Connectivity in directed and undirected infinite graphs

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Karl Heuer
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Vorsitzender der Prüfungskommission: Prof. Dr. Nathan Bowler

Erstgutachter: Prof. Dr. Reinhard Diestel

Zweitgutachter: Prof. Mathias Schacht, Ph.D.

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In loving memory of René Heuer

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Introduction

The notion of connectivity is probably one of the most fundamental ones in graph theory. Many variants of connectivity arose from the original graph invariant 'connectivity' and its aspects can be found in all areas of graph theory. This dissertation focuses on infinite graphs, directed ones as well as undirected ones, for which we study different aspects of connectivity. Infinite graph theory itself has many different branches which can be analysed. Three of them are mainly represented in this dissertation and, therefore, also form the chapters of it. Now we mention a few words about these three branches of infinite graph theory and how they appear in this dissertation, before we address the aspects of connectivity which we study within these branches.

Overview about the chapters

Chapter I deals with general infinite graph theory. By this we mean that we in general do not equip the graph with any additional structure, like a topology or an orientation for each of its edges. Furthermore, we do not restrict the cardinality of the studied graphs, neither directly by further assumptions nor indirectly via assumptions on the structure of the graphs, like being connected and locally finite implies being of countable size.

For Chapter II we work within topological infinite graph theory. Different from general infinite graph theory we only consider locally finite connected graphs, i.e., connected graphs where each vertex has finite degree, in this chapter. More precisely, in topological infinite graph theory we view such a graph G also as a 1-complex and then compactify it using the Freudenthal compactification [19], which yields the compact topological space $|G|$ (cf. [12, Section 8.5] and [13]). Analysing the space $|G|$ together with G, we can gain more information about the graph G than studying it solely, especially since we are able to consider the graph from another, namely topological point of view. Another benefit is that many concepts from finite graph theory can now more easily be generalised to infinite graphs by first interpreting them in a topological way and then studying their analogue in $|G|$. A main example to mention of this topological approach, initiated by Diestel and Kühn [17, 18], is the definition of cycles via topology. Cycles of a finite graph can be seen as homeomorphic images of the unit circle $S^1 \subseteq \mathbb{R}^2$ in the graph seen as a 1-complex. Generalising this definition slightly for any locally finite connected graph G by asking for homeomorphic images of the unit circle $S^1 \subseteq \mathbb{R}^2$ in $|G|$, we have obtained a definition for cycles which coincides with the usual definition for finite cycles, but does not necessarily restrict them to be finite anymore. So we have gained a sensible notion of infinite cycles for locally finite connected graphs.

The last chapter of this dissertation, Chapter III, is dedicated to directed infinite graphs. Apart from allowing more than one edge between the same pair of vertices, all edges are now also equipped with one of the two possible directions. Although in general considered in directed graph theory as well, we shall not work with edges that have only one endvertex, which are commonly called *loops*. The additional overlying directed structure on top of a multigraph allows to ask new questions about the graphs, but often brings new difficulties with it. While in general we do not want to restrict the cardinality of directed infinite graphs, we shall indirectly do this in the first section of that chapter. The reason for this is that we shall use methods from topological infinite graph theory in that section and, therefore, restrict our analysis to directed graphs, whose underlying undirected multigraph is locally finite and connected. Note that, as for undirected graphs, we call an undirected multigraph *locally finite* if each of its vertices is incident with only finitely many edges. Hence, the directed graphs considered in the first section of Chapter III will be of countable cardinality. In the other section of that chapter we shall again work with directed infinite graphs of arbitrary cardinality.

All chapters are subdivided by sections, each addressing a certain aspect of connectivity. For this reason we begin each section with its own rather specialised introduction. Let us now give an overview about the aspects of connectivity which are analysed in this dissertation.

The sections of Chapter I

One possibility to generalise the concept of connectivity of graphs is the following. Given a vertex set $A \subseteq V(G)$ of some graph G it might be that we cannot easily separate small subsets of the vertices of A from each other inside of G, although the graph $G[A]$ induced by A is not highly connected. More precisely, for $k \in \mathbb{N}$ we want to define a vertex set $A \subseteq V(G)$ of a graph G to be *k-connected in G*, if A contains at least k vertices and for any two subsets $A_1, A_2 \subseteq A$ of A of size $|A_1| = |A_2| \leqslant k$ there are $|A_1|$ many disjoint A_1–A_2 paths in G. As usual an X–Y path in G for two vertex set $X, Y \subseteq V(G)$ is a path in G that is disjoint from $X \cup Y$, except from its endvertices one of which lies in X while the other lies in Y. A remarkable consequence of this definition is that for each separation of the graph whose order $\ell \in \mathbb{N}$ is less than $k \in \mathbb{N}$, all but at most ℓ vertices of a k-connected vertex set lie on the same side of the separation.

Let us consider some examples of k-connected vertex sets. For integers $m, n \in \mathbb{N}$ we define the *$m \times n$ grid* as the following graph. The vertex set of the $m \times n$ grid consists of all pairs (i, j) of integers i, j satisfying $1 \leqslant i \leqslant m$ and $1 \leqslant j \leqslant n$. We define two of its vertices to be adjacent if their coordinates differ by precisely 1 in total. Similarly, we define the *full grid*, or more briefly $\mathbb{Z} \times \mathbb{Z}$, as the graph on all pairs of integers where the adjacency relation is defined as just before. Since the degree of all vertices in an $m \times n$ grid and in the full grid is at most 4, these graphs are at most 4-connected. However, if a graph G contains an $m \times n$ grid, then a vertex set forming a row or a column of that grid is k-connected in G for $k := \min\{m, n\}$. Furthermore, if the full grid is contained in G, then we even obtain a vertex set of countably infinite size which is k-connected in G for every $k \in \mathbb{N}$.

In Section A we restrict our attention to the graph $\mathbb{Z} \times \mathbb{Z}$ and the question which graphs contain $\mathbb{Z} \times \mathbb{Z}$ as a minor. We answer this question with Theorem A.1.4, which characterises these graphs in terms of the existence of certain set of disjoint rays, i.e., one-way infinite paths. Furthermore, we prove a duality theorem characterising the graphs that do not contain $\mathbb{Z} \times \mathbb{Z}$ as a minor by the existence of a certain tree-decomposition of the graph. While similar theorems (cf. Theorem A.1.2 due to Halin [30, Satz 4′] and Theorem A.1.3 due to Robertson, Seymour and Thomas [53, (2.6)]) were known for the *half grid*, also called $\mathbb{N} \times \mathbb{Z}$

and analogously defined as $\mathbb{Z} \times \mathbb{Z}$ above, nothing has been known for the full grid until Theorem A.1.4 has been proved.

Section B is dedicated to the characterisation of those graphs that contain a k-connected vertex set of a fixed but arbitrary infinite cardinality κ, where $k \in \mathbb{N}$. This is done in Theorem B.3.7. The characterisation is stated in terms of the existence of minors of certain, so-called *k-typical graphs* of size κ. We give an equivalent criterion via the existence of subdivisions of certain, so-called *generalised k-typical graphs* of size κ. Furthermore, we prove that the set all of k-typical graphs of size κ and generalised k-typical graphs of size κ is finite if we consider a fixed number $k \in \mathbb{N}$ and a fixed cardinality κ. Similarly as in the main result of Section A, we also prove a duality theorem characterising graphs without k-connected vertex set of size κ via the existence of a certain nested set of separations of the graph each of which has order less than k.

Related results have been proved for finite graphs by Geelen and Joeris [24, 39]. For infinite graphs and k-connected vertex set of infinite cardinality κ, not much has been known so far, especially not if κ is a singular cardinal. People rather studied the question which substructures arise in highly connected infinite graphs. For $k \in \mathbb{N}$ Halin [31] proved that every k-connected graph of size κ, for some regular cardinal κ, contains a subdivision of $K_{k,\kappa}$. In the case where $\kappa = \aleph_0$ Oporowski, Oxley and Thomas [49] refined this result with respect to the minor relation and proved that every countable infinite k-connected graph either contains $K_{k,\aleph_0}$ as a minor or another one out of a finite set of graphs. (In fact they used a slightly weaker condition as being k-connected, but we omit to state it here.) The $K_{k,\aleph_0}$ as well as the other graphs occurring in the result of Oporowski, Oxley and Thomas [49] will also occur as the k-typical graphs of countable size in our Theorem B.3.7.

In Section C, the last one of Chapter I, we answer a question proposed by Georgakopoulos [25, Problem 1]. He formulated this problem when he studied the same aspect of connectivity which is covered in all of Chapter II of this dissertation, namely Hamiltonicity for locally finite connected graphs. Before we focus on topological infinite graph theory and the relation between Hamiltonicity and connectivity, let us state the main result of Section C. Although the question of Georgakopoulos appeared in the context of topological infinite graph theory, the question itself is a purely graph theoretical one about infinite graphs. In order to state the question of Georgakopoulos, we have to give some definitions first.

In a graph G we call two rays *equivalent* if they cannot be separated by finitely many vertices. It is easy to check that this defines an equivalence relation on the set of all rays of G. The equivalence classes of this relation are called the *ends* of G. The elements of an end ω of G are called *ω-rays*. Now we call an end of G a *countable end*, if it does not contain uncountably many disjoint rays. For an end ω of G we, furthermore, say that a set of ω-rays $\mathcal{R}$ *devours* ω if every ω-ray in G has a non-empty intersection with $\bigcup \mathcal{R}$. Now we are able to state the question proposed by Georgakopoulos:

Question 1. *Let G be a graph, ω be a countable end of G and $\mathcal{R}'$ be any set of disjoint ω-rays. Does there exist a set $\mathcal{R}$ of ω-rays such that $\mathcal{R}$ devours ω and the set of startvertices of the rays in $\mathcal{R}$ equals the set of startvertices of the rays in $\mathcal{R}'$?*

We affirmatively answer Question 1 with Theorem C.1.2. Previously, Georgakopoulos [25, Lemma 10] had already proved the existence of such an ω-devouring set $\mathcal{R}$, but only in the special case where the set $\mathcal{R}'$ of ω-rays is finite. Nevertheless, this special case was a helpful ingredient in his proof [25, Thm. 3] that the square of every locally finite 2-connected graph is Hamiltonian (cf. Theorem D.1.2), which generalises a theorem of Fleischner [23] for finite graphs (cf. Theorem D.1.1).

The section in Chapter II

We proceed with the aspect of connectivity which is addressed by the sole section of Chapter II, namely Section D. Let us first consider finite undirected graphs. In general being 2-connected has an equivalent and quite illustrative reformulation, which says that any pair of vertices of a graph lies on a common cycle. However, for different pairs of vertices the cycles might also be different. Asking whether there exists one cycle in a graph which is a common witness of the 2-connectivity of the graph for all pairs of vertices at the same time, is a question that is difficult to answer. Such a cycle contains all vertices of the graph and is called a *Hamilton cycle*. Obviously, not all 2-connected graphs have a Hamilton cycle, but those that have one are called *Hamiltonian*.

Hamiltonicity of locally finite connected graphs is the connectivity related topic that is studied in Section D. As mentioned at the beginning of this introduction, the topological definition of cycles via homeomorphic images of the unit circles S^1 in the space $|G|$ for a locally finite connected graph G, now allows cycles that correspond to infinite subgraphs of G. So we are able to sensibly consider the question whether a locally finite connected graph G is Hamiltonian. Before we describe the content of Section D in more detail, let us briefly come back to the initial motivation of studying Hamiltonicity. In finite graphs, a Hamilton cycle is a witness for the 2-connectivity of the graph. At first sight it might not be clear whether the subgraph corresponding to a Hamilton cycle in an infinite locally finite connected graph could also suffice to prove the 2-connectivity of the graph, although it will not be a direct witness anymore containing two disjoint paths between any two vertices. However, this statement is true (cf. Corollary D.2.9).

The first two main results of Section D, Theorem D.1.5 and Theorem D.1.8, are extensions to locally finite connected graphs of theorems stating sufficient conditions for the Hamiltonicity of finite graphs. Instead of giving a precise formulation of them at this point, we refer to the introduction of Section D.

Let us mention the last result of Section D in more detail. Note that a Hamilton cycle in a finite graph might not be unique, if it exists at all. For locally finite connected graphs this remains the same. So the property of a graph to be *uniquely Hamiltonian*, i.e., there exists a unique Hamilton cycle for the graph, is very restrictive. For finite graphs, Sheehan [56] conjectured that no r-regular graph exists that is uniquely Hamiltonian, if $r > 2$ (cf. Conjecture D.1.10). Obviously, a cycle is a 2-regular graph that is uniquely Hamiltonian, justifying the condition $r > 2$. This conjecture is still open for finite graphs, but several partial results have been obtained [34, 61, 64].

Mohar [47] has asked an analogue question for infinite graphs (cf. Question D.1.11) and we answer it with Theorem D.1.12, which constructs a uniquely Hamiltonian cubic connected graph each of whose ends has an additional property as occurring in Mohar's question.

The sections of Chapter III

Although Chapter III is about directed graphs, which we briefly call *digraphs* from now on, the research in Section E has its beginning in undirected finite graph theory, namely with spanning trees and its connection to edge-connectivity. Spanning trees exist precisely when the corresponding multigraph is (edge-)connected. Furthermore, they have the remarkable property of being edge-minimal under the condition of meeting every non-empty cut of a connected multigraph. The existence of several, edge-disjoint spanning trees is characterised by the following famous result independently proved by Nash-Williams and Tutte.

Theorem 2. [48, 67], [12, Thm. 2.4.1] *A finite multigraph G has $k \in \mathbb{N}$ edge-disjoint spanning trees if and only if for every partition $\mathcal{P}$ of $V(G)$ there are at least $k(|\mathcal{P}| - 1)$ edges in G whose endvertices lie in different partition classes.*

This theorem has a qualitative corollary saying that every finite $2k$-edge-connected multigraph has k edge-disjoint spanning trees where $k \in \mathbb{N}$. So high edge-connectivity of a finite multigraph guarantees the existence of many edge-disjoint spanning trees.

In Section E we consider an analogue of Theorem 2 for finite digraphs, namely Edmond's Branching Theorem (cf. Theorem E.1.2). For a finite digraph D with a vertex $r \in V(D)$ we call a digraph obtained by taking an undirected spanning tree of the underlying undirected multigraph of D and then directing all its edges away from r a *spanning arborescence rooted in* r of D. Let us now state Edmond's Branching Theorem.

Theorem 3. [21], [3, Thm. 9.5.1] *A finite digraph G with a vertex $r \in V(G)$ has $k \in \mathbb{N}$ edge-disjoint spanning arborescences rooted in r if and only if there are k edges from X to Y for every bipartition (X, Y) of $V(G)$ with $r \in X$.*

So Edmond's Branching Theorem characterises, similarly as Theorem 2 for finite undirected multigraphs, which finite digraphs admit the existence of k edge-disjoint spanning arborescences with a common root where $k \in \mathbb{N}$.

We generalise this theorem to digraphs whose underlying undirected multigraph is locally finite and connected. We need this restriction on the structure of the digraph because we use methods from topological infinite graph theory similarly as in Chapter II. For our corresponding theorem we introduce the notion of *pseudo-arborescences*, which is a generalisation of ordinary arborescences in finite digraphs to infinite digraphs whose underlying undirected multigraph is locally finite and connected. The corresponding main result, Theorem E.4.3, characterises those digraphs that admit the existence of k edge-disjoint spanning pseudo-arborescences with a common root where $k \in \mathbb{N}$. Furthermore, we study the structure of spanning pseudo-arborescences, especially in which way they behave like trees and in which way they do not.

Section F is dedicated to an aspect of connectivity exclusively studied in digraphs, namely strong connectivity. In this section we again consider digraphs of arbitrary cardinality. Before we continue let us quickly state the definition of a digraph being strongly connected.

We call a digraph D *strongly connected* if for any two vertices $v, w \in V(D)$ there exist both in D, a directed path from v to w and a directed path from w to v. By name related we call a digraph *weakly connected* if its underlying undirected multigraph is connected.

An obvious obstruction for a digraph D to be strongly connected would be a cut of D where all edges of the cut have their head in a common side of that cut and their tail on the other. We call such a cut a *directed cut of* D, or briefly a *dicut of* D. It is easy to check that dicuts are in fact the only obstruction preventing a digraph from being strongly connected. So if we want to turn any weakly connected digraph D into a strongly connected one, we could achieve this by contracting an edge from every non-empty dicut of D. We call a set of edges of D which meets every non-empty dicut of D a *dijoin* of D. Now the minimum size of a dijoin of a digraph D measures how 'close' D is to being strongly connected. The following theorem of Lucchesi and Younger states an important fact about this parameter.

Theorem 4. [46, Thm.] *In every weakly connected finite digraph, the maximum number of disjoint dicuts equals the minimum size of a dijoin.*

With Theorem 4 in mind we shall study the relation between dijoins and dicuts of infinite digraphs in Section F. First of all we give an example that a direct extension to infinite digraphs of this theorem fails if we consider infinite dicuts as well. So we restrict our attention to finite dicuts and call an edge set of a digraph D

a *finitary dijoin* of D if it meets every finite dicut of D. Building up on this we state a conjecture (cf. Conjecture F.1.5) which, if verified, extends Theorem 4 to infinite digraphs in a way as the theorem of Menger for finite graphs (cf. Theorem F.1.1) has been extended to infinite graphs by Aharoni and Berger [1].

One of our main results in Section F is a reduction of Conjecture F.1.5 to countable digraphs, meaning that it is sufficient to verify Conjecture F.1.5 just for countable digraphs. Let us remark here that the proof of the infinite version of Menger's theorem is highly complicated, especially for graphs of uncountable cardinality. This might be an indication that extending Theorem 4 to infinite digraphs could be easier than the proof of the extension of Menger's theorem to infinite graphs. The other main results are verifications of Conjecture F.1.5 for several classes of digraphs.

Chapter I.

General infinite graph theory

A. Excluding a full grid minor

A.1. Introduction

In extremal graph theory it is common to analyse the structure of graphs which do not contain a certain minor or subdivision of some graph. This goes hand in hand with the search for conditions in terms of graph invariants, such as degree conditions, that force the existence of certain minors or subdivisions. Extending the scope of extremal questions to include infinite graphs, it is helpful to consider new graph invariants, which may not be defined for finite graphs, in order to gain more information about the structure of infinite graphs. For an overview of results in the field of extremal infinite graph theory see the surveys of Diestel [13] and of Stein [57].

One example for such a new invariant is the degree of an end of a graph. The *ends* of a graph are the equivalence classes of the *rays*, i.e., one-way infinite paths, where we say that two rays are equivalent if an only if they cannot be separated by finitely many vertices in the graph. Now the *degree* of an end is defined as the maximum number of disjoint rays in this end (including 'infinitely many'). The foundation of this definition, namely, that the end degree is well-defined, is provided by the following theorem of Halin.

Theorem A.1.1. [30, Satz 1] *If a graph contains n pairwise disjoint rays for every $n \in \mathbb{N}$, then it contains infinitely many pairwise disjoint rays.*

Furthermore, although without stating the term 'end degree' explicitly, Halin used the following theorem to show that an end of infinite degree forces the existence of an $\mathbb{N} \times \mathbb{N}$ grid minor. In fact he actually proved that it forces a subdivision of the graph H^∞ shown in Figure A.1. Then the statement about the $\mathbb{N} \times \mathbb{N}$ grid minor follows, since the graph H^∞ contains the $\mathbb{N} \times \mathbb{N}$ grid as a minor. Note that the question of whether a graph contains an $\mathbb{N} \times \mathbb{Z}$ grid minor is not more difficult than asking for an $\mathbb{N} \times \mathbb{N}$ grid minor since the $\mathbb{N} \times \mathbb{N}$ grid contains a subdivision of the $\mathbb{N} \times \mathbb{Z}$ grid.

Theorem A.1.2. [30, Satz 4′] *Whenever a graph contains infinitely many pairwise disjoint and equivalent rays, it contains a subdivision of H^∞.*

Beside Halin's proof of Theorem A.1.2, there is now also a shorter proof of this theorem by Diestel (see [14] or [12, Thm. 8.2.6]). Note that the converse of this implication is obviously true as well. So Theorem A.1.2 gives a characterisation of graphs without a subdivision of H^∞ and therefore also of graphs without an $\mathbb{N} \times \mathbb{Z}$ grid minor.

Robertson, Seymour and Thomas characterized the structure of graphs without $\mathbb{N} \times \mathbb{Z}$ grid minors as those that have tree-decompositions into finite parts and with finite adhesion. A tree-decomposition into finite parts has *finite adhesion* if along each ray of the tree the sizes of the adhesion sets corresponding to its edges are infinitely often less than some fixed finite number. Given such a tree-decomposition, an $\mathbb{N} \times \mathbb{Z}$ grid minor cannot be contained in a part because all of these are finite. The only other possibility where such a grid minor could lie in a graph would be in the union of the parts along a ray of the tree of the tree-decomposition. However, the finite adhesion prevents this possibility.

Theorem A.1.3. [53, (2.6)] *A graph has no $\mathbb{N} \times \mathbb{Z}$ grid minor if and only if it has a tree-decomposition into finite parts and with finite adhesion.*

While all the above theorems give characterisations for when graphs do or do not contain an $\mathbb{N} \times \mathbb{Z}$ grid minor, it was not clear whether a similar characterisation exists for $\mathbb{Z} \times \mathbb{Z}$ grids. The main theorem of Section A, Theorem A.1.4, and Corollary A.1.5 give characterisations for a $\mathbb{Z} \times \mathbb{Z}$ grid minor in the same spirit as the results above do for an $\mathbb{N} \times \mathbb{Z}$ grid minor. The key idea is to consider not just sets of disjoint equivalent rays but *bundles*, which are sets of disjoint equivalent rays having the additional property that there are infinitely many disjoint cycles that intersect with each ray of the bundle, but only in a path. Graphically, the cycles of a bundle can be viewed as concentric cycles around the common end in which the rays of the bundle lie. It is not difficult to see that graphs with a $\mathbb{Z} \times \mathbb{Z}$ grid minor contain arbitrarily large bundles. But it turns out that the converse is also true, and so containing arbitrarily large bundles is not only necessary for the existence of a $\mathbb{Z} \times \mathbb{Z}$ grid minor, but also sufficient. Now let us state the main theorem and its corollary precisely. See Section A.2 for the definitions of the involved terms.

Theorem A.1.4. *For a graph G the following are equivalent:*

(i) *There is an end ω of G and n-bundles B_n for every $n \in \mathbb{N}$ with $B_n \subseteq \omega$.*

(ii) *There is an ∞-bundle in G.*

(iii) *There is a consistent ∞-bundle in G.*

(iv) *G contains a subdivision of the Dartboard.*

(v) *G contains a $\mathbb{Z} \times \mathbb{Z}$ grid as a minor.*

(vi) *G contains a set $\mathcal{R}$ of infinitely many equivalent disjoint rays such that for every $R \in \mathcal{R}$ all rays in $\mathcal{R} \backslash \{R\}$ are still equivalent in $G - R$.*

Corollary A.1.5. *A graph has no $\mathbb{Z} \times \mathbb{Z}$ grid minor if and only if it has a bundle-narrow tree-decomposition into finite parts distinguishing all ends.*

The rest of Section A is organized as follows. In Section A.2 we state the definitions and notation that we need in all of Section A. Furthermore, we collect known results which we shall use in the proof of the main theorem and its corollary. The proofs of Theorem A.1.4 and of Corollary A.1.5 are the content of Section A.3.

A.2. Preliminaries

In this section, we list important definitions, notation and already known results needed for the rest of Section A. In general, we will use the graph theoretical notation of [12] in Section A. For basic facts about graph theory, especially for infinite graphs, the reader is referred to [12] as well.

All graphs we consider in Section A are undirected and simple. Furthermore, we do not assume a graph to be finite unless we state this explicitly.

For $n \geqslant 3$ we write C_n for the cycle with n vertices and for $m, k \in \mathbb{N}$ we denote by $K_{m,k}$ the complete bipartite graph with m vertices in one class and k in the other.

We define the $\mathbb{N} \times \mathbb{N}$ grid as the graph whose vertex set is $\mathbb{N} \times \mathbb{N}$ and two vertices are adjacent if and only if they differ in only one component by precisely 1. The $\mathbb{Z} \times \mathbb{Z}$ grid and the $\mathbb{N} \times \mathbb{Z}$ grid are defined in the same way but with vertex set $\mathbb{Z} \times \mathbb{Z}$ or $\mathbb{N} \times \mathbb{Z}$, respectively, instead of $\mathbb{N} \times \mathbb{N}$.

The graph H^∞ (see Fig. A.1) is the graph obtained in the following way: First take the $\mathbb{N} \times \mathbb{N}$ grid and delete the vertex $(0,0)$ together with all vertices (n,m) with $n > m$. Furthermore, delete all edges $(n,m)(n+1,m)$ when n and m have equal parity.

Now let us make some remarks on the graph H^∞. It follows from the definition of H^∞ that it is a subgraph of the $\mathbb{N} \times \mathbb{N}$ grid. However, H^∞ is still rich enough to contain the $\mathbb{N} \times \mathbb{N}$ grid as a minor. This fact is not so hard to prove and we omit a proof of it. Furthermore, every vertex in H^∞ has either degree 2 or 3. So having H^∞ as a minor in a graph is equivalent to containing a subdivision of it. So we can conclude that a graph has the $\mathbb{N} \times \mathbb{N}$ grid as a minor if and only if it contains a subdivision of H^∞.

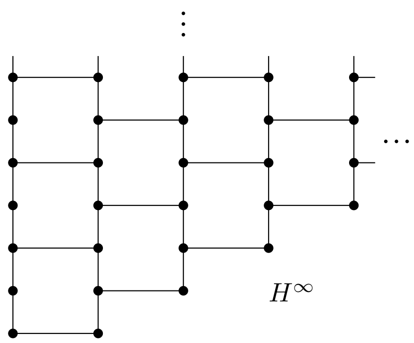

Figure A.1.: The graph H^∞.

A one-way infinite path in a graph G is called a *ray* of G. An equivalence relation can be defined on the set of all rays of G by saying that two rays in G are equivalent if they cannot be separated by finitely many vertices. It is straightforward to check that this relation really defines an equivalence relation. The corresponding equivalence classes of rays with respect to this relation are called the *ends* of G.

A ray which is contained in an end ω of the graph is called an *ω-ray*. The vertex of degree 1 in a ray is called the *startvertex* of the ray. A subgraph of a ray R which is itself a ray is called a *tail* of R.

For $n \in \mathbb{N}$ a set of n disjoint rays is called an *n-bundle* if there are infinitely many disjoint cycles each of which intersects with each ray, but only in a path. For every n-bundle, the cycles which witness that the n disjoint rays are an n-bundle can be chosen in such a way that they all run through the rays in the same cyclic order. We call such a set of cycles the *embracing cycles* of the n-bundle. Note that the definition of an n-bundle implies that an n-bundle is always a subset of one end. For the rest of Section A, we will implicitly assume by stating that $R_1, \ldots, R_n$ are the rays of an n-bundle that the embracing cycles traverse them in order $R_1, \ldots, R_{n-1}, R_n$.

An infinite set of disjoint rays $\{R_1, R_2, \ldots\}$ is called an ∞-*bundle* if there are disjoint cycles C_i and natural numbers c_i for every $i \in \mathbb{N}$ such that for all $i, j \in \mathbb{N}$ with $i < j$ we have $c_i < c_j$ and C_i intersects with each R_ℓ for $\ell \leqslant c_i$, but only in a path. Furthermore, we call an ∞-bundle *consistent* if for all $i, j \in \mathbb{N}$ with $i < j$ the cycles C_i and C_j run through the rays $R_1, \ldots, R_{c_i}$ in the same cyclic order. As for n-bundles we call the cycles C_i embracing cycles. Also note that the rays of an ∞-bundle are in the same end.

Now consider an n-bundle with rays $R_1, \ldots, R_n$ and a k-bundle whose rays are $R'_1, \ldots, R'_k$ where $n \leqslant k$. We say that the n-bundle can be *joined* to the k-bundle if there are vertices $r_i \in V(R_i)$ for every $i \in \{1, \ldots, n\}$ and $r'_j \in V(R'_j)$ for every $j \in \{1, \ldots, k\}$ together with n pairwise disjoint r_i–$r'_{\sigma(i)}$ paths, for some injection $\sigma : \{1, \ldots, n\} \longrightarrow \{1, \ldots, k\}$, each of which intersects $\bigcup_i R_i r_i \cup \bigcup_j r'_j R'_j$ only in its endvertices. The involved paths are called *joining paths*.

Finally, we call an n-bundle *infinitely joined* to a k-bundle if for every finite vertex set S of the graph the n-bundle can be joined to the k-bundle such that the joining paths do not intersect with S.

In order to define an archetypal example of a graph containing an ∞-bundle, we have to construct a sequence $(G_i)_{i \in \mathbb{N}}$ of graphs first. For this we need, furthermore, the function $f : \mathbb{N} \longrightarrow \mathbb{N}$ which is defined as follows:

$$f(i) = \begin{cases} 4, & \text{if } i = 1 \\ 2^i \cdot 3, & \text{if } i \geqslant 2. \end{cases}$$

Now we state the recursive definition of the graphs G_i. Let G_1 be a C_4. Next suppose G_i has already been defined. The construction yields that there is a unique cycle D_i in G_i which is isomorphic to $C_{f(i)}$ and contains all vertices that have degree 2 in G_i, of which there are $g(i) = \frac{1}{3} \cdot f(i+1)$ many. Enumerate these vertices according to the cyclic order in which they appear on D_i. Now we obtain G_{i+1}

by taking G_i together with a disjoint copy of $C_{g(i)}$ whose vertices we enumerate according to the cyclic order of this cycle too, adding an edge between the j-th vertex of D_i and the j-th of $C_{g(i)}$ for each j and subdividing each edge of $C_{g(i)}$ twice. Finally, we define the *Dartboard* (see Fig. A.2) as $\bigcup_i G_i$.

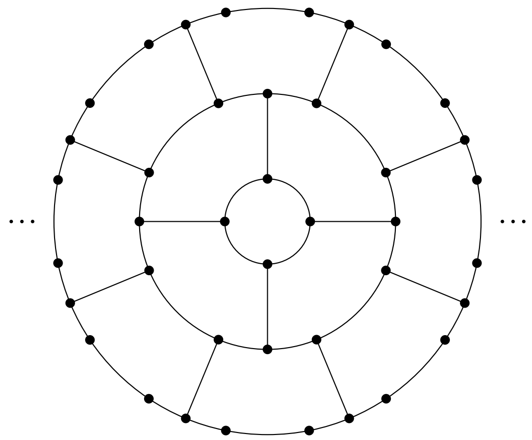

Figure A.2.: The Dartboard.

We continue with some remarks about normal spanning trees and tree-decompositions. Let T be a tree with root r and let $t \in V(T)$. Then we write $\lfloor t \rfloor$ for the up-closure of t with respect to the tree-order of T with root r. Similarly, we write $\lceil t \rceil$ for the down-closure of t.

A rooted spanning tree of a graph is *normal* if the endvertices of every edge in the graph are comparable in the tree-order.

The following theorem of Halin gives a very useful sufficient condition for the existence of a normal spanning tree.

Theorem A.2.1. [31, Thm. 10.1] *Every connected graph which does not contain a subdivision of a $K_{\aleph_0}$ has a normal spanning tree.*

Next let us recall the definition of a tree-decomposition. Let G be a graph, T be a tree and $(V_t)_{t \in V(T)}$ be a sequence of vertex sets of G. We call $(T, (V_t)_{t \in V(T)})$ a *tree-decomposition* of G if the following three properties hold:

1. $V(G) = \bigcup_{t \in V(T)} V_t$.

2. For each edge vw of G there is a $t \in V(T)$ such that $v, w \in V_t$.

3. For all $t_1, t_2, t_3 \in V(T)$ such that t_2 lies on the unique t_1–t_3 path in T the inclusion $V_{t_1} \cap V_{t_3} \subseteq V_{t_2}$ is true.

We call a tree-decomposition $(T, (V_t)_{t \in V(T)})$ *rooted* if the corresponding tree T is rooted. For a rooted tree-decomposition $(T, (V_t)_{t \in V(T)})$ whose tree T has root r, we write $(T, r, (V_t)_{t \in V(T)})$.

A graph has a tree-decomposition *into finite parts* if there is a tree-decomposition $(T, (V_t)_{t \in V(T)})$ of the graph with V_t finite for every $t \in V(T)$.

We say that a tree-decomposition $(T, (V_t)_{t \in V(T)})$ of a graph G into finite parts *distinguishes* all ends of G if for every ray $t_1 t_2 \ldots$ of T all rays of G that intersect all but finitely many V_{t_i} are equivalent. Since all parts of such a tree-decomposition are finite, there is an injection from the set of ends of G to the set of ends of T.

An easy observation shows that we always get a tree-decomposition into finite parts distinguishing all ends as soon as we have a normal spanning tree.

Lemma A.2.2. *Every graph with a normal spanning tree has a tree-decomposition into finite parts distinguishing all ends.*

Proof. Let T be a normal spanning tree of a graph G with root r. Then we define the desired tree-decomposition as $(T, r, (\lceil t \rceil)_{t \in V(T)})$. Let us briefly check that this really defines a tree-decomposition. It is obvious that each vertex v lies in some part, for example in $\lceil v \rceil$. Since T is normal, we know that the endvertices of every edge are comparable and must therefore lie in some common part. Note for the remaining property that for all $t_1, t_3 \in V(T)$ we have $\lceil t_1 \rceil \cap \lceil t_3 \rceil = \lceil t \rceil$ where t is the greatest vertex in the tree-order which is still comparable with t_1 and t_3. Since every vertex t_2 on the t_1–t_3 path in T is greater than t, we get that $\lceil t_1 \rceil \cap \lceil t_3 \rceil = \lceil t \rceil \subseteq \lceil t_2 \rceil$.

The definition of $(T, r, (\lceil t \rceil)_{t \in V(T)})$ ensures that every part is finite. So it remains to check that this tree-decomposition distinguishes all ends. Let us fix a ray $r = t_1 t_2 \ldots$ of T and suppose there are two rays in G which intersect with all but finitely many parts $\lceil t_i \rceil$. Since $\bigcup_{i \leqslant k} \lceil t_i \rceil \setminus \bigcup_{i \leqslant j} \lceil t_i \rceil$ always induces a connected subgraph for $k > j$, we get that the two rays cannot be separated by finitely many vertices, which means they are equivalent. □

A tree-decomposition $(T, (V_t)_{t \in V(T)})$ has *finite adhesion* if for every $t \in V(T)$ there is an integer $n \geqslant 0$ such that $|V_s \cap V_t| \leqslant n$ for every s being adjacent with t in T and additionally for every ray $t_1 t_2 \ldots$ of T there is an integer k such that $|V_{t_i} \cap V_{t_{i+1}}| \leqslant k$ holds for infinitely many $i \in \mathbb{N}$.

By Theorem A.1.3 a tree-decomposition of a graph G into finite parts and with finite adhesion is a witness that G does not contain an $\mathbb{N} \times \mathbb{Z}$ grid minor. Beside the requirement that each part shall be too small to contain a grid minor, which is done by requiring all parts to be finite, the possibility to distribute a grid minor along a branch in the tree-decomposition is prevented by making all branches too narrow for arbitrarily many rays to run through them. The latter goal is achieved by requiring the tree-decomposition to have finite adhesion.

Similar to the definition before we now introduce a property that prevents from distributing a $\mathbb{Z} \times \mathbb{Z}$ grid minor along a whole branch in a tree-decomposition. Unfortunately, verifying this property needs a closer look at the graph and the bundles in it, in contrast to the more abstract property of finite adhesion, which involves only the tree and parts of the decomposition.

A tree-decomposition $(T, (V_t)_{t \in V(T)})$ is called *bundle-narrow* if for every ray $t_1 t_2 \ldots$ of T there is an integer $k \geqslant 1$ such that there is no k-bundle in G whose rays intersect all but finitely many V_{t_i}.

We close this section with a well-known result about 2-connected graphs. We will need this lemma in the proof of Theorem A.1.4.

Lemma A.2.3. [12, Prop. 9.4.2] *For every positive integer k, there exists an integer n such that every 2-connected graph on at least n vertices contains a subgraph isomorphic to a subdivision of either $K_{2,k}$ or a cycle of length k.*

A.3. Proof of the main theorem

Before we can prove Theorem A.1.4 we have to make some observations about bundles. We start with the following lemma which tells us in our context of bundles that we can join a bundle to another one which is sufficiently large as soon as both are subsets of the same end.

Lemma A.3.1. *Let G be a graph, ω be an end of G and $k \geqslant n \geqslant 1$ be integers. Furthermore, let $\mathcal{R} = \{R_1, \ldots, R_n\}$ and $\mathcal{R}' = \{R'_1, \ldots, R'_k\}$ be sets of n and k pairwise disjoint ω-rays, respectively. Then there are vertices $r'_i \in V(R'_i)$ for each i with $1 \leqslant i \leqslant k$ such that there are n pairwise disjoint paths between the start vertices of the rays in $\mathcal{R}$ and the vertices $r'_1, \ldots, r'_k$ each of which intersects $\bigcup_i r'_i R'_i$ at most in r'_i.*

Proof. We want to work within a finite subgraph H of G in which we find the desired paths. To define H we take a set $\mathcal{P}$ of kn^2 pairwise disjoint paths such that for every $i \in \{1, \dots, n\}$ and every $j \in \{1, \dots, k\}$ there are n disjoint R_i–R'_j paths in $\mathcal{P}$. This is possible since all rays lie in the same end. For all i, j with $1 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant k$ let r_i be the last vertex on R_i which is an endvertex of one of the kn many R_i–R'_j paths from $\mathcal{P}$ and r'_j be the last vertex on the ray R'_j which is hit by any path from $\mathcal{P}$ or any $R_i r_i$. Next we define H as follows:

$$H := G\Big[\bigcup_{i=1}^{n} V(R_i r_i) \cup V\Big(\bigcup \mathcal{P}\Big) \cup \bigcup_{j=1}^{k} V(R'_j r'_j)\Big].$$

We complete the proof of this lemma by showing that there are n disjoint paths from the start vertices of the rays in $\mathcal{R}$ to n vertices of the set $\{r'_1, \dots, r'_k\}$ in the graph H. By Menger's Theorem it is sufficient to prove that there is no set S of less than n vertices which separates the start vertices of the rays in $\mathcal{R}$ from the vertices $r'_1, \dots, r'_k$ in H. Suppose for a contradiction that such a set S exists in H. Since S contains less than n vertices and the paths $R_i r_i$ are pairwise disjoint, we can find an index ℓ such that $R_\ell r_\ell$ does not contain any vertex of S. The same is true for the paths $R'_i r'_i$ with some index p. Furthermore, we can find an R_ℓ–R'_p path $P_{\ell p} \in \mathcal{P}$ that is disjoint from S because $\mathcal{P}$ contains n many R_ℓ–R'_p paths. Now we have a contradiction because the union of the paths $R_\ell r_\ell$, $P_{\ell p}$ and $R'_p r'_p$ contains a path from the startvertex of the ray R_ℓ to r'_p that avoids S. □

By iterating Lemma A.3.1, we obtain the following corollary.

Corollary A.3.2. *Let G be a graph and ω be an end of G. Then an n-bundle B_n is infinitely joined to a k-bundle B_k if $k \geqslant n$ and $B_n, B_k \subseteq \omega$.*

Proof. First we apply Lemma A.3.1 to the rays of B_n, say $\{R_1, \dots, R_n\}$, and B_k, say $\{R'_1, \dots, R'_k\}$. Let $\mathcal{P}_1$ be the resulting path system. Next we delete the finite subgraph H of G defined as in the proof of Lemma A.3.1 from G. By the definition of bundles, the tails of B_n and B_k in $G - H$ are still bundles and all of these tails are still equivalent. Next we apply Lemma A.3.1 to these tails and obtain a path system $\mathcal{P}_2$. By iterating this argument, we get path systems $\mathcal{P}_i$ for $i \in \mathbb{N}$ such that $P \cap Q = \varnothing$ for every $P, Q \in \bigcup_{i \in \mathbb{N}} \mathcal{P}_i$ with $P \neq Q$ and each path system $\mathcal{P}_i$ connects the n rays of B_n with n distinct rays of B_k. Since there is only a finite bounded number of possibilities on which rays the start- and endvertices of the paths of some path system $\mathcal{P}_i$ can be, we obtain by the pigeonhole principle that there is an infinite subset $\{\mathcal{P}'_j \ ; \ j \in \mathbb{N}\} \subseteq \{\mathcal{P}_i \ ; \ i \in \mathbb{N}\}$ of path systems and an injection $\sigma : \{1, \ldots, n\} \longrightarrow \{1, \ldots, k\}$ such that each path system $\mathcal{P}'_j$ contains a path from R_i to $R'_{\sigma(i)}$ for all $i \in \{1, \ldots, n\}$. So the set $\{\mathcal{P}'_j \ ; \ j \in \mathbb{N}\}$ of disjoint path systems witnesses that B_n is infinitely joined to B_k. $\square$

For n-bundles it follows from the pigeonhole principle that we can always find an infinite subset of the embracing cycles whose elements induce the same cyclic order on the rays of the n-bundle. So without loss of generality we could assume that the embracing cycles of an n-bundle run through the rays of the bundle always in the same cyclic order. We can do a similar thing for ∞-bundles, but it involves an application of the compactness principle rather than the pigeonhole principle. So before we make the corresponding statement about ∞-bundles precise, let us state a version of the compactness principle we will make use of, namely König's Lemma:

Lemma A.3.3. [12, Lemma 8.1.2] *Let $V_0, V_1, \ldots$ be an infinite sequence of disjoint non-empty finite sets, and let G be a graph on their union. Assume that every vertex in a set V_n with $n \geqslant 1$ has a neighbour in V_{n-1}. Then G contains a ray $v_0 v_1 \ldots$ with $v_n \in V_n$ for all n.*

Now the next lemma tells us that we always obtain a consistent ∞-bundle from an ∞-bundle.

Lemma A.3.4. *The rays of an ∞-bundle B_∞ form also a consistent ∞-bundle witnessed by an infinite subset of the embracing cycles of B_∞.*

Proof. Let $B_\infty = \{R_1, R_2, \ldots\}$ be an ∞-bundle of a graph and let $\{C_i \ ; \ i \in \mathbb{N}\}$ be the set of its embracing cycles. Furthermore, let the natural numbers c_i be given as in the definition of an ∞-bundle. Now we define an auxiliary graph G to apply König's Lemma. For every $n \geqslant 1$ let $V_n \subseteq V(G)$ be the set of all cyclic orders of how an embracing cycle C_j runs through the set of rays $\{R_1, \ldots, R_{c_n}\}$ for $j \geqslant n$. So each set V_n is finite and non-empty. Furthermore, let there be an edge in G between vertices $v_n \in V_n$ and $v_{n+1} \in V_{n+1}$ if the cyclic order v_{n+1} restricted to the set $\{R_1, \ldots, R_{c_n}\}$ is equal to v_n. With these definitions all requirements for König's Lemma (Lemma A.3.3) are fulfilled. So G contains a ray $v_1 v_2 \ldots$ with $v_n \in V_n$ for every $n \geqslant 1$. This allows us to take cycles C_{k_n} such that C_{k_n} induces v_n on the rays $\{R_1, \dots, R_{c_n}\}$ for every $n \geqslant 1$ where $k_n > k_{n'}$ holds for $n > n'$. These cycles witness that B_∞ is a consistent ∞-bundle. $\square$

Now we are prepared to prove Theorem A.1.4.

Proof of Theorem A.1.4. Using Lemma A.3.4 we get that the implication from (ii) to (iii) is true.

Showing that (iv) follows from (iii) is not difficult. We sketch the proof of this implication. Construct subdivisions of the defining subgraphs G_i of the Dartboard inductively. Start with an embracing cycle of the consistent ∞-bundle that runs through $f(1) = 4$ rays of the ∞-bundle as G_1. Now suppose we have already constructed a subdivision H_n of G_n and there are $f(n)$ tails $T_1, T_2, \dots, T_{f(n)}$ of rays of the ∞-bundle that intersect with H_n only in their startvertices. Pick $f(n)$ many embracing cycles $C'_1, C'_2, \dots, C'_{f(n)}$ of the ∞-bundle that are disjoint from H_n, each traversing the tails $T_1, T_2, \dots T_{f(n)}$, and another embracing cycle C which is disjoint from H_n, comes later in the enumeration of all embracing cycles than the ones we have picked so far and traverses at least $f(n+1)$ many rays of the ∞-bundle including the $f(n)$ tails T_i. Since the ∞-bundle is consistent, we can use the cycles $C, C'_1, \dots, C'_{f(n)}$ and the tails T_i to find a subdivision of H_{n+1} together with $f(n+1)$ many tails of rays of the ∞-bundle that intersect with H_{n+1} only in their startvertices. In this step we possibly have to reroute some of the tails T_i using the cycles C'_i in order to get compatible paths from H_n to $C \subseteq H_{n+1}$. Using this construction the union $\bigcup_n H_n$ gives us a subdivision of the Dartboard.

The implications from (iv) to (v) and from (v) to (vi) are easy and so we omit the details.

Now we look at the implication from (i) to (ii). Let ω be an end of a graph G such that there are n-bundles $B_n = \{R_1^n, \dots, R_n^n\} \subseteq \omega$ for every $n \in \mathbb{N}$. We construct an ∞-bundle inductively. In step i we shall have a graph H_i which satisfies the following properties:

1. H_i is the union of disjoint cycles $C_1, \dots, C_i$ and disjoint paths $P_1^i, \dots, P_i^i$.

2. The intersection of P_j^i with C_k is a path for all j, k with $j \leqslant k \leqslant i$.

3. $P_j^i \cap C_k = \varnothing$ holds for all j, k with $k < j \leqslant i$.

4. Each path P_j^i runs through the cycles $C_j, \dots, C_i$ in the order of their enumeration.

5. $H_i \cap H_{i-1} = H_{i-1}$ for $1 < i$.

6. P_j^{i-1} is an initial segment of P_j^i for every $j \leqslant i$ with $1 < i$.

7. In G there exist tails of rays of some n-bundle B_n such that every endvertex of a path P_j^i in $H_i - H_{i-1}$ with $j \leqslant i$ is a startvertex of one of these tails but apart from that the tails are disjoint from H_i.

For H_1 we take an embracing cycle of B_1 as C_1 and set $H_1 = C_1$. We define P_1^1 to be the trivial path which is the last vertex v of R_1^1 on C_1. So (1), (3), (4), (5) and (6) are obviously satisfied. Property (2) holds by the definition of embracing cycle. For (7) we can take the tail vR_1^1 of R_1^1.

Now suppose we have already defined H_i which satisfies the seven stated properties. Let B_n be the n-bundle which we get from property (7) for step i. By Corollary A.3.2 we get that B_n is infinitely joined to any k-bundle B_k if $k \geqslant n$. Let us fix an integer k with $k > n \geqslant i$. Since H_i is a finite graph, we can find joining paths $Q_1, \dots, Q_n$ from B_n to B_k which meet H_i only in the endvertices of the paths P_j^i. Now fix an embracing cycle C of B_k that is disjoint from H_i such that the tails of the rays of B_k starting from C are disjoint from H_i as well as from the joining paths Q_j. We set $C_{i+1} = C$. Furthermore, we define P_j^{i+1} for $j \leqslant i$ to be the concatenation of P_j^i with the joining path $Q_{j'}$ which it intersects and with the path $Q_{j'}^C$ where $Q_{j'}^C$ is the path which starts at the endvertex of $Q_{j'}$ which lies on a ray of B_k and follows that ray up to the last vertex that is in the intersection of this ray with C_{i+1}. Since $k > i$ holds, there is a ray R^* in B_k whose tail with startvertex in C_{i+1} does not intersect with any of the paths P_j^{i+1}. Now we set P_{i+1}^{i+1} to be the trivial path consisting of the last vertex on R^* which lies also on C_{i+1}. Finally, we set H_{i+1} to be the union of H_i with all paths P_j^{i+1}. It remains to check that the definitions we made for step $i+1$ ensure that the properties (1) to (7) are also true for H_{i+1}. Property (1), (5) and (6) are obviously true by definition. Since (5) and (6) hold and the paths Q_j and Q_j^C are chosen to be disjoint from H_i except for one starting vertex of each Q_j, we need to check property (2) just for the paths P_j^{i+1} and the cycle C_{i+1}. Note that the intersection of a path P_j^{i+1} with the cycle C_{i+1} is equal to the intersection of one of the rays of B_k with C_{i+1}. So this intersection is just a path because C_{i+1} is an embracing cycle of B_k. Property (3) and (4) are valid because of property (2) and since $P_j^{i+1} - P_j^i$ is disjoint from H_i. The bundle B_k together with suitable tails of its rays starting in C_{i+1} we chose in the construction for step $i+1$ witnesses that property (7) holds.

Using the sequence of graphs $(H_i)_{i \in \mathbb{N}}$, we are able to define an ∞-bundle B_∞. We set $R_j^\infty = \bigcup_{i \in \mathbb{N}} P_j^i$ for every $j \in \mathbb{N}$ and then $B_\infty = \{R_j^\infty \,;\, j \in \mathbb{N}\}$. Property (6) ensures that each R_j^∞ is a ray and the disjoint cycles C_i together with property (2) ensure that B_∞ is indeed an ∞-bundle. This completes the proof that (i) implies (ii).

It remains to prove the implication from (vi) to (i). Let ω be the end of G which contains $\mathcal{R}$ as a subset. Next let us fix an arbitrary $k \in \mathbb{N}$ and show that there is a k-bundle in the graph G all whose rays are elements of ω. For this purpose we choose n disjoint rays $R_1, \dots, R_n$ from the set $\mathcal{R}$ where n is as big as the integer n from Lemma A.2.3 with our fixed k as input. Next we define an auxiliary graph H to which we shall apply that lemma. First set $V(H) = \{R_1, \dots, R_n\}$. Furthermore, we say that there is an edge R_iR_j if and only if there exist infinitely many disjoint R_i–R_j paths in G which are disjoint from all rays in $\{R_1, \dots, R_n\} \backslash \{R_i, R_j\}$. In order to apply Lemma A.2.3 to H, we need to check that H is 2-connected. Suppose for a contradiction that there exists a ray R_ℓ such that $H - R_\ell$ is not connected. So we can find a bipartition (A, B) of $V(H) \backslash \{R_\ell\}$ which yields an empty cut of H. Now let us fix a ray $R \in A$ and $R' \in B$. We know by assumption that R and R' are equivalent in $G - R_\ell$. This implies that there are infinitely many disjoint R–R' paths in $G - R_\ell$. Using the pigeonhole principle and the fact that A and B contain less than n rays, infinitely many of these paths have a common last ray of A and a common first ray in B which they intersect, but this tells us that there exists an A–B edge in $H - R_\ell$. So we have a contradiction and can conclude that H is 2-connected. Now we apply Lemma A.2.3 to H. If the lemma tells us that H contains a subdivided cycle of length at least k, then we immediately also get a k-bundle in G all whose rays are elements of ω. So suppose there is a subdivision of $K_{2,k}$ in H. Without loss of generality let R_1^*, R_2^* and $R_1, \dots, R_k$ be branch vertices of the subdivided $K_{2,k}$ in H such that there are disjoint paths from R_1^* and R_2^* to R_i for every i with $1 \leqslant i \leqslant k$ in H. Now we use the rays R_1^* and R_2^* as distributing rays in G to build infinitely many disjoint cycles that witness $\{R_1, \dots, R_k\}$ being a k-bundle. The cycles can be built all in the same way: First pick a R_1–R_2^* path P_1^* which is disjoint from R_1^* and from each ray R_i for $1 \leqslant i \leqslant k$ and $i \neq 1$. Now start at the endvertex of P_1^* on R_1 and follow that ray until there is a R_1–R_1^* path P_1 which is disjoint from R_2^*, P_1^* and from each ray R_i for $1 \leqslant i \leqslant k$ and $i \neq 1$. Then follow P_1 and R_1^* afterwards until there is a R_1^*–R_2 path which is disjoint from R_2^*, P_1^*, P_1 and from each ray R_i for $1 \leqslant i \leqslant k$ and $i \neq 2$. Repeating this pattern we get a R_2^*–R_k path Q which meets every ray R_i for $1 \leqslant i \leqslant k$ only in a path. Then we can close Q to obtain a cycle by following R_k from the endvertex of Q on R_k until there is a R_k–R_2^* path P_2^* that is disjoint from R_1^*, from each ray R_i for $1 \leqslant i \leqslant k$ and $i \neq k$ and from each path we have used so far, then following P_2^* and finally using the P_2^*–P_1^* path on R_2^*. By deleting large enough initial segments from all rays, we can repeat the construction of such cycles infinitely often and obtain the desired sequence of disjoint cycles witnessing that $\{R_1, \ldots, R_k\} \subseteq \omega$ is a k-bundle. □

Using Theorem A.1.4 we prove now Corollary A.1.5, which describes the structure of graphs without $\mathbb{Z} \times \mathbb{Z}$ grid minor in terms of bundle-narrow tree-decompositions.

Proof of Corollary A.1.5. Let G be a graph and let us assume that it does not contain a $\mathbb{Z} \times \mathbb{Z}$ grid minor. So G cannot contain a subdivision of $K_{\aleph_0}$ either and we can apply Theorem A.2.1 telling us that G has a normal spanning tree. Using Lemma A.2.2 we obtain a tree-decomposition of G into finite parts distinguishing all ends. Now we know that for every ray $t_1 t_2 \ldots$ of T all rays of G that intersect all but finitely many of the parts V_{t_i} are equivalent in G. Using the equivalence of (i) and (v) in Theorem A.1.4, we can furthermore find for each end of G the least integer $k \geqslant 1$ such that no k-bundle exists in this end. Combining these two observations, we can find for every ray $t_1 t_2 \ldots$ of T the least integer $k \geqslant 1$ such that there is no k-bundle in G whose rays intersect with all but finitely many of the parts V_{t_i}. So our tree-decomposition of G into finite parts which distinguishes all ends is already bundle-narrow.

For the other direction let us assume that a graph graph G has a $\mathbb{Z} \times \mathbb{Z}$ grid minor and suppose for a contradiction that it also has a bundle-narrow tree-decomposition $(T, (V_t)_{t \in V(T)})$ into finite parts. Using that all parts V_t are finite, we can look at the last time a ray R of G leaves a part V_t. In this way R induces a ray $t_1 t_2 \ldots$ of T such that R intersects each part V_{t_i}. Note that equivalent rays in G induce rays in T which have a common tail, because they cannot be separated by finitely many vertices in G. By the equivalence of (i) and (v) in Theorem A.1.4, there exists an end of G which contains n-bundles for every $n \in \mathbb{N}$. We know that the rays of all these bundles induce rays of T that lie in the same end of T. Now any ray of T that belongs to this end of T contradicts our assumption that the tree-decomposition is bundle-narrow. $\square$

B. k-connected sets in infinite graphs: a characterisation by an analogue of the Star-Comb Lemma for higher connectivity

B.1. Introduction

It is a well-known and easy-to-prove fact that each connected finite graph contains a long path or a vertex of high degree. More precisely, for every $m \in \mathbb{N}$ there is an $n \in \mathbb{N}$ such that each connected graph with at least n vertices (or better say a graph with a component of size at least n) either contains a path P_m of length m or a star $K_{1,m}$ with m leaves (i.e. a complete bipartite graph with one vertex on one side and m vertices on the other side) as a subgraph (cf. [12, Prop. 9.4.1]). In a way, the existence of these 'unavoidable' subgraphs characterise graphs with large components, although not with a sharp equivalence: For the other direction, if a graph contains P_m or $K_{1,m}$ as a subgraph, then it obviously contains a component of size at least $m + 1$.

For 2-connected graphs there is an analogous result, which also is folklore: For every $m \in \mathbb{N}$ there is an $n \in \mathbb{N}$ such that each 2-connected finite graph with at least n vertices either contains a subdivision of a cycle C_m of length m or a subdivision of a complete bipartite graph $K_{2,m}$ (cf. [12, Prop. 9.4.2]). As before, these 'unavoidable' subdivisions yield some kind of characterisation: If a graph contains a subdivision of C_m or $K_{2,m}$, then it contains a 2-connected subgraph with at least m vertices.

In 1993, Oporowski, Oxley and Thomas [49] continued on this path and gave two finite lists of graphs that are similarly unavoidable subdivisions in k-connected graphs for $k \in \{3, 4\}$[1]. The graphs in these lists are not necessarily k-connected, but a slight modification of their result yields similar lists of unavoidable minors in k-connected graphs which are k-connected themself (cf. [12, Thm. 9.4.3 and Theorem 9.4.4]). For $k = 3$ the 'unavoidable' minors are the wheel or the complete bipartite graph $K_{3,m}$, while for $k = 4$ the number of 'unavoidable' minors is growing to four different minors, whose definition we omit here. Now a characterisation for the existence of large k-connected subgraphs fails for trivial reasons, as subdivisions (and similarly inflated subgraphs) of k-connected graphs for $k \geqslant 3$ are not necessarily more than 2-connected. To obtain a similar characterisation as before we need a different notion of a 'highly connected object' in a graph.

For $k \in \mathbb{N}$, a set X of at least k vertices of a graph G is called *k-connected in G*, if for all $Z_1, Z_2 \subseteq X$ with $|Z_1| = |Z_2| \leqslant k$ there are $|Z_1|$ many vertex disjoint paths from Z_1 to Z_2 in G. Note that any subset $Y \subseteq X$ with $|Y| \geqslant k$ is also k-connected in G. We often omit stating the graph in which X is k-connected if it is clear from the context. Now k-connected sets offer a solution for the other direction of a possible characterisation, since as it can be easily seen that any set containing precisely one vertex of each branch set of a k-connected minor of G is k-connected in G (cf. Lemma B.4.2).

Recently, Geelen and Joeris [24, 39] generalised these results to arbitrary $k \in \mathbb{N}$. They introduced so called generalised wheels (depending on k and m), which together with the complete bipartite graph $K_{k,m}$ are the 'unavoidable' minors in graphs with large k-connected sets. And for $k \in \{2, 3, 4\}$ they correspond precisely to the 'unavoidable' minors mentioned before. These generalised wheels and the $K_{k,m}$ are graphs that contain a k-connected set of size m. Hence as before, there is the converse direction: each graph that contains such a generalised wheel (depending on k and m) or $K_{k,m}$ as a minor also contains a k-connected set of size m.

Now let us consider infinite graphs. Again there is a well-known and easy-to-prove fact that each infinite connected graph contains either a *ray*, that is a one-way infinite path, or a vertex of infinite degree. Conversely, each graph that contains a ray or a vertex of infinite degree has an infinite component. There is also a more localised version of this result, which is known as the Star-Comb

[1]In fact, for $k = 4$ the authors show something slightly stronger by requiring the graph to have a property which is slightly weaker than being 4-connected.

Lemma (cf. Lemma B.2.4). In essence this lemma relates the subgraphs of the result from above to a given vertex set.

For 2-connected infinite graphs one can easily construct an analogous result. A *double ray* is a two-way infinite path. We say a vertex d *dominates* a ray R if they cannot be separated by deleting a finite set of vertices not containing d. An *end* of a graph is an equivalence class of rays, where two rays are equivalent, if they cannot be separated by deleting a finite set of vertices. Now it is a common exercise to prove that every infinite 2-connected graph contains either a double ray whose subrays belong to the same end, a ray which is dominated by a vertex, or a subdivision of a $K_{2,\aleph_0}$. With the advent of topological infinite graph theory, those results became an even more meaningful extension of the finite result. In *locally finite* graphs, that are graphs where each vertex has finite degree, a double ray whose subrays belong to the same end is the easiest example of an infinite *topological circle*, that is a homeomorphic image of the sphere S^1 in the Freudenthal compactification of the 1-complex of G (cf. [12, Section 8.5]). Moreover, a similar topological approach works in *finitely separable* graphs, that are graphs containing no subdivision of $K_{2,\aleph_0}$. In such a graph, a ray starting a at vertex dominating it is also an infinite topological circle [13, Section 5].

In 1978, Halin [31] studied such a problem for arbitrary $k \in \mathbb{N}$. He showed that every k-connected graph whose set of vertices has size at least κ for some uncountable regular cardinal κ contains a subdivision of $K_{k,\kappa}$. Hence for all those cardinals, $K_{k,\kappa}$ is the unique 'unavoidable' subdivision (or minor). The 'unavoidable' minors for graphs whose set of vertices has singular cardinality remained undiscovered.

Oporowski, Oxley and Thomas [49] also studied countably infinite graphs for arbitrary $k \in \mathbb{N}$. Together with the $K_{k,\aleph_0}$, the 'unavoidable' minors for countably infinite k-connected[2] graphs have the following structure. For $\ell, d \in \mathbb{N}$ with $\ell + d = k$, they consist of a set of ℓ disjoint rays, d vertices that dominate one of the rays (or equivalently all of those rays) and infinitely many edges connecting pairs of them in a tree-like way.

This leads to the first part of our main result. For $k \in \mathbb{N}$ and an infinite cardinal κ we will define certain graphs with a k-connected set of size κ in Section B.3, the so

[2] Again, the authors show something slightly stronger, requiring a slightly weaker property than k-connectivity, but still a stronger property than containing a k-connected set.

called *k-typical graphs*. These graphs will encompass complete bipartite graphs $K_{k,\kappa}$ as well as the graphs described by Oporowski, Oxley and Thomas [49] for $\kappa = \aleph_0$. We will moreover introduce such graphs even for singular cardinals κ. It will turn out that for fixed k and κ there are only finitely many k-typical graphs up to isomorphisms. We shall characterise graphs with a k-connected set of size κ via the existence of a minor of such a k-typical graph.

Moreover we will extend the definition of k-typical graphs to so called *generalised k-typical graphs*. As before for fixed k and κ there are only finitely many generalised k-typical graphs up to isomorphisms, and we shall extend the characterisation from before via the existence of subdivisions of such a generalised k-typical graph.

In finite graphs, k-connected sets have also been studied in connection to tree-width. Diestel, Gorbunov, Jensen and Thomassen [16, Prop. 3] showed that for any graph G and $k \in \mathbb{N}$, if G contains a $(k+1)$-connected set of size at least $3k$, then G has tree-width at least k, and conversely if G has no $(k+1)$-connected set[3] of size at least $3k$, then G has tree-width less than $4k$. As before with the minors, the characterisation is not possible as an exact equivalence.

In infinite graphs, different notions of decompositions of graphs in a tree-like way that extend the notion of tree-decompositions in finite graphs have been studied. Robertson, Seymour and Thomas [54] gave a survey of different results characterising the existence of different kinds of these decompositions via forbidden minors. In recent years, one of those decomposition notions, the notion of a *nested set of separations* has been studied in more detail [15]. They correspond to tree-decompositions of finite graphs in a natural way and offer a generalisation for infinite graphs. We define separations and the necessary terms, including the notion of parts for a nested set of separations, which provides some analogue of tree-width, in Section B.2.

This leads to the final characterisation in our main theorem.

Theorem B.1.1. *Let G be an infinite graph, let $k \in \mathbb{N}$ and let $\kappa \leqslant |V(G)|$ be an infinite cardinal. Then the following are equivalent.*

(a) $V(G)$ contains a subset of size κ that is k-connected in G.

(b) G contains a k-typical graph of size κ as a minor with finite branch sets.

[3] In fact, the authors show something slightly stronger, requiring for the second part a slightly weaker property than k-connectivity.

(c) G contains a subdivision of a generalised k-typical graph of size κ.

(d) There is no nested set of separations of order less than k of G such that every part has size less than κ.

In fact, we will prove a slightly stronger result which will require some more notation, Theorem B.3.7 in Subsection B.3.3. In the same vein as the Star-Comb Lemma, that result will relate the minors (or subdivisions) with a specific k-connected set in the graph.

After fixing some notation and recalling some basic definitions and simple facts in Section B.2, we will define the k-typical graphs and generalised k-typical graphs in Section B.3. In Section B.4 we will collect some basic facts about k-connected sets and their behaviour with minors or topological minors. Section B.5 deals with the structure of ends in graphs. Subsection B.5.1 is dedicated to extend a well-known connection between minimal separators and the degree of an end from locally finite graphs to arbitrary graphs. Afterwards, Subsection B.5.2 gives a construction on how to find disjoint rays in some end with additional structure between them. Sections B.6 and B.7 are dedicated to prove the characterisation via minors and topological minors for the case of regular cardinals in Section B.6 and, respectively, the case of singular cardinals in Section B.7. In Section B.8 we will talk about some applications of the minor characterisation, and in Section B.9 we shall finish the proof of the main theorem of Section B with the characterisation via nested sets of separations.

B.2. Preliminaries

For Section B let us explicitly note that we shall work in ZFC. For general notation about graph theory that we do not specifically introduce here we refer the reader to [12].

In Section B we consider both finite and infinite cardinals. As usual, for an infinite cardinal κ we define its *cofinality*, denoted by $\operatorname{cf}\kappa$, as the smallest infinite cardinal λ such that there is a set $X \subseteq \{Y \subseteq \kappa \mid |Y| < \kappa\}$ such that $|X| = \lambda$ and $\bigcup X = \kappa$. We distinguish infinite cardinals κ to *regular cardinals*, i.e. cardinals where $\operatorname{cf}\kappa = \kappa$, and *singular cardinals*, i.e. cardinals where $\operatorname{cf}\kappa < \kappa$. Note that $\operatorname{cf}\kappa$

is always a regular cardinal. For more information on infinite cardinals and ordinals, we refer the reader to [42].

Throughout Section B, let G denote an arbitrary simple and undirected graph with vertex set $V(G)$ and edge set $E(G)$. We call G *locally finite* if each vertex of G has finite degree.

Let G and H be two graphs. The *union* $G \cup H$ of G and H is the graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$. The *Cartesian product* $G \times H$ of G and H is the graph with vertex set $V(G) \times V(H)$ such that two vertices $(g_1, h_1), (g_2, h_2) \in V(G \times H)$ are adjacent if and only if either $h_1 = h_2$ and $g_1 g_2 \in E(G)$ or $g_1 = g_2$ and $h_1 h_2 \in E(H)$ holds.

Given two sets A and B, we denote by $K(A, B)$ the complete bipartite graph between the classes A and B. We also write $K_{\kappa,\lambda}$ for $K(A, B)$ if $|A| = \kappa$ and $|B| = \lambda$ for two cardinals κ and λ.

Unless otherwise specified, in Section B a path is a finite graph. The *length* of a path is the size of its edge set. A path is *trivial*, if it only contains only one vertex, which we will call its *endvertex*. Otherwise, the two vertices of degree 1 in the path are its *endvertices*. The other vertices are called the *inner vertices* of the path.

Let $A, B \subseteq V(G)$ be two (not necessarily disjoint) vertex sets. An *A–B path* is a path whose inner vertices are disjoint from $A \cup B$ such that one of its end vertices lies in A and the other lies in B. In particular, a trivial path whose endvertex is in $A \cap B$ is also an A–B path. An *A–B separator* is a set S of vertices such that $A \setminus S$ and $B \setminus S$ lie in different components of $G - S$. We also say *S separates* A and B. For convenience, by a slight abuse of notation, if $A = \{a\}$ (or $B = \{b\}$) is a singleton we will replace A by a (or B by b respectively) for these terms.

We shall need the following version of Menger's Theorem for finite parameter k in infinite graphs, which is an easy corollary of Menger's Theorem for finite graphs.

Theorem B.2.1. [12, Thm. 8.4.1] *Let $k \in \mathbb{N}$ and let A, $B \subseteq V(G)$. If A and B cannot be separated by less than k vertices, then G contains k disjoint A–B paths.*

We shall also need a trivial cardinality version of Menger's theorem, which is easily obtained from Theorem B.2.1 by noting that the union of less than κ many disjoint A–B paths for an infinite cardinal κ has size less than κ (cf. [12, Section 8.4]).

Theorem B.2.2. *Let κ be a cardinal and let A, $B \subseteq V(G)$. If A and B cannot be separated by less than κ vertices, then G contains κ disjoint A–B paths.* $\square$

Recall that a one-way infinite path R is called a *ray* and a two-way infinite path D is called a *double ray*. The unique vertex of degree 1 of R is its *start vertex*. A subgraph of R (or D) that is a ray itself is called a *tail* of R (or D respectively). Given $v \in R$, we write vR for the tail of R with start vertex v. A finite path $P \subseteq R$ (or $P \subseteq D$) is a *segment* of R (or D respectively). If v and w are the endvertices of P, then we denote P also by vRw (or vDw respectively). If v is the end vertex of vRw whose distance is closer to the start vertex of R, then v is called the *bottom vertex of* vRw and w is called the *top vertex of* vRw. If additionally v is the start vertex of R, then we call vRw an *initial segment* of R and denote it by Rw.

Recall that an *end* of G is an equivalence class of rays, where two rays are *equivalent* if they cannot be separated by deleting finitely many vertices of G. We denote the set of ends of G by $\Omega(G)$. A ray being an element of an end $\omega \in \Omega(G)$ is called an *ω-ray*. A double rays all whose tails are elements of ω is called an *ω-double ray*.

For an end $\omega \in \Omega(G)$ let $\deg(\omega)$ denote the *degree* of ω, that is the supremum of the set $\{|\mathcal{R}| \mid \mathcal{R}$ is a set of disjoint ω-rays$\}$. Note for each end ω there is in fact a set $\mathcal{R}$ of vertex disjoint ω-rays with $|\mathcal{R}| = \deg(\omega)$ [30, Satz 1].

Recall that a vertex $d \in V(G)$ *dominates* a ray R if d and some tail of R lie in the same component of $G - S$ for every finite set $S \subseteq V(G) \backslash \{d\}$. By Theorem B.2.2 this is equivalent to the existence of infinitely many d–R paths in G which are disjoint but for d itself. Note that if d dominates an ω-ray, then it also dominates every other ω-ray. Hence we also write that d *dominates* an end $\omega \in \Omega(G)$ if d dominates some ω-ray. Let $\mathrm{Dom}(\omega)$ denote the set of vertices dominating ω and let $\mathrm{dom}(\omega) = |\mathrm{Dom}(\omega)|$. If $\mathrm{dom}(\omega) > 0$, we call ω *dominated*, and if $\mathrm{dom}(\omega) = 0$, we call ω *undominated*.

For an end $\omega \in \Omega(G)$, let $\Delta(\omega)$ denote $\deg(\omega) + \mathrm{dom}(\omega)$, which we call the *combined degree* of ω. Note that the sum of an infinite cardinal with some other cardinal is just the maximum of the two cardinals.

The following lemma due to König about the existence of a ray is a weak version of the compactness principle in combinatorics.

Lemma B.2.3 (König's Infinity Lemma). [12, Lemma 8.1.2] *Let $(V_i)_{i\in\mathbb{N}}$ be a sequence of disjoint non-empty finite sets, and let G be a graph on their union. Assume that for every $n > 0$ each vertex in V_n has a neighbour in V_{n-1}. Then G contains a ray $v_0v_1\dots$ with $v_n \in V_n$ for all $n \in \mathbb{N}$.*

In Section B.9 we shall also use a stronger version of the compactness principle in combinatorics. We omit stating it here but refer to [12, Appendix A].

A *comb* C is the union of a ray R together with infinitely many disjoint finite paths each of which has precisely one vertex in common with R, which has to be an endvertex of that path. The ray R is the *spine* of C and the end vertices of the finite paths that are not on R together with the end vertices of the trivial paths are the *teeth* of C. A comb whose spine is in ω is also called an ω-comb. A *star* is the complete bipartite graph $K_{1,\kappa}$ for some cardinal κ, where the vertices of degree 1 are its *leaves* and the vertex of degree κ is its *centre*.

Next we state a version of the Star-Comb lemma in a slightly stronger way than elsewhere in the literature (e.g. [12, Lemma 8.2.2]). We also give a proof for the sake of completeness.

Lemma B.2.4 (Star-Comb Lemma). *Let $U \subseteq V(G)$ be infinite and let $\kappa \leqslant |U|$ be a regular cardinal. Then the following are equivalent.*

(a) There is a subset $U_1 \subseteq U$ with $|U_1| = \kappa$ such that U_1 is 1-connected in G.

(b) There is a subset $U_2 \subseteq U$ with $|U_2| = \kappa$ such that G either contains a subdivided star whose set of leaves is U_2 or a comb whose set of teeth is U_2.

(Note that if κ is uncountable, only the former can exist.)

Moreover, if these statements hold, we can choose $U_1 = U_2$.

Proof. Note that a set of vertices is 1-connected, if and only if it belongs to the same component of G. Hence if (b) holds, then U_2 is 1-connected and we can set $U_1 := U_2$ to satisfy (a).

If (a) holds, then we take a tree $T \subseteq G$ containing U_1 such that each edge of T lies on a path between two vertices of U_1. Such a tree exists by Zorn's Lemma since U_1 is 1-connected in G. We distinguish two cases.

If T has a vertex c of degree κ, then this yields a subdivided star with centre c and a set $U_2 \subseteq U_1$ of leaves with $|U_2| = \kappa$ by extending each incident edge of c to a c–U_1 path.

Hence we assume T does not contain a vertex of degree κ. Given some vertex $v_0 \in V$ and $n \in \mathbb{N}$, let D_n denote the vertices of T of distance n to v_0. Since T is connected, the union $\bigcup\{D_n \mid n \in \mathbb{N}\}$ equals $V(T)$. And while κ is regular, it follows that $\kappa = \aleph_0$, and therefore that T is locally finite. Hence each D_n is finite and, since T is still infinite, each D_n is non-empty. Thus T contains a ray R by Lemma B.2.3. If R does not already contain infinitely many vertices of U_1, then by the property of T there are infinitely many edges of T between $V(R)$ and $V(T - R)$. We can extend infinitely many of these edges to a set of disjoint R–U_1 paths, ending in an infinite subset $U_2 \subseteq U_1$, yielding the desired comb.

In both cases, U_2 is still 1-connected, and hence serves as a candidate for U_1 as well, yielding the "moreover" part of the claim. $\square$

The following immediate remark helps to identify when we can obtain stars by an application of the Star-Comb lemma.

Remark B.2.5. *If there is an ω-comb with teeth U and if v dominates ω, then there is also a set $U' \subseteq U$ with $|U'| = |U| = \aleph_0$ such that G contains a subdivided star with leaves U' and centre v.* $\square$

We say that an end ω is in the *closure* of a set $U \subseteq V(G)$, if there is an ω-comb whose teeth are in U. Note that this combinatorial definition of closure coincides with the topological closure when considering the topological setting of locally finite graphs mentioned in the introduction [12, Section 8.5] [13].

For an end ω of G and an induced subgraph G' of G we write $\omega\restriction G'$ for the set of rays $R \in \omega$ which are also rays of G'. The following remarks are immediate.

Remarks B.2.6. *Let $G' = G - S$ for some finite $S \subseteq V(G)$.*

1. *$\omega\restriction G'$ is an end of G' for every end $\omega \in \Omega(G)$.*

2. *For every end $\omega' \in \Omega(G')$ there is an end $\omega \in \Omega(G)$ such that $\omega\restriction G' = \omega'$.*

3. *The degree of $\omega \in \omega(G)$ in G is equal to the degree of $\omega\restriction G'$ in G'.*

4. $\mathrm{Dom}(\omega) = \mathrm{Dom}(\omega\restriction G') \cup (\mathrm{Dom}(\omega) \cap S)$ *for every end $\omega \in \Omega(G)$.* $\square$

Given an end $\omega \in \Omega(G)$, we say that an ω-ray R is ω-*devouring* if no ω-ray is disjoint from R. We need the following lemma about the existence of a single

ω-devouring ray for an end ω of at most countable degree, which is a special case of Theorem C.1.2. Note that Section C does not depend on Section B and its results.

Lemma B.2.7. *If* $\deg(\omega) \leqslant \aleph_0$ *for* $\omega \in \Omega(G)$, *then* G *contains an* ω*-devouring ray.*

A different way to prove this lemma arises from the construction of normal spanning trees, cf. [12, Prop. 8.2.4]. Imitating this proof according to an enumeration of the vertices of a maximal set of disjoint ω-rays yields that the normal ray constructed this way is ω-devouring.

Let us fix some notations regarding minors. Let G and M be graphs. We say M is a *minor* of G if G contains an *inflated subgraph* $H \subseteq G$ witnessing this, i.e. for each $v \in V(M)$

- there is a non-empty *branch set* $\mathfrak{B}(v) \subseteq V(H)$;
- $H[\mathfrak{B}(v)]$ is connected;
- $\{\mathfrak{B}(v) \mid v \in V(M)\}$ is a partition of $V(H)$; and
- there is an edge between $v, w \in V(M)$ in M if and only if there is an edge between some vertex in $\mathfrak{B}(v)$ and a vertex in $\mathfrak{B}(w)$ in H.

We call M a *finite branch set minor* or *fbs-minor* of G if each branch set is finite.

Without loss of generality we may assume that such an inflated subgraph H witnessing that M is a minor of G is minimal with respect to the subgraph relation. Then H has the following properties for all $v, w \in V(M)$:

- $H[\mathfrak{B}(v)]$ is a finite tree T_v;
- for each $v, w \in V(M)$ there is a unique edge e_{vw} in $E(H)$ between $\mathfrak{B}(v)$ and $\mathfrak{B}(w)$ if $vw \in E(M)$, and no such edge if $vw \notin E(M)$;
- each leaf of T_v is an endvertex of such an edge between two branch sets.

Given a subset $C \subseteq V(M)$ and a subset $A \subseteq V(G)$, we say that M *is an fbs-minor of* G *with* A *along* C, if M is an fbs-minor of G such that the map mapping each vertex of the inflated subgraph to the branch set it is contained in induces a bijection between A and the branch sets of C. As before, we assume without loss of generality that an inflated subgraph H witnessing that M is an fbs-minor of G is minimal with respect to the subgraph relation. We obtain the properties as above, but a leaf of T_v could be the unique vertex of A in $\mathfrak{B}(v)$ instead.

For $\ell, k \in \mathbb{N}$, we write $[\ell, k]$ for the *closed integer interval* $\{i \in \mathbb{N} \mid \ell \leqslant i \leqslant k\}$ as well as $[k, \ell)$ for the *half open integer interval* $\{i \in \mathbb{N} \mid \ell \leqslant i < k\}$. Given some set I, a *family* $\mathcal{F}$ indexed by I is a sequence of the form $(F_i \mid i \in I)$, where the *members* F_i are some not necessarily different sets. For convenience we sometimes use a family and the set of its members with a slight abuse of notation interchangeably, for example with common set operations like $\bigcup \mathcal{F}$. Given some $J \subseteq I$, we denote by $\mathcal{F} \restriction J$ the *subfamily* $(F_j \mid j \in J)$. A set T is a *transversal* of $\mathcal{F}$, if $|T \cap F_i| = 1$ for all $i \in I$. For a family $(F_i \mid i \in \mathbb{N})$ with index set $\mathbb{N}$ we say some property holds for *eventually all* members, if there is some $N \in \mathbb{N}$ such that the property holds for F_i for all $i \in \mathbb{N}$ with $i \geqslant N$.

The following lemma is a special case of the famous Delta-Systems Lemma, a common tool of infinite combinatorics.

Lemma B.2.8. [42, Thm. II.1.6] *Let κ be a regular cardinal, U be a set and $\mathcal{F} = (F_\alpha \subseteq U \mid \alpha \in \kappa)$ a family of finite subsets of U. Then there is a finite set $D \subseteq U$ and a set $I \subseteq \kappa$ with $|I| = \kappa$ such that $F_\alpha \cap F_\beta = D$ for all $\alpha, \beta \in I$ with $\alpha \neq \beta$.*

A *separation* of G is a tuple (A, B) of vertex sets such that $A \cup B = V(G)$ and such that there is no edge of G between $A \backslash B$ and $B \backslash A$. The set $A \cap B$ is the *separator of* (A, B) and the cardinality $|A \cap B|$ is called the *order* of (A, B). Given $k \in \mathbb{N}$, let $S_k(G)$ denote the set of all separations of G of order less than k.

Two separations (A, B) and (C, D) are *nested* if either $A \subseteq C$ and $D \subseteq B$, or $A \subseteq D$ and $C \subseteq B$ holds. A set N of separations of G is called a *nested separation system of* G if it is *symmetric*, i.e. $(B, A) \in N$ for each $(A, B) \in N$ and *nested*, i.e. the separations in N are pairwise nested.

An *orientation* O of a nested separation system N is a subset of N that contains precisely one of (A, B) and (B, A) for all $(A, B) \in N$. An orientation O of N is *consistent* if whenever $(A, B) \in O$ and $(C, D) \in N$ with $C \subseteq A$ and $B \subseteq D$, then $(C, D) \in O$. For each consistent orientation O of N we define a *part* P_O of N as the vertex set $\bigcap\{B \mid (A, B) \in O\}$. It is easy to check that the union of all parts cover the vertex set of G. Moreover, we allow the empty set $\emptyset$ as a nested separation system. In this case, we say that $V(G)$ is a part of $\emptyset$ (this can be viewed as the empty intersection of vertex sets of the empty set as an orientation of $\emptyset$).

A nested separation system N has *adhesion less than k* if all separations it contains have order less than k, i.e. $N \subseteq S_k(G)$.

Note that each oriented edge of the tree of a tree-decomposition of G induces a separation (A, B) where A is the union of the parts on one side of the edge while B is the union of the parts on the other side of the edge. It is easy to check that the set of separations induced by all those edges is a nested separation system. Moreover, properties like adhesion and the size of parts are transferred by this process.

For more information on nested separation systems and their connection to tree-decompositions we refer the interested reader to [15].

B.3. Typical graphs with k-connected sets

Throughout this section, let $k \in \mathbb{N}$ be fixed. Let κ denote an infinite cardinal.

In Subsection B.3.1 we shall describe an up to isomorphism finite class of graphs each of which contains a designated k-connected set of size κ. We call such a graph a *k-typical graph* and the designated k-connected set its *core*. These graphs will appear as the minors Theorem B.1.1(b).

In Subsection B.3.2 we shall describe based on these k-typical graphs a more general but still finite class of graphs each of which again contains a designated k-connected set of size κ. We call such a graph a *generalised k-typical graph* and the designated k-connected set its *core*. These graphs will appear as the topological minors Theorem B.1.1(c).

B.3.1. k-typical graphs

The most basic graph with a k-connected set of size κ is a complete bipartite graph $K_{k,\kappa} = K([0, k), Z)$ for any infinite cardinal κ and a set Z of size κ disjoint from $[0, k)$. Although in this graph the whole vertex set is k-connected, we only want to consider the infinite side Z as the *core* $C(K_{k,\kappa})$ of $K_{k,\kappa}$, cf. Figure B.1.

This is the first instance of a *k-typical graph* with a core of size κ. For uncountable regular cardinals κ, this is the only possibility for a k-typical graph with a core of size κ.

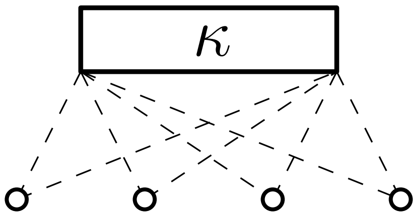

Figure B.1.: A stylised version of a $K_{4,\kappa}$, where the large box stands for the core of κ many vertices and the dashed lines from a vertex to the corners of the box represent that this vertex is connected to all vertices in the box.

A *k-blueprint* $\mathcal{B}$ is a tuple (B, D) such that

- B is a tree of order k; and
- D is a set of leaves of B with $|D| < |V(B)|$.

Take the ray $\mathfrak{N} := \big(\mathbb{N}, \{n(n+1) \mid n \in \mathbb{N}\}\big)$ and the Cartesian product $B \times \mathfrak{N}$. For a node $b \in V(B)$ and $n \in \mathbb{N}$ let

- b_n denote the vertex (b, n);
- $\mathfrak{N}_b$ denote the ray $(\{b\}, \emptyset) \times \mathfrak{N} \subseteq B \times \mathfrak{N}$; and
- B_n denote the subgraph $B \times (\{n\}, \emptyset) \subseteq B \times \mathfrak{N}$.

Then let $\mathfrak{N}(B/D) := (B \times \mathfrak{N})/\{\mathfrak{N}_d \mid d \in D\}$ denote the contraction minor of $B \times \mathfrak{N}$ obtained by contracting each ray $\mathfrak{N}_d$ for each $d \in D$ to a single vertex. We denote the vertex of $\mathfrak{N}(B/D)$ corresponding to the contracted ray $\mathfrak{N}_d$ by d for $d \in D$ and call such a vertex *dominating*. Using this abbreviated notation, we call the tree $B_n - D$ the *n-th layer of* $\mathfrak{N}(B/D)$.

A triple $\mathcal{B} = (B, D, c)$ is called a *regular k-blueprint* if (B, D) is a k-blueprint and $c \in V(B)\backslash D$. We name the graph $T_k(\mathcal{B}) := \mathfrak{N}(B/D)$ and the vertex set $C(T_k(\mathcal{B})) := V(\mathfrak{N}_c)$ is the *core* of $T_k(\mathcal{B})$, see Figure B.2 for an example.

Lemma B.3.1. *For a regular k-blueprint $\mathcal{B}$ the core of $T_k(\mathcal{B})$ is k-connected in $T_k(\mathcal{B})$.*

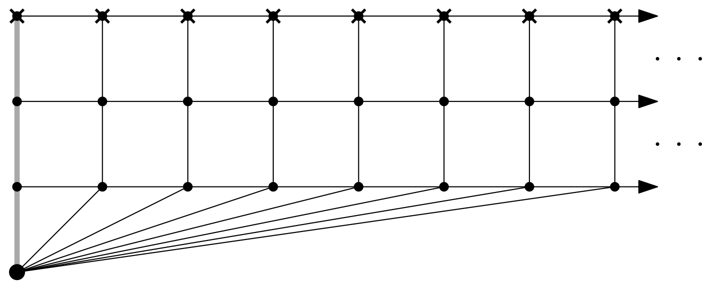

Figure B.2.: Image of $T_4(P, \{d\}, c)$ where $P = cabd$ is a path of length 3 between nodes c and d. P_0 is represented in gray. The crosses represent its core.

Proof. Let $\mathcal{B} = (B, D, c)$ and let $C = C(T_k(\mathcal{B}))$ denote the core of $T_k(\mathcal{B})$. Let $U, W \subseteq C$ with $|U| = |W| = k' \leqslant k$. Suppose for a contradiction there is a vertex set S of size less than k' separating U and W. Then there are $m, n \in \mathbb{N}$ with $c_m \in U \setminus S$, $c_n \in W \setminus S$ such that the n-th and m-th layer are both disjoint from S. Moreover there is a $b \in B$ such that $\mathfrak{N}_b$ (or $\{b\}$ if $b \in D$) are disjoint from U and W. Hence we can connect c_m and c_n with the path consisting of the concatenation of the unique c_m–b_m path in B_m, the unique b_m–b_n path in $\mathfrak{N}_b$ and the unique b_n–c_n path in B_n. This path avoids S, contradicting that S is a separator. By Theorem B.2.1 there are k' disjoint U–W paths, and hence C is k-connected in $T_k(\mathcal{B})$. □

For any regular k-blueprint $\mathcal{B} = (B, D, c)$ the graph $T_k(\mathcal{B})$ is a *k-typical graph* with a countable core. Such graphs are besides the complete bipartite graph $K_{k,\aleph_0}$ the only other k-typical graphs with a core of size $\aleph_0$.

Note that given two regular k-blueprints $\mathcal{B}_1 = (B_1, D_1, c_1)$ and $\mathcal{B}_2 = (B_2, D_2, c_2)$ such that there is an isomorphism φ between B_1 and B_2 that maps D_1 to D_2, then $T_k(\mathcal{B}_1)$ and $T_k(\mathcal{B}_2)$ are isomorphic. Moreover, if φ maps c_1 to c_2, then there is an isomorphism between $T_k(\mathcal{B}_1)$ and $T_k(\mathcal{B}_2)$ that maps the core of $T_k(\mathcal{B}_1)$ to the core of $T_k(\mathcal{B}_2)$. Hence up two isomorphism there are only finitely many k-typical graphs with a core of size $\aleph_0$.

Given a singular cardinals κ we have more possibilities for typical graphs with k-connected sets of size κ. We call a sequence $\mathcal{K} = (\kappa_\alpha < \kappa \mid \alpha \in \operatorname{cf} \kappa)$ of infinite cardinals a *good κ-sequence*, if

- it is cofinal, i.e. $\bigcup \mathcal{K} = \kappa$;

- it is strictly ascending, i.e. $\kappa_\alpha < \kappa_\beta$ for all $\alpha < \beta$ with $\alpha, \beta \in \operatorname{cf} \kappa$;
- $\operatorname{cf} \kappa < \kappa_\alpha < \kappa$ for all $\alpha \in \operatorname{cf} \kappa$; and
- κ_α is regular for all $\alpha \in \operatorname{cf} \kappa$.

Note that given any $I \subseteq \operatorname{cf} \kappa$ with $|I| = \operatorname{cf} \kappa$ there is a unique order-preserving bijection between $\operatorname{cf} \kappa$ and I. Hence we can relabel subsequence $\mathcal{K} \restriction I$ of a good κ-sequence $\mathcal{K}$ to a good κ-sequence $\overline{\mathcal{K} \restriction I}$. Moreover, note that any cofinal sequence can be made into a good κ-sequence by looking at an strictly ascending subsequence starting above the cofinality of κ, then replacing each element in the sequence by its successor cardinal and relabel as above. Here we use the fact that each successor cardinal is regular. Hence for every singular cardinal κ there is a good κ-sequence.

Let $\mathcal{K} = (\kappa_\alpha < \kappa \mid \alpha \in \operatorname{cf} \kappa)$ be a good κ-sequence and let $\ell \leqslant k$ be a non-negative integer. As a generalisation of the graph $K_{k,\kappa}$ we first consider the disjoint union of the complete bipartite graphs K_{k,κ_α}. Then we identify ℓ sets of vertices each consisting of a vertex of the finite side of each graph, and connect the other $k - \ell$ vertices of each with disjoint stars $K_{1,\operatorname{cf} \kappa}$. More formally, let $X = [\ell, k) \times \{0\}$, and for each $\alpha \in \operatorname{cf} \kappa$ let $Y^\alpha = \{\alpha\} \times [0, k) \times \{1\}$ and let $Z^\alpha = \{\alpha\} \times \kappa_\alpha \times \{2\}$. We denote the family $(Y^\alpha \mid \alpha \in \operatorname{cf} \kappa)$ with $\mathcal{Y}$ and the family $(Z^\alpha \mid \alpha \in \operatorname{cf} \kappa)$ with $\mathcal{Z}$. Then consider the union $\bigcup\{K(Y^\alpha, Z^\alpha) \mid \alpha \in \operatorname{cf} \kappa\}$ of the complete bipartite graphs and let ℓ–$K(k, \mathcal{K})$ denote the graph where for each $i \in [0, \ell)$ we identify the set $\operatorname{cf} \kappa \times \{i\} \times \{1\}$ to one vertex in that union. For this graph we fix some further notation. Let

- x_i denote $(i, 0) \in X$ for $i \in [\ell, k)$;
- $y_i = y_i^\alpha$ for all $\alpha \in \operatorname{cf} \kappa$ denote the vertex corresponding to $\operatorname{cf} \kappa \times \{i\} \times \{1\}$ for $i \in [0, \ell)$; we call such a vertex a *degenerate vertex of* ℓ–$K(k, \mathcal{K})$;
- y_i^α denote $(\alpha, i, 1)$ for $i \in [\ell, k)$; and
- $\mathcal{Y}_i$ denote $(y_i^\alpha \mid \alpha \in I)$ for $i \in [\ell, k)$.

Note that while the definition of ℓ–$K(k, \mathcal{K})$ formally depends on the choice of a good κ-sequence, the structure of the graph is independent of that choice.

Remark B.3.2. *ℓ–$K(k, \mathcal{K}_0)$ is isomorphic to a subgraph of ℓ–$K(k, \mathcal{K}_1)$, and vice versa, for any two good κ-sequences $\mathcal{K}_0$, $\mathcal{K}_1$.* $\square$

Given ℓ–$K(k, \mathcal{K})$ as above, let S_i denote the star $K(\{x_i\}, \bigcup \mathcal{Y}_i)$ for all $i \in [\ell, k)$. Consider the union of ℓ–$K(k, \mathcal{K})$ with $\bigcup_{i\in[\ell,k)} S_i$. We call this graph ℓ–$FK_{k,\kappa}(\mathcal{K})$, or an *$\ell$-degenerate frayed* $K_{k,\kappa}$ (with respect to $\mathcal{K}$). As before, any vertex y_i for $i \in [0, \ell)$ is called a *degenerate vertex of* ℓ–$FK_{k,\kappa}(\mathcal{K})$, and any x_i for $i \in [\ell, k)$ is called a *frayed centre of* ℓ–$FK_{k,\kappa}(\mathcal{K})$. The *core* $C(\ell$–$FK_{k,\kappa}(\mathcal{K}))$ of ℓ–$FK_{k,\kappa}(\mathcal{K})$ is the vertex set $\bigcup \mathcal{Z}$. As with $K_{k,\kappa}$ it is easy to see that $C(\ell$–$FK_{k,\kappa}(\mathcal{K}))$ is k-connected in ℓ–$FK_{k,\kappa}(\mathcal{K})$ and of size κ.

Note that Remark B.3.2 naturally extends to ℓ–$FK_{k,\kappa}(\mathcal{K})$. Hence for each κ we now fix a specific good κ-sequence and write just ℓ–$FK_{k,\kappa}$ when talking about an ℓ-degenerate frayed $K_{k,\kappa}$ regarding that sequence. Further note that k–$FK_{k,\kappa}$ is isomorphic to $K_{k,\kappa}$. We also call a 0-degenerate frayed $K_{k,\kappa}$ just a *frayed* $K_{k,\kappa}$ or $FK_{k,\kappa}$ for short, see Figure B.3 for an example.

For a singular cardinal κ and for any $\ell \in [0, k]$ the graph ℓ–$FK_{k,\kappa}$ is a *k-typical graph* with a core of size κ. These are besides the complete bipartite graph $K_{k,\kappa}$ the only other k-typical graphs with a core of size κ if κ has uncountable cofinality.

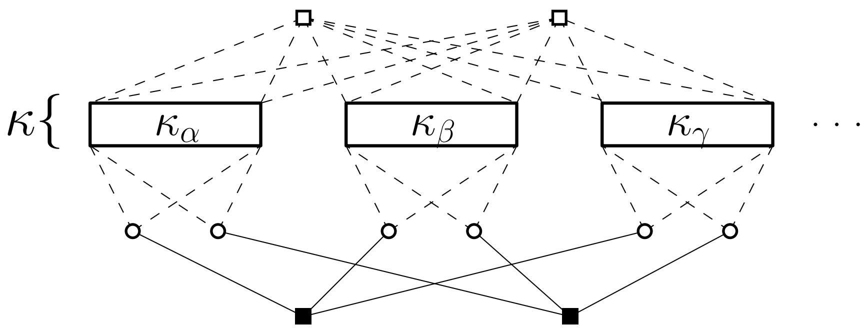

Figure B.3.: Image of $2-FK_{4,\kappa}$. The black squares represent the frayed centres and the white squares the degenerate vertices. Its core is represented by the union of the boxes (labelled according to the fixed good κ-sequence) and has size κ as illustrated by the bracket.

Next we will describe the other possibilities of k-typical graphs for singular cardinals with countable cofinality.

A *singular k-blueprint* $\mathcal{B}$ is a 5-tuple (ℓ, f, B, D, σ) such that

- $0 \leqslant \ell + f < k$;

- (B, D) is a $(k - \ell - f)$-blueprint with $2 \cdot |D| \leqslant |V(B)|$; and
- $\sigma : [\ell + f, k) \to V(B - D) \times \{0, 1\}$ is an injective map.

Let $\mathcal{B} = (\ell, f, B, D, \sigma)$ be a singular k-blueprint and let $\mathcal{K} = (\kappa_\alpha < \kappa \mid \alpha \in \aleph_0)$ be a good κ-sequence. We construct our desired graph $T_k(\mathcal{B})(\mathcal{K})$ as follows. We start with ℓ-$FK_{k,\kappa}(\mathcal{K})$ with the same notation as above. We remove the set $\{x_i \mid i \in [\ell + f, k)\}$ from the graph we constructed so far. Moreover, we take the disjoint union with $\mathfrak{N}(B/D)$ as above. We identify the vertices $\{y_i^\alpha \mid i \in [\ell + f, k)\}$ with distinct vertices of the $(2\alpha + |V(B)|)$-th and $(2\alpha + 1 + |V(B)|)$-th layer for every $\alpha \in \aleph_0$ as given by the map σ, that is

$$y_i^\alpha \sim \pi_0(\sigma(i))_{2\alpha + \pi_1(\sigma(i)) + |V(B)|}$$

where π_0 and π_1 denote the projection maps for the tuples in the image of σ. For convenience we denote a vertex originated via such an identification by any of its previous names. The *core* of $T_k(\mathcal{B})(\mathcal{K})$ is $C(T_k(\mathcal{B})(\mathcal{K})) := \bigcup \mathcal{Z}$. For an example we refer to Figure B.4.

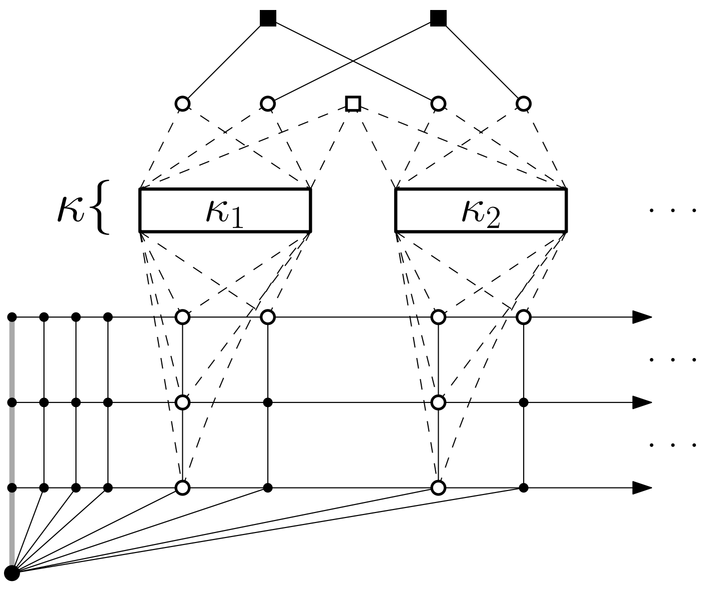

Figure B.4.: Image of $T_7(1, 2, P, \{d\}, \sigma)$ for P and $\{d\}$ as in Figure B.2.

As before, the information given by a specific good κ-sequence does not matter for the structure of the graph. Similarly, we get with Remark B.3.2 that two graphs $T_k(\mathcal{B})(\mathcal{K}_0)$ and $T_k(\mathcal{B})(\mathcal{K}_1)$ obtained by different good κ-sequences $\mathcal{K}_0$, $\mathcal{K}_1$ are isomorphic to fbs-minors of each other. Hence when we use the fixed good κ-sequence as before, we call the graph just $T_k(\mathcal{B})$.

Lemma B.3.3. *For a singular k-blueprint $\mathcal{B}$, the core of $T_k(\mathcal{B})$ is k-connected in $T_k(\mathcal{B})$.*

Proof. Let $\mathcal{B} = (\ell, f, B, D, \sigma)$ and let $C = C(T_k(\mathcal{B}))$ denote the core of $T_k(\mathcal{B})$. Let $U, W \subseteq C$ with $|U| = |W| = k' \leqslant k$. Suppose for a contradiction there is a vertex set S of size less than k' separating U and W. This separator needs to contain all degenerate vertices as well as block all paths via the frayed centres. Hence there are less than $k' - \ell - f$ many vertices of S on $\mathfrak{N}(B/D)$, and therefore there is either a $b \in V(B)\backslash D$ such that either $\mathfrak{N}_b$ does not contain a vertex of S or a $d \in D\backslash S$. Moreover, there are $m, n \in \mathbb{N}$ such that $u^m \in (U \cap Z^m)\backslash S$ and $w^n \in (W \cap Z^n)\backslash S$. Now $n \neq m$ since S cannot separate two vertices of $Z^n\backslash S$ in $K(Y^n, Z^n) \subseteq T_k(\mathcal{B})$. Since the vertices of $Y^n \cap \mathfrak{N}(B/D)$ lie on at least $(k - \ell - f)/2$ different rays of the form $\mathfrak{N}_x$ for $x \in V(B)\backslash D$, there is a vertex $v^n \in (Y^n \cap \mathfrak{N}(B/D))\backslash S$ such that the ray $\mathfrak{N}_x$ that contains v^n either has no vertices of S on its tail starting at v^n or on its initial segment upto v^n. Also, there is an $N \in \mathbb{N}$ with $N \geqslant n$ in the first case and $N \leqslant n$ in the second case (since $n \geqslant |V(B)|$) such that B_N does not contain a vertex of S. Hence we can find a path avoiding S starting at w^n and ending on the ray $\mathfrak{N}_b$ or the dominating vertex d. Analogously, we get $v^m \in (Y^m \cap \mathfrak{N}(B/D))\backslash S$, B_M and a respective path avoiding S. Hence we can connect u^m and w^n via a path avoiding S, contradicting the assumption. $\square$

For a singular cardinal κ with countable cofinality and for any singular k-blueprint $\mathcal{B}$ the graph $T_k(\mathcal{B})$ is a *k-typical graph* with core of size κ. These are the only remaining k-typical graphs.

Note that as before there are up to isomorphism only finitely many k-typical graphs with a core of size κ.

In summary we get for each $k \in \mathbb{N}$ and each infinite cardinal κ a finite list of k-typical graphs with a core of size κ:

κ	k-typical graph T	core $C(T)$
$\kappa = \operatorname{cf}\kappa > \aleph_0$	$K_{k,\kappa}$	Z
$\kappa = \operatorname{cf}\kappa = \aleph_0$	$K_{k,\kappa}$	Z
	$T_k(B, D, c)$	$V(\mathfrak{N}_c)$
$\kappa > \operatorname{cf}\kappa > \aleph_0$	$K_{k,\kappa}$	Z
	ℓ–$FK_{k,\kappa}$	$\bigcup \mathcal{Z}$
$\kappa > \operatorname{cf}\kappa = \aleph_0$	$K_{k,\kappa}$	Z
	ℓ–$FK_{k,\kappa}$	$\bigcup \mathcal{Z}$
	$T_k(\ell, f, B, D, \sigma)$	$\bigcup \mathcal{Z}$

Note that for the finiteness of this list we need the fixed good κ-sequence for a singular cardinal κ.

Lemma B.3.4. *The core of a k-typical graph is k-connected in that graph.* □

B.3.2. Generalised k-typical graphs

The k-typical graphs cannot serve for a characterisation for the existence of k-connected sets as in Theorem B.1.1(c) via subdivisions, as the following example illustrates. Consider two disjoint copies of the $K_{2,\aleph_0}$ together with a matching between the infinite sides, see Figure B.5. Now the vertices of the infinite side from one of the copies is a 4-connected set in that graph, but the graph does not contain any subdivision of a 4-typical graph, since it neither contains a path of length greater than 13 (and hence no subdivision of a $T_4(\mathcal{B})$ for some regular k-blueprint $\mathcal{B}$), nor a subdivision of a $K_{4,\aleph_0}$.

To solve this problem we introduce *generalised k-typical graphs*, where we 'blow up' some of the vertices of our k-typical graph to some finite tree, e.g. an edge in the previous example. This then will allow us to obtain the desired subdivisions for our characterisation.

Let G be a graph, $v \in V(G)$ be a vertex, T be a finite tree and $\gamma : N(v) \to V(T)$ be a map. We define the *(v, T, γ)-blow-up of v in G* as the operation where we delete v, add a vertex set $\{v\} \times V(T)$ disjointly and for each $w \in N(v)$ add the edge between w and $(v, \gamma(w))$. We call the resulting graph $G(v, T, \gamma)$.

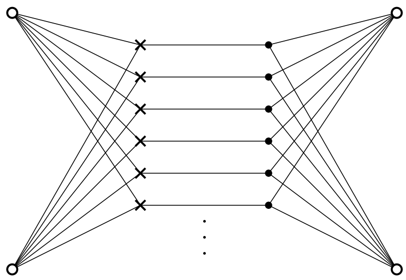

Figure B.5.: A graph with an infinite 4-connected set (marked by the cross vertices) containing no subdivision of a 4-typical graph.

Given blow-ups (v, T_v, γ_v) and (w, T_w, γ_w) in G, we can apply the blow-up of w in $G(v, T_v, \gamma_v)$ by replacing v in the preimage of γ_w by $(v, \gamma_v(w))$. We call this graph $G(v, T_v, \gamma_v)(w, T_w, \gamma_w)$. Note that the order in which we apply the blow-ups does not matter, that is $G(v, T_v, \gamma_v)(w, T_w, \gamma_w) = G(w, T_w, \gamma_w)(v, T_v, \gamma_v)$. We analogously define for a set $O = \{(v, T_v, \gamma_v) \mid v \in W\}$ of blow-ups for some $W \subseteq V(G)$ the graph $G(O)$ obtained by successively applying all the blow-ups in O. Note that if W is infinite, then $G(O)$ is still well-defined, since each edge gets each of its endvertices modified at most once.

A *type*-1 *k-template* $\mathcal{T}_1$ is a triple (T, γ, c) consisting of a finite tree T, a map $\gamma : [0, k) \to V(T)$ and a node $c \in V(T)$ such that each node of degree 1 or 2 in T is either c or in the image of γ. Note that for each k there are only finitely many type-1 k-templates up to isomorphisms of the trees, since their trees have order at most $2k + 1$.

Let $\mathcal{T}_1 = (T, \gamma, c)$ be a type-1 k-template and let $O_1 := \{(z, T, \gamma) \mid z \in C(K_{k,\kappa})\}$. We call the graph $K_{k,\kappa}(\mathcal{T}_1) := K_{k,\kappa}(O_1)$ a *generalised* $K_{k,\kappa}$. The *core* $C(K_{k,\kappa}(\mathcal{T}_1))$ is the set $C(K_{k,\kappa}) \times \{c\}$, see Figure B.6 for an example. Note that Figure B.5 is also an example.

Similarly, with $\mathcal{T}_1$ as above, let $O'_1 := \{(z, T, \gamma_\alpha) \mid \alpha \in \operatorname{cf} \kappa, z \in Z^\alpha\}$, where γ_α denotes the map defined by $y_i^\alpha \mapsto \gamma(i)$. The graph $\ell\text{–}K(k, \mathcal{K})(\mathcal{T}_1) := \ell\text{–}K(k, \mathcal{K})(O'_1)$ is a *generalised* $\ell\text{–}K(k, \mathcal{K})$. The vertex set $\bigcup \mathcal{Z} \times \{c\}$ is the *precore* of that graph.

Analogously, we obtain a *generalised* $\ell\text{–}FK_{k,\kappa}(\mathcal{K})$ for a good κ-sequence $\mathcal{K}$ as $\ell\text{–}FK_{k,\kappa}(\mathcal{K})(\mathcal{T}_1) := \ell\text{–}FK_{k,\kappa}(O'_1)$ with *core* $C(\ell\text{–}FK_{k,\kappa}(\mathcal{K})(\mathcal{T}_1)) := \bigcup \mathcal{Z} \times \{c\}$.

A *type*-2 *k-template* $\mathcal{T}_2$ for a k-blueprint (B, D) is a set $\{(b, P_b, \gamma_b) \mid b \in V(B) \setminus D\}$ of blow-ups in B such that for all $b \in V(B) \setminus D$

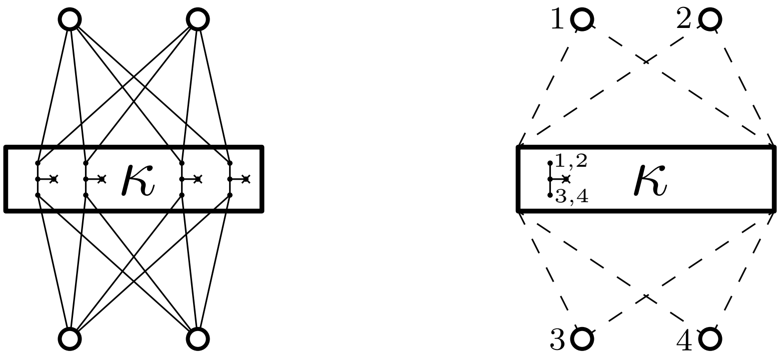

Figure B.6.: Image of a generalised $K_{4,\kappa}$ on the left. The crosses represent the core. On the right is how we represent the same graph in a simplified way by labelling the vertices according to their adjacencies.

- P_b is a path of length at most $k + 2$;
- the endnodes of P_b are called v_0^b and v_1^b;
- P_b contains nodes $v_\perp^b$ and $v_\top^b$;
- the nodes $v_0^b, v_\perp^b, v_\top^b, v_1^b$ need not be distinct;
- if $v_0^n \neq v_\perp^b$, then $v_0^b v_\perp^b \in E(P_b)$ and if $v_1^n \neq v_\top^b$, then $v_1^b v_\top^b \in E(P_b)$;
- $\gamma_b(N(b)) \subseteq v_\perp^b P_b v_\top^b$;

We say $\mathcal{T}_2$ is *simple* if $v_0^b = v_\perp^b$ and $v_1^b = v_\top^b$. Note that for each k there are only finitely many type-2 k-templates, up to isomorphisms of the trees in the k-blueprints and the paths for the blow-ups.

Let $\mathcal{T}_2 = \{(b, T_b, \gamma_b) \mid b \in V(B)\backslash D\}$ be a type-2 k-template for a k-blueprint (B, D). Then $O_2 := \{(b_n, T_b, \gamma_b^n) \mid n \in \mathbb{N}, b \in V(B)\backslash D\}$ is a set of blow-ups in $\mathfrak{N}(B/D)$, where γ_b^n is defined via

$$\gamma_b^n(v) = \begin{cases} \gamma_b(b') & \text{if } v = b'_n \text{ for } b' \in N(b); \\ v_\top^b & \text{if } v = b_{n+1}; \\ v_\perp^b & \text{if } n \geqslant 1 \text{ and } v = b_{n-1}. \end{cases}$$

Then $\mathfrak{N}(B/D)(\mathcal{T}_2) := \mathfrak{N}(B/D)(O_2)$ is a *generalised* $\mathfrak{N}(B/D)$.

Let $\mathcal{B} = (B, D, c)$ be a regular k-blueprint and let $\mathcal{T}_2 = \{(b, T_b, \gamma_b) \mid b \in V(B)\backslash D\}$ be a type-2 k-template for (B, D). We call $T_k(\mathcal{B})(\mathcal{T}_2) := T_k(\mathcal{B})(O_2)$ a *generalised* $T_k(\mathcal{B})$ with *core* $C(T_k(\mathcal{B})(\mathcal{T}_2)) := V(\mathfrak{N}_c) \times \{v_1^c\}$. For an example that generalises the graph of Figure B.2, see Figure B.7.

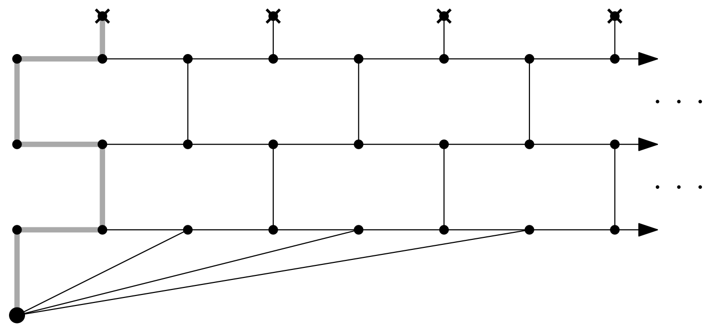

Figure B.7.: Image of a generalised $T_4(P, \{d\}, c)$ for P, $\{d\}$, c as in Figure B.2. In grey we represent the blow-up of P as given by some type-2 k-template. The crosses represent the core.

A *type-3 k-template* $\mathcal{T}_3$ for a singular k-blueprint $\mathcal{B} = (\ell, f, B, D, \sigma)$ is a tuple $(\mathcal{T}_1, \mathcal{T}_2)$ consisting of a type-1 $(\ell + f)$-template $\mathcal{T}_1$ and a type-2 $(k - \ell - f)$-template $\mathcal{T}_2$. Note that for each k there are only finitely many type-3 k-templates up to isomorphisms as discussed above for $\mathcal{T}_1$ and $\mathcal{T}_2$.

Let $\mathcal{T}_3 = (\mathcal{T}_1, \mathcal{T}_2)$ be a type-3 k-template with $\mathcal{T}_1 = (T, \gamma, c_1)$ for a singular k-blueprint $\mathcal{B} = (\ell, f, B, D, \sigma)$. Then for $(b_n, T_b, \gamma_b^n) \in O_2$ we extend γ_b^n to $\hat{\gamma}_b^n$ via

$$\hat{\gamma}_b^n(v) = \begin{cases} v_1^b & \text{if } v \in \{y_i^n \mid i \in [\ell + f, k)\} \text{ and } n \text{ even;} \\ v_0^b & \text{if } v \in \{y_i^n \mid i \in [\ell + f, k)\} \text{ and } n \text{ odd;} \\ \gamma_b^n(v) & \text{otherwise.} \end{cases}$$

Let $O_2' := \{(b_n, T_b, \hat{\gamma}_b^n) \mid (b_n, T_b, \gamma_b^n) \in O_2\}$ denote the corresponding set of blow-ups in $T_k(\mathcal{B})$ and let O_1' be for $\mathcal{T}_1$ as above. The graph $T_k(\mathcal{B})(\mathcal{T}_3) := T_k(\mathcal{B})(O_1' \cup O_2')$ is a *generalised* $T_k(\mathcal{B})$ with *core* $C(T_k(\mathcal{B})(\mathcal{T}_3)) := \bigcup \mathcal{Z} \times \{c_1\}$. For an example that generalises the graph of Figure B.4 see Figure B.8.

We call the graph from which a generalised graph is obtained via this process its *parent*. As before, Remark B.3.2 and its extensions extend to generalised k-typical graphs as well.

Remark B.3.5. *Every ℓ–$FK_{k,\kappa}(\mathcal{T}_1)(\mathcal{K})$ or $T_k(\mathcal{B})(\mathcal{T}_3)(\mathcal{K})$ for a singular k-blueprint $\mathcal{B}$, a type-1 k-template $\mathcal{T}_2$, a type-2 k-template $\mathcal{T}_3$ and a good κ-sequence $\mathcal{K}$, contains a subdivision of ℓ–$FK_{k,\kappa}(\mathcal{T}_1)$ or $T_k(\mathcal{B})(\mathcal{T}_3)$ respectively.*

A *generalised k-typical graph* is either $K_{k,\kappa}(\mathcal{T}_1)$, ℓ–$FK_{k,\kappa}(\mathcal{T}_1)$, $T_k(\mathcal{B})(\mathcal{T}_2)$ or $T_k(\mathcal{B}')(\mathcal{T}_3)$ for any type-1 k-template $\mathcal{T}_1$, any $\ell \in [0, k)$, any regular k-blueprint $\mathcal{B}$,

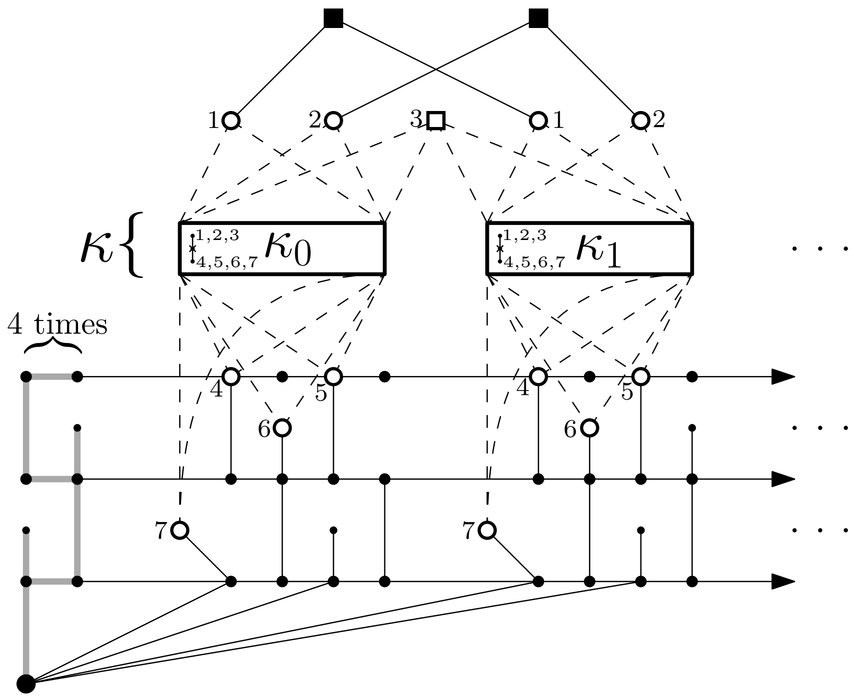

Figure B.8.: Image of a generalised $T_7(1, 2, P, \{d\}, \sigma)$ for P, $\{d\}$, σ as in Figure B.4.

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any type-2 k-template $\mathcal{T}_2$ for $\mathcal{B}$, any singular k-blueprint $\mathcal{B}'$ and any type-3 k-template $\mathcal{T}_3$ for $\mathcal{B}'$. As with the k-typical graphs we obtain that this list is finite.

Corollary B.3.6. *The core of a generalised k-typical graph is k-connected in that graph.* □

B.3.3. Statement of the Main Theorem

Now that we introduced all k-typical and generalised k-typical graphs, let us give the full statement of our main theorem of Section B.

Theorem B.3.7. *Let G be an infinite graph, let $k \in \mathbb{N}$, let $A \subseteq V(G)$ be infinite and let $\kappa \leqslant |A|$ be an infinite cardinal. Then the following are equivalent.*

(a) There is a subset $A_1 \subseteq A$ with $|A_1| = \kappa$ such that A_1 is k-connected in G.

(b) There is a subset $A_2 \subseteq A$ with $|A_2| = \kappa$ such that there is a k-typical graph which is a minor of G with finite branch sets and with A_2 along its core.

(c) There is a subset $A_3 \subseteq A$ with $|A_3| = \kappa$ such that there G contains a subdivided generalised k-typical graph with A_3 as its core.

(d) *There is no nested separation system $N \subseteq S_k(G)$ such that every part P of N can be separated from A by less than κ vertices.*

Moreover, if these statements hold, we can choose $A_1 = A_2 = A_3$.

Note that for $A = V(G)$ we obtain the simple version as in Theorem B.1.1 by forgetting the extra information about the core.

B.4. k-connected sets, minors and topological minors

In this section we will collect a few basic remarks and lemmas on k-connected sets and how they interact with minors and topological minors for future references. We omit some of the trivial proofs.

Remark B.4.1. *If $A \subseteq V(G)$ is k-connected in G, then any $A' \subseteq A$ with $|A'| \geqslant k$ is k-connected in G as well.* □

Lemma B.4.2. *If M is a minor of G and $A \subseteq V(M)$ is k-connected in M for some $k \in \mathbb{N}$, then any set $A' \subseteq V(G)$ with $|A'| \geqslant k$ consisting of at most one vertex of each branch set for the vertices of A is k-connected in G.* □

Lemma B.4.3. *For $k \in \mathbb{N}$, if G contains the subdivision of a generalised k-typical graph T with core A, then the parent of T is an fbs-minor with A along its core.* □

A helpful statement for the upcoming inductive constructions would be that for every vertex v of G, every large k-connected set in G contains a large subset which is $(k-1)$-connected in $G - v$. But while this is a true statement (cf. Corollary B.8.2), an elementary proof of it seems to be elusive if v is not itself contained in the original k-connected set. The following lemma is a simplified version of that statement and has an elementary proof.

Lemma B.4.4. *Let $k \in \mathbb{N}$ and let $A \subseteq V(G)$ be infinite and k-connected in G. Then for any finite set $S \subseteq V(G)$ with $|S| < k$ there is a subset $A' \subseteq A$ with $|A'| = |A|$ such that A' is 1-connected in $G - S$.*

Proof. Without loss of generality we may assume that A and S are disjoint. Take a sequence $(B_\alpha \mid \alpha \in |A|)$ of disjoint subsets of A with $|B_\alpha| = k$. For every

$\alpha \in |A| \setminus \{0\}$ there is at least one path from B_0 to B_α disjoint from S. By the pigeonhole principle there is some $v \in B_0$ such that $|A|$ many of these paths start in v. Now let A' be the set of endvertices of these paths. □

B.5. Structure within ends

This section studies the structure within an end of a graph.

In Subsection B.5.1 we will extend to arbitrary infinite graphs a well-known result for locally finite graphs relating end degree with a certain sequence of minimal separators, making use of the combined end degree.

Subsection B.5.2 is dedicated to the construction of a uniformly connecting structure between disjoint rays in a common end and vertices dominating that end.

B.5.1. End defining sequences and combined end degree

For an end $\omega \in \Omega(G)$ and a finite set $S \subseteq V(G)$ let $C(S, \omega)$ denote the unique component of $G - S$ that contains ω-rays. A sequence $(S_n \mid n \in \mathbb{N})$ of finite vertex sets of G is called an *ω-defining sequence* if $C(S_{n+1}, \omega) \subseteq C(S_n, \omega)$ for all $n \in \mathbb{N}$ and $\bigcap\{C(S_n, \omega) \mid n \in \mathbb{N}\} = \emptyset$. Note that for every ω-defining sequence $(S_n \mid n \in \mathbb{N})$ and every finite set $X \subseteq V(G)$ we can find an $N \in \mathbb{N}$ such that $X \subseteq G - C(S_N, \omega)$. Hence we shall also refer to the sets S_n in such a sequence as *separators* S_n and S_m. Given $n, m \in \mathbb{N}$ with $n < m$, let $G[S_n, S_m]$ denote $G[(S_n \cup C(S_n, \omega)) \setminus C(S_m, \omega)]$, the *graph between the separators*.

For ends of locally finite graphs there is a characterisation of the end degree given by the existence of certain ω-defining sequences. The degree of an end ω is equal to $k \in \mathbb{N}$, if and only if k is the smallest integer such that there is an ω-defining sequence of sets of size k, cf. [57, Lemma 3.4.2]. In this subsection we extend this characterisation to arbitrary graphs with respect to the combined degree. Recall the definition of the combined degree, $\Delta(\omega) := \deg(\omega) + \mathrm{dom}(\omega)$.

In arbitrary graphs ω-defining sequences need not necessarily exist, e.g. in $K_{\aleph_1}$. We start by characterising the ends admitting such a sequence.

Lemma B.5.1. *Let $\omega \in \Omega(G)$ be an end. Then there is an ω-defining sequence $(S_n \mid n \in \mathbb{N})$ if and only if $\Delta(\omega) \leqslant \aleph_0$.*

Moreover, this sequence can be chosen such that $S_i \cap S_j$ contains only vertices dominating ω for all $i \neq j$.

Proof. Note that for all finite $S \subseteq V(G)$, no $d \in \mathrm{Dom}(\omega)$ can lie in a component $C \neq C(S, \omega)$ of $G - S$. Hence for every ω-defining sequence $(S_n \mid n \in \mathbb{N})$ and every $d \in \mathrm{Dom}(\omega)$ there is an $N \in \mathbb{N}$ such that $d \in S_m$ for all $m \geqslant N$. Therefore, if $\mathrm{dom}(\omega) > \aleph_0$, no ω-defining sequence can exist, since the union of the separators is at most countable. Moreover, note that for every ω-defining sequence every ω-ray meets infinitely many distinct separators. It follows that $\deg(\omega)$ is at most countable as well if an ω-defining sequence exist.

For the converse, suppose $\Delta(\omega) \leqslant \aleph_0$. Let $\{d_n \mid n < \mathrm{dom}(\omega)\}$ be an enumeration of $\mathrm{Dom}(\omega)$. Let $R = r_0 r_1 \ldots$ be an ω-devouring ray, which exist by Lemma B.2.7. We build our desired ω-defining sequence $(S_n \mid n \in \mathbb{N})$ inductively. Set $S_0 := \{r_0\}$. For $n \in \mathbb{N}$ suppose S_n is already constructed as desired. Take a maximal set $\mathcal{P}_n$ of pairwise disjoint $N(S_n \backslash \mathrm{Dom}(\omega)) - R$ paths in $C(S_n, \omega)$. Note that $\mathcal{P}_n$ is finite since otherwise by the pigeonhole principle we would get a vertex $v \in S_n \backslash \mathrm{Dom}(\omega)$ dominating ω. Furthermore, $\mathcal{P}_n$ is not empty as $C(S_n, \omega)$ is connected. Define

$$\begin{aligned} S_{n+1} := \quad & (S_n \cap \mathrm{Dom}(\omega)) \cup \bigcup \mathcal{P}_n \\ & \cup \big\{ r_m \mid m \text{ is minimal with } r_m \in C(S_n, \omega) \big\} \\ & \cup \big\{ d_m \mid m \text{ is minimal with } d_m \in C(S_n, \omega) \big\}. \end{aligned}$$

By construction, $S_{n+1} \cap S_i$ contains only vertices dominating ω for $i \leqslant n$. Let P be any $S_n - C(S_{n+1}, \omega)$ path. We can extend P in $C(S_{n+1}, \omega)$ to a $S_n - R$ path. And since $C(S_{n+1}, \omega) \cap S_{n+1}$ is empty, we obtain $P \cap S_{n+1} \neq \varnothing$ by construction of S_{n+1}. Hence any $S_n - C(S_{n+1}, \omega)$ path meets S_{n+1}. While for any vertex $v \in C(S_{n+1}, \omega) \backslash C(S_n, \omega)$ there is a path to $C(S_n, \omega) \cap C(S_{n+1}, \omega)$ in $C(S_{n+1}, \omega)$, this path would meet a vertex $w \in S_n$. This vertex would be a trivial $S_n - C(S_{n+1}, \omega)$ path avoiding S_{n+1}, and hence contradicting the existence of such v. Hence $C(S_{n+1}, \omega) \subseteq C(S_n, \omega)$.

Suppose there is a vertex $v \in \bigcap \{C(S_n, \omega) \mid n \in \mathbb{N}\}$. By construction v is neither dominating ω nor is a vertex on R. Note that every $v - R$ path has to contain vertices from infinitely many S_n, hence it has to contain a vertex dominating ω. For each $d \in \mathrm{Dom}(\omega)$ let P_d be either the vertex set of a $v - \mathrm{Dom}(\omega)$ path containing d

if it exist, or $P_d = \emptyset$ otherwise. If $X := \bigcup\{P_d \mid d \in \mathrm{Dom}(\omega)\}$ is finite, we can find an $N \in \mathbb{N}$ such that $v \in X \subseteq G - C(S_N, \omega)$, a contradiction. Otherwise apply Lemma B.2.4 to $X \cap \mathrm{Dom}(\omega)$ in $G[X]$. Note that in $G[X]$ all vertices of $X \cap \mathrm{Dom}(\omega)$ have degree 1. Furthermore, we know that $V(R) \cap X \subseteq \mathrm{Dom}(\omega)$, since no P_d contains a vertex of R as an internal vertex. But then the centre of a star would be a vertex dominating ω in $X \setminus \mathrm{Dom}(\omega)$ and the spine of a comb would contain an ω-ray disjoint to R as a tail, again a contradiction. □

In the proof of the end-degree characterisation via ω-defining sequences we shall need the following fact regarding the relationship of $\deg(\omega)$ and $\mathrm{dom}(\omega)$.

Lemma B.5.2. *If* $\deg(\omega)$ *is uncountable for* $\omega \in \Omega(G)$*, then* $\mathrm{dom}(\omega)$ *is infinite.*

Proof. Suppose for a contradiction that $\mathrm{dom}(\omega) < \aleph_0$. For $G' := G - \mathrm{Dom}(\omega)$ let $\mathcal{R}$ be a set of disjoint $\omega{\restriction}G'$-rays of size $\aleph_1$, which exist by Remark B.2.6. Let T be a transversal of $\{V(R) \mid R \in \mathcal{R}\}$. Applying Lemma B.2.4 to T yields a subdivided star with centre d and uncountably many leaves in T. Now $d \notin \mathrm{Dom}(\omega)$ dominates $\omega{\restriction}G'$ in G' and hence ω in G by Remark B.2.6, a contradiction. □

Let $\omega \in \Omega(G)$ be an end with $\mathrm{dom}(\omega) = 0$, $(S_n \mid n \in \mathbb{N})$ be an ω-defining sequence and $\mathcal{R}$ be a set of disjoint ω-rays. We call $((S_n \mid n \in \mathbb{N}), \mathcal{R})$ a *degree witnessing pair* for ω, if for all $n \in \mathbb{N}$ and for each $s \in S_n$ there is a ray $R \in \mathcal{R}$ containing s and every ray $R \in \mathcal{R}$ meets S_n at most once for every $n \in \mathbb{N}$. Note that this definition only makes sense for undominated ends, since a ray that contains a dominating vertex meets eventually all separators not only in that vertex.

Lemma B.5.3. *Let* $\omega \in \Omega(G)$ *be an end with* $\mathrm{dom}(\omega) = 0$*. Then there is a degree witnessing pair* $((S_n \mid n \in \mathbb{N}), \mathcal{R})$*.*

Proof. By Lemma B.5.1 and Lemma B.5.2 there exists an ω-defining sequence $(S'_n \mid n \in \mathbb{N})$ with disjoint sets.

We want to construct an ω-defining sequence $(S_n \mid n \in \mathbb{N})$ with the property, that for all $n \in \mathbb{N}$ and for all $m > n$ there are $|S_n|$ many S_n–S_m paths in $G[S_n, S_m]$.

Let S_0 be an S'_0–$S'_{f(0)}$ separator for some $f(0) \in \mathbb{N}$ which is of minimum order among all candidates separating S'_0 from S'_m for any $m \in \mathbb{N}$. Suppose we already constructed the sequence up to S_n. Let S_{n+1} be an $S'_{f(n)+1}$–$S'_{f(n+1)}$ separator for some $f(n+1) > f(n) + 1$ which is of minimum order among all candidates separating $S'_{f(n)+1}$ and S'_m for any $m > f(n) + 1$.

Note that S_m and S_n are disjoint for all $m, n \in \mathbb{N}$ with $n \neq m$ and that $C(S_{n+1}, \omega) \subseteq C(S'_{f(n)+1}, \omega)$ for all $n \in \mathbb{N}$. Hence $(S_n \mid n \in \mathbb{N})$ is an ω-defining sequence with disjoint sets. Moreover, note that $|S_n| \leqslant |S_{n+1}|$ for all $n \in \mathbb{N}$, since S_{n+1} would have been a candidate for S_n as well. In particular, there is no S_n–S_{n+1} separator S of order less than $|S_n|$ for every $n \in \mathbb{N}$, since this would also have been a candidate for S_n. Hence by Theorem B.2.1 there is a set of $|S_n|$ many disjoint S_n–S_{n+1} paths $\mathcal{P}_n$ in $G[S_n, S_{n+1}]$.

Now the union $\bigcup\{\bigcup \mathcal{P}_n \mid n \in \mathbb{N}\}$ is by construction a union of a set $\mathcal{R}$ of rays, since the union of the paths in $\mathcal{P}_n$ intersect the union of the paths in $\mathcal{P}_m$ in precisely S_{n+1} if $m = n+1$ and are disjoint if $m > n+1$. These rays are necessarily ω-rays, meet every separator at most once and every $s \in S_n$ is contained in one of them, proving that $((S_n \mid n \in \mathbb{N}), \mathcal{R})$ is a degree witnessing pair for ω. □

Corollary B.5.4. *Let $k \in \mathbb{N}$ and let $\omega \in \Omega(G)$ with $\mathrm{dom}(\omega) = 0$. Then $\deg(\omega) \geqslant k$ if and only if for every ω-defining sequence $(S_n \mid n \in \mathbb{N})$ the sets S_n eventually have size at least k.*

Proof. Suppose $\deg(\omega) \geqslant k$. Let $(S_n \mid n \in \mathbb{N})$ be any ω-defining sequence. Then each ray out of a set of k disjoint ω-rays has to go through eventually all S_n. For the other direction take a degree witnessing pair $((S_n \mid n \in \mathbb{N}, \mathcal{R})$. Now $|\mathcal{R}| \geqslant k$, since eventually all S_n have size at least k. □

Corollary B.5.5. *Let $k \in \mathbb{N}$ and let $\omega \in \Omega(G)$ with $\mathrm{dom}(\omega) = 0$. Then $\deg(\omega) = k$ if and only if k is the smallest integer such that there is an ω-defining sequence $(S_n \mid n \in \mathbb{N})$ with $|S_n| = k$ for all $n \in \mathbb{N}$.* □

We can easily lift these results to ends dominated by finitely many vertices with the following observation based on Remark B.2.6.

Remark B.5.6. *Suppose $\mathrm{dom}(\omega) < \aleph_0$. Let G' denote $G - \mathrm{Dom}(\omega)$.*

(a) For every $\omega\restriction G'$-defining sequence $(S'_n \mid n \in \mathbb{N})$ of G' there is an ω-defining sequence $(S_n \mid n \in \mathbb{N})$ of G with $S'_n = S_n \setminus \mathrm{Dom}(\omega)$ for all $n \in \mathbb{N}$.

(b) For every ω-defining sequence $(S_n \mid n \in \mathbb{N})$ of G there is an $\omega\restriction G'$-defining sequence $(S'_n \mid n \in \mathbb{N})$ of G' with $S'_n = S_n \setminus \mathrm{Dom}(\omega)$ for all $n \in \mathbb{N}$. □

Corollary B.5.7. *Let $k \in \mathbb{N}$ and let $\omega \in \Omega(G)$. Then $\Delta(\omega) \geqslant k$ if and only if for every ω-defining sequence $(S_n \mid n \in \mathbb{N})$ the sets S_n eventually have size at least k.*

Proof. As noted before, each vertex dominating ω has to be in eventually all sets of an ω-defining sequence.

Suppose $\Delta(\omega) \geqslant k$. If $\operatorname{dom}(\omega) \geqslant \aleph_1$, then there is no ω-defining sequence and there is nothing to show. If $\operatorname{dom}(\omega) = \aleph_0$, then the sets of any ω-defining sequence eventually have all size at least k. If $\operatorname{dom}(\omega) < \aleph_0$, we can delete $\operatorname{Dom}(\omega)$ and apply Corollary B.5.4 to $G - \operatorname{Dom}(\omega)$ with $k' = \deg(\omega)$. With Remark B.5.6 (b) the claim follows.

If $\Delta(\omega) < k$, we can delete $\operatorname{Dom}(\omega)$ and apply Corollary B.5.4 with $k' = \deg(\omega)$. With Remark B.5.6 (a) the claim follows. □

Corollary B.5.8. *Let $k \in \mathbb{N}$ and let $\omega \in \Omega(G)$. Then $\Delta(\omega) = k$ if and only if k is the smallest integer such that there is an ω-defining sequence $(S_n \mid n \in \mathbb{N})$ with $|S_n| = k$ for all $n \in \mathbb{N}$.*

Proof. As before, we delete $\operatorname{Dom}(\omega)$ and apply Corollary B.5.5 with $k' = \deg(\omega)$ and Remark B.5.6. □

Finally, we state more remarks on the relationship between $\deg(\omega)$ and $\operatorname{dom}(\omega)$ similar to Lemma B.5.2 without giving the proof.

Remarks B.5.9. *Let κ_1, κ_2 be infinite cardinals and let k_1, $k_2 \in \mathbb{N}$.*

(1) If $\operatorname{dom}(\omega)$ is infinite, then so is $\deg(\omega)$ for every $\omega \in \Omega(G)$.

(2) If $\Delta(\omega)$ is uncountable, then both $\deg(\omega)$ and $\operatorname{dom}(\omega)$ are infinite for every $\omega \in \Omega(G)$.

(3) There is a graph with an end ω' such that $\deg(\omega') = \kappa_1$ and $\operatorname{dom}(\omega') = \kappa_2$, namely the union of the complete bipartite graph $K(A, B)$ with $|A| = \kappa_1$, $|B| = \kappa_2$ with the complete graph on A.

(4) There is a graph with an end ω' such that $\deg(\omega') = k_1$ and $\operatorname{dom}(\omega') = k_2$.

(5) There is a graph with an end ω' such that $\deg(\omega') = \aleph_0$ and $\operatorname{dom}(\omega') = k_2$.

B.5.2. Constructing uniformly connected rays

Let $\omega \in \Omega(G)$ and let I, J be disjoint finite sets with $1 \leqslant |I| \leqslant \deg(\omega)$ and $0 \leqslant |J| \leqslant \operatorname{dom}(\omega)$. Let $\mathcal{R} = (R_i \mid i \in I)$ be a family of disjoint ω-rays and let

$\mathcal{D} = (d_j \in \mathrm{Dom}(\omega) \mid j \in J)$ of distinct vertices disjoint from $\bigcup \mathcal{R}$. Let T be a tree on $I \cup J$ such that J is a set of leaves of T. Let $W := \bigcup \mathcal{R} \cup \mathcal{D}$ and $k := |I \cup J|$. We call a finite subgraph $\Gamma \subseteq G$ a $(T, \mathcal{T}_2)$-*connection*, if if $\mathcal{T}_2$ is a simple type-2 k-template for (T, J), and there is a set $\mathcal{P}$ of internally disjoint W–W paths such that $\Gamma \subseteq \bigcup \mathcal{R} \cup \bigcup \mathcal{P}$ and Γ is isomorphic to a subdivision of $T(\mathcal{T}_2)$. Moreover, the subdivision of $v^i_\perp P_i v^i_\top$ is the segment $R_i \cap \Gamma$ for all $i \in I$ such that $v^i_\top$ corresponds to the top vertex of that segment. Then $(\mathcal{R}, \mathcal{D})$ is called $(T, \mathcal{T}_2)$-*connected* if for every finite $X \subseteq V(G) \setminus \mathcal{D}$ there is a $(T, \mathcal{T}_2)$-connection avoiding X.

Lemma B.5.10. *Let $\omega \in \Omega(G)$, let $\mathcal{R} = (R_i \mid i \in I)$ be a finite family of disjoint ω-rays with $|I| \geqslant 1$ and let $\mathcal{D} = (d_j \in \mathrm{Dom}(\omega) \mid j \in J)$ be a finite family of distinct vertices disjoint from $\bigcup \mathcal{R}$ with $I \cap J = \emptyset$. Then there is a tree T on $I \cup J$ and a simple type-2 $|I \cup J|$-template $\mathcal{T}$ for (T, J) such that $(\mathcal{R}, \mathcal{D})$ is $(T, \mathcal{T})$-connected.*

Proof. Let $X \subseteq V(G) \setminus \mathcal{D}$ be any finite set. We extend X to a finite superset X' such that $R_i \cap X'$ is an initial segment of R_i for each $i \in I$, and such that $\mathcal{D} \subseteq X'$. As all rays in $\mathcal{R}$ are ω-rays, we can find finitely many $\bigcup \mathcal{R}$–$\bigcup \mathcal{R}$ paths avoiding X' which are internally disjoint such that their union with $\bigcup \mathcal{R}$ is a connected subgraph of G. Moreover it is possible to do this with a set $\mathcal{P}$ of $|I| - 1$ many such paths in a tree-like way, i.e. contracting a large enough finite segment avoiding X' of each ray in $\mathcal{R}$ and deleting the rest yields a subdivision Γ'_X of a tree on I whose edges correspond to the paths in $\mathcal{P}$. For each vertex d_j we can moreover find a d_j–$\bigcup \mathcal{R}$ path avoiding $V(\Gamma'_X) \cup X' \setminus \{d_j\}$ and all paths we fixed so far. This yields a tree T_X on $I \cup J$ and a simple type-2 k-template $\mathcal{T}_X$ for (T_X, J) such that J is a set of leaves and a $(T_X, \mathcal{T}_X)$-connection Γ_X avoiding X.

Now we iteratively apply this construction to find a family $(\Gamma_i \mid i \in \mathbb{N})$ of $(T_i, \mathcal{T}_i)$-connections such that $\Gamma_m - \mathcal{D}$ and $\Gamma_n - \mathcal{D}$ are disjoint for all $m, n \in \mathbb{N}$ with $m \neq n$. By the pigeonhole principle we now find a tree T on $I \cup J$, a type-2 $|I \cup J|$-template $\mathcal{T}$ and an infinite subset $N \subseteq \mathbb{N}$ such that $(T_n, \mathcal{T}_n) = (T, \mathcal{T})$ for all $n \in N$.

Now for each finite set $X \subseteq V(G) \setminus \mathcal{D}$ there is an $n \in N$ such that Γ_n and X are disjoint, hence $(\mathcal{R}, \mathcal{D})$ is $(T, \mathcal{T})$-connected. □

Corollary B.5.11. *Let $\omega \in \Omega(G)$, let $\mathcal{R} = (R_i \mid i \in I)$ be a finite family of disjoint ω-rays with $|I| \geqslant 1$ and let $\mathcal{D} = (d_j \in \mathrm{Dom}(\omega) \mid j \in J)$ be a finite family of distinct vertices disjoint from $\bigcup \mathcal{R}$ with $I \cap J = \emptyset$. Then there is a tree T such that G contains a subdivision of a generalised $\mathfrak{N}(T/J)$.*

Proof. By Lemma B.5.10 there is a tree T and a simple type-2 $|I \cup J|$-template $\mathcal{T}$ such that $(\mathcal{R}, \mathcal{D})$ is $(T, \mathcal{T})$-connected. Let $(\Gamma_i \mid i \in \mathbb{N})$ be a family of $(T, \mathcal{T})$-connections such that $\Gamma_m - \mathcal{D}$ and $\Gamma_n - \mathcal{D}$ are disjoint for all $m, n \in \mathbb{N}$ with $m \neq n$. Then $H = \bigcup\{\Gamma_n \mid n \in \mathbb{N}\} \cup \bigcup \mathcal{R}$ is the desired subdivision. □

Finally, this result can be lifted to the minor setting by Lemma B.4.3.

Corollary B.5.12. *Let $\omega \in \Omega(G)$, let $\mathcal{R} = (R_i \mid i \in I)$ be a finite family of disjoint ω-rays with $|I| \geqslant 1$ and let $\mathcal{D} = (d_j \in \mathrm{Dom}(\omega) \mid j \in J)$ be a finite family of distinct vertices disjoint from $\bigcup \mathcal{R}$ with $I \cap J = \varnothing$. Then there is a tree T such that G contains $\mathfrak{N}(T/J)$ as an fbs-minor.* □

B.6. Minors for regular cardinalities

This section is dedicated to prove the equivalence of (a), (b) and (c) of Theorem B.3.7 for regular cardinals κ.

B.6.1. Complete bipartite minors

In this subsection we construct the complete bipartite graph $K_{k,\kappa}$ as the desired minor (and a generalised version as the desired subdivision), if possible. The ideas of this construction differ significantly from Halin's construction [31, Thm. 9.1] of a subdivision of $K_{k,\kappa}$ in a k-connected graph of uncountable and regular order κ.

Lemma B.6.1. *Let $k \in \mathbb{N}$, let $A \subseteq V(G)$ be infinite and k-connected in G and let $\kappa \leqslant |A|$ be a regular cardinal. If*

- *either κ is uncountable;*
- *or there is no end in the closure of A;*
- *or there is an end ω in the closure of A with $\mathrm{dom}(\omega) \geqslant k$;*

then there is a subset $A' \subseteq A$ with $|A'| = \kappa$ such that $K_{k,\kappa}$ is an fbs-minor of G with A' along its core.

Moreover, the branch sets for the vertices of the finite side of $K_{k,\kappa}$ are singletons.

Proof. We iteratively construct a sequence of subgraphs H_i for $i \in [0,k)$ witnessing that $K_{i,\kappa}$ is a minor of G. Furthermore, we incorporate that the branch sets for the vertices of the finite side of $K_{i,\kappa}$ are singletons $\{v_j \mid j \in [0,i)\}$ and the branch sets for the vertices of the infinite side induce finite trees on H_i each containing a vertex of A. Moreover, we will guarantee the existence of a subset $A_i \subseteq A$ with $|A_i| = |A|$ which is 1-connected in $G_i := G - \{v_j \mid j \in [0,i)\}$ and such that each vertex of A_i is contained in a branch set of H_i and each branch set of H_i contains precisely one vertex of A_i.

Set $G_0 := G$, $A_0 := A$ and $H_0 = G[A]$. For any $i \in [0,k)$ we inductively apply Lemma B.2.4 (and in the third case also Remark B.2.5) to A_i in G_i to find a subdivided star S_i with centre v_i and κ many leaves $L_i \subseteq A_i$. Without loss of generality we can assume $v_i \notin V(H_i)$, since otherwise we could just remove the branch set containing v_i and from A_i the vertex contained in that branch set. Moreover, by Lemma B.4.4 we find a subset $L_i' \subseteq L_i$ with $|L_i'| = \kappa$ which is 1-connected in G_{i+1}. First we remove from H_i every branch set which corresponds to a vertex of the infinite side of $K_{i,\kappa}$ and does not contain a vertex of L_i'. Now each path in S_i from a neighbour of v_i to L_i eventually hits a vertex of one of the finite trees induced by one of the remaining branch sets of H_i. Since all these paths are disjoint, only finitely many of them meet the same branch set first. Thus κ many different of the remaining branch sets are met by those paths first. To get H_{i+1} we do the following. First we add $\{v_i\}$ as a new branch set. Then each of the κ many branch sets reached first as described above we extend by the path segment between v_i and that branch set of precisely one of those paths. Finally, we delete all remaining branch sets not connected to $\{v_i\}$. With $A_{i+1} := L_i' \cap V(H_{i+1})$ we now have all the desired properties.

Finally, setting $H := H_k$ and $A' := A_k$ finishes the construction. $\square$

Let H be an inflated subgraph witnessing that $K_{k,\kappa} = K([0,k), Z)$ is an fbs-minor of G with A along Z for some $A \subseteq V(G)$ where each branch set of $x \in [0,k)$ is a singleton. Given a type-1 k-template $\mathcal{T}_1 = (T, \gamma, c)$ we say H is $\mathcal{T}_1$-*regular* if for each $z \in Z$:

- there is an isomorphism $\varphi_z : T_z' \to T_z$ between a subdivision T_z' of T and the finite tree $T_z = H[\mathfrak{B}(z)]$;
- $x\varphi_z(\gamma(x)) \in E(H)$ for each $x \in [0,k)$; and

- $A \cap \mathfrak{B}(z) = \{\varphi_z(c)\}$.

We say G contains $K_{k,\kappa}$ as a $\mathcal{T}_1$*-regular fbs-minor with* A *along* Z if there is such a $\mathcal{T}_1$-regular H.

Lemma B.6.2. *Let $k \in \mathbb{N}$ and κ be a regular cardinal. If $K_{k,\kappa}$ is an fbs-minor of G with A' along its core where each branch set of $x \in [0, k)$ is a singleton, then there is type-1 k-template $\mathcal{T}_1$ and $A'' \subseteq A'$ with $|A''| = \kappa$ such that G contains $K_{k,\kappa}$ as a $\mathcal{T}_1$-regular fbs-minor with A'' along its core.*

Proof. Let H be the inflated subgraph witnessing that $K_{k,\kappa}$ is an fbs-minor as in the statement. Let x also denote the vertex of G in the branch set $\mathfrak{B}(x)$ of $x \in [0, k)$. Let $v^z_x \in \mathfrak{B}(z)$ denote the unique endvertex in $\mathfrak{B}(z)$ of the edge corresponding to $xz \in E(M)$ (cf. Section B.2). Let T_z denote a subtree of $H[\mathfrak{B}(z)]$ containing $B_z = \{v^z_x \mid x \in [0, k)\} \cup \{a_z\}$ for the unique vertex $a_z \in A \cap \mathfrak{B}(z)$. Without loss of generality assume that each leaf of T_z is in B_z. By suppressing each degree 2 node of T_z that is not in B_z, we obtain a tree suitable for a type-1 k-template where a_z is the node in the third component of the template.

By applying the pigeonhole principle multiple times there is a tree T such that there exist an isomorphism $\varphi_z : T'_z \to T_z$ for a subdivision T'_z of T for all $z \in Z'$ for some $Z' \subseteq Z$ with $|Z'| = \kappa$, such that $\{\varphi_z(v^z_x) \mid z \in Z'\}$ is a singleton $\{t_x\}$ for all $x \in [0, k)$ and $\{\varphi_z(a_z) \mid z \in Z'\}$ is a singleton $\{c\}$.

Therefore with $\gamma : [0, k) \to V(T)$ defined by $x \mapsto t_x$ and c defined as above, we obtain a type-1 k-template $\mathcal{T}_1 := (T, \gamma, c)$ such that the subgraph H' of H where we delete each branch set for $z \in Z \backslash Z'$ is $\mathcal{T}_1$-regular. $\square$

Hence, we also obtain a subdivision of a generalised $K_{k,\kappa}$.

Corollary B.6.3. *In the situation of Lemma B.6.1, there is $A'' \subseteq A'$ with $|A''| = \kappa$ such that G contains a subdivision of a generalised $K_{k,\kappa}$ with core A''.* $\square$

B.6.2. Minors for regular k-blueprints

In this subsection we construct the k-typical minors for regular k-blueprints, if possible. While these graphs are essentially the same minors given by Oporowski, Oxley and Thomas [49, Thm. 5.2], we give our own independently developed proof.

The first lemma constructs such a graph along some end of high combined degree.

Lemma B.6.4. *Let $\omega \in \Omega(G)$ be an end of G with $\Delta(\omega) \geqslant k \in \mathbb{N}$. Let $A \subseteq V(G)$ be a set with ω in its closure. Then there is a countable subset $A' \subseteq A$ and a regular k-blueprint $\mathcal{B}$ such that G contains a subdivision of a generalised $T_k(\mathcal{B})$ with core A'.*

Proof. Let I, J be disjoint sets with $|I \cup J| = k$ and $|I| \geqslant 1$. Let $\mathcal{R} = (R_i \mid i \in I)$ be a family of disjoint ω-rays and $\mathcal{D} = (d_j \in \mathrm{Dom}(\omega) \mid j \in J)$ be a family of distinct vertices disjoint from $\bigcup \mathcal{R}$. Applying Lemma B.5.10 yields a tree B on $I \cup J$ and a type-2 k-template $\mathcal{T}$ for (B, J) such that $(\mathcal{R}, \mathcal{D})$ is $(B, \mathcal{T})$-connected. Let $(\Gamma_i \mid i \in \mathbb{N})$ denote the family of $(B, \mathcal{T})$-connections as in the proof of Lemma B.5.10. Moreover, there is an infinite set of disjoint A–$\bigcup \mathcal{R}$ paths by Theorem B.2.2 since ω is in the closure of A. Now any infinite set of disjoint A–$\bigcup \mathcal{R}$ paths has infinitely many endvertices on one ray R_c for some $c \in I$. Let A'' denote the endvertices in A of such an infinite path system. Next we extend for infinitely many Γ_i the segment of R_c that it contains so that it has the endvertex of such an A''–R_c path as its top vertex and add that segment together with the path to Γ_i, while keeping them disjoint but for $\mathcal{D}$. Let A' denote the set of those endvertices of the paths in A'' we used to extend Γ_i for those infinitely many $i \in \mathbb{N}$. Finally, we modifying the type-2 k-template accordingly. We obtain the subdivision of the generalised $T_k(B, J, c)$ as in the proof of Corollary B.5.11. $\square$

The following lemma allows us to apply Lemma B.6.4 when Lemma B.6.1 is not applicable.

Lemma B.6.5. *Let $k \in \mathbb{N}$, let $A \subseteq V(G)$ be infinite and k-connected in G and let $\omega \in \Omega(G)$ be an end in the closure of A. Then $\Delta(\omega) \geqslant k$.*

Proof. We may assume that $\Delta(G)$ is finite. Hence without loss of generality A does not contain any vertices dominating ω. Let $(S_n \mid n \in \mathbb{N})$ be an ω-defining sequence, which exists by Lemma B.5.1. Take $N \in \mathbb{N}$ such that there is a set $B \subseteq A \backslash C(S_N, \omega)$ of size k. For every $n > N$ let $C_n \subseteq A \cap C(S_n, \omega)$ be a set of size k, which exists since ω is in the closure of A. Since A is k-connected in G, there are k disjoint B–C_n paths in G, each of which contains at least one vertex in S_n. Hence for all $n > N$ we have $|S_n| \geqslant k$ and by Corollary B.5.7 we have $\Delta(\omega) \geqslant k$. $\square$

We close this subsection with a corollary that is not needed in Section B, but provides a converse for Lemma B.6.5 as an interesting observation.

Corollary B.6.6. *Let $\omega \in \Omega(G)$ be an end of G with $\Delta(\omega) \geqslant k \in \mathbb{N}$. Then every subset $A \subseteq V(G)$ with ω in the closure of A contains a countable subset $A' \subseteq A$ which is k-connected in G.*

Proof. By Lemma B.6.4 we obtain a subdivision of a generalised $T_k(\mathcal{B})$ with core A' for some $A' \subseteq A$ in G for a regular k-blueprint $\mathcal{B}$. Corollary B.3.6 yields the claim. □

B.6.3. Characterisation for regular cardinals

Now we have developed all the necessary tools to prove the minor and topological minor part of the characterisation in Theorem B.3.7 for regular cardinals.

Theorem B.6.7. *Let G be a graph, let $k \in \mathbb{N}$, let $A \subseteq V(G)$ be infinite and let $\kappa \leqslant |A|$ be a regular cardinal. Then the following are equivalent.*

(a) There is a subset $A_1 \subseteq A$ with $|A_1| = \kappa$ such that A_1 is k-connected in G.

(b) There is a subset $A_2 \subseteq A$ with $|A_2| = \kappa$ such that

- *either $K_{k,\kappa}$ is an fbs-minor of G with A_2 along its core;*
- *or $T_k(\mathcal{B})$ is an fbs-minor of G with A_2 along its core for some regular k-blueprint $\mathcal{B}$.*

(c) There is a subset $A_3 \subseteq A$ with $|A_3| = \kappa$ such that

- *either G contains a subdivision of a generalised $K_{k,\kappa}$ with core A_3;*
- *or G contains the subdivision of a generalised $T_k(\mathcal{B})$ with core A_3 for some regular k-blueprint $\mathcal{B}$.*

Moreover, if these statements hold, we can choose $A_1 = A_2 = A_3$.

Proof. If (b) holds, then A_2 is k-connected in G by Lemma B.4.2 together with Lemma B.3.4.

If (a) holds, then we can find a subset $A_3 \subseteq A_1$ with $|A_3| = \kappa$ yielding (c) by either Lemma B.6.1 and Corollary B.6.3 or by Lemma B.6.5 and Lemma B.6.4.

If (c) holds, then so does (b) by Lemma B.4.3 with $A_2 := A_3$. Moreover, A_3 is a candidate for both A_2 and A_1. □

B.7. Minors for singular cardinalities

In this section we will prove the equivalence of (a), (b) and (c) of Theorem B.3.7 for singular cardinals κ.

B.7.1. Cofinal sequence of regular bipartite minors with disjoint cores.

In this subsection, given a k-connected set A of size κ, we will construct an ℓ–$K(k,\mathcal{K})$ minor in G for some suitable $\ell \in [0,k]$ and good κ-sequence $\mathcal{K}$ with a suitable subset of A along its precore. This minor is needed as an ingredient for any of the possible k-typical graphs but the $K_{k,\kappa}$ (which we obtain from the following lemma if $\ell = k$). Let $\mathcal{A} = (A^\alpha \subseteq A \mid \alpha \in \operatorname{cf}\kappa)$ be a family of disjoint subsets of A. We say that G contains ℓ–$K(k,\mathcal{K})$ as an fbs-minor with $\mathcal{A}$ along its precore $\mathcal{Z}$ if the map mapping each vertex of the inflated subgraph to its branch set induces a bijection between A^α and Z^α for all $\alpha \in \operatorname{cf}\kappa$.

Lemma B.7.1. *Let $k \in \mathbb{N}$, let $A \subseteq V(G)$ be infinite and k-connected in G and let $\kappa \leqslant |A|$ be a singular cardinal. Then there is an $\ell \in [0,k]$, a good κ-sequence $\mathcal{K} = (\kappa_\alpha < \kappa \mid \alpha \in \operatorname{cf}\kappa)$, and a family $\mathcal{A} = (A^\alpha \subseteq A \mid \alpha \in \operatorname{cf}\kappa)$ of pairwise disjoint subsets of A with $|A^\alpha| = \kappa_\alpha$ such that G contains ℓ–$K(k,\mathcal{K})$ as an fbs-minor with $\mathcal{A}$ along $\mathcal{Z}$. Moreover, the branch sets for the vertices in $\bigcup\mathcal{Y}$ are singletons*

Proof. We start with any good κ-sequence $\mathcal{K} = (\kappa_\alpha < \kappa \mid \alpha \in \operatorname{cf}\kappa)$. We construct the desired inflated subgraph by iteratively applying Lemma B.6.1.

For $\alpha \in \operatorname{cf}\kappa$ suppose we have already constructed for each $\beta < \alpha$ an inflated subgraph H^β witnessing that K_{k,κ_β} is an fbs-minor of G with some $A^\beta \subseteq A$ along its core. Furthermore, suppose that the branch sets of the vertices of the finite side are singletons and the branch sets of the vertices of the infinite side are disjoint to all branch sets of H^γ for all $\gamma < \beta$. We apply Lemma B.6.1 for κ_α to any set $A' \subseteq A \setminus \bigcup_{\beta<\alpha} A^\beta$ of size κ_α to obtain an inflated subgraph for K_{k,κ_α} with the properties as stated in that lemma. If any branch set for a vertex of the infinite side meets any branch set we have constructed so far, we delete it. Since κ_α is regular and $\kappa_\alpha > \operatorname{cf}\kappa$, the union of all inflated subgraphs we constructed so far has order less than κ_α. We obtain that the new inflated subgraph (after the deletions) still witnesses that K_{k,κ_α} is an fbs-minor of G with some $A^\alpha \subseteq A'$ along its core.

If a branch set for the finite side meets any branch set of a vertex for the infinite side for some $\beta < \alpha$, we delete that branch set and modify A^β accordingly. As the union of all branch sets for the finite side we will construct in this process has cardinality $\operatorname{cf}\kappa$, each A^β will loose at most $\operatorname{cf}\kappa < \kappa_\beta$ many elements, hence will remain at size κ_β for all $\beta \in \operatorname{cf}\kappa$. We denote the sequence $(A^\alpha \mid \alpha \in \operatorname{cf}\kappa)$ with $\mathcal{A}$.

By Lemma B.2.8 there is an $\ell \leqslant k$ and an $I \subseteq \operatorname{cf}\kappa$ with $|I| = \operatorname{cf}\kappa$ such that H^α and H^β have precisely ℓ branch sets for the vertices of the finite side in common for all $\alpha, \beta \in I$. Hence relabelling the subsequences $\mathcal{K}\restriction I$ and $\mathcal{A}\restriction I$ to $\overline{\mathcal{K}\restriction I}$ and $\overline{\mathcal{A}\restriction I}$ respectively as discussed in Section B.3 yields the claim, where the union of the respective subgraphs H^α is the witnessing inflated subgraph. $\square$

B.7.2. Frayed complete bipartite minors

In this subsection we will construct a frayed complete bipartite minor, if possible. We shall use an increasing amount of fixed notation in this subsection based on Lemma B.7.1, which we will fix as the situation in which we continue our construction.

Situation B.7.2. Let $k \in \mathbb{N}$, let $A \subseteq V(G)$ be infinite and k-connected in G and let $\kappa \leqslant |A|$ be a singular cardinal. Let $\ell \leqslant k$ and let

- $\mathcal{K} = (\kappa_\alpha < \kappa \mid \alpha \in \operatorname{cf}\kappa)$ be a good κ-sequence; and
- $\mathcal{A} = (A^\alpha \mid \alpha \in \operatorname{cf}\kappa)$ be a family of pairwise disjoint subsets of A such that $|A^\alpha| = \kappa_\alpha$.

Let H be an inflated subgraph witnessing that G contains ℓ–$K(k, \mathcal{K})$ as an fbs-minor with $\mathcal{A}$ along $\mathcal{Z}$ as in Lemma B.7.1. To simplify our notation, we denote the unique vertex of H in a branch set of y_i^α also by y_i^α for all $\alpha \in \operatorname{cf}\kappa$ and $i < k$. Similarly, we denote the set $\{y_i^\alpha \in V(H) \mid i \in [0, k)\}$ also with Y^α for all $\alpha \in \operatorname{cf}\kappa$, and denote the family $(Y^\alpha \subseteq V(H) \mid \alpha \in \operatorname{cf}\kappa)$ with $\mathcal{Y}$. Moreover, let H^α denote the subgraph of H witnessing that $K(Y^\alpha, Z^\alpha)$ is an fbs-minor of G with A^α along Z^α. Finally, let $D_\ell = \{y_i \mid i \in [0, \ell)\} = \bigcap\{V(H^\alpha) \mid \alpha \in \operatorname{cf}\kappa\}$ denote the set of degenerate vertices of ℓ–$K(k, \mathcal{K})$.

For a set $U \subseteq V(G)$ and $\alpha \in \operatorname{cf}\kappa$, a Y^α–U *bundle* P^α is the union $\bigcup\{P_i^\alpha \mid i \in [0, k)\}$ of k disjoint paths, where $P_i^\alpha \subseteq G$ is a (possibly trivial) Y^α–U

path starting in $y_i^\alpha \in Y^\alpha$ and ending in some $u_i^\alpha \in U$. A family $\mathcal{P} = (P^\alpha \mid \alpha \in \operatorname{cf}\kappa)$ of Y^α–U bundles is a $\mathcal{Y}$–U *bundle* if $P^\alpha - U$ and $P^\beta - U$ are disjoint for all $\alpha, \beta \in \operatorname{cf}\kappa$ with $\alpha \neq \beta$. Note that if a $\mathcal{Y}$–U bundle exists, then U contains D_ℓ.

A set $U \subseteq V(G)$ *distinguishes* $\mathcal{Y}$ if whenever y_i^α and y_j^β are in the same component of $G - U$ for $\alpha, \beta \in \operatorname{cf}\kappa$ and $i, j \in [0, k)$, then $\alpha = \beta$.

Lemma B.7.3. *If a set $U \subseteq V(G)$ distinguishes $\mathcal{Y}$, then there is a $\mathcal{Y}$–U bundle $\mathcal{P}$.*

Proof. Let $U \subseteq V(G)$ distinguish $\mathcal{Y}$. By definition every finite set separating Y^α from Y^β in G also has to separate A^α from A^β. Since A is k-connected in G, there are also k disjoint Y^α–Y^β paths in G by Theorem B.2.1. Hence we fix the initial Y^α–U segments of these paths for each $\alpha \in \operatorname{cf}\kappa$, which are disjoint outside of U by the assumption that U distinguishes $\mathcal{Y}$. This yields the desired $\mathcal{Y}$–U bundle. □

For a cardinal λ, a set $W \subseteq V(G)$ is λ-*linked* to a set $U \subseteq V(G)$, if for every $w \in W$ and every $u \in U$ there are λ many internally disjoint w–u paths in G.

The following lemma is the main part of the construction.

Lemma B.7.4. *In Situation B.7.2, suppose there is a set $U \subseteq V(G)$ such that*

- *there is a $\mathcal{Y}$–U bundle $\mathcal{P} = (P^\alpha \mid \alpha \in \operatorname{cf}\kappa)$; and*
- *there is a set $W \subseteq U$ with $|W| = k$ such that W is $\operatorname{cf}\kappa$-linked to U.*

Then there is an $I_0 \subseteq \operatorname{cf}\kappa$ with $|I_0| = \operatorname{cf}\kappa$ and a family $\mathcal{A}_0 = (A_0^\alpha \subseteq A^\alpha \mid \alpha \in I_0)$ with $|A_0^\alpha| = \kappa_\alpha$ for all $\alpha \in I_0$ such that ℓ–$FK_{k,\kappa}(\overline{\mathcal{K}\restriction I_0})$ is an fbs-minor of G with $\overline{\mathcal{A}_0}$ along $\overline{\mathcal{Z}\restriction I_0}$.

Proof. Let U, $\mathcal{P}$ and W be as above. By Lemma B.2.8 there is a $j \in [0, k]$ and a subset $I' \subseteq \operatorname{cf}\kappa$ with $|I'| = \operatorname{cf}\kappa$ such that (after possibly relabelling the sets Y^α for all $\alpha \in I'$ simultaneously) for every $\alpha, \beta \in I'$ with $\alpha \neq \beta$

- $y_i = u_i^\alpha = u_i^\beta$ for all $i \in [0, \ell)$;
- $x_i := u_i^\alpha = u_i^\beta$ for all $i \in [\ell, \ell + j)$; and
- $u_{i_0}^\alpha \neq u_{i_1}^\beta$ for all $i_0, i_1 \in [\ell + j, k)$.

Furthermore, after deleting at most j more elements from I' we obtain I'' such that

- $u_i^\alpha \neq y_i^\alpha$ for all $i \in [\ell, \ell + j)$ and all $\alpha \in I''$.

Note that if $|U| < \operatorname{cf} \kappa$, then $\ell + j = k$ and we set $I_0 := I''$ and $L := \emptyset$.

Otherwise we construct subdivided stars with distinct centres in W. We start with a $k - \ell - j$ element subset $W' = \{w_i \mid i \in [\ell + j, k)\} \subseteq W$ disjoint from D_ℓ and $\{x_i \mid i \in [\ell, \ell + j)\}$. A subgraph L of G is a *partial star-link* if there is a set $I(L) \subseteq I''$ such that L is the disjoint union of subdivided stars S_i for all $i \in [\ell + j, k)$ with centre w_i and leaves u_i^α, and L is disjoint to $P^\alpha - \{u_i^\alpha \mid i \in [\ell + j, k)\}$ for all $\alpha \in I(L)$. A partial star-link L is a *star-link* if $|I(L)| = \operatorname{cf} \kappa$. Note that the union of a chain of partial star-links (ordered by the subgraph relation) yields another partial star-link. Hence by Zorn's Lemma there is a maximal partial star-link M. Assume for a contradiction that M is not a star-link. Then the set $N = V(M) \cup \bigcup_{\alpha \in I(M)} V(P^\alpha)$ has size less than $\operatorname{cf} \kappa$. Take some $\beta \in I' \setminus I(M)$ such that M is disjoint to P^β. Since W is $\operatorname{cf} \kappa$-linked to U, we can find $k - \ell - j$ disjoint $W' - \{u_i^\beta \mid i \in [\ell + j, k)\}$ paths disjoint from $N \setminus W'$, contradicting the maximality of M (after possibly relabelling). Hence there is a star-link L, and we set $I_0 := I(L)$.

Let H_{I_0} denote the subgraph of H containing only the branch sets for vertices in $Y^\alpha \cup Z^\alpha$ for $\alpha \in I_0$. Since $L \cup \bigcup_{\alpha \in I_0} P^\alpha$ has size $\operatorname{cf} \kappa < \kappa_\alpha$ for all $\alpha \in I_0$, we can remove every branch set for some $z \in Z^\alpha$ meeting $L \cup \bigcup_{\alpha \in I_0} P^\alpha$ and obtain $A_0^\alpha \subseteq A^\alpha$ with $|A_0^\alpha| = \kappa_\alpha$. The union of the resulting subgraph with L and $\bigcup_{\alpha \in I_0} P^\alpha$ witnesses that $\ell\text{–}FK_{k,\kappa}(\overline{\mathcal{K} \restriction I_0})$ is an fbs-minor of G with $\overline{\mathcal{A}_0} := \overline{(A_0^\alpha \mid \alpha \in I_0)}$ along $\overline{\mathcal{Z} \restriction I_0}$. $\square$

As before, the previous lemma can be translated to find a desired subdivision of a generalised $\ell\text{–}FK_{k,\kappa}$.

Lemma B.7.5. *In the situation of Lemma B.7.4, there is an $I_1 \subseteq I_0$ with $|I_1| = \operatorname{cf} \kappa$ and a family $\mathcal{A}_1 = (A_1^\alpha \subseteq A_0^\alpha \mid \alpha \in I_1)$ with $|A_1^\alpha| = \kappa_\alpha$ for all $\alpha \in I_1$ such that G contains a subdivision of a generalised $\ell\text{–}FK_{k,\kappa}(\overline{\mathcal{K} \restriction I_1})$ with core $\bigcup \mathcal{A}_1$.*

Proof. Let H be the inflated subgraph witnessing that $\ell\text{–}FK_{k,\kappa}(\overline{\mathcal{K} \restriction I_0})$ is an fbs-minor of G with $\overline{\mathcal{A}_0}$ along its core. Let $H^\alpha \subseteq H$ be the subgraph corresponding to the subgraph $K(Y^\alpha, Z^\alpha)$ of $\ell\text{–}FK_{k,\kappa}(\overline{\mathcal{K} \restriction I_0})$ for each $\alpha \in I_0$. For each $\alpha \in I_0$ we apply Lemma B.6.2 to H^α. By the pigeonhole principle there is a set $I_1 \subseteq I_0$ with $|I_1| = \operatorname{cf} \kappa$ such that the type-1 k-template we got is the same for each $\alpha \in I_1$. This yields the desired subdivision as for Corollary B.6.3. $\square$

The remainder of this subsection is dedicated to identify when we can apply Lemma B.7.4.

Lemma B.7.6. *In Situation B.7.2, if either* $\operatorname{cf}\kappa$ *is uncountable or there is no end in the closure of some transversal* T *of* $\mathcal{A}$*, then there is a set* $U \subseteq V(G)$ *with the properties needed for Lemma B.7.4.*

Proof. We start with a transversal T of $\mathcal{A}$ (whose closure does not contain any end if $\operatorname{cf}\kappa$ is countable). We apply Lemma B.6.1 to T to obtain an inflated subgraph witnessing that $K_{k,\operatorname{cf}\kappa}$ is an fbs-minor of G with $T_0 \subseteq T$ along its core. We call the union of the singleton branch sets for the vertices of the finite side $W =: U_0$. By construction W is $\operatorname{cf}\kappa$-linked to U_0. Let I_0 denote the set $\{\alpha \in \operatorname{cf}\kappa \mid |T_0 \cap A^\alpha| = 1\}$. We construct U inductively.

For some $\alpha < \kappa$ we assume we already constructed a strictly $\subseteq$-ascending sequence $(U_\beta \mid \beta < \alpha)$ such that W is $\operatorname{cf}\kappa$-linked to U_β for all $\beta < \alpha$. If there is a subset $I \subseteq I_0$ with $|I| = \operatorname{cf}\kappa$ such that $U' := \bigcup_{\beta<\alpha} U_\alpha$ distinguishes $\overline{\mathcal{Y}\restriction I}$, then we are done by Lemma B.7.3 since by construction W is still $\operatorname{cf}\kappa$-linked to U'. Otherwise there is a component of $G - U'$ containing a transversal T_α of $\mathcal{Y}\restriction I_\alpha$ for some $I_\alpha \subseteq I_0$ with $|I_\alpha| = \operatorname{cf}\kappa$. Applying Lemma B.2.4 to T_α yields a subdivided star with centre u_α and $\operatorname{cf}\kappa$ many leaves $L_\alpha \subseteq T_\alpha$. We then set $U_\alpha := U' \cup \{u_\alpha\}$. By Theorem B.2.2 there are $\operatorname{cf}\kappa$ many internally disjoint w–u_α paths for all $w \in W$, since no set of size less than $\operatorname{cf}\kappa$ could separate u_α from L_α, L_α from T_0, or any subset of size $\operatorname{cf}\kappa$ of T_0 from w. Hence W is $\operatorname{cf}\kappa$-linked to U_α and we can continue the construction. This construction terminates at the latest if $U' = V(G)$. □

If $\operatorname{cf}\kappa$ is countable and there is an end in the closure of some transversal of $\mathcal{A}$, then there is still a chance to obtain an ℓ–$FK_{k,\kappa}$ minor. We just need to check whether G contains a $\mathcal{Y}$–$\operatorname{Dom}(\omega)$ bundle, since we have the following lemma.

Lemma B.7.7. *For every end* $\omega \in \Omega(G)$*, the set* $\operatorname{Dom}(\omega)$ *is* $\aleph_0$*-linked to itself.*

Proof. Suppose there are $u, v \in \operatorname{Dom}(\omega)$ with only finitely many internally disjoint u–v paths. Hence there is a finite separator $S \subseteq V(G)$ such that u and v are in different components of $G - S$. Then at least one of them is in a different component than $C(S, \omega)$, a contradiction. □

Hence, we obtain the final corollary of this subsection.

Corollary B.7.8. *In Situation B.7.2, suppose* $\operatorname{cf}\kappa$ *is countable and there is an end* ω *in the closure of some transversal of* $\mathcal{A}$ *with* $\operatorname{dom}\omega \geqslant k$ *such that* $\operatorname{Dom}(\omega)$ *distinguishes* $\mathcal{Y}$*. Then* $\operatorname{Dom}(\omega)$ *satisfies the properties needed for Lemma B.7.4.* □

B.7.3. Minors for singular k-blueprints

This subsection builds differently upon Situation B.7.2 in the case where we do not obtain the frayed complete bipartite minor. We incorporate new assumptions and notation, establishing a new situation, which we will further modify according to some assumptions that we can make without loss of generality during this subsection.

Situation B.7.9. Building upon Situation B.7.2, suppose $\operatorname{cf}\kappa$ is countable and there is an end ω in the closure of some transversal of $\mathcal{A}$, i.e. an ω-comb whose teeth are a transversal T of $\{A^i \mid i \in J\}$ for some infinite $J \subseteq \mathbb{N}$. Suppose that

(∗) there is no $\overline{\mathcal{Y}\restriction I}-\operatorname{Dom}(\omega)$ bundle for any infinite $I \subseteq \mathbb{N}$.

In particular $\operatorname{Dom}(\omega)$ does not distinguish $\overline{\mathcal{Y}\restriction J}$ by Lemma B.7.3. Hence there is a component C of $G - \operatorname{Dom}(\omega)$ containing a comb with teeth in $\mathcal{Y}\restriction J$, since a subdivided star would yield a vertex dominating ω outside $\operatorname{Dom}(\omega)$. This comb is an ω-comb since its teeth cannot be separated from T by a finite vertex set. Without loss of generality we may assume that $J = \mathbb{N}$ by redefining $\mathcal{K}$, $\mathcal{Y}$ and $\mathcal{A}$ as $\overline{\mathcal{K}\restriction J}$, $\overline{\mathcal{Y}\restriction J}$ and $\overline{\mathcal{A}\restriction J}$ respectively.

Let $G' := G[C]$ and let ω' be the end of G' containing the spine of the aforementioned ω-comb in G'. Let $\mathcal{S} = (S^n \mid n \in \mathbb{N})$ be an ω'-defining sequence in G' and let $\mathcal{R}$ be a family of disjoint ω'-rays in G' such that $(\mathcal{S}, \mathcal{R})$ witnesses the degree of the undominated end ω' of G', which exist by Lemma B.5.3. Moreover, we will modify this situation with some assumptions that we can make without loss of generality. We will fix them in some of the following lemmas and corollaries.

Lemma B.7.10. *In Situation B.7.9, we may assume without loss of generality that for all $n \in \mathbb{N}$ the following hold:*

- $S^n \cap \bigcup \mathcal{Y} = \emptyset$*; and*
- S^n *is contained in a component of* $G'[S^n, S^{n+1}]$.

Hence we include these assumptions into Situation B.7.9.

Proof. Given $x, y \in \mathbb{N}$ we can choose $n \in \mathbb{N}$ with $n \geqslant y$ and $m \in \mathbb{N}$ with $m > n$ such that S^n is contained in a component of $G'[S^n, S^m]$ and Y^x is disjoint to $S^n \cup S^m$. Note that it is possible to incorporate the first property since $(\mathcal{S}, \mathcal{R})$ is degree-witnessing in G'. Iteratively applying this observation yields subsequences of $\mathcal{S}$ and $\mathcal{Y}$. Taking the respective subsequences of $\mathcal{K}$ and $\mathcal{A}$ and relabelling all of them accordingly as before yields the claim. □

Lemma B.7.11. *In Situation B.7.9, we may assume without loss of generality that* $\emptyset \neq Y^n \setminus \mathrm{Dom}(\omega) \subseteq V(G'[S^n, S^{n+1}])$ *for all* $n \in \mathbb{N}$. *Hence we include this assumption into Situation B.7.9.*

Proof. Note that $Y^n \setminus \mathrm{Dom}(\omega) = \emptyset$ for only finitely many $n \in \mathbb{N}$ by $(*)$. Moreover, for all but finitely many $n \in \mathbb{N}$ there is an $x_n \in \mathbb{N}$ such that $Y^n \setminus \mathrm{Dom}(\omega)$ meets $V(G'[S^n, S^{n+1}])$ since ω is in the closure of $\mathcal{Y}$. Suppose that $Y^{x_n} \setminus \mathrm{Dom}(\omega)$ is not contained in $V(G'[S^n, S^{n+1}])$ for some $n \in \mathbb{N}$. Since for any $i, j \in [0, k)$ with $i \neq j$ there are κ_x many disjoint y_i^x–y_j^x paths in H^x, all but finitely many of them have to traverse $\mathrm{Dom}(\omega)$. In particular, there is an Y^x–$\mathrm{Dom}(\omega)$ bundle in H^x. Such a bundle trivially also exists if $Y^x \subseteq \mathrm{Dom}(\omega)$. If this happens for all x in some infinite $I \subseteq \mathbb{N}$, then there is a $\overline{\mathcal{Y} \restriction I}$–$\mathrm{Dom}(\omega)$ bundle in G, contradicting $(*)$. Hence this happens at most finitely often. Again, relabelling and taking subsequences yields the claim. □

The following lemma allows some control on how we can find a set of disjoint paths from Y^n to the rays in $\mathcal{R}$ and has two important corollaries.

Lemma B.7.12. *In Situation B.7.9, let* $\mathcal{R}' \subseteq \mathcal{R}$ *with* $|\mathcal{R}'| = \min(\deg(\omega'), k)$. *Then for all* $n > 2k$ *there is an* $M > n$ *such that for all* $m \geqslant M$ *there exists an* Y^n – $(\mathrm{Dom}(\omega) \cup (S^m \cap \bigcup \mathcal{R}'))$ *bundle* $P^{n,m}$ *with* $P^{n,m} - \mathrm{Dom}(\omega) \subseteq G'[S^{n-2k}, S^m]$.

Proof. Let $n > 2k$ be fixed. As in the proof of Lemma B.7.3 for each $x > 0$ there are k disjoint Y^n–Y^{x-1} paths in G, whose union contains a Y^n–$(\mathrm{Dom}(\omega) \cup S^x)$ bundle Q^x in $G[C \cup \mathrm{Dom}(\omega)]$. For Q^{n-2k}, let $M \in \mathbb{N}$ be large enough such that $Q^{n-2k} - \mathrm{Dom}(\omega) \subseteq G'[S^{n-2k}, S^M]$, and let $m \geqslant M$.

Suppose for a contradiction that there is a vertex set S of size less than k separating Y^n from $\mathrm{Dom}(\omega) \cup (S^m \cap \bigcup \mathcal{R}')$ in $G[V(G'[S^{n-2k}, S^m]) \cup \mathrm{Dom}(\omega)]$. Then for at least one $i \in [n-2k, n)$ the graph $G'[S^i, S^{i+1}]$ does not contain a vertex of S. We distinguish two cases.

Suppose $\deg(\omega') \geqslant k$. Then S contains a vertex from every path of Q^{n-2k} ending in $\mathrm{Dom}(\omega)$, but does not contain a vertex from every path of Q^{n-2k}. Let Q be such a Y^n–S^{n-2k} path avoiding S. Now Q meets S^i by construction. There is at least one ray $R \in \mathcal{R}'$ that does not contain a vertex of S. Since S^i is contained in a component of $G'[S^i, S^{i+1}]$ and $R \cap S^i \neq \varnothing$, we can connect Q with R and hence with $S^m \cap R$ in $G'[S^i, S^{i+1}]$ avoiding S, which contradicts the assumption.

Suppose $\deg(\omega') < k$, then $\mathcal{R}' = \mathcal{R}$ and hence $S^m \cap \bigcup \mathcal{R}' = S^m$. As before, there is a Y^n–S^m path Q in Q^m not containing a vertex of S. This path being contained in $G'[S^{n-2k}, S^m]$ would contradict the assumption. Hence we may assume the path meets S^j for every $j \in [n-2k, m]$ and in particular S^i. Let $Q_1 \subseteq Q$ denote Y^n–S^i path in $G'[S^i, S^m]$, and let $Q_2 \subseteq Q$ denote S^i–S^m path in $G'[S^i, S^m]$. As before, we can connect Q_1 and Q_2 in $G'[S^i, S^{i+1}]$ avoiding S, which again contradicts the assumption. □

Corollary B.7.13. *In Situation B.7.9, let $\mathcal{R}' \subseteq \mathcal{R}$ with $|\mathcal{R}'| = \min(\deg(\omega'), k)$. Without loss of generality for all $n \in \mathbb{N}$ there is a Y^n–$(\mathrm{Dom}(\omega) \cup (S^{n+1} \cap \bigcup \mathcal{R}'))$ bundle P^n such that $P^n - \mathrm{Dom}(\omega) \subseteq G'[S^n, S^{n+1}] - S^n$. Hence we include this assumption into Situation B.7.9.*

Proof. We successively apply Lemma B.7.12 to obtain suitable subsequences. Relabelling them yields the claim. □

Corollary B.7.14. *Situation B.7.9 implies* $\mathrm{dom}(\omega) < k$.

Proof. Suppose $\mathrm{dom}(\omega) \geqslant k$. Then for every $n > 2k$ there is no Y^n–$\mathrm{Dom}(\omega)$ separator S of size less than k by Lemma B.7.12, since m can be chosen such that $S \cap C(S^m, \omega') = \varnothing$. Hence we can extend a path of the bundle in $C(S^m, \omega')$. Therefore, for each $n > 2k$ there is an $m > n$ such that we can find a Y^n–$\mathrm{Dom}(\omega)$

bundle $P^{n,m}$ such that $P^{n,m} - \mathrm{Dom}(\omega) \subseteq G'[S^{n-2k}, S^m]$, and consequently an infinite subset $I' \subseteq \mathbb{N}$ such that $\overline{(P^{n,m} \mid n \in I')}$ is an $\overline{\mathcal{Y}}\restriction I'$ – $\mathrm{Dom}(\omega)$ bundle, contradicting the assumption (∗) in Situation B.7.9. □

This last corollary is quite impactful. From this point onwards, we know that $\omega' = \omega\restriction(G - \mathrm{Dom}(\omega))$ by Remark B.2.6.

Lemma B.7.15. *In Situation B.7.9, we may assume without loss of generality that for all $n \in \mathbb{N}$ the following hold:*

- $H^n - \mathrm{Dom}(\omega) \subseteq G'[S^n, S^{n+1}] - (S^n \cup S^{n+1})$*;*
- $H^n \cap \mathrm{Dom}(\omega) = D_\ell \subseteq Y^n$*.*

Hence we include these assumptions into Situation B.7.9.

Proof. Note that $H^n \cap G'[S^n, S^{n+1}] - (S^n \cup S^{n+1}) \neq \emptyset$ by Lemmas B.7.10 and B.7.11. We delete the finitely many branch sets of vertices corresponding to the infinite side of K_{k,κ_n} in H^n containing a vertex of $\mathrm{Dom}(\omega)$, S^n or S^{n+1}. Since the remaining inflated subgraph is connected, no branch set of the infinite side meets a vertex outside of $G'[S^n, S^{n+1}]$. Moreover, for all but finitely many $n \in \mathbb{N}$ the branch sets of vertices corresponding to the finite side of H^n that meet $\mathrm{Dom}(\omega)$ are precisely the singletons of the elements in D_ℓ by Corollary B.7.14. Deleting the exceptions and relabelling accordingly yields the claims. □

The next lemma reroutes some rays to find a bundle from Y^n to those new rays and dominating vertices with some specific properties.

Lemma B.7.16. *In Situation B.7.9, there is a set $\mathcal{R}''$ of $|\mathcal{R}'|$ disjoint ω'-rays in G' and a Y^n –$(\bigcup \mathcal{R}'' \cup \mathrm{Dom}(\omega))$ bundle Q^n for each $n \in \mathbb{N}$ such that for every $R'' \in \mathcal{R}''$*

- *there is an $R' \in \mathcal{R}'$ with $V(R'') \cap \bigcup \mathcal{S} = V(R') \cap \bigcup \mathcal{S}$; and*
- $|Q^n \cap R''| \leqslant 2$ *for every $n \in \mathbb{N}$.*

Hence we include references to these objects into Situation B.7.9.

Proof. Given $n \in \mathbb{N}$, let P^n be as in Corollary B.7.13. Let $\mathcal{P}$ be a set of $|\mathcal{R}'|$ disjoint S^n – S^{n+1} paths in $G'[S^n, S^{n+1}]$ each with end vertices $R' \cap (S^n \cup S^{n+1})$

for some $R' \in \mathcal{R}'$. We call such a set $\mathcal{P}$ *feasible*. For a feasible $\mathcal{P}$, let $P^n(\mathcal{P})$ denote the Y^n–$(\mathrm{Dom}(\omega) \cup \bigcup \mathcal{P})$ bundle contained in P^n and let $p^n(\mathcal{P})$ denote the finite parameter $|(P^n - P^n(\mathcal{P})) - \bigcup \mathcal{P}|$. Note that $\{R' \cap G'[S^n, S^{n+1}] \mid R' \in \mathcal{R}'\}$ is a feasible set. Now choose a feasible $\mathcal{P}^n$ such that $p^n(\mathcal{P}^n)$ is minimal and let $Q^n := P^n(\mathcal{P}^n)$.

Assume for a contradiction that there is a path $P \in \mathcal{P}^n$ with $|Q^n \cap P| > 2$. Let v_0, v_1 and v_2 denote vertices in this intersection such that $v_1 \in V(v_0 P v_2)$. We replace the segment $v_0 P v_2$ by the path consisting of the paths Q_i^n and Q_j^n that contain v_0 and v_2 respectively, as well as any y_i^n–y_j^n path in H^n avoiding the finite set $\mathrm{Dom}(\omega) \cup Q^n \cup S^n \cup S^{n+1}$. The resulting set $\mathcal{P}$ is again feasible and the parameter $p^n(\mathcal{P})$ is strictly smaller than $p^n(\mathcal{P}^n)$, contradicting the choice of $\mathcal{P}^n$.

Now let $\mathcal{R}''$ be the set of components in the union $\bigcup\{\mathcal{P}^n \mid n \in \mathbb{N}\}$. Indeed, this is a set of ω'-rays that together with the bundles Q^n satisfy the desired properties. $\square$

For $m, n \in \mathbb{N}$, we say Q^m and Q^n *follow the same pattern*, if for all $i, j \in [0, k)$

- Q_i^m and Q_i^n either meet the same ray in $\mathcal{R}''$ or the same vertex in $\mathrm{Dom}(\omega)$;
- if Q_i^m and Q_j^m both meet some $R \in \mathcal{R}''$ and Q_i^m meets R closer to the start vertex of R than Q_j^m, then Q_i^n meets R closer to the start vertex of R than Q_j^n.

Lemma B.7.17. *In Situation B.7.9, we may assume without loss of generality that*

- *there are $k_0, k_1, f \in \mathbb{N}$ with $1 \leqslant k_0 \leqslant \deg(\omega')$, $0 \leqslant \ell + f + k_1 \leqslant \mathrm{dom}(\omega)$ and $\ell + f + k_0 + k_1 = k$;*
- *there is a subset $\mathcal{R}_0 \subseteq \mathcal{R}''$ with $|\mathcal{R}_0| = k_0$; and*
- *there are disjoint $D_f, D_1 \subseteq \mathrm{Dom}(\omega) \setminus D_\ell$ with $|D_f| = f$ and $|D_1| = k_1$;*

such that for all $m, n \in \mathbb{N}$

(a) Q^n is a Y^n–$(\bigcup \mathcal{R}_0 \cup D_\ell \cup D_f)$ bundle;

(b) $Q^n \cap \mathrm{Dom}(\omega) = D_f \cup D_\ell$; and

(c) Q^m and Q^n follow the same pattern.

Hence we include these assumptions and references to the existing objects into Situation B.7.9.

Proof. Using the fact that $\mathrm{Dom}(\omega)$ is finite, we apply the pigeonhole principle to find a set $D_f \subseteq \mathrm{Dom}(\omega) \backslash D_\ell$ and an infinite subset $I \subseteq \mathbb{N}$ such that (b) hold for all $n \in I$. Set $f := |D_f|$. Applying it multiple times again, we find an infinite subset $I' \subseteq I$ such that (c) holds for all $m, n \in I'$. If $|\mathcal{R}''| \geqslant k - \ell - f$, then set $\mathcal{R}_0$ to be any subset of $\mathcal{R}''$ of size $k - \ell - f$ containing each ray that meets Q^n for any $n \in I'$. Otherwise set $\mathcal{R}_0 := \mathcal{R}''$ and set $k_0 := |\mathcal{R}_0| = \deg(\omega') = \deg(\omega)$. Now (a) holds by the choices of D_f and $\mathcal{R}_0$. Since $\Delta(\omega) \geqslant k$ by Lemma B.6.5, there is a set $D_1 \subseteq \mathrm{Dom}(\omega) \backslash (D_\ell \cup D_f)$ of size $k_1 := k - \ell - f - k_0$, completing the proof. $\square$

Finally, we construct the subdivision of a generalised k-typical graph for some singular k-blueprint.

Lemma B.7.18. *In Situation B.7.9, there exists a singular k-blueprint $\mathcal{B} = (\ell, f, B, D)$ for a tree B of order $k_0 + k_1$ with $|D| = k_0$, such that G contains a subdivision of a generalised $T_k(\mathcal{B})(\mathcal{K})$ with core $\bigcup \mathcal{A}$.*

Proof. We apply Lemma B.5.10 to $\mathcal{R}_0$ and D_1 to obtain a simple type-2 k-template $\mathcal{T}_2$, a tree B of order $k_0 + k_1$ and a set $D \subseteq V(B)$ with $|D| = k_1$ such that $(\mathcal{R}_0, D_1)$ is $(B, \mathcal{T}_2)$-connected. For each $n \in \mathbb{N}$ let Γ_n denote a $(B, \mathcal{T}_2)$-connection avoiding S^n, $\mathrm{Dom}(\omega) \backslash D_1$ as well as for each $R \in R_0$ its initial segment Rs for $s \in (S^n \cap V(R))$. Note that there is an $m > n$ such that $\Gamma_n - D_1 \subseteq G'[S^n, S^m] - S^m$. Hence Γ_n and Γ_{m+1} are disjoint to $G'[S^m, S^{m+1}] \supseteq Q^m$. For rays $R \in \mathcal{R}_0$ with $|Q^m \cap R| \geqslant 1$, we extend Γ_n on that ray to include precisely one vertex in the intersection as well as with the corresponding path in Q^m to Y^m. If furthermore $|Q^m \cap R| = 2$, we also extend Γ_{m+1} on that ray to include the other vertex of the intersection and with the corresponding path in Q^m to Y^m. Since Q^m and Q^n follow the same pattern for all $m, n \in \mathbb{N}$ by Lemma B.7.17, we can modify $\mathcal{T}_2$ to $\mathcal{T}_2'$ accordingly to have infinitely many $(B, \mathcal{T}_2')$-connections which pairwise meet only in D_1 and contain Y^n for each $n \in I$ for some infinite subset $I \subseteq \mathbb{N}$. After relabelling and setting $\mathcal{B} := (\ell, f, B, D)$, we obtain the subdivision of $T_k(\mathcal{B})(\mathcal{T}_2')$ as in the proof of Corollary B.5.11. $\square$

B.7.4. Characterisation for singular cardinals

Now we have developed all the necessary tools to prove the minor and topological minor part of the characterisation in Theorem B.3.7 for singular cardinals.

Theorem B.7.19. *Let G be a graph, let $k \in \mathbb{N}$, let $A \subseteq V(G)$ be infinite and let $\kappa \leqslant |A|$ be a singular cardinal. Then the following are equivalent.*

(a) There is a subset $A_1 \subseteq A$ with $|A_1| = \kappa$ such that A_1 is k-connected in G.

(b) There is a subset $A_2 \subseteq A$ with $|A_2| = \kappa$ such that

- *either G contains an ℓ-degenerate frayed $K_{k,\kappa}$ as an fbs-minor with A_2 along its core for some $0 \leqslant \ell \leqslant k$;*
- *or $T_k(\mathcal{B})$ is an fbs-minor of G with A_2 along its core for a singular k-blueprint $\mathcal{B}$.*

(c) There is a subset $A_3 \subseteq A$ with $|A_3| = \kappa$ such that

- *either G contains a subdivision of a generalised ℓ–$FK_{k,\kappa}$ with core A_3 for some $0 \leqslant \ell \leqslant k$;*
- *or G contains the subdivision of a generalised $T_k(\mathcal{B})$ with core A_3 for some singular k-blueprint $\mathcal{B}$.*

Moreover, if these statements hold, we can choose $A_1 = A_2 = A_3$.

Proof. If (b) holds, then A_2 is k-connected in G by Lemma B.4.2 together with Lemma B.3.4.

Suppose (a) holds. Then we can either find a subset $A_3 \subseteq A_1$ with $|A_3| = \kappa$ and a subdivision of ℓ–$FK_{k,\kappa}(\mathcal{K})$ with core A_3 for some good κ-sequence $\mathcal{K}$ by Lemma B.7.4 and either Lemma B.7.6 or Corollary B.7.8. Otherwise, we can apply Lemma B.7.18 to obtain $A_3 \subseteq A_1$ with $|A_3| = \kappa$ and a subdivision of $T_k(\mathcal{B})(\mathcal{K})$ with core A_3 for some singular k-blueprint $\mathcal{B}$ and a good κ-sequence $\mathcal{K}$. With Remark B.3.5 we obtain the subdivision of the respective generalised k-typical graph with respect to the fixed good κ-sequence.

If (c) holds, then so does (b) by Lemma B.4.3 with $A_2 := A_3$. Moreover, A_3 is a candidate for both A_2 and A_1. □

B.8. Applications of the minor-characterisation

In this section we will present some applications of the minor-characterisation of k-connected sets.

As a first corollary we just restate the theorem for $k = 1$, giving us a version of the Star-Comb Lemma for singular cardinalities. For this, given a singular cardinal κ, we call the graph $FK_{1,\kappa}$ a *frayed star*, whose *centre* is the vertex x_0 of degree $\operatorname{cf}\kappa$ and whose *leaves* are the vertices $\bigcup \mathcal{Z}$. Moreover, we call the graph $T_1(0, 0, (\{c\}, \varnothing), \varnothing, 0 \mapsto (c, 0))$ a *frayed comb* with *spine* $\mathfrak{N}_c$ and *teeth* $\bigcup \mathcal{Z}$. Note that each generalised frayed star or generalised frayed comb contains a subdivision of the frayed star or frayed comb respectively.

Corollary B.8.1 (Frayed-Star-Comb Lemma)**.** *Let $U \subseteq V(G)$ be infinite and let $\kappa \leqslant |U|$ be a singular cardinal. Then the following are equivalent.*

(a) There is a subset $U_1 \subseteq U$ with $|U_1| = \kappa$ such that U_1 is 1-connected in G.

(b) There is a subset $U_2 \subseteq U$ with $|U_2| = \kappa$ such that G either contains a subdivided star or a subdivided frayed star whose set of leaves is U_2 or a subdivided frayed comb whose set of teeth is U_2.

(Note that if $\operatorname{cf}\kappa$ is uncountable, only one of the former two can exist.)

Moreover, if these statements hold, we can choose $U_1 = U_2$. □

Even though this Frayed-Star-Comb Lemma has a much more elementary proof, we state it here only as a corollary of our main theorem.

Now Theorems B.6.7 and B.7.19 give us the tools to prove the statement we originally wanted to prove instead of Lemma B.4.4.

Corollary B.8.2. *Let $k \in \mathbb{N}$, let $A \subseteq V(G)$ be infinite and k-connected in G and let $\kappa \leqslant |A|$ be an infinite cardinal. Then for every $v \in V(G)$ there is a subset $A' \subseteq A$ with $|A'| = \kappa$ such that A' is $(k-1)$-connected in $G - v$.*

Proof. First we apply Theorem B.6.7 or Theorem B.7.19 to A to get a k-typical graph T and an inflated subgraph H witnessing that T is an fbs-minor of G with some $A'' \subseteq A$ along its core such that $|A''| = \kappa$. Let us call a vertex of T *essential*, if either

- it is a vertex of the finite side of $K_{k,\kappa}$ if $T = K_{k,\kappa}$;

- it is a degenerate vertex or frayed centre of ℓ–$FK_{k,\kappa}$ if $T = \ell$–$FK_{k,\kappa}$ for some $\ell \in [0, k]$; or
- it is a dominating vertex, a degenerate vertex or a frayed centre of $T_k(\mathcal{B})$ if $T = T_k(\mathcal{B})$ for some regular or singular k-blueprint $\mathcal{B}$.

We distinguish four cases.

If $v \notin V(H)$, then $H \subseteq G - v$ still witnesses that T is an fbs-minor of $G - v$ with $A' := A''$ along its core.

If v belongs to a branch set of a vertex c of the core, then the inflated subgraph obtained by deleting that branch set still yields a witness that T is an fbs-minor of $G - v$ with $A' := A'' \backslash \{c\}$ along its core.

If v belongs to a branch set of an essential vertex $w \in V(T)$, then the inflated subgraph where we delete this branch set from H witnesses that the obvious $(k-1)$-typical subgraph of $T - w$ is an fbs-minor of $G - v$ with $A' := A''$ along its core.

If v belongs to a branch set of a vertex $w \in V(\mathfrak{N}(B/D) - D)$, then we delete the branch sets of the layers (not including D) up to the layer containing w and relabelling accordingly (and modifying the κ-sequence if necessary). This yields a supergraph of an inflated subgraph witnessing that T is an fbs-minor of $G - v$ with A' along its core for some $A' \subseteq A''$ with $|A'| = \kappa$. Similar arguments yield the statement if v belongs to a branch set of a neighbour of a frayed centre.

In any case, with the other direction of Theorem B.6.7 or Theorem B.7.19 we get that A' is $(k-1)$-connected in $G - v$. □

As another corollary we prove that we are able to find k-connected sets of size κ in sets which cannot be separated by less than κ many vertices from another k-connected set. This will be an important tool for our last part of the characterisation in the main theorem of Section B.

Corollary B.8.3. *Let $k \in \mathbb{N}$, let $A, B \subseteq V(G)$ be infinite and let $\kappa \leqslant |A|$ be an infinite cardinal. If B is k-connected in G and A cannot be separated from B by less than κ vertices, then there is an $A' \subseteq A$ with $|A'| = \kappa$ which is k-connected in G.*

Proof. Let $\mathcal{P}$ be a set of κ many disjoint A–B paths as given by Theorem B.2.2. Let $B' = B \cap \bigcup \mathcal{P}$. Let $H \subseteq G$ be an inflated subgraph witnessing that a k-typical graph is an fbs-minor of G with B'' along its core for some $B'' \subseteq B'$ with $|B''| = \kappa$ as given by Theorem B.6.7 or Theorem B.7.19. Let $\mathcal{P}'$ denote the set of the A–H subpaths of the A–B'' paths in $\mathcal{P}$. We distinguish two cases.

If the k-typical graph is a $T_k(\mathcal{B})$ for some regular k-blueprint $\mathcal{B} = (T, D, c)$, then (since each branch set in H is finite) there is an infinite subset $\mathcal{P}'' \subseteq \mathcal{P}'$ and a node $c' \in V(T \backslash D)$ such that each branch set in H of vertices in $V(\mathfrak{N}_{c'})$ meets $\bigcup \mathcal{P}''$ at most once and no other branch set meets $\bigcup \mathcal{P}''$. Let $A' := \bigcup \mathcal{P}'' \cap A$. We extend each of these branch sets with the path from $\mathcal{P}''$ meeting it. This yields a subgraph H' witnessing that $T_k(T, D, c')$ is an fbs-minor of G with some A'' along its core with $A' \subseteq A''$.

Otherwise, since each branch set in H is finite, there is a subset $\mathcal{P}'' \subseteq \mathcal{P}'$ of size κ such that each branch set in H of vertices corresponding to the core meets $\bigcup \mathcal{P}''$ at most once and no other branch set meets $\bigcup \mathcal{P}''$. Let $A' := \bigcup \mathcal{P}'' \cap A$. Again, we extend each of these branch sets with the path from $\mathcal{P}''$ meeting it. This yields a subgraph H' witnessing that the same k-typical graph is an fbs-minor of G with some A'' along its core such that $A' \subseteq A'' \subseteq A' \cup B''$.

Now applying Theorem B.6.7 or Theorem B.7.19 again together with Remark B.4.1 yields the claim. $\square$

B.9. Nested separation systems

This section will finish the proof of Theorem B.3.7 by providing the last equivalence of the characterisation.

Given $k \in \mathbb{N}$, a tree-decomposition of adhesion less than k is called *k-lean* if for any two (not necessarily distinct) parts V_{t_1}, V_{t_2} of the tree-decomposition and vertex sets $Z_1 \subseteq V_{t_1}$, $Z_2 \subseteq V_{t_2}$ with $|Z_1| = |Z_2| = \ell < k + 1$ there are either ℓ disjoint Z_1–Z_2 paths in G or there is an edge tt' on the t_1–t_2 path in the tree such that $|V_{t_1} \cap V_{t_2}| < \ell$. In particular, given a k-lean tree-decomposition, each part V_t is $\min\{k, |V_t|\}$-connected in G.

In [9], the authors noted that the proof given in [4] of a theorem of Thomas [60, Thm. 5] about the existence of lean tree-decompositions witnessing the tree-width of a finite graph can be adapted to prove the existence of a k-lean tree-decomposition of that graph.

Theorem B.9.1. [9, Thm. 2.3] *Every finite graph has a k-lean tree-decomposition for any $k \in \mathbb{N}$.*

This definition can easily be lifted to nested separation systems. A nested separation system $N \subseteq S_k(G)$ is called *k-lean* if given any two (not necessarily distinct) parts P_1, P_2 of N and vertex sets $Z_1 \subseteq P_1$, $Z_2 \subseteq P_2$ with $|Z_1| = |Z_2| = \ell < k + 1$ there are either ℓ disjoint $Z_1 - Z_2$ paths in G or there is a separation (A, B) in N with $P_1 \subseteq A$ and $P_2 \subseteq B$ of order less than ℓ. Here, we specifically allow the empty set as a nested separation system to be k-lean if its part, the whole vertex set of G, is $\min\{k, |V(G)|\}$-connected. Again, we obtain that each part P of a k-lean nested separation system is $\min\{k, |P|\}$-connected in G. Moreover, note that the nested separation system that a k-lean tree-decomposition induces as described in Section B.2 is k-lean as well.

For a subset $X \subseteq V(G)$ consider the induced subgraph $G[X]$. Every separation of $G[X]$ is of the form $(A \cap X, B \cap X)$ for some separation (A, B) of G. We denote this separation also as $(A, B)\restriction X$. Given a set S of separations of G we write $S\restriction X$ for the set consisting of all separations $(A, B)\restriction X$ for $(A, B) \in S$ such that neither $A \cap X$ nor $B \cap X$ equals X.

Theorem B.9.2. *For every graph G and every $k \in \mathbb{N}$ there is a nested separation system $N \subseteq S_k(G)$ such that each part P of N is $\min(k, |P|)$-connected in G.*

Proof. For every finite $X \subseteq V(G)$ let $\mathcal{N}(X)$ denote the set of nested separation systems $N\restriction X$ of $G[X]$ such that there is a nested separation system $N \subseteq S_k(G[Z])$ that is k-lean for a finite $Z \subseteq V(G)$ containing X. Note that $\mathcal{N}(X)$ is not empty by Theorem B.9.1 for every finite $X \subseteq V(G)$. Moreover, for every finite $X \subseteq V(G)$, every $N \in \mathcal{N}(X)$ and every $Y \subseteq X$ we have $N\restriction Y \in \mathcal{N}(Y)$ by definition. Moreover, given any finite set $\mathcal{Y}$ of finite subsets of $V(G)$ we obtain for every $N \in \mathcal{N}(\bigcup \mathcal{Y})$ that $N\restriction Y \in \mathcal{N}(Y)$ for every $Y \in \mathcal{Y}$. Hence by the compactness principle there is a set N of separations of G such that $N\restriction X \in \mathcal{N}(X)$ for every finite $X \subseteq V(G)$. It is easy to check that each separation in N has order less than k and that N is a nested separation system.

For some part P of N we consider two arbitrary vertex sets $Z_1, Z_2 \subseteq P$ with $|Z_1| = |Z_2| = \ell \leqslant \min(k, |P|) =: k_P$. For a suitable finite set $X \subseteq V(G)$ containing $Z_1 \cup Z_2$ these sets are in the same part of the k-lean tree-decomposition of $G[X]$ inducing $N\restriction(Z_1 \cup Z_2)$. Hence there are k_P many disjoint Z_1-Z_2 paths in $G[X]$ and hence in G. Therefore P is k_P connected in G. □

Now we are able to prove the remaining equivalence of our main theorem of Section B.

Theorem B.9.3. *Let G be an infinite graph, let $k \in \mathbb{N}$, let $A \subseteq V(G)$ be infinite and let $\kappa \leqslant |A|$ be an infinite cardinal. Then the following are equivalent.*

(a) There is a subset $A_1 \subseteq A$ with $|A_1| = \kappa$ such that A_1 is k-connected in G.

(d) There is no nested separation system $N \subseteq S_k(G)$ such that every part P of N can be separated from A by less than κ vertices.

Proof. Assume that (a) does not hold. Let N be a nested separation system as obtained from Theorem B.9.2. Suppose for a contradiction that there exists a part P of N that cannot be separated from A by less than κ vertices. Then P is k-connected in G and has size at least κ. By Corollary B.8.3, there is a subset $A_1 \subseteq A$ of size κ which is k-connected in G, a contradiction. Hence every part of N can be separated from A by less than κ vertices, so (d) does not hold.

If (a) holds, let $N \subseteq S_k(G)$ be any nested separation system and let H be an inflated subgraph witnessing that a k-typical graph T is an fbs-minor of G with some $A' \subseteq A$ along its core for $|A'| = \kappa$ as in Theorem B.6.7 or Theorem B.7.19. Now there has to be a part of N containing at least one vertex from each branch set corresponding to the core of T, since no separation of size less than k can separate two distinct such branch sets from each other by Lemma B.3.4. This part has to have size at least κ, and the disjoint paths in each branch set from a vertex of A' to the part witness by Theorem B.2.2 that A cannot be separated by less than κ vertices from that part. □

Let us finish this section with an open problem regarding the question when it is possible to extend this characterisation to tree-decompositions.

Problem B.9.4. *For which class of infinite graphs is the existence of a k-connected set of size κ equivalent to the non-existence of a tree-decomposition of adhesion less than k where every part has size less than κ?*

We suspect that the class of locally finite connected graphs should be a solution for Problem B.9.4, where κ is necessarily equal to $\aleph_0$, since locally finite connected graphs are countable.

C. Infinite end-devouring sets of rays with prescribed start vertices

C.1. Introduction

Looking for spanning structures in infinite graphs such as spanning trees or Hamilton cycles often involves difficulties that are not present when one considers finite graphs. It turned out that the concept of *ends* of an infinite graph is crucial for questions dealing with such structures. Especially for *locally finite graphs*, i.e., graphs in which every vertex has finite degree, ends allow us to define these objects in a more general topological setting [13].

Nevertheless, the definition of an end of a graph is purely combinatorial: We call one-way infinite paths *rays* and the vertex of degree 1 in them the *start vertex* of the ray. For any graph G we call two rays *equivalent* in G if they cannot be separated by finitely many vertices. It is easy to check that this defines an equivalence relation on the set of all rays in the graph G. The equivalence classes of this relation are called the *ends* of G and a ray contained in an end ω of G is referred to as an *ω-ray*.

When we focus on the structure of ends of an infinite graph G, we observe that *normal spanning trees* of G, i.e., rooted spanning trees of G such that the endvertices of every edge of G are comparable in the induced tree-order, have a powerful property: For any normal spanning tree T of G and every end ω of G there is a unique ω-ray in T which starts at the root of T and has the property that it meets every ω-ray of G, see [12, Sect. 8.2]. For any graph G, we say that an ω-ray with this property *devours* the end ω of G. Similarly, we say that a set of ω-rays *devours* ω if every ω-ray in G meets at least one ray out of the set. Note that if a set of ω-rays devours ω, then every ω-ray R meets the union of that set infinitely often, since each tail of R meets at least one ray out of the set.

End-devouring sets of rays are helpful for the construction of spanning structures such as infinite Hamilton cycles. For example, in a one-ended locally finite graph after removing an end-devouring set of rays, each component is finite. Thomassen [63] used this fact to show that the square of each locally finite 2-connected one-ended graph contains a spanning ray. Georgakopoulos [25] generalised this to locally finite 2-connected graphs with arbitrary many ends by building some other kind of spanning structure with the help of an end-devouring set of rays, which he then used to construct an infinite Hamilton cycle in the square of such a graph. He proved the following proposition about the existence of finite sets of rays devouring any *countable end*, i.e., an end which does not contain uncountably many disjoint rays. Note that the property of an end being countable is equivalent to the existence of a finite or countably infinite set of rays devouring the end.

Proposition C.1.1. [25] *Let G be a graph and ω be a countable end of G. If G has a set $\mathcal{R}$ of $k \in \mathbb{N}$ disjoint ω-rays, then it also has a set $\mathcal{R}'$ of k disjoint ω-rays that devours ω. Moreover, $\mathcal{R}'$ can be chosen so that its rays have the same start vertices as the rays in $\mathcal{R}$.*

For the proof of this proposition Georgakopoulos recursively applies a construction similar to the one yielding normal spanning trees to find rays for the end-devouring set. However, this proof strategy does not suffice to give a version of this proposition for infinitely many rays. He conjectured that such a version remains true [25, Problem 1]. We confirm this conjecture with the following theorem, which also covers the proposition above.

Theorem C.1.2. *Let G be a graph, ω a countable end of G and $\mathcal{R}$ any set of disjoint ω-rays. Then there exists a set $\mathcal{R}'$ of disjoint ω-rays that devours ω and the start vertices of the rays in $\mathcal{R}$ and $\mathcal{R}'$ are the same.*

Note that, in contrast to the proposition, the difficulty of Theorem C.1.2 for an infinite set $\mathcal{R}$ comes from fixing the set of start vertices, since any inclusion-maximal set of disjoint ω-rays devours ω.

After introducing some additionally needed terminology in Section C.2, the proof of Theorem 1 will feature in Section C.3. In Section C.4 we will see why this theorem does not immediately extend to ends that contain an uncountable set of disjoint rays. There we discuss an additional necessary condition on the set of start vertices.

C.2. Preliminaries

All graphs in Section C are simple and undirected. For basic facts about finite and infinite graphs we refer the reader to [12]. If not stated differently, we also use the notation of [12].

We define the *union* $G \cup H$ of G and H as the graph $(V(G) \cup V(H), E(G) \cup E(H))$.

Any ray T that is a subgraph of a ray R is called a *tail* of R. For a vertex v and an end ω of a graph G we say that a vertex set $X \subseteq V(G)$ *separates* v from ω if there does not exist any ω-ray that is disjoint from X and contains v.

For a finite set M of vertices of a graph G and an end ω of G, let $C(M, \omega)$ denote the unique component of $G - M$ that contains a tail of every ω-ray.

Given a path or ray Q containing two vertices v and w we denote the unique $v-w$ path in Q by vQw. Furthermore, for Q being a $v-w$ path we write $v\bar{Q}$ for the path that is obtained from Q by deleting w.

For a ray R that contains a vertex v we write vR for the tail of R with start vertex v.

We use the following notion to abbreviate *concatenations* of paths and rays. Let P be a $v-w$ path for two vertices v and w, and let Q be either a ray or another path such that $V(P) \cap V(Q) = \{w\}$. Then we write PQ for the path or ray $P \cup Q$, respectively. We omit writing brackets when stating concatenations of more than two paths or rays.

The *degree* of an end ω of G, denoted by $\deg(\omega)$, is the maximum cardinality of a set of disjoint ω-rays. Halin [30, Satz 1] showed that the degree of an end is well-defined. Note that an end is countable if and only if its degree is either finite or countably infinite.

C.3. Theorem

For the proof of Theorem C.1.2 we shall use the following characterisation of ω-rays.

Lemma C.3.1. *Let G be a graph, ω an end of G and $\mathcal{R}_{max}$ an inclusion-maximal set of pairwise disjoint ω-rays. A ray $R \subseteq G$ is an ω-ray, if and only if it meets rays of $\mathcal{R}_{max}$ infinitely often.*

Proof. Let W denote the set $\bigcup\{V(R) \,;\, R \in \mathcal{R}_{max}\}$.

If R is an ω-ray, then each tail of R meets a ray of $\mathcal{R}_{max}$ since $\mathcal{R}_{max}$ is inclusion-maximal. Hence R meets W infinitely often.

Suppose for a contradiction that R is an ω'-ray for an end $\omega' \neq \omega$ of G that contains infinitely many vertices of W. Let M be a finite set of vertices such that the two components $C := C(M, \omega)$ and $C' := C(M, \omega')$ of $G - M$ are different. By the pigeonhole principle there is either one ω-ray of $\mathcal{R}_{max}$ containing infinitely many vertices of $V(C') \cap V(R) \cap W$, or infinitely many disjoint rays of $\mathcal{R}_{max}$ containing those vertices. In both cases we get an ω-ray with a tail in C', since we cannot leave C' infinitely often through the finite set M. But this contradicts the definition of C. □

A natural strategy for constructing up to infinitely many disjoint rays is to inductively construct them in countably many steps. In each step we fix only finitely many finite paths as initial segments instead of whole rays, while extending previously fixed initial segments and ensuring that they can be extended to rays. This strategy is for example used by Halin [30, Satz 1] to prove that the maximum number of disjoint rays in an end is well-defined. Our proof of Theorem C.1.2 is also based on that strategy. In order to guarantee that the set of rays we construct turns out to devour the end, we also fix an inclusion-maximal set of vertex disjoint rays of our specific end, so a countable set, and an enumeration of the vertices on these rays. Then we try in each step to either contain or separate the least vertex with respect to the enumeration that is not already dealt with from the end with appropriately chosen initial segments if possible. Otherwise, we extend a finite number of initial segments while still ensuring that all initial segments can be extended to rays. Although it is impossible to always contain or separate the considered vertex with our initial segments while being able to continue with the construction, it will turn out that the rays we obtain as the union of all initial segments actually do this.

Proof of Theorem C.1.2. Let us fix a finite or infinite enumeration $\{R_j \,;\, j < |\mathcal{R}|\}$ of the rays in $\mathcal{R}$. Furthermore, let s_j denote the start vertex of R_j for every $j < |\mathcal{R}|$ and define $S := \{s_j \,;\, j < |\mathcal{R}|\}$.

Next we fix an inclusion-maximal set $\mathcal{R}_{max}$ of pairwise disjoint ω-rays and an enumeration $\{v_i \,;\, i \in \mathbb{N}\}$ of the vertices in $W := \bigcup\{V(R) \,;\, R \in \mathcal{R}_{max}\}$. This is possible since ω is countable by assumption.

We do an inductive construction such that the following holds for every $i \in \mathbb{N}$:

1. P_s^i is a path with start vertex s for every $s \in S$.
2. $P_s^i = s$ for all but finitely many $s \in S$.
3. $P_s^{i-1} \subseteq P_s^i$ for every $s \in S$.
4. For every $s = s_j \in S$ with $j < \min\{i, |\mathcal{R}|\}$ there is a $w_s^i \in W \cap (P_s^i \backslash P_s^{i-1})$.
5. P_s^i and $P_{s'}^i$ are disjoint for all $s, s' \in S$ with $s \neq s'$.
6. For every $s \in S$ there exists an ω-ray R_s^i with P_s^i as initial segment and s as start vertex such that all rays R_s^i are pairwise disjoint.

If possible and not spoiling any of the properties (1) to (6), we incorporate the following property:

(∗) $\bigcup\limits_{s \in S} P_s^i$ either contains v_{i-1} or separates v_{i-1} from ω if $i > 0$.

We begin the construction for $i = 0$ by defining $P_s^0 := s =: P_s^{-1}$ for every $s \in S$. All conditions are fulfilled as witnessed by $R_{s_j}^0 := R_j$ for every $j < |\mathcal{R}|$.

Now suppose we have done the construction up to some number $i \in \mathbb{N}$. If we can continue with the construction in step $i + 1$ such that properties (1) to (6) together with (∗) hold, we do so and define all initial segments P_s^{i+1} and rays R_s^{i+1} accordingly. Otherwise, we set for all $s \in S$

$$\begin{aligned} P_s^{i+1} &:= sR_s^i w_s^i && \text{if } s = s_j \text{ for } j < \min\{i+1, |\mathcal{R}|\} \text{ and} \\ P_s^{i+1} &:= P_s^i && \text{otherwise,} \end{aligned}$$

where w_s^i denotes the first vertex of W on $R_s^i \backslash P_s^i$ which exist by Lemma C.3.1. With these definitions properties (1) up to (5) hold for $i + 1$. Witnessed by $R_s^{i+1} := R_s^i$ for every $s \in S$ we immediately satisfy (6) too. This completes the inductive part of the construction.

Using the paths P_s^i we now define the desired ω-rays of $\mathcal{R}'$. We set $R'_s := \bigcup_{i \in \mathbb{N}} P_s^i$ for every $s \in S$ and $\mathcal{R}' := \{R'_s \,;\, s \in S\}$. Properties (1), (3) and (4) ensure that R'_s is a ray with start vertex s for every $s \in S$, while we obtain due to property (5) that all rays R'_s are pairwise disjoint. Property (4) also ensures that all rays in $\mathcal{R}'$ are ω-rays by Lemma C.3.1.

It remains to prove that the set $\mathcal{R}'$ devours the end ω. Suppose for a contradiction that there exists an ω-ray R disjoint from $\bigcup \mathcal{R}'$. By the maximality of our chosen set of ω-rays $\mathcal{R}_{max}$, we know that R contains a vertex v_j for some $j \in \mathbb{N}$. In the next paragraph we will show how we could have proceeded in step $j+1$ to incorporate property $(*)$ as well. For an easier understanding of the technical definitions of that paragraph we refer to Figure C.1.

Without loss of generality, let v_j be the start vertex of R. Let P be an R–$\bigcup \mathcal{R}'$ path among those ones that are disjoint from $\bigcup_{s \in S} s\bar{P}_s^{j+1}$ for which $v_j R p$ is as short as possible where p denotes the common vertex of P and R. Such a path exists, because all rays in $\mathcal{R}' \cup \{R\}$ are equivalent and $\bigcup_{s \in S} s\bar{P}_s^{j+1}$ is finite by property (2). Let $t \in S$ and $q \in V(G)$ be such that $V(P) \cap V(R_t') = \{q\}$. Furthermore, let R^* be an ω-ray with start vertex $r^* \in R$ such that R^* is disjoint from $\bigcup_{s \in S} R_s'$ and $P(pRr^*) \cap R^* = \{r^*\}$ for which $v_j R r^*$ is as short as possible. Since p and pR are candidates for r^* and R^*, respectively, such a choice is possible. We define

$$\hat{P}_t^{j+1} := (tR_t'q)P(pRr^*) \quad \text{and} \quad \hat{R}_t^{j+1} := \hat{P}_t^{j+1} R^* \ ;$$

and replace in step $j+1$ the ray R_t^{j+1} by $\hat{R}_t^{j+1}$, the path P_t^{j+1} by $\hat{P}_t^{j+1}$ and for all $s \in S \backslash \{t\}$ the ray R_s^{j+1} by R_s' while keeping P_s^{j+1} as it was defined. By this construction properties (1) to (6) are satisfied.

Now we show that $(*)$ holds as well. Suppose for a contradiction that there exists an ω-ray Z disjoint from $\left(\bigcup_{s \in S \backslash \{t\}} P_s^{j+1}\right) \cup \hat{P}_t^{j+1}$ with start vertex v_j. First note that Z is disjoint from $r^* R p \subseteq \hat{P}_t^{j+1}$. Let us now show that Z is also disjoint from $pR \cup \bigcup_{s \in S} R_s'$. Otherwise, let z denote the first vertex along Z that lies in $pR \cup \bigcup_{s \in S} R_s'$. However, z cannot be contained in pR, as this would contradict the choice of r^*, and it cannot be an element of $\bigcup_{s \in S} R_s'$ since this would contradict the choice of p. But now with Z being not only disjoint from $pR \cup \bigcup_{s \in S} R_s'$ but also from $r^* R p$, we get again a contradiction to the choice of r^*. Hence, we would have been able to incorporate property $(*)$ without violating any of the properties (1) to (6) in step $j+1$ of our construction. This, however, is a contradiction since we always incorporated property $(*)$ under the condition of maintaining properties (1) to (6). So we arrived at a contradiction to the existence of the ray R since by $(*)$ every ray containing v_j meets the initial segments of rays fixed in our construction at step $j+1$. Therefore, the set $\mathcal{R}'$ devours the end ω. $\square$

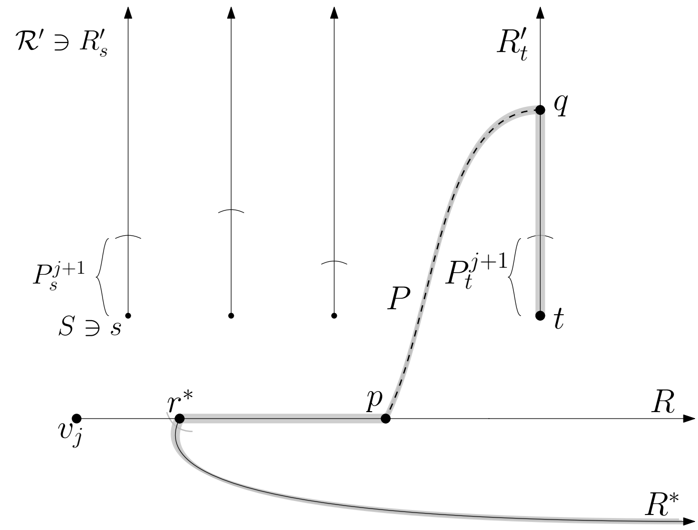

Figure C.1.: Sketch of the situation above. The rays in $\mathcal{R}'$ are drawn vertically, with their fixed initial segments from step $j+1$. Horizontally drawn is the ray R that is suppose to contradict that $\mathcal{R}'$ devours ω with its start vertex $v_j \in W$. The R–$\bigcup \mathcal{R}'$ path P is chosen with its vertex p on R as close to v_j as possible. The ray R^* is chosen disjoint to the rays in $\mathcal{R}'$ and except from its start vertex r^* on R disjoint from the initial segment of R upto p again with r^* as close to v_j as possible. The ray $\hat{R}_t^{j+1}$ is highlighted in grey with its initial segment fixed upto r^*.

C.4. Ends of uncountable degree

Given an end ω of some graph of uncountable degree, then by reasons of cardinality it cannot be devoured by a set of ω-rays which is strictly smaller than the degree of ω. But, unlike in the countable degree case, the the existence of a set of $\deg(\omega)$ many disjoint ω-rays is not sufficient for existence of a set of disjoint ω-rays devouring ω with the same start vertices. We illustrate an obvious obstruction.

A *separation of a graph* G is a tuple (A, B) with $A \cup B = V(G)$ such that there are no edges between $A \backslash B$ and $B \backslash A$. Suppose G contains a separation (A, B) such that both $G[A \backslash B]$ and $G[B \backslash A]$ contain a set of disjoint ω-rays of cardinality more than $|A \cap B|$. At least one of $G[A \backslash B]$ or $G[B \backslash A]$ contains a set $\mathcal{R}$ of $\deg(\omega)$ many disjoint ω-rays, say $G[B \backslash A]$. But no set $\mathcal{R}'$ of disjoint ω-rays with the same start vertices as the rays in $\mathcal{R}$ can devour ω since at most $|A \cap B|$ many rays meet vertices of $A \backslash B$ and hence cannot meet all ω-rays in $G[A \backslash B]$.

For an easy example of this obstruction consider two sets A and B of size $\kappa > \aleph_0$ such that $\aleph_0 \leqslant |A \cap B| < \kappa$ and let G be the union of the complete graphs on A and B. Then (A, B) a separation where both $G[A \backslash B]$ and $G[B \backslash A]$ contain a set of κ many disjoint rays to the unique end of G.

Hence we can state two necessary conditions for a set $S \subseteq V(G)$ to be a set of start vertices for a set of disjoint ω-rays devouring ω:

- there is a set $\mathcal{R}$ of disjoint ω-rays with S as its start vertices; and
- for each separation (A, B) of G, if $G[A \backslash B]$ contains a set of more than $|A \cap B|$ disjoint ω-rays, then $A \backslash B$ contains a vertex of S.

Problem C.4.1. *Are these conditions together also sufficient for the existence of a set of disjoint ω-rays devouring ω with S as its start vertices?*

Note that our construction for the proof of Theorem C.1.2, if continued transfinitely, might face numerous new problems at limit steps.

Chapter II.

Topological infinite graph theory

D. Hamiltonicity in locally finite graphs: two extensions and a counterexample

D.1. Introduction

Results about Hamilton cycles in finite graphs can be extended to locally finite graphs in the following way. For a locally finite connected graph G we consider its Freudenthal compactification $|G|$ [12, 13]. This is a topological space obtained by taking G, seen as a 1-complex, and adding certain points to it. These additional point are the *ends* of G, which are the equivalence classes of the rays of G under the relation of being inseparable by finitely many vertices. Extending the notion of cycles, we define *circles* [17, 18] in $|G|$ as homeomorphic images of the unit circle $S^1 \subseteq \mathbb{R}^2$ in $|G|$, and we call them *Hamilton circles* of G if they contain all vertices of G. As a consequence of being a closed subspace of $|G|$, Hamilton circles also contain all ends of G. Following this notion we call G *Hamiltonian* if there is a Hamilton circle in $|G|$.

One of the first and probably one of the deepest results about Hamilton circles was Georgakopoulos's extension of Fleischner's theorem to locally finite graphs.

Theorem D.1.1. [23] *The square of any finite 2-connected graph is Hamiltonian.*

Theorem D.1.2. [25, Thm. 3] *The square of any locally finite 2-connected graph is Hamiltonian.*

Following this breakthrough, more Hamiltonicity theorems have been extended to locally finite graphs in this way [5, 8, 25, 32, 35, 36, 43].

The purpose of Section D is to extend two more Hamiltonicity results about finite graphs to locally finite ones and to construct a graph which shows that another result does not extend.

The first result we consider is a corollary of the following theorem of Harary and Schwenk. A *caterpillar* is a tree such that after deleting its leaves only a path is left. Let $S(K_{1,3})$ denote the graph obtained by taking the star with three leaves, $K_{1,3}$, and subdividing each edge once.

Theorem D.1.3. [33, Thm. 1] *Let T be a finite tree with at least three vertices. Then the following statements are equivalent:*

(i) *T^2 is Hamiltonian.*

(ii) *T does not contain $S(K_{1,3})$ as a subgraph.*

(iii) *T is a caterpillar.*

Theorem D.1.3 has the following obvious corollary.

Corollary D.1.4. [33] *The square of any finite graph G on at least three vertices such that G contains a spanning caterpillar is Hamiltonian.*

While the proof of Corollary D.1.4 is immediate, the proof of the following extension of it, which is the first result of Section D, needs more work. We call the closure $\overline{H}$ in $|G|$ of a subgraph H of G a *standard subspace* of $|G|$. Extending the notion of trees, we define *topological trees* as topologically connected standard subspaces not containing any circles. As an analogue of a path, we define an *arc* as a homeomorphic image of the unit interval $[0, 1] \subseteq \mathbb{R}$ in $|G|$. Note that for standard subspaces being topologically connected is equivalent to being arc-connected by Lemma D.2.5. For our extension we adapt the notion of a caterpillar to the space $|G|$ and work with *topological caterpillars*, which are topological trees $\overline{T}$ such that $\overline{T - L}$ is an arc, where T is a forest in G and L denotes the set of vertices of degree 1 in T.

Theorem D.1.5. *The square of any locally finite connected graph G on at least three vertices such that $|G|$ contains a spanning topological caterpillar is Hamiltonian.*

The other two results of Section D concern the uniqueness of Hamilton circles. The first is about finite *outerplanar graphs*. These are finite graphs that can be embedded in the plane so that all vertices lie on the boundary of a common face. Clearly, finite outerplanar graphs have a Hamilton cycle if and only if they are 2-connected. In a 2-connected graph call an edge 2-*contractible* if its contraction leaves the graph 2-connected. It is also easy to see that any finite 2-connected outerplanar graph has a unique Hamilton cycle. This cycle consists precisely of the 2-contractible edges of the graph (except for the K^3), as pointed out by Sysło [58]. We summarise this with the following proposition.

Proposition D.1.6. (i) *A finite outerplanar graph is Hamiltonian if and only if it is* 2*-connected.*

(ii) [58, Thm. 6] *Finite* 2*-connected outerplanar graphs have a unique Hamilton cycle, which consists precisely of the* 2*-contractible edges unless the graph is isomorphic to a* K^3.

Finite outerplanar graphs can also be characterised by forbidden minors, which was done by Chartrand and Harary.

Theorem D.1.7. [11, Thm. 1] *A finite graph is outerplanar if and only if it contains neither a* K^4 *nor a* $K_{2,3}$ *as a minor.*[1]

In the light of Theorem D.1.7 we first prove the following extension of statement (i) of Proposition D.1.6 to locally finite graphs.

Theorem D.1.8. *Let* G *be a locally finite connected graph. Then the following statements are equivalent:*

(i) G *is* 2*-connected and contains neither* K^4 *nor* $K_{2,3}$ *as a minor.*[1]

(ii) $|G|$ *has a Hamilton circle* C *and there exists an embedding of* $|G|$ *into a closed disk such that* C *is mapped onto the boundary of the disk.*

Furthermore, if statements (i) and (ii) hold, then $|G|$ *has a unique Hamilton circle.*

From this we then obtain the following corollary, which extends statement (ii) of Proposition D.1.6.

Corollary D.1.9. *Let* G *be a locally finite* 2*-connected graph not containing* K^4 *or* $K_{2,3}$ *as a minor, and not isomorphic to* K^3*. Then the edges contained in the Hamilton circle of* $|G|$ *are precisely the* 2*-contractible edges of* G.

[1] Actually these statements can be strengthened a little bit by replacing the part about not containing a K^4 as a minor by not containing it as a subgraph. This follows from Lemma D.4.1.

We should note here that parts of Theorem D.1.8 and Corollary D.1.9 are already known. Chan [10, Thm. 20 with Thm. 27] proved that a locally finite 2-connected graph not isomorphic to K^3 and not containing K^4 or $K_{2,3}$ as a minor has a Hamilton circle that consists precisely of the 2-contractible edges of the graph. He deduces this from other general results about 2-contractible edges in locally finite 2-connected graphs. In our proof, however, we directly construct the Hamilton circle and show its uniqueness without working with 2-contractible edges. Afterwards, we deduce Corollary D.1.9.

Our third result is related to the following conjecture Sheehan made for finite graphs.

Conjecture D.1.10. [56] *There is no finite r-regular graph with a unique Hamilton cycle for any $r > 2$.*

This conjecture is still open, but some partial results have been proved [34, 61, 64]. For $r = 3$ the statement of the conjecture was first verified by C. A. B. Smith. This was noted in an article of Tutte [66] where the statement for $r = 3$ was published for the first time.

For infinite graphs Conjecture D.1.10 is not true in this formulation. It fails already with $r = 3$. To see this consider the graph depicted in Figure D.1, called the *double ladder*.

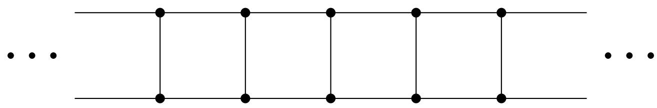

Figure D.1.: The double ladder

It is easy to check that the double ladder has a unique Hamilton circle, but all vertices have degree 3. Mohar has modified the statement of the conjecture and raised the following question. To state them we need to define two terms. We define the *vertex-* or *edge-degree* of an end ω to be the supremum of the number of vertex- or edge-disjoint rays in ω, respectively. In particular, ends of a graph G can have infinite degree, even if G is locally finite.

Question D.1.11. [47] *Does an infinite graph exist that has a unique Hamilton circle and degree $r > 2$ at every vertex as well as vertex-degree r at every end?*

Our result shows that, in contrast to Conjecture D.1.10 and its known cases, there are infinite graphs having the same degree at every vertex and end while being Hamiltonian in a unique way.

Theorem D.1.12. *There exists an infinite connected graph G with a unique Hamilton circle that has degree* 3 *at every vertex and vertex- as well as edge-degree* 3 *at every end.*

So with Theorem D.1.12 we answer Question D.1.11 positively and, therefore, disprove the modified version of Conjecture D.1.10 for infinite graphs in the way Mohar suggested by considering degrees of both, vertices and ends.

The rest of Section D is structured as follows. In Section D.2 we establish all necessary notation and terminology for Section D. We also list some lemmas that will serve as auxiliary tools for the proofs of the main theorems of Section D. Section D.3 is dedicated to Theorem D.1.5 where at the beginning of that section we discuss how one can sensibly extend Corollary D.1.4 and which problems arise when we try to extend Theorem D.1.3 in a similar way. In Section D.4 we present a proof of Theorem D.1.8. Afterwards we describe how a different proof of this theorem works which copies the ideas of a proof of statement (i) of Proposition D.1.6. We conclude this section by comparing the two proofs. Section D.5 contains the construction of a graph witnessing Theorem D.1.12.

D.2. Preliminaries

When we mention a graph during Section D we always mean an undirected and simple graph. For basic facts and notation about finite as well as infinite graphs we refer the reader to [12]. For a broader survey about locally finite graphs and a topological approach to them see [13].

Now we list important notions and concepts that we shall need in Section D followed by useful statements about them. In a graph G with a vertex v we denote by $\delta(v)$ the set of edges incident with v in G. Similarly, for a subgraph H of G or just its vertex set we denote by $\delta(H)$ the set of edges that have only one endvertex in H. Although formally different, we will not always distinguish between a cut $\delta(H)$ and the partition $(V(H), V(G)\setminus V(H))$ it is induced by. For two vertices $v, w \in V(G)$ let $d_G(v, w)$ denote the distance between v and w in G.

We call a finite graph *outerplanar* if it can be embedded in the plane such that all vertices lie on the boundary of a common face.

For a graph G and an integer $k \geqslant 2$ we define the *k-th power* of G as the graph obtained by taking G and adding additional edges vw for any two vertices $v, w \in V(G)$ such that $1 < d_G(v, w) \leqslant k$.

A tree is called a *caterpillar* if after the deletion of its leaves only a path is left.

We denote by $S(K_{1,3})$ the graph obtained by taking the star with three leaves $K_{1,3}$ and subdividing each edge once.

We call a graph *locally finite* if each vertex has finite degree.

A one-way infinite path in a graph G is called a *ray* of G, while we call a two-way infinite path in G a *double ray* of G. Every ray contains a unique vertex that has degree 1 it. We call this vertex the *start vertex* of the ray. An equivalence relation can be defined on the set of rays of a graph G by saying that two rays are equivalent if and only if they cannot be separated by finitely many vertices in G. The equivalence classes of this relation are called the *ends* of G. We denote the set of all ends of a graph G by $\Omega(G)$.

The union of a ray R with infinitely many disjoint paths P_i for $i \in \mathbb{N}$ each having precisely one endvertex on R is called a *comb*. We call the endvertices of the paths P_i that do not lie on R and those vertices v for which there is a $j \in \mathbb{N}$ such that $v = P_j$ the *teeth* of the comb.

The following lemma is a basic tool for infinite graphs. Especially for locally finite graphs it helps us to get a comb whose teeth lie in a previously fixed infinite set of vertex.

Lemma D.2.1. [12, Prop. 8.2.2] *Let U be an infinite set of vertices in a connected graph G. Then G contains either a comb with all teeth in U or a subdivision of an infinite star with all leaves in U.*

For a locally finite and connected graph G we can endow G together with its ends with a topology that yields the space $|G|$. A precise definition of $|G|$ can be found in [12, Ch. 8.5]. Let us point out here that a ray of G converges in $|G|$ to the end of G it is contained in. Another way of describing $|G|$ is to endow G with the topology of a 1-complex and then forming the Freudenthal compactification [19].

For a point set X in $|G|$, we denote its closure in $|G|$ by $\overline{X}$. We shall often write $\overline{M}$ for some M that is a set of edges or a subgraph of G. In this case we implicitly assume to first identify M with the set of points in $|G|$ which corresponds to the edges and vertices that are contained in M.

We call a subspace Z of $|G|$ *standard* if $Z = \overline{H}$ for a subgraph H of G.

A *circle* in $|G|$ is the image of a homeomorphism having the unit circle S^1 in $\mathbb{R}^2$ as domain and mapping into $|G|$. Note that all finite cycles of a locally finite connected graph G correspond to circles in $|G|$, but there might also be infinite subgraphs H of G such that $\overline{H}$ is a circle in $|G|$. Similar to finite graphs we call a locally finite connected graph G *Hamiltonian* if there exists a circle in $|G|$ which contains all vertices of G. Such circles are called *Hamilton circles* of G.

We call the image of a homeomorphism with the closed real unit interval $[0, 1]$ as domain and mapping into $|G|$ an *arc* in $|G|$. Given an arc α in $|G|$, we call a point x of $|G|$ an *endpoint* of α if 0 or 1 is mapped to x by the homeomorphism defining α. If the endpoint of an arc corresponds to a vertex of the graph, we also call the endpoint an *endvertex* of the arc. Similarly as for paths, we call an arc an x–y arc if x and y are the endpoints of the arc. Possibly the simplest example of a nontrivial arc is a ray together with the end it converges to. However, the structure of arcs is more complicated in general and they might contain up to $2^{\aleph_0}$ many ends. We call a subspace X of $|G|$ *arc-connected* if for any two points x and y of X there is an x–y arc in X.

Using the notions of circles and arc-connectedness we now extend trees in a similar topological way. We call an arc-connected standard subspace of $|G|$ a *topological tree* if it does not contain any circle. Note that, similar as for finite trees, for any two points x, y of a topological tree there is a unique x–y arc in that topological tree. Generalizing the definition of caterpillars, we call a topological tree $\overline{T}$ in $|G|$ a *topological caterpillar* if $\overline{T - L}$ is an arc, where T is a forest in G and L denotes the set of all leaves of T, i.e., vertices of degree 1 in T.

Now let ω be an end of a locally finite connected graph G. We define the *vertex-* or *edge-degree of ω in G* as the supremum of the number of vertex- or edge-disjoint rays in G, respectively, which are contained in ω. By this definition ends may have infinite vertex- or edge-degree. Similarly, we define the *vertex-* or *edge-degree of ω in a standard subspace X of $|G|$* as the supremum of vertex- or edge-disjoint arcs in X, respectively, that have ω as an endpoint. We should mention here that the supremum is actually an attained maximum in both definitions. Furthermore, when we consider the whole space $|G|$ as a standard subspace of itself, the vertex-degree in G of any end ω of G coincides with the vertex-degree in $|G|$ of ω. The same holds for the edge-degree. The proofs of these statements are nontrivial and since it is enough for us to work with the supremum, we will not go into detail here.

We make one last definition with respect to end degrees which allows us to distinguish the parity of degrees of ends when they are infinite. The idea of this definition is due to Bruhn and Stein [7]. We call the vertex- or edge-degree of an end ω of G in a standard subspace X of $|G|$ *even* if there is a finite set $S \subseteq V(G)$ such that for every finite set $S' \subseteq V(G)$ with $S \subseteq S'$ the maximum number of vertex- or edge-disjoint arcs in X, respectively, with ω as endpoint and some $s \in S'$ is even. Otherwise, we call the vertex- or edge-degree of ω in X, respectively, *odd*.

Next we collect some useful statements about the space $|G|$ for a locally finite connected graph G.

Proposition D.2.2. [12, Prop. 8.5.1] *If G is a locally finite connected graph, then $|G|$ is a compact Hausdorff space.*

Having Proposition D.2.2 in mind the following basic lemma helps us to work with continuous maps and to verify homeomorphisms, for example when considering circles or arcs.

Lemma D.2.3. *Let X be a compact space, Y be a Hausdorff space and $f : X \longrightarrow Y$ be a continuous injection. Then f^{-1} is continuous too.*

The following lemma tells us an important combinatorial property of arcs. To state the lemma more easily, let $\mathring{F}$ denote the set of inner points of edges $e \in F$ in $|G|$ for an edge set $F \subseteq E(G)$.

Lemma D.2.4. [12, Lemma 8.5.3] *Let G be a locally finite connected graph and $F \subseteq E(G)$ be a cut with sides V_1 and V_2.*

(i) *If F is finite, then $\overline{V_1} \cap \overline{V_2} = \emptyset$, and there is no arc in $|G| \setminus \mathring{F}$ with one endpoint in V_1 and the other in V_2.*

(ii) *If F is infinite, then $\overline{V_1} \cap \overline{V_2} \neq \emptyset$, and there may be such an arc.*

The next lemma ensures that connectedness and arc-connectedness are equivalent for the spaces we are mostly interested in, namely standard subspaces, which are closed by definition.

Lemma D.2.5. [20, Thm. 2.6] *If G is a locally finite connected graph, then every closed topologically connected subset of $|G|$ is arc-connected.*

We continue in the spirit of Lemma D.2.4 by characterising important topological properties of the space $|G|$ in terms of combinatorial ones. The following lemma deals with arc-connected subspaces. It will be convenient for us to use this in a proof later on.

Lemma D.2.6. [12, Lemma 8.5.5] *If G is a locally finite connected graph, then a standard subspace of $|G|$ is topologically connected (equivalently: arc-connected) if and only if it contains an edge from every finite cut of G of which it meets both sides.*

The next theorem is actually part of a bigger one containing more equivalent statements. Since we shall need only one equivalence, we reduced it to the following formulation. For us it will be helpful to check or at least bound the degree of an end in a standard subspace just by looking at finite cuts instead of dealing with the homeomorphisms that actually define the relevant arcs.

Theorem D.2.7. [13, Thm. 2.5] *Let G be a locally finite connected graph. Then the following are equivalent for $D \subseteq E(G)$:*

(i) *D meets every finite cut in an even number of edges.*

(ii) *Every vertex of G has even degree in $\overline{D}$ and every end of G has even edge-degree in $\overline{D}$.*

The following lemma gives us a nice combinatorial description of circles and will be especially useful in combination with Theorem D.2.7 and Lemma D.2.6.

Lemma D.2.8. [7, Prop. 3] *Let C be a subgraph of a locally finite connected graph G. Then $\overline{C}$ is a circle if and only if $\overline{C}$ is topologically connected, every vertex in $\overline{C}$ has degree 2 in $\overline{C}$ and every end of G contained in $\overline{C}$ has edge-degree 2 in $\overline{C}$.*

A basic fact about finite Hamiltonian graphs is that they are always 2-connected. For locally finite connected graphs this is also a well-known fact, although it has not separately been published. Since we shall need this fact later and can easily deduce it from the lemmas above, we include a proof here.

Corollary D.2.9. *Every locally finite connected Hamiltonian graph is* 2*-connected.*

Proof. Let G be a locally finite connected Hamiltonian graph and suppose for a contradiction that it is not 2-connected. Fix a subgraph C of G whose closure $\overline{C}$ is a Hamilton circle of G and a cut vertex v of G. Let K_1 and K_2 be two different components of $G - v$. By Theorem D.2.7 the circle $\overline{C}$ uses evenly many edges of each of the finite cuts $\delta(K_1)$ and $\delta(K_2)$. Since $\overline{C}$ is a Hamilton circle and, therefore, topologically connected, we also get that it uses at least two edges of each of these cuts by Lemma D.2.6. This implies that v has degree at least 4 in C, which contradicts Lemma D.2.8. $\square$

D.3. Topological caterpillars

In this section we close a gap with respect to the general question of when the k-th power of a graph has a Hamilton circle. Let us begin by summarizing the results in this field. We start with finite graphs. The first result to mention is the famous theorem of Fleischner, Theorem D.1.1, which deals with 2-connected graphs.

For higher powers of graphs the following theorem captures the whole situation.

Theorem D.3.1. [41, 55] *The cube of any finite connected graph on at least three vertices is Hamiltonian.*

These theorems leave the question whether and when one can weaken the assumption of being 2-connected and still maintain the property of being Hamiltonian. Theorem D.1.3 gives an answer to this question.

Now let us turn our attention towards locally finite infinite graphs. As mentioned in the introduction, Georgakopoulos has completely generalized Theorem D.1.1 to locally finite graphs by proving Theorem D.1.2. Furthermore, he also gave a complete generalization of Theorem D.3.1 to locally finite graphs with the following theorem.

Theorem D.3.2. [25, Thm. 5] *The cube of any locally finite connected graph on at least three vertices is Hamiltonian.*

What is left and what we do in the rest of this section is to prove lemmas about locally finite graphs covering implications similar to those in Theorem D.1.3, and mainly Theorem D.1.5, which extends Corollary D.1.4 to locally finite graphs.

Let us first consider a naive way of extending Theorem D.1.3 and Corollary D.1.4 to locally finite graphs. Since we consider spanning caterpillars for Corollary D.1.4, we need a definition of these objects in infinite graphs that allows them to contain infinitely many vertices. So let us modify the definition of caterpillars as follows: A locally finite tree is called a *caterpillar* if after deleting its leaves only a finite path, a ray or a double ray is left. Using this definition Theorem D.1.3 remains true for locally finite infinite trees T and Hamilton circles in $|T^2|$. The same proof as the one Harary and Schwenk [33, Thm. 1] gave for Theorem D.1.3 in finite graphs can be used to show this.

Corollary D.1.4 remains also true for locally finite graphs using this adapted definition of caterpillars. Its proof, however, is no trivial deduction anymore. The problem is that for a spanning tree T of a locally finite connected graph G the topological spaces $|T^2|$ and $|G^2|$ might differ not only in inner points of edges but also in ends. More precisely, there might be two equivalent rays in G^2 that belong to different ends of T^2. So the Hamiltonicity of T^2 does not directly imply the one of G^2. However, for T being a spanning caterpillar of G, this problem can only occur when T contains a double ray such that all subrays belong to the same end of G. Then the same construction as in the proof for the implication from (iii) to (i) of Theorem D.1.3 can be used to build a spanning double ray in T^2 which is also a Hamilton circle in $|G^2|$. The idea for the construction which is used for this implication is covered in Lemma D.3.4.

The downside of this naive extension is the following. For a locally finite infinite graph the assumption of having a spanning caterpillar is quite restrictive. Such graphs can especially have at most two ends since having three ends would imply that the spanning caterpillar must contain three disjoint rays. This, however, is impossible because it would force the caterpillar to contain a $S(K_{1,3})$. For this reason we have defined a topological version of caterpillars, namely topological caterpillars. Their definition allows graphs with arbitrary many ends to have a spanning topological caterpillar. Furthermore, it yields with Theorem D.1.5 a more relevant extension of Corollary D.1.4 to locally finite graphs.

We briefly recall the definition of topological caterpillars. Let G be a locally finite connected graph. A topological tree $\overline{T}$ in $|G|$ is a topological caterpillar if $\overline{T - L}$ is an arc, where T is a forest in G and L denotes the set of all leaves of T, i.e., vertices of degree 1 in T.

The following basic lemma about topological caterpillars is easy to show and so we omit its proof. It is an analogue of the equivalence of the statements (ii) and (iii) of Theorem D.1.3 for topological caterpillars.

Lemma D.3.3. *Let G be a locally finite connected graph. A topological tree $\overline{T}$ in $|G|$ is a topological caterpillar if and only if T does not contain $S(K_{1,3})$ as a subgraph and all ends of G have vertex-degree in $\overline{T}$ at most 2.*

Before we completely turn towards the preparation of the proof of Theorem D.1.5 let us consider statement (i) of Theorem D.1.3 again. A complete extension of Theorem D.1.3 to locally finite graphs using topological caterpillars seems impossible because of statement (i). To see this we should first make precise what the adapted version of statement (i) most possibly should be. In order to state it let G denote a locally finite connected graph and let $\overline{T}$ be a topological tree in $|G|$. Now the formulation of the adapted statement should be as follows:

(i*) *In the subspace $\overline{T^2}$ of $|G^2|$ is a circle containing all vertices of T.*

This statement does not hold if T has more than one graph theoretical component. Therefore, it cannot be equivalent to $\overline{T}$ being a topological caterpillar in $|G|$, which is the adapted version of statement (iii) of Theorem D.1.3 for locally finite graphs. Note that any two vertices of T lie in the same graph theoretical component of T if and only if they lie in the same graph theoretical component of T^2. Hence, we can deduce that statement (i*) fails if T has more than one graph theoretical component from the following claim.

Claim. *Let G be a locally finite connected graph and let $\overline{T}$ be a topological tree in $|G|$. Then there is no circle in the subspace $\overline{T^2}$ of $|G^2|$ that contains vertices from different graph theoretical components of T^2.*

Proof. We begin with a basic observation. The inclusion map from G into G^2 induces an embedding from $|G|$ into $|G^2|$ in a canonical way. Moreover, all ends of G^2 are contained in the image of this embedding. To see this note that any two non-equivalent rays in G stay non-equivalent in G^2 since G is locally finite. Furthermore, by applying Lemma D.2.1 it is easy to see that every end in G^2 contains a ray that is also a ray of G. This already yields an injection from $|G|$ to

$|G^2|$ whose image contains all of $\Omega(G^2)$. Verifying the continuity of this map and its inverse is immediate.

Now let us suppose for a contradiction that there is a circle C in $\overline{T^2}$ containing vertices v, v' from two different graph theoretical components K, K' of T^2. Say $v \in V(K)$ and $v' \in V(K')$. Let A_1 and A_2 denote the two v'–v arcs on C. Since A_1 and A_2 are disjoint except from their endpoints, they have to enter K via different ends ω_1^2 and ω_2^2 of G^2 that are contained in $\overline{K} \subseteq |G^2|$. Say $\omega_1^2 \in A_1$ and $\omega_2^2 \in A_2$. By the observation above ω_1^2 and ω_2^2 correspond to two different ends ω_1 and ω_2 of G. Only one of them, say ω_1, lies on the unique v'–v arc that is contained in the topological tree $\overline{T}$. Now we modify A_2 by replacing each edge uw of A_2 which is not in $E(T)$ by a u–w path of length 2 that lies in T. By Lemma D.2.6 this yields an arc-connected subspace of $\overline{T}$ that contains v and v'. By our observation above the unique v'–v arc in this subspace must contain the end ω_2. This, however, is a contradiction since we have found two different v'–v arcs in $\overline{T}$. □

Now we start preparing the proof of Theorem D.1.5. For this we define a certain partition of the vertex set of a topological caterpillar. Additionally, we define a linear order of these partition classes. Let G be a locally finite connected graph and $\overline{T}$ a topological caterpillar in $|G|$. Furthermore, let L denote the set of leaves of T. By definition, $\overline{T - L}$ is an arc, call it A. This arc induces a linear order $<_A$ of the vertices of $V(T) - L$. For consecutive vertices $v, w \in V(T) - L$ with $v <_A w$ we now define the set

$$P_w := \{w\} \cup (N_T(v) \cap L)$$

(cf. Figure D.2). If A has a maximal element m with respect to $<_A$, we define an additional set $P^+ = N_T(m) \cap L$. Should A have a minimal element s with respect to $<_A$, we define another additional set $P^- = \{s\}$. The sets P_w, possibly together with P^+ and P^-, form a partition $\mathcal{P}_T$ of $V(T)$. For any $v \in V(T)$ we denote the corresponding partition class containing v by V_v. Next we use the linear order $<_A$ to define a linear order $<_T$ on $\mathcal{P}_T$. For any two vertices $v, w \in V(T) - L$ with $v <_A w$ set $V_v <_T V_w$. If P^+ (resp. P^-) exists, set $P_v <_T P^+$ (resp. $P^- <_T P_v$) for every $v \in V(T) - L$. Finally we define for two vertices $v, w \in V(T)$ with $V_v \leqslant_T V_w$ the set

$$I_{vw} := \bigcup \{V_u \,;\, V_v \leqslant_T V_u \leqslant_T V_w\}.$$

The following basic lemma lists important properties of the partition $\mathcal{P}_T$ together

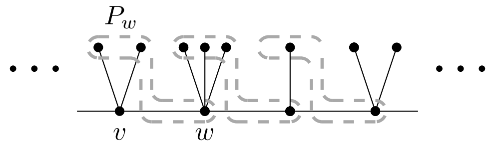

Figure D.2.: The partition classes P_w.

with its order $<_T$. The proof of this lemma is immediate from the definitions of $\mathcal{P}_T$ and $<_T$. Especially for Lemma D.3.5 and in the proof of Theorem D.1.5 the listed properties will be applied intensively. Furthermore, the proof that statement (iii) of Theorem D.1.3 implies statement (i) of Theorem D.1.3 follows easily from this lemma.

Lemma D.3.4. *Let $\overline{T}$ be a topological caterpillar in $|G|$ for a locally finite connected graph G. Then the partition $\mathcal{P}_T$ of $V(T)$ has the following properties:*

(i) *Any two different vertices belonging to the same partition class of $\mathcal{P}_T$ have distance 2 from each other in T.*

(ii) *For consecutive partition classes Q and R with $Q <_T R$, there is a unique vertex in Q that has distance 1 in T to every vertex of R. For $Q \neq P^-$, this vertex is the one of Q that is not a leaf of T.*

Proof. □

Referring to statement (ii) of Lemma D.3.4, let us call the vertex in a partition class $Q \in \mathcal{P}_T$ that is not a leaf of T the *jumping vertex* of Q.

We still need a bit of notation and preparation work before we can prove the main theorem of this section. Now let $\overline{T}$ denote a topological caterpillar with only one graph-theoretical component. Let $(\mathcal{X}_1, \mathcal{X}_2)$ be a bipartition of the partition classes V_v such that consecutive classes with respect to $\leqslant_T$ lie not both in $\mathcal{X}_1$, or in $\mathcal{X}_2$. Furthermore, let $v, w \in V(T)$ be two vertices, say with $V_v \leqslant_T V_w$, whose distance is even in T. We define a (v, w) *square string* S in T^2 to be a path in T^2 with the following properties:

1. S uses only vertices of partitions that lie in the bipartition class $\mathcal{X}_i$ in which V_v and V_w lie.

2. S contains all vertices of partition classes $V_u \in \mathcal{X}_i$ for $V_v <_T V_u <_T V_w$.

3. S contains only v and w from V_v and V_w, respectively.

Similarly, we define $(v, w]$, $[v, w)$ and $[v, w]$ square strings in T^2, but with the difference in (3) that they shall also contain all vertices of V_w, V_v and $V_v \cup V_w$, respectively. We call the first two types of square strings *left open* and the latter ones *left closed*. The notion of being *right open* and *right closed* is analogously defined. From the properties of $\mathcal{P}_T$ listed in Lemma D.3.4, it is immediate how to construct square strings.

The next lemma gives us two possibilities to cover the vertex set of a graph-theoretical component of a topological caterpillar $\overline{T}$ that contains a double ray. Each cover will consist of two, possibly infinite, paths of T^2. Later on we will use these covers to connect all graph-theoretical components of $\overline{T}$ in a certain way such that a Hamilton circle of G^2 is formed.

Lemma D.3.5. *Let G be a locally finite connected graph and let $\overline{T}$ be a topological caterpillar in $|G|$. Suppose T has only one graph-theoretical component and contains a double ray. Furthermore, let v and w be vertices of T with $V_v \leqslant_T V_w$.*

(i) *If $d_T(v, w)$ is even, then in T^2 there exist a v–w path P, a double ray D and two rays R_v and R_w with the following properties:*

- *P and D are disjoint as well as R_v and R_w.*
- *$V(T) = V(P) \cup V(D) = V(R_v) \cup V(R_w)$.*
- *v and w are the start vertices of R_v and R_w, respectively.*
- *$R_v \cap V_x = \varnothing$ for every $V_x >_T V_w$.*
- *$R_w \cap V_y = \varnothing$ for every $V_y <_T V_v$.*

(ii) *If $d_T(v, w)$ is odd, then in T^2 there exist rays R_v, R_w, R'_v, R'_w with the following properties:*

- *R_v and R_w are disjoint as well as R'_v and R'_w.*
- *$V(T) = V(R_v) \cup V(R_w) = V(R'_v) \cup V(R'_w)$.*
- *v is the start vertex of R_v and R'_v while w is the one of R_w and R'_w.*
- *$R_v \cap V_x = R'_w \cap V_x = \varnothing$ for every $V_x >_T V_w$.*
- *$R_w \cap V_y = R'_v \cap V_y = \varnothing$ for every $V_y <_T V_v$.*

Proof. We sketch the proof of statement (i). As v–w path P we take a square string S_{vw} in T^2 with v and w as endvertices. Depending whether v is a jumping vertex or not we take a left open or closed square string, respectively. Depending on w we take a right closed or open square string if w is a jumping vertex or not, respectively. Since $d_T(v, w)$ is even, we can find such square strings. To construct the double ray D start with a $(v^-, w^-]$ square string in T^2 where v^- and w^- denote the jumping vertices in the partition classes proceeding V_v and V_w, respectively. Using the properties (i) and (ii) of the partition $\mathcal{P}_T$ mentioned in Lemma D.3.4, the $(v^-, w^-]$ square string can be extend to a desired double ray D containing all vertices of T that do not lie in S_{vw} (cf. Figure D.3).

To define R_v we start with a square string S_v having v as one endvertex. For the definition of S_v we distinguish four cases. If v and w are jumping vertices, we set S_v as a path obtained by taking a $(v, w]$ square string and deleting w from it. If v is not a jumping vertex, but w is one, take a $[v, w]$ square string, delete w from it and set the remaining path as S_v. In the case that v is a jumping vertex, but w is none, S_v is defined as a path obtained from a (v, w) square string from which we delete w. In the case that neither v nor w is a jumping vertex, we take a $[v, w)$ square string, delete w from it and set the remaining path as S_v. Next we extend S_v using a square string to a path with v as one endvertex containing all vertices in partition classes V_u with $V_v <_T V_u <_T V_w$. We extend the remaining path to a ray that contains also all vertices in partition classes V_u with $V_u \leqslant_T V_v$, but none from partition classes V_x for $V_x >_T V_w$. The desired second ray R_w can now easily be build in $T^2 - R_v$.

The rays for statement (ii) are defined in a very similar way (cf. Figure D.3). Therefore, we omit their definitions here. □

The following lemma is essential for connecting the parts of the vertex covers of two different graph-theoretical components of $\overline{T}$. Especially, here we make use of the structure of $|G|$ instead of arguing only inside of $\overline{T}$ or $\overline{T^2}$. This allows us to build a Hamilton circle using square strings and to “jump over" an end to avoid producing an edge-degree bigger than 2 at that end.

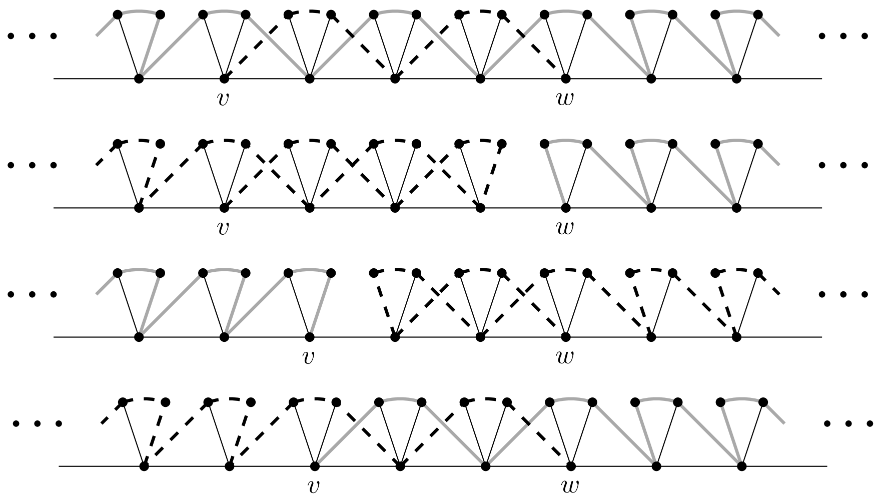

Figure D.3.: Examples for covering the vertices of a caterpillar as in Lemma D.3.5.

Lemma D.3.6. *Let $\overline{T}$ be a spanning topological caterpillar of a locally finite connected graph G and let $v, w \in V(G)$ where $V_v \leqslant_T V_w$. Then for any two vertices x, y with $V_v <_T V_x <_T V_w$ and $V_v <_T V_y <_T V_w$ there exists a finite x–y path in $G[I_{vw}]$.*

Proof. Let the vertices v, w, x and y be as in the statement of the lemma and, as before, let L denote the set of leaves of T. Now suppose for a contradiction that there is no finite x–y path in $G[I_{vw}]$. Then we can find an empty cut D of $G[I_{vw}]$ with sides M and N such that x and y lie on different sides of it. Since $\overline{T \cap G[I_{vw}]}$ contains an x–y arc, there must exist an end $\omega \in \overline{M} \cap \overline{N} \cap \overline{T - L}$.

Let us show next that there exists an open set O in $|G|$ that contains ω and, additionally, every vertex in O is an element of I_{vw}. To see this we first pick a set $O_A \subseteq \overline{T - L}$ so that it is open in the subspace $\overline{T - L}$, topologically connected and contains ω, but its closure does not contain the jumping vertices of V_v and V_w. Now let O' be an open set in $|G|$ witnessing that O_A is open in $\overline{T - L}$. We prove that O' contains only finitely many vertices of $V(G) \setminus I_{vw}$. Suppose for a contradiction that this is not the case. Then we would find an infinite sequence $(z_n)_{n \in \mathbb{N}}$ of different vertices in $O' \setminus I_{vw}$ that must converge to some point $p \in |G|$ by the compactness of $|G|$. Since $\overline{T}$ is a spanning topological caterpillar of G, it contains all the vertices z_n. Using that G is locally finite, we get that the jumping vertices of the sets V_{z_n} also form a sequence that converges to p. So we can deduce that $p \in \overline{T - L}$, because $\overline{T - L}$ is a closed subspace containing all jumping vertices. Hence, $p \in \overline{O'} \cap (\overline{T - L}) = \overline{O_A}$. This is a contradiction to our choice of

O_A ensuring $p \notin \overline{O_A}$. Hence, O' contains only finitely many vertices of $V(G) \backslash I_{vw}$, say $v_1, \ldots, v_n$ for some $n \in \mathbb{N}$. Before we define our desired set O using O', note that $O_v := |G| \backslash \{v\}$ defines an open set in $|G|$ for every vertex $v \in V(G)$. Therefore, $O := O' \cap \bigcap_{i=1}^{n} O_{v_i}$ is an open set in $|G|$ containing no vertex of $V(G) \backslash I_{vw}$.

Inside O we can find a basic open set B around ω, which contains a graph-theoretical connected subgraph with all vertices of B. Now B contains vertices of M and N as well as a finite path between them, which must then also exist in $G[I_{vw}]$. Such a path would have to cross D contradicting the assumption that D is an empty cut in $G[I_{vw}]$. □

To figure out which parts of the vertex covers of which graph-theoretical components of $\overline{T}$ we can connect such that afterwards we are still able to extend this construction to a Hamilton circle of G, we shall use the next lemma. For the formulation of the lemma, we use the notion of *splits*.

Let G be a multigraph and $v \in V(G)$. Furthermore, let $E_1, E_2 \subseteq \delta(v)$ such that $E_1 \cup E_2 = \delta(v)$ but $E_1 \cap E_2 = \emptyset$ where $E_i \neq \emptyset$ for $i \in \{1, 2\}$. Now we call a multigraph G' a *v-split* of G if

$$V(G') = V(G) \backslash \{v\} \cup \{v_1, v_2\}$$

with $v_1, v_2 \notin V(G)$ and

$$E(G') = E(G - v) \cup \{v_1 w \; ; \; wv \in E_1\} \cup \{v_2 u \; ; \; uv \in E_2\}.$$

We call the vertices v_1 and v_2 *replacement vertices* of v.

Lemma D.3.7. *Let G be a finite Eulerian multigraph and v be a vertex of degree 4 in G. Then there exist two v-splits G_1 and G_2 of G both of which are also Eulerian.*

Proof. There are $\frac{1}{2} \cdot \binom{4}{2} = 3$ possible non-isomorphic v-splits of G such that v_1 and v_2 have degree 2 in the v-split. Assume that one of them, call it G', is not Eulerian. This can only be the case if G' is not connected. Let (A, B) be an empty cut of G'. Note that $G - v$ has precisely two components C_1 and C_2 since G is Eulerian and v has degree 4 in G. So C_1 and C_2 must lie in different sides of (A, B), say $C_1 \subseteq A$. Since G was connected, we get that v_1 and v_2 lie in different sides of the cut (A, B), say $v_1 \in A$. Therefore, $A = C_1 \cup \{v_1\}$ and $B = C_2 \cup \{v_2\}$. If $\delta(v) = \{vw_1, vw_2, vw_3, vw_4\}$ and $\{v_1w_1, v_1w_2\}, \{v_2w_3, v_2w_4\} \subseteq E(G')$, set G_1 and

G_2 as v-splits of G such that the inclusions $\{v_1w_1, v_1w_3\}, \{v_2w_2, v_2w_4\} \subseteq E(G_1)$ and $\{v_1w_1, v_1w_4\}, \{v_2w_2, v_2w_3\} \subseteq E(G_2)$ hold. Now G_1 and G_2 are Eulerian, because every vertex has even degree in each of those multigraphs and both multigraphs are connected. To see the latter statement, note that any empty cut (X, Y) of G_i for $i \in \{1, 2\}$ would need to have C_1 and C_2 on different sides. If also v_1 and v_2 are on different sides, we would have $(A, B) = (X, Y)$, which does not define an empty cut of G_i by definition of G_i. However, v_1 and v_2 cannot lie on the same side of the cut (X, Y). This is because otherwise the cut (X, Y) would induce an empty cut in G after identifying v_1 and v_2 in G_i. Since G is Eulerian and therefore especially connected, we would have a contradiction. $\square$

Now we have all tools together to prove Theorem D.1.5. Before we start the proof, let us recall the statement of the theorem.

Theorem D.1.5. *The square of any locally finite connected graph G on at least three vertices such that $|G|$ contains a spanning topological caterpillar is Hamiltonian.*

Proof. Let G be a graph as in the statement of the theorem and let $\overline{T}$ be a spanning topological caterpillar of G. We may assume by Corollary D.1.4 that G has infinitely many vertices. Now let us fix an enumeration of the vertices, which is possible since every locally finite connected graph is countable. We inductively build a Hamilton circle of G^2 in at most ω many steps. We ensure that in each step $i \in \mathbb{N}$ we have two disjoint arcs $\overline{A^i}$ and $\overline{B^i}$ in $|G^2|$ whose endpoints are vertices of subgraphs A^i and B^i of G^2, respectively. Let a^i_ℓ and a^i_r (resp. b^i_ℓ and b^i_r) denote the endvertices of $\overline{A^i}$ (resp. $\overline{B^i}$) such that $V_{a^i_\ell} \leqslant_T V_{a^i_r}$ (resp. $V_{b^i_\ell} \leqslant_T V_{b^i_r}$). For the construction we further ensure the following properties in each step $i \in \mathbb{N}$:

1. The vertices a^i_r and b^i_r are the jumping vertices of $V_{a^i_r}$ and $V_{b^i_r}$, respectively.

2. The partition sets $V_{a^i_\ell}$ and $V_{b^i_\ell}$ as well as $V_{a^i_r}$ and $V_{b^i_r}$ are consecutive with respect to $\leqslant_T$.

3. If $V_v \cap V(A^i \cup B^i) \neq \emptyset$ holds for any vertex $v \in V(G)$, then $V_v \subseteq V(A^i \cup B^i)$.

4. If for any vertex $v \in V(G)$ there are vertices $u, w \in V(G)$ such that $V_u, V_w \subseteq V(A^i \cup B^i)$ and $V_u \leqslant_T V_v \leqslant_T V_w$, then $V_v \subseteq V(A^i \cup B^i)$ is true.

5. $A^i \cap A^{i+1} = A^i$ and $B^i \cap B^{i+1} = B^i$, but $V(A^{i+1} \cup B^{i+1})$ contains the least vertex with respect to the fixed vertex enumeration that was not already contained in $V(A^i \cup B^i)$.

We start the construction by picking two adjacent vertices t and t' in T that are no leaves in T. Then V_t and V_t' are consecutive with respect to $\leqslant_T$. Note that $G^2[V_t]$ and $G^2[V_{t'}]$ are cliques by property (i) of the partition $\mathcal{P}_T$ mentioned in Lemma D.3.4. We set A^1 to be a Hamilton path of $G^2[V_t]$ with endvertex t and B^1 to be one of $G^2[V_{t'}]$ with endvertex t'. This completes the first step of the construction.

Suppose we have already constructed A^n and B^n. Let $v \in V(G)$ be the least vertex with respect to the fixed vertex enumeration that is not already contained in $V(A^n \cup B^n)$. We know by our construction that either $V_v <_T V_x$ or $V_v >_T V_x$ for every vertex $x \in V(A^n \cup B^n)$. Consider the second case, since the argument for the first works analogously. Let $v' \in V(G)$ be a vertex such that $V_{v'}$ is the predecessor of V_v with respect to $\leqslant_T$. Further, let $w \in V(G)$ be a vertex such that $V_w >_T V_{a_r^n}, V_{b_r^n}$ and V_w is the successor of either $V_{a_r^n}$ or $V_{b_r^n}$, say $V_{b_r^n}$. By Lemma D.3.6 there exists a v'–w path P in $G[I_{b_r^n,v}]$. We may assume that $E(P) \backslash E(T)$ does not contain an edge whose endvertices lie in the same graph-theoretical component of T. Furthermore, we may assume that every graph-theoretical component of T is incident with at most two edges of $E(P) \backslash E(T)$. Otherwise we could modify the path P using edges of $E(T)$ to meet these conditions.

Next we inductively define a finite sequence of finite Eulerian auxiliary multigraphs $H_1, \ldots, H_k$ where H_k is a cycle for some $k \in \mathbb{N}$. Every vertex in each of these multigraphs will have either degree 2 or degree 4. Furthermore, we shall obtain H_{i+1} from H_i as a h-split for some vertex $h \in V(H_i)$ of degree 4 until we end up with a multigraph H_k that is a cycle.

As $V(H_1)$ take the set of all graph-theoretical components $T_1, \ldots, T_n$ of T that are incident with an edge of $E(P) \backslash E(T)$. Two vertices T_i and T_j are adjacent if either there is an edge in $E(P) \backslash E(T)$ whose endpoints lie in T_i and T_j or there is a t_i–t_j arc $\overline{A}$ in $\overline{T}$ for a subgraph A of T and vertices $t_i \in V(T_i)$ and $t_j \in V(T_j)$ such that no endvertex of any edge of $E(P) \backslash E(T)$ lies in $V(A) \cup N_T(A)$. Since $\overline{T}$ is a spanning topological caterpillar, the multigraph H_1 is connected. By definition of P, the multigraph H_1 is also Eulerian where all vertices have either degree 2 or degree 4.

Now suppose we have already constructed H_i and there exists a vertex $h \in V(H_i)$ with degree 4 in H_i. Since H_i is obtained from H_1 via repeated splitting operations, we know that h is incident with two edges d, e in H_i that correspond to edges d_P, e_P, respectively, of $E(P) \backslash E(T)$. Furthermore, h is incident with two edges f, g that correspond to arcs $\overline{A_f}$ and $\overline{A_g}$, respectively, of $\overline{T}$ for subgraphs A_f and A_g of T such that neither $V(A_f) \cup N_T(A_f)$ nor $V(A_g) \cup N_T(A_g)$ contain an endvertex of an edge of $E(P) \backslash E(T)$. Let T_j be the graph-theoretical component of T in which each of d_P and e_P has an endvertex, say w_d and w_e, respectively. Here we consider two cases:

Case 1. *The distance in T_j between w_d and w_e is even.*

In this case we define H_{i+1} as a Eulerian h-split of H_i such that one of the following two options holds for the edge d_{i+1} in H_{i+1} corresponding to d. The first option is that d_{i+1} is adjacent to the edge in H_{i+1} corresponding to e. The second options is that d_{i+1} is adjacent to the edge in H_{i+1} corresponding to either f or g with the property that the path in T_j connecting w_d and A_f (resp. A_g) does not contain w_e. This is possible since two of the three possible non-isomorphic v-splits of H_i are Eulerian by Lemma D.3.7.

Case 2. *The distance in T_j between w_d and w_e is odd.*

Here we set H_{i+1} as a Eulerian h-split of H_i such that the edge in H_{i+1} corresponding to d is not adjacent to the one corresponding to e. As in the first case, this is possible because two of the three possible non-isomorphic h-splits of H_i are Eulerian by Lemma D.3.7. This completes the definition of the sequence of auxiliary multigraphs.

Now we use the last auxiliary multigraph H_k of the sequence to define the arcs $\overline{A^{n+1}}$ and $\overline{B^{n+1}}$. Note that P is a w–v' path in $G[I_{b_r^n, v}]$ where v' and w lie in the same graph-theoretical components $T_{v'}$ and T_w of T as v and b_r^n, respectively. Since we may assume that $E(P) \backslash E(T) \neq \emptyset$ holds, let $e \in E(P) \backslash E(T)$ denote the edge which contains one endvertex w_e in T_w. Then either the distance between w_e and a_r^n or between w_e and b_r^n is even, say the latter one holds. Now we first extend B^n via a $(b_r^n, w_e]$ square string in T^2 and A^n by a $(a_r^n, w_e^+]$ square string in T^2 where $V_{w_e^+}$ is the successor of V_{w_e} with respect to $\leqslant_T$ and w_e^+ is the jumping vertex of $V_{w_e^+}$. Then we extend A^n further using a ray to contain all vertices of partition classes V_x with $V_x >_T V_{w_e^+}$ for $x \in T_w$. This is possible due to the properties (i) and (ii) of the partition $\mathcal{P}_T$ mentioned in Lemma D.3.4.

Next let P_1 and P_2 be the two edge-disjoint $T_{v'}$–T_w paths in H_k. Since every edge of $E(P)\backslash E(T)$ corresponds to an edge of H_k, we get that e corresponds either to P_1 or P_2, say to the former one. Therefore, we will use P_1 to obtain arcs to extend B^n and P_2 for arcs extending A^n. Now we make use of the definition of H_k via splittings. For any vertex T_j of H_1 of degree 4 we have performed a T_j-split. We did this in such a way that the partition of the edges incident with T_j into pairs of edges incident with a replacement vertex of T_j corresponds to a cover of $V(T_j)$ via two, possibly infinite, paths as in Lemma D.3.5. So for every vertex of H_1 of degree 4 we take such a cover. For every graph-theoretical component T_m of T such that there exist two consecutive edges T_iT_j and T_jT_ℓ of P_1 or P_2 that do not correspond to edges of $E(P)\backslash E(T)$ and $V_{t_i} <_T V_{t_m} <_T V_{t_j}$ or $V_{t_j} <_T V_{t_m} <_T V_{t_\ell}$ holds for every choice of $t_i \in T_i$, $t_j \in T_j$, $t_\ell \in T_\ell$ and $t_m \in T_m$, we take a spanning double ray of T_m^2. We can find such spanning double rays by using again the properties (i) and (ii) of the partition $\mathcal{P}_T$ mentioned in Lemma D.3.4. Since $H_k = P_1 \cup P_2$ is a cycle, we can use these covers and double rays to extend $\overline{A^n}$ and $\overline{B^n}$ to be disjoint arcs α^n and β^n with endvertices on $T_{v'}$. With the same construction that we have used for extending A^n and B^n on T_w, we can extend α^n and β^n to have endvertices v_j' and v_j which are the jumping vertices of $V_{v'}$ and V_v, respectively. Additionally, we incorporate that these extensions contain all vertices of partition classes V_y for $y \in T_{v'}$ and $V_y \leqslant V_v$. Then we take these arcs as $\overline{A^{n+1}}$ and $\overline{B^{n+1}}$ where A^{n+1} and B^{n+1} are the corresponding subgraphs of G^2 whose closures give the arcs. By setting a_r^{n+1} and b_r^{n+1} to be v_j' and v_j, depending on which of the two arcs $\overline{A^{n+1}}$ or $\overline{B^{n+1}}$ ends in these vertices, we have guaranteed all properties from (1) to (5) for the construction.

Now the properties (3) – (5) yield not only that $\overline{A}$ and $\overline{B}$ are disjoint arcs for $A = \bigcup_{i \in \mathbb{N}} A^i$ and $B = \bigcup_{i \in \mathbb{N}} B^i$, but also that $V(G) = V(A \cup B)$. If there exists neither a maximal nor minimal partition class with respect to $\leqslant_T$, the union $\overline{A \cup B}$ forms a Hamilton circle of G^2 by Lemma D.2.8. Should there exist a maximal partition class, say $V_{a_r^n}$ for some $n \in \mathbb{N}$ with jumping vertex a_r^n, the vertex a_r^n will also be an endvertex of $\overline{A}$. In this case we connect the endvertices a_r^n and b_r^n of $\overline{A}$ and $\overline{B}$ via an edge. Such an edge exists since $V_{a_r^n}$ and $V_{b_r^n}$ are consecutive with respect to $\leqslant_T$ by property (2) and a_r^n as well as b_r^n are jumping vertices by property (1). Analogously, we add an edge if there exists a minimal partition class. Therefore, we can always obtain the desired Hamilton circle of G^2. $\square$

D.4. Graphs without K^4 or $K_{2,3}$ as minor

We begin this section with a small observation which allows us to strengthen Theorem D.1.8 a bit by forbidding subgraphs isomorphic to a K^4 instead of minors.

Lemma D.4.1. *For graphs without $K_{2,3}$ as a minor it is equivalent to contain a K^4 as a minor or as a subgraph.*

Proof. One implication is clear. So suppose for a contradiction we have a graph without a $K_{2,3}$ as a minor that does not contain K^4 as a subgraph but as a subdivision. Note that containing a K^4 as a subdivision is equivalent to containing a K^4 as a minor since K^4 is cubic. Consider a subdivided K^4 where at least one edge e of the K^4 corresponds to a path P_e in the subdivision whose length is at least two. Let v be an interior vertex of P_e and a, b be the endvertices of P_e. Let the other two branch vertices of the subdivision of K^4 be called c and d. Now we take $\{a, b, c, d, v\}$ as branch vertex set of a subdivision of $K_{2,3}$. The vertices a and b can be joined to c and d by internally disjoint paths using the ones of the subdivision of K^4 except the path P_e. Furthermore, the vertex v can be joined to a and b using the paths vP_ea and vP_eb. So we can find a subdivision of $K_{2,3}$ in the whole graph, which contradicts our assumption. $\square$

Before we start with the proof of Theorem D.1.8 we need to prepare two structural lemmas. The first one will be very convenient for controlling end degrees because it bounds the size of certain separators.

Lemma D.4.2. *Let G be a 2-connected graph without $K_{2,3}$ as a minor and let K_0 be a connected subgraph of G. Then $|N(K_1)| = 2$ holds for every component K_1 of $G - (K_0 \cup N(K_0))$.*

Proof. Let K_0, G and K_1 be defined as in the statement of the lemma. Since G is 2-connected, we know that $|N(K_1)| \geqslant 2$ holds. Now suppose for a contradiction that $N(K_1) \subseteq N(K_0)$ contains three vertices, say u, v and w. Pick neighbours u_i, v_i

and w_i of u, v and w, respectively, in K_i for $i \in \{0,1\}$. Furthermore, take a finite tree T_i in K_i whose leaves are precisely u_i, v_i and w_i for $i \in \{0,1\}$. This is possible because K_0 and K_1 are connected. Now we have a contradiction since the graph H with $V(H) = \{u,v,w\} \cup V(T_0) \cup V(T_1)$ and $E(H) = \bigcup_{i=0}^{1}(\{uu_i, vv_i, ww_i\} \cup E(T_i))$ forms a subdivision of $K_{2,3}$. $\square$

Let G be a connected graph and H be a connected subgraph of G. We define the operation of *contracting H in G* as taking the minor of G which is attained by contracting in G all edges of H. Now let K be any subgraph of G. We denote by G_K the following minor of G: First contract in G each subgraph that corresponds to a component of $G - K$. Then delete all multiple edges.

Obviously G_K is connected if G was connected. We can push this observation a bit further towards 2-connectedness with the following lemma.

Lemma D.4.3. *Let K be a connected subgraph with at least three vertices of a 2-connected graph G. Then G_K is 2-connected.*

Proof. Suppose for a contradiction that G_K is not 2-connected for some G and K as in the statement of the lemma. Since K has at least three vertices, we obtain that G_K has at least three vertices too. So there exists a cut vertex v in G_K. If v is also a vertex of G and, therefore, does not correspond to a contracted component of $G - K$, then v would also be a cut vertex of G. This contradicts the assumption that G is 2-connected.

Otherwise v corresponds to a contracted component of $G - K$. Note that two vertices of G_K both of which correspond to contracted components of $G - K$ are never adjacent by definition of G_K. However, v being a cut vertex in G_K must have at least one neighbour in each component of $G_K - v$. So in particular we get that v separates two vertices, say x and y, of G_K that do not correspond to contracted components of $G - K$. This yields a contradiction because K is connected and, therefore, contains an x–y path. This path still exists in G_K and contradicts the statement that v separates x and y in G_K. $\square$

We shall need another lemma for the proof Theorem D.1.8. In that proof we shall construct an embedding of an infinite graph into a fixed closed disk D by first embedding a finite subgraph into D. Then we extend this embedding stepwise to bigger finite subgraphs so that eventually we define an embedding of the whole graph into D. The following lemma will allow us to redraw newly embedded edges as straight lines in each step while keeping the embedding of every edge that was already embedded as a straight line. Additionally, we will be able to keep the embedding of those edges that are mapped into the boundary of the disk.

Lemma D.4.4. *Let G be a finite 2-connected outerplanar graph and C be its Hamilton cycle. Furthermore, let $\sigma : G \longrightarrow D$ be an embedding of G into a fixed closed disk D such that C is mapped onto the boundary ∂D of D. Then there is an embedding $\sigma^* : G \longrightarrow D$ such that*

(i) *$\sigma^*(e)$ is a straight line for every $e \in E(G)\backslash E(C)$.*

(ii) *$\sigma^*(e) = \sigma(e)$ if $e \in E(C)$ or $\sigma(e)$ is a straight line.*

Proof. We prove the statement by induction on $\ell := |E(G)\backslash E(C)|$. For $\ell = 0$ we can choose the given embedding σ as our desired embedding σ^*. Now let $\ell \geqslant 1$ and suppose σ does not already fulfill all properties of σ^*. Then there exists an edge $e \in E(G)\backslash E(C)$ such that $\sigma(e)$ is not a straight line. Hence, $G - e$ is still a 2-connected outerplanar graph that contains C as its Hamilton cycle. Also $\sigma \restriction_{G-e}$ is an embedding of $G - e$ into D such that C is mapped onto ∂D. So by the induction hypothesis we get an embedding $\tilde{\sigma}^*$ satisfying (i) and (ii) with respect to $\sigma \restriction_{G-e}$. Now let $e = uv$ and suppose for a contradiction that we cannot additionally embed e as a straight line between u and v. Then there exists an edge $xy \in E(G - e)\backslash E(C)$ such that $\tilde{\sigma}^*(xy)$ is crossed by the straight line between u and v. Because $\tilde{\sigma}^*(xy)$ is a straight line between x and y by property (ii), we know that the vertices u, v, x and y are pairwise distinct. This, however, is a contradiction to G being outerplanar since the cycle C together with the edges uv and xy witness the existence of a K^4 minor in G with u, v, x and y as branch sets. So we can extend $\tilde{\sigma}^*$ by embedding $e = uv$ as a straight line between u and v, which yields our desired embedding of G into D. □

With the lemmas above we are now prepared to prove Theorem D.1.8. We recall the formulation of the theorem.

Theorem D.1.8. *Let G be a locally finite connected graph. Then the following statements are equivalent:*

(i) *G is 2-connected and contains neither K^4 nor $K_{2,3}$ as a minor.*

(ii) $|G|$ *has a Hamilton circle* C *and there exists an embedding of* $|G|$ *into a closed disk such that* C *is mapped onto the boundary of the disk.*

Furthermore, if statements (i) and (ii) hold, then $|G|$ *has a unique Hamilton circle.*

Proof. First we show that (ii) implies (i). Since G is Hamiltonian, we know by Corollary D.2.9 that G is 2-connected. Suppose for a contradiction that G contains K^4 or $K_{2,3}$ as a minor. Then G has a finite subgraph H which already has K^4 or $K_{2,3}$ as a minor. Now take any finite connected subgraph K_0 of G which contains H and set $K = G[V(K_0) \cup N(K_0)]$. Next let us take an embedding of $|G|$ as in statement (ii) of this theorem. It is easy to see using Lemma D.4.2 that our fixed embedding of $|G|$ induces an embedding of G_K into a closed disk such that all vertices of G_K lie on the boundary of the disk. This implies that G_K is outerplanar. So G_K can neither contain K^4 nor $K_{2,3}$ as a minor by Theorem D.1.7, which contradicts that H is a subgraph of G_K.

Now let us assume (i) to prove the remaining implication. We set K_0 as an arbitrary connected subgraph of G with at least three vertices. Next we define $K_{i+1} = G[V(K_i) \cup N(K_i)]$ for every $i \geqslant 0$. Inside G we define the vertex sets $L_i = \{v \in V(K_i)\ ;\ N(v) \subseteq V(K_i)\}$ for every $i \geqslant 1$. Let then $\tilde{K}_{i+1} = G_{K_{i+1}} - L_i$ for every $i \geqslant 1$. By Lemma D.4.3 we know that G_{K_i} is 2-connected for each $i \geqslant 0$. Furthermore, G_{K_i} contains neither K^4 nor $K_{2,3}$ as a minor for every $i \geqslant 0$ since it would also be a minor of G contradicting our assumption. So each G_{K_i} is outerplanar by Theorem D.1.7. Using statement (ii) of Proposition D.1.6 we obtain that each G_{K_i} has a unique Hamilton cycle C_i and that there is an embedding σ_i of G_{K_i} into a fixed closed disk D such that C_i is mapped onto the boundary ∂D of D. Set $E_i = E(C_i) \cap E(K_i)$ for every $i \geqslant 1$.

Next we define an embedding of G into D and extend it to the desired embedding of $|G|$. We start by taking σ_1. Note again that G_{K_1} is a finite 2-connected outerplanar graph by Lemma D.4.3. Furthermore, $\sigma_1(C_1) = \partial D$. So we can use Lemma D.4.4 to obtain an embedding $\sigma_1^* : G_{K_1} \longrightarrow D$ as in the statement of that lemma. Because of Lemma D.4.2 we can extend $\sigma_1^* \restriction_{K_1}$ using $\sigma_2 \restriction_{\tilde{K}_2}$, maybe after rescaling the latter embedding, to obtain an embedding $\varphi_2 : G_{K_2} \longrightarrow D$ such that $\varphi_2(C_2) = \partial D$. We apply again Lemma D.4.4 with φ_2, which yields an embedding $\sigma_2^* : G_{K_2} \longrightarrow D$ as in the statement of that lemma. Note that this construction ensures $\sigma_2^* \restriction_{K_1} = \sigma_1^* \restriction_{K_1}$. Proceeding in the same way, we get an embedding $\sigma^* : G \longrightarrow D$ by setting $\sigma^* := \bigcup_{i \in \mathbb{N}} \sigma_i^* \restriction_{K_i}$. The use of Lemma D.4.4 in the construction of σ^* ensures that all edges are embedded as straight lines unless they are contained in any E_i. However, all edges in the sets E_i, and therefore also all vertices of G, are embedded into ∂D. Furthermore, we may assure that σ^* has the following property:

Let $(M_i)_{i \geqslant 1}$ be any infinite sequence of components M_i of $G - K_i$ where $M_{i+1} \subseteq M_i$. Also, let $\{u_i, w_i\}$ be the neighbourhood of M_i in G. Then the sequences $(\sigma^*(u_i))_{i \geqslant 1}$ and $(\sigma^*(w_i))_{i \geqslant 1}$ converge to a common point on ∂D. $(*)$

It remains to extend this embedding σ^* to an embedding $\overline{\sigma}^*$ of all of $|G|$ into D. First we shall extend the domain of σ^* to all of $|G|$. For this we need to prove the following claim.

Claim 1. *For every end ω of G there exists an infinite sequence $(M_i)_{i \geqslant 1}$ of components M_i of $G - K_i$ with $M_{i+1} \subseteq M_i$ such that $\bigcap_{i \geqslant 1} \overline{M_i} = \{\omega\}$.*

Since K_i is finite, there exists a unique component of $G - K_i$ in which all ω-rays have a tail. Set this component as M_i. It follows from the definition that ω lies in $\overline{M_i}$. Furthermore, we get that $\bigcap_{i \geqslant 1} \overline{M_i}$ does neither contain any vertex nor an inner point of any edge. So suppose for a contradiction that $\bigcap_{i \geqslant 1} \overline{M_i}$ contains another end $\omega' \neq \omega$. We know there exists a finite set S of vertices such that all tails of ω-rays lie in a different component of $G - S$ than all tails of ω'-rays. By definition of the graphs K_i we can find an index j such that $S \subseteq V(K_j)$. So ω lies in $\overline{M_j}$ and ω' in $\overline{M'_j}$ where M'_j is the component of $G - K_j$ in which all tails of ω'-rays lie. Since G is locally finite, the cut $E(M_j, K_j)$ is finite. Using Lemma D.2.4 we obtain that $\overline{M_j} \cap \overline{M'_j} = \emptyset$. Therefore, $\omega' \notin \overline{M_j} \supseteq \bigcap_{i \geqslant 1} \overline{M_i}$. This contradiction completes the proof of the claim.

Now let us define the map $\overline{\sigma}^*$. For every vertex or inner point of an edge x, we set $\overline{\sigma}^*(x) = \sigma^*(x)$. For an end ω let $(M_i)_{i \geqslant 1}$ be the sequence of components M_i of $G - K_i$ given by Claim 1 and $\{u_i, w_i\}$ be the neighbourhood of M_i in G. Using property $(*)$ we know that $(\sigma^*(u_i))_{i \geqslant 1}$ and $(\sigma^*(w_i))_{i \geqslant 1}$ converge to a common point p_ω on ∂D. We use this to set $\overline{\sigma}^*(\omega) = p_\omega$. This completes the definition of $\overline{\sigma}^*$.

Next we prove the continuity of $\overline{\sigma}^*$. For every vertex or inner point of an edge x, it is easy to see that an open set around $\overline{\sigma}^*(x)$ in D contains $\overline{\sigma}^*(U)$ for some open set U around x in $|G|$. This holds because G is locally finite and so it follows from the definition of $\overline{\sigma}^*$ using the embeddings σ_i^*. Let us check continuity for ends. Consider an open set O around $\overline{\sigma}^*(\omega)$ in D, where ω is an end of G. Let $B_\varepsilon(\overline{\sigma}^*(\omega))$ denote the restriction to D of an open ball around $\overline{\sigma}^*(\omega)$ with radius $\varepsilon > 0$. Then $B_\varepsilon(\overline{\sigma}^*(\omega))$ is an open set and, for sufficiently small ε, contained in O. We fix such an ε for the rest of this proof. Let $(M_i)_{i\geqslant 1}$ be a sequence as in Claim 1 for ω and $\{u_i, w_i\}$ be the neighbourhood of M_i in G. By property $(*)$ and the definition of $\overline{\sigma}^*$, we get that $(\sigma^*(u_i))_{i\geqslant 1}$ and $(\sigma^*(w_i))_{i\geqslant 1}$ converge to $\overline{\sigma}^*(\omega)$ on ∂D. So there exists a $j \in \mathbb{N}$ such that $B_\varepsilon(\overline{\sigma}^*(\omega))$ contains $\sigma^*(u_i)$ and $\sigma^*(w_i)$ for every $i \geqslant j$. By the definitions of $\overline{\sigma}^*$ and σ^* using the embeddings σ_i^*, it follows that $\overline{\sigma}^*(\overline{M_j}) \subsetneqq B_\varepsilon(\overline{\sigma}^*(\omega)) \subseteq O$. At this point we use the property of σ^* that every edge of G is embedded as a straight line unless it is embedded into ∂D. Hence, if $vw \in E(G)$ and $\overline{\sigma}^*(v), \overline{\sigma}^*(w) \in B_\varepsilon(\overline{\sigma}^*(\omega))$, then $\overline{\sigma}^*(vw)$ is also contained in $B_\varepsilon(\overline{\sigma}^*(\omega))$ by the convexity of the ball. Since $\overline{M_j}$ together with the inner points of the edges of $E(M_j, K_j)$ is a basic open set in $|G|$ containing ω whose image under $\overline{\sigma}^*$ is contained in O, continuity holds for ends too.

The next step is to check that $\overline{\sigma}^*$ is injective. If x and y are each either a vertex or an inner point of an edge, then they already lie in some K_j. By the definition of $\overline{\sigma}^*$ we get that $\overline{\sigma}^*(x) = \overline{\sigma}^*(y)$ if and only if there exists a $j \in \mathbb{N}$ such that x and y are mapped to the same point by the embedding of K_j defined by $\bigcup_{i=1}^{j} \sigma_i^* \restriction_{K_i}$. So x and y need to be equal.

For an and ω of G, let $(M_i)_{i\geqslant 1}$ be a sequence of components of $G - K_i$ such that $\bigcap_{i\geqslant 1} \overline{M_i} = \{\omega\}$, which exists by Claim 1. Let $\{u_i, w_i\}$ be the neighbourhood of M_i in G. Since G is locally finite, there exists an integer j such that y lies in K_j if it is a vertex or an inner point of an edge, or y lies in $\overline{M_j'}$ for some component $M_j' \neq M_j$ of $G - K_j$ if y is an end of G that is different from ω. By the definition of $\overline{\sigma}^*$ and property $(*)$ we get that the arc on ∂D between $\sigma^*(u_j)$ and $\sigma^*(w_j)$ into which the vertices of M_j are mapped contains also $\overline{\sigma}^*(\omega)$ but not y. Hence, $\overline{\sigma}^*(\omega) \neq \overline{\sigma}^*(y)$ if $\omega \neq y$. This shows the injectivity of the map $\overline{\sigma}^*$.

To see that the inverse function of $\overline{\sigma}^*$ is continuous, note that $|G|$ is compact by Proposition D.2.2 and D is Hausdorff. So Lemma D.2.3 immediately implies that the inverse function of $\overline{\sigma}^*$ is continuous. This completes the proof that $\overline{\sigma}$ is an embedding.

It remains to show the existence of a unique Hamilton circle of G that is mapped onto ∂D by $\overline{\sigma}$. For this we first prove that $\partial D \subseteq \mathrm{Im}(\overline{\sigma})$. This then implies that the inverse function of $\overline{\sigma}^*$ restricted to ∂D is a homeomorphism defining a Hamilton circle of G since it contains all vertices of G. We begin by proving the following claim.

Claim 2. *For every infinite sequence $(M_i)_{i\geqslant 1}$ of components M_i of $G - K_i$ with $M_{i+1} \subseteq M_i$ there exists an end ω of G such that $\bigcap_{i\geqslant 1} \overline{M_i} = \{\omega\}$.*

Let $(M_i)_{i\geqslant 1}$ be any sequence as in the statement of the claim. Since for every vertex v there exists a $j \in \mathbb{N}$ such that $v \in K_j$, we get that $\bigcap_{i\geqslant 1} \overline{M_i}$ is either empty or contains ends of G. Using that each M_i is connected and that $M_{i+1} \subseteq M_i$, we can find a ray R such that every M_i contains a tail of R. Therefore, $\bigcap_{i\geqslant 1} \overline{M_i}$ contains the end in which R lies. The argument that $\bigcap_{i\geqslant 1} \overline{M_i}$ contains at most one end is the same as in the proof of Claim 1. This completes the proof of Claim 2.

Suppose a point $p \in \partial D$ does not already lie in $\mathrm{Im}(\sigma^*)$. Then it does not lie in $\mathrm{Im}(\sigma_i^* \restriction_{K_i})$ for any $i \geqslant 1$. So there exists an infinite sequence $(M_i)_{i\geqslant 1}$ of components M_i of $G - K_i$ with $M_{i+1} \subseteq M_i$ such that p lies in the arc A_i of ∂D between $\sigma^*(u_i)$ and $\sigma^*(w_i)$ into which the vertices of M_i are mapped, where $\{u_i, w_i\}$ denotes the neighbourhood of M_i in G. Using Claim 2 we obtain that there exists an end ω of G such that $\bigcap_{i\geqslant 1} \overline{M_i} = \{\omega\}$. By property $(*)$ of the map σ^* the sequences $(\sigma^*(u_i))_{i\geqslant 1}$ and $(\sigma^*(w_i))_{i\geqslant 1}$ converge to a common point on ∂D. This point must be p since the arcs A_i are nested. Now the definition of $\overline{\sigma}^*$ tells us that $\overline{\sigma}^*(\omega) = p$. Hence $\partial D \subseteq \mathrm{Im}(\overline{\sigma}^*)$ and G is Hamiltonian.

We finish the proof by showing the uniqueness of the Hamilton circle of G. Suppose for a contradiction that G has two subgraphs C_1 and C_2 yielding different Hamilton circles $\overline{C_1}$ and $\overline{C_2}$. Then there must be an edge $e \in E(C_1)\backslash E(C_2)$. Let $j \in \mathbb{N}$ be chosen such that $e \in E(K_j)$. By Lemma D.4.2 we obtain that $G_{K_j}[E(C_1) \cap E(G_{K_j})]$ and $G_{K_j}[E(C_2) \cap E(G_{K_j})]$ are two Hamilton cycles of G_{K_j} differing in the edge e. Note that G_{K_j} is a finite 2-connected outerplanar graph. The argument for this is the same as for G_K in the proof that (ii) implies (i). This yields a contradiction since G_{K_j} has a unique Hamilton cycle by statement (ii) of Proposition D.1.6. $\square$

Next we deduce Corollary D.1.9. Let us recall its statement first.

Corollary D.1.9. *The edges contained in the Hamilton circle of a locally finite 2-connected graph not containing K^4 or $K_{2,3}$ as a minor are precisely the 2-contractible edges of the graph unless the graph is isomorphic to a K^3.*

Proof. Let G be a locally finite 2-connected graph not isomorphic to a K^3 and not containing K^4 or $K_{2,3}$ as a minor. Further, let C be the subgraph of G such that $\overline{C}$ is the Hamilton circle of G. First we show that each edge $e \in E(C)$ is a 2-contractible edge. Note for this that the closure of the subgraph of G/e formed by the edge set $E(C) \backslash \{e\}$ is a Hamilton circle in $|G/e|$. Hence, G/e is 2-connected by Corollary D.2.9.

It remains to verify that no edge of $E(G) \backslash E(C)$ is 2-contractible. For this we consider any edge $e = uv \in E(G) \backslash E(C)$. Let K be a finite connected induced subgraph of G containing at least four vertices as well as $N(u) \cup N(v)$, which is a finite set since G is locally finite. Then we know by Lemma D.4.3 and by using the locally finiteness of G again that G_K is a finite 2-connected graph not containing K^4 or $K_{2,3}$ as a minor. So by Theorem D.1.7 and Proposition D.1.6 we get that G_K has a unique Hamilton cycle consisting precisely of its 2-contractible edges. However, as we have seen in the proof of Theorem D.1.8, $G_K[E(C) \cap E(G_K)]$ is the unique Hamilton cycle of G_K and does not contain e. Since G_K is outerplanar, we get that the vertex of G_K/e corresponding to the edge e is a cut vertex in G_K/e. By our choice of K containing $N(u) \cup N(v)$, we get that the vertex in G/e corresponding to the edge e is a cut vertex of G/e too. So e is not 2-contractible. □

The question arises whether one could prove the more complicated part of Theorem D.1.8, the implication $(i) \implies (ii)$, by mimicking a proof for finite graphs. To see the positive answer for this question, let us summarize the proof for finite graphs except the part about the uniqueness.

By Theorem D.1.7 every finite graph without K^4 or $K_{2,3}$ as a minor can be embedded into the plane such that all vertices lie on a common face boundary. Since every face of an embedded 2-connected graph is bounded by a cycle, we obtain the desired Hamilton cycle.

So for our purpose we would first need to prove a version of Theorem D.1.7 for $|G|$ where G is a locally finite connected graph. This can similarly be done in the way we have defined the embedding for the Hamilton circle in Theorem D.1.8 by decomposing the graph into finite parts using Lemma D.4.2. Since none of these parts contains a K^4 or a $K_{2,3}$ as a minor, we can fix appropriate embeddings of them and stick them together. However, in order to obtain an embedding of $|G|$ we have to be careful. We also need to ensure that the embeddings of finite parts that converge to an end in $|G|$ also converge to a point in the plane where we can map the corresponding end to.

The second ingredient of the proof is the following lemma pointed out by Bruhn and Stein, but which is a corollary of a stronger and more general result of Richter and Thomassen [52, Prop. 3].

Lemma D.4.5. [6, Cor. 21] *Let G be a locally finite 2-connected graph with an embedding $\varphi : |G| \longrightarrow S^2$. Then the face boundaries of $\varphi(|G|)$ are circles of $|G|$.*

These observations show that the proof idea for finite graphs is still applicable for locally finite graphs.

Let us compare the proof for the implication $(i) \implies (ii)$ of Theorem D.1.8 that we sketched right above, with the one we outlined completely. The two proofs share a big similarity. Both need to show first that $|G|$ can be embedded into the plane such that all vertices lie on a common face boundary if G is a connected or 2-connected, respectively, locally finite graph without K^4 or $K_{2,3}$ as a minor. At this point the proof we outlined completely already incorporates further properties into the embedding without too much additional effort. Especially, we use the 2-connectedness of the graph there by finding suitable finite 2-connected contraction minors. Then we apply Proposition D.1.6 for these. The embeddings we obtain for the contraction minors allow us to define an embedding of $|G|$ into a fixed closed disk. Furthermore, this embedding of $|G|$ has the additional property that its restriction onto the boundary of the disk directly witnesses the existence of a Hamilton circle. The second proof, however, takes a step backward and argues more general. There the 2-connectedness of G is used to apply Lemma D.4.5, which, as noted before, is a corollary of a more general result of Richter and Thomassen [52, Prop. 3]. At this point we forget about the special embedding of $|G|$ into the plane that we had to construct before. We continue the argument with an arbitrary one given that G is a 2-connected locally finite graph. So for the purpose of proving the implication $(i) \implies (ii)$ of Theorem D.1.8, the outlined proof is more straightforward and self-contained.

D.5. A cubic infinite graph with a unique Hamilton circle

This section is dedicated to Theorem D.1.12. We shall construct an infinite graph with a unique Hamilton circle where all vertices in the graph have degree 3. Furthermore, all ends of that graph have vertex-degree 3 as well as edge-degree 3. The main ingredient in our construction is the finite graph T depicted in Figure D.4. This graph has three distinguished vertices of degree 1, which we denote by u, l and r as in Figure D.4. For us, the important feature of T is that we know where all *Hamilton paths*, i.e., spanning paths, of $T - u$ and $T - r$ proceed. Tutte [66] came up with the graph T to construct a counterexample to Tait's conjecture [59], which said that every 3-connected cubic planar graph is Hamiltonian. The crucial observation of Tutte in [66] was that $T - u$ does not contain a Hamilton path. We shall use this observation as well, but we need more facts about T, which are covered in the following lemma. The proof is straightforward, but involves several cases that need to be distinguished.

Lemma D.5.1. *There is no Hamilton path in $T - u$, but there are precisely two in $T - r$ (see Figure D.4).*

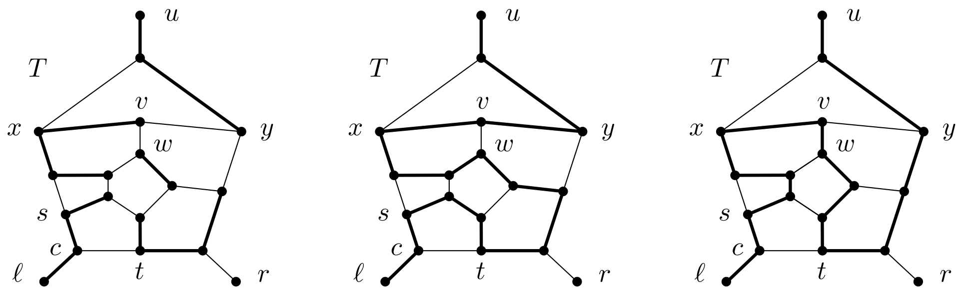

Figure D.4.: The fat edges in the most left picture are in every Hamilton path of $T - r$. The fat edges in the other two pictures mark the two Hamilton paths of $T - r$.

Proof. As mentioned already by Tutte [66], the graph $T - u$ does not have a Hamilton path. It remains to show that $T - r$ has precisely two Hamilton paths. For this we need to check several cases, but afterwards we can precisely state the Hamilton paths. For convenience, we label each edge with a number as depicted in Figure D.5 and refer to the edges just by their labels for the rest of the proof.

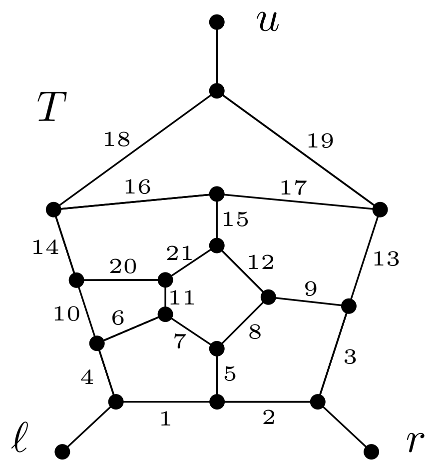

Figure D.5.: Our fixed labelling of the relevant edges of T.

Obviously, the edges incident with ℓ and u would need to be in every Hamilton path of $T - r$ since these vertices have degree 1. Furthermore, the edges 2 and 3 need to be in every Hamilton path of $T - r$ since the vertex incident with 2 and 3 has degree 2 in $T - r$.

Claim 1. *The edge* 4 *needs to be in every Hamilton path of* $T - r$.

Suppose for a contradiction that there is a Hamilton path P in $T - r$ that does not use 4. Then it needs to contain 1. Since it also contains 2, we know $5 \notin E(P)$. This implies further that $7, 8 \in E(P)$. We can use $4 \notin E(P)$ also to deduce that $6, 10 \in E(P)$ holds. Now we get $11 \notin E(P)$ since $6, 7 \in E(P)$. This implies $20, 21 \in E(P)$. But now $14 \notin E(P)$ holds because $10, 20 \in E(P)$. From this we get then $16, 18 \in E(P)$. So 19 cannot be contained in P, which implies $13, 17 \in E(P)$. Now we arrived at a contradiction since the edges incident with l and u together with the edges of the set $\{1, 2, 3, 13, 17, 16, 18\}$ form a ℓ-u path in $T - r$ that is contained in P and needs therefore to be equal to P. Then, however, P would not be a Hamilton path $T - r$. This completes the proof of the Claim 1.

We immediately get from Claim 1 that 5 needs to be in every Hamilton path of $T - r$ and since 8 and 9 can not both be contained in any Hamilton path of $T - r$, because they would close a cycle together with $5, 2$ and 3, we also know that 12 needs to be in every Hamilton path of $T - r$.

Claim 2. *The edges* 14 *and* 16 *lie in every Hamilton path of* $T - r$.

Suppose for a contradiction that the claim is not true. Then there is a Hamilton path P of $T - r$ containing 18. So P cannot contain 19, which implies $13, 17 \in E(P)$. Since $3, 13 \in E(P)$, we obtain $9 \notin E(P)$, from which we follow that $8 \in P$ holds. Furthermore, 15 cannot be contained in P, because then the edges $15, 17, 13, 3, 2, 5, 8, 12$ would form a cycle in P. Therefore, 16 is an edge of P. From $5, 8 \in E(P)$ we can deduce that $7 \notin E(P)$ holds. So 6 and 11 are edges of P, which that implies $10 \notin E(P)$. Then $14, 20 \in E(P)$ needs to be true. Now, however, we have a contradiction, because P would have a vertex incident with three vertices, namely $14, 16$ and 18. This completes the proof of Claim 2

It follows from Claim 2 that 19 is contained in every Hamilton path of $T - r$. We continue with another claim.

Claim 3. *The edges* 6 *and* 20 *lie in every Hamilton path of* $T - r$.

Suppose for a contradiction that the claim is not true. Then there is a Hamilton path P of $T - r$ containing 10. This immediately implies that $6 \notin E(P)$, yielding $7, 11 \in E(P)$, and $20 \notin E(P)$, yielding $21 \in E(P)$. We note that 8 cannot be an edge of P since P would then contain a cycle spanned by the edge set $\{8, 7, 11, 21, 12\}$. Therefore, $9 \in E(P)$ must hold. Here we arrive at a contradiction, since P now contains a cycle spanned by the edge set $\{9, 3, 2, 5, 7, 11, 21, 12\}$. This completes the proof of Claim 3

Using all the observations we have made so far, we can now show that $T - r$ has precisely two Hamilton paths and state them by looking at the edge 11. Assume that 11 is contained in a Hamilton path P_1 of $T - r$. Then $7, 21 \notin E(P_1)$ follows, because $6, 20 \in E(P_1)$ holds by Claim 3. Since we could deduce from Claim 1 that $5, 12 \in E(P_1)$ holds, we get furthermore $8, 15 \in E(P_1)$. This now yields $9, 17 \notin E(P_1)$ and, therefore, $13 \in E(P_1)$. As we can see, the assumption that 11 is contained in a Hamilton path P_1 of $T - r$ is true. Also, P_1 is uniquely determined with respect to this property and consists of the fat edges in the most right picture of Figure D.4.

Next assume that there is a Hamilton path P_2 of $T - r$ that does not contain the edge 11. Then 7 and 21 have to be edges of P_2. Using again that $5, 12 \in E(P_2)$ holds, we deduce $8, 15 \notin E(P_2)$. Then, however, we get $9, 17 \in E(P)$ and have already uniquely determined P_2, which corresponds to the fat edges in the middle picture of Figure D.4. $\square$

Using Lemma D.5.1 we shall now prove Theorem D.1.12 by constructing a prescribed graph. During the construction we shall often refer to certain distinguished vertices of T that are named as depicted in Figure D.4. Let us recall the statement of the theorem.

Theorem D.1.12. *There exists an infinite connected graph G with a unique Hamilton circle that has degree 3 at every vertex and vertex- as well as edge-degree 3 at every end.*

Proof. We construct a sequence of graphs $(G_n)_{n\in\mathbb{N}}$ inductively and obtain the desired one G as a limit of the sequence. We start with $G_0 = T_0^1 = T$.

Now suppose we have already constructed G_n for $n \geqslant 0$. Furthermore, let $\{T_n^i\ ;\ 1 \leqslant i \leqslant 2^n\}$ be a specified set of disjoint subgraphs of G_n each of which each is isomorphic to T. We define G_{n+1} as follows. Take G_n and two copies T_c and T_v of T for each $T_n^i \subseteq G_n$. Then identify for every i the vertices of T_c that correspond to u, ℓ and r, respectively, with the vertices of the related $T_n^i \subseteq G_n$ corresponding to ℓ, s and t, respectively. Also identify for every i the vertices of T_v corresponding to u, ℓ and r, respectively, with the ones of the related $T_n^i \subseteq G_n$ corresponding to w, x and y, respectively. Finally, delete in each $T_n^i \subseteq G_n$ the vertices corresponding to c and v, see Figure D.6. This completes the definition of G_{n+1}. It remains to fix the set of 2^{n+1} many disjoint copies of T that occur as disjoint subgraphs in G_{n+1}. For this we take the set of all copies T_c and T_v of T that we have inserted in the subgraphs T_n^i of G_n.

Using the graphs G_n we define a graph $\hat{G}$ as a limit of them. We set

$$\hat{G} = G[\hat{E}] \text{ where } \hat{E} = \left\{ e \in \bigcup_{n\in\mathbb{N}} E(G_n)\ ;\ \exists N \in \mathbb{N} : e \in \bigcap_{n\geqslant N} E(G_n) \right\}.$$

Note that an edge $e \in E(G_n)$ is an element of $\hat{E}$ if and only if it was not deleted during the construction of G_{n+1} as an edge incident with one of the vertices that correspond to c or v in T_n^i for some i. Finally, we define G as the graph obtained from $\hat{G}$ by identifying the three vertices that correspond to u, ℓ and r of T_0^1.

Next let us verify that every vertex of G has degree 3 and that every end of G has vertex- as well as edge-degree 3 in G. Since every vertex of T except u, ℓ

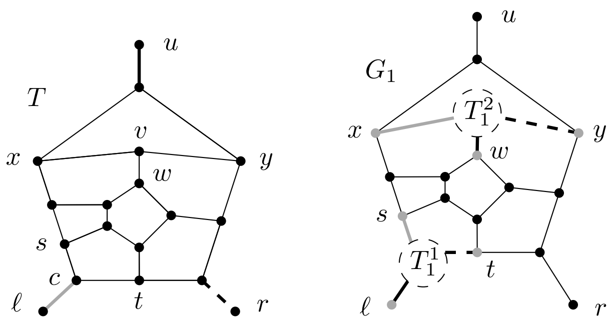

Figure D.6.: A sketch of the construction of G_1. The fat black, grey and dashed edges incident with the grey vertices in the right picture correspond to the ones in the left picture.

and r has degree 3, the construction ensures that every vertex of G has degree 3 too. In order to analyse the end degrees, we have to make some observations first. The edges of G that are adjacent to vertices corresponding to u, ℓ and r of any T_n^i define a cut $E(A_n^i, B_n^i)$ of G. Note that for any finite cut of a graph all rays in one end of the graph have tails that lie completely on one side of the cut. Therefore, the construction of G ensures that for every end ω of G there exists a function $f : \mathbb{N} \longrightarrow \mathbb{N}$ with $f(n) \in \{1, \ldots, 2^n\}$ such that all rays in ω have tails in $B_n^{f(n)}$ for each $n \in \mathbb{N}$ and $B_n^{f(n)} \supseteq B_{n+1}^{f(n+1)}$ with $\bigcap_{n \in \mathbb{N}} B_n^{f(n)} = \emptyset$. Using that $|E(A_n^i, B_n^i)| = 3$ for every n and i, this implies that every end of G has edge-degree at most 3. Since there are three disjoint paths from $\{u, \ell, r\}$ to $\{s, \ell, t\}$ as well as to $\{x, w, y\}$ in T, we can also easily construct three disjoint rays along the cuts $E(A_n^i, B_n^i)$ that belong to an arbitrary chosen end of G. So every end of G has vertex-degree 3. In total this yields that every end of G has vertex- as well as edge-degree 3 in G.

It remains to prove that G has precisely one Hamilton circle. We begin by stating the edge set of the subgraph C defining the Hamilton circle $\overline{C}$ of G. Let $E(C)$ consist of those edges of $E(G) \cap T_n^i$ for every n and i that correspond to the fat edges of T in the most right picture of Figure D.4. Now consider any finite cut D of G. The construction of G yields that there exists an $N \in \mathbb{N}$ such that D is already a cut of the graph obtained from G_n by identifying the vertices corresponding to u, ℓ and r of $T_0^1 \subseteq G_n$ for all $n \geqslant N$. Using this observation we can easily see that every vertex of G has degree 2 in $\overline{C}$. We also obtain that every finite cut is met at least twice, but always in an even number of edges of C. By

Lemma D.2.6 we get that $\overline{C}$ is topologically and also arc-connected. Therefore, every end of G has edge-degree at least 1 and at most 3 in $\overline{C}$. Together with Theorem D.2.7 this implies that every end of G has edge-degree 2 in $\overline{C}$. Hence, Lemma D.2.8 tells us that $\overline{C}$ is a circle, which is Hamiltonian since it contains all vertices of G.

We finish the proof by showing that $\overline{C}$ is the unique Hamilton circle of G. Since any Hamilton circle $\overline{H}$ of G meets each cut $E(A_n^i, B_n^i)$ precisely twice, $\overline{H}$ induces a path through T that contains all vertices of T except one out of the set $\{u, \ell, r\}$. By Lemma D.5.1 we know that such paths must contain the edge adjacent to u. Let us consider any T_n^i in G_n. Now let T_{n+1}^j be the copy of T whose vertices of degree 1 we have identified with the vertices corresponding to the neighbours of c in T_n^i during the construction of G_{n+1}. The way we have identified the vertices implies that the path induced by $\overline{H}$ through T_n^i must also use the edge adjacent to ℓ since the induced path in T_{n+1}^j must use the edge adjacent to u. With a similar argument we obtain that the induced path inside T_n^i must use the edge corresponding to vw. We know from Lemma D.5.1 that there is a unique Hamilton path in $T - r$ that uses the edges ℓc and vw, namely the one corresponding to the fat edges in the most right picture of Figure D.4. So the edges which must be contained in every Hamilton circle are precisely those of C. $\square$

Remark D.5.2. After reading a preprint covering the content of Section D Max Pitz [51] carried further some ideas of Section D. Also using the graph T, he recently constructed a two-ended cubic graph with a unique Hamilton circle where both ends have vertex- as well as edge-degree 3. He further proved that every one-ended Hamiltonian cubic graph whose end has edge-degree 3 (or vertex-degree 3) admits a second Hamilton circle.

Chapter III.

Directed infinite graphs

E. An analogue of Edmonds' Branching Theorem for infinite digraphs

E.1. Introduction

Studying how to force spanning structures in finite graphs is a basic task. The most fundamental spanning structure is a spanning tree, whose existence is already characterised by the connectedness of the graph. Moving on and characterising the existence of a given number of edge-disjoint spanning trees via an immediately necessary condition, Nash-Williams [48] and Tutte [67] independently proved the following famous theorem.

Theorem E.1.1. [48, 67], [12, Thm. 2.4.1] *A finite multigraph G has $k \in \mathbb{N}$ edge-disjoint spanning trees if and only if for every partition $\mathcal{P}$ of $V(G)$ there are at least $k(|\mathcal{P}| - 1)$ edges in G whose endvertices lie in different partition classes.*

Later, Edmonds [21] generalised Theorem E.1.1 to finite digraphs, also involving a condition which is immediately seen to be necessary for the existence of the spanning structures. In his theorem, Edmonds considers as spanning structures *out-arborescences* rooted in a vertex r, i.e., spanning trees whose edges are directed away from the root r. His theorem immediately implies a corresponding result for *in-arborescences* rooted in r, i.e., spanning trees directed towards r, via reversing every edge in the digraph. For this reason we shall focus in Section E only on out-arborescences and denote them just by *arborescences*.

Theorem E.1.2. [21], [3, Thm. 9.5.1] *A finite digraph G with a vertex $r \in V(G)$ has $k \in \mathbb{N}$ edge-disjoint spanning arborescences rooted in r if and only if there are k edges from X to Y for every bipartition (X, Y) of $V(G)$ with $r \in X$.*

One of the main results of Section E is to extend Theorem E.1.2 to a certain class of infinite digraphs. There has already been work in this area. In order to mention two important results about this let us call a one-way infinite path all whose edges are directed away from the unique vertex incident with only one edge a *forward directed ray*. Similarly, we call the digraph obtained by reversing all edges of a forward directed ray a *backwards directed ray*. Thomassen [64] extended Theorem E.1.2 to infinite digraphs that do not contain a backwards directed ray, while Joó [40] obtained an extension for infinite digraphs without forward directed rays using different methods. In contrast to these two results we shall demand a local property for our digraphs by considering *locally finite digraphs*, i.e., digraphs where every vertex has finite in- and out-degree. Similarly, undirected multigraphs are called *locally finite* if every vertex has finite degree.

When trying to extend Theorem E.1.2 to infinite digraphs it is important to know that a complete extension is not possible. The reason for this is that Oxley [50, Ex. 2] constructed a locally finite graph without two edge-disjoint spanning trees but fulfilling the necessary condition in Theorem E.1.1. Following up, Aharoni and Thomassen [2, Thm.] gave a construction for further counterexamples to Theorem E.1.2, which are all locally finite and can even be made $2k$-edge-connected for arbitrary $k \in \mathbb{N}$. Hence, using ordinary spanning trees for an extension of Theorem E.1.1 to locally finite graphs does not work. This immediately implies that extending Theorem E.1.2 to locally finite digraphs fails as well if ordinary arborescences are used. While Thomassen and Joó could overcome this problem by forbidding certain one-way infinite paths, for us it is necessary to additionally change the notion of arborescence since the counterexamples to direct extensions of Theorem E.1.1 and Theorem E.1.2 to infinite (di)graphs are locally finite.

For undirected locally finite (connected) multigraphs G the problem of how to extend Theorem E.1.1 has successfully been overcome. The key was to not just consider G but the Freudenthal compactification $|G|$ [12, 13] of the 1-complex of G. Instead of ordinary spanning trees, now packings of topological spanning trees of G are considered. We call a topologically connected subspace of $|G|$ which is the closure of a set of edges of G, contains all vertices of G but contains no homeomorphic image of the unit circle $S^1 \subseteq \mathbb{R}^2$, a *topological spanning tree* of G. There is an equivalent but more combinatorial, and especially finitary, way of defining topological spanning trees of G. They are precisely the closures in $|G|$ of the minimal edge sets that meet every finite cut of G [12]. As already observed by

Tutte, this finitary condition can be used to obtain the following packing theorem for disjoint edge sets each meeting every finite cut, via the compactness principle.

Theorem E.1.3. [67] *A locally finite multigraph G has $k \in \mathbb{N}$ disjoint edge sets each meeting every finite cut of G if and only if for every finite partition $\mathcal{P}$ of $V(G)$ there are at least $k(|\mathcal{P}| - 1)$ edges in G whose endvertices lie in different partition classes.*

By the equivalence noted above, Theorem E.1.3 implies a packing result for topological spanning trees:

Theorem E.1.4. [12, Thm. 8.5.7] *A locally finite multigraph G has $k \in \mathbb{N}$ edge-disjoint topological spanning trees if and only if for every finite partition $\mathcal{P}$ of $V(G)$ there are at least $k(|\mathcal{P}| - 1)$ edges in G whose endvertices lie in different partition classes.*

In the spirit of Tutte's approach, we prove the following packing theorem generalising Theorem E.1.2 to locally finite digraphs for what we call spanning pseudo-arborescences rooted in some vertex r. For a locally finite weakly connected digraph G and $r \in V(G)$ we define a *spanning pseudo-arborescences rooted in r* as a minimal edge set $F \subseteq E(G)$ such that F contains an edge directed from X to Y for every bipartition (X, Y) of $V(G)$ with $r \in X$ and finitely many edges between X and Y in either direction.

Theorem E.1.5. *A locally finite weakly connected digraph G with $r \in V(G)$ has $k \in \mathbb{N}$ edge-disjoint spanning pseudo-arborescences rooted in r if and only if there are k edges from X to Y for every bipartition (X, Y) of $V(G)$ with $r \in X$ and finitely many edges between X and Y in either direction.*

In fact we shall prove a slightly stronger version of this theorem, Theorem E.4.3, which requires more notation.

While minimal edges sets meeting every finite cut in an undirected multigraph turn out to be topological extensions of finite trees, there is no analogous topological interpretation of spanning pseudo-arborescences on terms of the Freudenthal compactification of the underlying multigraph. In Section E.5 we give an example of a digraph G with underlying multigraph H where the closure in $|H|$ of the underlying undirected edges of any spanning pseudo-arborescence of G contains a homeomorphic image of S^1. We shall be able to extend to pseudo-arborescences, in a suitable topological setting, the property of finite arborescences of being edge-minimal such that each vertex is still reachable by a directed path from the root. While in finite arborescences such directed paths are unique, however, their analogues in pseudo-arborescences are not in general unique. This will be illustrated by an example given in Section E.5.

Finally, we prove the following structural characterisation for spanning pseudo-arborescences.

Theorem E.1.6. *Let G be a locally finite weakly connected digraph and $r \in V(G)$. Then the following statements are equivalent for an edge set $F \subseteq E(G)$ containing an edge from X to Y for every bipartition (X, Y) of $V(G)$ with $r \in X$ and finitely many edges between X and Y in either direction.*

(i) *F is a spanning pseudo-arborescences rooted in r.*

(ii) *For every vertex $v \neq r$ of G there is a unique edge in F whose head is v, and no edge in F has r as its head.*

(iii) *For every weak component T of $G[F]$ the following holds: If $r \in V(T)$, then T is an arborescence rooted in r. Otherwise, the underlying multigraph of T is a tree, T contains a backwards directed ray and all other edges of T are directed away from that ray.*

We prove a slightly more general version of Theorem E.1.6 in Section E.5 (cf. Theorem E.5.3).

The structure of Section E is as follows. In Section E.2 we give basic definitions and fix our notation for directed and undirected (multi)graphs. We especially refer to the topology we consider on locally finite (weakly) connected digraphs and (undirected) multigraphs, and state some basic lemmas that we shall need for our main results of Section E. In Section E.3 we extend some lemmas about directed walks and paths in finite digraphs to locally finite (weakly) connected digraphs. Section E.4 is dedicated to the proof of Theorem E.1.5. Section E is completed with Section E.5 containing the proof of Theorem E.1.6 and a discussion about how much pseudo-arborescences resemble finite arborescences or topological trees.

E.2. Preliminaries

For basic facts about finite and infinite graphs we refer the reader to [12]. As a source especially for facts about directed graphs we refer to [3].

Throughout all of Section E we shall often write $G = (V, E)$ for a digraph. Then $V(G)$ will denote its vertex set V and $E(G)$ its set of directed edges E. As for undirected graphs, we shall call the elements of $E(G)$ just edges. In general, we allow our digraphs to have parallel edges, but no loops. We view the edges of a digraph G as ordered pairs (a, b) of vertices $a, b \in V(G)$ and shall write ab instead of (a, b), although this might not uniquely determine an edge. For an edge $ab \in E(G)$ we furthermore denote the vertex a as the *tail* of ab and b as the *head* of ab.

For two disjoint vertex sets X, Y of a digraph G we denote by $E(X, Y)$ the set of all edges of G having not both, their head and their tail, in just one of the sets X and Y. By $\overrightarrow{E}(X, Y)$ we denote the set of edges of G that have their tail in X and their head in Y. For a multigraph or digraph G we call the edge set $E(X, Y)$ a *cut* if (X, Y) is a bipartition of $V(G)$. If we introduce a cut $E(X, Y)$, then we implicitly want (X, Y) to be the corresponding bipartition of $V(G)$ defining the cut. For a vertex set $X \subseteq V(G)$ we set $d^+(X) = |\overrightarrow{E}(X, V(G) \backslash X)|$ and $d^-(X) = |\overrightarrow{E}(V(G) \backslash X, X)|$. If $X = \{v\}$ for some vertex $v \in V(G)$, we write $d^+(v)$ instead of $d^+(\{v\})$ and call it the *out-degree* of v. Similarly, we write $d^-(v)$ instead of $d^-(\{v\})$ and call it the *in-degree* of v.

For a finite non-trivial directed path P we call the vertex of out-degree 1 and in-degree 0 in P the *start vertex of* P. Similarly, the vertex of in-degree 1 and out-degree 0 in P the *endvertex of* P. If P consists only of a single vertex, we call that vertex the *endvertex* of P.

We define a finite directed walk as a tuple $(\mathcal{W}, <_{\mathcal{W}})$ with the following properties:

1. $\mathcal{W}$ is a weakly connected graph with at least one vertex on the edge set $E(\mathcal{W}) = \{e_1, e_2, \ldots, e_n\}$ for some $n \in \mathbb{N}$ such that the head of e_{i-1} is the tail of e_i for every $i \in \mathbb{N}$ satisfying $1 \leqslant i \leqslant n$.

2. $<_{\mathcal{W}}$ is a linear order on $E(\mathcal{W})$ stating that $e_i <_{\mathcal{W}} e_j$ if and only if $i < j$ for all $i, j \in \{1, \ldots, n\}$.

We call a directed walk without edges *trivial* and call its unique vertex its *endvertex*.

Otherwise, we call the tail of e_1 the *start vertex* of $(\mathcal{W}, <_{\mathcal{W}})$ and the tail of e_n the *endvertex* of $(\mathcal{W}, <_{\mathcal{W}})$. If the start vertex and the endvertex of finite directed walk are equal, we call it *closed*. Lastly, we call $(\mathcal{W}, <_{\mathcal{W}})$ a finite directed *s*–*t* walk for two vertices $s, t \in V(\mathcal{W})$ if s is the start vertex of $(\mathcal{W}, <_{\mathcal{W}})$ and t is the endvertex of $(\mathcal{W}, <_{\mathcal{W}})$. We might call a finite graph $\mathcal{W}$ a finite directed walk and implicitly assume that there exists a linear order $<_{\mathcal{W}}$, which we then also fix, such that $(\mathcal{W}, <_{\mathcal{W}})$ is a finite directed walk. Especially, we will say that a finite directed walk $(\mathcal{W}, <_{\mathcal{W}})$ is contained in a graph G' if $\mathcal{W}$ is a subgraph of G'. Note that directed paths are directed walks when equipped with the obviously suitable linear order.

We define a *ray* to be an undirected one-way infinite path. Any subgraph of a ray R that is itself a ray is called a *tail* of R.

We call a weakly connected digraph R a *backwards directed ray* if there is a unique vertex $v \in V(R)$ with $d^-(v) = 1$ and $d^+(v) = 0$ while $d^-(w) = d^+(w) = 1$ holds for every other vertex $w \in V(R) \setminus \{v\}$. A *forward directed ray* is analogously defined by interchanging d^- and d^+.

For an undirected multigraph G we define an equivalence relation on the set of all rays in G. We call two rays in G *equivalent* if they cannot be separated by finitely many vertices in G. An equivalence class with respect to this relation is called an *end* of G. We denote the set of all ends of G by $\Omega(G)$. We define the *ends of a digraph D* precisely as the ends of its underlying multigraph. The set of all ends of D is also denoted by $\Omega(D)$. We say that a backwards directed ray R of D is *contained* in some end $\omega \in \Omega(D)$ if the underlying ray of R is contained in the end ω of the underlying multigraph of D.

We call a digraph A an *out-arborescence rooted in r* if $r \in V(A) \cup \Omega(A)$ and the underlying multigraph of A is a tree such that $d^-(v) = 1$ holds for every vertex $v \in V(A) \setminus \{r\}$ and additionally $d^-(r) = 0$ in the case that $r \in V(A)$, while we demand that r contains a backwards directed ray if $r \in \Omega(A)$.

Note that if $r \in V(A)$, then A does not contain a backwards directed ray. In the case where $r \in \Omega(A)$, then r is the unique end of A containing a backwards directed ray, since a second one would yield a vertex with in-degree bigger than 1 by using that the underlying multigraph of A is a tree. Also note that if A is a finite digraph, the condition $d^-(r) = 0$ for $r \in V(A)$ in the definition of an out-arborescence rooted in r is redundant, because it is implied by the tree structure of A.

Similarly, an *in-arborescence rooted in r* is defined with d^- replaced by d^+. Corresponding results about in-arborescences are immediate by reversing the orientations of all edges. For both types of arborescences we call r the *root* of the arborescence. In Section E we shall only work with out-arborescences. Hence, we shall drop the prefix 'out' and just write arborescence from now on.

A multigraph is called *locally finite* if each vertex has finite degree. We further call a digraph *locally finite* if its underlying multigraph is locally finite.

For a vertex set X in a locally finite connected multigraph G we define its *combinatorial closure* $\overline{X} \subseteq V(G) \cup \Omega(G)$ as the set X together with all ends of G that contain a ray which we cannot separate from X by finitely many vertices. Note that for a finite cut $E(X, Y)$ of G we obtain that $(\overline{X}, \overline{Y})$ is a bipartition of $V(G) \cup \Omega(G)$, because every end in $\overline{X}$ can be separated from Y by the finitely many vertices of X that are incident with edges of $E(X, Y)$, and, furthermore, each ray contains a subray that is either completely contained in X or in Y since $E(X, Y)$ is finite. The *combinatorial closure* of a vertex set in a digraph is just defined as the combinatorial closure of that set in the underlying undirected multigraph.

Let G be a locally finite digraph and $Z \subseteq V(G)\backslash\{r\}$ where $r \in V(G) \cup \Omega(G)$. An edge set $F \subseteq E(G)$ is called *r-reachable for Z* if $|F \cap \overrightarrow{E}(X, Y)| \geqslant 1$ holds for every finite cut $E(X, Y)$ of G where $r \in \overline{X}$ and $Y \cap Z \neq \varnothing$. Furthermore, if F is an r-reachable set for $Z = V(G)\backslash\{r\}$, we call F a *spanning r-reachable set.* We continue with a very basic remark about spanning r-reachable sets.

Remark E.2.1. *Let G be a locally finite digraph with a spanning r-reachable set F where $r \in V(G) \cup \Omega(G)$. Then $|F \cap \overrightarrow{E}(V(G)\backslash M, M)| \geqslant 1$ holds for every non-empty finite set $M \subseteq V(G)$ with $r \notin M$.*

Proof. Since G is locally finite and M is finite, we know that the cut $E(V(G)\backslash M, M)$ is finite. The assumption $r \notin M$ ensures that $r \in \overline{V(G)\backslash M}$. Using that F is a spanning r-reachable set and that M, as a non-empty set, contains a vertex different from r, we get the desired inequality $|F \cap \overrightarrow{E}(V(G)\backslash M, M)| \geqslant 1$ by the definition of spanning r-reachable sets. $\square$

Note that for a locally finite digraph G with a spanning r-reachable set F the digraph $G[F]$ is spanning. This follows by applying Remark E.2.1 to the set $M = \{v\}$ for every vertex $v \in V(G)$. Furthermore, note that if G is finite, the subgraph induced by a spanning r-reachable set contains a spanning arborescence rooted in $r \in V(G)$.

We conclude this section with a last definition. We call an inclusion-wise minimal r-reachable set F for a set $Z \subseteq V(G)\backslash\{r\}$ a *pseudo-arborescence for Z rooted in r*. Moreover, if F is spanning, we call it a *spanning pseudo-arborescence rooted in r*.

E.2.1. Topological notions for undirected multigraphs

For this subsection let $G = (V, E)$ denote a locally finite connected multigraph. We can endow G together with its ends with a topology which yields the topological space $|G|$. A precise definition of $|G|$ can be found in [12, Ch. 8.5]. However, this concept and definition directly extends to locally finite connected multigraphs. For a better understanding we should point out here that a ray of G converges in $|G|$ to the end of G that it is contained in. An equivalent way of describing $|G|$ is by first endowing G with the topology of a 1-complex and then compactifying this space using the Freudenthal compactification [19].

For an edge $e \in E$ let $\mathring{e}$ denote the set points in $|G|$ that correspond to inner points of the edge e. For an edge set $F \subseteq E$ we define $\mathring{F} = \bigcup\{\mathring{e}\ ;\ e \in F\} \subseteq |G|$. Given a point set X in $|G|$, we denote the *closure* of X in $|G|$ by $\overline{X}$. To ease notation we shall also use this notation when X denotes an edge set or a subgraph of G, meaning that we apply the closure operator to the set of all points in $|G|$ that correspond to X. Note that for a vertex set its closure coincides with its combinatorial closure in locally finite connected multigraphs. Hence, we shall use the same notation for these two operators. Further we call a subspace $Z \subseteq |G|$ *standard* if $Z = \overline{H}$ for a subgraph H of G.

Let $W \subseteq |G|$ and $<_W$ be a linear order on $\mathring{E} \cap W$. We call the tuple $(W, <_W)$ a *topological walk* in $|G|$ if there exists a continuous map $\sigma : [0, 1] \longrightarrow |G|$ such that the following hold:

1. W is the image of σ,
2. each point $p \in \mathring{E} \cap W$ has precisely one preimage under σ, and
3. the linear order $<_W$ equals the linear order $<_\sigma$ on $\mathring{E} \cap W$ defined via $p <_\sigma q$ if and only if $\sigma^{-1}(p) <_{\mathbb{R}} \sigma^{-1}(q)$ where $<_{\mathbb{R}}$ denotes the natural linear order of the reals.

We call such a map σ a *witness* of $(W, <_W)$. When we talk about a topological walk $(W, <_W)$ we shall often omit stating its linear order $<_W$ explicitly and just refer to the topological walk by writing W. Especially, we might say that a topological walk $(W, <_W)$ is contained in some subspace X of $|G|$ if $W \subseteq X$ holds. Further, we call a point x of $|G|$ an *endpoint* of W if 0 or 1 is mapped to x by a witness of W. Similar to finite walks in graphs we call an endpoint x of W an *endvertex* of W if x corresponds to a vertex of G. Further, we denote W as an x–y topological walk, if x and y are endpoints of W. If W has just one endpoint, which then has to be an end or a vertex by definition, we call it *closed*. Note that an x–y topological walk is a standard subspace for any $x, y \in V \cup \Omega(G)$. We say that a witness σ of a topological walk W *pauses* at a vertex $v \in V$ if the preimage of v under σ is a disjoint union of closed nontrivial intervals.

We define an *arc* in $|G|$ as the image of a homeomorphism mapping into $|G|$ and with the closed real unit interval $[0, 1] \subseteq \mathbb{R}$ as its domain. Note that arcs in $|G|$ are also topological walks in $|G|$ if we equip them with a suitable linear order, of which there exist only two. Since the choice of such a linear order does not change the set of endpoints of the arc if we then consider it as a topological walk, we shall use the notion of endpoints and endvertices also for arcs. Furthermore, note that finite paths of G which contain at least one edge correspond to arcs in $|G|$, but again there might be infinite subgraphs, for example rays, whose closures form arcs in $|G|$. We now call a subspace X of $|G|$ *arc-connected* if there exists an x–y arc in X for any two points $x, y \in X$.

Lastly, we define a *circle* in $|G|$ as the image of a homeomorphism mapping into $|G|$ and with the unit circle $S^1 \subseteq \mathbb{R}^2$ as its domain. It is easy to check that any circle needs to contain a vertex. Hence, we might also consider any circle as a closed topological walk if we equip it with a suitable linear order, which, however, depends on the point on the circle that we choose as the endpoint for the closed topological walk, and on choosing one of the two possible orientations of S^1. Similar as for finite paths, note that finite cycles in G correspond to circles in $|G|$, but there might be infinite subgraphs of G whose closures are circles in $|G|$ as well.

Using these definitions we can now formulate a topological extension of the notion of trees. We define a *topological tree* in $|G|$ as an arc-connected standard subspace of $|G|$ that does not contain any circle. Note that in a topological tree there is a unique arc between any two points of the topological tree, which resembles a property of finite trees with respect to its vertices and finite paths. Furthermore, we denote by a *topological spanning tree of* G a topological tree in $|G|$ that contains all vertices of G. Since topological spanning trees are closed subspaces of $|G|$, they need to contain all ends of G as well.

E.2.2. Topological notions for digraphs

In this subsection we extend some of the notions of the previous subsection to directed graphs. Throughout this subsection let G denote a locally finite weakly connected digraph and let H denote its underlying multigraph. We define the topological space $|G|$ as $|H|$. Additionally, every edge $e = uv \in E(G)$ defines a certain linear order $<_e$ on $\overline{\{e\}} \subseteq |G|$ via its direction. For the definition of $<_e$ we first take any homeomorphism $\varphi_e : [0,1] \longrightarrow \overline{\{e\}} \subseteq |G|$ with $\varphi_e(0) = u$ and $\varphi_e(1) = v$. Now we set $p <_e q$ for arbitrary $p, q \in \overline{\{e\}}$ if $\varphi_e^{-1}(p) <_{\mathbb{R}} \varphi_e^{-1}(q)$ where $<_{\mathbb{R}}$ is the natural linear order on the real numbers. Note that the definition of $<_e$ does not depend on the choice of the homeomorphism φ_e.

Let $(W, <_W)$ be a topological walk in $|G|$ with witness σ. We call $(W, <_W)$ *directed* if $<_e \restriction \mathring{e}$ equals $<_W \restriction \mathring{e}$ for every edge $e \in E(G)$ with $\mathring{e} \cap W \neq \emptyset$. If $(W, <_W)$ is directed and $\sigma(0) = s \neq t = \sigma(1)$ for $s, t \in |G|$, then there is no linear order $<'_W$ such that $(W, <'_W)$ is a directed topological walk with a witness σ' satisfying $\sigma'(0) = t$ and $\sigma'(1) = s$, because every topological s–t walk uses inner points of some edge. Hence, if we consider a directed topological s–t walk $(W, <_W)$ for $s, t \in |G|$, we implicitly assume that $\sigma(0) = s \neq t = \sigma(1)$ holds for every witness σ of $(W, <_W)$.

As arcs and circles can be seen as special instances of topological walks, *directed arcs* and *directed circles* are analogously defined. Note that if we can equip an arc with a suitable linear order such that it becomes a directed topological walk, then this linear order is unique. Hence, when we call an arc directed we implicitly associate this unique linear order with it.

E.2.3. Basic lemmas

The proofs of two lemmas (Lemma E.3.1 and Lemma E.4.1) rely to some extend on compactness arguments. At those points it will be sufficient for us to use the following lemma, which is known as König's Infinity Lemma.

Lemma E.2.2. [12, Lemma 8.1.2] *Let $(V_i)_{i \in \mathbb{N}}$ be a sequence of disjoint non-empty finite sets, and let G be a graph on their union. Assume that for every $n > 0$ each vertex in V_n has a neighbour in V_{n-1}. Then G contains a ray $v_0 v_1 \ldots$ with $v_n \in V_n$ for all $n \in \mathbb{N}$.*

We shall heavily work with the topological space $|G|$ of a locally finite multigraph G appearing as the underlying graph of digraphs we consider. Therefore, we shall make use of some basic statements and properties of the space $|G|$, especially such involving connectivity. Although the following lemmas are only stated for locally finite graphs, their proofs immediately extend to locally finite multigraphs.

Proposition E.2.3. [12, Lemma 8.5.1] *If G is a locally finite connected multigraph, then $|G|$ is a compact Hausdorff space.*

The next lemma is essential for decoding the topological property of arc-connectedness of standard subspaces of $|G|$ into a combinatorial one.

Lemma E.2.4. [12, Lemma 8.5.3] *Let G be a locally finite connected multigraph and $F \subseteq E(G)$ be a cut with sides V_1 and V_2.*

(i) *If F is finite, then $\overline{V_1} \cap \overline{V_2} = \emptyset$, and there is no arc in $|G| \setminus \mathring{F}$ with one endpoint in V_1 and the other in V_2.*

(ii) *If F is infinite, then $\overline{V_1} \cap \overline{V_2} \neq \emptyset$, and there may be such an arc.*

Note that for a finite cut $E(X, Y)$ of G we obtain that $(\overline{X}, \overline{Y})$ is a bipartition of $V(G) \cup \Omega(G)$.

The following lemma captures the equivalence of arc-connectedness and connectedness for standard subspaces of $|G|$.

Lemma E.2.5. [12, Lemma 8.5.4] *If G is a locally finite connected multigraph, then every connected standard subspace of $|G|$ is arc-connected.*

We conclude with a convenient lemma which combines the essences of the previous two.

Lemma E.2.6. [12, Lemma 8.5.5] *If G is a locally finite connected multigraph, then a standard subspace of $|G|$ is connected if and only if it contains an edge from every finite cut of G of which it meets both sides.*

E.3. Fundamental statements about topological directed walks in locally finite digraphs

In this section we lift several facts about topological walks and arcs to their directed counterparts. Most of the involved techniques and proof ideas are similar to the ones used in undirected locally finite connected multigraphs. Nevertheless, because of overlying directed structure on the multigraph, some adjustments and additional arguments are needed in the proofs. We start with a statement that combinatorially characterises the existence of directed topological walks in a standard subspace via finite cuts.

Lemma E.3.1. *Let G be a locally finite weakly connected digraph, $s, t \in V(G) \cup \Omega(G)$ with $s \neq t$ and $F \subseteq E(G)$. Then the following are equivalent:*

(i) *$\overline{F}$ contains a directed topological s–t walk.*

(ii) *$|F \cap \overrightarrow{E}(X, Y)| \geqslant 1$ for every finite cut $E(X, Y)$ of G with $s \in \overline{X}$ and $t \in \overline{Y}$.*

(iii) *$|F \cap \overrightarrow{E}(X, Y)| = |F \cap \overrightarrow{E}(Y, X)| + 1$ for every finite cut $E(X, Y)$ of G with $s \in \overline{X}$ and $t \in \overline{Y}$.*

Proof. First we prove the implication from (i) to (iii). Let $E(X, Y)$ be any finite cut of G with $s \in \overline{X}$ and $t \in \overline{Y}$. Since $\overline{F}$ contains a directed topological s–t walk $(\overline{W}, <_{\overline{W}})$ for an edge set $W \subseteq E(G)$, we know that $F \cap E(X, Y) \neq \varnothing$ by Lemma E.2.6. Note furthermore that $\overline{X} \cap \overline{Y} = \varnothing$ by Lemma E.2.4. As $\overline{X}$ and $\overline{Y}$ are closed and $|G|$ is compact by Proposition E.2.3, we get that $\overline{X}$ and $\overline{Y}$ are compact too. Now let φ be a witness of $\overline{W}$. Since $\overline{Y}$ is compact and φ is continuous, there exists a smallest number $q \in [0, 1]$ such that $\varphi(q) \in \overline{Y}$. Furthermore, there is a biggest number $p \in [0, q]$ such that $\varphi(p) \in \overline{X}$. Note that $p \neq q$ since $\overline{X} \cap \overline{Y} = \varnothing$. Now let $M = \{\varphi(r) \in |G| \,;\, p < r < q\}$. Obviously, M contains only inner points of edges in $E(X, Y)$. Since M is connected, we obtain $M = \mathring{e}$ for some edge $e \in E(X, Y)$. Using that $<_{\overline{W}} \restriction \mathring{e}$ equals $<_e \restriction \mathring{e}$ because $(\overline{W}, <_{\overline{W}})$ is a directed s–t walk, we see that $e \in W \cap \overrightarrow{E}(X, Y)$. Next we consider $\varphi \restriction [q, 1]$ and iterate the previous argument. Using that $E(X, Y)$ contains only finitely many edges, we get inductively that $|F \cap \overrightarrow{E}(X, Y)| = |F \cap \overrightarrow{E}(Y, X)| + 1$ is true.

The implication from (iii) to (ii) is immediate.

It remains to show that (ii) implies (i). For this we first fix a sequence $(S_n)_{n\in\mathbb{N}}$ of finite vertex sets $S_n \subseteq V(G)$ such that $S_n \subsetneq S_{n+1}$ for every $n \in \mathbb{N}$ and $\bigcup_{n\in\mathbb{N}} S_n = V(G)$. For every $n \in \mathbb{N}$ let G_n denote the digraph which arises by contracting $E(G - S_n)$ in G. Since G is locally finite, we know that each G_n is a finite digraph. We call the vertices of G_n that are not contained in S_n *dummy vertices*. Note that each dummy vertex of G_n corresponds to a unique weak component of $G - S_n$.

If some $v \in V(G) \cup \Omega(G)$ is not contained in S_n, there exists a unique component C_n of $G - S_n$ such that $v \in \overline{C_n}$. This is obviously true if v is a vertex of G, but also holds if v is an end of G. To see the latter statement suppose $v \in \Omega(G)$ is contained in $\overline{C_n}$ for a component C_n of $G - S_n$. Then the cut $E(V(C_n), V(G)\backslash V(C_n))$ is finite as S_n is finite and G is locally finite. Hence $\overline{V(C_n)} \cap (\overline{V(G)\backslash V(C_n)}) = \varnothing$ by Lemma E.2.4, which means that v cannot lie in the closure of another component of $G - S_n$. We refer to the dummy vertex of G_n corresponding to C_n by a slight abuse of notation as v.

Since for each $n \in \mathbb{N}$ every cut of G_n corresponds to a finite cut of G, we obtain by Theorem E.1.2 that $F \cap E(G_n)$ contains the edge set of a finite directed s–t walk in the digraph G_n. Furthermore, any finite directed s–t walk $(\mathcal{W}_{n+1}, <_{\mathcal{W}_{n+1}})$ in G_{n+1} induces a finite directed s–t walk $(\mathcal{W}_n, <_{\mathcal{W}_n})$ in G_n via $E(\mathcal{W}_n) := E(\mathcal{W}_{n+1}) \cap E(G_n)$ and defining $<_{\mathcal{W}_n}$ as $<_{\mathcal{W}_{n+1}} \restriction E(\mathcal{W}_n)$. Note that each maximal interval with respect to $<_{\mathcal{W}_{n+1}}$ of $E(\mathcal{W}_{n+1})\backslash E(\mathcal{W}_n)$ corresponds to some v–w walk where v and w are the same dummy vertex of G_n. Hence each time a dummy vertex of G_n appears as the head of some edge $e \in E(\mathcal{W}_n)$ there is a corresponding, possibly trivial, walk $\mathcal{W}^e_{n+1}$ using edges of of such a maximal interval with the induced order $<_{\mathcal{W}_{n+1}} \restriction E(\mathcal{W}^e_{n+1})$.

For every $n \in \mathbb{N}$ let V_n denote the set of all finite directed s–t walks in G_n that use only edges from F. Obviously, each set V_n is finite as G_n is a finite digraph. By the previously given arguments, none of the sets V_n is empty and each element of V_{n+1} induces one of V_n. Hence, we get a sequence $((\mathcal{W}_n, <_{\mathcal{W}_n}))_{n\in\mathbb{N}}$ of finite directed s–t walks where $(\mathcal{W}_n, <_{\mathcal{W}_n}) \in V_n$ such that $E(\mathcal{W}_{n+1}) \cap E(\mathcal{W}_n) = E(\mathcal{W}_n)$ and $<_{\mathcal{W}_{n+1}} \restriction E(\mathcal{W}_n)$ equals $<_{\mathcal{W}_n}$ for every $n \in \mathbb{N}$ by Lemma E.2.2. We define $W_n := E(\mathcal{W}_n)$ for every $n \in \mathbb{N}$. Next we set $W := \bigcup_{n\in\mathbb{N}} W_n$ and $<_W := \bigcup_{n\in\mathbb{N}} <_{\mathcal{W}_n}$.

Further, we define a linear order $<_{\overline{W}}$ on $\mathring{W}$ as follows for $p, q \in \mathring{W}$ with $p \neq q$:

$$p <_{\overline{W}} q \quad \text{iff} \quad \begin{cases} p \in \mathring{e} \text{ and } q \in \mathring{f} \text{ with } e <_W f \text{ for some } e, f \in W \text{ with } e \neq f, \text{ or} \\ p, q \in \mathring{e} \text{ and } p <_e q \text{ for some } e \in W. \end{cases}$$

Now we claim that $(\overline{W}, <_{\overline{W}})$ is a directed topological s–t walk in $|G|$. In order to show this we first have to define a witness φ for $(\overline{W}, <_{\overline{W}})$. We shall obtain φ as a limit of countably many certain witnesses φ_n of directed topological walks $(\overline{W_n}, <_{\overline{W_n}})$ in $|G_n|$ that we define inductively, where $<_{\overline{W_n}}$ is analogously defined as $<_{\overline{W}}$ but with respect to W_n.

For $n = 0$ we start with a witness φ_0 of the directed topological s–t walk $(\overline{W_0}, <_{\overline{W_0}})$ in $|G_0|$ which pauses at every dummy vertex of G_0 contained in $\overline{W_0}$.

Now suppose that the witness φ_n of $(\overline{W_n}, <_{\overline{W_n}})$ has already been defined such that it pauses at every dummy vertex of G_n that is contained in $\overline{W_n}$. Then we define φ_{n+1} as some witness of $(\overline{W_{n+1}}, <_{\overline{W_{n+1}}})$ as follows. For every edge $e \in W_n$ whose head is a dummy vertex of G_n, let W^e_{n+1} be the edge set of the walk $\mathcal{W}^e_{n+1}$ as above and let φ^e_{n+1} be a witness that $\overline{\mathcal{W}^e_{n+1}}$ is the corresponding directed topological walk that pauses at every dummy vertex of G_{n+1} that is contained in $\overline{W^e_{n+1}}$. Starting with φ_n, each time we enter some dummy vertex d of G_n by an edge e, we replace the image of the interval that is mapped to d with a rescaled version of φ^e_{n+1}.

Using the maps φ_n we are able to define φ as follows: For every $q \in [0, 1]$ for which there exists an $n \in \mathbb{N}$ such that $\varphi_n(q) \in S_n$, we set $\varphi(q) := \varphi_n(q)$. Otherwise, $\varphi_n(q)$ corresponds to a contracted component C_n of $G - S_n$ for every $n \in \mathbb{N}$. Since $S_n \subsetneq S_{n+1}$ for every $n \in \mathbb{N}$ and $\bigcup_{n \in \mathbb{N}} S_n = V(G)$, it is easy to check that $\bigcap_{n \in \mathbb{N}} \overline{C_n} = \{\omega\}$ for some end ω of G. In this case, we define $\varphi(q) := \omega$. This completes the definition of φ. It is straightforward to verify that φ is continuous and also onto $\overline{W}$ because each φ_n is onto $\overline{W_n}$ and $W := \bigcup_{n \in \mathbb{N}} W_n$. This ensures that it is a witness of $(\overline{W}, <_{\overline{W}})$ being a topological s–t walk. Note that the linear order $<_{\overline{W}} \restriction \mathring{e}$ equals $<_e \restriction \mathring{e}$ for each edge $e \in W$ since each linear order $<_{\overline{W_n}}$ has this property. Hence, φ witnesses that $(\overline{W}, <_{\overline{W}})$ is a directed topological s–t walk in $|G|$ with $W \subseteq F$. $\square$

We proceed with a lemma which encodes the existence of directed arcs in standard subspaces in the same combinatorial way as Lemma E.3.1 did this for directed topological walks.

Lemma E.3.2. *Let G be a locally finite weakly connected digraph, $s, t \in V(G) \cup \Omega(G)$ with $s \neq t$ and $A \subseteq E(G)$. Then the following are equivalent:*

(i) *$\overline{A}$ is a directed s–t arc.*

(ii) *A is inclusion-wise minimal such that $|A \cap \overrightarrow{E}(X, Y)| \geqslant 1$ holds for every finite cut $E(X, Y)$ of G with $s \in \overline{X}$ and $t \in \overline{Y}$.*

(iii) *A is inclusion-wise minimal such that $|A \cap \overrightarrow{E}(X, Y)| = |A \cap \overrightarrow{E}(Y, X)| + 1$ holds for every finite cut $E(X, Y)$ of G with $s \in \overline{X}$ and $t \in \overline{Y}$.*

Proof. First we show the implication from (i) to (iii). As $\overline{A}$ is a directed s–t arc, it is also a directed topological s–t walk. So by Lemma E.3.1, we only need to check the minimality of A for property (iii). Since $\overline{A}$ is an s–t arc, we know that s and t are in different topological components of $\overline{A \backslash \{e\}}$ for any edge $e \in A$. So no proper subset of A has the property that its closure in $|G|$ contains a directed topological s–t walk. Again by Lemma E.3.1 we know that no proper subset of A satisfies statement (iii) of Lemma E.3.1. This proves the minimality of A and hence statement (iii).

Next let us verify that (iii) implies (ii). Assume for a contradiction that statement (iii) holds, but (ii) does not. Then there must exist a proper subset $A' \subsetneq A$ that fulfils property (ii), maybe except from being minimal. By Lemma E.3.1 we get that A' satisfies also statement (iii) of Lemma E.3.1. This contradicts the minimality of A.

It remains to prove the implication from (ii) to (i). By assuming (ii) we know from Lemma E.3.1 that $\overline{A}$ contains a directed topological s–t walk and by the minimality of A we know that $\overline{A}$ is in fact a directed topological s–t walk, say witnessed by $\varphi : [0, 1] \longrightarrow |G|$. Now suppose for a contradiction that $\overline{A}$ is not a directed s–t arc. Then there exists a point $a \in V(G) \cup \Omega(G)$ that spoils injectivity for φ. Note that $\overline{A}$ is compact because it is a closed set in $|G|$ that is a compact space by Proposition E.2.3. Since φ is continuous and $\overline{A}$ is compact, there exists a smallest number $x \in [0, 1]$ and a largest number $y \in [0, 1]$ such that $\varphi(x) = \varphi(y) = a$. We obtain from this that the image of $\varphi \restriction [0, x]$ is a directed topological s–a walk and the image of $\varphi \restriction [y, 1]$ is a directed topological a–t walk. Concatenating these two walks yields another directed topological s–t walk, which is the closure in $|G|$ of some edge set $A' \subseteq A$. Knowing that $x \neq y$, we get that

$A' \subsetneq A$ since the image of $\varphi\restriction[x,y]$ contains points that correspond to inner points of edges. This is a contradiction to the minimality of A. $\square$

We conclude this section with the following corollary which allows us to extract a directed s–t arc from a directed topological s–t walk for distinct points s, t of $|G|$.

Corollary E.3.3. *Let $s, t \in V(G) \cup \Omega(G)$ with $s \neq t$ for some locally finite weakly connected digraph G. Then every directed topological s–t walk in $|G|$ contains a directed s–t arc.*

Proof. Let $\overline{W}$ be a directed topological s–t walk where $W \subseteq E(G)$. So W has property (ii) of Lemma E.3.1. Now consider the set $\mathcal{W}$ of all subsets of W that also have property (ii) of Lemma E.3.1. This set is ordered by inclusion and not empty since $W \in \mathcal{W}$. Next let us check that every decreasing chain $C \subseteq \mathcal{W}$ is bounded from below by $\bigcap C$, which is an element of $\mathcal{W}$. Obviously, $\bigcap C \subseteq c$ holds for every $c \in C$. To see that $\bigcap C$ is an element of $\mathcal{W}$ note that for every finite cut $E(X,Y)$ of G with $s \in \overline{X}$ and $t \in \overline{Y}$ all but finitely many $c \in C$ contain the same edges from $E(X,Y)$. As every $c \in C$ has also at least one edge from $E(X,Y)$, we know that the same is true for $\bigcap C$, which shows that $\bigcap C \in \mathcal{W}$ holds. Now Zorn's Lemma implies that $\mathcal{W}$ has a minimal element, which is a directed s–t arc by Lemma E.3.2. $\square$

E.4. Packing pseudo-arborescences

We begin this section with a lemma characterising when a packing of $k \in \mathbb{N}$ many edge-disjoint spanning r-reachable sets is possible in a locally finite weakly connected digraph G with $r \in V(G) \cup \Omega(G)$. This lemma is the main ingredient to prove our first main result of Section E. The proof is mainly based on a compactness argument.

Lemma E.4.1. *A locally finite weakly connected digraph G with $r \in V(G) \cup \Omega(G)$ has $k \in \mathbb{N}$ edge-disjoint spanning r-reachable sets if and only if every bipartition (X, Y) of $V(G)$ with $r \in \overline{X}$ and $|E(X,Y)| < \infty$ satisfies $d^-(Y) \geqslant k$.*

Proof. The condition that every bipartition (X,Y) of $V(G)$ with $r \in \overline{X}$ and $|E(X,Y)| < \infty$ satisfies $d^-(Y) \geqslant k$ is obviously necessary for the existence of k edge-disjoint spanning r-reachable sets.

Let us now prove the converse. First we fix a sequence $(S_n)_{n\in\mathbb{N}}$ of finite vertex sets $S_n \subseteq V(G)$ such that $\bigcup_{n\in\mathbb{N}} S_n = V(G)$. For every $n \in \mathbb{N}$ let G_n denote the digraph which arises by contracting, inside of G, each weak component of $G - S_n$ to a single vertex. Here we keep multiple edges, but delete loops that arise. Since G is locally finite, we know that each G_n is a finite digraph.

Note that, as in the proof of Lemma E.3.1, if $r \notin S_n$, there exists a unique component C_n of $G - S_n$ such that $r \in \overline{C_n}$ and we refer to the vertex of G_n corresponding to C_n as r.

Now we define V_n as the set of all k-tuples consisting of k edge-disjoint spanning r-reachable sets of G_n. As every cut of G_n is finite and also corresponds to a cut of G, our labelling with r ensures that each G_n has k edge-disjoint arborescences rooted in r by Theorem E.1.2. So none of the V_n is empty. Furthermore, each V_n is finite as G_n is a finite digraph.

Next we show that every spanning r-reachable set F_{n+1} of G_{n+1} induces one for G_n via $F_n := F_{n+1} \cap E(G_n)$. So let F_{n+1} be given and consider a cut $E(X_n, Y_n)$ of G_n with $r \in X_n$. As each component of $G - S_{n+1}$ is contained in a component of $G - S_n$, we can find a cut $E(X_{n+1}, Y_{n+1})$ of G_{n+1} with $r \in X_{n+1}$ such that $\overrightarrow{E}(X_n, Y_n) = \overrightarrow{E}(X_{n+1}, Y_{n+1})$ (and in fact also $\overrightarrow{E}(Y_n, X_n) = \overrightarrow{E}(Y_{n+1}, X_{n+1})$). Since F_{n+1} is a spanning r-reachable set of G_{n+1}, we obtain that F_n is one of G_n.

Now we can apply Lemma E.2.2 to the graph defined on the vertex set $\bigcup_{n\in\mathbb{N}} V_n$ where two vertices $v_{n+1} \in V_{n+1}$ and $v_n \in V_n$ are adjacent if the i-th spanning r-reachable set in v_n is induced by the i-th one of v_{n+1} for every i with $1 \leqslant i \leqslant k$. So we obtain a ray $r_0 r_1 \ldots$ with $r_n \in V_n$ and set $\mathcal{F} := (F^1, \ldots, F^k)$ where $F^i := \bigcup_{n\in\mathbb{N}} r_n^i$ and r_n^i denotes the i-th entry of the k-tuple r_n for every i with $1 \leqslant i \leqslant k$. Let us now check that each F^i is a spanning r-reachable set of G. As $\bigcup_{n\in\mathbb{N}} S_n = V(G)$ holds, we can find for every finite cut $E(X, Y)$ of G with $r \in \overline{X}$ an $n \in \mathbb{N}$ such that all endvertices of edges of $E(X, Y)$ are contained in S_n. Hence, there exists a cut $E(X_n, Y_n)$ of G_n with $r \in X_n$ such that $\overrightarrow{E}(X_n, Y_n) = \overrightarrow{E}(X, Y)$ and $\overrightarrow{E}(Y_n, X_n) = \overrightarrow{E}(Y, X)$. Since each F^i contains the edges of r_n^i, which is a spanning r-reachable set of G_n and, therefore, contains an edge of $\overrightarrow{E}(X_n, Y_n)$, we know that each F^i is a spanning r-reachable set of G. Finally, we get that all the F^i are pairwise edge-disjoint because for every $n \in \mathbb{N}$ the r_n^i are pairwise edge-disjoint. $\square$

The next lemma ensures the existence of pseudo-arborescences for a set $Z \subseteq V(G)\backslash\{r\}$ in the sense that every r-reachable set for Z contains one. The proof of this lemma works by an application of Zorn's Lemma and is very similar to the proof of Corollary E.3.3. Therefore, we omit stating its proof.

Lemma E.4.2. *Let G be a locally finite weakly connected digraph, $Z \subseteq V(G)\backslash\{r\}$ with $r \in V(G) \cup \Omega(G)$. Then every r-reachable set for Z in G contains a pseudo-arborescences for Z rooted in r.* $\square$

Combining Lemma E.4.1 and Lemma E.4.2 we now obtain one of our main results of Section E.

Theorem E.4.3. *A locally finite weakly connected digraph G with $r \in V(G) \cup \Omega(G)$ has $k \in \mathbb{N}$ edge-disjoint spanning pseudo-arborescences rooted in r if and only if every bipartition (X, Y) of $V(G)$ with $r \in \overline{X}$ and $|E(X, Y)| < \infty$ satisfies $d^-(Y) \geqslant k$.* $\square$

E.5. Structure of pseudo-arborescences

The following lemma characterises r-reachable sets in terms of directed arcs. Additionally, it justifies the naming of r-reachable sets.

Lemma E.5.1. *Let G be a locally finite weakly connected digraph with sets $F \subseteq E(G)$ and $Z \subseteq V(G)\backslash\{r\}$ where $r \in V(G) \cup \Omega(G)$. Then F is an r-reachable set for Z in G if and only if there exists a directed r–z arc inside $\overline{F}$ for every $z \in \overline{Z}$.*

Proof. Let us first assume that F is an r-reachable set for Z in G. We fix some $z \in \overline{Z}$ and prove next that $|F \cap \overrightarrow{E}(X, Y)| \geqslant 1$ holds for each finite cut $E(X, Y)$ where $r \in \overline{X}$ and $z \in \overline{Y}$. If z is a vertex, this follows immediately from the definition of an r-reachable set for Z. In the case that $z \in \Omega(G)$, we also get that some vertex of Z lies in Y. This follows, because z is contained in the closed and, therefore, compact set $\overline{Z}$, which implies the existence of a sequence S of vertices of Z converging to z. Since $E(X, Y)$ is a finite cut and $z \in \overline{Y}$, we can find inside $\overline{G[Y]}$ a basic open set O containing z. Now O must contain a vertex of S and hence Y must do so as well. Therefore, the desired inequality follows again by the definition of an r-reachable set for Z.

Now we are able to use Lemma E.3.1, which yields that $\overline{F}$ contains a directed topological r–z walk. We complete the argument by applying Corollary E.3.3 telling us that $\overline{F}$ contains also a directed r–z arc.

Conversely, consider any finite cut $E(X, Y)$ where $r \in \overline{X}$ and $Y \cap Z \neq \varnothing$, say $z \in Y \cap Z$. The assumption ensures the existence of a directed r–z arc in $\overline{F}$. By Lemma E.3.2 we obtain that $|F \cap \overrightarrow{E}(X, Y)| \geqslant 1$ holds as desired. □

Now let us turn our attention towards spanning pseudo-arborescences rooted in some vertex or end in a locally finite weakly connected digraph. The question arises how similar these objects behave compared to spanning arborescences rooted in some vertex in a finite graph. A basic property of finite arborescences is the existence of a unique directed path in the arborescence from the root to any other vertex of the graph. Closely related is the absence of any cycle, directed or undirected, in a finite arborescence since its underlying graph is a tree. Although we know by Lemma E.5.1 that the closure of a spanning pseudo-arborescences contains a directed arc from the root to any other vertex (or even end) of the graph, we shall see in the following example that we can neither guarantee the uniqueness of such arcs nor avoid infinite circles (directed or undirected ones).

Example E.5.2. Consider the graph depicted in Figure E.1. This graph contains spanning r-reachable sets, for example the bold black edges together with the bold grey edges. However, every spanning r-reachable set of this graph must contain all bold black edges because for any head of such an edge there is no other edge of which it is a head. As this graph has only one end, namely ω, we see that there are directed and undirected infinite circles containing only bold black edges. This shows already that, in general, it is not possible to find spanning r-reachable sets that do not contain directed or undirected infinite circles. So there does not exist a stronger version of Theorem E.4.3 in the sense that the edges of the underlying multigraph of every spanning pseudo-arborescences form a topological spanning tree in the Freudenthal compactification of the underlying multigraph.

The graph in Figure E.1 shows furthermore that, in general, we cannot find spanning r-reachable sets F such that there exists a unique directed arc from r to every vertex and every end of the graph inside $\overline{F}$. In the example we have two different directed arcs from r to the end ω that contain only bold black edges and are therefore in every spanning r-reachable set of this graph. Hence, we also get

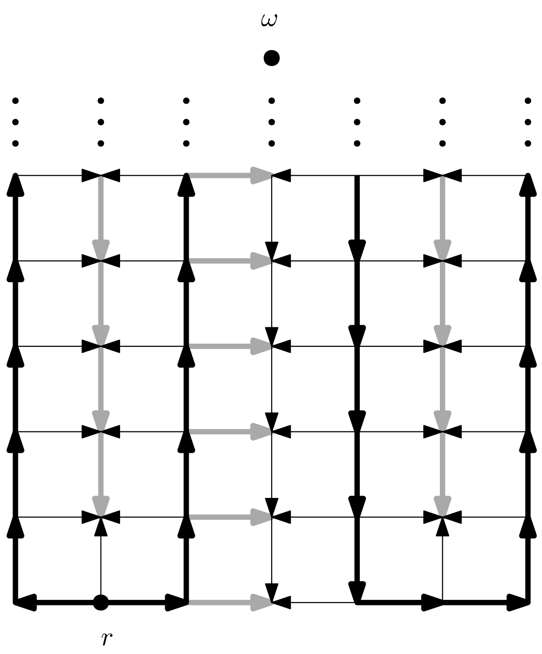

Figure E.1.: An example of a graph with a marked vertex r where the closure of any spanning r-reachable set contains an infinite circle and multiple arcs to the end ω and certain vertices.

two different directed arcs from r to every vertex on the infinite directed circle that consists only of bold black edges.

Although, in general, spanning pseudo-arborescences do not behave tree-like in the sense that their underlying graphs correspond to topological spanning trees, they do so in a local sense. We conclude this section with a main result about characterising those spanning r-reachable that are inclusion-wise minimal via some local tree-like properties. Especially, we obtain the absence of finite cycles (directed or undirected ones) in any spanning pseudo-arborescences.

Theorem E.5.3. *Let G be a locally finite weakly connected digraph and further let $r \in V(G) \cup \Omega(G)$. Then the following are equivalent for a spanning r-reachable set F of G:*

(i) *F is a spanning pseudo-arborescences rooted in r.*

(ii) *For every vertex $v \neq r$ of G there is a unique edge in F whose head is v, and no edge in F has r as its head.*

(iii) *For every weak component T of $G[F]$ the following holds: If $r \in V(T)$, then T is an arborescence rooted in r. Otherwise, T is an arborescence rooted in some end of T.*

Proof. We start by proving the implication from (i) to (ii). Let us first suppose for a contradiction that F contains an edge e whose head is r. Obviously, there is no finite cut $E(X, Y)$ of G such that $r \in \overline{X}$ and $e \in \overrightarrow{E}(X, Y)$. Hence, $F \backslash \{e\}$ is a smaller spanning r-reachable set of G contradicting the minimality of F.

Next let us consider an arbitrary vertex $v \neq r$ of G. We know by Remark E.2.1 that F contains at least one edge of $\overrightarrow{E}(V(G) \backslash \{v\}, \{v\})$. So F contains at least one edge whose head is v.

Now suppose for a contradiction that there exists some vertex $v \neq r$ of G which is the head of at least two edges of F, say e and f. We know by Lemma E.5.1 that $\overline{F}$ contains a directed r–v arc A. Since the cut $E(V(G) \backslash \{v\}, \{v\})$ is finite and A is a directed r–v arc, we get that A must contain precisely one edge of $\overrightarrow{E}(V(G) \backslash \{v\}, \{v\})$. Hence, one of the edges e, f is not contained in A, say e. By the minimality of F, we obtain that $F \backslash \{e\}$ cannot be a spanning r-reachable set of G. So there must exist a finite cut $E(X, Y)$ of G with $r \in \overline{X}$ such that e is the only edge in $F \cap \overrightarrow{E}(X, Y)$. Now we have a contradiction since the head of e is v and lies in Y, which means that the directed arc A contains at least one edge of $\overrightarrow{E}(X, Y)$ by Lemma E.3.2, but such an edge is different from e. Therefore, e was not the only edge in $F \cap \overrightarrow{E}(X, Y)$.

We continue with the proof that statement (ii) implies statement (iii). For this let us fix an arbitrary weak component T of $G[F]$. We now show that T is a tree. Suppose for a contradiction that T contains a directed or undirected cycle C.

If C is a directed cycle, each vertex on C would already be a head of some edge of the cycle. Hence, r cannot be a vertex on C. Applying Remark E.2.1 with the finite set $V(C)$, we obtain that there needs to be an edge uv of F with $v \in V(C)$ and $u \in V(G) \backslash V(C)$. So v is the head of two edges of F, which contradicts statement (ii).

In the case that C is a cycle, but not a directed one, take a maximal directed path on C. Its endvertex is the head of two edges of C. So we get again a contradiction to statement (ii). We can conclude that T is a tree.

If r is be a vertex of T, then it is immediate from statement (ii) that T is an arborescence rooted in r. Otherwise, there needs to be a backwards directed ray R

in T as each vertex different from r is the head of a unique edge of F. Let ω be the end of T which contains R. Hence, T is an arborescence rooted in ω, completing the proof of this implication.

It remains to show the implication from (iii) to (i). For this we assume statement (iii) and suppose for a contradiction that F is not minimal with respect to inclusion. Hence, $F' = F \backslash \{e\}$ is a spanning r-reachable set as well for some $e = uv \in F$. Let T be the weak component of $G[F]$ which contains v. As T is an arborescence rooted in r or some end of T, we get that no edge of F' has v as its head. Note that $r \neq v$ because of the edge $uv \in F$. Now we get a contradiction by applying Remark E.2.1 with F' and the set $\{v\}$, which tells us that F' needs contains an edge whose head is v. □

The question might arise whether we can be more specific in statement (iii) of Theorem E.5.3 in the case when r is an end of G. Unfortunately, it is not true that there has to exist a weak component of $G[F]$ whose unique backwards directed ray lies in r. The reason for this is that the end r might be an accumulation point of a sequence of infinitely many different weak components of $G[F]$ in $|G|$ each of which contains a backwards directed ray to a different end of G. It is not difficult to construct an example for this situation and so we omit such a description here. On the other hand if the end $r \in \Omega(G)$ is not an accumulation point of different ends of G, then there exists at least one weak component of $G[F]$ whose backwards directed ray is contained in r. To see this fix an arbitrary directed r–v arc A inside $\overline{F}$ for some vertex v. Since F is a spanning r-reachable set of G, we can find such an arc. If among all of the weak components of $G[F]$ which are met by A, there is a first one with respect to the linear order of A, then a backwards directed ray of this component is an initial segment of A and, therefore, contained in r. Note for the other case that tails of the backwards directed rays of each component of $G[F]$ that is met by A must be contained A. Since A is an arc, all these backwards directed rays must be contained in different ends of G. These ends, however, would then have r as an accumulation point in $|G|$ contradicting the assumption on r.

F. On the Infinite Lucchesi-Younger Conjecture

F.1. Introduction

In finite graph theory there exist a lot of theorems which relate the maximum number of disjoint substructures of a certain type in a graph with the minimal size of another substructure in that graph, which bounds the number of disjoint objects of the first type that can exist. Often there is no gap between such numbers. Some results of this type even have a reformulation in the language of linear programming.

Probably the most well-known example of such a result is the theorem of Menger for finite graphs. In order to state the theorem more easily, let us make the following definition. For two vertex sets $A, B \subseteq V(G)$ in a graph G we call a path P an A–B *path* in G if one endvertex of P lies in A, the other in B and except from these two vertices, P is disjoint from the set $A \cup B$. Note that a vertex in $A \cap B$ is also an A–B path.

Theorem F.1.1. [12, Thm. 3.3.1] *Let G be a finite graph and $A, B \subseteq V(G)$. Then the maximum number of disjoint A–B paths in G equals the minimum size of a vertex set separating A from B in G.*

This theorem has the following immediate corollary.

Corollary F.1.2. *Let G be a finite graph and $A, B \subseteq V(G)$. Then there exists a tuple $(S, \mathcal{P})$ such that the following holds*

(i) *$\mathcal{P}$ is a set of disjoint A–B paths in G.*

(ii) *$S \subseteq V(G)$ separates A from B in G.*

(iii) *$S \subseteq \bigcup \mathcal{P}$.*

(iv) $|S \cap P| = 1$ *for every* $P \in \mathcal{P}$*.*

However, this corollary is not weaker than Theorem F.1.1 because Theorem F.1.1 is conversely implied by Corollary F.1.2. The crucial point of Corollary F.1.2 is that the elements of the tuple $(S, \mathcal{P})$ make certain optimality assertions about each other: The set $\mathcal{P}$ and the way it interacts with S proves that the separator S has minimum size. Conversely, the size of S bounds the size of any set of disjoint A–B paths. Hence, S and its interaction with $\mathcal{P}$ shows that $\mathcal{P}$ is of maximum size.

The benefit of the formulation of Corollary F.1.2 is that it avoids talking about maximality and minimality in terms of sizes or cardinalities. In infinite graphs this now becomes much more meaningful. An extension of Theorem F.1.1 which only asks for the existence of κ many disjoint A–B paths and a set of size κ separating A from B for some cardinality κ is quite easy to prove. In contrast to this, the extension of Corollary F.1.2 which asks for the same tuple, but in a graph of arbitrary cardinality, is probably one of the deepest theorems in infinite graph theory and due to Aharoni and Berger [1]. While the proof of this theorem is already challenging for countable graphs, it becomes much more complicated in graphs of higher cardinality.

We want to consider a theorem about finite digraphs which has a similar formulation as Theorem F.1.1. To state the theorem we have do give some definitions first. In a weakly connected directed graph D we call a cut of D *directed*, or a *dicut* of D, if all of its edges have their head in a common side of the cut. Now we call a set of edges of D a *dijoin* of D if it meets every non-empty dicut of D. Now we can state the mentioned theorem, which is due to Lucchesi and Younger.

Theorem F.1.3. [46, Thm.] *In every weakly connected finite digraph D, the maximum number of disjoint dicuts of D equals the minimum size of a dijoin of D.*

Beside the proof Theorem F.1.3 of Lucchesi and Younger [46, Thm.], further ones appeared by Lovász [45, Thm. 2] and Frank [22, Thm. 9.7.2]. As for Theorem F.1.1 we state a reformulation of Theorem F.1.3 which avoids talking about maximality and minimality in terms of sizes or cardinalities.

Corollary F.1.4. *Let D be a finite weakly connected digraph. Then there exists a tuple $(F, \mathcal{B})$ such that the following holds*

(i) $\mathcal{B}$ *is a set of disjoint dicuts of* D*.*

(ii) $F \subseteq E(D)$ *is a dijoin of* D*.*

(iii) $F \subseteq \bigcup \mathcal{B}$*.*

(iv) $|F \cap B| = 1$ *for every* $B \in \mathcal{B}$*.*

Now we consider the question whether Corollary F.1.4 extends to infinite digraphs as Corollary F.1.2 did for infinite graphs. Let us first show that a direct extension of this formulation to arbitrary infinite digraphs fails. To do this we define a *double ray* to be an undirected two-way infinite path. Now consider the digraph depicted in Figure F.1. Its underlying undirected graph is the Cartesian product of a double ray with an edge. Then we orient all edges corresponding to one copy of the double ray in one direction and all edges of the other copy in the different direction. Finally, we direct all remaining edges such that they have their tail in the same copy of the double ray.

This digraph does not have any finite dicut, but infinite ones. Note that every dicut of this digraph contains at most one horizontal edge, which corresponds to a oriented one of some copy of the double ray, and all vertical edges left to some vertical edge. So we cannot even find two disjoint dicuts. Next let us look at dijoins of the digraph depicted in Figure F.1. In order to hit every dicut which contains a horizontal edge, a dijoin must contain infinitely many vertical edges left to some vertical edge. So we obtain that each dijoin hits every dicut infinitely often in this digraph. Therefore, neither the statement of Corollary F.1.4 nor the statement of Theorem F.1.3 using cardinalities remains true if we consider arbitrary dicuts in infinite digraphs.

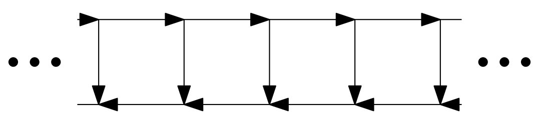

Figure F.1.: A counterexample to an extension of Corollary F.1.4 to infinite digraphs where infinite dicuts are considered too.

In order to overcome the problem of this example let us again consider the situation in Corollary F.1.2. There, all elements of the set $\mathcal{P}$ are just finite paths. So we might need to restrict our attention to finite dicuts when extending

Corollary F.1.4 to infinite digraphs. Hence, we make the following definitions. In a weakly connected digraph D we call an edge set $F \subseteq E(D)$ a *finitary dijoin* of D if it intersects every non-empty finite dicut of D. Building up on this definition we call a tuple $(F, \mathcal{B})$ as in Corollary F.1.4, but where F is now only a finitary dijoin and $\mathcal{B}$ a set of disjoint finite dicuts of D, an *optimal pair* for D. Furthermore, we call an optimal pair *nested* if the elements of $\mathcal{B}$ are pairwise, i.e., any two finite dicuts $E(X_1, X_2), E(Y_1, Y_2) \in \mathcal{B}$ either satisfy $X_1 \subseteq Y_1$ or $Y_1 \subseteq X_1$.

Not in contradiction to the example given above, we make following conjecture, which we call the Infinite Lucchesi-Younger Conjecture.

Conjecture F.1.5. *There exists an optimal pair for every weakly connected digraph.*

Apparently, an extension of Theorem F.1.3 as in Conjecture F.1.5 has independently been conjectured by Aharoni [44].

The three mentioned proofs [46, Thm.] [45, Thm. 2] [22, Thm. 9.7.2] of Theorem F.1.3 even show a slightly stronger result.

Theorem F.1.6. [46, Thm.] *There exists a nested optimal pair for every weakly connected finite digraph.*

Hence, we also make the following conjecture.

Conjecture F.1.7. *There exists a nested optimal pair for every weakly connected digraph.*

An indication why Conjecture F.1.7 might be properly stronger than Conjecture F.1.5 is the following. Different from finite digraphs, not every finitary dijoin that is part of an optimal pair for a given weakly connected infinite digraph can also feature as part of some nested optimal pair for that digraph. As an example for this, consider the infinite digraph depicted twice in Figure F.2. Its underlying graph consists of a ray R together with an additional vertex $v \notin V(R)$ which is precisely adjacent to every second vertex along R, beginning with the unique vertex on R of degree 1. Then we orient all edges incident with v towards v. Each remaining edge is oriented towards the unique neighbour of v to which it is incident with.

Considering Figure F.2 it is easy to check that the grey edges F_L in the left instance of the digraph are a finitary dijoin. Furthermore, we can easily find a

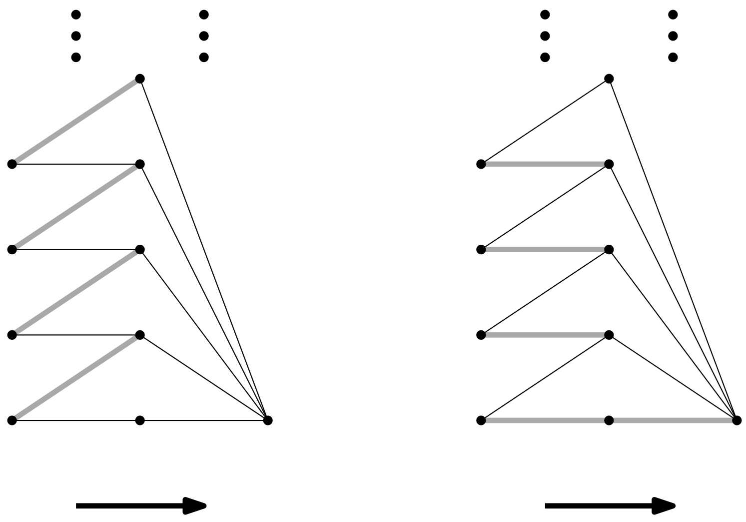

Figure F.2.: All edges are meant to be directed from left to right. The grey edges in the left picture feature in a finitary dijoin of a nested optimal pair. The grey edges in the right picture feature in a finitary dijoin of an optimal pair, but not in any finitary dijoin of a nested optimal pair.

nested optimal pair in which F_L features. In the right instance of the digraph, the grey edges F_R also form a finitary dijoin and we can also easily find an optimal pair in which F_R features. However, no matter which finite dicut we choose on which the rightmost grey edge lies, it cannot be nested with all the finite dicuts we choose for all the other edges of F_R.

One of the main results here is that we verify Conjecture F.1.7 for several classes of digraphs. We gather all these results in the following theorem. Before we can state the theorem, we have to give some further definitions. We call a minimal non-empty dicut of a digraph a *dibond.* Furthermore, we call an undirected one-way infinite path a *ray.* An undirected multigraph which does not contain a ray, is called *rayless.*

Theorem F.1.8. *Conjecture F.1.7 holds for a weakly connected digraph D if it has any of the following properties:*

(i) *There exists a finitary dijoin of D of finite size.*

(ii) *There is a finite maximal number of disjoint finite dicuts of D.*

(iii) *There is a finite maximal number of disjoint and pairwise nested finite dicuts of D.*

(iv) *Every edge of D lies in only finitely many finite dibonds of D.*

(v) *D has no infinite dibond.*

(vi) *The underlying multigraph of D is rayless.*

The other main result of Section F is that we can reduce Conjecture F.1.5 and Conjecture F.1.7 to countable digraphs with a certain separability property and whose underlying multigraph is 2-connected. In order to state the theorem, we have to make a further definition. We call a digraph D *finitely diseparable* if for any two vertices $v, w \in V(D)$ there is a finite dicut of D such that v and w lie in different sides of that finite dicut.

Theorem F.1.9.

(i) *If Conjecture F.1.5 holds for all countable finitely diseparable digraphs whose underlying multigraph is* 2*-connected, then Conjecture F.1.5 holds for all weakly connected digraphs.*

(ii) *If Conjecture F.1.7 holds for all countable finitely diseparable digraphs whose underlying multigraph is* 2*-connected, then Conjecture F.1.7, respectively, holds for all weakly connected digraphs.*

The structure of Section F is as follows. In Section F.2 we introduce our needed notation. Furthermore, we state and prove several lemmas we shall need to prove the main theorems of Section F. Section F.3 is dedicated to the proof of Theorem F.1.9. In the last part, Section F.4, shall prove Theorem F.1.8 via several lemmas.

F.2. Preliminaries

For basic facts about finite and infinite graphs we refer the reader to [12]. Several proofs, especially in Section F.4, base on certain compactness arguments using the compactness principle in combinatorics. We omit stating it here but refer to [12, Appendix A]. Especially for facts about directed graphs we refer to [3].

In general, we allow our digraphs to have parallel edges, but no loops if we do not explicitly mention them. Similarly, all undirected multigraphs we consider do not have loops if nothing else is explicitly stated.

Throughout this section let $D = (V, E)$ denote a digraph. Similarly as in undirected graphs we shall call the elements of E just *edges*. We view the edges of D as ordered pairs (u, v) of vertices $u, v \in V$ and shall write uv instead of (u, v), although this might not uniquely determine an edge. In parts where a finer distinction becomes important we shall clarify the situation. For an edge $uv \in E$ we furthermore denote the vertex u as the *tail* of uv and v as the *head* of uv. We denote the underlying multigraph of D by $\mathrm{Un}(D)$.

In an undirected non-trivial path we call the vertices incident with just one edge the *endvertices* of that path. For the trivial path consisting just of one vertex, we call that vertex also an *endvertex* of that path. If P is an undirected path with endvertices v and w, we call P a *v–w path*. For a path P containing two vertices $x, y \in V(P)$ we write xPu for the x–u subpath contained in P. Should P additionally be a directed path where v has out-degree 1, then we call P a *directed v–w path*. We also allow to call the trivial path with endvertex v a *directed v–v path*. For two vertex sets $A, B \subseteq V$ we call an undirected path $P \subseteq D$ an *A–B path* if P is an a–b path for some $a \in A$ and $b \in B$ but is disjoint from $A \cup B$ except from its endvertices. Similarly, we call an directed path that is an A–B path a *directed A–B path*.

We call an undirected graph a *star* if it is isomorphic to the complete bipartite graph $K_{1,\kappa}$ for some cardinal κ, where the vertices of degree 1 are its *leaves* and the vertex of degree κ is its *centre*.

We define a *ray* to be an undirected one-way infinite path. Any subgraph of a ray R that is itself a ray is called a *tail* of R. An undirected multigraph that does not contain a ray is called *rayless*.

A *comb* C is an undirected graph that is the union of a ray R together with infinitely many disjoint undirected finite paths each of which has precisely one vertex in common with R, which has to be an endvertex of that path. The endvertices of the finite paths that are not on R together with the endvertices of the trivial paths are the *teeth* of C.

For two vertex sets $X, Y \subseteq V$ we define $E(X, Y) \subseteq E$ as the set of those edges that have their head in $X \backslash Y$ and their tail in $Y \backslash X$, or their head in $Y \backslash X$ and their tail in $X \backslash Y$. Further we define $\overrightarrow{E}(X, Y) = \{uv \in E(X, Y) \; ; \; u \in X \text{ and } v \in Y\}$. If $X \cup Y = V$ and $X \cap Y = \varnothing$, we call $E(X, Y)$ a *cut* of D and refer to X and Y as the *sides* of the cut. Moreover, by writing $E(M, N)$ and calling it a cut of D we implicitly assume M and N to be the sided of that cut. We call two cuts $E(X_1, Y_1)$ and $E(X_2, Y_2)$ of D *nested* if either $X_1 \subseteq X_2$ and $Y_1 \supseteq Y_2$ holds or $X_2 \subseteq X_1$ and $Y_2 \supseteq Y_1$ is true. Moreover, we call a set or sequence of cuts of D nested if its elements are pairwise nested. If two cuts of D are not nested, we call them *crossing* (or say that they *cross*). A cut is said to *separate* two vertices $v, w \in V$ if v and w lie on different sides of that cut. We call a cut $E(X, Y)$ *directed*, or briefly a *dicut*, if all edges of $E(X, Y)$ have their head in one common side of the cut. We call D *finitely separable* if for any two different vertices $v, w \in V$ there exists a finite cut of D such that v and w are separated by that cut. If furthermore any two different vertices $v, w \in V$ can even be separated by a finite dicut of D, we call D *finitely diseparable*. A minimal non-empty cut is called a *bond*. Note that $D[X]$ and $D[Y]$ are weakly connected digraphs for a bond $E(X, Y)$. A bond that is also a dicut is called a *dibond*. For a vertex set $Y \subseteq V$ we define $\delta^-(Y) = \overrightarrow{E}(V \backslash Y, Y)$. Analogously, we set $\delta^+(Y) = \overrightarrow{E}(Y, V \backslash Y)$. Given a dicut $B = \overrightarrow{E}(X, Y)$ with sides $X, Y \in V$, we call Y the *in-shore* of B and X the *out-shore* of B. We shall writ $\mathrm{in}(B)$ for the in-shore of the dicut B and $\mathrm{out}(B)$ for the out-shore of B.

For undirected multigraphs *cuts*, *bonds*, *sides*, the notion of being *nested* and the notion of *separating* two vertices are analogously defined. Hence, we call an undirected multigraph *finitely separable* if any two vertices can be separated by a finite cut of the multigraph. Furthermore, in an undirected multigraph G with $X, Y \subseteq V(G)$ we write $E(X, Y)$ for the set of those edges of G that have one endvertex in $X \backslash Y$ and the other in $Y \backslash X$.

Let us mention two very basic but important observations with respect to dicuts.

Remark F.2.1. *Let D be a digraph and let X_n be an in-shore of a dicut of D for each $n \in \mathbb{N}$ such that $\bigcap_{n \in \mathbb{N}} X_n \neq \emptyset$. Then $\bigcap_{n \in \mathbb{N}} X_n$ and $\bigcup_{n \in \mathbb{N}} X_n$ are in-shores of dicuts of D as well.* $\square$

Note that $\bigcap_{n \in \mathbb{N}} X_n$ and $\bigcup_{n \in \mathbb{N}} X_n$ might be infinite dicuts of D, even if each X_n is finite. Furthermore, note that if X_1 and X_2 are in-shores of dibonds, $X_1 \cap X_2$ does not need to be an in-shore of a dibond, even if $X_1 \cap X_2 \neq \emptyset$.

Remark F.2.2. *Let D be a digraph and let X_1 and X_2 be in-shores of dicuts of D such that $X_1 \cap X_2 \neq \emptyset$. Then $\delta^-(X_1) \cup \delta^-(X_2) = \delta^-(X_1 \cup X_2) \cup \delta^-(X_1 \cap X_2)$.*

Moreover, if $\delta^-(X_1)$ and $\delta^-(X_2)$ are disjoint, then $\delta^-(X_1 \cup X_2)$ and $\delta^-(X_1 \cap X_2)$ are disjoint as well. $\square$

For a set $N \subseteq E$ let D/N denote the contraction minor of D which is obtained by contracting inside D all edges of N and deleting all loops that might occur. Similar, we define $D.N := D/(E \setminus N)$. For a vertex $v \in V$ and any contraction minor D/N with $N \subseteq E$ let $\dot{v}$ denote the vertex in D/N which corresponds to the contracted (possibly trivial) weak component of $D[N]$ containing v.

We state the following basic remark without proof.

Remark F.2.3. *Let D be a digraph and $v, w \in V(D)$. Then the following holds.*

(i) *If B is a cut or dicut in D, then it is also a cut or dicut, respectively, in $D.N$ for every $N \supseteq B$.*

(ii) *If B is a cut or dicut in $D.N$ for some $N \supseteq B$, then it is also a cut or dicut, respectively, in D.*

(iii) *If B is a cut or dicut in $D.M$ for some $M, N \subseteq E(D)$ with $N \supseteq M \supseteq B$, then it is also a cut or dicut, respectively, in $D.N$.*

(iv) *If B is a cut in D and separates v and w in D, then B separates $\dot{v}$ and $\dot{w}$ in $D.N$ for every $N \supseteq B$.*

(v) *If B is a cut in $D.N$ and separates $\dot{v}$ and $\dot{w}$ in $D.N$ for some $N \supseteq B$, then B separates v and w in D.*

(vi) *If B is a cut in $D.M$ and separates $\dot{v}$ and $\dot{w}$ in $D.M$ for some $M, N \subseteq E(D)$ with $N \supseteq M \supseteq B$, then B separates $\dot{v}$ and $\dot{w}$ in $D.N$.* □

For a multigraph G we call a subgraph $X \subseteq G$ a 2-*block* of G if X either consists of a set of pairwise parallel edges in G or is a maximal 2-connected subgraph of G. In a digraph D we call a subdigraph X a 2-*block* of D if $\mathrm{Un}(X)$ is a 2-block of $\mathrm{Un}(D)$.

We call an edge set $F \subseteq E$ a *dijoin* of D if $F \cap B \neq \emptyset$ holds for every non-empty dicut B of D. Similarly, we call an edge set $F \subseteq E$ a *finitary dijoin* of D if $F \cap B \neq \emptyset$ holds for every non-empty finite dicut B of D. Note that an edge set $F \subseteq E$ is already a (finitary) dijoin if $F \cap B \neq \emptyset$ holds for every (finite) dibond of D since every (finite) dicut is a disjoint union of (finite) dibonds. We call a pair $(F, \mathcal{B})$ consisting of a finitary dijoin F and a set of disjoint finite dicuts $\mathcal{B}$ an *optimal pair* for D if $F \subseteq \bigcup \mathcal{B}$ and $|F \cap B| = 1$ for every $B \in \mathcal{B}$. Furthermore, we call an optimal pair $(F, \mathcal{B})$ for D *nested*, if $\mathcal{B}$ is nested.

We state a basic remark about optimal pairs.

Remark F.2.4. *If $(F, \mathcal{B})$ is an optimal pair for a weakly connected digraph D, then each $B \in \mathcal{B}$ is a finite dibond of D.*

Proof. Suppose for a contradiction that there is some $B \in \mathcal{B}$ such that B is not a finite dibond of D. Since B is the disjoint union of finite dibonds of D, we find two finite dibonds B_1 and B_2 of D such that $B_1, B_2 \subseteq B$. By the property of $(F, \mathcal{B})$ is an optimal pair for D, we know that $|F \cap B| = 1$. This, however, implies that $F \cap B_j = \varnothing$ for some $j \in \{1, 2\}$. Now we have a contradiction to F being a finitary dijoin of D. $\square$

The following lemma is a basic tool in infinite graph theory. We shall only apply it for vertex sets of cardinality $\aleph_0$ and $\aleph_1$ in this section.

Lemma F.2.5. *B.2.4 Let G be an infinite connected undirected multigraph and let $U \subseteq V(G)$ be such that $|U| = \kappa$ for some regular cardinal κ. Then there exists a set $U' \subseteq U$ with $|U'| = |U|$ such that G either contains a comb whose set of teeth is U' or a subdivided star whose set of leaves is U'.*

Using Lemma F.2.5 let us now prove the following lemma.

Lemma F.2.6. *In a finitely separable rayless multigraph all* 2*-blocks are finite.*

Proof. Let G be a finitely separable rayless multigraph and suppose for a contradiction that there exists a 2-block X of G such that $V(X)$ is infinite. Let $U \subseteq V(X)$ be such that $|U| = \aleph_0$. Applying Lemma F.2.5 to U in X, we obtain a subdivided star S_1 in X whose set of leaves L_1 satisfies $|L_1| = |U|$ since G is rayless. Let c_1 be the centre of S_1. Using that X is 2-connected, we now apply Lemma F.2.5 to L_1 in $G - c_1$, which is still a connected rayless multigraph. Hence, we obtain a subdivided star S_2 in $G - c_1$ whose set of leaves L_2 satisfies $|L_2| = |L_1| = \aleph_0$ and $L_2 \subseteq L_1$. Let c_2 denote the centre of S_2. Now we get a contradiction to G being finitely separable because S_1 and S_2 have infinitely many common leaves in L_2. So $G[V(S_1) \cup V(S_2)]$ contains infinitely many disjoint c_1–c_2 paths, witnessing that c_1 and c_2 cannot be separated by a finite cut of G.

To complete the proof we still need to consider for a contradiction a 2-block X of G whose vertex set is finite but whose edge set is infinite. Since there are only finitely many two-element subsets of $V(X)$, we find by the pigeonhole principle two vertices $x, y \in V(X)$ such that infinitely many edges of X have x and y as their endvertices. Now these infinitely many edges witness that x and y cannot be separated by a finite cut in G, contradicting again that G in finitely separable. $\square$

We obtain the following immediate corollary.

Corollary F.2.7. *A finitely separable rayless multigraph has no infinite bond.*

Proof. By considering the 2-block-cutvertex tree (cf. [12, Lemma 3.1.4]) of a given multigraph we can easily deduce that each bond of that multigraph is contained in precisely one of its 2-blocks. Hence, the statement follows from Lemma F.2.6. $\square$

The following lemma makes a similar assertion as Lemma F.2.6 but without the assumption of being rayless. The proof strategy is the same as in Lemma F.2.6: We apply Lemma F.2.5 twice and use our assumption to ensure that we do not get a comb by the application of Lemma F.2.5. We state the proof for the sake of completeness here.

Lemma F.2.8. *Every* 2*-block of a finitely separable multigraph is countable.*

Proof. Let G be a finitely separable multigraph. Suppose for a contradiction that X is a 2-block of some finitely separable multigraph such that $V(X)$ is uncountable. We obtain that X is also finitely separable, and by definition that X is 2-connected. Let $U \subseteq V(X)$ be a set of cardinality $\aleph_1$. By applying Lemma F.2.5 with U in X we have to find a subdivided star S_1 whose set of leaves is some $U' \subseteq U$ with $|U'| = \aleph_1$. Let c_1 denote the centre of S_1. Using the 2-connectedness of X we know that $X - c_1$ is still connected. So we can again apply Lemma F.2.5, this time with U' in $X - c_1$. We obtain a subdivided star S_2 whose set of leaves is some $U'' \subseteq U'$ with $|U''| = \aleph_1$. Let c_2 be the centre of S_2. Since X is finitely separable, there exists a finite dicut B of X which separates c_1 from c_2. However, the subdivided stars S_1 and S_2, which have uncountably many common leaves in U'', witness that B cannot be finite. This is a contradiction.

It remains to consider for a contradiction a 2-block X of some finitely separable multigraph such that $V(X)$ is countable but $E(X)$ in uncountable. As before we know that X is finitely separable. Since there exist only countably many two-element subsets of $V(X)$, we have to find uncountably many edges in X that have pairwise the same endvertices, say x and y. Now we have again a contradiction to X being finitely separable since any dicut separating x and y would need to contain uncountably many edges. □

F.2.1. Quotients

For G being a digraph or a multigraph with $v, w \in V(G)$ let us write $v \equiv w$ if and only if we cannot separate v from w by a finite cut in G. It is easy to check that $\equiv$ defines an equivalence relation. For $v \in V(G)$ we shall write $[v]_\equiv$ for the equivalence class with respect to $\equiv$ containing v.

Let $G/{\equiv}$ denote the di- or multigraph which is formed from G by identifying for each equivalence class of $\equiv$ all vertices contained in it while keeping all edges that did not become loops. For any vertex $v \in V(G)$ let (v) denote the vertex of $V(G/{\equiv})$ corresponding to $[v]_\equiv$. Furthermore, let $\hat{X} := \{(x) \, ; \, x \in X\}$ for every set $X \subseteq V(D)$.

The proofs for the statements (i)-(iv) in the following proposition work analogously to those for the statements in Proposition F.2.12. Hence we only carry out the proof of Proposition F.2.12. The proof of statement (v) in the following proposition works via a proof by contradiction and using a straightforward inductive construction. Therefore, we omit to state it as well.

Proposition F.2.9. *Let G be a digraph or a multigraph. Then the following holds.*

(i) *$G/{\equiv}$ is (weakly) connected if G is (weakly) connected.*

(ii) *For every finite cut $E(X, Y)$ of G we get that $E(\hat{X}, \hat{Y})$ is a finite cut of $G/{\equiv}$ with $E(X, Y) = E(\hat{X}, \hat{Y})$.*

(iii) *For every finite cut $E(M, N)$ of $G/{\equiv}$ we get that $M = \hat{X}$ and $N = \hat{Y}$ for some finite cut $E(X, Y)$ of G with $E(\hat{X}, \hat{Y}) = E(X, Y)$.*

(iv) *$G/{\equiv}$ is finitely separable.*

(v) *$G/{\equiv}$ is rayless if $\mathrm{Un}(G)$ or G, respectively, is rayless.* □

Let D be any digraph. We define a relation $\sim$ on $V(D)$ by saying that $v \sim w$ for $v, w \in V(D)$ if and only if there is no finite dicut separating v and w. It is easy to check that $\sim$ defines an equivalence relation and so we omit a proof for this. Let $[v]_\sim$ denote the equivalence class of $\sim$ containing v.

We define the digraph $D/\sim$ in the same way as we defined the quotient $D/\equiv$ but now with respect to the relation $\sim$. For any vertex $v \in V(D)$ let $[v]$ denote the vertex of $V(D/\sim)$ which corresponds to $[v]_\sim$. Further, set $\tilde{X} = \{[x]\ ;\ x \in X\}$ for every set $X \subseteq V(D)$.

Next we prove some basic lemmas about the relation $\sim$ that we shall need later. The first lemma will help us to work with the relation $\sim$ more easily. More precisely, the lemma characterises the relation $v \sim w$ for any two vertices v, w of the digraph by the existence of a certain edge set working as a witness. For any finite cut separating v and w it will be enough to consider this edge set to see that this cut is not a dicut.

Lemma F.2.10. *Let D be a digraph and $v, w \in V(D)$. Then $v \sim w$ if and only if there is an edge set $C \subseteq E(D)$ such that $|C \cap \vec{E}(X,Y)| = |C \cap \vec{E}(Y,X)|$ holds, with $C \cap \vec{E}(X,Y) \neq \emptyset$ if $E(X,Y)$ separates v and w, for every finite cut $E(X,Y)$ of D.*

Moreover, $C = \emptyset$ is satisfies the properties for $v \sim w$ precisely when $v \equiv w$.

Proof. If an edge set C as in the statement of the lemma exists, then obviously $v \sim w$ holds.

For the converse we assume $v \sim w$. We prove the existence of the desired set C via a compactness argument. Let $\mathcal{B}$ be a finite set of finite cuts of D. Now we consider the finite contraction minor $D.(\bigcup \mathcal{B})$. Since $v \sim w$ and using Remark F.2.3, there is no dicut in $D.(\bigcup \mathcal{B})$ separating $\dot{v}$ and $\dot{w}$. This, however, implies the existence of a directed $\dot{v}$–$\dot{w}$ path and a directed $\dot{w}$–$\dot{v}$ path in $D.(\bigcup \mathcal{B})$. So the union of these paths yields an edge set $C_\mathcal{B} \subseteq \bigcup \mathcal{B}$ such that for each cut $E(X_\mathcal{B}, Y_\mathcal{B})$ of $D.(\bigcup \mathcal{B})$ we have $|C_\mathcal{B} \cap \vec{E}(X_\mathcal{B}, Y_\mathcal{B})| = |C_\mathcal{B} \cap \vec{E}(Y_\mathcal{B}, X_\mathcal{B})|$ with $C_\mathcal{B} \cap \vec{E}(X_\mathcal{B}, Y_\mathcal{B}) \neq \emptyset$ if $E(X_\mathcal{B}, Y_\mathcal{B})$ separates $\dot{v}$ and $\dot{w}$.

Now let $\mathcal{B}' \subseteq \mathcal{B}$ and let $C_\mathcal{B}$ be any subset of $\bigcup \mathcal{B}$ with the properties mentioned above. Then we get that $C_{\mathcal{B}'} := C_\mathcal{B} \cap \bigcup \mathcal{B}'$ satisfies the properties mentioned above as well but with respect to $D.(\bigcup \mathcal{B}')$ by Remark F.2.3.

By the compactness principle there exists an edge set $C \subseteq E(D)$ such that the equation $|C \cap \vec{E}(X,Y)| = |C \cap \vec{E}(Y,X)|$ holds for every finite cut $E(X,Y)$ of D as $E(X,Y)$ is also a cut of the finite contraction minor $D.(E(X,Y))$ by Remark F.2.3. Similarly, $C \cap \vec{E}(X,Y) \neq \varnothing$ if $E(X,Y)$ separates v and w, because $E(X,Y)$ separates $\dot{v}$ and $\dot{w}$ in the finite contraction minor $D.(E(X,Y))$ again by Remark F.2.3. Hence, C is as desired in the statement of the lemma.

For the last assertion of the lemma let us first assume $v \equiv w$. Then there is no finite dicut of D separating v and w by definition of $\equiv$. Therefore, $C = \varnothing$ satisfies all desired conditions and $v \sim w$.

For the converse we assume that $C = \varnothing$ satisfies all desired conditions and $v \sim w$. This implies that there is no finite cut of D separating v and w. Hence, we know $v \equiv w$. $\square$

For two vertices $v, w \in V(D)$ such that $v \sim w$ let us call any edge set $C \subseteq E(D)$ with the properties as in Lemma F.2.10 a *witness for* $v \sim w$. Note that there exists always an inclusion-minimal witness for $v \sim w$ by Zorn's Lemma.

The following lemmas tells us that given a minimal witness C for $v \sim w$, all vertices incident with an edge of C are also equivalent to v with respect to $\sim$.

Lemma F.2.11. *Let D be a digraph and $v \sim w$ for two vertices $v, w \in V(D)$. Then a minimal edge set C of D witnessing $v \sim w$ does also witness $v \sim y$ for any $y \in V(D[C])$.*

Proof. Let C be as in the statement of the lemma. Now suppose for a contradiction that there is a $y \in V(D[C])$ which is separated from v by a finite dicut $B = E(X,Y)$ of D and $C \cap B = \varnothing$. Without loss of generality let $y \in Y$. Since C witnesses $v \sim w$, both vertices v and w have to lie on the same side of B, namely X. We claim that $C' := C \cap E(D[X])$ does also witness $v \sim w$. This would be a contradiction to the minimality of C as y is incident with an edge of C both of which endvertices lie in Y because $C \cap B = \varnothing$.

Let us first consider any finite cut $E(M,N)$ of D. Since $E(X \cap M, Y \cup N)$ is also a finite cut, but $C \cap E(X \cap M, Y) = \varnothing$, we obtain the desired equation $|C' \cap \vec{E}(M,N)| = |C' \cap \vec{E}(N,M)|$.

Especially, if $E(M,N)$ separates v and w, then $E(X \cap M, Y \cup N)$ does so as well. Hence, the same argument yields $C' \cap \vec{E}(M,N) \neq \varnothing$. $\square$

We continue by collecting some properties of $D/\sim$ in the following proposition.

The proof of statement (v) needs a bit more preparation. Therefore, we shall postpone it until we have proved two further lemmas.

Proposition F.2.12. *Let D be a digraph. Then the following holds.*

(i) *$D/\sim$ is weakly connected if D is weakly connected.*

(ii) *For every finite dicut $E(X, Y)$ of D we get that $E(\tilde{X}, \tilde{Y})$ is a finite dicut of $D/\sim$ with $E(X, Y) = E(\tilde{X}, \tilde{Y})$.*

(iii) *For every finite dicut $E(M, N)$ of $D/\sim$ we get that $M = \tilde{X}$ and $N = \tilde{Y}$ for some finite dicut $E(X, Y)$ of D with $E(\tilde{X}, \tilde{Y}) = E(X, Y)$.*

(iv) *$D/\sim$ is finitely diseparable.*

(v) $\mathrm{Un}(D/\sim)$ *is rayless if* $\mathrm{Un}(D)$ *is rayless.*

Proof of statements (i)–(iv). Statement (i) is immediate.

If $E(X, Y)$ is a finite dicut of D, then for every $x \in X$ all vertices of $[x]_\sim$ are contained in X by definition of $\sim$. Analogously, all vertices of $[y]_\sim$ lie in Y for each $y \in Y$. Hence, $E(\tilde{X}, \tilde{Y})$ is a finite dicut of $D/\sim$ proving statement (ii).

Next let us verify statement (iii). Let $E(M, N)$ be a finite dicut of $D/\sim$. Then set $X = \bigcup\{m \in V(D) \ ; \ [m] \in M\}$ and $Y = \bigcup\{n \in V(D) \ ; \ [n] \in N\}$. By definition of $\sim$ we obtain that $E(X, Y)$ is a finite dicut of D as well as $M = \tilde{X}$ and $N = \tilde{Y}$ yielding $E(X, Y) = E(\tilde{X}, \tilde{Y})$.

In order to show statement (iv), let $[v]$ and $[w]$ be two different vertices of $V(D/\sim)$. Since v and w are not contained in the same equivalence class, there must exist a finite dicut $E(X, Y)$ of D separating them. By statement (ii) we get that $E(\tilde{X}, \tilde{Y})$ is a finite dicut of $D/\sim$ and it separates $[v]$ from $[w]$ by definition of $\sim$. $\square$

Before we can complete the proof of Proposition F.2.12, we have to prepare some lemmas. The first is about inclusion-minimal edge sets witnessing the equivalence of two vertices with respect to $\sim$ in digraphs whose underlying multigraph is rayless.

Lemma F.2.13. *Let D be a digraph such that* $\mathrm{Un}(D)$ *is rayless and let $v \sim w$ for two vertices $v, w \in V(D)$. Then any inclusion-minimal edge set of D witnessing $v \sim w$ is finite.*

Proof. Let $C \subseteq E(D)$ be an inclusion-minimal edge set witnessing that $v \sim w$. Due to the minimality of C we know that each element of C lies on a finite cut of D separating v and w. As each cut is a disjoint union of bonds, each edge in C is contained in a finite bond of D separating v and w. Using Proposition F.2.9 we get that $C \subseteq E(D/\equiv)$ where each edge in C lies on a finite bond of $D/\equiv$ separating (v) and (w).

Next we consider the 2-block-cutvertex tree T of $D/\equiv$ (cf. [12, Lemma 3.1.4]). Let P denote the finite path in T whose endvertices are the 2-blocks of $D/\equiv$ containing (v) and (w), respectively. Now we use the basic fact that each bond of a di- or multigraph is also a bond of a unique 2-block of that di- or multigraph, respectively, and therefore completely contained in that 2-block. Hence, each bond of $D/\equiv$ separating (v) and (w) is a bond of the finitely many 2-blocks corresponding to the vertices of P. This implies that all edges in C are contained in the finitely many 2-blocks which correspond to vertices of P. However, each 2-block of $D/\equiv$ is finite because $\mathrm{Un}(D/\equiv)$ is finitely separable and rayless by Proposition F.2.9 and such multigraphs do not have infinite 2-blocks by Lemma F.2.6. So C is contained in a finite set and thus itself finite. $\square$

The next lemma builds up on Lemma F.2.13 and is the last one we shall need to complete the proof of Proposition F.2.12.

Lemma F.2.14. *Let D be a digraph such that $\mathrm{Un}(D)$ is rayless and let $v \sim w$ for two vertices $v, w \in V(D)$. Then any minimal edge set of D witnessing $v \sim w$ is in $D/\equiv$ a strongly connected finite edge-disjoint union of directed cycles.*

Proof. Let C be a minimal edge set of D witnessing $v \sim w$. Since $\mathrm{Un}(D)$ is rayless, we know by Lemma F.2.13 that C is finite. Let $v_1, v_2, \ldots, v_n$ be the endvertices of all edges in C where $n \in \mathbb{N}$. By Lemma F.2.11 we know that $v_i \sim v$ holds for all i with $1 \leqslant i \leqslant n$. Next consider the set $M = \{(v_i)\ ;\ 1 \leqslant i \leqslant n\} \subseteq V(D/\equiv)$, whose size is at most n. Because of Remark F.2.3 we get that C is also an inclusion-minimal witness for $(v) \sim (w)$ and a witness for $(v) \sim (v_i)$ for every $(v_i) \in M$. We fix for each pair of vertices in M a cut of $D/\equiv$ that separates these two vertices, which is possible since $D/\equiv$ is finitely separable by Proposition F.2.9. Let $\mathcal{B}$ denote the set of all these cuts. As C witnesses $(v_i) \sim (v_j)$ for all $(v_i), (v_j) \in M$, we obtain that C intersects each cut in $\mathcal{B}$. Especially, $C \subseteq \bigcup \mathcal{B}$ as each edge in C has two vertices of M as its endvertices.

Next we consider the finite contraction minor $K := (D/{\equiv}).(\bigcup\mathcal{B})$. We observe, similarly as in the proof of Lemma F.2.10, that $K[C]$ is a finite edge-disjoint union of directed cycles. Furthermore, it contains a directed $(\dot{v})$–$(\dot{v_i})$ path and a directed $(\dot{v_i})$–$(\dot{v})$ path for every $(v_i) \in M$. Therefore, $K[C]$ is also strongly connected. Due to our choice of $\mathcal{B}$ we know that C is still a strongly connected finite edge-disjoint union of directed cycles in $D/{\equiv}$. $\square$

We are now able to prove the last statement of Proposition F.2.12.

Proof of statement (v) *of Proposition F.2.12.* Suppose for a contradiction that $\mathrm{Un}(D)$ is rayless but $\mathrm{Un}(D/{\sim})$ contains a ray $R = [v_0][v_1]\dots$ with vertices $[v_i] \in V(D/{\sim})$ for all $i \in \mathbb{N}$. For each $i \in \mathbb{N}$ let $v_i' \in [v_i]_\sim$ and $v_{i+1}'' \in [v_{i+1}]_\sim$ be the endvertices of the edge $[v_i][v_{i+1}]$ seen in D. Furthermore, let C_i be an inclusion-minimal edge set witnessing $v_i'' \sim v_{i+1}'$ for every $i \in \mathbb{N}$ with $i \geqslant 1$. We know by Lemma F.2.11 that each C_i is completely contained in $[v_i]_\sim$.

Next we consider the graph $D/{\equiv}$. Since $\mathrm{Un}(D)$ is rayless, we obtain that $\mathrm{Un}(D/{\equiv})$ is rayless as well by statement (v) of Proposition F.2.9. Therefore, we know by Lemma F.2.14 that each C_i is a strongly connected finite edge-disjoint union of directed cycles in $D/{\equiv}$. Since each C_i is completely contained in $[v_i]_\sim$, we get that $(D/{\equiv}[C_i]) \cap (D/{\equiv}[C_j]) = \emptyset$ holds for all $i, j \in \mathbb{N}$ with $i \neq j$. Similarly, v_i' and v_{i+1}'' lie in different equivalence classes with respect to $\equiv$ for every $i \in \mathbb{N}$ because they do so as well with respect to $\sim$. Let $P_i \subseteq D/{\equiv}$ be a directed (v_i'')–(v_{i+1}') path that is contained in C_i for every $i \in \mathbb{N}$ with $i \geqslant 1$. We define the edge set $M := \bigcup_{i \geqslant 1} E(P_i) \cup \bigcup_{i \geqslant 0} (v_i')(v_{i+1}'') \subseteq E(D/{\equiv})$. Now we derive a contradiction because the graph $D/{\equiv}[M]$ is a ray in $\mathrm{Un}(D/{\equiv})$. $\square$

Let us close this subsection with the following observation.

Lemma F.2.15. *For every digraph D each 2-block of $D/{\sim}$ is countable.*

Proof. We know by Proposition F.2.12 that each 2-block of $D/{\sim}$ is finitely diseparable. Hence, $\mathrm{Un}(X)$ is a 2-connected finitely separable multigraph. So Lemma F.2.8 implies the statement of this lemma. $\square$

F.3. Reductions for the Infinite Lucchesi-Younger Conjecture

In this section we prove some reductions for Conjecture F.1.5 and Conjecture F.1.7 in the sense that it suffices to solve these conjectures on a smaller class of digraphs. We begin by reducing these conjectures to finitely diseparable digraphs via the following lemma.

Lemma F.3.1. *Let D be a weakly connected digraph. Then the following statements are true:*

(i) *If $(F, \mathcal{B})$ is a (nested) optimal pair for D, then $(F, \tilde{\mathcal{B}})$ is a (nested) optimal pair, respectively, for $D/\sim$, where $\tilde{\mathcal{B}} := \{E(\tilde{X}, \tilde{Y})\ ;\ E(X, Y) \in \mathcal{B}\}$.*

(ii) *If $(F, \mathcal{B}')$ is a (nested) optimal pair for $D/\sim$, then there is a (nested) optimal pair $(F, \mathcal{B})$, respectively, for D such that $\mathcal{B}' = \tilde{\mathcal{B}}$ holds where we define $\tilde{\mathcal{B}} := \{E(\tilde{X}, \tilde{Y})\ ;\ E(X, Y) \in \mathcal{B}\}$.*

Proof. Note first that by Proposition F.2.12 $D/\sim$ is weakly connected because D is so. We now start with the proof of statement (i). Since $\mathcal{B}$ is a set of disjoint finite dicuts of D we obtain by Proposition F.2.12 that $\tilde{\mathcal{B}}$ is a set of disjoint finite dicuts of $D/\sim$. Furthermore, if $\mathcal{B}$ is nested, then so is $\tilde{\mathcal{B}}$ since the definition of $D/\sim$ ensures that we never identify two vertices of D that lie on different sides of a finite dicut of D. We also obtain $F \subseteq \bigcup \tilde{\mathcal{B}}$ and $|F \cap B'| = 1$ for every $B' \in \tilde{\mathcal{B}}$ because $(F, \mathcal{B})$ is a optimal pair for D and because of the definition of $\tilde{\mathcal{B}}$. In order to see that F is a finitary dijoin of $D/\sim$, consider any finite dicut $E(M, N)$ of $D/\sim$. We know by Proposition F.2.12 that $M = \tilde{X}$ and $N = \tilde{Y}$ holds for some finite dicut $E(X, Y)$ of D. Since F is a finitary dijoin of D, we know that F intersects with $E(X, Y)$. So F intersects with $E(M, N)$ as well.

Now we prove statement (ii). By Proposition F.2.12 we know that for each finite dicut $E(M, N)$ of $D/\sim$ we have $M = \tilde{X}$ and $N = \tilde{Y}$ for some finite dicut $E(X, Y)$ of D. Hence, $\mathcal{B}' = \tilde{\mathcal{B}}$ for some set $\mathcal{B}$ of finite dicuts of D. Since the elements of $\tilde{\mathcal{B}}$ are pairwise disjoint, we know that the elements of $\mathcal{B}$ are also pairwise disjoint. Furthermore, if $\tilde{\mathcal{B}}$ is nested, then $\mathcal{B}$ is nested as well. We directly obtain that $F \subseteq \bigcup \mathcal{B}$ and $|F \cap B| = 1$ holds for every $B \in \mathcal{B}$ since $(F, \tilde{\mathcal{B}})$ is an optimal pair for $D/\sim$. It remains to verify that F is a finitary dijoin of D. Using Proposition F.2.12

again we know that for any finite dicut $E(X,Y)$ of D the set $E(\tilde{X},\tilde{Y})$ is a finite dicut of $D/\sim$. Since F intersects $E(\tilde{X},\tilde{Y})$ as F is a finitary dijoin of $D/\sim$, we get that F intersects $E(X,Y)$ as well. So F is a finitary dijoin of D. □

The next reduction of Conjecture F.1.5 and Conjecture F.1.7 tells us that we can restrict our attention also to digraphs whose underlying multigraphs is 2-connected.

Lemma F.3.2. *Let D be a weakly connected digraph. Then the following statements are true:*

(i) *If $(F,\mathcal{B})$ is a (nested) optimal pair for D, then $(F \cap E(X), \mathcal{B}{\restriction}X)$ defines a (nested) optimal pair, respectively, for every 2-block X of D, where we set $\mathcal{B}{\restriction}X := \{B \in \mathcal{B}\,;\ B \subseteq E(X)\}$.*

(ii) *If $(F_X, \mathcal{B}_X)$ is a (nested) optimal pair for every $X \in \mathcal{X}$ of D, then $(\bigcup_{X\in\mathcal{X}} F_X, \bigcup_{X\in\mathcal{X}} \mathcal{B}_X)$ is a (nested) optimal pair, respectively, for D, where $\mathcal{X}$ denotes the set of all 2-blocks of D.*

Proof. We first prove statement (i). Let X be a 2-block of D. We assume that $(F,\mathcal{B})$ is an optimal pair for D. This implies that each element of $\mathcal{B}$ is a finite dibond of D by Remark F.2.4. By considering the 2-block-cutvertex tree of D (cf. [12, Lemma 3.1.4]) we can easily deduce that for any dibond $B = E(M,N)$ of D we have either $B \cap E(X) = \emptyset$ or $B \subseteq E(X)$. In the later case B is also a dibond of X, but with sides $M \cap V(X)$ and $N \cap V(X)$. Hence, if $\mathcal{B}$ is a set of disjoint dibonds of D, we get that $\mathcal{B}{\restriction}X$ is a set of disjoint dibonds of X. Furthermore, $\mathcal{B}{\restriction}X$ is nested if $\mathcal{B}$ is nested. We also directly obtain from our observation that $F \cap E(X) \subseteq \bigcup \mathcal{B}{\restriction}X$ and $|(F \cap E(X)) \cap B| = 1$ for every $B \in \mathcal{B}{\restriction}X$ since $(F,\mathcal{B})$ is an optimal pair for D. What remains is to check that $(F \cap E(X)$ is a finitary dijoin of X. It is easy to see using the 2-block-cutvertex tree of D (cf. [12, Lemma 3.1.4]) that any dibond of X is also a dibond of D, although with adapted sides. Hence, F intersects every finite dibond of X as F is a finitary dijoin of D. So F is also a finitary dijoin of X.

Now we show that statement (ii) is true. So let us assume that $(F_X, \mathcal{B}_X)$ is a optimal pair for every $X \in \mathcal{X}$. We know by Remark F.2.4 that all elements of some $\mathcal{B}_X$ are finite dibonds of the 2-block X of D. As noted in the proof of statement (i) all these dibonds are also finite dibonds of D. Hence, $\bigcup_{X\in\mathcal{X}} \mathcal{B}_X$ is a set of disjoint dibonds of D. Furthermore, if each $\mathcal{B}_X$ is nested, then it is easy to deduce that

$\bigcup_{X\in\mathcal{X}}\mathcal{B}_X$ is a set of nested dibonds of D using the 2-block-cutvertex tree of D (cf. [12, Lemma 3.1.4]). Using that for each $X \in \mathcal{X}$ the pair $(F_X, \mathcal{B}_X)$ is an optimal one, we immediately get $\bigcup_{X\in\mathcal{X}} F_X \subseteq \bigcup\bigcup_{X\in\mathcal{X}}\mathcal{B}_X$ and $|B \cap \bigcup_{X\in\mathcal{X}} F_X| = 1$ for every $B \in \bigcup_{X\in\mathcal{X}}\mathcal{B}_X$. To see that $\bigcup_{X\in\mathcal{X}} F_X$ is a finitary dijoin of D let B be any finite dicut of D. Then B contains a finite dibond B' of D, which needs to intersect with some 2-block of D, say with $X \in \mathcal{X}$. As noted in the proof of statement (i), we know that B' is also a finite dibond of X. Since $(F_X, \mathcal{B}_X)$ is an optimal pair for X, we get that $\bigcup_{X\in\mathcal{X}} F_X$ intersects B' and, therefore, also B. □

We can now close this section by proving Theorem F.1.9. In order to do this we basically only need to combine Lemma F.3.1 and Lemma F.3.2. Let us restate the theorem.

Theorem F.1.9.

(i) *If Conjecture F.1.5 holds for all countable finitely diseparable digraphs whose underlying multigraph is* 2*-connected, then Conjecture F.1.5 holds for all weakly connected digraphs.*

(ii) *If Conjecture F.1.7 holds for all countable finitely diseparable digraphs whose underlying multigraph is* 2*-connected, then Conjecture F.1.7, respectively, holds for all weakly connected digraphs.*

Proof. Let us prove statement (i) and assume that Conjecture F.1.5 holds for all countable finitely diseparable digraphs whose underlying multigraph is 2-connected. Now let D be any weakly connected digraph. We know by Proposition F.2.12 that $D/\sim$ is a weakly connected finitely diseparable digraph, and so is every 2-block of it. Furthermore, Lemma F.2.15 yields that each 2-block of $D/\sim$ is countable. By our assumption we know that Conjecture F.1.5 holds for every countable 2-block of $D/\sim$. So using Lemma F.3.2 we obtain an optimal pair for $D/\sim$. Then we also obtain an optimal pair for D by Lemma F.3.1.

The proof for statement (ii) works completely analogously to the one for statement (i). □

F.4. Special cases

In this section we prove some special cases of Conjecture F.1.7. Before we come to the first special case, we state a basic observation.

Lemma F.4.1. *In a weakly connected digraph D the following are equivalent:*

(i) *There is finitary dijoin of D of finite size.*

(ii) *There is a finite maximal number of disjoint finite dicuts of D.*

(iii) *There is a finite maximal number of disjoint and pairwise nested finite dicuts of D.*

Proof. We start by proving the implication from (i) to (ii). Let F be a finitary dijoin of D of finite size. Then, by definition, we can find at most $|F|$ many disjoint finite dicuts of D.

The implication from (ii) to (iii) is immediate.

Finally, we assume statement (iii) and prove statement (i). Let $\mathcal{B}$ be a finite set of maximal size containing disjoint and pairwise nested finite dicuts of D. We claim that $F := \bigcup \mathcal{B}$ is a finite finitary dijoin of D.

Suppose this is not the case. Then there exists a finite dicut B_0 of D which is disjoint to F. By our choice of $\mathcal{B}$ we know that B_0 is not nested with each element of $\mathcal{B}$. Let $\mathcal{B}'_0 = \{B'_0, \dots, B'_k\}$ with $k \in \mathbb{N}$ be the set of those elements of $\mathcal{B}$ which are crossing with B_0. Further, let B'_i with $i \in \{0, \dots, k\}$ be such that either $\mathrm{in}(B'_i)$ or $\mathrm{out}(B'_i)$ is inclusion-minimal among all sides of the dicuts $B'_j \in \mathcal{B}'_0$. If $\mathrm{in}(B'_i)$ is inclusion-minimal among all sides of the elements $B'_j \in \mathcal{B}'_0$, set $B''_i := \delta^-(\mathrm{in}(B'_i) \cap \mathrm{in}(B_0))$ and $B_1 := \delta^-(\mathrm{in}(B'_i) \cup \mathrm{in}(B_0))$. Otherwise, define $B''_i := \delta^-(\mathrm{in}(B'_i) \cup \mathrm{in}(B_0))$ as well as $B_1 := \delta^-(\mathrm{in}(B'_i) \cap \mathrm{in}(B_0))$. We also define $\mathcal{B}'_1 = \mathcal{B}'_0 \backslash \{B'_i\}$. By Remark F.2.1 and Remark F.2.2 we know that B_1 and B''_i are nested finite dicuts of D and the elements of the set $\{B_1, B''_i\} \cup \mathcal{B}'_1$ are pairwise disjoint. Furthermore, B''_i is nested with each element of $\mathcal{B}$ and B_1 is nested with each element of $\mathcal{B} \backslash \mathcal{B}'_1$.

We can repeat the argument with B_1 instead of B_0 and with $\mathcal{B}'_1$ instead of $\mathcal{B}'_0$. Iterating this procedure we obtain after $k+1$ steps the set $\mathcal{B}'' = \{B''_0, \dots, B''_k\}$ and the finite dicut B_k of D such that $(\mathcal{B} \backslash \mathcal{B}'_0) \cup \mathcal{B}'' \cup \{B_k\}$ is a nested set of disjoint finite dicuts of D. This, however, is a contradiction to the maximality of the set $\mathcal{B}$. Hence, F is a finite finitary dijoin of D. $\square$

Now we prove a first special case for Conjecture F.1.7 about digraphs that admit a finitary dijoin of finite size.

Lemma F.4.2. *Let D be a weakly connected digraph with one of the following properties:*

(i) *D has a finitary dijoin of finite size.*

(ii) *There is a finite maximal number of disjoint finite dicuts of D.*

(iii) *There is a finite maximal number of disjoint and pairwise nested finite dicuts of D.*

Then Conjecture F.1.7 holds for D.

Proof. We know by Lemma F.4.1 that properties (i), (ii) and (iii) are equivalent. So let us fix a set $\mathcal{B}$ of maximum size which consists of pairwise nested and disjoint finite dicuts of D. By assumption $|\mathcal{B}|$ is finite.

Let $N \subseteq E(D)$ be a finite set of edges such that $\bigcup \mathcal{B} \subseteq N$ holds and $D.N$ is weakly connected. Since $D.N$ is a finite weakly connected digraph, there exists a nested optimal pair $(F_N, \mathcal{B}_N)$ for $D.N$ by Theorem F.1.6. By the choice of N we know that each element of $\mathcal{B}$ is also a finite dicut of $D.N$. Furthermore, each finite dicut in $D.N$ is also one in D and, thus, $\mathcal{B}_N$ is a set of disjoint finite dicuts in D. Hence, $|\mathcal{B}| = |\mathcal{B}_N| = |F_N|$. Using that the elements in $\mathcal{B}$ are pairwise nested and disjoint finite dicuts, we get that $(F_N, \mathcal{B})$ is a nested optimal pair for $D.N$ as well. Given a finite edge set $M \supseteq N$ with a nested optimal pair $(F_M, \mathcal{B}_M)$ in $D.M$ we obtain that $(F_M, \mathcal{B})$ is also a nested optimal pair for $D.N$.

Note that for any finite edge set $N \subseteq E(D)$ satisfying $\bigcup \mathcal{B} \subseteq N$ there are only finitely many possible edge sets $F_N \subseteq \bigcup \mathcal{B}$ such that $(F_N, \mathcal{B})$ is a nested optimal pair for $D.N$. Hence, we get via the compactness principle an edge set $F \subseteq \bigcup \mathcal{B}$ with $|F \cap B| = 1$ for every $B \in \mathcal{B}$ such that $(F, \mathcal{B})$ is a nested optimal pair for $D.M$ for every finite edge set $M \subseteq E(D)$ satisfying $\bigcup \mathcal{B} \subseteq M$.

We claim that $(F, \mathcal{B})$ is a nested optimal pair for D. We already know by definition that $\mathcal{B}$ is a nested set of disjoint finite dicuts of D and that $F \subseteq \bigcup \mathcal{B}$ with $|F \cap B| = 1$ for every $B \in \mathcal{B}$. It remains to check that F is a finitary dijoin of D. So let B' be any finite dicut of D. Then the set $N' := B' \cup \bigcup \mathcal{B}$ is also finite and B' is a finite dicut of $D.N'$. Since $(F, \mathcal{B})$ is also a nested optimal pair for $D.N'$, we know that $F \cap B' \neq \emptyset$ holds, which proves that F is a finitary dijoin of D. $\square$

We continue with another special case. Its proof is also based on a compactness argument. However, we need to choose the set up for the argument more carefully.

Lemma F.4.3. *Conjecture F.1.7 holds for weakly connected digraphs in which every edge lies in only finitely many finite dibonds.*

Proof. Let D be a weakly connected digraph where every edge lies in only finitely many finite dibonds. For an edge $e \in E(D)$ let $\mathcal{B}_e$ denote the set of finite dibonds of D that contain e. Our assumption on D implies that $\mathcal{B}_e$ is a finite set. For a finite set $\mathcal{B}$ of finite dibonds of D we define $\hat{\mathcal{B}} = \bigcup\{\mathcal{B}_e \;;\; e \in \bigcup \mathcal{B}\}$. Again our assumption on D implies that $\hat{\mathcal{B}}$ is finite. Note that $\mathcal{B} \subseteq \hat{\mathcal{B}}$ holds.

Given a finite set $\mathcal{B}$ of finite dibonds of D, we call $(F_{\mathcal{B}}, \mathcal{B}')$ a *nested pre-optimal pair* for $D.(\bigcup \mathcal{B})$ if the following hold:

1. $F_{\mathcal{B}}$ intersects every element of $\mathcal{B}$,
2. $\mathcal{B}' \subseteq \hat{\mathcal{B}}$,
3. the elements of $\mathcal{B}'$ are pairwise nested,
4. $F_{\mathcal{B}} \subseteq \bigcup \mathcal{B}'$, and
5. $|F_{\mathcal{B}} \cap B'| = 1$ for every $B' \in \mathcal{B}'$.

We know that for every finite set $\mathcal{B}$ of finite dibonds of D there exists a nested pre-optimal pair for $D.(\bigcup \mathcal{B})$, since a nested optimal pair for $D.(\bigcup \hat{\mathcal{B}})$ is one and it exists by Theorem F.1.6. However, there can only be finitely many nested pre-optimal pairs for $D.(\bigcup \mathcal{B})$ as $\bigcup \hat{\mathcal{B}}$ is finite.

Now let $\mathcal{B}_1$ and $\mathcal{B}_2$ be two finite sets of finite dibonds of D with $\mathcal{B}_1 \subseteq \mathcal{B}_2$, and let $(F_{\mathcal{B}_2}, \mathcal{B}_2')$ be a nested pre-optimal pair for $D.(\bigcup \mathcal{B}_2)$. Then $(F_{\mathcal{B}_2} \cap \bigcup \mathcal{B}_1, \mathcal{B}_2' \cap \hat{\mathcal{B}}_1)$ is a nested pre-optimal pair for $D.(\bigcup \mathcal{B}_1)$. Now we get by the compactness principle an edge set $F_D' \subseteq E(D)$ and a set $\mathcal{B}_D$ of finite dibonds of D such that $(F_D' \cap \bigcup \mathcal{B}, \mathcal{B}_D \cap \hat{\mathcal{B}})$ is a nested pre-optimal pair for $D.(\bigcup \mathcal{B})$ for every finite set $\mathcal{B}$ of finite dibonds of D. Further let $F_D \subseteq F_D'$ be such that each element of F_D lies on a finite dibond of D and $(F_D \cap \bigcup \mathcal{B}, \mathcal{B}_D \cap \hat{\mathcal{B}})$ is still a nested pre-optimal pair for $D.(\bigcup \mathcal{B})$ for every finite set $\mathcal{B}$ of finite dibonds of D.

We claim that $(F_D, \mathcal{B}_D)$ is a nested optimal pair for D. First we verify that F_D is a finitary dijoin of D. Let B be any finite dibond of D. Then F_D intersects B, because $(F_D \cap B, \mathcal{B}_D \cap \widehat{\{B\}})$ is a nested pre-optimal pair for $D.B$. So F_D is a finitary dijoin of D.

Next consider any element $e \in F_D$. By definition of F_D we know that $e \in B_e$ holds for some finite dibond B_e of D. Using again that $(F_D \cap B_e, \mathcal{B}_D \cap \widehat{\{B_e\}})$ is a nested pre-optimal pair for $D.B_e$, we get that $e \in \bigcup \mathcal{B}_D$. So the inclusion $F_D \subseteq \bigcup \mathcal{B}_D$ is valid.

Given any $B_D \in \mathcal{B}_D$ we know that $(F_D \cap B_D, \mathcal{B}_D \cap \widehat{\{B_D\}})$ is a nested pre-optimal pair for $D.B_D$. Hence, $|F_D \cap B| = 1$ holds for every $B \in \mathcal{B}_D \cap \widehat{\{B_D\}}$. Especially, $|F_D \cap B_D| = 1$ is true because $B_D \in \mathcal{B}_D \cap \widehat{\{B_D\}}$.

Finally, let us consider two arbitrary but different elements B_1 and B_2 of $\mathcal{B}_D$. We know that $(F_D \cap (B_1 \cup B_2), \mathcal{B}_D \cap \widehat{\{B_1, B_2\}})$ is a nested pre-optimal pair for $D.(B_1 \cup B_2)$. Therefore, B_1 and B_2 are disjoint and nested. This shows that $(F_D, \mathcal{B}_D)$ is a nested optimal pair for D and completes the proof of this lemma. $\square$

Before we can continue proving further special cases of Conjecture F.1.7, we have to state the following lemma, which is due to Thomassen and Woess. This lemma is a helpful tool in infinite graph theory. For us it will be especially useful in connection with Lemma F.4.3.

Lemma F.4.4. [65, Prop. 4.1] *Let G be a connected graph, $e \in E(G)$ and $k \in \mathbb{N}$. Then there are only finitely many bonds of G of size k that contain e.*

The next lemma can be used together with Lemma F.4.3 to deduce that Conjecture F.1.7 holds for weakly connected digraphs without infinite dibonds.

Lemma F.4.5. *In a weakly connected digraph without infinite dibonds each edge lies in only finitely many finite dibonds.*

Proof. Let D be a weakly connected digraph and $e \in E(D)$ such that it lies on infinitely many finite dibonds. We shall prove that e lies on some infinite dibond of D. Since e lies on only finitely many dibonds of D with size k for every $k \in \mathbb{N}$ by Lemma F.4.4, we can pick a sequence $(B''_n)_{n \in \mathbb{N}}$ of finite dibonds of D all containing e such that $|B''_n| < |B''_{n+1}|$ holds for every $n \in \mathbb{N}$. Iteratively using Remark F.2.1 we can obtain a nested sequence $(B'_n)_{n \in \mathbb{N}}$ of finite dibonds of D all containing e such that $|B'_n| < |B'_{n+1}|$ holds for every $n \in \mathbb{N}$, where the inequality can again be achieved due to Lemma F.4.4. Now we can find an infinite set $I \subseteq \mathbb{N}$ such that either $\text{in}(B'_i) \supseteq \text{in}(B'_j)$ holds for all $i, j \in I$ with $i \leqslant j$ or $\text{in}(B'_i) \subseteq \text{in}(B'_j)$ is true for all $i, j \in I$ with $i \leqslant j$. Since the following argument is symmetric with respect to in- or out-shores, we assume without loss of generality that the first case holds for the sequence $(B'_i)_{i \in I}$.

We inductively find an edge set $E^* = \{e_i \in E(D) \ ; \ i \in \mathbb{N}\}$ together with a subsequence $(B_n)_{n \in \mathbb{N}}$ of $(B'_i)_{i \in I}$ in the sense that there is an order preserving bijection $\sigma : I \longrightarrow \mathbb{N}$ such that $B'_i = B_{\sigma(i)}$ holds for every $i \in I$, such that the following properties are fulfilled:

1. $e_0 = e$ holds.
2. $E^*_n := \{e_0, \ldots, e_n\} \subseteq B_n$ holds for every $n \in \mathbb{N}$.
3. $\text{in}(B_n)$ contains an undirected tree T_n that covers all heads of edges in E^*_n and satisfies $E(T_n) \cap B_m = \emptyset$ for all $n, m \in \mathbb{N}$ with $m \geqslant n$.

We start by setting $B_0 := B'_{k_0}$ for some $k_0 \in I$ such that B_0 contains an edge $e'_1 \notin E^*_0$. This is possible since $|B'_i| < |B'_j|$ holds for all $i, j \in I$. Further, set T_0 as the head of e_0. Let v'_1 be the head of e'_1. Now let P'_1 be an undirected $\{v'_1\}$–$V(T_0)$ path in $D[\text{in}(B_0)]$. Such a path exists since B_0 is a dibond of D and so $D[\text{in}(B_0)]$ is weakly connected. We now define e_1 to be the last edge on P'_1 in the direction from v'_1 to T_0 which lies in infinitely many dibonds in $(B'_i)_{i \in I}$ if it exists, and $e_1 := e'_1$ otherwise. Note that there needs to be an edge in $E(P'_1) \cup \{e'_1\}$ which lies in infinitely many dibonds in $(B'_i)_{i \in I}$ because $\text{in}(B'_i) \subseteq \text{in}(B'_{k_0})$ holds for all $i \in I$ with $i \geqslant k_0$ and so $B'_i \cap (E(P'_1) \cup \{e'_1\}) \neq \emptyset$ holds for all $i \geqslant k_0$. Let v_1 be the head of e_1. Now we set $I_1 \subseteq I \backslash \{k_0\}$ to be an infinite index set such that $e_1 \in B'_i$ for all $i \in I_1$ and $E(P_1) \cap B'_i = \emptyset$ for all $i \in I_1$ where P_1 is the $\{v_1\}$–$V(T_0)$ path contained in P'_1. Also we set $T_1 := T_0 \cup P_1$ and $B_1 := B'_{k_1}$ for some $k_1 \in I_1$ such that B_1 contains an edge $e'_2 \notin E^*_1$. Note that $T_1 \subseteq D[B'_i]$ for each $i \in I_1$ by construction. Now we repeat the argument with k_1 instead of k_0 and with T_1 instead of T_0, etc. Iterating this construction infinitely often yields our desired sequence $(B_n)_{n \in \mathbb{N}}$ of finite dibonds of D.

Let B' be the dicut of D whose in-shore is defined via $\text{in}(B') := \bigcap_{n \in \mathbb{N}} \text{in}(B_n)$. Remark F.2.1 ensures that B' is in fact a dicut of D. Note that the equality $\text{out}(B') = \bigcup_{n \in \mathbb{N}} \text{out}(B_n)$ holds and each $\text{out}(B_n)$ induces a weakly connected subdigraph of D as B_n is a dibond of D. So we know that $D[\mathrm{out}(B')]$ is weakly connected as well. Now we set $T := \bigcup_{n\in\mathbb{N}} T_n$. Property (3) ensures that $V(T) \subseteq \mathrm{in}(B')$ holds. Let K be the vertex set of the weak component of $D[\mathrm{in}(B')]$ that contains $V(T)$. Then $B := E(V(D)\backslash K, K)$ is a dicut of D whose in-shore is K. By definition $D[K]$ is a weakly connected subdigraph of D and $B \subseteq B'$ holds. Let $\mathcal{C}$ be the set of weak components of $D[\mathrm{in}(B')]$. Since each element of $\mathcal{C}$ is adjacent with $\mathrm{out}(B')$, we obtain that $D[\mathrm{out}(B)]$ is also a weakly connected digraph. Hence, B is a dibond of D. Finally, property (2) together with property (3) ensure that $E^* \subseteq B$ holds. Especially, $e = e_0 \in B$ by property (1). So B is an infinite dibond of D containing e. □

As noted before, we obtain the following corollary.

Corollary F.4.6. *Conjecture F.1.7 holds for weakly connected digraphs without infinite dibonds.* □

We close this section with a last special case where we can show that Conjecture F.1.7 holds.

Lemma F.4.7. *Conjecture F.1.7 holds for weakly connected digraphs whose underlying multigraph is rayless.*

Proof. Let D be a weakly connected digraph such that $\mathrm{Un}(D)$ is rayless. We know by Proposition F.2.12 that $\mathrm{Un}(D/\sim)$ is rayless as well, and that $D/\sim$ is weakly connected and finitely diseparable. So we obtain from Corollary F.2.7 that $D/\sim$ has no infinite dibond. Now Corollary F.4.6 implies that Conjecture F.1.7 is true in the digraph $D/\sim$. Using again that $D/\sim$ is finitely diseparable, any nested optimal pair for $D/\sim$ directly translates to one for D by Lemma F.3.1. Hence, Conjecture F.1.7 is true for D as well. □

Apendix

Summary

Different aspects of connectivity in infinite directed and undirected graphs are studied in this dissertation, which consists of three chapters.

In Chapter I, graphs are studied whose cardinality is not necessarily bounded by further assumptions. In Section A of this chapter, those graphs are characterised that contain $\mathbb{Z} \times \mathbb{Z}$ as a minor. This is done in terms of the existence of a certain collection of rays. Furthermore, we prove a duality theorem characterising those graphs not containing $\mathbb{Z} \times \mathbb{Z}$ as a minor by the existence of a certain tree-decomposition for them.

Section B is dedicated to a characterisation of graphs containing a k-connected vertex set of fixed but arbitrary infinite cardinality, where $k \in \mathbb{N}$. This is done via the existence of minors of certain, so-called *k-typical* graphs as well as via the existence of subdivisions of certain, so-called *generalised k-typical* graphs. Given the number $k \in \mathbb{N}$ and fixing the cardinality of the k-connected vertex set, there are only finitely many k-typical graphs and also only finitely many generalised k-typical ones, which is also proved in that section. Finally, we prove a duality theorem characterising the absence of a k-connected vertex set of a certain cardinality in a graph by the existence of a certain nested set of separations of order less than $k \in \mathbb{N}$ of the graph.

The last section of Chapter I, Section C, answers a question of Georgakopoulos [25]. Given a countable end ω of some graph, i.e., an end which does not contain uncountably many disjoint rays, that section contains the proof for the existence of a set of ω-devouring rays in that graph whose set of startvertices can be arbitrarily chosen as long as it is the set of startvertices of any set of disjoint rays in ω.

In contrast to Chapter I, Chapter II focusses on locally finite connected graphs, which must be of countable cardinality. By considering the Freudenthal compactifi-

cation of such graphs, a topological definition of infinite cycles can be given, which coincides with the usual definition for finite cycles, but also allows infinite ones. Cycles are defined as homeomorphic images of the unit circle $S^1 \subseteq \mathbb{R}^2$ into the Freudenthal compactification of a locally finite connected graph. Now questions regarding Hamiltonicity of infinite graphs can be asked in a more meaningful way.

In Section D, two sufficient conditions for Hamiltonicity of such graphs are proved. These results extend to locally finite connected graphs corresponding theorems for finite graphs. First it is shown that the square of any locally finite connected graph on at least three vertices which contains the edge set of a spanning topological caterpillar is Hamiltonian.

The second result says that for locally finite connected graphs being 2-connected and neither containing a subdivision of a K^4 nor of a $K_{2,3}$ is equivalent to the existence of an embedding of the Freudenthal compactification of the graph into a closed disk such that the boundary of that disk witnesses the Hamiltonicity of the graph. Moreover, it is shown that such graphs are uniquely Hamiltonian.

The last result of Section D affirmatively answers the question of Mohar [47] whether an uniquely Hamiltonian cubic infinite graph exists all whose ends have vertex- or edge-degree 3. Such a graph with vertex- and edge-degree 3 at every end of the graph is constructed in that section.

The last chapter of this dissertation, Chapter III, deals with directed infinite graphs. In the first section of that chapter Edmonds' Branching Theorem is extended to locally finite directed graphs. For this the notion of pseudo-arborescences is defined, which generalises ordinary arborescences in finite directed graphs. There we prove a corresponding packing result for pseudo-arborescences. In order to do this, methods from infinite directed graph theory and topological infinite graph theory, as used in Chapter II, are combined. For this reason, the directed graphs appearing in this section are countable. Furthermore, the structure of pseudo-arborescences is studied in Section E as well.

In Section F, the second and also last section of Chapter III, a conjecture is made which, if affirmatively proved, would generalise the theorem of Lucchesi and Younger for finite directed graphs to infinite directed graphs in a way as it has been done with the theorem of Menger. Different from Section E, directed graphs of arbitrary cardinality are considered in Section F. One of the main results there, however, is that it suffices to verify the conjecture for countable directed graphs.

Apart from that, the conjecture is verified for several classes of directed graphs in Section F, which forms the other main result of that section.

Zusammenfassung

Diese Dissertation, mit dem ins Deutsche übersetzten Titel "Zusammenhang in gerichteten und ungerichteten unendlichen Graphen", welche in drei Kapitel unterteilt ist, behandelt verschiedene Aspekte von Zusammenhang in unendlichen gerichteten sowie ungerichteten Graphen.

Im ersten Kapitel werden Graphen studiert, deren Kardinalität nicht notwendiger Weise durch weitere Annahmen beschränkt ist. In Abschnitt A werden diejenigen Graphen charakterisiert, welche $\mathbb{Z} \times \mathbb{Z}$ als Minor enthalten. Dies geschieht mittels der Existenz einer bestimmten Menge von Strahlen. Außerdem beweisen wir einen Dualitätssatz, welcher die Nichtexistenz von $\mathbb{Z} \times \mathbb{Z}$ als Minor in einem Graphen durch die Existenz einer bestimmten Baumzerlegung des Graphen charakterisiert.

Abschnitt B ist der Charakterisierung solcher Graphen gewidmet, welche eine k-zusammenhängende Eckenmenge fester aber beliebiger unendlicher Kardinalität enthalten, wobei $k \in \mathbb{N}$. Dies geschieht mittels der Existenz von Minoren von bestimmten, sogenannten *k-typischen* Graphen und außerdem durch die Existenz von Unterteilungen von bestimmten, sogenannten *verallgemeinerten k-typischen* Graphen. Bei gegebenem $k \in \mathbb{N}$ und fixierter Kardinalität der k-zusammenhängenden Eckenmenge gibt es bloß endlich viele k-typische Graphen und auch nur endlich viele verallgemeinerte k-typische Graphen. Dies wird ebenfalls in jenem Abschnitt bewiesen. Schließlich zeigen wir einen Dualitätssatz, welcher die Nichtexistenz einer k-zusammenhängenden Eckenmenge fixer Kardinalität in einem Graphen durch die Existenz einer bestimmten geschachtelten Menge von Teilungen des Graphen der Ordnung kleiner als $k \in \mathbb{N}$ charakterisiert.

Der letzte Abschnitt des ersten Kapitels, Abschnitt C, beantwortet eine Frage von Georgakopoulos [25]. Für ein abzählbares Ende ω eines beliebigen Graphen, was bedeutet, dass ω nicht überabzählbar viele disjunkte Strahlen enthält, wird in diesem Abschnitt folgendes bewiesen: Es existiert eine Menge ω-verschlingender Strahlen im Graphen mit beliebig wählbarer Menge von Startecken, sofern diese Menge von Startecken für irgendeine Menge disjunkter Strahlen aus ω auch die Menge der Startecken darstellt.

Im Kontrast zum ersten Kapitel konzentriert sich das zweite Kapitel auf lokal endliche zusammenhängende Graphen, welche somit abzählbare Kardinalität haben müssen. Mittels der Freudenthal-Kompaktifizierung solcher Graphen kann eine topologische Definition von Kreisen angegeben werden, welche für endliche Kreise mit der gängigen Definition übereinstimmt, nun aber auch unendliche Kreise zulässt. Kreise werden definiert als homöomorphe Bilder des Einheitskreises $S^1 \subseteq \mathbb{R}^2$ in der Freudenthal-Kompaktifizierung eines lokal endlichen zusammenhängenden Graphen. Damit können nun Fragen bezüglich der Hamiltonizität unendlicher Graphen in einem aussagekräftigeren Kontext gestellt werden.

In Abschnitt D werden zwei hinreichende Bedingungen für die Hamiltonizität solcher Graphen bewiesen. Diese Resultate verallgemeinern entsprechende Sätze über die Hamiltonizität endlicher Graphen auf lokal endliche zusammenhängende Graphen. Zuerst wird gezeigt, dass das Quadrat eines jeden lokal endlichen zusammenhängenden Graphen mit mindestens drei Ecken, welcher die Kantenmenge einer aufspannenden topologischen Raupe enthällt, hamiltonisch ist.

Das zweite Resultat besagt, dass 2-zusammenhängend zu sein und weder einen K^4, noch einen $K_{2,3}$ als Minor zu enthalten, für lokal endliche zusammenhängende Graphen dazu äquivalent ist, dass eine Einbettung der Freudenthal-Kompaktifizierung des Graphen in eine abgeschlossene Kreisscheibe existiert, deren Rand gleichzeitig die Hamiltonizität des Graphen bezeugt. Darüber hinaus wird gezeigt, dass solche Graphen auf eindeutige Weise hamiltonisch sind.

Das letzte Resultat von Abschnitt D gibt eine positive Antwort auf die Frage von Mohar [47], ob ein kubischer unendlicher Graph existiert, der auf eindeutige Weise hamiltonisch ist und nur Enden besitzt, deren Ecken- oder Kantengrad 3 ist. Solch ein Graph, bei dem jedes seiner Enden sowohl Ecken- als auch Kantengrad 3 hat, wird in jenem Abschnitt konstruiert.

Das dritte und letzte Kapitel dieser Dissertation behandelt gerichtete unendliche Graphen. Im ersten Abschnitt dieses Kapitels, Abschnitt E, wird Edmonds' Branching Theorem auf lokal endliche gerichtete Graphen erweitert. Dafür wird der Begriff der Pseudo-Arboreszenz definiert, welcher den Begriff der herkömmlichen Arboreszenz in endlichen Graphen generalisiert, und ein entsprechender Satz bewiesen, der die Existenz von $k \in \mathbb{N}$ vielen kanten-disjunkten Pseudo-Arboreszenzen in einem gerichteten Graphen charakterisiert. Hierbei werden Methoden für unendlichen gerichtete Graphen mit Methoden aus der topologischen unendlichen Graphentheorie, wie im zweiten Kapitel dieser Dissertation verwendet, kombiniert. Aus diesem Grund haben die in diesem Abschnitt betrachteten gerichteten Graphen abzählbare Kardinalität. Schließlich wird die Struktur von Pseudo-Arboreszenzen studiert.

In Abschnitt F, dem zweiten und letzten Abschnitt des dritten Kapitels, wird eine Vermutung aufgestellt, die, sofern sie bestätigt wird, den Satz von Lucchesi und Younger für endliche gerichtete Graphen auf solche Weise auf unendliche gerichtete Graphen verallgemeinert, wie es bereits mit dem Satz von Menger passierte. Anders als in Abschnitt E werden in Abschnitt F gerichtete Graphen beliebiger Kardinalität betrachtet. Eines der Hauptresultate hier ist allerdings, dass es ausreichend ist die Vermutung für abzählbare gerichtete Graphen zu verifizieren. Das andere Hauptresultat jenes Abschnitts beinhaltet die Bestätigung der Vermutung für diverse Klassen von gerichteten Graphen.

Publications related to this dissertation

The following articles are related to this dissertation:

Chapter I:

1. Section A is based on [37].
2. Section B is based on [28].
3. Section C is based on [27].

Chapter II:

Section D is based on [38].

Chapter III:

1. Section E is based on [26].
2. Section F is based on [29].

Declaration on my contributions

I am the sole contributor for the whole Section A in Chapter I and for all of Chapter II. All other sections are based on articles together with J. Pascal Gollin as my only co-author, as mentioned in the section 'Publications related to this dissertation'. On the articles J. Pascal Gollin and I worked together, we share an equal amount of work.

The research which led to the results in Section B of Chapter I was inspired by and started after a research seminar talk of Prof. Dr. Reinhard Diestel at the University of Hamburg, in which J. Pascal Gollin and I both participated.

The main result in Section C of Chapter I answers a question of Georgakopoulos [25]. I studied the corresponding article of Georgakopoulos [25] for my work in the field of topological infinite graph theory, especially during my work concerning the content of Chapter II. I came up with a first proof idea for locally finite graphs. Later J. Pascal Gollin and I improved it and generalised it to arbitrary graphs.

Prof. Dr. Reinhard Diestel encouraged J. Pascal Gollin and me to extend Edmonds' Branching Theorem to infinite digraphs, which is the topic of Section E in Chapter III. After common work and research on this topic I drafted a first version of the article this section is based on.

During my studies regarding directed graphs I learned about the theorem of Lucchesi and Younger F.1.3 for finite directed graphs. On my own I tried to prove a version of this theorem for infinite directed graphs. I came up with the example in Section F whose conclusion is that a direct extension, also respecting infinite dicuts, is not possible. Then I made Conjecture F.1.5 and proved some special cases, covered in Lemma F.4.2. Afterwards I worked together with J. Pascal Gollin on this topic, which led to the mentioned article this section is based on.

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Eidesstattliche Versicherung

Hiermit versichere ich an Eides statt, die vorliegende Dissertation selbst verfasst und keine anderen als die angegebenen Hilfsmittel benutzt zu haben. Darüber hinaus versichere ich, dass diese Dissertation nicht in einem früheren Promotionsverfahren eingereicht wurde.