\chi_j,\psi) = \omega(\chi_j,\chi_i,\psi) \qquad \forall\, \psi \in \widehat{G}$$
$$\omega(\bar{\chi}_i,\chi_i,\psi) = 1 \qquad \forall\, \psi \in \widehat{G}.$$

Lemma 5.0.4. *Let $G = \bigoplus_{i=1}^n \mathbb{Z}_{m_i}$ be a finite abelian group with generators $\mathfrak{g}_i$, $i = 1,\dots n$. Every abelian 3-cocycle $(\omega,\sigma) \in Z^3_{ab}(G)$ is cohomologous to a nice abelian 3-cocycle. An explicit representative is given by:*

$$\omega(a,b,c) := \prod_{i=1}^n Q(g_i)^{m_i \delta_{a_i+b_i \geq m_i} c_i}.$$

Proof. This is the second part of Prop. 2.2.4. □

Example 5.0.5. *Let $(\omega,\sigma) \in Z^3_{ab}(\widehat{G})$ be a nice abelian 3-cocycle in the sense of Def. 5.0.3. Then, the quasi-Hopf algebra $u(\omega,\sigma)$ is generated by elements E_i, F_i and δ_χ for $1 \leq i \leq n$ and $\chi \in \widehat{G}$. We have the same quasi-Hopf algebra relations as for an arbitrary abelian 3-cocycle, except that the braided commutator simplifies to:*

$$[E_i K_{\chi_i}, F_j]_\sigma = \delta_{ij}\sigma(\chi_i,\bar{\chi}_i)\left(1 - \bar{K}_{\chi_i} K_{\chi_i}\right).$$

[1] In contrast to the $(\omega = 1)$-case, $D(u(\omega,\sigma)^{\leq})$ contains $uu(\omega,\sigma)^{\leq}$ as a quasi-Hopf subalgebra, but not $(u(\omega,\sigma)^{\leq})^*$, which is a coquasi-Hopf algebra

Example 5.0.6. *We now fix a datum $(\mathfrak{g}, q, \Lambda, \Lambda_1, \Lambda_2, \Lambda', f)$ as in Section 4. On $\widehat{G} := \widehat{\Lambda/\Lambda'}$, we define a bihomomorphism σ as in Remark 4.0.2 and set $\omega = 1$. Moreover, we define $\chi_i := q^{(\alpha_i, \_)}$ for a choice of simple roots $\alpha_i \in \Lambda_R$. From Ex. 5.0.5 it is easy to see that in this case, $u(\omega, \sigma)$ turns out to be the ordinary extended small quantum group $u_q(\mathfrak{g}, \Lambda)$ as introduced in Section 4.*

We now construct the quasi-Hopf algebra from the previous Theorem step-by-step, starting with the Yetter-Drinfeld module V over the quasi-Hopf algebra $k_\omega^{\widehat{G}}$.

5.1 A Yetter-Drinfeld module

We start with the definition of a Yetter-Drinfeld module over a quasi-Hopf algebra H.

Definition 5.1.1. *[Maj98][Sch02] Let H be a quasi-Hopf algebra. Let $\rho : H \otimes V \to V$ be a left H-module and let $\delta : V \to H \otimes V$, $v \mapsto v_{[-1]} \otimes v_{[0]}$ be a linear map, s.t.*

1. $(\epsilon \otimes id) \circ \delta = id$
2. $X^1(Y^1.v)_{[-1](1)}Y^2 \otimes X^2(Y^1.v)_{[-1](2)}Y^3 \otimes X^3.(Y^1.v)_{[0]}$
   $= X^1 v_{[-1]} \otimes (X^2.v_{[0]})_{[-1]}X^3 \otimes (X^2.v_{[0]})_{[0]}$
3. $h_{(1)}v_{[-1]} \otimes h_{(2)}v_{[0]} = (h_{(1)}.v)_{[-1]}h_{(2)} \otimes (h_{(1)}.v)_{[0]}$,

where $\phi = X^1 \otimes X^2 \otimes X^3 = Y^1 \otimes Y^2 \otimes Y^3$ denotes the associator of H. Then, the triple (V, ρ, δ) is called a Yetter-Drinfeld module over H.

Obviously, for $\phi = 1 \otimes 1 \otimes 1$ this matches the usual definition of a Yetter-Drinfeld module. As in this case, we have the following:

Proposition 5.1.2. *[Maj98] [BN02] Let H be a quasi-Hopf algebra. The category ${}^H_H\mathcal{YD}$ of Yetter-Drinfeld modules over H is a braided monoidal category, with usual tensor product $V \otimes W$ of H-modules $V, W \in {}^H_H\mathcal{YD}$. The comodule structure on $V \otimes W$ is given by*

$$\begin{aligned}\delta_{V\otimes W}(v \otimes w) = &X^1(x^1Y^1 \cdot v)_{[-1]}x^2(Y^2 \cdot v)_{[-1]}X^3 \\ &\otimes X^2 \cdot (x^1Y^1 \cdot v)_{[0]} \otimes X^3x^3(Y^2 \cdot v)_{[0]}.\end{aligned}$$

The associator in ${}^H_H\mathcal{YD}$ is the same as in Rep_H and the braiding given by

$$c_{V,W}(v \otimes w) = v_{[-1]}.w \otimes v_{[0]}.$$

If we plug in the data of the twisted dual group algebra k_ω^G from example 2.1.4, the above definition simplifies significantly [Maj98]:

Lemma 5.1.3. *For A a finite abelian group, a Yetter-Drinfeld module over k_ω^G consists of the following data:*

- *a A-graded vector space $V = \bigoplus_{a\in A} V_a$ with basis $\{F_i\}_{i\in I}$ in degrees $a_i := |F_i|$*
- *a map $\rho : kA \otimes V \to V$, $a \otimes F_i \mapsto a.F_i$, s.t.*

$$a_1.(a_2.F_i) = \frac{\omega(a_i,a_1,a_2)\omega(a_1,a_2,a_i)}{\omega(a_1,a_i,a_2)}(a_1+a_2).F_i, \qquad F_i = 0.F_i, \qquad |a.F_i| = |F_i|. \tag{5.1}$$

The first condition in the previous lemma is very similar to the defining relations of an abelian 3-cocycle (see 2.2.1). This implies

Corollary 5.1.4. *Let $(\sigma,\omega) \in Z^3_{ab}(\widehat{G})$ be an abelian 3-cocycle and $\{\chi_i\}_{i\in I} \subseteq \widehat{G}$ a subset. Then, setting $V := \bigoplus_{i\in I} k \cdot F_i$, with homogeneous degrees $|F_i| = \bar{\chi}_i$ and action*

$$\begin{aligned} \rho :\ & k\widehat{G} \otimes V \longrightarrow V \\ & \chi \otimes F_i \longmapsto \sigma(\bar{\chi}_i,\chi)\, F_i, \end{aligned}$$

indeed defines a Yetter-Drinfeld module over $k^{\widehat{G}}_\omega$. Note that substituting σ by $(\sigma^T)^{-1}$ would also work.

Remark 5.1.5. *Note that a dual of V is given by $V^\vee = V^* = \bigoplus_{i\in I} k \cdot F_i^*$, with homogeneous degrees $|F_i| = \chi_i$ and action*

$$\rho^\vee :\ \chi \otimes F_i^* \mapsto \theta(\bar{\chi})(\chi\chi_i,\bar{\chi}_i) d\sigma(\chi_i,\chi,\bar{\chi})^{-1}\sigma(\bar{\chi}_i,\bar{\chi})F_i^* = \sigma(\chi_i,\chi)F_i^*,$$

where $\theta(\chi)(\psi_1,\psi_2) \in Z^2(G,k^G)$ is the 2-cocycle defined by

$$\theta(\chi)(\psi_1,\psi_2) := \frac{\omega(\chi,\psi_1,\psi_2)\omega(\psi_1,\psi_2,\chi)}{\omega(\psi_1,\chi,\psi_2)}.$$

The evaluation $V^\vee \otimes V \to k$ is given by $F_i^\vee \otimes F_j \mapsto \delta_{i,j}$.

5.2 A Nichols algebra

We now give relations for the corresponding Nichols algebra $B(V) \in {}^{k^G_\omega}_{k^G_\omega}\mathcal{YD}$ of the Yetter-Drinfeld module V constructed in Corollary 5.1.4.
As in the Hopf-case (see [Hec08]), the Nichols algebra $B(V) = \bigoplus_{n\geq 0} B^n V := T(V)/I$ of V is a quotient of the tensor (Hopf-)algebra $T(V) := \bigoplus_{n\geq 0} T^n V$ in ${}^{k^G_\omega}_{k^G_\omega}\mathcal{YD}$ by the maximal Hopf ideal I, s.t. $B^1V := V$ are exactly the primitive elements in $B(V)$. A brief introduction to Nichols algebras in arbitrary abelian braided monoidal categories is given in App. B. Details can be found in [BB13]. Since we are dealing with a non-trivial associator, we have to fix a bracketing $T^nV := T^{n-1}V \otimes V$ and $T^0V := k$. Accordingly, we define $F_i^n := F_i^{n-1}F_i$ for primitive generators $F_i \in V \subseteq B(V)$. In order to compare our results with [Ros98], we use the short-hand notation $q_{ij} := \sigma(\bar{\chi}_i,\bar{\chi}_j)$. In particular, we will see that the resulting relation for the adjoint representation on $B(V)$ depends only on q_{ij} and not on the 3-cocycle ω.

Lemma 5.2.1. *For $k,l \in \mathbb{Z}_{\geq 0}$, let $a^i_{k,l} \in k^\times$ denote the elements defined by $F_i^k F_i^l = a^i_{k,l} F_i^{k+l}$. They are given explicitly by $a^i_{k,l} := \prod_{r=0}^{l-1} \omega(\bar{\chi}_i^k, \bar{\chi}_i^r, \bar{\chi}_i)^{-1}$ and satisfy the following identities:*

1. $\frac{a^i_{k,l+m} a^i_{l,m}}{a^i_{k+l,m} a^i_{k,l}} = \omega(\bar{\chi}_i^k, \bar{\chi}_i^l, \bar{\chi}_i^m)^{-1}$
2. $a^i_{k,l} = \frac{\sigma(\bar{\chi}_i^k, \bar{\chi}_i^l)}{q_{ii}^{kl}} a^i_{l,k}$.

Proof. We prove the first part by induction in m. The case $m=0$ is trivial. For

$$
\begin{aligned}
\frac{a^i_{k,l+m+1} a^i_{l,m+1}}{a^i_{k+l,m+1} a^i_{k,l}} &= \frac{a^i_{k,l+m} a^i_{l,m}}{a^i_{k+l,m} a^i_{k,l}} \frac{\omega(\bar{\chi}_i^{l+k}, \bar{\chi}_i^m, \bar{\chi}_i)}{\omega(\bar{\chi}_i^k, \bar{\chi}_i^{l+m}, \bar{\chi}_i)\omega(\bar{\chi}_i^l, \bar{\chi}_i^m, \bar{\chi}_i)} \\
&= \frac{\omega(\bar{\chi}_i^{k+l}, \bar{\chi}_i^m, \bar{\chi}_i)}{\omega(\bar{\chi}_i^k, \bar{\chi}_i^l, \bar{\chi}_i^m)\omega(\bar{\chi}_i^k, \bar{\chi}_i^{l+m}, \bar{\chi}_i)\omega(\bar{\chi}_i^l, \bar{\chi}_i^m, \bar{\chi}_i)} \\
&= \omega(\bar{\chi}_i^k, \bar{\chi}_i^l, \bar{\chi}_i^{m+1})^{-1}.
\end{aligned}
$$

For the second part, we compute:

$$
\begin{aligned}
a_{k+1,l} &= \prod_{r=0}^{l-1} \omega(\bar{\chi}_i^{k+1}, \bar{\chi}_i^r, \bar{\chi}_i)^{-1} \\
&= \prod_{r=0}^{l-1} \frac{\omega(\bar{\chi}_i, \bar{\chi}_i^k, \bar{\chi}_i^{r+1})}{\omega(\bar{\chi}_i, \bar{\chi}_i^k, \bar{\chi}_i^r)\omega(\bar{\chi}_i^k, \bar{\chi}_i^r, \bar{\chi}_i)\omega(\bar{\chi}_i, \bar{\chi}_i^{k+r}, \bar{\chi}_i)} \\
&= \omega(\bar{\chi}_i, \bar{\chi}_i^k, \bar{\chi}_i^l) a_{k,l} \prod_{r=0}^{l-1} \omega(\bar{\chi}_i, \bar{\chi}_i^{k+r}, \bar{\chi}_i)^{-1} \\
&= \omega(\bar{\chi}_i, \bar{\chi}_i^k, \bar{\chi}_i^l) a_{k,l} \prod_{r=0}^{l-1} \frac{\sigma(\bar{\chi}_i^{k+r}, \bar{\chi}_i)\sigma(\bar{\chi}_i, \bar{\chi}_i)}{\sigma(\bar{\chi}_i^{k+r+1}, \bar{\chi}_i)} \\
&= \omega(\bar{\chi}_i, \bar{\chi}_i^k, \bar{\chi}_i^l) a_{k,l} \frac{\sigma(\bar{\chi}_i^k, \bar{\chi}_i) q_{ii}^l}{\sigma(\bar{\chi}_i^{k+l}, \bar{\chi}_i)}.
\end{aligned}
$$

We want to prove the second part by induction in l. For $l=1$, we obtain

$$
a^i_{1,k} = \prod_{r=0}^{k-1} \omega(\bar{\chi}_i, \bar{\chi}_i^r, \bar{\chi}_i)^{-1} = \prod_{r=0}^{k-1} \frac{\sigma(\bar{\chi}_i^r, \bar{\chi}_i)\sigma(\bar{\chi}_i, \bar{\chi}_i)}{\sigma(\bar{\chi}_i^{r+1}, \bar{\chi}_i)} = \frac{q_{ii}^k}{\sigma(\bar{\chi}_i^k, \bar{\chi}_i)} \cdot 1 = \frac{q_{ii}^k}{\sigma(\bar{\chi}_i^k, \bar{\chi}_i)} a^i_{k,1}.
$$

For $l+1$ we obtain

$$
\begin{aligned}
a^i_{k,l+1} &= a^i_{k,l} \omega(\bar{\chi}_i^k, \bar{\chi}_i^l, \bar{\chi}_i)^{-1} \\
&= \frac{\sigma(\bar{\chi}_i^k, \bar{\chi}_i^l)}{q_{ii}^{kl}} \omega(\bar{\chi}_i^k, \bar{\chi}_i^l, \bar{\chi}_i)^{-1} a^i_{l,k} \\
&= \frac{\sigma(\bar{\chi}_i^k, \bar{\chi}_i^l)}{q_{ii}^{kl}} \omega(\bar{\chi}_i^k, \bar{\chi}_i^l, \bar{\chi}_i)^{-1} \omega(\bar{\chi}_i, \bar{\chi}_i^l, \bar{\chi}_i^k)^{-1} \frac{\sigma(\bar{\chi}_i^{k+l}, \bar{\chi}_i)}{\sigma(\bar{\chi}_i^l, \bar{\chi}_i) q_{ii}^k} \\
&= \frac{\sigma(\bar{\chi}_i^{l+1}, \bar{\chi}_i^k)}{q_{ii}^{k(l+1)}}.
\end{aligned}
$$

In the last line, we used the identity $\omega \cdot \omega^T = d\sigma^{-1}$. $\square$

Lemma 5.2.2. *Let $ad_c(X)(Y) = \mu \circ (id - c)(X \otimes Y)$ be the adjoint representation of the associative algebra $B(V) \in {}^{k^G_\omega}_{k^G_\omega}\mathcal{YD}$. We have*

$$ad_c^n(F_i)(F_j) = \sum_{k=0}^{n} \mu_n(k) \left(F_i^k F_j\right) F_i^n - k \text{ , where}$$

$$\mu_n(k) = (-1)^{n-k}\sigma(\bar{\chi}_i^{n-k}, \bar{\chi}_j)\frac{\omega(\bar{\chi}_i^k, \bar{\chi}_i^{n-k}, \bar{\chi}_j)}{\omega(\bar{\chi}_i^k, \bar{\chi}_j, \bar{\chi}_i^{n-k})} q_{ii}{}^{\binom{n-k}{2} - \binom{n}{2}} \prod_{r=0}^{n} \frac{\sigma(\bar{\chi}_i, \bar{\chi}_i^r)}{\omega(\bar{\chi}_i, \bar{\chi}_i^r, \bar{\chi}_j)} a_{k,n-k}^{-1} \binom{n}{k}_{q_{ii}}$$

Proof. For the sake of readability, we use the short-hand notation i for $\bar{\chi}_i$ during this proof. For $n = 1$, we obtain $ad_c(F_i)(F_j) = F_iF_j - q_{ij}F_jF_i$. For larger n, we first want to find an inductive expression for the coefficients in the following expansion:

$$ad_c^n(F_i)(F_j) = \sum_{k=0}^{n} \mu_n(k)\,(F_i^k F_j)F_i^{(n-k)}.$$

To this end, we compute

$$\begin{aligned}
ad_c^n(F_i)(F_j) &= ad_c(F_i)(ad_c^{n-1}(F_i)(F_j)) \\
&= ad_c(F_i)\left(\sum_{k=0}^{n-1} \mu_{n-1}(k)\,(F_i^k F_j)F_i^{(n-k-1)}\right) \\
&= \sum_{k=0}^{n-1} \mu_{n-1}(k) \left(q_{ii}^k(\sigma(ki,i)\omega(i,ki,j)\omega(i,ki+j,(n-k-1)i))^{-1}\,(F_i^{k+1}F_j)F_i^{n-k-1}\right. \\
&\quad \left. - \sigma(i,(n-1)i+j)\omega(ki+j,(n-k-1)i,i)\,(F_i^kF_j)F_i^{n-k}\right).
\end{aligned}$$

From this, we obtain

$$\begin{aligned}
\mu_n(n) &= \prod_{r=0}^{n-1} \frac{\sigma(i,ri)}{q^r\omega(i,ri,j)} =: c_n \\
\mu_n(0) &= (-1)^n\sigma(ni,j)q_{ii}^{\frac{(n-1)n}{2}} c_n \\
\mu_n(k) &= \mu_{n-1}(k-1)q_{ii}^{(k-1)}\sigma((k-1)i,i)\omega(i,(k-1)i,j)^{-1}\omega(i,(k-1)i+j,(n-k)i)^{-1} \\
&\quad - \mu_{n-1}(k)\sigma(i,(n-1)i+j)\omega(ki+j,(n-k-1)i,1).
\end{aligned}$$

This allows us to prove the following formula by induction, even though we are going to omit the proof here, since it is long and tedious without any interesting inputs except an exhaustive use of the abelian 3-cocycle conditions and q-binomial coefficients. The hard part was rather to find the formula by tracing down the inductive formula to $n = 1$, than to prove it.

$$\mu_n(k) = (-1)^{n-k}c_n a_{k,n-k}^{-1}\sigma((n-k)i,j)\frac{\omega(ki,(n-k)i,j)}{\omega((n-k)i,ki,j)}q_{ii}^{\binom{n-k}{2}}\binom{n}{k}_{q_{ii}}.$$

$\square$

Proposition 5.2.3. *The coproducts of F_i^n and $ad_c^n(F_i)(F_j)$ are given by*

$$\Delta(F_i^n) = \sum_{k=0}^{n} (a^i_{k,n-k})^{-1} \binom{n}{k}_{q_{ii}} F_i^k \otimes F_i^{n-k}$$

$$\begin{aligned}\Delta(ad_c^n(F_i)(F_j)) &= (ad_c^n(F_i)(F_j)) \otimes 1 + 1 \otimes (ad_c^n(F_i)(F_j)) \\ &+ \sum_{k=0}^{n} \mu_n(k) \frac{\left(q_{ii}^{2k} q_{ij} q_{ji}\right)^{n-k}}{\sigma(\bar\chi_i^{n-k}, \bar\chi_i^k \bar\chi_j)} \sum_{m=0}^{n-k-1} \frac{\omega(\bar\chi_i^{n-k-m}, \bar\chi_i^m, \bar\chi_i^k \bar\chi_j)}{\sigma(\bar\chi_i^k \bar\chi_j, \bar\chi_i^m)} (a^i_{n-k-m,m})^{-1} \binom{n-k}{m}_{q_{ii}} \\ &\times \prod_{r=0}^{n-k-m-1} \left(1 - \frac{q^{k+m}}{q_{ii}^r q_{ij} q_{ji}}\right) F_i^{n-k-m} \otimes \left(F_i^k F_j\right) F_i^m\end{aligned}$$

Proof. During this proof, we use the abbreviation i for $\bar\chi_i$. We first proof the equation for $\Delta(F_i^n)$. We set $\Delta(F_i^n) := \sum_{k=0}^{n} f_n(k)\, F_i^k \otimes F_i^{n-k}$. First, we want to find an inductive relation between the coefficients $f_n(k)$. We have

$$\begin{aligned}\Delta(F_i^n) &= \Delta(F_i^{n-1})\Delta(F_i) \\ &= \Delta(F_i^{n-1})(F_i \otimes 1 + 1 \otimes F_i) \\ &= \sum_{k=0}^{n-1} f_{n-1}(k)\, (F_i^k \otimes F_i^{n-k})(F_i \otimes 1 + 1 \otimes F_i)\end{aligned}$$

After computing the product, we obtain

$$\begin{aligned}f_n(k) &= f_{n-1}(k)\omega(ki, (n-1-k)i, i) \\ &\quad + f_{n-1}(k-1)\frac{\omega((k-1)i, (n-k)i, i)}{\omega((k-1)i, i, (n-k)i)}\sigma((n-k)i, i).\end{aligned}$$

Moreover, we have $f_n(0) = f_{n-1}(0) = 1$ and $f_n(n) = f_{n-1}(n-1) = 1$. Now, we want to show the following formula by induction:

$$f_n(k) = \prod_{r=0}^{(n-k)-1} \omega(ki, ri, i) \binom{n}{k}_{q_{ii}}.$$

For $n = 1$, we obtain $\binom{1}{0}_{q_{ii}} = 1 = f_1(0)$ and $\binom{1}{1}_{q_{ii}} = 1 = f_1(1)$. Now, we assume that the formula holds for $n-1$. Then,

$$\begin{aligned}f_n(k) &= f_{n-1}(k)\omega(ki, (n-1-k)i, i) \\ &\quad + f_{n-1}(k-1)\frac{\omega((k-1)i, (n-k)i, i)}{\omega((k-1)i, i, (n-k)i)}\sigma((n-k)i, i) \\ &= \prod_{r=0}^{(n-1-k)-1} \omega(ki, ri, i) \binom{n-1}{k}_{q_{ii}} \omega(ki, (n-1-k)i, i) \\ &+ \prod_{r=0}^{(n-k)-1} \omega((k-1)i, ri, i) \binom{n-1}{k-1}_{q_{ii}} \frac{\omega((k-1)i, (n-k)i, i)}{\omega((k-1)i, i)}\sigma((n-k)i, (n-k)i, i)\end{aligned}$$

$$
\begin{aligned}
&= \prod_{r=0}^{(n-k)-1} \omega(ki,ri,i) \binom{n-1}{k}_{q_{ii}} \\
&+ \prod_{r=0}^{(n-k)} \frac{\omega(ki,ri,i)\omega(i,(k-1)i,(r+1)i)}{\omega(i,(k-1)i,ri)\omega(i,(k+r-1)i,i)} \frac{\sigma((n-k)i,i)}{\omega((k-1)i,i,(n-k)i)} \binom{n-1}{k-1}_{q_{ii}} \\
&= \prod_{r=0}^{(n-k)-1} \omega(ki,ri,i) \binom{n-1}{k}_{q_{ii}} \\
&+ \prod_{r=0}^{(n-k)-1} \omega(ki,ri,i) \frac{\omega(ki,(n-k)i,i)\omega(i,(k-1)i,(n-k+1)i)}{\omega(i,(k-1)i,i)} \frac{q_{ii}^{(n-k)}\sigma(ki,i)}{\sigma(ni,i)} \\
&\times \frac{\sigma((n-k)i,i)}{\omega((k-1)i,i,(n-k)i)} \binom{n-1}{k-1}_{q_{ii}} \\
&= \prod_{r=0}^{(n-k)-1} \omega(ki,ri,i) \left(\binom{n-1}{k}_{q_{ii}} - \right. \\
&+ \left. \frac{\omega(ki,(n-k)i,i)\omega(i,ki,(n-k)i)}{\omega(ki,i,(n-k)i)} \frac{\sigma(ki,i)\sigma((n-k)i,i)}{\sigma(ni,i)} q_{ii}^{(n-k)} \binom{n-1}{k-1}_{q_{ii}} \right) \\
&= \prod_{r=0}^{(n-k)-1} \omega(ki,ri,i) \left(\binom{n-1}{k}_{q_{ii}} - q_{ii}^{(n-k)} \binom{n-1}{k-1}_{q_{ii}} \right) \\
&= \prod_{r=0}^{(n-k)-1} \omega(ki,ri,i) \binom{n}{k}_{q_{ii}},
\end{aligned}
$$

where we used the abelian 3-cocycle conditions exhaustively.

Instead of giving a rigorous proof for the second part, which would take a few pages, we describe instead what we did. First, we computed the coefficients A, B of the following expression:

$$
\begin{aligned}
\Delta\left((F_i^k F_j)F_i^{(n-k)}\right) &= \left(\Delta\left(F_i^k\right)\Delta(F_j)\right)\Delta\left(F_i^{(n-k)}\right) \\
&= \sum_{l=0}^{k}\sum_{m=0}^{n-k} f_k(l) f_{n-k}(m) \left(A \left(F_i^l F_j\right) F_i^m \otimes F_i^{n-m-l}\right. \\
&\left. + B\, F_i^{m+l} \otimes \left(F_i^{(k-l)} F_j\right) F_i^{n-k-m}\right).
\end{aligned}
$$

Then we plugged this in $\Delta(\mathrm{ad}_c^n(F_i)(F_j))$, using the expansion for $\mathrm{ad}_c^n(F_i)(F_j)$ from the previous proposition:

$$
\begin{aligned}
\Delta(\mathrm{ad}_c^n(F_i)(F_j)) = &\sum_{k=0}^{n}\sum_{l=0}^{k}\sum_{m=0}^{n-k} \mu_n(k) \binom{k}{l}_{q_{ii}} \binom{n-k}{m}_{q_{ii}} (a^i_{l,k-l} a^i_{m,n-k-m})^{-1} \\
&\times \left(A \left(F_i^l F_j\right) F_i^m \otimes F_i^{n-m-l} + B\, F_i^{m+l} \otimes \left(F_i^{(k-l)} F_j\right) F_i^{n-k-m}\right)
\end{aligned}
$$

After changing the order of summation and plugging in our expression for the coefficients $\mu_n(k)$ from the previous propsition, we see that the B-summand cancels completely, whereas the A-summand can be brought in to a form, where we can apply the q-binomial coefficient theorem. This gives the above result. $\square$

From the above coproducts we can read off the following relations:

Corollary 5.2.4. *For any $n \in \mathbb{N}$,*

1. *$F_i^n \neq 0$ if and only if $(n)_{q_{ii}}! \neq 0$.*
2. *$ad_c^n(F_i)(F_j) \neq 0$ if and only if $(n)_{q_{ii}}! \prod_{r=0}^{n-1} (1 - q_{ii}^r q_{ij} q_{ji})$.*

Remark 5.2.5. *Note that the above relations do not depend on the 3-cocycle ω and are identical with the ones given in [Ros98], Lemma 14.*

5.3 A Radford biproduct

For a general quasi-Hopf algebra H and a braided Hopf algebra $B \in {}^H_H\mathcal{YD}$ in the category of Yetter-Drinfeld modules over H, the Radford biproduct $B\#H$ was defined in [BN02]. It is again a quasi-Hopf algebra.

The Nichols algebra $B(V)$ constructed in the previous section is a Hopf algebra in the category ${}^{k^{\widehat{G}}_\omega}_{k^{\widehat{G}}_\omega}\mathcal{YD}$ of Yetter-Drinfeld modules over $k^{\widehat{G}}_\omega$ and thus the definition in [BN02] applies. We collect the relevant relations in the following proposition:

Proposition 5.3.1. *As a vector space, the quasi-Hopf algebra $B(V)\#k^{\widehat{G}}_\omega$ is given by $B(V) \otimes k^{\widehat{G}}_\omega$. A general product of generators is given by*

$$(F_i\#\delta_{\psi_1})(F_j\#\delta_{\psi_2}) = \delta_{\psi_1\chi_j,\psi_2}\omega(\bar{\chi}_i, \bar{\chi}_j, \psi_2)^{-1}(F_iF_j)\#\delta_{\psi_2}.$$

The inclusion $k^{\widehat{G}}_\omega \hookrightarrow B(V)\#k^{\widehat{G}}_\omega$, $\delta_\psi \mapsto 1\#\delta_\psi$ is a homomorphism of quasi-Hopf algebras. This legitimises the short-hand notation $\delta_\chi = 1\#\delta_\chi \in B(V)\#k^{\widehat{G}}_\omega$. Moreover, we set $F_i = F_i\#1_{k^{\widehat{G}}_\omega}$. Then, the following algebra relations hold:

- *$\delta_\chi \cdot (F_i\#\delta_\psi) = \delta_{\chi\chi_i,\psi}(F_i\#\delta_\psi)$, in particular $\delta_\chi \cdot F_i = F_i\#\delta_{\chi\chi_i}$*
- *$(F_i\#\delta_\chi) \cdot \delta_\psi = \delta_{\chi,\psi}(F_i\#\delta_\psi)$, in particular $F_i \cdot \delta_\chi = F_i\#\delta_\chi$*
- *$F_iF_j = (F_iF_j)\#\left(\sum_{\chi\in\widehat{G}} \omega(\bar{\chi}_i, \bar{\chi}_j, \chi)^{-1}\,\delta_\chi\right)$*

The comultiplication is given by

$$\Delta(F_i\#\delta_\psi) = \sum_{\psi'\in G} \frac{\omega(\psi', \bar{\chi}_i, \psi\bar{\psi}')}{\omega(\bar{\chi}_i, \psi', \psi\bar{\psi}')}\sigma(\bar{\chi}_i, \psi')\delta_{\psi'} \otimes (F_i\#\delta_{\psi\bar{\psi}'}) + \sum_{\psi'\in\widehat{G}} \omega(\bar{\chi}_i, \psi', \psi\bar{\psi}')^{-1}(F_i\#\delta_{\psi'}) \otimes \delta_{\psi\bar{\psi}'},$$

in particular

$$\Delta(F_i) = \sum_{\chi,\psi\in\hat{G}} \frac{\omega(\chi,\bar{\chi}_i,\psi)}{\omega(\bar{\chi}_i,\chi,\psi)}\sigma(\bar{\chi}_i,\chi)\delta_\chi \otimes (F_i\#\delta_\psi) + \sum_{\chi,\psi\in\widehat{G}} \omega(\bar{\chi}_i,\chi,\psi)^{-1}(F_i\#\delta_\chi)\otimes\delta_\psi.$$

The antipode is given by

$$S(F_i\#\delta_\psi) = -\omega(\psi,\bar{\chi}_i,\chi_i\bar{\psi})\sigma(\bar{\chi}_i,\psi)(F_i\#\delta_{\chi_i\bar{\psi}}).$$

In the later chapters we will also need the following formula:

$$\begin{aligned}(\Delta\otimes\mathrm{id})\circ\Delta(F_i) = \sum_{\psi_1,\psi_2,\psi_3\in G} & \alpha_1^i(\psi_1,\psi_2,\psi_3)(F_i\#\delta_{\psi_1})\otimes\delta_{\psi_2}\otimes\delta_{\psi_3} \\ & + \alpha_2^i(\psi_1,\psi_2,\psi_3)\delta_{\psi_1}\otimes(F_i\#\delta_{\psi_2})\otimes\delta_{\psi_3} \\ & + \alpha_3^i(\psi_1,\psi_2,\psi_3)\delta_{\psi_1}\otimes\delta_{\psi_2}\otimes(F_i\#\delta_{\psi_3}),\end{aligned}$$

where

$$\begin{aligned}\alpha_1^i(\psi_1,\psi_2,\psi_3) &= \omega(\bar{\chi}_i,\psi_1\psi_2,\psi_3)^{-1}\omega(\bar{\chi}_i,\psi_1,\psi_2)^{-1} \\ \alpha_2^i(\psi_1,\psi_2,\psi_3) &= \frac{\omega(\psi_1,\bar{\chi}_i,\psi_2)}{\omega(\bar{\chi}_i,\psi_1\psi_2,\psi_3)\omega(\bar{\chi}_i,\psi_1,\psi_2)}\sigma(\bar{\chi}_i,\psi_1) \\ \alpha_3^i(\psi_1,\psi_2,\psi_3) &= \frac{\omega(\psi_1\psi_2,\bar{\chi}_i,\psi_3)}{\omega(\bar{\chi}_i,\psi_1\psi_2,\psi_3)}\sigma(\bar{\chi}_i,\psi_1\psi_2).\end{aligned}$$

We now want to consider twists of the Radford biproduct as defined in Def. 2.1.3. We are mainly interested in twists coming from elements $J = \sum_{\chi,\psi\in\widehat{G}} \zeta(\chi,\psi)\,\delta_\chi\otimes\delta_\psi \in k_\omega^{\widehat{G}}\otimes k_\omega^{\widehat{G}}$ in the group part of $(B(V)\#k_\omega^{\widehat{G}})^{\otimes 2}$. The corresponding coproduct $\Delta^J(F_i)$ is then given by:

$$\begin{aligned}\Delta^J(F_i) = & \sum_{\chi,\psi\in\widehat{G}} \frac{\omega(\chi,\bar{\chi}_i,\psi)}{\omega(\bar{\chi}_i,\chi,\psi)}\sigma(\bar{\chi}_i,\chi)\frac{\zeta(\chi,\psi\bar{\chi}_i)}{\zeta(\chi,\psi)}\delta_\chi\otimes(F_i\#\delta_\psi) \\ & + \sum_{\chi,\psi\in\widehat{G}} \omega(\bar{\chi}_i,\chi,\psi)^{-1}\frac{\zeta(\chi\bar{\chi}_i,\psi)}{\zeta(\chi,\psi)}(F_i\#\delta_\chi)\otimes\delta_\psi.\end{aligned}$$

Not surprisingly, the corresponding associator is given by

$$\phi^J = \sum_{\psi_1,\psi_2,\psi_3} \omega(\psi_1,\psi_2,\psi_3)d\zeta(\psi_1,\psi_2,\psi_3)\,\delta_{\psi_1}\otimes\delta_{\psi_2}\otimes\delta_{\psi_3}.$$

Lemma 5.3.2. *The identity on $k_{\omega d\zeta}^{\widehat{G}}$ extends to an isomorphism of quasi-Hopf algebras defined by:*

$$\begin{aligned}f_\zeta : \left(B(V)\#k_\omega^{\widehat{G}}\right)^J &\longrightarrow B(V)\#k_{\omega d\zeta}^{\widehat{G}} \\ F_i &\longmapsto \sum_{\chi\in\widehat{G}} \zeta(\bar{\chi}_i,\chi)\,F_i\#\delta_\chi\end{aligned}$$

Proof. This is a special case of Thm. 5.1. in [BN02]. $\square$

Example 5.3.3. *We continue with the example from section 4. The corresponding Yetter-Drinfeld module for the group $\widehat{G}/T$ is given by $V := k \cdot F$, where $|F| = \bar{\chi}^{-2}$. Using the abelian 3-cocycle $(\bar{\omega}, \bar{\sigma})$ on $\widehat{G}/T$ as defined in Example 4.0.3, we obtain for the twisted coproduct on $B(V)\#k_{\bar{\omega}}^{\widehat{G}/T}$:*

$$\begin{aligned}
\Delta^J(F) &= \sum_{i,j=0}^{\ell-1} q^{\frac{i(j-2-[j-2])}{2}} q^{-i} \frac{\zeta(\bar{\chi}^i, \bar{\chi}^{[j-2]})}{\zeta(\bar{\chi}^i, \bar{\chi}^j)} \delta_{\bar{\chi}^i} \otimes F\#\delta_{\bar{\chi}^j} + \sum_{i,j=0}^{\ell-1} \frac{\zeta(\bar{\chi}^{[i-2]}, \bar{\chi}^j)}{\zeta(\bar{\chi}^i, \bar{\chi}^j)} F\#\delta_{\bar{\chi}^i} \otimes \delta_{\bar{\chi}^j} \\
&= K^{-1} \otimes F \cdot \sum_{i,j=0}^{\ell-1} q^{\frac{i(j-2-[j-2])}{2}} \frac{\zeta(\bar{\chi}^i, \bar{\chi}^{[j-2]})}{\zeta(\bar{\chi}^i, \bar{\chi}^j)} \delta_{\bar{\chi}^i} \otimes \delta_{\bar{\chi}^j} \\
&+ F \otimes 1 \cdot \sum_{i,j=0}^{\ell-1} \frac{\zeta(\bar{\chi}^{[i-2]}, \bar{\chi}^j)}{\zeta(\bar{\chi}^i, \bar{\chi}^j)} \delta_{\bar{\chi}^i} \otimes \delta_{\bar{\chi}^j},
\end{aligned}$$

where $K^{-1} := \sum_{i=0} q^{-i} \delta_{\bar{\chi}^i}$. If we now set $\zeta = \zeta_t$ from Example 4.0.3, we obtain

$$\Delta^J(F) = K^{-1} \left(q^t \sum_{2 \nmid i} \delta_{\bar{\chi}^i} + \sum_{2|i} \delta_{\bar{\chi}^i} \right) \otimes F + F \otimes 1$$

$$\phi^J = \sum_{i,j,k=0}^{\ell-1} \bar{\omega}_t(\bar{\chi}^i, \bar{\chi}^j, \bar{\chi}^k) \delta_{\bar{\chi}^i} \otimes \delta_{\bar{\chi}^j} \otimes \delta_{\bar{\chi}^k}.$$

5.3.1 Dualization of $B(V)\#k_\omega^{\widehat{G}}$

Since $B(V)\#k_\omega^{\widehat{G}}$ is a finite-dimensional quasi-Hopf algebra, the dual space $\left(B(V)\#k_\omega^{\widehat{G}}\right)^*$ carries the structure of a coquasi-Hopf algebra, which is the dual analogue of a quasi-Hopf algebra. In particular, a coquasi-Hopf algebra is a coassociative coalgebra and has an algebra structure which is only associative up to an element $\Psi \in (H \otimes H \otimes H)^*$.

We define dual elements $\tilde{E}_i := (F_i\#\delta_1)^*$ and $\tilde{K}_\psi := \delta_\psi^*$ in $(B(V)\#k_\omega^{\widehat{G}})^*$. We find the following relations for these elements:

- $\Delta(\tilde{E}_i) = \tilde{K}_i^{-1} \otimes \tilde{E}_i + \tilde{E}_i \otimes 1$
- $\Delta(\tilde{K}_\psi) = \tilde{K}_\psi \otimes \tilde{K}_\psi$
- $\tilde{K}_\psi \tilde{E}_i = \sigma(\bar{\chi}_i, \psi)(F_i\#\delta_\psi)^* = \sigma(\bar{\chi}_i, \psi)\tilde{E}_i\tilde{K}_\psi$.

From the last relation we see immediately that the product cannot be associative, since σ is not a bihomomorphism on $\widehat{G}$.

5.4 A Drinfeld double

From now on, we assume the Nichols algebra $B(V)$ and hence $B(V)\#k_\omega^{\widehat{G}}$ to be finite dimensional.

5.4.1 The general case

In the following, we recall the definition of the Drinfeld double $D(H)$ of a finite-dimensional quasi-Hopf algebra as introduced in [HN99a, HN99b]. As an algebra, Hausser and Nill defined the Drinfeld double as a special case of a diagonal crossed product $H^* \bowtie_\delta M$ (see [HN99a], Def. 10.1) with a so-called two-sided coaction (δ, Ψ) (see [HN99a], Def. 8.1) of the quasi-Hopf algebra H on the algebra $M = H$ as input data. In this case, we have

$$\delta = (\Delta \otimes \mathrm{id}) \circ \Delta : H \to H^{\otimes 3}$$
$$\Psi = ((\mathrm{id} \otimes \Delta \otimes \mathrm{id})(\phi) \otimes 1)\,(\phi \otimes 1 \otimes 1)\,\big((\delta \otimes \mathrm{id} \otimes \mathrm{id})(\phi^{-1})\big)\,.$$

The multiplication on $D(H)$ is given by

$$(\varphi \bowtie m)(\psi \bowtie n) := \big((\Omega^1 \rightharpoonup \varphi \leftharpoonup \Omega^5)(\Omega^2 m_{(1)(1)} \rightharpoonup \psi \leftharpoonup S^{-1}(m_{(2)})\Omega^4)\big) \bowtie \Omega^3 m_{(1)(2)} n, \qquad (5.2)$$

where

$$\Omega = (\mathrm{id} \otimes \mathrm{id} \otimes S^{-1} \otimes S^{-1})((1 \otimes 1 \otimes 1 \otimes f)\Psi^{-1}) \in H^{\otimes 5}.$$

The Drinfeld twist $f \in H \otimes H$ is defined in Eq. 2.7. We summarize some of the main results in [HN99a, HN99b] in the following theorem, even though we state them in a more explicit way:

Theorem 5.4.1 (Hausser, Nill)**.** *We have an algebra inclusion $\iota : H \to D(H)$ and a linear map $\Gamma : H^* \longrightarrow D(H)$ given by*

$$\iota : H \longrightarrow D(H) \qquad\qquad \Gamma : H^* \longrightarrow D(H)$$
$$h \longmapsto (1 \bowtie h) \qquad\qquad \varphi \longmapsto \Big(p^1_{(1)} \rightharpoonup \varphi \leftharpoonup S^{-1}(p^2)\Big) \bowtie p^1_{(2)},$$

s.t. the algebra $D(H)$ is generated by the images $\iota(H)$ and $\Gamma(H^*)$. In particular, we have

$$\varphi \bowtie h = \iota(q^1)\Gamma(\varphi \leftharpoonup q^2)\iota(h).$$

The elements $p_R = p^1 \otimes p^2, q_R = q^1 \otimes q_2 \in H \otimes H$ were defined in Eq. 2.3.

Without going into detail here, in [HN99a], Ch. 11, Hausser and Nill showed that it is possible to define coproduct and antipode on the diagonal crossed product $D(H)$, such that it becomes a quasi-Hopf algebra. The above theorem implies that it is sufficient to define the coproduct on $D(H)$ on elements $\iota(h)$ and $\Gamma(\varphi)$. It is given by

$$\Delta(\iota(h)) : = (\iota \otimes \iota)(\Delta(h))$$
$$\Delta(\Gamma(\varphi)) : = \big(\varphi \otimes \mathrm{id}_{D(H)\otimes D(H)}\big)\,\big(\phi_{312}^{-1}\Gamma_{13}\phi_{213}\Gamma_{12}\phi^{-1}\big)\,,$$

where the inclusions in Eq. 5.3 are understood. The element $\Gamma \in H \otimes D(H)$ is defined by $\Gamma := e_i \otimes \Gamma(e^i)$, where e_i and e^i are a dual pair of bases on H and H^*.

Remark 5.4.2. *Note, that the definition of $\Delta(\Gamma(\varphi))$ in Eq. 5.3 is different from the one in [HN99b], Thm. 3.9. Using their terminology, we checked both coherency and normality for the resulting $\lambda\rho$-intertwiner*

$$T := \phi_{312}^{-1}\Gamma_{13}\phi_{213}\Gamma_{12}\phi^{-1},$$

but we don't know if the definition in Eq. (11.4) in [HN99a] is false or just another possibility. We needed the one in Eq. 5.3 in order to make the next Proposition work.

The antipode on $D(H)$ is defined by

$$S(\iota(h)) := \iota(S(h)) \tag{5.3}$$

$$S(\Gamma(\varphi)) := (1 \bowtie f^1)\left(p^1_{(1)}g^{(-1)} \rightharpoonup \varphi \circ S^{-1} \leftharpoonup f^2 S^{-1}(p^2)\right) \bowtie p^1_{(2)}g^{(-2)}. \tag{5.4}$$

As in Eq. 5.3, we gave the action of S on $\Gamma(\varphi)$ explicitly instead of defining it in terms of generating matrices as in [HN99b], Thm. 3.9.
The associator on $D(H)$ as well as the elements α and β are simply inherited from H by the inclusion $\iota : H \to D(H)$, which becomes then an inclusion of quasi-Hopf algebras.
Unit and counit are given by:

$$1_{D(H)} := 1_{H^*} \bowtie 1_H, \qquad \epsilon_{D(H)}(\iota(h)) := \epsilon_H(h), \qquad \epsilon_{D(H)}(\Gamma(\varphi)) := \varphi(1_H).$$

Finally, we recall that a two-sided coaction (δ, Ψ) of H on an algebra M can be twisted by an element $U \in H \otimes M \otimes H$, giving rise to a twist-equivalent two-sided coaction (δ', Ψ') on M (see [HN99a], Dfn. 8.3):

$$\begin{aligned} \delta'(h) &:= U\delta(h)U^{-1} \\ \Psi' &:= (1 \otimes U \otimes 1)(\mathrm{id} \otimes \delta \otimes \mathrm{id})(U)\Psi(\Delta \otimes \mathrm{id}_M \otimes \Delta)(U^{-1}) \end{aligned} \tag{5.5}$$

In [HN99a] Prop. 10.6.1., Hausser and Nill show that twist equivalent two-sided coactions give rise to equivalent diagonal crossed products $H^* \bowtie_\delta M$ and $H^* \bowtie_{\delta'} M$.

On the other hand, for any twist $J \in H \otimes H$, the pair (δ, Ψ^J) is a two-sided coaction of the twisted quasi-Hopf algebra H^J (see Def. 2.1.3) on M, where

$$\Psi^J := \Psi(J^{-1} \otimes J^{-1}). \tag{5.6}$$

Again, in [HN99a] Prop. 10.6.2., Hausser and Nill show that for two-sided coactions (δ, Ψ) of H and (δ, Ψ^J) of H^J, we get $H^* \bowtie M = (H^J)^* \bowtie M$ with trivial identification.

Proposition 5.4.3. *Let $J \in H \otimes H$ be a twist on H and $\tilde{J} := (\iota \otimes \iota)(J) \in D(H) \otimes D(H)$ the corresponding twist on $D(H)$. For $U_L = (J \otimes 1)(\Delta \otimes id)(J) \in H \otimes H \otimes H$, the following map is an isomorphism of quasi-Hopf algebras:*

$$\begin{aligned} F_J : D(H)^{\tilde{J}} &\longrightarrow D(H^J) \\ \varphi \bowtie a &\longmapsto \left(U_L^1 \rightharpoonup \varphi \leftharpoonup S^{-1}\left(U_L^3\right)\right) \bowtie U_L^2 a \end{aligned}$$

Proof. As in Eq. 5.5, the element $U_L \in H^{\otimes 3}$ defines a twisted two-sided coaction (δ', Ψ') of H on H. On the other hand, we can twist (δ', Ψ') in the sense of Eq. 5.6 via the twist $J \in H \otimes H$, giving rise to a two-sided coaction (δ', Ψ') of H^J on the algebra H. This defines a diagonal crossed product $(H^J) \bowtie_{\delta'} H$, serving as the underlying algebra of $D(H^J)$. The fact that the map F_J is an algebra isomorphism follows then simply from [HN99a] Prop. 10.6. Note that Hausser and Nill showed this for crossed products of the form $M \bowtie H^*$, but this is no problem due to Thm. 10.2 in [HN99a] relating $M \bowtie H^*$ and $H^* \bowtie M$. Using their terminology, we simply choose the left δ-implementer $\tilde{L} = L' \prec U_L$ instead of the right δ-implementer $\tilde{R} = U_L^{-1} \succ R'$ in [HN99a], Eq. (10.46).
Next, we want to show $(F_J \otimes F_J) \circ \Delta = \Delta \circ F_J$. By Thm. 5.4.1 it remains to prove this for elements of the form $\iota(h)$ and $\Gamma(\varphi)$. It is easy to see that F_J is the identity on H, thus

$$\begin{aligned} (F_J \otimes F_J) \circ \Delta_{D(H)^{\tilde{J}}}(\iota(h)) &= (F_J \otimes F_J)(\tilde{J}\Delta(\iota(h))\tilde{J}^{-1}) \\ &= (F_J \otimes F_J)(\tilde{J}(\iota(h_{(1)}) \otimes \iota(h_{(2)})\tilde{J}^{-1}) \\ &= (F_J \otimes F_J)(\iota(J^1 h_{(1)} K^{(-1)}) \otimes \iota(J^2 h_{(2)} K^{(-2)})) \\ &= \iota(J^1 h_{(1)} K^{(-1)}) \otimes \iota(J^2 h_{(2)} K^{(-2)}) = \Delta_{D(H^J)} \circ F_J(\iota(h)). \end{aligned}$$

In order to show $(F_J \otimes F_J) \circ \Delta(\Gamma(\varphi)) = \Delta \circ F_J(\Gamma(\varphi))$, we use the identities

$$\begin{aligned} \Gamma_J := \Gamma_{D(H^J)}(\varphi) &= (1 \bowtie J^1) F_J(\Gamma(K^{(-1)} \rightharpoonup \varphi \leftharpoonup J^2))(1 \bowtie K^{(-2)}) \\ F_J \circ \Gamma(\varphi) &= (1 \bowtie K^{(-1)}) \Gamma_J(J^1 \rightharpoonup \varphi \leftharpoonup K^{(-2)})(1 \bowtie J^2). \end{aligned}$$

Then,

$$\begin{aligned} \Delta_{D(H^J)} \circ F_J(\Gamma(\varphi)) &= \Delta_{D(H^J)}((1 \bowtie K^{(-1)}) \Gamma_J(J^1 \rightharpoonup \varphi \leftharpoonup K^{(-2)})(1 \bowtie J^2)) \\ &= (\iota \otimes \iota)(\Delta(K^{(-1)})) \Delta_{D(H)^J}(\Gamma_J(J^1 \rightharpoonup \varphi \leftharpoonup K^{(-2)}))(\iota \otimes \iota)(\Delta(J^2)) = (\star) \end{aligned}$$

Using the definition in Eq. 5.3 we can show

$$\begin{aligned} \Delta_{D(H)}(\Gamma(\varphi)) = (1 \bowtie x^1 X^1) &\Gamma(y^1 \rightharpoonup \varphi_{(2)})(1 \bowtie y^2) \\ &\otimes (1 \bowtie x^2) \Gamma(X^2 \rightharpoonup \varphi_{(1)} \leftharpoonup x^3)(1 \bowtie X^3 y^3). \end{aligned}$$

Hence,

$$\begin{aligned} (F_J \otimes F_J) \circ \Delta_{D(H)^{\tilde{J}}}(\Gamma(\varphi)) = (1 \bowtie J^1 x^1 X^1) &F_J \Gamma(y^1 \rightharpoonup \varphi_{(2)})(1 \bowtie y^2 K^{(-1)}) \\ &\otimes (1 \bowtie J^2 x^2) F_J \Gamma(X^2 \rightharpoonup \varphi_{(1)} \leftharpoonup x^3)(1 \bowtie X^3 y^3 K^{(-2)}) \end{aligned}$$

Using the identity

$$a_{(1)}F_J\Gamma(b_{(1)} \rightharpoonup \varphi a_{(2)})b_{(2)} = F_J\Gamma((ab)_{(1)} \rightharpoonup \varphi)(ab)_{(2)},$$

a simple but tedious calculation shows that this equals $(\star)$.
For the antipode, it is again sufficient to show $S \circ F_J = F_J \circ S$ on generators $\iota(h)$ and $\Gamma(\varphi)$, where the former is trivial by the same argument as above. Using the definition in Eq. 5.4, we obtain

$$F_J \circ S(\Gamma(\varphi)) = (1 \bowtie f^1)\left((U^1 p^1_{(1)} g^{(-1)} \rightharpoonup \varphi \circ S^{-1} \leftharpoonup f^2 S^{-1}(U^3 p^2)) \bowtie U^2 p^1_{(2)} g^{(-2)}\right) \tag{5.7}$$

and

$$\begin{aligned} S \circ F_J(\Gamma(\varphi)) &= (1 \bowtie S(J^2) f^1_J) \\ &\times \left(\left(p^1_{J(1)(1)} g_J^{(-1)} S(K^{(-2)}) \rightharpoonup \varphi \circ S^{-1} \leftharpoonup S(J^1) f^2_J S^{-1}(p^2_J)\right) \bowtie p^1_{J(1)(2)} g_J^{(-2)} S(K^{(-1)})\right). \end{aligned}$$

The Drinfeld twist $f = g \in H \otimes H$ is defined in 2.7. Using the identities in 2.8, a simple but tedious calculation shows that both terms are equal.
Finally, since F_J is the identity on $H \subseteq D(H)$ it is easy to see that F_J preserves the associator and the elements α and β, which are inherited from H. This proves the proposition. $\square$

The following proposition shows that retractions of quasi-Hopf algebras induce monomorphisms between the Drinfeld doubles:

Proposition 5.4.4. *Let $i : K \to H$ be a split monomorphism of quasi-Hopf algebras, i.e. there exists a homomorphism of quasi-Hopf algebras $p : H \to K$, s.t. $p \circ i = id_K$. Then the map $\Phi_i : D(K) \to D(H)$ defined by*

$$\begin{aligned} \Phi_i(\iota_K(h)) &:= \iota_H(i(h)) \\ \Phi_i(\Gamma_K(\varphi)) &:= \Gamma_H(\varphi \circ p) \end{aligned}$$

is a monomorphism of quasi-Hopf algebras with left inverse $\Phi_p : D(H) \to D(K)$ being a coalgebra homomorphism defined by

$$\begin{aligned} \Phi_p(\iota_H(h)) &:= \iota_K(p(h)) \\ \Phi_p(\Gamma_H(\varphi)) &:= \Gamma_K(\varphi \circ i). \end{aligned}$$

Proof. The fact that Φ_i is an algebra isomorphism follows from the universal property of the map Γ (see Thm. II in [HN99a]). In their terminology, we choose a normal element $T := e_i \otimes \Gamma_H(e^i \circ p) \in K \otimes D(H)$. Using the identities given in Eq. 2.4 and $p \circ i = \mathrm{id}$, it is a straight-forward calculation to check that this element satifies both conditions (6.2) and (6.3) in the first part of the theorem. Since i is a homomorphism of quasi-Hopf algebras, Φ_i preserves coproduct and antipode on elements of the form $\iota_K(h)$ as well as the associator ϕ and the elements α and β. For the image of Γ_K we have

$$\begin{aligned}
(\Phi_i\otimes\Phi_i)\circ\Delta_{D(K)}(\Gamma_K(\varphi)) &= (\Phi_i\otimes\Phi_i)((\varphi\otimes\mathrm{id})(T_K))\\
&= (\Phi_i\otimes\Phi_i)(x^1X^1\Gamma_K(y^1\rightharpoonup\varphi_{(2)})y^2\otimes x^2\Gamma_K(X^2\rightharpoonup\varphi_{(1)}\leftharpoonup x^3)X^3y^3)\\
&= i(x^1X^1)\Gamma_H((y^1\rightharpoonup\varphi_{(2)})\circ p)i(y^2)\\
&\otimes i(x^2)\Gamma_H((X^2\rightharpoonup\varphi_{(1)}\leftharpoonup x^3)\circ p)i(X^3y^3))\\
&= \tilde{x}^1\tilde{X}^1\Gamma_H(\tilde{y}^1\rightharpoonup(\varphi\circ p)_{(2)})\tilde{y}^2\otimes\tilde{x}^2\Gamma_H(\tilde{X}^2\rightharpoonup(\varphi\circ p)_{(1)}\leftharpoonup\tilde{x}^3)\tilde{X}^3\tilde{y}^3\\
&= (\varphi\circ p\otimes\mathrm{id})(T_H)\\
&= \Delta_{D(H)}(\Gamma_H(\varphi\circ p)) = \Delta_{D(H)}(\Phi_i(\Gamma_K(\varphi))),
\end{aligned}$$

where T is the element defined in Remark 5.4.2 and $\phi_H=\tilde{X}_1\otimes\tilde{X}_2\otimes\tilde{X}_3=\tilde{Y}_1\otimes\tilde{Y}_2\otimes\tilde{Y}_3=i\otimes i\otimes i(\phi_K)$ and $\Phi_H^{-1}=\tilde{x}_1\otimes\tilde{x}_2\otimes\tilde{x}_3$. We omit the proof that Φ_i preserves the antipode on the image of Γ, since it is completely analogous.
Using $p\circ i=\mathrm{id}_K$, it is easy to see that Φ_p is a left inverse of Φ_i. By analogous arguments as above, Φ_p is a coalgebra homomorphism. $\square$

5.4.2 The Drinfeld double of $B(V)\#k_\omega^G$

In order to state the defining relations for the Drinfeld double of $B(V)\#k_\omega^{\widehat{G}}$, we compute the element $\Omega\in(B(V)\#k_\omega^{\widehat{G}})^{\otimes 5}$ from the previous subsection for this case:

Lemma 5.4.5. *In the case $H=B(V)\#k_\omega^{\widehat{G}}$, the element $\Omega\in(B(V)\#k_\omega^{\widehat{G}})^{\otimes 5}$ is given by*

$$\Omega=\sum_{\chi_i} f(\chi_1,\chi_2,\chi_3,\chi_4,\chi_5)\,(1\#\delta_{\chi_1})\otimes(1\#\delta_{\chi_2})\otimes(1\#\delta_{\chi_3})\otimes(1\#\delta_{\chi_4})\otimes(1\#\delta_{\chi_5}),\ \textit{where}$$

$$f(\chi_1,\chi_2,\chi_3,\chi_4,\chi_5)=\frac{\omega(\chi_1\chi_2\chi_3,\bar{\chi}_4,\bar{\chi}_5)\omega(\chi_5,\chi_4,\bar{\chi}_4)}{\omega(\chi_1,\chi_2\chi_3,\bar{\chi}_4)\omega(\chi_1,\chi_2,\chi_3)\omega(\chi_4\chi_5,\bar{\chi}_4,\bar{\chi}_5)}.$$

In particular, we obtain $f(\psi_1,\psi_2,\chi,\psi_2,\psi_1)^{-1}=\theta(\chi)(\psi_1,\psi_2)d\nu(\psi_1,\psi_2)(\chi)$, where $\theta\in Z^2(\widehat{G},k^{\widehat{G}})$ is the 2-cocycle from Remark 5.1.5 and $\nu(\psi)(\chi):=\omega(\psi,\bar{\chi},\chi)$. From this, it can be seen that $D(k_\omega^{\widehat{G}})$ is indeed isomorphic to the double $D^\omega(\widehat{G})$ in the sense of [DPR92].

Before we are going to derive the braided commutator relations, we define:

$$E_i:=\Gamma\left((F_i\#\delta_1)^*\right),\qquad F_j:=\iota(F_j),$$

$$\bar{L}_\chi:=\iota(L_\chi),\qquad L_\chi:=\iota(\bar{L}_\chi),\qquad \bar{K}_\chi:=\Gamma((1\#\delta_{\bar{\chi}})^*)\bowtie\left(\sum_\psi\theta(\psi|\chi,\bar{\chi})\,\iota(\delta_\psi)\right).$$

Lemma 5.4.6. *Let $a_i, b_j : \widehat{G} \to k^\times$ be solutions to the equation*

$$\frac{a_i(\psi\bar{\chi}_j)b_j(\psi)}{a_i(\psi)b_j(\psi\chi_i)} = \frac{\omega(\chi_i, \bar{\chi}_j, \psi)}{\omega(\bar{\chi}_j, \chi_i, \psi)}. \tag{5.8}$$

For $\chi, \psi \in \widehat{G}$, we set

$$E_i^\chi := c_\chi E_i L_{\bar{\chi}_i} \iota \left(\sum_\xi a_i(\xi) \left(1 \# \delta_\xi\right) \right) \qquad F_j^\psi := c_\psi F_j \iota \left(\sum_\xi b_j(\xi) \left(1 \# \delta_\xi\right) \right).$$

Here, the elements $c_\chi \in \left(k^{\widehat{G}}\right)^* \otimes k^{\widehat{G}}$ are defined by

$$c_\chi := \bar{K}_{\bar{\chi}} \bar{L}_\chi^{-1} = (1 \# \delta_\chi)^* \bowtie \left(\sum_{\psi \in \widehat{G}} \frac{\sigma(\psi, \chi)}{\omega(\psi, \chi, \bar{\chi})} \left(1 \# \delta_\psi\right) \right).$$

Let $[E_i^\chi, F_j^\psi]_\sigma := E_i^\chi F_j^\psi - \sigma(\chi_i, \bar{\chi}_j) F_j^\psi E_i^\chi$ denote the braided commutator in $D\left(B(V) \# k_\omega^{\widehat{G}}\right)$. Then we have

$$[E_i^\chi, F_j^\psi]_\sigma = \delta_{ij} \sigma(\chi_i, \bar{\chi}_i) c_{\chi\psi} \left(1 - \bar{K}_{\chi_i} L_{\bar{\chi}_i}\right) \iota \left(\sum_\xi \frac{a_i(\xi) b_i(\xi\chi_i)}{\omega(\bar{\chi}_i, \chi_i, \xi)} (1 \# \delta_\xi) \right) \tag{5.9}$$

Proof. For $\lambda_i, \mu_j : \widehat{G} \times \widehat{G} \to k$ arbitrary non-zero maps, we define elements

$$\begin{aligned} E_i^\lambda &:= \sum_{\chi, \psi} \lambda_i(\chi, \psi) (F_i \# \delta_\chi)^* \bowtie (1 \# \delta_\psi) \\ F_j^\mu &:= \sum_{\chi, \psi} \mu_j(\chi, \psi) (1 \# \delta_\chi)^* \bowtie (F_j \# \delta_\psi). \end{aligned}$$

Using the general multiplication formula 5.2 for the Drinfeld double, we obtain

$$\begin{aligned} E_i^\lambda F_j^\mu &= \sum_{\chi, \psi} \left(\sum_\xi r_{\chi, \psi}(\xi) \mu_j(\chi\bar{\xi}, \psi) \lambda_i(\xi, \psi\bar{\chi}_j) \right) (F_i \# \delta_\chi)^* \bowtie (F_j \# \delta_\psi), \text{ where} \\ r_{\chi, \psi}(\xi) &= \frac{f(\xi, \chi\bar{\xi}, \psi\bar{\chi}_j, \chi\bar{\xi}, \xi\bar{\chi}_i)}{\omega(\bar{\chi}_i, \xi, \chi\bar{\xi})} \end{aligned}$$

For $F_j^\mu E_i^\lambda$, we obtain

$$F_j^\mu E_i^\lambda = \delta_{ij}\left(\sum_{\chi,\psi}\lambda_i(\chi,\psi)\left(\sum_{\xi}\alpha_1^i(\chi,\psi,\chi_i\bar{\chi})f(\xi,\chi\bar{\chi}_i,\psi,\chi\bar{\chi}_i,\xi)\mu_j(\xi,\psi\chi_i)\,(1\#\delta_{\chi\xi\bar{\chi}_i})^*\right)\bowtie(1\#\delta_\psi)\right.$$
$$\left.-\sum_{\chi,\psi}\lambda_i(\chi,\psi)\left(\sum_{\xi}\frac{\alpha_3^i(\chi,\psi,\chi_i\bar{\chi})f(\xi,\chi,\psi,\chi,\xi)}{\omega(\chi,\bar{\chi}_i,\chi_i\bar{\chi})\sigma(\bar{\chi}_i,\chi)}\mu_j(\xi,\psi\chi_i)\,(1\#\delta_{\chi\xi})^*\right)\bowtie(1\#\delta_\psi)\right)$$
$$+\sum_{\chi,\psi}\left(\sum_{\xi}s_{\chi,\psi}(\xi)\mu_j(\chi\bar{\xi},\psi\chi_i)\lambda_i(\xi,\psi)\right)(F_i\#\delta_\chi)^*\bowtie(F_j\#\delta_\psi),\ \text{where}$$
$$s_{\chi,\psi}(\xi) = r_{\chi,\psi}(\xi)\alpha_2^j(\xi,\psi,\chi_i\bar{\xi})\frac{\sigma(\chi_i,\xi\bar{\chi})\omega(\chi\psi\bar{\xi}\bar{\chi}_j,\chi_i,\xi\bar{\chi})}{\omega(\chi\psi\bar{\xi}\bar{\chi}_j,\xi\bar{\chi},\chi_i)\omega(\chi\bar{\xi},\psi\bar{\chi}_j,\chi_i)}.$$

The functions α_k^i were defined after Prop. 5.3.1. For the following choices of μ_j and λ_i, the above formulas simply significantly:

$$\lambda_i(\chi,\psi) = \delta_{\chi,\xi}a_i^\xi(\psi) \qquad\qquad \mu_j(\chi,\psi) = \delta_{\chi,\xi}b_j^\xi(\psi),$$

where $a_i^\xi, b_j^\xi : \widehat{G}\to k^\times$. Or goal is to find functions $\lambda_i,\mu_j : \widehat{G}\times\widehat{G}\to k$, s.t. that the braided commutator $[E_i^\lambda, F_j^\mu]_\sigma$ takes values in $\left(k^{\widehat{G}}\right)^*\bowtie k^{\widehat{G}}$. The necessary and sufficient condition for this is

$$0 = \sum_{\xi} r_{\chi,\psi}(\xi)$$
$$\times\left(\mu_j(\chi\bar{\xi},\psi)\lambda_i(\xi,\psi\bar{\chi}_j) - \sigma(\chi_i,\bar{\chi}_j)\mu_j(\chi\bar{\xi},\psi\chi_i)\lambda_i(\xi,\psi)\frac{\alpha_2^j(\xi,\psi,\chi_i\bar{\xi})\sigma(\chi_i,\xi\bar{\chi})\omega(\chi\psi\bar{\xi}\bar{\chi}_j,\chi_i,\xi\bar{\chi})}{\omega(\chi\psi\bar{\xi}\bar{\chi}_j,\xi\bar{\chi},\chi_i)\omega(\chi\bar{\xi},\psi\bar{\chi}_j,\chi_i)}\right),$$

At least for the above choices of λ_i and μ_j, this is equivalent to the existence of solutions $a_i, b_j : \widehat{G}\to k^\times$ to the equation:

$$\frac{a_i(\psi\bar{\chi}_j)b_j(\psi)}{a_i(\psi)b_j(\psi\chi_i)} = \frac{\omega(\chi_i,\bar{\chi}_j,\psi)}{\omega(\bar{\chi}_j,\chi_i,\psi)}.$$

This can be seen by setting

$$a_i^\xi(\psi) = \frac{\sigma(\psi,\xi)\sigma(\chi_i,\psi)}{\omega(\psi,\xi,\chi_i\bar{\xi})}a_i(\psi) \qquad b_j^\xi(\psi) = \frac{\omega(\bar{\xi},\xi,\psi\bar{\chi}_j)}{\sigma(\psi\bar{\chi}_j\xi,\bar{\xi})}b_j(\psi)$$

and using the abelian 3-cocycle conditions. Plugging in these solutions for λ_i and μ_j in the braided commutator and using that $c_{\chi\psi} = c_\chi c_\psi$ yields the claimed result. □

Corollary 5.4.7. *Let $\omega\in Z^3(\widehat{G})$ be a nice 3-cocycle. Then, the braided commutator in $D\left(B(V)\#k^{\widehat{G}}_{\omega d\zeta}\right)$ is given by:*

$$[E_i^\chi, F_j^\psi]_\sigma = \delta_{ij}\sigma(\chi_i,\bar{\chi}_i)c_{\chi\psi}\left(1-\bar{K}_{\chi_i}L_{\bar{\chi}_i}\right). \tag{5.10}$$

Proof. In case of a nice 3-cocycle, we can set $a_i, b_j = 1$. Together with the second niceness condition this implies

$$\iota\left(\sum_{\xi} \frac{a_i(\xi)b_i(\xi\chi_i)}{\omega(\bar{\chi}_i, \chi_i, \xi)}(1\#\delta_\xi)\right) = 1.$$

$\square$

The remaining relations simplify drastically after taking the quotient in the next chapter, so we wait until then before we state them.

5.5 A quotient of the Drinfeld double

In this subsection we define the small quasi-quantum group as a quotient of the Drinfeld double $D := D\left(B(V)\#k_\omega^{\widehat{G}}\right)$ (see Section 5.4.2) by the biideal $I \subseteq D$ induced by the following map:

Proposition 5.5.1. *Let $D := D\left(B(V)\#k_\omega^{\widehat{G}}\right)$ be the Drinfeld double from Section 5.4.2. Moreover, let $k\widehat{G}$ be the group algebra of $\widehat{G}$.*

1. *The following map is an algebra inclusion into the center of D:*

$$\begin{aligned} j : k\widehat{G} &\longrightarrow Z(D) \\ \chi &\longmapsto c_\chi := (1\#\delta_\chi)^* \bowtie \left(\sum_{\psi\in\widehat{G}} \frac{\sigma(\psi,\chi)}{\omega(\psi,\chi,\bar{\chi})}(1\#\delta_\psi)\right). \end{aligned}$$

   Moreover, we have $\Delta \circ j = j \otimes j \circ \Delta$ and $\epsilon = \epsilon \circ j$.

2. *Set $N^+ := ker\left(\epsilon|_{j(k\widehat{G})}\right)$ and $I := N^+D$. Then, $I \subseteq D$ is a biideal.*

3. *The quotient D/I is a quasi-Hopf algebra with quasi-Hopf structure induced by the quotient map $D \twoheadrightarrow D/I$.*

Proof. We first show that j is an algebra homomorphism. We have

$$
\begin{aligned}
c_{\chi\psi} &= (1\#\delta_{\chi\psi})^* \bowtie \left(\sum_{\xi\in\widehat{G}} \frac{\sigma(\xi,\psi\chi)}{\omega(\xi,\chi\psi,\bar{\chi}\bar{\psi})}\,(1\#\delta_\xi) \right) \\
&= (1\#\delta_\chi)^*(1\#\delta_\psi)^* \bowtie \left(\sum_{\xi\in\widehat{G}} \theta(\xi|\chi,\psi)^{-1}\frac{\sigma(\xi,\psi)\sigma(\xi,\chi)}{\omega(\xi,\chi\psi,\bar{\chi}\bar{\psi})}\,(1\#\delta_\xi) \right) \\
&= \sum_{\xi\in\widehat{G}} \theta(\xi|\chi,\psi)^{-1}\frac{\sigma(\xi,\psi)\sigma(\xi,\chi)}{\omega(\xi,\chi\psi,\bar{\chi}\bar{\psi})}\,(1\#\delta_\chi)^*(1\#\delta_\psi)^* \bowtie (1\#\delta_\xi) \\
&= \sum_{\xi,\nu\in\widehat{G}} \frac{\sigma(\xi,\psi)\sigma(\xi,\chi)}{\omega(\xi,\chi,\bar{\chi})\omega(\xi,\psi,\bar{\psi})}\,((1\#\delta_\chi)^* \bowtie (1\#\delta_\xi))((1\#\delta_\psi)^* \bowtie (1\#\delta_\nu)) \\
&= c_\chi c_\chi.
\end{aligned}
$$

Obviously, j preserves the unit. We continue with the coproduct:

$$
\begin{aligned}
\Delta(c_\chi) &= \Delta(\Gamma(1\#\delta_\chi)^*))\left(\sum_{\psi\in\widehat{G}} \sigma(\psi,\chi)\,\Delta(1\#\delta_\psi) \right) \\
&= \Gamma((1\#\delta_\chi)^*)\otimes\Gamma((1\#\delta_\chi)^*)\left(\sum_{\psi\in\widehat{G}} \theta(\chi|\psi,\chi)^{-1}\sigma(\psi\xi,\chi)\,(1\#\delta_\psi)\otimes(1\#\delta_\xi) \right) \\
&= \Gamma((1\#\delta_\chi)^*)\otimes\Gamma((1\#\delta_\chi)^*)\left(\sum_{\psi\in\widehat{G}} \sigma(\psi,\chi)\sigma(\xi,\chi)\,(1\#\delta_\psi)\otimes(1\#\delta_\xi) \right) \\
&= c_\chi \otimes c_\chi.
\end{aligned}
$$

The counit is again trivial. Next, we check that j takes values in the center. It suffices to show that c_χ commutes with elements of the form $\Gamma((F_i\#\delta_\chi)^*)$ and $\iota(F_j\#\delta_\psi)$:

$$
\begin{aligned}
c_\chi\Gamma((F_i\#\delta_\xi)^*) &= \Gamma((1\#\delta_\chi)^*) \bowtie \left(\sum_{\psi\in\widehat{G}} \sigma(\psi,\chi)\,(1\#\delta_\psi)\Gamma((F_i\#\delta_\xi)^*) \right) \\
&= \Gamma((1\#\delta_\chi)^*) \bowtie \left(\sum_{\psi\in\widehat{G}} \sigma(\psi,\chi)\,\Gamma((F_i\#\delta_\xi)^*)(1\#\delta_{\bar{\chi}_i\psi}) \right) \\
&= \Gamma((1\#\delta_\chi)^*)\Gamma((F_i\#\delta_\xi)^*) \bowtie \left(\sum_{\psi\in\widehat{G}} \sigma(\psi\chi_i,\chi)\,(1\#\delta_\psi) \right) \\
&= \sigma(\chi_i,\chi)^{-1}\Gamma((F_i\#\delta_\xi)^*)\Gamma((1\#\delta_\chi)^*) \bowtie \left(\sum_{\psi\in\widehat{G}} \theta(\chi|\psi,\chi_i)\sigma(\psi\chi_i,\chi)\,(1\#\delta_\psi) \right)
\end{aligned}
$$

$$
\begin{aligned}
&= \Gamma((F_i\#\delta_\xi)^*)\Gamma((1\#\delta_\chi)^*) \bowtie \left(\sum_{\psi\in\widehat{G}} \sigma(\psi,\chi)\,(1\#\delta_\psi)\right) \\
&= \Gamma((F_i\#\delta_\xi)^*)c_\chi.
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
c_\chi(F_i\#\delta_\xi) &= \Gamma((1\#\delta_\chi)^*) \bowtie \left(\sum_{\psi\in\widehat{G}} \sigma(\psi,\chi)\,(1\#\delta_\psi)(F_i\#\delta_\xi)\right) \\
&= \Gamma((1\#\delta_\chi)^*) \bowtie \left(\sum_{\psi\in\widehat{G}} \sigma(\psi,\chi)\,(F_i\#\delta_\xi)(1\#\delta_{\psi\chi_i})\right) \\
&= \Gamma((1\#\delta_\chi)^*) \bowtie (F_i\#\delta_\xi) \left(\sum_{\psi\in\widehat{G}} \sigma(\psi\bar{\chi}_i,\chi)\,(1\#\delta_\psi)\right) \\
&= \sigma(\bar{\chi}_i,\chi)^{-1}(F_i\#\delta_\xi)\Gamma((1\#\delta_\chi)^*) \left(\sum_{\psi\in\widehat{G}} \theta(\chi|\psi,\bar{\chi}_i)^{-1}\sigma(\psi\bar{\chi}_i,\chi)\,(1\#\delta_\psi)\right) \\
&= (F_i\#\delta_\xi)\Gamma((1\#\delta_\chi)^*) \left(\sum_{\psi\in\widehat{G}} \sigma(\psi,\chi)\,(1\#\delta_\psi)\right) \\
&= (F_i\#\delta_\xi)c_\chi.
\end{aligned}
$$

This proves the first part of the proposition. We now come to the second part. Since $j(k\widehat{G}) \subseteq Z(D)$, we have $DN^+ = N^+D$, hence $I \subseteq D$ is an both-sided ideal. As a kernel of a coalgebra homomorphism, $N^+ \subseteq D$ is a coideal and so is $I = HN^+$. The fact that we are dealing with non-coassociative coalgebras plays no role so far. We have shown that I is a biideal
As it is stated in [Sch05], Section 2, this is equivalent to D/I being a quotient quasi-bialgebra. Since D is a finite dimensional quasi-Hopf algebra, we can apply Thm. 2.1 in [Sch05] in order to prove that D/I is a quasi-Hopf algebra. □

Definition 5.5.2. *Let G be a finite abelian group, $(\omega,\sigma) \in Z^3_{ab}(\widehat{G})$ an abelian 3-cocycle on the dual group $\widehat{G}$ and $\{\chi_i \in \widehat{G}\}_{1\leq i\leq n} \subseteq \widehat{G}$ a subset of $\widehat{G}$, s.t. the corresponding Nichols algebra $B(V)$ is finite dimensional. Then we define the small quasi-quantum group corresponding to that data to be the quotient*

$$u(\omega,\sigma) := D\left(B(V)\#k^{\widehat{G}}_\omega\right)/I.$$

It is clear that in the quotient $u(\omega,\sigma)$, we have $\bar{L}_\chi = \bar{K}_{\bar{\chi}}$. In order to get rid of L's, we also define $K_\chi := L_{\bar{\chi}}$.

Proposition 5.5.3. *The following map is a monomorphism of quasi-Hopf algebras, which is split as a coalgebra homomorphism:*

$$\begin{aligned}\xi : k^{\widehat{G}}_\omega &\longrightarrow u(\omega,\sigma)\\ \delta_\chi &\longmapsto [\iota(1\#\delta_\chi)]\,.\end{aligned}$$

Proof. It is clear that the inclusion $i : k^{\widehat{G}}_\omega \to B(V)\#k^{\widehat{G}}_\omega$, $\delta_\chi \mapsto 1\#\delta_\chi$ is a split monomorphism, with left inverse denoted by p. We saw in Prop. 5.4.4 that it must therefore induce a quasi-Hopf algebra monomorphism $\Phi_i : D\left(k^{\widehat{G}}_\omega\right) \to D\left(B(V)\#k^{\widehat{G}}_\omega\right)$ with left inverse Φ_p being a coalgebra homomorphism. Since the inclusion $j : k\widehat{G} \to D\left(B(V)\#k^{\widehat{G}}_\omega\right)$ factors through $D\left(k^{\widehat{G}}_\omega\right)$, the following diagram commutes:

$$\begin{tikzcd}
D\left(k^{\widehat{G}}_\omega\right) \arrow[r, hook, "\Phi_i"] \arrow[d, two heads] & D\left(B(V)\#k^{\widehat{G}}_\omega\right) \arrow[r, "\Phi_p"] \arrow[d, two heads] & D\left(k^{\widehat{G}}_\omega\right) \arrow[d, two heads] \\
D\left(k^{\widehat{G}}_\omega\right)/\tilde{I} \arrow[r, hook, "{[\Phi_i]}"] & D\left(B(V)\#k^{\widehat{G}}_\omega\right)/I \arrow[r, "{[\Phi_p]}"] & D\left(k^{\widehat{G}}_\omega\right)/\tilde{I},
\end{tikzcd}$$

where $\tilde{I} := N^+D\left(k^{\widehat{G}}_\omega\right)$. If we show, that the map $f : k^{\widehat{G}}_\omega \to D\left(k^{\widehat{G}}_\omega\right)/\tilde{I}$ given by $\delta_\chi \mapsto [\iota(\delta_\chi)]$ is an isomorphism, we can define a left inverse of ξ by $f^{-1}\circ[\Phi_p]$.
It is clear that f is surjective, since for an arbitrary element $\sum_{\chi,\psi} a(\chi,\psi)\Gamma((1\#\delta_\chi)^*)\iota(\delta_\psi) \in D\left(k^{\widehat{G}}_\omega\right)$, we obtain

$$\begin{aligned}\left[\sum_{\chi,\psi} a(\chi,\psi)\Gamma((1\#\delta_\chi)^*)\iota(\delta_\psi)\right] &= \left[\sum_{\chi,\psi} a(\chi,\psi)c_\chi\bar{K}_\chi^{-1}\iota(\delta_\psi)\right] = \left[\sum_{\chi,\psi}\frac{a(\chi,\psi)}{\sigma(\psi,\chi)}c_\chi\iota(\delta_\psi)\right]\\ &= \left[\sum_{\chi,\psi}\frac{a(\chi,\psi)}{\sigma(\psi,\chi)}\iota(\delta_\psi)\right] = f\left(\sum_{\chi,\psi}\frac{a(\chi,\psi)}{\sigma(\psi,\chi)}\delta_\psi\right),\end{aligned}$$

where we used that $[c_\chi]=[1]$ holds in the quotient. Moreover, we have

$$dim\left(D\left(k^{\widehat{G}}_\omega\right)/\tilde{I}\right) = dim\left(D\left(k^{\widehat{G}}_\omega\right)\right)/dim\left(k\widehat{G}\right) = |G|$$

by the quasi-Hopf algebra version of the Nichols-Zoeller theorem (see [Sch04]). Since $dim\left(k^{\widehat{G}}_\omega\right) = |G|$, f must be an isomorphism. □

In the following, we will usually omit the map ξ.

Remark 5.5.4. *1. We can identify the group part $\left(k^{\widehat{G}}\right)^*\otimes k^{\widehat{G}} \subseteq D$ with $D^\omega(\widehat{G})$ from Exp. 2.1.5 via*

$$\begin{aligned}D^\omega(\widehat{G}) &\longrightarrow \left(k^{\widehat{G}}\right)^*\otimes k^{\widehat{G}} \subseteq D\\ \chi\otimes\delta_\psi &\longmapsto \Gamma\left((1\#\delta_\chi)^*\right)\iota(\delta_\psi).\end{aligned}$$

2. *Note that the elements K_χ and $\bar{K}_\psi$ do not necessarily generate the group part $k^{\widehat{G}} \subseteq u(\omega,\sigma)$ and are not grouplike in general.*

In the quotient $u(\omega,\sigma)$, we have the following relations:

$$\Delta(F_i) = K_{\bar{\chi}_i} \otimes F_i \left(\sum_{\chi,\psi} \theta(\chi|\bar{\chi}_i,\psi)\omega(\bar{\chi}_i,\psi,\chi)^{-1}\, \delta_\chi \otimes \delta_\psi\right) + F_i \otimes 1 \left(\sum_{\chi,\psi} \omega(\bar{\chi}_i,\chi,\psi)^{-1}\, \delta_\chi \otimes \delta_\psi\right)$$

$$\Delta(E_i) = \left(\sum_{\chi,\psi} \theta(\psi|\chi\bar{\chi}_i,\chi_i)^{-1}\omega(\psi,\chi,\bar{\chi}_i)^{-1}\, \delta_\chi \otimes \delta_\psi\right) E_i \otimes \bar{K}_{\chi_i} + \left(\sum_{\chi,\psi} \omega(\chi,\psi,\bar{\chi}_i)^{-1}\, \delta_\chi \otimes \delta_\psi\right) 1 \otimes E_i$$

$$\Delta(K_\chi) = (K_\chi \otimes K_\chi)P_\chi^{-1} \qquad \Delta(\bar{K}_\chi) = (\bar{K}_\chi \otimes \bar{K}_\chi)P_\chi, \qquad P_\chi := \sum_{\psi,\xi} \theta(\chi|\psi,\xi)\, \delta_\psi \otimes \delta_\xi$$

$$[E_i^a K_{\chi_i}, F_j^b]_\sigma = \delta_{ij}\sigma(\chi_i,\bar{\chi}_i)\left(1 - K_{\chi_i}\bar{K}_{\chi_i}\right)\left(\sum_\xi \frac{a_i(\xi)b_i(\xi\chi_i)}{\omega(\bar{\chi}_i,\chi_i,\xi)}\, \delta_\xi\right), \text{ where}$$

$$E_i^a := E_i \left(\sum_\xi a_i(\xi)\, \delta_\xi\right) \qquad F_j^b := F_j \left(\sum_\xi b_j(\xi)\, \delta_\xi\right), \quad \text{with } a_i, b_j \text{ solutions to Eq. 5.8.}$$

$$K_\chi E_i = \sigma(\chi,\chi_i)E_i K_\chi Q_{\chi,\chi_i}^{-1}, \qquad \bar{K}_\chi E_i = \sigma(\chi_i,\chi)E_i\bar{K}_\chi Q_{\chi,\chi_i}, \qquad Q_{\chi,\psi} := \sum_\xi \theta(\chi|\xi,\psi)\, \delta_\xi$$

$$K_\chi F_i = \sigma(\chi,\bar{\chi}_i)F_i K_\chi Q_{\chi,\bar{\chi}_i}^{-1}, \qquad \bar{K}_\chi F_i = \sigma(\bar{\chi}_i,\chi)F_i\bar{K}_\chi Q_{\chi,\bar{\chi}_i}$$

$$S(F_i) = -\left(\sum_\psi \omega(\bar{\psi},\bar{\chi}_i,\chi_i\psi)d\sigma(\chi_i,\psi,\bar{\psi})\theta(\bar{\psi}|\psi\chi_i,\bar{\chi}_i)^{-1}\delta_\psi\right) K_{\chi_i}F_i$$

$$S(E_i) = -E_i\bar{K}_{\chi_i}^{-1}\left(\sum_\psi \frac{\omega(\bar{\chi}_i\bar{\xi},\chi_i,\xi)}{\omega(\bar{\xi},\bar{\chi}_i,\chi_i)}\, \delta_\xi\right)$$

$$\epsilon(K_\chi) = \epsilon(\bar{K}_\chi) = 1, \qquad \epsilon(E_i) = \epsilon(F_i) = 0, \qquad 1_{u(\omega,\sigma)} = K_1$$

Here, we have omitted the inclusion $\iota : B(V)\#k^{\widehat{G}} \to u(\omega,\sigma)$ and the quotient map $[\_] : D\left(B(V)\#k_\omega^{\widehat{G}}\right) \to u(\omega,\sigma)$.

Remark 5.5.5. *Another interesting form of E_i, F_j-commutator is the following: If we set*

$$M_{ij} := \sum_{\chi\in\widehat{G}} \omega(\bar{\chi}_j,\xi\chi_j,\bar{\chi}_i)^{-1}\, \delta_\chi,$$

then we obtain

$$E_iF_j - M_{ij}F_jE_i = \delta_{ij}(K_{\bar{\chi}_i} - \bar{K}_{\bar{\chi}_i}^{-1}).$$

5.6 The R-matrix of $u(\omega,\sigma)$

By Theorem 3.9 in [HN99b], the Drinfeld double $D(H)$ of a quasi-Hopf algebra H has the structure of a quasi-triangular quasi-Hopf algebra. Using their formula, we compute the R-matrix for the case $H = B(V)\#k_\omega^{\widehat{G}}$:

$$\tilde{R} = \sum_{b\in\mathcal{B}} \sum_{\chi\in\widehat{G}} \iota(b\#\delta_\chi) \otimes \Gamma\left((b\#\delta_\chi)^*\right).$$

Here, $\mathcal{B}$ is a basis of the Nichols algebra $B(V)$. In order to match our results with the Hopf case described in [Lus93], we will work with the reverse R-matrix

$$R := \left(\tilde{R}^T\right)^{-1}.$$

The following Lemma is an easy exercise:

Lemma 5.6.1. *Let H and H' be quasi-Hopf algebras and $\varphi : H \to H'$ a surjective homomorphism of quasi-Hopf algebras. If $R \in H \otimes H$ is an R-matrix in H, then $R' := (\varphi \otimes \varphi)(R)$ is an R-matrix in H'. Moreover, if $\nu \in H$ is a ribbon in (H, R), then $\nu' := \varphi(\nu)$ is a ribbon element in (H', R').*

By the previous Lemma, we can transport the R-matrix from $D(H)$ to an R-matrix of $u(\omega,\sigma)$. By abuse of notation, we will denote this R-matrix also by R.

Remark 5.6.2. *We saw in Section 5.1 that instead of σ, we could have taken $(\sigma^T)^{-1}$ in order to define our Yetter-Drinfeld module. If we would have defined $u(\omega,\sigma)$ as an algebra over $k_\sigma := k\left(\sigma(\chi,\psi)|\chi,\psi\in\widehat{G}\right)$, then the (well-defined) involution $k_\sigma \to k_\sigma$ given by $\sigma(\chi,\psi) \mapsto \sigma(\psi,\chi)^{-1}$ would induce an involution $i : u(\omega,\sigma) \to u(\omega,(\sigma^T)^{-1})$ with $i(F_i) = F_i$, $i(E_i) = E_i$ and $i(K_\chi) = \bar{K}_\chi^{-1}$.*

From now on, we will omit the quotient map $[\_] : D\left(B(V)\#k_\omega^{\widehat{G}}\right) \to u(\omega,\sigma)$.

Proposition 5.6.3. *We define elements in $u(\omega,\sigma)^{\otimes 2}$:*

$$\Theta := \sum_{b\in\mathcal{B}} \left(\Gamma((b\#\delta_1)^*) \otimes \iota(b)\right)\gamma_{|b|}, \quad \textit{where} \quad \gamma_{|b|} := \sum_{\chi,\psi\in\widehat{G}} \omega(\chi|\bar{b}|,|b|,\psi)\delta_\chi \otimes \delta_\psi$$

$$R_0 := \sum_{\chi,\psi\in\widehat{G}} \sigma(\chi,\psi)\,\delta_\chi \otimes \delta_\psi.$$

They have the following properties:

1. *$\tilde{R}$ decomposes as $\tilde{R} = \Theta^T(R_0^T)^{-1}$, in particular $R = R_0\Theta^{-1}$.*
2. *Let $\bar{\Delta}$ denote the coproduct in $u(\omega,(\sigma^T)^{-1})$. Then*

$$\Delta^{op}(h)R_0 = R_0\bar{\Delta}(h).$$

3. *The element Θ is a quasi-R-matrix in the sense of [Lus93], i.e.*

$$\Delta(h)\Theta = \Theta\bar{\Delta}(h).$$

Moreover, we have

$$\Theta\Delta(h) = \bar{\Delta}(h)\Theta \qquad \textit{and} \qquad \Theta^2 = 1.$$

4. *The Drinfeld element $u \in u(\omega,\sigma)$ (see Lemma 2.1.8) is given by*

$$u = u_0\left(\sum_{b\in\mathcal{B}} \iota(S^2_{B(V)}(b))\bar{K}_{|b|}\Gamma((b\#\delta_1)^*)\right) = u_0\left(\sum_{b\in\mathcal{B}} \lambda_b\, \iota(b)\bar{K}_{|b|}\Gamma((b\#\delta_1)^*)\right),$$

where $u_0 := \sum_\chi \sigma(\chi,\bar{\chi})\,\delta_\chi$ and

$$\lambda_b = \prod_{k<l} \mathrm{B}(\bar{\chi}_{i_k},\bar{\chi}_{i_l}) \qquad \textit{for} \qquad b = (\dots(F_{i_1}F_{i_2})\dots)F_{i_n}.$$

Proof. We start with (1): We have a general product formula

$$\Gamma((b\#\delta_\chi)^*)\Gamma((b'\#\delta_\psi)^*) = \sum_{\pi\in\widehat{G}} \frac{\omega(\pi|\bar{b}'||\bar{b}|,|b|,\chi)\omega(\pi|\bar{b}'|,|b'|,\psi)}{\omega(\pi|\bar{b}'||\bar{b}|,|b|,|b'|)\omega(\pi|\bar{b}'||\bar{b}|,|b||b'|,\chi\psi)}\frac{\sigma(\pi,\chi\psi)}{\sigma(\pi,\psi)\sigma(\pi|b'|,\chi)}$$
$$\times\,\Gamma\left((b*b'\#\delta_{\chi\psi})^*\right)\iota(\delta_\pi),$$

where $b*b'$ is the quantum shuffle product in $B(V)$ as introduced Appendix B. In particular, we have

$$\begin{aligned}\Gamma((b\#\delta_\chi)^*) &= \Gamma((b\#\delta_1)^*)\Gamma((1\#\delta_\chi)^*)\left(\sum_{\psi\in\widehat{G}}\omega(\psi|\bar{b}|,|b|,\chi)\,\iota(\delta_\psi)\right)\\ &= \Gamma((b\#\delta_1)^*)\bar{K}_\chi^{-1}\left(\sum_{\psi\in\widehat{G}}\omega(\psi|\bar{b}|,|b|,\chi)\,\iota(\delta_\psi)\right),\end{aligned}$$

where we used in the second line that $c_\chi = 1$ holds in the quotient $u(\omega,\sigma)$. Thus,

$$\begin{aligned}\tilde{R} &= \sum_{b\in\mathcal{B}}\sum_{\chi\in\widehat{G}} \iota(b\#\delta_\chi)\otimes\Gamma\left((b\#\delta_\chi)^*\right)\\ &= \sum_{b\in\mathcal{B}}\sum_{\chi\in\widehat{G}} \iota(b\#\delta_\chi)\otimes\Gamma((b\#\delta_1)^*)\bar{K}_\chi^{-1}\left(\sum_{\psi\in\widehat{G}}\omega(\psi|\bar{b}|,|b|,\chi)\,\iota(\delta_\psi)\right)\\ &= \sum_{b\in\mathcal{B}}(\iota(b)\otimes\Gamma((b\#\delta_1)^*))\gamma_{|b|}\left(\sum_{\chi,\psi\in\widehat{G}}\sigma(\psi,\chi)^{-1}\iota(\delta_\chi)\otimes\iota(\delta_\psi)\right)\\ &= \Theta^T(R_0^T)^{-1}.\end{aligned}$$

We now prove (2): It is sufficient to prove the formula for $h = E_i, F_j, \iota(\delta_\chi)$, since these elements generate $u(\omega,\sigma)$. We will only show the computation for E_i, since for δ_χ it is trivial and for F_j it is very similar.

$$\begin{aligned}
\Delta^{op}(E_i)R_0 &= \left(\sum_{\chi,\psi} \theta(\psi|\chi\bar{\chi}_i,\chi_i)^{-1}\omega(\psi,\chi,\bar{\chi}_i)^{-1}\,\delta_\psi\otimes\delta_\chi\right)(\bar{K}_{\chi_i}\otimes E_i)R_0 \\
&+ \left(\sum_{\chi,\psi} \omega(\chi,\psi,\bar{\chi}_i)^{-1}\,\delta_\psi\otimes\delta_\chi\right)(E_i\otimes 1)R_0 \\
&= \left(\sum_{\chi,\psi} \theta(\psi|\chi\bar{\chi}_i,\chi_i)^{-1}\sigma(\psi,\chi_i)\sigma(\psi,\chi\bar{\chi}_i)\omega(\psi,\chi,\bar{\chi}_i)^{-1}\,\delta_\psi\otimes\delta_\chi\right)(1\otimes E_i) \\
&+ \left(\sum_{\chi,\psi} \sigma(\psi,\bar{\chi}_i)\omega(\chi,\psi,\bar{\chi}_i)^{-1}\,\delta_\psi\otimes\delta_\chi\right)(E_i\otimes 1) \\
&= \left(\sum_{\chi,\psi} \sigma(\psi,\chi)\omega(\psi,\chi,\bar{\chi}_i)^{-1}\,\delta_\psi\otimes\delta_\chi\right)(1\otimes E_i) \\
&+ \left(\sum_{\chi,\psi} \theta(\chi|\psi\bar{\chi}_i,\chi_i)^{-1}\frac{\sigma(\psi,\chi)}{\sigma(\chi_i,\chi)}\omega(\chi,\psi,\bar{\chi}_i)^{-1}\,\delta_\psi\otimes\delta_\chi\right)(E_i\otimes 1) \\
&= R_0\left(\left(\sum_{\chi,\psi} \omega(\chi,\psi,\bar{\chi}_i)^{-1}\,\delta_\chi\otimes\delta_\psi\right)(1\otimes E_i)\right. \\
&+ \left.\left(\sum_{\chi,\psi} \theta(\psi|\chi\bar{\chi}_i,\chi_i)^{-1}\omega(\psi,\chi,\bar{\chi}_i)^{-1}\,\delta_\chi\otimes\delta_\psi\right)(E_i\otimes K_{\chi_i}^{-1})\right) = R_0\bar{\Delta}(E_i).
\end{aligned}$$

We continue with (3): Since $R = R_0\Theta^{-1}$ is an R-matrix, we have

$$R_0\Theta^{-1}\Delta(h) = \Delta^{op}(h)R_0\Theta^{-1} = R_0\bar{\Delta}(h)\Theta^{-1},$$

where we used (2) in the second equation. This proves the first claim of (2). If we would have used $(\sigma^T)^{-1}$ instead of σ in our construction of $u(\omega,\sigma)$, the element Θ would be exactly the same, whereas R_0 would change to R_0^{-1}. Since $\Theta^T R_0$ is then an R-matrix in $u(\omega,(\sigma^T)^{-1})$, we have

$$\bar{\Delta}^{op}(h)\Theta^T R_0 = \Theta^T R_0\bar{\Delta}(h) = \Theta^T\Delta^{op}(h)R_0,$$

which implies $\bar{\Delta}(h)\Theta = \Theta\Delta(h)$. In particular, Θ^{-1} is a quasi-R-matrix as well. By an analogous argument as given in [Lus93] for the quasi-Hopf case, a quasi-R-matrix is unique, hence $\Theta^{-1} = \Theta$. Finally, we prove (4): We want to compute the Drinfeld element for the R-matrix R, but it is easier to compute it in terms of the R-matrix $\tilde{R} = \tilde{R}^1\otimes\tilde{R}^2$. Using graphical calculus it is not hard to find the following formula for u:

$$u = S^2(\tilde{q}^2\tilde{R}^1\tilde{p}^1)\tilde{q}^1\tilde{R}^2\tilde{p}^2.$$

After simplifying, we obtain

$$u = \sum_{b\in\mathcal{B}} S^2(\iota(b))u_0 K_{|b|}^{-1}\Gamma((b\#\delta_1)^*).$$

The square of the antipode is given by

$$\begin{aligned} S^2(\iota(b)) &= S\left(\sum_{\chi\in\widehat{G}} \omega(\bar{\psi}|\bar{b}|, |b|, \psi)\sigma(|b|, \bar{\psi}|\bar{b}|)\left(S_{B(V)}(b)\#\delta_\psi\right)\right) \\ &= u_0\iota\left(S^2_{B(V)}(b)\right)K_{|b|}\bar{K}_{|b|}u_0^{-1}. \end{aligned}$$

Hence,

$$u_0\left(\sum_{b\in\mathcal{B}} \iota(S^2_{B(V)}(b))\bar{K}_{|b|}\Gamma((b\#\delta_1)^*)\right).$$

The antipode in the Nichols algebra $S_{B(V)}(b)$ is given by

$$S_{B(V)}(b) = (-1)^{tr|b|}\mu_b\, b^T,$$

where

$$\mu_b = \prod_{j=1}^{n-1}\sigma\left(\prod_{k=1}^{j}\bar{\chi}_{i_k}, \bar{\chi}_{i_{j+1}}\right)\prod_{j=2}^{n-1}\omega\left(\prod_{k=j+1}^{n}\bar{\chi}_{i_k}, \bar{\chi}_{i_j}, \prod_{l=1}^{j-11}\bar{\chi}_{i_l}\right) \quad \text{for} \quad b = (\ldots(F_{i_1}F_{i_2})\ldots)F_{i_n}.$$

A tedious calculation shows that $\mu_b\mu_{b^T} = \lambda_b$, hence $S^2_{B(V)}(b) = \lambda_b\, b$. This proves the claim. □

Remark 5.6.4. *Since $(\omega,\sigma)\in Z^3_{ab}(\widehat{G})$ is an abelian 3-cocycle, it is clear that $R_0 = \sum_{\chi,\psi\in\widehat{G}}\sigma(\chi,\psi)\,\delta_\chi\otimes\delta_\psi$ is an R-matrix for the quasi-Hopf algebra $k^{\widehat{G}}_\omega$, so that the monomorphism $\xi: k^{\widehat{G}}_\omega \to u(\omega,\sigma)$ from Prop. 5.5.3 becomes a homomorphism of quasitriangular quasi-Hopf algebras.*

Chapter 6. Modularization

In the following definition due to [Shi16], we specify the class of categories we want to consider in this chapter.

Definition 6.0.1. *By a (braided) finite tensor category $\mathcal{C}$, we mean a k-linear category that is equivalent to Rep_A for some finite dimensional k-algebra A. Here, we assume k to be algebraically closed. Moreover, $\mathcal{C}$ should be rigid, monoidal (and braided). A functor $F : \mathcal{C} \to \mathcal{D}$ is called a (braided) tensor functor if it preserves this structure.*

We suggest a definition for a modularization of a non-semisimple premodular tensor category:

Definition 6.0.2. *Let $\mathcal{C}$ be a finite braided tensor category. In this case, the coend $\mathbb{F}_{\mathcal{C}} = \int^{X \in \mathcal{C}} X^{\vee} \otimes X$ exists and has the canonical structure of a Hopf algebra in $\mathcal{C}$. Moreover, there is a symmetric Hopf pairing $\omega_{\mathcal{C}} : \mathbb{F}_{\mathcal{C}} \otimes \mathbb{F}_{\mathcal{C}} \to \mathbb{I}$ on $\mathbb{F}_{\mathcal{C}}$. In [Shi16], Shimizu showed that the following conditions are equivalent.*

1. *The Hopf pairing $\omega_{\mathcal{C}} : \mathbb{F}_{\mathcal{C}} \otimes \mathbb{F}_{\mathcal{C}} \to \mathbb{I}$ on the coend $\mathbb{F}_{\mathcal{C}} = \int^{X \in \mathcal{C}} X^{\vee} \otimes X$ is non-degenerate.*
2. *Every transparent object is isomorphic to the direct sum of finitely many copies of the unit object $\mathbb{I} \in \mathcal{C}$. Equivalently, the Müger center $\mathcal{C}'$ of $\mathcal{C}$, which is defined as the full subcategory of transparent objects, is braided equivalent to Vect_k.*

If these conditions are satisfied, the category $\mathcal{C}$ is called modular. In general, we refer to a braided finite tensor category $\mathcal{C}$ as a premodular category.

Remark 6.0.3. *The reader might be surprised that we didn't include a ribbon structure in our definition of a premodular category. This is due to the fact that it is still possible to define a Hopf structure on the coend $\mathbb{F}$, where the ribbon structure is usually used to define the antipode. For the case of $\mathcal{C} = \mathsf{Rep}_H$, where H is a quasi-triangular quasi-Hopf algebra, the antipode on $\mathbb{F} = \underline{H^*}$ is induced by the Drinfeld element $u \in H$ (see App. C).*

We suggest a definition for a modularization in the non-semisimple case:

Definition 6.0.4. *A braided tensor functor between premodular categories $F : \mathcal{C} \to \mathcal{D}$ is a modularization of $\mathcal{C}$, if $\mathcal{D}$ is modular and*

$$F\left(\mathbb{F}_{\mathcal{C}}/Rad(\omega_{\mathcal{C}})\right) \cong \mathbb{F}_{\mathcal{D}}.$$

as braided Hopf algebras.

Remark 6.0.5. *We should point out here, that there are other approaches in order to define a non-semisimple modularization. For example, the notion of a dominant functor still makes sense in this case. It is therefore tempting to define a modularization of $\mathcal{C}$ as an exact sequence $\mathcal{C}' \to \mathcal{C} \xrightarrow{F} \mathcal{D}$ of tensor categories in the sense of [BN11], where $\mathcal{C}'$ is the Müger center of $\mathcal{C}$ and $\mathcal{D}$ is modular. At least in the setting of Thm. 6.0.6, both definitions coincide. It would also be interesting to generalize the actual construction of a modularization in [Bru00] to the non-semisimple case. A step in this direction has been made in [BN11], where the authors show that a dominant functor $F : \mathcal{C} \to \mathcal{D}$ with exact right adjoint is equivalent to the free module functor $\mathcal{C} \to mod_{\mathcal{C}}(A)$ for some commutative algebra A in the center of $\mathcal{C}$. One of the reasons for choosing the given definition is its closeness to one of the equivalent definitions of a modular tensor category. We conjecture that it reduces to the definition of Bruguierès in the semisimple case.*

Let $u := (u_q(\mathfrak{g}, \Lambda), R_0\bar{\Theta})$ be the quasi-triangular Hopf algebra as described in Section 4 with Cartan part $u^0 = \mathbb{C}[G]$, where $G = \Lambda/\Lambda'$. In particular, the category Rep_u is a non-semisimple premodular category. As we have seen before, we have an equivalence

$$\mathsf{Rep}_{(u^0,R_0)} \longrightarrow \mathsf{Vect}_{\widehat{G}}^{(1,\sigma)},$$

where $\sigma : \widehat{G} \times \widehat{G} \to \mathbb{C}^\times$ is given as in Remark 4.0.2.

Assumption: From now on, we assume that $\mathsf{Vect}_{\widehat{G}}^{(1,\sigma)}$ is modularizable, i.e. the quadratic form Q associated to σ is trivial on $T := Rad(\sigma\sigma^T)$. Let $\mathsf{Vect}_{\widehat{G}/T}^{(\bar{\sigma},\bar{\omega})}$ be the modularized category from Prop. 3.0.1.

The aim of this chapter is to modularize the category Rep_u. To this end, we first define a quasi-Hopf algebra $\bar{u}$ and an algebra monomorphism $M : \bar{u} \to u$. Then we show that restriction along this algebra inclusion defines a modularization of Rep_u.
Let $\bar{u} := (u(\bar{\omega}, \bar{\sigma}), \bar{R})$ denote the quasi-Hopf algebra from Thm. 5.0.1, associated to the data $(\widehat{\bar{G}}, \bar{\sigma}, \bar{\omega}, \chi_i := q^{(\alpha_i,\_)}|_{\bar{G}})$. Here, $\bar{G} := Ann(T) \subseteq G$ is the subgroup introduced in Section A and $(\bar{\omega}, \bar{\sigma})$ is the abelian 3-cocycle on $\widehat{G}/T$ associated to a set-theoretic section $s : \widehat{G}/T \to \widehat{G}$ as defined in Section A.2.1. Note that $\widehat{\bar{G}} \cong \widehat{G}/T$. In particular, $\bar{u}$ has all the necessary structure to endow $\mathsf{Rep}_{\bar{u}}$ with a premodular structure.

We now state the main result of this chapter:

Theorem 6.0.6. *1. The category $\mathsf{Rep}_{\bar{u}}$ is a modular tensor category.*

2. The restriction along the algebra monomorphism M from Prop. 6.0.10 defines a modularization $\mathcal{F} : \mathsf{Rep}_u \to \mathsf{Rep}_{\bar{u}}$ in the sense of Def. 6.0.2. The monoidal structure of this functor is *given by*

$$\tau_{V,W} : \mathcal{F}(V) \otimes \mathcal{F}(W) \longrightarrow \mathcal{F}(V \otimes W)$$
$$v \otimes w \longmapsto \sum_{\chi,\psi \in \widehat{G}} \kappa(\chi,\psi)\, \delta_\chi.v \otimes \delta_\psi.w,$$

where the 2-cochain $\kappa \in C^2(\widehat{G})$ is defined in Lemma A.2.1. It satisfies $\pi^*(\bar{\omega},\bar{\sigma}) = (1,\sigma)$. The natural duality isomorphism is given by

$$\xi_V : \mathcal{F}(V^\vee) \longrightarrow \mathcal{F}(V)^\vee$$
$$f \longmapsto (f \leftharpoonup S(\kappa^1)\kappa^2),$$

where $\kappa = \kappa^1 \otimes \kappa^2$ by abuse of notation.

The rest of this chapter is devoted to the proof of this theorem. We will need the following proposition.

Proposition 6.0.7. *Let $F : \mathcal{C} \to \mathcal{D}$ be a cocontinuous left exact braided tensor functor between finite braided tensor categories. Let $\mu : (\_)^\vee \otimes (\_) \Rightarrow \mathbb{F}_\mathcal{C}$ and $\nu : (\_)^\vee \otimes (\_) \to \mathbb{F}_\mathcal{D}$ denote the coends in $\mathcal{C}$ and $\mathcal{D}$. Then there is a braided Hopf algebra epimorphism $p : F(\mathbb{F}_\mathcal{C}) \to \mathbb{F}_\mathcal{D}$ in $\mathcal{D}$, s.t. the Hopf pairings on the coends are related as follows:*

$$\omega_\mathcal{D} \circ (p \otimes p) = F(\omega_\mathcal{C}) \circ \tau_{\mathbb{F}_\mathcal{C},\mathbb{F}_\mathcal{C}}.$$

If $\mathcal{C}$ is a modular tensor category, then F is a modularization in the sense of Def. 6.0.2, with isomorphism $F(\mathbb{F}_\mathcal{C}/Rad(\omega_\mathcal{C})) \cong \mathbb{F}_\mathcal{D}$ induced by p.

Proof. We first note that since F is cocontinuous, it preserves the coend and hence $F(\mathbb{F}_\mathcal{C})$ is the coend over the functor $F((\_)^\vee \otimes (\_))$ with dinatural transformation $F(\mu)$. We also have a Hopf pairing on $F(\mathbb{F}_\mathcal{C})$ given by $\omega_{F(\mathbb{F}_\mathcal{C})} = F(\omega_\mathcal{C}) \circ \tau_{\mathbb{F}_\mathcal{C},\mathbb{F}_\mathcal{C}}$, where τ denotes the monoidal structure on F. Let $m : F((\_)^\vee \otimes (\_)) \Rightarrow F(\_)^\vee \otimes F(\_)$ denote the natural isomorphism constructed from the structure isomorphisms of the monoidal dual preserving functor F. Then,

$$\zeta_V := \nu_{F(V)} \circ m_V : F(V^\vee \otimes V) \to \mathbb{F}_\mathcal{D}$$

defines a dinatural transformation $F((\_)^\vee \otimes (\_)) \to \mathbb{F}_\mathcal{D}$. Hence, due to the universal property of $F(\mathbb{F}_\mathcal{C})$, there is a unique morphism $p : F(\mathbb{F}_\mathcal{C}) \to \mathbb{F}_\mathcal{D}$, s.t.

$$\nu_{F(V)} \circ m_V = p \circ F(\mu_V).$$

Again by the universal property, p must be an epimorphism. If we can show that

$$\omega_\mathcal{D} \circ (p \otimes p) \circ (F(\mu_V) \otimes F(\mu_W)) = \omega_{F(\mathbb{F}_\mathcal{C})} \circ (F(\mu_V) \otimes F(\mu_W)),$$

for all $V, W \in \mathcal{C}$, then $\omega_{\mathcal{D}} \circ (p \otimes p) = \omega_{F(\mathbb{F}_{\mathcal{C}})}$ holds again by the universal property of $F(\mathbb{F}_{\mathcal{C}})$. We have

$$\begin{aligned}\omega_{\mathcal{D}} \circ (p \otimes p) \circ (F(\mu_V) \otimes F(\mu_W)) &= \omega_{\mathcal{D}} \circ (\nu_{F(V)} \circ m_V \otimes \nu_{F(W)} \circ m_W) \\ &= \omega_{\mathcal{D}\, F(V),F(W)} \circ (m_V \otimes m_W) \\ &= \left(F(\omega_{\mathcal{C}\, V,W}) \circ \tau_{V^\vee \otimes V, W^\vee \otimes W} \circ (m_V \otimes m_W)^{-1}\right) \circ (m_V \otimes m_W) \\ &= F(\omega_{\mathcal{C}\, V,W}) \circ \tau_{V^\vee \otimes V, W^\vee \otimes W} \\ &= \omega_{F(\mathbb{F}_{\mathcal{C}})} \circ (F(\mu_V) \otimes F(\mu_W)),\end{aligned}$$

where $\omega_{\mathcal{D}\, X,Y}$ and $\omega_{\mathcal{C}\, V,W}$ denote the bi-dinatural transformations given, morally speaking, by $(ev \otimes ev) \circ (\mathrm{id} \otimes c^2 \otimes \mathrm{id})$. We used the equality

$$\omega_{\mathcal{D}\, F(V),F(W)} = F(\omega_{\mathcal{C}\, V,W}) \circ \tau_{V^\vee \otimes V, W^\vee \otimes W} \circ (m_V \otimes m_W)^{-1},$$

which holds since F is a braided tensor functor. The fact that p is a morphism of braided Hopf algebras follows from the very same arguments. We now prove the second part of the statement. By Lemma 5.2.1 in [KL01], we can identify $Rad(\omega_{\mathcal{C}})$ with the kernel of the adjunct $\omega^!_{\mathcal{C}} : \mathbb{F}_{\mathcal{C}} \to \mathbb{F}^\vee_{\mathcal{C}}$. Since F is left exact, we have $F(ker(\omega_{\mathbb{F}^!_{\mathcal{C}}})) = ker(\omega^!_{F(\mathbb{F}_{\mathcal{C}})})$. As we have seen, $\omega^!_{F(\mathbb{F}_{\mathcal{C}})}$ is given by $p^\vee \circ \omega_{\mathbb{F}^!_{\mathcal{D}}} \circ p$. If $\mathcal{D}$ is modular, then $p^\vee \circ \omega_{\mathbb{F}^!_{\mathcal{D}}}$ is a monomorphism and hence $ker(\omega^!_{F(\mathbb{F}_{\mathcal{C}})}) = ker(p)$. As a cocontinuous functor, F preserves quotients and hence $F(\mathbb{F}_{\mathcal{C}}/ker(\omega^!_{\mathcal{C}})) \cong F(\mathbb{F}_{\mathcal{C}})/ker(p) \overset{p}{\cong} \mathbb{F}_{\mathcal{D}}$. □

We recall that the braided monoidal category $\mathsf{Vect}^{(\omega,\sigma)}_{\widehat{G}}$ embeds into the braided monoidal category of Yetter-Drinfeld modules over $k^{\widehat{G}}_\omega$ with $k^{\widehat{G}}_\omega$-coaction on simple objects $\mathbb{C}_\chi$ given by $1_\chi \mapsto L_{\bar{\chi}} \otimes 1_\chi$. From now on, we will treat $\mathsf{Vect}^{(\omega,\sigma)}_{\widehat{G}}$ as a braided monoidal subcategory of Yetter-Drinfeld modules over $k^{\widehat{G}}_\omega$.

Let $F : \mathsf{Vect}^{(\sigma,1)}_{\widehat{G}} \to \mathsf{Vect}^{(\bar{\sigma},\bar{\omega})}_{\widehat{G}/T}$ be the braided monoidal modularization functor from Section 3.0.1 and $\tau_{V,W} : F(V) \otimes F(W) \to F(V \otimes W)$ the corresponding monoidal structure. For the moment, we forget about the ribbon structure. Moreover, let $V = \oplus_i F_i \mathbb{C} \in \mathsf{Vect}^{(\sigma,\omega)}_{\widehat{G}}$ be the Yetter-Drinfeld module with $|F_i| = \bar{\chi}_i = q^{-(\alpha_i, \_)}$. If $B(V)$ is the Nichols algebra of V, then we clearly have a braided algebra structure on $F(B(V))$ with multiplication given by $m_{F(B(V))} = F(m_{B(V)}) \circ \tau_{B(V),B(V)}$. Also, we have a braided algebra isomorphism $T : B(F(V)) \longrightarrow F(B(V))$ induced by the map $T := \oplus_{n \geq 0} T^n$, where $T^n : F(V)^{\otimes n} \to F(V^{\otimes n})$ is inductively defined by

$$T^n := \tau_{V^{\otimes n-1},V} \circ (T^{n-1} \otimes \mathrm{id}_{F(V)}), \qquad T^0 := \mathrm{id}_{\mathbb{C}}.$$

From now on, we assume that the set-theoretic section $s : \widehat{G}/T \to \widehat{G}$ of the projection $\pi : \widehat{G} \to \widehat{G}/T$ has the property $s([|b|]) = |b|$ for homogeneous vectors $b \in B(V)$.

Lemma 6.0.8. *Let $\kappa : \widehat{G} \times \widehat{G} \to \mathbb{C}^\times$ be such that $\pi^*\bar{\omega} = d\kappa^{-1}$ and $\pi^*\bar{\sigma} = \kappa/\kappa^T$. The following map is an algebra inclusion:*

$$U : B(F(V))\# k_{\bar{\omega}}^{\widehat{G}/T} \longrightarrow B(V)\# k^{\widehat{G}}$$
$$b\#\delta_\chi \longmapsto \sum_{\tau\in T} \kappa(|b|, s(\chi)\tau)\, T(b)\#\delta_{s(\chi)\tau}.$$

Here, $b = (\dots(F_{i_1}F_{i_2})\dots)F_{i_n}$ is a PBW-basis element of the Nichols algebra $B(F(V))$. Moreover, we used the assumption $s([|b|]) = |b|$ implicitly.

Proof. We know that $\kappa(1,\chi) = 1$, $T(1) = 1$ and

$$\sum_{\chi\in\widehat{G}/T}\sum_{\tau\in T}\delta_{s(\chi)\tau} = \sum_{\psi\in\widehat{G}}\delta_\psi = 1_{k^{\widehat{G}}}.$$

Hence, U preserves the unit. Moreover, we have

$$\begin{aligned}
U((b\#\delta_\chi)(b'\#\delta_\psi)) &= \delta_{\chi,[|b'|]\psi}\,\bar{\omega}([|b|],[|b'|],\psi)^{-1}U(bb'\#\delta_\psi)\\
&= \delta_{\chi,[|b'|]\psi}\sum_{\tau\in T}\bar{\omega}([|b|],[|b'|],\psi)^{-1}\kappa(|b||b'|, s(\psi)\tau)\,T(bb')\#\delta_{s(\psi)\tau}\\
&= \delta_{\chi,[|b'|]\psi}\sum_{\tau\in T}\bar{\omega}([|b|],[|b'|],\psi)^{-1}\kappa(|b||b'|, s(\psi)\tau)\kappa(|b|,|b'|)\,T(b)T(b')\#\delta_{s(\psi)\tau},
\end{aligned}$$

where we used $T(bb') = \kappa(|b|,|b'|)\,T(b)T(b')$ in the last line. On the other hand, we have

$$\begin{aligned}
U(b\#\delta_\chi)U(b'\#\delta_\psi) &= \sum_{\tau,\tau'\in T}\kappa(|b|, s(\chi)\tau')\kappa(|b'|, s(\psi)\tau)\,(T(b)\#\delta_{s(\chi)\tau'})(T(b')\#\delta_{s(\psi)\tau})\\
&= \sum_{\tau,\tau'\in T}\delta_{s(\chi)\tau', |b'|s(\psi)\tau}\,\kappa(|b|, s(\chi)\tau')\kappa(|b'|, s(\psi)\tau)\,T(b)T(b')\#\delta_{s(\psi)\tau}\\
&= \delta_{\chi,[|b'|]\psi}\sum_{\tau\in T}\kappa(|b|, |b'|s(\psi)\tau)\kappa(|b'|, s(\psi)\tau)\,T(b)T(b')\#\delta_{s(\psi)\tau}.
\end{aligned}$$

Since $\psi = [s(\psi)\tau]$ and $\pi^*\bar{\omega} = (d\kappa)^{-1}$, we have an equality. □

Remark 6.0.9. *Note that the above algebra homomorphism U has a linear left inverse, given by*

$$Q : B(V)\# k^{\widehat{G}} \longrightarrow B(F(V))\# k_{\bar{\omega}}^{\widehat{G}/T}$$
$$b\#\delta_\psi \longmapsto \kappa(|b|,\psi)^{-1}\,T^{-1}(b)\# s^*\delta_\psi.$$

It is easy to see that this map preserves the unit. Moreover we have

$$Q((b\#\delta_\chi)(b'\#\delta_\psi)) = Q(b\#\delta_\chi)Q(b'\#\delta_\psi)$$

if and only if $s([|b'|\psi]) = |b'|\psi$.

We now show that there is an algebra homomorphism between the corresponding small quantum groups of $\bar{u}^{\leq 0} := B(F(V))\#k_{\bar{\omega}}^{\widehat{G}/T}$ and $u^{\leq 0} := B(V)\#k^{\widehat{G}}$ extending U:

Proposition 6.0.10. *The following map defines an algebra inclusion:*

$$\begin{aligned} M : \bar{u} &\longrightarrow u \\ \Gamma_{\bar{u}}((b\#\delta_1)^*) &\longmapsto \Gamma_u((T(b)\#\delta_1)^*)\left(\sum_{\chi\in\widehat{G}} \kappa(\chi|\bar{b}|,|b|)^{-1}\,\delta_\chi\right) \\ \iota_{\bar{u}}(b\#\delta_\chi) &\longmapsto \iota_u(U(b\#\delta_\chi)). \end{aligned}$$

Proof. It is not hard to see that elements of the form $\Gamma_{\bar{u}}((b\#\delta_1)^*)$ and $\iota_{\bar{u}}(b\#\delta_\chi)$ generate the algebra $\bar{u}$, since we have for a general small quasi-quantum group $u(\omega,\sigma)$

$$\Gamma((b\#\delta_\chi)^*) = \Gamma((b\#\delta_1)^*)\left(\sum_\psi \frac{\omega(\psi|\bar{b}|,|b|,\chi)}{\sigma(\psi,\chi)}\delta_\psi\right)c_\chi = \Gamma((b\#\delta_1)^*)\left(\sum_\psi \frac{\omega(\psi|\bar{b}|,|b|,\chi)}{\sigma(\psi,\chi)}\delta_\psi\right),$$

where we used that $c_\chi = 1$ holds in the quotient. Moreover, the elements $\Gamma((b\#\delta_1)^*)\iota(\tilde{b}\#\delta_\chi) \in u(\omega,\sigma)$ form a basis. Hence, we need to show that M preserves products of the form

$$\Big(\Gamma((b\#\delta_1)^*)\iota(\tilde{b}\#\delta_\chi)\Big)\cdot\Big(\Gamma((b'\#\delta_1)^*)\iota(\tilde{b}'\#\delta_\psi)\Big)$$

Since we know that $M|_{\bar{u}^{\leq 0}} = U$ is a quasi-Hopf inclusion, for this it is sufficient to prove the following relations:

- $M\left(\Gamma((b\#\delta_1)^*)\Gamma((b'\#\delta_1)^*)\right) = M\left(\Gamma((b\#\delta_1)^*)\right)M\left(\Gamma((b'\#\delta_1)^*)\right)$,
- $M\left(\iota(\tilde{b}\#\delta_\chi)\Gamma((b\#\delta_1)^*)\right) = M\left(\iota(\tilde{b}\#\delta_\chi)\right)M\left(\Gamma((b\#\delta_1)^*)\right)$.

We start with the first relation. We have a general formula for the product $\Gamma((b\#\delta_1)^*)\Gamma((b'\#\delta_1)^*)$ given by

$$\Gamma((b\#\delta_1)^*)\Gamma((b'\#\delta_1)^*) = \sum_{\xi\in\widehat{G}/T} \bar{\omega}(\xi[|\bar{b}'||\bar{b}|],[|b|],[|b'|])^{-1}\,\Gamma((b*b'\#\delta_1)^*)\delta_\xi,$$

where $b * b' \in B(F(V))$ denotes the quantum shuffle product as introduced in Appendix B. Hence,

$$\begin{aligned}
M\left(\Gamma((b\#\delta_1)^*)\Gamma((b'\#\delta_1)^*)\right) &= M\left(\sum_{\xi\in\widehat{G}/T} \bar{\omega}(\xi[|\bar{b}'||\bar{b}|],[|b|],[|b'|])^{-1}\,\Gamma((b*b'\#\delta_1)^*)\iota(\delta_\xi)\right)\\
&= \sum_{\xi\in\widehat{G}/T} \bar{\omega}(\xi[|\bar{b}'||\bar{b}|],[|b|],[|b'|])^{-1}\,M\left(\Gamma((b*b'\#\delta_1)^*)\right)M\left(\delta_\xi\right)\\
&= \sum_{\chi\in\widehat{G}} \bar{\omega}([\chi|\bar{b}'||\bar{b}|],[|b|],[|b'|])^{-1}\kappa(\chi|\bar{b}||\bar{b}'|,|b||b'|)^{-1}\,\Gamma((T(b*b')\#\delta_1)^*)\iota\left(\delta_\chi\right)\\
&= \sum_{\chi\in\widehat{G}} \bar{\omega}([\chi|\bar{b}'||\bar{b}|],[|b|],[|b'|])^{-1}\kappa(\chi|\bar{b}||\bar{b}'|,|b||b'|)^{-1}\kappa(|b|,|b'|)^{-1}\\
&\times\ \Gamma((T(b)*T(b')\#\delta_1)^*)\iota\left(\delta_\chi\right),
\end{aligned}$$

where we used $T(b*b') = \kappa(|b|,|b'|)^{-1}T(b)*T(b')$ in the last equality. On the other hand, we have

$$\begin{aligned}
M\left(\Gamma((b\#\delta_1)^*)\right)M\left(\Gamma((b'\#\delta_1)^*)\right) &= \sum_{\xi,\psi\in\widehat{G}} \kappa(\xi|\bar{b}|,|b|)^{-1}\kappa(\psi|\bar{b}'|,|b'|)^{-1}\\
&\times\ \Gamma((T(b)\#\delta_1)^*)\delta_\xi\Gamma((T(b')\#\delta_1)^*)\delta_\psi\\
&= \sum_{\psi\in\widehat{G}} \kappa(\psi|\bar{b}'||\bar{b}|,|b|)^{-1}\kappa(\psi|\bar{b}'|,|b'|)^{-1}\,\Gamma((T(b)*T(b')\#\delta_1)^*)\delta_\psi.
\end{aligned}$$

Since $\pi^*\bar{\omega} = d\kappa^{-1}$, we have an equality. For the second relation, it suffices to prove $M(\bar{F}_j\bar{E}_i) = M(\bar{F}_j)M(\bar{E}_i)$, where the bars indicate that the generators live in $\bar{u}$. By Remark 5.5.5, we have

$$\begin{aligned}
M\left(\bar{F}_j\bar{E}_i\right) &= M\left(\left(\bar{E}_i\bar{F}_j - \delta_{ij}\left(K_{[\bar{\chi}_i]} - \bar{K}^{-1}_{[\bar{\chi}_i]}\right)\right)\iota\left(\sum_{\xi\in\widehat{G}/T}\bar{\omega}([\bar{\chi}_j],\xi[\chi_i],[\bar{\chi}_i])\delta_\xi\right)\right)\\
&= \left(M(\bar{E}_i)M(\bar{F}_j) - \delta_{ij}\left(M\left(K_{[\bar{\chi}_i]}\right) - M\left(\bar{K}^{-1}_{[\bar{\chi}_i]}\right)\right)\right)\iota\left(\sum_{\xi\in\widehat{G}}\bar{\omega}([\bar{\chi}_j],[\xi\chi_i],[\bar{\chi}_i])\,\delta_\xi\right)\\
&= \left(E_iF_j - \delta_{ij}\left(K_{\bar{\chi}_i} - \bar{K}^{-1}_{\bar{\chi}_i}\right)\right)\iota\left(\sum_{\xi\in\widehat{G}}\frac{\kappa(\bar{\chi}_j,\xi)}{\kappa(\xi\bar{\chi}_j\chi_i,\bar{\chi}_i)}\bar{\omega}([\bar{\chi}_j],[\xi\chi_i],[\bar{\chi}_i])\,\delta_\xi\right)\\
&= \left(E_iF_j - \delta_{ij}\left(K_{\bar{\chi}_i} - \bar{K}^{-1}_{\bar{\chi}_i}\right)\right)\iota\left(\sum_{\xi\in\widehat{G}}\frac{\kappa(\bar{\chi}_j,\xi\chi_i)}{\kappa(\xi\chi_i,\bar{\chi}_i)}\,\delta_\xi\right)\\
&= F_jE_i\,\iota\left(\sum_{\xi\in\widehat{G}}\frac{\kappa(\bar{\chi}_j,\xi\chi_i)}{\kappa(\xi\chi_i,\bar{\chi}_i)}\,\delta_\xi\right) = M(\bar{F}_j)M(\bar{E}_i).
\end{aligned}$$

Finally, it is easy to see that M preserves the unit. This proves the claim. $\square$

Remark 6.0.11. *Without proof, we simply remark that M is a homomorphism of quasitriangular quasi-Hopf algebras if we replace u by its twisted version u^J (see Def. 2.1.3), where $J = \sum_{\chi\in\widehat{G}} \kappa(\chi,\psi)\,\delta_\chi \otimes \delta_\psi$.*

Lemma 6.0.12. *Let $\mathcal{C}$ denote the representation category of $u(1,\sigma)$. Then the Müger center $\mathcal{C}'$ is equivalent to Vect_T, where $T = Rad(B)$ is the radical of the associated bihomomorphism $B = \sigma\sigma^T$.*

Proof. In the case $u(1,\sigma) = (u_q(\mathfrak{g}), R_0)$ with simple Lie algebra $\mathfrak{g}$ and R_0 coming from a symmetric bihomomorphism f, this follows from Cor. 5.4.6. The general case follows from Thm. 6.2. in [Shi16]. The theorem says that the Müger center $\mathcal{C}'$ of a finite braided category $\mathcal{C}$ is equivalent to the Müger center $\mathcal{YD}(\mathcal{C})_B^{B\prime}$ of the category $\mathcal{YD}(\mathcal{C})_B^B$ of Yetter-Drinfeld modules over a braided Hopf algebra B in $\mathcal{C}$. In our case, we can set $\mathcal{C} = \mathsf{Vect}_{\hat{G}}^{(1,\sigma)}$, $B = B(V)$. As it is pointed out in Chapter 6.5 in [Shi16], we have $\mathcal{YD}(\mathcal{C})_B^B \cong \mathsf{Rep}_u$ in this case. Moreover, by Prop. 3.0.1 we have $\mathcal{C}' \cong \mathsf{Vect}_T$, which proves the claim. □

We now proof the main theorem:

Proof of Thm. 6.0.6. We first prove the second part of the theorem. Since the restriction functor is the identity on morphisms, it is additive, linear and even exact. In order to show that τ is a monoidal structure, we choose $u\otimes v\otimes w \in (\mathcal{F}(U)\otimes\mathcal{F}(V))\otimes\mathcal{F}(W)$. For $\bar{\alpha}_{X,Y,Z}$ and $\alpha_{U,V,W}$ denoting the associators in $\mathsf{Rep}_{\bar{u}}$ and Rep_u, respectively, we obtain

$$\begin{aligned}
&\tau_{U,V\otimes W}\circ(\mathrm{id}_{\mathcal{F}(U)}\otimes\tau_{V,W})\circ\bar{\alpha}_{\mathcal{F}(U),\mathcal{F}(V),\mathcal{F}(W)}(u\otimes v\otimes w)\\
&=\sum_{\xi_k\in\widehat{G}/T}\sum_{\psi_l\in\widehat{G}}\kappa(\psi_1,\psi_2\psi_3)\kappa(\psi_2,\psi_3)\bar{\omega}(\xi_1,\xi_2,\xi_3)\,\delta_{\psi_1}M(\delta_{\xi_1}).u\otimes\delta_{\psi_2}M(\delta_{\xi_2}).v\otimes\delta_{\psi_3}M(\delta_{\xi_3}).w\\
&=\sum_{\psi_l\in\widehat{G}}\kappa(\psi_1,\psi_2\psi_3)\kappa(\psi_2,\psi_3)\bar{\omega}([\psi_1],[\psi_2],[\psi_3])\,\delta_{\psi_1}.u\otimes\delta_{\psi_2}.v\otimes\delta_{\psi_3}.w\\
&=\sum_{\psi_l\in\widehat{G}}\kappa(\psi_1\psi_2,\psi_3)\kappa(\psi_1,\psi_2)\,\delta_{\psi_1}.u\otimes\delta_{\psi_2}.v\otimes\delta_{\psi_3}.w\\
&=\mathcal{F}(\alpha_{U,V,W})\circ\tau_{U\otimes V,W}\circ(\tau_{U,V}\otimes\mathrm{id}_{\mathcal{F}(W)}).
\end{aligned}$$

This proves the associativity axiom. The unitality axiom follows from the fact that κ is a normalized 2-cochain. Hence, τ is a monoidal structure. We now show that $(\mathcal{F},\tau)$ preserves the braiding. For this, we first notice that

$$
\begin{aligned}
&(M \otimes M)(\bar{\Theta}) \\
&= \sum_{b \in \mathcal{B}} \left(M\left(\Gamma((b \# \delta_1)^*)\right) \otimes M\left(\iota(b)\right)\right) \cdot \left(\sum_{\chi,\psi \in \widehat{G}/T} \bar{\omega}(\chi[|\bar{b}|], [|b|], \psi)\, M(\delta_\chi) \otimes M(\delta_\psi) \right) \\
&= \sum_{b \in \mathcal{B}} \left(\Gamma((T(b) \# \delta_1)^*) \otimes \iota(T(b))\right) \left(\sum_{\pi,\nu \in \widehat{G}} \frac{\kappa(|b|,\nu)}{\kappa(\pi|\bar{b}|,|b|)}\, \bar{\omega}([\pi|\bar{b}|], [|b|], [\psi])\, \delta_\pi \otimes \delta_\nu \right) \\
&= \sum_{b \in \mathcal{B}} \left(\Gamma((T(b) \# \delta_1)^*) \otimes \iota(T(b))\right) \left(\sum_{\pi,\nu \in \widehat{G}} \frac{\kappa(\pi,\nu)}{\kappa(\pi|\bar{b}|,\nu|b|)}\, \delta_\pi \otimes \delta_\nu \right) \\
&= \left(\sum_{\chi,\psi \in \widehat{G}} \kappa(\chi,\psi)^{-1}\, \delta_\chi \otimes \delta_\psi \right) \left(\sum_{b \in \mathcal{B}} \left(\Gamma((T(b) \# \delta_1)^*) \otimes \iota(T(b))\right) \right) \left(\sum_{\pi,\nu \in \widehat{G}} \kappa(\pi,\nu)\, \delta_\pi \otimes \delta_\nu \right) \\
&= \left(\sum_{\chi,\psi \in \widehat{G}} \kappa(\chi,\psi)^{-1}\, \delta_\chi \otimes \delta_\psi \right) \Theta \left(\sum_{\pi,\nu \in \widehat{G}} \kappa(\pi,\nu)\, \delta_\pi \otimes \delta_\nu \right).
\end{aligned}
$$

where $\bar{\Theta}$ and Θ denote the quasi-R-matrices in $\bar{u}$ and u, as defined in Prop. 5.6.3. Now, let $\bar{c}_{X,Y}$ and $c_{V,W}$ denote the braiding in $\mathsf{Rep}_{\bar{u}}$ and Rep_u, respectively. We set $\kappa := \kappa^1 \otimes \kappa^2 = \sum_{\chi,\psi} \kappa(\chi,\psi)\, \delta_\chi \otimes \delta_\psi$. By $\bar{R} = \bar{R}^1 \otimes \bar{R}^2 = \bar{R}^1_0 \bar{\Theta}^{(-1)} \otimes \bar{R}^2_0 \bar{\Theta}^{(-2)}$ and $R = R^1 \otimes R^2 = R^1_0 \Theta^{(-1)} \otimes R^2_0 \Theta^{(-2)}$, we denote the R-matrices of $\bar{u}$ and u as defined in Section 5.6. Again, we choose an arbitrary element $v \otimes w \in \mathcal{F}(V) \otimes \mathcal{F}(W)$. We have

$$
\begin{aligned}
\tau_{W,V} \circ \bar{c}_{\mathcal{F}(V),\mathcal{F}(W)}(v \otimes w) &= \kappa^1 M(\bar{R}^2).w \otimes \kappa^2 M(\bar{R}^1).v \\
&= \kappa^1 M(\bar{R}^2_0) M(\bar{\Theta}^{(-2)}).w \otimes \kappa^2 M(\bar{R}^1_0) M(\bar{\Theta}^{(-1)}).v \\
&= \kappa^1 M(\bar{R}^2_0) \tilde{\kappa}^{(-2)} \Theta^{(-2)} \tilde{\tilde{\kappa}}^2.w \otimes \kappa^2 M(\bar{R}^1_0) \tilde{\kappa}^{(-1)} \Theta^{(-1)} \tilde{\tilde{\kappa}}^1.v \\
&= R^2_0 \Theta^{(-2)} \tilde{\tilde{\kappa}}^2.w \otimes R^1_0 \Theta^{(-1)} \tilde{\tilde{\kappa}}^1.v \\
&= R^2 \tilde{\tilde{\kappa}}^2.w \otimes R^1 \tilde{\tilde{\kappa}}^1.v \\
&= \mathcal{F}(c_{V,W}) \circ \tau_{V,W}(v \otimes w).
\end{aligned}
$$

Here, we used the notation $\kappa^1 \otimes \kappa^2 = \tilde{\kappa}^1 \otimes \tilde{\kappa}^2 = \tilde{\tilde{\kappa}}^1 \otimes \tilde{\tilde{\kappa}}^2 = \kappa$. The equality $\kappa^1 M(\bar{R}^2_0) \tilde{\kappa}^{(-2)} \otimes \kappa^2 M(\bar{R}^1_0) \tilde{\kappa}^{(-1)} = R^1_0 \otimes R^2_0$ follows from $\pi^* \bar{\sigma} = \sigma \cdot \kappa / \kappa^T$.

The fact that $\xi_V : \mathcal{F}(V^\vee) \to \mathcal{F}(V)^\vee$ is a natural $\bar{u}$-module isomorphism follows from the Remark 6.0.11 that M is a homomorphism of quasi-Hopf algebras, if we replace u by the twisted quasi-Hopf algebra u^J for $J = \sum \kappa(\chi,\psi)\, \delta_\chi \otimes \delta_\psi$. So far, we showed that $\mathcal{F}$ is a ribbon functor.

By Lemma 6.0.12, we know that the radical of the pairing ω_{Rep_u} is k^T. Since $\mathcal{F}$ preserves the radical and $\mathcal{F}(k^T)$ is trivial, $\mathsf{Rep}_{\bar{u}}$ must be modular. As our functor $\mathcal{F}$ is a restriction functor it satisfies the conditions in Prop. 6.0.7 and hence it is a modularization in the sense of Def. 6.0.2. $\square$

Appendix A. Some tools in finite abelian groups

We start with a small quantum group $u := u_q(\mathfrak{g}, \Lambda)$ with R-matrix $R = R_0(f)\bar{\Theta}$ as in Section 4. The Cartan part is given by $u_0 = \mathbb{C}[\Lambda/\Lambda']$, where $\Lambda' = \mathrm{Cent}_{\Lambda_R}(\Lambda)$. The non-degenerate bihomomorphism $f : G_1 \times G_2 \to \mathbb{C}^\times$ defines a braiding on $\mathsf{Vect}_{\widehat{G}}$ given by:

$$\sigma(\chi, \psi) := \chi|_{G_1}\left(f^{-1}\left(\bar{\psi}|_{G_2}\right)\right).$$

From now on, we assume $G = G_1 + G_2$. We set $Rad_0 := Rad\left(f \cdot f^T|_{G_1 \cap G_2}\right) \subseteq G_1 \cap G_2 =: G_{12}$ and $T := Rad\,(\mathrm{B}) \subseteq \widehat{G}$.

Lemma A.0.1. *The following map is an isomorphism:*

$$\begin{aligned} \Phi : Rad_0 &\longrightarrow T \\ \mu &\longmapsto \left(\nu = \nu_1 + \nu_2 \mapsto \frac{f(\nu_1, \mu)}{f(\mu, \nu_2)}\right). \end{aligned}$$

Proof. By definition of T the map Φ is well-defined. It is injective, since $\Phi(\mu) = 1$ implies $f(\nu_1, \mu) = 1$ for all $\nu_1 \in G_1$ and $f(\mu, \nu_2) = 1$ for all $\nu_2 \in G_2$. By the non-degeneracy of f, we have $\mu = 0$. Finally, we show that Φ is surjective. For $\chi \in T$, we have elements $\mu_1 \in G_1$, $\mu_2 \in G_2$, s.t. $\chi|_{G_2} = f(\mu_1, \_\,)$ and $\chi|_{G_1} = f(\,\_\,, \mu_2)$ by non-degeneracy of f. Since χ is in the radical of $\sigma \cdot \sigma^T$, we have

$$\sigma(\chi, \psi)\sigma(\psi, \chi) = f(f^{-1}(\psi|_{G_2}), \mu_2)\psi|_{G_1}(\mu_1) = \psi|_{G_2}(\mu_2)\psi|_{G_1}(\mu_1) = 1.$$

Thus, $\psi(\mu_1) = \psi(-\mu_2)$ for all $\psi \in \widehat{G}$ and hence $\mu_1 = -\mu_2 =: \mu$. This implies

$$\chi = \chi|_{G_1}\chi|_{G_2} = f(\,\_\,, -\mu)(\mu, \_\,) = \Phi(\mu).$$

$\square$

We define two more important groups:

$$\overline{G} := Ann(T) \subseteq G \qquad \overline{G_1 \times G_2} := \{\, (\mu_1, \mu_2) \in G_1 \times G_2 \mid f(\mu_1, \nu) = f(\nu, \mu_2) \quad \forall \nu \in G_{12} \,\}.$$

Corollary A.0.2. *The isomorphism $\Phi : Rad_0 \to T$ induces an isomorphism of exact sequences:*

$$\begin{tikzcd}
Rad_0 \arrow[r, hook, "\iota"] \arrow[d, "\Phi"] & \overline{G_1 \times G_2} \arrow[r, two heads, "\pi"] \arrow[d] & \overline{G} \arrow[d, "\Psi"] \\
T \arrow[r, hook] & \widehat{G} \arrow[r, two heads] & \widehat{G}/T
\end{tikzcd}$$

Here $\iota(\mu) = (\mu, -\mu)$ and $\pi(\mu_1, \mu_2) = \mu_1 + \mu_2$.

Proof. The map $F : \overline{G_1 \times G_2} \to \widehat{G}$ is given by $(\mu_1, \mu_2) \mapsto (\nu = \nu_1 + \nu_2 \mapsto f(\nu_1, \mu_2)f(\mu_1, \nu_2))$ and does not depend on the splitting $\nu = \nu_1 + \nu_2$.
We show that the map π is surjective. Let $\mu = \mu_1 + \mu_2 \in \overline{G}$. We choose a set-theoretic section $\tilde{s} : G_{12}/Rad_0 \to G_{12}$. We can push $f \cdot f^T$ down to a non-degenerate symmetric bihomomorphism $\overline{f \cdot f^T}$ on G_{12}/Rad_0. Hence, there must be an element $x \in G_{12}/Rad_0$, s.t.

$$\frac{f(\mu_1, \_\,)}{f(\,\_\,, \mu_2)} = \overline{f \cdot f^T}(x, \_\,) = f \cdot f^T(\tilde{s}(x), \_\,)$$

as characters on G_{12}/Rad_0. For $s(\mu) = (s(\mu)_1, s(\mu)_2) := (\mu_1 - s(x), \mu_2 + s(x)) \in \overline{G_1 \times G_2}$ we obtain $\pi(s(\mu)) = \mu$. The map Ψ is the well-defined isomorphism given by $\Psi(\mu) = [F(s(\mu))]$. $\square$

Example A.0.3. *Let $G_{12} = G$ and f symmetric and $\mu \in \overline{G}$. We have*

$$f(s(\mu)_1, \_\,) = f(\,\_\,, s(\mu)_2) = f(s(\mu)_2, \_\,)$$

on G and since f is non-degenerate this implies $s(\mu)_2 = s(\mu)_1 =: \tilde{\mu}$. Since $s : \overline{G} \to \overline{G_1 \times G_2}$ is a section, this implies $\mu = 2\tilde{m}u$. On the other hand, for $\nu = 2\tilde{\nu} \in 2G$, by Lemma A.0.1 we have

$$\chi(\nu) = \frac{f(\tilde{\nu}, \mu)}{f(\mu, \tilde{\nu})} = \frac{f(\tilde{\nu}, \mu)}{f(\tilde{\nu}, \mu)} = 1 \qquad \forall \chi \in T.$$

Hence, $\overline{G} = 2G$ in this case.

A.1 Grouplike elements

Let $(\bar{\omega}, \bar{\sigma}) \in Z^3_{ab}\left(\widehat{G}/T\right)$ be an abelian 3- cocycle on $\widehat{G}/T$. The following elements replace the grouplike elements of the Radford biproduct $B(V)\#k^{\widehat{G}/T}$ for the case of a non-trivial 3-cocycle $\bar{\omega}$. For every $\chi \in \widehat{G}/T$, we set

$$L_\chi := \sum_{\psi \in \widehat{G}/T} \bar{\sigma}(\bar{\chi}, \psi)\, \delta_\psi \in k^{\widehat{G}/T}$$

$$\bar{L}_\chi := \sum_{\psi \in \widehat{G}/T} \bar{\sigma}(\psi, \bar{\chi})\, \delta_\psi \in k^{\widehat{G}/T}$$

It is easy to see that the element $L_\chi \bar{L}_\chi \in k^{\widehat{G}/T}$ is grouplike, since $\mathrm{B} = \bar{\sigma}\bar{\sigma}^T$ is a bihomomorphism.

Lemma A.1.1. *The element L_χ is grouplike if and only if the 2-cocycle $\theta(\chi) \in Z^2(\widehat{G}/T)$ from 5.1.5 is trivial.*

Proof. We have

$$\begin{aligned}\Delta(L_\chi) &= \sum_{\psi \in \widehat{G}/T} \bar{\sigma}(\bar{\chi}, \psi)\, \Delta(\delta_\psi) \\ &= \sum_{\psi_1, \psi_2 \in \widehat{G}/T} \bar{\sigma}(\bar{\chi}, \psi_1\psi_2)\, \delta_{\psi_1} \otimes \delta_{\psi_1} \\ &= \sum_{\psi_1, \psi_2 \in \widehat{G}/T} \theta(\bar{\chi})(\psi_1, \psi_2)\bar{\sigma}(\bar{\chi}, \psi_1)\bar{\sigma}(\bar{\chi}, \psi_2)\, \delta_{\psi_1} \otimes \delta_{\psi_1}.\end{aligned}$$

This proves the claim. □

A.2 A particular representative $(\bar{\omega}, \bar{\sigma}) \in Z^3(\widehat{G}/T)$

Let σ be a bihomomorphism on the dual of a finite abelian group $\widehat{G}$, such that the associated quadratic form $Q(\chi) = \sigma(\chi, \chi)$ vanishes on the radical $T = Rad(\mathrm{B})$. Starting with an arbitrary set-theoretic section $s : \widehat{G}/T \to \widehat{G}$ for the quotient map $\pi : \widehat{G} \to \widehat{G}/T$, we want to define an abelian 3-cocycle $(\bar{\omega}, \bar{\sigma})$, such that

$$\pi^*\bar{\omega} = d\kappa^{-1}, \qquad \pi^*\bar{\sigma} = \kappa/\kappa^T.$$

Before we define this abelian 3-cocycle, we notice that $\sigma|_T$ is an alternating bihomomorphism and thus we have $\sigma|_T := \eta/\eta^T$ for some 2-cocycle $\eta : T \times T \to \mathbb{C}^\times$.

Lemma A.2.1. *Let $r(\chi, \psi) := s(\chi)s(\psi)s(\chi\psi)^{-1}$ denote the corresponding 2-cocycle to the set-theoretic section $s : \widehat{G}/T \to \widehat{G}$. Moreover, for $\chi \in \widehat{G}$, we define $\tau_\chi := \chi s(\pi(\chi))^{-1} \in T$. We set $\bar{\sigma} := s^*\sigma$. Together with*

$$\bar{\omega}(\chi, \psi, \xi) := \sigma(s(\xi), r(\chi, \psi))df(s(\chi), s(\psi), s(\xi)), \qquad f(\chi, \psi) := \eta(r(\pi(\chi), \pi(\psi)), \tau_\chi\tau_\psi),$$

this defines an abelian 3-cocycle. Explicitly, we have

$$df(s(\chi), s(\psi), s(\xi)) = \frac{\eta(r(\chi, \psi\xi), r(\psi, \xi))}{\eta(r(\chi\psi, \xi), r(\chi, \psi))}.$$

The 2-cocochain $\kappa \in C^2(\widehat{G})$ satisfying $\pi^*(\bar{\omega}, \bar{\sigma}) = d_{ab}k \cdot (1, \sigma)$ is given by

$$\kappa(\chi, \psi) = (\sigma(\tau_\chi, \psi)\eta(\tau_\psi, \tau_\chi)f(\chi, \psi))^{-1}.$$

Proof. Before we check that $(\bar{\omega}, \bar{\sigma})$ is an abelian 3-cocycle, we show that $\pi^*(\bar{\omega}, \bar{\sigma}) = d_{ab}k \cdot (1, \sigma)$ holds. We have

$$\frac{\kappa(\chi, \psi)}{\kappa(\psi, \chi)}\sigma(\chi, \psi) = \frac{\eta(\tau_\chi, \tau_\psi)}{\sigma(\tau_\chi, \psi)}\frac{\sigma(\tau_\psi, \chi)}{\eta(\tau_\psi, \tau_\chi)}\sigma(\chi, \psi) = \frac{\sigma(\tau_\chi, \tau_\psi)\sigma(\chi, \psi)}{\sigma(\tau_\chi, \psi)\sigma(\chi, \tau_\psi)} = \sigma(\chi\bar{\tau}_\chi, \psi\bar{\tau}_\psi) = \bar{\sigma}(\pi(\chi), \pi(\psi)).$$

Here, we used that σ is a bihomomorphism satisfying $\sigma|_T = \eta/\eta^T$ and $\sigma(\tau,\chi) = \sigma(\chi,\tau)^{-1}$ for $\tau \in T$. For $\pi^*\bar{\omega}$, the following relations are not very hard to check:

$$\begin{aligned} df(\chi,\psi,\xi) &= s^*df(\pi(\chi),\pi(\psi),\pi(\xi))\sigma(r(\chi,\psi),\tau_\xi)dg(\chi,\psi,\xi)^{-1} \\ \sigma(\xi, r(\pi(\chi),\pi(\psi))) &= \sigma(s([\xi]), r(\pi(\chi),\pi(\psi)))\sigma(\tau_\xi, r(\pi(\chi),\pi(\psi))), \end{aligned}$$

where $g(\chi,\psi) = \eta(\tau_\psi,\tau_\chi)$. Thus, we have

$$\pi^*\bar{\omega}(\chi,\psi,\xi) = df(\chi,\psi,\xi)\sigma(\xi, r(\pi(\chi),\pi(\psi)))dg(\chi,\psi,\xi).$$

Hence, $\pi^*\bar{\omega} = d\kappa^{-1}$.

We now want to show that $\bar{\omega}$ as defined above is a 3-cocycle. For this, we compute $d(s^*df)$ and dm, where $m(a,b,c) := \sigma(s(c), r(a,b))$. We start with $d(s^*df)$:

$$\begin{aligned} d(s^*df)(a,b,c,d) &= \frac{s^*df(a,b,c)s^*df(a,bc,d)s^*df(b,c,d)}{s^*df(ab,c,d)s^*df(a,b,cd)} \\ &= \frac{\eta(r(a,bc),r(b,c))}{\eta(r(ab,c),r(a,b))}\frac{\eta(r(a,bcd),r(bc,d))}{\eta(r(abc,d),r(a,bc))}\frac{\eta(r(b,cd),r(c,d))}{\eta(r(bc,d),r(b,c))} \\ &\times \frac{\eta(r(abc,d),r(ab,c))}{\eta(r(ab,cd),r(c,d))}\frac{\eta(r(ab,cd),r(a,b))}{\eta(r(a,bcd),r(b,cd))} \\ &= \frac{\eta(r(ab,cd),r(a,b))}{\eta(r(ab,cd),r(c,d))}\frac{\eta(r(ab,cd)r(a,b),r(c,d))}{\eta(r(ab,cd)r(c,d),r(a,b))} \\ &= \frac{\eta(r(a,b),r(c,d))}{\eta(r(c,d),r(a,b))} \\ &= \sigma(r(a,b),r(c,d)). \end{aligned}$$

Here, we only used the fact that η is a 2-cocycle and that $\sigma|_T = \eta/\eta^T$. On the other hand,

$$\begin{aligned} dm(a,b,c,d) &= \frac{\sigma(s(c),r(a,b))\sigma(s(d),r(a,bc))\sigma(s(d),r(b,c))}{\sigma(s(d),r(ab,c))\sigma(s(cd),r(a,b))} \\ &= \frac{\sigma(s(c),r(a,b))\sigma(s(d),r(a,bc))\sigma(s(d),r(b,c))}{\sigma(s(d),r(ab,c))\sigma(s(cd),r(a,b))}\frac{\sigma(s(d),r(a,b))}{\sigma(s(d),r(a,b))} \\ &= \sigma(s(d),dr(a,b,c))\frac{\sigma(s(c),r(a,b)\sigma(s(d),r(a,b)}{\sigma(s(cd),r(a,b)} \\ &= \sigma(r(c,d),r(a,b)) \\ &= \sigma(r(a,b),r(c,d))^{-1}, \end{aligned}$$

where we used that σ is a bihomomorphism and η is a 2-cocycle. Combining both results, we see that $\bar{\omega}$ is a 3-cocycle. We now want to show that $(\bar{\omega}, \bar{\sigma})$ satifies the hexagon equations. We have

$$\begin{aligned}
\frac{\bar{\omega}(a,b,c)\bar{\omega}(c,a,b)}{\bar{\omega}(a,c,b)} &= \frac{\sigma(s(c), r(a,b))\sigma(s(b), r(c,a))}{\sigma(s(b), r(a,c))} \\
&\times \frac{\eta(r(a,bc), r(b,c))\eta(r(c,ab), r(a,b))\eta(r(ac,b), r(a,c))}{\eta(r(ab,c), r(a,b))\eta(r(ca,b), r(c,a))\eta(r(a,cb), r(c,b))} \\
&= \sigma(s(c), r(a,b)) = \sigma(r(a,b), s(c))^{-1} = \frac{\bar{\sigma}(ab,c)}{\bar{\sigma}(a,c)\bar{\sigma}(b,c)}.
\end{aligned}$$

The second hexagon equation follows from the fact that $\bar{\sigma}\bar{\sigma}^T$ is a bihomomorphism. $\square$

Appendix B. Nichols algebras in braided monoidal categories

In this section, we briefly recall the notion of a Nichols algebra in an abelian rigid braided monoidal category (see [BB13] for details). Moreover, we give a categorical definition of the quantum shuffle product.

Let $V \in \mathcal{C}$ be an object in an abelian braided monoidal category $\mathcal{C}$ with associator α and braiding c. We define $V^n := V^{n-1} \otimes V$ with $V^0 = \mathbb{I}$. The tensor algebra $T(V) := \bigoplus_{i\geq 0} V^i$ has a free algebra structure in $\mathcal{C}$ induced by the multiplications $m_{i,n-i} : V^i \otimes V^{n-i} \to V^n$, given by

$$m_{i,n-i} := (\alpha^{-1}_{V^i,V,V} \otimes \mathrm{id}_{V^{\otimes(n-(i+2))}}) \circ \cdots \circ \alpha^{-1}_{V^i,V^{(n-(i+1))},V}.$$

Let $d_1, d_2 : V \to T(V)\otimes T(V)$ be the canonical inclusions and set $\Delta_1 := d_1 + d_2 : V \to T(V)\otimes T(V)$. Then there is a unique extension $\Delta : T(V) \to T(V) \otimes T(V)$ of Δ_1, s.t. Δ is an algebra homomorphism in $\mathcal{C}$. Moreover, we define a counit on $T(V)$ by $\epsilon|_{\mathbb{I}} = id_{\mathbb{I}}$ and $\epsilon_{V^n} = 0$ for $n \geq 1$. Similarly to the coproduct, we can uniquely extend the map $S_1 := -id_V : V \to V \subseteq T(V)$ to an anti-algebra homomorphism $S : T(V) \to T(V)$, which turns $T(V)$ into a Hopf algebra in $\mathcal{C}$.

It is clear that the braid group B_n with generators σ_i acts as automorphisms of V^n via

$$\begin{aligned}
\sigma_i.(v_1 \otimes \cdots \otimes v_n) &= \left(A_V^{i,n}\right)^{-1} \circ ((\mathrm{id}_{V^{\otimes(i-1)}} \otimes c_{V,V}) \otimes \mathrm{id}_{V^{\otimes(n-(i+1))}}) \circ A_V^{i,n}(v_1 \otimes \cdots \otimes v_n), \text{ where} \\
A_V^{i,n} &:= (A_V^i \otimes \mathrm{id}_{V^{\otimes(n-(i+2))}}) \circ \cdots \circ (A_V^1 \otimes \mathrm{id}_{V^{\otimes(n-3)}}) \\
A_V^i &:= (\mathrm{id}_V \otimes A^{i-1}) \circ \alpha_{V,V^{\otimes n},V}, \qquad A_V^1 := \alpha_{V,V,V}.
\end{aligned}$$

Here, the paranthesis of the tensor powers $V^{\otimes n}$ in the indices is understood. Let $\rho : S_n \to B_n$ be the Matsumoto section of the canonical epimorphism $B_n \twoheadrightarrow S_n$. We define the so-called Woronowicz symmetrizer:

$$Wor(c)_n := \sum_{\sigma\in S_n} \rho(\sigma) \in End(V^n) \qquad Wor(c) := \bigoplus_{n=1}^{\infty} Wor(c)_n \in End(T(V)).$$

Definition B.0.1. *The Nichols algebra $B(V)$ of V in $\mathcal{C}$ is defined as the quotient Hopf algebra $T(V)/ker(Wor(c))$.*

Remark B.0.2. *The Nichols algebra $B(V)$ has two important equivalent characterizations:*

- *We can extend the evaluation map $ev_V : V^\vee \otimes V \to \mathbb{I}$ to a unique Hopf pairing $T(V^\vee) \otimes T(V) \to \mathbb{I}$. This Hopf pairing factors through a non-degenerate Hopf pairing $B(V^\vee) \otimes B(V) \to \mathbb{I}$.*
- *It can be shown that $B(V)$ is the unique quotient Hopf algebra of $T(V)$, s.t. $V \subseteq B(V)$ and $ker(\Delta_n - (1 \otimes id_{V^n} + id_{V^n} \otimes 1)) = 0$ for $n > 1$.*

In order to define the quantum shuffle-algebra, we define a different Hopf structure on $T(V)$. Similar to the free algebra structure from above, we can endow $T(V)$ with the cofree coalgebra structure. Moreover, we define a multiplication on $T(V)$ as follows:

Definition B.0.3 (Quasi-quantum shuffle product)**.** *A permutation $\sigma \in S_n$ is called an i-shuffle if*

$$\sigma(1) < \cdots < \sigma(i), \qquad \sigma(i+1) < \cdots < \sigma(n).$$

We define a multiplication $\mu_{i,n-i} : V^i \otimes V^{n-i} \to V^n$ by

$$\mu_{i,n-i} := \sum_{\sigma:\, i-shuffle} \rho(\sigma).m_{i,n-i}. \tag{B.1}$$

The induced product on $T(V)$ is denoted by $* : T(V) \otimes T(V) \to T(V)$. We call this the braided shuffle product. In the case $\mathcal{C} = {}^H_H\mathcal{YD}$, where H is a quasi-Hopf algebra, we call this the quasi-quantum shuffle product.

Again, we can define a corresponding unit and antipode uniquely in order to turn $T(V)$ into a Hopf algebra in $\mathcal{C}$, which we now denote by $t(V)$. The Hopf algebras $B(V)$ and $t(V)$ are related as follows:

Proposition B.0.4. *The Woronowicz symmetrizer $Wor(c) : B(V) \to t(V)$ is a monomorphism of Hopf algebras in $\mathcal{C}$. The image of $Wor(c)$ is simply the Hopf subalgebra of $t(V)$ generated by V as an algebra.*

Appendix C. Factorizable quasi-Hopf algebras

In [BT04], the authors define the notion of factorizability for quasi-Hopf algebras:

Definition C.0.1. *Let (H, R) be a quasi-triangular quasi-Hopf algebra and H^* the corresponding coquasi-Hopf algebra. We will say that H is factorizable if the following linear map is bijective:*

$$\begin{aligned} Q : H^* &\longrightarrow H \\ f &\longmapsto f\left(S(X^2_{(2)}\tilde{p}^2)f^1R^2r^1g^1S(q^2)X^3\right)X^1S(X^2_{(1)}\tilde{p}^1)f^2R^1r^2g^2S(q^1). \end{aligned}$$

Here $R = R^1 \otimes R^2 = r^1 \otimes r^2$, $f = f^1 \otimes f^2$ and $f^{-1} = g^1 \otimes g^2$ (see Sec. 2.1).

They also defined braided Hopf structures on H and H^* in the braided monoidal category Mod_H (see [BT04], Sec.4). To avoid confusion, they denoted these braided Hopf algebras by $\underline{H}$ and $\underline{H}^*$. They furthermore showed that Q is a braided Hopf algebra homomorphism.
In [FGR17], the authors gave an alternative interpretation for $\underline{H}$, $\underline{H}^*$ and Q by proving the following statements:

Proposition C.0.2. *Let H be a finite dimensional quasi-triangular quasi-Hopf algebra. Furthermore, let $\underline{H}, \underline{H}^* \in \mathsf{Mod}_H$ be the braided Hopf algebras as defined in [BT04], Sec.4. Then we have*

1. *$\underline{H}$ is the end over the functor $(\_) \otimes (\_)^\vee$ with dinatural transformation given by $\pi_X(h) = h.e_i \otimes e^i$.*

2. *$\underline{H}^*$ is the coend over the functor $(\_)^\vee \otimes (\_)$ with dinatural transformation given by $\iota_X(f \otimes x) := f(\_.x)$.*

3. *The morphism $Q : \underline{H}^* \to \underline{H}$ is uniquely determined by*

$$\pi_Y \circ Q \circ \iota_X = (ev_X \otimes id_{Y \otimes Y^\vee}) \circ (id \otimes c^2_{X,Y} \otimes id) \circ (id_{X^\vee \otimes X} \otimes coev_Y),$$

where we omitted the associators in Mod_H.

Remark C.0.3. *Note that in [BT04] the authors did not assume a ribbon structure on H in order to define the braided Hopf structure on the coend $\underline{H}^*$. However, up to the antipode it is exactly the* *same Hopf structure as for example defined in [KL01]. One can show that the antipode in [BT04] is uniquely determined by*

$$S \circ \iota_X = (\iota_{X^\vee} \otimes ev_X) \circ (c^{-1}_{X^\vee, X^{\vee\vee} \otimes X^\vee} \otimes id_X) \circ (coev_{X^\vee} \otimes id_{X^\vee \otimes X}),$$

where we again omitted the associators.

In [BPVO00], the authors give a more general interpretation for the underline ($\underline{\quad}$):

Proposition C.0.4. *Let H be a quasi-Hopf algebra, A an associative algebra and $f : H \to A$ an algebra homomorphism. Then we can define a new multiplication on A via*

$$a \underline{\cdot} b := f(X^1) a f(S(x^1 X^2) \alpha x^2 X^3_1) b f(S(x^3 X^3_2)).$$

With this multiplication, unit given by β and left H-action given by $h.a := f(h_1) a f(S(h_2))$, A becomes a left H-module algebra, i.e. an algebra in Rep_H, which we denote by $\underline{A}$.

It is easy to see that $f : \underline{H} \to \underline{A}$ then becomes a left H-module algebra homomorphism. Note that this is exactly our situation in Section 6, where we have an algebra homomorphism $M : \bar{u} \to u$. Similarly, if H and A are quasitriangular quasi-Hopf algebras and $f : H \to A$ is a homomorphism of such, we can endow A with a braided Hopf algebra structure in Rep_H with H-module algebra structure as above and comultiplication, counit and antipode given by

$$\begin{aligned}
\underline{\Delta}(a) &: = f(x^1 X^1) a_1 f(g^1 S(x^2 R^2 y^3 X^3_2)) \otimes x^3 R^1.(f(y^1 F^2) a_2 f(g^2 S(y^2 X^3_1))) \\
\underline{\epsilon}(a) &: = \epsilon(a) \\
\underline{S}(a) &: = f(X^1 R^2 p^2) S(f(q^1)(X^2 R^1 p^1.a) S(f(q^2)) f(X^3)).
\end{aligned}$$

Conversely, if $f : A \to H^*$ is a homomorphism of coquasi-Hopf algebras, we can endow A with the structure of a braided Hopf algebra, denoted by $\underline{A}$, in the category of right H^*-comodules (see Thm. 3.5 in [BT04]) which can be identified with the category of left H-modules Rep_H.

Summary of results

1. In the first part, we reinterpret the conditions in [Mül98] on the Lusztig ansatz $R = R_0\bar{\Theta}$ to give an R-matrix on a small quantum group u (see Section 0.4) in terms of the non-degeneracy of a certain bihomomorphism $\hat{f} : G_1 \times G_2 \to \mathbb{C}^\times$, respectively $a_g^\ell : H_1 \times H_2 \to \mathbb{C}^\times$, leading to an extension of the results in [LN15] (see Cor. 3.2.7).
   We show that irreducible transparent objects in Rep_u are 1-dimensional and classify them (see Cor. 5.4.6). In particular, we show that factorizability of the above R-matrix is equivalent to the non-degeneracy of the symmetrization $Sym(f)$ (see Def. 5.1.2) of the non-degenerate bihomomorphism f (see Thm. 5.1.6).
   We compute all possible R-matrices of the above form and the corresponding groups of transparent objects for all small quantum groups and collect them in Table 1.
   Finally, we construct a ribbon structure on every quasi-triangular small quantum group of the above form (see Thm. 6.0.1).

2. In the second part, we explicitly construct a modularization of $\mathsf{Vect}_G^{(\omega,\sigma,\eta)}$ using a different approach than [Bru00] (see Prop. 3.0.1).
   We construct a family of finite-dimensional quasi-triangular quasi-Hopf algebras, generalizing extended small quantum groups. Moreover, we compute the relevant relations for them (see Thm. 5.0.1).
   We give sufficient conditions for the modularizability of representation categories of small quantum groups. If they are fulfilled, we define an explicit modularization functor (see Thm. 6.0.6).

The first part of this thesis is based on the publication [LO17].

Zusammenfassung der Resultate

1. Im ersten Teil interpretieren wir die Bedingungen in [Mül98] für den Lusztig-Ansatz $R = R_0\bar{\Theta}$ einer R-Matrix auf einer kleinen Quantengruppe neu (siehe Kap. 0.4). Eine wesentliche Rolle spielt dabei die Nicht-Degeneriertheit eines Bihomomorphismus $\hat{f} : G_1 \times G_2 \to \mathbb{C}^\times$, bzw. $a_g^\ell : H_1 \times H_2 \to \mathbb{C}^\times$. Dies führt zu einer Erweiterung der Ergebnisse in [LN15] (siehe Kor. 3.2.7).
   Wir zeigen, dass die transparenten Objekte in Rep_u 1-dimensional sind und klassifizieren sie (siehe Kor. 5.4.6). Insbesondere zeigen wir, dass die Nicht-Degeneriertheit der obigen R-Matrix äquivalent zur Nicht-Degeneriertheit der Symmetrisierung $Sym(f)$ (siehe Def. 5.1.2) des nicht-degenerierten Bihomomorphismus f ist (siehe Thm. 5.1.6).
   Wir berechnen alle möglichen R-Matrizen der obigen Form und die jeweilige Gruppe der transparenten Objekte für alle kleinen Quantengruppen und sammeln diese in Tabelle 1.
   Schliesslich konstruieren wir ein Band-Element auf jeder quasi-triangulären kleinen Quantengruppe der obigen Form (siehe Thm. 6.0.1).

2. Im zweiten Teil konstruieren wir eine explizite Modularisierung von $\mathsf{Vect}_G^{(\omega,\sigma,\eta)}$ unter Verwendung eines anderen Ansatzes als in [Bru00] (siehe Prop. 3.0.1).
   Wir konstruieren eine Familie endlich-dimensionaler quasi-triangulärer Quasi-Hopf Algebren, welche erweiterte kleine Quantengruppen verallgemeinern. Weiterhin berechnen wir die relevanten Relationen dieser Quasi-Hopf Algebren (siehe Thm. 5.0.1).
   Wir geben notwendige Bedingungen für die Modularisierbarkeit der Darstellungskategorien kleiner Quantengruppen. Falls diese erfüllt sind, definieren wir einen expliziten Modularisierungsfunktor (siehe Thm. 6.0.6).

Der erste Teil dieser Thesis basiert auf der Publikation [LO17].

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Declaration on oath/Eidestattliche Erklärung

I hereby declare, on oath, that I have written the present dissertation by my own and have not used other than the acknowledged resources and aids.

Hiermit erkläre ich an Eides statt, dass ich die vorliegende Dissertationsschrift selbst verfasst und keine anderen als die angegebenen Quellen und Hilfsmittel benutzt habe.

Hamburg, 2018-06-20 Signature