Defects in Conformal Field Theories

Dissertation

zur Erlangung des Doktorgrades
an der Fakultät für Mathematik,
Informatik und Naturwissenschaften
Fachbereich Physik
der Universität Hamburg

vorgelegt von

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Hamburg

2018

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Datum der Disputation:	02.11.2018
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Diese Arbeit ist meiner Mutter Christine gewidmet.

Abstract

Extended objects such as line or surface operators, interfaces or boundaries play an important role in conformal field theory. Here we propose a systematic approach to the relevant conformal blocks which are argued to coincide with the wave functions of an integrable multi-particle Calogero-Sutherland problem. This generalizes a recent observation for four-point blocks and makes extensive mathematical results from the modern theory of multi-variable hypergeometric functions available for studies of conformal defects. Applications range from several new relations with scalar four-point blocks to a Lorentzian inversion formula for defect correlators.

Zusammenfassung

Ausgedehnte Objekte wie Linien- oder Flächenoperatoren, Schnittstellen oder Begrenzungen spielen eine wichtige Rolle in konformen Feldtheorien. Hier schlagen wir einen systematischen Zugang zu den relevanten konformen Blöcken vor, die mit den Wellenfunktionen eines integrablen Mehrteilchen-Calogero-Sutherland-Problems übereinstimmen. Dies verallgemeinert eine kürzliche Beobachtung für Vierpunktblöcke und macht umfassende mathematische Befunde der modernen Theorie über hypergeometrische Funktionen für das Studium von konformen Blöcken verfügbar. Anwendungen reichen von neuen Zusammenhängen mit skalaren Vierpunktblöcken zu einer lorentzischen Inversionsformel für Defektkorrelatoren.

This thesis is based on the publications:

- P. Liendo, Y. Linke, and V. Schomerus, *Lorentzian Inversion Formula for Defect Blocks, in preparation.*
- M. Isachenkov, P. Liendo, Y. Linke, and V. Schomerus, *Calogero-Sutherland Approach to Defect Blocks*, `arXiv:1806.09703`.

Other publications by the author:

- W. Mader, Y. Linke, M. Mader, L. Sommerlade, J. Timmer, and B. Schelter, *A numerically efficient implementation of the expectation maximization algorithm for state space models*, *Applied Mathematics and Computation* **241** (2014) 222 – 232.
- M. Lenz, M. Musso, Y. Linke, O. Tüscher, J. Timmer, C. Weiller, and B. Schelter, *Joint eeg/fmri state space model for the detection of directed interactions in human brains—a simulation study*, *Physiological Measurement* **32** (2011), no. 11 1725.
- B. Schelter, Y. Linke, D. Saur, V. Glauche, R. Lange, C. Weiller, and J. Timmer, *P18-19 brain connectivity: Improvements of fmri data analysis techniques*, *Clinical Neurophysiology* **121** (2010) S213.

Contents

Chapter 1

Introduction

Quantum field theory is an extremely successful framework to describe various phenomena in modern physics. Its applications include the description of the fundamental interactions at high energies and small scales which has resulted in the *Standard Model* of particle physics. It has been completed with discovery of the Higgs boson in 2012. At low energies, quantum field theory has been found to be very useful in effectively describing condensed matter systems.

In thesis we focus on an important class of quantum field theories, the so-called *conformal field theories* which are theories that are invariant under scalings and special conformal transformations. They have been first studied by Dirac [1] but did not receive much attention before the 70s [2–6]. Especially, the seminal paper by Belavin, Polyakov and Zamolodchikov [7] led to a break-through in understanding two-dimensional conformal field theories. Even though conformal invariance may look like a strange symmetry to begin with - certainly, our universe is not conformal invariant - conformal field theories have many interesting applications. First of all, many physical systems become scale-invariant when they undergo second order phase transitions [8]. The critical exponents describing the power-law behavior of certain observables at the critical point can be predicted by conformal field theory calculations. In general, conformal field theories lie at the fixed points of the renormalization group (RG) flow [9], serving as lampposts to understand any quantum field theory. The world-sheet dynamics in string theory is also described by a two-dimensional conformal field theory [10]. Further applications include turbulences [11–13], chemistry [14] and even finance [15].

The reason why conformal field theories are interesting from a theoretical point of view stems from the fact that the enhanced symmetry provides a non-perturbative handle to understand these theories. Besides a few examples, generic quantum field theories can only be understood perturbatively. Conformal symmetry, however, tightly constrains the allowed space of theories, making an attempt to solve these theories feasible without even relying to a Lagrangian description. This is even more fascinating since many quantum field theories without a Lagrangian description were discovered (for instance [16, 17]). Studying a theory by its symmetries only is called the *conformal bootstrap*. It was used successfully to analytically solve certain models in two dimensions [4–6]. In general it is quite hard to solve the associated bootstrap equations analytically because they compromise a non-linear system of infinitely many equations. However, in the last years it was possible to use these equations to put constraints on allowed conformal field theories numerically, initiated by [18]. A striking example is the Ising model in three dimensions whose most accurate theoretical predictions come from the numerical bootstrap analysis [19–24].

Despite the notoriously difficult bootstrap equations there has been progress in obtaining analytic results. The so-called lightcone limit allows to extract the spectrum and operator product coefficients of large spin operators [25, 26]. Using the numerical data of the three-dimensional Ising model as input it was possible to derive bounds in the large spin limit [24, 27]. Surprisingly, it turned out that large spin includes spins as low as spin two. This was explained in [28] where it was shown that there is a delicate balance between operators of spin larger than one. The reason is that these operators organize themselves in families analytic in spin.

Extended objects such as line or surface operators, defects, interfaces, and boundaries are important probes of the dynamics in quantum field theory. They give rise to observables that can detect a wide range of phenomena including phase transitions and non-perturbative dualities. In two-dimensional conformal field theories they also turned out to play a vital role for modern formulations of the bootstrap program. In fact, in the presence of extended objects the usual crossing symmetry becomes part of a much larger system of sewing constraints [29]. While initially the two-dimensional bootstrap started from the crossing symmetry of bulk four-point functions to gradually bootstrap correlators involving extended objects, better strategies were adopted later which depart from some of the sewing constraints involving extended objects. The usual crossing symmetry constraint is then solved at a later stage to find the bulk spectrum and operator product expansion, see e.g. [30].

The bootstrap program, whether in its original formulation, or in the presence of extended objects, relies on conformal block expansions [31,32] that decompose physical correlation functions into kinematically determined blocks and dynamically determined coefficients. These conformal blocks for a four-point correlator are functions of two cross-ratios and the coefficients are those that appear in the operator product expansion of local fields. Such conformal partial wave expansions thereby separate very neatly the dynamical meat of a conformal field theory from its kinematical bones.

In order to perform a conformal block expansion one needs a good understanding of the relevant conformal blocks. While they are in principle determined by conformal symmetry alone, it is still a highly non-trivial challenge to identify them in the zoo of special functions. In the case of scalar four-point functions much progress has been made in the conformal field theory literature starting with [33–35]. If the dimension d is even, one can actually construct the conformal blocks from products of two hypergeometric functions each of which depends on one of the cross-ratios. For more generic dimensions many important properties of the scalar blocks have been understood, these include their detailed analytical structure and various series expansions [36–39].

Extended objects give rise to new families of blocks. Previous work on this subject has focused mostly on local operators in the presence of a defect. This includes correlators and blocks for boundary or defect conformal field theory [40–44], and also bootstrap studies using a combination of numerical and analytical techniques [45–50].[1] Even in this relatively simple context that involves no more than two cross-ratios, the relevant conformal blocks were only identified in some special cases. More general situations, such as e.g. the correlation function of two (Wilson- or 't Hooft) line operators in a d-dimensional conformal field theory, often possess more than two conformal invariant cross-ratios. Two conformal line operators in a four-dimensional theory, for example, give rise to three cross-ratios. For a configuration of a p- and a q-dimensional object in a d-dimensional theory, the number of cross-ratios is given by $N = min(d-p, q+2)$ if $p \geq q$ [54]. So clearly, the study of such defect correlation functions involves new types of special functions which depend on more than two variables.

In order to explore the features of these new functions, understand their analytical properties or find useful expansions one could try to follow the same route that was used for four-point blocks, see e.g. [55, 56] for some recent work in this direction. It is the central message of this thesis, however, that there is another route that gives a much more direct access to defect blocks. It relies on a generalization of an observation in [57] that four-point blocks are wave functions of certain integrable two-particle Hamiltonians of Calogero-Sutherland type [58, 59]. The solution theory for this quantum mechanics problem is an important subject of modern mathematics, starting with the seminal work of Heckman-Opdam [60], see [39] for a recent review in the context of conformal blocks. Much of the development in mathematics is not restricted to the two-particle case and it has given rise to an extensive branch of the modern theory of multi-variable hypergeometric functions.

In order to put all this mathematical knowledge to use in the context of defect blocks, all that is missing is the link between the corresponding conformal blocks, which

[1] Related work includes studies using Mellin space [51, 52], and "alpha space" [53].

depend on N variables, to the wave functions of an N-particle Calogero-Sutherland model. Establishing this link is the main goal of this thesis. Following a general route through harmonic analysis on the conformal group that was proposed in [61], we construct the relevant Calogero-Sutherland Hamiltonian, i.e. we determine the parameters of the potential in terms of the dimensions p, q of the defects and the dimension d. In the special case of correlations of bulk fields in the presence of a defect, the parameters also depend on the conformal weights of the external fields. Eventually, the Lorentzian inversion formula is derived. All these results will be stated in chapter 4.

Calogero-Sutherland models possess a number of fundamental symmetries that can be composed to produce an exhaustive list of relations between defect blocks. We will present these as a first application of our approach. Special attention will be paid to relations involving scalar four-point blocks for which we produce a complete list that significantly extends previously known constructions of defect blocks.

As interesting as such relations are, they provide only limited access to defect blocks. We develop the complete solution theory for defect blocks with $N = 2$ and $N > 2$ cross-ratios in chapter 5 by exploiting known mathematical results on the solutions of Calogero-Sutherland eigenvalue equations. In particular, we shall review the concept of Harish-Chandra scattering states, discuss the issue of series expansions, poles and their residues, as well as global analytical properties such as cuts and their monodromies. The thesis concludes with an outlook and a list of important open problems.

Chapter 2

Conformal symmetry

This chapter gives an introduction to conformal symmetries in d-dimensions and its implications. It has been an active field of research for the last decades and consequently, there is a huge amount of literature. Gentle introductions can be found in [62, 63].

2.1 Conformal group in d-dimensions

Consider $\mathbb{R}^d$ with the Euclidean metric $g_{\mu\nu} = \delta_{\mu\nu}$. A conformal transformation is defined to be a change of coordinates that leaves the metric invariant up to a scale factor

$$g_{\mu\nu}(x) \to g'_{\mu\nu}(x') = \Omega(x) g_{\mu\nu}(x)\,, \quad \Omega(x) > 0\,. \tag{2.1.1}$$

Under an infinitesimal transformation $x^\mu \mapsto x^\mu + \epsilon^\mu$, this corresponds to the *conformal Killing equation*

$$\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu = \frac{2}{d}(\partial \cdot \epsilon)\eta_{\mu\nu}\,, \tag{2.1.2}$$

where the constant of proportionality can be found by contracting both sides with $\delta_{\mu\nu}$. This equation has the following vector fields as solutions:

- translations $p_\mu = \partial_\mu$,
- rotations $m_{\mu\nu} = x_\nu \partial_\mu - x_\mu \partial_\nu$,
- scale transformations $d = x \cdot \partial$ and
- special conformal transformations $k_\mu = 2x_\mu(x \cdot \partial) - x^2 \partial_\mu$.

As we will see shortly, these transformations form the conformal algebra $\mathfrak{so}(1, d+1)$ and are present in any space-time dimension. In $d = 2$, there exist an infinite number of solutions leading to the powerful Virasoro symmetry (see [64] for reference).

However, we will leave the dimension d arbitrary and therefore will not make use of this.

Exponentiated to finite transformations, we find

- translations $x \mapsto x + a$,
- rotations $x \mapsto \Lambda x$, where $\Lambda \in \mathrm{SO}(d)$,
- scale transformations $x \mapsto \lambda x$ and
- special conformal transformations $x \mapsto \frac{x+bx^2}{1+2b\cdot x+b^2x^2}$.

A conformal field theory is defined to be a quantum field theory invariant under conformal transformations. By Noether's theorem, each solution to the Killing equation (2.1.2) has a corresponding charge, denoted by P^μ, $M^{\mu\nu}$, D and K^μ, respectively. They obey the following non-vanishing commutation relations, inherited from the commutator of vector fields:

$$\begin{aligned} [M_{\mu\nu}, P_\rho] &= \delta_{\nu\rho} P_\mu - \delta_{\mu\rho} P_\nu \,, \\ [M_{\mu\nu}, K_\rho] &= \delta_{\nu\rho} K_\mu - \delta_{\mu\rho} K_\nu \,, \\ [M_{\mu\nu}, M_{\rho\sigma}] &= \delta_{\nu\rho} M_{\mu\sigma} - \delta_{\mu\rho} M_{\nu\sigma} + \delta_{\nu\sigma} M_{\rho\mu} - \delta_{\mu\sigma} M_{\rho\nu} \,, \\ [D, P_\mu] &= P_\mu \,, \\ [D, K_\mu] &= -K_\mu \,, \\ [P_\mu, K_\nu] &= 2\delta_{\mu\nu} D - 2M_{\mu\nu} \,. \end{aligned} \tag{2.1.3}$$

The last three lines say that P_μ and K_μ define raising and lowering operators for D. A fact that we will use in the next section to construct representations.

To show that the charges indeed form the algebra $\mathfrak{so}(1, d+1)$, define the generators

$$\begin{aligned} L_{\mu\nu} &= M_{\mu\nu} \,, \\ L_{-1,0} &= -L_{0,-1} = D \,, \\ L_{0,\mu} &= -L_{\mu,0} = \frac{1}{2}(P_\mu + K_\mu) \,, \\ L_{-1,\mu} &= -L_{\mu,-1} = \frac{1}{2}(P_\mu - K_\mu) \,. \end{aligned} \tag{2.1.4}$$

A straightforward calculation shows that they obey the commutation relations of $\mathfrak{so}(1, d+1)$. This suggests that there is a linear realization of the conformal action on $\mathbb{R}^{1,d+1}$. We will make use of this idea, dubbed *embedding space formalism* [1, 65–69], to define cross-ratios in correlation functions for defects.

2.2 Conformal representations

In order to build representations of the conformal group, it is enough to consider operators at the origin because we can always use a translation,

$$\mathcal{O}(x) = e^{x\cdot P}\mathcal{O}(0)e^{-x\cdot P}\,. \tag{2.2.1}$$

In any QFT we assume that local operators at the origin transform as finite irreducible representations of the rotation group, that is

$$M_{\mu\nu}\mathcal{O}^a(0) = (R_{\mu\nu})^a_b\mathcal{O}^b(0)\,, \tag{2.2.2}$$

where a, b are indices of some representation r of $\mathfrak{so}(d)$ and $R_{\mu\nu}$ are its generators. We might suppress the spin indices whenever it is clear from the context.

Furthermore, it is convenient to diagonalize the action of the dilatation operator in a scale invariant theory,

$$D\mathcal{O}(0) = \Delta\mathcal{O}(0)\,, \tag{2.2.3}$$

where Δ is the *conformal dimension* of $\mathcal{O}$. The representation r and the dimension Δ are the main quantum numbers of any operator $\mathcal{O}$.

We are left with the generator of special conformal transformations K_μ. As anticipated, it acts as a lowering operator for the conformal weight,

$$DK_\mu\mathcal{O}(0) = (K_\mu D + [D, K_\mu])\mathcal{O}(0) = (\Delta - 1)K_\mu\mathcal{O}(0)\,. \tag{2.2.4}$$

In physical theories we demand that the spectrum is bounded from below (in unitary theories, this follows from unitarity bounds). Repetitively acting with K_μ annihilates any operator eventually, i. e. there exits an operator such that

$$K_\mu\mathcal{O}(0) = 0\,. \tag{2.2.5}$$

Such operators are called *primary operators*. Acting with the corresponding raising operator, P_μ, we can build a tower of *descendants*,

$$\begin{aligned}\mathcal{O}(0) &\mapsto P_{\mu_1}\ldots P_{\mu_n}\mathcal{O}(0)\,,\\ \Delta &\mapsto \Delta + n\,.\end{aligned} \tag{2.2.6}$$

The collection of the dimensions of the primary operators is called the *spectrum* of a CFT. It is typically discrete in $d \geq 3$ dimensions which follows from demanding a finite thermal partition function [63].

The above construction is called a *parabolic Verma module* for the primary $\mathcal{O}$, which plays the role as the highest weight vector. More formally, let $\mathfrak{g} = \mathfrak{so}(1, d+1)$ be the full conformal algebra. We can decompose it as $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$, where $\mathfrak{k}$ is the Lie algebra of a maximal compact subgroup K, $\mathfrak{a}$ are scale transformations and $\mathfrak{n}$ are special conformal transformations. This is known as Iwasawa decomposition. Now let $\mathfrak{m} = C_{\mathfrak{a}}(\mathfrak{k}) = \mathfrak{so}(d)$ be the centralizer of $\mathfrak{a}$ in $\mathfrak{k}$ and $\mathfrak{g}_x = \mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n}$, i. e. the subalgebra generated by $M_{\mu\nu}$, D and K_μ. Note that its Lie group,

$$G_x = \mathrm{SO}(1,1) \times \mathrm{SO}(d)\,, \tag{2.2.7}$$

is the stabilizer of a point x in coordinate space, i. e. the support of a local operator. Then, G/G_x is the space of all inequivalent operator insertions, i. e. the compactification S^d of the coordinate space. The stabilizer G_x, also known as the little group, labels the representations. Indeed, given a finite dimensional representation (π, V) of its Lie algebra $\mathfrak{g}_x$, where the action of elements in $\mathfrak{n}$ is trivial, the parabolic Verma module $M_{\mathfrak{g}_x}(V)$ is defined as the induced representation

$$M_{\mathfrak{g}_x}(V) := \mathrm{Ind}_{\mathfrak{g}_x}^{\mathfrak{g}}(\pi) = U(\mathfrak{g}) \otimes_{\mathfrak{g}_x} V\,, \tag{2.2.8}$$

where $U(\mathfrak{g})$ is the universal enveloping algebra of $\mathfrak{g}$. Highest weights of $M_{\mathfrak{g}_x}(V)$ are the highest weights of V and we will use both interpretations interchangeably. These Verma modules are irreducible in general but sometimes they are reducible. This happens if certain linear combinations of descendants vanish, e. g. for conserved currents.

Let us briefly outline the fairly standard representation theory of the rotation group $\mathfrak{so}(d)$. Consider first the case of odd dimensions $d = 2k+1$, where $k \in \mathbb{N}$ is the rank of $\mathfrak{so}(d)$. Pick a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{so}(d)$ with basis H_i, $i = 1, \dots, k$. Denote the set of roots by $\Phi \subset \mathfrak{h}^*$ and the set of positive and negative roots by $\Phi^\pm$ (fig. 2.1a). The elements E_α, $\alpha \in \Phi$, of the corresponding root spaces and the generators H_i fulfill the following algebra,

$$[H_i, E_\alpha] = \alpha(H) E_\alpha\,, \qquad [E_\alpha, E_{-\alpha}] = \frac{2}{\langle\alpha, \alpha\rangle} H_\alpha\,, \tag{2.2.9}$$

where H_α is the dual vector to α w. r. t. the Killing form. A (bosonic) highest weight vector $|\ell = (l_1, \dots, l_k)\rangle$ is characterized by

$$H_i|\ell\rangle = l_i|\ell\rangle\,, \qquad E_\alpha|\ell\rangle = 0\,, \quad \forall\alpha \in \Phi^+\,. \tag{2.2.10}$$

The representation itself is the Verma module for the heighest weight $|\ell\rangle$, i. e. it is constructed by applying the negative roots E_α, $\alpha \in \Phi^-$. Irreducible, finite-dimensional representations are labeled by Young tableaux:

$$
\begin{array}{|c|c|c|c|c|c|}
\hline
\mu_1^1 & \cdots & \cdots & \cdots & \cdots & \mu_{l_1}^1 \\
\hline
\end{array}
$$
$$\vdots$$
$$
\begin{array}{|c|c|c|}
\hline
\mu_1^k & \cdots & \mu_{l_k}^k \\
\hline
\end{array}
$$

It is an ordered set of integers $\ell = (l_1, \ldots, l_k)$, $l_1 \geq \cdots \geq l_k \geq 0$. The corresponding weight diagram is depicted in fig. 2.1b. Each l_i counts the number of boxes in the i-th row. The boxes are filled with a tensor index, which are symmetric in each row and antisymmetric in each column.

The analysis is similar for even dimensions $d = 2k$, where again $k \in \mathbb{N}$ is the rank of $\mathfrak{so}(d)$. However, there are two highest weight representations associated to a given Young tableaux with $l_k > 0$: one representation with eigenvalue l_k of H_k and one with eigenvalue $-l_k$. Consequently, we can label the representations by $\ell = (l_1, \ldots, l_k)$, $l_1 \geq \cdots \geq |l_k| \geq 0$.

To summarize, a highest weight vector of the full conformal group is labeled by $[\Delta, \ell]$,

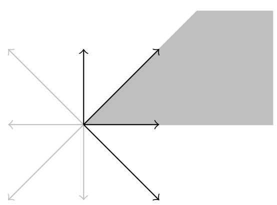
$$
\begin{aligned}
H_0|\Delta, \ell\rangle \ &\equiv D|\Delta, \ell\rangle = \Delta|\Delta, \ell\rangle \, , \\
H_i|\Delta, \ell\rangle &= l_i|\Delta, \ell\rangle \, .
\end{aligned}
\tag{2.2.11}
$$

For later convenience, we note that the quadratic Casimir element is given by

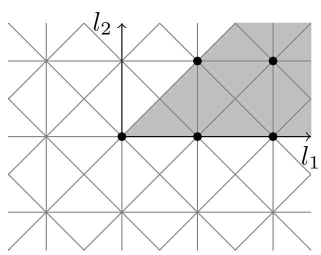
$$
\mathcal{C} = -\frac{1}{2}L^2 = \sum_{i=0}^{k} H_i H_i + \frac{1}{2}\sum_{\alpha\in\Phi} \langle \alpha, \alpha \rangle E_\alpha E_{-\alpha} \, , \tag{2.2.12}
$$

where α runs over the roots Φ of $\mathfrak{so}(1, d+1)$. Its action on a primary state can be evaluated to

$$
\mathcal{C}|\Delta, \ell\rangle = C_{\Delta,\ell}|\Delta, \ell\rangle \, , \qquad C_{\Delta,\ell} = \left[\Delta(\Delta - d) + \sum_{i=1}^{k} l_i(l_i + d - 2i)\right] . \tag{2.2.13}
$$



(a) Root system of $\mathfrak{so}(5)$. The set of positive roots Φ^+ is depicted by black arrows. The Weyl chamber is shown in gray.



(b) Weight diagram of $\mathfrak{so}(5)$. The highest weights are depicted by dots.

Figure 2.1

To complete the discussion, let us remark that there exists a natural invariant pairing between representations $[\Delta, \ell]$ and $[d-\Delta, \ell]$,

$$\langle f_1 | f_2 \rangle = \int\limits_{G/G_x} d^d x\, f_1(x) \cdot f_2(x)\,, \tag{2.2.14}$$

where $\cdot$ is the inner product in the representation of $\mathfrak{so}(d)$. This defines an inner product because the combination $[\Delta, \ell] \times [d-\Delta, \ell]$ transforms as a d-dimensional scalar, compensating for the Jacobian. Thus, $[d-\Delta, \ell]$ can be seen as the dual representation to $[\Delta, \ell]$ and is called the *shadow representation.* For $\Delta = \frac{d}{2} + ic$, $c \in \mathbb{R}$, the shadow representation becomes the complex conjugate and eq. (2.2.14) becomes the L^2-norm on G. These unitary representations compromise the *principal series representations.* Note that this differs from unitarity of the Lorentzian conformal group analytically continued to Euclidean signature, which requires $\Delta \in \mathbb{R}_{\geq 0}$.

2.3 Defects in conformal field theories

Extended operators or *defects* in conformal field theories do not preserve the $G = \mathrm{SO}(1, d+1)$ symmetry of the conformal group. However, if we consider a p-dimensional flat defect it is clear that its support is left invariant by the subgroup

$$G_p = \mathrm{SO}(1, p+1) \times \mathrm{SO}(d-p) \subset G\,. \tag{2.3.1}$$

Here, the first factor describes conformal transformations of the *world-volume* of the defect and the second factor accounts for rotations of the transverse space. This is the maximal amount of conformal symmetry a defect can preserve. A spherical defect is obtained from a flat defect by special conformal transformations in the orthogonal directions.

Elements of the d-dimensional conformal group G that are not contained in the subgroup G_p act as transformations on the defect. The number of such non-trivial transformations is given by the dimension of the space of defect configurations, i. e. the quotient G/G_p,

$$\dim G/G_p = (p+2)(d-p)\ . \tag{2.3.2}$$

For $p = 0$, the defect $\mathcal{D}^{(p=0)}$ consists of a pair of points and the $2d$-dimensional quotient G/G_0 describes their configuration space. When we set $p = d-1$, i. e. consider a *boundary* of codimension $d - p = 1$, the quotient G/G_p has dimension $\dim G/G_{d-1} = d + 1$. A $(d-1)$-dimensional conformal defect is localized along a sphere in the d-dimensional background and the $d+1$ parameters provided by the surface G/G_{d-1} represent the position of its center and the radius.

The defect supports a CFT on its world-volume. In addition to *bulk operators* there are operators with support on the defect only, so-called *defect-local operators*. This defect-local theory almost behaves like any (bulk) CFT, in particular they have a convergent product expansion. However, the defect system is coupled to the bulk, i. e. defect-local operators can have non-vanishing two-point functions with bulk operators. This has an important consequence: as energy is freely exchanged with the bulk, there is no stress-energy tensor on the defect. Furthermore, Ward identities for bulk currents get modified in presence of defects, e. g. for the stress-energy tensor in presence of a p-dimensional flat defect $\mathcal{D}$:

$$\partial_\mu T^{\mu A}(x) = D^A(x)\delta_{\mathcal{D}}(x)\,, \qquad \mu = 1,\dots,d\,, \quad A = 1,\dots,d-p\,, \tag{2.3.3}$$

where A labels the orthogonal directions and $\delta_{\mathcal{D}}$ is the delta-function with support on the defect.

The orthogonal rotations $\mathrm{SO}(d-p)$ preserve the defect point-wise and label the spin of the defect. On the other hand, we require the defect to transform as a scalar under the parallel rotations $\mathrm{SO}(1,p+1)$, i. e. in a one-dimensional representation. For $p \neq 0$, the only one-dimensional representation is the trivial one. If p vanish, one can have a non-trivial one-dimensional representation for which the generator of dilations is represented by a complex number. Along the lines of the previous section, the defect is represented as the parabolic Verma module $M_{\mathfrak{g}_p}(V)$, where (π, V) is a finite dimensional representation of the Lie algebra $\mathfrak{g}_p = \mathrm{Lie}(G_p)$. However, in this thesis we will analyze scalar defects only.

2.4 Embedding space formalism

We briefly review the embedding formalism, which is a standard approach frequently used to study correlators in conformal field theory. While some aspects become easier in embedding space (especially defect configurations, see next subsection), some are more convenient in physical space and we will use both point of views interchangeably. For details on the embedding space formalism see for example [70].

Because the Euclidean conformal group in d dimensions is $\mathrm{SO}(1,d+1)$ it is natural to represent its action linearly on an embedding space $\mathbb{R}^{1,d+1}$. In order to retrieve the usual non-linear action of the conformal group on the d-dimensional Euclidean space we must get rid of the two extra dimensions. This is done by restricting the coordinates to the projective null cone, i.e. we demand $X^2 = 0$ for $X \in \mathbb{R}^{1,d+1}$ and identify $X \sim gX$ for $g \in \mathbb{R}$. It is useful to work in lightcone coordinates with dot product given by

$$X \cdot Y = (X^+, X^-, X^i) \cdot (Y^+, Y^-, Y^i) = -\frac{1}{2}(X^+Y^- + X^-Y^+) + X^iY^i \,. \tag{2.4.1}$$

In other words, points on the physical space $x \in \mathbb{R}^d$ are represented by elements of the projective lightcone of the embedding space. It is common to use the projective identification $X \sim gX$ in order to fix a particular section of the cone given by

$$X = (1, x^2, x^\mu)\,. \tag{2.4.2}$$

This is called the Poincaré section. Note that this section is invariant under $\mathrm{SO}(1, d+1)$ only up to projective identifications. The point at infinity is lifted to $\Omega = (0, 1, 0^\mu)$. The Poincaré section has the useful property that the distance between two points in physical space is then given by $|x - y|^2 = -2X \cdot Y$, where X and Y are the Poincaré lifts of x and y, respectively.

2.4.1 Defects in embedding space

The adaptation of the embedding space to the defect setup can be found in [41, 54]. A $p + 2$-dimensional hyperplane in embedding space $\mathbb{R}^{1,d+1}$ with a time-like direction preserves the subgroup $\mathrm{SO}(1, p + 1) \times \mathrm{SO}(d - p)$ of the conformal group, see fig. 2.2. Furthermore, it can be shown that the intersection of such a hyperplane with the Poincaré section projects down to a p-sphere (or p-hyperplane) in $\mathbb{R}^d$ [54], the locus of the defect in Euclidean space. Hence, one can parametrize the position of the defect through $(d - p)$ orthonormal vectors $P_\alpha, \alpha = 1, \ldots, d - p$, one for each transverse direction. In order to do so, we first pick any $p + 2$ points x_k, $k = 1, \ldots, p + 2$, on the defect $\mathcal{D}^{(p)} \subset \mathbb{R}^d$ and consider their lift $X_k = (1, x_k^2, x_k)$ to the Poincaré section. This uniquely defines the $(p + 2)$-dimensional hyperplane. To select a set of vectors P_α, which are of course not unique, we demand that

$$X_k \cdot P_\alpha = 0\,, \quad P_\alpha \cdot P_\beta = \delta_{\alpha\beta}\,. \tag{2.4.3}$$

Besides conformal transformations, there also exists an $\mathrm{O}(d - p)$-gauge symmetry which acts on the index α, i. e. it transforms the vectors P_α into each other. This becomes important later when we discuss cross-ratios.

Let us conclude this section with two examples. Consider a p-dimensional flat defect $\mathcal{D}^{(p)}$ spanning the directions $e_1, \ldots, e_p$, where e_i is the unit vector in the i-th direction. Its lift to the null-cone can be described by the points

$$X_1 = (1, 1, e_1)\,, \quad \ldots, \quad X_p = (1, 1, e_p)\,, \tag{2.4.4}$$

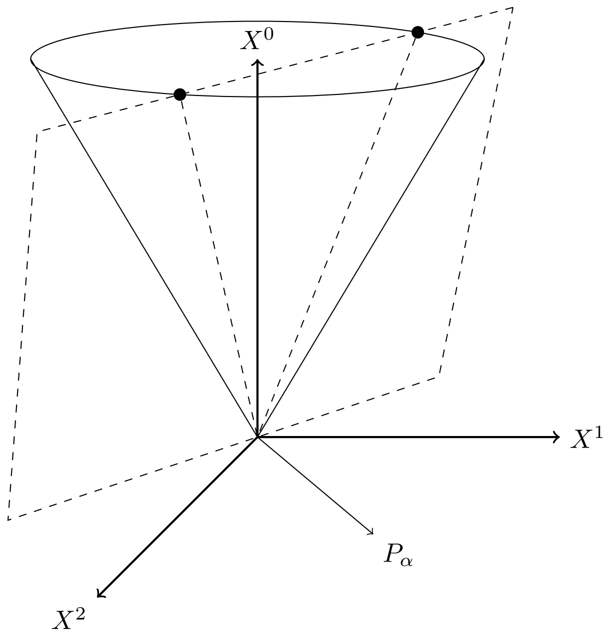


Figure 2.2: Point-like defect in embedding space. Two points are lifted from physical space to the Poincaré-section defining a two-dimensional hyperplane going through the origin. The hyperplane is characterized by the orthonormal vectors P_α.

in addition to the origin $(1, 0, \vec{0})$ and the point at infinity Ω. A convenient set of orthogonal vectors P_α, i.e. satisfy the conditions $X \cdot P = 0$, is given by

$$P_1 = (0, 0, e_{p+1}) \,, \quad \ldots \,, \quad P_{d-p} = (0, 0, e_d) \,. \tag{2.4.5}$$

We see that any vector $Y \in \mathbb{R}^{1,d+1}$ (and hence $y \in \mathbb{R}^d$) nicely splits in to a parallel part and an orthogonal part, $Y = (Y^{\parallel}, Y^{\perp})$.

Now, consider a p-dimensional spherical defect of radius R centered at the origin, immersed in the directions e_1, …, e_{p+1}. In embedding space, it runs through the points

$$X_1 = (1, R^2, Re_1) \,, \quad \ldots \,, \quad X_{p+1} = (1, R^2, Re_{p+1}) \,, \quad X_{p+2} = (1, R^2, -Re_1) \,. \tag{2.4.6}$$

We choose the orthogonal vectors

$$P_1 = \left(\frac{1}{R}, -R, \vec{0}\right) \,, \quad P_2 = (0, 0, e_d) \,, \quad \ldots \,, \quad P_{d-p} = (0, 0, e_{p+2}) \,. \tag{2.4.7}$$

In general the orthogonal projection of a vector $Y \in \mathbb{R}^{1,d+1}$ is defined as

$$Y_\alpha^{\perp} = P_\alpha \cdot Y \,.$$

Note that it is conformal invariant but not $\mathrm{O}(d-p)$-gauge invariant.

2.5 Correlators and cross-ratios

2.5.1 *n*-point functions without defects

Conformal invariance put severe constraints on correlation functions in CFTs with and without defects. While the functional form of lower point functions of local operators are fixed completely, the situation changes when we consider four-point functions of local operators or defect correlators.

The standard one-point function is fixed by dimensional analysis,

$$\langle \mathcal{O}(x) \rangle = \begin{cases} 1 & \text{if } \mathcal{O} \text{ is the unit operator,} \\ 0 & \text{otherwise.} \end{cases} \tag{2.5.1}$$

The two-point function is fixed completely up to a constant. For two primaries $\mathcal{O}_{\Delta_1,\ell_1}$ and $\mathcal{O}_{\Delta_2,\ell_2}$, it is only non-vanishing iff $\Delta_1 = \Delta_2$ and $\ell_1 = \ell_2$. For example, for scalar primaries $\ell = (0, \dots, 0)$ of dimension Δ it is given by

$$\langle \phi(x_1)\phi(x_2) \rangle = \frac{c_\phi}{x_{12}^{2\Delta}} \,, \tag{2.5.2}$$

where c_ϕ is a constant and normally normalized to one. In general, primaries in traceless symmetric representations $\ell = (l, 0, \dots, 0)$ have the two-point function

$$\langle \mathcal{O}^{\mu_1,\dots,\mu_l}(x_1)\mathcal{O}_{\nu_1,\dots,\nu_l}(x_2) \rangle = \frac{c_\mathcal{O}}{x_{12}^{2\Delta}} \left(I^{(\mu_1}_{\nu_1}(x_{12}) \dots I^{\mu_l)}_{\nu_l}(x_{12}) - \text{traces} \right) \,, \tag{2.5.3}$$

where the terms in the brackets are called *tensor structures* consisting of $I^\mu_\nu(x)$,

$$I^\mu_\nu(x) = \delta^\mu_\nu - 2\frac{x^\mu x_\nu}{x^2} \,. \tag{2.5.4}$$

Sometimes traceless symmetric operators come with their canonical normalization, i. e. demanding that certain Ward identities are satisfied as it is the case for the stress tensor $T^{\mu\nu}$. In this case the normalization is physical meaningful.

For later convenience, consider a scalar operator $\phi(x)$ of dimension Δ with its lift $\Phi(X) = \phi(x)$ to the Poincaré section. Its two-point function is given by eq. (2.5.2),

$$\langle \Phi(X)\Phi(Y) \rangle = \frac{1}{(-2X \cdot Y)^\Delta} \,. \tag{2.5.5}$$

In order to respect the action of $\mathrm{SO}(1, d+1)$, $\Phi(X)$ should be a function of X/X^+ only. It is convenient to demand the scaling property $\Phi(X) = (X^+)^\Delta \Phi(X/X^+)$ such that the two-point function (2.5.5) holds on the complete projective null cone.

As with two-point functions, three-point functions are fixed completely up to a constant, too. This time however the constant is physical meaningful. Together with the spectrum it comprises the *local CFT data* because they determine all higher correlation functions in flat space (see chapter 3).

The three-point function of scalar primaries is [2]

$$\langle \phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\rangle = \frac{c_{123}}{x_{12}^{\Delta_1+\Delta_2-\Delta_3} x_{13}^{\Delta_1+\Delta_3-\Delta_2} x_{23}^{\Delta_2+\Delta_3-\Delta_1}}\,, \tag{2.5.6}$$

where Δ_i denotes the dimension of ϕ_i.

Generalizing to two scalars and one traceless symmetric operator [71],

$$\langle \phi_1(x_1)\phi_2(x_2)\mathcal{O}_{\Delta_3}^{\mu_1\ldots\mu_l}(x_3)\rangle = \frac{c_{123}\,(Z^{\mu_1}\ldots Z^{\mu_l} - \text{traces})}{x_{12}^{\Delta_1+\Delta_2-\Delta_3+l} x_{13}^{\Delta_1+\Delta_3-\Delta_2-l} x_{23}^{\Delta_2+\Delta_3-\Delta_1-l}}\,, \tag{2.5.7}$$

where

$$Z^\mu(x) = \frac{x_{13}^\mu}{x_{13}^2} - \frac{x_{23}^\mu}{x_{23}^2}\,. \tag{2.5.8}$$

Note that only traceless symmetric operators can have non-vanishing three-point functions with two scalars. Given three points in $\mathbb{R}^d$, $d \geq 2$, we can use conformal invariance to move them on a line, say

$$x_1 = (0,\ldots,0)\,, \qquad x_2 = (1,0,\ldots,0)\,, \qquad x_3 = \infty\,. \tag{2.5.9}$$

The correlation function is non-zero iff the configuration is invariant under the residual symmetry $\mathrm{SO}(d-1)$, i. e. if the operators appearing in the correlator transform in representations R_1, R_2 and R_3 of $\mathrm{SO}(d)$, then $R_1 \otimes R_2 \otimes R_3$ must contain a singlet of $\mathrm{SO}(d-1)$. For $R_1 = R_2$ being the trivial representation, this is only the case if R_3 is a traceless symmetric representation (see, for example, [72]). It follows that for operators in general representations of $\mathrm{SO}(d)$, there can be more than one tensor structure appearing. Its number is given by

$$N(R_1,R_2,R_3) = \dim(V_1 \otimes V_2 \otimes V_3)^{\mathrm{SO}(d-1)}\,, \tag{2.5.10}$$

where V_i denotes the vector space carrying the representation R_i.

Conformal invariance is not powerful enough to restrict the four-point function completely. For scalar primaries, it takes the general form

$$\langle \phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\phi_4(x_4)\rangle = \underbrace{\frac{1}{x_{12}^{\Delta_1+\Delta_2} x_{34}^{\Delta_3+\Delta_4}} \left(\frac{x_{14}^2}{x_{24}^2}\right)^a \left(\frac{x_{14}^2}{x_{13}^2}\right)^b}_{=:\mathbf{K}_4} \mathcal{G}(u,v)\,, \tag{2.5.11}$$

where $2a = \Delta_2 - \Delta_1$ and $2b = \Delta_3 - \Delta_4$. The factor $\mathbf{K}_4$ is chosen such that the *stripped* *amplitude* $\mathcal{G}(u,v)$ only depends on the conformally invariant *cross-ratios*,

$$u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}\,, \qquad v = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}\,. \tag{2.5.12}$$

This can be understood geometrically. As we saw in eq. (2.5.9), we can move three points on a line. This is why the two- and three-point functions are so restricted. Four points however can only be moved in a plane, say,

$$x_4 = (\tau, \sigma, 0, \ldots, 0)\,. \tag{2.5.13}$$

It is useful to define

$$z = \sigma + i\tau\,, \qquad \bar{z} = \sigma + i\tau\,. \tag{2.5.14}$$

In Euclidean space, z and $\bar{z}$ are complex conjugates. If we analytically continue to Lorentzian signature, $\tau \to it$, they become real and independent.

Plugging eqs. (2.5.9) and (2.5.13) into eq. eq. (2.5.12), we see the connection to the conformal cross-ratios,

$$u = z\bar{z}\,, \qquad v = (1-z)(1-\bar{z})\,. \tag{2.5.15}$$

A choice of coordinates like in eqs. (2.5.9) and (2.5.13) is called *conformal frame*. It is enough to know a correlator in a conformal frame since the whole correlator can be recovered from conformal invariance [73]. This explains why the four-point function depends only on two parameters.

The four-point function for identical scalars is manifestly invariant under permutations of the points x_i, leading to the following conditions on the stripped amplitude $\mathcal{G}(u,v)$,

$$\mathcal{G}(u,v) = \mathcal{G}\left(\frac{u}{v}, \frac{1}{v}\right) \qquad \text{(from swapping } 1 \leftrightarrow 2 \text{ or } 3 \leftrightarrow 4\text{)}, \tag{2.5.16}$$

$$\mathcal{G}(u,v) = \left(\frac{u}{v}\right)^{\Delta_\phi} \mathcal{G}(v,u) \qquad \text{(from swapping } 1 \leftrightarrow 3 \text{ or } 2 \leftrightarrow 4\text{)}. \tag{2.5.17}$$

Let us remark that even though $\mathcal{G}(u,v)$ is not fixed by symmetries alone, it can be derived from the three-point function by using the operator product expansion, see chapter 3.

Higher point functions are even less constraint by conformal symmetry. As mentioned before, we will see in chapter 3 that they are given by the CFT data, i. e. the spectrum and three-point functions.

2.5.2 n-point functions with defect

In presence of defects, operators can acquire a one-point value. We assume the defects to be normalized, i. e.

$$\langle \mathcal{D}^{(p)}(P_\alpha) \rangle = 1\,. \tag{2.5.18}$$

Local operators in the bulk acquire a one-point function which form is fixed by conformal invariance. Consider a scalar operator $\Phi(X)$ with dimension Δ_Φ in embedding space and a defect $\mathcal{D}^{(p)}(P_\alpha)$[1]. Note that correlator must be independent of a particular $\mathrm{O}(d-p)$-gauge of the P_α. The only combination invariant under conformal and gauge transformations is

$$\langle \mathcal{D}^{(p)}(P_\alpha)\Phi(X) \rangle = \mathcal{C}^{\mathcal{D}}_{\Phi} \left(X^\perp \cdot X^\perp \right)^{-\frac{\Delta_\Phi}{2}}\,. \tag{2.5.19}$$

Since the defect and the local operator are normalized, the constant $\mathcal{C}^{\mathcal{D}}_{\Phi}$ is physically meaningful.[2]

In this thesis we will look at scalar external operators only. However, since exchanged operators in intermediate channels can have arbitrary spin, it is convenient to keep in mind the following observations. A defect breaks the conformal symmetry down to $G_p = \mathrm{SO}(1, p+1) \times \mathrm{SO}(d-p)$ which is further broken down to $\mathrm{SO}(p+1) \times \mathrm{SO}(d-p-1)$ by a bulk operator $\mathcal{O}$ [54]. Let the defect and the operator transform in representations $R_{\mathcal{D}}$ and $R_{\mathcal{O}}$ of $\mathrm{SO}(d-p)$ and $\mathrm{SO}(d)$, respectively. Then the one-point function can only be non-vanishing if $R_{\mathcal{D}} \otimes R_{\mathcal{O}}$ contains a singlet under $\mathrm{SO}(p+1) \times \mathrm{SO}(d-p-1)$. As mentioned before, we are only interested in scalar defects.

For example, an operator in presence of a boundary must be a singlet under the whole $\mathrm{SO}(d)$ rotation group, i. e. only scalars couple to boundaries. For point-like defects ($p = 0$) or codimension-2 defects ($p = d-2$), we need the operator to be a singlet under $\mathrm{SO}(d-1)$. Hence, it can only be a traceless symmetric tensor.

In general, more complicated representations can appear. Consider the case of a line operator ($p = 1$) in four dimensions ($d = 4$). We are looking for singlets of $\mathrm{SO}(2)\times\mathrm{SO}(2)$ in representations of $\mathrm{SO}(4)$. The corresponding Lie algebra $\mathfrak{so}(2)\times\mathfrak{so}(2)$ can be chosen as the Cartan subalgebra of $\mathfrak{so}(4)$. Since the action of the Cartan subalgebra on the weight-space 0 is trivial, highest-weight representations which include the weight 0 contain a singlet of $\mathrm{SO}(2) \times \mathrm{SO}(2)$. This happens for Young tableaux $\ell = (l_1, l_2)$ with an even number of boxes.

Next consider the correlation function of the bulk scalar $\Phi(X)$ with a defect-local scalar $\hat{\mathcal{O}}(\hat{Y})$ of dimension $\hat{\Delta}_{\hat{\mathcal{O}}}$. Defect-local quantities will always carry a hat except if stated otherwise. Since the operator $\hat{\mathcal{O}}$ lives on the defect, the coordinates' orthogonal

[1] The discussion can be generalized to bulk operators with spin, see [41].
[2] The notation a_Φ for $\mathcal{C}^{\mathcal{D}}_{\Phi}$ is also common in the literature.

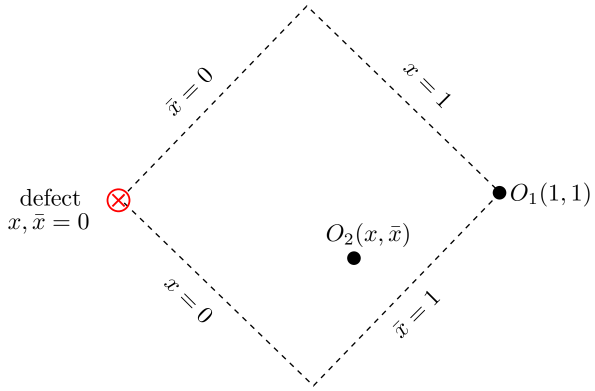


Figure 2.3: Two-point function configuration in a plane orthogonal to the defect. The defect is at the origin while the operators O_1 and O_2 are at points $(1,1)$ and $(x,\bar{x})$, respectively.

projection $\hat{Y}^\perp$ is zero. Again, invariance under conformal and gauge transformations dictates

$$\langle \mathcal{D}^{(p)}(P_\alpha)\hat{\mathcal{O}}(\hat{Y})\Phi(X)\rangle = \mathcal{C}^{\mathcal{D}}_{\Phi,\hat{\mathcal{O}}}\left(X^\perp\cdot X^\perp\right)^{\frac{\hat{\Delta}_{\hat{\mathcal{O}}}-\Delta_\Phi}{2}}(-2X\cdot\hat{Y})^{-\Delta_{\hat{\mathcal{O}}}}\,. \tag{2.5.20}$$

In particular, $\mathcal{C}^{\mathcal{D}}_{\Phi} = \mathcal{C}^{\mathcal{D}}_{\Phi,\hat{\mathbf{1}}}$, where $\hat{\mathbf{1}}$ denotes the defect identity.[3]

Again, representation theory tells us which representations $R_{\hat{\mathcal{O}}}$ of $\mathrm{SO}(p)\times\mathrm{SO}(d-p) \subset \mathrm{SO}(d)$ can appear for $\hat{\mathcal{O}}$. Since the defect $\mathcal{D}$ and the bulk scalar $\Phi(X)$ transform in the trivial representation, $R_{\hat{\mathcal{O}}}$ must contain a singlet of $\mathrm{SO}(p+1)\times\mathrm{SO}(d-p-1)$. This means that $\hat{\mathcal{O}}$ is a scalar under the $\mathrm{SO}(p)$ parallel rotation and a traceless symmetric tensor with spin s under the $\mathrm{SO}(d-p)$ transverse rotation [41].

Let us turn our attention to the case of two scalar operators in presence of a defect. Cconformal invariance is not powerful enough to fix the entire correlator. There are again two cross-ratios (see [41, 54] or the discussion in section 2.5.3), which we choose according to fig. 2.3. Its precise relation to the insertion points $X_{1,2}$ has been worked out [49],

$$\frac{(1-x)(1-\bar{x})}{(x\bar{x})^{\frac{1}{2}}} = -\frac{2X_1\cdot X_2}{(X_1^\perp\cdot X_1^\perp)^{\frac{1}{2}}(X_2^\perp\cdot X_2^\perp)^{\frac{1}{2}}}\,,\qquad \frac{x+\bar{x}}{2(x\bar{x})^{\frac{1}{2}}} = \frac{X_1^\perp\cdot X_2^\perp}{(X_1^\perp\cdot X_1^\perp)^{\frac{1}{2}}(X_2^\perp\cdot X_2^\perp)^{\frac{1}{2}}}\,. \tag{2.5.21}$$

The correlator of two scalars $\Phi_{1,2}(X_{1,2})$ of dimensions $\Delta_{1,2}$ with a p-dimensional defect $\mathcal{D}^{(p)}(P_\alpha)$ takes the general form

$$\langle \mathcal{D}^{(p)}(P_\alpha)\Phi_1(X_1)\Phi_2(X_2)\rangle = \frac{\mathcal{F}(x,\bar{x})}{(X_1^\perp\cdot X_1^\perp)^{\frac{\Delta_1}{2}}(X_2^\perp\cdot X_2^\perp)^{\frac{\Delta_2}{2}}}\,, \tag{2.5.22}$$

where $\mathcal{F}(x,\bar{x})$ is the *stripped amplitude* with dependence on the cross-ratios only.

[3] The notation $b_{\Phi,\hat{\mathcal{O}}}$ for $\mathcal{C}^{\mathcal{D}}_{\Phi,\hat{\mathcal{O}}}$ is also common in the literature.

2.5.3 Defect-defect correlators

While some of our new results do concern the configurations considered in the previous section, our approach covers a more general setup involving two defects of dimension p and q, respectively. The first systematic discussion of such defect correlators can be found in [54]. That paper determined the number N of cross-ratios and also introduced a particular set of coordinates on the space of these cross-ratios. Here we shall review the latter before we discuss an alternative, and more geometric choice of coordinates in the next chapter.

In order to study the two-point function of two defect operators $\mathcal{D}^{(p)}(P_\alpha)$ and $\mathcal{D}^{(q)}(Q_\beta)$ that are inserted along surfaces associated with P_α and Q_β, respectively, we need to single out the invariant cross-ratios. Consider the matrix with elements $M_{\alpha\beta} = P_\alpha \cdot Q_\beta$ of *conformal* invariants. The residual gauge symmetries $\mathrm{SO}(d-p)$ and $\mathrm{SO}(d-q)$ which act on the matrix M through left- and right multiplication, respectively, can be used to diagonalize M. The non-trivial eigenvalues provide a complete set of independent cross-ratios.

To determine their number we need a bit more detail. First, let us consider the case in which the hyperplanes that are spanned by P_α and Q_β have no directions in common. This requires that $2d-p-q \leq d+2$ or equivalently $d-p \leq q+2$. If we assume $p \geq q$ from now on, the number of cross-ratios is given by $N = d-p$,

$$M = \left.\underbrace{\begin{pmatrix} * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \end{pmatrix}}_{d-q}\right\}{\scriptstyle d-p} \;\xrightarrow[\mathrm{SO}(d-q)]{\mathrm{SO}(d-p)}\; \begin{pmatrix} \underbrace{\begin{matrix} * & 0 & 0 \\ 0 & * & 0 \\ 0 & 0 & * \end{matrix}}_{d-p} & \begin{matrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{matrix} \end{pmatrix}. \tag{2.5.23}$$

If $d-p > q+2$, on the other hand, the two hyperplanes spanned by P_α and Q_β must intersect in $d-2-(p+q)$ directions. Hence $d-2-(p+q)$ of the scalar products are invariant and there are only $d-p-(d-2-(p+q)) = q+2$ nontrivial eigenvalues,

$$M = \left.\underbrace{\begin{pmatrix} * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \end{pmatrix}}_{d-q}\right\}{\scriptstyle d-p} \;\xrightarrow[\mathrm{SO}(d-q)]{\mathrm{SO}(d-p)}\; \begin{pmatrix} \underbrace{\begin{matrix} * & 0 \\ 0 & * \\ 0 & 0 \end{matrix}}_{q+2} & \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{matrix} \end{pmatrix}. \tag{2.5.24}$$

In total, the number of invariant cross-ratios is therefore $N = \min(d-p, q+2)$. To be precise, we point out that the full gauge group is actually given by $\mathrm{O}(d-p) \times \mathrm{O}(d-q)$ and hence the values on the diagonal are only meaningful up to a sign. One way to construct fully invariant cross-ratios is to consider

$$\eta_a = tr(MM^T)^a \tag{2.5.25}$$

where $a = 1, \ldots, N$. This is the set of cross-ratios introduced in [54].

After having identified the variables, we can write down the two-point function of defects $\mathcal{D}^{(p)}(P_\alpha)$ and $\mathcal{D}^{(q)}(Q_\beta)$,

$$\langle \mathcal{D}^{(p)}(P_\alpha)\mathcal{D}^{(q)}(Q_\beta)\rangle = \mathcal{F}(\eta_a)\,. \tag{2.5.26}$$

2.6 Defect cross-ratios

Before we discuss a few examples in free field theory, we want to consider a second, alternative set of coordinates, that is more geometric and also will turn out to possess a very simple relation with the coordinates of the Calogero-Sutherland Hamiltonian.

Roughly, our new parameters consist of the ratio R/r of radii of the spherical defects along with $N-1$ tilting angles θ_i of the lower $(q-)$dimensional defect in the space that is transverse to the higher $(p-)$dimensional defect. To be more precise, we place our two spherical defects of dimensions p and q, respectively, such that they are both centered at the origin $\mathbb{R}^d$. Without restriction we can assume that the $p-$dimensional defect of radius R is immersed in the subspace spanned by the first $p+1$ basis vectors $e_1, \ldots, e_{p+1}$ of the d-dimensional Euclidean space. The radius of the second, $q-$dimensional defect, we denote by r. To begin with, we insert this defect in the subspace spanned by the first $q+1$ basis vectors $e_1, \ldots, e_{q+1}$. Then we tilt the second defect by angles $\theta_1, \ldots, \theta_{N-1}$ in the $e_1 - e_d, \ldots, e_{N-1} - e_{d+2-N}$ planes, respectively. In other words we act on the locus of the second sphere with 2-dimensional rotation matrices $R_{(i-1,d+2-i)}(\theta_i)$ in the plane spanned by the basis vectors e_{i-1} and e_{d+2-i} for $i = 1, \ldots, N-1$. This gives a well-defined configuration of defects, because we have $N-1 \leq q+1 \leq p+1 < d+2-N$ for $p \geq q$. With a little bit of work it is possible to compute the matrix M of scalar products explicitly, see appendix A for a derivation,

$$M = \left(\begin{array}{ccccc|c} \cosh\vartheta & & & & & \\ & \cos\theta_1 & & & & \\ & & \ddots & & & 0 \\ & & & \cos\theta_{N-1} & & \\ & & & & I & \end{array}\right) \quad \text{where} \quad \cosh\vartheta = \frac{1}{2}\left(\frac{r}{R} + \frac{R}{r}\right)\,. \tag{2.6.1}$$

We shall pick ϑ to be a positive real number. Using the general prescription in eq. (2.5.25) the cross-ratios η_a that were introduced in [54] take the form

$$\eta_a = \cosh^{2a}\vartheta + \cos^{2a}\theta_1 + \cdots + \cos^{2a}\theta_{N-1}\,, \quad a = 1, \ldots, N\,. \tag{2.6.2}$$

From now on we shall adopt the parameters ϑ and $\theta_i, i = 1, \ldots, N-1$ as the fundamental conformal invariants for $N \geq 3$. While ϑ can be any non-negative real number, the variables θ_i take values in the interval $\theta_i \in [0, \pi[$.

Let us stress once again, that our geometric parameters R/r and θ_i represent just one convenient choice. In the special case with $p = q = d-2$, the variables η_1 and η_2 possess a direct geometric interpretation that is based on a slightly different setup in which one defect is assumed to be flat while the second is kept at finite radius but displaced and tilted with respect to the first, see [54]. Another important special case appears for $q = 0$, i. e. when two bulk fields are placed in the background of a defect, which we discussed at length in the previous chapter. In particular, we have introduced a geometric parametrization of the two cross-ratios, namely through the parameters x and $\bar{x}$, see eq. (2.5.21). It is not too difficult to work out, see appendix A, that these are related to the parameters ϑ and $\theta \equiv \theta_1$ through

$$x = \tanh^{-2}\frac{\vartheta + i\theta}{2}\,, \quad \bar{x} = \tanh^{-2}\frac{\vartheta - i\theta}{2}\,. \tag{2.6.3}$$

We will use the coordinates x, $\bar{x}$ as the fundamental conformal invariants for $N = 2$. Equation (2.6.3) also shows that the variables ϑ and θ_i generalize the radial coordinates that were introduced for $N = 2$ in [42].

2.7 A free field example

Strongly coupled CFTs can normally be described by the conformal data only, since explicit expressions in terms of renormalized elementary fields are not available. On the other hand, free field theory can serve as a starting point in perturbatively accessible theories, see for example [45]. In free field theories, conformal invariance is completely determined by dimensional analysis since there are no running coupling constants.

Consider a free scalar field in d dimensions with two-point function

$$\langle\phi(x_1)\phi(x_2)\rangle = \frac{1}{|x_1 - x_2|^{2\Delta_\phi}}\,, \quad \Delta_\phi = \frac{d}{2} - 1\,. \tag{2.7.1}$$

The following two defects exist in any (not necessarily free) theory,

$$\mathbf{I}^{(p)} = \mathbf{I} \qquad \text{(trivial defect)}, \tag{2.7.2}$$

$$\mathcal{D}^{(0)} = |x_1 - x_2|^{2\Delta_\phi} \phi(x_1)\phi(x_2) \qquad \text{(point-like defect)}. \tag{2.7.3}$$

More interesting is the following defect,

$$\mathcal{D}^{(\Delta_\phi)} = \exp\left(\lambda \int\limits_{S^p_R} d^p\sigma\, \phi(\sigma)\right), \quad p = \Delta_\phi = \frac{d}{2} - 1\,, \tag{2.7.4}$$

where R is the radius of $\mathcal{D}^{(\Delta_\phi)}$. For example, this is the scalar Wilson line in four dimensions [74]. The one-point functions can be calculated by expanding the free exponential,

$$\langle \mathcal{D}^{(\Delta_\phi)} \phi(0) \rangle = \lambda \int\limits_{S^p_R} d\sigma\, \langle \phi(\sigma)\phi(0)\rangle = \lambda \frac{\Omega_p}{R^{\Delta_\phi}} \equiv \frac{\mathcal{C}_\phi}{(2R)^{\Delta_\phi}} \quad \text{with } \mathcal{C}_\phi = \lambda \frac{\Omega_{\Delta_\phi}}{2^{\Delta_\phi}}\,, \tag{2.7.5}$$

where $\Omega_{n-1} = 2\pi^{n/2}/\Gamma(n/2)$ is the volume of S^{n-1}. Likewise, the two-point function is

$$\begin{aligned} \langle \mathcal{D}^{(\Delta_\phi)} \phi(x_1)\phi(x_2)\rangle &= \langle \phi(x_1)\phi(x_2)\rangle + \langle \mathcal{D}^{(\Delta_\phi)}\phi(x_1)\rangle \langle \mathcal{D}^{(\Delta_\phi)}\phi(x_2)\rangle \\ &= \frac{1}{|x_{12}|^{2\Delta_\phi}} + \frac{\mathcal{C}_\phi^2}{(x_1^\perp)^{\Delta_\phi}(x_2^\perp)^{\Delta_\phi}} = \frac{1}{(x_1^\perp)^{\Delta_\phi}(x_2^\perp)^{\Delta_\phi}} \mathcal{F}(x,\bar{x}) \end{aligned} \tag{2.7.6}$$

with the stripped amplitude

$$\mathcal{F}(x,\bar{x}) = \left(\frac{(1-x)(1-\bar{x})}{(x\bar{x})^{\frac{1}{2}}}\right)^{-\frac{\Delta_1+\Delta_2}{2}} + \mathcal{C}_\phi^2\,. \tag{2.7.7}$$

We notice that the two-point function of ϕ is only mildly deformed by the defect.

Defect-defect correlators are harder to compute due to the integrals involved, but in four dimensions the line-line correlator can be evaluated to

$$\langle \mathcal{D}^{(\Delta_\phi)} \mathcal{D}^{(\Delta_\phi)} \rangle = \mathcal{F}^{\mathcal{D}}(\vartheta, \theta_1, \theta_2) = \exp\left[\lambda^2 \int \frac{d\sigma_1 d\sigma_2}{|\sigma_1 - \sigma_2|^2}\right] \tag{2.7.8}$$

$$= \exp\left[\frac{4\pi\lambda^2}{\sqrt{\sinh^2\vartheta + \sin^2\theta_1}} \mathrm{K}\left(\frac{\sin^2\theta_1 - \sin^2\theta_2}{\sinh^2\vartheta + \sin^2\theta_1}\right)\right], \tag{2.7.9}$$

where $K(m) = \frac{\pi}{2}{}_2F_1(\frac{1}{2},\frac{1}{2};1;m)$ is the elliptic integral of the first kind. As promised, the correlator depends on three cross-ratios.

Chapter 3

Conformal blocks

Conformal blocks in presence of defects are the central objects in this thesis. This chapter is devoted to introduce them in case of four local operators and presents the various ways to calculate them. In particular, we will show how a Calogero-Sutherland model arises which paves the way to derive the blocks for defects.

3.1 Operator product expansion

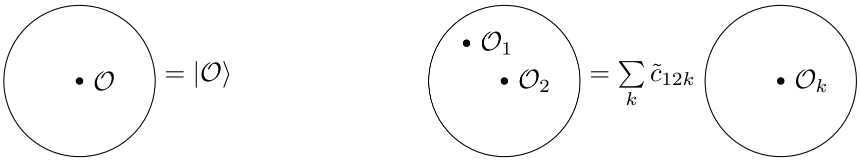
The operator product expansion is one of the main tools to analyze any CFT. In this section we will see why and how it works.

In any QFT, a quantization consists of foliating the space into codimension one slices, each endowed with its own Hilbert space. There is a unitary operator between those Hilbert spaces, the *time evolution operator*, $U(\Delta t) = e^{iH\Delta t}$, where H is the Hamiltonian. In an Euclidean CFT, it is convenient to choose quantization slices that respects the symmetry, i. e. $(d-1)$-dimensional spheres S^{d-1} located at the origin. This is called *radial quantization*. The role of the Hamiltonian is now played by the dilation operator, $U(\Delta t) = e^{iD\Delta t}$, with the origin corresponding to past infinity.

Consider an operator $\mathcal{O}_{\Delta,\ell}(0)$ at the origin with dimension Δ and in some representation ℓ of $\mathrm{SO}(d)$. Enclose the operator by some quantization sphere and perform the path integral over its interior, which results in a state on the boundary. Using conformal symmetry, we can shrink the quantization sphere, resulting in a state $|\mathcal{O}_{\Delta,\ell}\rangle$. We can write

$$\mathcal{O}_{\Delta,\ell}(0)|0\rangle = |\mathcal{O}_{\Delta,\ell}\rangle \,. \tag{3.1.1}$$

This is called the *state-operator correspondence*. The operator corresponding to the vacuum is the identity operator $\mathbf{1}$. Of course, we are not restricted to operators at the origin. Equation (2.2.1) tells us that an operator $\mathcal{O}_{\Delta,\ell}(x)$ creates an infinite linear combination of descendant states.



(a) **(b)**

Figure 3.1: **(a)** *State-operator correspondence*: surrounding the operator $\mathcal{O}$ by a sphere and performing the path integral gives the state $|\mathcal{O}\rangle$. **(b)** *Operator product expansion*: A state created by two operators can be expanded in terms of local operators.

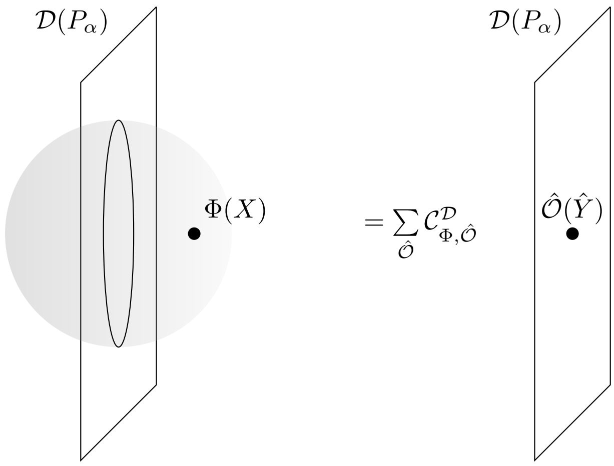


Figure 3.2: *Bulk-to-defect expansion*: The quantization surface cuts through the defect. The state induced on the defect is expanded in terms of defect-local operators $\mathcal{O}$.

Now, repeat the discussion with two operators $\mathcal{O}_1(x)$ and $\mathcal{O}_2(y)$ inside some quantization sphere. The state-operator correspondence allows us to expand the state on the boundary in terms of local operators,

$$\mathcal{O}_1^a(x)\mathcal{O}_2^b(y) = \sum_k \tilde{c}_{12k}^{abc}\mathcal{O}_k^c(z)\,, \tag{3.1.2}$$

where we made the dependence on the spin indices a, b, c explicit. Grouping together descendants gives the conformal *operator product expansion (OPE)*,

$$\mathcal{O}_1(x)\mathcal{O}_2(y) = \sum_k c_{12k} C_a(x, y, z, \partial_z)\mathcal{O}_k^a(z)\,, \tag{3.1.3}$$

where c_{12k} are the constants appearing in the three-point function and the functions $C_a(x, y, z, \partial_z)$ are fixed by conformal symmetry. This can be seen from demanding that the OPE reproduces the three-point function.

What happens if we introduce defects? Consider a bulk operator $\Phi(X)$ near a defect $\mathcal{D}$. Surround the operator with a quantization sphere that cuts through the defect, see fig. 3.2. The resulting state on the boundary is *not* part of the bulk Hilbert space but rather a Hilbert space decorated with the defect. Using scale

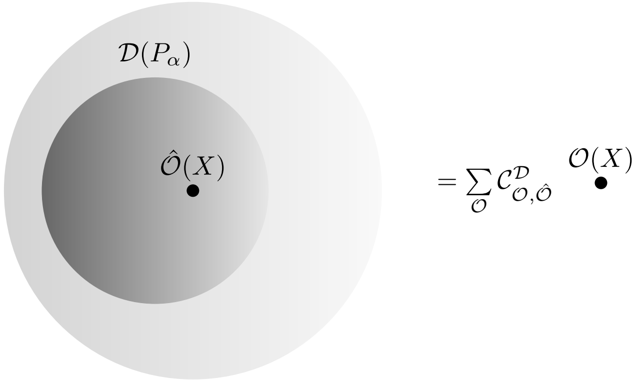


Figure 3.3: *Defect-expansion*: The quantization surface encloses the defect. The state induced in the bulk is expanded in terms of bulk operators $\mathcal{O}$.

transformations, the quantization sphere can be scaled down to a point on the defect. Hence, we can expand a bulk operator in terms of defect-local operators.

The other possibility is to expand the defect itself in terms of bulk fields. Consider a defect (where we allow for a defect-local excitation) and surround it completely with quantization sphere, see fig. 3.3. The corresponding state on the boundary can be expanded in terms of bulk data in the usual way. This is called the *defect expansion*. We are mostly interested in the case with no defect-local excitation. The expansion then takes the form

$$\mathcal{D}(P_\alpha) = \sum_{\mathcal{O}} C^{\mathcal{D}}_{\mathcal{O}} D_{\Delta_{\mathcal{O}}}(P_\alpha, X, \partial_X)\mathcal{O}(X)\,. \tag{3.1.4}$$

Similar to the bulk OPE, the functions $D_{\Delta_\Phi}(P_\alpha, X, \partial_X)$ are fixed by conformal invariance and encode the contributions of the descendants.

3.2 Conformal blocks

The magic happens when we use the OPE inside an n-point function. The conformal OPE converges whenever we find a sphere separating the two operators from any other. This allows use to write an n-point function as a sum over $(n-1)$-point functions,

$$\begin{aligned}&\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\mathcal{O}_3(x_3)\ldots\mathcal{O}_n(x_n)\rangle \\ &\quad= \sum_k c_{12k} C(x_1, x_2, \partial_2)\langle \mathcal{O}_k(x_2)\mathcal{O}_3(x_3)\ldots\mathcal{O}_n(x_n)\rangle\,.\end{aligned} \tag{3.2.1}$$

This way any correlation function can be calculated in principle once we know the CFT data and one-point functions. In practice, this can be quite hard.

For concreteness, let us look at the four-point function of four scalar primaries,

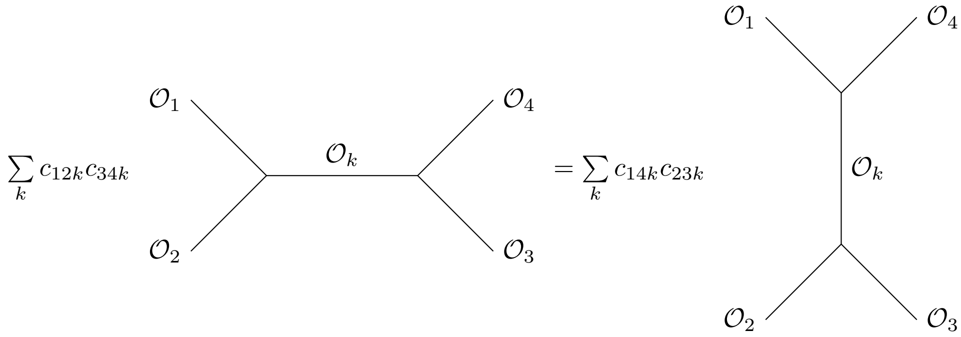


Figure 3.4: *Crossing symmetry* of the four-point function.

eq. (2.5.11). Using the OPE we get

$$\langle \phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\phi_4(x_4)\rangle = \mathbf{K}_4 \sum_k c_{12k}c_{34k}g_{\Delta,\ell}(u,v) \tag{3.2.2}$$

i. e.

$$\mathcal{G}(u,v) = \sum_k c_{12k}c_{34k}g_{\Delta,\ell}(u,v)\,, \tag{3.2.3}$$

where the sum runs over primaries and

$$g_{\Delta,\ell}(u,v) = \mathbf{K}_4^{-1} C_a(x_1,x_2,\partial_2)C_b(x_3,x_4,\partial_4)\frac{I^{ab}(x_{24})}{x_{24}^{2\Delta}}\,. \tag{3.2.4}$$

are called *conformal blocks.* They encode all the kinematical information of the exchanged primary operator in representation $[\Delta,\ell]$ and its descendants and are fixed by conformal symmetry (up to a normalization). In other words, the decomposition in eq. (3.2.4) separates these dynamical data from the kinematical skeleton of the correlation function. Note that only operators in traceless symmetric representations of $\mathrm{SO}(d)$ can be exchanged (see the discussion of the three-point function in section 2.5). Equation (3.2.4) fixes the asymptotic behavior of the blocks. In this work, we will use the normalization of [28],

$$g_{\Delta,\ell}(z,\bar{z}) \overset{z,\bar{z}\to 0}{\to} (z\bar{z})^{\frac{\Delta-\ell}{2}}(z+\bar{z})^{\ell} + \dots\,. \tag{3.2.5}$$

where $z,\bar{z}$ are defined in eq. (2.5.15). The quantity $\tau = \Delta - \ell$ is called *twist* of the given operator. It will play an important role in the light-cone bootstrap in section 3.6.

The two possible ways to apply the OPE inside the correlation function are depicted in fig. 3.4. Since they describe the same four-point function, the two expressions must agree whenever their region of convergence overlaps. This is called *crossing symmetry* and is the key idea of the conformal bootstrap. Decomposing eq. (2.5.17)

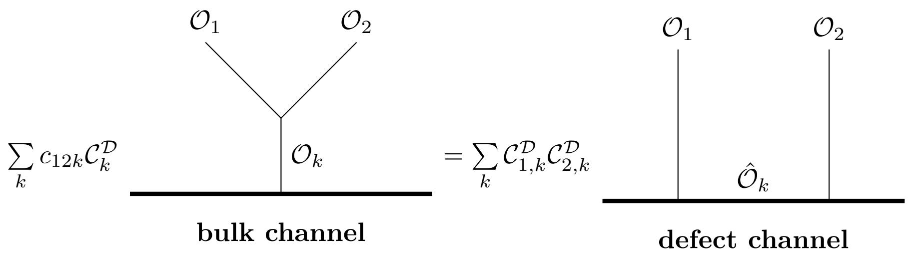


Figure 3.5: Crossing symmetry of two bulk fields in presence of a defect.

into conformal blocks yields the famous crossing symmetry equation for a four-point function of scalar operators ϕ,

$$(z\bar{z})^{-\Delta_\phi}\sum_{\mathcal{O}} c^2_{\phi\phi\mathcal{O}}\, g_{\Delta,\ell}(z,\bar{z}) = [(1-z)(1-\bar{z})]^{-\Delta_\phi}\sum_{\mathcal{O}} c^2_{\phi\phi\mathcal{O}}\, g_{\Delta,\ell}(1-z,1-\bar{z})\,. \quad (3.2.6)$$

This is the starting point for the numerical and the light-cone bootstrap program which require a good understanding of the conformal blocks numerically and analytically. This thesis is devoted to provide a better understanding of the blocks in cases which involve defects, hopefully to pave the way for bootstrapping.

Even though we could evaluate eq. (3.2.4) directly, this can be quite hard. There are more elegant ways to determine the conformal blocks, making the physical and group theoretical structure more accessible.

Turning our attention to the case with defects, we recall the two-point function,

$$\langle \mathcal{D}^{(p)}(P_\alpha)\Phi_1(X_1)\Phi_2(X_2)\rangle = \frac{\mathcal{F}(x,\bar{x})}{(X_1^\perp\cdot X_1^\perp)^{\frac{\Delta_1}{2}}(X_2^\perp\cdot X_2^\perp)^{\frac{\Delta_2}{2}}}\,. \quad (3.2.7)$$

It has two conformal block expansions: the bulk channel and the defect channel, depicted in fig. 3.5 and to be described below.

The bulk channel expansion is obtained by using the standard operator product expansion for two local bulk fields (only traceless symmetric tensors can appear) before evaluating the one-point functions of the resulting bulk fields in the background of the defect,

$$\mathcal{F}(x,\bar{x}) = \left(\frac{(1-x)(1-\bar{x})}{(x\bar{x})^{\frac{1}{2}}}\right)^{-\frac{\Delta_1+\Delta_2}{2}} \sum_k c_{12k}\mathcal{C}_k^{\mathcal{D}}\, f\begin{pmatrix} p,a,d \\ \Delta_k,\ell_k \end{pmatrix}; x,\bar{x}\Bigg)\,, \quad (3.2.8)$$

where we made the dependence on the defect dimension p, the relevant information about the external scalars $a = (\Delta_2-\Delta_1)/2$, and the dimension d explicit. We normalize the blocks such that[1]

$$f\begin{pmatrix} p,a,d \\ \Delta,\ell \end{pmatrix}; x,\bar{x}\Big) \overset{x,\bar{x}\to 1}{\longrightarrow} [(1-x)(1-\bar{x})]^{\frac{\Delta-\ell}{2}}(2-x-\bar{x})^{\ell}\,. \tag{3.2.9}$$

The conformal field theory data in this channel corresponds to the bulk three-point coupling c_{12k} multiplied with the coefficients $C_k^{\mathcal{D}}$ of the one-point function of scalar operators. The general form of the bulk channel blocks cannot be found in closed-form in the existing literature, see however [42] for efficient power series expansions. For some selected cases the defect block can be mapped to the conformal blocks for four scalars in standard bulk conformal field theory, see sections 2.5.3, 4.4 and 4.5 and appendix B. Our results in chapter 5 generalize these isolated results and thereby fill an important gap.

Local operators in the bulk of a defect conformal field theory may be expanded in terms of operators that are inserted along the defect. Applying the bulk-to-defect expansion to the external operators (only transverse traceless symmetric tensors can appear) results in the following conformal block expansion

$$\mathcal{F}(x,\bar{x}) = \sum_k \mathcal{C}^{\mathcal{D}}_{1,k}\mathcal{C}^{\mathcal{D}}_{2,k}\,\hat{f}\left(\begin{matrix} p,a,d \\ \widehat{\Delta}_k, s_k \end{matrix}; x,\bar{x}\right)\,, \tag{3.2.10}$$

where k runs through the set of all intermediate fields $\widehat{O}_k$ of weight $\widehat{\Delta}_k$ and transverse spin s_k. The blocks $\hat{f}(x,\bar{x})$ factorize in terms of the $\mathrm{SO}(d-1,1)\times\mathrm{SO}(d-p)$ symmetry group. This simplifies the analysis significantly and it is possible to write $\hat{f}(x,\bar{x})$ as a product of hypergeometric functions [41],

$$\begin{aligned}\hat{f}\left(\begin{matrix} p,a,d \\ \widehat{\Delta}_k, s_k \end{matrix}; x,\bar{x}\right) = x^{\frac{\widehat{\Delta}-s}{2}}\bar{x}^{\frac{\widehat{\Delta}+s}{2}}\,{}_2F_1\left(-s,\frac{d-p}{2}-1,2-\frac{d-p}{2}-s,\frac{x}{\bar{x}}\right) \\ \times\,{}_2F_1\left(\widehat{\Delta},\frac{p}{2},\widehat{\Delta}+1-\frac{p}{2},x\bar{x}\right)\,.\end{aligned} \tag{3.2.11}$$

In the following we shall mostly focus on the bulk channel and its generalizations.

As an aside let us comment on the boundary case which is special, since the transverse space is one-dimensional ($p=d-1$). In this case the two-point function depends only on the first invariant in eq. (2.5.21),

$$\langle \mathcal{D}^{(d-1)}(P_\alpha)\Phi_1(X_1)\Phi_2(X_2)\rangle = \frac{1}{(X_1^\perp\cdot X_1^\perp)^{\frac{\Delta_1}{2}}(X_2^\perp\cdot X_2^\perp)^{\frac{\Delta_2}{2}}}\mathcal{F}\left(\frac{(1-x)(1-\bar{x})}{(x\bar{x})^{\frac{1}{2}}}\right)\,. \tag{3.2.12}$$

The conformal block expansion of this correlator was originally studied in [40], and

[1]Note that the normalization differs from [41], i.e. $f^{there} = 2^{-\ell} f^{here}$. For the scalar four-point blocks, we adopt a normalization of [28]. To switch to conventions of [35,39], one should multiply our scalar blocks by $(d/2-1)_{\ell'}/(d-2)_{\ell'}$.

the boundary bootstrap was implemented in [46–48].

Finally, let us generalize eq. (2.5.22) and the bulk channel blocks in eq. (3.2.8) to an arbitrary pair of defects. Using the defect expansion in eq. (3.1.4) we can relate the correlator to one-point data,

$$\langle \mathcal{D}^{(p)}(P_\alpha)\mathcal{D}^{(q)}(Q_\beta)\rangle = \mathcal{F}(\vartheta,\theta_i) = \sum_k C_k^{\mathcal{D}^{(p)}} C_k^{\mathcal{D}^{(q)}}\, f_D\begin{pmatrix} p,q,d \\ \Delta_k,\ell_k \end{pmatrix};\vartheta,\theta_i\Big) \,, \tag{3.2.13}$$

where the spin ℓ is labeled by a set of even integers $\ell = (l_1,\ldots,l_{N-1})$ with $l_1 \geq \cdots \geq l_{N-1} \geq 0$. Using the defect expansion it is also possible to infer the asymptotic behavior of the defect blocks f_D. In the coordinates we introduced in section 2.6, we normalize the blocks such that

$$f_D\left(\begin{matrix} p,q,d \\ \Delta,\ell \end{matrix};\vartheta,\theta_i\right) \overset{\vartheta\to\infty}{\to} 4^{\Delta} e^{-\Delta\vartheta}\prod_{i=1}^{N-1}(-2\cos\theta_i)^{l_i}\,. \tag{3.2.14}$$

The kinematical information that enters through the *defect blocks* $f_D(\vartheta,\theta_i)$ which are the main objects of interests for the present work. Although they share a similar name, they should not be confused with the defect channel blocks introduced in the last section. It is not known yet how defect channel blocks generalize to defect-defect correlators.

3.3 Computation of four-point conformal blocks

In this section we will describe the various ways that have been found to compute four-point conformal blocks.

3.3.1 Shadow formalism

Historically, the first technique to access the conformal blocks was introduced in the 70s [32]. They used the shadow operator $\tilde{\mathcal{O}}_{d-\Delta,\ell}$ (see section 2.2),

$$\tilde{\mathcal{O}}_{d-\Delta,\ell} = \int d^dY\,\frac{1}{(-2X\cdot Y)^{d-\Delta+\ell}}\mathcal{O}_{d-\Delta,\ell}(Y)\,, \tag{3.3.1}$$

to define the following projection operator

$$\begin{aligned} \tilde{\mathcal{P}}_{\Delta,\ell} &= \frac{1}{\mathcal{N}_{\mathcal{O}}}\int d^dX\,|\mathcal{O}_{\Delta,\ell}(X)\rangle\langle\tilde{\mathcal{O}}_{d-\Delta,\ell}(X)|\,,\\ \mathcal{N}_{\mathcal{O}} &= \pi^d\frac{(\Delta-1)_\ell\Gamma\left(\Delta-\frac{d}{2}\right)}{\Gamma(\Delta+\ell)}\frac{(d-\Delta-1)_\ell\Gamma\left(\frac{d}{2}-\Delta\right)}{\Gamma(d-\Delta+\ell)}\,. \end{aligned} \tag{3.3.2}$$

Inserting this operator in into the four-point function eq. (3.2.2),

$$\langle\phi_1(x_1)\phi_2(x_2)|\tilde{\mathcal{P}}_{\Delta,l}|\phi_3(x_3)\phi_4(x_4)\rangle = \mathbf{K}_4 c_{12\mathcal{O}} c_{34\mathcal{O}} F_{\Delta,\ell}(u,v)\,, \tag{3.3.3}$$

and using the OPE, we can see that $F_{\Delta,\ell}(u,v)$ can almost be identified with the conformal block in eq. (3.2.4). Actually, the symmetry $\Delta \leftrightarrow d-\Delta$ of $\tilde{\mathcal{P}}_{\Delta,\ell}$ implies that $F_{\Delta,\ell}(u,v)$ is a linear combination of the block $g_{\Delta,\ell}(u,v)$ and its shadow $g_{d-\Delta,\ell}(u,v)$ [35]. The latter contribution can be removed by noting that they transform differently under a monodromy transformation (c. f. eq. (3.2.5)) [75],

$$g_{\Delta,l}(z,\bar{z}) \to e^{2i\pi\Delta} g_{\Delta,l}(z,\bar{z})\,, \qquad \text{for } z \to e^{4\pi i} z\,, \quad \bar{z} \text{ fixed.} \tag{3.3.4}$$

The linear combination of blocks appearing in $F_{\Delta,\ell}(u,v)$,

$$\begin{gathered} F_{\Delta,\ell}(u,v) = g_{\Delta,\ell} + \frac{K_{\Delta,\ell}}{K_{d-\Delta,\ell}}\, g_{d-\Delta,\ell}\,, \\ K_{\Delta,\ell} = \frac{\Gamma(\Delta-1)}{\Gamma\left(\Delta-\frac{d}{2}\right)}\kappa_{\Delta+\ell}\,, \qquad \kappa_\beta = \frac{\Gamma\left(\frac{\beta}{2}+a\right)\Gamma\left(\frac{\beta}{2}-a\right)\Gamma\left(\frac{\beta}{2}+b\right)\Gamma\left(\frac{\beta}{2}-b\right)}{2\pi^2\Gamma(\beta-1)\Gamma(\beta)}\,, \end{gathered} \tag{3.3.5}$$

is called *conformal partial wave*[2] and is a useful quantity on its own (see section 5.3 for an alternative derivation of the constant $K_{\Delta,\ell}$). Unlike the individual block $g_{\Delta,\ell}(z,\bar{z})$, the partial wave $F_{\Delta,\ell}(u,v)$ is single-valued on the Euclidean domain $\bar{z} = z^*$ [76] and is therefore used to invert the expansion eq. (3.2.2) on the principal series [28].

In [55] this approach was generalized to certain defect configurations. However, the projection operator in eq. (3.3.2) is quite easy to write down, though the resulting integral can be hard to evaluate.

3.3.2 Casimir equation

This approach is due to Dolan and Osborn [33]. Consider another projection operator on the conformal multiplet,

$$\mathcal{P}_{\Delta,l} = \sum_{\alpha,\beta=\mathcal{O},P\mathcal{O},P^2\mathcal{O},\dots} |\alpha\rangle \mathcal{N}^{-1}_{\alpha\beta}\langle\beta|\,, \tag{3.3.6}$$

where $\mathcal{N}_{\alpha\beta}$ is the Gram matrix of the multiplet. Inserting $\mathcal{P}_{\Delta,l}$ in the four-point function and performing the OPE, we see that the result is the conformal block,

$$\langle\phi_1(x_1)\phi_2(x_2)|\mathcal{P}_{\Delta,l}|\phi_3(x_3)\phi_4(x_4)\rangle = \mathbf{K}_4 c_{12\mathcal{O}} c_{34\mathcal{O}} g_{\Delta,l}(u,v)\,. \tag{3.3.7}$$

[2]There different meanings of conformal partial wave in the literature. Here the nomenclature of [28] is adopted.

Recall from eq. (2.2.13) that the quadratic Casimir acts on a traceless symmetric primary states as

$$\mathcal{C}|\Delta, l\rangle = C_{\Delta,l}|\Delta, l\rangle\,, \qquad C_{\Delta,l} = \Delta(\Delta-d) + l(l+d-2)\,. \tag{3.3.8}$$

and therefore

$$\mathcal{C}\mathcal{P}_{\Delta,l} = \mathcal{P}_{\Delta,l}\mathcal{C} = C_{\Delta,l}\mathcal{P}_{\Delta,l}\,. \tag{3.3.9}$$

Finally, we need the action of the Casimir on $\phi_3(x_3)\phi_4(x_4)$,

$$\mathcal{C}|\phi_3(x_3)\phi_4(x_4)\rangle = \mathcal{D}_{34}|\phi_3(x_3)\phi_4(x_4)\rangle\,, \qquad \mathcal{D}_{34} = -\frac{1}{2}(\mathcal{L}_3^{ab} + \mathcal{L}_4^{ab})(\mathcal{L}_{3,ab} + \mathcal{L}_{4,ab}) \tag{3.3.10}$$

where $\mathcal{L}_i^{ab}$ is the differential operator giving the action of L^{ab} on $\phi_i(x_i)$.

Now, acting with $\mathcal{D}_{34}$ on eq. (3.3.7) yields the *Casimir equation*, a differential equation for the conformal blocks,

$$\mathcal{D}g_{\Delta,l}(z,\bar{z}) = \frac{1}{2}C_{\Delta,l}g_{\Delta,l}(z,\bar{z})\,, \tag{3.3.11}$$

where

$$\begin{aligned}\mathcal{D} &= z^2(1-z)\partial_z^2 + \bar{z}^2(1-\bar{z})\partial_{\bar{z}}^2 - (a+b+1)(z^2\partial_z + \bar{z}\partial_{\bar{z}}) - ab(z+\bar{z}) \\ &\quad + (d-2)\frac{z\bar{z}}{z-\bar{z}}((1-z)\partial_z - (1-\bar{z})\partial_{\bar{z}})\,.\end{aligned} \tag{3.3.12}$$

The asymptotic behavior of the blocks in eq. (3.2.5) severs as a boundary condition for the solutions of eq. (3.3.11).

In two dimensions, the Casimir equations factorizes and the solution is [33, 34]

$$\begin{aligned}g_{\Delta,l}^{(2d)} &= k_{\Delta+l}(z)k_{\Delta-l}(\bar{z}) + k_{\Delta-l}(z)k_{\Delta+l}(\bar{z})\,, \\ k_\beta(x) &= x^{\frac{\beta}{2}}{}_2F_1\left(\begin{matrix}\frac{\beta}{2}+a,\ \frac{\beta}{2}+b \\ \beta\end{matrix};x\right),\end{aligned} \tag{3.3.13}$$

where $2a = \Delta_2 - \Delta_1$ and $2b = \Delta_3 - \Delta_4$. Using shift operators for the dimension, $d \to d+2$, one arrives at closed form expressions in even dimensions [34]. For example, in 4d,

$$g_{\Delta,l}^{(4d)} = \frac{z\bar{z}}{z-\bar{z}}(k_{\Delta+l}(z)k_{\Delta-l-2}(\bar{z}) - k_{\Delta-l-2}(z)k_{\Delta+l}(\bar{z}))\,. \tag{3.3.14}$$

In odd dimensions, no closed form for the solutions is known. However, one can write down series expansions, which we will introduce next.

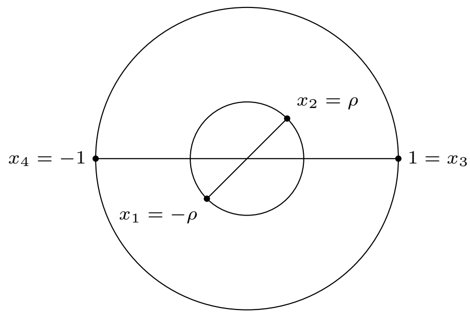


Figure 3.6: *Radial coordinates*

3.3.3 Radial expansion

In order to write down a series expansion for the conformal blocks, we need a suitable basis of functions to expand in. One physically motivated basis is given by the radial quantization [37]. Consider the conformal frame depicted in fig. 3.6, where each two operators live on a quantization slice. Let $\mathbf{n}$ and $\mathbf{n}'$ be the unit vectors pointing to x_2 and x_3, respectively. We can introduce the *radial coordinates* [36],

$$\rho = re^{i\theta}, \qquad \cos\theta = \mathbf{n}\cdot\mathbf{n}', \tag{3.3.15}$$

which are related to z, $\bar{z}$ via

$$z = \frac{4\rho}{(1+\rho)^2}, \quad \bar{z} = \frac{4\bar{\rho}}{(1+\bar{\rho})^2}. \tag{3.3.16}$$

The l. h. s. of eq. (3.3.7) now becomes

$$\begin{aligned}&\langle\phi_3(1,\mathbf{n}')\phi_4(1,-\mathbf{n}')|\mathcal{P}_{\Delta,l}|\phi_1(r,-\mathbf{n})\phi_2(r,\mathbf{n})\rangle\\ &\quad= \langle\phi_3(1,\mathbf{n}')\phi_4(1,-\mathbf{n}')|\mathcal{P}_{\Delta,l}r^D|\phi_1(1,-\mathbf{n})\phi_2(1,\mathbf{n})\rangle\,,\end{aligned} \tag{3.3.17}$$

where we used that dilation operator plays the of the Hamiltonian in radial quantization.

The conformal multiplet $[\Delta, l]$ consists of descendants $|\Delta+m, j\rangle$ of dimension $\Delta+m$, $m \geq 0$, and in traceless symmetric representations $j = \max(l-m, 0), \ldots, l+m$. The latter implies that the dependence of a descendant on the vector $\mathbf{n}$ must be $(\mathbf{n}^{\mu_1}\ldots\mathbf{n}^{\mu_j} - \text{traces})$. Contracting two such tensor structures in eq. (3.3.17) results in a Gegenbauer polynomial $C_j^{\frac{d-2}{2}}(\mathbf{n}\cdot\mathbf{n}' = \cos\theta)$.

Now we are a ready to write down an ansatz,

$$g_{\Delta,l}(\rho,\bar{\rho}) = \sum_{m=0}^{\infty} r^{\Delta+m} \sum_{j=\max(l-m,0)}^{l+m} w(m,j) C_j^{\frac{d-2}{2}}(\cos\theta)\,, \tag{3.3.18}$$

with coefficients $w(m,j)$. The OPE limit eq. (3.2.5) implies $w(0,l) = \frac{l!}{(d/2-1)!}$.

Higher $w(m,l)$ can be calculated recursively by plugging this ansatz into the Casimir eq. (3.3.11) [37, 77].

In [42] a similar radial expansion for defects blocks was derived. However, in this thesis we will focus on the Calogero-Sutherland approach described in the next section.

3.3.4 Calogero-Sutherland model

Actually, the Casimir eq. (3.3.11) has a surprising connection to a well-known problem in quantum mechanics. Applying a change of coordinates [57],

$$z = -\sinh^{-2}\frac{u_1}{2}\,, \quad \bar{z} = -\sinh^{-2}\frac{u_2}{2}\,, \tag{3.3.19}$$

yields

$$\begin{aligned}\Bigg[-\frac{\partial^2}{\partial u_1^2} - \frac{\partial^2}{\partial u_1^2} + \left((a+b)^2 - \frac{1}{4}\right)\left(\frac{1}{\sinh^2 u_1} + \frac{1}{\sinh^2 u_2}\right) - ab\left(\frac{1}{\sinh^2\frac{u_1}{2}} + \frac{1}{\sinh^2\frac{u_2}{2}}\right) \\ + (d-2)(d-4)\left(\frac{1}{\sinh^2\frac{u_1-u_2}{2}} + \frac{1}{\sinh^2\frac{u_1+u_2}{2}}\right)\Bigg]\psi_\epsilon(u_1,u_2) = \epsilon\psi_\epsilon(u_1,u_2)\,,\end{aligned} \tag{3.3.20}$$

where

$$\psi_\epsilon(u_1,u_2) = 2^{d-2\Delta}\frac{[(1-z)(1-\bar{z})]^{\frac{a+b}{2}+\frac{1}{4}}}{(z\bar{z})^{\frac{d-1}{2}}}|z-\bar{z}|^{\frac{d-2}{2}}g_{\Delta,l}(z,\bar{z})\,, \tag{3.3.21}$$

$$2a = \Delta_2 - \Delta_1\,, \quad 2b = \Delta_3 - \Delta_4\,, \quad \epsilon = -\frac{1}{2}C_{\Delta,l} - \frac{d^2-2d+2}{4}\,. \tag{3.3.22}$$

This differential equation is the Hamiltonian of the two-variable Calogero-Sutherland model. It is an integrable model that has been extensively analyzed in the literature [58, 59, 78]. Especially, its solution theory provides us with conformal blocks that are analytical in spin, which comes in handy when using the Lorenztian inversion formula in [28].

In general the Calogero-Sutherland Hamiltonian we will consider in this thesis is of the form

$$\begin{aligned}H_{\mathrm{CS}} = -\sum_{i=1}^{N}\frac{\partial^2}{\partial\tau_i^2} + \frac{k_3(k_3-1)}{2}\sum_{i<j}^{N}\left[\sinh^{-2}\left(\frac{\tau_i+\tau_j}{2}\right) + \sinh^{-2}\left(\frac{\tau_i-\tau_j}{2}\right)\right] \\ + \sum_{i=1}^{N}\left[k_2(k_2-1)\sinh^{-2}(\tau_i) + \frac{k_1(k_1+2k_2-1)}{4}\sinh^{-2}\left(\frac{\tau_i}{2}\right)\right]\,.\end{aligned} \tag{3.3.23}$$

The coupling constants $k_i, i = 1,2,3$ that appear in the potential are referred to as multiplicities in the mathematical literature. In principle, these can assume complex values though we will mostly be interested in cases in which they are real. The coordinates τ_i may also be complex in general. Later we will describe their values for defect configurations in more detail. The case $N = 1$ is a bit special since it involves only two coupling constants.

The Calogero-Sutherland Hamiltonian possesses two different interpretations. We can think of it as describing a system of N interacting particles that move on a one-dimensional half-line with external potential. The external potential is given by the terms in the second line of eq. (3.3.23). These terms contains two of the three coupling constants, namely k_1 and k_2. The interaction terms, on the other hand, involve the third coupling constant k_3. Alternatively, we can also think of a scattering problem for a single particle in an $N-$dimensional space. We will mostly adopt the second view.

Is is not by accident that we recovered the Calogero-Sutherland model. By a procedure described in [79,80], we are guaranteed to get it from Hamiltonian reduction of the conformal group under certain assumptions. We will redo this analysis in presence of defects in chapter 4, and we will describe the solution theory in detail in chapter 5.

3.4 Known blocks for defects

As we mentioned before, these blocks are known in a few examples where they can be related to the blocks of four scalar bulk fields. Furthermore, in [42] a radial expansion similar to section 3.3.3 is described. The task of computing conformal blocks for defects will be the main point of this thesis and will be handled in the next chapters.

The first example we want to discuss here is taken from [43]. It applies to the case in which two bulk fields in $d = 4$ dimensions are inserted into the background of a line defect, i. e. $p = 1$ and $q = 0$. In order to relate the defect block $f(x, \bar{x})$ to the blocks $g(\gamma, \bar{\gamma})$ of four scalar fields, let us consider the following change of coordinates

$$\gamma = \left(\frac{1-x}{1+x}\right)^2 , \quad \bar{\gamma} = \left(\frac{1-\bar{x}}{1+\bar{x}}\right)^2 . \tag{3.4.1}$$

which maps the Euclidean region of the defect coordinates $x, \bar{x}$ to the Euclidean region of the four-point cross-ratios γ, $\bar{\gamma}$. Given this change the following identity holds [43]

$$f\begin{pmatrix} 1,0,4 \\ \Delta,\ell \end{pmatrix}; x, \bar{x}\Big) \propto (\gamma\bar{\gamma})^{-\frac{1}{4}}\, g\begin{pmatrix} \frac{1}{4}, -\frac{1}{4}, 3 \\ \frac{\Delta+1}{2}, \frac{\ell}{2} \end{pmatrix}; \gamma, \bar{\gamma}\Big) . \tag{3.4.2}$$

The lower indices on the block g refer to the conformal weight and spin of the intermediate field. The upper indices $(a, b, d) = (1/4, -1/4, 3)$ contain the relevant information about the external i. e. the parameters $a = (\Delta_2 - \Delta_1)/2, b = (\Delta_3 - \Delta_4)/2$ and the dimension d. Note that the four-point block on the right hand side is the one with $\Delta_1 - \Delta_2 = \Delta_3 - \Delta_4 = -1/2$ and dimension $d = 3$ even though the original defect setup is in $d = 4$ dimensions and involves two bulk fields of the same weight.

A second example for a relation between defects and scalar four-point was pointed out in [54]. Conformal blocks for the two-point function of defects of dimension $p = q = d - 2$ can be mapped to the four-point function of scalars with the following relation between the different variables

$$\eta_1 = \frac{2(1+v)}{u}, \quad \eta_2 = \frac{2(1+6v+v^2)}{u^2} \tag{3.4.3}$$

where u and v are related to the usual cross-ratios z and $\bar{z}$ as $u = z\bar{z}$ and $v = (1-z)(1-\bar{z})$.

With this change of variables the relation of [54] reads

$$f_D \begin{pmatrix} d-2, d-2, d \\ \Delta, \ell \end{pmatrix} ; \eta_a \Bigg) = g \begin{pmatrix} 0, 0, d \\ \Delta, \ell \end{pmatrix} ; z, \bar{z} \Bigg) . \tag{3.4.4}$$

As in the previous example, the Euclidean region of the defect block is mapped to a pair of complex conjugate variables $z, \bar{z}$ and hence to the Euclidean region of the four-point blocks. The scalar block on the right hand side is the one with $a = b = 0$ and the same dimension d as on the left hand side.

Another relation between blocks was suggested in [41]. These authors considered two bulk fields, i. e. $q = 0$, in the presence of a defect of dimension $p = d - 2$ and proposed the following relation between the corresponding defect blocks in the bulk channel with four-point blocks

$$f \left(\begin{matrix} d-2, 0, d \\ \Delta, \ell \end{matrix} ; x, \bar{x} \right) \sim g \left(\begin{matrix} 0, 0, d \\ \Delta, \ell \end{matrix} ; 1-x, 1-\frac{1}{\bar{x}} \right) . \tag{3.4.5}$$

Let us point out, however, that the relation (3.4.5) does not map the Euclidean region of the defect block to the Euclidean region of the scalar four-point block. In fact, it maps to a Lorentzian region where the bulk fields become light-like separated. Hence, any relation of the form (3.4.5) involves an analytic continuation. Since the blocks possess branch cuts, this continuation requires additional choices. As it stands, eq. (3.4.5) only describes a relation between the parameters of the relevant Casimir equations. This is why we put a $\sim$ in between the left- and the right- hand side.

As we will see, the technology presented in the next chapter will explain all these relations and vastly generalize them, through a (re-)interpretation as symmetries of Calogero-Sutherland models.

3.5 Inversion formulas for scalar four-point blocks

The idea behind the Euclidean inversion formula in [81] is to extract the operator product coefficients from the correlator. While the Euclidean inversion formula is not analytical in spin, it can be transformed into a Lorentzian formula that is [28]. Analyticity in spin provides control over individual OPE coefficients in the contexts of large spin expansions [24–27, 82, 83]. Here, we outline the results of [28].

First we need to turn the sum over the conformal dimensions Δ in eq. (3.2.3) into an integral such that the single-valued conformal partial waves $F_{\Delta,\ell}$ of section 3.3.1 appear [76],

$$\mathcal{G}(z,\bar{z}) = \mathbf{1}_{12}\mathbf{1}_{34} + \sum_{\ell=0}^{\infty} \int\limits_{d/2-i\infty}^{d/2+i\infty} \frac{d\Delta}{2\pi i}\, c(\Delta,\ell) F_{\Delta,\ell}(z,\bar{z})\,. \tag{3.5.1}$$

The *shadow coefficients* $c(\Delta,\ell)$ are defined such that[3]

$$c_{12\mathcal{O}}c_{34\mathcal{O}} = -\operatorname*{Res}_{\Delta'=\Delta} c(\Delta',\ell) \qquad (\Delta \text{ generic})\,, \tag{3.5.2}$$

where the operator $\mathcal{O}$ in the intermediate channel has conformal weight Δ and spin ℓ. Assuming orthogonality of the partial waves $F_{\Delta,\ell}$, we can integrate against them and obtain the *Euclidean inversion formula*,

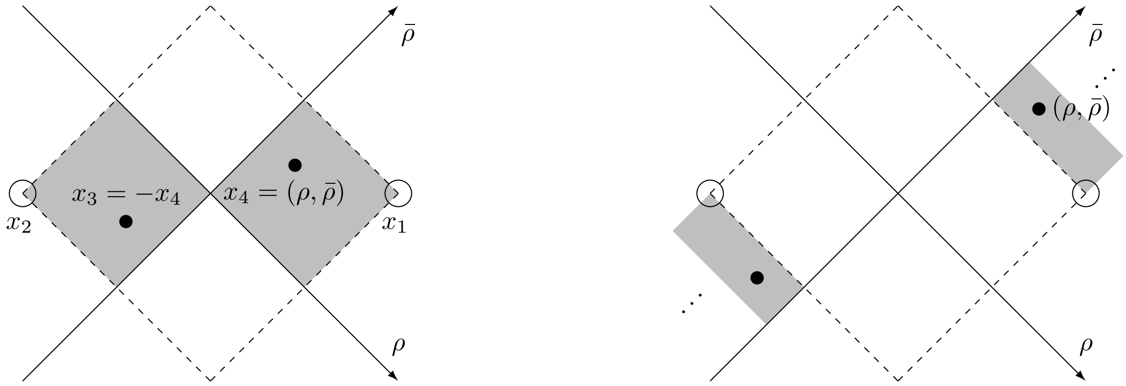
$$\begin{aligned}
c(\Delta,\ell) &= N(\Delta,\ell) \int\limits_{\mathbb{C}} d^2z\, \mu(z,\bar{z}) F_{\Delta,\ell}(z,\bar{z}) \mathcal{G}(z,\bar{z})\,,\\
N(\Delta,\ell) &= \frac{K_{\Delta,\ell}}{K_{d-\Delta,\ell}} \frac{\Gamma\left(\ell+\frac{d-2}{2}\right)\Gamma\left(\ell+\frac{d}{2}\right)}{2\pi\Gamma\left(\ell+1\right)\Gamma\left(\ell+d-2\right)}\,,\\
\mu(z,\bar{z}) &= \left|\frac{z-\bar{z}}{z\bar{z}}\right|^{d-2} \frac{\left[(1-z)(1-\bar{z})\right]^{a+b}}{(z\bar{z})^2}\,,
\end{aligned} \tag{3.5.3}$$

where $K_{\Delta,\ell}$ is defined in eq. (3.3.5). There are some subtleties regarding convergence and contours which one has to take of [28]. However, they are of no concern to us in what follows.

In this thesis we describe the necessary steps to derive the blocks and partial waves for defect configurations which can be eventually used to write down the Euclidean inversion formula in this case [84].

In order to derive the *Lorentzian inversion formula*, we need to look at the Lorentzian kinematics, where $z, \bar{z}$ become real and independent. This region is most conveniently understood in terms of the radial coordinates of eq. (3.3.15). In the region $0 < \rho, \bar{\rho} < 1$ all points are space-like separated and the physics is essential Euclidean, depicted in fig. 3.7a. The region $0 < \rho < 1 < \bar{\rho}$, depicted in fig. 3.7b, is

[3]This formula has to be corrected for non-generic Δ, see [28] for details.



(a) Euclidean-like domain $0 < \rho, \bar\rho < 1$.

(b) Lorentzian domain $0 < \rho < 1 < \bar\rho$.

Figure 3.7: Domains in Lorentzian kinematics. In order to pass from the Euclidean-like region **(a)** to the Lorentzian region **(b)**, we need to cross two light-cones, once of x_1 and once of x_2.

more interesting: the distances $x_4 - x_1$ and $x_2 - x_3$ become time-like separated. Now we wish to analytic continue the inversion formula (in terms of the cross-ratios $\rho, \bar\rho$) in eq. (3.5.3) to the Lorentzian region. This is a quite involved task carried out in [28], see also [85] for a derivation in terms of space-time positions. Either way, the result is

$$c(\Delta, \ell) = c^t(\Delta, \ell) + (-1)^\ell c^u(\Delta, \ell)\,, \tag{3.5.4}$$

$$c^t(\Delta, \ell) = \frac{\kappa_{\Delta+\ell}}{4} \int_0^1 d^2z\, \mu(z, \bar z) g_{\ell+d-1, \Delta-d+1}(z, \bar z)\, \mathrm{dDisc}\, \mathcal{G}(z, \bar z)\,. \tag{3.5.5}$$

where κ_β is defined in eq. (3.3.5). A few comments are in order. First of all, the *double discontinuity* of the stripped amplitude appears,

$$\begin{aligned} \mathrm{dDisc}\, \mathcal{G}(\rho, \bar\rho) &= \cos(\pi(a+b)) \mathcal{G}(\rho, 1/\bar\rho) \\ &\qquad - \frac{1}{2} e^{+i\pi(a+b)} \mathcal{G}(\rho, \bar\rho - i0) - \frac{1}{2} e^{-i\pi(a+b)} \mathcal{G}(\rho, \bar\rho + i0)\,. \end{aligned} \tag{3.5.6}$$

This comes from the fact that the amplitude posses a branch-point on the light-cone along the path of analytic continuation. Because we simultaneously move $x_{3,4}$ across the light-cones of $x_{2,1}$, respectively, we collect the *double* discontinuity of the amplitude. Physically, this can be understood as the double commutator,

$$\mathrm{dDisc}\, \mathcal{G}(\rho, \bar\rho) = -\frac{1}{2} \langle [\mathcal{O}_2(-1), \mathcal{O}_3(-\rho)][\mathcal{O}_1(1), \mathcal{O}_4(\rho)] \rangle \geq 0\,. \tag{3.5.7}$$

Positivity follows either from the so-called Rindler positivity (valid in any QFT) [86–88] or positivity of OPE coefficients in unitary CFTs [28, 89, 90].

The u-channel contribution in eq. (3.5.4) comes from exchanging operators $\mathcal{O}_1 \leftrightarrow$

$\mathcal{O}_2$. Equivalently, we can change the integration region to $(-\infty, 0)$ in eq. (3.5.5) and taking the double discontinuity around ∞.

Finally, eq. (3.5.5) is manifestly analytic in spin.[4] The integral converges if $\operatorname{Re}\ell$ is large enough, in unitary theories convergence is guaranteed for $\ell > 1$ [28]. In the next section we will discuss its application to the light-cone bootstrap.

3.6 Light-cone bootstrap

The *light-cone bootstrap* allows to extract data analytically from the crossing-symmetry equations at large spin, showing the existence of double-twist operators [25, 26]. In this section we want to outline the basic results for bulk fields in order to apply the techniques to defects in section 4.7.

The light-cone limit is given by $z, 1-\bar{z} \ll 1$.[5] Consider the crossing-equation for four identical scalars, eq. (3.2.6), in the light-cone limit,

$$z^{-\Delta_\phi} + \cdots = \sum_{\mathcal{O}} c^2_{\phi\phi\mathcal{O}} (1-\bar{z})^{\frac{\Delta-\ell}{2}-\Delta_\phi} k_{\Delta+\ell}(1-z) + \ldots . \tag{3.6.1}$$

We are assuming a twist gap above the identity which always the case for unitary theories in $d > 2$. Then the left-hand side is dominated by the unit operator $z^{-\Delta_\phi}(1 + O(1-\bar{z}))$ because the small z limit singles out lowest twist operators, see eq. (3.2.5). On the right-hand side no single term dominates in the small z limit. Instead, we have expanded the blocks in the small $1-\bar{z}$ limit [33, 34],

$$\begin{aligned} g_{\Delta,\ell}(1-z, 1-\bar{z}) &= (1-\bar{z})^{\frac{\Delta-\ell}{2}} \left(k_{\Delta+\ell}(1-z) + O(1-\bar{z})\right) , \\ k_\beta(x) &= x^{\frac{\beta}{2}} {}_2F_1\left(\begin{matrix}\frac{\beta}{2}, \frac{\beta}{2} \\ \beta\end{matrix}; x\right) . \end{aligned} \tag{3.6.2}$$

We see that each block on the right-hand side has a logarithmic singularity in the small z limit, whereas the unit operator has a power-law behavior. We see that we need an infinite number of blocks in the limit $\Delta + \ell \to \infty$ while $\Delta - \ell \to 2\Delta_\phi$ (because the left-hand side is independent of $\bar{z}$). This is the family of *double-twist operators* $[\phi\phi]_0$. The OPE coefficients are given by [25]

$$c^2_{\phi\phi[\phi\phi]_0} \overset{\ell\to\infty}{\approx} \frac{2^{3-2\Delta_\phi-2\ell}\sqrt{\pi}}{\Gamma(\Delta_\phi)^2} \ell^{2\Delta_\phi - \frac{3}{2}} . \tag{3.6.3}$$

Precisely speaking, this is only true for asymptotic density of the coefficients. The density could be distributed in many ways. The analyticity in spin of the Lorentzian

[4] In [39] an explicit expression for the conformal blocks analytic in spin is derived.

[5] Sometimes, this is called the *double light-cone limit*, whereas the *light-cone limit* is $z \ll 1 - \bar{z} \ll 1$. We will not make this distinction here.

inversion formula on the other hand shows that the OPE coefficients are distributed in the simplest way. For instance, this is required in the context of *conformal Regge theory* [76, 91].

One may consider subleading contributions in twist beyond the unit operator on the left-hand side of eq. (3.6.1). This results in asymptotic corrections in $1/\ell$ to the conformal dimension and OPE coefficients of the double-twist operators. Again, using the inversion formula one may compute the corrections at finite spin ℓ, at least numerically. Furthermore, the convergence of the integral for $\mathrm{Re}\,\ell > 1$ explains why the light-cone bootstrap matches the numerical bootstrap results for spins down to $\ell = 2$.

Let us analyze the light-cone limit $z, 1-\bar{z} \ll 1$ using the Lorentzian inversion formula in detail. The spectrum arises from singularities in Δ in eq. (3.5.2) which originate from the $z \to 0$ limit of the Lorentzian inversion formula in eq. (3.5.5). Expanding the integrand in z suggests the definition of a *generating function*[6] [28],

$$C^t(z,\beta) \equiv \int_z^1 \frac{d\bar{z}(1-\bar{z})^{a+b}}{\bar{z}^2} \kappa_\beta k_\beta(\bar{z})\, \mathrm{dDisc}\, \mathcal{G}(z,\bar{z}) \,, \tag{3.6.4}$$

where the poles of the shadows coefficients can be expressed in terms of the generating function as

$$c^t(\Delta,\ell)\Big|_{\text{poles}} = \int_0^1 \frac{dz}{2z} z^{\frac{\ell-\Delta}{2}} \left(\sum_{m=0}^{\infty} z^m \sum_{k=-m}^{m} B^{(m,k)}_{\Delta,\ell} C^t(z, \Delta+\ell+2k) \right) . \tag{3.6.5}$$

The coefficients $B^{(m,k)}_{\Delta,\ell}$ are fixed by conformal invariance with $B^{(0,0)}_{\Delta,\ell} = 1$. They can be either calculated recursively using the quadratic Casimir equation or from the known solution of the $N = 2$ CS-Hamiltonian that is analytic in spin [39].

As before we are interested in the contribution of the identity operator in the t-channel,

$$\mathcal{G}(u,v) = \left(\frac{u}{v}\right)^{\Delta_\phi} . \tag{3.6.6}$$

Plugging the identity into eq. (3.6.4) yields in the limit $z \to 0$

$$\begin{aligned} C^t(z,\beta) &= z^{\Delta_\phi} \int_0^1 \frac{d\bar{z}(1-\bar{z})^{a+b}}{\bar{z}^2} \kappa_\beta k_\beta(\bar{z})\, \mathrm{dDisc} \left(\frac{1-\bar{z}}{\bar{z}}\right)^{\Delta_\phi} \\ &= \frac{z^{\Delta_\phi}}{\Gamma^2(\Delta_\phi)} \frac{\Gamma^2\left(\frac{\beta}{2}\right)}{\Gamma(\beta-1)} \frac{\Gamma\left(\frac{\beta}{2}+\Delta_\phi-1\right)}{\Gamma\left(\frac{\beta}{2}-\Delta_\phi+1\right)} . \end{aligned} \tag{3.6.7}$$

[6] The integration range is restricted to $\bar{z} > z$ in order to avoid over-counting.

Note that the integral is dominated by $\bar{z} \to 1$ at large spin ℓ. This justifies the consideration of the unit operator only, as we did in light-cone limit in eq. (3.6.1). Furthermore, the contribution of the unit operator cannot be canceled by any other operator at large spin ℓ.

We see that eq. (3.6.5) has poles in z whenever $\Delta - \ell = 2\Delta_\phi$. Including the identical u-channel contribution, eq. (3.5.2) yields for the OPE coefficients

$$c^2_{\phi\phi[\phi\phi]_0} = [1 + (-1)^\ell] \frac{(\Delta_\phi)^2_\ell}{l!\,(2\Delta_\phi + \ell - 1)_\ell}\,, \qquad \ell \to \infty\,. \tag{3.6.8}$$

We conclude that the OPE coefficients of the double-twist operators at large spin are given by their values in mean field theory [92].

Chapter 4

Calogero-Sutherland approach to conformal blocks

In this chapter we want to describe a fully systematic framework for the Casimir equations of conformal blocks for correlation functions of two defects. Rather than working with the popular embedding space, we shall realize all blocks as functions on the conformal group itself. If the latter is equipped with an appropriate set of coordinates, the Casimir equations assume a universal form. In fact, they can be phrased as an eigenvalue problem for an N-particle Calogero-Sutherland system. This follows the ideas of [93–95] who showed how to obtain a Calogero-Sutherland Hamiltonian from the harmonic analysis on Lie groups. See also [79, 80] and [39, 57, 61, 96] for applications to four-point blocks. We will review the derivation in the first section, relate the abstract coordinates to physical setup in the second and discuss some immediate consequences of the equations and their symmetries afterwards. Finally, we will derive the Lorentzian inversion formula for the defect blocks.

4.1 Harmonic approach to defect blocks

In order to understand the connection between conformal blocks and the Calogero-Sutherland model, we will employ a more group theoretic approach to the blocks. We will follow the discussion in [61] but adjust it to the case involving defects.

As we have stated before, a p-dimensional conformal defect breaks the conformal group $G = \mathrm{SO}(1, d+1)$ down to the subgroup

$$G_p = \mathrm{SO}(1, p+1) \times \mathrm{SO}(d-p) \ \subset \ G \ . \tag{4.1.1}$$

In order to define the space of blocks for two defects $\mathcal{D}^{(p)}$ and $\mathcal{D}^{(q)}$ we must first choose two finite dimensional irreducible (unitary) representations π_L and π_R of the groups G_p and G_q. Here we shall restrict to scalar blocks from the very beginning which means that π_L and π_R are assumed to be one-dimensional. For $p, q \neq 0$, the only one-dimensional representation is the trivial one. Only if either q or even p and q vanish, one can have a non-trivial one-dimensional representation for which the generator of dilations is represented by a complex number. We shall denote these parameters by b and a, respectively. If $p, q \neq 0$ the space of conformal blocks is given by

$$\Gamma_{pq} \;=\; \{\, f : G \to \mathbb{C} \,|\, f(h_L g h_R) = f(g)\,;\, h_L \in G_p, h_R \in G_q \,\} \;, \tag{4.1.2}$$

i.e. it consists of all complex valued functions on the conformal group that are invariant with respect to left translations by elements $h_L \in G_p$ and to right right translations by elements $h_R \in G_q$. When $q = 0$ but $p \neq 0$, translations with elements

$$d(\lambda) = \begin{pmatrix} \cosh\lambda & \sinh\lambda \\ \sinh\lambda & \cosh\lambda \end{pmatrix} \tag{4.1.3}$$

of the subgroup $D = \mathrm{SO}(1,1) \in G_0$ are accompanied by a non-trivial phase shift

$$\Gamma^a_{\ p} \;=\; \{\, f : G \to \mathbb{C} \,|\, f(h_L g d h'_R) = e^{-2a\lambda} f(g)\,;\, h_L \in G_p, h'_R \in \mathrm{SO}(d) \,\} \;. \tag{4.1.4}$$

In case both p and q vanish, finally, the resulting space of scalar four-point blocks is given by [61]

$$\Gamma^{ba} \;=\; \{\, f : G \to \mathbb{C} \,|\, f(h_L g d h'_R) = e^{2(b-a)\lambda} f(g)\,;\, h'_L, h'_R \in \mathrm{SO}(d) \,\} \;. \tag{4.1.5}$$

In all three cases, the elements of the space Γ are uniquely determined by the values they take on the double quotient $G_p \backslash G / G_q$. This two-sided coset parametrizes the space of cross-ratios. The precise relation between cross-ratios and coordinates on the conformal groups will be discussed below. For the moment let us only check that the double quotient is N-dimensional. In order to see that, we anticipate from our discussion of coordinates below that a point on the double quotient is stabilized by the subgroup

$$B_{pq} = \mathrm{SO}(p-q) \times \mathrm{SO}(|d-p-q-2|) \subset G_p, G_q \subset G \;. \tag{4.1.6}$$

Once this is taken into account, it is is straightforward to compute the dimension of the double coset space,

$$\dim G_p \backslash G / G_q = \dim G - \dim G_p - \dim G_q + \dim B_{pq} = N \;.$$

All this is valid for any choice of p, q including $p = q = 0$. In the latter case, the double coset coincides with the one that was introduced in the context of scalar four-point blocks [61].

The space Γ of conformal blocks comes equipped with an action of several differential operators. In fact, the Casimir elements of the conformal group G give rise to differential operators for functions on the conformal group with the usual Laplacian associated to the quadratic Casimir element. Higher order differential operators come with the higher order Casimir elements. These differential operators on the group commute with both left and right translation and hence they descend to a set of commuting differential operators on the space Γ. By definition conformal blocks are eigenfunctions of these differential operators, i. e. we obtain a decomposition in to intermediate channels

$$\Gamma = \bigoplus_{\Delta,\ell} \Gamma^{(\Delta,\ell)} \,. \tag{4.1.7}$$

In deriving the results of this thesis our main task is to evaluate the quadratic Casimir element on the quotient $G_p\backslash G/G_q$. This is facilitated by a choice of coordinates on the conformal group that is adapted to the geometrical setup. More precisely, via the Cartan decomposition we can parametrize elements $g \in G$ of the conformal group as

$$g = h'_L a(\tau) h_R \qquad h_R \in G_q\,,\ h'_L \in G_p/B_{pq}\ . \tag{4.1.8}$$

The choice of coordinates for elements $h_R \in G_q$ of the subgroup G_q is not important. In order to parametrize the subgroup G_p one should first choose coordinates on the subgroup B_{pq} and then extend these to coordinates of G_p. Elements h'_L of the $(\dim G_p - \dim B_{pq})$-dimensional quotient G_p/B_{pq} do not depend on the coordinates on B_{pq}. In order to factorise elements g of the conformal group as in eq. (4.1.8), we need N additional coordinates which parametrize the factor $a = a(\tau)$ in the middle. This takes the form

$$a(\tau) = e^{\tau_i M_{i-1,p+1+i}} \in A_{pq} \qquad \text{where } M_{i-1,p+1+i}\,, \quad i = 1, \dots, N$$

are the usual generators of $\mathrm{SO}(1, d+1)$. In particular, the generators $M_{i-1,p+1+i}$ with $i \geq 3$ are generators of rotations in the $(i-2, p+i)$-plane while

$$M_{0,p+2} = \frac{1}{2}\left(P_{p+2} - K_{p+2}\right) \quad , \quad M_{1,p+3} = \frac{1}{2}\left(P_{p+3} + K_{p+3}\right)$$

are linear combinations of infinitesimal translations and special conformal transformations. The various subgroups and the generators $M_{i-1,p+1+i}$ of the torus A are illustrated in figure 4.1. Let us note that the generators $M_{i-1,p+1-i}$ commute with elements in the subgroup B_{pq}, a result we anticipated above.

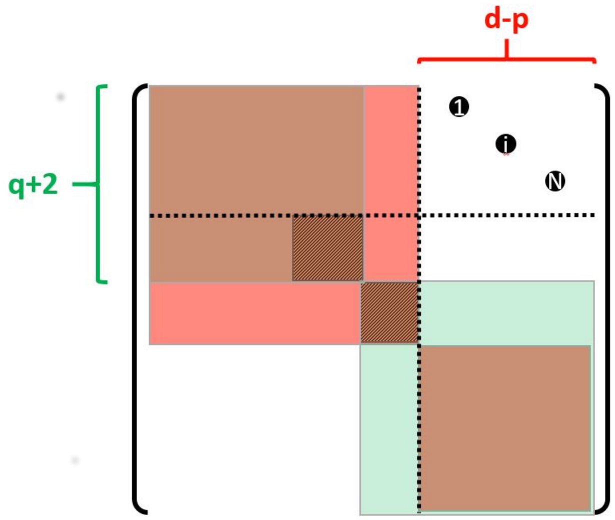


Figure 4.1: The figure illustrates our choice of coordinates on the conformal group. The blocks in red/green correspond to the left/right group G_p/G_q while the additional generators $M_{i-1,p+1+i}$ are represented by block dots. The subgroup B_{pq} of elements that commute with $M_{i-1,p+1+i}, i = 1, \dots, N$ is shown as the shaded area. Obviously, it is contained in the intersection of G_p and G_q (brown area).

Once we have fixed our coordinates on G it is straightforward to compute first the metric and then the Laplace-Beltrami operator Δ_{LB} on G. The metric on the group manifold G is given in terms of the Killing form. In local coordinates,

$$g_{\alpha\beta}(x) = -2\,\mathrm{tr}\, h^{-1}\partial_\alpha h h^{-1}\partial_\beta h\,, \qquad h \in G\,. \tag{4.1.9}$$

The formula of the Laplace-Beltrami operator on any Riemannian manifold is in local coordinates

$$\Delta_{\mathrm{LB}} = |\det g|^{-\frac{1}{2}}\partial_\alpha g^{\alpha\beta}|\det g|^{\frac{1}{2}}\partial_\beta\,. \tag{4.1.10}$$

The Laplace-Beltrami operator commutes with left and right actions of G and can be written in terms of a quadratic expression of invariant vector fields on G contracted with the Killing form. Hence, it coincides with the quadratic Casimir element.

The resulting expression is a second order differential operator that contains derivatives with respect to all the coordinates x_i on the conformal group, including the coordinates τ_i on the torus A_{pq} and the parameters λ_R and λ_L on the subgroups $D = \mathrm{SO}(1,1)$ of dilations in case $q = 0$ or $p = q = 0$. In order to descend to the space of conformal blocks, i. e. functions on the Cartan subgroup A_{pq}, we need to introduce a scalar product and thus the invariant Haar measure on G,

$$d\mu_G = \sqrt{\det g}\prod_i dx_i\,. \tag{4.1.11}$$

The Laplace-Beltrami operator is densely defined on the associated space of square integrable functions L^2_G and is Hermitian with respect to the scalar product.

The measure on A_{pq} is given by

$$d\mu_A(\tau_i) = m(\tau_i) \prod_i d\tau_i = \frac{1}{Z} \fint_{G_p/B_{pq}\times G_q} d\mu_G \,. \tag{4.1.12}$$

where Z is the (infinite) volume of $G_p/B_{pq} \times G_q$: The integration domain contains the non-compact factor D associated to dilations and has to be regularized, e. g.

$$\frac{1}{\mathrm{vol}(\mathbb{R})} \fint_{\mathbb{R}} d\lambda = \lim_{L\to\infty} \frac{1}{2L} \int_{-L}^{L} d\lambda \,. \tag{4.1.13}$$

Equipped with the measures on G and A_{pq} and a regularization procedure, we can define the Laplacian on A_{pq}. Functions f_A on A_{pq} uniquely extend to covariantly constant functions f on G,

$$f(h'_L a h_R) := e^{2b\lambda_l + 2a\lambda_r} f_A(a)\,, \qquad h_R \in G_q\,, \quad h'_L \in G_p/B_{pq}\,. \tag{4.1.14}$$

In case $p \neq 0$ or $p, q \neq 0$ we set $b = 0$ or $a = b = 0$, respectively. The Laplacian Δ^A_{LB} on A is then defined by

$$\int_A d\mu_A\, f_{1A}(a)\, \Delta^A_{\mathrm{LB}} f_{2A}(a) = \frac{1}{Z} \fint_G d\mu_G\, f_1(g)\, \Delta_{\mathrm{LB}} f_2(g)\,. \tag{4.1.15}$$

Effectively, we have set all other derivatives to zero so that we end up with a second order differential operator Δ^A_{LB} in τ_i. In case $q = 0$ or $p = q = 0$ the derivatives with respect to λ_R and λ_L are replaced by $-2a$ and $2b$, respectively. The operator Δ^A_{LB} still turns out to contain some first order terms which come from the non-trivial measure $m(\tau_i)d\tau_i$ on the Cartan subgroup. The latter can be removed by an appropriate "gauge transformation"

$$\omega = m(\tau_i)^{-\frac{1}{2}}\,. \tag{4.1.16}$$

The Casimir operator L^2 is given by

$$L^2 = \omega^{-1}\, \Delta^A_{\mathrm{LB}}\, \omega\,, \tag{4.1.17}$$

and indeed takes the form of a Calogero-Sutherland type Hamiltonian. The above procedure can be effectively implemented in Mathematica [97]. Before we state the results, we will give a geometric explanation of the coordinates τ_i.

4.2 Defect cross-ratios revisited

It remains to relate the group theoretic variables τ_i we introduced through our parametrization of the conformal group G to the cross-ratios. As we explained in section 2.4.1, the location of the defect operators $\mathcal{D}^{(p)}(P_\alpha)$ and $\mathcal{D}^{(q)}(Q_\beta)$ can be characterized by a set of orthonormal vectors $P_\alpha, \alpha = p+2, \dots, d+1$, and $Q_\beta, \beta = q+2, \dots, d+1$, which are transverse to the defect in embedding space, respectively. We can complete these two sets to an orthonormal basis $\mathcal{P}$, $\mathcal{Q}$ of the full embedding space by adding vectors $\tilde{P}_\alpha, \alpha = 0, 1, \dots, p+1$, and $\tilde{Q}_\beta, \beta = 0, 1, \dots, q+1$. Let us now combine these systems of orthonormal vectors into two matrices

$$\mathcal{P} = (\tilde{P}, P) \in G = SO(1, d+1)\,, \quad \mathcal{Q} = (\tilde{Q}, Q) \in G\,. \tag{4.2.1}$$

By construction, both $\mathcal{P}$ and $\mathcal{Q}$ carry a left action of the conformal group (since the columns are vectors in embedding space) and a right with respect to G_p and G_q, respectively. The latter respects the split of the columns into vectors tangential and transverse to the defect. For the two $\mathrm{SO}(1, d+1)$ matrices $\mathcal{P}$ and $\mathcal{Q}$ we can now form the matrix $\mathcal{P}^T\mathcal{Q} \in \mathrm{SO}(1, d+1)$. Obviously, $\mathcal{P}^T\mathcal{Q}$ is invariant under conformal transformations, but it transforms non-trivially under the action of G_p and G_q. In this way, any configuration of two defects of dimension p and q gives rise to an orbit $G_p\mathcal{P}^T\mathcal{Q}G_q$ in the double quotient $G_p\backslash G/G_q$.

In section 2.5.3 we considered the matrix $M = P^TQ$ in order to construct the cross-ratios η_i of the defect configurations. Now we see that M appears as the lower right matrix block of the matrix $a(\tau)$ we introduced in eq. (4.1.8). From the explicit construction in terms of the generators $M_{i-1,p+1+i}$ we can see that the lower right corner of $a(\tau)$ takes the form

$$\left(\begin{array}{ccccc|c} \cosh\frac{\tau_1}{2} & & & & & \\ & \cosh\frac{\tau_2}{2} & & & & \\ & & \ddots & & & 0 \\ & & & \cosh\frac{\tau_N}{2} & & \\ & & & & I & \end{array}\right). \tag{4.2.2}$$

Comparison with our discussion of the cross-ratios allows us to read off the relation (4.3.9) between the group theoretic variables and cross-ratios.

The last task is to relate the Calogero-Sutherland eigenfunctions to the conformal blocks. In case of $p, q > 0$, the Casimir equation for the correlator is the same as for the block (see eq. (3.2.13)). Hence we just need to undo the gauge transformation (4.1.16). In case the defect configuration includes local fields, i.e. when $q = 0$ or

$p, q = 0$, there are non-trivial prefactors, c. f. eqs. (2.5.11) and (2.5.22). However, the Casimir equations have already been worked out in [34, 41] and we can just compare. This concludes the derivation of the Calogero-Sutherland model for defect blocks. In the next section we will state the results of this analysis.

4.3 Calogero-Sutherland models for defects

We have argued that the Casimir equations for conformal blocks of two defects can be restated as an eigenvalue problem for the Calogero-Sutherland Hamiltonian of the form

$$\begin{aligned} H_{\mathrm{CS}} = &-\sum_{i=1}^{N} \frac{\partial^2}{\partial \tau_i^2} + \frac{k_3(k_3-1)}{2} \sum_{i<j}^{N} \left[\sinh^{-2}\left(\frac{\tau_i+\tau_j}{2}\right) + \sinh^{-2}\left(\frac{\tau_i-\tau_j}{2}\right) \right] \\ &+ \sum_{i=1}^{N} \left[k_2(k_2-1)\sinh^{-2}(\tau_i) + \frac{k_1(k_1+2k_2-1)}{4} \sinh^{-2}\left(\frac{\tau_i}{2}\right) \right] . \end{aligned} \tag{4.3.1}$$

Let us note that the multiplicities are not defined uniquely, i.e. different choices of the multiplicities k_i can give rise to identical Casimir equations. This is partly due to the fact that the multiplicities appear quadratically in the potential. In addition, one may show that a simultaneous shift of all coordinates $\tau_i \to \tau_i + i\pi$ for $i = 1, \ldots, N$ leads to a Calogero-Sutherland Hamiltonian of the form (4.3.1) with different multiplicities. The complete list of symmetries is given in table 4.1. Later we see that these innocent looking replacements have remarkable consequences, since they produce non-trivial relations between the blocks of various (defect) configurations.

Table 4.1: Symmetries of the Calogero-Sutherland model for generic values of the multiplicities. The last symmetry also involves a shift $\tau_i' = \tau_i \pm i\pi$ of the coordinates.

	k_1'	k_2'	k_3'
ϱ_1	$1-k_1-2k_2$	k_2	k_3
ϱ_2	$-k_1$	$1-k_2$	k_3
ϱ_3	k_1	k_2	$1-k_3$
$\tilde{\varrho}$	k_1	$1-k_1-k_2$	k_3

Let us now describe the main new results of this work. The first case to look at is the case of two defects of dimension $p \geq q$ with $q \neq 0$. The corresponding Casimir equation for conformal blocks is an eigenvalue equation for the operator

$$L^2 = H_{CS} + \epsilon_0\,, \quad \epsilon_0 = \frac{N}{8}\left(\frac{d(d+2)}{2} - N(d+1) + \frac{2N^2+1}{3}\right) \tag{4.3.2}$$

with the following choice of parameters

$$N = \min(d-p, q+2)\,, \quad k_1 = \frac{d}{2} - (p-q) - N + 1\,, \quad k_2 = \frac{p-q}{2}\,, \quad k_3 = \frac{1}{2}\,. \tag{4.3.3}$$

Recall from eq. (2.2.13) that in a representation of spin ℓ and weight Δ, the operator L^2 assumes the value

$$C_{\Delta,J} = \Delta(\Delta-d) + \sum_{i=1}^{N-1} l_i(l_i + d - 2i)\,, \tag{4.3.4}$$

where the spin ℓ is labeled by a set of even integers $\ell = (l_1, \dots, l_{N-1})$ with $l_1 \geq \dots \geq l_{N-1} \geq 0$. The wave function $\psi(\tau)$ is given by the Schrödinger-like equation

$$H_{CS}\psi_\epsilon(\tau) = \epsilon\psi_\epsilon(\tau) \tag{4.3.5}$$

and is related to the conformal block by[1]

$$f_D\begin{pmatrix} p,q,d \\ \Delta,\ell \end{pmatrix}; \tau\Bigg) = 2^{2\Delta - \frac{1}{2}N(d-N+1)}\omega(\tau)\psi_\epsilon(\tau)\,, \quad \epsilon = -\frac{1}{4}C_{\Delta,\ell} - \epsilon_0\,, \tag{4.3.6}$$

where the "gauge transformation" $\omega(\tau)$ (c. f. eq. (4.1.16)) is given by

$$\omega(\tau) = \prod_{i=1}^{N} \sinh^{N-\frac{d}{2}+\frac{p-q}{2}-1}\left(\frac{\tau_i}{2}\right)\cosh^{-\frac{p-q}{2}}\left(\frac{\tau_i}{2}\right)\prod_{i<j}\sinh^{-\frac{1}{2}}\left(\frac{\tau_i \pm \tau_j}{2}\right). \tag{4.3.7}$$

Here and throughout the entire text below we use the shorthand

$$\sinh\left(\frac{x \pm y}{2}\right) = \sinh\left(\frac{x+y}{2}\right)\sinh\left(\frac{x-y}{2}\right). \tag{4.3.8}$$

Equation (4.3.5) is to be considered on a subspace of the semi-infinite hypercuboid A_N^E that is parametrized by the coordinates

$$\tau_1 = 2\vartheta = 2\log\frac{R}{r} \in [0,\infty)\,, \quad \tau_{j+1} = 2i\theta_j \in i[0,2\pi]\,, \tag{4.3.9}$$

for $j = 1, \dots, N-1$. We shall discuss the domain in much more detail in section 5.1. Of course, the choice of multiplicities k_i is not unique since we can apply any of the transformations listed in table 4.1. We will discuss the consequences in the next section.

If $q = 0$ while $0 < p \leq d-2$, the setup describes two scalar bulk fields in the presence of a p-dimensional defect of co-dimension greater or equal to two. In this

[1] We postpone the normalization of $\psi_\epsilon(\tau)$ to section 5.2.

case, the conformal Casimir operator takes the form

$$L^2 = H_{CS} + \epsilon_0\,, \quad \epsilon_0 = \frac{d^2 - 2d + 2}{8} \tag{4.3.10}$$

with parameters

$$N = 2\,, \quad k_1 = \frac{d}{2} - p - 1\,, \quad k_2 = \frac{p}{2}\,, \quad k_3 = \frac{1}{2} + a\,. \tag{4.3.11}$$

Here, the parameter a is related to the conformal weights Δ_1 and Δ_2 of the two bulk fields through $2a = \Delta_2 - \Delta_1$. The range of the variables x_i is the same as in eq. (4.3.9) for $N = 2$. If we set the parameter a to zero, we recover the Casimir operator (4.3.2) with parameters (4.3.3) for $q = 0$ and $p \leq d - 2$. Hence, the parameter a may be regarded as a deformation that exists for $q = 0$.

If $p = d - 1$, while $q = 0$ as in the previous paragraph, we are dealing with a correlator of two bulk fields in the presence of a boundary or conformal interface. In this case $N = \min(d - p, q + 2) = \min(1, 2) = 1$ so that there is a single cross-ratio only, as is well known from [40]. The Casimir operator takes the simple form

$$L^2 = H_{CS} + \epsilon_0\,, \quad \epsilon_0 = \frac{d^2}{16} \tag{4.3.12}$$

with parameters

$$N = 1\,, \quad k_1 = 1 - 2a - \frac{d}{2}\,, \quad k_2 = \frac{d-1}{2}\,. \tag{4.3.13}$$

Note that the Calogero-Sutherland model from $N = 1$ contains only two multiplicities. The corresponding eigenvalue equation can be mapped to the hypergeometric differential equation. Once again, for $a = 0$ we recover the Casimir problem (4.3.2) for two defects of dimension $p = d - 1$ and $q = 0$.

For reference, we conclude this list of results with the case $p = q = 0$ which is associated with correlations of four scalar bulk fields and was studied within the context of Calogero-Sutherland models in [57, 61]. In this case the Casimir operator is known to take the form

$$L^2 = \frac{1}{2}H'_{CS} + \epsilon_0\,, \quad \epsilon_0 = \frac{d^2 - 2d + 2}{8} \tag{4.3.14}$$

with

$$N = 2\,, \quad k_1 = -2b\,, \quad k_2 = a + b + \frac{1}{2}\,, \quad k_3 = \frac{d-2}{2}\,, \tag{4.3.15}$$

where the parameters $2a = \Delta_2 - \Delta_1$ and $2b = \Delta_3 - \Delta_4$ are determined by the conformal weights of four external scalar fields. We put a prime $'$ on the Hamiltonian to indicate that it actually depends on two variables u_1 and u_2 that are complex conjugates of each other and belong to the range

$$\Re u_i \in [0, \infty[\quad \Im u_1 = -\Im u_2 \in [0, \pi[\,. \tag{4.3.16}$$

In contrast to the previous cases, the gauge transformation is now given by

$$\omega'(u_1, u_2) = \prod_{i=1}^{2} \sinh^{a+b-\frac{1}{2}}\left(\frac{u_i}{2}\right) \cosh^{-(a+b)-\frac{1}{2}}\left(\frac{u_i}{2}\right) \sinh^{-\frac{d-2}{2}}\left(\frac{u_1 \pm u_2}{2}\right) , \tag{4.3.17}$$

and the eigenvalues ϵ' of the Calogero-Sutherland Hamiltonian H' are related to the conformal weight Δ and the spin ℓ of the intermediate field by $\epsilon' = -\frac{1}{2}C_{\Delta,\ell} - 2\epsilon_0$.

Of course, when we send the two parameters a and b to $a = b = 0$ we expect to recover the Casimir problem (4.3.2) for $p = q = 0$. This is indeed true but it requires to perform a non-trivial linear transformation on the coordinates and the multiplicities. We shall denote this transformation by σ_2. It maps the coordinates τ_1 and τ_2 to u_1 and u_2 as

$$\sigma_2 : \quad u_1 = \frac{\tau_1 + \tau_2}{2} \,, \quad u_2 = \frac{\tau_1 - \tau_2}{2} \tag{4.3.18}$$

and the multiplicities $k_1, k_2 = 0$ and k_3 to

$$\sigma_2 : \quad k_1' = 0 \,, \quad k_2' = k_3 \,, \quad k_3' = k_1 \,. \tag{4.3.19}$$

We note that σ_2 maps the range (4.3.9) of the variables τ_i to the range (4.3.16). Let us stress that we defined the transformation σ_2 only on Calogero-Sutherland Hamiltonians (4.3.1) with multiplicity $k_2 = 0$. It is not difficult to verify that upon acting with σ_2 on the Hamiltonian (4.3.1) we obtain a Hamiltonian H'_{CS} of the same form iff[2] $k_2 = 0$ (up to an overall factor of 2) but with multiplicities k_i' instead of k_i. For the case of interest here, i.e. when $p = q = 0$, the condition $k_2 = 0$ is indeed satisfied as one can infer from eq. (4.3.3). After applying the transformation (4.3.19) to the multiplicities we find $(k_1', k_2', k_3') = (0, 1/2, d/2 - 1)$. As we have claimed, we end up with the set of parameters (4.3.15) for $a = b = 0$. This is what we wanted to show.

As a small corollary of the previous discussion let us briefly mention that the transformation (4.3.19) can be inverted in case $N = 2$ and $k_1 = 0$. On the coordinates, the inverse reads

$$\sigma_1 : \quad v_1 = \tau_1 + \tau_2 \,, \quad v_2 = \tau_1 - \tau_2 \,, \tag{4.3.20}$$

[2]Or, equivalently, $k_2 - 1 = 0$, but this is already captured by symmetry ρ_2 in table 4.1.

while it acts on the multiplicities as

$$\sigma_1: \quad k_1' = k_3\,, \quad k_2' = 0\,, \quad k_3' = k_2\,. \tag{4.3.21}$$

The maps σ_1 and σ_2 describe two symmetries of Calogero-Sutherland model with $k_1 = 0$ and $k_2 = 0$, respectively, that exist for $N = 2$ only and act on multiplicities as well as coordinates. These symmetries are not included in table 4.1 but will play some role in our discussion below. Unlike the dualities displayed in table 4.1 which generalize Euler-Pfaff symmetries of Gauss hypergeometric function, the transformations (4.3.18,4.3.20) represent special cases of quadratic transformations of Calogero-Sutherland wave functions, generalizing classical quadratic transformations of Gauss hypergeometric functions.[3]

4.4 Relations between defect blocks with $q \neq 0$

As we stressed before, the Calogero-Sutherland Hamiltonian (4.3.1), i.e. the quadratic Casimir operator for the block, possesses some obvious symmetries which we listed in table 4.1. In the previous section we have explained how the coupling constants k_i of the Calogero-Sutherland model are determined by the dimension p and q of the two defects and the dimension d. Putting this together, we can rephrase the symmetries from table 4.1 in terms of the parameters $(p, q; d)$. The result is stated in table 4.2. The first two symmetry transformations ϱ_1 and ϱ_2 give rise to non-trivial relations between the parameters while the third one acts trivially on the coupling constants of our Calogero-Sutherland model since $k_3 = 1/2 = 1 - k_3 = k_3'$. Let us also note that the reconstruction of p, q and d from the multiplicities is not unique since they depend on p and q only through N and $p - q$. The ambiguity is described by the following duality

$$p' = d - q - 2\,, \quad q' = d - p - 2\,, \tag{4.4.1}$$

which we included as the final row of the table. It makes up for the trivial third row. As in table, 4.1, the forth row describes a symmetry for which the action on parameters is accompanied by a shift of coordinates $\tau_i \to \tau_i \pm i\pi$. These innocent looking relations have remarkable consequences of which we have seen a very special case before when we reviewed the results from [54]. Namely, in section 3.4 we discussed the blocks for a two point function for defects of dimension $p = q = d - 2$. If we plug these values into the relation (4.4.1) we find $p' = 0 = q'$, i. e. the blocks for two point functions of defects of dimension $p = d - 2 = q$ are related to four-point blocks of

[3]See also [98] for further results and a state-of-art discussion of quadratic transformations among wave functions in the trigonometric case and e.g. [99] for elliptically-deformed analogues.

Table 4.2: The action of symmetries in table 4.1 on the parameters $(p, q; d)$ that characterize a configuration of two defects. As in table 4.1 the symmetry transformation $\tilde{\varrho}$ is accompanied by a shift of coordinates. The last row is new and results from the fact it is not possible to reconstruct the parameters $(p, q; d)$ uniquely from the coupling constants k_i.

	p'	q'	d'
ϱ_1	$N + (p - q) - 2$	$N - 2$	$4N - d + 2(p - q) - 2$
ϱ_2	$N - (p - q)$	$N - 2$	$4N - d$
ϱ_3	p	q	d
$\tilde{\varrho}$	$3N - d + (p - q) - 2$	$N - 2$	$4N - d$
ϱ_0	$d - q - 2$	$d - p - 2$	d

scalar bulk fields. As we explained in the previous section, the relation between the two Calogero-Sutherland problems involves the coordinate transformations (4.3.18) and

$$z = -\sinh^{-2}\left(\frac{u_1}{2}\right) , \quad \bar{z} = -\sinh^{-2}\left(\frac{u_2}{2}\right) . \tag{4.4.2}$$

Using the relations (4.3.9) and (2.6.2), we recover the relation (3.4.3) observed in [54]. More generally, any relation between Calogero-Sutherland models that can be obtained by applying one or several of the symmetries in table 4.2 leads to a relation between solutions. In case one does not need to apply the symmetry $\tilde{\rho}$, the Euclidean region of one system is mapped to the Euclidean of the other and hence one can also match boundary conditions so that all symmetries other than $\tilde{\rho}$ actually map blocks to blocks. Thereby, our table 4.2 provides a vast generalization of eq. (3.4.4).

4.5 Defect configurations with $q = 0$ and four-point blocks

The other two relations between defect blocks and those for scalar four-point functions that we discussed in section 3.4 involve configurations with $q = 0$. We have determined the coupling constants of the associated Calogero-Sutherland model in eqs. (4.3.11). Once again we can apply the symmetries from table 4.1 to find the symmetry relations listed in table 4.3.

Table 4.3: The action of symmetries in table 4.1 on the parameters $(p, a; d)$ that characterize a configuration of two scalar bulk fields in the presence of a single defect. As in table 4.1 the symmetry transformation $\tilde{\varrho}$ is accompanied by a shift of coordinates.

	p'	a'	d'
ϱ_1	p	a	$2p - d + 6$
ϱ_2	$2 - p$	a	$8 - d$
ϱ_3	p	$-a$	d
$\tilde{\varrho}$	$4 - d + p$	a	$8 - d$

Let us re-derive and generalize the relation (3.4.2) between two identical scalars in the presence of a line defect in $d = 4$ dimensions and scalar four-point blocks from [43]. We actually want to consider two scalar fields whose weights differ by $a = (\Delta_2 - \Delta_1)/2$ in the presence of a $(d/2 - 1)$-dimensional defect in d dimensions. According to the general results, the corresponding Calogero-Sutherland model has $N = 2$ coordinates τ_1, τ_2 and its coupling constants are determined by the parameters $(p, a; d) = (\frac{d}{2} - 1, a; d)$ of the configuration through eq. (4.3.11), i.e. $k_1' = 0$. This means that we can apply the symmetry σ_1 that we introduced at the end of the previous section. The resulting triple of multiplicities can be interpreted as a set of multiplicities (4.3.15) in the Calogero-Sutherland model for scalar four-point block with weights

$$a' = \frac{1}{2}(\Delta_2' - \Delta_1') = -\frac{1}{4} + \frac{a}{2} \quad , \quad b' = \frac{1}{2}(\Delta_3' - \Delta_4') = -\frac{1}{4} - \frac{a}{2}$$

in a $(d/2 + 1)$-dimensional Euclidean space. In order to compare with the duality (3.4.2) found in [43] we need to flip the sign of a' by applying $\tilde{\varrho}$. So, in order to match the parameters we have applied the symmetry transformations σ_1 and $\tilde{\varrho}$.

Let us now see how these transformations act on the coordinates. Since both σ_1 and $\tilde{\varrho}$ act on them non-trivially, the map between the parameters $x, \bar{x}$ of the original configuration and the cross-ratios $\gamma, \bar{\gamma}$ of the four-point blocks will be non-trivial as well. Recall the relations (2.6.3) and (4.3.9) between the coordinates $x, \bar{x}$ and our coordinates τ_1, τ_2. After applying σ_1 we pass to the cross-ratios $y, \bar{y}$ using eq. (4.4.2) to obtain

$$y = -\frac{(1-x)^2}{4x} \, , \quad \bar{y} = -\frac{(1-\bar{x})^2}{4\bar{x}} \, . \tag{4.5.1}$$

Next we need to apply $\tilde{\varrho}$, i.e. shift the coordinates v_1, v_2 by $i\pi$ to obtain[4]

$$\gamma = \frac{y}{y-1} = \left(\frac{1-x}{1+x}\right)^2 \, , \quad \bar{\gamma} = \frac{\bar{y}}{\bar{y}-1} = \left(\frac{1-\bar{x}}{1+\bar{x}}\right)^2 \, , \tag{4.5.2}$$

which is precisely the relation between the relevant cross-ratios that was found in [43].

It remains to identify the weight and spin of the exchanged field in the scalar four-point blocks. In order to do so we only need to impose the correct asymptotics of the blocks on both sides. This is done in two steps. First, we obtain the gauge transformation between the defect block f and the corresponding four-point block g

[4]We need to exploit the $2\pi i$-periodicity of the potential and shift v_2 by $-2\pi i$ in order to ensure that v_1, v_2 stay complex conjugates.

by using (4.3.7) and (4.3.17). Then we impose the limits in eqs. (3.2.5) and (3.2.9)

$$f\begin{pmatrix} p,a,d \\ \Delta,\ell \end{pmatrix}; x,\bar{x}\Bigg) \xrightarrow{x,\bar{x}\to 1} [(1-x)(1-\bar{x})]^{\frac{\Delta-\ell}{2}}(2-x-\bar{x})^{\ell}\,, \tag{4.5.3}$$

$$g\begin{pmatrix} a',b',d' \\ \Delta',\ell' \end{pmatrix}; z,\bar{z}\Bigg) \xrightarrow{z,\bar{z}\to 0} (z\bar{z})^{\frac{\Delta'-\ell'}{2}}(z+\bar{z})^{\ell'}\,, \tag{4.5.4}$$

which fixes Δ', ℓ'. The final result that we obtain from our symmetries and the comparison of asymptotics is

$$f\left(\begin{matrix} \frac{d}{2}-1,a,d \\ \Delta,\ell \end{matrix}; x,\bar{x}\right) = (-1)^{-\frac{\ell}{2}}2^{\Delta}(y\bar{y})^{-\frac{1}{4}}\, g\left(\begin{matrix} -\frac{1}{4}+\frac{a}{2}, -\frac{1}{4}-\frac{a}{2}, \frac{d}{2}+1 \\ \frac{\Delta+1}{2}, \frac{\ell}{2} \end{matrix}; y,\bar{y}\right) \tag{4.5.5}$$

$$= 2^{\Delta}(\gamma\bar{\gamma})^{-\frac{1}{4}}\,[(1-\gamma)(1-\bar{\gamma})]^{-\frac{a}{2}}\, g\left(\begin{matrix} \frac{1}{4}-\frac{a}{2}, -\frac{1}{4}-\frac{a}{2}, \frac{d}{2}+1 \\ \frac{\Delta+1}{2}, \frac{\ell}{2} \end{matrix}; \gamma,\bar{\gamma}\right). \tag{4.5.6}$$

The first line corresponds to the application of σ_1 only. To pass to the second line we used that the scalar four-point blocks transform under $\tilde{\varrho}$ as

$$g\left(\begin{matrix} a',b',d' \\ \Delta',\ell' \end{matrix}; z,\bar{z}\right) = (-1)^{\ell'}\,[(1-z)(1-\bar{z})]^{-b'}\, g\left(\begin{matrix} -a',b',d' \\ \Delta',\ell' \end{matrix}; \frac{z}{z-1},\frac{\bar{z}}{\bar{z}-1}\right) \tag{4.5.7}$$

for integer ℓ'. The resulting formula indeed reduces to eq. (3.4.2) when we choose $d=4$ and $a=0$ and hence provides a rather non-trivial extension. There are three other dualities between defect and four-point blocks that can be derived along the same route, one more involving the symmetry σ_1,

$$\begin{aligned} f\left(\begin{matrix} p,a,d=4 \\ \Delta,\ell \end{matrix}; x,\bar{x}\right) &= (-1)^{-\frac{\ell-p+1}{2}}2^{\Delta}(y\bar{y})^{-\frac{1}{4}}\left|\sqrt{\frac{y-1}{y}}-\sqrt{\frac{\bar{y}-1}{\bar{y}}}\right|^{p-1} \\ &\quad\times g\left(\begin{matrix} -\frac{1}{4}+\frac{a}{2}, -\frac{1}{4}-\frac{a}{2}, p+2 \\ \frac{\Delta+p}{2}, \frac{\ell-p+1}{2} \end{matrix}; y,\bar{y}\right), \end{aligned} \tag{4.5.8}$$

and two involving σ_2,

$$f\left(\begin{matrix} p=0,a,d \\ \Delta,\ell \end{matrix}; x,\bar{x}\right) = (x\bar{x})^{\frac{a}{2}}\, g\left(\begin{matrix} a,0,d \\ \Delta,\ell \end{matrix}; 1-x,1-\bar{x}\right), \tag{4.5.9}$$

$$f\left(\begin{matrix} p=2,a,d \\ \Delta,\ell \end{matrix}; x,\bar{x}\right) = \frac{(1-x)(1-\bar{x})}{1-x\bar{x}}(x\bar{x})^{\frac{a}{2}}\, g\left(\begin{matrix} a,0,d-2 \\ \Delta-1,\ell+1 \end{matrix}; 1-x,1-\bar{x}\right). \tag{4.5.10}$$

Note that eq. (4.5.9) applies to $p = 0$ and hence it maps four-point blocks to four-point blocks, as was already discussed for $a = 0$ in section 4.3. The prefactor $(x\bar{x})^{\frac{a}{2}}$ on the right hand side stems from different gauge choices used in the literature.

Finally, let us comment on the duality (3.4.5) from [41] that relates two-point functions in presence of a $d-2$-dimensional defect to four-point blocks in the same dimension. It is not difficult to identify the symmetries that are needed to relate the parameters on the left and the right hand side. In fact, one simply needs to apply the symmetry $\tilde{\varrho}$ in table 4.3 before passing to the four-point case using σ_2. Allowing once again for non-vanishing a one obtains

$$f(p = d-2, a, d) \sim g(a, 0, d) \quad \text{and} \quad f(p = d-4, a, d) \sim g(a, 0, d+2)\,. \qquad (4.5.11)$$

Here, we have only displayed the parameters in the first row of the defect blocks f and the four-point blocks g, i.e. we suppressed the dependence on conformal weights and cross-ratios. As in our discussion above, one can apply the symmetries to the cross ratios only to find that the resulting transformation does not map the Euclidean domain of the defect cross-ratios to the Euclidean domain of the four-point block, but instead to a Lorentzian domain. Hence, eq. (4.5.11) does not provide a relation between blocks. Nevertheless, we will be able to construct the relevant defect blocks directly in chapter 5, without passing through four-point blocks. Let us stress again that in this section we did not only recover all previously known relations between blocks form the symmetries of the Calogero-Sutherland model, but we also extended them vastly, see in particular the relations (4.5.5)-(4.5.10).

4.6 Lorentzian inversion formula for defects

In this section we want to state some of the results of our upcoming paper [100]. In section 3.5 we described inversion formulas for Euclidean and Lorentzian four-point correlators. It is interesting to extend such a formula to defects, and in particular to correlation functions of two bulk fields in the presence of a defect. In [49], a Lorentzian inversion formula was derived for the *defect channel* of a single defect with two bulk fields, i.e. for $q = 0$. This defect channel inversion formula allowed to extract information on defect operators from the bulk. Through a Lorentzian inversion formula for the *bulk channel* of the kind described above it is possible to go in the other direction, i.e. to infer properties of the bulk from information of the defect fields. One way to obtain the missing Lorentzian inversion formula for (the bulk channel of) defects is to closely follow the steps in [28]. One should also be able to determine the kernel of the Lorentzian inversion formula algebraically, as explained in [39], starting from the characterization of the Euclidean kernel in [84]. However, here we want to present a shortcut by exploiting the dualities described in the previous section.

Recall that the two-point function with a point-like defect is essentially a four-point function with $2b = \Delta_3 - \Delta_4 = 0$. Concretely, using the setup defined in section 2.6, we can interpret the correlator as a defect or a four-point amplitude,

$$\begin{aligned}\langle \mathcal{D}^{(0)}(P_\alpha)\phi_1(Y_1)\phi_2(Y_2)\rangle &= (2r)^{-(\Delta_1+\Delta_2)}\left(\frac{(1-x)(1-\bar{x})}{\sqrt{x\bar{x}}}\right)^{\frac{\Delta_1+\Delta_2}{2}}\mathcal{F}(x,\bar{x}) \\ &= (2R)^{2\Delta_0}\langle\phi_0(X_1)\phi_0(X_2)\phi_1(Y_1)\phi_2(Y_2)\rangle \\ &= (2r)^{-(\Delta_1+\Delta_2)}[(1-z)(1-\bar{z})]^{\frac{a}{2}}\mathcal{G}(z,\bar{z})\,,\end{aligned} \tag{4.6.1}$$

where the factor $(2R)^{2\Delta_0}$ appears because we have normalized the defect to $\langle\mathcal{D}^{(0)}\rangle = 1$. Solving for $\mathcal{G}(z,\bar{z})$ and using the Lorentzian inversion formula eq. (3.5.5) results in

$$c^t(\Delta,\ell) = \frac{\kappa_{\Delta+\ell}}{4}\int_0^1 d^2x\,\mu(x,\bar{x})\,f\begin{pmatrix} p,a,d \\ \ell+d-1,\Delta-d+1\end{pmatrix}; x,\bar{x}\Big)\,\mathrm{dDisc}\,\mathcal{F}(\rho,\bar{\rho})\,, \tag{4.6.2}$$

where

$$\begin{aligned}\mathrm{dDisc}\,\mathcal{F}(x,\bar{x}) &= \cos(\pi a)\mathcal{F}(\rho,1/\bar{\rho}) \\ &\quad -\frac{1}{2}e^{+i\pi\frac{\Delta_1+\Delta_2}{2}}\mathcal{F}(\rho,\bar{\rho}+i0) - \frac{1}{2}e^{-i\pi\frac{\Delta_1+\Delta_2}{2}}\mathcal{F}(\rho,\bar{\rho}-i0)\,,\end{aligned} \tag{4.6.3}$$

and the measure $\mu(x,\bar{x})$ is given by

$$\mu(x,\bar{x})\big|_{p=0} = \left(\frac{(1-x)(1-\bar{x})}{\sqrt{x\bar{x}}}\right)^{\frac{\Delta_1+\Delta_2}{2}}\frac{|x-\bar{x}|^{d-2}}{[(1-x)(1-\bar{x})]^d}\,. \tag{4.6.4}$$

For the analytic continuation to the Lorentzian domain it did not matter that $\mathcal{G}$ is an actual four-point amplitude, as long as there is a decomposition into four-point conformal blocks. Using the dualities derived in the previous subsection, eqs. (4.5.5) to (4.5.10), we can write down an effective four-point amplitude $\mathcal{G}'$ for the defect amplitude $\mathcal{F}$ whose coefficients can then be determined by the inversion formula. In detail, the dualities have the form

$$f\left(\begin{matrix} p,a,d \\ \Delta,\ell\end{matrix}; x,\bar{x}\right) = d'_{\Delta,\ell}\,\omega(z',\bar{z}')\,g\left(\begin{matrix} a',b',d' \\ \Delta',\ell'\end{matrix}; z',\bar{z}'\right)\,, \tag{4.6.5}$$

where the prefactor factorizes into a coordinate dependent part $\omega(z',\bar{z}')$ and a constant part that may dependent on the quantum numbers of exchanged operator $\mathcal{O}_{\Delta,\ell}$.

Plugging this into the conformal block decomposition,

$$\begin{aligned}\langle \mathcal{D}^{(p)}(P_\alpha)\phi_1(Y_1)\phi_2(Y_2)\rangle &= (2r)^{-(\Delta_1+\Delta_2)} \left(\frac{(1-x)(1-\bar{x})}{\sqrt{x\bar{x}}}\right)^{\frac{\Delta_1+\Delta_2}{2}} \mathcal{F}(x,\bar{x}) \\ &= (2r)^{-(\Delta_1+\Delta_2)} \sum_k c_{12k}\mathcal{C}^{\mathcal{D}}_k\, f\begin{pmatrix} p, a, d \\ \Delta_k, \ell_k \end{pmatrix}; x, \bar{x}\Bigg)\end{aligned} \tag{4.6.6}$$

we can define the effective amplitude $\mathcal{G}'$ by

$$\begin{aligned}\langle \mathcal{D}^{(p)}(P_\alpha)\phi_1(Y_1)\phi_2(Y_2)\rangle &= (2r)^{-(\Delta_1+\Delta_2)} \sum_k c_{12k}\mathcal{C}^{\mathcal{D}}_k d'_k \omega(z',\bar{z}')\, g\left(\begin{matrix} a', b', d' \\ \Delta'_k, \ell'_k \end{matrix}; z', \bar{z}'\right) \\ &= (2r)^{-(\Delta_1+\Delta_2)}\omega(z',\bar{z}') \sum_k c_{12k}c'_{34k}\, g\left(\begin{matrix} a', b', d' \\ \Delta'_k, \ell'_k \end{matrix}; z', \bar{z}'\right) \\ &= (2r)^{-(\Delta_1+\Delta_2)}\omega(z',\bar{z}')\mathcal{G}'(z',\bar{z}')\,,\end{aligned} \tag{4.6.7}$$

where $c'_{34k} \equiv \mathcal{C}^{\mathcal{D}}_k d'_k$. Using the Lorentzian inversion formula, we can solve for $c_{12k}c'_{34k}$ and hence $c_{12k}\mathcal{C}^{\mathcal{D}}_k$. Doing so yields the same formula (4.6.2), except that the residue now calculates

$$c_{12\mathcal{O}}\mathcal{C}^{\mathcal{D}}_{\mathcal{O}} = -\operatorname*{Res}_{\Delta'=\Delta} c(\Delta',\ell) \qquad (\Delta \text{ generic})\,, \tag{4.6.8}$$

and the measures are given by

$$\mu(x,\bar{x})|_{p=2} = \left(\frac{(1-x)(1-\bar{x})}{\sqrt{x\bar{x}}}\right)^{\frac{\Delta_1+\Delta_2}{2}} \frac{|x-\bar{x}|^{d-4}(1-x\bar{x})^2}{[(1-x)(1-\bar{x})]^d}\,, \tag{4.6.9}$$

$$\mu(x,\bar{x})|_{p=\frac{d}{2}-1} = \left(\frac{(1-x)(1-\bar{x})}{\sqrt{x\bar{x}}}\right)^{\frac{\Delta_1+\Delta_2}{2}} \frac{|x-\bar{x}|^{\frac{d}{2}-1}(1-x\bar{x})^{\frac{d}{2}-1}}{[(1-x)(1-\bar{x})]^d}\,. \tag{4.6.10}$$

We notice that the measures are the measures of the Euclidean inversion formula [84],

$$\mu(x,\bar{x}) = \left(\frac{(1-x)(1-\bar{x})}{\sqrt{x\bar{x}}}\right)^{\frac{\Delta_1+\Delta_2}{2}} \frac{|x-\bar{x}|^{d-p-2}(1-x\bar{x})^p}{[(1-x)(1-\bar{x})]^d}\,, \tag{4.6.11}$$

as it happens in the four-point inversion formula. Furthermore, the analytic structure of the conformal blocks does not depend on the defect dimension p, see chapter 5 and [84]. Hence, we are confident that eq. (4.6.2) generalizes to any p beyond the cases $p = 0$, 2 and $\frac{d}{2} - 1$.

Let us conclude with an interpretation of the integration range $0 < x, \bar{x} < 1$ of the Lorentzian inversion formula. In terms of the physical coordinates (2.5.21) it reads

$\theta = \pi - i\tilde{\theta}$, $\tilde{\theta} \in \mathbb{R}$. Plugging it into the coordinates of the scalars ϕ yields

$$\begin{aligned} y_1 &= (-r\cos(\theta_1)e_1 + r\sin(\theta_1)e_d) = (r\cosh(\tilde{\theta})e_1 + ir\sinh(\tilde{\theta})e_d)\,, \\ y_2 &= (r\cos(\theta_1)e_1 - r\sin(\theta_1)e_d) = (-r\cosh(\tilde{\theta})e_1 - ir\sinh(\tilde{\theta})e_d)\,. \end{aligned} \tag{4.6.12}$$

We see that the points $y_1 = (re_1)$ and $y_2 = (-re_1)$ are boosted by $\tilde{\theta}$ after interpreting ie_d as the time coordinate. As expected we are in the setting of fig. 3.7. The Euclidean-like region is $0 \leq |\tilde{\theta}| < \vartheta$. Crossing the light-cones of the defects corresponds to $\tilde{\theta} > \vartheta > 0$.

4.7 Light-cone bootstrap for defects

In this section we want to argue along the lines of the four-point light-cone bootstrap and infer the properties of bulk operators from assumptions on the defect spectrum. We consider two identical scalars ϕ of conformal dimension Δ_ϕ in presence of a p-dimensional defect $\mathcal{D}^{(p)}$. For simplicity, we assume that there is the identity exchanged on the defect. To justify the light-cone bootstrap, we further assume that it is the lowest twist operator and that there is a twist gap above it.

The light-cone limit is given by $1-x, \bar{x} \ll 1$. Again, we may consider the *generating function*,

$$C^t(x,\beta) = (1-x)^{\Delta_\phi}\kappa_\beta \int_0^x d\bar{x}\,\bar{x}^{-\frac{\Delta_\phi}{2}}(1-\bar{x})^{\Delta_\phi - 2}k_\beta(1-\bar{x})\,\mathrm{dDisc}\,\mathcal{F}(x,\bar{x})\,. \tag{4.7.1}$$

Remarkably, it is independent of p. The dependence on p will reenter through the coefficients $B^{(m,k)}_{\Delta,\ell}$ in the expansion

$$c^t(\Delta,\ell)\Big|_{poles} \simeq \int_0^1 \frac{dx}{2(1-x)}(1-x)^{\frac{\ell-\Delta}{2}}\sum_{m=0}^{\infty}(1-x)^m \sum_{k=-m}^{m} B^{(m,k)}_{\Delta,\ell}C^t(x,\Delta+\ell+2k)\,. \tag{4.7.2}$$

The coefficients $B^{(m,k)}_{\Delta,\ell}$ are fixed by conformal invariance with $B^{(0,0)}_{\Delta,\ell} = 1$. They can be either calculated recursively using the quadratic Casimir equation or from the known solution of the $N = 2$ Calogero-Sutherland Hamiltonian that is analytic in spin, see [39] and chapter 5.

Now, consider the defect identity,

$$\mathcal{F}(x,\bar{x}) = 1\,. \tag{4.7.3}$$

Using the generating function in eq. (4.7.1) results in

$$\begin{aligned}
C^t(x,\beta) &= (1-x)^{\Delta_\phi}\kappa_\beta \int_0^x d\bar{x}\,\bar{x}^{-\frac{\Delta_\phi}{2}}(1-\bar{x})^{\Delta_\phi-2}k_\beta(1-\bar{x})\,\mathrm{dDisc}\,1 \\
&= (1-x)^{\Delta_\phi}\frac{\sin^2\left(\pi\frac{\Delta_\phi}{2}\right)}{\pi^2}\frac{\Gamma^4\left(\frac{\beta}{2}\right)}{\Gamma(\beta-1)\Gamma(\beta)}\frac{\Gamma\left(1-\frac{\Delta_\phi}{2}\right)\Gamma\left(\frac{\beta}{2}+\Delta_\phi-1\right)}{\Gamma\left(\frac{\beta}{2}+\frac{\Delta_\phi}{2}\right)} \\
&\quad\times {}_3F_2\left(\begin{matrix}\frac{\beta}{2},\ \frac{\beta}{2},\ \frac{\beta}{2}+\Delta_\phi-1 \\ \beta,\ \frac{\beta}{2}+\frac{\Delta_\phi}{2}\end{matrix};1\right) \\
&= \frac{(1-x)^{\Delta_\phi}}{2^{\frac{\beta}{2}-\Delta_\phi+1}\Gamma^2\left(\frac{\Delta_\phi}{2}\right)}\frac{\Gamma^2\left(\frac{\beta}{4}\right)}{\Gamma\left(\frac{\beta}{2}-\frac{1}{2}\right)}\frac{\Gamma\left(\frac{\beta}{4}+\frac{\Delta_\phi}{2}-\frac{1}{2}\right)}{\Gamma\left(\frac{\beta}{4}-\frac{\Delta_\phi}{2}+1\right)}\,,
\end{aligned} \tag{4.7.4}$$

where we used Watson's theorem to write the generalized hypergeometric function ${}_3F_2$ in terms of gamma functions [101]. We see that eq. (4.7.4) has poles in $1-x$ whenever $\Delta-\ell=2\Delta_\phi$. Like in the four-point case, the family of double-twist operators emerge in the bulk spectrum at large spin. Including the identical u-channel contribution, eq. (4.6.8) yields for the combined OPE coefficients

$$\mathcal{C}^{\mathcal{D}}_{[\phi\phi]_0}c_{\phi\phi[\phi\phi]_0} = \frac{(1+(-1)^\ell)}{2^{\ell+1}\left(\frac{\ell}{2}\right)!}\frac{\left(\frac{\Delta_\phi}{2}\right)^2_{\frac{\ell}{2}}}{\left(\Delta_\phi+\frac{\ell}{2}-\frac{1}{2}\right)_{\frac{\ell}{2}}}\,,\qquad \ell\to\infty\,, \tag{4.7.5}$$

where $c_{\phi\phi[\phi\phi]_0}$ is given by the mean field coefficients, see eq. (3.6.8). Since the latter are only determined up to a sign, so are the one-point coefficients $\mathcal{C}^{\mathcal{D}}_{[\phi\phi]_0}$. The final result is

$$\left(\mathcal{C}^{\mathcal{D}}_{[\phi\phi]_0}\right)^2 = \frac{(1+(-1)^\ell)}{8}\frac{\ell!}{\left[\left(\frac{\ell}{2}\right)!\right]^2}\frac{\left(\frac{\Delta_\phi}{2}\right)^4_{\frac{\ell}{2}}}{(\Delta_\phi)^2_{\frac{\ell}{2}}}\frac{1}{(2\Delta_\phi+\ell-1)_l}\,,\qquad \ell\to\infty\,. \tag{4.7.6}$$

Let us stress again that obtained coefficients are completely universal at large spin. They do not depend on the details of the defect theory, thus are valid in theories like 3D Ising model or $\mathcal{N}=4$ SYM. Furthermore, the analyticity in spin shows that every operator in the double-twist family $[\phi\phi]_0$ couples to the defect.

Chapter 5

Calogero-Sutherland scattering states

Here we present a review of the solution theory. We introduce the fundamental domain of the Calogero-Sutherland problem and its fundamental (monodromy) group, Harish-Chandra scattering states, the monodromy representations and physical (monodromy free) wave functions.

5.1 Symmetries and fundamental domain

It is useful to consider the Calogero-Sutherland potential (4.3.1) as a function of N complex variables first and to impose reality conditions a bit later. As a function of complex coordinates $\tau_i \in \mathbb{C}$, the potential possesses a few important symmetries. These include independent shifts of the coordinates τ_i by $2\pi i$ in the imaginary direction as well as two types of reflections, namely the inversion symmetries $\tau_i \leftrightarrow -\tau_i$ and the particle exchange symmetry $\tau_i \leftrightarrow \tau_j$. Together these form a non-abelian group that mathematicians refer to as affine Weyl group $\mathcal{W}_N$. The reflections actually generate a usual Weyl group and the shifts make this affine. The affine Weyl group is known to possess a so-called Coxeter representation through $N+1$ generators $w_i, i = 0, \ldots, N$ with relations

$$w_i w_j \;=\; w_j w_i \quad \text{for } |i-j| \geq 2\,, \tag{5.1.1}$$

$$w_i w_{i+1} w_i \;=\; w_{i+1} w_i w_{i+1} \quad \text{for } i = 1, \ldots N-2\,, \tag{5.1.2}$$

$$w_0 w_1 w_0 w_1 = w_1 w_0 w_1 w_0 \quad , \quad w_{N-1} w_N w_{N-1} w_N = w_N w_{N-1} w_N w_{N-1}\,. \tag{5.1.3}$$

and

$$w_i^2 = 1 \quad \textit{for all} \quad i = 0, \ldots, N\,. \tag{5.1.4}$$

In this presentation of the affine Weyl group, the generators of the shifts in the imaginary direction are a bit hidden, but they can be reconstructed from the w_i, see [102, 103].

The fundamental domain for the Calogero-Sutherland model is given by the quotient of the configuration space $\mathbb{C}^N$ with respect to the symmetries, i.e.

$$D_N = \mathbb{C}^N/\mathcal{W}_N \ . \tag{5.1.5}$$

We have depicted a 3-dimensional projection of the fundamental domain for $N = 2$ in figure 5.1. Inside the wedge-shaped domain, the Calogero-Sutherland potential is finite but it diverges along the edges. We will refer to the hyperplanes of singularities as "walls" of the Calogero-Sutherland model. It turns out that the model possesses $N + 1$ different walls $\omega_i, i = 0, \ldots, N$, one for each generator w_i of the affine Weyl group. For $N = 2$ there are three such walls which are shown in figure 5.1. The possible real domains A_N^α of the model are given by the various faces of the domain D_N. Mathematicians usually study the Schroedinger problem in the real wedge A_N^+ which is given by $\tau_i \in \mathbb{R}$ with $\tau_i > \tau_j > 0$ for all $i < j$.

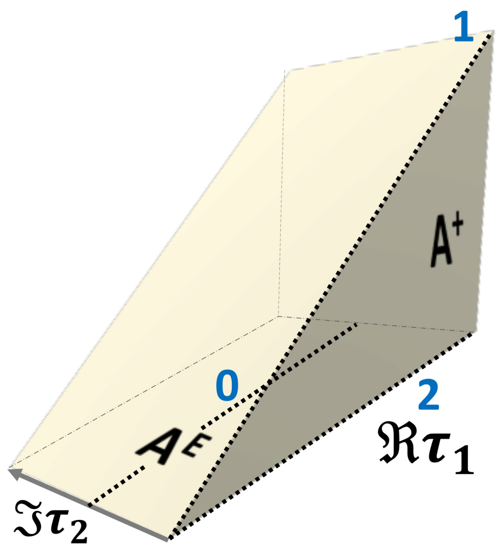


Figure 5.1: A 3-dimensional slice of the fundamental domain D_2 for the BC_2 Calogero-Sutherland model in τ-space with $\Im\tau_1 = 0$. Front and back side of the wedge should be identified. The fixed points (walls) under the action of w_2 and w_1 are shown as bold dashed lines. Fixed points of w_2 fall into two disconnected components which carry the labels 0 and 2. The shaded area in front is the Weyl chamber A_2^+. It is bounded by the walls ω_1 and ω_2. The subset A_2^E is the 2-dimensional semi-infinite strip of width 2π on the bottom of the wedge. It is bounded by the wall ω_2, whereas wall ω_0 cuts through its middle.

The fundamental group $\pi_1(D_N)$ of the fundamental domain plays an important role in Calogero-Sutherland theory. It is generated by $N+1$ generators g_i subject to the relations (5.1.1)-(5.1.3) with w_i replaced by g_i. On the other hand, the generators g_i of the fundamental group do not satisfy relation (5.1.4). The fundamental group of the domain D_N is also referred to as affine braid group. Its relation to the affine Weyl group is like the relation between the braid group and the permutation group. Let us note that the generators $w_a, a = 1, \ldots, N-1$ generate a subgroup $S_N \subset W_N$ of the affine Weyl group that is isomorphic to the symmetric group S_N. The corresponding generators $g_a, a = 1, \ldots, N-1$, within the monodromy group generate Artin's braid group. In addition, the full monodromy contains two more generators, g_0 and g_N which satisfy some fourth order 'reflection type' equations with g_1 and g_{N-1}, respectively.

5.2 Harish-Chandra scattering states

Before we enter our discussion of wave functions, it is advantageous to introduce a bit of notation. We shall denote by $e_i, i = 1, \ldots, N$, the i^{th} unit vector in $\mathbb{C}^N$, i.e. the vector that is zero everywhere except in the i^{th} entry which is one instead. From these unit vectors we build the following set Σ^+ of vectors in $\mathbb{C}^N$,

$$\Sigma^+ = \{e_i, 2e_i, e_i \pm e_j | 1 \leq i, j \leq N; i < j\} \ . \tag{5.2.1}$$

As one can easily count, the set contains $N(N+1)$ elements. Looking back at our Calogero-Sutherland potentials we observe that they contain one summand for each element in Σ^+. In fact, we can also write the potential as

$$V^{\text{CS}}(\tau_i) = \sum_{\alpha \in \Sigma^+} \frac{k_\alpha (k_\alpha + 2k_{2\alpha} - 1)\langle \alpha, \alpha \rangle}{4 \sinh^2 \frac{\langle \alpha, \tau \rangle}{2}} \ . \tag{5.2.2}$$

where the scalar product $\langle \cdot, \cdot \rangle$ is normalized such that $\langle e_i, e_j \rangle = \delta_{i,j}$ and we assembled all the coordinates $\tau_i \in \mathbb{C}$ into a vector $\tau = \sum_i \tau_i e_i$ with

$$k_{e_i} = k_1 \quad , \quad k_{2e_i} = k_2 \quad , \quad k_{e_i \pm e_j} = k_3 \ .$$

Let us agree to extend the definition of k_α to arbitrary elements $\alpha \in \mathbb{R}^N$ such that is vanishes whenever $\alpha \notin \Sigma^+$. Just as in the case of the potential, many formulas below will turn out to become much simpler when written as sums or products over the set Σ^+.

With these notations set up let us come to our main subject, namely the study of wave functions. Since the Calogero-Sutherland potential falls off at $\tau_i \to \infty$, any wave function becomes a superposition of plane waves in this asymptotic regime.

In mathematics is it customary to factor off the ground state wave function of the trigonometric Calogero-Sutherland model, i.e. of the Hamiltonian that is obtained when all the τ_i are purely imaginary. This ground state wave function Θ is explicitly known,

$$\Theta(\tau_i) = \prod_{\alpha\in\Sigma^+} \left(2\sinh\frac{\langle\alpha,\tau\rangle}{2}\right)^{k_\alpha} \quad . \tag{5.2.3}$$

For the wave function of the the Calogero-Sutherland model on the domain A_N^+ we make the Ansatz

$$\Psi(\lambda, k; \tau) = \Theta(k;\tau)\Phi(\lambda,k;\tau)\ . \tag{5.2.4}$$

Let us note in passing that the function $\Theta(k,\tau)$ possesses the following asymptotics for large τ,

$$\begin{aligned}\Theta(k;\tau) \quad &\sim \quad e^{\langle\rho_k,\tau\rangle} + \dots \quad \textit{where} \qquad (5.2.5)\\ \rho_k \quad &:= \quad \left(\frac{k_1}{2} + k_2 + (N-1)k_3, \frac{k_1}{2} + k_2 + (N-2)k_3, \dots, \frac{k_1}{2} + k_2\right)\ .\end{aligned}$$

So-called Harish-Chandra wave functions $\Phi(\lambda, k;\tau)$ are W_N symmetric solutions of the Calogero-Sutherland Hamiltonian for which Φ possesses the following simple asymptotic behavior

$$\Phi(\lambda, k;\tau) \ \sim\ e^{\langle\lambda-\rho_k,\tau\rangle} + \dots \quad \text{for}\ \ \tau\to\infty \text{ in } A_N^+ = WC_N \tag{5.2.6}$$

where $\lambda = \sum_i \lambda_i e_i$ and $\tau \to \infty$ in A_N^+ means that all components become large while preserving the order $\tau_N < \tau_{N-1} < \cdots < \tau_1$. Imposing W_N symmetry implies that as a function of τ_i, Φ is reflection symmetric and invariant under any permutation of the τ_i. The condition (5.2.6) selects a unique solution of the scattering problem describing a single plane wave. It is analytic in the wedge A_N^+. The corresponding eigenvalue of the Calogero-Sutherland Hamiltonian is given by

$$\varepsilon = \varepsilon(\lambda) = -\sum \lambda_i^2\ .$$

When we required the Harish-Chandra functions to be symmetric, we used the action of the Weyl group W_N on the coordinate space. On the other hand, the Weyl group also acts in a natural way on the asymptotic data λ of the Harish-Chandra functions by sending any choice of λ through a sequence of Weyl reflections to $w\lambda, w \in W_N$. In particular, the generators $w_j, j = 1,\dots,N$ act as

$$w_a\lambda_i = \delta_{a+1,i}\lambda_{i-1} + (1-\delta_{a,i})\left(1-\delta_{a+1,i}\right)\lambda_i + \delta_{a,i}\lambda_{i+1} \quad , \quad w_N\lambda_i = (-1)^{\delta_{N,i}}\lambda_i \tag{5.2.7}$$

for $a = 1, \ldots, N-1$ and $i = 1, \ldots, N$. Since the eigenvalue ε is invariant under exchange and reflection of the momenta λ_i, our Harish-Chandra functions come in families. For generic choices of λ, one obtains $|W_N| = N!2^N$ solutions $\Phi(w\lambda, k; \tau)$ which all possess the same eigenvalue of the Hamiltonian.

At least for sufficiently generic values of the momenta,[1] Harish-Chandra functions possess a series expansion in the variables $x_i = \exp \tau_i$

$$\Phi(\lambda, k; \tau) = \sum_{\mu \in Q_+} \Gamma_\mu(\lambda, k) e^{\langle \lambda - \rho_k - \mu, \tau \rangle}, \qquad \Gamma_0(\lambda, k) = 1, \tag{5.2.8}$$

where we adopt $|\Im \tau_i| < \pi$ for $i = 1, \ldots, N$ on the principal branch of BC_N Harish-Chandra functions and we sum over elements μ of the integer cone

$$Q_+ = \{\mu = \sum_{a=1}^{N-1} n_a(e_a - e_{a+1}) + n e_N \,|\, n_a, n \geq 0 \text{ for } a = 1, \ldots, N-1\} \ .$$

By inserting this formal expansion into the Calogero-Sutherland eigenvalue equations one can easily derive equations for the expansion coefficients Γ_μ that may be solved recursively, at least for generic eigenvalues λ_i. In a few cases, explicit formulas for Γ_μ are also known. For $N = 2$, for example, the series expansion of Harish-Chandra functions with generic eigenvalues λ_i was recently worked out in [39], generalizing earlier expressions by Dolan and Osborn that were only valid for cases in which $\lambda_1 - \lambda_2 - k_3$ is non-negative integer. The procedure that was employed in [39] can in principle be extended to $N > 2$. This remains an interesting challenge for future work.

In Heckman-Opdam theory many properties of the Harish-Chandra functions have been obtained without knowing the explicit series expansions. In particular let us mention that the functions $\exp(\langle -\lambda + \rho(k), \tau \rangle)\Phi(\lambda; k; \tau)$ are known to be entire functions of the multiplicities k_i and meromorphic functions of asymptotic data λ_i, for any fixed choice of τ in the fundamental domain. They are known to possess simple poles whenever the set of λ_i satisfies one of the following conditions

$$\langle \lambda_*, \alpha \rangle = \frac{s}{2} \langle \alpha, \alpha \rangle \quad \text{for} \quad s = 1, 2, \ldots \quad , \quad \alpha \in \Sigma^+ \ . \tag{5.2.9}$$

For the poles at $\lambda_* = \lambda_{\alpha,n}$, the residues are given by (see e.g. [104])

$$\mathrm{Res}_{(\alpha,s)} \Phi(\lambda, k; \tau) \sim \Phi(w(\alpha)\lambda_{\alpha,s}, k; \tau) \ . \tag{5.2.10}$$

where $\sim$ indicates that the relation with the Harish-Chandra function on the right hand side holds only up to a constant factor. The latter is not known in general, but

[1] A precise formulation of the condition is stated in [103].

it can be found from the series expansion as in [39] for $N = 2$. The Harish-Chandra function on the right hand side is related to the one on the left by acting with an element $w(\alpha) \in W_N$ of the Weyl group on the set of momenta λ_i, defined in (5.2.7). A complete discussion of poles and residues for $N = 2$, including non-generic momenta λ_i can be found in [39].

5.3 Monodromy representation and wave functions

The scattering states we have discussed in the previous section fail to be good wave functions for the various real slices one may consider. In fact, at infinity Harish-Chandra function contains a single plain wave. On the other hand, the latter are not regular at the walls of the scattering problem. Finding true wave functions requires to impose regularity conditions at the walls and hence forces us to consider certain linear combinations of the $2^N N!$ Harish-Chandra functions with given energy ε.

The behavior of all wave functions at the walls is encoded in the monodromy representation of the fundamental group. As we saw above, the fundamental group, which in our case has been identified as the affine braid group, contains one generator $g_i, i = 0, \ldots, N$ for each of the walls. The representation of this generator encodes how wave functions behave as we continue along a curve that surrounds the wall. Note that all walls possess real co-dimension two since they are defined by one complex linear equation. The $2^N N!$-dimensional space of Harish-Chandra functions $\Phi(w\lambda, \tau), w \in W_N$ carries a representation of the monodromy group. The representation matrices $M_i = M(g_i)$ are explicitly known from the work of Heckman and Opdam, see [39] for explicit formulas. In the special case of $N = 2$, expressions for two of the three monodromy matrices were also worked out in the conformal field theory literature [28]. Let us stress that these matrices satisfy the relations (5.1.1)-(5.1.3) that are the defining relation of the affine braid group. In addition they turn out to obey the following set of Hecke relations

$$\begin{aligned} (M_r - 1)(M_r - \gamma_r) &= 0 \quad , \quad \text{where} \\ \gamma_0 = e^{\pi i(2k_2-1)} \; , \; \gamma_i &= e^{\pi i(2k_3-1)} \; , \; \gamma_N = e^{\pi i(2k_1+2k_2-1)} \end{aligned} \tag{5.3.1}$$

for $r = 0, \ldots, N$ and $i = 1, \ldots, N-1$. These may be considered as a deformation of the relations (5.1.4). In this sense, this monodromy representation of the affine braid group is rather close to being a representation of the affine Weyl group. For generic values of the multiplicities k and momenta λ, the monodromy representation of the affine braid group on Harish-Chandra functions is irreducible. The precise condition is

$$2\frac{\langle \lambda, \alpha \rangle}{\langle \alpha, \alpha \rangle} \notin \mathbb{Z} \quad and \quad 2\frac{\langle \lambda, \alpha \rangle}{\langle \alpha, \alpha \rangle} + \frac{k_{\alpha/2}}{2} + k_\alpha \notin \mathbb{Z} \tag{5.3.2}$$

for all elements $\alpha \in \Sigma^+$. When one of these conditions is violated, the monodromy representation may contain non-trivial subrepresentations.

In terms of these monodromy matrices, regularity of the wave function Φ at a wall ω_i is equivalent to Φ being an eigenfunction of the corresponding monodromy matrix $M_i = M(g_i)$ with unit eigenvalue, i.e. Φ is regular along ω_i if and only if $M_i\Phi = \Phi$. There exists a very simple prescription how to build a function Φ that is analytic at some subset $\omega_{i_1}, \dots, \omega_{i_r}$ consisting of $r \leq N$ of the N walls that bound A_N^+, i.e $i_\nu \neq 0$. For each of these walls there is a generator w_{i_ν} of the Weyl group and so our set of r walls is associated with a subgroup $V \subset W_N$ of the Weyl group that is generated by $w_{i_1}, \dots, w_{i_r}$. Given this subgroup we now define the following superposition of Harish-Chandra functions

$$\Phi^V(\lambda, k; \tau) = \sum_{w \in V} c(w\lambda, k)\Phi(w\lambda, k; \tau) \tag{5.3.3}$$

where the so-called Harish-Chandra c-function reads

$$c(\lambda, k) \quad = \quad \frac{\gamma(\lambda, k)}{\gamma(\rho(k), k)} \quad , \quad \gamma(\lambda, k) = \prod_{\alpha \in \Sigma^+} \gamma_\alpha(\lambda, k) \quad , \tag{5.3.4}$$

$$\gamma_\alpha(\lambda, k) = \frac{\Gamma\left(\frac{1}{2}k_{\alpha/2} + \langle \lambda, \alpha^\vee \rangle\right)}{\Gamma\left(\frac{1}{2}k_{\alpha/2} + k_\alpha + \langle \lambda, \alpha^\vee \rangle\right)}. \tag{5.3.5}$$

For future convenience, let us also introduce

$$\gamma^*_\alpha(\lambda, k) = \frac{\Gamma\left(1 - \frac{1}{2}k_{\alpha/2} - k_\alpha - \langle \lambda, \alpha^\vee \rangle\right)}{\Gamma\left(1 - \frac{1}{2}k_{\alpha/2} - \langle \lambda, \alpha^\vee \rangle\right)}. \tag{5.3.6}$$

Any wave function of the form (5.3.3) turns out be be regular at the walls $\omega_{i_1}, \dots, \omega_{i_r}$. Physical wave functions on the Weyl chamber A_N^+ are obtained when $V = W_N$ is the entire Weyl group, the most well studied case in the mathematical literature. For this choice of V we end up with one unique linear combination of Harish-Chandra functions for each Weyl-orbit of λ. The functions $F_N^+ = \Phi^{W_N}$ are known as Heckman-Opdam hypergeometric function. They are close cousins of the Lorentzian hypergeometric functions that were introduced in [39]. The set of true wave functions $F_N^+(\lambda, \tau)$ of the Calogero-Sutherland model gives rise to an orthonormal basis of functions on the wedge A_N^+. Let us note, however, that, while the functions F_N^+ are analytic in a neighborhood of A_N^+, they fail to be analytic at the wall ω_0. Other real domains whose boundary contains the wall ω_0, are associated with different subgroups of the affine

Weyl group. Which subgroup one has to sum over in order to obtain an orthonormal basis of wave functions and the precise form of coefficients in this sum depend on the chosen domain for the Calogero-Sutherland scattering problem.

5.4 Defect blocks

The Heckman-Opdam hypergeometric functions we described briefly in the final paragraph of the previous section, provide physical wave functions for the domain A_N^+. Their construction is well known in the mathematical literature. We are mostly interested in the physical wave functions for the Euclidean domain A_N^E that was introduced in eq. (4.3.9). As far as we know, there exists no general theory for these functions, but for the specific example of $N = 2$ that is associated to scalar four-point blocks, such wave functions have been known in the context of conformal field theory for a long time, see e.g. [28,76] for explicit formulas in the recent literature. In [84] these functions were generalized to $N \geq 2$ using the characterization that was proposed in [39].

Before we can characterize the physical wave functions we need to introduce a bit of notation. In eq. (4.3.9) we have introduced the domain A_N^E. Of course, there are quite a few walls within A_N^E. When we consider the Calogero-Sutherland problem it is natural to first formulate it in a smaller domain that is bounded by walls but does not have walls in the interior. Here we shall describe such a small domain D_N^E and then explain how to glue A_N^E from the small domain D_N^E and some of its images under the action of the affine Weyl group. In order to do so we first define the simplex $\triangle_{N-1}$ that is parametrized by an ordered set of $N-1$ angles θ_i

$$\triangle_{N-1} := \{(\theta_i, \dots, \theta_{N-1}) \,|\, \theta_i \in [0, \pi/2[; \theta_i \geq \theta_j \text{ for } i < j\} \,. \tag{5.4.1}$$

We can then introduce the domain D_N^E as a semi-infinite cylinder over $\triangle_{N-1}$, i.e.

$$D_N^E = \{(\vartheta, \theta_i) \,|\, \vartheta \in \mathbb{R}_0^+ \,;\, (\theta_i) \in \triangle_{N-1}\} \,. \tag{5.4.2}$$

The hypercubic base of our the Euclidean domain A_N^E that was introduced in eq. (4.3.9) can be triangulated into a disjoint union of the simplex $\triangle_{N-1}$ an its reflections under the following subgroup W_N^B of the Weyl group W_N,

$$W_N^B := \{w_2, \dots, w_{N-1}, w_N \mid \text{ relations of } W_N\} \subset W_N \,. \tag{5.4.3}$$

More precisely, our Euclidean domain A_N^E can be decomposed as

$$A_N^E = \bigsqcup_{w \in \varkappa \cdot W_N^B} w D_N^E, \tag{5.4.4}$$

where $\varkappa$ is an element of affine Weyl group which simultaneously shifts all the angular variables. Explicitly, $\varkappa$ acts on the coordinates as $\varkappa : \theta_j \mapsto \theta_j + \pi/2$ for $j = 1, \dots, N-1$ or, equivalently, in terms of the variables τ_j, it is given by $\varkappa : \tau_{j+1} \mapsto \tau_{j+1} + i\pi$, $j = 1, \dots, N-1$, while leaving τ_1 invariant. Let us stress that in the decomposition formula (5.4.4) the Weyl group elements w act on coordinates, not on momenta as in most other formulas.

The boundary of $D_N^E \subset \mathbb{R}^N$ runs along various walls of our Calogero-Sutherland problem. In fact, the simplex $\triangle_{N-1}$ which appears at $\tau_1 = 0$, runs along the wall ω_N acted upon with the Weyl reflection $w_1 w_2 \cdots w_{N-1}$. There are also two semi-infinite cells of the boundary defined by $\tau_N = 0$ and $\tau_2 = i\pi$ which are part of the wall ω_N, and of its image under the Weyl reflection $w_2 \cdots w_{N-1}$, respectively. Finally, the boundary components at $\tau_A = \tau_{A+1}, A = 2, \dots, N-1$ run along the walls ω_A for $A = 2, \dots, N-1$.

Our goal is to construct the blocks that we introduced through the expansion (3.2.13) in terms of Harish-Chandra functions. As in the case of four-point blocks, all we need to do is to decompose the monodromy-free conformal partial waves into a sum of a block and its shadow, see [84] for the explicit construction. Here we just state the result. We denote

$$\Sigma_\star^+ := \{e_1 - e_j \,|\, j = 2, \dots, N\}, \tag{5.4.5}$$

and

$$\gamma^E(\lambda, k) := \prod_{\alpha \in \Sigma_\star^+} \gamma_\alpha^*(\lambda, k) \prod_{\alpha \in \Sigma^+ \setminus \Sigma_\star^+} \gamma_\alpha(\lambda, k). \tag{5.4.6}$$

The desired blocks F^B are then obtained by summing Harish-Chandra functions over the subgroup W_N^B,

$$F^B(\lambda_i; k_a; \tau_i) = \sum_{w \in W_N^B} \gamma^E(w\lambda, k) \Phi(w\lambda; k_a; \tau_1, \dots, \tau_N). \tag{5.4.7}$$

If we take care of all prefactors and gauge transformations, we arrive at the following expressions for the blocks we introduced through the decomposition (3.2.13),

$$f_D\begin{pmatrix} p, q, d \\ \Delta_k, \ell_k \end{pmatrix}; \vartheta, \theta_i\Bigg) = \frac{4^{\frac{d}{2} - 2\lambda_1}}{\gamma^E(\lambda, k)} \cdot F^B(\lambda_i; k_a; \tau_i) \tag{5.4.8}$$

where the multiplicities k_a on the right hand side are related to the parameters p, q, d on the left through eq. (4.3.3). Moreover, the Calogero-Sutherland momenta λ_i on the right hand side are determined by the conformal weight Δ and the spin $\ell = (l_1, \ldots, l_{N-1})$ of the intermediate channel of the defect block as

$$\lambda_1 = \frac{d}{4} - \frac{\Delta}{2} \qquad \lambda_{j+1} = \frac{d}{4} + \frac{l_j - j}{2}, \; j = 1, \ldots N-1 \; . \tag{5.4.9}$$

Formulas (5.4.7) and (5.4.8) describe conformal blocks for configurations of two defects as a linear combination of $2^{N-1}(N-1)!$ Harish-Chandra functions. All coefficients are given explicitly in eq. (5.4.6). This extends the construction of four-point blocks from pure functions that was spelled out in [28] to an arbitrary number N of cross ratios.

In the case $q = 0$, the blocks can contain an additional parameter a that also enters the normalization. Here we will adopt the following normalization

$$f\begin{pmatrix} p, a, d \\ \Delta, \ell \end{pmatrix}; x, \bar{x} \Bigg) = \frac{4^{\frac{d}{2}+a-2\lambda_1}}{\gamma^E(\lambda, k)} \cdot \sinh^a \frac{\tau_1 \pm \tau_2}{2} \, F^B_{N=2}(\lambda_i; k_a; \tau_i) \tag{5.4.10}$$

which reduces to eq. (5.4.8) with $q = 0$ when $a = 0$, and behaves as

$$f\begin{pmatrix} p, a, d \\ \Delta, \ell \end{pmatrix}; x, \bar{x} \Bigg) \xrightarrow{x \to 1, \bar{x} \to 1} \left[(1-x)(1-\bar{x})\right]^{\frac{\Delta-\ell}{2}} (2 - x - \bar{x})^l \; . \tag{5.4.11}$$

Hence, our conventions match those in the literature. Note, however, that our normalization differs from those in [41]. In order to obtain their blocks one has to multiply our blocks by a factor $2^{-\ell}$. Formulas (5.4.7) and (5.4.10) provide an explicit construction of blocks for the bulk channel of configurations with $q = 0$, i.e. when we deal with two local fields in the presence of a defect of dimension $p < d - 1$. In chapter 4 we described a few cases in which such blocks can be obtained through the relation with scalar four-point blocks. The results of this chapter, derived through the solution theory of Calogero-Sutherland models, do not use this connection to four-point blocks. See, however, our discussion of another class of such formulas in appendix B.

Chapter 6

Outlook

In this work we developed a systematic theory of conformal blocks for a pair of defects in a d-dimensional Euclidean space. By extending the harmonic analysis approach that was initiated in [61, 96] we were able to derive the associated Casimir equations systematically. These were shown to take the form of an eigenvalue problem for an N-particle Calogero-Sutherland Hamiltonian, generalizing the observation of [57] for four-point blocks. We exploited known symmetries of the Calogero-Sutherland models to obtain a large set of relations between blocks, of which only a few special cases were known before. Using these dualities we obtained the Lorentzian inversion formula for the bulk channel. The latter generalizes the inversion formula for scalar four-point blocks in [28]. Finally, we gave a lightning review of Heckman-Opdam theory for the Calogero-Sutherland scattering problem and applied it to the constructions of defect blocks.

It would be interesting to combine the Lorentzian inversion formula for the *bulk channel* with the one for the *defect channel* found in [49]. The former allows to infer properties of the bulk from information on the defect fields, whereas the latter allows to extract information on defect operators from the bulk. This process could then be iterated [24].

Another interesting direction concerns the extension to spinning blocks, i.e. to non-trivial representations of the rotation groups $\mathrm{SO}(d-p)$ and $\mathrm{SO}(d-q)$. When $q = 0$, these can be used to expand correlation functions of two fields with spin, such as e.g. the stress tensor, in the presence of the defect. The harmonic analysis approach that we used in chapter 4 to derive our results on the relation with Calogero-Sutherland Hamiltonians was recently extended to the case of four bulk fields with arbitrary spin [61, 96], i.e. of $p = 0 = q$, see also [79]. It is rather straightforward to include defects into such an analysis. Going through the relevant group theory, one can see that the stabilizer subgroup of any given point on the double coset is given by $B = \mathrm{SO}(p-q) \times \mathrm{SO}(|d-p-q-2|)$ which is non-trivial unless the two defects possess the same dimension $p = q$ and $d = 2p + 2$. Consequently, the analysis of spinning defect blocks is similar to the cases studied in [96]. In any case, the corresponding Casimir equations will take the form of Calogero-Sutherland eigenvalue equations with a matrix valued potential. It should be rewarding to work these out, at least in a few examples.

As we mentioned in the introduction, extensions of the conformal bootstrap programme including correlation functions of two bulk fields in the presence of a defect, have played some role already both for $d = 2$ and higher dimensions. Constraint equations on dynamical data of the theory arise from the comparison of the two different channels that exist for $q = 0$, the bulk and the defect channel. While the defect channel is entirely determined by the expansion of bulk fields near the defect, the bulk channel also contains information about the bulk operator product expansions. It is a relevant challenge to compute dynamical data for defect two-point functions and to formulate appropriate consistency conditions these quantities need to satisfy. In this context it might also be interesting to include correlators in non-trivial geometries [105] and at finite temperature [106–108].

Let us finally stress, that the Heckman-Opdam theory we needed in chapter 5 is only a very small part of what is known about Calogero-Sutherland models. In fact, the most remarkable property of the Calogero-Sutherland model is its (super-)integrability. It furnishes a wealth of additional and very powerful algebraic structure. So far, the only algebra we have seen above was the Hecke algebra that appeared in the context of the monodromy representation. It acts in the $2^N N!$-dimensional spaces of Harish-Chandra functions $\Phi(w\lambda; z), w \in W_N$, i.e. in finite dimensional subspaces of functions which all possess the same eigenvalue of the Hamiltonian. This is just the tip of a true iceberg of algebraic structure that involves e.g. Ruijsenaars-Schneider models and double affine Hecke algebras, see comments in the conclusions of [39].

Appendix A

Derivation of coordinates

A.1 τ-coordinates

Let us carry out the steps that we outlined in section 2.6 for a pair of defects of dimension p and q. In embedding space, the location of the p-dimensional spherical defect of radius R is described by the points

$$X_i = (1, R^2, Re_i)\,, \quad X_{p+2} = (1, R^2, -Re_1)\,, \qquad i = 1, \ldots, p+1\,. \tag{A.1.1}$$

Similarly, the tilted q-dimensional spherical defect of radius r runs through the following set of $q+2$ points

$$\begin{aligned} Y_i &= (1, r^2, -r\cos(\theta_i)e_i + r\sin(\theta_i)e_{d-i+1})\,, \qquad i = 1, \ldots, q+1\,, \\ Y_{q+2} &= (1, r^2, r\cos(\theta_1)e_1 - r\sin(\theta_1)e_d)\,, \end{aligned} \tag{A.1.2}$$

where we set $\theta_i = 0$ for $i \geq N = \min(d-p, q+2)$. A convenient set of orthonormal vectors P_α and Q_β that are transverse to the two defects, i.e. satisfy the conditions $X \cdot P = Y \cdot Q = 0$, is given by

$$P_1 = \left(\frac{1}{R}, -R, \vec{0}\right)\,, \quad P_i = (0, 0, e_{d-i+2})\,, \qquad i = 2, \ldots, d-p\,, \tag{A.1.3}$$

$$\begin{aligned} Q_1 &= \left(\frac{1}{r}, -r, \vec{0}\right)\,, \\ Q_j &= (0, 0, \sin(\theta_{j-1})e_{j-1} + \cos(\theta_{j-1})e_{d-j+2})\,, \qquad j = 2, \ldots, d-q\,. \end{aligned} \tag{A.1.4}$$

From these explicit expressions it is easy to compute the matrix M of conformal invariants. It takes the form

$$M = P^T Q = \left(\begin{array}{ccccc|c} \cosh\vartheta & & & & & \\ & \cos\theta_1 & & & & \\ & & \ddots & & & 0 \\ & & & \cos\theta_{N-1} & & \\ & & & & I & \end{array}\right), \tag{A.1.5}$$

where $\cosh\vartheta = \frac{1}{2}\left(\frac{r}{R} + \frac{R}{r}\right)$. We recovered our formula (2.6.1).

A.2 x, $\bar{x}$-coordinates

Next we want to determine how the coordinates x, $\bar{x}$ in (2.5.21) that we used for configurations with $N = 2$ cross-ratios relate to our variables ϑ, $\theta \equiv \theta_1$. The former are defined through two local bulk fields ($q = 0$) in presence of a p-dimensional defect. In order to apply eq. (2.5.21), we need to project Y_1 and Y_2 onto the transverse space, i.e. the space spanned by $P_1, \ldots, P_{d-p}$:

$$Y_1^\perp = \left(\frac{1}{2}\left(1 - \frac{r^2}{R^2}\right), \frac{1}{2}(r^2 - R^2), r\sin(\theta)e_d\right), \tag{A.2.1}$$

$$Y_2^\perp = \left(\frac{1}{2}\left(1 - \frac{r^2}{R^2}\right), \frac{1}{2}(r^2 - R^2), -r\sin(\theta)e_d\right). \tag{A.2.2}$$

Eq. (2.5.21) yields

$$\frac{(1-x)(1-\bar{x})}{(x\bar{x})^{\frac{1}{2}}} = -\frac{2Y_1 \cdot Y_2}{(Y_1^\perp \cdot Y_1^\perp)^{\frac{1}{2}}(Y_2^\perp \cdot Y_2^\perp)^{\frac{1}{2}}} = \frac{4}{\sinh^2\vartheta + \sin^2\theta}, \tag{A.2.3}$$

$$\frac{x + \bar{x}}{2(x\bar{x})^{\frac{1}{2}}} = \frac{\tilde{Y_1} \cdot Y_2^\perp}{(Y_1^\perp \cdot Y_1^\perp)^{\frac{1}{2}}(Y_2^\perp \cdot Y_2^\perp)^{\frac{1}{2}}} = \frac{\sinh^2\vartheta - \sin^2\theta}{\sinh^2\vartheta + \sin^2\theta}. \tag{A.2.4}$$

We can solve these two equations for x, $\bar{x}$ to obtain the expressions we have anticipated in eq. (2.6.3). In case of four local operators ($p = q = 0$) this construction corresponds to the radial coordinates

$$\rho = \frac{r}{R}e^{i(\pi-\theta)} = -e^{-(\vartheta+i\theta)}, \quad \bar{\rho} = \frac{r}{R}e^{-i(\pi-\theta)} = -e^{-(\vartheta-i\theta)}, \tag{A.2.5}$$

and therefore we get

$$z = \frac{4\rho}{(1+\rho)^2} = -\sinh^{-2}\frac{\vartheta + i\theta}{2} \equiv 1 - x, \quad \bar{z} = \frac{4\bar{\rho}}{(1+\bar{\rho})^2} = -\sinh^{-2}\frac{\vartheta - i\theta}{2} \equiv 1 - \bar{x}. \tag{A.2.6}$$

Appendix B

General relations with scalar four-point blocks

In this appendix we want to discuss some formulas that can be used to relate any defect block with $N = 2$ cross ratios to blocks for scalar four-point function. Let us stress, however, that the two relations we are about to discuss involve a continuation of the four-point block beyond the Euclidean domain, see discussion below. As we have seen before, a situation with $N = 2$ cross ratios arises when the dimension p of the first defect is $p = d - 2$ and the dimension q takes any value $q \leq d - 2$. In this case we can relate relevant defect blocks to scalar four-point blocks through

$$f_D\begin{pmatrix} d-2, q, d \\ \Delta, \ell \end{pmatrix}; x, \bar{x}\Big) \sim (-4)^{\frac{\Delta+\ell}{2}} \left[(1-x)(1-\bar{x})\right]^{\frac{d}{2}-2} (\bar{x}-x)^{2-\frac{d}{2}}$$

$$\times\, g\left(\begin{matrix} \frac{d-2q-2}{4}, \frac{d-4}{4}, 3 \\ \frac{\Delta-\ell}{2} - \frac{d}{2} + 2, -\frac{\Delta+\ell}{2} \end{matrix}; -\frac{(1-x)(1-\bar{x})}{(\sqrt{x}-\sqrt{\bar{x}})^2}, -\frac{(1-x)(1-\bar{x})}{(\sqrt{x}+\sqrt{\bar{x}})^2}\right) . \quad \text{(B.1.1)}$$

Recall that the parameters in the upper row of the argument of g are the parameters a, b and d of the scalar four-point block while the parameters in the lower row are the weight Δ and the spin l of the exchanged field. If the pair $(x, \bar{x})$ describes a point in the Euclidean domain, i.e. if x and $\bar{x}$ are complex conjugate to each other, then cross-ratios in the scalar four-point block g are real, but not inside the unit interval $[0, 1]$:

$$z = \sin^{-2}\theta \in [1, \infty) , \qquad \bar{z} = -\sinh^{-2}\vartheta \in (-\infty, 0) . \qquad \text{(B.1.2)}$$

This means that the four-point block in the right hand side is neither in the Euclidean nor in the Lorentzian domain, i.e. it is related to the usual four-point block only through analytic continuation to negative real cross-ratios. Conformal blocks, however, possess branch cuts along the wall ω_1. Since the monodromy along this wall is non-trivial, the result of the analytic continuation on the path along which we continue from positive to negative real cross-ratios is not unique. The $\sim$ between the left and the right side is meant to remind us of this continuation. Formula (B.1.1) does correctly encode the match of parameters in the Casimir equations, though, and the identification of eigenvalues up to the action of the Weyl group. In other words, the defect block on the left hand side can be written through a linear combination of Harish-Chandra (or 'pure' functions in the terminology of [28]) with eigenvalues Δ, l running through all the images of

$$\Delta_g := \frac{\Delta - \ell}{2} - \frac{d}{2} + 2 \quad , \quad \ell_g := -\frac{\Delta + \ell}{2} \tag{B.1.3}$$

under the replacements $\ell_g \leftrightarrow 2 - d_g - \ell_g$, $\Delta_g \leftrightarrow d_g - \Delta_g$ and $\Delta_g \leftrightarrow 1 - \ell_g$ with $d_g = 3$.

A similar discussion applies to the second setup with two cross-ratios, namely when we have two local operators whose weights differ by $\Delta_{12} = -2a$ in presence of a p-dimensional defect. In this case one finds that

$$\begin{aligned} f\begin{pmatrix} p, a, d \\ \Delta, \ell \end{pmatrix}; x, \bar{x} \end{pmatrix} &\sim (-4)^{\frac{\Delta+\ell}{2}+a} (x\bar{x})^{\frac{a}{2}} \left[(1-x)(1-\bar{x})\right]^{\frac{d}{2}-a-2} (\bar{x}-x)^{2-\frac{d}{2}} \\ &\times g\begin{pmatrix} -\frac{d-2p-2}{4}, \frac{d-4}{4}, 3+2a \\ \frac{\Delta-\ell}{2} - \frac{d}{2} + a + 2, -\frac{\Delta+\ell}{2} - a \end{pmatrix}; -\frac{(1-x)(1-\bar{x})}{(\sqrt{x}-\sqrt{\bar{x}})^2}, -\frac{(1-x)(1-\bar{x})}{(\sqrt{x}+\sqrt{\bar{x}})^2} \end{pmatrix} . \end{aligned} \tag{B.1.4}$$

The $\sim$ between the left and the right hand side has the same meaning as in eq. (B.1.1). In some sense, our relations (B.1.1) and (B.1.4) extend the relation (3.4.5) from [41]. While the latter applies to the very special case of $p = d - 2$ and $a = 0$ only, our relations cover any setup with two cross-ratios. While the relation between the cross-ratios $x, \bar{x}$ and the arguments of g is a little different in eq. (3.4.5), one central feature is the same: it maps the Euclidean domain of the defect correlator to a different domain and hence, the function g on the right hand side of eq. (3.4.5) should also be interpreted as some linear combination of Harish-Chandra functions with eigenvalues $\Delta_g = \Delta$ and $\ell_g = \ell$ running over the full orbit of the Weyl group.

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Eidesstattliche Erklärung

Hiermit versichere ich an Eides statt, die vorliegende Dissertationsschrift selbst verfasst und keine anderen als die angegebenen Hilfsmittel und Quellen benutzt zu haben.

Die eingereichte schriftliche Fassung entspricht der auf dem elektronischen Speichermedium.

Die Dissertation wurde in der vorgelegten oder einer ähnlichen Form nicht schon einmal in einem früheren Promotionsverfahren angenommen oder als ungenügend beurteilt.

Hamburg, den 5. Oktober 2018 Unterschrift