Equivariant Transversality Theory Applied to Hamiltonian Relative Equilibria

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Introduction

Motivation

The existence of symmetries is a fundamental assumption of theoretical mechanics.

In the classical Newton model, the laws of motion do not depend on a particular choice of a linear coordinate system. Therefore, they are unchanged by translations, rotations, and reflections. These transformations generate the *Euclidean group* $\mathrm{E}(3)$, the group of isometries of 3-dimensional Euclidean space $\mathbb{R}^3$. The motion of n particles within $\mathbb{R}^3$ is described by a dynamical system given by a vector field X on the tangent space $T\mathbb{R}^{3n} = \mathbb{R}^{3n} \times \mathbb{R}^{3n}$. The diagonal $\mathrm{E}(3)$-action on $\mathbb{R}^3$ induces an action of $\mathrm{E}(3)$ on the tangent space, and the invariance of the laws of motion is reflected by the fact that the flow φ_X of the vector field X is $\mathrm{E}(3)$-*equivariant*, that is $\varphi_X(gx) = g\varphi_X(x)$ for any $g \in \mathrm{E}(3)$.

More generally, it is often assumed in classical mechanics that unknown external forces constrain the motion to a submanifold $Q \subset \mathbb{R}^{3n}$. In that case, the symmetries of the system are given by a Lie group G that acts on Q.

For example, the motion of a rigid body may be described by a curve in the Euclidean group. If $(R(t), b(t))$ is such a curve, where $R(t)$ is an element of the orthogonal group and $b(t)$ is a translation, the motion of any particle of the rigid body is given by the curve $x_i(t) = R(t)X_i + b(t)$, where X_i is the position of the particle in a fixed reference body. The Euclidean group acts by left multiplication on the configuration space. If the rigid body has some rotational symmetry that is given by a subgroup $K \subset \mathrm{SO}(3)$ and preserves the reference body, there is an additional K-action by right multiplication.

The equations of motion for the dynamical system on the manifold have a particularly simple form in the Hamiltonian formalism. A curve $q : \mathbb{R} \supset I \to Q$ corresponds to a curve $(q(t), p(t))$ in the cotangent bundle, where $p(t) \in T^*_{q(t)}Q$ is an expression of $q(t)$ and $\dot{q}(t)$. The equations of motion are determined by the Hamiltonian function $h \in C^\infty(T^*Q, \mathbb{R})$.

The Hamiltonian function h defines the Hamiltonian vector field X_h by

$$\mathrm{d}h(p) = \omega_p(X_h(p), \cdot)$$

for any $p \in T^*Q$, where ω denotes the canonical symplectic form of T^*Q. A G-action on Q induces a G-action on T^*Q, which lifts the G-action on Q and leaves ω invariant. The Hamiltonian system on T^*Q has *G-symmetry* if the Hamiltonian function h is G-invariant. Then, the Hamiltonian vector field is G-equivariant and hence the G-action commutes with the flow of X_h.

Theoretical mechanics is the original motivation for studying Hamiltonian dynamical systems, which is nowadays a research field in its own right. A

Hamiltonian system with symmetry consists of a symplectic manifold P with a Lie group G acting by symplectomorphisms on P and a G-invariant Hamiltonian function.

One of the main objects of interest in the study of Hamiltonian systems with symmetry are trajectories that are contained in a single group orbit. A point of P whose trajectory has this property is a *relative equilibrium*. The G-action on P defines a map $\xi \mapsto \xi_P$ from the Lie algebra $\mathfrak{g}$ of G into the space of vector fields on P such that $\cup_{\xi\in\mathfrak{g}}\xi_P(p)$ coincides with the tangent space T_pGp. If $p \in P$ is a relative equilibrium, there is an element $\xi \in \mathfrak{g}_P$ such that $X_h(p) = \xi_P(p)$. ξ is a *generator* of the relative equilibrium p. For free actions, the generator is uniquely defined, but in general, it is only unique modulo the Lie algebra $\mathfrak{g}_p$ of the isotropy subgroup G_p.

The question of determining the relative equilibria of a given system has a long tradition in the study of mechanical systems. The relative equilibria of a mechanical system are often configurations of constant shape that rotate about a fixed principal axis. In particular, this holds for the n-body problem, which was described in the beginning.

The investigation of relative equilibria goes back to Riemann and even further. In [Rie61], Riemann finds all solutions of constant shape for a model of ellipsoidal liquid drops, which was proposed by Dirichlet ([Dir60]). Moreover, he shows that all of these motions of constant shape consist of a rotation about one of the principal axis of the ellipsoid and an oval motion of the fluid particles within the ellipsoid, which corresponds to a rotation of a spherical reference body.

A modern treatment of the same problem may be found in [RdS99]. Here, the problem is described as a Hamiltonian dynamical system on $T^*\mathrm{GL}_+$, the cotangent space of the group of 3×3-matrices with positive determinant.

Outline

In chapter 1, we introduce the basic concepts and notions of the theory of Hamiltonian systems with symmetry. In particular, we define the momentum map, which is a basic tool for the most modern approaches to determine the relative equilibria of a Hamiltonian system:

Noether's famous theorem of classical mechanics states that symmetries correspond to conserved quantities. For general Hamiltonian systems with symmetry, this is true at least locally. In this thesis, we will always suppose that these conserved quantities are given by globally defined functions on the symplectic space P. Putting these conserved quantities together in an appropriate way yields the momentum map $\mathbf{J} : P \to \mathfrak{g}^*$. The map $\mathbf{J}^\xi := \langle \mathbf{J}(\cdot), \xi \rangle$ is a Hamiltonian function of the vector field ξ_P. Thus, the relative equilibria with generator ξ coincide with the critical points of the *augmented Hamiltonian* $h_\xi := (h - \mathbf{J}^\xi)$.

In many relevant cases, the momentum map is equivariant with respect to the coadjoint action on $\mathfrak{g}^*$. For example if G is compact, an equivariant momentum map may be obtained from any given momentum map by averaging over G.

For simplicity, this thesis focuses on actions of compact groups. The more general case is only formulated if the statement does not become more complicated. Otherwise, we refer to the literature.

In particular, we will assume the equivariance of the momentum map with respect to the coadjoint action.

The equivariance of the momentum map implies in general that $\mathbf{J}(p) = \mathbf{J}(gp)$ holds if and only if $g \in G_\mu$ for $\mu = \mathbf{J}(p)$. Hence if p is a relative equilibrium, p is also a relative equilibrium with respect to the G_μ-action. If ξ is a generator of p, we obtain $\xi \in \mathfrak{g}_\mu$ or equivalently $\operatorname{coad}_\xi \mu = 0$.

This commutation relation occurs in different forms in most of the approaches to analyse the structure of Hamiltonian relative equilibria presented in this thesis. For instance, it is equivalent to the first of two equations that occur in [MR-O13] and [MR-O15] and characterize relative equilibria. These equations are derived in chapter 2, in which the theory of the local structure of the system near a given G-orbit is treated. They are stated in the coordinates of the Marle-Guillemin-Sternberg normal form, which is a local model of the symplectic space P near a given G-orbit. The Marle-Guillemin-Sternberg normal form is often considered as the symplectic version of the Slice Theorem. For simplicity, let us assume that G is compact. Then the normal form theorem states that for $p \in P$, there is a tubular neighbourhood U of Gp and a G-equivariant symplectomorphism to a G-invariant neighbourhood of $[e, 0, 0]$ in the space

$$Y = G \times_{G_p} (\mathfrak{m}^* \times N),$$

where N is a symplectic G_p-representation and $\mathfrak{m}$ is a G_p-invariant complement of $\mathfrak{g}_p$ within $\mathfrak{g}_\mu$ for $\mu = \mathbf{J}(p)$. The symplectic form on Y is determined by the symplectic form on N and the group G. The momentum map on Y is of particularly simple form: $\mathbf{J}_Y([g, \rho, v]) = \operatorname{Coad}_g(\mu + \rho + \mathbf{J}_N(v))$, where $\mathbf{J}_N$ is the momentum map of N.

With respect to this normal form, Montaldi and Rodríguez-Olmos characterize relative equilibria as follows: Let $\bar{h}$ denote the induced function of h on the slice $\mathfrak{m}^* \times N$. The point $[e, \rho, v]$ corresponds to a relative equilibrium p' near p with momentum $\mu' = \mathbf{J}_Y([e, \rho, v]) = \mu + \rho + \mathbf{J}_N(v)$ if and only if there is an $\eta \in \mathfrak{g}_p$ such that $\xi' = \mathrm{d}_{\mathfrak{m}^*} \bar{h}(\rho, v) + \eta$ satisfies

$$\operatorname{coad}_{\xi'} \mu' = 0$$

and

$$\bar{h}_\eta(\rho, v) := (\bar{h} - \mathbf{J}_N^\eta)(\rho, v) = 0.$$

In this case, ξ' is a generator of p'. Montaldi and Rodríguez-Olmos deduce these equations from the bundle equations, which lift the Hamiltonian vector field on Y to the space $G \times (\mathfrak{m}^* \times N)$. The bundle equations have been discovered by Roberts and de Sousa Dias ([RdSD97]) for actions of compact groups and have been generalized by Roberts, Wulff, and Lamb ([RWL02]) and others. In these publications, a particular lift to $G \times (\mathfrak{m}^* \times N)$ is chosen. In contrast, Montaldi and Rodríguez-Olmos consider all possible lifts and obtain the bundle equations with isotropy, which imply the above equations for relative equilibria.

To understand the local dynamics near an equilibrium or a relative equilibrium, it is also often useful to consider the linearization of the Hamiltonian vector field. The theory of linear Hamiltonian systems with symmetry will be presented in chapter 3. In particular, we investigate the possible forms of the centre space of the linearization $\mathrm{d}X_h(0)$ and the generic eigenvalue structure of $\mathrm{d}X_h(0)$.

Subsequently in chapter 4, we consider free actions of compact connected groups. For free actions, the level sets $\mathbf{J}^{-1}(\mu)$ of the momentum map form manifolds. Moreover, the G-action induces a G_μ-action on $\mathbf{J}^{-1}(\mu)$, and we obtain a flow on the *reduced space* $P_\mu := \mathbf{J}^{-1}(\mu)\big/G_\mu$. P_μ is a symplectic manifold and the flow coincides with the flow of the Hamiltonian vector field for the function h_μ induced of h. Consequently, relative equilibria correspond to critical points of h_μ in this case. Thus, an investigation of the topology of the reduced spaces leads to an estimation of the number of G_μ-orbits of relative equilibria with momentum μ. This is the approach in [Mon97]. In that article, Montaldi considers the local picture near a relative equilibrium p with momentum value α such that $\mathrm{d}^2 h_\alpha(p) \neq 0$. For μ near α, he relates the number of critical points of h_μ to the cardinality of the Weyl group orbits of μ and α.

Patrick and Roberts ([PR00]) analyse the generic structure of the set of relative equilibria for free actions in a different way: They formulate a transversality condition, which is generic within the set of G-invariant Hamiltonian functions with respect to the Whitney C^∞-topology. If h satisfies this condition, the relative equilibria form a Whitney stratified set: The adjoint and the coadjoint action together give a G-action on $\mathfrak{g}^* \oplus \mathfrak{g}$. The strata are given by the sets of relative equilibria whose momentum generator pairs (ξ, μ) have the same isotropy type.

The main idea of the proof is as follows: Patrick and Roberts consider vector subbundles $\mathcal{T}$ and $\mathcal{K}$ of the tangent bundle TP given by $\mathcal{T}_p = \mathfrak{g} \cdot p$ and $\mathcal{K}_p = \ker \mathrm{d}\mathbf{J}(p)$. They observe that p is a relative equilibrium if and only if $X_h(p) \in \mathcal{T}_p \cap \mathcal{K}_p$. Moreover, they consider the set of pairs (ξ, μ) with $\operatorname{coad}_\xi \mu = 0$ and show that the stratification by isotropy type of this set induces a stratification of $\mathcal{T}_p \cap \mathcal{K}_p$. A relative equilibrium p is called *transverse* if the Hamiltonian vector field is transverse to this stratification at p.

From a generalization of Patrick's and Roberts' approach, we also obtain an alternative deduction of the above two equations, which characterize relative equilibria: The vector bundles $\mathcal{K}^\circ$ and $\mathcal{T}^\circ$ that occur in the formulation of the transversality condition in [PR00] on the cotangent space are still semi-analytic sets if the action is not necessarily free. Moreover, it is true in general that p is a relative equilibrium if and only if $\mathrm{d}h(p) \in \mathcal{K}^\circ \cap \mathcal{T}^\circ$. An evaluation of this condition on the Marle-Guillemin-Sternberg normal form gives the above equations of relative equilibria.

We will come to these ideas in chapter 6.

Prior to that, we discuss some results for the case of non-trivial isotropy subgroups in chapter 5. In particular, we permit isotropy subgroups of positive dimension. Most of the results that we discuss here adapt ideas from bifurcation theory. Indeed, the augmented Hamiltonians $h_\xi = h - \mathbf{J}^\xi$ can be considered as a family parameterized by $\xi \in \mathfrak{g}$. Similarly, the left hand side of the second one of the above equations, that is $\mathrm{d}\bar{h}(\rho, v) - \mathbf{J}^\eta(v)$, can be considered as a family with parameters η and ρ. This point of view is explicitly formulated in [CLOR03]. There, Chossat, Lewis, Ortega, and Ratiu consider equivalent formulations of the above two equation. Similar methods are used in [MR-O15]. Most of these results require conditions that assure that the commutation relation $\operatorname{coad}_\xi \mu = 0$ is automatically satisfied, such that only the second equation has to be considered. Its solutions for a fixed $\rho \in (\mathfrak{m}^*)^{G_p}$, correspond to relative equilibria of the Hamiltonian $\bar{h}(\rho, \cdot)$ on the symplectic slice N.

The special case of Hamiltonian systems on a representation with an equilibrium at 0 is treated in [OR04(a)]. In that context, to find the relative equilibria near $0 \in V$, we have to solve $\mathrm{d}h_\xi(v) = 0$. If we consider $\xi \in \mathfrak{g}$ as a parameter, in contrast to ordinary bifurcation theory, we have a non-trivial action on the parameter space. It is given by the adjoint action. To circumvent this problem, we will fix a maximal torus $T \subset G$ and search for relative equilibria with generators in its Lie algebra $\mathfrak{t}$. If ξ is a generator of the relative equilibrium p, then $\mathrm{Ad}_g\,\xi$ is generator of gp for $g \in G$. Since all adjoint orbits intersect $\mathfrak{t}$, the G-orbits of relative equilibria with generators in $\mathfrak{t}$ contain all relative equilibria.

This approach combined with the main theorem in [OR04(a)] yields the generic structure of Hamiltonian relative equilibria near 0 for representations of connected compact groups of rank 1, i.e. for the groups SO(3) and SU(2). For groups of rank 1, all relative equilibria are contained in a periodic orbit. Therefore, these conclusions alternatively follow from a simple application of the equivariant Weinstein-Moser theorem of [MRS88]. These consideration will also be discussed in chapter 5.

Nevertheless, the idea may be generalized to groups of higher rank using equivariant transversality theory. In mechanics, symmetry groups of higher rank often occur for approximations of the Hamiltonians. An example will be discussed in section 6.6 of chapter 6. There might also be applications to modern physics since there occur symmetry groups of rank greater than 1.

In chapter 6, we introduce equivariant transversality and applications to the theory of Hamiltonian relative equilibria:

One of my own results in this thesis is the observation that the transversality condition from [PR00] may be extended to the case of non-free actions using equivariant transversality. Indeed, Field has developed a genericity theory for relative equilibria in 1-parameter families of equivariant vector fields (see [Fie96]). An adaption of Fields approach to Hamiltonian systems leads to a transversality condition, which can be formulated in terms of G_p-equivariant transversality to the semi-analytic set $\mathcal{K}^\circ \cap \mathcal{T}^\circ$. For free actions, the definition coincides with the Patrick's and Roberts' definition.

The main implication of this observation is that Patrick's and Roberts' theory generically holds for the fixed point subspace P^H of an isotropy subgroup $H \subset G$ and the free action of the identity component of $N(H)/H$. The density of this property within the space $C^\infty(P)$ of G-invariant smooth functions on P with the Whitney C^∞-topology follows directly from the results in [PR00]. The equivariant transversality theory approach shows that this condition is open and that the entire set of relative equilibria is Whitney stratified.

For the second application, we consider representations and generalize the ideas indicated in chapter 5. We first investigate torus representations. Again, we adapt Fields method for equivariant bifurcation problems, but this time the space $\mathfrak{t}$ is considered as a parameter space. This way, we obtain that generically the topological structure of the relative equilibria coincides with that of the linearized vector field. In the generic situation, the set of relative equilibria near 0 is a union of manifolds that are tangent to sums of weight spaces with linearly independent weights of the centre space of $\mathrm{d}X_h(0)$. Conversely, there is such a manifold for any set of linearly independent weights.

These results may be applied to representations of a general connected compact group G with maximal torus T by restricting the action to T. This way, we do not obtain the whole generic structure of the relative equilibria, but we still predict branches that generically exist: Generically, the real parts of the sums of the eigenspaces of $dX_h(0)$ for each pair of purely imaginary eigenvalues are irreducible G-symplectic spaces. They may be regarded as irreducible complex G-representations. Consider the set of weights of one of these irreducible representations. For each affine subset of $\mathfrak{t}^*$ that contains only a linearly independent subset of these weights (counted with multiplicity), there is a manifold of relative equilibria tangent to the sum of the corresponding weight spaces. Moreover, if we join subsets of weights of these kind of different eigenspaces and the union is linearly independent, we generically obtain a manifold tangent to the sum of the corresponding weight spaces, too.

In particular, this result implies that there generically are non-trivial relative equilibria in any neighbourhood of the origin if the G-action on the centre space of $dX_h(0)$ is non-trivial. This seems to contradict a result of Birtea et al ([BPRT06]) at first glance: These authors also use equivariant transversality theory to investigate relative equilibria in Hamiltonian systems with symmetry, but they consider 1-parameter families of Hamiltonian functions. They claim that under some – quite hard to check – assumption on the G-symplectic representation, the Hamiltonian relative equilibria form curves in the orbit space that only approach the origin at parameter values with a degenerate linearization of the Hamiltonian vector field at the origin. However, it is not clear if their condition holds in relevant cases and moreover, the proof has an essential flaw. Anyhow, other results in the literature, which we discuss in this thesis, also indicate that this is not the typical situation.

Since the theory of Hamiltonian dynamical systems is a wide field, the presentation in this thesis is restricted to a small aspect: The structure of the set of relative equilibria. In general, these sets may be of arbitrarily complicated form. Therefore, the emphasis is placed on the generic case within the set of invariant Hamiltonian functions.

Since the new results of this thesis only consider the structure of the set of relative equilibria, we usually omit stability results. For these, we refer to [PRW04] and [MR-O15]. Montaldi ([Mon00]) also contains an introduction to the topic. Moreover, it is a good overview of the research field of Hamiltonian relative equilibria in general and influenced the selection of results presented in this thesis. An introduction to related themes including relative periodic equilibria can be found in [MBLP05].

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Chapter 1

Relative equilibria in symmetric Hamiltonian systems

1.1 Hamiltonian dynamics

Hamiltonian vector fields are defined on symplectic manifolds. The basic definitions and results are given in the following. Proofs can be found for example in [McDS98].

A local model of symplectic manifolds is given by symplectic vector spaces:

Definition 1.1. A *symplectic vector space* is a vector space V together with a *symplectic bilinear form* ω that is a non-degenerate skew-symmetric bilinear form.

A linear map $A : (V, \omega) \to (V', \omega')$ between symplectic vector spaces with $A^*\omega' := \omega'(A\cdot, A\cdot) = \omega$ is a *linear symplectomorphism.*

A vector subspace U of V is *isotropic* iff ω vanishes on U. A *Lagrangian subspace* is a maximal isotropic subspace.

Example 1.2. $\mathbb{R}^{2n}$ is a symplectic vector space with the symplectic form $\omega_0 := \langle \cdot, J_0 \cdot \rangle$, where $\langle \cdot, \cdot \rangle$ denotes the standard inner product and

$$J_0 = \begin{pmatrix} 0 & -\mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}.$$

This example even gives all isomorphism classes of symplectic vector spaces:

Theorem 1.3. *Every finite dimensional symplectic vector space is isomorphic to* $(\mathbb{R}^{2n}, \langle \cdot, J_0 \cdot \rangle)$ *for some* $n \in \mathbb{N}$.

Symplectic manifolds are endowed with a smooth 2-form that defines a symplectic bilinear form on each tangent space and is in addition closed:

Definition 1.4. Let P be a smooth manifold. A non-degenerate closed smooth 2-form ω on P is called *symplectic form.* The pair (P, ω) forms a *symplectic manifold.*

A smooth diffeomorphism $\Phi : (P, \omega) \to (P', \omega')$ is called a *symplectomorphism* iff $\Phi^* \omega' = \omega$, where $\Phi^* \omega'_p(\cdot, \cdot) = \omega'_{\Phi(p)}(\mathrm{d}\Phi(p)\cdot, \mathrm{d}\Phi(p)\cdot)$ for $p \in P$.

In classical mechanics, the phase spaces are given by cotangent bundles:

Example 1.5. Let Q be a smooth manifold. Then the cotangent bundle T^*Q is a symplectic manifold in a natural way: In coordinates $(q_1, \dots, q_n, p_1, \dots, p_n)$ induced by a chart $(q_1, \dots, q_n) : U \to \mathbb{R}^n$ defined on $U \subset Q$ such that $(p_1, \dots, p_n)$ denotes the element $\sum_i p_i \mathrm{d}q_i$ of the fibre $T^*_q Q$, the *canonical symplectic form* is given by $\omega = \sum_i \mathrm{d}q_i \wedge \mathrm{d}p_i$. ω is exact: Let $\pi : T^*Q \to Q$ be the projection. Then for $(q, \alpha_q) \in T^*Q$, the *canonical 1-form* θ is defined by

$$\theta_{(q,\alpha_q)} = \alpha_q \circ \mathrm{d}\pi(q, \alpha_q) : T_{(q,\alpha_q)} T^*Q \to \mathbb{R}.$$

With respect to the above coordinates, $\theta = \sum_i p_i \mathrm{d}q_i$. Thus $\omega = -\mathrm{d}\theta$.

Darboux's theorem states that symplectic manifolds are indeed locally symplectomorphic to symplectic representations:

Theorem 1.6 (Darboux's theorem)**.** *Let (P, ω) be a symplectic manifold of dimension $2n$ and $p \in P$. Then there is an open neighbourhood U of p which is symplectomorphic to an open subset U' of $\mathbb{R}^{2n}$ together with the form ω_0.*

Coordinates corresponding to a symplectomorphism $U \to U' \subset \mathbb{R}^{2n}$ are called *canonical coordinates.*

If (P, ω) is a symplectic manifold, every smooth function $h : P \to \mathbb{R}$ defines a vector field on P: The *Hamiltonian vector field* X_h is given by

$$\mathrm{d}h(p) = \omega_p(X_h(p), \cdot) = (i_{X_h}\omega)(p).$$

Since ω is non-degenerate, X_h is unique. The function h is called the *Hamiltonian function* and the triple (P, ω, h) defines a *Hamiltonian system.*

The flow φ^h_t of the Hamiltonian vector field has some remarkable properties:

For any smooth function $f : P \to \mathbb{R}$, the time-derivative of $f(\varphi^h_t(p))$ at $t = 0$ is given by

$$\frac{\mathrm{d}}{\mathrm{d}t} f(\varphi^h_t(p)) = \mathrm{d}f(\varphi^h_t(p)) X_h(\varphi^h_t(p)) = \omega_{\varphi^h_t(p)}(X_f(\varphi^h_t(p)), X_h(\varphi^h_t(p))).$$

Thus the function f is constant along the trajectories of X_h, iff the *Poisson bracket* $\{f, h\} := \omega(X_f, X_h)$ vanishes on P. The Poisson bracket defines a Lie algebra structure on $C^\infty(P) := C^\infty(P, \mathbb{R})$. In particular, the level sets of the Hamiltonian function h itself are flow invariant.

Moreover, the flow φ^h_t consists of symplectomorphisms, since by Cartan's formula

$$\frac{\mathrm{d}}{\mathrm{d}t} (\varphi^h_t)^* \omega = (\varphi^h_t)^* (L_{X_h}\omega) = (\varphi^h_t)^* (i_{X_h} \mathrm{d}\omega + \mathrm{d} i_{X_h} \omega) = 0.$$

If $\dim P = 2n$, the symplectic form yields the volume form $\bigwedge^n \omega$, with respect to which the flow of the of the Hamiltonian vector field is volume preserving. Thus in Hamiltonian dynamics, there are for instance no asymptotically stable equilibria.

1.2 Group actions

This section contains basic notions and facts on group actions that will be used in the next chapters. If no other reference is given, proofs may be found in [DK00].

Definition 1.7. Let G be a Lie group and M be a smooth manifold. An *action* of G on M is a smooth map $\Phi : G \times M \to M$, such that

1. $\Phi(e, m) = m$, where e denotes the neutral element, and
2. $\Phi(g, \Phi(h, m)) = \Phi(gh, m)$.

The manifold M together with the G-action is called a *G-manifold* or *G-space*.

Remark 1.8. In the literature, this is often called a *left action*.

A G-action on a manifold M induces actions on its tangent and cotangent bundle:

Example 1.9. Set $\Phi_g := \Phi(g, \cdot)$. If Φ defines a G-action on M, the map

$$(g, (m, v_m)) \mapsto (gm, \mathrm{d}\Phi_g(m)v_m)$$

defines an action on TM.

Example 1.10. In the above setting,

$$(g, (m, \alpha_m)) \mapsto (gm, ((\mathrm{d}\Phi_g(m))^{-1})^*\alpha_m) = (gm, \alpha_m((\mathrm{d}\Phi_g(m)^{-1})\cdot))$$

defines an action on T^*M.

Definition 1.11. Let M be a G-manifold. A function $f : M \to \mathbb{R}$ is *G-invariant* iff $f(gm) = f(m)$ for every $g \in G$ and $m \in M$. The set of smooth G-invariant functions on M is denoted by $C^\infty(M)^G$.

Definition 1.12. Let M, N be G-manifolds. A map $f : M \to N$ is *G-equivariant* iff $f(gm) = gf(m)$ for every $g \in G$ and $m \in M$. The set of smooth G-equivariant maps from M to N is denoted by $C^\infty_G(M, N)$. M and N are *isomorphic* G-manifolds iff there is an equivariant diffeomorphism from M to N.

In particular vector fields on G-manifolds are called *G-equivariant* iff they are equivariant with respect to the induced G-action on the tangent space. G-equivariance of 1-forms is defined similarly. A short calculation shows that the derivative $\mathrm{d}f$ of a G-invariant function f defined on a G-manifold is a G-equivariant 1-form.

A special case of a G-manifold is a representation:

Definition 1.13. Let V be $\mathbb{K}$-vector space, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. A G-representation on V is a Lie group homomorphism

$$\rho : G \to \mathrm{GL}(V).$$

A *morphism between G-representations* V and W is an equivariant linear map $V \to W$. The vector space of morphisms is denoted by $\mathrm{Hom}_G(V, W)$. Accordingly, $\mathrm{End}_G(V) := \mathrm{Hom}_G(V, V)$. An invertible morphism is an *isomorphism* of representations.

A G-invariant subspace of V is called a *subrepresentation*. V is irreducible iff V and $\{0\}$ are the only subrepresentations of V.

Obviously, the only morphism between non-isomorphic irreducible representations is the zero morphism. The endomorphisms of irreducible representations are characterized as follows:

Lemma 1.14 (Schur's Lemma, [BtD85, Chapter I, Theorem 1.10])**.** *Let V be an irreducible complex representations of a group G. Then every $f \in \mathrm{End}_G(V)$ is of the form $\lambda\mathbb{1}$ for some $\lambda \in \mathbb{C}$.*

Lemma 1.15 (Schur's Lemma for real representations, [Fie07, Theorem 2.7.2])**.** *Let V be an irreducible real representations of a group G. Then $\mathrm{End}_G(V)$ is isomorphic as an $\mathbb{R}$-algebra to one of the algebras $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$.*

Depending on which case occurs, V is of *type* $\mathbb{R}$, $\mathbb{C}$ of $\mathbb{H}$. An irreducible real representation of type $\mathbb{R}$ is also called *absolutely irreducible*.

Remark 1.16. If the representation admits a G-invariant inner product ($\mathbb{K} = \mathbb{R}$) or Hermitian product ($\mathbb{K} = \mathbb{C}$), the orthogonal complement of any G-invariant subspace is G-invariant, too. Thus the representation V is isomorphic to a direct sum $\bigoplus_i U_i^{p_i}$ of irreducible representations U_i in this case. The spaces $U_i^{p_i}$ correspond to unique subrepresentation of V called the *isotypic components*.

If the group G is compact, one obtains an invariant inner product or Hermitian product from an arbitrary one denoted by $\langle\cdot,\cdot\rangle$ by averaging over G: $\int_{g\in G}\langle g\cdot, g\cdot\rangle$ is invariant. Thus for a representation of a compact group, we assume w.l.o.g. in the following that the action is given by orthogonal or unitary transformations.

For the isotypic composition $V = \bigoplus_i U_i^{p_i}$, we have

$$\mathrm{End}_G(V) = \bigoplus_i \mathrm{End}_G(U_i^{p_i}).$$

If V is real representation and U_i is irreducible of type $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$, then the $\mathbb{R}$-algebra $\mathrm{End}_G(U_i^{p_i})$ is isomorphic to the $\mathbb{R}$-algebra of $p_i \times p_i$-matrices with entries in $\mathbb{K}$ (see [Fie07, Proposition 2.7.3]).

We will abbreviate $\Phi(g, m)$ by gm in the following.

Definition 1.17. For $m \in M$, the *isotropy subgroup* or *stabilizer* of m is given by

$$G_m := \{g \in G \mid gm = m\}.$$

The *isotropy type* of m is the conjugacy class (G_m).

It is easy to see that points of the same G-orbit have the same isotropy type.

Definition 1.18. A G-action on a manifold M is *transitive* iff $M = Gm$ for some $m \in M$.

Definition 1.19. A G-action on a manifold M is *free* iff $G_m = \{e\}$ for every $m \in M$.

Definition 1.20. A continuous map $f : X \to Y$ between locally compact Hausdorff spaces is *proper* iff the preimage of every compact set is compact.

Definition 1.21. An action $\Phi : G \times M \to M$ is *proper* iff the map

$$\begin{aligned} \tilde{\Phi} : G \times M &\to M \times M \\ (g, m) &\mapsto (gm, m) \end{aligned}$$

is proper.

Example 1.22. If G is a compact Lie group, any G-action is proper: Let $K \subset M \times M$ be compact. Then K is closed and thus $\tilde{\Phi}^{-1}(K)$ is closed. Moreover, the projection $p_2(K)$ of K onto the second copy of M is compact and $\tilde{\Phi}^{-1}(K)$ is contained in $G \times p_2(K)$. Thus $\tilde{\Phi}^{-1}(K)$ is a closed subset of a compact subset and thus compact.

Example 1.23. For any closed subgroup H of a Lie group G, the action of H on G given by $\Phi^H(h, g) = gh^{-1}$ is proper: If $H = G$, the map $\tilde{\Phi}^G$ is a homeomorphism and thus proper. Thus for a general closed subgroup $H \subset G$ and a compact subset $K \subset G \times G$, the set

$$(\tilde{\Phi}^H)^{-1}(K) = (\tilde{\Phi}^G)^{-1}(K) \cap H \times G$$

is the intersection of a compact subset and a closed subset. Hence $(\tilde{\Phi}^H)^{-1}(K)$ is compact.

If the action is proper, obviously all isotropy subgroups are compact. In this thesis, we will consider proper group actions with the emphasis on actions of compact groups. The following results characterize the G-manifold structure for proper actions:

Theorem 1.24 (Bochner's linearization theorem)**.** *Let M be a G-manifold and suppose that G_m is compact for some $m \in M$. Then there is a G_m-invariant neighbourhood U and a G_m-invariant open subset O of $0 \in T_mM$ (with respect to the linear G_m-representation on T_mM) such that U and O are isomorphic G_m-manifolds.*

Theorem 1.25 (Free proper actions)**.** *Let G act freely and properly on the smooth manifold M. Then the orbit space $M/G := M/\sim$, where $m \sim n$ iff $m = gn$ for some $g \in G$, has a unique structure of a smooth manifold of dimension $\dim M - \dim G$ such that the projection $M \to M/G$ is a submersion and M is a fibre bundle over M/G.*

In particular for a closed subgroup $H \subset G$, the space of left cosets G/H is smooth, since it is the quotient space with respect to the free, proper H-action on G given by $(h, g) \mapsto gh^{-1}$.

If G acts properly on M, it can be shown that for $m \in M$ with $G_m = H$ the orbit Gm is a smooth submanifold of M diffeomorphic to G/H. Moreover, there is a G-invariant neighbourhood of m that is isomorphic to a fibre bundle over G/H:

Definition 1.26. Let $H \subset G$ be a compact subgroup and S be an H-manifold. Then H acts freely on the product space $G \times S$ by $h(g, s) = (gh^{-1}, hs)$. The orbit space with respect to this action forms the *twisted product* $G \times_H S$.

$G \times_H S$ is a smooth manifold. The G-action from the left on G yields a G-action on $G \times S$, which induces a G-action on $G \times_H S$. Thus $G \times_H S$ is a G-manifold. The projection $G \times S \to G$ induces a map $G \times_H S \to {}^{G}\!/_{H}$, which defines a fibre bundle over ${}^{G}\!/_{H}$.

Definition 1.27. Let M be a G-manifold and $m \in M$. A *slice* S_m for the G-action at m is a G_m-invariant smooth submanifold of M transverse to the G-orbit Gm such that

1. $S_m \cap Gm = m$,
2. $GS_m := \bigcup_{g \in G} gS_m$ is an open neighbourhood of Gm, and
3. $gS_m \cap S_m \neq \emptyset$ iff $g \in G_m$.

Theorem 1.28 (Slice Theorem/ Tube theorem)**.** *Let G act properly on the manifold M. Then for $m \in M$ with $H := G_m$, there is a G-invariant open neighbourhood U and a slice S_m at m such that $U \simeq G \times_H S_m$ as G-manifolds.*

Remark 1.29.

1. Such a neighbourhood of U is called a *tubular neighbourhood* of the orbit Gm.
2. By Bochner's linearization theorem, the G_m-manifold S_m is locally isomorphic to the G_m-representation on $V = T_m S_m$. Thus the vector bundle $G \times_H V$ is a local model for the G-manifold structure near Gm, in the sense that U is isomorphic to a neighbourhood of the zero section of $G \times_H V \to {}^{G}\!/_{H}$.
3. If N is a G-manifold and $f : S_m \to N$ is a smooth H-equivariant map, f may be extended to a smooth G-equivariant map from U to N. (See [Fie07, Theorem 3.5.1 and Exercise 3.5.1])

We define isotropy subspaces of the G-manifold M as follows:

Definition 1.30. For a closed subgroup $H \subset G$, the *fixed point space* is given by

$$M^H := \{m \in M \mid hm = m \quad \forall h \in H\}.$$

M_τ for $\tau = (H)$ is the set of points of M of isotropy type τ. M_H denotes the set of points with isotropy subgroup H. If H is compact

$$M_H = M_\tau \cap M^H$$

(see [OR04(b), Proposition 2.4.4]).

If V is a representation, the fixed point subspaces of V are obviously subspaces. If G acts properly on the smooth manifold M, Bochner's linearization theorem yields that the connected components of M^H are smooth manifolds. The Slice Theorem implies the same for connected components of M_τ and M_H. Moreover, an equivariant map $f : M \to N$ between G-manifolds obviously maps the fixed point space M^H into N^H. In particular, equivariant vector fields are tangent to the fixed point spaces, which thus form invariant submanifolds.

Theorem 1.31. *If G acts properly on M, the partition of M into connected components of the spaces M_τ corresponding to the isotropy types of the action is a Whitney stratification.*

(For the definition of a Whitney stratification, see appendix A.2.)

For a closed subgroup $H \subset G$, the *normalizer* is given by

$$N(H) = N_G(H) = \{ g \in G \mid gHg^{-1} = H \} .$$

The submanifolds M^H and M_H are obviously $N(H)$-invariant.

The main objects of interest of this thesis are relative equilibria:

Definition 1.32. Let X be a G-equivariant smooth vector field defined on a G-manifold M. A point $m \in M$ is a *G-relative equilibrium* (or simply *relative equilibrium*) of X iff the G-orbit Gm is invariant with respect to the flow of the vector field X.

Suppose that $H = G_m$ for the relative equilibrium $m \in M$. Then $(Gm)^H = N(H)m$. Since fixed point spaces are invariant subspaces, the trajectory of m is even contained in the orbit $N(H)m$.

Alternatively, relative equilibria can be characterized with respect to the action of the Lie algebra $\mathfrak{g}$ on M:

Definition 1.33. An element $\xi \in \mathfrak{g}$ defines a vector field ξ_M on a G-manifold M by

$$\xi_M(m) = \frac{\mathrm{d}}{\mathrm{d}t} \exp(t\xi)m \Big|_{t=0} .$$

We also write $\xi \cdot m$ for $\xi_M(m)$ and $\mathfrak{g} \cdot m := \bigcup_{\xi \in \mathfrak{g}} \xi \cdot m = T_m(Gm)$.

Remark 1.34. The map $\xi \mapsto \xi_M$ is a Lie algebra anti-homomorphism from $\mathfrak{g}$ to the Lie algebra of vector fields on M, i.e. it is linear and $[\xi, \eta]_M = -[\xi_M, \eta_M]$. In addition, the map $(\xi, m) \mapsto \xi_M(m)$ is a smooth map from $\mathfrak{g} \times M$ to TM. A map with these properties is called a *(left) Lie algebra action.* See [OR04(b), Definition 2.2.6].

A point $m \in M$ is a relative equilibrium of X iff there is a $\xi \in \mathfrak{g}$ such that m is a zero of the vector field $X - \xi_M$. In this case, ξ is called a *generator* or *velocity* of the relative equilibrium m. The generator ξ is not unique in general. It is only unique modulo the Lie algebra $\mathfrak{h}$ of the group $H := G_m$. Since the generator is contained in the Lie algebra of the group $N(H)$, it may be regarded as an element of the Lie algebra of the group $N(H)/H$.

If m is a relative equilibrium, this holds also for any other element gm of the G-orbit of m. If $\xi \in \mathfrak{g}$ is a generator of m, $\operatorname{Ad}_g \xi$ is a generator of gm, where $\operatorname{Ad}_g M$ is defined as follows:

Definition 1.35. A Lie group G acts on itself by conjugation: $\Phi : G \times G \to G$ is given by

$$\Phi(g, h) = c(g)h := ghg^{-1}.$$

The neutral element $e \in G$ is a fixed point of this action. Thus the action induces a representation of G at the tangent space $T_eG = \mathfrak{g}$. This representation is called the *adjoint representation* of G and the corresponding map is denoted by $\operatorname{Ad} : G \to \operatorname{GL}(\mathfrak{g})$. The dual representation of G on $\mathfrak{g}^*$ is called coadjoint representation and denoted by $\operatorname{Coad} : G \to \operatorname{GL}(\mathfrak{g}^*)$. Explicitly, $\operatorname{Coad}_g \mu = \operatorname{Ad}^*_{g^{-1}} \mu = \mu(\operatorname{Ad}_{g^{-1}} \cdot)$ for $\mu \in \mathfrak{g}^*$ and $g \in G$.

The adjoint and the coadjoint representation induce corresponding representations of the Lie algebra, that is Lie algebra homomorphisms ad and coad to the Lie algebras $\mathfrak{gl}(\mathfrak{g})$ and $\mathfrak{gl}(\mathfrak{g}^*)$ of $\mathrm{GL}(\mathfrak{g})$ and $\mathrm{GL}(\mathfrak{g}^*)$ respectively. The homomorphism ad coincides with the homomorphism $\mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$, i.e. $\mathrm{ad}_\xi = [\xi, \cdot]$.

Let us assume in the following that G is compact and connected. We will need some facts about maximal tori, that are stated in the following. Proofs can be found in for instance in [BtD85].

Lemma 1.36. *Any compact connected Abelian Lie group is isomorphic to a torus* $T = T^n = \mathbb{R}^n/\mathbb{Z}^n$ *for some* $n \in \mathbb{N}$.

In the following, an Abelian connected closed subgroup of G will be called a *torus* of G.

Definition 1.37. A torus T of G is *maximal* iff $T \subset T'$ for any torus $T' \subset G$ implies that $T = T'$.

Theorem 1.38. *Each element* $g \in G$ *is contained in a maximal torus.*

Theorem 1.39. *All maximal tori of* G *are conjugate.*

Fix a maximal torus $T \subset G$. The action by conjugation of G on itself restricts to an action of the normalizer $N(T)$ on T, which again induces an action of $N(T)/T$ on T.

Definition 1.40. $W = W(G) = N(T)/T$ is called the *Weyl group* of G.

(Since all maximal tori are conjugate, the isomorphism class of W is independent of the choice of T.)

Theorem 1.41. *The Weyl group* W *is finite.*

Lemma 1.42. *For any* $t \in T$ *and* $g \in G$ *with* $gtg^{-1} \in T$, *there is an element* $w \in W$ *such that* $gtg^{-1} = wtw^{-1}$.

Definition 1.43. A maximal Abelian Lie subalgebra of $\mathfrak{g}$ is called a *Cartan subalgebra* of $\mathfrak{g}$.

By [Hal03, Proposition 11.7], every Cartan subalgebra is the Lie algebra $\mathfrak{t}$ of a maximal torus T.

The adjoint representation induces a representation of W on $\mathfrak{t}$. Similarly, the coadjoint representation induces a representation of W on $\mathfrak{t}^*$. Choosing a G-invariant inner product on $\mathfrak{g}$ yields an isomorphism of the adjoint representation and its dual, the coadjoint representation. The W representations on $\mathfrak{t}$ and $\mathfrak{t}^*$ are isomorphic as well.

The infinitesimal version of Lemma 1.42 is as follows:

Lemma 1.44. *For* $\xi \in \mathfrak{t}$ ($\mu \in \mathfrak{t}^*$)*, the intersection of the adjoint orbit* $G\xi$ ($G\mu$) *with* $\mathfrak{t}$ ($\mathfrak{t}^*$) *coincides with the Weyl group orbit* $W\xi$ ($W\mu$).

Since the representation of W on $\mathfrak{t}$ is faithful, W can be identified with a subgroup of $\mathrm{GL}(\mathfrak{t})$.

Theorem 1.45. *The Weyl group* W *is generated by reflections of* $\mathfrak{t}$.

The fixed point spaces of the reflections in $\mathfrak{t}$ and $\mathfrak{t}^*$ are called *Weyl walls*. The isotropy subgroup G_ξ of an element $\xi \in \mathfrak{t}$ is determined by the Weyl walls that contain ξ. $G_\xi = T$ iff ξ is not contained in a Weyl wall. The same is true for the coadjoint representation.

Lemma 1.46. *Let G be a compact Lie group, M be a smooth G-manifold, X be a smooth G-equivariant vector field on M, and $p \in M$ be a relative equilibrium. Then there is a torus $T \subset G$ such that the X-orbit $\mathcal{O}_p$ is contained in Tp.*

Proof. Let $\xi \in \mathfrak{g}$ be a generator of p. Then there is a maximal torus $T \subset G$ such that ξ is contained in its Lie algebra $\mathfrak{t}$. Thus p is also a T-relative equilibrium. □

1.3 Momentum maps

Definition 1.47. Let $\Phi : G \times P \to P$ be an action of the Lie group G on the symplectic manifold (P, ω). The action Φ is *canonical* iff $\Phi_g^* \omega = \omega$ for every $g \in G$. Then ω is *G-invariant* and P is a *G-symplectic manifold.*

An equivariant symplectomorphism $\Phi : P \to P'$ between G-symplectic spaces P and P' is an *isomorphism.*

Example 1.48. If Q is a G-manifold, the cotangent bundle T^*Q is a G-symplectic space.

Example 1.49. Let $\mathcal{O}_\mu$ be the coadjoint orbit of $\mu \in \mathfrak{g}^*$. If the coadjoint action is proper, $\mathcal{O}_\mu$ is a smooth submanifold diffeomorphic to G/G_μ. (If not, $\mathcal{O}_\mu$ is the image of an immersion $G/G_\mu \hookrightarrow \mathfrak{g}^*$ and we endow $\mathcal{O}_\mu$ with the differentiable structure of G/G_μ.) The *Kostant-Kirillov-Souriau symplectic form* $\omega_{\mathcal{O}_\mu}$ is given by

$$\omega_{\mathcal{O}_\mu}(\nu)(\xi_{\mathfrak{g}^*}(\nu), \eta_{\mathfrak{g}^*}(\nu)) = \langle \nu, [\xi, \eta] \rangle.$$

Since $T_\nu \mathcal{O}_\mu \simeq T_\nu \mathcal{O}_\nu \simeq \mathfrak{g}/\mathfrak{g}_\nu$, $\omega_{\mathcal{O}_\mu}$ is a well-defined 2-form on $\mathcal{O}_\mu$.

$$\begin{aligned}(\operatorname{Coad}_g^* \omega_{\mathcal{O}_\mu})(\nu)(\xi_{\mathfrak{g}^*}(\nu), \eta_{\mathfrak{g}^*}(\nu)) &= \omega_{\mathcal{O}_\mu}(\operatorname{Coad}_g \nu)((\operatorname{Ad}_g \xi)_{\mathfrak{g}^*}(\nu), (\operatorname{Ad}_g \eta)_{\mathfrak{g}^*})(\nu)) \\ &= \langle \nu, [\xi, \eta] \rangle,\end{aligned}$$

implies that $\omega_{\mathcal{O}_\mu}$ is G-invariant. Thus $L_{\xi_{\mathfrak{g}^*}} \omega_{\mathcal{O}_\mu} = 0$ holds for every $\xi \in \mathfrak{g}$. Inserting this into the contraction axiom for the Lie derivative yields

$$\xi_{\mathfrak{g}^*}(\omega_{\mathcal{O}_\mu}(\zeta_{\mathfrak{g}^*}(\nu), \eta_{\mathfrak{g}^*})) = \omega_{\mathcal{O}_\mu}([\xi_{\mathfrak{g}^*}, \zeta_{\mathfrak{g}^*}(\nu)], \eta_{\mathfrak{g}^*}) + \omega_{\mathcal{O}_\mu}(\zeta_{\mathfrak{g}^*}, [\xi_{\mathfrak{g}^*}, \eta_{\mathfrak{g}^*}(\nu)])$$

for $\xi, \zeta, \eta \in \mathfrak{g}$. If this again is inserted into the formula for $d\omega_{\mathcal{O}_\mu}$, the Jacobi identity for the Lie bracket on $\mathfrak{g}$ yields that $\omega_{\mathcal{O}_\mu}$ is closed. Since $\langle \nu, [\xi, \cdot] \rangle = \operatorname{coad}_\xi \nu = 0$ iff $\xi \in \mathfrak{g}_\nu$, the form $\omega_{\mathcal{O}_\mu}$ is non-degenerate. Thus $(\mathcal{O}_\mu, \omega_{\mathcal{O}_\mu})$ is a G-symplectic space.

Similarly we define a *G-symplectic representation*, which is a special case of a G-symplectic manifold:

Definition 1.50. Let G be a Lie group and V be G-representation together with a G-invariant symplectic form ω. Then we call V a *G-symplectic representation* (or just *symplectic representation*). A G-symplectic representation is *irreducible* if it contains no proper non-trivial G-symplectic subrepresentations. Two G-symplectic representations are *isomorphic* if there is a G-equivariant symplectic linear isomorphism that maps one into the other.

A G-symplectic manifold P together with a G-invariant Hamiltonian function $h : P \to \mathbb{R}$ form a *Hamiltonian system with G-symmetry.* The G-invariant symplectic form ω yields a G-equivariant isomorphism of vector bundles $\omega^\# : T^*P \to TP$ given by the inverse of the map $(p, X_p) \mapsto (p, \omega_p(X_p, \cdot))$. Since the 1-form $\mathrm{d}h : P \to T^*P$ is a G-equivariant section of the bundle map $T^*P \to P$, the Hamiltonian vector field

$$X_h = \omega^\# \circ \mathrm{d}h$$

is G-equivariant as well.

A fundamental concept of classical mechanics is that symmetries of the system give rise to conserved quantities:

For each $\xi \in \mathfrak{g}$, the form $\omega(\xi_P, \cdot) = i_{\xi_P}\omega$ is closed by Cartan's formula:

$$0 = \frac{\mathrm{d}}{\mathrm{d}t}\Phi^*_{\exp(t\xi)}\omega\bigg|_{t=0} = L_{\xi_P}\omega = i_{\xi_P}\mathrm{d}\omega + \mathrm{d}i_{\xi_P}\omega = \mathrm{d}i_{\xi_P}\omega.$$

Vector fields with this property are called *locally Hamiltonian*, since they are Hamiltonian on simply connected neighbourhoods. Let us assume that ξ_P is even globally Hamiltonian with Hamiltonian function $\mathbf{J}^\xi$, that is

$$\mathrm{d}\mathbf{J}^\xi = \omega(\xi_P, \cdot). \tag{1.1}$$

Theorem 1.51 (Noether's theorem)**.** *Let (P, ω, h) be a Hamiltonian system with G-symmetry. Suppose that $\mathbf{J}^\xi$ satisfies equation* (1.1) *for $\xi \in \mathfrak{g}$. Then $\mathbf{J}^\xi$ is constant along the trajectories of X_h.*

Proof.
$$X_h(\mathbf{J}^\xi) = \{h, \mathbf{J}^\xi\} = -\{\mathbf{J}^\xi, h\} = \xi_P(h) = 0.$$ □

Definition 1.52. A canonical action of a Lie group G on a symplectic manifold P is *weakly Hamiltonian* iff for each $\xi \in \mathfrak{g}$, the vector field ξ_P is globally Hamiltonian.

Let us suppose that G acts on P in a weakly Hamiltonian way. The functions $\mathbf{J}^\xi$ are defined by equation (1.1) up to a constant on each connected component of P. Choosing a basis $\xi_1, \dots, \xi_n$ of $\mathfrak{g}$ and functions $\mathbf{J}^{\xi_1}, \dots \mathbf{J}^{\xi_n}$, we may extend the definition linearly to $\mathfrak{g}$ and obtain a linear map $\mathfrak{g} \to C^\infty(P)$, where $\xi \mapsto \mathbf{J}^\xi$. This yields a map $\mathbf{J} : P \to \mathfrak{g}^*$, defined by

$$\langle \mathbf{J}(p), \xi \rangle = \mathbf{J}^\xi(p).$$

$\mathbf{J}$ is called a *momentum map*, since it can be considered as a generalization of the linear and angular momentum (see examples below). The definition of the momentum map goes back to Lie, see [MR99, section 11.2] for an overview of the history of the momentum map.

In this thesis, the existence of a momentum map will always be assumed. For generalizations in the case that no momentum map as defined above is available, consider [OR04(b)].

Given a canonical action of a Lie group G on a symplectic manifold P, each of the following conditions implies that the action is weakly Hamiltonian and thus a momentum map can be constructed:

1. If $H^1(P) = 0$, the closed vector fields $i_{\xi_P}\omega$ for $\xi \in \mathfrak{g}$ are exact and thus the action is weakly Hamiltonian.

2. Canonical actions of semi-simple Lie groups are always weakly Hamiltonian, see [GS84, chapter II, section 24]. A Lie group is *semi-simple* iff its Lie algebra has no non-trivial Abelian ideals. A compact Lie group is semi-simple iff its centre is finite (see [DK00, Corollary 3.6.3]).

3. If $\omega = \mathrm{d}\theta$ for a G-invariant 1-form θ, then by Cartan's formula
$$0 = L_{\xi_P}\theta = \mathrm{d}i_{\xi_P}\theta + i_{\xi_P}\mathrm{d}\theta$$
and thus $i_{\xi_P}\omega = \mathrm{d}i_{\xi_P}\theta$ is exact and the momentum map is given by
$$\langle \mathbf{J}(p), \xi\rangle = i_{\xi_P}\theta(p) = \theta_p(\xi_P(p)).$$

In the following, we suppose that P is connected. The momentum map is an equivariant map from the G-manifold P to $\mathfrak{g}^*$ with respect a G-action on $\mathfrak{g}^*$, which often coincides with the coadjoint action:

The map $\xi \mapsto \xi_P$ from $\mathfrak{g}$ to the space of vector fields on P satisfies $\Phi_g^*\xi_P = (\mathrm{Ad}_{g^{-1}}\xi)_P$:
$$\Phi_g^*\xi_P(p) = \frac{\mathrm{d}}{\mathrm{d}t}(\Phi_g^{-1} \circ \Phi_{\exp(t\xi)}\Phi_g)(p)\bigg|_{t=0} = (\mathrm{Ad}_{g^{-1}}\xi)_P(p).$$
Since $\xi_P = X_{\mathbf{J}^\xi}$ and $\Phi_g^* X_{\mathbf{J}^\xi} = X_{\mathbf{J}^\xi\circ\phi_g}$ by the G-invariance of ω, this yields
$$X_{\mathbf{J}^\xi\circ\phi_g} = X_{\mathbf{J}^{\mathrm{Ad}_{g^{-1}}\xi}}.$$
Thus, for fixed $g \in G$ and $\xi \in \mathfrak{g}$, the function
$$\sigma(g) = \mathbf{J}\circ\phi_g - \mathrm{Coad}_g\,\mathbf{J} : P \to \mathfrak{g}^*$$
is constant.

Definition 1.53. If there is a choice of the momentum map such that σ vanishes, we call the action *Hamiltonian.*

Remark 1.54. The definition of an Hamiltonian action is quite inconsistent in the literature. Sometimes the term *Hamiltonian action* is used for weakly Hamiltonian actions as defined above.

A weakly Hamiltonian G-action on the connected manifold P is Hamiltonian if one of the following condition holds:

1. P is compact: ω^n defines a G-invariant volume form on P, where $\dim P = 2n$. If we require $\int_P \mathbf{J}^\xi(p)\omega^n = 0$, we obtain a unique choice of $\mathbf{J}^\xi$, which is linear in ξ. Then for any $g \in G$, the integral over P vanishes for the two functions $\mathbf{J}^\xi \circ \Phi_g$ and $\mathbf{J}^{\mathrm{Ad}_{g^{-1}}\xi}$, which have the same derivative. Thus they coincide.

2. G is compact, and $\mathbf{J}$ is a momentum map: Averaging over G yields an equivariant momentum map $\bar{\mathbf{J}}$:

$$\bar{\mathbf{J}}(p) := \int_{g \in G} \operatorname{Coad} \mathbf{J}(g^{-1}p).$$

   This argument is taken from [Mon97].

3. G is semi-simple, as in condition 2 above. Again we refer to [GS84, chapter II., section 24].

4. $\omega = \mathrm{d}\theta$ for a G-invariant 1-form θ (condition 3 above): Since $\mathbf{J}^\xi = i_{\xi_P}\theta$ and θ is G-invariant

$$\mathbf{J}^\xi(gp) = \theta_{gp}(\xi_P(gp)) = (\Phi_g^*\theta)(\Phi_g^*\xi_P)(p) = \theta(\operatorname{Ad}_{g^{-1}}\xi)_P(p) = \mathbf{J}^{\operatorname{Ad}_{g^{-1}}\xi}(p).$$

Since in this thesis the emphasis is placed on action of compact groups, the momentum map will be assumed to be equivariant with respect to the coadjoint action. Nevertheless in general, $\mathbf{J}$ is equivariant with respect to the action on $\mathfrak{g}^*$ given by

$$(g, \mu) \mapsto \operatorname{coad}_g \mu + \sigma(g).$$

This action has been introduced by Souriau, [Sou69]). A simple calculation shows that this is indeed an action, see for example [MR99, Proposition 12.3.1]. If the coadjoint action is replaced by this modified action, many of the results for Hamiltonian actions still hold for weakly Hamiltonian actions.

Now, we calculate the momentum map for some basic examples:

Example 1.55. Let $P = T^*Q$ be the cotangent bundle of the G-manifold Q. Then $\omega = \mathrm{d}\theta$ for the G-invariant canonical 1-form θ. Since for $(q, \alpha_q) \in T^*Q$ we have

$$\mathrm{d}\pi\xi_P(q, \alpha_q) = \xi_Q(q),$$

this yields

$$\langle \mathbf{J}(q, \alpha_q), \xi \rangle = \alpha_q(\xi_Q(q)).$$

In particular, we obtain:

Example 1.56. Let $G = \mathbb{R}^3$ act by translations on $Q = \mathbb{R}^3$. Then $\mathfrak{g}^* = \mathfrak{g} = \mathbb{R}^3$, $\xi_Q(q) = \xi$, and the momentum map for the action on $T^*Q = \mathbb{R}^3 \times \mathbb{R}^3$ is given by

$$\langle \mathbf{J}(q, p), \xi \rangle = \langle p, \xi \rangle.$$

Thus $\mathbf{J}(q, p) = p$ is just the linear momentum.

Example 1.57. Let $Q = \mathbb{R}^3$ be the standard representation of $\mathrm{SO}(3) = G$. Then $\mathfrak{g}^* = \mathfrak{g} = \mathfrak{so}(3)$ consists of the Lie algebra of skew symmetric linear maps, which may be identified with $\mathbb{R}^3$ via the isomorphism $\xi \in \mathbb{R}^3 \mapsto (\xi \times \cdot)$, where $\times$ denotes the cross product on $\mathbb{R}^3$. Then

$$\langle \mathbf{J}(q, p), \xi \rangle = \langle p, \xi \times q \rangle = \langle \xi, q \times p \rangle,$$

and thus $\mathbf{J}(q, p) = q \times p$ is the angular momentum.

Given a momentum map for the G-action, relative equilibria of the Hamiltonian vector field are considered as critical points in different ways:

p is a relative equilibrium with generator $\xi \in \mathfrak{g}$ iff $X_h - \xi_P(p) = 0$. Since $\mathbf{J}^\xi$ is a Hamiltonian function for the Hamiltonian vector field ξ_P, this is equivalent to $\mathrm{d}(h - \mathbf{J}^\xi)(p) = 0$.

Definition 1.58. The functions

$$h_\xi := h - \mathbf{J}^\xi$$

are called *augmented Hamiltonian functions.*

Another approach is to fix the value of the momentum: By Noether's theorem, the momentum level sets $\mathbf{J}^{-1}(\mu)$ for $\mu \in \mathfrak{g}^*$ are invariant subsets. Thus each set $\mathbf{J}^{-1}(\mu)$ may be considered as a dynamical system on its own. In general, this is difficult, since the momentum level sets are not necessarily manifolds. For free actions the momentum map is a submersion, as follows from the following lemma:

Lemma 1.59. *For any $p \in P$, the kernel and image of* $\mathrm{d}\mathbf{J}(p)$ *are given by*

$$\ker \mathrm{d}\mathbf{J}(p) = (\mathfrak{g} \cdot p)^{\perp_\omega},$$
$$\operatorname{im} \mathrm{d}\mathbf{J}(p) = \operatorname{ann} \mathfrak{g}_p,$$

where $V^{\perp_\omega}$ denotes the ω_p-orthogonal complement of a vector subspace V of T_pP and ann *denotes the annihilator.*

Proof. If $v \in T_pP$,

$$\mathrm{d}\langle \mathbf{J}(p), \xi \rangle v = 0 \Leftrightarrow \omega_p(v, \xi \cdot p) \quad \forall \xi \in \mathfrak{g}.$$

Thus $\ker \mathrm{d}\mathbf{J}(p) = (\mathfrak{g} \cdot p)^{\perp_\omega}$. Since $\operatorname{im} \mathrm{d}\mathbf{J}(p) \subset \operatorname{ann} \mathfrak{g}_p$ holds obviously,

$$\dim \ker \mathrm{d}\mathbf{J}(p) = \dim T_pP - \dim(\mathfrak{g} \cdot p) = \dim T_pP - \dim \mathfrak{g} + \dim \mathfrak{g}_p$$

yields the equality. □

Thus $\mathbf{J}^{-1}(\mu)$ is a manifold if the action is free. Then $p \in P$ is a relative equilibrium iff p is a critical point of the restriction $h\big|_{\mathbf{J}^{-1}(\mu)}$: This is equivalent to the existence of a Lagrange multiplier $\xi \in \mathfrak{g}$ such that $\mathrm{d}h(p) = \langle \mathrm{d}\mathbf{J}(p), \xi \rangle$.

Suppose in addition that the action is proper. Then the *(Marsden-Weinstein) reduced space*

$$P_\mu := \mathbf{J}^{-1}(\mu) \big/ G_\mu$$

is a manifold, too. Let $\pi : \mathbf{J}^{-1}(\mu) \to \mathbf{J}^{-1}(\mu) \big/ G_\mu$ be the projection and ω_μ denote the restriction of ω to the tangent space of $\mathbf{J}^{-1}(\mu)$. The form $\omega_\mu(\mathrm{d}\pi X, \mathrm{d}\pi Y) = \omega(X, Y)$ is well-defined and symplectic on P_μ. (We postpone the proof to the next chapter.) The restricted Hamiltonian function induces a function h_μ on P_μ called the *reduced Hamiltonian function.* The associated Hamiltonian vector field X_{h_μ} is called the *reduced (Hamiltonian vector) field.* This way the equations of motion may be reduced to the ones of Hamiltonian systems of smaller dimension which carry all information except from the motion along the group orbit. The formalization of this process first appeared in [MW74]. In particular, $p \in P$ with $\mathbf{J}(p) = \mu$ is a relative equilibrium iff $[p] := \pi(p)$ is a critical point of h_μ, equivalently iff $[p]$ is an equilibrium of the reduced field.

Alternatively, the reduced spaces may be defined by

$$P_\mu := \mathbf{J}^{-1}(\mathcal{O}_\mu)\big/G \, ,$$

where $\mathcal{O}_\mu$ denotes the coadjoint orbit of μ. The two definitions yield isomorphic symplectic spaces, see [OR04(b), Theorem 6.4.1].

If the G-action is not free, the structures of the momentum level sets and the reduced spaces are in general complicated. Nevertheless, the relative equilibria may be considered as critical points if G is compact:

The following lemma yields that the fixed point submanifolds P^H for an isotropy subgroup $H \subset G$ together with the restricted invariant Hamiltonian function form a Hamiltonian system with L-symmetry, where $L := N(H)\big/H$ with Lie algebra $\mathfrak{l}$:

Lemma 1.60. *Let $G \times P \to P$ be a proper Hamiltonian action and $H \subset G$ be a closed subgroup with Lie algebra $\mathfrak{h}$. Then P^H is an L-symplectic manifold. If $\mathfrak{n}(H)$ denotes the Lie algebra of $N(H)$ and $\mathbb{P} : \mathfrak{g}^* \to \mathfrak{n}(H)^*$ is the projection,*

$$\mathbb{P} \circ \mathbf{J}(P^H) \subset \operatorname{ann} \mathfrak{h} = \mathfrak{l}.$$

Thus $\mathbb{P} \circ \mathbf{J}$ is a momentum map for the L-action.

Proof. P^H is a manifold, since the action is proper. For $p \in P^H$, we have $(T_pP)^H = T_p(P^H)$ and $(T_p^*P)^H = T_p^*(P^H)$. By equivariance of $\omega^\#$, we have $\omega^\# : T_p(P^H) \to T_p^*(P^H)$ and $(\omega^\#)^{-1} : T_p^*(P^H) \to T_p(P^H)$. Thus P^H is a symplectic submanifold. $\mathbb{P} \circ \mathbf{J}(P^H) \subset \operatorname{ann} \mathfrak{h}$ is obvious. □

As Sjamaar and Lerman have shown ([SL91]), if G is compact, the momentum level sets $\mathbf{J}^{-1}(\mu)$ are stratified by the connected components $\mathbf{J}^{-1}(\mu)_\tau$ of the same isotropy type τ, and their images in P_μ form symplectic manifolds. If $\tau = (H)$, the image of $\mathbf{J}^{-1}(\mu)_\tau$ in P_μ may be identified with the quotient of $(\mathbb{P} \circ \mathbf{J})^{-1}(\mathbb{P}(\mu)) \subset P_H$ with respect to the $L_{\mathbb{P}(\mu)}$-action. We obtain $\mathbf{J}^{-1}(\mu)_\tau\big/G_\mu \simeq (P_H)_{\mathbb{P}(\mu)}$, where the right hand side is a symplectic manifold, since the L-action on P_H is free. $p \in P_H \cap \mathbf{J}^{-1}(\mu)$ is a relative equilibrium for X_h iff it is a relative equilibrium for the Hamiltonian system on P_H. Moreover, $\mathrm{d}h(p) = 0$ iff this holds for the projection of $\mathrm{d}h(p)$ to $T_p^*(P_H)$. Thus p is a relative equilibrium iff it is a critical point of the restriction of h to the corresponding stratum in $\mathbf{J}^{-1}(\mu)_\tau$, or equivalently iff $[p]$ is a critical point of the restriction of h_μ to the corresponding stratum in P_μ.

For a Hamiltonian action, we obtain a further restriction on the generator of a relative equilibrium from Noether's theorem and the equivariance of the momentum map: If p is a relative equilibrium with generator ξ and momentum μ,

$$0 = \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{J}(\exp(t\xi)p)\bigg|_{t=0} = \operatorname{coad}_\xi \mu.$$

Chapter 2

Local theory

Here we consider the Hamiltonian analogues to some methods to investigate the dynamics in equivariant dynamical systems near a given relative equilibrium. The G-action is always to assumed to be proper. We start with the characterization of the G-symplectic structure near a G-orbit in the first two sections:

In section 2.1, we cite an equivariant version of Darboux's theorem.

A symplectic version of the Slice Theorem is presented in section 2.2. This yields a normal form for a tubular neighbourhood of the relative equilibrium. As shown by Krupa ([Kru90]), a general vector field that is equivariant with respect to the action of a compact group may be decomposed as the sum of the *tangential vector field*, which is tangent to the orbit, and the *normal vector field*. Similarly in section 2.3, the Hamiltonian vector field is written in the coordinates given by the symplectic normal form. Such a normal form exists in general for a proper group action (see for instance [OR04(b)].) It has, however, a simpler form if $\mathfrak{g}$ has an inner product that is invariant with respect to the isotropy subgroup G_μ of the momentum μ of some element p of the orbit. We will impose this condition, which holds in particular for compact group actions.

The presentation in sections 2.2 and 2.3 follows [OR04(b)].

From the normal form of the Hamiltonian vector field, we obtain two equations that characterize relative equilibria. Some solutions of the second equation correspond to relative equilibria near an equilibrium in a Hamiltonian system with G_p-symmetry. One of the aims of this thesis is to understand this special case, in which the symmetry is given by the action of a compact group and we are in a small neighbourhood of an equilibrium. Section 2.4 is devoted to this aim. Here, a method to reduce the search of relative equilibria to the kernels of the Hessians at the equilibrium of the augmented Hamiltonians is illustrated. It can be seen as a special form of Lyapunov-Schmidt reduction, which is a standard technique of (equivariant) bifurcation theory.

2.1 Equivariant Darboux theorem

The equivariant version of Darboux's theorem states in particular, that G-symplectic manifolds are locally isomorphic to G-symplectic representations near fixed points of the G-action if the action is proper. It is a special case of the

Darboux-Weinstein theorem, see [GS84, Theorem 22.2].

Theorem 2.1 (Equivariant Darboux theorem, [OR04(b), Theorem 7.3.1]). *Let G act properly on the smooth manifold P. Suppose that there are two G-invariant symplectic forms ω_0 and ω_1 and a point $p \in P$ with $\omega_0\big|_{Gp} = \omega_1\big|_{Gp}$. Then there are G-invariant neighbourhoods U_0 and U_1 of Gp and a G-equivariant diffeomorphism $\phi : U_0 \to U_1$ such that $\phi\big|_{Gp} = \mathbb{1}_{Gp}$ and $\phi^*\omega_1 = \omega_0$.*

As Dellnitz and Melbourne point out in [DM93b], there are incorrect equivariant generalizations of the Darboux theorem stated in the literature, for example in [GS84, Theorem 22.2]: In contrast to the non-symmetric case, it is not true in general that all G-invariant symplectic forms on a given G-representation can be transformed into each other by an equivariant linear coordinate change. We will come back to this in chapter 3.

In particular, two G-symplectic manifolds with a fixed point of the G-action on each one such that the underlying G-manifolds are locally isomorphic near the fixed points are not in general locally isomorphic as G-symplectic manifolds.

2.2 Marle-Guillemin-Sternberg normal form

Given a proper Hamiltonian action of a Lie group G on a symplectic manifold P, Marle ([Mar85]) and Guillemin and Sternberg ([GS84]) classify the local isomorphism classes of G-invariant neighbourhoods of G-orbits in P:

The local structure of the G-manifold P is characterized by the Slice Theorem 1.28, which states the existence of a tubular neighbourhood of $p \in P$ isomorphic to the normal form $G \times_{G_p} V$. To describe the G-symplectic space P locally, we have to include the symplectic form and the corresponding momentum map.

We start with a splitting of the tangent space, which in this general formulation first appeared in [MRS88]. It is often called Witt-Artin decomposition.[1]

In the Slice Theorem, we use a splitting of the tangent space T_pP into the G_p-invariant spaces $\mathfrak{g} \cdot p$ and V. For the Hamiltonian version, we choose a particular form of V:

Since level sets of the momentum map are invariant sets of the flow, the Hamiltonian vector field is tangent to the level sets, hence $X_h(p) \in \ker \mathrm{d}\mathbf{J}(p)$. We fix a G_p-invariant inner product and split the tangent space into $\ker \mathrm{d}\mathbf{J}(p)$ and its normal space $(\ker \mathrm{d}\mathbf{J}(p))^\perp$. Then we consider the intersections $T_0 = \ker \mathrm{d}\mathbf{J}(p) \cap \mathfrak{g} \cdot p$ and $T_1 = (\ker \mathrm{d}\mathbf{J}(p))^\perp \cap \mathfrak{g} \cdot p$ and their complements N_1 and N_0 within $\ker \mathrm{d}\mathbf{J}(p)$ and $(\ker \mathrm{d}\mathbf{J}(p))^\perp$ respectively. We obtain the G_p-invariant orthogonal splitting

$$T_pP = T_0 \oplus T_1 \oplus N_0 \oplus N_1, \tag{2.1}$$

where $T_0 \oplus T_1 = \mathfrak{g} \cdot p$, $T_0 \oplus N_1 = \ker \mathrm{d}\mathbf{J}(p)$, and $T_1 \oplus N_0 = (\ker \mathrm{d}\mathbf{J}(p))^\perp$. (The notation is as in [PR00].) In the following, we often denote $N_0 \oplus N_1$ by V since this space is a G_p-invariant complement of $\mathfrak{g} \cdot p$ as in the Slice Theorem.

[1] The name refers to the decomposition of a symplectic vector space (V, ω) defined by any subspace $U \subset V$ into pairwise ω-orthogonal symplectic subspaces: V can be written as $V = X \oplus Y \oplus Z$, where X is a complement of $U \cap U^{\perp\omega}$ within U, Y is a complement of $U \cap U^{\perp\omega}$ within $U^{\perp\omega}$ and Z can be written as the sum of $U \cap U^{\perp\omega}$ and a complement of $U + U^{\perp\omega}$. This splitting in principle occurs in Artin's theory of vector spaces with a non-degenerate symmetric or skew-symmetric form ([Art57]), which is based on Witt's investigation of symmetric forms ([Wit37]).

If $\mathbf{J}(p) = \mu$,

$$T_0 = \mathfrak{g}_\mu \cdot p \simeq \mathfrak{g}_\mu \big/ \mathfrak{g}_p \quad \text{and}$$
$$T_1 \simeq \mathfrak{g} \big/ \mathfrak{g}_\mu \, .$$

For $\xi \in \mathfrak{g}$ and $\mu \in \mathfrak{g}^*$, let $\mathcal{O}_\xi$ and $\mathcal{O}_\mu$ denote the G-orbits with respect to the adjoint and coadjoint action respectively.

Lemma 2.2. $T_\mu \mathcal{O}_\mu = \operatorname{ann} \mathfrak{g}_\mu$.

Proof. Obviously, these two spaces are of the same dimension. Moreover, for $\eta \in \mathfrak{g}_\mu$ and $\xi \in \mathfrak{g}$, we obtain

$$\langle \operatorname{coad}_\xi \mu, \eta \rangle = -\langle \mu, [\xi, \eta] \rangle = \langle \mu, [\eta, \xi] \rangle = \langle - \operatorname{coad}_\eta \mu, \xi \rangle = 0. \qquad \square$$

Remark 2.3. If G is compact, we identify $\mathfrak{g}$ and $\mathfrak{g}^*$ via a G-invariant inner product. For any subspace $\mathfrak{k} \subset \mathfrak{g}$, the dual space $\mathfrak{k}^*$ is identified with the image of $\mathfrak{k}$ under this isomorphism $\mathfrak{g} \simeq \mathfrak{g}^*$. In this case, Lemma 2.2 can be formulated as follows:

Lemma 2.4. *Suppose that G is compact. With respect to any choice of a G-invariant inner product on $\mathfrak{g}$,*

$$\mathfrak{g}_\xi = T_\xi \mathcal{O}_\xi^\perp \quad \text{and} \quad \mathfrak{g}_\mu^* = T_\mu \mathcal{O}_\mu^\perp .$$

In general, there is no G-invariant inner product on $\mathfrak{g}$ and $\mathfrak{g}^*$. Anyhow, since G_p is compact, there is a G_p-invariant inner product. The equivariance of $\mathbf{J}$ implies $G_p \subset G_\mu$ and thus $\mathfrak{g}_p \subset \mathfrak{g}_\mu$. From the G_p-invariant inner product, we obtain a G_p-invariant splitting $\mathfrak{g}_\mu = \mathfrak{m} \oplus \mathfrak{g}_p$. This yields a corresponding splitting $\mathfrak{g}_\mu^* = \mathfrak{m}^* \oplus \mathfrak{g}_p^*$.

Lemma 2.5.
1. *The restriction of $\mathrm{d}\mathbf{J}(p)$ composed with the projection to the quotient by $\mathfrak{g}_p$ yields an isomorphism $N_0 \simeq (\mathfrak{g}_\mu \big/ \mathfrak{g}_p)^* \simeq \mathfrak{m}^*$ between G_p-representations, where the G_p-action on $(\mathfrak{g}_\mu \big/ \mathfrak{g}_p)^*$ is given by the quotient of the coadjoint action.*
2. *N_1 and T_1 are symplectic and ω-orthogonal to each other. T_0 is an isotropic subspace of the ω-orthogonal complement of $N_1 \oplus T_1$.*
3. *$(\mathfrak{g} \big/ \mathfrak{g}_\mu, \omega_\mu)$ with $\omega_\mu = \langle \mu, [\cdot, \cdot] \rangle$ is a symplectic space isomorphic to T_1.*
4. *For some suitable choice of the inner product, $T_0 \oplus N_0$ is a symplectic subspace of $T_p P$, which is ω-orthogonal to $T_1 \oplus N_1$ and isomorphic to $\mathfrak{g}_\mu \big/ \mathfrak{g}_p \oplus (\mathfrak{g}_\mu \big/ \mathfrak{g}_p)^*$ together with the canonical symplectic form.*

Proof. 1. Since the restriction $\mathrm{d}\mathbf{J}(p)\big|_{T_1 \oplus N_0}$ is injective,

$$T_1 \oplus N_0 \simeq \operatorname{im} \mathrm{d}\mathbf{J}(p) = \operatorname{ann} \mathfrak{g}_p$$

by Lemma 1.59. Moreover,

$$\mathrm{d}\mathbf{J}(p) T_1 = T_\mu \mathcal{O}_\mu = \operatorname{ann} \mathfrak{g}_\mu .$$

Thus, $\mathrm{d}\mathbf{J}(p)N_0$ forms a G_p-invariant complement of $\operatorname{ann}\mathfrak{g}_\mu$ within $\operatorname{ann}\mathfrak{g}_p$. Hence

$$N_0 \simeq \mathrm{d}\mathbf{J}(p)N_0 \simeq (\mathfrak{g}_\mu/\mathfrak{g}_p)^* \simeq \mathfrak{m}^*.$$

2. Since $\ker \mathrm{d}\mathbf{J}(P) = (\mathfrak{g}\cdot p)^{\perp_\omega}$, the space $T_0 = \mathfrak{g}\cdot p \cap (\mathfrak{g}\cdot p)^{\perp_\omega}$ is isotropic, $T_1 \simeq \mathfrak{g}\cdot p/T_0$ and $N_1 \simeq (\mathfrak{g}\cdot p)^{\perp_\omega}/T_0$ are symplectic, $T_1 \perp_\omega N_1$, and $T_0 \subset (T_1 \oplus N_1)^{\perp_\omega}$.

3. ω_μ is the Kostant-Kirillov-Souriau symplectic form on $T_\mu\mathcal{O}_\mu \simeq \mathfrak{g}/\mathfrak{g}_\mu$, see Example 1.49. The isomorphism is given by $\mathrm{d}\mathbf{J}(p)$:

$$\omega_p(\xi\cdot p, \eta\cdot p) = \langle \mathrm{d}\mathbf{J}(p)(\eta\cdot p), \xi\rangle = \langle \operatorname{coad}_\eta \mathbf{J}(p), \xi\rangle = \langle \mu, [-\eta, \xi]\rangle.$$

4. Choose a G_p-invariant Lagrangian complement of T_0 within $(T_1 \oplus N_1)^{\perp_\omega}$ for N_0. Such a complement exists: There is a G_p-invariant inner product on $(T_1 \oplus N_1)^{\perp_\omega}$ such that ω is represented by an orthogonal skew-symmetric endomorphism J which is obviously G_p-equivariant. Then $N_0 = JT_0$ is a possible choice. $T_0 \oplus N_0$ is symplectic, since $(T_1 \oplus N_1)^{\perp_\omega}$ is symplectic.

   The isomorphism $\mathrm{d}\mathbf{J}(p) : N_0 \to (\mathfrak{g}_\mu/\mathfrak{g}_p)^*$ corresponds to the isomorphism $T_0 \to T_0^*$ defined by $\omega(p)\big|_{T_0}$: For $[\xi] \in (\mathfrak{g}_\mu/\mathfrak{g}_p)$ and $n \in N_0$,

$$\langle \mathrm{d}\mathbf{J}(p)n, [\xi]\rangle = \omega(p)(\xi\cdot p, n).$$

   Hence the symplectic space $T_0 \oplus N_0$ is isomorphic to $\mathfrak{g}_\mu/\mathfrak{g}_p \oplus (\mathfrak{g}_\mu/\mathfrak{g}_p)^*$ together with the canonical symplectic form. □

Remark 2.6. In the case of a free action, N_1 is obviously isomorphic to the tangent space $T_{[p]}P_\mu$ of the reduced space and $\pi^*\omega$ coincides with the restriction of ω to N_1. Thus $(P_\mu, \pi^*\omega)$ is indeed a symplectic manifold.

In the following we will always assume that in addition to the G_p-invariant splitting $\mathfrak{g}_\mu = \mathfrak{g}_p \oplus \mathfrak{m}$, there is a G_μ-invariant splitting $\mathfrak{g} = \mathfrak{g}_\mu \oplus \mathfrak{q}$. If we set $\mathfrak{g}_\mu^* := \operatorname{ann}\mathfrak{q}$ and $\mathfrak{q}^* = \operatorname{ann}\mathfrak{g}_\mu$, we obtain a G_μ-invariant splitting $\mathfrak{g}^* = \mathfrak{g}_\mu^* \oplus \mathfrak{q}^*$.

The Slice Theorem together with the Artin-Witt decomposition of the tangent space gives rise to a symplectic normal form of the tubular neighbourhood: By the equivariant Darboux Theorem 2.1, the isomorphism class of the tubular neighbourhood as a symplectic space is determined by the restriction of ω to the G-orbit of p. In principle, to obtain a normal form, it is sufficient to identify the restriction $TP\big|_V$ of the tangent space to the slice with $\mathfrak{g}/\mathfrak{g}_p \times V$ via a local section of the projection $G \to G/G_p$ and to extend the form ω_p, which is computed in the Artin-Witt decomposition theorem, constantly to $TP\big|_V$ and equivariantly to $T(G \times_{G_p} V)$. This choice of normal form would depend on the choice of the section. It is more natural and more practicable to have a normal form for ω that comes from a G_p-invariant closed two-form on $G \times V$ which is degenerate on the tangent spaces to the G_p-orbits. Then ω may be expressed with respect to coordinates $(g, \rho, v) \in G \times \mathfrak{m}^* \times N_1 = G \times V$. For convenience, we denote N_1 by N from now on.

To construct such a natural normal form for ω, we start with the subbundle of $G \times_{G_p} V$ over the G_μ-orbit of p consisting of the copies of $N_0 \simeq \mathfrak{m}^* \simeq (\mathfrak{g}_\mu / \mathfrak{g}_p)^*$. We obtain the symplectic bundle

$$G_\mu \times_{G_p} (\mathfrak{g}_\mu / \mathfrak{g}_p)^*,$$

which is isomorphic to $T^*(G_\mu / G_p)$: For $\xi \in T_e G$, $\mu \in T_e^* G$ and $g \in G$, e abbreviate $\mathrm{d}\Phi_g \xi \in TG$ by $g \cdot \xi$ and $\mathrm{d}\Phi^*_{g^{-1}} \mu \in T^*G$ by $g \cdot \mu$.

For $h \in G$, the derivative of the right multiplication

$$\begin{aligned} R_h : G &\to G \\ g &\mapsto gh^{-1} \end{aligned}$$

is given by $\mathrm{d}R_h(g \cdot \xi) = gh^{-1} \cdot \mathrm{Ad}_h \xi$, and hence for $g \cdot \mu \in T^*G$, we obtain

$$((\mathrm{d}R_{h^{-1}})^*(g \cdot \mu))(gh^{-1} \cdot \xi) = (g \cdot \mu)(\mathrm{d}R_{h^{-1}}(gh^{-1} \cdot \xi)) = (g \cdot \mu)(g \cdot \mathrm{Ad}_{h^{-1}} \xi).$$

Thus $(\mathrm{d}R_{h^{-1}})^*(g \cdot \mu) = gh^{-1} \cdot \mathrm{Coad}_h \mu$. Therefore there is a submersion

$$G_\mu \times (\mathfrak{g}_\mu / \mathfrak{g}_p)^* \to T^*(G_\mu / G_p),$$

such that for $g \in G_\mu$ and $\mu \in (\mathfrak{g}_\mu / \mathfrak{g}_p)^* = \operatorname{ann} \mathfrak{g}_p$, all elements of the form $(gh^{-1}, \mathrm{Coad}_h \mu)$ with $h \in G_p$ are identified.

Hence a natural choice for the restriction of ω to this subbundle is the canonical symplectic form of $T^*(G_\mu / G_p)$, whose restriction to the zero-section coincides with the symplectic form determined by the Artin-Witt decomposition theorem.

The cotangent space $T^*(G_\mu / G_p)$ is isomorphic to the reduced space $(T^*G_\mu)_0$ with respect to the right G_p-action and the corresponding momentum map, see for example the last theorem in [Arn78, appendix 5 B]. This is a motivation to consider the bundle $T^*G_\mu \simeq G_\mu \times \mathfrak{g}_\mu^*$. By invariant extension and by adding forms on new parts of the tangent space, we will obtain a symplectic form defined on $G \times \mathfrak{g}_\mu^* \times N$, which is invariant with respect to a suitable G_p-action, such that $G \times_{G_p} (\mathfrak{m}^* \oplus N)$ is isomorphic to the reduced space for the momentum value $0 \in \mathfrak{g}_p^*$:

Let ω_0 be the restriction of the canonical symplectic form of $T^*G = G \times \mathfrak{g}$ to $G \times \mathfrak{g}_\mu^*$. Then the 2-form ω_0 extends the the canonical symplectic form of T^*G_μ and is G-invariant with respect to the left G-action on G and trivial action on $\mathfrak{g}_\mu^*$.

The form ω_0 is given by

$$\omega_0(g,\nu)((g \cdot \xi, \rho), (g \cdot \eta, \sigma)) = \langle \sigma, \xi \rangle - \langle \rho, \eta \rangle + \langle \nu, [\xi, \eta] \rangle,$$

see [OR04(b), Example 4.1.20]. (Recall that $\mathfrak{g}_\mu^*$ is identified with the annihilator of $\mathfrak{q}$.)

For each tangent space $T_g(gG_\mu)$ to a coset of G_μ, the space $(T_g(gG_\mu) \times \mathfrak{g}_\mu^*, \omega_0)$ is symplectic. Clearly, ω_0 vanishes on the subspace $(g \cdot \mathfrak{q}) \times \{0\} \in T_{(g,0)}(G \times \mathfrak{g}_\mu^*)$.

Moreover, for points $p = (g, 0) \in G \times \mathfrak{g}_\mu^*$ of the zero section, the space $\mathfrak{q} \cdot p \subset T_p(G \times \mathfrak{g}_\mu^*)$ corresponds to the space $T_1 = \mathfrak{q} \cdot p \subset T_{(g,\nu)}(G \times \mathfrak{g}_\mu^*)$ of the Artin-Witt decomposition, which is symplectic with the Kostant-Kirillov-Souriau symplectic form $\omega_1(\xi \cdot p, \eta \cdot p) = \mu([\xi, \eta])$. If we define the extension ω_1 to $G \times \mathfrak{g}_\mu^*$ via

$$\omega_1(g, \nu)((g \cdot \xi, \rho), (g \cdot \eta, \sigma)) = \mu([\xi, \eta]),$$

the subspace $g \cdot \mathfrak{q} \times \{0\} \in T_{(0,\nu)}(G \times \mathfrak{g}_\mu^*)$ together with ω_1 is symplectic. Since ω_1 vanishes on $T_g(gG_\mu) \times \mathfrak{g}_\mu^*$, the form $\Omega = \omega_0 + \omega_1$ is non-degenerate on the zero section of the vector bundle $G \times \mathfrak{g}_\mu^* \to G$. Therefore this holds as well on a neighbourhood of the zero section. Ω is closed, since ω_0 and ω_1 are closed. Thus, Ω is symplectic. Since for $h \in G$, the derivative of the right multiplication

$$\begin{aligned} R_h : G &\to G \\ g &\mapsto gh^{-1} \end{aligned}$$

is given by $\mathrm{d}R_h(g \cdot \xi) = gh^{-1} \cdot \mathrm{Ad}_h \xi$ and $G_p \subset G_\mu$, the 2-form Ω is invariant with respect to the G_p-action $\mathcal{R}$ with

$$\mathcal{R}(h)(g, \nu) = (gh^{-1}, \mathrm{coad}_h \nu) \quad \text{for } h \in G_p, g \in g, \nu \in \mathfrak{g}_\mu^*.$$

A corresponding momentum map $\mathbf{J}_\mathcal{R} : G \times \mathfrak{g}_\mu^* \to \mathfrak{g}_p^*$ is given by

$$\mathbf{J}_\mathcal{R}(g, \nu) = -\mathbb{P}_{\mathfrak{g}_p^*}\nu,$$

where $\mathbb{P}_{\mathfrak{g}_p^*}$ denotes the projection to $\mathfrak{g}_p^*$: The form ω_1 vanishes on the tangent spaces to the G_p-orbits. Hence for $\xi \in \mathfrak{g}_p$,

$$\begin{aligned} \langle \mathrm{d}\mathbf{J}_\mathcal{R}(g, \nu)(g \cdot \eta, \sigma), \xi \rangle &= \omega_0(g, \nu)((g \cdot (-\xi), \mathrm{coad}_\xi \nu), (g \cdot \eta, \sigma)) \\ &= \langle \sigma, -\xi \rangle - \langle \mathrm{coad}_\xi \nu, \eta \rangle + \langle \nu, [-\xi, \eta] \rangle = \langle -\sigma, \xi \rangle. \end{aligned}$$

Thus, $\mathrm{d}\mathbf{J}_\mathcal{R}(g, \nu)(g \cdot \eta, \sigma) = -\mathbb{P}_{\mathfrak{g}_p^*}\sigma$. Hence $\mathbf{J}_\mathcal{R}$ is also given by $-\mathbb{P}_{\mathfrak{g}_p^*}$.

Next, we consider the product action of G_p on $G \times \mathfrak{g}_\mu^* \times N$ and add the G_p-invariant symplectic form ω_N defined on N to Ω to obtain an invariant symplectic form on $G \times \mathfrak{g}_\mu^* \times N$. The momentum map for the product action is given by $\mathbf{K} := \mathbf{J}_\mathcal{R} + \mathbf{J}_N$, where $\mathbf{J}_N : N \to \mathfrak{g}_p$ denotes the momentum map of N. The space $G \times \mathfrak{m}^* \times N$ may be identified with $\mathbf{K}^{-1}(0)$ via the diffeomorphism

$$\begin{aligned} l : G \times \mathfrak{m}^* \times N &\to \mathbf{K}^{-1}(0) \subset G \times \mathfrak{m}^* \times \mathfrak{g}_p^* \times N \\ (g, \rho, v) &\mapsto (g, \rho, \mathbf{J}_N(v), v). \end{aligned}$$

Thus, the space $Y := G \times_{G_p} (\mathfrak{m}^* \times N)$ is diffeomorphic to the reduced space $\mathbf{K}^{-1}(0)/G_p$. Since the G_p-action on $G \times \mathfrak{m}^* \times \mathfrak{g}_p^* \times N$ is free, the reduced space is a symplectic manifold. Hence Y is a symplectic manifold with the induced symplectic form

$$\begin{aligned} &\omega_Y([g, \rho, v])(\mathrm{d}\pi(g \cdot \xi_1, \dot\rho_1, \dot v_1), (\mathrm{d}\pi(g \cdot \xi_2, \dot\rho_2, \dot v_2)) \\ &= \langle \dot\rho_2 + \mathrm{d}\mathbf{J}_N(v)\dot v_2, \xi_1 \rangle - \langle \dot\rho_1 + \mathrm{d}\mathbf{J}_N(v)\dot v_1, \xi_2 \rangle \\ &\quad + \langle \mu + \rho + \mathbf{J}_N(v), [\xi_1, \xi_2] \rangle + \omega_N(\dot v_1, \dot v_2), \end{aligned} \tag{2.2}$$

where $\pi : G \times \mathfrak{m}^* \times N \to Y$ denotes the projection. (Note that in fact $\mathrm{d}\mathbf{J}_N(v)$ is constant in v, since N is a representation).

Since ω_Y is G-invariant and coincides at any point of the orbit $G[e,0,0]$ with the symplectic form computed in the Artin-Witt decomposition theorem, the equivariant Darboux theorem yields that there is a G-equivariant symplectomorphism defined on a tubular neighbourhood of $G[e,0,0]$ that maps $[e,0,0]$ to p and whose image is a tubular neighbourhood of Gp. Hence we have obtained a symplectic version of the Slice Theorem: a local description of the symplectic G-manifold about some G-orbit.

A real benefit of the Marle-Guillemin-Sternberg normal form is the existence of a quite simple momentum map:

Lemma 2.7.

$$\begin{aligned}\mathbf{J}_Y : Y = G \times_{G_p} (\mathfrak{m}^* \times N) &\to \mathfrak{g}^* \\ [g,\rho,v] &\mapsto \mathrm{Coad}_g(\mu + \rho + \mathbf{J}_N(v))\end{aligned}$$

is a momentum map for the G-action on (Y,ω_Y).

Proof. Clearly, $\mathbf{J}_Y$ is well-defined.

For $\eta \in \mathfrak{g}$ and $y = [g,\rho,v] \in Y$,

$$\eta\cdot y = \frac{\mathrm{d}}{\mathrm{d}t}[\exp(t\eta)g,\rho,v]\Big|_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t}[gg^{-1}\exp(t\eta)g,\rho,v]\Big|_{t=0} = \mathrm{d}\pi(g\cdot \mathrm{Ad}_{g^{-1}}\eta, 0, 0).$$

Thus,

$$\begin{aligned}\omega_Y(y)(\eta\cdot y, \mathrm{d}\pi(g\cdot\xi,\dot\rho,\dot v)) &= \omega_Y([g,\rho,v])(\mathrm{d}\pi(g\cdot \mathrm{Ad}_{g^{-1}}\eta,0,0), \mathrm{d}\pi(g\cdot\xi,\dot\rho,\dot v)) \\ &= \langle \dot\rho + \mathrm{d}\mathbf{J}_N(v)\dot v, \mathrm{Ad}_{g^{-1}}\eta\rangle + \langle \mu+\rho+\mathbf{J}_N(v), [\mathrm{Ad}_{g^{-1}}\eta,\xi]\rangle.\end{aligned}$$

Since

$$\begin{aligned}\mathrm{d}\mathbf{J}_Y^\eta(y)\mathrm{d}\pi(g\cdot\xi,\dot\rho,\dot v) &= \frac{\mathrm{d}}{\mathrm{d}t}\langle \mathbf{J}_Y([g\exp(t\xi),\rho+t\dot\rho, v+t\dot v]),\eta\rangle\Big|_{t=0} \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\langle \mathrm{Coad}_{g\exp(t\xi)}(\mu+\rho+t\dot\rho+\mathbf{J}_N(v+t\dot v)),\eta\rangle\Big|_{t=0} \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\langle \mathrm{Coad}_{\exp(t\xi)}(\mu+\rho+t\dot\rho+\mathbf{J}_N(v+t\dot v)),\mathrm{Ad}_{g^{-1}}\eta\rangle\Big|_{t=0} \\ &= \langle \mu+\rho+\mathbf{J}_N(v), [\mathrm{Ad}_{g^{-1}}\eta,\xi]\rangle + \langle \dot\rho+\mathrm{d}\mathbf{J}_N(v)\dot v, \mathrm{Ad}_{g^{-1}}\eta\rangle,\end{aligned}$$

$\mathbf{J}_Y$ is a momentum map for the G-action on Y. □

2.3 Bundle equations

The *bundle equations* or *reconstruction equations* give a normal form of X_h with respect to coordinates of the Marle-Guillemin-Sternberg normal form. Thus they characterize the dynamics locally near a G-orbit, in particular near an orbit of relative equilibria. For a (local) lift of X_h to $G\times\mathfrak{m}^*\times N$, the bundle equations describe the components $X_\mathfrak{g}$, $X_{\mathfrak{m}^*}$ and X_N. In [RdSD97], they are computed for the case of a compact group action. A comprehensive generalization may be found in [RWL02].

As in the last section, we will only discuss the case that a G_μ-invariant splitting $\mathfrak{g} = \mathfrak{g}_\mu \oplus \mathfrak{q}$ exists. Recall that $\mathfrak{m}$ denotes a G_p-invariant complement of $\mathfrak{g}_p$ within $\mathfrak{g}_\mu$. Let $X_\mathfrak{g} = X_{\mathfrak{g}_p} + X_\mathfrak{m} + X_\mathfrak{q}$ be the sum of the corresponding components. Since h is G-invariant,

$$\mathrm{d}h([g,\rho,v])\mathrm{d}\pi(g\cdot\xi,\dot\rho,\dot v) = \mathrm{d}_{\mathfrak{m}^*}(h\circ\pi)(g,\rho,v)\dot\rho + \mathrm{d}_N(h\circ\pi)(g,\rho,v)\dot v.$$

For $\xi = \xi_{\mathfrak{g}_p} + \xi_\mathfrak{m} + \xi_\mathfrak{q}$,

$$\begin{aligned}
&\mathrm{d}h([g,\rho,v])\mathrm{d}\pi(g\cdot\xi,\dot\rho,\dot v)\\
&= \omega_Y([g,\rho,v])(\mathrm{d}\pi((g\cdot X_\mathfrak{g}, X_{\mathfrak{m}^*}, X_N), \mathrm{d}\pi(g\cdot\xi,\dot\rho,\dot v))\\
&= \langle\dot\rho, X_\mathfrak{m}\rangle + \langle \mathrm{d}\mathbf{J}_N(v)\dot v, X_{\mathfrak{g}_p}\rangle - \langle X_{\mathfrak{m}^*}, \xi_\mathfrak{m}\rangle - \langle \mathrm{d}\mathbf{J}_N(v)X_N, \xi_{\mathfrak{g}_p}\rangle\\
&\quad + \langle \mu+\rho+\mathbf{J}_N(v), [X_{\mathfrak{g}_p} + X_\mathfrak{m} + X_\mathfrak{q}, \xi_{\mathfrak{g}_p} + \xi_\mathfrak{m} + \xi_\mathfrak{q}]\rangle + \omega_N(X_N,\dot v).
\end{aligned}$$

Since

$$\langle \mathrm{d}\mathbf{J}_N(v)\dot v, X_{\mathfrak{g}_p}\rangle = \omega_N(X_{\mathfrak{g}_p}\cdot v, \dot v)$$

and $\mu([\eta,\xi])$ vanishes for $\eta\in\mathfrak{g}_\mu$ or $\xi\in\mathfrak{g}_\mu$, this is equivalent to the following five equations:

$$X_\mathfrak{m} = \mathrm{d}_{\mathfrak{m}^*}(h\circ\pi), \tag{2.3}$$

$$\omega_N(X_N + X_{\mathfrak{g}_p}\cdot v, \cdot) = \mathrm{d}_N(h\circ\pi), \tag{2.4}$$

$$\mathbb{P}_{\mathfrak{q}^*}(-\operatorname{coad}_{X_{\mathfrak{g}_p}+X_\mathfrak{m}+X_\mathfrak{q}}(\mu+\rho+\mathbf{J}_N(v))) = 0, \tag{2.5}$$

$$\mathbb{P}_{\mathfrak{m}^*}(-\operatorname{coad}_{X_{\mathfrak{g}_p}+X_\mathfrak{m}+X_\mathfrak{q}}(\rho+\mathbf{J}_N(v))) = X_{\mathfrak{m}^*}, \tag{2.6}$$

$$\mathbb{P}_{\mathfrak{g}_p^*}(-\operatorname{coad}_{X_{\mathfrak{g}_p}+X_\mathfrak{m}+X_\mathfrak{q}}(\rho+\mathbf{J}_N(v))) = \mathrm{d}\mathbf{J}_N(v)(X_N). \tag{2.7}$$

Lemma 2.8. *Equation* (2.5) *is equivalent to* $X_\mathfrak{g}\in\mathfrak{g}_\mu$.

Proof. If $X_\mathfrak{q} = 0$, equation (2.5) is obviously satisfied. For the converse, consider the linear map

$$\begin{aligned}
L([e,\rho,v]) : \mathfrak{q} &\to \mathfrak{q}^*\\
\zeta &\mapsto \mathbb{P}_{\mathfrak{q}^*}(\operatorname{coad}_\zeta(\mu+\rho+\mathbf{J}_N(v))
\end{aligned}$$

parameterized by $\mathfrak{m}^*\times N$. Since $\operatorname{coad}_\zeta(\mu)$ is contained in the annihilator of $\mathfrak{g}_\mu$, which we identify with $\mathfrak{q}^*$, $L([e,0,0])\zeta = 0$ implies $\zeta\in\mathfrak{q}\cap\mathfrak{g}_\mu = \{0\}$. Therefore $L([e,\rho,v])$ is an isomorphism for small ρ and v. Equation (2.5) yields that near $[e,0,0]$ the value of $X_\mathfrak{q}$ must be equal to 0. □

Now, we search for solutions of equations (2.3) to (2.7): The $X_\mathfrak{m}$-component is determined by equation (2.3). Since the vector field X_h is defined on the quotient by G_p, we may choose the $\mathfrak{g}_p$-component and set $X_{\mathfrak{g}_p} = 0$. Then equation (2.4) yields

$$X_N = (\omega_N^\#)(\mathrm{d}_N(h\circ\pi)),$$

where $\omega_N^\# : N^*\to N$ denotes the isomorphism given by ω_N. By Lemma 2.8, equation (2.5) is equivalent to $X_\mathfrak{q} = 0$. At last, the component $X_{\mathfrak{m}^*}$ is given by equation (2.6). Hence, the components of the preimage of X_h under the projection $\pi : G\times(\mathfrak{m}^*\times N)\to Y$ are given by

$$X_\mathfrak{g}(g,\rho,v) = \mathrm{d}_{\mathfrak{m}^*}\bar h(\rho,v) \tag{2.8}$$

$$X_{\mathfrak{m}^*}(g,\rho,v) = -\mathbb{P}_{\mathfrak{m}^*}(\operatorname{coad}_{\mathrm{d}_{\mathfrak{m}^*}\bar h(\rho,v)}(\rho+\mathbf{J}_N(v))) \tag{2.9}$$

$$X_N(g,\rho,v) = (\omega_N^\#)(\mathrm{d}_N\bar h)(\rho,v), \tag{2.10}$$

where $\bar{h} = h \circ \pi\big|_{\mathfrak{m}\times N}$ is locally defined on the quotient ${}^{P}\!/_{G}$.

We only have to verify that equation (2.7) is automatically satisfied for this choice of components. To do this, we consider both sides evaluated at an arbitrary element $\xi \in \mathfrak{g}_p$. We start with the right-hand side:

$$\begin{aligned}\langle \mathrm{d}\mathbf{J}_N(v)(X_N), \xi\rangle &= \omega_N(v)(\xi \cdot v, X_N)\\ &= -\mathrm{d}_N \bar{h}(\rho, v)\xi \cdot v\\ &= -\frac{\mathrm{d}}{\mathrm{d}t}\bar{h}(\rho, \exp(t\xi)\cdot v)\bigg|_{t=0}\\ &= -\frac{\mathrm{d}}{\mathrm{d}t}\bar{h}(\exp(-t\xi)\rho, v)\bigg|_{t=0}\\ &= \mathrm{d}_{\mathfrak{m}^*}\bar{h}(\rho, v)\,\mathrm{coad}_\xi\,\rho = \langle \mathrm{coad}_\xi\,\rho, X_{\mathfrak{m}}\rangle.\end{aligned}$$

For the left-hand side, we obtain

$$\begin{aligned}\langle(-\,\mathrm{coad}_{\mathrm{d}_{\mathfrak{m}^*}\bar{h}(\rho,v)}(\rho + \mathbf{J}_N(v))), \xi\rangle &= \langle \rho + \mathbf{J}_N(v), \mathrm{ad}_{\mathrm{d}_{\mathfrak{m}^*}\bar{h}(\rho,v)}\,\xi\rangle\\ &= \langle \rho + \mathbf{J}_N(v)), -\,\mathrm{ad}_\xi\,\mathrm{d}_{\mathfrak{m}^*}\bar{h}(\rho, v)\rangle\\ &= \langle \mathrm{coad}_\xi\,\rho, X_{\mathfrak{m}}\rangle,\end{aligned}$$

where the last equation holds, since $-\,\mathrm{ad}_\xi\,\mathrm{d}_{\mathfrak{m}^*}\bar{h}(\rho, v) \in \mathfrak{m}$ and $\mathbf{J}_N(v)$ is contained in $\mathfrak{g}_p^*$, the annihilator of $\mathfrak{m}$.

Thus the equations (2.8) to (2.10) describe a vector field on $G \times \mathfrak{m}^* \times N$ that projects to the Hamiltonian vector field X_h on $Y = G \times_{G_p} \mathfrak{m}^* \times N$. As Montaldi and Rodríguez-Olmos suggest in [MR-O13] and [MR-O15], it can be helpful to omit the condition $X_{\mathfrak{g}_p} = 0$ and consider all possible preimages of $X_h(0)$ in $G \times \mathfrak{m}^* \times N$ instead. To do this, one has to add all lifts of the zero vector field. These are of the form

$$X(g, \rho, v) = (g \cdot \eta, \mathrm{coad}_\eta\,\rho, -\eta \cdot v).$$

G_p-invariance of the splitting $g_\mu^* = g_p^* \oplus \mathfrak{m}^*$ yields

$$\mathrm{coad}_\eta\,\rho \in \mathfrak{m}^* \quad \text{and} \quad \mathrm{coad}_\eta\,\mathbf{J}_N(v) \in \mathfrak{g}_p^*.$$

Thus adding $\mathrm{coad}_\eta\,\rho$ to (2.9) is the same as adding $\mathrm{coad}_\eta(\rho+\mathbf{J}_N(v))$ to the term inside the parentheses. Moreover, $-\eta \cdot v$ is equal to $(\omega_N^{\#})(\mathrm{d}_N J_N^\eta(\rho, v)v)$. This yields the bundle equations with isotropy as given in [MR-O13] and [MR-O15]:

$$X_{\mathfrak{g}}(g, \rho, v) = \mathrm{d}_{\mathfrak{m}^*}\bar{h}(\rho, v) + \eta \tag{2.11}$$

$$X_{\mathfrak{m}^*}(g, \rho, v) = -\mathbb{P}_{\mathfrak{m}^*}(\mathrm{coad}_{\mathrm{d}_{\mathfrak{m}^*}\bar{h}(\rho,v)+\eta}(\rho + \mathbf{J}_N(v))) \tag{2.12}$$

$$X_N(g, \rho, v) = (\omega_N^{\#})^{-1}(\mathrm{d}_N(\bar{h} - \mathbf{J}_N^\eta)(\rho, v)) \tag{2.13}$$

An advantage of this approach is that relative equilibria can be characterized easily: $[g, \rho, v]$ is a relative equilibrium iff there is an $\eta \in \mathfrak{g}_p$ such that

$$\begin{aligned}\mathbb{P}_{\mathfrak{m}^*}(\mathrm{coad}_{\mathrm{d}_{\mathfrak{m}^*}\bar{h}(\rho,v)+\eta}(\rho + \mathbf{J}_N(v))) &= 0\\ \mathrm{d}_N(\bar{h} - \mathbf{J}_N^\eta)(\rho, v) &= 0.\end{aligned}$$

In this case, $d_{\mathfrak{m}^*+\eta}\bar{h}(\rho,v)+\eta$ is a generator of $[g,\rho,v]$. Inserting $X_N = 0$ into equation (2.7) yields that these two equations are equivalent to

$$\mathrm{coad}_{d_{\mathfrak{m}^*}\bar{h}(\rho,v)+\eta}(\rho + \mathbf{J}_N(v)) = 0 \tag{2.14}$$

$$d_N(\bar{h} - \mathbf{J}_N^\eta)(\rho, v) = 0. \tag{2.15}$$

Let (ρ, v) correspond to a relative equilibrium with generator $\xi' = d_{\mathfrak{m}^*+\eta}\bar{h}(\rho,v)+\eta$ and momentum

$$\mu' = \mathbf{J}_Y(e,\rho,v) = \mu + \rho + \mathbf{J}_N(v).$$

Since $\xi \in \mathfrak{g}_\mu$, equation (2.14) is equivalent to $\mathrm{coad}_{\xi'}\,\mu' = 0$. Thus we will call equation (2.14) the *commutation equation* in the following. Equation (2.15) will be called the *symplectic slice equation*. For $\rho \in (\mathfrak{m}^*)^{G_p}$, solutions correspond to relative equilibria of the Hamiltonian systems with G_p-symmetry on the symplectic slice N with Hamiltonian functions $h(\cdot,\rho)$ parameterized by ρ.

Thus, we are in particular interested in the structure of relative equilibria near the origin in Hamiltonian systems on symplectic representations of compact groups. A method to investigate this situation is illustrated in the next section.

2.4 Splitting Lemma reduction

Given a symplectic representation V of a compact Lie group G and a G-invariant Hamiltonian function $h : V \to \mathbb{R}$ with $dh(0) = 0$, all relative equilibria near 0 with a generator near some element $\xi \in \mathfrak{g}$ can be characterized by an equation on $\ker d^2(h - \mathbf{J}^\xi)(0)$. This reduced equation may be obtained from the Splitting Lemma or Lyapunov-Schmidt reduction. Both methods yield the same equation, which will be shown in Remark 2.12.

In the following, we will denote a locally defined function f between topological spaces V and W by $f : (V, v) \to W$. This means that f is defined in a neighbourhood of $v \in V$. If we write $(V, v) \to (W, w)$, we require in addition that $f(v) = w$. We are often mainly interested in the *germ* of f, that is the equivalence class of functions defined in a neighbourhood of v that coincide with f on a possibly smaller neighbourhood. Correspondingly, we are often interested in *germs* of sets near $v \in V$, also called *local sets*, that is an equivalence class of sets with respect to the equivalence relation that two sets coincide in some neighbourhood of v.

We start with a proof of the Splitting Lemma. The proof is in principle the proof in [PS78] combined with an argument of the proof of the equivariant Morse lemma given in [Arn76].

Lemma 2.9 (Equivariant Splitting Lemma)**.** *Let G be compact, W, Λ be G-representations, and $f : W \times \Lambda \to \mathbb{R}$ be a smooth G-invariant function with critical point $(0, \lambda_0)$. If for $\lambda_0 \in \Lambda^G$ the Hessian $d_W^2 f(0, \lambda_0)$ is non-degenerate, then there is a G-equivariant local diffeomorphism $\varphi : (W \times \Lambda, (0, \lambda_0)) \to (W \times \Lambda, (0, \lambda_0))$ of the form $\varphi(w, \lambda) = (\varphi_1(w, \lambda), \lambda)$, such that*

$$f \circ \varphi(w, \lambda) = g(\lambda) + Q(w),$$

where g is a smooth G-invariant function and Q is the non-degenerate quadratic form $Q(w) = d_W^2 f(0, \lambda_0)(w, w)$.

If Λ coincides with the kernel of $\mathrm{d}^2 f(0,\lambda_0)$ (i.e. $\mathrm{d}^2 f(0,\lambda_0)(\cdot,\lambda) = 0$ for every $\lambda \in \Lambda$), then $\mathrm{d}_\Lambda \varphi_1(0,\lambda_0) = 0$ and the functions g and $f(0,\cdot)$ have the same Taylor polynomial of third order at λ_0.

Proof. Since $\mathrm{d}_W f(0,\lambda_0) = 0$ and $\mathrm{d}^2_W f(0,\lambda_0)$ is non-degenerate, there is a local implicit function w of λ with

$$\mathrm{d}_W f(w(\lambda),\lambda) = 0.$$

By uniqueness of the implicit function, the function w is equivariant. Hence $\psi : W\times\Lambda, (0,\lambda_0) \to W\times\Lambda, (0,\lambda_0)$ with $\psi(x,\lambda) := (x+w(\lambda),\lambda)$ is an equivariant diffeomorphism such that $\mathrm{d}_W(f\circ\psi)(0,\lambda) = 0$. Thus we can assume without loss of generality that $\mathrm{d}_W f(0,\lambda) = 0$ for every λ. Then

$$f(w,\lambda) = Q(w) + F(w,\lambda)$$

for some smooth function F such that 0 is a critical point of $F(\cdot,\lambda)$ for every λ, and the Hessian of $F(\cdot,\lambda_0)$ vanishes at 0. In order to construct a diffeomorphism with the desired properties, we search for a family $\varphi(\cdot,\lambda,t)$ of local diffeomorphisms of W, defined for $t\in[0,1]$ and λ in some neighbourhood of λ_0, such that $\varphi(0,\lambda,t) = 0$ for every λ and t and

$$Q(\varphi(w,t,\lambda)) + tF(\varphi(w,t,\lambda),\lambda) = Q(w) + tF(0,\lambda). \tag{2.16}$$

Denote $F^\lambda := F(\cdot,\lambda)$ and $\varphi^\lambda_t := \varphi(\cdot,\lambda,t)$. For fixed λ near λ_0, we show the existence of a family of time-dependent vector fields ξ^λ_t that generates the family φ^λ_t. Differentiating equation (2.16) with respect to t yields a linear equation for ξ^λ_t:

$$\mathrm{d}(Q + tF^\lambda)\xi^\lambda_t = F^\lambda(0) - F^\lambda \tag{2.17}$$

in some neighbourhood of 0. A solution ξ^λ_t of equation (2.17) that satisfies $\xi^\lambda_t(0) = 0$ generates a solution to (2.16) with $\varphi^\lambda_t(0) = 0$ for $t\in[0,1]$. Now we construct such a vector field ξ^λ_t, that is defined in some neighbourhood of 0: Since Q is non-degenerate, there is a linear change of coordinates such that $Q(x) = \pm x_1^2 \pm\cdots\pm x_k^2$, $k = \dim W$. If $\xi^\lambda_t = \sum_i \eta^\lambda_i(x,t)\partial_{x_i}$, equation (2.17) is equivalent to

$$F^\lambda(0) - F^\lambda(x) = \sum_i \eta^\lambda_i(x,t) y^\lambda_i(x,t), \quad y^\lambda_i(x,t) := \pm 2x_i + t\partial_{x_i}F^\lambda(x).$$

$\mathrm{d}F^\lambda(0) = 0$ yields $y^\lambda(0,t) = 0$. For $\lambda = \lambda_0$ and $t\in[0,1]$, $\mathrm{d}^2 F^{\lambda_0}(0) = 0$ implies $\det(\mathrm{d}_x y^{\lambda_0}(0,t)) = \pm 2^k$. Hence $(x,t,\lambda)\mapsto (y^\lambda(x,t),t,\lambda)$ is a diffeomorphism in some neighbourhood of $(0,\lambda_0)\times[0,1]$. Since the left-hand side vanishes at 0 and for fixed λ and t the functions $y^{\lambda,t}_i := y^\lambda_i(\cdot,t)$ are local coordinates on V, the Hadamard lemma yields functions η^λ_i with

$$F^\lambda(0) - F^\lambda(y^{\lambda,t}) = \sum_i \eta^\lambda_i(y^{\lambda,t}) y^{\lambda,t}_i.$$

Therefore there is a (possibly non-equivariant) vector field solving equation (2.17).

Next, we construct an equivariant solution of (2.17), i.e. a family of time-dependent vector fields $\bar\xi^\lambda_t$, such that $(\bar\xi^{g\lambda}_t)(gv) = g(\xi^\lambda_t)(v)$ for every $g\in G$:

Since $\lambda_0 \in W^G$, $f(\cdot, \lambda_0)$ is G-invariant. Therefore the quadratic form $Q(w) = \mathrm{d}_V^2 f(0, \lambda_0)(w, w)$ is G-invariant, and hence F and $Q + tF$ are G-invariant functions. Assume that the family ξ_t^λ solves equation (2.17) and is defined on some G-invariant neighbourhood of $(0, \lambda_0) \times [0, 1]$. For $g \in G$, define

$$((g\xi)_t^{g\lambda})(gw) := g(\xi_t^\lambda)(w).$$

Then $(g\xi)$ solves (2.17) as well:

$$\begin{aligned} \mathrm{d}(Q + tF^{g\lambda})(gw)((g\xi)_t^{g\lambda})(gw) &= \mathrm{d}(Q + tF^\lambda)(w)g^{-1}g(\xi_t^\lambda)(w) \\ &= F^\lambda(0) - F^\lambda(w) = F^{g\lambda}(0) - F^{g\lambda}(gw). \end{aligned}$$

Therefore, averaging over G yields an equivariant solution $\bar{\xi} := \int_{g \in G} (g\xi) \mathrm{d}g$ of (2.17). The family $\bar{\xi}_t^\lambda$ generates a family of equivariant local diffeomorphisms φ_t^λ. The local diffeomorphism $\varphi_1(x, \lambda) = \varphi_1^\lambda(x)$ solves

$$f(\varphi_1(x, \lambda), \lambda) = Q(v) + F(0, \lambda).$$

For the last part, notice that under the assumption $w(\lambda) = 0$, we have $\varphi_1(0, \lambda) = 0$. In general, we have to consider this change of coordinates, then $\varphi_1(0, \lambda) = w(\lambda)$. Hence,

$$\mathrm{d}_W f(\varphi_1(0, \lambda), \lambda) = 0$$

for every λ and therefore

$$\begin{aligned} 0 &= \mathrm{d}_\Lambda(\mathrm{d}_W f(\varphi_1(0, \lambda), \lambda))\big|_{\lambda = \lambda_0} \\ &= \underbrace{\mathrm{d}^2 f(0, \lambda_0)(\cdot\big|_\Lambda, \cdot\big|_W)}_{=0} + \mathrm{d}_W^2 f(0, \lambda_0)(\mathrm{d}_\Lambda \varphi_1(0, \lambda_0)\cdot, \cdot). \end{aligned}$$

Since $\mathrm{d}_W^2 f(0, \lambda_0)$ is non-degenerate, $\mathrm{d}_\Lambda \varphi_1(0, \lambda_0) = 0$. For the Taylor polynomials, we calculate:

$$\begin{aligned} g(\lambda) &= f \circ \varphi(0, \lambda) = f(w(\lambda), \lambda) \\ \Rightarrow \mathrm{d}g(\lambda) &= \underbrace{\mathrm{d}_W f(w(\lambda), \lambda)}_{=0} \mathrm{d}_\Lambda w(\lambda) + \mathrm{d}_\Lambda f(w(\lambda), \lambda) \\ \Rightarrow \mathrm{d}^2 g(\lambda) &= \mathrm{d}_W \mathrm{d}_\Lambda f(w(\lambda), \lambda) \mathrm{d}_\Lambda w(\lambda) + \mathrm{d}_\Lambda^2 f(w(\lambda), \lambda) \\ &= \mathrm{d}_\Lambda(\underbrace{\mathrm{d}_W f(w(\lambda), \lambda)}_{=0} \mathrm{d}_\Lambda w(\lambda)) - \mathrm{d}_W^2 f(w(\lambda), \lambda)(\mathrm{d}_\Lambda w(\lambda)\cdot, \mathrm{d}_\Lambda w(\lambda)\cdot) \\ &\quad + \mathrm{d}_\Lambda^2 f(w(\lambda), \lambda) \end{aligned}$$

Since $\mathrm{d}_\Lambda w(\lambda_0) = \mathrm{d}_\Lambda \varphi_1(0, \lambda_0) = 0$, all terms that contain a first derivative of w vanish in λ_0. Hence the derivatives up to second order coincide in λ_0 (all vanish) and in addition

$$\mathrm{d}^3 g(\lambda_0) = \mathrm{d}_\Lambda^3 f(0, \lambda_0). \qquad \square$$

By applying the Splitting Lemma to the augmented Hamiltonian, we may reduce the search for critical points near 0 to some gradient equation on the kernel of the Hessian at 0:

Let V be a symplectic G-representation and $h : V \to \mathbb{R}$ be a smooth G-invariant Hamiltonian function. Suppose that for some $\xi_0 \in \mathfrak{g}$

$$\ker \mathrm{d}^2(h - \mathbf{J}^{\xi_0})(0) =: V_0 \neq \{0\}.$$

Let $K \subset G$ be a subgroup such that $\xi_0 \in \mathfrak{g}^K$. Then V_0 is K-invariant. Let V_1 be a K-invariant complement to V_0. Define $\mathcal{H} : V_1 \times V_0 \times \mathfrak{g} \to \mathbb{R}$ by

$$\mathcal{H}(v_1, v_0, \xi) = (h - \mathbf{J}^\xi)(v_1 + v_0).$$

$(0, 0, \xi_0)$ is a critical point of $\mathcal{H}$, because $\mathrm{d}_V(h - \mathbf{J}^\xi)(0) = 0$ and $\mathrm{d}_\xi \mathbf{J}^\xi(0) = 0$. Since $\mathrm{d}^2_{V_1}\mathcal{H}(0, 0, \xi)$ is non-degenerate, the Splitting Lemma yields a K-equivariant local diffeomorphism

$$\varphi : V_1 \times V_0 \times \mathfrak{g}, (0, 0, \xi_0) \to V_1 \times V_0 \times \mathfrak{g}, (0, 0, \xi_0)$$

of the form

$$\varphi(v_1, v_0, \xi) = (\varphi_1(v_1, v_0, \xi), v_0, \xi)$$

with

$$\mathcal{H} \circ \varphi(v_1, v_0, \xi) = Q(v_1) + g(v_0, \xi)$$

for some K-invariant quadratic form Q and some K-invariant function g. Since $\varphi(\cdot, \cdot, \xi)$ is a diffeomorphism, $(\varphi_1(v_1, v_0, \xi), v_0)$ is a critical point of $\mathcal{H}(\cdot, \cdot, \xi)$ iff (v_1, v_0) is a critical point of $\mathcal{H} \circ \varphi(\cdot, \cdot, \xi)$. The latter is equivalent to $v_1 = 0$ and $\mathrm{d}_{V_0} g(v_0, \xi) = 0$. Hence the solutions (v_0, ξ) of $\mathrm{d}_{V_0} g(v_0, \xi) = 0$ near $(0, \xi_0)$ are in one-to-one correspondence to the pairs (v, ξ) of relative equilibria v near 0 with generators ξ near ξ_0. Moreover, since φ is a diffeomorphism, the local sets of such pairs can be mapped diffeomorphically into each other (if both are considered as subsets of $V \times \mathfrak{g}$).

Furthermore, the kernel of $\mathrm{d}^2\mathcal{H}(0, 0, \xi_0)$ coincides with $V_0 \times \mathfrak{g}$: We have $\mathrm{d}^2_V \mathcal{H}(0, 0, \xi_0)\big|_{V_0} = 0$ by assumption. The second derivatives that involve at least one derivative in the direction of $\mathfrak{g}$ coincide with those of the function $(v, \xi) \mapsto \mathbf{J}^\xi(v)$. Since this function is linear in ξ and quadratic in v, these derivatives vanish at $(0, \xi)$ for any ξ, in particular at $(0, \xi_0)$. Therefore the second part of the Splitting Lemma applies and

$$\mathrm{d}_{V_0}\varphi_1(0, 0, \xi_0) = \mathrm{d}_\mathfrak{g}\varphi_1(0, 0, \xi_0) = 0.$$

Hence the relative equilibria near 0 with generators near ξ_0 are in some sense tangent to V_0 at 0. (We will see later that they generically form a manifold tangent to V_0 if G is a torus.) Moreover, the Taylor polynomial of third order of g at $(0, \xi_0)$ coincides with that of the restriction of $\mathcal{H}$ to $V_0 \times \mathfrak{g}$ (both are homogeneous of degree 3).

Remark 2.10. If in the above setting $V_0 \subset V^L$ for some subgroup $L \subset K$, $\mathbf{J}^\xi(v_0)$ is constant in the direction of the Lie algebra $\mathfrak{l}$ of L for every $v_0 \in V_0$. Hence for every $\eta \in \mathfrak{g}$ with $\eta - \xi_0 \in \mathfrak{l}$, we have the same kernel, i.e. $\ker \mathrm{d}^2(h - \mathbf{J}^\eta)(0) = V_0$. This leads to the question how the local diffeomorphisms φ and local functions g at $(0, \xi_0)$ and $(0, \eta)$ are related to each other. Let $v_1 : V_0 \times \mathfrak{g} \to V_1$ be the local K-equivariant function $\varphi_1(0, \cdot, \cdot)$ at $(0, \xi_0)$, hence v_1 solves

$$\mathrm{d}_{V_1}\mathcal{H}(v_1(v_0, \xi), v_0, \xi) = 0$$

uniquely. Assume that for some $x \in V_0$, $v_1(x,\eta)$ and $v_1(x,\xi_0)$ are defined. Since $\mathcal{H}\big|_{V^L}$ is invariant in the direction of $\mathfrak{l}$ and $v_1(x,\xi_0) \in V^L$,

$$\mathrm{d}_{V_1}\mathcal{H}(v_1(x,\xi_0),x,\eta) = 0$$

and by uniqueness

$$v_1(x,\eta) = v_1(x,\xi_0).$$

Thus v_1 is also locally constant in the direction of $\mathfrak{l}$ and hence can be extended to a function defined on a neighbourhood of $(0,\xi_0) + \mathfrak{l}$ such that the function is constant on $\mathfrak{l}$. Then v_1 is equivalent to a local function defined on a neighbourhood of $(0,[x_0]) \in V_0 \times \mathfrak{g}/\mathfrak{l}$. Thus φ and g can also be considered as maps on a neighbourhood of $(0,[x_0]) \in V_0 \times \mathfrak{g}/\mathfrak{l}$. We obtain the same φ and g, when we consider h as a function defined on $V^L \times \mathfrak{g}/\mathfrak{l}$ and perform the reduction at $(0,[x_0])$.

Remark 2.11. More generally, if we are interested in the solutions near $(0,\xi_0)$ that are fixed by L for some subgroup $L \subset K$, we can restrict the system to the fixed point space V^L. Then $\ker(h - \mathbf{J}^\eta)(0) \cap V^L = V_0^L$ for every $\eta \in (0,\xi_0) + \mathfrak{l}$. As in Remark 2.10, v_1 and g can be extended constantly in $\mathfrak{l}$ to a neighbourhood $(0,\xi_0) + \mathfrak{l}$. By uniqueness of v_1, the germs of v_1 and g at $(0,\xi_0)$ are equivalent to those of the restrictions to V_0^L of the corresponding functions defined on V_0.

Remark 2.12. Alternatively, we may apply a Lyapunov-Schmidt reduction to obtain an equation on the kernel of $\mathrm{d}^2(h - \mathbf{J}^\xi)(0)$:

Let f, W and Λ be as in the Splitting Lemma. To find the critical points of f, we have to find the zeros of

$$\mathrm{d}f : W \times \Lambda \to (W \times \Lambda)^*.$$

As in the proof of the Splitting Lemma, the implicit function theorem yields the local function v with

$$\mathrm{d}_W f(w(\lambda),\lambda) = 0.$$

If $\mathbb{P}_{W^*} : W^* \times \Lambda^* \to W^*$ is the projection, the critical points are then given by the solutions of

$$\mathbb{P}_{W^*} \circ \mathrm{d}f(w(\lambda),\lambda) = 0.$$

For $f = \mathcal{H}$ as above, $W = V_1$, and $\Lambda = V_0 \times \mathfrak{g}$, we obtain the equation

$$\mathbb{P}_{V_0^*} \circ \mathrm{d}(h - \mathbf{J}^\xi)(v_1(v_0,\xi) + v_0) = 0.$$

The application of Splitting Lemma yields that the relative equilibria correspond to the critical points of the function

$$(h - \mathbf{J}^\xi)(v_1(v_0,\xi) + v_0).$$

As argued in [CLOR03], both formulations are equivalent: Choose an invariant inner product $\langle \cdot,\cdot \rangle$ on $V = V_0 + V_1$. Then for any $w_0 \in V_0$:

$$\begin{aligned}
&\langle \nabla_{V_0} g(v_0,\xi), w_0 \rangle \\
&= \mathrm{d}(h - \mathbf{J}^\xi)(v_1(v_0,\xi) + v_0)(w_0 + \mathrm{d}_{V_0} v_1(v_0,\xi) w_0) \\
&= \langle \nabla_V (h - \mathbf{J}^\xi)(v_1(v_0,\xi) + v_0), \mathbb{P}_{V_0} w_0 + (\mathbb{1} - \mathbb{P}_{V_0}) \mathrm{d}_{V_0} v_1(v_0,\xi) w_0 \rangle \\
&= \langle \mathbb{P}_{V_0} \nabla_V (h - \mathbf{J}^\xi)(v_1(v_0,\xi) + v_0), w_0 \rangle,
\end{aligned}$$

since $\mathbb{P}_{V_0}$ is self-adjoint and

$$\langle (\mathbb{1} - \mathbb{P}) \nabla_V (h - \mathbf{J}^\xi)(v_1(v_0,\xi) + v_0), \cdot \rangle = \mathrm{d}_{V_1}(h - \mathbf{J}^\xi)(v_1(v_0,\xi) + v_0) = 0.$$

Chapter 3

Linear theory

In this section, we study the derivative of the Hamiltonian vector field near some equilibrium. Suppose that the Hamiltonian system has a symmetry given by a proper Lie group action. By the equivariant Darboux theorem and Bochner's linearization theorem, we can assume w.l.o.g. that the symplectic manifold is a symplectic representation V of a compact group G and the equilibrium point is 0.

Since

$$\begin{aligned}\omega(\mathrm{d}X_h(0)v, w) = \mathrm{d}^2h(0)(v, w) = \mathrm{d}^2h(0)(w, v) &= \omega(\mathrm{d}X_h(0)w, v) \\ &= -\omega(v, \mathrm{d}X_h(0)w),\end{aligned}$$

$\mathrm{d}X_h(0)$ is contained in

$$\mathfrak{sp}(V) = \{A \in \mathrm{End}_{\mathbb{R}}(V) \mid \omega(A\cdot, \cdot) + \omega(\cdot, A\cdot) = 0\},$$

which is the Lie algebra of the group $\mathrm{SP}(V)$ of linear symplectomorphisms $V \to V$.

Since V is G-symplectic and h is G-invariant, $\mathrm{d}X_h(0)$ is contained in

$$\mathfrak{sp}_G(V) := \mathfrak{sp}(V) \cap \mathrm{End}_G(V),$$

the Lie algebra of the subgroup $\mathrm{SP}_G(V) \subset \mathrm{SP}(V)$ of G-equivariant elements.

The elements of $\mathfrak{sp}(V)$ are investigated in [Wil36], where Williamson studies the eigenvalue structure, the Jordan normal form, and the corresponding normal form of the matrix representing ω and gives normal forms for symplectic linear coordinate changes. The theory is transferred to the equivariant case in [DM93a]. To do this, Dellnitz and Melbourne use the fact that the equivariant homomorphisms preserve the isotypic components and the restrictions to the isotypic components can be identified with $\mathbb{K}$-linear endomorphisms of finite dimensional $\mathbb{K}$-vector spaces for $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$. Hence they have to extend the results for $\mathbb{K} = \mathbb{R}$ to include the cases $\mathbb{K} = \mathbb{C}$ and $\mathbb{K} = \mathbb{H}$.

The eigenvalue structure of elements of $\mathfrak{sp}_G(V)$ and 1-parameter families in $\mathfrak{sp}_G(V)$ had already studied in [GS87] and [DMM92] before, where the emphasis is placed on the generic case.

See also [MRS88] for the classification of G-symplectic irreducible representations and implications of the components of V on the eigenvalues of elements of $\mathrm{SP}_G(V)$ (and hence $\mathfrak{sp}_G(V)$). In particular, the authors classify *cyclospectral representations* V, for which eigenvalues of any linear map of $\mathrm{SP}_G(V)$ are all contained in the unit circle, and therefore elements of $\mathfrak{sp}_G(V)$ admit only purely imaginary eigenvalues.

Below, these results are presented as far as needed for the theory developed later on.

Let V be a symplectic vector space.

Proposition 3.1. *If λ is an eigenvalue of $A \in \mathfrak{sp}(V)$, $\bar{\lambda}$, $-\lambda$, and $-\bar{\lambda}$ are also eigenvalues of A with the same geometric and algebraic multiplicities and the same sizes of Jordan blocks.*

Proof. Since A is a real matrix, we only have to show the claim for $-\lambda$. Choose an inner product $\langle\cdot,\cdot\rangle$ and suppose $\omega = \langle\cdot, J\cdot\rangle$. Then $A \in \mathfrak{sp}(V)$ implies $A^T J = -JA$. Thus, A^T and $-A$ are similar matrices and the same is true for A and $-A$. □

The proposition yields a fundamental difference between the generic dynamic behaviour near equilibria in Hamiltonian systems and general dynamical systems:

Corollary 3.2. *If $\mathrm{d}X_h(0)$ has a pair of algebraically simple purely imaginary eigenvalues, there is some neighbourhood $\mathcal{N} \subset C^\infty(V)$ of h (C^∞-topology or Whitney C^∞-topology) such that $\mathrm{d}X_{\tilde{h}}(0)$ has the same property for any $\tilde{h}$ in $\mathcal{N}$.*

Proof. If $\tilde{h}$ is close to h, the map $\mathrm{d}X_{\tilde{h}}(0)$ is close to $\mathrm{d}X_h(0)$ within the space of linear maps. If $\alpha\mathrm{i}$ is an algebraically simple eigenvalue of $\mathrm{d}X_h(0)$ with $\alpha \in \mathbb{R} \setminus \{0\}$, the implicit function theorem yields a single solution $c = c(A) \in \mathbb{C}$ to the equation

$$\det(A - c\mathbb{1}) = 0$$

for any complex linear endomorphism A in some neighbourhood of $\mathrm{d}X_h(0)$ such that $c(\mathrm{d}X_h(0)) = \alpha\mathrm{i}$. Since $-\overline{c(\mathrm{d}X_{\tilde{h}}(0))}$ is also an eigenvalue of $\mathrm{d}X_{\tilde{h}}(0)$, by uniqueness,

$$-\overline{c(\mathrm{d}X_{\tilde{h}}(0))} = c(\mathrm{d}X_{\tilde{h}}(0)),$$

and thus, $c(\mathrm{d}X_{\tilde{h}}(0))$ is purely imaginary. □

Hence, in contrast to general dynamical systems, equilibrium points in Hamiltonian systems are not generically hyperbolic.

Moreover, the quadruplets of eigenvalues give rise to a symplectic splitting of V:

Definition 3.3. For $A \in \mathfrak{sp}(V)$ and any $\lambda \in \mathbb{C}$, let E_λ denote the real part of the sum of the generalized eigenspaces of λ and $\bar{\lambda}$ and set $E_{\pm\lambda} := E_\lambda + E_{-\lambda}$.

Proposition 3.4. *For any $A \in \mathfrak{sp}(V)$ and $\lambda \in \mathbb{C}$, the space $E_{\pm\lambda}$ is symplectic. If $\mu \notin \{\lambda, \bar{\lambda}, -\lambda, -\bar{\lambda}\}$, $E_{\pm\lambda}$ and $E_{\pm\mu}$ are ω-orthogonal.*

Proof. See [GS87, Proposition 3.1] for a direct proof. Alternatively, this follows from the normal form theory in [Wil36]. □

The following result can be shown in a similar way:

Proposition 3.5. *For $A \in \mathfrak{sp}_G(V)$ and any $\lambda \in \mathbb{C}$ with $\mathrm{Re}\lambda \neq 0$, the space E_λ is a Lagrangian subspace of $E_{\pm\lambda}$.*

Proof. Set
$$B := (A - \lambda\mathbb{1})(A - \bar{\lambda}\mathbb{1}) = A^2 - 2\mathrm{Re}\lambda A + |\lambda|^2\mathbb{1}.$$

Then there is some $n \in \mathbb{N}$ such that $B^n\big|_{E_\lambda} = 0$. Consider B^n as a polynomial in A and let $(B^n)_o$ be the sum of the odd terms in A and $(B^n)_e$ the sum of the even ones. Then $B^n = (B^n)_o + (B^n)_e$. We claim that the restrictions of $(B^n)_o$ and $(B^n)_e$ to the space E_λ are both invertible: It is

$$\begin{aligned} B^n &= \sum_{k=0}^{n} (-1)^k \binom{n}{k} 2^k (\mathrm{Re}\lambda)^k A^k (A^2 + |\lambda|^2\mathbb{1})^{n-k} \\ &= \sum_{k=0}^{n} (-1)^k \binom{n}{k} 2^k (\mathrm{Re}\lambda)^k A^k ((A-\lambda\mathbb{1})(A-\bar{\lambda}\mathbb{1}) + 2\mathrm{Re}\lambda A)^{n-k}, \end{aligned}$$

and $(B^n)_o$ is given by the sum over the odd numbers k, while $(B^n)_e$ is given by the sum over the even ones. Now, $(B^n)_o$ and $(B^n)_e$ have a factorization in the polynomial ring with variable A into linear factors of the form $A - c\mathbb{1}$, $c \in \mathbb{C}$. For every $c \neq \lambda, \bar{\lambda}$, these factors are invertible on the space E_λ. Thus, we only have to show that $A - \lambda\mathbb{1}$ and $A - \bar{\lambda}\mathbb{1}$ do not occur as factors of $(B^n)_o$ and $(B^n)_e$. To show this, we insert $\lambda\mathbb{1}$ and $\bar{\lambda}\mathbb{1}$ in the place of A in the polynomial representations of $(B^n)_o$ and $(B^n)_e$. We obtain a product of the term $2^n(\mathrm{Re}\lambda)^n\lambda^n\mathbb{1}$ and $2^n(\mathrm{Re}\lambda)^n\bar{\lambda}^n\mathbb{1}$ respectively and a negative real number in the case $(B^n)_o$ and a positive one in the case $(B^n)_e$. Thus, in all cases, the result does not vanish and hence no factor $\lambda\mathbb{1}$ or $-\lambda\mathbb{1}$ occurs. This proves the claim.

Now, the bilinear form $\omega(\cdot, B^n\cdot)$ vanishes on E_λ. Thus on E_λ,

$$\omega(\cdot, (B^n)_o\cdot) = -\omega(\cdot, (B^n)_e\cdot).$$

Since the form on the left hand side is symmetric while the on the right hand side is skew-symmetric, both forms vanish. This yields that ω vanishes on E_λ, since $(B^n)_o$ and $(B^n)_e$ are automorphisms of E_λ. □

Next, we consider the symmetry given by the action of the compact group G.

Lemma 3.6. *V admits a G-invariant inner product such that the endomorphism J that represents ω commutes with the group action and defines a complex structure, i.e. $J^2 = -\mathbb{1}$.*

Proof. Let $\tilde{J}$ denote the endomorphism which represents the symplectic form ω with respect to some G-invariant inner product $\langle\cdot,\cdot\rangle$. Then $\tilde{J}$ commutes with the G-action: $\forall g \in G$

$$\langle g\cdot, g\tilde{J}\cdot\rangle = \langle\cdot, \tilde{J}\cdot\rangle = \langle g\cdot, \tilde{J}g\cdot\rangle.$$

Since $\tilde{J}$ is skew-symmetric, $\tilde{J}$ is normal and has only purely imaginary eigenvalues. Thus there is an orthogonal splitting $W = \oplus E_{\pm\beta\mathrm{i}}$ into the sums of the eigenspaces for pairs $\pm\beta\mathrm{i}$. By the equivariance of $\tilde{J}$, the spaces $E_{\pm\beta\mathrm{i}}$ are G-invariant. Hence rescaling $\langle\cdot,\cdot\rangle$ on each of the $E_{\pm\beta\mathrm{i}}$ components yields a G-invariant inner product such that the representing endomorphism J with respect to this inner product has the eigenvalues $\pm\mathrm{i}$ and $J^2 = -\mathbb{1}$. □

Theorem 3.7 ([GS87, Theorem 2.1])**.** *Any G-symplectic representation splits into a direct sum of pairwise ω-orthogonal irreducible G-symplectic subrepresentations.*

Theorem 3.8. *An irreducible G-symplectic representation V is of one of the three following types:*

1. *a sum of the form $W \oplus W$, where W is an absolutely irreducible G-representation such that each of the two summands is a Lagrangian subspace of $W \oplus W$,*

2. *an irreducible representation of complex type, or*

3. *an irreducible representation of quaternionic type.*

In the cases 1 and 3 two such symplectic representations are isomorphic iff the underlying G-representations are isomorphic. In the complex case 2, each representation admits exactly two isomorphism classes of G-symplectic structures. Each pair ω and $-\omega$ of G-invariant symplectic forms represents both isomorphism classes.

Proof. By Lemma 3.6, ω can be represented by a G-equivariant complex structure. In this way, V can be considered as an irreducible complex G-representation. The underlying real representation of a complex irreducible one is of one of the forms 1, 2, or 3. Furthermore, two complex irreducible representation of the form 1 or 3 are isomorphic iff they are isomorphic as real representations, while in case 2, there are the two isomorphism classes of V and $\overline{V}$. (See for example [BtD85, chapter II, Theorem 6.7, Table 6.2, and Proposition 6.1] and use the uniqueness of the isotypic decomposition of $e^{\mathbb{C}}_{\mathbb{R}} r^{\mathbb{C}}_{\mathbb{R}} V_1 \simeq e^{\mathbb{C}}_{\mathbb{R}} r^{\mathbb{C}}_{\mathbb{R}} V_2$ if $r^{\mathbb{C}}_{\mathbb{R}} V_1 \simeq r^{\mathbb{C}}_{\mathbb{R}} V_2$ for complex representations V_1 and V_2). □

Definition 3.9. Two irreducible G-symplectic representations of complex type are *dual* to each other iff they are isomorphic as (real) G-representations, but non-isomorphic as G-symplectic representations.

Again, we are interested in the stable occurrence of non-hyperbolic equilibria. Non-hyperbolicity is necessary for interesting dynamic behaviour like the existence of nearby relative equilibria.

In contrast to general dynamical systems, in Hamiltonian systems an equilibrium needs to be non-hyperbolic to be stable: If the equilibrium is stable, even all eigenvalues have to lie on the imaginary axis, since every eigenvalue with non-vanishing real part implies the existence of one with positive real part.

Definition 3.10. The real part of the sum of the eigenspaces corresponding to purely imaginary eigenvalues of $\mathrm{d}X_h(0)$ is called the *centre space* $\mathbb{E}_\mathrm{c}$ of $\mathrm{d}X_h(0)$.

In the following, we investigate the centre space $\mathbb{E}_\mathrm{c}$ of $\mathrm{d}X_h$. In particular, we are interested in isomorphism classes of $\mathbb{E}_\mathrm{c}$ that are stable under perturbation of h.

We start with the following well-known observation on continuity of eigenvalues:

Lemma 3.11. *Consider the action of the symmetric group S_n on $\mathbb{C}^n$ by permutation of coefficients. Let $F : \mathbb{C}^n/_{S_n} \to \mathbb{C}^n$ be the map from the roots to the coefficients of monic polynomials of degree n (which can be computed explicitly be factoring out $p(x) = (x - x_{n-1}) \cdot \dots \cdot (x - x_0)$). Then F is a homeomorphism.*

Proof. F is continuous and bijective. Moreover, F is proper: We only need an estimation in the coefficients which yields an $M > 0$ such that $|x^n + a_{n-1}x^{n-1} + \dots + a_0| > 0$ if $|x| > M$. For example, we may take

$$M = n \max(|a_{n-1}|, \sqrt{|a_{n-2}|}, \sqrt[3]{|a_{n-3}|}, \dots, \sqrt[n]{|a_0|})$$

Thus F is a continuous proper bijection into a compactly generated Hausdorff space and hence a homeomorphism. □

Therefore, the eigenvalues of a matrix depend continuously on its entries, which determine the characteristic polynomial.

Now, we consider a generalization of Corollary 3.2 to the symmetric case:

Lemma 3.12. *Let V be a G-symplectic vector space and $h : V \to \mathbb{R}$ be a G-invariant Hamiltonian function. If $\mathrm{d}X_h(0)$ has a pair of purely imaginary eigenvalues $\pm\alpha\mathrm{i}$, $\alpha \in \mathbb{R} \setminus \{0\}$ such that $E_{\alpha\mathrm{i}}$ is an irreducible G-symplectic subrepresentation, there is a neighbourhood $\mathcal{N} \subset C^\infty(V)^G$ such that $\mathrm{d}X_{\tilde{h}}(0)$ has a pair of purely imaginary eigenvalues $\pm\tilde{\alpha}I$, $\alpha \in \mathbb{R} \setminus \{0\}$ with $E_{\tilde{\alpha}\mathrm{i}} \simeq E_{\alpha\mathrm{i}}$ for every $\tilde{h} \in \mathcal{N}$.*

Proof. By Lemma 3.6, V may be considered as a complex G-representation. Then $E_{\alpha\mathrm{i}}$ is a complex irreducible subrepresentation. Let W denote the isotypic component of $E_{\alpha\mathrm{i}}$. For any invariant Hamiltonian $\tilde{h}$, the space W is $\mathrm{d}X_{\tilde{h}}(0)$-invariant. If $\tilde{h}$ is close to h, $\mathrm{d}X_{\tilde{h}}(0)\big|_W$ is close to $\mathrm{d}X_{\tilde{h}}(0)\big|_W$. Then $\mathrm{d}X_{\tilde{h}}(0)\big|_W$ has an eigenvalue λ near $\alpha\mathrm{i}$. Since generalized eigenspaces of G-equivariant linear maps are G-invariant, all eigenvalues of $\mathrm{d}X_{\tilde{h}}(0)\big|_W$ and $\mathrm{d}X_h(0)\big|_W$ occur with multiplicities that are multiples of $\dim_{\mathbb{R}} E_{\alpha I}$. (Note that the generalized eigenspaces of $\mathrm{d}X_{\tilde{h}}(0)\big|_W$ and $\mathrm{d}X_{\tilde{h}}(0)\big|_W$ are subspaces of $W \otimes_{\mathbb{R}} \mathbb{C}$, where W is considered as a real vector space.) Thus, for $\tilde{h}$ near h, all eigenvalues of $\mathrm{d}X_{\tilde{h}}(0)\big|_W$ in some neighbourhood of $\alpha\mathrm{i}$ coincide. This yields $-\bar{\lambda} = \lambda$. □

Corollary 3.13. *If all the spaces E_λ contained in $\mathbb{E}_{\mathrm{c}}$ are G-symplectic irreducible and $\mathrm{d}X_h(0)$ is non-degenerate, the centre space of $\mathrm{d}X_{\tilde{h}}(0)$ is isomorphic to $\mathbb{E}_{\mathrm{c}}$ for every $\tilde{h}$ close enough to h.*

Proof. By Lemma 3.12, the centre space of $\mathrm{d}X_{\tilde{h}}(0)$ contains a space isomorphic to $\mathbb{E}_{\mathrm{c}}$. Since the other eigenvalues of $\tilde{h}$ are bounded away from the imaginary axis and the sum of their multiplicities coincides with that of the eigenvalues with non-vanishing real part of $\mathrm{d}X_h(0)$, the dimensions of both centre spaces are the same. □

By the following theorem of Dellnitz, Melbourne, and Marsden, the assumption of Corollary 3.13 is generic:

Theorem 3.14. *([DMM92, Theorem 3.1]) Let G be a compact Lie group and V be a G-symplectic representation. There is a dense open subset $\mathcal{O} \subset \mathfrak{sp}_G(V)$ such that for every $A \in \mathcal{O}$ and any $\beta > 0$ the space $E_{\pm\beta\mathrm{i}}$ of A is symplectic irreducible.*

Theorem 3.14 yields the following result about generic normal forms, which is also a corollary of the normal form theory of [DM93a]:

Theorem 3.15. *If $A \in \mathcal{O}$ (as in Theorem 3.14), then there are G-invariant inner products on the eigenspaces $E_{\pm\beta\mathrm{i}}$ of A such that the matrix that represents A commutes with the matrix J that represents ω and J satisfies $J^2 = -\mathbb{1}$.*

Proof. The form $Q = \omega(\cdot, A\cdot)$ is G-invariant and symmetric. If $E_{\pm\beta\mathrm{i}}$ is irreducible, the representing endomorphism JA of Q with respect to any G-invariant inner product is a real multiple of the identity (since JA is G-equivariant and has only real eigenvalues) and hence commutes with J. Then A commutes with J as well. Thus we may choose any appropriate inner product such that $J^2 = -\mathbb{1}$.

Otherwise $E_{\pm\beta\mathrm{i}}$ is a sum of two isomorphic absolutely irreducible representations by Theorem 3.8. For any choice of an invariant inner product, the endomorphism representing $\omega(\cdot, A\cdot)$ has absolutely irreducible perpendicular eigenspaces $W_1 \simeq W_2$ and $V = W_1 \oplus W_2$. By Theorem 3.8, W_1 and W_2 are isotropic. Hence for any invariant inner product, the representing endomorphism of ω maps W_1 into W_2 and vice versa. Thus for any choice of a G-invariant inner product, ω is represented by a matrix of the form

$$\begin{pmatrix} 0 & a\mathbb{1} \\ -a\mathbb{1} & 0 \end{pmatrix}$$

with $a \in \mathbb{R}$. Choosing an appropriate inner product, we may suppose $a = 1$.

If the eigenvalues of $J \circ A$ are $\lambda_1, \lambda_2 \in \mathbb{R}$, A is a multiple of

$$\begin{pmatrix} 0 & -\mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \circ \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} 0 & -\lambda_2 \\ \lambda_1 & 0 \end{pmatrix}.$$

Since $\det A = \beta^2 > 0$, $\lambda_1\lambda_2 > 0$ (hence Q is definite). An appropriate scaling of the inner products on W_1 and W_2 yields a new invariant inner product, with respect to which Q is represented by a multiple of the identity and the matrix J is preserved. □

From Theorem 3.14 and Theorem 3.15 we obtain immediately:

Corollary 3.16. *There is an open, dense subset $\mathcal{W}$ of $C^\infty(V)^G$ (C^∞-topology or Whitney C^∞-topology) such that for $h \in \mathcal{W}$, there is an inner product such that the eigenspaces of the matrix representing $\mathrm{d}^2h(0)\big|_{\mathbb{E}_\mathrm{c}}$ are G-symplectic irreducible subrepresentations and ω is represented by a matrix J with $J^2 = -\mathbb{1}$. Namely, $\mathcal{W}$ consists of the functions $h \in C^\infty(V)^G$ with $\mathrm{d}X_h(0) \in \mathcal{O}$ (as in Theorem 3.14).*

Here, $\mathcal{W}$ is the set of Hamiltonian functions h with $\mathrm{d}X_h(0) \in \mathcal{O}$.

As pointed out in [MRS88], in some cases, V and $\mathbb{E}_\mathrm{c}$ coincide for all Hamiltonian functions. Montaldi, Roberts, and Stewart call these symplectic representations *cyclospectral representations* and characterize their isotypic compositions:

Theorem 3.17. *A G-symplectic representation is cyclospectral iff it contains no representation of real type and no pair of dual G-symplectic representations of complex type and each isotypic component of a representation of quaternionic type is irreducible.*

Proof. Assume that V is a G-symplectic representation of this form. Using Lemma 3.6, the symplectic form on V can be represented by a complex structure J. If we split V into G-symplectic irreducible components, each component is J-invariant. Furthermore, J commutes with any $A \in \mathfrak{sp}_G(V)$:

Since A and J are G-equivariant endomorphisms and hence preserve isotypic components, we only have to show this for the restrictions of A and J to isotypic components of V.

If $W \subset V$ is irreducible of complex type, $J\big|_W$ is a complex structure on W which commutes with the G-action. Since $\mathrm{End}_G(W) \simeq \mathbb{C}$, $J\big|_W$ corresponds to i or $-\mathrm{i}$ under this identification. The isotypic component associated to W does not contain a dual of the G-symplectic irreducible representation W. Therefore, either $J\big|_W$ acts as multiplication by i or by $-\mathrm{i}$ on all G-symplectic subrepresentations of V that are isomorphic to W as (real) G-representations (if the isomorphism $\mathrm{End}_G(W) \simeq \mathbb{C}$ is fixed). Hence, J is a complex multiple of the identity on any isotypic component of complex type and commutes with restriction of A.

If $W \subset V$ is irreducible of quaternionic type, W coincides with the corresponding isotypic component. Since $JA \in \mathrm{End}_G(W) \simeq \mathbb{H}$ is symmetric, JA is a real multiple of the identity: If $\langle\cdot,\cdot\rangle$ is a G-invariant inner product, this is also true for $\langle h\cdot, h\cdot\rangle$ for any $h \in \mathrm{Aut}_G(W)$. Hence, by averaging over the unit quaternions, we may assume that the unit quaternions correspond to orthogonal endomorphisms of $\mathrm{End}_G(W)$. Then, i, j and k correspond to skew-symmetric matrices and hence the symmetric matrices in $\mathrm{End}_G(W)$ are the real multiplies of the identity. Therefore, J commutes with JA and hence with A.

Since J and JA commute, the eigenvalues of their product $-A$ are given by products of eigenvalues of J and JA. Since J has the eigenvalues i and $-\mathrm{i}$ and the symmetric matrix JA has real eigenvalues, the eigenvalues of A are purely imaginary.

For the contrary, it suffices to give examples of endomorphisms in $\mathfrak{sp}_G(V)$ with eigenvalues with non-vanishing real part in the case that V is a sum of two absolutely irreducible representations, two irreducible representations of quaternionic type, or two dual G-symplectic representations of complex type. These examples are shown in the following table, where we choose an isomorphism $\mathrm{End}_G(W) \simeq \mathrm{End}_{\mathbb{K}}(\mathbb{K}^2)$, $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$, an inner product, and an orthonormal base with respect to which J is of this form:

$W \oplus W$ W absolutely irreducible	$W_1 \oplus W_2$ W_1,W_2 complex duals	$W \oplus W$ W irreducible of type $\mathbb{H}$
$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$	$J = \begin{pmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{pmatrix}$	$J = \begin{pmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{pmatrix}$
$A = \begin{pmatrix} 0 & a \\ a & 0 \end{pmatrix}$	$A = \begin{pmatrix} 0 & a\mathrm{i} \\ -a\mathrm{i} & 0 \end{pmatrix}$	$A = \begin{pmatrix} 0 & a\mathrm{i} \\ -a\mathrm{i} & 0 \end{pmatrix}$

□

For our applications, we are interested in general how the centre space may look like for a given G-symplectic representation V. More precisely, we would like to have a list of all isotypic decompositions of $\mathbb{E}_\mathrm{c}$ that occur. Such a list will be given in the following with the additional property that for each isotypic composition of this list, there is a non-empty open subset of Hamiltonian functions in $C^\infty(V)$ with $\mathbb{E}_\mathrm{c}$ of this isomorphism class.

The main part is to determine the "minimum" of $\mathbb{E}_c$, i.e. a G-symplectic representation $V_{\min}$ such that the centre space always contains a G-symplectic subrepresentation isomorphic to $V_{\min}$. This is done in [DM93a]: The authors call a representation *weakly cyclospectral* iff $\mathbb{E}_c \neq \{0\}$ holds for every Hamiltonian function. They classify the weakly cyclospectral representations by deducing the isotypic composition of $V_{\min}$ from the normal forms calculated in [DM93a]. We give a more direct proof here:

Theorem 3.18. *Let V be a G-symplectic representation and $V = \oplus_i V_i$ its isotypic decomposition. Let $V_{\min}$ be the sum of the following spaces:*

1. *U_i for each i with $V_i \simeq U_i^u$, where U_i is irreducible of type $\mathbb{H}$ and u is odd.*
2. *For each i corresponding to an irreducible representation U_i of type $\mathbb{C}$, we distinguish the two isomorphism classes U_i and $\bar{U}_i$ of G-symplectic representations such that V_i decomposes as $V_i = U_i^k \oplus \bar{U}_i^l$. W.l.o.g. suppose $k \geq l$. Then $V_{\min}$ contains a copy of U_i^{k-l}.*

Then $\mathbb{E}_c$ contains a G-symplectic subrepresentation isomorphic to $V_{\min}$ for every Hamiltonian function.

Proof. Since every isotypic component of V is a $\mathrm{d}X_h(0)$-invariant G-symplectic subrepresentation, we may suppose $V = V_i$ and consider the two cases one after another.

1. Let c be an eigenvalue of $\mathrm{d}X_h(0)$ with nonzero real part. Recall that E_c denotes the real part of the sum of the generalized eigenspaces of the pair c and $\bar{c}$. Then by Proposition 3.1, $\dim E_c = \dim E_{-c}$. Moreover, both spaces are G-invariant and hence isomorphic to U_i^l for some number $l \in \mathbb{N}$. Thus, $E_{\pm c} \simeq U_i^{2l}$. Since u is odd, $\mathrm{d}X_h(0)$ must have at least one purely imaginary eigenvalue if $V = U_i^u$.
2. Again, let c be an eigenvalue of $\mathrm{d}X_h(0)$ with nonzero real part. By Proposition 3.5, E_c is a Lagrangian subspace of the symplectic space $E_{\pm c}$. Choose an appropriate G-invariant inner product such that ω is represented by a G-equivariant endomorphism J with $J^2 = \mathbb{1}$. Then $E_{\pm c} = E_c \oplus J(E_c)$. Split E_c into a sum $\bigoplus_i U_i$ of irreducible G-representations. Then $E_{\pm c} = \bigoplus_i (U_i \oplus J(U_i))$. If we consider J as a complex structure, each space $U_i \oplus J(U_i)$ is just the complexification of the real representation U_i. By [BtD85, chapter II, Proposition 6.6], the complexification of a real representation is isomorphic to a sum of complex duals. Thus $E_{\pm c}$ splits into a sum of pairs of complex duals and hence both types occur with the same multiplicity in $E_{\pm c}$. □

We now state a theorem which yields a list as described above. This implies the minimality of $V_{\min}$.

Theorem 3.19. *For a G-symplectic representation V and a Hamiltonian function $h \in C^\infty(V)^G$, the decomposition of the centre space $\mathbb{E}_c$ of $\mathrm{d}X_h(0)$ into pairwise ω-orthogonal G-symplectic irreducible subrepresentations is given by a sum consisting of*

- *pairs of absolutely irreducible G-representations,*

- *an even number of every isomorphism class of irreducible G-representations of type $\mathbb{H}$,*
- *the same number of every isomorphism class of irreducible G-representations of type $\mathbb{C}$ and its dual, and*
- *the decomposition of $V_{\min}$ as described in Theorem 3.18.*

Moreover, for each sum of this form, there is a non-empty open set of Hamiltonian functions such that $\mathbb{E}_{\mathrm{c}}$ is of this isomorphism class.

Proof. The proof of 3.18 yields immediately that $\mathbb{E}_{\mathrm{c}}$ has to be of this form.

For the converse, consider a splitting of V into pairwise ω-orthogonal G-symplectic irreducible subrepresentations and choose a subspace given by a sum $\bigoplus_i U_i$ of components of this splitting of the described form. Fix an inner product as in Lemma 3.6. If with respect to this inner product, for each i, the space U_i coincides with the α_i-eigenspace of $\mathrm{d}^2h(0)$ for some $\alpha_i \in \mathbb{R} \setminus \{0\}$, $\bigoplus_i U_i$ is an invariant subspace of $\mathrm{d}X_h(0)$. If in addition on a $\mathrm{d}X_h(0)$-invariant complement of $\bigoplus_i U_i$, the matrix $\mathrm{d}X_h(0)$ consists of block-matrices of the form described in the proof of Theorem 3.17, the centre space of $\mathrm{d}X_h(0)$ coincides with $\bigoplus_i U_i$. Thus the set of Hamiltonian functions h with this property is non-empty. By Corollary 3.13, small perturbations of a function h with $\mathrm{d}X_h(0)$ of this form preserve the isomorphism class of $\mathbb{E}_{\mathrm{c}}$ if $\alpha_i \neq \alpha_j$ for $i \neq j$. □

Chapter 4

Free actions

If the Hamiltonian action of the connected compact group G on the phase space P is free, the structure of relative equilibria is quite well-understood. This situation occurs in the dynamics of a rigid body moving in space without the action of external forces.

The starting point of the survey was a persistence result for non-degenerate relative equilibria with regular momenta, which are defined as follows:

Definition 4.1. A relative equilibrium for a free action is *non-degenerate* iff its image in the reduced space is a non-degenerate critical point of the reduced Hamiltonian.

Definition 4.2. An element $\xi \in \mathfrak{g}$ ($\mu \in \mathfrak{g}^*$) is *regular* iff its isotropy with respect to the adjoint (coadjoint) action of G is a maximal torus.

The set of regular elements is open and dense in $\mathfrak{g}$ ($\mathfrak{g}^*$) (see for example [BtD85, chapter V, Proposition 2.6]).

Let P be a symplectic phase space with a free Hamiltonian action of the connected compact Lie group G. The following theorem is known as Arnol'd's theorem, since the main ideas can be found in principle in [Arn78]. (There, see Theorem 8 of appendix E and appendix 5.)

Theorem 4.3. *If $p \in P$ is a non-degenerate relative equilibrium and $\alpha = \mathbf{J}(p)$ is regular, there is a neighbourhood of p in which the set of relative equilibria forms a smooth manifold such that for any momentum value μ near α, there is exactly one relative equilibrium in the reduced space P_μ close to p.*

Proof. Since the action is free, the momentum map $\mathbf{J}$ is a submersion. For any smooth path γ from α to μ, the isotopy theorem of transversality theory implies that there is a smooth isotopy of diffeomorphisms $\phi_t : \mathbf{J}^{-1}(\alpha) \to \mathbf{J}^{-1}(\gamma(t))$ (see [AR67]). If μ is close to α, the isotropy subgroups G_μ and G_α are conjugate. We assume w.l.o.g. that $G_\mu = G_\alpha$ and choose a path in $(\mathfrak{g}^*)^{G_\mu}$. Then we can construct equivariant diffeomorphisms ϕ_t. (This follows from the proof of the isotopy theorem given in [AR67]). Hence we obtain a family of diffeomorphisms $\bar{\phi}_t : P_\alpha \to P_{\gamma(t)}$. The result follows from the implicit function theorem applied to the family $h \circ \bar{\phi}_t$. □

Arnol'd's theorem was generalized in the 90s up to 2000 by Montaldi, Patrick, and Roberts to the case of non-regular momentum values. In [Pat95, Theorem 17, part (i)], Patrick proves another smoothness result:

Theorem 4.4. *Let $p \in P$ be a non-degenerate relative equilibrium with generator ξ and momentum μ. If μ or ξ is regular, the set of relative equilibria is a smooth manifold of dimension* $\dim G + \operatorname{rank} G$.

Moreover, Patrick shows that this manifold is symplectic if μ is regular and the spectrum of the linearization of the reduced vector field is disjoint from the spectrum of the linear map ad_ξ (see [Pat95, Theorem 17, part (ii)]). Both results are generalized and improved in [PR00], see below.

The results in [Mon97] and [PR00] show the occurrence of bifurcation of the structure of relative equilibria, in the sense that the set is not smooth at each element. In [Mon97], Montaldi investigates the structure of the set of relative equilibria near a non-degenerate equilibrium with regular generator but possibly non-regular momentum. He shows that the number of relative equilibria in the reduced spaces is related to the cardinality of the Weyl group orbit of the corresponding momentum and hence changes in the neighbourhood of non-regular momentum values.

Patrick and Roberts generalize Patrick's Theorem 4.4 in [PR00] and obtain a result about the generic global structure of the set of relative equilibria, which holds for all Hamiltonian functions that satisfy some transversality condition: For these, the set of relative equilibria is stratified by the conjugacy class of $G_\mu \cap G_\xi$.

We start with a sketch of the results of Montaldi ([Mon97]) about free actions and study the results of Patrick and Roberts ([PR00]) afterwards.

4.1 Bifurcation at non-regular momentum values

In [Mon97], Montaldi generalizes Arnol'd's observation to the case non-regular momentum by analysing the topology of the reduced spaces in the case of free group actions. This yields an estimate of the number of relative equilibria near a given relative equilibrium p with momentum value α in the reduced space P_μ for μ near α. Let $w(\mu)$ denote the cardinality of the Weyl group orbit of μ with respect to a maximal torus T with $\mu \in \mathfrak{t}^*$. Moreover, $\mathfrak{g}_\alpha^*$ contains a slice in α for the coadjoint action (see Lemma 2.4) and thus the coadjoint orbit of μ contains an element of $\mathfrak{g}_\alpha^*$ if μ is close to α. W.l.o.g. we assume $\mu \in \mathfrak{g}_\alpha^*$. Let $\mathcal{O}_\mu^\alpha$ be the coadjoint G_α-orbit of μ in $\mathfrak{g}_\alpha^*$.

Theorem 4.5. *Let P be a symplectic manifold with a free Hamiltonian action of a compact Lie group G. If $p \in P$ is a non-degenerate relative equilibrium with $\mathbf{J}(p) = \alpha$, there is a G-invariant neighbourhood U of p such that for $\mu \in \mathfrak{g}^*$ near α, there are at least $1 + \frac{1}{2}\dim \mathcal{O}_\mu^\alpha$ relative equilibria in P_μ. If all relative equilibria in P_μ are non-degenerate, this number is at least $\frac{w(\mu)}{w(\alpha)}$.*

Theorem 4.6. *If in addition the generator $\xi \in \mathfrak{g}$ of p is regular, all relative equilibria near p are non-degenerate and this estimate is exact.*

For the proof, Montaldi shows first that we may assume $\alpha = 0$: As can be seen from the Guillemin-Sternberg normal form, the local isomorphism class of the orbit space near $[p] \in P_\alpha$ is determined by G_p and G_α (hence in the free case by G_α). In fact, P is locally isomorphic to $G \times_{G_\alpha} Q_\alpha$, where $Q_\alpha = \mathbf{J}^{-1}(S_\alpha)$ for some G_α-invariant slice $S_\alpha \subset \mathfrak{g}^*$ to the coadjoint orbit of α (for details see [Mon97]). Q_α is a manifold, since it follows from the G-equivariance of $\mathbf{J}$ that $\operatorname{im} \mathrm{d}\mathbf{J}(p)$ contains the tangent space to the coadjoint orbit and hence $\mathbf{J}$ is transverse to S_α at p. The local isomorphism $P \simeq G \times_{G_\alpha} Q_\alpha$ yields locally $P/G \simeq Q_\alpha/G_\alpha$.

The space Q_α is symplectic: With respect to an appropriate choice of the decomposition

$$T_pP = T_0 \oplus T_1 \oplus N_0 \oplus N_1$$

(see section 2.2), the tangent space T_pQ_α is given by $T_0 \oplus N_0 \oplus N_1$:

$$\mathrm{d}\mathbf{J}(p)T_1 = \mathrm{d}\mathbf{J}(p)(\mathfrak{g} \cdot p) \subset T_\alpha \mathcal{O}_\alpha,$$

and $\mathrm{d}\mathbf{J}(p)$ is injective on T_1. Hence $\mathrm{d}\mathbf{J}(p)^{-1}(S_\alpha) \cap T_1 = \{0\} \in T_pP$. Furthermore, $T_0 \oplus N_1 = \ker \mathrm{d}\mathbf{J}(p)$ and $\mathrm{d}\mathbf{J}(p)N_0 \simeq \mathfrak{m}^*$ is a complement of $T_\alpha\mathcal{O}_\alpha$ within $\operatorname{im} \mathrm{d}\mathbf{J}(p) = \operatorname{ann}(\mathfrak{g} \cdot p)$, see Lemma 2.4 and Lemma 2.5, 1. A suitable choice of N_0 yields

$$\mathrm{d}\mathbf{J}(p)(T_0 \oplus N_0 \oplus N_1) = T_pS_\alpha.$$

By 2.5, $T_0 \oplus N_0 \oplus N_1$ is symplectic, thus Q_α is a G_α-symplectic space.

The projection of the restriction of $\mathbf{J}$ to Q_α to $\mathfrak{g}_\alpha^*$ defines a momentum map on Q_α, which we modify by adding $-\alpha$. Since $\alpha \in \mathfrak{g}_\alpha^{G_\alpha}$, we obtain a G_α-equivariant momentum map on Q_α such that the momentum of p is 0. Then reduced space $(Q_\alpha)_0$ is symplectomorphic to the reduced space P_α.

To complete the reduction to the case $\alpha = 0$, Montaldi shows that the cardinality $w_\alpha(\mu)$ of the Weyl group orbit of $\mu \in \mathfrak{g}_\alpha^*$ with respect to the G_α-action is equal to $\frac{w(\mu)}{w(\alpha)}$: If W and W_α denote the Weyl groups of G and G_α respectively, $W_\alpha \subset W$. Fix a Cartan subalgebra $\mathfrak{t} \subset \mathfrak{g}_\alpha$ containing μ. The slice theorem applied to the W-action on $\mathfrak{t}$ in the point α yields

$$W\mu \simeq W \times_{W_\alpha} (W_\alpha \mu),$$

and hence

$$w(\mu) = \#(W/W_\alpha) \cdot w_\alpha(\mu) = w(\alpha) \cdot w_\alpha(\mu).$$

Now the result follows from the topology of the reduced spaces: By the Guillemin-Sternberg normal form theorem, for free actions, any point with momentum 0 has a neighbourhood isomorphic to $G \times N \times \mathfrak{g}^*$. Hence the orbit space is locally isomorphic to $N \times \mathfrak{g}^*$ and the reduced spaces are of the form $P_\mu = N \times \mathcal{O}_\mu$. The Hamiltonian function h decomposes as $h = \bar{h} \circ \pi$, where $\pi : P \to P/G$ denotes the projection map. If p is a non-degenerate equilibrium, the Hessian of $\bar{h}(\cdot, 0) = h_0$ is non-degenerate. An application of the implicit function theorem or the Splitting Lemma as described in section 2.4 yields that for every μ in some neighbourhood of 0, there is a unique $v(\mu) \in N$ such that $\mathrm{d}_N \bar{h}(v(\mu), \mu) = 0$. Moreover, it follows that $(v(\mu), \mu)$ is a critical point of $\bar{h}\big|_{N \times \mathcal{O}_\mu}$ iff μ is a critical point of the function

$$\begin{aligned} H_\mu : \mathcal{O}_\mu &\to \mathbb{R} \\ \nu &\mapsto \bar{h}(v(\nu), \nu). \end{aligned}$$

Hence, we only have to estimate the number of critical points of a real valued function defined on $\mathcal{O}_\mu$. The most general estimate follows from Lyusternik-Schnirelmann-theory, which yields that a lower bound of the number of critical points of a function on a manifold is given by its cup lengths plus 1. For a symplectic space of dimension $2n$, ω^n defines a volume form and hence n is a lower bound of the cup length. Since $\mathcal{O}_\mu$ together with the Kirillov-Kostant-Souriau-form $\langle \mu, [\cdot,\cdot] \rangle$ is symplectic, a function on $\mathcal{O}_\mu$ has at least $1 + \frac{1}{2}\dim(\mathcal{O}_\mu)$ critical points.

For Morse functions, we obtain a sharper estimate: The Morse inequality yields that the number of critical points of a Morse function defined on $\mathcal{O}_\mu$ is at least $w(\mu)$. To obtain this estimate, one has to compute the homology of $\mathcal{O}_\mu$. A way to do this is to fix a regular element $\xi \in \mathfrak{g} = \mathfrak{g}^{**}$ and to restrict the linear function $\nu \mapsto \nu(\xi)$ to $\mathcal{O}_\mu$. It can be shown that ξ is a Morse function and a computation of the indices of the critical points implies the above estimation.

The main observation for the proof that ξ defines a Morse function is that the Hamiltonian vector field of this function with respect to the Kostant-Kirillov-Souriau symplectic form is given by $\nu \mapsto \operatorname{coad}_\xi \nu$: For any $\eta \in \mathfrak{g}$, the directional derivative along the vector $\eta \cdot \nu = \operatorname{coad}_\eta \mu$ is

$$\operatorname{coad}_\eta \nu(\xi) = \nu([\xi, \eta]).$$

Therefore, the critical points of the function are exactly the points of $(\mathfrak{g}^*)^\xi \cap \mathcal{O}_\mu$ where $(\mathfrak{g}^*)^\xi$ is the zero set of coad_ξ. If we identify $\mathfrak{g}^*$ and $\mathfrak{g}$ via an invariant inner product, this becomes $\mathfrak{g}_\xi \cap \mathcal{O}_\mu$. Since ξ is regular, $\mathfrak{g}_\xi = \mathfrak{t}$ for the maximal torus T whose Lie algebra contains ξ and hence the set of critical points coincides with a Weyl group orbit within $\mathcal{O}_\mu$. For the non-degeneracy of the critical points, Montaldi gives a short argument: Any critical point $\nu \subset \mathcal{O}_\mu$ is fixed by the torus $T(\xi)$ generated by ξ and hence its tangent space $T_\nu \mathcal{O}_\mu$ is $T(\xi)$-invariant and locally $T(\xi)$-symplectomorphic to $\mathcal{O}_\mu$ at μ. The evaluation at ξ is the Hamiltonian function of the Hamiltonian vector field on the symplectic space $T_\nu \mathcal{O}_\mu \subset \mathfrak{g}^*$ defined by the coadjoint action of ξ. Therefore the evaluation map coincides with the momentum map $\mathbf{J}^\xi$ for the linear $T(\xi)$-action and for the $T(\xi)$-action on $\mathcal{O}_\mu$. For a linear action, the momentum map is a homogeneous quadratic function. The same holds for the momentum map on $\mathcal{O}_\mu$ with respect to appropriate local coordinates in a neighbourhood of μ. Hence a critical point is non-degenerate iff it is isolated. Since the points of $\mathfrak{t} \cap \mathcal{O}_\mu$ are isolated, ν is a non-degenerate critical point.

Montaldi also uses this fact to prove Theorem 4.6:

The derivative of the function

$$\begin{aligned} H : \mathfrak{g}^* &\to \mathbb{R} \\ \nu &\mapsto \bar{h}(v(\nu), \nu) \end{aligned}$$

in 0 coincides with the restriction of

$$\mathrm{d}h(p) : T_p P \simeq \mathfrak{g} \times \mathfrak{g}^* \times N \to \mathbb{R}$$

to $\mathfrak{g}^*$. In [Mon97], it is shown that $\mathrm{d}h(p)\big|_{\mathfrak{g}^*}$ is given by the evaluation at ξ. (This is also contained in the characterization of relative equilibria derived from the bundle equation, see section 2.3.) Hence $\mathrm{d}H(0)$ restricted to $\mathcal{O}_\mu$ has exactly

$w(\mu)$ critical points for every $\mu \in \mathfrak{g}^*$. A blow up argument shows that the function H_μ has the same number of critical points if μ is small:

Let S be the unit sphere in $\mathfrak{g}^*$ and consider H as a smooth function defined on $\mathbb{R} \times S$ by $H(r,\theta) := H(r\theta)$. W.l.o.g., we assume $H(0) = 0$ and hence $H(0,\theta) = 0$. Then, there is a smooth function H_1 with

$$H(r,\theta) = rH_1(r,\theta). \tag{4.1}$$

To find critical points of H on $\mathcal{O}_\mu$, we search for critical points of H_1 on $\mathcal{O}_\mu$, since $\mathcal{O}_\mu$ is contained in a sphere. For H_1, equation (4.1) and the Taylor theorem yield

$$\begin{aligned} H_1(r,\theta) &= \partial_r H(r,\theta) - r\partial_r H_1(r,\theta) = \partial_r H(0,\theta) + r\tilde{H}(r,\theta) - r\partial_r H_1(r,\theta) \\ &= \mathrm{d}H(0)\theta + rR(r,\theta) = \langle \xi, \theta \rangle + rR(r,\theta) \end{aligned}$$

for appropriate smooth functions $\tilde{H}$ and R. Now, we consider the restriction of H_1 to $\mathbb{R} \times \mathcal{O}_\theta$. If $\nu_0 \in \mathcal{O}_\theta$ is a critical point of the evaluation at ξ, ν_0 is non-degenerate. The implicit function theorem yields that there is a smooth function $\nu : (-\delta,\delta) \to \mathcal{O}_\theta$ such that $\nu(0) = \nu_0$,

$$\mathrm{d}_{\mathcal{O}_\theta} H_1(r, \nu(r)) = 0,$$

and $\nu(r)$ is a non-degenerate critical point. Then $r\nu(r)$ is a non-degenerate critical point of $H_{r\theta}$ if $r \neq 0$. Moreover, $r\nu(r)$ is unique in rU for some neighbourhood U of $\nu_0 \in \mathcal{O}_\theta$. By compactness of $\mathcal{O}_\theta$, there are exactly $w(\theta) = w(r\theta)$ critical points of $H_{r\theta}$ for small $r > 0$.

4.2 Transverse relative equilibria

In [PR00], Patrick and Roberts extend Patrick's Theorem 4.4 to arbitrary isotropy subgroups $K = G_\xi \cap G_\mu$ of $(\xi,\mu) \in \mathfrak{g} \oplus \mathfrak{g}^*$. The non-degeneracy condition is replaced by a transversality condition, which is generic with respect to the Whitney C^∞-topology and a weaker condition than non-degeneracy in the case $K = T$ (not in general). The main idea is that the Whitney stratification by isotropy type of $\mathfrak{g}^* \oplus \mathfrak{g}$ induces a stratification of $P \times \mathfrak{g}$. Moreover, $P \times \mathfrak{g}$ is isomorphic to the subbundle $\mathcal{T} \subset TP$ with $\mathcal{T}_p = \mathfrak{g} \cdot p$. The image of X_h is contained in the subbundle $\mathcal{K} \subset TP$ with $\mathcal{K}_p = \ker \mathrm{d}\mathbf{J}(p)$. It is shown that the intersection $\mathcal{T}^c := \mathcal{T} \cap \mathcal{K}$ corresponds to the set $(\mathfrak{g}^* \oplus \mathfrak{g})^c = \{(\mu,\xi) \mid \mathrm{coad}_\xi\, \mu = 0\}$, which coincides with the points whose isotropy type contains a maximal torus. Hence, the set $\mathcal{T}^c$ is Whitney stratified. The authors call a relative equilibrium p *transverse*[1] iff $X_h \pitchfork \mathcal{T}^c$ at p within $\mathcal{K}$ with respect to this Whitney stratification.

Alternatively, the transversality condition may be expressed in terms of $\mathrm{d}h$ and a stratification of a subbundle of T^*P which corresponds to the stratified subbundle of TP via the isomorphism $TP \simeq T^*P$ induced by the symplectic form ω.

For all this, the condition that the action is free (at least locally free) is essential: First of all, the stratification of $P \times \mathfrak{g}$ is given by the preimages of the

[1] Originally, Patrick and Roberts use the term *transversal* here. Since this is a noun, which is often confused with the adjective *transverse*, it is better to say *transverse*.

strata of $\mathfrak{g}^* \oplus \mathfrak{g}$ under the map $\tilde{\mathbf{J}}$ defined by the momentum map via

$$\begin{aligned}\tilde{\mathbf{J}} : P \times \mathfrak{g} &\to \mathfrak{g}^* \oplus \mathfrak{g}\\ (p, \xi) &\mapsto (\mathbf{J}(p), \xi).\end{aligned}$$

Since $\operatorname{im} \mathrm{d}\mathbf{J}^\xi(p) = \operatorname{ann}(\mathfrak{g}_p) \subset \mathfrak{g}^*$, $\mathbf{J}$ is a submersion iff $\mathfrak{g}_p = \{0\}$ and so is $\tilde{\mathbf{J}}$. Therefore, in the case of a free action, the preimages of the strata form a Whitney stratification.

Moreover, $\mathcal{K}$ and $\mathcal{T}$ are vector bundles in the free case, and the map $I : (p, \xi) \mapsto \xi_p$ is an isomorphism of $P \times \mathfrak{g}$ and $\mathcal{T}$.

The same holds, of course, for the dual formulation: Here we consider the subbundles of T^*P that correspond to $\mathcal{T}$ and $\mathcal{K}$ under the isomorphism $TP \simeq T^*P$ via $X_p \mapsto \omega(X_p, \cdot)$. Since $\mathcal{T}_p = \mathfrak{g} \cdot p$ and

$$\mathcal{K}_p = \ker \mathrm{d}\mathbf{J}(p) = (\mathfrak{g} \cdot p)^{\perp_\omega} = T_p^{\perp_\omega},$$

the image of $\mathcal{T}$ is given by the bundle $\mathcal{K}^\circ$ with $\mathcal{K}_p^\circ = \operatorname{ann}(\mathcal{K}_p)$ and the image of $\mathcal{K}$ coincides with $\mathcal{T}^\circ$ which is given by $\mathcal{T}_p^\circ = \operatorname{ann}(\mathcal{T}_p)$. The map

$$I^\circ : (p, \xi) \mapsto \omega(\xi_p, \cdot) = \mathrm{d}\mathbf{J}^\xi(p)$$

is an isomorphism of $P \times \mathfrak{g}$ and $\mathcal{K}^\circ$ if the action is free.

Now, let us consider the results about the stratifications on $\mathfrak{g}^* \oplus \mathfrak{g}$ and $P \times \mathfrak{g}$ in detail.

A basic observation is that $(\mathfrak{g}^* \oplus \mathfrak{g})^c = \{(\mu, \xi) \mid \operatorname{coad}_\xi \mu = 0\}$ coincides with the image of $\mathcal{T}^c = \mathcal{T} \cap \mathcal{K}$ and $\mathcal{K}^{\circ c} := \mathcal{T}^\circ \cap \mathcal{K}^\circ$ under $\tilde{\mathbf{J}} \circ I^{-1}$ and $\tilde{\mathbf{J}} \circ (I^\circ)^{-1}$ respectively: If $\mathbf{J}(p) =: \mu$, $\xi \cdot p \in \ker \mathrm{d}\mathbf{J}(p)$ is equivalent to $\xi \in \mathfrak{g}_\mu$ due to the equivariance of $\mathbf{J}$.

Moreover, the isotropy subgroup of an element $(\mu, \xi) \in \mathfrak{g}^* \oplus \mathfrak{g}$ is given by $G_\mu \cap G_\xi$, and in the case $\xi \in \mathfrak{g}_\mu$, this coincides with the isotropy subgroup $(G_\mu)_\xi$ of ξ with respect to the adjoint action of G_μ. Since $(G_\mu)_\xi$ contains a maximal torus T of G_μ which is also maximal in G, $(\mathfrak{g}^* \oplus \mathfrak{g})^c$ is contained in the set of points whose isotropy type contains a maximal torus. For the contrary, assume that a maximal torus T is contained in $G_\mu \cap G_\xi$ and note that $(\mathfrak{g}^* \oplus \mathfrak{g})^T = \mathfrak{t}^* \oplus \mathfrak{t}$, where $\mathfrak{t}$ is the Lie algebra of T. This yields $\operatorname{coad}_\xi \mu = 0$. Hence $(\mathfrak{g}^* \oplus \mathfrak{g})^c$ consists of a substratification of the isotropy type stratification of $\mathfrak{g}^* \oplus \mathfrak{g}$ and thus $\mathcal{T}^c$ and $\mathcal{K}^{\circ c}$ are stratified subsets with the induced stratifications.

Next, we investigate the stratification by isotropy type on $(\mathfrak{g}^* \oplus \mathfrak{g})^c$. Consider $(\mu, \xi) \in \mathfrak{g}^* \oplus \mathfrak{g}^c$ with isotropy subgroup $K = G_\mu \cap G_\xi$ with Lie algebra $\mathfrak{k}$. In [PR00, Proposition 1], it is shown using the Slice Theorem, that there is a neighbourhood of (μ, ξ) in $(\mathfrak{g}^* \oplus \mathfrak{g})^c$ which is isomorphic to

$$G \times_K (\mathfrak{k}^* \oplus \mathfrak{k})^c.$$

Hence, the local stratification of $(\mathfrak{g}^* \oplus \mathfrak{g})^c$ by isotropy type near (μ, ξ) is determined by that of $(\mathfrak{k}^* \oplus \mathfrak{k})^c$.

Let $Z(K)$ denote the centre of K with Lie algebra $\mathfrak{z}$ and $\mathfrak{l}$ be the Lie algebra of $L = K/Z(K)$. The next observation is that the K-representation $\mathfrak{k}^* \oplus \mathfrak{k}$ splits into

$$\mathfrak{k}^* \oplus \mathfrak{k} = (\mathfrak{z}^* \oplus \mathfrak{z}) \oplus (\mathfrak{l}^* \oplus \mathfrak{l})$$

and $(\mathfrak{k}^* \oplus \mathfrak{k})^K = \mathfrak{z}^* \oplus \mathfrak{z}$. Altogether, the dimension of the stratum of isotropy type (K) in $(\mathfrak{g}^* \oplus \mathfrak{g})^c$ is equal to

$$2 \dim \mathfrak{z} + \dim({}^{G}\!/_{K}) = 2 \dim Z(K) + \dim G - \dim K. \tag{4.2}$$

Straightforward calculations with codimensions show that near a transverse relative equilibrium, the relative equilibria for which the pair (μ, ξ) of momentum and generator is of isotropy type (K) form a manifold of the same dimension.

In addition, the above observations allow to determine the local structure of the set of relative equilibria near a transverse relative equilibrium: For any point x contained in a stratum S of a locally closed Whitney stratified set Z, there is a cone C such that Z and $C \times S$ are locally homeomorphic near x, see [GWPL76, chapter II, Corollary (5.5)]. If Z is a subset of a manifold M and C is contained in a vector space $\mathbb{E}$ such that there is an open subset U of a Euclidean space and a smooth local embedding $\mathbb{E} \times U \to M$ that maps $C \times U$ to Z, Patrick and Roberts say that Z has *singularity type* C at x. The above observations show that $(\mathfrak{g}^* \oplus \mathfrak{g})^c \subset \mathfrak{g}^* \oplus \mathfrak{g}$ is of singularity type $(\mathfrak{l}^* \oplus \mathfrak{l})^c$ at x if $G_x = K$. As is easily verified, if $f : N \to M$ is transverse to Z at $n \in f^{-1}(x)$, the preimage $f^{-1}(Z)$ has also singularity type C at n. Applied to the maps $\tilde{\mathbf{J}}$ and X_h, this yields that at a transverse relative equilibrium with generator ξ and momentum μ and corresponding isotropy subgroup K, the set of relative equilibria is of singularity type $(\mathfrak{l}^* \oplus \mathfrak{l})^c$.

As may be deduced from transversality theory, the set of Hamiltonian functions h that have only transverse relative equilibria is open and dense in $C^\infty(P)^G$ with respect to the Whitney C^∞-topology: Since $\mathcal{K}^{\circ c}$ is a closed Whitney stratified subset of T^*P, openness follows from the Thom-Mather transversality Theorem A.21. The Jet-transversality theorem (see [Hir76, Theorem 2.8]) implies that the set of Hamiltonian functions h with $\mathrm{d}h$ transverse to any single stratum of $\mathcal{K}^{\circ c}$ is residual. By the local finiteness, a Whitney stratified set has at most countable strata. Thus the set of those h for those $\mathrm{d}h$ is transverse to all strata of $\mathcal{K}^{\circ c}$ is a countable intersection of residual sets. Hence it is residual and hence dense.

Patrick and Roberts also give an explicit characterization of transversality at a relative equilibrium p with generator via the derivative of $X_{h_\xi} = X_h - \xi_M$ at p (recall that $\xi_M(q) := \xi \cdot q$ for any $q \in P$ and $h_\xi = h - \mathbf{J}^\xi$): To do this, they use another equivalent formulation of their transversality condition:

Let $\psi : P \times \mathfrak{g} \to TP$ be the map $(p, \xi) \mapsto X_h(p) - \xi \cdot p = X_{h_\xi}(p)$ and $\psi^\circ : P \times \mathfrak{g} \to T^*P$ its dual $(p, \xi) \mapsto \mathrm{d}h(p) - \mathrm{d}\mathbf{J}^\xi(p) = \mathrm{d}h_\xi(p)$. The subscript $_{(K)}$ denotes the restriction to $(P \times \mathfrak{g})^c_{(K)}$, whose image is contained in $\mathcal{K}$.

Lemma 4.7. *A relative equilibrium $p \in P$ with momentum μ, generator ξ, and $G_\mu \cap G_\xi = K$ is transverse iff one of the following equivalent conditions is satisfied:*

1. *$\psi_{(K)} : (P \times \mathfrak{g})^c_{(K)} \to \mathcal{K}$ is transverse to the zero section of $\mathcal{K}$ at (p, ξ);*

2. *$\psi^\circ_{(K)} : (P \times \mathfrak{g})^c_{(K)} \to \mathcal{T}^\circ$ is transverse to the zero section of $\mathcal{T}^\circ$ at (p, ξ).*

Proof. The two conditions are obviously dual to each other with respect to the isomorphism $TP \simeq T^*P$ via ω and hence are equivalent. Either condition is equivalent to transversality of p by the following lemma, for example for the first we have to choose $E = \mathcal{K}$, $M = ((P \times \mathfrak{g})^c_{(K)})$, $f = I : (p, \xi) \mapsto \xi \cdot p$, $f_0 : (p, \xi) \mapsto p$, and $X = X_h$. $\square$

Lemma 4.8. *For a vector bundle $\pi : E \to P$ with smooth section X, a smooth embedding f and a smooth map f_0 as in the commutative diagram*

$$\begin{array}{ccc} & & E \\ & \overset{f}{\nearrow} & \downarrow \pi \\ M & \xrightarrow{f_0} & P \end{array},$$

X is transverse to $f(M)$ iff $X \circ f_0 - f$ is transverse to the zero section $Z(E)$ of E.

Proof. If X is transverse to $f(M)$ and $X(p) = f(m)$ (and hence $p = f_0(m)$), in particular

$$T(E_p) \subset \mathrm{d}X(p)T_pP + \mathrm{d}f(m)T_mM. \tag{4.3}$$

In coordinates with respect to a local splitting $\pi^{-1}(U) \cong U \times E_p$ for a neighbourhood U of p, the maps X and f are of the form $(\mathbb{1}, X_e)$ and (f_0, f_1) respectively. For $v_p \in T_pP$ and $v_m \in T_mM$,

$$\mathrm{d}X(p)v_p + \mathrm{d}f(m)v_m = (v_p + \mathrm{d}f_0(m)v_m, \mathrm{d}X_e(p)v_p + \mathrm{d}f_1(m)v_m).$$

The right hand side is contained in $T(E_p)$ iff $\mathrm{d}f_0(m)v_m = -v_p$. Thus,

$$\mathrm{d}(X_e \circ f_0 - f_1)(m)T_mM = T(E_p) \tag{4.4}$$

and hence $X \circ f_0 - f$ is transverse to $Z(E)$.

For the converse, note that by the above argument (4.4) $\Leftrightarrow$ (4.3) and hence $X \circ f_0 - f$ is transverse to $Z(E)$ iff (4.3) holds. Since $\mathrm{d}X(p)T_pP$ contains a complement of $T(E_p)$, this yields transversality of X to $f(M)$. $\square$

Let

$$\mathrm{d}X_{h_\xi}(p) : T_pP \to T_pP$$

denote the linear Hamiltonian vector field of the quadratic form

$$v \mapsto \mathrm{d}^2 h_\xi(p)(v, v)$$

on T_pP, which is well-defined, since p is a critical point of h_ξ. To justify this definition, consider the whole map

$$\mathrm{d}X_{h_\xi} : TP \to T(TP)$$

and a local trivialization $TP \supset \pi^{-1}(U) \simeq U \times V$, where U is an open neighbourhood of p, $\pi : TP \to P$ is the projection, and $V \simeq T_pP$. Then

$$T(p^{-1}(U)) \simeq TU \times TV \simeq U \times V \times V \times V.$$

With respect to corresponding coordinates

$$\mathrm{d}X_{h_\xi}(p)v = (p, v, (X_{h_\xi})_p, A(v)),$$

where A is the derivative of the map $p \mapsto (X_{h_\xi})_p$ from U to V. Similarly $T(T^*P)$ is locally isomorphic to $U \times V \times V^* \times V^*$ and

$$\mathrm{d}^2 h_\xi(p) = (p, v, \mathrm{d}(h_\xi)_p, B(v)),$$

where B is the derivative of $p \mapsto \mathrm{d}(h_\xi)_p$ from U to V^*. Since $X_{h_\xi}(p) = \omega_p^\# \circ \mathrm{d}(h_\xi)_p$ for the linear isomorphism $\omega_p^\# : V^* \to V$ defined by the symplectic form ω_p on V, this also holds for the derivatives A and B.

As shown in [Pat99], with respect to the splitting of the tangent space introduced in section 2.2,

$$T_pP \simeq \mathfrak{g}_\mu^\perp \oplus \mathfrak{g}_\mu \oplus N \oplus \mathfrak{g}_\mu^*,$$

the linear map $\mathrm{d}X_{h_\xi}(p)$ is of the form

$$\begin{pmatrix} -\operatorname{ad}_\xi \big|_{\mathfrak{g}_\mu^\perp} & 0 & 0 & 0 \\ 0 & -\operatorname{ad}_\xi \big|_{\mathfrak{g}_\mu} & C^* & D \\ 0 & 0 & \mathrm{d}X_{h_\mu}([p]) & C \\ 0 & 0 & 0 & -\operatorname{coad}_\xi \big|_{\mathfrak{g}_\mu^*} \end{pmatrix}, \tag{4.5}$$

where the map $\mathrm{d}X_{h_\mu}([p])$ is the derivative of the Hamiltonian vector field of the reduced system. Hence $\mathrm{d}X_{h_\mu}([p])$ corresponds to $\mathrm{d}_N^2 \bar{h}(0,0)$ via the symplectic form on N. Furthermore, D is symmetric and $(\omega_N^\#)^{-1} \circ C$ and C^* are dual to each other, where $\omega_N^\# : N^* \to N$ is the isomorphism defined by the symplectic form ω_N on N.

Remark 4.9. In [Pat99], the complement of $\mathfrak{g}_\mu \cdot p$ within $\mathfrak{g} \cdot p$ is identified via $\mathrm{d}\mathbf{J}(p)$ with the tangent space of the μ-orbit in $\mathfrak{g}^*$ and the corresponding component of $\mathrm{d}X_{h_\xi}(p)$ is given by $-\operatorname{coad}_\xi$. This is the same as $-\operatorname{ad}_\xi$ for our identification $\mathfrak{g}_\mu^\perp p \simeq \mathfrak{g}_\mu^\perp$ (which is also used in [PR00]), since

$$\mathrm{d}\mathbf{J}(p)\eta_p = \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{J}(\exp(t\eta)p)\bigg|_{t=0} = \operatorname{coad}_\eta \mu$$

and

$$\operatorname{coad}_{[\xi,\eta]} \mu = \operatorname{coad}_\xi(\operatorname{coad}_\eta \mu) - \operatorname{coad}_\eta(\underbrace{\operatorname{coad}_\xi \mu}_{=0}).$$

Nowadays, the normal form (4.5) can be deduced directly from the bundle equations (2.8) to (2.10), which in the free case represent $X_h(g, \rho, v)$ by

$$\begin{aligned} \dot{g} &= g \cdot \mathrm{d}_{\mathfrak{g}_\mu^*} \bar{h}(\rho, v) \\ \dot{v} &= (\omega_N^\#)(\mathrm{d}_N \bar{h}(\rho, v)) \\ \dot{\rho} &= -\operatorname{coad}_{\mathrm{d}^*_{\mathfrak{g}_\mu} \bar{h}(\rho,v)} \rho. \end{aligned}$$

Recap that $g\cdot$ denotes the lift of the left multiplication by g to TG.

Since

$$\exp(t\xi)(g, \rho, v) = (\exp(t\xi)g, \rho, v)$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}\exp(t\xi)g\bigg|_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t}g(g^{-1}\exp(t\xi)g)\bigg|_{t=0} = g \cdot \operatorname{Ad}_{g^{-1}} \xi,$$

we obtain for $\xi_P(g,\rho,v)$:

$$\begin{aligned}\dot g &= g\cdot \mathrm{Ad}_{g^{-1}}\xi\\ \dot v &= 0\\ \dot\rho &= 0.\end{aligned}$$

Hence, with respect to the local trivialization $TG \simeq G\times\mathfrak{g}$ given by left multiplication, the $\mathfrak{g}$-component of X_{h_ξ} equals $\mathrm{d}_{\mathfrak{g}_\mu^*}\bar h(\rho,v) - \mathrm{Ad}_{g^{-1}}\xi$. Since

$$(\omega_N^\#)(\mathrm{d}_N^2\bar h(\rho,v)) = \mathrm{d}_N X_{h_\mu}(\rho,v)$$

and

$$\mathrm{d}_{\mathfrak{g}_\mu^*}\bar h(0,0) = \xi,$$

we obtain the matrix (4.5) for $\mathrm{d}X_{h_\xi}(e,0,0)$, where

$$C^* = \mathrm{d}_{N,\mathfrak{g}_\mu^*}\bar h(\rho,v),\quad C = (\omega_N^\#)^{-1}(\mathrm{d}_{N,\mathfrak{g}_\mu^*}\bar h(\rho,v)),\quad D = \mathrm{d}^2_{\mathfrak{g}_\mu^*}\bar h(\rho,v).$$

Here, the linear maps have to be considered as maps between the following spaces:

$$C:\mathfrak{g}_\mu^*\to N,\quad C^*:N\to\mathfrak{g}_\mu,\quad D:\mathfrak{g}_\mu^*\to\mathfrak{g}_\mu.$$

Remark 4.10. Since $\mathrm{d}_{\mathfrak{g}_\mu^*}\bar h(\rho,v)$ coincides with the generator in the direction of the group orbit, D describes in some sense the change of the $\mathfrak{g}\cdot p$-component of the generator, when the momentum is varied in orthogonal direction to the orbit $\mathcal{O}_\mu$.

Next, Patrick states that $\mathrm{d}X_{h_\xi}(p)$ can be written as the sum of the semi-simple and the nilpotent part of its Jordan normal form as follows:

$$\begin{pmatrix} -\mathrm{ad}_\xi\big|_{\mathfrak{g}_\mu^\perp} & 0 & 0 & 0\\ 0 & -\mathrm{ad}_\xi\big|_{\mathfrak{g}_\mu} & 0 & 0\\ 0 & 0 & S & 0\\ 0 & 0 & 0 & -\mathrm{coad}_\xi\big|_{\mathfrak{g}_\mu^*}\end{pmatrix} + \begin{pmatrix} 0&0&0&0\\ 0&0&C^*&D\\ 0&0&Z&C\\ 0&0&0&0\end{pmatrix},$$

where S and Z denote the semi-simple and the nilpotent part of the Jordan normal form of $\mathrm{d}X_{h_\mu}([p])$ respectively.

Remark 4.11. If the spectrum of $\mathrm{d}X_{h_\xi}([p])$ is disjoint from the spectrum of ad_ξ, $C = C^* = 0$. This case is investigated in [Pat95].

Since the semi-simple part and the nilpotent part of the Jordan normal form of $\mathrm{d}X_{h_\xi}(p)$ commute, we then obtain

$$SC = -C\,\mathrm{coad}_\xi\big|_{\mathfrak{g}_\mu^*} \tag{4.6}$$

$$C^*S = -\mathrm{ad}_\xi\big|_{\mathfrak{g}_\mu} C^* \tag{4.7}$$

$$\mathrm{ad}_\xi\big|_{\mathfrak{g}_\mu} D = D\,\mathrm{coad}_\xi\big|_{\mathfrak{g}_\mu^*}. \tag{4.8}$$

Despite the fact that the matrix on the left hand side is semi-simple and the matrix on the right hand side is obviously nilpotent, I do not see why they have to coincide with he corresponding part of the Jordan decomposition in general.

Nevertheless, there is a choice of an Artin-Witt decomposition such that the equations (4.6) to (4.8) are satisfied:

First of all, equation (4.8) follows from the fact that $\mathrm{coad}_\xi = \mathrm{ad}_\xi^*$ and $D = D^*$. Equation (4.7) holds for an appropriate choice of N:

$\ker \mathrm{d}\mathbf{J}(p) \simeq \mathfrak{g}_\mu \oplus N$ and $\mathfrak{g}_\mu \cdot p$ are invariant subspaces of $\mathrm{d}X_{h_\xi}(p)$. The complement N of $\mathfrak{g}_\mu \cdot p$ within $\ker \mathrm{d}\mathbf{J}(p)$ may be chosen in such a way that C^* forms the corresponding component of the nilpotent part of the restriction $\mathrm{d}X_{h_\xi}(p)\big|_{\ker \mathrm{d}\mathbf{J}(p)}$. Then equation (4.7) is true. With the identification $N^* \simeq N$ via $\omega_N^\#$, equation (4.6) follows from equation (4.7) by taking duals.

For this choice of the normal form, Patrick and Roberts state the following characterization of transverse relative equilibria:

Theorem 4.12 (as stated in [PR00, Theorem 4], corrected version see below)**.** *A relative equilibrium $p \in P$ with momentum μ and generator ξ, $G_\mu \cap G_\xi = K$ is transverse iff all the following conditions are satisfied (with C, C^* and D as above):*

1. *Either p is non-degenerate or 0 is a semi-simple eigenvalue of $\mathrm{d}X_\mu([p])$.*
2. *C maps the dual $\mathfrak{z}^*$ of the Lie algebra of $Z(K)$ onto $\ker \mathrm{d}X_\mu([p])$.*
3. *$C^*(\ker \mathrm{d}X_\mu([p])) + D(\ker C \cap \mathfrak{z}^*) + \mathfrak{z} = \mathfrak{k}$.*

In Patrick's and Roberts' proof ([PR00]), the equations (4.6), (4.7), and (4.8) are used with S replaced by $\mathrm{d}X_{h_\mu}$. Thus the theorem is true in the case of a semi-simple reduced derivative $\mathrm{d}X_{h_\mu}$ but probably not in general. To obtain a more general version, it has to be slightly modified. Nevertheless, most conclusions of this theorem given in [PR00] are true:

To describe transverse relative equilibria in terms of the matrices $\mathrm{d}X_{h_\xi}(p)$, C, C^*, and D, we consider the matrix

$$M = \begin{pmatrix} \mathrm{d}X_{h_\mu}([p]) & C \\ C^* & D \end{pmatrix},$$

which maps $N \oplus \mathfrak{g}_\mu^*$ to $N \oplus \mathfrak{g}_\mu$. Note that this matrix coincides with $\mathrm{d}^2\bar{h}([p])$ if we omit the isomorphism $(\omega_N^\#) : N^* \to N$ given by the symplectic form. The following corrected theorem gives two conditions, which are both equivalent to transversality of the relative equilibrium. The second one depends on the choice of the splitting, for we assume that the above commutation equations hold true:

Theorem 4.13. *A relative equilibrium $p \in P$ with momentum μ and generator ξ with $G_\mu \cap G_\xi = K$ is transverse iff the following equivalent conditions are satisfied, where $\mathfrak{z}$ denotes the Lie algebra of $Z(K)$:*

1. *$N^* \oplus \mathfrak{g}_\mu$ coincides with the sum of $\mathfrak{z}$, $\mathfrak{g}_\mu\xi$ and the image of the restriction $\mathrm{d}^2\bar{h}([p]) : N \oplus \mathfrak{z}^* \to N^* \oplus \mathfrak{g}_\mu$.*
2. *If E_0 denotes the generalized 0-eigenspace of $\mathrm{d}X_{h_\mu}([p])$,*
$$E_0 \oplus \mathfrak{k} = \mathrm{im}\, M\big|_{E_0 \oplus \mathfrak{z}^*} + \mathfrak{z}.$$

The proof of 1 is just the correct first part of the proof of [PR00, Theorem 4], which is sketched in the following. Alternatively, 1 can be shown using ideas from chapter 6, see Remark 4.14 below. The equivalence of 1 and 2 is in principle the last part of the proof, the argument is just slightly corrected.

Proof of Theorem 4.13. First, the authors unravel the condition that

$$\mathrm{d}\psi(p,\xi)T_{(p,\xi)}(P\times\mathfrak{g})^c_{(K)} = \ker \mathrm{d}\mathbf{J}(p) = \mathfrak{g}_\mu \oplus N.$$

Calculating the P and $\mathfrak{g}$-derivatives of ψ yields together

$$\mathrm{d}\psi(p,\xi)(v,\eta) = (v, \mathrm{d}X_{h_\xi}(p)v - \eta\cdot p). \tag{4.9}$$

As the above investigation of the isotropy structure of $(\mathfrak{g}^*\oplus\mathfrak{g})^c$ shows, the pair $(v,\eta)\in T_pP\times\mathfrak{g}$ is contained in $T_{(p,\xi)}(P\times\mathfrak{g})^c_{(K)}$ iff

$$(\mathrm{d}\mathbf{J}(p)v,\eta)\in(\mathfrak{z}^*\oplus\mathfrak{z})\oplus\mathfrak{g}\cdot(\mu,\xi).$$

Hence, the $\mathfrak{g}_\mu^*$-component μ_0 of v is contained in $\mathfrak{z}^*$ (recap that the corresponding part of the tangent space was identified with $\mathfrak{g}_\mu$ via $\mathrm{d}\mathbf{J}$). Moreover, inserting the normal form of $\mathrm{d}X_{h_\xi}(p)v$ and evaluation yields

$$\mathfrak{g}_\mu\oplus N = \left\langle\left\{C^*w + D\mu_0 + \mathfrak{g}_\mu\cdot\xi + \mathfrak{z}, \mathrm{d}X_{h_\mu}([p])w + C\mu_0 \;\middle|\; w\in N, \mu_0\in\mathfrak{z}^*\right\}\right\rangle$$

(for details, see [PR00]). If we compose the second component with the inverse of the isomorphism $\omega_N^\# : N^*\simeq N$, this is just the same as condition 1. Note that we have not used the equations (4.6) to (4.8), hence the result is independent of the concrete choice of coordinates.

We now proceed with the equivalence of 1 and 2: Since $\mathfrak{z}^*\subset\mathfrak{k}^*\subset\mathfrak{g}_\xi^*$, equation (4.6) shows that $\operatorname{im} C\big|_{\mathfrak{z}^*}$ is contained in the generalized eigenspace $E_0\subset N$ of $\mathrm{d}X_{h_\mu}([p])$. If E_1 denotes the sum of the other generalized eigenspaces of $\mathrm{d}X_{h_\mu}([p])$, the restriction $\mathrm{d}X_{h_\mu}([p]) : E_1\to E_1$ is invertible. Hence the map $(\mathrm{d}X_{h_\mu}([p]), C) : N\oplus\mathfrak{z}^*\to N$ is surjective iff its restriction to $E_0\oplus\mathfrak{z}^*$ maps onto E_0. By equation (4.7), C^* maps E_0 into $\mathfrak{k}$ and E_1 into $\mathfrak{g}_\mu\cdot\xi$, and by equation (4.8), D maps $\mathfrak{k}^*$ (and hence $\mathfrak{z}^*$) into $\mathfrak{k}^*$. Moreover, $\mathfrak{g}_\mu\cdot\xi$ is a complement of $\mathfrak{k}$ within $\mathfrak{g}_\mu$, see Lemma 2.4. Therefore, 1 holds iff

$$\operatorname{im} M\big|_{E_0\oplus\mathfrak{z}^*} + \mathfrak{z} = \mathfrak{k} + E_0. \qquad \square$$

Remark 4.14. The first part of Theorem 4.13 (i.e. the equivalence of condition 1 and transversality of the relative equilibrium) can also be shown using ideas that will be presented in chapter 6. As we will see there, a relative equilibrium p with generator ξ and momentum μ is transverse iff for the restriction $\bar h$ of h to $V=\mathfrak{g}_\mu^*\oplus N$ the map $\mathrm{d}\bar h : V\to T^*V$ is transverse to the Whitney stratified set $\mathcal{K}^{\circ c}\big|_V$, which is equal to $\mathcal{K}^{\circ c}\cap T^*V$. Since

$$\mathcal{K}^{\circ c}_x = \langle \mathrm{d}\mathbf{J}(x)\cdot, \mathfrak{g}_{\mathbf{J}(x)}\rangle$$

and by Lemma 2.7, in the case of a free action

$$\mathbf{J}([e,\rho,v]) = \mu+\rho,$$

it is

$$\langle \mathrm{d}\mathbf{J}([e,\rho,v])(0,\dot\rho,\dot v),\eta\rangle = \dot\rho(\eta).$$

Hence for $x=[e,\rho,v]$, the set $\mathcal{K}^{\circ c}_x\subset T^*V = (\mathfrak{g}_\mu^*\oplus N)\oplus(\mathfrak{g}_\mu\oplus N^*)$ is given by $(\rho,v)\oplus\mathfrak{g}_{\mu+\rho}$. If ρ is small, $G_{\mu+\rho}\subset G_\mu$ and hence $G_{\mu+\rho}=(G_\mu)_\rho$. Rearranging the components of T^*V, we obtain

$$\mathcal{K}^{\circ c}\big|_V = N\oplus(\mathfrak{g}_\mu^*\oplus\mathfrak{g}_\mu)^c.$$

Therefore, $\mathcal{K}^{\circ c}\big|_V$ is a Whitney stratified set which is stratified by isotropy type of the momentum and generator pair. If $K = G_\mu \cap G_\xi$, the relative equilibrium is transverse iff $\mathrm{d}\bar{h}$ is transverse to $N \oplus (\mathfrak{g}_\mu^* \oplus \mathfrak{g}_\mu)^c_{(K)}$ at p. The tangent space of $(\mathfrak{g}_\mu^* \oplus \mathfrak{g}_\mu)^c_{(K)}$ is given by

$$(\mathfrak{g}_\mu \cdot (\mu, \xi)) \oplus (\mathfrak{z}^* \oplus \mathfrak{z}) = \mathfrak{z}^* \oplus (\mathfrak{z} + \mathfrak{g}_\mu \cdot \xi).$$

The map

$$\begin{aligned} \mathrm{d}\bar{h} : \mathfrak{g}_\mu^* \oplus N &\to (\mathfrak{g}_\mu^* \oplus N) \oplus (\mathfrak{g}_\mu \oplus N^*) \\ (\rho, v) &\mapsto ((\rho, v), \mathrm{d}\bar{h}_{(\rho,v)}) \end{aligned}$$

is transverse to $N \oplus \mathfrak{z}^* \oplus (\mathfrak{z} + \mathfrak{g}_\mu \cdot \xi)$ at $p = (\rho, v)$ iff $\mathfrak{z}$, $\mathfrak{g}_\mu \cdot \xi$ and the image of restriction of $\mathrm{d}^2\bar{h}_{(\rho,v)}$ to $N \oplus \mathfrak{z}^*$ together span $\mathfrak{g}_\mu \oplus N^*$.

Theorem 4.13 yields a relation between non-degeneracy and transversality:

Corollary 4.15. *1. A non-degenerate relative equilibrium p is transverse iff $D\mathfrak{z}^* + \mathfrak{z} = \mathfrak{k}$.*

2. If p is a non-degenerate equilibrium and K is a maximal torus, p is transverse.

Proof. 1. $E_0 = 0$ yields $C\big|_{\mathfrak{z}^*} = 0$ by equation (4.6) and hence $C^*\big|_{\mathfrak{z}^*} = 0$.

2. In this case $\mathfrak{k} = \mathfrak{z}$ and $D\mathfrak{z}^* + \mathfrak{z} = \mathfrak{k}$ is automatically satisfied. □

Remark 4.16. Corollary 4.15 illustrates a connection to the results in [Mon97]: If p is a non-degenerate relative equilibrium with generator ξ and $G_\xi = T$ is a maximal torus, p is transverse and all relative equilibria near p have a momentum generator pair of isotropy type (T). Since p is transverse, the map

$$\begin{aligned} \mathrm{d}\bar{h} : N \oplus \mathfrak{t}^* &\to (N \oplus \mathfrak{t}^*) \oplus (N^* \oplus \mathfrak{g}_\mu) \\ (v, \rho) &\mapsto ((v, \rho), \mathrm{d}\bar{h}_{(v,\rho)}) \end{aligned}$$

is transverse to

$$(N \oplus \mathfrak{t}^*) \oplus (\mathfrak{t} + \mathfrak{g}_\mu \cdot \xi) = (N \oplus \mathfrak{t}^*) \oplus \mathfrak{g}_\mu,$$

see Lemma 2.4 for the equality. Thus, the set of relative equilibria with momentum generator pair of type (T) forms a $\dim T$-dimensional submanifold of $N \oplus \mathfrak{t}^*$. Moreover, since p is non-degenerate, there is locally a unique $v(\rho) \in N$ for any $\rho \in \mathfrak{g}_\mu^*$ such that the N^*-component of $\mathrm{d}\bar{h}_{(v(\rho),\rho)}$ vanishes. The set of relative equilibria of isotropy type (T) forms a $(\dim G + \dim T)$-dimensional manifold (see equation (4.2) and below), hence its intersection with the slice $N \oplus \mathfrak{g}_\mu^*$ is a manifold of dimension $\dim T$, too. Thus the set of relative equilibria of $(N \oplus \mathfrak{g}_\mu^*)$ is locally given by the pairs $(v(\rho), \rho)$ for $\rho \in \mathfrak{t}^*$. The reduced space P_ρ within some neighbourhood of p is represented by $N \oplus \mathcal{O}_\rho$. Therefore, the set of relative equilibria in P_ρ is given by the pairs $(v(\alpha), \alpha)$ for every element α of the set $\mathfrak{t} \cap \mathcal{O}_\rho$, which is of cardinality $w(\rho)$. ($\rho \in \mathfrak{t}^*$ is no real restriction, since every element of $\mathfrak{g}_\mu^*$ is conjugate to one in $\mathfrak{t}^*$.)

Possibly in a similar way, Montaldi's Theorem 4.6 may be generalized to relative equilibria that are both non-degenerate and transverse with non-regular generators to compute the exact number of orbits of relative equilibria for a given momentum value near the momentum of p.

Next, Patrick and Roberts calculate the tangent space at the transverse relative equilibrium p to the corresponding stratum $\mathcal{E}_{(K)}$ of the set $\mathcal{E}$ of relative equilibria and obtain

$$T_p\mathcal{E}_{(K)} = \left\{ (\xi_0, \xi_1, w, \mu_0) \;\middle|\; \begin{array}{c} \xi_0 \in \mathfrak{g}_\mu^\perp, \\ \xi_1 \in \mathfrak{g}_\mu, \\ w \in \ker \mathrm{d}X_{h_\mu}([p]), \\ \mu_0 \in \mathfrak{z}^* \cap \ker C, \\ C^* w + D\mu_0 \in \mathfrak{z} \end{array} \right\}. \tag{4.10}$$

The proof is similar to that of Theorem 4.13. $T_p\mathcal{E}_{(K)}$ is given by the zero set of $\mathrm{d}\psi(p,\xi)\big|_{T_{(p,\xi)}(P\times\mathfrak{g})^c_{(K)}}$. Now again the normal form is inserted in equation (4.9) and everything is evaluated. At one point, the proof uses $\operatorname{im} C \subset \ker \mathrm{d}X_{h_\mu}([p])$, which is probably wrong in general, but the argument may be replaced by one which holds in general: The normal form yields that

$$\mathrm{d}X_{h_\mu}([p])w + C\mu_0 = 0 \tag{4.11}$$

for $(\xi_0, \xi_1, w, \mu_0) \in T_p\mathcal{E}_{(K)}$. From this, we may conclude $\mathrm{d}X_{h_\mu}([p])w = 0$ and $C\mu_0 = 0$ in the following way: $\operatorname{im} C \subset E_0$ holds true in general. After a possible reordering of the base, $\mathrm{d}X_{h_\mu}([p])\big|_{E_0}$ and C are submatrices of the nilpotent part of the Jordan normal form of $\mathrm{d}X_{h_\xi}(p)$. Thus

$$\operatorname{im} C \cap \operatorname{im} \mathrm{d}X_{h_\mu}([p])\big|_{E_0} = \{0\}$$

and hence equation (4.11) yields $\mathrm{d}X_{h_\mu}([p])w = 0$ and $C\mu_0 = 0$.

Using the description of the tangent space (4.10), Patrick and Roberts prove a generalization and partial converse of a result in [Pat95], see our Theorem 4.4 and below:

Theorem 4.17. *If the relative equilibrium p as in Theorem 4.13 is transverse, then the manifold $\mathcal{E}_{(K)}$ is symplectic in a neighbourhood of p iff p is non-degenerate and G_μ is a maximal torus.*

Proof. Since $\mathfrak{g}_\mu \subset T_p\mathcal{E}_K$, if $T_p\mathcal{E}_K$ is symplectic, $\mathfrak{g}_\mu^* = \mathfrak{z}^* \cap \ker C$, hence G_μ is Abelian and coincides with a maximal torus. Moreover, $C = 0$. Since $\mathrm{d}X_{h_\mu}([p])E_0 + \operatorname{im} C = E_0$, $\mathrm{d}X_{h_\mu}([p])$ is invertible.

Conversely, if G_μ is a maximal torus, $\mathfrak{g}_\mu = \mathfrak{z}$ and $C^*w + D\mu_0 \in \mathfrak{z}$ is automatically satisfied. Moreover, $\mathrm{ad}_\xi\big|_{\mathfrak{g}_\mu} = 0$. Thus by Remark 4.11, $C = 0$ if p is non-degenerate. In this case,

$$T_p\mathcal{E}_K = \mathfrak{g}_\mu^\perp \oplus \mathfrak{g}_\mu \oplus \{0\} \oplus \mathfrak{g}_\mu^*$$

is symplectic. ☐

Remark 4.18. One might wonder what happens in the case of non-connected compact groups. Later on, this case will occur during the reduction to isotropy subspaces P^{G_p}, on which the group $N(G_p)/G_p$ acts freely.

Requiring the group G to be connected is no real restriction: Suppose that a non-connected group G acts freely on the space P. Then for any $p \in P$, the G-orbit Gp is non-connected. By the Slice Theorem, there is a slice S and

G-invariant neighbourhood of Gp of the form $U \simeq G \times S$. If G° denotes the identity component of G, U is a G°-space which consists of several copies of the G°-invariant subspace $G^\circ \times S$. Moreover, a G°-invariant smooth function on $G^\circ \times S$ can be extended uniquely to a G-invariant function on U and this extension defines a homeomorphism between the corresponding function spaces (with the Whitney C^∞-topology). Hence, for the investigation of the generic structure of relative equilibria near the orbit Gp, it is sufficient to consider the G°-action.

Furthermore, the G-action and the G°-action both yield Whitney stratifications of $\mathfrak{g}^* \oplus \mathfrak{g}$, which induce Whitney stratifications of $\mathcal{K}^{\circ c}$. Hence the question arises which one is the "right" or more natural one. Obviously, the stratification induced by the G-action refines the stratification by G°-isotropy type of the momentum-generator pair. Moreover, we will show later that the stratification corresponding to the G°-action coincides with the canonical stratification. Hence it is in some sense the more natural stratification and we will stick to this one. Thus in the above setting, we will call a relative equilibrium $p \in P$ *transverse* iff it is transverse with respect to the G°-action. Nevertheless, the G-isotropy type of the pair (μ, ξ) also defines a stratification into manifolds, which are submanifolds of the canonical strata.

Chapter 5

Continuous isotropy

For both approaches to investigate the structure of relative equilibria in Hamiltonian systems with a free group action that we have seen in the last chapter, the condition that $\mathfrak{g}_p$ is finite is essential. Montaldi's results can be generalized to locally free actions, see [MoR99].

Ortega and Ratiu suggest in [OR97] to consider the subspaces P_H of points with the same isotropy subgroup $H \subset G$. Since the spaces P_H are symplectic and the group $N(H)/H$ acts freely on P_H, the results for free actions can be applied to these spaces. Nevertheless, this way we obtain no information about the overall structure of the relative equilibria. Moreover, genericity statements may not be transferred to general isotropy groups. In particular, we cannot conclude that for generic G-invariant Hamiltonian functions all relative equilibria in P_H are transverse with respect to the $N(H)/H$-action. It is easy to see that the set of Hamiltonians whose restriction to any fix point space has only transverse relative equilibria is dense in $C^\infty(P)^G$, but openness may not be deduced this way. We will give a proof of openness in chapter 6.

Another result that applies in the case of actions of compact groups with non-finite isotropy groups is given in [Mon97]: A relative equilibrium is called *extremal* iff its equivalence class is an extremum of the reduced Hamiltonian. (The reduced space is not necessarily a manifold here.) Simple topological arguments show that an extremal relative equilibrium has a neighbourhood U such that every non-empty intersection of U and a momentum level set contains an extremal relative equilibrium.

In this chapter another approach is presented, which adapts ideas from bifurcation theory: Relative equilibria are characterized by equation (2.14) (commutation equation) and (2.15) (symplectic slice equation) or equivalent (possibly more general) ones. In principle, the solutions of the symplectic slice equation are studied, while some assumptions assure that the commutation equation is satisfied. The symplectic slice equation is considered as a parameter dependent family on the symplectic slice N. Using the reduction techniques known from bifurcation theory as described in section 2.4, the symplectic slice equation may be reduced to the kernel of Hessian of the augmented Hamiltonian.

5.1 Bifurcation theory perspective

The search for relative equilibria in symmetric Hamiltonian systems may be considered as a bifurcation problem: Bifurcation theory is the study of parameter-dependent dynamical systems and the change of the occurrence and structure of specific dynamical phenomena like (relative) equilibria and periodic orbits at particular parameter values. A single Hamiltonian system with symmetry may be seen as a parameterized family of dynamical systems in several ways:

The parameter value may be the momentum, in the sense that the whole system is considered as the collection of the reduced systems of the reduced Hamiltonians h_μ on the reduced spaces P_μ. This way, the symmetry is eliminated and the investigation of the structure of relative equilibria reduces to the investigation of the change of the set of critical points. In contrast to ordinary bifurcation theory, the geometry of the underlying space may change. Thus, there are two reasons for a change of the local structure of relative equilibria: Degeneracy of the relative equilibrium or a bifurcation due to the geometry of momentum level sets of the phase space. The investigation of the second case is the approach in [Mon97] that was presented in section 4.1. We have seen the simplest case, in which all spaces are isomorphic and can be identified, in the proof of Arnol'd's observation (Theorem 4.3). In the case of non-free actions, this approach may be difficult, since the reduced spaces are not necessarily manifolds and their structure may be rather complicated.

A different point of view, which is more close to ordinary bifurcation theory, is to consider the augmented Hamiltonian functions h_ξ as families parameterized by $\xi \in \mathfrak{g}$. In this case, all functions are defined on the same space. In contrast to the reduced Hamiltonians, the augmented Hamiltonians still have symmetries. In ordinary bifurcation theory involving symmetry it is usually assumed that every family element has the same symmetry properties, but here we have in general a non-trivial action on the parameter space, which causes different symmetry groups for the functions with fixed parameter values. This difficulty does not occur if the adjoint action on the parameter space $\mathfrak{g}$ is trivial, i.e. if G is Abelian. For this reason, the treatment of Abelian groups is usually much simpler than the general case. Most results for non-free group actions require at least some commutativity assumptions.

This also holds for a third approach based on the Marle-Guillemin-Sternberg model and the corresponding equations for relative equilibria (i.e. the commutation equation (2.14) and the symplectic slice equation (2.15) derived from the bundle equations, or equivalent ones): The solutions of the slice equation (2.15) are investigated, where the function $\bar{h}$ is considered as a family of functions defined on N with parameters $\rho \in \mathfrak{m}^*$ and $\eta \in \mathfrak{g}_p$, while commutativity assumptions make sure that the commutation equation (2.14) is satisfied and in addition simplify the symmetry properties of the parameter-dependent family.

An example of the last approach is presented in [CLOR03], which contains a Hamiltonian version of the famous Equivariant Branching Lemma of bifurcation theory and a similar result concerning bifurcation of relative equilibria of maximal isotropy type. Another example of this approach based on the bundle equations with isotropy can be found in [MR-O15]. These results will be sketched in section (5.3). The results in [CLOR03] are developed independently from the bundle equations: Using the implicit function theorem several times, the authors derive analogues to equations (2.14) and (2.15), which hold in a more general context. Even though the bundle equations were known in 2003, the bundle equations with isotropy, which yield the equations (2.14) and (2.15), were first considered in [MR-O13] and [MR-O15]. In the following presentation, the results in [CLOR03] will also be considered in view of the equations (2.14) and (2.15). If the action is proper and there is a G_μ-invariant complement of $\mathfrak{g}_\mu$ and hence the equations (2.14) and (2.15) are valid, the equations given in [CLOR03] are equivalent but in a more implicit expression. An application of Lyapunov-Schmidt reduction to the analogue to the symplectic slice equation (2.15) yields an equation on the kernel of $\mathrm{d}^2 h_\xi(p)$, where ξ is a generator of the relative equilibrium p. This equation is called the *bifurcation equation*. The analogue to the commutation equation is called *rigid residual equation*.

In a second paper, [OR04(a)], Ortega and Ratiu use the bifurcation equation to investigate the structure of relative equilibria near an ordinary equilibrium with the full symmetry. In this case, the rigid residual equation is trivial and in addition, we only have the bifurcation parameter $\xi \in \mathfrak{g}$ of the augmented Hamiltonian. Thus, the last two of the bifurcation theory points of view illustrated above coincide here. We will present the results of Ortega and Ratiu ([OR04(a)]) in the next section and come back to [CLOR03] in more detail first:

The theory presented in [CLOR03] is formulated in a quite general context, such that the proceeding in principle may be applied to general Hamiltonian Lie group actions and many ideas may be adapted to infinite dimensional systems. The main assumption is the existence of a *slice mapping*:

Let P be a G-symplectic manifold and $\mathbf{J} : P \to \mathfrak{g}^*$ an equivariant momentum map. For $p \in P$ with $\mathbf{J}(p) = \mu$, let $\mathfrak{g} = \mathfrak{g}_\mu \oplus \mathfrak{q}$ and $\mathfrak{g}_\mu = \mathfrak{m} \oplus \mathfrak{g}_p$.

Definition 5.1. A *slice mapping* for $p \in P$ is an embedding Ψ of a neighbourhood of $(0, 0)$ in $\mathfrak{m}^* \times N$ into P, where N is a vector space, such that $\psi(0, 0) = p$,

$$T_p P = (\mathfrak{m} \oplus \mathfrak{q}) \cdot \Psi(\rho, v) + \mathrm{d}\Psi(\rho, v)(\mathfrak{m}^* \times N),$$

and $\mathbf{J}_Y := \mathbf{J} \circ \Psi$ satisfies $\mathrm{d}\mathbf{J}_Y(0, 0)(\dot{\rho}, \dot{v}) = \dot{\rho}$.

For the main results in [CLOR03], P is assumed to be finite-dimensional and the action is assumed to be proper. In this case, the existence of a slice mapping is given by the Marle-Guillemin-Sternberg normal form and $\mathbf{J}_Y$ is given by

$$\mathbf{J}_Y(\rho, v) = \mu + \rho + \mathbf{J}_N(v), \tag{5.1}$$

see Lemma 2.7.

Using the slice mapping, the authors split the equation $\mathrm{d}h_\xi(x) = 0$ into four equations.

For $x = \Psi(\rho, v)$, each equation is obtained from the derivative in one of the four directions $\mathfrak{q} \cdot x$, $\mathfrak{m} \cdot x$, $\mathrm{d}\psi(\rho, v)\mathfrak{m}^*$, and $\mathrm{d}\psi(\rho, v)N$. Since h is G-invariant and $\mathbf{J}$ is G-equivariant with respect to the coadjoint action, the derivative in the direction on the G-orbit yields the relation $\mathrm{coad}_\xi \mathbf{J}(x) = 0$. For $\mathfrak{q} \cdot x$ and $\mathfrak{m} \cdot x$, we obtain the corresponding restricted relations. Thus, $x = \Psi(\rho, v)$ is a relative equilibrium with generator ξ' iff

$$\mathrm{coad}_{\xi'} \mathbf{J}_Y(\rho, v)\big|_{\mathfrak{q}} = 0, \tag{5.2}$$

$$\mathrm{coad}_{\xi'} \mathbf{J}_Y(\rho, v)\big|_{\mathfrak{m}} = 0, \tag{5.3}$$

$$\mathrm{d}_{\mathfrak{m}^*} \bar{h}_{\xi'}(\rho, v) = 0, \tag{5.4}$$

$$\mathrm{d}_N \bar{h}_{\xi'}(\rho, v) = 0, \tag{5.5}$$

where $\bar{h}_{\xi'} := h_{\xi'} \circ \Psi$.

Next, the system of equations is simplified step by step near the relative equilibrium p with generator ξ and momentum μ. In a first step, equation (5.2) is solved using the implicit function theorem such that the $\mathfrak{q}$-component of ξ' is given as a function of ρ, v, and the $\mathfrak{m}$- and $\mathfrak{g}_p$-components of ξ'. If we assume that $\mathfrak{q}$ is G_μ-invariant (which is always a possible choice in the case of a compact group G), this function is just the zero function and the first step may be omitted.

In a second step, equation (5.4) is solved for the $\mathfrak{m}$-component of ξ', again using the implicit function theorem. In the case of a proper action, the Marle-Guillemin-Sternberg normal form is valid and $\mathbf{J}_Y$ is given by equation (5.1). In this case, we obtain an explicit expression of the projection $\mathbb{P}_\mathfrak{m}\xi'$ of ξ' to $\mathfrak{m}$:

$$0 = \mathrm{d}_{\mathfrak{m}^*}\bar{h}(\rho, v) - \mathrm{d}_{\mathfrak{m}^*}\mathbf{J}_Y^{\xi'}(e, \rho, v) = \mathrm{d}_{\mathfrak{m}^*}\bar{h}(\rho, v) - \mathbb{P}_\mathfrak{m}\xi'.$$

If in addition $\mathfrak{q}$ is G_μ-invariant, there is some $\eta \in \mathfrak{g}_p$ with $\xi' = \mathrm{d}_{\mathfrak{m}^*}\bar{h}(\rho, v) + \eta$. Inserting this expression into the equations (5.3) and (5.5) and using (5.1) yields the commutation equation (2.14) and the symplectic slice equation (2.15). Equation (5.3) with the implicit function for the $\mathfrak{m}$-component of ξ' inserted is called *rigid residual equation* in [CLOR03]. If there is no G_μ-invariant splitting $\mathfrak{g} \oplus \mathfrak{q} = \mathfrak{g}$, $\xi(\rho, v, \eta)$ is at least defined implicitly near $(0, 0, \mathbb{P}_{\mathfrak{g}_p}\xi)$. This is inserted into (5.5). In a last step, the resulting equation, which is the symplectic slice equation in the above case, is reduced to the kernel V_0 of $\mathrm{d}_N^2\bar{h}_{\xi'}(0, 0)$ via Lyapunov-Schmidt reduction. Since the $\mathfrak{g}$-derivative of the augmented Hamiltonian is given by the momentum map, the $\mathfrak{g}$-derivative of $\mathrm{d}_N\bar{h}_{\xi'}(0, 0)$ is equal to $\mathrm{d}_N\mathbf{J}_N(0, 0) = 0$ for any ξ'. Thus,

$$V_0 = \ker \mathrm{d}_N^2\bar{h}_{\xi'}(0, 0) = \ker \mathrm{d}_N^2(\bar{h}_{\xi'(\rho,v,\eta)})(0, 0)$$

and hence the Lyapunov-Schmidt reduction step indeed yields an equation on $\mathfrak{m}^* \times V_0 \times \mathfrak{g}_p$.

The resulting equation is called the *bifurcation equation*. It is of the form

$$B(\rho, v_0, \eta) = 0,$$

where B is a smooth function defined on a neighbourhood of $(0, 0, \mathbb{P}_{\mathfrak{g}_p}\xi)$ in $\mathfrak{m} \times V_0 \times \mathfrak{g}_p$ with values in V_0. In the case of an equilibrium with symmetry G, the bifurcation equation is a gradient equation, see Remark 2.12 or [CLOR03, Remark 3.1]. If the action is proper and a G_μ-invariant complement exists, this holds as well if some $\rho \in (\mathfrak{m}^*)^{G_p}$ is fixed: Then the symplectic slice equation determines the relative equilibria near $0 \in N$ for the G_p-invariant Hamiltonian system on the symplectic slice N with the Hamiltonian function $\bar{h}(\rho, \cdot)$. Thus, the argument given in [CLOR03, Remark 3.1] applies. (As explained in Remark 2.12, the Splitting Lemma yields the same equation in this case.) Set $G_{p,\xi} := G_p \cap G_\xi$. If all occurring subspaces are $G_{p,\xi}$-invariant (such a choice is always possible for proper actions), the functions on the left hand side of the rigid residual equation and the bifurcation equation are $G_{p,\xi}$-equivariant.

First, the authors apply this formalism to actions of Abelian groups and obtain a generalization of theorem of Lerman and Singer, which is valid for torus actions (at least G_μ has to be torus, see [LS98, Theorem 1.5]) to proper actions:

Theorem 5.2. *If the Abelian group G acts properly on P and $p \in P$ is a relative equilibrium with generator ξ such that $\mathrm{d}_N^2 h_\xi(0)$ is non-degenerate, p is contained in a symplectic manifold of relative equilibria with the same isotropy type (H) of dimension $2(\dim G - \dim H)$.*

(This result was already presented in Ortega's PhD thesis [Ort98].)

The proof relies on the fact that in this case the rigid residual equation holds due to the commutativity of the group and the bifurcation equation is also trivially satisfied, since the function B takes values in $V_0 = \{0\}$. Thus, all relative equilibria near p with generators near ξ are contained in the image of a smooth function S. S is defined in a tubular neighbourhood of $(e, 0) \in G \times \mathfrak{g} = G \times (\mathfrak{m} \oplus \mathfrak{h})$ by

$$(g, \rho, \eta) \mapsto [g, \rho, v(\rho, \eta)],$$

where the right hand side denotes a point in the Marle-Guillemin-Sternberg normal form and $v : \mathfrak{m} \oplus \mathfrak{h} \to N$ is an implicitly defined H-equivariant smooth function. Here, the Lyapunov-Schmidt-reduction step is just a trivial application of the implicit function theorem. An examination of this step and the normal form of the momentum map yield together that the rank of $\mathrm{d}_\mathfrak{h} v((g, \rho, \eta)$ is given by $\dim H - \dim H_{v(\rho,\eta)}$. (For a detailed computation, see [CLOR03].) Thus the rank of S at a point (g, ρ, η) is given by

$$2(\dim G - \dim H) + (\dim H - \dim H_{v(\rho,\eta)}).$$

Therefore the image of the restriction of S to a tubular neighbourhood of $(e, 0) \in G \times \mathfrak{m}$ is a manifold of dimension $2(\dim G - \dim H)$: Since v is H-equivariant and H acts trivially on $\mathfrak{m}$, $H_{v(\rho,0)} = H$ and the claim follows from the constant rank theorem. A straightforward computation with the standard symplectic form of the Marle-Guillemin-Sternberg model shows that the manifold is symplectic.

Remark 5.3. Curiously, the authors do not mention the fact that H acts trivially on the whole Lie algebra $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h}$ and hence $H_{v(\rho,\eta)} = H$ for every pair (ρ, η) which implies that v and S are constant in η. Thus, the $2(\dim G - \dim H)$-dimensional manifold contains all relative equilibria near p with velocities near ξ.

The case of non-trivial kernels of the Hessian of the augmented Hamiltonian is also investigated in [CLOR03]. In this case, the solution set of the bifurcation equation may bifurcate. The existence of non-trivial branches of relative equilibria is shown for maximal isotropy types with 1-dimensional fix point spaces in V_0 under the assumption that the rigid residual equation is satisfied on the corresponding isotropy subspace of $\mathfrak{m}^* \times V_0$. For the Hamiltonian version of the classical Equivariant Branching Lemma, the fix point space has to be real 1-dimensional, while another theorem deals with the case that the corresponding normalizer induces an S^1-action on the fix point space.

For the Hamiltonian analogue to the Equivariant Branching Lemma, the fix point subspace of $\mathfrak{m}^*$ of the maximal isotropy subgroup is considered as the parameter subspace. Thus this space has to be non-trivial. In particular the theorem does not apply for relative equilibria near a true equilibrium with the full symmetry. Moreover, the value of the momentum is varied along each branch. The precise statement is as follows:

Theorem 5.4. *Let P be a symplectic manifold and $p \in P$ be a relative equilibrium with respect to a proper Hamiltonian action of the Lie group G with generator $\xi \in \mathfrak{g}$ and momentum $\mu \in \mathfrak{g}^*$. Suppose that* $\ker \mathrm{d}^2 h_\xi(p) = \mathfrak{g}_\mu \oplus V_0$*. Generically, for any subgroup $K \subset G_{\xi,p}$ with $\dim V_0^K = 1$, $\dim(\mathfrak{m}^*)^K \geq 1$, and the property that the rigid residual equation is satisfied on $(\mathfrak{m}^*)^K \times V_0^K \times \{\mathbb{P}_{\mathfrak{g}_p}\xi\}$, a branch of relative equilibria with isotropy subgroup K bifurcates from p.*

Proof. If $L := N_{G_{\xi,p}}(K)\big/K$, L acts freely on $V_0^K \setminus \{0\} \simeq \mathbb{R} \setminus \{0\}$ and thus $L \simeq \{e\}$ or $L \simeq \mathbb{Z}_2$.

From a general property of equations obtained by Lyapunov-Schmidt reduction, it follows that $\mathrm{d}_{V_0} B(0, 0, \mathbb{P}_{\mathfrak{g}_p}\xi) = 0$:

Doing a Lyapunov-Schmidt reduction, the local solutions of an equation $F(x, \lambda) = 0$ near a given solution $(0, \lambda_0) \in V \times \Lambda$ are expressed as the zeros of a map of the form $f(x_0, \lambda) = \mathbb{P}F(x_0 + x_1(x_0, \lambda), \lambda)$. Here $\mathbb{P}$ is a projection whose kernel coincides with the range of $\mathrm{d}_V F(0, \lambda_0)$ and x_0 and x_1 are coordinates with respect to a splitting $V_0 \oplus V_1$ with $V_0 := \ker \mathrm{d}_V F(0, \lambda_0)$. The implicit function x_1 satisfies $x_1(0, \lambda_0) = 0$. Thus, $\mathrm{d}_{V_0} f(0, \lambda_0) = 0$. In our special case, f is given by B.

Set $\eta_0 := \mathbb{P}_{\mathfrak{g}_p}\xi$. $\mathrm{d}_{V_0} B(0, 0, \mathbb{P}_{\mathfrak{g}_p}\xi) = 0$ yields that in the two cases $L \simeq \{e\}$ and $L \simeq \mathbb{Z}_2$ the Taylor expansion of $B^K := B\big|_{(\mathfrak{m}^*)^K \times V_0^K \times \{\eta_0\}}$ is of the following form:

$$B^K(\rho, v_0, \eta_0) = \begin{cases} \langle \kappa, \rho \rangle + c v_0^2 + \dots & L \simeq \{e\} \\ v_0(\langle \kappa, \rho \rangle + c v_0^2 + \dots) & L \simeq \mathbb{Z}_2 \end{cases},$$

where the dots denote terms of at least second order, κ is an element of $\mathfrak{m}^*$, $\langle \cdot, \cdot \rangle$ is an inner product on $\mathfrak{m}^*$, and c is a constant. By [GS85, chapter IV, Table 2.1 and Table 2.3], in the generic case $\kappa \neq 0$ and $c \neq 0$, we have a saddle-node bifurcation if $L \simeq \{e\}$ and a pitchfork bifurcation if $L \simeq \mathbb{Z}_2$, when the parameter ρ is varied in the direction of κ. □

Next, the authors investigate the solution set of the bifurcation equation on the vector space N within isotropy subspaces of maximal isotropy subgroups $K \subset N_{G_{p,\xi}}$ such that $L := L(K) := N_{G_{p,\xi}}(K)\big/K$ is isomorphic to S^1 or $S^1 \times \mathbb{Z}_2$. Such an isotropy subgroup is called *complex isotropy subgroup*. Indeed, they fix a K-invariant complement of the Lie algebra $\mathfrak{k}$ of K within the Lie algebra of $N_{G_{p,\xi}}$. Since it is isomorphic to the Lie algebra of L, the complement will be denoted by $\mathfrak{l}$ in the following. Then, they restrict their search for solutions of the bifurcation equation to the set $\{0\} \times V_0^K \times \mathfrak{l}$. This means that they search for relative equilibria that correspond to periodic orbits of the Hamiltonian vector field on N for the Hamiltonian function $\bar{h}(0, \cdot)$: The solutions correspond to relative equilibria in N^K with generators in the Lie algebra of $N_{G_{p,\xi}}$. Thus their $N^\circ_{G_{p,\xi}}$-orbits are invariant, which correspond to the orbits of the induced L-action on N^K. Hence the trajectories form periodic orbits.

Theorem 5.5 (as stated in [CLOR03, Theorem 5.2], see remark below for a correction)**.** *Let $p \in P$ be a relative equilibrium with generator $\xi \in \mathfrak{g}$ with respect to a proper Hamiltonian action of the Lie group G. Suppose that* $\ker \mathrm{d}^2 h_\xi(p) = \mathfrak{g}_\mu \oplus V_0$ *and $V_0^{G_{p,\xi}} = \{0\}$. Then for each maximal complex isotropy subgroup K of the $G_{p,\xi}$-action on V_0 for which the $L = L(K)$-action on $\mathfrak{g}_p^K$ is trivial and the rigid residual equation is satisfied on $\{0\} \times V_0^K \times \mathfrak{l}$, there are generically at* *least* $\frac{1}{2}\dim V_0^K$ *branches of relative equilibria bifurcating from* p *if* $L \simeq S^1$ *and* $\frac{1}{4}\dim V_0^K$ *ones if* $L \simeq S^1 \times \mathbb{Z}_2$*.*

Remark 5.6. As far as I can see, there is a definiteness condition missing in the statement of the theorem. We will come back to this below in a sketch of the proof. For example, the assumption that $\mathrm{d}^2 h(0)$ is definite on V_0^K suffices.

Moreover, it is not completely clear what "generically" and "branch" mean in this context. I do not expect smoothness at the origin in general. A more cautious formulation is that there are at least $\frac{1}{2}V_0^K$ ($\frac{1}{4}V_0^K$) relative equilibria such that the v_0-component has norm $\varepsilon > 0$ for small values of ε and the norm $\|v_0\| = \pm\frac{1}{2}\mathrm{d}^2 h(0)(v_0, v_0)$. Then an additional genericity assumption is unnecessary.

Let us broadly sketch the proof of Theorem 5.5:

Sketch of the proof of 5.5. Since by assumption the rigid residual equation is satisfied on the set $\{0\}\times V_0^K \times \mathfrak{l}$, the solutions of the restriction of the bifurcation equation to this set correspond to relative equilibria.

In a first step, the bifurcation equation is solved partly: For any choice of an invariant inner product $\langle\cdot,\cdot\rangle$, a blow up argument together with the implicit function yields a smooth function $\alpha : V_0^K \to \mathfrak{l}$ such that

$$\langle B(0, u, \alpha(u)), u\rangle = 0.$$

To apply the implicit function theorem, the lower order terms of $\langle B(0, u, \alpha), u\rangle$ have to be computed. One obtains that $\langle B(0, u, \alpha), u\rangle$ is of the form

$$\langle B(0, u, \alpha), u\rangle = -\mathbb{P}\mathrm{d}^2_N \mathbf{J}^\alpha_Y(0, 0)(u, u) + \langle u, L_1(\alpha)u + g(u, \alpha)\rangle, \tag{5.6}$$

where $\mathbb{P}$ is the projection to V_0, $L_1(\alpha)$ is a linear endomorphism of V_0^K with $L_1(0) = L_1'(0) = 0$, $g(0, \alpha) = 0$, and $\mathrm{d}_u g(0, \alpha) = 0$ for all $\alpha \in \mathfrak{l}$. (Note that $\mathrm{d}^2_N \mathbf{J}^\alpha_Y(0, 0) = \mathrm{d}^2 \mathbf{J}^\alpha_N(0)$ if the Marle-Guillemin-Sternberg form is valid.)

Now, the authors argue that $\mathbb{P}\mathrm{d}^2_N \mathbf{J}^\alpha_Y(0, 0)$ is a multiple of the identity. This is not true in general, but follows from the additional definiteness assumption of Remark 5.6: Since $\mathrm{d}^2_N \mathbf{J}^\alpha_Y(0, 0)$ is linear in $\alpha \in \mathfrak{l}$ and $\mathfrak{l}$ is 1-dimensional, $\mathrm{d}^2_N \mathbf{J}^\alpha_Y(0, 0)$ is definite for all $\alpha \neq 0$ if this is true for a particular choice of $\alpha \neq 0$. Now, $V_0^K \neq \{0\}$ implies that ξ is contained in the Lie algebra of $N_{G_\xi}(K)$, since V_0^K is an invariant subspace with respect to the linear map $\mathrm{d}^2_N(\bar{h} - \mathbf{J}^\xi_Y)(0, 0)$ and the elements of V_0^K are relative equilibria of the linearized vector field with generator ξ. Since the restriction of $\mathrm{d}^2_N \bar{h}(0, 0)$ to V_0^K is definite and $\mathrm{d}^2_N \bar{h}(0, 0) = \mathrm{d}^2_N \mathbf{J}^\xi_Y(0, 0)$ on V_0, $\mathrm{d}^2_N \mathbf{J}^\alpha_Y(0, 0)$ is definite for all $\alpha \neq 0$. Moreover, since L acts freely on V_0^K, all irreducible subrepresentations of V_0^K are given by an isomorphism $L \simeq \mathrm{SO}(2)$. Definiteness of $\mathrm{d}^2_N \mathbf{J}^\alpha_Y(0, 0)$ on V_0^K yields that only one of the two possible isomorphisms occurs. Hence the restriction of $\mathrm{d}^2_N \mathbf{J}^\alpha_Y(0, 0)$ to V_0^K is given by $\pm\alpha\mathbb{1}$.

Thus the right hand side of equation (5.6) is of the form $f(u, \alpha)\cdot\|u\|^2$, where f is smooth and $\mathrm{d}_\alpha f(0, 0) = \pm\mathbb{1}$. The implicit function theorem yields a locally defined function α with $f(u, \alpha(u)) = 0$.

For the second step, the authors set

$$X_\varepsilon(u) := B(0, \varepsilon u, \alpha(\varepsilon u)).$$

Then X_ε defines an L-equivariant vector field on the unit sphere S^{2n-1}, $2n = \dim V_0^K$, such that the zeros of X_ε correspond to solutions of the bifurcation equation of norm ε^2. As shown in [CLOR03, Lemma 5.3], the assumption that $N_{G_\xi}(K)$ acts trivially on $\mathfrak{g}_p^K$ implies that X_ε is orthogonal to the L-orbits. In the case that the action is proper and there is a G_μ-invariant complement of $\mathfrak{g}_p$ in $\mathfrak{g}_\mu$, this follows even without this assumption, since in this case X_ε is a gradient vector field of an invariant function. Since this function is constant on L-orbits, its gradient is perpendicular to them. (This argument is from [CLOR03, Remark 5.5].) Thus the L-orbits of zeros of X_ε correspond to the zeros of the induced field $\tilde{X}_\varepsilon$ on S^{2n-1}/L. By the Poincaré-Hopf theorem (see for example [Mil65]), this number is generically not less than the Euler characteristic $\chi(S^{2n-1}/L)$. In the case $L \simeq S^1$, S^{2n-1}/L is homeomorphic to the projective space $\xi(\mathbb{CP}^{n-1})$ and thus $\chi(S^{2n-1}/L) = n$. If $L \simeq S^1 \times \mathbb{Z}_2$, the space S^{2n-1}/L is homeomorphic to $\mathbb{CP}^{n-1}/\mathbb{Z}_2$ and thus $\chi(S^{2n-1}/L) = \frac{n}{2}$. $\square$

Since the orbits correspond to periodic orbits of the reduced field, this result is related to the equivariant Weinstein-Moser-theorem of [MRS88], which will be discussed in more detail in section 5.2.3: For simplicity, suppose that $p = 0$ is an equilibrium in the symplectic representation V. In this case, the assumption that the L-action on $\mathfrak{g}_p^K$ is trivial is not necessary. Moreover, to exclude bifurcation of equilibria, we suppose that $\mathrm{d}^2 h(0)$ is non-degenerate. The relative equilibria predicted by Theorem 5.5 are contained in the fix point space V^K, on which we have an L-action. Let u be the trajectory of a relative equilibrium in $V^K \setminus \{0\}$. Then u forms a periodic orbit and an L°-orbit. Let T be the minimal period of u. Then for each $t \in \mathbb{R}$, there is a map $\alpha : L^\circ \to S^1 \simeq \mathbb{R}/\mathbb{Z}$ such that

$$lu(t) = u(t + \alpha(l)T). \tag{5.7}$$

By L-equivariance of the flow of X_h, α is independent of t and defines a group homomorphism. Since the L-action on $V_0^K \setminus \{0\}$ is free, α is an isomorphism.

As explained above in the sketch of the proof, the definiteness condition yields that the restriction of $\mathrm{d}^2\mathbf{J}^\xi(0)$ to V_0^K is a multiple of the identity. Thus the same holds for $\mathrm{d}^2 h(0)\big|_{V_0^K}$. Hence V_0^K is contained in the real part $E_{\pm\lambda}$ of the sum of the eigenspaces of a pair $\pm\lambda \in \mathrm{i}\mathbb{R}$ of $\mathrm{d}X_h(0)$.

All orbits of $\mathrm{d}X_h(0)$ in $E_{\pm\lambda}$ are periodic with the same period and this yields a free S^1-action on $E_{\pm\lambda}$. This action commutes with the G-action on $E_{\pm\lambda}$. In particular, we obtain a $G \times S^1$-action on $E_{\pm\lambda}$. Consider the subgroup A of $G \times S^1$ formed by the elements (n, s) with $n \in N_{G_\xi}(K)^\circ$ and $s = -\alpha([n])$, where $[n] \in L^\circ$ is the equivalence class of n. Note that $K = K \times \{e\} \subset A$, and thus $E_{\pm\lambda}^A$. The equivariant Weinstein-Moser theorem (with the weaker definiteness assumption as in [MRS88, Remark 1.2(c)]) states that there are at least $\frac{1}{2} \dim E_{\pm\lambda}^A$ periodic orbits, which satisfy equation (5.7). Here we use that $E_{\pm\lambda}^A = V_\lambda^A$, where V_λ is the resonance space of λ as defined in [MRS88], see also section 5.2.3. This follows easily from the fact that by maximality of K, L° acts freely on $V_\lambda^K \setminus \{0\}$. Moreover, we claim that $E_{\pm\lambda}^A = V_0^K$. Thus, Theorem 5.5 predicts the same number of group orbits of relative equilibria on each energy level (if $L \simeq S^2 \times \mathbb{Z}_2$, two periodic orbits correspond to the same group orbit). To be precise, the "energy levels" are defined a bit different in

Theorem 5.5, see remark 5.6, but this should not make much difference. To prove the claim, note that there is an element $\eta_\lambda \in \mathbb{R}$ of the Lie algebra of S_1 such that $\mathrm{d}X_h(0)x = \eta_\lambda x$ for every $x \in E_{\pm\lambda}$ and the S^1-action on $E_{\pm\lambda}$. If $x \in V_0$, then $\mathrm{d}^2h(0)x = \mathrm{d}^2\mathbf{J}^\xi(0)(x) = \mathrm{d}^2\mathbf{J}^\xi(x)$ and hence $\mathrm{d}X_h(0)x = \xi x$. Since $V_0^K \subset E_{\pm\lambda}$, we have $\eta_\lambda x = \xi x$ for $x \in V_0^K$. This defines a Lie algebra isomorphism between $\mathfrak{l}$ and $\mathbb{R}$ and because the S^1-action and the L°-action are free on V_0^K, we obtain a group automorphism $\alpha : L^\circ \to S^1$ which transforms the two actions into each other. Hence $V_0^K \subset E_{\pm\lambda}^A$ for the corresponding group A. On the contrary, for $x \in E_{\pm\lambda}^A \subset V^K$, the vector $\eta_\lambda x$ coincides with ηx for $\eta = \mathrm{d}\alpha(0)^{-1}\eta_\lambda \in \mathfrak{l}$. Since $V_0^K \subset E_{\pm\lambda}^A$ is non-empty, the equivalence class of ξ in $\mathfrak{l}$ coincides with η and thus $E_{\pm\lambda}^A \subset V_0^K$.

5.2 Representations

One of the main goals of this thesis is the survey of the structure of relative equilibria near 0 in a symplectic representation of the compact group G.

In this section, some known results for the case of representations are presented and some conclusions are added afterwards: We start in section 5.2.1 with the paper [OR04(a)], in which Ortega and Ratiu apply the reduction method presented in [CLOR03] to this case in order to find relative equilibria near a possibly non-extremal equilibrium. Their main result predicts branches of relative equilibria tangent to a kernel V_0 of the Hessian $\mathrm{d}^2(h - \mathbf{J}^\xi)(0)$ of the augmented Hamiltonian if $\mathrm{d}^2h(0)$ is definite on V_0 and the group G_ξ acts transitively on the unit sphere of V_0.

We will discuss which part of the result remains true if the transitivity condition is omitted. In fact, we obtain a lower bound for G_ξ-orbits of relative equilibria on each energy level which have a generator that is a multiple of ξ. (In contrast, in chapter 6, the structure of the set of all relative equilibria with arbitrary generators is investigated. For future work, it might be helpful to combine both approaches.)

Afterwards in 5.2.2, the main result in [OR04(a)] is applied to the case of groups of rank 1. In this case, all relative equilibria consist of periodic orbits and thus the theory of periodic orbits near 0 in symplectic representations with symmetry may also be helpful to predict relative equilibria (thanks to James Montaldi for pointing this out). Indeed, the conclusions of 5.2.2 may also be deduced directly from the equivariant Weinstein-Moser theorem of [MRS88]. This is sketched in 5.2.3. Nevertheless, the ideas presented in 5.2.2 may be generalized using equivariant transversality theory. This way, we will obtain one of the main results of this thesis, which will be presented in chapter 6.

5.2.1 Bifurcation equation for representations

In order to obtain the main result in [OR04(a)], which predicts relative equilibria near an equilibrium, which is not necessarily stable, Ortega and Ratiu first consider the stable case with definite Hessian $\mathrm{d}^2h(0)$ and generalize the result to the case that the restriction of $\mathrm{d}^2h(0)$ to the kernel V_0 of some augmented Hamiltonian $\mathrm{d}(h - \mathbf{J}^\xi)(0)$ is definite. The theorem for the stable case is stated as follows (note the explanation of the notions below): For the whole section 5.2, let G denote a compact group.

Theorem 5.7. *Let (V,ω) be a Hamiltonian G-representation with momentum map $\mathbf{J}$ and h a G-invariant Hamiltonian function such that $h(0)=0$, $\mathrm{d}h(0)=0$, and the quadratic form $Q:=\mathrm{d}^2h(0)$ is definite. If $\mathrm{d}^2\mathbf{J}^\xi(0)$ is definite for $\xi\in\mathfrak{g}$ and $\varepsilon>0$ is small enough, there are at least*

$$\mathrm{Cat}_{G_\xi}(h^{-1}(\varepsilon))=\mathrm{Cat}_{G_\xi}(Q^{-1}(\varepsilon))$$

G_ξ-distinct relative equilibria in $h^{-1}(\varepsilon)$ whose generators are multiples of ξ.

Here, relative equilibria are called G_ξ*-distinct* iff their G_ξ-orbits are distinct. Hence, the number of G_ξ-distinct relative equilibria whose generators are multiples of ξ coincides with the number of G-orbits of relative equilibria whose generators are given by a G-orbit of a multiple of ξ.

Cat_G denotes the equivariant Lyusternik-Schnirelmann category with respect to the G-action, which is a lower bound for the number of critical orbits of a G-invariant smooth function, see [Mar89].

The proof is quite simple and relies on the fact that $\mathbf{J}^\xi$ is linear in ξ: For any critical point $x\in Q^{-1}(\varepsilon)$ of the function $\mathbf{J}^\xi$ restricted to the sphere $Q^{-1}(\varepsilon)$, there is a Lagrange multiplier λ such that $\mathrm{d}\mathbf{J}^\xi(x)=\lambda\mathrm{d}h(x)$. Since $\mathrm{d}^2\mathbf{J}^\xi(0)$ is definite, $\lambda\neq 0$ for small values of ε. Thus $\mathrm{d}\mathbf{J}^{\frac{\xi}{\lambda}}(x)=\mathrm{d}h(x)$.

We now discuss the literal formulation of the generalization to the case that the quadratic form is definite only on the kernel of $\mathrm{d}^2(h-\mathbf{J}^\xi)(0)$ as stated in [OR04(a)]: (The quintuple $(V,\omega,h,G,\mathbf{J})$ is called a Hamiltonian G-vector space iff (V,ω) is a Hamiltonian G-representation with momentum map $\mathbf{J}$ and h is a G-invariant Hamiltonian function.)

Theorem 5.8. *Let $(V,\omega,h,G,\mathbf{J})$ be a Hamiltonian G-vector space with G a compact Lie group. Suppose that $h(0)=0$ and $\mathrm{d}h(0)=0$. Let $\xi\in\mathfrak{g}$ be a root of the polynomial equation*

$$\det(\mathrm{d}^2(h-\mathbf{J}^\xi)(0))=0.$$

Define

$$V_0:=\ker(\mathrm{d}^2(h-\mathbf{J}^\xi)(0))$$

and suppose that:

1. *The restricted quadratic form $Q:=\mathrm{d}^2h(0)\big|_{V_0}$ on V_0 is definite.*
2. *Let $\|\cdot\|$ be the norm on V_0 defined by $\|v_0\|:=\mathrm{d}^2h(0)(v_0,v_0)$, $v_0\in V_0$. This map is indeed a norm due to the definiteness assumption on $\mathrm{d}^2h(0)\big|_{V_0}$ (if $\mathrm{d}^2h(0)\big|_{V_0}$ is negative definite, a minus sign is needed in the definition). Let $l=\dim V_0$ and S^{l-1} be the unit sphere in V_0. The function $j\in C^\infty(S^{l-1})$ defined by $j(u):=\frac{1}{2}\mathrm{d}^2\mathbf{J}^\xi(0)(u,u)$ is G^ξ-Morse with respect to the G^ξ-action on S^{l-1}.*

Then there are at least

$$\mathrm{Cat}_{G_\xi}(h\big|_{V_0}^{-1}(\varepsilon))=\mathrm{Cat}_{G_\xi}(Q^{-1}(\varepsilon))$$

G_ξ-distinct relative equilibria of h on each of its energy levels near zero. These relative equilibria appear in smooth branches when the energy is varied and their velocities are close to ξ.

First, let us consider condition 2 more closely: Since $V_0 := \ker(\mathrm{d}^2(h-\mathbf{J}^\xi)(0))$, j is just the constant function $\frac{1}{2}$ and thus condition 2 is true, iff the G_ξ-action on S^{l-1} is transitive. Seemingly, this is not noticed by the authors (but all examples in [OR04(a)] satisfy the transitivity condition). Moreover, this implies that $\mathrm{Cat}_{G_\xi}(Q^{-1}(\varepsilon)) = 1$. In addition, condition 2 implies condition 1, since Q is constant on G_ξ-orbits.

We will discuss some suggestions for generalizations later. Nevertheless, the result for the transitive case is also very useful as the examples in [OR04(a)] illustrate. We will only need this version for the application presented in 5.2.2.

Let us outline the proof and meanwhile make the meaning of “branch” more precise: As in [CLOR03], Ortega and Ratiu start with the mapping $F : V \times \mathfrak{g}^{G_\xi} \to V$ given by $F(v, \alpha) = \mathrm{d}_V(h - \mathbf{J}^{\xi+\alpha})(v)$ and perform a Lyapunov-Schmidt reduction. Choosing a G_ξ-invariant splitting $V = V_0 \oplus V_1$ with corresponding equivariant projection $\mathbb{P} : V \to V_1$ and defining the map $v_1 : V_0 \times \mathfrak{g}^{G_\xi} \to V_1$ implicitly by

$$\mathbb{P}F(v_0 + v_1(v_0, \alpha), \alpha) = 0 \quad \forall v_0 \in V_0,$$

they obtain the bifurcation equation

$$(\mathbb{1} - \mathbb{P})F(v_0 + v_1(v_0, \alpha), \alpha) = 0$$

on V_0, which characterizes relative equilibria near 0 with generators near ξ in $\mathfrak{g}^{G_\xi}$. This the same proceeding as in [CLOR03] for the special case of representations. As mentioned in the discussion in [CLOR03], (x, α) is a solution of the bifurcation equation iff $\mathrm{d}_{V_0} g(x, \alpha) = 0$, where

$$g(x, \alpha) = (h - \mathbf{J}^{\xi+\alpha})(v_0 + v_1(v_0, \alpha)).$$

Alternatively, this may also be obtained using the Splitting Lemma.

Now, the main idea of the proof is to define the functions h_α and $\mathbf{J}^\beta_\alpha$ on V_0 by

$$h_\alpha(v_0) := h(v_0 + v_1(v_0, \alpha)) \text{ and } \mathbf{J}^\beta_\alpha(v_0) := \mathbf{J}^\beta(v_0 + v_1(v_0, \alpha))$$

and to consider critical points of the restriction $\mathbf{J}^{\xi+\beta}_\alpha$ to the level sets on h_α. If v_0 is such a critical point,

$$\mathrm{d}\mathbf{J}^{\xi+\beta}_\alpha(v_0) = \Lambda(v_0)\mathrm{d}h_\alpha(v_0). \tag{5.8}$$

Then v_0 is a relative equilibrium iff in addition

$$\frac{\xi + \beta}{\Lambda(v_0)} = \xi + \alpha. \tag{5.9}$$

As a first step, the existence of smooth functions $v_0 : \mathbb{R} \times \mathfrak{g}^{G_\xi} \times \mathfrak{g}^{G_\xi} \to V_0$ such that $v_0(\varepsilon, \alpha, \beta)$ is a critical point of $\mathbf{J}^{\xi+\beta}_\alpha\big|_{h_\alpha^{-1}(\varepsilon)}$ is shown. Using the equivariant Splitting Lemma 2.9, there is a G_ξ-equivariant α-dependent change of coordinates ψ_α such that $h_\alpha(\psi_\alpha(v_0)) = \|v_0\|^2 + f(\alpha)$ for some smooth function $f : \mathfrak{g}^{G_\xi} \to \mathbb{R}$. (If the quadratic form Q is negative definite, a minus has to be added.) Moreover, since $\psi_\alpha(0)$ is a local minimum of h_α, we have $\psi_\alpha(0) = 0$ and $f \cong 0$. Hence for fixed α, the level sets of h_α are diffeomorphic to spheres. Since equation (5.9) holds for the pair of functions $\mathbf{J}^{\xi+\beta}_\alpha$ and h_α iff it is true for the pair $\mathbf{J}^{\xi+\beta}_\alpha \circ \psi_\alpha$ and $h_\alpha \circ \psi_\alpha$, we may consider the latter pair instead.

From now on, we replace the variable ε by $r = \left\|\psi_\alpha^{-1}(v_0)\right\|$ depending on α.

In case of a transitive G_ξ-action on S^{l-1}, the existence of a smooth branch v_0 is trivial: We may just set $v_0(r,\alpha,\beta) = ru$ for any $u \in S^{l-1}$. Anyway, the argument in [OR04(a)] does not rely directly on the transitivity condition and hence it may possibly be adapted for a proof of a generalization. Ortega and Ratiu use a blow-up to obtain branches corresponding to the non-degenerate critical G_ξ-orbits of the function j. (Hence the G_ξ-Morse property of j is used, but the equation $V_0 = \ker \mathrm{d}^2(h - \mathbf{J}^\xi)(0)$ does not enter here). To do this, the function $\mathbf{J}_\alpha^{\xi+\beta}$ is composed with the mapping

$$\begin{aligned} \mathbb{R} \times S^{l-1} &\to V_0 \\ (r,u) &\mapsto \psi_\alpha(ru) \end{aligned}$$

to obtain a function $\bar{\mathbf{J}}_\alpha^{\xi+\beta}$ depending on r and u. Since $\psi_\alpha(0) = 0$, the resulting function and its r-derivative vanish in 0. Hence, it is given by a product of r^2 and third smooth function $\hat{\mathbf{J}}_\alpha^{\xi+\beta}$. For fixed $\alpha = \beta = 0$ and $r = 0$, we have $\hat{\mathbf{J}}_0^\xi(0,\cdot) = j$:

$$\begin{aligned} 2\hat{\mathbf{J}}_\alpha^{\xi+\beta}(0,u) = \partial_r^2 \bar{\mathbf{J}}_\alpha^{\xi+\beta}(0,u) = \frac{\mathrm{d}}{\mathrm{d}r}^2 \mathbf{J}_\alpha^{\xi+\beta}(\psi_\alpha(ru))\Big|_{r=0} \\ = \frac{\mathrm{d}}{\mathrm{d}r}\mathrm{d}\mathbf{J}_\alpha^{\xi+\beta}(\psi_\alpha(ru))\mathrm{d}\psi_\alpha(ru)u\Big|_{r=0} \\ = \mathrm{d}^2\mathbf{J}_\alpha^{\xi+\beta}(0)(\mathrm{d}\psi_\alpha(0)u, \mathrm{d}\psi_\alpha(0)u) + \underbrace{\mathrm{d}\mathbf{J}_\alpha^{\xi+\beta}(0)}_{=0}\mathrm{d}^2\psi_\alpha(0)(u,u) \end{aligned}$$

Moreover, since $\mathrm{d}\mathbf{J}^\xi(0)$ and $\mathrm{d}h(0)$ vanish and $\mathrm{d}_{V_0}v_1(0,0) = 0$ (as is quite standard and easy to see),

$$\mathrm{d}^2\mathbf{J}_0^\xi(0)(\cdot,\cdot) = \mathrm{d}^2\mathbf{J}^\xi(0)((\mathbb{1} + \mathrm{d}_{V_0}v_1(0,0))\cdot, (\mathbb{1} + \mathrm{d}_{V_0}v_1(0,0))\cdot) = \mathrm{d}^2\mathbf{J}^\xi(0)(\cdot,\cdot) \tag{5.10}$$

holds on V_0. Thus, we only have to show that $\mathrm{d}^2\mathbf{J}^\xi(0)(\cdot,\cdot)$ is $\mathrm{d}\psi_0(0)$-invariant. Since $\mathrm{d}^2\mathbf{J}^\xi(0) = \mathrm{d}^2h(0)$, we may as well show this for $\mathrm{d}^2h(0)$. Now from a calculation analogous to equation (5.10), we obtain $\mathrm{d}^2h(0) = \mathrm{d}^2h_0(0)$. The identity

$$h_0 \circ \psi_0(v_0) = \|v_0\|^2 = \mathrm{d}^2h(0)(v_0,v_0)$$

implies

$$\begin{aligned} 2\mathrm{d}^2h(0)(\cdot,\cdot) &= \mathrm{d}^2(h_0 \circ \psi_0)(0)(\cdot,\cdot) && (5.11) \\ &= \mathrm{d}^2h_0(0)(\mathrm{d}\psi_0(0)\cdot, \mathrm{d}\psi_0(0)\cdot) = \mathrm{d}^2h(0)(\mathrm{d}\psi_0(0)\cdot, \mathrm{d}\psi_0(0)\cdot), && (5.12) \end{aligned}$$

which completes the proof of the claim.

Locally near a critical point $u_0 \in S^{l-1}$ of j, an application of the Slice Theorem yields coordinates z and s of u, where s is a point of a $(G_\xi)_{u_0}$-invariant slice and $z \in G_\xi u_0$. By G_ξ-invariance, $\hat{\mathbf{J}}_\alpha^{\xi+\beta}$ is independent of z. Since the critical G_ξ-orbits of j are non-degenerate, $\mathrm{d}_s^2\hat{\mathbf{J}}_\alpha^{\xi+\beta}(0,0)$ is invertible for $\alpha = \beta = 0$. Thus, the implicit function theorem yields a function $s(r,\alpha,\beta)$ such that

$$r \cdot (z, s(r,\alpha,\beta))$$

is a critical point of $\mathbf{J}_\alpha^{\xi+\beta}$ restricted to the sphere of norm r for any z. Set

$$u_0(r,\alpha,\beta) := (0, s(r,\alpha,\beta))$$

and $v_0 = r \cdot u_0$.

Next, we may deduce that the Lagrange multiplier

$$\Lambda(r,\alpha,\beta) := \Lambda(v_0(r,\alpha,\beta))$$

depends smoothly on (r,α,β):

$$\Lambda(r,\alpha,\beta) = \frac{\mathrm{d}(\mathbf{J}_\alpha^{\xi+\beta} \circ \psi_\alpha)(v_0(r,\alpha,\beta)) \cdot v_0(r,\alpha,\beta)}{\mathrm{d}(h_\alpha \circ \psi_\alpha)(v_0(r,\alpha,\beta)) \cdot v_0(r,\alpha,\beta))}$$

and the denominator does not vanish if $r \neq 0$. Moreover Λ can be extended smoothly to the set $r = 0$: If

$$\bar{\mathbf{J}}_\alpha^{\xi+\beta}(r,u) = \mathbf{J}_\alpha^{\xi+\beta}(\psi_\alpha(ru)) \quad \text{and} \quad \bar{h}_\alpha(r,u) = h_\alpha(\psi_\alpha(ru)) = r^2$$

for $r \in \mathbb{R}$ and $u \in S^{l-1}$, we have

$$\begin{aligned} \mathrm{d}(\mathbf{J}_\alpha^{\xi+\beta} \circ \psi_\alpha)(ru)u &= \partial_r \bar{\mathbf{J}}_\alpha^{\xi+\beta}(r,u) = 2r\hat{\mathbf{J}}_\alpha^{\xi+\beta}(r,u) + r^2 \partial_r \hat{\mathbf{J}}_\alpha^{\xi+\beta}(r,u) \\ \mathrm{d}(h_\alpha \circ \psi_\alpha)(ru)u &= \partial_r \bar{h}_\alpha(r,u) = 2r. \end{aligned}$$

Thus, if we abbreviate $u_0(r,\alpha,\beta)$ by u_0,

$$\Lambda(r,\alpha,\beta) = \frac{2r\hat{\mathbf{J}}_\alpha^{\xi+\beta}(r,u_0) + r^2\partial_r \hat{\mathbf{J}}_\alpha^{\xi+\beta}(r,u_0)}{2r} = \hat{\mathbf{J}}_\alpha^{\xi+\beta}(r,u_0) + \frac{1}{2} r \partial_r \hat{\mathbf{J}}_\alpha^{\xi+\beta}(r,u_0).$$

Hence Λ is smooth for $r = 0$ and $\Lambda(0,0,0) = 1$.

Now, to complete the proof, the zeros of the local mapping

$$\begin{aligned} E : \mathbb{R} \times \mathfrak{g}^{G_\xi} \times \mathfrak{g}^{G_\xi} &\to \mathfrak{g}^{G_\xi} \\ (r,\alpha,\beta) &\mapsto \xi + \beta - \Lambda(r,\alpha,\beta)(\xi + \alpha) \end{aligned}$$

near the zero in $(0,0,0)$ are investigated via Lyapunov-Schmidt reduction. In a first step, the authors show that $W_0 := \ker \mathrm{d}_\beta E(0,0,0) = \mathbb{R}\xi$. Hence, choosing a complement W_1 of W_0 within $\mathfrak{g}^{G_\xi}$, the implicit function theorem yields a smooth locally defined function $\rho : \mathbb{R} \times \mathfrak{g}^{G_\xi} \times \mathbb{R}\xi \to W_1$ that solves

$$(\mathbb{1} - \mathbb{P}_{W_0})E(r,\alpha,w_0 + w_1) = 0,$$

where $w_0 \in W_0$, $w_1 \in W_1$, and $\mathbb{P}_{W_0}$ is the projection to W_0 associated to the splitting $\mathfrak{g}^{G_\xi} = W_0 \oplus W_1$. Next,

$$g(r,\alpha,w_0) := \mathbb{P}_{W_0} E(r,\alpha,w_0 + \rho(r,\alpha,w_0)) = 0$$

has to be solved. As calculated in [OR04(a)], the directional derivative

$$\mathrm{d}_\alpha g(0,0,0)\big|_{\mathbb{R}\xi \subset \mathfrak{g}^{G_\xi}}$$

is given by -1. Thus, there is an implicitly defined local function $\lambda : \mathbb{R} \times W_1 \times \mathbb{R}\xi \to \mathbb{R}$ with $\lambda(0,0,0) = 0$ and

$$g(r, \lambda(r,\nu,w_0)\xi + \nu, w_0) = 0.$$

Hence, in case of a transitive G_ξ-action on S^{l-1}, we obtain a *branch* of relative equilibria parameterized by $(r, \nu, w_0) \in \mathbb{R} \times W_1 \times \mathbb{R}\xi$: The smooth function

$$\begin{aligned} v : (r, \nu, w_0) \mapsto\, & v_0(r, \alpha(r, \nu, w_0), \beta(r, \nu, w_0)) \\ & + v_1(v_0(r, \alpha(r, \nu, w_0), \beta(r, \nu, w_0)), \lambda(r, \nu, w_0)\xi + \nu) \end{aligned}$$

gives the location of a relative equilibrium for (r, ν, w_0), whose generator is given by $\xi + \lambda(r, \nu, w_0)\xi + \nu$ and whose energy level (value of the function h) is r^2.

Remark 5.9. We now investigate these branches more closely: Consider the sets $\mathfrak{g}^{G_\xi}$. Since G_ξ contains a maximal torus T

$$\mathfrak{g}^{G_\xi} \subset \mathfrak{g}^T = \mathfrak{t},$$

where the last equation follows from the maximality of T. Thus,

$$\mathfrak{g}^{G_\xi} = \mathfrak{t}^{G_\xi}.$$

By [BtD85, chapter V, Proposition 2.3, part (ii)], $\mathfrak{t}^{G_\xi}$ is given by $\mathfrak{t}$ if ξ is regular and $\mathfrak{t}^{G_\xi}$ coincides with the intersection of all Weyl walls containing ξ otherwise (where $\mathfrak{t}^*$ and $\mathfrak{t}$ are identified). As will be shown in section 6.4, the set of elements $\eta \in \mathfrak{t}$ with $\det(\mathrm{d}^2(h - \mathbf{J}^\eta)(0)) = 0$ forms a union of affine hyperplanes of $\mathfrak{t}$. Moreover, each of these hyperplanes is associated to an irreducible subrepresentation of T such that $\ker(\mathrm{d}^2(h - \mathbf{J}^\eta)(0))$ consists of the sum of the irreducible subrepresentations corresponding to the hyperplanes that contain η. The underlying subspace of an affine hyperplane coincides with the Lie algebra $\mathfrak{t}_x$ of the stabilizer T_x of any nonzero element x of the associated irreducible representation. By the non-degeneracy of $\mathrm{d}^2h(0)$, these affine hyperplanes do not contain 0. Thus the intersection of each of these affine hyperplanes with $\mathfrak{g}^{G_\xi} = \mathfrak{t}^{G_\xi}$ is an affine hyperplane of $\mathfrak{t}^{G_\xi}$ or empty. We now take the condition into account that G_ξ acts transitively on S^{l-1}. This implies that $(G_\xi)_{v_0}$ and $(G_\xi)_{w_0}$ are G_ξ-conjugate for any two non-zero elements $v_0, w_0 \in V_0$. If the corresponding Lie algebras are denoted by $\mathfrak{g}_{\xi,v_0}$ and $\mathfrak{g}_{\xi,w_0}$ respectively, this yields

$$\mathfrak{t}_{v_0} \cap \mathfrak{g}^{G_\xi} = \mathfrak{g}_{\xi,v_0} \cap \mathfrak{g}^{G_\xi} = \mathfrak{g}_{\xi,w_0} \cap \mathfrak{g}^{G_\xi} = \mathfrak{t}_{w_0} \cap \mathfrak{g}^{G_\xi}.$$

Hence, ξ is contained in only one of these affine hyperplanes of $\mathfrak{t}^{G_\xi}$: The underlying subspaces of these hyperplanes are of the form $\mathfrak{t}_{v_0} \cap \mathfrak{g}^{G_\xi}$ for some $v_0 \in V_0 \setminus \{0\}$ and hence are all equal.

Thus, we may choose W_1 to be the underlying subspace of the affine hyperplane of $\mathfrak{t}^{G_\xi}$ that contains ξ. Indeed, since 0 is not an element of this affine hyperplane, in this case, W_1 is a complement of the space W_0 generated by ξ. Then W_1 is contained in the Lie algebra of the stabilizer of any element of V_0.

If we perform the Lyapunov-Schmidt reduction with respect to the symmetry, i.e. we choose a G_ξ-invariant complement V_1, the implicitly defined function v_1 is G_ξ-equivariant. Hence, we may assume that the stabilizers of v_0 and $v_0 + v_1(v_0, \alpha)$ within G_ξ coincide for any $\alpha \in \mathfrak{g}^{G_\xi}$. Thus, W_1 is contained in $\mathfrak{t}^{G_\xi} \cap \mathfrak{g}_{v(r,\nu,w_0)}$ for every triple (r, ν, w_0). This implies that for every $\nu \in W_1$, the Lie algebra element $\xi + \lambda(r, 0, w_0)\xi + \nu$ is a generator of the relative equilibrium $v(r, 0, w_0)$. Since the proof of Theorem 5.8 yields all pairs of a relative equilibrium near 0 and a generator in $\mathfrak{g}^{G_\xi}$ near ξ and the pair $(v(r, \nu, w_0), \xi + \lambda(r, \nu, w_0)\xi + \nu)$ is the unique one with W_1-component ν, the functions v and λ are constant in ν. Alternatively, this can be argued in a more explicit way: Since W_1 is contained in the Lie algebra of the stabilizer of $v_0 + v_1(v_0, \alpha)$ for any v_0 and α, $\mathrm{d}\mathbf{J}^\nu(v_0 + v_1(v_0, \alpha))$ vanishes for any $\nu \in W_1$. Hence, $v_1(v_0, \alpha) = v_1(v_0, \alpha + \nu)$ and the function $\mathbf{J}^\beta_\alpha$ is constant in the W_1-component of both α and β. Going through the construction of v and λ, we obtain that these functions are constant in $\nu \in W_1$.

Thus, the branches can be parameterized by r and the 1-dimensional variable w_0.

Now, we discuss some ideas how to generalize the transitivity condition:

First, we replace the condition by the assumption that G_ξ acts irreducibly on V_0. If we define

$$\|v\|_0 = \pm \mathrm{d}^2 h(0)\big|_{V_0}(v_0, v_0) = \pm \mathrm{d}^2 \mathbf{J}^\xi(0)\big|_{V_0}(v_0, v_0),$$

it is not hard to see that the function $\bar{h}(u, r) := h(ru)$ is of the form

$$\bar{h}(u, r) = r^2\left(\frac{1}{2} + r f(u, r)\right)$$

for some C^∞-function $f : S^{l-1} \times \mathbb{R} \to \mathbb{R}$.

Thus, we have to consider terms of at least order 3 to obtain non-constant restrictions to the spheres. Due to symmetry, even higher order terms may be forced to be constant. It seems to be reasonable to consider the lowest order terms that are not constant. Then a blow-up argument might be possible if an additional condition is satisfied: Suppose that the lowest order term that is not constant on spheres is a Morse-function. In this case, we may proceed in a similar way as Ortega and Ratiu ([OR04(a)]) or Field ([Fie07, chapter 4]), who presents a blow-up method for bifurcation theory developed by Field and Richardson. Unfortunately, the condition on the lowest order term without spherical symmetry is presumably not generic for every irreducible representation.

Moreover, as far as I can see, the assumption that V_0 is an irreducible G_ξ-representation is necessary for a blow-up argument that yields a non-constant function on the 1-sphere for $r = 0$: For simplicity, search for relative equilibria with generators in the space W_0 of multiples of ξ. Let us call the corresponding 1-dimensional parameter λ (instead of w_0). Then the Lyapunov-Schmidt reduction may cause terms that depend on λ and u in a non-constant way and are only quadratic in r. In this case, a blow-up cannot be performed. If $V_0 := \ker \mathrm{d}^2(h - \mathbf{J}^\xi(0))$ is irreducible, the G_ξ-equivariance of the derivatives of the invariant functions $v_0 \mapsto h(v_0 + v_1(v_0, \lambda))$, $v_0 \mapsto \mathbf{J}^{\xi+\lambda}(v_0 + v_1(v_0, \lambda))$, and similar ones prohibits such terms.

Thus, a generalization in this way would not gain much if we aim to investigate the structure of the set of relative equilibria near the equilibrium 0 in general.

Therefore, it seems to be impossible to prove the existence of branches that are smooth at the origin under genericity assumptions this way. Smoothness at the origin is probably not a generic phenomenon in general. Nevertheless, there is another approach to generalize the transitive condition, which yields a theorem in spirit of the original intention of Theorem 5.8 without a smoothness result. In some way, it is even closer to the original idea of Theorem 5.7:

For simplicity, we search only for solutions with generators of the form $\lambda\xi$ for λ in $\mathbb{R}$. Set

$$h_\lambda(v_0) = h_{\lambda\xi}(v_0), \quad \mathbf{J}_\lambda(v_0) = \mathbf{J}^{\lambda\xi}_{\lambda\xi}(v_0), \quad \Psi_\lambda(v_0) = \Psi_{\lambda\xi}(v_0),$$

where the right hand sides are defined in the proof of Theorem 5.8. Then $h_\lambda \circ \Psi_{\lambda\xi}(v_0) = \|v_0\|^2$. In order to find solutions of

$$\mathrm{d}(h_\lambda \circ \Psi_{\lambda\xi} - \mathbf{J}_\lambda \circ \Psi_{\lambda\xi}) = 0,$$

we proceed in a similar way as the authors of [CLOR03] in the proof of Theorem 5.5:

First we use the implicit function theorem to obtain a locally defined function λ such that

$$\langle \nabla(\mathbf{J}_{\lambda(v_0)} \circ \Psi_{\lambda(v_0)})(v_0), v_0 \rangle = 0. \tag{5.13}$$

To do this, we observe that for $r \in \mathbb{R}$ and an element u of the unit sphere of V_0

$$\begin{aligned}\langle \nabla(\mathbf{J}_\lambda \circ \Psi_\lambda)(ru), u \rangle &= \mathrm{d}(\mathbf{J}_\lambda \circ \Psi_\lambda)(ru)u \\ &= \partial_r \bar{J}^{\lambda\xi}_{\lambda\xi}(r,u) \\ &= r(2\hat{\mathbf{J}}^{\lambda\xi}_{\lambda\xi}(r,u) + r\partial_r \hat{\mathbf{J}}^{\lambda\xi}_{\lambda\xi}(r,u)).\end{aligned}$$

A similar calculation as in the proof of Theorem 5.8 yields that the λ-derivative of the term in the parenthesis at $\lambda = 1$ and $r = 0$ is given by $2j(u,u) = 1$. Thus there is a locally defined function λ such that equation (5.13) holds. Then the critical points of

$$\mathbf{J}_{\lambda(v_0)} \circ \Psi_{\lambda(v_0)}$$

restricted to the spheres of norm ε correspond to relative equilibria within the energy level $h^{-1}(\varepsilon^2)$. Since this function is G_ξ-invariant, we obtain Theorem 5.8 with condition 2 and the smoothness statement omitted.

Example 5.10. Consider the T-representation on $\mathbb{C} \oplus \mathbb{C}$ with $T = S^1 \times S^1$ such that the first factor acts on the first summand and the second factor on the second one by multiplication in $\mathbb{C}$. Suppose that $h : \mathbb{C} \oplus \mathbb{C} \to \mathbb{R}$ is a T-invariant Hamiltonian function with $\mathrm{d}h(0) = 0$ and $\mathrm{d}^2h(0)$ definite. Any G-invariant real-valued function on $\mathbb{C} \oplus \mathbb{C}$ is of the form $(z_1, z_2) \mapsto a|z_1|^2 + b|z_2|^2$ for some real numbers a and b. Thus the minimum number of critical orbits on the unit sphere of such a function is 2. It is easy to see that there is a $\xi \in \mathfrak{t}$ with $\mathrm{d}^2(h - \mathbf{J}^\xi)(0) = 0$. Hence from the above variant of Theorem 5.8 follows that there are at least 2 orbits of relative equilibria on each energy level. Anyhow, every point of $\mathbb{C} \oplus \mathbb{C}$ is a relative equilibrium, since the momentum level sets coincide with the T-orbits, see section 6.4.

Anyway, if we search for relative equilibria with periodic orbits, the transitivity condition is often satisfied for the relevant kernels and thus we obtain smooth branches. This will be presented in the next section.

5.2.2 Implications for groups of rank 1

Again, we suppose that V is a symplectic G-representation and $0 \in V$ a non-degenerate critical point of a G-invariant Hamiltonian function h. We will see that Theorem 5.8 predicts a branch of relative equilibria for every 2-dimensional kernel $V_0 = \ker \mathrm{d}^2(h - \mathbf{J}^\xi)(0)$. In particular, if G is a connected compact group of rank 1, this application of Theorem 5.8 yields the generic structure of all relative equilibria near 0.

Indeed, let $T \subset G_\xi$ be a maximal torus with ξ contained in the Lie algebra $\mathfrak{t}$ of T and consider $V_0 = \ker \mathrm{d}^2(h - \mathbf{J}^\xi)(0)$ as a T-representation. This representation is non-trivial: Otherwise, $\mathbf{J}^\xi\big|_{V_0}$ and hence $\mathrm{d}^2\mathbf{J}^\xi(0)\big|_{V_0} = \mathrm{d}^2h(0)\big|_{V_0}$ vanishes, which contradicts the assumption that $\mathrm{d}^2h(0)$ is non-degenerate. Thus V_0 is 2-dimensional iff V_0 is a non-trivial irreducible T-representation and in this case, $T \subset G_\xi$ acts transitively on the unit sphere of V_0. Then Theorem 5.8 applies and we obtain a branch of relative equilibria tangent to V_0 with multiples of ξ as generators.

We now consider an approach to find all relative equilibria near 0:

A necessary condition for the local existence of relative equilibria near 0 is the occurrence of purely imaginary eigenvalues of $\mathrm{d}X_h(0)$. Otherwise, 0 is a hyperbolic equilibrium of X_h. As presented in chapter 3, purely imaginary eigenvalues can occur in a stable way in Hamiltonian systems.

Indeed, using the Splitting Lemma or equivalently Lyapunov-Schmidt reduction, we may reduce our search for relative equilibria to the union of the kernels $\ker \mathrm{d}^2(h - \mathbf{J}^\xi)(0)$ for $\xi \in \mathfrak{g}$, and the next lemma shows that all these kernels are contained in the centre space $\mathbb{E}_\mathrm{c}$ of $\mathrm{d}X_h(0)$:

Lemma 5.11. $\ker \mathrm{d}^2(h - \mathbf{J}^\xi)(0) \subset \mathbb{E}_\mathrm{c}$ *for every* $\xi \in \mathfrak{g}$.

Proof. Consider the linearization of X_h at 0. The vector field $x \mapsto \mathrm{d}X_h(0)x$ is the Hamiltonian vector field of the quadratic part Q_h of h. Since $\mathbf{J}^\xi$ is also quadratic, $\mathrm{d}(Q_h - \mathbf{J}^\xi)(x)$ vanishes for $x \in V_0 := \ker \mathrm{d}^2(h - \mathbf{J}^\xi)(0)$ and hence V_0 consists of relative equilibria of X_{Q_h}. Obviously, all compact trajectories of the linearization have to be contained in the centre space, in particular relative equilibria. □

By Theorem 3.16, generically there is a choice of an inner product such that the eigenspaces of the restriction $\mathrm{d}^2h(0)$ to $\mathbb{E}_\mathrm{c}$ are irreducible G-symplectic representations and the symplectic form defines a complex structure J. In the following, we assume that this genericity assumption is satisfied and fix an appropriate inner product and a base. Since the eigenspaces are G-symplectic, they are $\mathrm{d}^2\mathbf{J}^\xi(0)$-invariant for any choice of $\xi \in \mathfrak{g}$. Thus, we can search for solutions ξ of

$$\det \mathrm{d}^2(h - \mathbf{J}^\xi)(0) = 0$$

on any restriction to an eigenspace of $\mathrm{d}^2h(0)$ separately. Generically, we obtain disjoint solution sets for each eigenspace of $\mathrm{d}^2h(0)$. (From the following calculation of the eigenvalues of $\mathrm{d}^2\mathbf{J}^\xi(0)$ in the rank 1 case, it will follow that this is satisfied if the purely imaginary eigenvalues of $X_h(0) = J\mathrm{d}^2h(0)$ are *non-resonant in an equivariant sense*, which means that for each pair $\pm\alpha\mathrm{i}$ of purely imaginary eigenvalues, no other integer multiples of $\alpha\mathrm{i}$ occur and the space $E_{\alpha\mathrm{i}}$ is an irreducible G-symplectic subrepresentation.)

Hence, for each eigenvalue $c \in \mathbb{R} \setminus \{0\}$ of $\mathrm{d}^2h(0)$, we have to identify the values of $\xi \in \mathfrak{g}$ such that c is an eigenvalue of $\mathrm{d}^2\mathbf{J}^\xi(0)$. Since the set of those ξ consists of G-orbits of the adjoint representation on $\mathfrak{g}$, we may restrict our search to the Lie algebra of a maximal torus of G.

If J denotes the linear map representing ω with respect to our choice of an invariant inner product, we consider each eigenspace as an irreducible complex representation by setting $\mathrm{i}x := Jx$.

We now suppose that G is a connected compact group of rank 1. By [BtD85, chapter V, Corollary 1.6], there are only two groups of this kind: SO(3) and SU(2). Hence we consider a symplectic G-representation (V, ω), where $G = \mathrm{SO}(3)$ or $G = \mathrm{SU}(2)$.

In case $G = \mathrm{SU}(2)$, the irreducible complex representations are given by the spaces S_n, $n \in \mathbb{N}$, where S_0 is the trivial representation, S_1 is the standard representation on $\mathbb{C}^2$, and S_n is the nth symmetric power of S_1. (The theory of SO(3)- and SU(2)-representations is presented for example in [BtD85]. The relevant facts are summarized in this paragraph.) S_n may be identified with the space of homogeneous polynomials of power n. Then $g \in \mathrm{SU}(2)$ acts on a polynomial $P \in S_n$ by

$$(gP)(z) = P(zg),$$

where z is considered as a line vector and the product zg denotes matrix multiplication. The polynomials

$$P_k(z_1, z_2) = z_1^k z_2^{n-k}, \quad 0 \leq k \leq n$$

form a basis of S_n. Moreover, each of these basis elements generates an irreducible representation of the maximal torus of SU(2) formed by the elements of the form

$$g_\theta = \begin{pmatrix} \theta & 0 \\ 0 & \theta^{-1} \end{pmatrix}, \quad \theta \in U(1) \simeq S^1 :$$

Since $g_\theta P_k = \theta^{2k-n} P_k$, the representation is isomorphic to the irreducible $U(1)$-representation $U(1) \to U(1)$ given by $\theta \mapsto \theta^{2k-n}$. Moreover, since these representations are pairwise different, the elements P_k are pairwise orthogonal with respect to any invariant Hermitian product. The irreducible subrepresentations P_k of the maximal torus of SU(2) are called *weight spaces*. Now, let us compute $\mathrm{d}^2\mathbf{J}^\xi(0)$ for ξ contained in the Lie algebra $\mathfrak{t}$ of this maximal torus: By definition,

$$\mathrm{d}\mathbf{J}^\xi(x)y = \omega(\xi \cdot x, y) = \langle \xi \cdot x, \mathrm{i}y \rangle$$

for our fixed invariant (real-valued) inner product $\langle \cdot, \cdot \rangle$. Thus $\mathrm{d}^2\mathbf{J}^\xi(0)$ is given by $-\mathrm{i}A_\xi$, where A_ξ is the matrix corresponding to the action of ξ:

$$A_\xi x = \xi \cdot x = \frac{\mathrm{d}}{\mathrm{d}t} \begin{pmatrix} e^{2\pi \mathrm{i}t\xi} & 0 \\ 0 & e^{-2\pi \mathrm{i}t\xi} \end{pmatrix} x \bigg|_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t} g_{e^{2\pi \mathrm{i}t\xi}} x \bigg|_{t=0}$$

Thus the vectors P_k are complex eigenspaces of A_ξ with corresponding eigenvalues $2\pi\mathrm{i}(2k-n)\xi$ and hence the eigenvalues of $\mathrm{d}^2\mathbf{J}^\xi(0)$ are $2\pi(2k-n)\xi$.

Let U be an eigenspace U of $\mathrm{d}^2h(0)$ contained in $\mathbb{E}_\mathrm{c}$ with eigenvalue $c \neq 0$ isomorphic to S_n. Then the solutions $\xi \in \mathfrak{t}$ of $\det \mathrm{d}^2(h - \mathbf{J}^\xi)(0) = 0$ are given by $\frac{1}{2k-n} \cdot \frac{c}{2\pi}$ for $k = 0, \ldots, n$ if n is odd and $k = 0, \ldots, \frac{n}{2} - 1, \frac{n}{2} + 1, \ldots, n$ if n is even. Since ξ and $-\xi$ belong to the same adjoint orbit, we obtain $\frac{n+1}{2}$ adjoint orbits in the Lie algebra of SU(2) with a non-invertible restriction of $\mathrm{d}^2(h - \mathbf{J}^\xi)(0)$ to U for odd n and $\frac{n}{2}$ orbits for even n. For each element ξ of these orbits, the kernel is isomorphic to $\mathbb{C}$ and the stabilizer $G_\xi = \mathrm{SU}(2)_\xi$ is isomorphic to the maximal torus S^1 which acts transitively on the unit sphere of the kernel. Thus we generically obtain $\frac{n+1}{2}$ branches of relative equilibria for each summand V_n of $\mathbb{E}_c$ with n odd and $\frac{n}{2}$ branches for each summand V_n of $\mathbb{E}_c$ with n even. Each branch is the SU(2)-orbit of an $\mathbb{R}$-2-dimensional symplectic manifold.

The analysis in the case $G = \mathrm{SO}(3)$ is similar: SU(2) can be expressed as a two-fold covering of SO(3) and hence the irreducible representations of SO(3) can be considered as irreducible SU(2)-representations. This way, we obtain exactly those irreducible SU(2)-representations, whose kernel contains $-\mathbb{1}$. These are the representations S_n with n even. (See [BtD85] for details.) Thus, we also generically obtain l branches for each summand S_{2l} of $\mathbb{E}_c$. Since the covering epimorphism $\mathrm{SU}(2) \to \mathrm{SO}(3)$ may be restricted to a two-fold map between the maximal tori which induces multiplication by 2 on the corresponding Lie algebras, the corresponding generators on a c-eigenspace of $\mathrm{d}^2h(0)$, $c \neq 0$ are given by the adjoint orbits of $\frac{1}{k-l} \cdot \frac{c}{2\pi}$ for $k = 0, \dots, l-1$.

5.2.3 Equivariant Weinstein-Moser theorem

Alternatively, the results for SO(3)- and SU(2)-representations may be deduced by a simple application of the equivariant Weinstein-Moser theorem of [MRS88]. We first state the theorem, which is an adaption of a theorem proved by Weinstein and Moser to the symmetric case, and prove the results for the rank 1 case as a corollary. A broad sketch of the proof and history of the Weinstein-Moser theorem and its equivariant version is given afterwards.

As before, let G be a compact Lie group, V be a G-symplectic representation, and $h : V \to \mathbb{R}$ be a G-invariant Hamiltonian.

Definition 5.12. For any non-zero purely imaginary eigenvalue $\nu\mathrm{i}$ (i.e. $\nu \in \mathbb{R} \setminus \{0\}$) of $L := \mathrm{d}X_h(0)$, let the *resonance space* $V_{\nu\mathrm{i}}$ be the real part of the sum of eigenspaces with eigenvalues of the form $k\nu\mathrm{i}$, $k \in \mathbb{Z}$.

Any resonance space $V_{\nu\mathrm{i}}$ is G-invariant. Moreover, if the restriction of the linearization L to $V_{\nu\mathrm{i}}$ is diagonizable over $\mathbb{C}$, there is an S^1-action on $V_{\nu\mathrm{i}}$: The trajectories of the flow of the linearized vector field $x \mapsto Lx$ are all $\frac{2\pi}{|\lambda|}$-periodic, and thus the flow defines an S^1-action which commutes with the G-action by equivariance of the flow. This yields a $G \times S^1$-action on $V_{\nu\mathrm{i}}$.

The equivariant Weinstein-Moser theorem requires the following two conditions on the Hamiltonian function and the resonance space, which Montaldi, Robert, and Stewart call H1 and H2:

- (H1) $\mathrm{d}^2h(0)$ is a non-degenerate quadratic form.
- (H2) $\mathrm{d}^2h(0)\big|_{V_{\nu\mathrm{i}}}$ is positive definite.

Condition H2 implies the semi-simplicity of $L\big|_{V_{\nu\mathrm{i}}}$: There is an inner product $\langle\cdot,\cdot\rangle$ with respect to which $\mathrm{d}^2h(0)$ is represented by a multiple of the identity. Thus L is a multiple of the matrix J with $\omega = \langle\cdot, J\cdot\rangle$, which is skew-symmetric and hence diagonizable.

Thus $V_{\nu\mathrm{i}}$ may be considered as a $G \times S^1$-symplectic representation as explained above.

The set of periodic orbits may also considered as a G-space: Let $\sigma : \mathbb{R} \to V$ be a periodic trajectory with minimal period τ. With the identification $S^1 =$

$\mathbb{R}/\mathbb{Z}$, for $(g,\theta) \in G \times S^1$, we define

$$((g,\theta)\sigma)(s) = g\sigma(s+\theta\tau).$$

For each isotropy subgroup of the $G \times S^1$-action on $V_{\nu\mathrm{i}}$, the equivariant Weinstein-Moser theorem predicts periodic orbits that have this symmetry with respect to the $G \times S^1$-action on the set of periodic orbits:

Theorem 5.13 (Equivariant Weinstein-Moser theorem, [MRS88, Theorem 1])**.** *Suppose (H1) and (H2) hold. Then for every isotropy subgroup $K \subset G \times S^1$ of the $G \times S^1$-action on $V_{\nu\mathrm{i}}$, there exist at least*

$$\frac{1}{2}\dim V_{\nu\mathrm{i}}^K$$

periodic trajectories of X_h with periods near $\frac{2\pi}{|\lambda|}$ and symmetry group containing K on each level set of h near 0.

To be precise, for our application, we need the following special case, which is a generalization of the Lyapunov centre theorem. (See [AM67, appendix C] for a version by Kelley, which coincides with the theorem below for the case of a trivial representation if the word "smooth" is replaced by "C^1". The original version of the Lyapunov centre theorem, that appeared in [Lya49], misses the smoothness statement and applies only for analytic Hamiltonians.) In this case, the proof also yields smooth branches of relative equilibria:

Theorem 5.14 (Equivariant Lyapunov centre theorem, Remark 1.2 (b) in [MRS88])**.** *Suppose that in Theorem 5.13, $\dim V_{\nu\mathrm{i}}^K = 2$. Then there is a smooth 2-dimensional submanifold of V through 0 tangent to $V_{\nu\mathrm{i}}^K$ which consists of periodic orbits with symmetry group K.*

In addition, we need the following completeness statement, which follows from the proof of Theorem 5.14:

Theorem 5.15. *In the setting of Theorem 5.14, the 2-dimensional manifold contains all periodic orbits near 0 with symmetry group K and periods near $\frac{2\pi}{|\lambda|}$.*

Lemma 1.46 implies immediately:

Corollary 5.16. *If in the setting of Lemma 1.46, the group G has rank 1, p is contained in a periodic orbit.*

(Here, we consider an equilibrium as a special case of a periodic orbit.)

Thus for groups of rank 1, we only have to find periodic orbits with additional symmetry properties to obtain all relative equilibria:

Suppose that p is a relative equilibrium, but not an equilibrium. Fix a torus T such that the trajectory of p is contained in Tp. Then the trajectory of p is periodic and coincides with Tp. Let τ denote its minimal period.

If $S^1 = \mathbb{R}/\mathbb{Z}$ and $\sigma : \mathbb{R} \to V$ denotes the trajectory of p, there is a map $\theta_\alpha : T \to S^1$ such that for any $t \in T$ and $s \in S^1$

$$t\sigma(s) = \sigma(s + \theta_\alpha(t)\tau).$$

By T-equivariance of the flow, $\theta_\alpha(t)$ is independent of s and forms a group homomorphism. Then the isotropy subgroup of σ contains the group

$$K_\alpha := \left\{(t, -\theta_\alpha(t)) \in G \times S^1 \;\middle|\; t \in T\right\}.$$

On the contrary, any periodic orbit fixed by a group of this type is a relative equilibrium iff θ_α is surjective, which is equivalent to θ_α being non-trivial.

If the resonance space $V_{\nu\mathrm{i}}$ of any purely imaginary eigenvalue $\nu\mathrm{i}$ of L is considered as a complex representation by extending the S^1-action $\mathbb{R}$-linearly to $\mathbb{C}$, the space $V_{\nu\mathrm{i}}^{K_\alpha}$ coincides with the weight space corresponding to θ_α of $V_{\nu\mathrm{i}}$. (Note that in case $\mathrm{d}X_h(0) \in \mathcal{O}$ and hence $V_{\nu\mathrm{i}} = E_{\pm\nu\mathrm{i}}$, this complex structure coincides with the one that we obtain from theorem 3.15.)

Now, the equivariant Lyapunov centre Theorem 5.14 yields the following generic structure of relative equilibria:

Corollary 5.17. *Suppose that each resonance space $V_{\nu\mathrm{i}}$ forms an irreducible complex G-representation and hence $V_{\nu\mathrm{i}} \simeq S_n$ for some n. Moreover, suppose that (H1) and (H2) are satisfied for each $V_{\nu\mathrm{i}}$. Then there are $\frac{n+1}{2}$ smooth branches of G-orbits of relative equilibria bifurcating from 0 if n is odd and $\frac{n}{2}$ such branches if n is even. The trajectories form periodic orbits with periods near $\frac{|\lambda|}{2\pi}$. The union of these branches for all resonance spaces contains all relative equilibria near 0.*

Proof. The result follows from the above arguments, the decomposition of S_n into weight spaces as presented in the last section and Theorems 5.14 and 5.15. □

Remark 5.18. Suppose that G is a compact connected group, not necessarily of rank 1 and $T \subset G$ is a maximal torus. A *(global) weight* of a complex G-representation V is a complex 1-dimensional T-subrepresentation given by a map $\theta_\alpha : T \to U(1) \simeq S^1$. As in the last chapter, we obtain a branch of relative equilibria (with periodic trajectories) for each non-trivial weight θ_α of the centre space $\mathbb{E}_\mathrm{c}$ (counted with multiplicities): Let $K_\alpha = \{(t, \theta_\alpha(t)) \mid t \in T\}$ be the corresponding subgroup of $G \times S^1$. For every resonance space $V_{\nu\mathrm{i}}$, the space $V_{\nu\mathrm{i}}^{K_\alpha}$ coincides with the weight space of $V_{\nu\mathrm{i}}$ corresponding to the weight θ_α. Moreover, the periodic orbits with isotropy group containing K_α are obviously relative equilibria. Thus, the the Equivariant Weinstein-Moser Theorem implies the existence of a family of relative equilibria that intersects each energy level in a G-orbit. If θ_α occurs with multiplicity one, this family is smooth. Obviously, these relative equilibria are contained in the fixed point space of the group $\ker \theta_\alpha$, which coincides with the weight space in V corresponding to α. Conversely, all relative equilibria in this weight space are contained in such a family. This follows from the following implication of corollary 5.16.

Corollary 5.19. *If in the setting of Lemma 1.46, M is a representation and p is contained in an irreducible real subrepresentation of a maximal torus of G, p is contained in a periodic orbit.*

Proof. Since irreducible torus representations are at most of dimension 2, the isotropy subgroup $H = G_p$ contains a torus whose dimension is at least $\operatorname{rank} G - 1$. Thus the group $N(H)/H$ is at most of rank 1. V^H is an X-invariant subspace, and the restriction of X to V^H is $N(H)/H$-equivariant. Hence p is either a relative equilibrium with respect to a finite group action, which is an equilibrium, or corollary 5.16 applies. □

The proof of Theorem 5.13 in principle follows [Wei78], which was preceded by several publications on persistence of periodic orbits, starting with [Mos70] and [Wei73].

These articles deal with manifolds consisting of periodic orbits contained in a level set of the Hamiltonian function. If these manifolds satisfy some non-degeneracy assumption, it is shown that a finite number of periodic orbits persists under perturbations of the Hamiltonian function and the number of these orbits is estimated with respect to the topology of the manifold. By considering the Hamiltonian function as a perturbation of its linearization and using a blow-up argument, this yields periodic solutions near an equilibrium point with definite Hessian of the Hamiltonian function.

The reasoning in [Wei78] is similar to the proof of the Hopf theorem known in bifurcation theory: The problem of finding periodic orbits in a symplectic manifold P is reformulated on the infinite dimensional loop space of maps $S^1 \to P$. Then the periodic orbits correspond to the zeros of some closed 1-form defined on the loop space (and thus in the exact case, to the critical points of some function called the action integral). Afterwards, one reduces to finite-dimensions again using some kind of Lyapunov-Schmidt reduction. In principle, this proceeding is suggested by Moser in [Mos76], where the existence of periodic orbits near an equilibrium point is investigated, but it is implemented in a quite indirect way there. In [Wei78], Weinstein deals with the technical difficulties to give an appropriate notion of a Hamiltonian system on the loop space such that the periodic orbits correspond to the zeros of the Hamiltonian vector field on the loop space and hence to the zeros of a 1-form. Moreover, Weinstein gives a definition of non-degeneracy for this setting, which allows to perform a Lyapunov-Schmidt reduction:

In case of infinite-dimensional manifolds, we cannot expect the tangent space at a point to be isomorphic to its dual space in general. Thus, the symplectic form defined by Weinstein on the loop space induces an isomorphism of the tangent space of a point to a dense subspace of its dual. Accordingly, it is called a *weak symplectic structure*.

Indeed, let the loop space $\bigwedge P$ be the function space $C^1(S^1, P)$. (This way, it is a Banach manifold.) The tangent space $T_c \bigwedge P$ at $c \in \bigwedge P$ is given by the space of $\Gamma^1(c^*TP)$ of C^1 vector fields along c. Its dual is given by some space of distributions. If ω is the symplectic form on P, Weinstein endows $\bigwedge P$ with the weak symplectic structure given by its lift $\bigwedge \omega$, where the *lift* $\bigwedge \alpha$ of an n-form α on P is given by

$$\langle \bigwedge \alpha, (v_1, \dots v_n) \rangle = \int_0^1 \langle \omega(c(t)), (v_1(t), \dots v_n(t)) \rangle \mathrm{d}t.$$

(Here S^1 is identified with $\mathbb{R}/\mathbb{Z}$, thus c can be considered as a curve parameterized $t \in [0,1]$.) For each $c \in \bigwedge P$, $\bigwedge \omega$ yields an isomorphism from $T_c \bigwedge P = \Gamma^1(c^*TP)$ to $\Gamma^1(c^*T^*P)$, which is a dense subspace of $T_c^* P$.

Next, Weinstein determines a 1-form, whose zeros correspond to periodic orbits of X_h in P: Let $\mathcal{D}$ denote the vector field on $\bigwedge P$ defined by $\mathcal{D}(c) = \frac{\mathrm{d}c}{\mathrm{d}t}$.

(To be precise, $\mathcal{D}(c)$ is in general only a continuous vector field along c and thus $\mathcal{D}(c)$ is not a section of $T \bigwedge P$ but of its super-bundle $\check{T} \bigwedge P$, where $\check{T}_c \bigwedge P = \Gamma^0(c^*TP)$. We will come back to this later.)

A loop $c \in \bigwedge P$ forms a periodic orbit of X_h of period 1 iff $\bigwedge X_h(c) := X_h \circ c = \mathcal{D}(c)$. More generally, c corresponds to a periodic orbit of period $\tau > 0$ iff $\mathcal{D}(c) = \tau \bigwedge X_h(c)$. Now, we consider the dual formulation:

$$\bigwedge \omega(\mathcal{D}, \cdot) = \tau \bigwedge \omega(X_h, \cdot) = \tau \mathrm{d} \bigwedge h,$$

where $\bigwedge h(c) = \int_0^1 h(c(t))\mathrm{d}t$. The periodic orbits contained in an energy level set $h^{-1}(E)$ with $E \in \mathbb{R}$ may be expressed as the zeros of a single closed 1-form α_E^h defined on the space $\bigwedge P \times \mathbb{R}^+$:

$$\langle \alpha_E^h(c, \tau), (v, a) \rangle = \bigwedge \omega(\mathcal{D}, v) - \tau \mathrm{d} \bigwedge h - a \bigwedge (h - E)(c).$$

As Weinstein shows, $\bigwedge \omega(\mathcal{D}, v)$ is exact if ω is exact. More precisely, $\omega = \mathrm{d}\theta$ implies $\bigwedge \omega(\mathcal{D}, v) = \mathrm{d}A$ for $A(c) = \int_c \omega$. In this case, $\alpha_E^h = \mathrm{d}\psi_E^h$ for

$$\psi_E^h(c, \tau) = (A - \tau(h - E))(c).$$

Since $\mathcal{D}(c)$ is an element of $\Gamma^0(c^*TP)$, $\bigwedge \omega(\mathcal{D}, \cdot)$ can be considered as an element of $\Gamma^0(c^*T^*P)$, which is a subspace of the dual space T_c^*P of $T_cP = \Gamma^0(c^*TP)$. Thus α_E^h is a section of the dense subbundle $\check{T}^* \bigwedge P$, whose fibre $T_c^* \bigwedge P$ is $\Gamma^0(c^*T^*P)$.

Now, Weinstein develops a theory of *weak non-degeneracy* for zero sets of 1-forms, whose image is contained in a fixed dense subbundle $\check{T}^*B$ of the cotangent bundle of a manifold B modelled over a reflexive Banach space: If $\alpha : B \to T^*B$ is a closed 1-form on B and $Z \subset B$ is a compact manifold with $\alpha(b) = 0$ for every $b \in Z$, Z is called *non-degenerate* (due to definitions by Bott and others) iff for every $b \in Z$, the derivative $\mathrm{d}_b\alpha : T_bB \to T_b^*B$ has closed range and $\ker \mathrm{d}_b\alpha = T_bZ$. In the case that the image of α is contained in $\check{T}^*B$ this is impossible, since $\mathrm{d}_b\alpha$ cannot have a closed range and a finite dimensional kernel. Then, $\alpha = i \circ \check{\alpha}$, where $\check{\alpha} : B \to \check{T}^*B$ and $i : \check{T}^*B \to T^*B$ is the inclusion. In this case, Weinstein calls Z a *weakly non-degenerate* zero manifold of α iff $\check{\alpha}$ has closed range and $\ker(\mathrm{d}_b\check{\alpha})^* = i^*T_bZ$. Since the domain of $(\mathrm{d}_b\check{\alpha})^*$ contains the domain of $(\mathrm{d}_b\alpha)^* = \mathrm{d}_b\alpha$, the elements of $\ker(\mathrm{d}_b\check{\alpha})^*$ may be seen as weak solutions of $\mathrm{d}_b\check{\alpha}v = 0$.

Using this concept, Weinstein is able to perform a Lyapunov-Schmidt reduction, formulated in terms of transversality theory, to obtain the following result:

Theorem 5.20. *If α and ϕ are 1-forms on B with image in $\check{T}^*B$, α is closed, and $Z \subset B$ is a compact weakly non-degenerate zero manifold for α, then there is a neighbourhood $U of Z$ and a family of embeddings $e_\varepsilon : Z \to U$ defined for small $\varepsilon > 0$ such that the zero set of $\alpha + \varepsilon\phi$ is given by the zero set of its pullback to $e_\varepsilon(Z)$.*

In addition, Weinstein argues that by homotopy invariance of de-Rham-cohomology, the pullback α to $e_\varepsilon(Z)$ is exact, since its pullback to $e_0(Z) = Z$ is 0. If ϕ is exact, the zeros of $\alpha + \varepsilon\phi$ consequently coincide with the critical points of a function.

To obtain a result for periodic orbits, this theorem is applied to the closed 1-form α_E^h on the space $\bigwedge P \times \mathbb{R}^+$. A submanifold $Z \subset \bigwedge P \times \mathbb{R}^+$ is called a *non-degenerate* periodic manifold iff Z is a weakly non-degenerate zero manifold of α_E^h and none the of the corresponding periodic orbits is a point curve. As shown in [Wei78], this definition is equivalent to former characterization of *non-degenerate* periodic manifolds given in [Wei73]: For a periodic orbit c in P with period τ, let $\mathcal{L} : T_{c(0)}P \to T_{c(0)}P$ be the linearization at $c(0)$ of the of the time-τ-map defined by the flow of the Hamiltonian vector field. Then the zero manifold Z of α_E^h is weakly non-degenerate iff

$$\dim Z = \dim \left\{ x \in T_{c(0)}(h^{-1}(E)) \;\middle|\; x - \mathcal{L}x \text{ is a multiple of } X_h(c(0)) \right\}. \quad (5.14)$$

As Weinstein points out, the Lyapunov-Schmidt reduction may be performed with respect to symmetry if there is a continuous action by diffeomorphisms of a compact group K on B and α and ϕ are K-invariant. In [Wei78], this is applied to the S^1-action on the loop space, whereas Montaldi, Roberts, and Stewart use this more generally for a $G \times S^1$-action on the loop space induced by a symplectic G-action on P.

If $h+\varepsilon h_1$ is a G-invariant perturbation of a G-invariant Hamiltonian function, the embedding $e_\varepsilon : Z \to \bigwedge Z \times \mathbb{R}^+$ and the form $\alpha_E^{h+\varepsilon h_1}$ are G-equivariant. Moreover, $\alpha_E^{h+\varepsilon h_1} = \alpha_E^h + \varepsilon \mathrm{d} \bigwedge h_1$ and thus the pullback $\alpha_E^{h+\varepsilon h_1}$ is exact on $e_\varepsilon(Z)$. Thus the periodic orbits of the Hamiltonian vector field $h + \varepsilon h_1$ near Z on the energy level E coincide with the critical points of a $G \times S^1$-invariant function φ defined on $e_\varepsilon(Z)$. Hence for any subgroup $\Sigma \subset G \times S^1$, the periodic orbits whose isotropy subgroup contains Σ correspond to the critical points of φ contained in $e_\varepsilon(Z)^\Sigma$, which in turn coincide with the critical points of the restriction $\varphi|_{e_\varepsilon(Z)^\Sigma}$.

To predict periodic orbits near an equilibrium, Montaldi, Roberts, and Stewart use a blow-up argument similar to the one used in the proof of the non-equivariant version of the theorem (see for example [Wei73]): If $x = \varepsilon y$, the equation

$$\dot{x} = X_h(x)$$

is equivalent to

$$\dot{y} = Ly + \varepsilon \bar{X}(y),$$

where $\bar{X}(y) = \varepsilon^{-2}(X_h(\varepsilon y) - \varepsilon L y)$ is smooth. The vector field $L + \varepsilon \bar{X}$ is Hamiltonian with Hamiltonian function $y \mapsto \varepsilon^{-2} h(\varepsilon y)$, and periodic orbits of X_h with energy ε^2 correspond to periodic orbits of $L + \varepsilon \bar{X}$ with the same period and energy 1. For $\varepsilon = 0$, V_λ consists of periodic solutions of period $\frac{2\pi}{|\lambda|}$. Thus V_λ can be embedded in a $G \times S^1$-equivariant way into the loop space such that the image of the energy-1-level set can be identified with a compact submanifold Z of

$$\bigwedge V_\lambda \times \{\frac{2\pi}{|\lambda|}\} \subset \bigwedge V_\lambda \times \mathbb{R}^+.$$

Since this periodic manifold coincides with the level set of energy 1, it is a non-degenerate periodic manifold (and in fact for any closed curve C, $x - \mathcal{L}x = 0$ for every element x of the tangent space at $c(0)$). Thus the periodic orbits fixed by Σ on the energy level $h^{-1}(\varepsilon^2)$ of the original Hamiltonian function correspond to the critical points of a $G \times S^1$-invariant function on $e_\varepsilon(Z)^\Sigma$. $e_\varepsilon(Z)^\Sigma$ is homeomorphic to a sphere. The restriction can be considered as an

S^1-invariant function on $e_\varepsilon(Z)^\Sigma$. As shown in [Wei73, Theorem 1.3], such a function has at least

$$\frac{1}{2}(\dim e_\varepsilon(Z)^\Sigma + 1) = \frac{1}{2}(V_\lambda)^\Sigma$$

critical orbits. The proof of [Wei73, Theorem 1.3] uses a theorem of Krasnosel'skii, which gives the Lyusternik-Schnirelmann category of some quotient space. For a hint to a simpler approach see [Mos76], end of the proof of Theorem 4: Similarly as Chossat et al. in their proof of Theorem 5.5, Moser suggests to use that the quotient space is isomorphic to a complex projective space if the action is free. A footnote indicates that Weinstein has extended this argument to the general case, but the proof is not given.

For the equivariant version of the Lyapunov centre theorem, we note that in the case $\dim(V_\lambda)^\Sigma = 2$, each space $e_\varepsilon(Z)^\Sigma$ is a 1-dimensional sphere and thus forms a critical orbit. The Lyapunov-Schmidt reduction argument in [Wei78] yields the completeness statement of Theorem 5.15 and that the embeddings depend differentiably on ε. Thus we obtain a C^1-manifold of periodic orbits in this case. It may be shown that this manifold is even smooth, see for example [Mos76], there the remark after Lemma 2 on page 739 and the last paragraph of the proof of Theorem 3 on page 742).

5.3 Some results derived from the bundle equations

The bundle equations yield two equations that characterize relative equilibria. For the bundle equations with isotropy, these are the commutation equation (2.14) and the symplectic slice equation (2.15).

Most of the results that describe solutions of the two equations require conditions that assure that the commutation relation is automatically solved. One theorem of this type is [RdSD97, Theorem 3.2]. In [MR-O15], this result is generalized and similar theorems are added. As Montaldi and Rodríguez-Olmos point out, the following condition suffices to solve the commutation equation:

Definition 5.21. Let H be a compact subgroup of G with Lie algebra $\mathfrak{h}$. $\mathfrak{h}$ is a *co-central* subalgebra of $\mathfrak{g}$ iff there is a splitting $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ with $\mathfrak{m} \subset \mathfrak{z}(\mathfrak{g})$, where $\mathfrak{z}(\mathfrak{g})$ denotes the centre of $\mathfrak{g}$.

If $\mathfrak{g}_p$ is a co-central subalgebra of $\mathfrak{g}_\mu$, for every triple (ρ, v, η), the equation

$$\mathbb{P}_{\mathfrak{m}^*}(\mathrm{coad}_{\mathrm{d}_{\mathfrak{m}^*}\bar{h}(\rho,v)+\eta}(\rho + \mathbf{J}_N(v))) = 0$$

is satisfied, which together with the symplectic slice equation (2.15) implies the commutation relation (2.14): This follows from $\xi' := \mathrm{d}_{\mathfrak{m}^*}\bar{h}(\rho, v) + \eta \in \mathfrak{g}_\mu$ and

$$(\mathrm{coad}_{\xi'}\,\nu)(\zeta) = -\nu([\xi', \zeta]) = 0$$

for any $\zeta \in \mathfrak{m}$.

Moreover, to obtain solutions of a specific isotropy type (K), we only need the fixed point subalgebra $\mathfrak{g}_p^K$ to be co-central in $\mathfrak{g}_\mu^K$. (Since the adjoint representation is induced by Lie group automorphisms, it consists of Lie algebra automorphisms and thus the fix point subspaces are indeed Lie subalgebras.) This is how the condition occurs in the theorems in [MR-O15].

To obtain solutions of the symplectic slice equation, Montaldi and Rodríguez-Olmos use two lemmas. The first one is just an implication of the implicit function theorem and may be applied to a relative equilibrium p with a generator such that p is a non-degenerate critical point of the augmented Hamiltonian:

Lemma 5.22 ([MR-O15, Lemma 2.8])**.** *Let $H \subset G_p$ be a closed subgroup and $f \in C^\infty(\mathfrak{g}_p \times \mathfrak{m}^* \times N)^H$. Suppose that $K \subset H$ is a closed subgroup such that*

$$\mathrm{d}_N f(0,0,0) = 0 \text{ and}$$
$$\mathrm{d}^2_N f(0,0,0)\big|_{N^K} \text{ is non-degenerate.}$$

Then there is a unique locally defined $N_H(K)$-equivariant smooth map

$$v : ((\mathfrak{m}^*)^K \times \mathfrak{g}_p^K, (0,0)) \to (N^K, 0)$$

such that

$$\mathrm{d}_N f(\eta', \rho, v(\rho, \eta')) = 0.$$

For $m = (\rho, v(\rho, \eta'))$, the isotropy subgroup $(G_p)_m$ satisfies

$$K \subset (G_p)_m \subset (G_p)_\rho.$$

The second lemma may be considered as a version of the Equivariant Branching Lemma (see for example [Fie07]) combined with a reduction argument. The proof is based on the Splitting Lemma and a Taylor expansion argument as in the standard proof of the Equivariant Branching Lemma.

If K is a compact group and V is a K-representation, V is of *cohomogeneity one* iff for any choice of a K-invariant inner product, K acts transitively on the norm-1-sphere. (Note that this is the transitivity condition of Theorem 5.8.) A representation of cohomogeneity one is obviously irreducible.

A continuous real-valued function σ on a topological space *crosses 0* at u_0 iff $\sigma(u_0) = 0$ and any neighbourhood of u_0 contains points u_1 and u_2 with $\sigma(u_1) > 0$ and $\sigma(u_2) < 0$.

Lemma 5.23 ([MR-O15, Lemma 2.9])**.** *Let W and N be representations of the compact Lie group K and f be contained in $C^\infty(W \times N)^K$. Let $L \subset K$ be a closed subgroup and Λ a path-connected open neighbourhood of the origin in W^L such that*

$$\mathrm{d}_N f(\lambda, 0) = 0 \quad \forall \lambda \in \Lambda.$$

Suppose that $\mathcal{N} := \ker \mathrm{d}^2_N f(0,0) \cap N^L \neq \{0\}$ and

1. *the $N_K(L)$-representation $\mathcal{N}$ is of cohomogeneity one,*
2. *the eigenvalue $\sigma(\lambda)$ of $\mathrm{d}^2_N f(\lambda, 0)\big|_{\mathcal{N}}$ crosses 0 at $\lambda = 0$.*

If $N^L = \mathcal{N} \oplus S$ is a $N_K(L)$-invariant decomposition and $v \in \mathcal{N}$ is small, there is $\lambda_v \in \Lambda$ and $s_v \in S$ such that

$$\mathrm{d}_N f(\lambda_v, v, s_v) = 0.$$

If L is an isotropy subgroup of N, condition 1 implies that L is *maximal*, i.e. the only isotropy subgroup of N that contains L as a proper subgroup is K. Thus in this case for $v \neq 0$ and $m = (\lambda_v, v, s_v)$, the isotropy subgroup K_m coincides with L.

In the following, we sketch some of the results given in [MR-O15]. We always consider a proper Hamiltonian action of a Lie group G on a symplectic space P and suppose that $p \in P$ is a relative equilibrium with generator $\xi \in \mathfrak{g}_\mu$ and momentum $\mathbf{J}(p) = \mu$ such that $\mathfrak{g}_\mu$ has a G_μ-invariant complement in $\mathfrak{g}$. We write $\xi = \xi^\perp + \eta$ with $\xi^\perp \in \mathfrak{m}$ and $\eta \in \mathfrak{g}_p$.

We start with a generalization of [RdSD97, Theorem 3.2], which itself is deduced from the bundle equations given in [RdSD97]:

Theorem 5.24 ([MR-O15, Theorem 4.3]). *Suppose that $K \subset (G_p)_\eta$ is a closed subgroup such that*

1. *$\mathfrak{g}_p^K$ is a co-central subalgebra of $\mathfrak{g}_\mu^K$ and*
2. *$\mathrm{d}^2 h_\xi(p)\big|_{N^K}$ is non-degenerate.*

Then there is a smooth $N_{(G_p)_\eta}(K)$-equivariant local map

$$\begin{aligned} \bar{p} : (\mathfrak{m}^{*K} \times \mathfrak{g}_p^K, (0,0)) &\to (P^K, p) \\ (\rho, \eta') &\mapsto \bar{p}(\rho, \eta') \end{aligned}$$

such that $\bar{p}(\cdot, \eta')$ is an immersion for each η' and $\bar{p}(\rho, \eta')$ is a relative equilibrium with generator $\xi + \eta'$. Moreover

$$K \subset G_{\bar{p}(\rho,\eta')} \subset (G_p)_\rho.$$

The image of $\bar{p}$ contains every relative equilibrium in P^k near p with generator ξ' near ξ such that $\xi' - \xi \in \mathfrak{g}_p^K$.

The proof is an application of Lemma 5.22 with $f(\eta', \rho, v) = \bar{h}_{\eta+\eta'}$. By [MR-O15, Lemma 3.5], condition 2 yields that $\mathrm{d}^2_N f(0,0,0)_{N^K}$ is non-degenerate.

Clearly, $\bar{p}(\rho, \cdot)$ is not an embedding, since $\bar{p}(\rho, \eta') = \bar{p}(\rho, \eta'')$ if $\eta' - \eta'' \in \mathfrak{g}_{\bar{p}(\rho,\eta')}$. Thus in general, the image of $\bar{p}$ is not a smooth manifold.

As is shown in [MR-O15, Theorem 4.4], if condition 1 is replaced by the assumption that the Lie algebra $\mathfrak{l}$ of $L := N_{G_\mu}(G_p)\big/G_p$ is Abelian and we choose $K = G_p$, then we obtain a manifold of relative equilibria of dimension $\dim G\big/G_p + \dim(\mathfrak{m}^*)^{G_p}$, all of which have the same isotropy type as p. If L is compact, this follows from the results discussed in section 4.2 applied to the L-action on P_{G_p}: Since L is Abelian and $p \in P_{G_p}$ is a non-degenerate equilibrium, p is transverse then. As an easy computation shows, $(\mathfrak{m}^*)^{G_p} \simeq (\mathfrak{m}^{G_p})^* \simeq \mathfrak{l}^*$ (compare Lemma 6.38). Thus the set of relative equilibria within P_{G_p} forms a manifold of dimension $2 \dim L$. Hence its G-orbit is a manifold of dimension

$$2 \dim L + \dim G\big/G_p - \dim L = \dim G\big/G_p + \dim L.$$

[MR-O15, Theorem 4.4] contains a generalization of this conclusion to the non-compact case. Moreover, Montaldi and Rodríguez-Olmos show that the manifold is symplectic if G_p is a normal subgroup of G_μ, which in the case of compact

L may also be deduced from the results for free actions and the fact that $\mathfrak{q} \cdot p$ is a symplectic subspace for the complement $\mathfrak{q}$ of $\mathfrak{g}_\mu$ in $\mathfrak{g}_p$.

For generators ξ of p such that $\mathrm{d}^2 h_\xi(p)$ is degenerate, the second one of the above lemmas, Lemma 5.23, yields relative equilibria of a different isotropy type if there is an admissible isotropy subgroup $L \subset (G_p)_\eta$. In particular, one obtains branches that bifurcate from a given manifold of relative equilibria:

Montaldi and Rodríguez-Olmos say that two relative equilibria p and p' with $\mathbf{J}(p) = \mu$ and $\mathbf{J}(p') = \mu'$ are of the same *symplectic type* iff there is a $g \in G$ with $gG_pg^{-1} = G_{p'}$ and $gG_\mu g^{-1} = G_{\mu'}$. A *parameterized branch of relative equilibria* is an injective smooth map $\bar{p}$ from a neighbourhood W of 0 in some vector space to P whose image consists of relative equilibria such that the G-orbits of $\bar{p}(w)$ and $\bar{p}(w')$ are disjoint for $w \neq w'$.

Theorem 5.25 ([MR-O15])**.** *Let $K \subset (G_p)_\eta$ be a closed subgroup, $W \subset \mathfrak{m}^* \times \mathfrak{g}_p^K$ be an open neighbourhood of 0 of a vector subspace and $\bar{p} : W \to P$ be a parameterized branch of relative equilibria of the same symplectic type such that $\bar{p}(0,0) = p$, $\mathbf{J}(\bar{p}(\rho, \eta')) = \mu + \rho$, $G_{\mu+\rho} = G_\mu$, and $G_{\bar{p}(\rho,\eta')} = G_p$. Choose generators $\xi(\rho, \eta')$ for $\bar{p}(\rho, \eta')$ with $\mathfrak{g}_p$-component $\eta + \eta'$. Suppose that there is a closed subgroup $L \subset K$ such that*

1. *$\mathfrak{g}_p^L$ is co-central in $\mathfrak{g}_\mu$,*
2. *$N_{G_p}(L)$ acts on $\mathcal{N} := \ker \mathrm{d}^2 h_\xi(0)\big|_{N^L}$ with cohomogeneity one, and*
3. *the eigenvalue $\sigma(\rho, \eta)$ of $\mathrm{d}^2 h_\xi(\bar{p}(\rho, \eta))\big|_{\mathcal{N}}$ crosses 0 at $0 \in W$.*

Then for every $0 \neq v \in \mathcal{N}$ near 0, there is a relative equilibrium p_v near p, which is not contained in the G-orbit of any point of the original branch, with generator $\xi_v \in \mathfrak{g}_\mu$ near ξ. Its isotropy subgroup contains L.

Remark 5.26. If L is an isotropy subgroup, it is maximal and Lemma 5.23 implies $G_{p_v} = L$.

So far, all cited results of Montaldi and Rodríguez-Olmos ([MR-O15]) are of the form described in the beginning: A commutativity condition assures that every solution of the symplectic slice equation solves the commutation equation and the focus of the theorem lies mainly in the solution set of the symplectic slice equation. There is also another persistence result of a different type given in [MR-O15] whose proof is based on an approach that resembles the method in [Mon97] for free actions of compact groups. As indicated in [MR-O15, Remark 2.7], this approach might possibly yield a generalization of these results to actions with isotropy subgroups of positive dimension. The given result deals with a *formally stable* relative equilibrium p, that means that there is a generator ξ such that $\mathrm{d}^2 h_\xi(p)$ is definite. The main stability theorem in [MR-O15] states that a formally stable relative equilibrium is *stable modulo G_μ*, that is every G_μ-invariant neighbourhood U of the G_μ-orbit $G_\mu p$ contains an open neighbourhood O of p such that the integral curves starting in O are contained in U. This also shows that it is reasonable to take all possible choices of generators of a relative equilibrium into account.

In particular, a formally stable relative equilibrium is extremal in the sense of the definition given in [Mon97], see the introduction of this chapter. Thus the following result may be seen as a part generalization of the results on extremal relative equilibria in [Mon97] and also [MT03], even though formal stability is a slightly stronger condition than extremality.

Theorem 5.27 ([MR-O15, Theorem 4.6])**.** *Let p be a formally stable relative equilibrium with momentum $\mu = \mathbf{J}(p)$. Suppose that $\mathfrak{g}_\mu$ has a G_μ-invariant inner product. Then there is a G-invariant neighbourhood U of p such that for every μ' near μ with $\mu' \in \mathbf{J}(U)$ there is a relative equilibrium p' near p with momentum $\mathbf{J}(p') = \mu'$.*

The proof is based on the following observation: Identify p with the point $[e, 0, 0]$ of the local model. The commutation equation

$$\operatorname{coad}_{\mathrm{d}_{\mathfrak{m}^*}\bar{h}(\rho,v)+\eta}(\rho + \mathbf{J}_N(v)) = 0$$

may equivalently be written as

$$(\mathrm{d}_{\mathfrak{m}^*}\bar{h}(\rho, v) + \eta)\big|_{T_{\rho+\mathbf{J}_N(v)}\mathcal{O}} = 0, \tag{5.15}$$

where $\mathcal{O}$ denotes the coadjoint orbit of $\rho + \mathbf{J}_N(v)$ within $\mathfrak{g}_\mu^*$: The left hand side of the commutation equation is contained in $\mathfrak{g}_\mu^* = \operatorname{ann}\mathfrak{q}$, where $\mathfrak{q}$ is a G_μ-invariant complement. Moreover, for any $\zeta \in \mathfrak{g}_\mu$ with $\xi := \mathrm{d}_{\mathfrak{m}^*}\bar{h}(\rho, v) + \eta$ and $\nu := \rho + \mathbf{J}_N(v)$, we have

$$\langle \operatorname{coad}_\xi \nu, \zeta\rangle = -\langle \nu, [\xi, \zeta]\rangle = \langle \nu, [\zeta, \xi]\rangle = -\langle \operatorname{coad}_\zeta \nu, \xi\rangle.$$

For a fixed $\gamma \in \mathfrak{g}_p$, let $f \in C^\infty(\mathfrak{g}_\mu^* \times N)$ and $\phi \in C^\infty(\mathfrak{g}_\mu^* \times N, \mathfrak{g}_p^*)$ be defined by

$$f(\alpha, v) := \bar{h}_\gamma(\alpha\big|_{\mathfrak{m}}, v) - \bar{h}_\gamma(0, 0) + \langle \alpha, \gamma\rangle$$
$$\phi(\alpha, v) := \mathbf{J}_N(v) - \alpha\big|_{\mathfrak{g}_p},$$

where $\bar{h}_\gamma(\rho, v) := \bar{h}(\rho, v) - \mathbf{J}_N^\gamma(v)$ is the augmented Hamiltonian of $\bar{h}(\rho, \cdot)$. Denote the restrictions to $\mathcal{O} \times N$ by $f_\mathcal{O}$ and $\phi_\mathcal{O}$ respectively. Then equation (5.15) and the symplectic slice equation

$$\mathrm{d}_N\bar{h}_\eta(\rho, v) = 0$$

are both satisfied for some $\eta \in \mathfrak{g}_p$ iff the restriction of $f_\mathcal{O}$ to $\phi^{-1}(0)$ has a critical point at $(\rho + \mathbf{J}_N(v), v)$: This holds iff there is a Lagrange multiplier $\eta' \in \mathfrak{g}_p$ such that

$$\mathrm{d}_N(f - \langle \phi, \eta'\rangle)(\rho + \mathbf{J}_N(v), v) = 0,$$
$$\mathrm{d}_{\mathfrak{g}_\mu^*}(f - \langle \phi, \eta'\rangle)\big|_{T_{\rho+\mathbf{J}_N(v)}\mathcal{O}} = 0.$$

In this case, $\eta := \gamma + \eta'$, ρ, and v solve equation (5.15) and the symplectic slice equation.

Thus it only has to be shown that for small values of ρ and v and some choice of γ, the function $f_\mathcal{O}$ has a critical point on $\phi_\mathcal{O}^{-1}(0)$: Recall that

$$\mathbf{J}_Y([g, \rho, v]) = \operatorname{Coad}_g(\mu + \rho + \mathbf{J}_N(v)).$$

If the relative equilibrium p' corresponds to the critical point in the coadjoint orbit $\mathcal{O} \subset \mathfrak{g}_\mu^*$ of $\rho + \mathbf{J}_N(v)$, then $\mathbf{J}(Gp') = G\mathcal{O} \subset \mathfrak{g}^*$ is the coadjoint orbit of $\rho + \mathbf{J}_N(v)$ within $\mathfrak{g}^*$.

The proof of Theorem 5.27 proceeds as follows: $\gamma \in \mathfrak{g}_p$ is chosen such that $\mathrm{d}^2_N \bar{h}_\gamma(0,0)$ is definite. Since there is a G_μ-invariant product on $\mathfrak{g}^*_\mu$, $\mathcal{O}$ is compact. Together with a Splitting Lemma argument, this yields that $f_{\mathcal{O}}$ is proper and bounded from below or above depending on whether $\mathrm{d}^2_N \bar{h}_\gamma(0,0)$ is positive or negative definite. Thus $f_{\mathcal{O}}$ has a minimum or maximum and hence a critical point. (See [MR-O15] for details.)

Investigating the topology of the G_p-space $\phi_{\mathcal{O}}^{-1}(0)$ might yield an estimation of the number of critical G_p-orbits of the G_p-invariant function $f_{\mathcal{O}}$. In the case of a free action, $\phi_{\mathcal{O}}^{-1}(0)$ coincides with $\mathcal{O} \times N$ and we obtain the setting investigated in [Mon97], see section 4.1.

Chapter 6

Equivariant transversality approach

Equivariant transversality was defined by Field and independently by Bierstone in 1977 ([Bie77] and [Fie77a]). In the followup paper [Fie77b], Field shows that the definitions of equivariant general position and equivariant transversality given in the two papers are equivalent.

The two authors have developed the theory with different intentions: Field applies the theory to equivariant dynamical systems in [Fie90] and gives for example an equivariant version of the Kupka-Smale theorem.

In [Bie76], Bierstone develops the theory further to include jets of equivariant functions in order to generalize one of Mather's results on the stability of C^∞-mappings to equivariant maps. (The result is the equivalence of C^∞-stability to an infinitesimal condition.)

The theory has turned out to be very useful for applications in equivariant bifurcation theory. In [FR89], Field and Richardson show that the equilibrium sets of smooth 1-parameter families of G-equivariant vector fields generically consist of finitely many C^1-curves and are topologically stable under small perturbations. In [Fie89], Field defines a notion of determinacy of a representation that respects derivatives at the points of the branches. Using Bierstone's generalization to jets of equivariant maps, he shows the genericity of the non-degeneracy of the bifurcating branches. In later publications, he also considers relative equilibria of the vector fields. (See [Fie96] for a comprehensive presentation or [Fie07] for a more elementary one.) We will sketch the bifurcation theory results in section 6.2.

As will be presented in this chapter, equivariant transversality theory may also be a powerful tool for the investigation of relative equilibria in equivariant Hamiltonian systems. We will apply the theory in two different ways: The first one combines Field's approach to the local investigation of relative equilibria in equivariant systems with Patrick's and Roberts' theory of transverse relative equilibria and yields a partial generalization of the results in [PR00] to the non-free case, see section 6.3.

The second way is presented in section 6.4. As it was suggested by Chossat et al. ([CLOR03]), we consider the family of augmented Hamiltonians $h_\xi = h - \mathbf{J}^\xi$ on a symplectic representation as a family of functions parameterized by ξ. In case of a torus action, the group action on the parameter space is trivial and we may proceed similarly as Field and Richardson in [FR89]. These results apply to G-symplectic representations for compact connected groups G by considering a maximal torus.

The approach of this thesis is not the first application of equivariant transversality theory to the theory of Hamiltonian relative equilibria: Birtea et al ([BPRT06]) give a definition of equivariant transversality for families of Hamiltonian systems. However, the proof of their main result has faults that seem to be irreparable. We will discuss this in section 6.2. In contrast, in this thesis we investigate the structure of relative equilibria in a single Hamiltonian system with symmetry given by a Hamiltonian action of a compact Lie group G. We obtain results about their generic structure, which are valid for a quite general class of Hamiltonian systems with symmetry.

6.1 Equivariant transversality theory

The goal of equivariant transversality theory is to develop an appropriate notion of transversality to G-invariant submanifolds for smooth G-equivariant maps $f : M \to N$, where G is a compact Lie group and M and N are smooth G-manifolds. If $P \subset N$ is a G-invariant manifold, we recap the classical definition of transversality and the properties of this definition:

Definition 6.1. f is *transverse to P at $x \in M$ ($f \pitchfork_x P$)* if $f(x) \notin P$ or

$$\mathrm{d}f(x)(T_x M) + T_{f(x)}P = T_{f(x)}N.$$

f is *transverse to P along $A \subset M$ ($f \pitchfork_A P$)* if f is transverse to P at every $x \in A$. f is *transverse to P ($f \pitchfork P$)* iff f is transverse to P along M.

This yields the following implications, see [AR67]:

Theorem 6.2 (Transversality Theorem).

1. *(smoothness) $f^{-1}(P)$ is a manifold with*
$$\operatorname{codim} f^{-1}(P) = \operatorname{codim} P.$$
2. *(openness) If $f \pitchfork_x P$ for $x \in M$, there is a neighbourhood $U \subset M$ of x such that $f \pitchfork_y P$ for every $y \in U$.*
3. *(density) If M is compact, the subset $\mathcal{T}$ of maps that are transverse to P is residual in $C^\infty(M, N)$ with respect to the C^∞-topology.*
4. *(stability) If $A \subset M$ is compact and P is closed, the set*
$$\mathcal{T}_A := \{f \in C^\infty(M, N) \mid f \pitchfork_A P\}$$
is C^∞-open.
5. *(isotopy) If M is compact, $f : M \times [0, 1] \to N$ smooth, and $f_t := f(\cdot, t) \pitchfork P$ for all $t \in [0, 1]$, then there is an isotopy of diffeomorphisms $K : M \times [0, 1] \to M$, $K_t := K(\cdot, t)$ with*
$$K_t(f_t^{-1}(P)) = f_0^{-1}(P) \quad \textit{and} \quad K_0 = \mathbb{1}_M.$$

Remark 6.3. If M is not compact, the implication 3 is also true with respect to the Whitney C^∞-topology instead of the C^∞-topology. If A is not compact but closed in 4, $\mathcal{T}_A$ is Whitney C^1-open and hence Whitney C^∞-open. See appendix A.1 for the definitions of the topologies.

A good definition of equivariant transversality should transfer these properties as far as possible to equivariant maps. For the properties 3 and 4, this means that for compact M and closed P, the subset of equivariant maps f with $f \pitchfork P$ should be generic (open and dense) within the set of equivariant maps. The density property 3 is the reason why a new definition is necessary for the equivariant case. Transverse equivariant maps may not even exist:

Example 6.4. If M and N are representations with $M^G = 0$ and $N^G = 0$, $P = \{0\}$, and there is a subgroup $H \subset G$ such that $\operatorname{codim} M^H < \operatorname{codim} N^H$, then there is no equivariant map that is transverse to P.

The analogues to the properties 3 and 4 also imply that there is no chance to have the same smoothness property as implication 1: It is well-known in bifurcation theory that generically zero sets may not be manifolds. For example, if $\mathbb{R}_{\mathbb{Z}_2}$ denotes the non-trivial $\mathbb{Z}_2$-representation on $\mathbb{R}$ and $\mathbb{R}$ the trivial one, for an equivariant map

$$\begin{aligned} f : \mathbb{R}_{\mathbb{Z}_2} \times \mathbb{R} &\to \mathbb{R}_{\mathbb{Z}_2} \\ (x, \lambda) &\mapsto f(x, \lambda) \end{aligned}$$

generically near a point $(0, \lambda_0)$ with $\partial_x f(0, \lambda_0) = 0$, the zero-set has the form of a pitchfork (see for example [GSS88]).

Anyhow, there is a more general notion of transversality with a suitable analogue to property 1: transversality to Whitney stratified sets. A Whitney stratified set is a subset of a manifold together with a partition into submanifolds which fit together in a nice way. For the precise definition see appendix A.2. A smooth map is *transverse to a Whitney stratified set* Q iff it is transverse to each stratum. In that case, $f^{-1}(Q)$ is a Whitney stratified set, too. If in property 1 of Theorem 6.2, we only require P to be Whitney stratified and we replace the word “diffeomorphisms” in property 5 by “homeomorphisms”, we obtain true statements, which form the Thom-Mather-transversality-theorem, see appendix A.2.

In fact, the definition of equivariant transversality relies on the theory of transversality to Whitney stratified sets and the transversality condition implies that the preimage $f^{-1}(P)$ is a Whitney stratified set.

For equivariant transversality, we also want to have an isotopy statement as property 5 of Theorem 6.2 such that an equivariant isotopy of homeomorphisms exists. This implies that the preimages in each isotropy subspace are homeomorphic.

Before we give the definition of Field ([Fie07]), we consider a definition which seems to be an apparent attempt:

For an isotropy subgroup $H \subset G$ and the conjugacy class $\tau = [H]$, set $M_H := M^H_\tau := \left\{ x \in M^H \;\middle|\; G_x = H \right\}$.

Definition 6.5. $f : M \to N$ is *stratumwise transverse* to P iff for all isotropy subgroups $H \subset G$ and $[H] = \tau$ the map

$$f^H := f\big|_{M_H} : M_H \to N^H$$

is transverse to P^H.

If M is compact or alternatively if M is a representation and P is closed, the set of equivariant maps $f : M \to N$ that are stratumwise transverse to P is residual in $C^\infty_G(M,N)$: For every compact subset $K \subset M_H$, the set $\mathcal{T}^H_K$ of maps f for which f^H is transverse to P^H along K is open and dense. If $K_1 \subset K_2 \subset \cdots \subset M_H$ is a countable system of compact subsets of M_H with $\bigcup_{i\in\mathbb{N}} K_i = M$, the intersection $\bigcap_{i\in\mathbb{N}} \mathcal{T}^H_{K_i}$ is equal to the set of maps f with $f^H \pitchfork P^H$ which hence is residual. If H' is conjugate to H, $f^H \pitchfork P^H$ is equivalent to $f^{H'} \pitchfork P^{H'}$. Since M has only finitely many isotropy types, the stratumwise transverse maps form a residual subset of the equivariant maps.

The problem with this definition is that the set of stratumwise transverse maps to P may not be open in the set of equivariant maps:

Example 6.6. Let $M = N$ be the non-trivial $\mathbb{Z}_2$-representation on $\mathbb{R}$ and $P = 0$. $f_\varepsilon(x) = x(x+\varepsilon)^2(x-\varepsilon)^2$ is stratumwise transverse to 0 iff $\varepsilon = 0$.

Nevertheless, it seems to be reasonable that an appropriate definition of G-transversality should imply stratumwise transversality: If f is G-transverse to P, for any small perturbation $\tilde{f}$, the preimages $(f^H)^{-1}(P)$ and $(\tilde{f}^H)^{-1}(P)$ should be homeomorphic. If it was possible to perturb f^H locally near some point $x \in M_H$ such that locally near x the topology of the preimage $(f^H)^{-1}(P)$ changes, this perturbation could be extended via the Slice Theorem to a perturbation of f. Hence, we want to exclude the possibility of such perturbations. This can be done by requiring f to be stratumwise transverse to P.

In fact, equivariant transversality implies stratumwise transversality, as is shown in [Bie77].

6.1.1 Definition of equivariant transversality

The definition of equivariant transversality which was given in [Bie77] relies on some basic results about G-spaces and G-representations, which are essential for the understanding of equivariant maps: the Slice Theorem, Bochner's linearization theorem, and the theory of polynomial generators of invariant and equivariant maps.

By the Slice Theorem, for every $x \in M$, there is a tubular neighbourhood $U \simeq G \times_{G_x} S_x$, where S_x is a G_x-invariant slice, such that the set of smooth equivariant maps $C^\infty_G(M,N)$ is in one-to-one correspondence to $C^\infty_{G_x}(S_x,N)$: The inverse of the restriction map $f \mapsto f\big|_{S_x}$ for $f \in C^\infty_G(S_x,N)$ is given by the unique equivariant extension $h \mapsto \tilde{h}$ of $g \in C^\infty_{G_x}(S_x,N)$ with $\tilde{h}(gx) = gh(x)$ for $x \in S$ and $g \in G$.

Together with Bochner's linearization theorem (Theorem 1.24), the Slice Theorem (Theorem 1.28) shows that locally G-equivariant maps can be considered as equivariant maps between representations: By Bochner's theorem for any $x \in m$, there is a neighbourhood U of $f(x)$ that is locally $G_{f(x)}$-equivariantly diffeomorphic to the $G_{f(x)}$-representation on $T_{f(x)}N$. If f is G-equivariant, $G_x \subset G_{f(x)}$ and $T_{f(x)}N$ can be considered as a G_x-representation.

Furthermore, we can choose S_x such that $f(S_x) \subset U$. Let S_x be locally G_x-diffeomorphic to the G_x-representation V. Since a small perturbation of f also maps S_x into U, we only need to consider G_x-equivariant smooth maps from V into $T_{f(x)}N$ to study the stability of the set $f^{-1}(P)$ locally at x. Suppose $f(x) \in P$, otherwise we define f to be G-transverse to P at X. Since P is G-invariant, $T_{f(x)}P$ is a G_x-invariant subspace of $T_{f(x)}N$, hence it has a G_x-invariant complement W.

In this way, we reduce the definition to the case of representations and the invariant submanifold 0: We will define G_x-transversality to $0 \in W$ at a point in V^{G_x} and extend the definition to the general case by calling f G-transverse to $P \subset N$ at $x \in M$ if the composition of f with the projection to W is G_x-transverse to $0 \in W$. Of course, it has to be shown that this definition is independent of the choice of the slice. We will omit this proof and refer to the literature.

To define transversality to 0 in the case of representations, we need the theory of polynomial generators of invariant and equivariant smooth maps on representations. An introduction to the theory and proofs can be found in [GSS88, chapter XII, §4–6]. (The proof of Schwarz's theorem is only sketched. For a complete proof, we refer to Schwarz's original paper [Schw75].)

Recall that for G-spaces M and N, $C^\infty(M)^G$ denotes the ring of G-invariant smooth real-valued functions on M and $C^\infty_G(M,N)$ denotes the $C^\infty(M)^G$-module of G-equivariant smooth maps from M to N. Similarly for G-representations V and W, we write $P(V)^G$ for the ring of G-invariant real-valued polynomials on V and $P_G(V,W)$ for the $P(V)^G$-module of G-equivariant polynomial maps from V to W, where *polynomial* means that the map can be expressed in polynomials in the coordinates with respect to some (and hence any) choice of bases.

First, we need some facts about invariant polynomials and equivariant polynomial maps, which can be derived from Hilbert's basis theorem:

Theorem 6.7 (Hilbert-Weyl Theorem)**.** *Let G be a compact Lie group and V be a G-representation. Then $P(V)^G$ is finitely generated.*

Obviously, the generators $p_1, \dots, p_l$ of $P_G(V)$ can be chosen to be homogeneous. If such a set is minimal (no proper subset is a generating set), it is called a *minimal set of homogeneous generators* of $P(V)^G$.

Theorem 6.8. *$P_G(V,W)$ is a finitely generated $P(V)^G$-module.*

As for the invariant polynomials, it is possible to choose homogeneous generators of $P_G(V,W)$: Take the homogeneous parts of any set of generators. Again, if such a set is minimal, it is a *minimal set of homogeneous generators* of $P_G(V,W)$.

Although we do not need it for the definition of equivariant transversality, we start with Schwarz's theorem, since it will be necessary later on for the higher order version. Moreover, the theorem for equivariant maps may be derived from it.

Theorem 6.9 (Schwarz's Theorem, [Schw75])**.** *Let G be a compact Lie group and V be a G-representation. If $p_1, \dots, p_l$ generate $P(V)^G$, any $f \in C^\infty(M)^G$ can be written in the form $f = h \circ P$, where $P = (p_1, \dots, p_l) : V \to \mathbb{R}^l$ and $h \in C^\infty(\mathbb{R}^l)$.*

The next theorem is due to Malgrange and appeared first in [Poe76]. A possible proof uses Schwarz's theorem (see [Poe76] or [GSS88]). Another proof is based on an equivariant version of the Stone-Weierstraß theorem (see [Fie07]):

Theorem 6.10. *If the equivariant polynomial maps $F_1, \dots, F_k$ generate the $P(V)^G$-module $P_G(V,W)$, they also generate the $C^\infty(V)^G$-module $C^\infty_G(V,W)$. Hence, every $f \in C^\infty_G(V,W)$ can be written as*

$$f(x) = \sum_{i=1}^{k} g_i(x) F_i(x),$$

where the g_i are invariant functions.

Example 6.11. Let $V = W$ be the non-trivial $\mathbb{Z}_2$-representation $\mathbb{R}_{\mathbb{Z}_2}$ on $\mathbb{R}$. Then every equivariant map is a product of an invariant function and the function $F_1(x) = x$.

Example 6.12. Let $V = \mathbb{R}_{\mathbb{Z}_2} \times \mathbb{R}$, where $\mathbb{Z}_2$ acts trivially on the component $\mathbb{R}$, and $W = \mathbb{R}_{\mathbb{Z}_2}$. Then every equivariant map is also a product of an invariant function and the function $F_1(x, \lambda) = x$.

To define equivariant transversality, we fix a minimal set $F_1, \dots, F_k$ of homogeneous generators of $P_G(V,W)$. With respect to this choice, we define the algebraic function

$$\vartheta(x, t_1, \dots, t_k) = \Sigma t_i F_i(x).$$

$\Sigma := \theta^{-1}(0)$ is an algebraic set. Any algebraic set admits a canonical Whitney stratification. Hence, there is an appropriate definition of transversality to Σ such that the Thom-Mather transversality theorem applies.

For any function $f \in C^\infty_G(V,W)$, we choose a representation

$$f(x) = \sum_{i=1}^{k} g_i(x) F_i(x)$$

and set

$$\Gamma_f(x) = (x, g_1(x), \dots, g_k(x)) \in V \times \mathbb{R}^k.$$

Then $f = \vartheta \circ \Gamma_f$.

We use this to define G-transversality to $0 \in W$ at $0 \in V$:

Definition 6.13. $f \in C^\infty_G(V,W)$ is *G-transverse* to $0 \in W$ at $0 \in V$ ($f \pitchfork_G 0$ at 0) iff Γ_f is transverse to Σ at 0.

Of course, it has to be shown that this is well-defined. In particular, one has to prove the independence of the choice of the F_i and the choice of the representation of f as a linear combination of the F_i, which is not always unique. Again, we refer to [Bie77], [Fie96], and [Fie07].

Example 6.14. In Example 6.11, $\Sigma \subset \mathbb{R}_{\mathbb{Z}_2} \times \mathbb{R}$ is given by the union of the two lines $x = 0$ and $t = 0$. The canonical stratification consists of the point $(0,0)$ and each of the lines with the point $(0,0)$ omitted. Hence $f : \mathbb{R}_{\mathbb{Z}_2} \to \mathbb{R}_{\mathbb{Z}_2}$ is $\mathbb{Z}_2$-transverse to 0 at 0 iff $\partial_x f(0,0) = g(0,0) \neq 0$.

Example 6.15. In Example 6.12, $\Sigma \subset (\mathbb{R}_{\mathbb{Z}_2} \times \mathbb{R}) \times \mathbb{R}$ consists of the product of the two lines $x = 0$ and $t = 0$ with the λ-axis. The canonical stratification is the same but each stratum is multiplied with the λ-axis. In this case, Γ_f is transverse to Σ at $(0,0)$ if $\partial_x f(0,0) = g(0,0) \neq 0$ or $\partial_x f(0,0) = g(0,0) = 0$ and $\partial_\lambda \partial_x f(0,0) = \partial_\lambda g(0,0) \neq 0$. In the second case, the local preimage of the zero set of f looks like a pitchfork.

Example 6.15 illustrates the application of equivariant transversality theory to bifurcation theory. We will come back to this in section 6.2.

As explained above, Definition 6.13 is extended to the general case by choosing a slice at $x \in M$ and a complement to $T_{f(x)}P$ if $f(x) \in P$. Independence of choices is proved in [Bie77].

Remark 6.16. For any $x \in V$, $S = (\mathfrak{g}x)^{\perp}$ contains a slice for the G-action. Hence a G-equivariant map $f : V \to W$ is G-transverse to $0 \in W$ at x iff its restriction to S is G_x-transverse to $0 \in W$. Since generators of $P_G(V, W)$ restrict to generators of $P_{G_x}(S, W)$, this is equivalent to $\Gamma_f \pitchfork \Sigma$ at x, see [Bie77].

Based on the theory of transversality to Whitney stratified sets, the transversality Theorem 6.2 can be transferred to the equivariant case:

Theorem 6.17 ([Fie07, Proposition 6.14.2 and Theorem 6.14.1])**.** *Let M and N be smooth G-manifolds and $P \subset N$ be a smooth G-invariant submanifold.*

1. *If $f : M \to N$ is a smooth G-equivariant map and $f \pitchfork_G P$, then $f^{-1}(P)$ is a Whitney stratified subset of M.*
2. *If $f : M \to N$ is a smooth G-equivariant map and $f \pitchfork_G P$ at $x \in M$, there is a neighbourhood $U \subset M$ of x such that $f \pitchfork_G P$ at every $y \in U$.*
3. *If M is compact, the subset $\mathcal{T}$ of maps that are G-transverse to P is residual in $C^\infty_G(M, N)$ with respect to the C^∞-topology.*
4. *If $A \subset M$ is compact and P is closed, the set*
$$\mathcal{T}_A := \{f \in C^\infty_G(M, N) \mid f \pitchfork_G P \text{ along } A\}$$
is C^∞-open.
5. *If M is compact, $f : M \times [0, 1] \to N$ smooth, $f_t := f(\cdot, t) \pitchfork_G P$ for all $t \in [0, 1]$, then there is an isotopy of G-equivariant homeomorphisms $K : M \times [0, 1] \to M$, $K_t := K(\cdot, t)$, with*
$$K_t(f_t^{-1}(P)) = f_0^{-1}(P) \quad \text{and} \quad K_0 = \mathbb{1}_M.$$

Remark 6.18. Again we may omit the compactness assumption on M in 3 and A in 4 if we replace the C^∞-topology with the Whitney C^∞-topology. Then we require A to be closed.

For our application to Hamiltonian relative equilibria, we will need the generalization of the theory to jets of functions, which is also used in bifurcation theory to predict stability properties of the branches:

6.1.2 Higher order version

In [Bie76], Bierstone develops a theory of equivariant general position, which includes the derivatives of an equivariant map, in order to overcome the following problems:

1. In the definition of equivariant transversality, the degrees of the homogeneous generators are not taken into account. In particular, one obtains no information about a possible degeneracy of the derivative at a specified point.

2. Let $J^q(M,N)$ denote the space of q-jets, which forms a bundle over $M \times N$ (see Definition A.3 in appendix A.1). The classical transversality theory naturally extends to the case of a closed analytic submanifold (or more general, a closed semi-analytic subset) $Q \subset J^q(M,N)$: The set of maps f for which $j^q f : M \to J^q(M,N)$ is transverse to Q at a given point $x \in M$ is open and dense in $C^\infty(M,N)$.

   If M and N are G-manifolds, there is an induced G-action on $J^q(M,N)$: $gj^q f(x) = j^q(gfg^{-1})(gx)$. Hence, if f is equivariant, $j^q f : M \to J^q(M,N)$ is an equivariant section. If $Q \subset J^q(M,N)$ is a G-invariant analytic submanifold, the set of equivariant maps f with $j^q f$ G-transverse to Q at $x \in M$ is not always dense in $C^\infty_G(M,N)$.

Bierstone ([Bie77]) gives a suitable definition of G-transversality of $j^q f$ to Q and puts the definitions for the q-jet spaces together to give a definition of general position of equivariant maps. For our purpose, the definition for the q-jet suffices. For this, we adapt the term *G-q-jet-transversality*, which was introduced by Field.

The definition of G-q-jet-transversality again splits into two parts, where one part is the definition for maps between representations and the second one is the extension to the general case via the Slice Theorem.

For the application to Hamiltonian systems, we will only need the case $q = 1$ and $N = \mathbb{R}$ with the trivial G-action. Hence in the presentation of the theory, the emphasis is placed on this case.

We start with the first part: Assume that V and W are representations. Let $p_1, \ldots, p_l$ and $F_1, \ldots, F_k$ form minimal sets of homogeneous polynomial generators of $P(V)^G$ and $P_G(V,W)$ respectively. Set $P = (p_1, \ldots, p_l)$. P is also called the *orbit map*. An equivariant map f can be written as

$$f(x) = \sum_{i=1}^{k} l_i(p_1(x), \ldots, p_l(x))F_i(x) = \sum_{i=1}^{k} l_i(P(x))F_i(x).$$

This yields an expression of $j^q f(x)$ in terms of $L = (l_1, \ldots, l_k) : \mathbb{R}^l \to \mathbb{R}^k$, P, and the maps F_i. For example $j^1 f(x) = (x, f(x), \mathrm{d}f(x))$ is given by

$$(x, \sum_{i=1}^{k} l_i(P(x))F_i(x), \sum_{i=1}^{k} (\mathrm{d}l_i(P(x)) \circ \mathrm{d}P(x)\cdot)F_i(x) + l_i(P(x))\mathrm{d}F_i(x))$$

Note that in the case $W = \mathbb{R}$, the module $P_G(V,\mathbb{R})$ coincides with the ring $P(V)^G$ and hence it is generated by the constant map $F_1(x) = 1$. In this case, L is a real valued function and

$$j^1 f(x) = (x, L \circ P(x), \sum_{i=1}^{l} \partial_i L(P(x))\mathrm{d}p_i(x)).$$

Let $P_q(\mathbb{R}^l, \mathbb{R}^k)$ denote the space of polynomial maps of degree $\le q$ from $\mathbb{R}^l$ to $\mathbb{R}^k$. $P_q(\mathbb{R}^l, \mathbb{R}^k)$ is a vector space and $J^q(\mathbb{R}^l, \mathbb{R}^k) = \mathbb{R}^l \times P_q(\mathbb{R}^l, \mathbb{R}^k)$. The map $j^q f : V \to J^q(V,W)$ can be decomposed as

$$j^q f = U_q \circ (\mathbb{1}, \tilde{j}^q L \circ P),$$

where

$$\tilde{j}^q L : \mathbb{R}^l \to P_q(\mathbb{R}^l, \mathbb{R}^k)$$

is the q-jet of L with the $\mathbb{R}^l$-component omitted and the map

$$U_q : V \times P_q(\mathbb{R}^l, \mathbb{R}) \to J^q(V, \mathbb{R})$$

can be written in terms of derivatives of P and the F_i and is in particular independent of f. We give the explicit forms of U_q and $\tilde{j}^q L$ in the case $q = 1$ and $W = \mathbb{R}$:

We write $(t_0, t_1^1, \ldots, t_1^l)$ for the coordinates of $P_1(\mathbb{R}^l, \mathbb{R}) \simeq \mathbb{R} \times \mathbb{R}^l$. Then

$$\begin{aligned} U_1 : V \times P_1(\mathbb{R}^l, \mathbb{R}) &\to J^1(V, \mathbb{R}) \\ (x, t_0, t_1^1, \ldots, t_1^l) &\mapsto (x, t_0, \sum t_1^i \mathrm{d}p_i(x)) \end{aligned}$$

and

$$\begin{aligned} \tilde{j}^1 L : \mathbb{R}^l &\to \mathbb{R} \times V^* \\ y &\mapsto (L(y), \mathrm{d}L(y)). \end{aligned}$$

With these maps, we may formulate the definition for the case of representations:

Definition 6.19. If $Q \subset J^q(V, W)$ is a G-invariant closed semi-algebraic subset, $f \in C_G^\infty(V, W)$ is *G-q-jet-transverse* to $Q \subset J^q(V, \mathbb{R})$ iff $(\mathbb{1}, \tilde{j}^q L \circ P)$ is transverse to $U_q^{-1}(Q)$.

For the definition of a semi-algebraic subset of a real vector space, see appendix A.2.

Of course, it has to be shown that the definition is independent of the choice of generators and the representation of f. The proof is given in [Bie76].

By abuse of notation, we write $j^q f \pitchfork_G Q$ in the case of equivariant jet-transversality, even though this is not consistent with our former definition of this notation.

Example 6.20. Suppose that we are interested in the critical points of some G-invariant function f defined on the G-representation V. Then we have to investigate the preimage $(j^1 f)^{-1}(Q)$, where

$$Q = V \times \mathbb{R} \times \{0\} \subset V \times \mathbb{R} \times V^* = J^1(V, \mathbb{R}).$$

In this case, $U_1^{-1}(Q) \subset V \times \mathbb{R} \times \mathbb{R}^l$ is given by the product of $\mathbb{R}$ and the zero set in $V \times \mathbb{R}^l$ of the function

$$(x, t_1^1, \ldots, t_1^l) \mapsto \sum_{i=1}^{l} t_1^i \mathrm{d}p_i(x).$$

We omit the constant index 1 and denote this function by ϑ and its zero set by Σ to emphasize the analogy to ordinary equivariant transversality. Similarly (again by abuse of notation), Γ_f denotes the map

$$x \mapsto (x, \partial_1 L(P(x)), \ldots, \partial_l L(P(x))) \in V \times \mathbb{R}^l.$$

Then $j^1 f \pitchfork_G Q$ at x iff $\Gamma_f \pitchfork \Sigma$.

Now let us consider the general case. Let us assume for simplicity that G is algebraic. Since we are interested in local properties of an equivariant map $M \to N$ near some point $x \in M$, we can assume w.l.o.g. that $M = G \times_H V$, where $H = G_x$ and V is an H-representation that is locally H-equivariantly isomorphic to a slice at x, and define G-q-jet-transversality with respect to a semi-algebraic subset $Q \subset J^q(G \times_H V, N)$ at the point $[e, 0]$.

In addition, for any $y \in N^H$, there is an H-representation W and an H-equivariant local diffeomorphism $(W, 0) \to (N, y)$ with the property that any $f \in C^\infty_H(V, W)$ can be extended to a map $\tilde{f} \in C^\infty_G(G \times_H V, N)$, where $\tilde{f}([g, x]) = gf(x)$. It is easy to see that the map $j^q f(x) \mapsto j^q \tilde{f}(x)$ is well-defined on the set of jets of H-equivariant maps $V \to W$ and that the image of the set $j^q_0(C^\infty_H(V, W))$ of jets of H-equivariant smooth maps at 0 is an open subset of the set

$$j^q_{[e,0]}(C^\infty_G(G \times_H V, N))$$

of jets of G-equivariant smooth maps at $[e, 0]$.

If we choose a smooth local section σ of $G \to {}^G/_H$ defined in a neighbourhood of H, $\tilde{f}$ can be expressed as

$$\tilde{f}[\sigma(gH), x] = \sigma(gH)f(x).$$

Via this expression, $\tilde{f}$ can locally be defined for any $f \in C^\infty(V, W)$. This yields a map

$$\begin{aligned} \mathcal{A}_\sigma : J^q(V, W) &\to J^q(G \times_H V, N) \\ j^q f(x) &\mapsto j^q \tilde{f}(x), \end{aligned}$$

which depends on the choice of σ for jets of non-equivariant functions.

Let $\iota : V \to G \times_H V$ be the inclusion. The map

$$\mathcal{A}_\sigma \circ U_q : V \times P^q(\mathbb{R}^l, \mathbb{R}^k) \to J^q(G \times_H V, N)$$

has a unique G-equivariant extension $\mathfrak{U}_q$ to $(G \times_H V) \times P^q(\mathbb{R}^l, \mathbb{R}^k)$:

$$\begin{array}{ccccc} V \times P^q(\mathbb{R}^l, \mathbb{R}^k) & \xrightarrow{U_q} & J^q(V, W) & \xrightarrow{\mathcal{A}_\sigma} & J^q(G \times_H V, N) \\ \big\downarrow{\scriptstyle \iota \times \mathbb{1}} & & & \nearrow{\scriptstyle \mathfrak{U}_q} & \\ (G \times_H V) \times P^q(\mathbb{R}^l, \mathbb{R}^k) & & & & \end{array}$$

If $\mathcal{A}_\sigma \circ U_q(x, t) = j^q \tilde{f}(x)$, then $\mathfrak{U}_q([g, x], t) = j^q \tilde{f}(gx) = g \cdot (j^q \tilde{f}(x))$, since $\tilde{f}$ is G-equivariant.

In addition, the H-invariant orbit map $P : V \to \mathbb{R}^k$ extends uniquely to a G-invariant map $G \times_H V \to \mathbb{R}^k$, which we also denote by P. This yields the map

$$(\mathbb{1}, \tilde{j}^q L \circ P) : G \times_H V \to (G \times_H V) \times P^q(\mathbb{R}^l, \mathbb{R}^k).$$

Using this extension, the G-equivariant map

$$j^q f : G \times_H V \to J^q(G \times_H V, N)$$

is equal to the composition

$$\mathfrak{U}_q \circ (\mathbb{1}, \tilde{j}^q L \circ P).$$

This motivates the general definition of G-q-jet-transversality: Suppose that for $Q \subset J^q(P,N)$ and $p \in P$ with $H := G_p$, tubular neighbourhoods $G \times_H V$ of p and T of $f(p)$ can be chosen in such a way that $Q \cap J^q(G \times_H V, T)$ is semi-algebraic.

W.l.o.g. we assume $Q \subset J^q(G \times_H V, T)$.

Definition 6.21. If $Q \subset J^q(G \times_H V, T)$ is a closed G-invariant semi-analytic subset, $f \in C_G(G \times_H V, T)$ is *G-q-jet-transverse* to Q ($j^q \pitchfork_G Q$) at $[e,0]$ iff $(\mathbb{1}, \tilde{j}^q L \circ P)$ is transverse to $\mathfrak{U}_q^{-1}(Q)$ at $[e,0]$.

Remark 6.22. Even if G is not algebraic, $J^q(G \times_H V, N)$ is an analytic manifold: By [BtD85, chapter III, Theorem 4.1], G is isomorphic to a closed subgroup of $\mathrm{GL}(V)$ for some vector space V. Thus the exponential map yields an analytic structure on G and hence $G \times_H V$, T, and $J^q(G \times_H V, N)$ are analytic manifolds. In [Bie76], Q is required to be an analytic submanifold. In fact, the necessary assumption is that $\mathfrak{U}_q^{-1}(Q)$ has a canonical Whitney stratification. This holds as well for a semi-analytic Q, since $\mathfrak{U}_q^{-1}(Q)$ is also semi-analytic in that case and semi-analytic sets admit a canonical Whitney stratification as well.

For simplicity, we restrict ourselves to the semi-algebraic case. As will be discussed below, there is an equivalent definition based on the definition for representations. This way, we do not need an algebraic structure on G.

As is shown in [Bie76], G-q-jet-transversality is well-defined and in particular independent of the choice of a local isomorphism of $G \times_H V$ and a neighbourhood of $x \in M$ that sends $[e,0]$ to x. Furthermore, the definition admits the following generalization of Thom's transversality theorem:

Let M and N be smooth G-manifolds and $Q \subset J^q(M,N)$ be an admissible G-invariant subset such that G-q-jet-transversality to Q can be defined, i.e. there are appropriate choices of tubular neighbourhoods such that the corresponding sets $\mathfrak{U}_q^{-1}(Q)$ have a canonical Whitney stratification.

Theorem 6.23 ([Bie76])**.** *Let Q be closed and be $A \subset M$ be a closed G-invariant subset.*

1. *$\{f \in C_G^\infty(M,N) \mid j^q f \pitchfork_G Q \text{ along } A\}$ is open and dense in $C_G^\infty(M,N)$ with respect to the Whitney C^∞-topology.*
2. *Let M be compact, $f : M \times [0,1] \to N$ be smooth and set $f_t := f(\cdot, t)$. If $j^q f_t \pitchfork_G Q$ holds for every $t \in [0,1]$, there is an isotopy of G-equivariant homeomorphisms $h_t : M \to M$, $t \in [0,1]$ such that $h_0 = \mathbb{1}$ and $h_t(j^q f_t^{-1}(Q)) = j^q f_0^{-1}(Q)$.*

Remark 6.24. For any $x \in G \times_H V$, $j^q f \pitchfork_G Q$ at x iff $(\mathbb{1}, \tilde{j}^q L \circ P)$ is transverse to $\mathfrak{U}_q^{-1}(Q)$ at x, see [Bie76, Proposition 7.4].

For $q = 0$, G-0-jet-transversality to $V \times \{0\} \subset V \times W$ is just G-transversality to $0 \in W$: If $\Gamma_f = (\mathbb{1}, \gamma_f)$ is transverse to Σ at $0 \in V$, this is also true if we extend γ_f G-invariantly to $G \times_H V$ and replace Σ by $G\Sigma \subset G \times_H V$. Moreover, $\gamma_f = L \circ P$, $\vartheta = U_0$, and $j^0 f(x) = j^0 \tilde{f}(x)$ imply $\mathfrak{U}_0\big|_V = U_0$. Therefore, $\mathfrak{U}_0^{-1}(0) = G\Sigma$.

Remark 6.25. The other way round, we now reformulate the general definition of G-jet-transversality in a way such that the analogy to the definition of ordinary equivariant transversality theory becomes clearer: We consider the restriction of the map $j^q f : V \times_H G \to J^q(V \times_H G, N)$ to V. This map decomposes as

$$\mathcal{A}_\sigma \circ U_q \circ (\mathbb{1}, \tilde{j}^q L \circ P),$$

compare the following diagram:

$$\begin{array}{ccccccc}
V & \xrightarrow{(\mathbb{1}, \tilde{j}^q L \circ P)} & V \times P^q(\mathbb{R}^l, \mathbb{R}^k) & \xrightarrow{U_q} & J^q(V, W) & \xrightarrow{\mathcal{A}_\sigma} & J^q(G \times_H V, N) \\
\big\downarrow \iota & & \big\downarrow \iota \times \mathbb{1} & & & \nearrow \mathfrak{U}_q & \\
(G \times_H V) & \xrightarrow{(\mathbb{1}, \tilde{j}^q L \circ P)} & (G \times_H V) \times P^q(\mathbb{R}^l, \mathbb{R}^k) & & & &
\end{array}$$

Since $\mathfrak{U}_q^{-1}(Q)$ is G-invariant, the map

$$(\mathbb{1}, \tilde{j}^q L \circ P) : G \times_H V \to (G \times_H V) \times P^q(\mathbb{R}^l, \mathbb{R}^k)$$

is transverse to $\mathfrak{U}_q^{-1}(Q)$ at $[e, 0]$ iff it is transverse to the intersection

$$\mathfrak{U}_q^{-1}(Q) \cap (V \times P^q(\mathbb{R}^l, \mathbb{R}^k)),$$

which equals $(\mathcal{A}_\sigma \circ U_q)^{-1}(Q)$. Moreover, the derivative of $\tilde{j}^q L \circ P$ at $[e, 0]$ vanishes in the direction of the G-orbit, since P is G-invariant. Hence, $j^q f \pitchfork_G Q$ iff the map

$$(\mathbb{1}, \tilde{j}^q L \circ P) : V \to V \times P^q(\mathbb{R}^l, \mathbb{R}^k)$$

is transverse to $(\mathcal{A}_\sigma \circ U_q)^{-1}(Q)$ at 0, equivalently, iff $f\big|_V$ is H-q-jet-transverse to the H-invariant subset $\mathcal{A}_\sigma^{-1}(Q)$.

The image of $\mathcal{A}_\sigma \circ U_q$ coincides with the set of jets of equivariant maps (i.e. the jets that have a representation as a jet of an equivariant function) with base points in V. Hence the image of the G-equivariant extension $\mathfrak{U}_q$ of $\mathcal{A}_\sigma \circ U_q$ is given by the set of jets of equivariant maps. That means that Q may be replaced with its intersection with this set without any difference.

In the case $q = 1$ and $N = \mathbb{R}$ with the trivial G-action, the set of jets of equivariant maps is contained in the set of jets that vanish in the direction of the G-orbits.

More precisely, assume $M = G \times_H V$ and let $\mathcal{T} \subset TM$ and $\mathcal{T}^\circ \subset T^*M$ denote the sets given by

$$\mathcal{T}_x = \mathfrak{g}x \quad \text{and} \quad \mathcal{T}_x^\circ = \operatorname{ann}(\mathcal{T}_x).$$

Then, we have

$$\operatorname{im}(\mathcal{A}_\sigma \circ U_q) = (\mathcal{T}^\circ \times \mathbb{R})\big|_V \subset T^*M \times \mathbb{R} = J^1(M, \mathbb{R}).$$

In our applications, Q is of the form $Q' \times \mathbb{R}$ with $Q' \subset T^*M$ and we can omit the $\mathbb{R}$-component. Then we only have to consider the intersection $Q' \cap \mathcal{T}^\circ$. The preimage of this intersection under $\mathcal{A}_\sigma$ consists of the derivatives in V-direction and f is G-1-jet-transverse to Q iff $f\big|_V$ is H-q-jet-transverse to the projection of $Q' \cap \mathcal{T}^\circ$ to T^*V.

6.2 Application to bifurcation theory

As Field and Richardson observe, equivariant transversality theory is quite useful to describe the "branching pattern" of generic 1-parameter families $f : (x, \lambda) \mapsto f_\lambda(x)$ of equivariant vector fields. Here, the term "branching pattern" initially means the germ of the zero set of f near a *bifurcation point*: A zero (x_0, λ_0) of f is a *bifurcation point* iff $\mathrm{d}_x f(x_0, \lambda_0)$ is not invertible. (If $\mathrm{d}_x f(x_0, \lambda_0)$ is invertible, the implicit function theorem implies that the zero set can be locally parameterized by λ.)

Obviously, the zero set of f corresponds to the equilibria of the vector fields f_λ. Later on, the points (x, λ) near (x_0, λ_0) such that x is a relative equilibrium of f_λ will also be considered as part of the "branching pattern".

This section gives a broad sketch of the main ideas of this application of equivariant transversality theory.

Via Lyapunov-Schmidt-reduction or a centre manifold reduction, the search for the local zeros of f can be reduced to the search for the zeros of an equivariant map from $\ker \mathrm{d}_x f(x_0, \lambda_0) \times \mathbb{R}$ to $\ker \mathrm{d}_x f(x_0, \lambda_0)$ (where the group acts trivially on $\mathbb{R}$). As is well-known, for equivariant 1-parameter-families generically the kernels $\ker \mathrm{d}_x f(x_0, \lambda_0)$ at bifurcation points are absolutely irreducible (see for example [Fie07, section 7.1.1]).

Thus, as far as we are interested in bifurcation of equilibria, we only have to consider G-equivariant maps $f : V \times \mathbb{R} \to V$, where V is an absolutely irreducible representation and $\mathbb{R}$ the trivial representation, such that $f(0, 0) = 0$ and $\mathrm{d}_x f(0, 0) = 0$. Since the non-symmetric case is well-known, we suppose in addition that V is non-trivial. Then $V^G = \{0\}$ implies $f(0, \lambda) = 0$.

$$\mathrm{d}_x f(0, \lambda) = \sigma(\lambda)\mathbb{1}, \quad \sigma(\lambda) \in \mathbb{R},$$

since V is absolutely irreducible. In the following, we assume in addition that the genericity condition $\sigma'(0) \neq 0$ is satisfied.

The set of these maps f is denoted by $C^\infty_G(V \times \mathbb{R}, V)_*$. This set of functions is considered in [FR89]. In his later publications, Field uses a reparameterization to assume $\sigma(\lambda) = \lambda$. The set of maps f with this property is denoted by $\mathcal{V}_0$. Both function spaces are endowed with the Whitney C^∞-topology.

Now, an equivalent characterization of equivariant transversality, which is closer to the original definition of Field ([Fie77a]), turns out to be useful:

We first consider a slight modification, that is also considered in [Bie77]: Let $F_1, \dots, F_k$ be a set of homogeneous polynomial generators of the $P(V)^G$-module $P_G(V, V)$. Note that the maps

$$(x, \lambda) \mapsto F_i(x)$$

form a homogeneous set of generators of the $P(V)^G$-module $P_G(V \times \mathbb{R}, V)$. By abuse of notation, these generators will also be denoted by F_i in the following. Suppose $f = \sum_i g_i(x, \lambda) F_i(x)$.

As illustrated in Example 6.15, we may replace the maps

$$\begin{aligned} \theta : V \times \mathbb{R} \times \mathbb{R}^k &\to V \\ (x, \lambda, t) &\mapsto \sum_i t_i F_i(x, \lambda) = \sum_i t_i F_i(x) \end{aligned}$$

and

$$\Gamma_f : V \times \mathbb{R} \to V \times \mathbb{R} \times \mathbb{R}^k (x, \lambda) \mapsto (x, \lambda, g_1(x, \lambda), \ldots, g_k(x, \lambda))$$

(again by abuse of notation) by

$$\begin{aligned} \theta : V \times \mathbb{R}^k &\to V \\ (x, t) &\mapsto \sum_i t_i F_i(x) \end{aligned}$$

and

$$\Gamma_f : V \times \mathbb{R} \to V \times \mathbb{R}^k (x, \lambda) \mapsto (x, g_1(x, \lambda), \ldots, g_k(x, \lambda)).$$

If we set $\Sigma = \theta^{-1}(0)$ for our new definition of θ, then $f \pitchfork_G 0 \in V$ at $(0, 0) \in V \times \mathbb{R}$ still holds iff $\Gamma_f \pitchfork \Sigma$ at $(0, 0)$.

Now, consider the map

$$\begin{aligned} \gamma_f : V \times \mathbb{R} &\to \mathbb{R}^k \\ (x, \lambda) &\mapsto (g_1(x, \lambda), \ldots, g_k(x, \lambda)). \end{aligned}$$

Note that $\Gamma_f(x) = (x, \gamma_f(x))$ is the graph of γ_f.

Moreover the canonical stratification $\mathcal{S}$ of $\Sigma \subset V \times \mathbb{R}^K$ yields a stratification of $\mathbb{R}^k = \Sigma^G = \Sigma_{(G)}$: For each isotropy type τ, set $\mathcal{S}_\tau = \cup_{s \in \mathcal{S}} S_\tau$. By [Fie07, Theorem 6.10.1], $\mathcal{S}_\tau$ is a Whitney stratification of Σ_τ consisting of $\mathcal{S}$-strata.

Thus, $f \pitchfork_G 0 \in V$ at $(0, 0)$ is equivalent to $\gamma_f \pitchfork \mathcal{S}_{(G)}$. The next lemma gives some insight into the stratification $\mathcal{S}_{(G)}$. From now on, we fix a minimal set of homogeneous polynomial generators $F_1, \ldots, F_k$ with $F_1(x) = x$. Then F_k is of order ≥ 2 for $x \geq 2$.

Lemma 6.26. *Every stratum $S \subset \mathcal{S}_{(G)}$ with $\operatorname{codim} S > 0$ in $\mathbb{R}^k$ is contained in $\mathbb{R}^{k-1} := \{t \in \mathbb{R}^k \mid t_1 = 0\}$.*

Proof. Consider a point $(0, t) \in \mathbb{R}^k = \{0\} \times \mathbb{R}^k \subset V \times \mathbb{R}^k$. Since $\theta(x, t) = \sum_i t_i F_i(x)$, we obtain $\mathrm{d}_x \theta(0, t) = t_1 \mathbb{1}$. If $t_1 \neq 0$, the implicit function theorem yields that there is a neighbourhood U of $(0, t)$ whose intersection with the set $\Sigma = \theta^{-1}(0) \subset V \times \mathbb{R}^k$ coincides with $U \cap \mathbb{R}^k$. By the construction of the canonical stratification, $U \cap \mathbb{R}^k$ is contained in a single stratum of $\mathcal{S}$. ☐

This alternative characterization of G-transversality has some advantages:

- For a fixed choice of a minimal set of homogeneous polynomial generators $F_1, \ldots, F_k$, the map γ_f is independent of the choice of the representation $f(x, \lambda) = \sum_i g_i(x, \lambda) F_i(x)$ (see [Fie07, Lemma 6.6.3] and the definition of γ after his Corollary 6.6.1). This yields an elegant alternative to Bierstone's proof of the independence of the choice of the representation. As in Bierstone's proof, a general G-representation V is considered as a product $V \times \mathbb{R}^s$, where $V^G = \{0\}$ and G acts trivially on the parameter space $\mathbb{R}^s$.

- We would like to prove openness and density of the property $f \pitchfork_G 0 \in V$ at $(0, 0)$ within the function space that we choose for the investigation of bifurcation problems. Openness follows from the openness in the space

$C^\infty_G(V \times \mathbb{R}, \mathbb{R})$. For the proof of density, we give another equivalent description of G-transversality:

Consider the map $\gamma : f \mapsto \gamma_f$ from $C^\infty_G(V \times \mathbb{R}, V)_*$ or $\mathcal{V}_0$ to $C^\infty(\mathbb{R}, \mathbb{R}^k)$. γ is continuous ([Fie07, Lemma 6.6.7]). The image is given by the set of functions whose first coordinate is given by $\sigma(\lambda)$ or λ respectively. Thus, $\gamma_f(0) \in \mathbb{R}^{k-1}$ and γ_f is transverse to $\mathbb{R}^{k-1}$ at 0. By Lemma 6.26, intersecting the strata of $\mathcal{S}_{(G)}$ with $\mathbb{R}^{k-1}$ yields a Whitney stratification $\mathcal{A}$ of $\mathbb{R}^{k-1}$ and γ_f is transverse to $\mathcal{S}_{(G)}$ if $\gamma_f(0)$ is contained in a stratum of $\mathcal{A}$ of codimension 0.

Since this holds for a dense subset of the image of γ, the preimage of this set in $C^\infty_G(V \times \mathbb{R}, \mathbb{R})_*$ or $\mathcal{V}_0$ is also dense.

- This alternative description of G-transversality confirms that a vector field $f \in C^\infty_G(V \times \mathbb{R}, \mathbb{R})_*$ is G-transverse to 0 at $(0,0)$ if this is true for the d-jet of f_0, where d is the maximal degree of the F_i. We will see below that in this case the homeomorphism class of the zeros of f is stable under perturbations. This yields Field's finite (weak) determinacy result. (In Field's terminology, broadly speaking, a representation is weakly d-determined if for a generic equivariant smooth family f_λ, the topological properties of the zero set are stable under perturbations and determined by $j^d f_0$. Determinacy requires also hyperbolicity of the non-trivial zeros.)

If $f \in C^\infty_G(V \times \mathbb{R}, \mathbb{R})_*$ is G-transverse to 0 at $(0,0)$, the local zero set of f is a Whitney stratified subset of $V \times \mathbb{R}$ whose structure may be deduced from that of Σ.

The following smoothness result for the isotropy components Σ_τ is essential: For any isotopy type $\tau = (H)$, set

$$g_\tau := \dim G/H = \dim G - \dim H,$$
$$n_\tau := \dim N(H)/H = \dim N(H) - \dim H.$$

Lemma 6.27 ([Fie07, Lemma 6.9.2]). *For each isotropy type τ of V, the set Σ_τ is a smooth manifold with*

$$\dim \Sigma_\tau = k + g_\tau - n_\tau.$$

If $f \pitchfork_G 0$ at $(0,0)$, Γ_f is transverse to the canonical stratification $\mathcal{S}$ of Σ along a neighbourhood U of $(0,0)$. Since for each τ, the stratification $\mathcal{S}_\tau$ consists of $\mathcal{S}$-strata, Γ_f is transverse to $\mathcal{S}_\tau$ and hence to Σ_τ along U as well. Thus the zeros of f of isotropy type τ contained in N form a $\dim g_\tau - n_\tau + 1$-dimensional smooth manifold of $V \times \mathbb{R}$, whose closure contains the origin $(0,0)$ if they exist.

We first consider the case of a finite group G: In this case, we obtain 1-dimensional smooth manifolds, whose boundaries consist of the origin. Each of these curves is Whitney regular over the origin. Let us call each union of the origin and one of these curves a *branch*. Applying a result of Pawlucki on regularity of Whitney stratified semi-algebraic sets to Σ_τ and a stratum in $\bar{\Sigma}_\tau \cap \mathbb{R}^k$, Field and Richardson even show that each branch may be parameterized as a C^1-curve $[0, \delta) \to V \times \mathbb{R}$.

If $\dim G > 0$, the zeros of isotropy type τ consist of G-orbits of dimension g_τ. Thus, if $n_\tau > 1$, we do not expect any zeros of isotropy type τ near 0. If

$n_\tau = 1$, connected components of zeros of isotropy type τ in N consist of single G-orbits. Since the zero set of f is Whitney stratified, by local finiteness, they are bounded away from zero. Hence in the local sense, there are no zeros of isotropy type τ near zero. Only in the case $n_\tau = 0$, we obtain a "branch" of G-orbits of zeros: If $p_1, \ldots, p_l$ form a minimal set of homogeneous generators of the ring $P(V)^G$, the orbit map $P \times \mathbb{1} = (p_1, \ldots, p_l, \lambda)$ maps each of these orbits to a single point of $\mathbb{R}^l \times \mathbb{R}$. As in the finite case, it may be shown that the image in $\mathbb{R}^l$ may be expressed as a union of C^1-curves starting at $(0, 0)$.

For a non-finite group G, it seems to be more natural to include relative equilibria. This is illustrated by the above results: Local branches of zeros of isotropy type τ only occur in the case $n_\tau = 0$. In this case, all relative equilibria are in fact equilibria, since the trajectory of a relative equilibrium with isotropy subgroup H is contained in its $N(H)$-orbit. Moreover, in the case $n_\tau = 1$, isolated orbits of zeros may occur. As examples show, these are usually embedded in branches of relative equilibria. The analysis of the local structure of relative equilibria yields that this is indeed the behaviour we generically expect.

The generalization of the results for finite groups to the bifurcation of relative equilibria first appeared in [Fie96]. For the generalization, we have to consider also bifurcations that are caused by a pair of purely imaginary eigenvalues $\pm\alpha\mathrm{i}$, $\alpha > 0$. Centre manifold reduction or a Lyapunov-Schmidt reduction in a way used for the proof of Hopf bifurcation theorems yields an equivariant family f_λ of vector fields on the real part V of the sum of the generalized eigenspaces for $\pm\alpha\mathrm{i}$. Generically, the generalized eigenspaces coincide with the eigenspaces such that $\mathrm{d}f_0(0)^2 = -\alpha^2\mathbb{1}$. Thus, $\frac{1}{\alpha}\mathrm{d}f_0(0)^2$ defines a complex structure on V. Moreover, it may be shown that V with respect to this complex structure forms a complex irreducible G-representation. Using the theory of Birkhoff normal forms, we may assume that the Taylor polynomial $T_r f$ of order r commutes with the S^1-action defined by the complex structure for an arbitrary finite r. Field proves that for r large enough, the branches of relative equilibria of $T_r f$ persist if the higher order terms of f are added. (But there may be additional relative equilibria for f.) Thus, Field restricts his analysis to the case of complex irreducible representations of compact groups of the form $G = K \times S^1$. For these, he considers the set of normalized families

$$\mathcal{V}_0(V, G) = \{ f \in C^\infty_G(V \times \mathbb{R}, V) \mid \mathrm{d}f_\lambda(0) = (\lambda + \mathrm{i})\mathbb{1} \} .$$

(The normalization consists of a scaling of time which corresponds to a scaling of f and a reparameterization in the variable λ afterwards.) The rest of the argument is similar to the one for bifurcations of equilibria:

The set Σ is replaced by the set

$$\Sigma^* = \{ (x, t) \mid \theta(x, t) \in T_x Gx \} .$$

Note that Σ^* is an algebraic set, since $(x, t) \in \Sigma^*$ iff $\mathrm{d}P(x)\theta(x, t) = 0$:

Suppose that x has isotropy type τ. Since $\theta(\cdot, t)$ is an equivariant vector field, $\theta(x, t) \in V^{G_x} \subset T_x V_\tau$.

For every connected component $V_\tau^i \subset V_\tau$, the image $P(V_\tau^i) = V_\tau^i / G$ is a smooth manifold and $P : V_\tau^i \to V_\tau^i / G$ is a submersion (see [DK00, Remark 2.7.5]). Hence for $x \in V_\tau$, the kernel of $\mathrm{d}P(x)\big|_{T_x V_\tau}$ coincides with $T_x Gx$.

The isotropy components of Σ^* also form smooth manifolds:

Lemma 6.28 ([Fie07, Lemma 10.2.2]). *For each isotropy type τ of V, the set Σ^*_τ is a smooth manifold with*

$$\dim \Sigma^*_\tau = k + g_\tau.$$

As before, each Σ^*_τ is a union of strata of the canonical stratification of Σ^*.

Thus, for any $f \in \mathcal{V}_0(V,G)$ with $\Gamma_f \pitchfork \Sigma^*$ at $(0,0)$, the set of pairs (x,λ) such that x is a relative equilibrium of f_λ consists of branches of G-orbits (in the same sense as above).

It only remains to show that this transversality condition is open and dense in $\mathcal{V}_0(V,G)$. Openness is clear. Density is proved in a similar way as above: We choose a minimal set of homogeneous equivariant generators $F_1,\dots,F_k$ with $F_1(x) = x$ and $F_2(x) = \mathrm{i}x$. Again, $\overline{\Sigma^*_\tau} \cap \mathbb{R}^k$ is contained in $\mathbb{R}^{k-1} = \{t \in \mathbb{R}^k \mid t_1 = 0\}$. Thus, again we obtain a Whitney stratification $\mathcal{A}^*$ of $\mathbb{R}^{k-1}$ such that the transversality condition is satisfied if

$$\gamma_f(0) = (0, \mathrm{i}, g_3(0,0), \dots g_k(0,0))$$

is contained in a stratum of $\mathcal{A}$ of codimension 0. Moreover, using $\mathrm{i}x = T_x S^1 x \subset T_x Gx$, it may be shown that the stratification $\mathcal{A}^*$ is invariant under translation in the direction of the t_2-axis. As above, this yields the density of our transversality condition.

Remark 6.29. The transversality condition can be formulated in terms of equivariant transversality: Consider the algebraic subset

$$T := \{(x,v) \mid \mathrm{d}P(x)v = 0\} \subset V \times V$$

and the map

$$\begin{aligned} \mathbb{1} \times \theta : V \times \mathbb{R}^k &\to V \times V \\ (x,t) &\mapsto (x, \sum_i t_i F_i(x)). \end{aligned}$$

Then $\Sigma^* = (\mathbb{1} \times \theta)^{-1}(T)$. The space $V \times V$ may be identified with the space $J^0(V,V)$ and $\mathbb{1} \times \theta$ coincides with the map U_0. The composition $U_0 \circ \Gamma_f$ yields the graph of f that may be seen as the 0-jet of f. Thus, in the notation of equivariant 0-jet-transversality, the condition is $j^0 f \pitchfork_G T$. Equivariant 0-jet-transversality is just a slight generalization of equivariant transversality.

Field also considers an equivariant 1-jet-transversality condition in order to show that the bifurcating equilibria and relative equilibria are generically hyperbolic and *normally hyperbolic* respectively. For the characterization of normal hyperbolicity in this context, the splitting $f = f_T + f_N$ of a vector field $f \in C^\infty_G(V,V)$ within a tubular neighbourhood is used, where f_T is tangential to the G-orbits and f_N is tangential to the slices. The existence of such a splitting is a result of Krupa [Kru90]. A G-orbit of relative equilibria is a *normally hyperbolic* submanifold for f iff each of its elements is a hyperbolic zero of f_N restricted to the corresponding slice. Equivalently, for any element x of the orbit and any equivariant vector field $\tilde{f}$ with $\tilde{f}(x) = f(x)$ that is tangential to the G-orbits, the centre space of $\mathrm{d}(f - \tilde{f})(x)$ has dimension $\dim Gx$.

Let $H(V) \subset \mathrm{End}_{\mathbb{R}}(V)$ be the semi-algebraic subset consisting of hyperbolic manifolds. In the case of a finite group G, Field sets

$$Z_1 := \left\{(x, 0, A) \in J^1(V, V) = V \times V \times \mathrm{End}_{\mathbb{R}}(V) \;\middle|\; A \notin H(V)\right\}.$$

For the non-finite case, the definition of a corresponding set Z_1^* is more involved. Starting with the semi-algebraic set

$$Z_0^*(\tau) = \{(x, v) \in V \times V \mid x \in V_\tau, v \in T_x N(G)x\}$$

for any isotropy type τ, a map Ξ from $Z_0^*(\tau)$ into the set of vector fields $C_G^\infty(V, V)$ is constructed such that $(\Xi(x, v))$ is tangential to the G-orbits and $(\Xi(x, v))(x) = v$. Based on these maps, one obtains a semi-algebraic set Z_1^* with the property that the G-orbit of relative equilibrium x is a normally hyperbolic submanifold for f iff $j^1 f(x) \notin Z_1^*$.

In both cases, Field proves that for dimensional reasons, $j^1 f \pitchfork_G Z_1$ and $j^1 f \pitchfork_G Z_1^*$ are equivalent to $j^1 f(x) \notin Z_1^*$ and $j^1 f(x) \notin Z_1^*$ respectively. Again, openness and density of this condition can also be shown for the sets $\mathcal{V}_0(V, G)$.

These methods of analysing questions in bifurcation theory have strongly inspired the proceedings of this thesis.

As mentioned in the beginning, Birtea et al ([BPRT06]) also build on Field's ideas to study bifurcations in 1-parameter families of equivariant vector fields. In particular, they investigate bifurcations of relative equilibria in the Hamiltonian case. Their analysis relies on a variant of Field's method proposed by Kœnig and Chossat ([KC94]): For a given equivariant vector field X on a G-representation V, we consider the projection $\tilde{X}$ to the orbit space ${}^{V}\!/_{G} \subset \mathbb{R}^k$, given by $\tilde{X}(P(x)) = \mathrm{d}P(x)X(x)$. The space ${}^{V}\!/_{G}$ is Whitney stratified, where the strata are given by the images of the subsets of the same isotropy type. The projection $\tilde{X}$ is a vector field on ${}^{V}\!/_{G}$, in the sense that it is tangent to the strata. If the family f is represented by $f(x, \lambda) = \sum_i g_i(P(x), \lambda)F_i(x)$ then we have

$$\tilde{f}(P(x), \lambda) = \sum_i g_i(P(x), \lambda)\tilde{F}_i(x).$$

Instead of the map $\theta : V \times \mathbb{R}^k \to V$, we consider the induced map $\tilde{\theta} : {}^{V}\!/_{G} \times \mathbb{R}^k \to {}^{V}\!/_{G}$ given by

$$\tilde{\theta}(P(x), t) = \sum_i t_i \tilde{F}_i(x).$$

Then $\tilde{\Sigma} := \tilde{\theta}^{-1}(0)$ coincides with the projection of the set $\Sigma^* \subset V \times \mathbb{R}^k$ to ${}^{V}\!/_{G} \times \mathbb{R}^k$. Moreover, the projections of the strata of Σ^* form a stratification of $\tilde{\Sigma}$. The induced stratifications of $\mathbb{R}^k$ are the same, when we identify $\mathbb{R}^k$ with the subsets $\{0\} \times \mathbb{R}^k$ of Σ^* and $\tilde{\Sigma}$ respectively. Kœnig and Chossat say that the projected family $\tilde{f}$ is *G-transverse* to $0 = P(0) \in {}^{V}\!/_{G}$ at $(0, 0) \in {}^{V}\!/_{G} \times \mathbb{R}$ iff the map γ with $\gamma(\lambda) = (g_1(0, \lambda), \ldots, g_k(0, \lambda))$ is transverse to this stratification of $\mathbb{R}^k$. This is equivalent to Field's condition.

Now, Birtea et al formulate a Hamiltonian analogue of equivariant transversality theory: They note that for a G-invariant Hamiltonian function h with a representation $h = g \circ P$, the Hamiltonian vector field X_h is given by

$$X_h(x) = \Sigma_{i=1}^l \partial_i g(P(x)) J \nabla p_i(x),$$

where $\omega = \langle \cdot, J\cdot \rangle$. Thus, if we redefine θ by $\theta(x,t) := \sum_{i=1}^{l} t_i J\nabla p_i(x)$ for $(x,t) \in V \times \mathbb{R}^l$ and Γ_h by $\Gamma_h(x) := (x, \partial_1 g(P(x)), \ldots, \partial_l g(P(x)))$, then X_h can be decomposed as $\theta \circ \Gamma_h$. This is in principle what we will do in section 6.3 of this chapter. We will give our definition in terms of 1-jet-transversality in order to deduce directly from Bierstone's theory that our transversality condition is well-defined and generic – a matter that Birtea et al take no notice of when they give their definition.

For a family f of Hamiltonian functions $f(\cdot,\lambda)$, we obtain an analogous decomposition. Alternatively, one could transfer Kœnig's and Chossat's formulation. This is the approach of Birtea et al ([BPRT06]). They observe that the projections of the vector fields $J\nabla p_i$ are given by

$$\begin{pmatrix} \{p_1, p_i\} \\ \{p_2, p_i\} \\ \cdots \\ \{p_l, p_i\} \end{pmatrix}.$$

Thus the map $\tilde{\theta} : {}^{V}\!/_{G} \times \mathbb{R}^l \to {}^{V}\!/_{G} \subset \mathbb{R}^l$ induced by θ is given by $(x,t) \mapsto A(x)t$, where the $l \times l$-matrix $A(x)$ has the entries $(A(x))_{ij} = \{p_i, p_j\}(x)$. We also define $\tilde{\Sigma}$, and $\mathcal{S}_{(G)}$ in an analogous way as for general equivariant vector fields; i.e. $\tilde{\Sigma} := \tilde{\theta}^{-1}(0)$ and $\mathcal{S}_{(G)}$ is the stratification of $\mathbb{R}^l \subset \tilde{\Sigma}$ induced by the canonical stratification of $\tilde{\Sigma}$.

Birtea et al consider families of Hamiltonian functions parametrized by $\lambda \in \mathbb{R}$ such that at $\lambda = 0$ the derivative of the Hamiltonian vector field at the origin has either an eigenvalue 0 or a pair of non-zero purely imaginary eigenvalues $\pm\beta\mathrm{i}$ such that $E_{\pm\beta\mathrm{i}}$ is not irreducible as a G-symplectic representation. In the first case, generically this can be reduced to the study of families $f : V \times \mathbb{R} \to \mathbb{R}$, where V is an irreducible symplectic representation and $\mathrm{d}X_{f(\cdot,\lambda)}(0) = \sigma(\lambda)J$ with $\sigma(0) = 0$ and $\sigma'(0) \neq 0$. In the second case, the generic situation can be reduced to the case that V is a sum of a pair of complex duals and $\mathrm{d}X_{f(\cdot,\lambda)}(0)$ is given by a sum of four matrices, each of which is a product of a constant matrix and a function in λ. One of these functions is also called σ and satisfies $\sigma(0) = 0$, $\sigma'(0) \neq 0$. (This is a result from [COR02]. See [BPRT06] or [COR02] for the precise form.)

If $f : V \times \mathbb{R} \to \mathbb{R}$ is such a family of Hamiltonian functions with

$$X_{f(\cdot,\lambda)}(x) = \Sigma_{i=1}^{l} \partial_i g(P(x), \lambda) J\nabla p_i(x),$$

equivalently

$$\tilde{X}_{f(\cdot,\lambda)}(x) = A(x)\nabla_{P(x)} g(P(x), \lambda),$$

Birtea et al set $\gamma(\lambda) := (\partial_{p_1} g(0,\lambda), \ldots, \partial_{p_l} g(0,\lambda)) \in \mathbb{R}^l)$ and call the family $\tilde{X}_{f(\cdot,\lambda)}$ *transverse* to $0 \in {}^{V}\!/_{G}$ at $\lambda = 0$ iff γ is transverse to $\mathcal{S}_{(G)}$ at 0.

Then, they consider such families that satisfy this transversality condition and state that for a special class of symplectic representations V, the relative equilibria form 'branches' as they do for generic equivariant families of general vector fields. That is, they locally form C^1-curves in the orbit space ${}^{V}\!/_{G} \times \mathbb{R}$ that start at $(0,0)$ and contain no point $(0,\lambda)$ with $\lambda \neq 0$. The condition on V is, that there are indices $i_0, i_1 \in \{1, \ldots, l\}$, such that the function $\{p_{i_0}, p_{i_1}\}$ does not vanish identically and there is $x_0 \in V$ with $\lim_{x \to x_0} \frac{\{p_{i_0}, p_i\}(x)}{\{p_{i_0}, p_{i_1}\}(x)}$ for all $i \neq I_1$ and x in the domain of definition. The proof, however, is erroneous: The authors claim, that this condition implies that all strata of $\mathcal{S}_{(G)}$ of codimension ≥ 1 are contained in the subspace $\{t_{i_1} = 0\} \subset \mathbb{R}^l$ and that it is hence possible to adapt Field's argument. First, such an adaption of the arguments is only possible if σ coincides with the i_1-th entry of the function γ. Second, the proof of the claim about the stratification relies on the assertion, that if the i_0-component of map $\tilde{\theta}$ vanishes for some pair $(P(x_0), t_0)$, then the i_0-component of $\tilde{\theta}(\cdot, t_0)$ vanishes identically. This is obviously not true.

Moreover, if there was a G-symplectic representation V such that the relative equilibria of a generic family of Hamiltonian systems with G-symmetry on V form branches of this kind, a single generic Hamiltonian function would have no non-trivial relative equilibria in some neighbourhood of the origin. This is not the behaviour that we expect. Indeed, as we have shown in chapter 3, if the connected component G° of the identity acts non-trivially on V, there is an open set of Hamiltonian functions h such that G° acts non-trivially on the corresponding centre space $\mathbb{E}_c$ of $\mathrm{d}X_h(0)$. In this case, the results discussed in chapter 5 show that we have to expect relative equilibria near $0 \in V$. For example, this follows from remark 5.18.

Nevertheless, it is an interesting observation that $\tilde{\theta}$ takes this particular simple form. It may lead to a better understanding of the structure of Hamiltonian relative equilibria in symplectic representations in future work. The approach of the following sections, however, is different:

In section 6.3, we consider an algebraic set similar to that considered by Field for his investigation of the bifurcation of relative equilibria. We only need a suitable notion for equivariant transversality of Hamiltonian vector fields which will be given in terms of equivariant 1-jet-transversality. As mentioned, this is in principle equivalent to the definition given by Birtea et al ([BPRT06]).

The analysis of torus representations of section 6.4 is similar to Field's original method of analysing the bifurcation of zero sets: As suggested by Chossat et al. in [CLOR03], we search for zeros of the augmented Hamiltonian and consider the $\xi \in \mathfrak{g}$ as parameter. If $\mathfrak{g}$ is Abelian, the action on the parameter space is trivial and thus the problem can be handled as above. The only difference is the dimension of the parameter space, which is not an obstacle. As mentioned, more dimensional parameter spaces even occur in the proof of the independence of choices.

6.3 Transverse relative equilibria with continuous isotropy

In an analogous way as Field's approach in the context of bifurcation theory for equivariant vector fields, equivariant transversality theory may be used to study the generic structure of relative equilibria of Hamiltonian vector fields for invariant Hamiltonian functions. In this way, the theory for free actions can be generalized to some extent. We will see that the definition of a transverse relative equilibrium given in [PR00] is a special case of a more general definition, which involves equivariant transversality.

In particular, we obtain that generically the results of Patrick and Roberts ([PR00]) for transverse relative equilibria hold within isotropy submanifolds P_H for any isotropy subgroup $H \subset G$ with respect to the free action of the group

$\bar{N}(H) := ({}^{N(H)}\!/_{H})^\circ$. If we identify the groups $\bar{N}(H)$ of relative equilibria of the same isotropy type via conjugation, generically the subsets of the same type (K) within $\bar{N}(H)$ of the momentum generator pair form a smooth manifold of dimension

$$\dim G - \dim H + 2\dim Z(K) - \dim K.$$

In addition, the set of these relative equilibria within the set of phase space points of the same isotropy type has singularity type $(\mathfrak{l}^* \oplus \mathfrak{l})^c$, where $\mathfrak{l}$ is a K-invariant complement of the Lie algebra $\mathfrak{z}$ of $Z(K)$ within $\mathfrak{k}$. That these properties hold for a residual set of G-invariant Hamiltonian functions, can also be deduced directly from the theory presented in [PR00], but equivariant transversality theory yields the openness of this set. Moreover, we can conclude in addition that the whole set of relative equilibria forms a Whitney stratified set.

There are several possible ways how to obtain these generalizations of the theory of transverse relative equilibria. A possible approach might be to generalize the fact that equivariant transversality implies stratumwise transversality to the jet version of transversality theory and invariant algebraic subsets of the jet space. Then the results of Patrick and Roberts ([PR00]) may be applied to the fixed point spaces. But for algebraic sets, the subsets of the same isotropy type are no longer manifolds and this may cause difficulties with the implications between the transversality properties with respect to the different occurring stratifications. (It might be necessary to modify the definition and use a stratification which respects the isotropy type.)

However, we will show stratumwise transversality only in the relevant special case and conclude the results in a more direct way here, which is a bit closer to Field's proceeding for general equivariant vector fields.

We start with a generalization of the definition of transverse relative equilibria: For finding relative equilibria in the Hamiltonian case, we search for solutions (p, ξ) of the equation

$$X_h(p) - \xi_p = 0,$$

which is the same equation as for equivariant vector fields with the only difference that we vary the vector field within the class of Hamiltonian vector fields. This leads to the question how equivariant transversality to the subset $\mathcal{T} = \bigcup_{p \in P} \mathfrak{g} \cdot p$ of TP should be defined for Hamiltonian vector fields.

This question can be answered by considering the dual equation

$$\mathrm{d}h(p) - \mathrm{d}\mathbf{J}^\xi(p) = 0:$$

If $\mathcal{K}^\circ \subset T^*P$ denotes the set $\mathcal{K}^\circ = \bigcup_{p \in P, \xi \in \mathfrak{g}} \mathrm{d}\mathbf{J}^\xi(p)$, the product $\mathbb{R} \times \mathcal{K}^\circ$ can be considered as a subset of $J^1(P, \mathbb{R}) \simeq \mathbb{R} \times T^*P$. Then we only have to apply equivariant jet-transversality theory. More precisely:

Definition 6.30. A relative equilibrium $p \in P$ is *transverse* iff h is G-1-jet-transverse to $\mathcal{K}^\circ \times \mathbb{R}$ at p.

If w.l.o.g. $P \simeq G \times_H V$, the set $\mathcal{K}^\circ$ is a semi-analytic subset of T^*P and hence transversality of p is well-defined: The analytic isomorphism $\omega^\# : T^*P \to TP$ induced by the symplectic form ω maps $\mathcal{K}^\circ$ to the subset $\mathcal{T} \subset TP$ with $\mathcal{T}_x = \mathfrak{g} \cdot x$. Since P is locally analytically isomorphic to ${}^{\mathfrak{g}}\!/_{\mathfrak{h}} \times V$, TP is locally isomorphic to the analytic manifold $T(\mathfrak{g}/\mathfrak{h}) \times TV$ and the subset $\mathcal{T}$ corresponds to the product of $T(\mathfrak{g}/\mathfrak{h})$ and the subset $\mathcal{T}' \subset TV$ given by $\mathcal{T}'_x = \mathfrak{h} \cdot x$. The set $\mathcal{T}'$ is even a semi-algebraic subset of TV. This follows in a similar way as Field's argument, why the set Σ^* is algebraic: If $p_1, \dots, p_l$ are polynomial generators of $P(V)^H$ and $P = (p_1, \cdots, p_l)$ is the orbit map, $\mathcal{T}'$ is equal to union of the zero sets of the maps $\mathrm{d}P : TV_\tau \to \mathbb{R}^l$, $(x, v) \mapsto \mathrm{d}P(x)v$, defined on the tangent bundles of the isotropy subspaces. We only have to show that the tangent bundles TV_τ are semi-algebraic subsets of TV. By [Fie07, Lemma 6.8.1] the subsets $V_\tau \subset V$ are semi-algebraic. In addition, the points of V_τ are non-singular points of their real Zariski closure, in the sense of [BCR98, Chapter 3, Section 3]. (This was shown by G. Schwarz, see [Fie96, Lemma 9.6.1, part (1)].) Therefore it follows from [BCR98, Proposition 3.3.8] that V_τ can be covered by semi-algebraic subsets whose tangents spaces are semi-algebraic. Hence TV_τ is semi-algebraic itself. Thus, $\mathcal{K}^\circ$ is semi-analytic. If G is an algebraic group and hence is H, $\mathcal{K}^\circ$ is even semi-algebraic.

Remark 6.31. In order to avoid to use the – far from trivial – result shown by Schwarz, we could alternatively proceed analogously to Field: Instead of $\mathcal{K}^\circ$, we can consider the image $\tilde{\mathcal{K}}^\circ$ under $\omega^\#$ of the G-invariant set $\tilde{T}$, where $\tilde{T}_{[e,x]} \simeq T(\mathfrak{g}/\mathfrak{h}) \times \ker \mathrm{d}P(p)$ with respect to the local isomorphism $(P, p) \simeq (\mathfrak{g}/\mathfrak{h} \times V, (e, 0))$ near p and the orbit map $P : V \to V/H \subset \mathbb{R}^l$ for the representation of $H := G_p$ on the tangent space V to a slice in p. In the case of a free action, the sets $\mathcal{K}^\circ$ and $\tilde{\mathcal{K}}^\circ$ coincide. In general, their intersections with the set $\mathcal{T}^\circ$ with $\mathcal{T}^\circ_q = \operatorname{ann} \mathfrak{g}q$ coincide: The vectors in the preimage $(\omega^\#)^{-1}(\mathcal{T}^\circ_q)$ can locally be extended to Hamiltonian vector fields of G-invariant Hamiltonian functions. Thus, if $q = [e, x]$ and $K := H_x$, then $(\omega^\#)^{-1}(\mathcal{T}^\circ_q)$ is contained in $TP^K \subset TP_{(K)}$ and the latter is locally equivariantly diffeomorphic to $\mathfrak{g}/\mathfrak{h} \times V_{(K)}$. As argued above, if $v \in T_x V_{(K)}$, then $\mathrm{d}P(x)v = 0$ is equivalent to $v \in \mathfrak{h}x$.

From the equality of these intersections, it follows that we obtain the same transversality condition if we replace $\mathcal{K}^\circ$ by $\tilde{\mathcal{K}}^\circ$.

This implies, that the set $\mathfrak{U}_1^{-1}(\mathcal{K}^\circ)$ is closed and in particular locally closed, even though this does not hold in general for $\mathcal{K}^\circ$. Thus, the Thom-Mather-transversality theorem A.21 applies.

To justify the definition, we first consider the free case:

Example 6.32. If G acts freely on P, P is locally of the form $G \times V$, where V is a vector space (only with a trivial group action). The corresponding map U^1 is just the identity $J^1(V, \mathbb{R}) \to J^1(V, \mathbb{R})$. Hence the map $\mathfrak{U}_1$ is given by the embedding $G \times J^1(V, \mathbb{R}) \hookrightarrow J^1(G \times V, \mathbb{R})$ that maps $(g, j^1 f_x)$ to $j^1 \tilde{f}_{gx}$ where $\tilde{f}$ is the invariant extension of f. In local coordinates corresponding to local coordinates on G and V, the embedding is just the extension by 0 in the coordinates of the G-derivative. This means that $G \times J^1(V, \mathbb{R})$ can be identified with the set $\mathcal{T}^\circ \times \mathbb{R}$, where $T^*(G \times V) \supset \mathcal{T}^\circ = \bigcup_{p \in G \times V} \operatorname{ann}(\mathfrak{g} \cdot p)$. If we omit the $\mathbb{R}$-factors, $\mathfrak{U}_1^{-1}(\mathcal{K}^\circ)$ is just the intersection $\mathcal{K}^\circ \cap \mathcal{T}^\circ$.

Since V and $\mathbb{R}$ are trivial representations, invariant and equivariant polynomial generators are given by the coordinate functions on V and $\mathbb{R}$. Thus, $l = \dim V$, $\mathbb{R}^l \simeq V$, $k = 1$, $V \times P^1(\mathbb{R}^l, \mathbb{R})$ can be identified with $J^1(V, \mathbb{R})$ and $(\mathbb{1}, \tilde{j}^1 h \circ P) : V \to J^1(V, \mathbb{R})$ is just the 1-jet $j^1 h$. The extension to $G \times V$ of this map may be considered as a map to $\mathcal{T}^\circ \times \mathbb{R}$ and again we can omit the $\mathbb{R}$-component and accordingly the function value of $j^1 h$. Then, if we assume w.l.o.g.

$G \times V \simeq P$, a relative equilibrium p is transverse iff $\mathrm{d}h : P \to \mathcal{T}^\circ$ is transverse to the Whitney stratified set $\mathcal{K}^{\circ c} = \mathcal{K}^\circ \cap \mathcal{T}^\circ$ at p. This is just the definition given in [PR00]. We only have to show that the canonical stratification of $\mathcal{K}^{\circ c}$ coincides with the stratification by isotropy type. This will be done with the next lemma and theorem.

Remark 6.33. Note the relation of the subsets $\mathcal{T}^\circ$, $\mathcal{K}^\circ$ of T^*P and their intersection $\mathcal{K}^{oc}$ to the Witt-Artin decomposition of the tangent space at p: We have $T_pP = T_0 \oplus T_1 \oplus N_0 \oplus N_1$ and the corresponding splitting of the dual space $T_p^*P = T_0^* \oplus T_1^* \oplus N_0^* \oplus N_1^*$. The subspace $\mathfrak{g} \cdot p = \mathcal{T}_p \subset T_pP$ coincides with $T_0 \oplus T_1$. Thus $\mathcal{T}_p^\circ = \operatorname{ann} \mathcal{T}_p$ is identified with $N_0^* \oplus N_1^*$. Since

$$\mathcal{K}_p^\circ = \omega^\#(\mathcal{T}_p) = \omega^\#(T_0 \oplus T_1) = N_0^* \oplus T_1^*$$

(see Lemma 2.5 for the last equation), we obtain $\mathcal{K}_p^{oc} = N_0^* \simeq \mathfrak{m}$.

Lemma 6.34. *Let K be a connected compact Lie group with Lie algebra $\mathfrak{k}$ and $\mathfrak{l}$ be a K-invariant complement of the Lie algebra $\mathfrak{z}$ of the centre $Z(K)$ with respect to the adjoint action. Suppose that a closed subset $U \subset \mathfrak{l}^* \oplus \mathfrak{l}$ contains the point $(0,0)$, all points of isotropy strata of dimension d for some $d > 0$, and no points of isotropy strata of dimension greater than d. If U is a smooth manifold, $U = \mathfrak{l}^* \oplus \mathfrak{l}$.*

Proof. Let S be the union of isotropy strata of dimension d. S is a d-dimensional smooth manifold. Moreover, any $x \in S$ has a neighbourhood within U which is contained in S. If U is a manifold, for any sequence $(s_i) \subset S$ approaching $(0,0)$, the sequence of tangent spaces $T_{s_i}S$ converges to $T_{(0,0)}U$. Since in particular for any $s \in S$, the sequence $\frac{1}{n}s$ converges to $(0,0)$ and the corresponding sequence of tangent spaces is the constant sequence $a_i = T_sS$, all tangent spaces of the manifold S are the same. Thus, S is an open subset of a d-dimensional subspace E of $\mathfrak{l}^* \oplus \mathfrak{l}$ and $U = E$. We only have to show that S spans $\mathfrak{l}^* \oplus \mathfrak{l}$: Let $L := K\big/Z(K)$ and $\tau = (H)$ be an isotropy type of the L-representation $\mathfrak{l}^* \oplus \mathfrak{l}$. Then for any isotropy subgroup $H \subset L$, $\tau := (H)$, the dimension of the isotropy stratum $(\mathfrak{l}^* \oplus \mathfrak{l})_\tau$ is given by $\dim L - \dim H + 2 \dim Z(H)$, see [PR00].

Let $\mathfrak{t}$ be the Lie algebra of a maximal torus T of L with $Z(H) \subset T$. If we identify $\mathfrak{t}$ and $\mathfrak{t}^*$, the set $\mathfrak{t}^H$ is given by the intersection W of some Weyl walls. Moreover, $\dim H$ is determined by $\dim W = \dim Z(H)$ ([BtD85, chapter V, Proposition 2.3]). Hence, within $\mathfrak{t} = \{0\} \oplus \mathfrak{t} \subset \mathfrak{l}^* \oplus \mathfrak{l}$, all points of intersections of some specific number of Weyl walls are contained in S. Since $\dim S > 0$, all edges of some Weyl chamber belong to S. Thus S spans $\mathfrak{t}$ and by L-invariance, S spans $\mathfrak{l}$. Similarly, S spans $\mathfrak{l}^*$. Thus, $U = \mathfrak{l}^* \oplus \mathfrak{l}$. □

Theorem 6.35. *Let P be a symplectic manifold with a free Hamiltonian action of a connected compact group G.*

1. *The canonical stratification of $\mathcal{K}^{\circ c}$ coincides with the stratification by isotropy type of the momentum generator pair.*

2. *Any G-invariant Whitney stratification of $\mathcal{K}^{\circ c}$ is a refinement of the canonical stratification.*

Proof. 1. As presented in section 4.2, near a point of $\mathcal{K}^{\circ c}_{(K)}$, the stratified set $\mathcal{K}^{\circ c}$ is locally diffeomorphic to a product of $(\mathfrak{l}^* \oplus \mathfrak{l})^c$ and a vector space E. Therefore the canonical stratification locally coincides with the product of the canonical stratification of $(\mathfrak{l}^* \oplus \mathfrak{l})^c$ and E.

To construct the canonical stratification of $\mathcal{K}^{\circ c}$, we start with the subset of highest dimension, which consists of all points for which a neighbourhood of $\mathcal{K}^{\circ c}$ forms a manifold of dimension $\dim G + \dim T$, where $T \subset G$ is a maximal torus. By Lemma 6.34, these are exactly the points whose generator momentum pair is of isotropy type (T): For any isotropy subgroup K of $(\mathfrak{g}^* \oplus \mathfrak{g})^c$ with $(K) \neq (T)$, $(\mathfrak{l}^* \oplus \mathfrak{l})^c \subsetneq \mathfrak{l}^* \oplus \mathfrak{l}$ is not regular at $(0,0)$ and thus the set $\mathcal{K}^{\circ c}$ is not regular at points of isotropy (K). Next, we consider the set $\mathcal{K}^{\circ c}$ with the points of type (T) omitted. Again by Lemma 6.34 the set of regular points consists of the sets $\mathcal{K}^{\circ c}_{\tau}$ of maximal dimension, where τ is an isotropy type of $(\mathfrak{g}^* \oplus \mathfrak{g})^c \setminus (\mathfrak{g}^* \oplus \mathfrak{g})^c_{(T)}$. Moreover, since the stratification by isotropy type is a Whitney stratification, the set $\mathcal{K}^{\circ c}_{(T)}$ is Whitney regular over the set of regular points of $\mathcal{K}^{\circ c}_{\tau} \setminus \mathcal{K}^{\circ c}_{(T)}$. Going on, the construction of the canonical stratification yields step by step the stratification by isotropy type of (μ, τ).

2. Let $\mathcal{S}$ be the canonical stratification of $\mathcal{K}^{\circ c}$ and $\mathcal{S}'$ be another Whitney stratification of $\mathcal{K}^{\circ c}$. Suppose that $(\mathcal{K}^{\circ c})_j = (\mathcal{K}^{\circ c})'_j$ for all $j > i$ and $(\mathcal{K}^{\circ c})_i \subset (\mathcal{K}^{\circ c})'_i$ (notation as in appendix A.2). There are two possible cases:

   (a) The set $(\mathcal{K}^{\circ c})'_i \setminus (\mathcal{K}^{\circ c})_i$ consists of $\mathcal{S}'$-strata each of which is contained in an $\mathcal{S}$-stratum.

   (b) There is an $\mathcal{S}'$-stratum S' with $S' \cap ((\mathcal{K}^{\circ c})'_i \setminus (\mathcal{K}^{\circ c})_i) \neq \emptyset$ which intersects at least two $\mathcal{S}$-strata.

   In the first case, we modify the stratification $\mathcal{S}'$ by joining all strata together that are contained in the same $\mathcal{S}$-stratum of dimension $i+1$. This way, we obtain a new Whitney stratification $\mathcal{S}''$ with $(\mathcal{K}^{\circ c})_j = (\mathcal{K}^{\circ c})''_j$ for all $j > i-1$. If we continue with this procedure, we either eventually obtain the second case or $\mathcal{S}'$ was a refinement of $\mathcal{S}$.

   In the second case, the dimension of an $\mathcal{S}$-stratum that intersects S' is at most $i+1$ since the higher dimensional strata coincide for both stratifications. For two $(i+1)$-dimensional $\mathcal{S}$-strata form a disconnected set, S' intersects $(\mathcal{K}^{\circ c})_i$. Let $x \in S' \cap (\mathcal{K}^{\circ c})_i$ and $y \in S' \cap ((\mathcal{K}^{\circ c})'_i \setminus (\mathcal{K}^{\circ c})_i)$. By topological triviality of the stratification $\mathcal{S}'$ along the stratum S', there are homeomorphic neighbourhoods of x and y within $(\mathcal{K}^{\circ c})'_{i+1} = (\mathcal{K}^{\circ c})_{i+1}$. At $y \in (\mathcal{K}^{\circ c})_{i+1} \setminus (\mathcal{K}^{\circ c})_i$, the set $(\mathcal{K}^{\circ c})_{i+1}$ is locally a manifold. Thus, x has a neighbourhood U whose intersection with $(\mathcal{K}^{\circ c})_{i+1}$ is a topological manifold. Hence $(\mathcal{K}^{\circ c})_{i+1}$ is locally a topological manifold at any $\tilde{x} \in U \cap (\mathcal{K}^{\circ c})_i$ and we may assume w.l.o.g. that $\tilde{x}$ is contained in a stratum of $(\mathcal{K}^{\circ c})_i$ of locally maximal dimension. As shown in [PR00], the set $\mathcal{K}^{\circ c}$ is locally diffeomorphic at $\tilde{x}$ to a product of a vector space and $(\mathfrak{l}^* \oplus \mathfrak{l})^c$ at 0 for the Lie algebra $\mathfrak{l}$ of some Lie group L. For $\tilde{x}$ is contained in a stratum S of locally maximal dimension within $(\mathcal{K}^{\circ c})_i$, there is locally no stratum whose dimension is greater than $\dim S$ and less than $i+1$. The isotropy type by an element ξ of $\mathfrak{l}$ contained in a Cartan subalgebra of $\mathfrak{l}$ is determined by the number of Weyl walls of the Cartan subalgebra that contain ξ (see [BtD85, chapter V, Proposition 2.3, part (ii)]). Since $\tilde{x}$ corresponds to $(0,0) \in (\mathfrak{l}^* \oplus \mathfrak{l})^c$, $(\mathcal{K}^{\circ c})_{i+1}$ corresponds to the set $\mathcal{W}$ of pairs (μ, ξ) such that μ and ξ are both contained in 1-dimensional intersections of Weyl walls within the Lie algebra $\mathfrak{t}$ of a maximal torus $T \subset L_\xi \cap L_\mu$. We now show that $\mathcal{W} \setminus \{(0,0)\}$ is disconnected. This is a contradiction to the assumption that $(\mathcal{K}^{\circ c})_{i+1}$ was locally a manifold. To do this, we investigate the projection of $\mathcal{W}$ to $\mathfrak{l}^*$ which consists of the L-orbits of the 1-dimensional edges of Weyl walls of $\mathfrak{t}^*$. Choose some Weyl chamber C of $\mathfrak{t}^*$ and consider its 1-dimensional edges. By [BtD85, chapter V, Lemma 4.3], if $x, y \in \overline{C}$, w is a Weyl group element, and $wx = y$, then $w = 1$ holds. Hence the L-orbits of the interiors of the 1-dimensional edges of C are disjoint. Thus the projection of $\mathcal{W} \setminus \{(0,0)\}$ to $\mathfrak{l}^*$ is disconnected and so is $\mathcal{W} \setminus \{(0,0)\}$. □

By the first part of the theorem, our definition is a natural extension of the definition of Patrick and Roberts ([PR00]) to the case of possibly non-free actions. Genericity of the property that all relative equilibria are transverse within the space of invariant C^∞-functions with respect to the Whitney C^∞-topology follows immediately from the theory presented in [Bie76]. Moreover, if this genericity assumption is satisfied, the relative equilibria form a Whitney stratified subset.

Remark 6.36. Let us unravel the condition $\mathrm{d}h(x) \in \mathcal{K}^{\circ c} = \mathcal{K}^\circ \cap \mathcal{T}^\circ$ for x near p. We identify p with $[e, 0, 0]$ in the corresponding Guillemin-Sternberg normal form $Y = G \times_H (\mathfrak{m}^* \times N)$, where $H = G_p$. Since $\mathrm{d}h(x) \in \mathcal{K}^{\circ c}$ holds for x iff it holds for gx for any $g \in G$, we may assume w.l.o.g. that $x \in V = (\mathfrak{m}^* \times N)$. (Recall that to simplify the notation, the symplectic normal space N_1 of the Witt-Artin decomposition is denoted by N in the Guillemin-Sternberg normal form). The set $(\mathcal{K}^\circ \cap \mathcal{T}^\circ)_x$ is given by the forms $\mathrm{d}\mathbf{J}^\xi(x)$ with $\mathrm{d}\mathbf{J}^\xi(x) \in \mathrm{ann}(\mathfrak{g}x)$. Therefore we investigate for which $\xi \in \mathfrak{g}$ the derivative $\mathrm{d}\mathbf{J}^\xi(x)$ is contained in $\mathrm{ann}(\mathfrak{g}x)$. The momentum map is given by

$$\mathbf{J} = \mathbf{J}_Y : G \times_H (\mathfrak{m}^* \times N) \to \mathfrak{g}^* \tag{6.1}$$

$$[g, \rho, \nu] \mapsto \mathrm{Coad}_g(\mu + \rho + \mathbf{J}_N(\nu)), \tag{6.2}$$

where $\mathbf{J}_N$ is the momentum map on N. Hence, for a point $x = [e, \rho, \nu]$ of $V = \mathfrak{m}^* \times N$, the derivative of $\mathbf{J}$ in the direction of the group orbit is equal to

$$\mathrm{d}\mathbf{J}(x)\eta_x = \mathrm{coad}_\eta(\mu + \rho + \mathbf{J}_N(\nu))$$

for any $\eta \in \mathfrak{g}$. Thus,

$$\mathrm{d}\mathbf{J}^\xi(x)\eta_x = (\mu + \rho + \mathbf{J}_N(\nu))(-[\eta, \xi]) = -\,\mathrm{coad}_\xi(\mu + \rho + \mathbf{J}_N(\nu))(\eta).$$

Therefore, $\mathrm{d}\mathbf{J}^\xi(x)$ vanishes on $\mathfrak{g} \cdot x$ iff

$$\mathrm{coad}_\xi(\mu + \rho + \mathbf{J}_N(\nu)) = 0. \tag{6.3}$$

If x is close to p, $\mu' := \mathbf{J}(x) = \mu + \rho + \mathbf{J}_N(\nu)$ is close to μ, and hence we may assume that $G_{\mu'} \subset G_\mu$. Then equation (6.3) is equivalent to $\xi \in \mathfrak{g}_\mu$ and

$$\mathrm{coad}_\xi(\rho + \mathbf{J}_N(\nu)) = 0. \tag{6.4}$$

Next, we assume that $\mathrm{d}\mathbf{J}^\xi(x) \in \operatorname{ann}(\mathfrak{g}\cdot p)$ and investigate the condition $\mathrm{d}h(x) = \mathrm{d}\mathbf{J}^\xi(x)$. Since $\mathrm{d}h(x) \in \operatorname{ann}(\mathfrak{g}\cdot p)$, this is true iff it holds for the restrictions $\bar{h}$ and $\bar{\mathbf{J}}^\xi$ to $V = \mathfrak{m}^* \times N$, i.e.

$$\mathrm{d}_{\mathfrak{m}^*}\bar{h}(\rho,\nu) + \mathrm{d}_N\bar{h}(\rho,\nu) = \mathbb{P}_{\mathfrak{m}}\xi + \langle \mathrm{d}_N\mathbf{J}_N(\nu)\cdot, \mathbb{P}_{\mathfrak{h}}\xi\rangle,$$

where $\mathbb{P}_{\mathfrak{m}}$ and $\mathbb{P}_{\mathfrak{h}}\xi$ denote the projections to $\mathfrak{m}$ and $\mathfrak{h}$ respectively. Since $\xi \in \mathfrak{g}_\mu$, this yields $P_{\mathfrak{m}}\xi = \mathrm{d}_{\mathfrak{m}^*}\bar{h}(\rho,\nu)$ and for $\eta := P_{\mathfrak{h}^*}\xi$

$$\mathrm{d}_N\bar{h}(\rho,\nu) - \mathrm{d}_N\mathbf{J}_N^\eta(\nu) = 0.$$

Thus we obtain the commutation equation (2.14) and the symplectic slice equation (2.15), which characterize relative equilibria.

Now, we keep the notation of remark 6.36 and suppose in addition, that p is a relative equilibrium. As illustrated in section 6.1.2, p is a transverse relative equilibrium if $\mathrm{d}h(p) \in \mathcal{K}^{\circ c} = \mathcal{K}^\circ \cap \mathcal{T}^\circ$ and the restriction $h\big|_V$ is H-transverse to the projection of $\mathcal{K}^{\circ c}$ to T^*V, which coincides with the intersection $\mathcal{K}^{\circ c} \cap T^*V$.

The structure of this set may be complicated in general, but the intersection with T^*V_H is easier to analyse:

As argued above, for $(\rho,\nu) \in V = \mathfrak{m}^* \oplus N$, we have that

$$\mathcal{K}^{\circ c}_{(\rho,\nu)} = \bigcup_{\xi \in (\mathfrak{g}_\mu)_{\mathbf{J}(\rho,\nu)}} \mathrm{d}\mathbf{J}^\xi(\rho,\nu).$$

Recall that

$$\mathbf{J}(\rho,\nu) = \mathbf{J}_Y([e,\rho,\nu]) = \rho + \mathbf{J}_N(\nu).$$

Lemma 6.37. *The momentum map $\mathbf{J}_N : N \to \mathfrak{h}^*$ vanishes on N^H.*

Proof. In general, for a Hamiltonian G-action on a phase space P with momentum map $\mathbf{J} : P \to \mathfrak{g}^*$, the kernel of $\mathrm{d}\mathbf{J}(p)$ is given by $(\mathfrak{g}\cdot p)^{\perp_\omega}$ for $p \in P$. In particular, $\mathrm{d}\mathbf{J}(p) = 0$ if $p \in P^G$. Hence $\mathbf{J}$ is constant on P^G. On the representation N, we choose $\mathbf{J}_N(0) = 0$ (otherwise equation (6.1) for the momentum on the model space $G \times_H (\mathfrak{m}^* \oplus N)$ would contradict the requirement $\mathbf{J}(e,0,0) = \mu$). □

Let $\mathfrak{n}(H)$ be the Lie algebra of the normalizer $N(H)$ within G, $\mathfrak{c}(H)$ be the Lie algebra of the centralizer $C(H)$ within G, $\mathfrak{n}$ be the Lie algebra of $\bar{N}(H) = (N(H)/H)^\circ$, and $\mathfrak{n}_\mu$ be the Lie algebra of $(\bar{N}(H))_\mu = N_{G_\mu}(H)/H$.

Lemma 6.38. *If $\mathfrak{m}$ is an $N_{G_\mu}(H)$-invariant complement of $\mathfrak{h}$ within $\mathfrak{g}_\mu$ (and hence the corresponding splitting $\mathfrak{g}_\mu^* = \mathfrak{m}^* \oplus \mathfrak{h}^*$ is $N_{G_\mu}((H)$-invariant), then $\mathfrak{m}^H \simeq \mathfrak{n}_\mu$ and $(\mathfrak{m}^*)^H \simeq \mathfrak{n}_\mu^*$ as $\bar{N}(H)_\mu$-spaces.*

Proof. Since we may identify $\mathfrak{g}$ and $\mathfrak{g}^*$ via a G-invariant inner product, we only have to show $\mathfrak{m}^H \simeq \mathfrak{n}_\mu$. Since $\mathfrak{m}$ is $N(H)$-invariant, $\mathfrak{m} \simeq \mathfrak{g}_\mu/\mathfrak{h}$ as $N(H)$-spaces. We show $(\mathfrak{g}/\mathfrak{h})^H = \mathfrak{n}$. Then the restriction to the subgroup G_μ yields the required result. We have

$$[\xi] \in (\mathfrak{g}/\mathfrak{h})^H \Leftrightarrow \mathrm{Ad}_h\,\xi - \xi \in \mathfrak{h} \quad \forall h \in H.$$

The right hand side is equivalent to $\xi \in \mathfrak{n}(H)$, as can be shown using the Baker-Campbell-Hausdorff-formula, see [OR04(b), Lemma 2.1.13]. Alternatively, we may use the fact that $\mathfrak{n}(H) = \mathfrak{h} + \mathfrak{c}(H)$ (for a proof, see for example [Fie07, Corollary 3.10.1]): The right hand side is obviously true for $\xi \in \mathfrak{h} + \mathfrak{c}(H)$. On the contrary, suppose that the right hand side holds for $\xi \in \mathfrak{g}$ and choose an H-invariant complement $\mathfrak{l}$ of $\mathfrak{h}$. If $\xi = \eta + \lambda$, $\eta \in \mathfrak{h}$, and $\lambda \in \mathfrak{l}$, then for any $h \in \mathfrak{h}$, the element $\mathrm{Ad}_h \lambda - \lambda$ is contained in $\mathfrak{l} \cap \mathfrak{h} = \{0\}$. Thus, $\xi \in \mathfrak{h} + \mathfrak{c}(H)$. □

Lemma 6.38 shows that

$$T^*V^H = (N^H \oplus (\mathfrak{m}^*)^H) \oplus ((N^H)^* \oplus \mathfrak{m}^H) = N^H \oplus \mathfrak{n}_\mu^* \oplus (N^H)^* \oplus \mathfrak{n}_\mu.$$

For $(\rho, \nu) \in V^H = \mathfrak{n}_\nu^* \oplus N^H$, we have $\mathbf{J}(\rho, \nu) = \rho$ and thus $\mathrm{d}\mathbf{J}(\rho, \nu)\big|_{V^H}$ coincides with the projection to $\mathfrak{n}_\nu^*$. Hence $\langle \mathrm{d}\mathbf{J}(\rho, \nu)\big|_{V^H}, \xi \rangle$ only depends on the $\mathfrak{n}_\nu$-component $\xi_{\mathfrak{n}_\nu}$ of ξ for any

$$\xi \in \mathfrak{g}_\mu = \mathfrak{m}^H \oplus (\mathfrak{m}^H)^\perp = \mathfrak{n}_\mu \oplus \mathfrak{n}_\mu^\perp$$

with respect to an $N(H)$-invariant inner product on $\mathfrak{g}_\mu$. Indeed,

$$\mathrm{d}\mathbf{J}^\xi(\rho, \nu)\big|_{V^H}(\dot{\rho}, \dot{\nu}) = \dot{\rho}(\xi_{\mathfrak{n}_\nu}).$$

Hence the subspace $\mathcal{K}^{\circ c}_{(\rho,\nu)} \cap T^*_{(\rho,\nu)}(V^H)$ is given by $(\mathfrak{n}_\nu)_\rho \subset N^* \oplus \mathfrak{n}_\nu = T^*_{(\rho,\nu)}(V^H)$. Thus,

$$(\mathcal{K}^{\circ c}\big|_{T^*V_H}) = (N^H \oplus (\mathfrak{n}_\mu^* \oplus \mathfrak{n}_\mu)^c)_H \subset T^*V_H. \tag{6.5}$$

Let $\Sigma^* \subset V \times \mathbb{R}^k$ be the preimage of $\mathcal{K}^{\circ c}\big|_{T^*V}$ under U_1 (where we omit the factor $\mathbb{R}$). For an isotropy subgroup $K \subset (\bar{N}(H))_\mu$ of $(\mathfrak{n}_\mu^* \oplus \mathfrak{n}_\mu)^c$, let $(\mathcal{K}^{\circ c}\big|_{T^*V_H})_{(K)}$ be the subset $\mathcal{K}^{\circ c}_{(K)}$ with respect this action and $(\Sigma^*)^H_{(K)}$ be its preimage under U_1.

Let $p_1, \dots, p_l$ be a minimal set of homogeneous generators of $P(V)^H$.

We now obtain a Hamiltonian analogue to [Fie07, Lemma 6.9.2]:

Theorem 6.39. *The set $(\Sigma^*)^H$ is a Whitney stratified subset with strata $(\Sigma^*)^H_{(K)}$ of dimension $2\dim Z(K) - \dim K + l$, where $Z(K)$ is the centralizer of K. If $\mathfrak{l}$ is a K-invariant complement of the Lie algebra $\mathfrak{z}$ of $Z(K)$ within the Lie algebra $\mathfrak{k}$ of K, the set $(\Sigma^*)^H$ has singularity type $(\mathfrak{l}^* \oplus \mathfrak{l})^c$ at any point of $(\Sigma^*)^H_{(K)}$.*

Proof. The map $U_1 : V \times \mathbb{R}^l \to T^*V = V \times V^*$ is given by

$$(x, t) \mapsto (x, \sum_{i=1}^{l} t_i \mathrm{d}p_i(x)).$$

Since the functions p_i are H-invariant, $\mathrm{d}p_i(x) \in (V^*)^H$ for $x \in V^H$ with respect to the dual H-action on V^*. Therefore U_1 maps $V^H \times \mathbb{R}^l$ to T^*V^H. Moreover, the map $U_1 : V^H \times \mathbb{R}^l \to T^*V^H$ is a submersion: For any $x \in V^H$ and $\alpha \in (V^*)^H$, $\alpha(\cdot)$ is an H-invariant function and hence $\alpha = a \circ P$ for some function $a : \mathbb{R}^l \to \mathbb{R}$. Thus,

$$\alpha = \mathrm{d}\alpha(x) = \mathrm{d}a(P(x)) \circ \mathrm{d}P(x) = \sum_{i=1}^{l} \partial_i a(P(x)) \mathrm{d}p_i(x).$$

By equation (6.5) and the theory presented in [PR00], $(\mathcal{K}^{\circ c}|_{T^*V_H})$ is a Whitney stratified set with strata $(\mathcal{K}^{\circ c}|_{T^*V_H})_{(K)}$ of dimension

$$\dim N^H + \dim(\bar{N}(H)_\mu) + 2\dim Z(K) - \dim K = \dim V^H + 2\dim Z(K) - \dim K.$$

Since U_1 is a submersion, its preimage $(\Sigma^*)^H_{(K)}$ is a submanifold of the same codimension $\dim V^H - 2\dim Z(K) + \dim K$, hence

$$\dim(\Sigma^*)^H_{(K)} = 2\dim Z(K) - \dim K + l.$$

Moreover, since the singularity type of $(\mathcal{K}^{\circ c}|_V)^H$ at a point of $(\mathcal{K}^{\circ c}|_{T^*V})^H_{(K)}$ is given by $(\mathfrak{l}^* \oplus \mathfrak{l})^c$, this is also true for the preimages. □

The next result is completely analogous to the first part of [Fie07, Theorem 6.10.1]: Let $\mathcal{S}$ be the canonical stratification of Σ^*. By uniqueness of the canonical stratification, every $S \in \mathcal{S}$ is G-invariant. Thus for any isotropy type of the H-representation V, the subset $S_\tau = S \cap \Sigma^*_\tau$ is an isotropy subspace of the G-space S. Hence S_τ is a manifold. Let $\mathcal{S}_\tau$ denote the stratification of Σ^*_τ consisting of the sets S_τ for $S \in \mathcal{S}$.

Theorem 6.40. *For every isotropy type τ of the H-representation T^*V, $\mathcal{S}_\tau$ is a Whitney stratification of Σ^*_τ.*

Since we will only need the case $\tau = (H)$ of the theorem, we will only prove this case. To do this, we just reproduce the proof of Field ([Fie07]). The proof of the general case is similar.

Proof of Theorem 6.40 for $\tau = (H)$. Suppose $S, T \in \mathcal{S}_{(H)}$ and $x \in \partial S \cap T$. We have to show Whitney regularity: Let $(p_n) \subset S$ and $(q_n) \subset T$ be sequences converging to x such that $T_{p_n}S$ converges to a $\dim(S)$-dimensional subspace E of $V \times \mathbb{R}^l$ and $\mathbb{R}(p_n - q_n)$ converges to a line L. We show $L \subset E$: Suppose $S = \tilde{S} \cap (V^H \times \mathbb{R}^l)$ and $T = \tilde{T} \cap (V^H \times \mathbb{R}^l)$ for some $\tilde{S}, \tilde{T} \in \mathcal{S}$. There is a subsequence of $T_{p_n}\tilde{S}$ that converges to a $\dim \tilde{S}$-dimensional subspace $\tilde{E}$ of $V \times \mathbb{R}^l$. Since $\mathcal{S}$ is a Whitney stratification, $L \subset \tilde{E}$. If $V = W \oplus V^H$ is an H-invariant splitting and $\pi : (W \oplus V^H) \times \mathbb{R}^l \to V^H \times \mathbb{R}^l$ is the projection, π maps any H-invariant subspace to its intersection with $V^H \times \mathbb{R}^l$. By H-invariance of $\tilde{S}$, the spaces $T_{p_n}\tilde{S}$ are H-invariant. Thus $\pi(T_{p_n}\tilde{S}) = T_{p_n}S$ and hence $E = \pi(\tilde{E}) = \tilde{E} \cap (V^H \times \mathbb{R}^l)$. $L \subset V^H \times \mathbb{R}^l$ yields $L \subset E$. □

Let $h = h(p_1, \dots, p_l)$ be the H-invariant restriction of the Hamiltonian function to V. The relative equilibrium p is transverse if the map

$$\begin{aligned} \Gamma_h : V &\to V \times \mathbb{R}^l \\ x &\mapsto (x, \partial_1 h(P(x)), \dots, \partial_l h(P(x)) \end{aligned}$$

is transverse to Σ^* in $0 \in V$. As discussed, we may suppose $V^H = 0$ by considering

$$\begin{aligned} \Gamma_h : V \times \mathbb{R}^s &\to V \times \mathbb{R}^l \\ (x, s) &\mapsto (x, \partial_1 h(P(x, s)), \dots, \partial_l h(P(x, s)). \end{aligned}$$

H-equivariant transversality in $(0,s)$ may also be described in terms of the map

$$\begin{aligned}\gamma_h : \mathbb{R}^s &\to \mathbb{R}^l \\ s &\mapsto (\partial_1 h(P(0,s)), \ldots, \partial_l h(P(0,s))).\end{aligned}$$

As in [Fie07], we obtain:

Lemma 6.41. *Γ_h is transverse to Σ^* in $(0,s) \in V \times \mathbb{R}^s$ iff γ_h is transverse to $\Sigma^*_{(H)} = (\Sigma^*)^H$ with respect to the stratification $\mathcal{S}_{(H)}$ in $s \in \mathbb{R}^s$.*

Proof. Suppose $\Gamma_h(0,s) = y \in S \in \mathcal{S}$.

Since $\mathrm{d}_x P(\cdot, s) : V \to V^*$ is H-equivariant, $\mathrm{d}_x P(0,s) = 0$ and hence the image of the x-derivative of Γ_h in $(0,s)$ is equal to V. Thus Γ_h is transverse to S in $(0,s)$ iff $\mathrm{d}\gamma_h(s)\mathbb{R}^s + (T_y S \cap \mathbb{R}^l) = \mathbb{R}^l$. By the H-invariance of S, we have $T_y S \cap \mathbb{R}^l = T_y(S^H)$, see the proof of Theorem 6.40. □

For $\tau = H$, set

$$\begin{aligned} g_\tau &= \dim G - \dim H = \dim {}^{G}\!/_{H} , \\ n_\tau &= \dim \bar{N}(H) = \dim {}^{N(H)}\!/_{H} . \end{aligned}$$

Again, $\mathcal{E}$ denotes the set of relative equilibria. If p is a relative equilibrium with isotropy group $G_p = H$, its generator may naturally be considered as an element in $\mathfrak{n}$, this way, it is unique. Lemma 6.37 and Lemma 6.38 show that its momentum may also be considered as an element of $\mathfrak{n}^*$. Moreover, at least locally μ is also naturally an element of the quotient by $\mathfrak{h}$: In a tubular neighbourhood $G \times_H V$ of p, the momentum map might be modified by adding an element of $\mathfrak{h}^*$ to the restriction to V and G-equivariant extension such that the resulting map is still equivariant.

Let $\mathcal{E}_{H,(K)}$ be the set of relative equilibria with isotropy subgroup H and isotropy type (K) of the momentum generator pair, where $K \subset \bar{N}(H)$. Moreover, for $\tau = (H)$, let $\mathcal{E}_{\tau,(K)}$ denote the G-orbit of $\mathcal{E}_{H,(K)}$. Similarly, let $\mathcal{K}^{\circ c}_{H,(K)} \subset \mathcal{K}^{\circ c}$ be the set of pairs $(x, \xi \cdot x)$ with $G_x = H$ and $(\bar{N}(H)_\xi \cap \bar{N}(H)_{\mathbf{J}(x)}) = (K)$, and let $\mathcal{K}^{\circ c}_{\tau,(K)}$ be its G-orbit.

Theorem 6.35, Theorem 6.39, Theorem 6.40, and Lemma 6.41 yield together

Theorem 6.42. *If $p \in P$ is a transverse relative equilibrium of type (K), $K \subset {}^{N(H)}\!/_{H}$, the sets $\mathcal{E}_{H,(K)}$ and $\mathcal{E}_{\tau,(K)}$ form smooth manifolds of dimension $n_\tau + 2\dim Z(K) - \dim K$ and dimension $g_\tau + 2\dim Z(K) - \dim K$ respectively in a neighbourhood of p. Moreover, the sets $\mathcal{E}^H$ and $\mathcal{E}_{(\tau)}$ have singularity type $(\mathfrak{l}^* \oplus \mathfrak{l})^c$ at p.*

Proof. By Lemma 6.41, the restriction of Γ_h to the H-isotropy subspace is transverse to $\Sigma^*_{(H)}$ with respect to the stratification $\mathcal{S}_{(H)}$. By Theorem 6.40, $\mathcal{S}_{(H)}$ is a Whitney stratification of $\Sigma^*_{(H)}$. As we have shown in the proof of Theorem 6.40, $\Sigma^*_{(H)}$ is the preimage of $(\mathcal{K}^{\circ c}\big|_{T^*V})^H$ under a submersion. Hence, $\Sigma^*_{(H)}$ is locally isomorphic to a product of a vector space and $\mathcal{K}^{\circ c}_{(H)}$. By Theorem 6.35, any Whitney stratification of $\mathcal{K}^{\circ c}_{(H)}$ is a refinement of the stratification into the strata $(\mathcal{K}^{\circ c})^H_{(K)}$ for isotropy subgroups $K \subset \bar{N}(H)$ of the $\bar{N}(H)$-representation $\mathfrak{n}^* \oplus \mathfrak{n}$. Similarly, any Whitney stratification of $(\Sigma^*)^H$ is a refinement of the stratification into the manifolds $(\Sigma^*)^H_{(K)}$. In particular, Γ_h is transverse to the strata $(\Sigma^*)^H_{(K)}$. Hence, $\mathcal{E} \cap V^H$ has singularity type $(\mathfrak{l}^* \oplus \mathfrak{l})^c$ at p and the same is true for the set of relative equilibria in a neighbourhood in $G \times_H V^H$ of p in P_τ. Furthermore, the intersection $\mathcal{E}_{H,(K)} \cap V_H$ is a smooth manifold of dimension $2 \dim Z(K) - \dim K$. Since

$$\mathcal{E}_{H,(K)} \simeq N(H) \times_H (\mathcal{E}_{H,(K)} \cap V^H)$$

and

$$\mathcal{E}_{\tau,(K)} \simeq G \times_H (\mathcal{E}_{H,(K)} \cap V^H),$$

both are smooth manifolds and

$$\begin{aligned} \dim \mathcal{E}_{H,(K)} &= n_\tau + 2 \dim Z(K) - \dim K, \\ \dim \mathcal{E}_{\tau,(K)} &= g_\tau + 2 \dim Z(K) - \dim K. \end{aligned}$$

□

Remark 6.43. Note that near a transverse relative equilibrium, all relative equilibria are transverse, even those of a different isotropy type.

Theorem 6.42 shows that the conjugacy class of the pair $(H, \bar{N}(H)_\xi \cap \bar{N}(H)_\mu)$ contains important information about the local structure of relative equilibria near a relative equilibrium p with generator $\xi \in \mathfrak{n}$ and momentum $\mu \in \mathfrak{n}^*$.

Definition 6.44. Two relative equilibria p and q are of the same *type* iff $G_p = gG_qg^{-1}$ for some $g \in G$ and the isomorphism $N(G_p) \simeq N(G_q)$ that is induced by the conjugation with g identifies the isotropy subgroups $K \subset \bar{N}(G_p)$ and $L \subset \bar{N}(G_q)$ of the momentum generator pairs.

Thus, each of the sets $\mathcal{E}_{\tau,(K)}$ is a set of relative equilibria of the same type.

Remark 6.45. There is another apparent generalization of the definition of transverse relative equilibria for free actions to the case of non-trivial isotropy subgroups using equivariant transversality theory: As shown in Lemma 4.7, in the free case, transversality of a relative equilibrium p with generator ξ is equivalent to the map $\psi^\circ_{(K)} : (P \times \mathfrak{g})^c_{(K)} \to \mathcal{T}^\circ$ being transverse to the zero section of $\mathcal{T}^\circ$ at (p, ξ), where $\psi^\circ_{(K)} = \psi^\circ\big|_{(P\times\mathfrak{g})^c_{(K)}}$ and $\psi^\circ(x, \eta) = \mathrm{d}h_\eta(x)$. This is equivalent to G-1-jet-transversality of

$$\begin{aligned} \Psi : P \times \mathfrak{g} &\to \mathbb{R} \\ (x, \eta) &\mapsto h_\eta(x) \end{aligned}$$

to the product of $\mathbb{R}$ and zero section Q of $T^*P \times \mathfrak{g}$, where we consider $T^*P \times \mathfrak{g}$ as a subbundle of $T^*(P \times \mathfrak{g})$ over $P \times \mathfrak{g}$: The G-action on $P \times \mathfrak{g}$ is given by the product of the G-action on P and the adjoint action on $\mathfrak{g}$. Since P is a free G-space, this is also true for $P \times \mathfrak{g}$. Ψ is a G-invariant function with respect to this action.

If V is a slice at p for the G-action on P, $V \times \mathfrak{g}$ forms a slice at (p, ξ) transverse to the G-orbit $G(p, \xi)$.

As we have seen in Remark 6.25, G-1-jet-transversality of Ψ to $\mathbb{R} \times Q$ is equivalent to transversality of the derivative of the restriction $\Psi\big|_{V\times\mathfrak{g}}$ to the intersection $Q \cap T^*(V \times \mathfrak{g})$, i.e. the subset of Q consisting of cotangent vectors that vanish in the direction of the G-orbit. A cotangent vector $\mathrm{d}f_{(p,\xi)}$ is an element of $T^*(V \times \mathfrak{g})$ iff

$$\mathrm{d}f_{(p,\xi)}(\eta \cdot p, [\eta, \xi]) = 0 \quad \forall \eta \in \mathfrak{g}.$$

Moreover, $\mathrm{d}f_{(p,\xi)} \in Q$ implies

$$\mathrm{d}f_{(p,\xi)}(\eta \cdot p, 0) = 0 \quad \forall \eta \in \mathfrak{g}.$$

Hence $\mathrm{d}f_{(p,\xi)} \in Q \cap T^*(V \times \mathfrak{g})$ is equivalent to

$$\mathrm{coad}_\xi \, \mathrm{d}_\mathfrak{g} f_{(p,\xi)} = 0$$

and

$$\mathrm{d}_P f_{(p,\xi)} = 0.$$

Thus

$$Q \cap T^*(V \times \mathfrak{g}) = (V \oplus \{0\}) \times (\mathfrak{g} \oplus \mathfrak{g}^*)^c \subset (V \oplus V^*) \times (\mathfrak{g} \oplus \mathfrak{g}^*) = T^*(V \times \mathfrak{g}).$$

Suppose that p is a relative equilibrium with generator $\xi \in \mathfrak{g}$ and momentum μ and $G_\mu \cap G_\xi = K$. The map Ψ is G-1-jet-transverse to $\mathbb{R} \times Q$ at (p, ξ) iff $\mathrm{d}(\Psi\big|_{V \times \mathfrak{g}})$ is transverse to $(V \oplus \{0\}) \times (\mathfrak{g} \oplus \mathfrak{g}^*)^c_{(K)}$ at (p, ξ). Since $\mathrm{d}_\mathfrak{g} \Psi_{(x,\xi)} = \mathbf{J}(x)$ and $\mathrm{d}\mathbf{J}(p)\big|_V$ is a submersion to $\mathfrak{g}^*$, this is equivalent to the transversality of the restriction $\mathrm{d}_V \Psi\big|_{(V \times \mathfrak{g})^c_{(K)}}$ to $V \oplus \{0\}$, where $(V \times \mathfrak{g})_{(K)} := (P \times \mathfrak{g})^c_{(K)} \cap (V \times \mathfrak{g})$. By G-invariance of ψ, equivalently $\psi^\circ_{(K)} = \mathrm{d}_P \Psi\big|_{(P \times \mathfrak{g})^c_{(K)}}$ is transverse to the zero section of $\mathcal{T}^\circ$.

In the next section, we use this generalization to investigate relative equilibria near 0 in symplectic representations of connected compact groups (but we restrict the action to the action of the maximal torus). It might be interesting to analyse how both generalizations are related. Anyway, both transversality properties are generic in the class of invariant Hamiltonian functions.

6.4 Representations

We now consider the case of a symplectic representation of a connected compact group and investigate the structure of the relative equilibria near an equilibrium at 0. Understanding this special case may also be helpful for the theory of bifurcations of Hamiltonian relative equilibria in general, since it is related to the symplectic slice equation (2.15): For a fixed $\rho \in \mathfrak{m}^{G_p}$, the G_p-relative equilibria for the Hamiltonian system for $\bar{h}(\rho, \cdot)$ on the slice N at a relative equilibrium p correspond to solutions of equation (2.15).

As we have seen in the case of groups of rank 1, it can be fruitful to consider the action of the maximal torus, since the adjoint representation of tori is trivial. Therefore we start with torus representations and deduce implications for general connected compact groups afterwards.

For rank ≥ 2, the structure of generators which possibly admit bifurcations is more complex and we will need equivariant transversality theory to handle this issue.

6.4.1 Torus representations

Since symplectic representations admit a complex structure that commutes with the group action, we can consider them as complex representations. Then symplectic subrepresentations correspond to complex subrepresentations. The irreducible complex representations of a torus $T = T^n := \mathbb{R}^n/\mathbb{Z}^n$ are of dimension 1 and are determined by the elements $\alpha \in \mathfrak{t}^*$ that map $\mathbb{Z}^n$ to $\mathbb{Z}$. These elements are called *integral forms.* Under the identification $\mathfrak{t}^* \cong \mathfrak{t} = \mathbb{R}^n$ via an inner product, the set of integral forms is given by $\mathbb{Z}^n$. An integral form α defines an irreducible representation on $\mathbb{C}$ by

$$\theta_\alpha : \exp(\xi) \mapsto \mathrm{e}^{2\pi \mathrm{i}\alpha(\xi)} \quad \xi \in \mathfrak{t},$$

here the exponential map $\exp : \mathfrak{t} \to T$ coincides with the projection $\mathbb{R}^n \to \mathbb{R}^n/\mathbb{Z}^n$. We denote this representation by $\mathbb{C}_\alpha$ and consider it as a symplectic representation on $\mathbb{R}^2$ with the symplectic form $\omega = \langle\cdot, \mathrm{i}\cdot\rangle$, where $\langle\cdot,\cdot\rangle$ is the standard (real) inner product on $\mathbb{R}^2$. For the momentum map on $\mathbb{C}_\alpha$, we compute:

$$-\mathrm{d}\mathbf{J}^\xi(x) = \omega(\cdot, (\xi \cdot x)) = \langle\cdot, (\mathrm{i} \cdot 2\pi\mathrm{i}\alpha(\xi)x)\rangle = -2\pi\alpha(\xi)\langle\cdot, x\rangle.$$

Hence $\mathbf{J}(x) = \pi|x|^2\alpha$ is a momentum map, which is obviously equivariant with respect to the (trivial) coadjoint action.

Now let V be a symplectic T-representation. The integral forms α that correspond to the irreducible components of V are called the *(infinitesimal) weights* of V. (The infinitesimal weight α defines the global weight θ_α. In the following, a *weight* is an infinitesimal weight.) Our goal is to understand the structure of the relative equilibria for a generic T-invariant Hamiltonian h. We start with some simple examples:

Example 6.46. If V is irreducible and hence $V = \mathbb{C}_\alpha$, every point of V is a relative equilibrium, since the level sets of the momentum map are the circles centred at the origin and hence coincide with the T-orbits. (Alternatively, note that for $x \in V$, the $h(x)$-level set of h consists of the norm circle of x or $\nabla h(x) = 0$.)

Example 6.47. The same holds for a sum $\mathbb{C}_\alpha \oplus \mathbb{C}_\beta$ with α and β linearly independent or more generally for sums $\bigoplus \mathbb{C}_{\alpha_i}$ for linearly independent α_i: Since for $x = \sum x_{\alpha_i} \in \bigoplus \mathbb{C}_{\alpha_i}$

$$\mathbf{J}(x) = \sum_i \mathbf{J}_{\mathbb{C}_{\alpha_i}}(x_{\alpha_i}) = \pi|x_{\alpha_i}|^2\alpha_i,$$

the preimages $\mathbf{J}^{-1}(\mu)$ and the group orbits coincide and are given by the tori that are products of norm spheres in the $\mathbb{C}_{\alpha_i}$.

Since in these examples the set of relative equilibria does not even depend on the Hamiltonian, the set seems to be rather stable.

We now investigate the set of relative equilibria near an equilibrium at the origin of a general T-symplectic representation. By the implicit function theorem, a necessary condition for the occurrence of relative equilibria near the origin with generators ξ near some given $\xi_0 \in \mathfrak{t}$ is that $V_0 := \ker \mathrm{d}^2(h - \mathbf{J}^{\xi_0})(0)$ is non-trivial.

Suppose now that V_0 has only linearly independent weights. Splitting Lemma reduction yields a T-invariant germ $g : V_0 \times \mathfrak{t} \to \mathbb{R}$ at $(0, \xi_0)$ whose Taylor expansion at $(0, \xi_0)$ coincides with that of $\mathcal{H}(v_0, \xi) := h - \mathbf{J}^\xi(v_0)$ up to third order (see section 2.4). If $\mathrm{d}_{V_0} g(v_0, \xi) = 0$, v_0 corresponds to a relative equilibrium with generator v_0. This leads to the conjecture that there is a branch of relative equilibria tangent to V_0.

Remark 6.48. The conjecture is supported by the following observation: Factor out the kernel of the torus action on $V_0 := \bigoplus \mathbb{C}_{\alpha_i}$ with $\alpha_1, \ldots, \alpha_n$ linearly independent to obtain a faithful action of $T := T^n$. We consider the Hamiltonian system on V_0 with Hamiltonian function $h\big|_{V_0}$. For each $x = \sum_i x_{\alpha_i} \in \bigoplus \mathbb{C}_{\alpha_i}$ in which no component x_{α_i} vanishes, the isotropy subgroup T_x is trivial. Hence there is a unique generator $\xi(x) \in \mathfrak{t}$ of the relative equilibrium x. Since $\mathcal{H}\big|_{V_0 \times \mathfrak{t}}$ is a T-invariant function, $\mathrm{d}_{V_0}\mathcal{H}(x, \xi)$ is contained in the annihilator $\mathrm{ann}(\mathfrak{t}x)$ of $\mathfrak{t}x$. The ξ-derivative of $\mathrm{d}_{V_0}\mathcal{H}(x, \xi)$ is given by

$$\begin{aligned} \mathrm{d}_\xi \mathrm{d}_{V_0}\mathcal{H}(x, \xi) = \mathrm{d}_\xi \mathrm{d}_{V_0}\mathbf{J}^\xi(x) : \mathfrak{t} &\to \mathrm{ann}(\mathfrak{t}x) \subset V_0^* \\ \eta &\mapsto \sum_{i=1}^n 2\pi\alpha_i(\eta)\langle \cdot, x_{\alpha_i}\rangle. \end{aligned}$$

Since $\mathrm{ann}(\mathfrak{t}x)$ is n-dimensional and $\alpha_1, \ldots, \alpha_n$ are linearly independent,

$$\mathrm{d}_\xi \mathrm{d}_{V_0}\mathcal{H}(x, \xi) : \mathfrak{t} \to \mathrm{ann}(\mathfrak{t}x)$$

is invertible, in particular for $\xi = \xi(x)$. If we consider the restrictions of C^1-maps

$$A : V_0 \times \mathfrak{t} \to \mathrm{ann}(\mathfrak{t}x)$$

to some compact neighbourhood of $(x, \xi(x))$ together with the C^1-norm, we obtain a Banach space and the map $(A, v_0, \xi) \mapsto A(v_0, \xi)$ is C^1. By the implicit function theorem, for A close enough to $\mathrm{d}_{V_0}\mathcal{H}$, there is a unique $\xi = \xi(A, x)$ with $A(\xi, x) = 0$. Since the Splitting Lemma reduction yields a local map $g : V_0 \times \mathfrak{t}, (0, \xi_0) \to \mathbb{R}$ that is in some sense close to $\mathcal{H}$ near $(0, x_0)$, there is hope that we obtain a manifold of relative equilibria tangent to V_0. Anyway, this heuristic reasoning is by no means a proof, since we do not know if $\mathrm{d}_{V_0} g$ is close enough to $\mathrm{d}_{V_0}\mathcal{H}$ at any point with trivial isotropy. g is only locally defined near the singular point $(0, x_0)$. Equivariant transversality theory is designed for handling difficulties of such kind and will be our way to solve this problem.

Before proving the conjecture, we will survey the set of those $\xi \in \mathfrak{t}$ with singular Hessian $\mathrm{d}^2(h - \mathbf{J}^\xi)(0)$:

By Lemma 5.11, we may consider the restriction to the centre space $\mathbb{E}_\mathrm{c}$ of $\mathrm{d}X_h(0)$. By Theorem 3.14, generically $\mathbb{E}_\mathrm{c}$ splits into irreducible T-symplectic representations, each of which is the real part of the sum of the eigenspaces corresponding to a pair of purely imaginary eigenvalues. By Corollary 3.16, then there is an appropriate choice of an inner product of $\mathbb{E}_\mathrm{c}$ such that the eigenspaces of the restriction of $\mathrm{d}^2 h(0)$ are irreducible symplectic representations and consequently are given by weight spaces.

Since we will often suppose this condition on the centre space, we give it a name:

Definition 6.49. The G-invariant Hamiltonian function $h : V \to \mathbb{R}$ satisfies the *generic centre space condition* (GC) iff $\mathrm{d}X_h(0)$ is non-degenerate and $\mathrm{d}X_h(0) \in \mathcal{O}$ as in Theorem 3.14.

We now assume that (GC) is satisfied, where $G = T$. Let c_i, $i = 1, \dots, n$ denote the eigenvalues of $\mathrm{d}^2 h(0)$ with corresponding eigenspaces $\mathbb{C}_{\alpha_i}$. Since on $\mathbb{C}_{\alpha_i}$ the matrix that represents $\mathrm{d}^2\mathbf{J}^\xi(0)$ is equal to $2\pi\alpha_i(\xi)\mathbb{1}$, the Hessian $\mathrm{d}^2(h - \mathbf{J}^\xi)(0)$ is singular iff ξ solves at least one of the equations

$$c_i - 2\pi\alpha_i(\xi) = 0.$$

Equivalently, ξ is a zero of the product of the left-hand sides. The solution sets of these equations form affine hyperplanes that are parallel to the kernels of the α_i. For any ξ, the kernel of $\mathrm{d}^2(h - \mathbf{J}^\xi)(0)$ is equal to the sum of the $\mathbb{C}_{\alpha_i}$ for those i for which ξ solves the ith equation. Thus each intersection of hyperplanes forms an affine subspace of generators that correspond to the same kernel. The following lemma shows that generically for all these kernels the α_i are linearly independent. (A geometric formulation of this property is that no k hyperplanes intersect in an affine subspace of $\mathfrak{t}^n$ dimension greater that $n - k$.)

Lemma 6.50. *Let T be a vector space and let $\alpha_1, \dots, \alpha_n$ be linearly dependent elements of T^*. For any $a \in \mathbb{R}$, let $X_i(a)$ denote the affine subspace of T of solutions of*

$$\alpha_i(x) = a.$$

There is an open and dense subset $O_S \subset \mathbb{R}^n$ such that for $c = (c_1, \dots, c_n) \in O_S$ the set $\bigcap_i X_i(c_i)$ is empty.

Proof. Consider the linear map

$$\begin{aligned} &T \to \mathbb{R}^n \\ &x \mapsto (\alpha_1(x), \dots, \alpha_n(x)). \end{aligned}$$

This map is not surjective, because the α_i are linearly dependent. For any $c = (c_1, \dots, c_n) \in \mathbb{R}^n$ contained in the complement of the image of this map, the intersection $\bigcap_i X_i(c_i)$ is empty. □

From now on, we assume this generic condition which we call non-resonance condition (NR):

Definition 6.51. Suppose that condition (GC) holds for h. Then the *non-resonance condition* (NR) is satisfied iff for every $\xi \in \mathfrak{t}^n$, the kernel of Hessian $\mathrm{d}^2(h - \mathbf{J}^\xi)(0)$ consists of a sum of spaces $\mathbb{C}_{\alpha_i}$, $1 = 1, \dots, k$, such that the weights $\alpha_1, \dots, \alpha_k$ are linearly independent.

Let Q_h denote the quadratic part of h:

$$Q_h(x) := \frac{1}{2}\mathrm{d}^2 h(0)(x, x).$$

Then X_{Q_h} is a linear vector field and coincides with linearization of X_h at the origin.

Lemma 6.52. *An element $x \in V$ is a relative equilibrium of the vector field X_{Q_h} with generator $\xi \in \mathfrak{t}$ iff $x \in \ker \mathrm{d}^2(h - \mathbf{J}^\xi)(0)$.*

Proof. Since $\mathbf{J}^\xi$ and Q_h are quadratic forms and $\mathrm{d}^2Q_h(0) = \mathrm{d}^2h(0)$, the critical points of $Q_h - \mathbf{J}^\xi$ are given by $\ker \mathrm{d}^2(h - \mathbf{J}^\xi)(0)$. □

The above considerations give the structure of nontrivial kernels $\mathrm{d}^2(h-\mathbf{J}^\xi)(0)$ in the generic case, which correspond to relative equilibria of X_{Q_h}:

Theorem 6.53. *Suppose that the conditions (GC) and (NR) are satisfied for* $h : V \to \mathbb{R}$.

The zero set of $\xi \mapsto \det \mathrm{d}^2(h - \mathbf{J}^\xi)(0)$ *consists of a union of affine hyperplanes, whose underlying subspaces correspond to the kernels of the weights of* $\mathbb{E}_\mathrm{c}$*, such that there is a bijection between the hyperplanes and the weight spaces* $\mathbb{C}_{\alpha_i}$ *of some specific splitting* $\mathbb{E}_\mathrm{c} = \bigoplus_{i\in I} \mathbb{C}_{\alpha_i}$.

For each $\xi \in \mathfrak{t}$*, the space* $\ker \mathrm{d}^2(h - \mathbf{J}^\xi)(0)$ *is given by the sum of the weight spaces that are associated to the hyperplanes that contain* ξ*. The corresponding weights are linearly independent.*

Conversely, for each linearly independent subset $\{\alpha_i\}_{i\in J}$ *for some* $J \subset I$*, there is a* $\xi \in \mathfrak{t}$ *with* $\ker \mathrm{d}^2(h - \mathbf{J}^\xi)(0) = \bigoplus_{i\in J} \mathbb{C}_{\alpha_i}$.

If $x \in \bigoplus_{i\in K} \mathbb{C}_{\alpha_i}$ *for some minimal set* $K \subset I$*, then the isotropy subgroup of* x *coincides with the intersection of the kernels of the* θ_{α_i} *with* $i \in K$.

Proof. The first two statements follow from the above reasoning. If the set $\{\alpha_i\}_{i\in J}$ is linearly independent, then the linear system

$$c_i - 2\pi\alpha_i(\xi) = 0 \quad \forall i \in J$$

(where, as above, the c_i are the corresponding eigenvalues of $\mathrm{d}^2h(0)$) has a non-empty solution set $X \subset \mathfrak{t}$. For every superset $L \supsetneq J$, either L is linearly independent and thus the solution set for L is a lower dimensional subset of X or L is linearly dependent and thus the corresponding solution set is empty by the non-resonance condition (NR). Since there are only finitely many subsets of the finite set I, there is $\xi \in \mathfrak{t}$ such that

$$c_i - 2\pi\alpha_i(\xi) = 0$$

is satisfied iff $i \in J$.

The statement about the isotropy subgroups is obvious. □

Since for a generic T-invariant function h, the kernels of all augmented Hamiltonians have linearly independent weights, the above conjecture suggests, that the set of relative equilibria of X_h near the origin is locally homeomorphic to the one of its linearization X_{Q_h}. Indeed, we will prove the following theorem:

Theorem 6.54. *Let* T *be a torus and* V *be a* T*-symplectic representation. Let* $h : V \to \mathbb{R}$ *be a smooth* T*-invariant Hamiltonian function with equilibrium at* 0*. Suppose that* h *satisfies the genericity assumptions (GC) and (NR), that is:*

1. $\mathrm{d}X_h(0)$ *is non-singular.*
2. *For each pair* $\pm\beta\mathrm{i}$ *of purely imaginary eigenvalues of* $\mathrm{d}X_h(0)$*, the space* $E_{\pm\beta\mathrm{i}}$ *is an irreducible* T*-symplectic representation.*
3. *For every* $\xi \in \mathfrak{t}$*, the weights of* $\ker \mathrm{d}^2(h - \mathbf{J}^\xi)(0)$ *are linearly independent.*

Then the local set $\mathcal{E}$ of relative equilibria near 0 *has the same structure as the one of the Hamiltonian $Q_h : V \to \mathbb{R}$ defined by $Q_h(v) = \mathrm{d}^2h(0)(v,v)$. More precisely, for every $\xi \in \mathfrak{t}$, there is a T-invariant manifold of relative equilibria tangent to* $\ker \mathrm{d}^2(h - \mathbf{J}^\xi)(0)$ *with generators near ξ, and $\mathcal{E}$ consists of the union of these manifolds. (Note that the set of critical points of $Q_h - \mathbf{J}^\xi$ is given by* $\ker \mathrm{d}^2(h - \mathbf{J}^\xi)(0)$*.) Moreover $\mathcal{E}$ is locally homeomorphic to the local set of relative equilibria of X_{Q_h} via a T-equivariant local homeomorphism.*

Theorem 6.54 relies on the following lemma which is proved with the help of equivariant transversality theory:

Lemma 6.55. *Suppose that for some $\xi_0 \in \mathfrak{t}$, we have*

$$V_0 := \ker \mathrm{d}^2(h - \mathbf{J}^{\xi_0})(0) = \oplus_{i=1}^{l} \mathbb{C}_{\alpha_i}$$

such that $\alpha_1, \ldots, \alpha_l$ are linearly independent. Let V_1 be a T-invariant complement of V_0 in V.

1. *The set of pairs of relative equilibria of X_h and their generators is locally T-equivariantly homeomorphic to the one of X_{Q_h} at $(0, \xi_0) \in V \times \mathfrak{t}$.*

2. *Moreover, there are neighbourhoods $U_0 \subset V_0$ of $0 \in V_0$, U_1 of $0 \in V_1$, and $O \subset \mathfrak{t}$ of ξ_0 and a smooth T-equivariant map $m_{V_0} : U_0 \to V$ of the form*

$$v_0 \mapsto v_0 + v_1(v_0, \xi(v_0)),$$

   where $\xi : U_0 \to O$ and $v_1 : U_0 \times O \to U_1$ are smooth, such that the image of m_{V_0} coincides with the set of relative equilibria of X_h in $(U_0 \times U_1) \times O$.

3. *If there is some η_0 with $W_0 := \ker \mathrm{d}^2(h - \mathbf{J}^{\eta_0})(0) \subset V_0$, then the germs of $m_{V_0}\big|_{W_0 \cap U_0}$ and m_{W_0} coincide.*

Proof. 1. We restrict the function $\mathcal{H} : (v, \xi) \mapsto h - \mathbf{J}^\xi(0)$ to $V_0 \times \mathfrak{t}$. Then we check for equivariant (1-jet)-transversality of $\mathcal{H}$ to $0 \in V_0^*$ at $(0, \xi_0)$. To be precise, we consider the subset

$$Q := \mathbb{R} \times \{0\} \subset \mathbb{R} \times V_0^* = J^1(V_0, \mathbb{R})$$

and characterize the functions h such that $\mathcal{H}$ is T-1-jet-transverse to Q at 0, but we omit the factor $\mathbb{R}$ and the $\mathbb{R}$-components of the reduced 1-jet $\tilde{j}^1L$ and the map U_1. To stress the similarity to ordinary T-transversality, we denote the remaining components of the maps $\mathbb{1} \times (\tilde{j}^1 L \circ P)$ and U_1 by Γ_h and ϑ respectively.

As is common in bifurcation theory, we moreover can omit the $\mathfrak{t}$-component and replace the space $V_0 \times \mathfrak{t} \times \mathbb{R}^l$ by $V_0 \times \mathbb{R}^l$, since T acts trivially on $\mathfrak{t}$.

A minimal generating set of invariant homogeneous polynomials on V_0 is given by

$$p_i : x = (x_{\alpha_1}, \ldots, x_{\alpha_l}) \mapsto |x_{\alpha_i}|^2.$$

Since T acts trivially on $\mathfrak{t}$, the functions $p_i : (x, \xi) \mapsto p_i(x)$ form such a set on $V_0 \times \mathfrak{t}$.

We identify V_0^* and V_0 via the invariant inner product and since $\nabla p_i(x) = 2x_{\alpha_i}$, we obtain the universal polynomial

$$\begin{aligned}\vartheta : V_0 \times \mathbb{R}^l &\to V_0\\ \vartheta(x,t) &= 2\sum_{i=1}^{l} t_i x_{\alpha_i}\end{aligned}$$

and the universal variety $\Sigma := \vartheta^{-1}(0)$. Let P be the function

$$P(x) := (p_1(x), \dots, p_l(x)).$$

By abuse of notation, let $\mathcal{H}$, h, and $\mathbf{J}^\xi$ also denote the functions on $\mathbb{R}^l$ whose composition with P is equal to the corresponding functions on V_0. For $\Gamma_\mathcal{H}$, we obtain

$$\begin{aligned}\Gamma_\mathcal{H} : V_0 \times \mathfrak{t} &\to V_0 \times \mathbb{R}^l\\ \Gamma_\mathcal{H}(x,\xi) &= (x, \partial_{p_1}\mathcal{H}(P(x),\xi), \dots, \partial_{p_l}\mathcal{H}(P(x),\xi)).\end{aligned}$$

Now, we test for T-1-jet-transversality of $\mathcal{H}$ to 0 at $(0,\xi)$ for some arbitrary $\xi \in \mathfrak{t}$:

By definition, we have to check transversality of $\Gamma_\mathcal{H}$ to the canonical stratification of Σ. As usual, the image of the x-derivative of $\Gamma_\mathcal{H}$ is equal to V_0.

Since for $(x,\xi) \in V_0$, we have

$$\begin{aligned}\mathcal{H}(x,\xi) &= h(x) - \mathbf{J}^\xi(x)\\ &= h(P(x)) - \pi \sum p_i(x)\alpha_i(\xi),\end{aligned}$$

the ξ-derivative of $\Gamma_\mathcal{H}$ is independent of the Hamiltonian function and we obtain

$$\mathrm{d}_\xi \partial_{p_i} \mathcal{H}(P(x),\xi) = -\mathrm{d}_\xi \partial_{p_i} \mathbf{J}^\xi(P(x)) = -\pi\alpha_i.$$

Since the α_i are linearly independent, this yields $\mathrm{d}_\xi \Gamma_\mathcal{H}(0,\xi)\mathfrak{t} = \{0\} \times \mathbb{R}^l$. Thus the image of the total derivative $\mathrm{d}\Gamma_\mathcal{H}(0,\xi)$ is the whole space $V_0 \times \mathbb{R}^l$. Hence $\mathcal{H}$ is always T-1-jet-transverse to $0 \in V_0$ at $(0,\xi_0)$.

As presented in chapter 3, by the Splitting Lemma or Lyapunov-Schmidt-reduction, for an invariant complement V_1 of V_0, we obtain an equivariant smooth local map $v_1 : V_0 \times \mathfrak{t}^n \to V_1$ and a local function $g : V_0 \times \mathfrak{t}^n \to \mathbb{R}$ at $(0,\xi_0)$ such that v_0 is a critical point of $g(\cdot,\xi)$ iff $v_1(v_0,\xi) + v_0$ is a critical point of $\mathcal{H}(\cdot,\xi) : V \to \mathbb{R}$ and the 3-jets of g and the restriction of $\mathcal{H}$ to $V_0 \times \mathfrak{t}^n$ at $(0,\xi_0)$ coincide. Since the local functions $t\mathcal{H} + (1-t)g$ all have the same 3-jet at $(0,\xi_0)$, they form an isotopy of functions that are T-1-jet-transverse to 0 at $(0,\xi_0)$. Thus from the Thom-Mather Transversality Theorem (Theorem A.21), part 4, we obtain a local T-equivariant homeomorphism between the zero sets of of the families $(x,\xi) \mapsto \mathrm{d}_{V0} g(x,\xi)$ and $(x,\xi) \mapsto \mathrm{d}_{V0}\mathcal{H}(x,\xi)$ in $V_0 \times \mathfrak{t}$ near $(0,\xi_0)$. Altogether, we have a T-equivariant continuous embedding $(V_0,0) \hookrightarrow (V,0)$ that maps the local set of pairs of relative equilibria of X_h and their generators $(v_0,\xi) \subset V_0 \times \mathfrak{t}$ of the Hamiltonian vector field of the restriction of h to V_0 to the local set of these pairs of X_h. Since $\mathcal{H}$ is always T-1-jet-transverse to $0 \in V_0$ at $(0, \xi_0)$, this also holds, if we replace h by $th + (1-t)Q_h$ for any $t \in [0,1]$. Thus, the local sets of these pairs in $V_0 \times \mathfrak{t}$ are also locally T-equivariantly homeomorphic for the restrictions of h and Q_h to V_0. Hence the vector fields X_h and X_{Q_h} have locally homeomorphic sets of pairs $(x, \xi) \in V \times \mathfrak{t}$ of relative equilibria and their generators.

2. Looking closer at the differentiable structure, we can even show slightly more: The above calculation yields that $\Gamma_{\mathcal{H}}$ (respectively Γ_g) is not only transverse to $\Sigma \subset \mathbb{R}^l$ but also to $0 \in \mathbb{R}^l$. Thus the preimage $\Gamma_g^{-1}(0)$ forms a manifold. W.l.o.g. we assume that $T = T^l$ (otherwise we factor out the kernel of the T-action on V_0, see Remark 2.10). By the implicit function theorem, $\Gamma_g^{-1}(0)$ is the graph of a local smooth function $\xi : V_0 \to \mathfrak{t}$. If $v_1 : V \times \mathfrak{t} \to V_1$ is as in the Splitting Lemma 2.9, the map

$$m_{V_0} : v_0 \mapsto v_1(v_0, \xi(v_0)) + v_0$$

is a local equivariant smooth embedding $(V_0, 0) \hookrightarrow (V, 0)$, which maps a neighbourhood U_0 of $0 \in V_0$ to a manifold of relative equilibria, which is tangent to V_0. Actually, this manifold consists of all relative equilibria near 0 that admit a generator near ξ_0: As we have seen in Remark 2.11, if $v_0 \in V_0^S$ for some subtorus $S \subset T$ with Lie algebra $\mathfrak{s}$, then v_0 is also a critical point of $g(\cdot, \eta)$ for any $\eta \in \xi(v_0) + \mathfrak{s}$ and $v_1(v_0, \eta) = v_1(v_0, \xi(v_0))$. Now we only have to show that locally $\mathrm{d}_{V_0} g(v_0, \xi) = 0$ implies $\xi \in \mathfrak{t}_{v_0} + \xi(v_0)$. To see this, notice that the isotropy subspaces of $V_0 = \oplus_{i=1}^l \mathbb{C}_{\alpha_i}$ are given by sums $\oplus_{i \in M} \mathbb{C}_{\alpha_i}$ for subsets $M \subset \{1, \ldots, l\}$. In particular $x = \sum x_{\alpha_i}$ has trivial isotropy iff $x_{\alpha_i} \neq 0$ for every i. In this case, $\Gamma_{\mathcal{H}}(x, \xi) \in \Sigma$ implies $\Gamma_{\mathcal{H}}(x) = (x, 0)$ and hence $\xi = \xi(x)$. The same reasoning holds for x in some proper isotropy subspace $V_0^{T_x}$, we only have to restrict to this subspace and factor out the kernel T_x of the T-action again, hence we can assume that T_x is trivial. Since there are neighbourhoods $U_1 \subset V_1$ of $0 \in V_1$ and $O \subset \mathfrak{t}$ of ξ_0 such that all relative equilibria of X_h in $U_0 \times U_1$ with a generator in $\xi \in O$ are of the form $v_0 + v_1(v_0, \xi)$ with $v_0 \in U_0$ and $v_1(v_0, \xi) = v_1(v_0, \xi(v_0))$, all pairs of relative equilibria and their generators in $U_0 \times U_1 \times O$ are contained in the image of m_{V_0}.

3. First, we argue that the germ m_{V_0} does not depend on the choice of ξ_0 with $\ker \mathrm{d}^2(h - \mathbf{J}^{\xi_0})(0) = V_0$: The image of the corresponding map contains all relative equilibria in $U_0 \times U_1$ with a generator in a neighbourhood O of ξ_0. Suppose that ξ_1 also satisfies $\ker \mathrm{d}^2(h - \mathbf{J}^{\xi_1})(0) = V_0$. Then $\mathrm{d}^2 \mathbf{J}^{\xi_1 - \xi_0}(0)$ vanishes on V_0 and hence the Lie algebra element $\xi_1 - \xi_0$ acts trivially on V_0. Thus every relative equilibrium in the image of m_{V_0} has the generator $\xi \in O$ iff it also has the generator $\xi - \xi_0 + \xi_1$, which is contained in the neighbourhood $O - \xi_0 + \xi_1$ of ξ_1. Hence it holds as well, that m_{V_0} maps $U_0 \times (O - \xi_0 + \xi_1)$ onto the set of all relative equilibria in $U_0 \times U_1$ with generator in $O - \xi_0 + \xi_1$.

   Thus for $\ker \mathrm{d}^2(h - \mathbf{J}^{\eta_0})(0) = W_0 \subset V_0$, we may assume w.l.o.g. that $\eta_0 \subset O$. By Theorem 6.53, $W_0 = V_0^S$ for some isotropy subgroup $S \subset T$. Thus by equivariance, the image of m_{W_0} is contained in V^S. A complement of W_0 in V^S is given by V_1^S. Therefore, if $w_0 \in W_0$ is small, $m_{W_0}(w_0)$ is a relative equilibrium that has a generator in O and the V_0-component w_0 with respect to the splitting $V = V_0 \oplus V_1$. Hence $m_{W_0}(w_0) = m_{V_0}(w_0)$.

□

Proof of Theorem 6.54. Since h satisfies the genericity assumptions (GC) and (NR), the set of $\xi \in \mathfrak{t}$ with $\ker \mathrm{d}^2(h - \mathbf{J}^\xi)$ non-trivial consists of a union of hyperplanes as in theorem 6.53. We assume w.l.o.g. that the T-action on $\mathbb{E}_c$ is faithful. (Otherwise we consider the quotient of $\mathfrak{t}$ and the Lie algebra of the subgroup that acts trivially on $\mathbb{E}_c$.) Then there are finitely many points $\xi_1, \ldots, \xi_k$ in $\mathfrak{t}$ which coincide with the intersection of a particular subset of hyperplanes. By Lemma 6.55, for each ξ_i, there is an $\varepsilon_i > 0$ such that the map $m_{V_0^i} : v_0 \mapsto v_0 + v_1(v_0, \xi(v_0))$ is defined on an ε_i-neighbourhood U_i of $0 \in V_0^i := \ker \mathrm{d}^2(h - \mathbf{J}^{\xi_i})(0)$ and $m_{V_0^i}(U_i)$ contains all relative equilibria of norm less than ε_i that have a generator ξ with $\|\xi - \xi_i\| < \varepsilon_i$. We can find such an ε_i for any choice of the norm on $\mathfrak{t}$ and here we choose the following one: The weights of V_0^i define a linear coordinate system on $\mathfrak{t}$, where the coordinates are given by the values of the evaluation maps divided by 2π. We choose the supremum norm with respect to these coordinates. Thus our choice of the norm depends on i; we denote it by $\|\cdot\|_i$. Then $\|\xi - \xi_i\|_i < \varepsilon_i$ iff all eigenvalues of $\mathrm{d}^2(h - \mathbf{J}^\xi)(0)\big|_{V_0^i}$ have absolute value less than ε_i.

Now, we set $\varepsilon := \min_i \varepsilon_i$ and define a map Ψ from the set of relative equilibria of X_{Q_h} within $B_\varepsilon(0)$ to the set of relative equilibria of X_h: The set of relative equilibria of X_{Q_h} is given by the union of the kernels $\ker \mathrm{d}^2(h - \mathbf{J}^\xi)$. If $x \in V_0 = \ker \mathrm{d}^2(h - \mathbf{J}^\xi) \cap B_\varepsilon(0)$, we set

$$\Psi(x) = m_{V_0}(x).$$

Since $V_0 \subset (V_0^i)$ for some i and the germ $m_{V_0^i}$ restricts to m_{V_0} by Lemma 6.55, the map m_{V_0} can indeed be defined on $B_\varepsilon(0) \cap V_0$ and moreover, the map Ψ is well-defined since $m_{V_0}(x)$ is independent of the choice of the kernel V_0 that contains x. By Theorem 6.53, there is an isotropy subgroup $S \subset T$ such that $V_0 = (V_0^i)^S$. Since $m_{V_0^i}$ is T-equivariant, the image of its restriction m_{V_0} to the fixed point set of S contains all relative equilibria in $B_\varepsilon(0)^S$ that have a generator ξ with the property that all eigenvalues of $\mathrm{d}^2(h - \mathbf{J}^\xi)(0)\big|_{V_0^i}$ have absolute value less than ε.

We now consider the eigenvalue structure of $\mathrm{d}^2(h - \mathbf{J}^{\xi+\eta})(0)\big|_{V_0^i}$ corresponding to alternative generators of the form $\xi + \eta$ with η contained in $\mathfrak{s}$, the Lie algebra of S: The group S is given by the intersection of the kernels of representations defined by the weights of V_0. With respect to the coordinates on $\mathfrak{t}$ given by the weights of V_0^i, the subset $\mathfrak{s}$ hence corresponds to the subspace on which the coordinates given by the weights of V_0 vanish. Thus all diagonal matrices for complex linear maps $V_0^i \to V_0^i$ that vanish on V_0 are of the form $\mathrm{d}^2\mathbf{J}^\eta(0)\big|_{V_0^i}$ for some $\eta \in \mathfrak{s}$. Hence $m_{V_0}(B_\varepsilon(0)^S)$ contains all relative equilibria of X_h in $B_\varepsilon(0)^S$ with generators ξ, such that the eigenvalues of $\mathrm{d}^2(h - \mathbf{J}^\xi)(0)\big|_{V_0}$ have absolute value less than ε.

Ψ is obviously a T-equivariant homeomorphism onto its image. Thus we only have to show, that there is a $\delta > 0$ such that all relative equilibria of X_h in $B_\delta(0)$ are contained in the image of Ψ.

To see this, we note that

$$\mathrm{d}(h - \mathbf{J}^\xi)(x) = \mathrm{d}^2(h - \mathbf{J}^\xi)(0)x + R_h(x), \tag{6.6}$$

for some map $R_h : V \to V^*$ with $R_h(0) = 0$ and $\lim_{x\to 0} \frac{R_h(x)}{\|x\|} = 0$. If $\frac{\left\|\mathrm{d}^2(h-\mathbf{J}^\xi)(0)x\right\|}{\|x\|}$ has a positive lower bound for all $x \neq 0$, there is thus a neighbourhood of the origin that contains no relative equilibria with generator ξ.

Let $\mathbb{E}_{\mathrm{su}}$ denote the real part of the sum the eigenspaces of $\mathrm{d}X_h(0)$ corresponding to eigenvalues with non-vanishing real part. Then $\mathbb{E}_{\mathrm{su}}$ is a T-invariant, $\mathrm{d}^2(h - \mathbf{J}^\xi)(0)$-invariant complement of $\mathbb{E}_{\mathrm{c}}$. We now argue, that there is an $\varepsilon' > 0$, which is independent of ξ, such that for $x \in \mathbb{E}_{\mathrm{su}}$ with $\|x\| = 1$ we have $\left\|\mathrm{d}^2(h - \mathbf{J}^\xi)(0)x\right\| > \varepsilon'$: We split $\mathbb{E}_{\mathrm{su}}$ into isotypic components for the T-action. Each isotypic component is $\mathrm{d}^2h(0)$-invariant and $\mathrm{d}\mathbf{J}^\xi(0)$-invariant. Choosing a T-invariant inner product and the corresponding norm, we only have to find such a number on every isotypic component, then ε' is given by the minimum of these numbers. On each isotypic component, $\mathrm{d}^2\mathbf{J}^\xi(0) = \lambda\mathbb{1}$ for some $\lambda = \lambda(\xi) \in \mathbb{R}$. Obviously $\left\|\mathrm{d}^2h(0)x - \lambda x\right\|$ is bounded away from 0 for large absolute values of λ and x in the unit sphere S^k of the isotypic component (of dimension k). Since by Lemma 5.11, the linear map $\mathrm{d}^2(h-\mathbf{J}^\xi)(0)$ is invertible on $\mathbb{E}_{\mathrm{su}}$ for every ξ, we can also find a lower bound for $\mathrm{d}^2(h - \mathbf{J}^\xi)(0)x$ for all x in S^k and ξ such that $\mathrm{d}^2\mathbf{J}^\xi(0)x = \lambda x$ for some λ contained in a given compact interval. Hence there is indeed such an ε'. If $\mathrm{d}^2(h-\mathbf{J}^\xi)(0)$ has eigenvalues of absolute value less than ε', the sum of the corresponding eigenspaces is hence contained in $\mathbb{E}_{\mathrm{c}}$. Replacing ε' by a possibly smaller ε'', we can force the corresponding sum of eigenspaces to be contained in one of the V_0^i.

Now, we choose $\delta \leq \varepsilon$ such that $\frac{\|R_h(x)\|}{\|x\|} < \varepsilon''' := \min(\varepsilon'', \varepsilon)$ for every $x \in B_\delta(0)$. Suppose that $x \in B_\delta(0)$ is a relative equilibrium with generator ξ.

We consider the sum V_0 of the eigenspaces corresponding to eigenvalues of $\mathrm{d}^2(h - \mathbf{J}^\xi)(0)$ of absolute value $< \varepsilon'''$. By $\varepsilon''' \leq \varepsilon''$ and the assumption on ε'', we have $V_0 \subset V_0^i$ for some i.

Let $S \subset T$ be an isotropy subgroup with $V_0 = (V_0^i)^S$. For any choice of a T-invariant inner product, the vector field $\nabla(h - \mathbf{J}^\xi)$ is T-equivariant and thus sends V^S to V^S. Let W denote the orthogonal complement of V^S and consider only the W-component of equation 6.6 (where we identify V and V^* via the inner product). It depends only on the W-component x_W of x. Since all eigenvalues of $\mathrm{d}^2(h-\mathbf{J}^\xi)(0)\big|_W$ have absolute value greater than ε''' and $\frac{\|R_h(x_W)\|}{\|x_W\|} < \varepsilon'''$, we conclude $x_W = 0$. Thus, $x \in V^S$. Since ξ satisfies that the eigenvalues of the restriction of $\mathrm{d}^2(h - \mathbf{J}^\xi)(0)$ to $V_0 = (V_0^i)^S$ all have absolute value less then ε, the relative equilibrium x is contained in the image of m_{V_0}, which is a subset of the image of Ψ. □

Definition 6.56. If $V_0 = \ker \mathrm{d}^2(h - \mathbf{J}^{\xi_0})(0)$ for some ξ_0 and Ψ is the T-equivariant local homeomorphism between the local set of relative equilibria of X_{Q_h} and $\mathcal{E}$, we call $\Phi(V_0)$ the *manifold that bifurcates* at V_0, or at ξ_0, or at the affine set $X := \left\{\xi \in \mathfrak{t} \;\middle|\; V_0 \subset \ker \mathrm{d}^2(h - \mathbf{J}^\xi)(0)\right\}$. Similarly, we refer to the stratum of minimal isotropy of $\Phi(V_0)$ as the *stratum that bifurcates* at V_0, ξ_0, or X.

Remark 6.57. Let M_0 be the stratum that bifurcates at the affine subset $X \subset \mathfrak{t}$. Then for each relative equilibrium in M_0 the corresponding set of generators is a parallel translation of X: Let V_0 denote the corresponding subspace and $\mathfrak{s}$ be the underlying subspace of X. Then for any $\eta \in \mathfrak{s}$, the linear map $\mathrm{d}^2\mathbf{J}^\eta(0)$ vanishes on V_0 and $\mathfrak{s}$ is the maximal subset of $\mathfrak{t}$ with this property. Equivalently, $\mathfrak{s}$ is the Lie algebra of the minimal isotropy subgroup $S \subset T$ of elements of V_0. Thus the isotropy subgroup of any element of M_0 is S and equivalently the underlying subspace of its set of generators is $\mathfrak{s}$.

Moreover, M_0 contains all relative equilibria in some neighbourhood of 0 with this property whose set of generators is contained in a particular $\mathfrak{s}$-invariant neighbourhood of X. Indeed, if $V_0 = \ker \mathrm{d}^2(h - \mathbf{J}^{\xi_0})(0)$, then by Lemma 6.55 $\Phi(V_0)$ contains all relative equilibria near 0 with a generator contained in a neighbourhood O of $\xi_0 \in X$. Thus $M_0 = \Phi(V_0)_{(S)}$ contains all relative equilibria near zero with with isotropy subgroup S whose set of generators is contained in $O + \mathfrak{s}$.

6.4.2 Representations of connected compact groups

Fortunately, our results about torus representations are derived from explicitly given generic conditions. This is not always the case when dealing with equivariant transversality, because it is often hard to specify the open and dense subset of equivariant smooth maps that are G-transverse to some invariant set. In that case, the results for some specific group do not help to understand the generic behaviour of maps with higher symmetry. But in our results for tori, our assumptions on the Hessian of the augmented Hamiltonian imply the necessary T-transversality condition. This circumstance makes it possible to apply the method to representations of connected compact groups. In general, we will not be able to investigate the generic structure of all relative equilibria, but nevertheless, the method yields some branches of relative equilibria that generically exist:

Let G be a connected compact Lie group and V be a G-representation. To apply the results about torus representations, fix a maximal torus $T \subset G$ and consider V as a T-representation. If ξ is a generator of a relative equilibrium x, then every element of the adjoint orbit of ξ is a generator of a relative equilibrium in the G-orbit of x. Since all adjoint orbits intersect the Lie algebra $\mathfrak{t}$ of T, we only have to find the solutions of $\mathrm{d}(h - \mathbf{J}^\xi)(x) = 0$ with $x \in V$ and $\xi \in \mathfrak{t}$. Then all relative equilibria are contained in the G-orbits of these solutions.

Again, we start with an investigation of the kernels of the Hessians of the augmented Hamiltonians for generic Hamiltonian functions. The reasoning is similar as for torus representations: By Lemma 5.11, for any $\xi \in \mathfrak{t}$, the kernel of $\mathrm{d}^2(h - \mathbf{J}^\xi)(0)$ is contained in the centre space $\mathbb{E}_\mathrm{c}$ of $\mathrm{d}X_h(0)$. Moreover, we assume that $\mathbb{E}_\mathrm{c}$ splits into G-symplectic irreducible spaces corresponding to pairs of purely imaginary eigenvalues. Recall that this is our generic centre space condition (GC). Then $\mathbb{E}_\mathrm{c}$ admits a complex structure i and a (real) inner product $\langle \cdot, \cdot \rangle$ such that ω is represented by the multiplication with i and the quadratic form $\mathrm{d}^2h(0)$ is represented by a real multiple of the identity on each space $E_{\pm\beta\mathrm{i}}$ of $\mathrm{d}X_h(0)$ (see Theorem 3.15 and Corollary 3.16). We fix this choice of an inner product in the following.

Recall that for torus representations the generic situation is that exactly each sum of weight spaces corresponding to linearly independent weights forms such a kernel (see Theorem 6.53). Here, the structure of the kernels is more complicated and depends on the weights of the G-symplectic irreducible subrepresentations of $\mathbb{E}_c$. The kernels are characterized by the following lemmas:

Lemma 6.58. *Suppose that the generic centre space condition (GC) is satisfied. Then $\mathbb{E}_c$ splits into a sum of G-symplectic irreducible eigenspaces U_i of $\mathrm{d}^2h(0)$. For each $\xi \in \mathfrak{t}$, we have*

$$\ker \mathrm{d}^2(h - \mathbf{J}^\xi)(0) = \bigoplus_i \ker \mathrm{d}^2(h - \mathbf{J}^\xi)(0)\big|_{U_i}.$$

For each U_i the kernel of the restriction $\mathrm{d}^2(h - \mathbf{J}^\xi)(0)\big|_{U_i}$ is of the form $\bigoplus_{\alpha \in S_i} \mathbb{C}_\alpha$ for some subset S_i of the weights of U_i which is maximal within $\mathrm{aff}(S_i)$ and such that $0 \notin \mathrm{aff}(S_i)$. (Here, we allow multiple weights in S_i. Maximality of S_i in $\mathrm{aff}(S_i)$ implies that every weight of S_i occurs with the same multiplicity in S_i as in the set of all weights of U_i.)

Proof. Consider each eigenspace U_i of $\mathrm{d}^2h(0)|_{\mathbb{E}_c}$ as an irreducible complex G-representation. Then U_i splits into irreducible T-representations $\mathbb{C}_{\alpha^i_j}$, its weight spaces with corresponding weights α^i_j. The momentum map $\mathbf{J}$ restricted to $\mathfrak{t}$ is the same as the momentum map corresponding to the T-action on V. Thus for any $\xi \in \mathfrak{t}$, the endomorphism that represents $\mathrm{d}^2\mathbf{J}^\xi(0)$ on $\mathbb{E}_c$ with respect to our choice of the inner product has the eigenspaces $\mathbb{C}_{\alpha^i_j}$ with eigenvalues $2\pi\alpha^i_j(\xi)$. Since the function $h - \mathbf{J}^\xi$ is T-invariant, $V_0 := \ker \mathrm{d}^2(h - \mathbf{J}^\xi)(0)$ is T-invariant and splits into a sum of weight spaces. Let c_i be the eigenvalue of $\mathrm{d}^2h(0)$ on the eigenspace U_i. Then V_0 is given by the sum of those weight spaces $\mathbb{C}_{\alpha^i_j}$ of the eigenspaces of $\mathrm{d}^2h(0)\big|_{\mathbb{E}_c}$ such that

$$2\pi\alpha^i_j(\xi) = c_i$$

is satisfied. For each ξ and each eigenvalue c_i, the set of solutions $\alpha \in \mathfrak{t}^*$ of the equation $2\pi\alpha(\xi) = c_i$ is affine. Since $\mathrm{d}^2h(0)$ is non-degenerate, 0 is not contained in any of these affine subsets of $\mathfrak{t}^*$. Thus the solution set S_i of weights of U_i is maximal in its affine span $\mathrm{aff}(S_i)$, which does not contain 0. ☐

For torus representations, the non-resonance condition (NR) is generic. This is no longer true for representations of general connected compact groups G, as we will see soon. We replace condition (NR) by a more general condition:

Definition 6.59. Let T be a real vector space and $S = \bigcup_{i=1}^n S_i$ be a union of subsets $S_i \subset T^*$. Then S is *full* iff for every vector $(c_1, \ldots, c_n) \in \mathbb{R}^n$, there is an $x \in T$ with

$$\forall i : \forall \alpha \in S_i : \alpha(x) = c_i.$$

Remark 6.60. Equivalently, S is full iff for every $k \in \{1, \ldots, n\}$ there is a solution for the k-th standard vector e_k of $\mathbb{R}^n$. Let W_k be the underlying subspace of $\mathrm{aff}(S_k)$. Then $x \in T$ satisfies $\alpha(x) = 0$ for all $\alpha \in \bigcup_{i \neq k} S_j$ and $\alpha(x) = \beta(x)$ for all $\alpha, \beta \in S_k$ iff

$$x \in (\langle \bigcup_{i \neq k} S_i \rangle + W_k)^\perp,$$

where $\langle \cdot \rangle$ denotes the span and $A^\perp \subset T$ the zero set of a subspace $A \subset T^*$. Hence there is a solution for e_k iff

$$(\langle \bigcup_{i \neq k} S_i \rangle + W_k)^\perp \nsubseteq \langle S_k \rangle^\perp,$$

equivalently,

$$\langle S_k \rangle \nsubseteq \langle \bigcup_{i \neq k} S_i \rangle + W_k.$$

In other words, in the quotient T/W_k, the projection of $\langle S_k \rangle$, which is a single element of T/W_k, is not contained in the span of the image of $\bigcup_{i \neq k} S_i$ under the projection.

Definition 6.61. Suppose that G-invariant Hamiltonian function h satisfies condition (GC). Then h satisfies the generalized non-resonance condition (NR') iff for each union $S = \bigcup_i S_i$ of sets S_i of linearly independent weights of eigenspaces U_i of $\mathrm{d}^2 h(0)$ and the vector $c = (c_0, \dots c_n)$ of the corresponding eigenvalues, there is an $x \in T$ with

$$\forall i : \forall \alpha \in S_i : \alpha(x) = c_i$$

iff S is full.

Since there are only finitely many such sets S, condition (NR') obviously holds for all h contained in a dense open subset of $C^\infty_G(V)$ (with the C^∞-topology or Whitney C^∞-topology).

Now we can formulate a converse of Lemma 6.58, which holds for generic G-invariant Hamiltonian functions h:

Lemma 6.62. *Suppose that $h : V \to \mathbb{R}$ satisfies conditions (GC) and (NR'). Let $U \subset \mathbb{E}_\mathrm{c}$ be an eigenspace of $\mathrm{d}^2 h(0)$ with corresponding eigenvalue c and $S \subset \mathfrak{t}$ be a subset of the weights of U with the property that S is maximal in $\operatorname{aff}(S)$ and $0 \notin \operatorname{aff}(S)$. (See the comment in Lemma 6.58 about the multiplicity of weights.) Then there is a $\xi \in \mathfrak{t}$ such that*

$$\ker \mathrm{d}^2(h - \mathbf{J}^\xi)(0) = \bigoplus_{\alpha \in S} \mathbb{C}_\alpha \subset U.$$

Proof. Use a G-invariant inner product to identify $\mathfrak{t}^*$ and $\mathfrak{t}$ and consider the perpendicular s from $0 \in \mathfrak{t}$ to $\operatorname{aff}(S)$. The inner product with any point of s is constant along $\operatorname{aff}(S)$. Moreover, if we vary the point of s, we may obtain any real number as the result of the inner product. In particular, there is a $\tilde{\xi}$ with $2\pi\alpha(\tilde{\xi}) = c$ for any $\alpha \in S \subset \operatorname{aff}(S)$. Thus,

$$\bigoplus_{\alpha \in S} \mathbb{C}_\alpha \subset \ker \mathrm{d}^2(h - \mathbf{J}^{\tilde{\xi}})(0). \tag{6.7}$$

The same inclusion holds if we add any point of the orthogonal complement of the span $\langle S \rangle$ to $\tilde{\xi}$. If $S \subsetneq \tilde{S}$ for some subset $\tilde{S}$ of the set of weights of U, the maximality of S yields $\operatorname{aff}(S) \subsetneq \operatorname{aff}(\tilde{S})$. Suppose that $0 \in \operatorname{aff}(\tilde{S})$. Then by Lemma 6.62, $\bigoplus_{\alpha \in \tilde{S}} \mathbb{C}_\alpha$ is not contained in $\ker \mathrm{d}^2(h - \mathbf{J}^{\tilde{\xi}})(0)$ for any $\xi \in \mathfrak{t}$. If $0 \notin \operatorname{aff}(\tilde{S})$, then $\langle S \rangle \subsetneq \langle \tilde{S} \rangle$. Thus, we obtain a lower dimensional solution set for $\tilde{\xi}$ if we replace S by $\tilde{S}$ in the inclusion 6.7. Hence, for ξ contained in an open dense subset of solutions of the above inclusion, we obtain

$$\bigoplus_{\alpha \in S} \mathbb{C}_\alpha = \ker \mathrm{d}^2(h - \mathbf{J}^\xi)(0)\big|_U.$$

If we consider the whole kernel $\ker \mathrm{d}^2(h - \mathbf{J}^\xi)(0)$, it is a sum of the kernels of the restrictions of $\mathrm{d}^2(h - \mathbf{J}^\xi)(0)$ to irreducible subrepresentations U_i of $\mathbb{E}_\mathrm{c}$ with $\mathrm{d}^2h(0)\big|_{U_i} = c_i\mathbb{1}$, where w.l.o.g. $U = U_1$. By Lemma 6.58, for each U_i, the kernel $\ker \mathrm{d}^2(h - \mathbf{J}^\xi)(0)\big|_{U_i}$ is given by a sum $\bigoplus_{\alpha\in S_i} \mathbb{C}_\alpha$, where S_i is a set of weights which is maximal within its affine span $\mathrm{aff}(S_i)$ and $0 \notin \mathrm{aff}(S_i)$. Now, genericity assumption (NR') implies that the dimension of the set of solutions ξ of the system

$$\forall i : \forall \alpha \in S_i : 2\pi\alpha(\xi) = c_i$$

is less then the dimension of the set of solutions ξ of the system

$$\forall \alpha \in S_1 : 2\pi\alpha(\xi) = c_1.$$

Since there are only finitely many choices for $\bigcup_i S_i$, there are values of $\xi \in \mathfrak{g}$ with

$$\bigoplus_{\alpha\in S} \mathbb{C}_\alpha = \ker \mathrm{d}^2(h - \mathbf{J}^\xi)(0). \qquad \square$$

The following corollary now shows, that non-resonance condition (NR) is not generic in general. Again, we consider V as a complex G-representation. If G has rank ≥ 2, all but a finite number of the irreducible complex G-representations have linearly dependent sets of weights whose affine span does not contain 0. See (See [Hal03, Theorem 10.1]).

Corollary 6.63. *Let $U \subset V$ be a complex irreducible subrepresentation and suppose that $S \subset \mathfrak{t}$ is set of weights of S with S maximal in $\mathrm{aff}(S)$ and $0 \notin \mathrm{aff}(S)$. Then there is an open subset $\mathcal{U} \subset C^\infty(V)^G$ (C^∞- or Whitney C^∞-topology) such that for every $h \in \mathcal{U}$ there is a $\xi \in \mathfrak{t}$ with*

$$\mathrm{d}^2(h - \mathbf{J}^\xi)(0) \simeq \bigoplus_{\alpha\in S} \mathbb{C}_\alpha.$$

Proof. By Theorem 3.19, there is an open subset $\tilde{\mathcal{U}} \subset C^\infty(V)^G$ such that for every $h \in \tilde{\mathcal{U}}$, the centre space $\mathbb{E}_\mathrm{c}$ of $\mathrm{d}X_h(0)$ contains a subspace isomorphic to U. Then Lemma 6.62 implies the existence of such a ξ for h in a dense open subset $\mathcal{U}$ of $\tilde{\mathcal{U}}$. $\square$

However, our results for torus representations apply to kernels of $\mathrm{d}^2(h - \mathbf{J}^\xi)$ with linearly independent weights. We obtain:

Theorem 6.64. *Let G be a connected compact Lie group with maximal torus T and V be a symplectic G-representation. Suppose that $h : V \to \mathbb{R}$ is a smooth G-invariant Hamiltonian function with critical point at 0 that satisfies the genericity assumptions (GC) and (NR'), that is:*

1. $\mathrm{d}X_h(0)$ *is non-degenerate.*

2. $\mathbb{E}_\mathrm{c} = \bigoplus_{i\in I} U_i$, *where each U_i is irreducible and $U_i = E_{\pm\beta_i \mathrm{i}}$ for a pair of purely imaginary eigenvalues $\pm\beta_i\mathrm{i}$ of $\mathrm{d}X_h(0)$.*

3. *Consider subspaces $U_i \subset \mathbb{E}_\mathrm{c}$ and the vector $c = (c_1, \dots, c_n)$ such that $\mathrm{d}^2h(0)\big|_{U_i} := -J\mathrm{d}X_h(0)\big|_{U_i} = c_i\mathbb{1}_{U_i}$ for a representation matrix J of ω with respect to a G-invariant inner product on V such that $J^2 = -\mathbb{1}$. For*

each union $S = \bigcup_i S_i$ of sets S_i of linearly independent weights of U_i, we have a solution x of

$$\forall i : \forall \alpha \in S_i : \alpha(x) = c_i.$$

iff S is full (see Definition 6.59 and Remark 6.60).

Consider (possibly empty) subsets S_i of weights of U_i such that each S_i is maximal in $\operatorname{aff}(S_i)$ (in particular, the elements of S_i occur with multiplicity 1 in the set of weights of U_i). If $S := \bigcup_{i \in I} S_i$ is linearly independent, there is a T-invariant manifold of T-relative equilibria whose tangent space at 0 is given by the sum of the corresponding weight spaces of the elements of S: For $\alpha \in S_i$, we obtain the summand $\mathbb{C}_\alpha \subset U_i$.

Proof. We show first that for a linear independent set S, there is indeed a $\xi \in \mathfrak{t}$ such that $\ker \mathrm{d}^2(h - \mathbf{J}^\xi)(0)$ consists of the corresponding weight spaces of the weights contained in S: Since the S_i are linearly independent, we have $0 \notin \operatorname{aff}(S_i)$. Thus by Lemma 6.58, there are nonempty subsets $X_i \subset \mathfrak{t}$ such $\bigoplus_{\alpha \in S_i} \mathbb{C}_\alpha$ is contained in $\ker \mathrm{d}^2(h - \mathbf{J}^\xi)(0)$ if $\xi \in X_i$. Since $S = \bigcup_{i=1}^r S_i$ is linearly independent, the intersection $X := \bigcap_i X_i$ is non-empty. Moreover, as argued in the proof of Lemma 6.58, the condition 3 (which is condition (NR')) implies, that there is a $\xi \in X$ with $\ker \mathrm{d}^2(h - \mathbf{J}^\xi)(0) = \bigoplus_{\alpha \in S} \mathbb{C}_\alpha$.

Now the result follows from Lemma 6.55. □

Let us consider again the affine subset $X \subset \mathfrak{t}$ related to the subspace $V_0 := \ker \mathrm{d}^2(h - \mathbf{J}^{\xi_0})(0)$. We have $X := \{\xi \in \mathfrak{t} \mid V_0 \subset \ker \mathrm{d}^2(h - \mathbf{J}^\xi)(0)\}$. Suppose that X is contained in k hyperplanes of $\mathfrak{t}$ consisting of zeros of the map $\xi \mapsto \det \mathrm{d}^2(h - \mathbf{J}^\xi)(0)$ and that $\dim \mathfrak{t} = n$. (Here, we distinguish hyperplanes corresponding to different weight spaces even if they coincide.) Then the weights of V_0 are linearly independent iff $\dim X = n - k$. In this case, we call X *regular*. Moreover, we use the same terminology as in Definition 6.56 then.

As explained above, the G-orbits of these bifurcating manifolds of T-relative equilibria consist of G-relative equilibria. Now, we investigate the structure of these group orbits.

For torus representations, points of $V_0 = \bigoplus \mathbb{C}_{\alpha_i}$ for linearly independent α_i are contained in the same group orbit exactly if their momenta coincide. Hence by equivariance of m_{V_0}, the orbits of relative equilibria in the stratum tangent to $\bigoplus \mathbb{C}_{\alpha_i}$ are given by the preimage of the canonical stratification of $\mathbf{J}(\bigoplus \mathbb{C}_{\alpha_i})$. In contrast, when we consider the G-action, different T-orbits may be contained in the same G-orbit.

For example, this happens in the following case:

Lemma 6.65. *Let $X, Y \subset \mathfrak{t}$ be regular affine subsets of*

$$\mathcal{Z} := \{\xi \in \mathfrak{t} \mid \det \mathrm{d}^2(h - \mathbf{J}^\xi)(0) = 0\}$$

with the same W-orbit, where $W = W(G)$ is the Weyl group of G. Then the manifolds that bifurcate at X and Y respectively are contained in the same G-orbit.

Proof. If $Y = wX$ for $w \in W$ and $\tilde{w} \in N(T)$ is a representative of w, multiplication by $\tilde{w}$ maps the manifold M_0 of T-relative equilibria that bifurcates at X to a manifold $\tilde{w}M_0$ of T-relative equilibria with generators near the elements of wX. By uniqueness (see Remark 6.57), $\tilde{w}M_0$ coincides with the manifold that bifurcates at wX. □

Remark 6.66. Note that the Weyl group W permutes the weights of $\mathbb{E}_c$ and hence the corresponding affine hyperplanes of $\mathfrak{t}$, which form together the zero set of $\xi \mapsto \det \mathrm{d}^2(h - \mathbf{J}^\xi)(0)$. Obviously, an intersection X of some of these affine hyperplanes is regular iff this holds for every wX for $w \in W$.

To identify the relative equilibria of the same G-orbit in general, we consider them together with their generators. The set of pairs $(v, \xi) \in V \times \mathfrak{g}$ of a relative equilibrium and an admissible generator is given by the G-orbit of its intersection with $(v, \xi) \in V \times \mathfrak{t}$.

If two elements (v, ξ) and (v', ξ') of this intersection are contained in the same G-orbit, we have in particular $\xi' = \mathrm{Ad}_g \xi$ for some $g \in G$. Since the intersection of the adjoint orbit $G\xi$ with $\mathfrak{t}$ coincides with the Weyl group orbit $W\xi$, there is a $w \in W$ with $\xi' = \mathrm{Ad}_w \xi$. We already know that for regular affine subsets that have the same W-orbit, the bifurcating strata have the same G-orbit; we only want to find additional relations between the G-orbits of strata. Thus we choose some representative $\tilde{w} \in G$ of W and replace (v', ξ') by $(\tilde{w}^{-1}v', \mathrm{Ad}_w^{-1} \xi') = (\tilde{w}v', \xi)$, which is also contained in the G-orbit. Hence we assume in the following that $\xi = \xi'$. Then v and v' are contained in the same G_ξ-orbit.

$G_\xi \neq T$ iff ξ is contained in some Weyl wall ([BtD85, chapter V, Theorem 2.3]). If we consider only relative equilibria whose generator ξ is contained in some small neighbourhood of some ξ_0 with $\det \mathrm{d}^2(h - \mathbf{J}^{\xi_0})(0) = 0$, we may deduce that ξ_0 must also be contained in this Weyl wall and $G_\xi = G_{\xi_0}$. Hence, if two relative equilibria v and v' near 0 with generators in $\mathfrak{t}$ are contained in the same G-orbit but not in the same T-orbit, they must necessarily be contained in the manifold tangent to $\ker \mathrm{d}^2(h - \mathbf{J}^{\xi_0})(0)$ for some Weyl wall element ξ_0.

We now investigate the intersection of group orbits with such a manifold: Since $h - \mathbf{J}^{\xi_0}$ is a G_ξ-invariant function, $V_0 := \ker \mathrm{d}^2(h - \mathbf{J}^{\xi_0})(0)$ is G_ξ-invariant. Hence, the Splitting Lemma yields a G_ξ-equivariant local map

$$v_1 : (V_0 \times \mathfrak{g}_\xi, (0, \xi_0)) \to V_1,$$

where V_1 is a G_ξ-invariant complement of V_0, such that v_1 solves

$$\mathrm{d}_{V_1}(h - \mathbf{J}^\xi)(v_1 + v_0) = 0.$$

By uniqueness, v_1 extends the corresponding local map

$$v_1 : (V_0 \times \mathfrak{t}, (0, \xi_0)) \to V_1.$$

Similarly, the G_ξ-invariant function $g : (V_0 \times \mathfrak{g}_\xi, (0, \xi_0)) \to \mathbb{R}$ given by

$$g(v_0, \xi) = (h - \mathbf{J}^\xi)(v_1(v_0, \xi) + v_0)$$

is an extension of the T-invariant function g defined on a neighbourhood of $(0, \xi_0)$ in $V_0 \times \mathfrak{t}$. Since V_1 and V_0 are G_ξ-invariant and v_1 is G_ξ-equivariant, the pairs $(v_1(v_0, \xi) + v_0, \xi)$ and $(v_1(v_0', \xi) + v_0', \xi)$ are contained in the same G_ξ-orbit iff this is true for (v_0, ξ) and (v_0', ξ). Hence we only have to understand the G_ξ-action on V_0.

(Note that we do not require that V_0 has linearly independent weights at this point. Nevertheless, we will investigate the G_ξ-action on V_0 later on only for the case hat the weights of V_0 are linearly independent. It will turn out, that this suffices to describe the G-orbits of the strata of T-relative equilibria that bifurcate at regular affine sets.)

Remark 6.67. If two pairs $(x, \xi), (x', \xi') \in V \times \mathfrak{t}$ are contained in the same G-orbit, this is obviously also true for the points x and x'. Conversely, if two T-relative equilibria x and x' are contained in the same G-orbit and $\xi \in \mathfrak{t}$ is a generator of x, it is an interesting question if there is generator a $\xi' \in \mathfrak{t}$ of x' such that the pairs (x, ξ) and (x', ξ') are contained in the same G-orbit. Since $x = gx'$ for some $g \in G$ implies that $\mathrm{Ad}_g \xi'$ is a generator of x, this is true if there is an element $h \in G_{x'}$ with $\mathrm{Ad}_h(\mathrm{Ad}_g \xi') \in \mathfrak{t}$. The following lemma describes a situation when this happens.

Lemma 6.68. *Let $\tilde{X} \subset \mathfrak{g}$ be the set of generators of the G-relative equilibrium x and $X := \tilde{X} \cap \mathfrak{t}$. If X is not contained in a Weyl wall, then each G_x-adjoint orbit of an element of $\tilde{X}$ contains an element of X.*

Proof. To see this, note that $\tilde{X}$ is a G_x-invariant affine subset of $\mathfrak{g}$ of the form $\eta_0 + \mathfrak{g}_x$, where we can choose η_0 to be contained in $\mathfrak{g}_x^{\perp}$ (with respect to an invariant inner product on $\mathfrak{g}$). Since $\tilde{X}$ is G_x-invariant, G_x fixes η_0 and hence $[\xi, \eta_0] = 0 \quad \forall \xi \in \mathfrak{g}_x$. Consider the Lie algebra $\mathfrak{l} = \mathfrak{g}_x \oplus \langle \eta_0 \rangle$. Suppose that $\mathfrak{c}$ is a Cartan subalgebra of $\mathfrak{l}$. Then all G_x-orbits in $\mathfrak{l}$ intersect $\mathfrak{c}$: There is a splitting $\mathfrak{c} = \mathfrak{c}' \oplus \langle \eta_0 \rangle$, where $\mathfrak{c}'$ is a Cartan subalgebra of $\mathfrak{g}_x$ and thus intersects all adjoint orbits in $\mathfrak{g}_x$.

Since $\mathfrak{l} \cap \mathfrak{t}$ is an Abelian subalgebra of $\mathfrak{l}$, it is contained in a Cartan subalgebra $\mathfrak{c} \subset \mathfrak{l}$. If $\mathfrak{l} \cap \mathfrak{t} = \mathfrak{c}$, every element of $\tilde{X}$ has an element of $\mathfrak{l} \cap \mathfrak{t}$ in its G_x-orbit. Then the G_x-invariance of $\tilde{X}$ implies that it is contained in $\tilde{X} \cap \mathfrak{t} = X$. If $\mathfrak{l} \cap \mathfrak{t} \subsetneq \mathfrak{c}$, there is a Cartan subalgebra $\mathfrak{t}' \neq \mathfrak{t}$ of $\mathfrak{g}$ that contains $\mathfrak{c}$. Thus the elements of $\mathfrak{l} \cap \mathfrak{t}$ are contained in the two different Cartan subalgebras $\mathfrak{t}$ and $\mathfrak{t}'$ of $\mathfrak{g}$ and hence they have to be contained in a Weyl wall of $\mathfrak{t}$ (see [BtD85, Proposition 2.3, part (iv)]). □

Together with the consideration in Remark 6.67 we obtain:

Corollary 6.69. *Let $X \subset \mathcal{Z}$ be a regular affine subset that is not contained in a Weyl wall and M be the stratum that bifurcates at X. Then M intersects the G-orbit of a T-relative equilibrium x iff for every generator ξ of x there is $g \in G$ with $gx \in M$ and* $\mathrm{Ad}_g x \in \mathfrak{t}$.

Remark 6.70. Notice that G_ξ is the union of all tori whose Lie algebra contains ξ and hence G_ξ is connected ([BtD85, chapter IV, Theorem 2.3, part (ii)]). $T \subset G_\xi$ is also a maximal torus of G_ξ and hence the subspaces of V_0 of the form $\mathbb{C}_\alpha$ are also weight spaces of the G_ξ-representation V_0. The Weyl group $W(G_\xi)$ is generated by a subset of the reflections of $W(G)$.

Since V_0 is G_ξ-invariant, the set of weights of V_0 is $W(G_\xi)$-invariant. The following lemma shows, that there are two possible cases:

Lemma 6.71. *Let $S := \{\alpha_1, \dots, \alpha_k\} \subset \mathfrak{t}^*$ be a subset of the weights of some complex G-representation V, $c \in \mathbb{R} \setminus \{0\}$, and $X \subset \mathfrak{t}$ maximal with*

$$\alpha_i(\xi) = c \quad \forall \xi \in X, \forall i = 1, \dots, k$$

such that S is the maximal set of weights of V with this property for X. Let $Z \subset G$ be a connected subgroup with $T \subset Z$ and $W(Z)$ its Weyl group. The following conditions are equivalent:

1. $W(Z)S = S$,

2. $W(Z)X = X$.

If the conditions hold, each reflection $w \in W(Z)$ acts trivially on exactly one of the sets S and X. Hence either S or X is contained in the corresponding Weyl wall.

Proof. $1 \Rightarrow 2$: By invariance of S,

$$\alpha_i(\mathrm{Ad}_w(\xi)) = \mathrm{Coad}_{w^{-1}}\, \alpha_i(\xi) = c \quad \forall \xi \in X, \forall i, \forall w \in W(Z).$$

By maximality of X, $W(Z)X = X$.

$2 \Rightarrow 1$: Similar.

For the last statement, we identify $\mathfrak{t}$ and $\mathfrak{t}^*$ via a W-invariant inner product. Let $\langle S\rangle$ be the span of S and consider the splitting $\mathfrak{t} = \langle S\rangle \oplus \langle S\rangle^\perp$. If $w \in W(Z)$ is a reflection with $wS = S$, then $w\langle S\rangle = \langle S\rangle$ and $w\langle S\rangle^\perp = \langle S\rangle^\perp$. Then $X = x_0 + \langle S\rangle^\perp$ for some $x_0 \in \langle S\rangle^w$. Since w is a reflection about a hyperplane, w fixes either $\langle S\rangle$ or $\langle S\rangle^\perp$ but not both. Hence either $X \in \mathfrak{t}^w$ or $S \in \mathfrak{t}^w$. □

Hence, if $G_\xi = G_{\xi_0}$ is non-Abelian, either the set X for V_0 is contained in some Weyl wall or X is perpendicular to a Weyl wall, ξ_0 is contained in the intersection of X with the Weyl wall, and the weights of V_0 are all contained in the corresponding Weyl wall.

Since we can predict bifurcations if the set $\{\alpha_1, \dots, \alpha_k\}$ is linearly independent, we investigate the G_ξ-action on V_0 in this case now.

Now, let us investigate the $W(G_\xi)$-action on the linear independent set S if $V_0 = \bigoplus_{\alpha \in S} \mathbb{C}_\alpha$.

Lemma 6.72. *Assume a set of linearly independent weights of some symplectic G-representation is left invariant by a Weyl group reflection w. Then there is at most one pair of different weights α and $w\alpha$ contained in the set and all the other elements are fixed by w.*

Proof. Assume that α, $w\alpha$, β, and $w\beta$ are pairwise different weights. Since w fixes a hyperplane, the image of $\mathbb{1} - w$ is 1-dimensional. Hence, $\alpha - w\alpha$ and $\beta - w\beta$ are multiples of each other. Thus α, $w\alpha$, β, and $w\beta$ are linearly dependent. □

Hence, given a reflection $w \in W(G_\xi)$, there are two possible cases for the linearly independent set S of weights of V_0:

1. w fixes every element of S.

2. w fixes every element of S but some pair of different weights β and $w\beta$ contained in S.

The main observation on the orbit structure of V_0 is contained in the following lemma. Here, we assume that V_0 is contained in some symplectic irreducible G-representation. To prove the general case, we will split V_0 into G_ξ-invariant sums of weight spaces that are contained in different irreducible G-symplectic representations.

Lemma 6.73. *Suppose that* $V_0 = \ker \mathrm{d}^2(h - \mathbf{J}^\xi)(0)$ *is contained in an irreducible* G*-symplectic subrepresentation of* V *and has a linearly independent set of weights* S*. Consider a Weyl group reflection* $w \in W(G_\xi)$*. Then there is a connected closed subgroup* $Z \subset G_\xi$ *with* $T \subset Z$ *and* $W(Z) = \langle w \rangle$*. The* Z*-orbits are as follows, depending on the above two cases:*

1. *In case 1, the* Z*-orbits coincide with the* T*-orbits.*
2. *In case 2, the space* $\mathbb{C}_\beta \oplus \mathbb{C}_{w\beta}$ *is* Z*-invariant. The* Z*-orbits in* $\mathbb{C}_\beta \oplus \mathbb{C}_{w\beta}$ *coincide with the norm spheres. The complement* $\bigoplus_{\alpha \in S \setminus \{\beta, w\beta\}} \mathbb{C}_\alpha$ *is also* Z*-invariant. Within this space, the* G_ξ*-orbits coincide with the* T*-orbits.*

With the help of Lemma 6.73 we can deduce the relations between the G-orbits of strata that bifurcate at regular intersections of hyperplanes in $\mathfrak{t}$:

Theorem 6.74. *Assume that the* G*-invariant Hamiltonian function* $h : V \to \mathbb{R}$ *satisfies the genericity assumptions (GC) and (NR'). Consider the strata of* T*-relative equilibria of* X_h *that bifurcate at regular affine subsets of* $\mathcal{Z}$*. Then the union of these strata is contained in the* G*-orbit of the union of those strata that bifurcate at regular affine subsets of* $\mathcal{Z}$ *that are not contained in a Weyl wall.*

If $X, Y \subset \mathcal{Z}$ *are regular affine subsets not contained in a Weyl wall, the strata that bifurcate at* X *and* Y *respectively have the same* G*-orbit iff the Weyl group orbits of* X *and* Y *coincide. Otherwise, their* G*-orbits are disjoint.*

Proof. Let $X \subset \mathcal{Z}$ be a regular affine subset contained in a Weyl wall $\mathfrak{t}^w$ fixed by the Weyl group reflection $w \in W$. Then X is an intersection of hyperplanes consisting of the solutions ξ of equations of the form

$$2\pi\alpha_j^i(\xi) = c_i,$$

where c_i is an eigenvalue of $\mathrm{d}^2h(0)\big|_{\mathbb{E}_c}$ and α_j^i a weight of the corresponding eigenspace U_i. Let X_i denote the intersection of solution sets corresponding to the eigenvalue c_i and S_i be the corresponding set of weights. Since X is w-invariant and $X \subset X_i$, we also have $X \subset wX_i$ and thus also $X_i = wX_i$ by minimality of X_i. Thus by Lemma 6.71, each set S_i is w-invariant, too. Moreover, for each i, X_i is either contained in $\mathfrak{t}^w$ or perpendicular to $\mathfrak{t}^w$. Since $X = \bigcap_i X_i \subset \mathfrak{t}^w$ and X is non-empty, there is at least one particular i^* with $X_{i^*} \subset \mathfrak{t}^w$. Then again by Lemma 6.71, S_{i^*} contains a pair $\beta \neq w\beta$ of weights. By Lemma 6.72, all weights in $\alpha \in S \setminus \{\beta, w\beta\}$ satisfy $w\alpha = \alpha$. Now we choose $\xi \in X$ with

$$\ker \mathrm{d}^2(h - \mathbf{J}^\xi) = \bigoplus_i \underbrace{\bigoplus_{\alpha \in S_i} \mathbb{C}_\alpha}_{\subset U_i} =: V_0.$$

Since $w\xi = \xi$ we have $w \in W(G_\xi)$. By Lemma 6.73 and the G_ξ-invariance of V_0, we have $V_0 = G_\xi W_0$ with

$$W_0 := \Big(\bigoplus_{i \neq i^*} \underbrace{\bigoplus_{\alpha \in S_i} \mathbb{C}_\alpha}_{\subset U_i} \oplus \underbrace{\bigoplus_{\alpha \in S_{i^*} \setminus \{\beta, w\beta\}} \mathbb{C}_\alpha}_{\subset U_{i^*}} \Big) \oplus \mathbb{C}_\beta.$$

Thus the above reasoning shows that the stratum that bifurcates at V_0 is contained in the G-orbit of the stratum that bifurcates at W_0.

On the contrary, let $Y \subset \mathcal{Z}$ be a regular affine subsets that is not contained in any Weyl wall. By Lemma 6.65 all strata that bifurcate at any image of Y under the action of a Weyl group element have the same G-orbit.

Suppose that $Y' \subset \mathcal{Z}$ is also a regular affine subset not contained in any Weyl wall such that the intersection of the G-orbits of the strata M and M' that bifurcate at Y and at Y' respectively is non-empty. We have to show, that there is a Weyl group element that maps Y to Y'. Suppose that $x \in M$ and $x' \in M'$ are contained in the same G-orbit and choose a generator $\xi \in \mathfrak{t}$ of x that is not contained in any Weyl wall. By Corollary 6.69, there is $g \in G$ with $gx = x'$ and a generator $\xi' \in \mathfrak{t}$ of x' with $\mathrm{Ad}_g \xi = \xi'$. As argued above, we may assume that $g \in G_\xi$; we just replace Y by one of its images under the action of the Weyl group $W(G)$. Now, since $G_\xi = T$ and the strata are T-invariant, x and x' are contained in the same stratum then. □

The content of remaining part of this section is the proof of Lemma 6.73.

In order to investigate the G_ξ-action on V_0, we will need the following fact about compact Lie groups:

Lemma 6.75. *Let $Z \subset G$ be a connected closed subgroup with $T \subset Z$. For $w \in W(Z)$, let $(\mathfrak{t}^*)^w$ be the corresponding Weyl wall and let $G(w) := \bigcap_{\xi \in (\mathfrak{t}^*)^w} G_\xi$ denote the maximal subgroup that acts trivially on $(\mathfrak{t}^*)^w$. Then the Weyl group of $G(w)$ of the identity is generated by w. Moreover, the group Z is generated by the groups $G(w)$.*

Remark 6.76. Note that $G(w)$ is the centralizer of the group connected component of 1 of the group $\ker \theta_\alpha$, where α is a root corresponding to w (see below). Thus, as for G_ξ, we obtain from [BtD85, chapter IV, Theorem 2.3, part (ii)] that $G(w)$ is connected.

To prove the lemma, we will use the following observation:

Lemma 6.77. *If $G' \subset G$ is a closed connected subgroup of the compact connected Lie group G such that G' contains a maximal torus T of G and $W(G') = W(G)$, then $G' = G$.*

Proof. The Weyl group of G is determined by the *roots* of G, i.e. the nontrivial weights of the complexified adjoint representation on $\mathfrak{g}$. The roots form a *root system*, see [BtD85, chapter V, section 3] for the definition and proofs. The Weyl group $W(G)$ is generated by the orthogonal reflections that map a pair of roots of the form α and $-\alpha$ into each other, where we have chosen an appropriate inner product. There are no other roots than α and $-\alpha$ contained in the span of α. Thus each of these reflections corresponds to a single pair of roots. Since $W(G) = W(G')$, all (such pairs of) roots of G are also roots of G'. Moreover, the weight space corresponding to weight 0 of the complexified adjoint representation is $(\mathfrak{g} \otimes \mathbb{C})^T = \mathfrak{t} \otimes \mathbb{C} \subset \mathfrak{g}' \otimes \mathbb{C}$. Thus $\mathfrak{g}' \otimes \mathbb{C} = \mathfrak{g} \otimes \mathbb{C}$, which implies $\mathfrak{g}' = \mathfrak{g}$ and hence $G' = G$. □

Proof of Lemma 6.75. In principle, this is just [BtD85, chapter V, Proposition 2.3, part (ii)]: The above observation shows that Z is uniquely determined by $W(Z)$: If $Z' \subset G$ is a closed connected subgroup with $T \subset Z'$ and $W(Z') = W(Z)$, this implies $W(Z' \cap Z) = W(Z)$ and thus $Z = Z' \cap Z = Z'$.

Let α be a root of G and $w_\alpha \in W$ be the reflection associated to the pair of roots α and $-\alpha$ (see the proof of the above lemma). Let $\theta_\alpha : T \to U(1)$ be the homomorphism given by

$$\exp(\xi) \mapsto e^{2\pi \mathrm{i}\alpha(\xi)}$$

and set $U_\alpha := \ker \theta_\alpha \subset T$. Then w_α fixes the Weyl wall $\ker \alpha$, which coincides with the Lie algebra of U_α.

Suppose that $u_\alpha \in U_\alpha$ and $u_\alpha \notin U_\beta$ for any root $\beta \neq \alpha$ of G and $u_\alpha = \exp \xi_\alpha$ for some small ξ_α. Then the centralizer $Z(u_\alpha) := \{g \in G \mid g u_\alpha g^{-1} = u_\alpha\}$ satisfies $Z(u_\alpha) = G_{\xi_\alpha}$. Thus [BtD85, chapter V, Proposition 2.3, part (ii)] yields that $G(w_\alpha) = G_{\xi_\alpha}$ and that the Lie algebra of $G(w_\alpha)$ is given by $\mathfrak{t} \oplus M_\alpha$, where M_α is the real part of the sum of the weight spaces of α and $-\alpha$ in $\mathfrak{g}$. Thus the Weyl group of $G(w_\alpha)$ is generated by w_α.

Let R^+ contain exactly one of each pair $\alpha, -\alpha$ of roots of Z and suppose $t \in \bigcap_{\alpha \in R^+} U_\alpha$. Then by [BtD85, chapter V, Proposition 2.3, part (ii)], the Lie algebra of $Z(t)$ is given by $\mathfrak{t} \bigoplus_{\alpha \in R^+} M_\alpha$. Thus the Lie algebra of $Z(t)$ is the sum of the Lie algebras of the groups $G(w_\alpha)$ for $\alpha \in R^+$. Hence the groups $G(w_\alpha)$ generate $Z(t)$ (see [BtD85, chapter I, equation (3.5)]). Moreover, $Z(t)$ and Z have the same roots (and hence the same Weyl groups). Thus $Z = Z(t)$. □

Lemma 6.78. *Assume the genericity assumption (GC) is satisfied. Suppose*

$$V_0 = \ker \mathrm{d}^2(h - \mathbf{J}^\xi)(0) \subset U$$

for some $\xi \in \mathfrak{t}^*$ and some complex irreducible eigenspace $U \subset V$ of $\mathrm{d}^2 h(0)$ such that $V_0 = \bigoplus_{\alpha \in S} \mathbb{C}_\alpha$ for a linearly independent set of weights of S. Suppose further that there is some connected subgroup $Z \subset G_\xi$ containing T such that S is $W(Z)$-invariant. Then for any $W(Z)$-invariant subset $S' \subset S$, the space $\bigoplus_{\alpha \in S'} \mathbb{C}_\alpha$ is Z-invariant.

Proof. By Lemma 6.75, it suffices to show that for any $w \in W(Z)$ and any w-invariant subset $S' \subset S$ the space $\bigoplus_{\alpha \in S'} \mathbb{C}_\alpha$ is $G(w)$-invariant.

To show this, partition S into its $\langle w \rangle$-orbits. Each orbit has either 1 or 2 elements. Let $\mathrm{d}^2 h(0)\big|_U = c\mathbb{1}$, where $c \in \mathbb{R} \setminus \{0\}$. For each of these orbits S_i, consider the maximal affine subset $X_i \subset \mathfrak{t}$ of solutions ξ of

$$2\pi\alpha(\xi) = c$$

for all α contained S_i. By the linear independence of S, $X_i \not\subset X_j$ if $i \neq j$. X_i is perpendicular to the Weyl wall $\mathfrak{t}^w$ if S_i has 1 element, and X_i is contained in $\mathfrak{t}^w$ if S_i has 2 elements. In both cases, the intersection $X_i \cap \mathfrak{t}^w$ is a hyperplane of $\mathfrak{t}^w$. By $X_i \not\subset X_j$ for $i \neq j$, no two of these hyperplanes coincide. Thus, for any i, there is an $\eta_i \in X_i \cap \mathfrak{t}^w$ that is not contained in X_j for any $j \neq i$. Thus,

$$\ker \mathrm{d}^2(h - \mathbf{J}^{\eta_i})(0) \cap V_0 = \bigoplus_{\alpha \in S_i} \mathbb{C}_\alpha.$$

Since $G(w) \subset Z$, V_0 is $G(w)$-invariant. Moreover, by definition, $G(w) \subset G_{\eta_i}$. The G_{η_i}-invariance of $\ker \mathrm{d}^2(h - \mathbf{J}^{\eta_i})(0)$ yields the $G(w)$-invariance of $\bigoplus_{\alpha \in S_i} \mathbb{C}_\alpha$. □

We are now in the position to prove Lemma 6.73

Proof of Lemma 6.73. Set $Z := G(w)$. Let us investigate the Z-orbits in both cases:

1. According to Lemma 6.78, $W(Z)\alpha = \alpha$ implies $Z\mathbb{C}_\alpha = \mathbb{C}_\alpha$ for every $\alpha \in S$. Since T acts transitively on the spheres of $\mathbb{C}_{\alpha_i}$, the Z-orbits of V_0 coincide with the T-orbits.

2. Since $\tilde{w}\mathbb{C}_\beta = \mathbb{C}_{w\beta}$ for some representative $\tilde{w}$ of w in Z and Z is connected, there is a path contained in a Z-orbit that connects a point of the unit sphere of $\mathbb{C}_\beta$ with a point of the unit sphere of $\mathbb{C}_{w\beta}$. Because

$$Z(\mathbb{C}_\beta \oplus \mathbb{C}_{w\beta}) = \mathbb{C}_\beta \oplus \mathbb{C}_{w\beta}$$

holds by Lemma 6.78, the path is contained in $\mathbb{C}_\beta \oplus \mathbb{C}_{w\beta}$. On the unit-sphere of $\mathbb{C}_\beta \oplus \mathbb{C}_{w\beta}$, the T-orbits coincide with the level sets of $|x_\beta|^2$. Since the path connects a point in the 1-level set to a point of the 0-level set, it intersects all T-orbits. Hence Z acts transitively on the spheres of $\mathbb{C}_\beta \oplus \mathbb{C}_{w\beta}$. $G(w)\mathbb{C}_{\alpha_i} = \mathbb{C}_{\alpha_i}$ for every $\alpha_i \neq \beta, w\beta$ yields

$$\bigoplus \mathbb{C}_{\alpha_i} = Z(\bigoplus_{\alpha_i \neq w\beta} \mathbb{C}_{\alpha_i}) = Z(\bigoplus_{\alpha_i \neq \beta} \mathbb{C}_{\alpha_i}).$$

□

Remark 6.79. More generally, suppose that $V_0 = \ker \mathrm{d}^2(h - \mathbf{J}^\xi)(0)$ is contained in a complex irreducible eigenspace of $\mathrm{d}^2 h(0)$ and let the set S of weights of V_0 be linearly independent. If a subset $S' \subset S$ is an orbit of the action of the Weyl group $W(Z)$ of a connected closed subgroup $T \subset Z \subset G_\xi$, Z acts transitively on the spheres of the sum $\bigoplus_{\alpha \in S'} \mathbb{C}_\alpha$ of the weight spaces: This follows from Kostant's theorem, which states that for a connected compact group, the projection of a coadjoint orbit to $\mathfrak{t}$ coincides with the convex hull of the corresponding Weyl group orbit (see for example [Ati82]): $\bigoplus_{\alpha \in S'} \mathbb{C}_\alpha$ is Z-invariant by Lemma 6.78. Furthermore, for any point x of the unit-sphere of $\mathbb{C}_\alpha$ with momentum value $\mathbf{J}(x) = \mu$, the projection of the orbit $Z\mu$ to $\mathfrak{t}$ must contain the convex hull of $\pi S'$. Hence Zx intersects all T-orbits in the unit sphere of $\bigoplus_{\alpha \in S'} \mathbb{C}_\alpha$.

6.5 Examples

In the following, we consider some representations of SU(3) and compute the tangent spaces at which manifolds of relative equilibria bifurcate. Consider the centre space $\mathbb{E}_c$ of $\mathrm{d}X_h(0)$ as a complex SU(3)-representation. The diagonal matrices with entries in $S^1 \subset \mathbb{C}$ and determinant 1 form a maximal torus T of SU(3). Its Lie algebra $\mathfrak{t}$ consists of diagonal matrices with entries in $\mathrm{i}\mathbb{R}$ and trace 0. We identify $\mathfrak{t}$ and $\mathfrak{t}^*$ with the space

$$\left\{\xi \in \mathbb{R}^3 \mid x_1 + x_2 + x_3 = 0\right\} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}^\perp.$$

Then the restriction of an element in the dual space $(\mathbb{R}^3)^*$ to $\mathfrak{t}$ corresponds to the orthogonal projection to $(1,1,1)^\perp$.

The Weyl group is the symmetric group S_3, which acts on $\mathfrak{t} = \mathfrak{t}^*$ by permutations of the coordinates (see [BtD85, chapter IV, Theorem 3.3]).

Example 6.80. Let $\mathbb{E}_c = \mathbb{C}^3$ be the standard representation of SU(3). The weights of V are given by the projections of $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$ to $(1, 1, 1)^\perp$. Thus the weights are $\alpha_1 = (\frac{2}{3}, -\frac{1}{3}, -\frac{1}{3})$, $\alpha_2 = (-\frac{1}{3}, \frac{2}{3}, -\frac{1}{3})$, and $\alpha_3 = (-\frac{1}{3}, -\frac{1}{3}, \frac{2}{3})$. Each pair of weights forms the orbit of a reflection of the Weyl group. Thus if $c \in \mathbb{R}\setminus\{0\}$ is the eigenvalue of $\mathrm{d}^2h(0)\big|_{\mathbb{E}_c}$, the corresponding affine lines of solutions to $-2\pi\alpha(\xi) = c$ intersect only in the Weyl walls. See Figure 6.1: The black lines represent the Weyl walls in $\mathfrak{t}^*$. The black dots mark the weights. The dashed lines show a possible configuration of the affine lines when $\mathfrak{t}^*$ and $\mathfrak{t}$ are identified.

Thus we obtain one branch of SU(3)-orbits of relative equilibria, whose trajectories form periodic solutions. The branch forms a manifold tangent to the centre space $\mathbb{E}_c$. Alternatively, we may deduce this from the fact that each 1-dimensional complex subspace is the fix point subspace of a group isomorphic to SU(2).

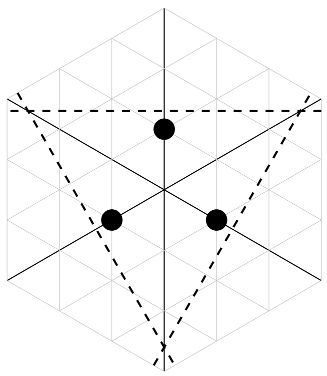

Figure 6.1: $\mathbb{C}^3$ (Example 6.80)

Example 6.81. Let $\mathbb{E}_c = \mathbb{C}^3 \oplus \mathbb{C}^3$ be the sum of two copies of the standard representation. Suppose the genericity assumption that $\mathrm{d}^2h(0)\big|_{\mathbb{E}_c}$ has two different non-zero eigenvalues c_1, c_2 with $c_1 \neq -2c_2$ and $c_2 \neq -2c_1$. This situation is shown in Figure 6.2. The dashed lines represent the affine lines for the first eigenvalue, the dotted lines correspond to the second one.

Then we obtain two manifolds of relative equilibria, each of which is tangent to one copy of $\mathbb{C}^3$. Moreover, for any pair of two weight spaces contained in different copies of $\mathbb{C}^3$ and corresponding to different weights, there is a manifold of T-relative equilibria tangent to their sum. All of these additional relative equilibria are contained in one branch of SU(3)-orbits.

By representation theory of compact connected Lie groups G, the irreducible representations G are uniquely determined by the Weyl group orbit of their *maximal weight*. A weight is *maximal* iff the convex hull of its Weyl group orbit contains all other weights of the representation. We now consider the SU(3)-representation with maximal weight $(1, 0, -1)$. This is a complex representation of dimension 8. The weights are given by the Weyl group orbit of $(1, 0, -1)$ and

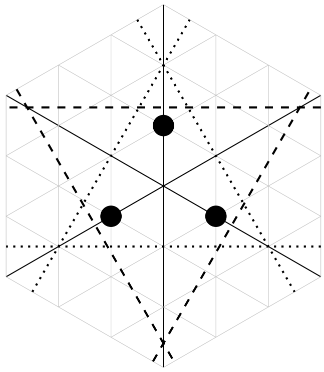

Figure 6.2: $\mathbb{C}^3 \oplus \mathbb{C}^3$ (Example 6.81)

the weight 0, which occurs with multiplicity 2. (See [Hal03, section 6.5]. Note that Hall chooses a different isomorphism $\mathfrak{t}^* \simeq \mathbb{R}^2$ there, such that the weight $(1,1)$ in [Hal03] corresponds to our weight $(1,0,-1)$.)

Example 6.82. Let $\mathbb{E}_c = \mathbb{C}^8$ be the irreducible representation with maximal weight $(1,0,-1)$ and suppose that $d^2h(0)$ is non-degenerate. First of all, we obtain for each non-trivial weight a manifold of T-relative equilibria with periodic trajectories which is tangent to the corresponding weight space. Since all these weights are contained in the same Weyl group orbit, the G-orbits of these manifolds coincide. Although we have only one Weyl group orbit of non-trivial weights here, there are pairs of weights that do not form an orbit of some reflection. A possible configuration of the affine lines is shown in Figure 6.3.

Thus in addition to the relative equilibria contained in the above G-orbit of manifolds, we obtain a manifold of T-relative equilibria for each such pair that is tangent to the sum of the corresponding weight spaces. Again, all these additional manifolds are contained in the same G-orbit.

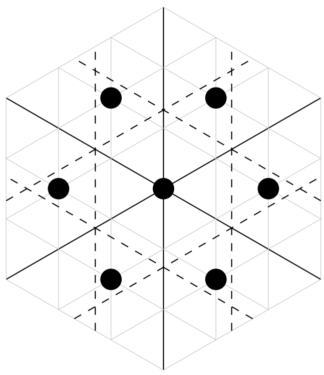

Figure 6.3: $\mathbb{C}^8$ (Example 6.82)

6.6 Application to Birkhoff normal forms

Although in mechanical systems the symmetries of nature typically correspond to isotropy subgroups of the states of at most rank 1, approximations of the Hamiltonian functions near an equilibrium often have symmetry groups of higher rank.

Consider for example the harmonic oscillator in n degrees of freedom. It consists of the quadratic part h_2 of the Hamiltonian h near an equilibrium p such that $\mathrm{d}^2 h(p)$ is positive definite. There are canonical coordinates

$$(q_1, \dots q_n, p_1, \dots p_n)$$

such that $q_i(p) = p_i(p) = 0$ and

$$h_2 = \frac{1}{2} \sum_{i=1}^{n} \lambda_i (q_i^2 + p_i^2) \quad \lambda_i > 0, i = 1, \dots n.$$

For each symplectic subspace $\mathbb{C}_i$ generated by the elements of the form

$$(0, \dots, 0, q_i, \dots, p_i, 0, \dots, 0),$$

the standard action of $S^1 \simeq \mathrm{SO}(2) \simeq U(1)$ is canonical and leaves h_2 invariant. Thus the Hamiltonian system for the harmonic oscillator h_2 has T^n-symmetry.

Typically, the Hamiltonian function h is not T^n-invariant. Nevertheless, we often obtain canonical coordinates from the theory of Birkhoff normal forms, with respect to which also higher order approximations of h are invariant for an action of a subtorus:

The eigenvalues $\lambda_1, \dots, \lambda_n$ of $\mathrm{d}^2 h(0)$ are called *rationally independent* iff the only integer solution of $\sum_{i=1}^{n} a_i \lambda_i = 0$ is $a_i = 0$ for all i. In this case, the Birkhoff normal form theorem implies that of any $r \in \mathbb{N}$, there are canonical coordinates such that the Taylor polynomial h_r of h up to order r is T^n-invariant (see [Arn78, appendix 7 A]). Then all solutions of the Hamiltonian dynamical system for h_r are relative equilibria. Hence they are *quasi-periodic*, that is, their trajectories are dense in some torus. This is in general not true for the dynamical system for the vector field X_h. Nevertheless, for small time intervals, the integral curves for h_r are good approximations of the integral curves for h. Moreover, KAM theory implies that generically there is a set of invariant tori for the Hamiltonian system of the original Hamiltonian function h and the ratio of the measure of its intersection with a neighbourhood of the equilibrium and the full measure of the neighbourhood is arbitrarily close to 1 for small neighbourhoods, see [Arn78, appendix 8 D, section 4].

If the eigenvalues $\lambda_1, \dots, \lambda_n$ are rationally dependent, this is no longer true for arbitrary $r \in N$ in general. For example, this may occur if symmetries of the mechanical system cause multiple eigenvalues. Nevertheless, there are canonical coordinates such that h_r is invariant with respect to a subtorus of T^n:

Set $\lambda := (\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n$. The integer vectors $j \in \mathbb{Z}^n$ with $\langle j, \lambda \rangle = 0$ form a $\mathbb{Z}$-submodul J. Let k be the rank of the free $\mathbb{Z}$-modul J.

As shown in the proof of [Mos68, Theorem 3], there are canonical coordinates Q_i, P_i with the following properties: The quadratic part of h_r is given by

$$\frac{1}{2} \sum_{i=1}^{n} \lambda_i (Q_i^2 + P_i^2).$$

We may choose $(n-k)$ integer vectors $\gamma_1, \dots, \gamma_r$ that form a basis of $J^\perp \cap \mathbb{Z}^n$. For each $\gamma_j = (\gamma_1^j, \dots, \gamma_n^j)$, we obtain the function

$$G_j = \frac{1}{2} \sum_{i=1}^{n} \gamma_i^j (Q_i^2 + P_i^2).$$

Then we have $\{G_j, h_r\} = 0$ for every j and hence the G_j are conserved quantities of the Hamiltonian system for h_r.

The flow φ_j of the Hamiltonian vector field X_{G_j} is obviously periodic. Moreover, we have $\{G_j, G_k\} = 0$ for every pair i, j. Thus, the flows φ_i and φ_j commute. Hence they define a linear T^{n-k}-action on

$$\mathbb{R}^{2n} = \{(Q_1, \dots, Q_n, P_1, \dots, P_n)\},$$

which leaves h_r invariant.

The weight spaces of the representation are the subspaces $\mathbb{C}_i$ of vectors of the form $(0, \dots, 0, Q_i, \dots, P_i, 0, \dots, 0)$. The weights are given by

$$\alpha_i = (\gamma_i^1, \dots, \gamma_i^{n-k}).$$

Since $\lambda \in J^\perp$, λ is contained in the image of the matrix with columns γ_j, which coincides with the matrix with rows α_i. Let I be a subset of $\{1, \dots n\}$. For an element ξ of the Lie algebra $\mathfrak{t}$ of T^{n-k},

$$\ker \mathrm{d}^2(h - \mathbf{J}^\xi) = \oplus_{i \in I} \mathbb{C}_i$$

holds iff

$$-2\pi\alpha_i(\xi) = \lambda_i$$

is satisfied exactly for $i \in I$. Since λ is contained in the image of the matrix formed by the α_i, there is a such a ξ if the linear spans $\langle \alpha_i \rangle_{i \in I}$ and $\langle \alpha_i \rangle_{i \notin I}$ are complements within the span $\langle \alpha_1, \dots, \alpha_n \rangle$. This is the case if the λ_i for $i \in I$ are not part of any resonance relation, i.e. for any $j = (j_1, \dots, j_n) \in J$, we have $j_i = 0$ for $i \in I$. If there is a set I with this property and in addition the set $\{\alpha_i \mid i \in I\}$ is linearly independent, then the Hamiltonian dynamical system for h_r has a manifold of relative equilibria. This is not true for the original Hamiltonian function h anymore. Nevertheless for small time scales, some solutions resemble quasi-periodic motion. Moreover, the system on the symplectic manifold of relative equilibria may be considered as a perturbation of the quadratic system on its tangent space. Thus I also expect that generically a dense subset of this manifold consists of tori, which have in some sense a counterpart in the original system.

Chapter 7

Prospects

For future work, it might be interesting to determine the isotropy subgroups of the relative equilibria and their momenta. Then for a given relative equilibrium p with isotropy subgroup $G_p = H$, the isotropy subgroups of the momentum generator pair within the group $N = \left({}^{N(H)}\!\big/_{H} \right)^{\circ}$ may be computed. The following thought is a first idea to compute the group G_μ: Let M_0 be the manifold of T-relative equilibria that bifurcates at V_0. By the equivariant Darboux theorem, M_0 is locally T-symplectomorphic to V_0, its tangent space at the point 0. If $\phi : M_0 \to V_0$ is a T-equivariant symplectomorphism and p is a element of M_0 with $\mathbf{J}(m) = \mu$, the $\mathfrak{t}^*$-components of μ and $\mathbf{J}(\phi(p))$ coincide. For any connected closed subgroup $K \subset G$, Kostant's theorem (see for example [Ati82]) implies that the projection of the orbit $K\mu$ to $\mathfrak{t}^*$ contains the Weyl group orbit of the $\mathfrak{t}^*$-component $\mu\big|_{\mathfrak{t}}$ of μ. Thus the Weyl group of K fixes $\mu\big|_{\mathfrak{t}}$. Consequently, K fixes $\mu\big|_{\mathfrak{t}}$. Hence G_μ must be contained in the isotropy subgroup $G_{\mu|_{\mathfrak{t}}} = G_{\mathbf{J}(\phi(m))|_{\mathfrak{t}}}$ and the same is true for G_p.

Another approach for the future can be to consider parameter families of Hamiltonians on torus representations. This is interesting for two reasons:

The results for torus representations yield the local structure of the set of relative equilibria near a non-degenerate relative equilibrium for proper actions of Abelian groups. The global set of relative equilibria in general also contains degenerate relative equilibria. To analyse this situation, it might be helpful to introduce additional parameters that correspond to ρ.

Moreover, this may also give some insight into systems of representations of non-Abelian groups. In this case, the Hessian $\mathrm{d}^2(h - \mathbf{J}^\xi)(0)$ can have kernels that consist of sums of weight spaces corresponding to linearly dependent weights. In this case, there may be branches of relative equilibria tangent to the kernel, but the results of this thesis do not predict them. Kernels of this type generically do not occur for torus representations, but this changes if we consider parameter families of Hamiltonians. Thus, an analysis of this situation might yield generalizations of the theory in several ways.

As Patrick and Roberts point out in [PR00], the Hamiltonian functions that occur in mechanics are usually contained in a much more restricted set of functions. It may be valuable to investigate genericity properties within a smaller set of functions than $C^\infty(P)^G$.

Appendix A

Thom-Mather transversality theorem

This appendix is a translated part of my diploma thesis, which again follows [GWPL76].

A.1 C^∞- and Whitney C^∞-topology

In this appendix the definitions and properties of the topologies on spaces of differentiable functions used in this thesis are presented. In the case of smooth functions, the spaces are mostly considered with the C^∞-topology or Whitney C^∞-topology and we are usually not too strict which one to choose. The reason is that for the investigation of local phenomena, it does not matter: We will see that the quotient topologies on the spaces of germs are the same.

We start with the C^∞-topology. Let V and W be finite-dimensional $\mathbb{R}$-vector spaces. For $n \in \mathbb{N}$ and a compact subset $K \subset V$ we define the semi-norm

$$\|f\|_n^K := \sup_{x \in K} \|\mathrm{d}^n f(x)\| .$$

Set $D_n := \overline{B}_n(0) \subset V$. We obtain a metric

$$d(f,g) = \sum_{n=0}^{\infty} 2^{-n} \frac{\|f-g\|_n^{D_n}}{1 + \|f-g\|_n^{D_n}},$$

which induces the *C^∞-topology.*

Definition A.1. A locally convex topological vector X space is called *Fréchet space* iff its topology is induced by a metric with respect to which X is complete.

As is easy to see, $C^\infty(V,W)$ together with the C^∞-topology is a Fréchet space. Thus for mappings between such function spaces, the open mapping theorem applies:

Theorem A.2 (Open Mapping Theorem, [Rud73, Theorem 2.11])**.** *Let V, W be Fréchet spaces and $f : V \to W$ a surjective continuous linear map. Then f is open.*

In a similar way, the definition can be extended to $C^\infty(M,N)$ for smooth manifolds M and N: Given a coordinate chart (U,ϕ) on M, a compact subset $K \subset U$, a coordinate chart (V,ψ) on N, a function $f \in C^\infty(M,N)$ such that $f(K) \subset V$, $n \in \mathbb{N}$, and $\varepsilon > 0$, we consider the set $\mathcal{N}(\phi,\psi,K,f,n,\varepsilon)$ of maps $g \in C^\infty(M,N)$ with $g(K) \subset U$ and $\|\psi \circ (f-g) \circ \phi\|_n^{\phi^{-1}(K)} < \varepsilon$. Then the sets of this type form a base of the C^∞-topology.

Alternatively, we will often use the Whitney C^∞-topology, which coincides with the C^∞-topology if M is compact. Following [GG73], we introduce the Whitney C^k-topologies on $C^\infty(M,N)$. For $k=\infty$, we obtain the finest one of these topologies.

Definition A.3. Let M and N be smooth manifolds, $p \in M$, and $f,g : M \to N$ smooth maps with $f(p) = g(p) =: q$.

1. f and g are in *contact of first order* at p iff $\mathrm{d}f_p = \mathrm{d}g_p$. f and g are in *contact of k-th order* at p, denoted $f \sim_k g$ at p, if $\mathrm{d}f, \mathrm{d}g : TM \to TN$ are in contact of $(k-1)$-th order in all points of T_pM.

2. Let $J^k(M,N)_{p,q}$ be the set of equivalence classes of mappings $f : M \to N$ with respect to the relation $\sim_k$ at p. Then we define
$$J^k(M,N) := \cup_{(p,q)\in M\times N} J^k(M,N)_{p,q}.$$
An element $\sigma \in J^k(M,N)$ is called a *k-jet* from M to N.

3. Let $\alpha : J^k(M,N) \to M$ denote the projection $J^k(M,N)_{p,q} \ni \sigma \mapsto p$. Accordingly, let $\beta : J^k(M,N) \to M$ be given by $J^k(M,N)_{p,q} \ni \sigma \mapsto q$.

For open sets $U \subset \mathbb{R}^n$, $V \subset \mathbb{R}^m$, there is a bijection between $J^k(U,V)$ and the open subset $U \times V \times P^{(k)}(\mathbb{R}^n,\mathbb{R}^m)_0 \subset \mathbb{R}^n \times \mathbb{R}^m \times P^{(k)}(\mathbb{R}^n,\mathbb{R}^m)_0$. For smooth manifolds M and N, the sets $J^k(U,V) \subset J^k(M,N)$, where $U \subset M$ and $V \subset N$ are domains of charts, may be used to define the structure of a differentiable manifold on $J^k(M,N)$ ([GG73, chapter II, Theorem 2.2]).

Every smooth map $f : M \to N$ defines a smooth map
$$\begin{aligned} j^k f : M &\to J^k(M,N) \\ p &\mapsto j^k f(p), \end{aligned}$$
that maps $p \in M$ to the equivalence class of f in $J^k(M,N)$.

Definition A.4. For smooth manifolds M and N, $k \in \mathbb{N}_0$ and $U \subset J^k(M,N)$, set
$$M(U) := \left\{ f \in C^\infty(M,N) \;\middle|\; j^k f(M) \subset U \right\}.$$

Since $M(U) \cap M(V) = M(U \cap V)$, the sets $M(U)$ for open subsets $U \subset J^k(M,N)$ form a base of a topology on $C^\infty(M,N)$.

Definition A.5. The topology on $C^\infty(M,N)$ generated by
$$\left\{ M(U) \;\middle|\; U \subset J^k(M,N) \text{ open} \right\}$$
is called *Whitney C^k-topology.* W_k denotes the set of Whitney C^k-open subsets of $C^\infty(M,N)$.

For $k \le l$, we have $W_k \subset W_l$: Let $\pi^l_k : J^l(M,N) \to J^k(M,N)$ map the equivalence class of a function f in $J^l(M,N)_{p,q}$ to the equivalence class of f in $J^k(M,N)$. Then $M(U) = M((\pi^l_k)^{-1}(U))$ for any $U \subset J^k(M,N)$. Thus $\cup_{k=0}^\infty W_k$ is a base of a topology.

Definition A.6. The topology on $C^\infty(M,N)$ generated by the base $\cup_{k=0}^\infty W_k$ is called *Whitney C^∞-topology*

Since $J^k(M,N)$ is a smooth manifold, it is metrizable. Fix a metric d that induces the topology of $J^k(M,N)$.

For $f \in C^\infty(M,N)$ and a continuous function $\delta : M \to \mathbb{R}^+$, set

$$B^k_\delta(f) := \left\{ g \in C^\infty \;\middle|\; d(j^k f(x), j^k g(x)) < \delta(x)\ \forall x \in M \right\}.$$

The sets $B^k_\delta(f)$ form a neighbourhood system of f with respect to the C^k-topology: $B^k_\delta(f) = M(U)$, where

$$U := \left\{ \sigma \in J^k(M,N) \;\middle|\; d(j^k f(\alpha(\sigma)), \sigma) < \delta(\alpha(\sigma)) \right\}.$$

Since U is open, $B^k_\delta(f)$ is Whitney C^k-open. Moreover, we show that for each open set $V \subset J^k(M,N)$ with $f \in M(V)$, there is a continuous function $\delta : M \to \mathbb{R}^+$ such that $B^k_\delta(f) \subset M(V)$. Set

$$m(x) := \inf \left\{ d(\sigma, j^k f(x)) \;\middle|\; \sigma \in \alpha^{-1}(x) \cap (J^k(M,N) \setminus V) \right\}.$$

Since m is bounded below by a positive number, there is a continuous function $\delta : M \to \mathbb{R}^+$ such that $\delta(x) < m(x)$ for every $x \in M$. This yields $B^k_\delta(f) \subset M(V)$.

If M is compact, every continuous function $\delta : M \to \mathbb{R}^+$ is bounded below. Thus the sets $B_{\frac{1}{n}}(f)$ form a neighbourhood system for $f \in C^\infty(M,N)$ of the Whitney C^k-topology. Thus, a sequence f_n converges to f iff all partial derivatives up to order k converge uniformly to the partial derivatives of f.

If M is not compact, convergence with respect to the Whitney C^k-topology is a stronger condition than uniform convergence of the partial derivatives up to order k: A sequence f_n converges to f iff there are a compact set K and a number $N \in \mathbb{N}$ such that f_n and f coincide outside of K for $n > N$ and all partial derivatives up to order k converge uniformly on K to the partial derivatives of f ([GG73, chapter II, §3]).

Definition A.7. Let X be a topological space. A subset of X is *residual* iff it is the intersection of countably many dense open subsets of X. X is a *Baire space* iff every residual subset of X is dense.

Theorem A.8 ([GG73, Proposition 3.3])**.** *For smooth manifolds M and N, the space $C^\infty(M,N)$ with the Whitney C^∞-topology is a Baire space.*

Now, let us come back to the local case: Let $C^\infty(D_k, W)$ be the set of functions from D_k to W that may be extended smoothly to V. Obviously, the restriction map $C^\infty(V,W)$ to $C^\infty(D_k,W)$ induces for the Whitney C^∞- and the C^∞-topology on $C^\infty(V,W)$ the same quotient topology on $C^\infty(D_k,W)$. Thus for the investigation of local phenomena, it does not matter, which of both topologies is chosen.

A.2 Transversality to Whitney stratified subsets

Here, we collect the definitions and most important facts of the theory of Whitney stratified subsets. For details, we refer to [GWPL76].

Definition A.9. A *stratification* $\mathcal{S}$ of a subset P of a smooth manifold M is a partition of P into smooth submanifolds of M such that each point of P has a neighbourhood that intersects only finitely many elements of $\mathcal{S}$. An element of $\mathcal{S}$ is called *stratum.* The pair $(P, \mathcal{S})$ forms a *stratified set.*

To define a Whitney stratification, we need a topology on the set $G(k, n)$ of k-dimensional vector subspaces of $\mathbb{R}^n$ for every pair of natural numbers $k \geq n$.

Suppose $W \subset G(k, n)$ and let $\{w_1, \dots, w_k\}$ be a basis of W. If $\{w'_1, \dots, w'_k\}$ is another basis of W, the elements $w'_1 \wedge \dots \wedge w'_k \in \bigwedge^k(\mathbb{R}^n)$ and $w_1 \wedge \dots \wedge w_k \in \bigwedge^k(\mathbb{R}^n)$ are multiples of each other. Moreover, $w \in W$ is equivalent to $w \wedge w_1 \wedge \dots \wedge w_k = 0 \in \bigwedge^{k+1}(\mathbb{R}^n)$.

Let $\mathbb{P}(\bigwedge^k(\mathbb{R}^n))$ be the projective space of $\bigwedge^k(\mathbb{R}^n)$. Then we obtain an injective map $\psi : G(k, n) \to \mathbb{P}(\bigwedge^k(\mathbb{R}^n))$, which is called *Plücker-embedding.*

Definition A.10. The image $\psi(G(k, n))$ of the Plücker-embedding together with the subset topology of $\mathbb{P}(\bigwedge^k(\mathbb{R}^n))$ is called *Graßmann space* $\mathbb{G}(k, n)$.

Remark A.11. For $k = 1$, the Graßmann space coincides with the projective space $\mathbb{P}^{n-1}(\mathbb{R})$.

To begin with, we define Whitney regularity for pairs of strata of a stratified set that is contained in an open subset of $\mathbb{R}^n$.

Definition A.12. For smooth submanifolds $X, Y \subset \mathbb{R}^n$, we call Y *Whitney regular over* X *at* $x \in X$ iff for all sequences $(x_i) \subset X$, $(y_i) \subset Y$ with $x_i \to x$ and $y_i \to x$ the following condition is satisfied: Set $k := \dim(Y)$. If the sequence of tangent spaces $(T_{y_i}Y)$ converges in the Graßmann space $\mathbb{G}(k, n)$ to a vector subspace $T \subset \mathbb{R}^n$ and the sequence of lines passing trough the origin spanned by $x_i - y_i$ has a limit L in $\mathbb{G}(1, n) = \mathbb{P}^{n-1}(\mathbb{R})$, then $L \subseteq T$.

Remark A.13. This condition is often called *Whitney condition (b).* It implies the *Whitney condition (a)*: With the notation of the definition, $T_{y_i}Y \to T$ implies $T_xX \subseteq T$.

For a triple (X, Y, x) as in the definition, the Whitney conditions (a) and (b) are obviously local conditions that are preserved under local diffeomorphisms. For smooth submanifolds X and Y of a smooth manifold, we call Y *Whitney regular over* X *at* $x \in X$ iff this holds with respect to local coordinates. Y is *Whitney regular over* X iff Y is Whitney regular over X at x for every $x \in X$.

Definition A.14. A stratification is called a *Whitney stratification* iff every stratum is Whitney regular over every other stratum.

For subsets $P \subset M$ and $Q \subset N$ of smooth manifolds M and N with Whitney stratifications $\mathcal{S}$ of P and $\mathcal{T}$ of Q, the stratification

$$\mathcal{S} \times \mathcal{T} := \{S \times T \mid S \in \mathcal{S}, T \in \mathcal{T}\}$$

is a Whitney stratification of $P \times Q \subset M \times N$. For $S \in \mathcal{S}$, let $S \times \mathcal{T} := \{S \times T \mid T \in \mathcal{T}\}$ be the *induced stratification* of $S \times Q$. If U is an open subset of P, the stratification $U \cap \mathcal{S} := \{S \cap U \mid S \in \mathcal{S}\}$ is a Whitney stratification of U.

In the literature, the definition of a Whitney stratification often contains an additional requirement, the *frontier condition*: If $X \cap \overline{Y} \neq \emptyset$ for a pair of strata X, Y, then $X \subset \overline{Y}$.

We do not require the frontier condition here. Nevertheless, for a Whitney stratification of a locally closed subset of a smooth manifold consisting of connected strata, the frontier condition is always satisfied ([GWPL76, chapter 2, Corollary 5.7]). (We may suppose connectedness of the strata w.l.o.g., since the connected components of the strata of a Whitney stratification form a locally finite partition and hence are a Whitney stratification, too ([GWPL76, chapter 2, Theorem 5.6]).)

Moreover, the local topological properties of the stratification are constant along a stratum of a Whitney stratification of a locally closed set. More precisely:

Definition A.15. A stratification $\mathcal{S}$ of a subset P of a smooth manifold is *topologically locally trivial at* $x \in P$ with x contained in the stratum $X \in \mathcal{S}$, iff there are a neighbourhood U of x in P, a stratified set $(F, \mathcal{F})$, a point $y \in F$ with $\{y\} \in \mathcal{F}$, and a homeomorphism h from U to a neighbourhood W of $(x, y) \in X \times F$ such that $h(x) = (x, y)$ and h maps each stratum of $U \cap \mathcal{S}$ to a stratum of $W \cap (X \times \mathcal{F})$. $\mathcal{S}$ is *topologically locally trivial* iff this holds at every $x \in P$

Theorem A.16 ([GWPL76, chapter II, Corollary 5.5])**.** *Any Whitney stratification of a locally closed set is topologically locally trivial.*

Given a set which admits Whitney stratifications, the question arises which one to choose. Of course, we would like to have one which is as coarse as possible in the sense that any other Whitney stratification is a refinement. Such a stratification does not always exist (see the counterexample in [GWPL76, chapter 1, below Result 1.4]). Nevertheless, in many cases, there is an apparent choice that is minimal in some sense:

Definition A.17. Let $\mathcal{S}$ be a stratification of a subset P of a smooth manifold. Then the *associated filtration by dimension* of P is given by $P = \cup_{i \geq 0} P_i$, where P_i denotes the union of strata of dimension $\leq i$.

Definition A.18. A Whitney stratification of a subset P of a smooth manifold is called *canonical* iff the associated filtration by dimension is given as follows: For every $i \geq 0$, the set $P_i \setminus P_{i-1}$ is the maximal subset of P_i that forms a smooth submanifold over which the set $P_j \setminus P_{j-1}$ is Whitney regular for every $j > i$.

Obviously, the canonical stratification satisfies the following minimality property: If $\mathcal{S}$ and $\mathcal{S}'$ are Whitney stratifications of the same set, let P_i and P_i' be the associated filtrations by dimension. We say that $\mathcal{S} \leq \mathcal{S}'$ iff there is some i such that $P_j = P_j'$ for all $j > i$ and $P_i \supsetneq P_i'$. Then the canonical stratification is minimal with respect to this partial order.

Now, we define a class of subsets of $\mathbb{R}^n$, which admit a canonical Whitney stratification:

Definition A.19. A subset M of $\mathbb{R}^n$ is called *semi-algebraic* iff it is a finite union of sets of the form $\{x \in \mathbb{R}^n \mid p(x) = 0, q(x) > 0\}$, where p and q are polynomials in the coordinates of $\mathbb{R}^n$.

Complements, finite intersections, and finite unions of semi-algebraic sets are obviously semi-algebraic. Moreover, the following holds (see [Loj65]): If $M \subset \mathbb{R}^n$ is semi-algebraic, $\overline{M}$, M°, and ∂M are semi-algebraic. For any polynomial map $p \in P(\mathbb{R}^n, \mathbb{R}^m)$, the image $p(M)$ is semi-algebraic. Each semi-algebraic set consists of finitely many connected components.

Theorem A.20 ([GWPL76, chapter 1, Result 2.7])**.** *Every semi-algebraic subset of $\mathbb{R}^n$ has a canonical Whitney stratification.*

To construct the canonical Whitney stratification for $P \subset \mathbb{R}^n$, the sets $P_n \setminus P_{n-1}$, $P_{n-1} \setminus P_{n-1}$, $\ldots$, $P_1 \setminus P_0$, and P_0 are defined one after another as the maximal set with the above property.

In the following, we require the strata of the canonical stratification to be connected. This way, it is unique.

A smooth map $f : M \to N$ between smooth manifolds M and N is *transverse* to a stratified set $(P, \mathcal{P}) \subset N$ or to the stratification $\mathcal{P}$ iff f is transverse to every stratum of S. For transverse maps to Whitney stratified sets, the Thom-Mather transversality theorem applies:

Theorem A.21 (Thom-Mather transversality theorem)**.** *Let M, N be smooth manifolds, $(P, \mathcal{P})$ a closed Whitney stratified subset of N, and $f \in C^\infty(M, N)$.*

1. *The set $T := \{x \in M \mid f \pitchfork P \text{ in } x\}$ is an open subset of M.*

2. *$\mathcal{T} := \{f \in C^\infty(M, N) \mid f \pitchfork P\}$ is a dense subset $C^\infty(M, N)$ with respect to the Whitney C^∞-topology. (This holds also if P is not closed.)*

3. *For a closed subset $A \subset M$, the set*
$$\mathcal{T}_A := \{f \in C^\infty(M, N) \mid f \pitchfork P \text{ along } A\}$$
is Whitney C^1-open.

4. *If M is compact and $g \in C^\infty(M \times (-\delta, \delta), N)$ such that $g_t := g(\cdot, t)$ is transverse to P for every $t \in (-\delta, \delta)$, there is an isotopy of homeomorphisms $h : M \times (-\delta, \delta) \to M$, $h_t := h(\cdot, t)$ such that $h_0 = \mathbb{1}_M$ and*
$$h_t(g_t^{-1}(P)) = P$$
for every $t \in (-\delta, \delta)$.

5. *If $x \in M$ (M not necessarily compact) and $g \in C^\infty(M \times (-\varepsilon, \varepsilon), N)$ for some $\varepsilon > \delta$ such that $g_t := g(\cdot, t)$ is transverse to P in x for every $t \in (-\varepsilon, \varepsilon)$, there is a compact neighbourhood K of x and an isotopy of continuous embeddings $h : K \times (-\delta, \delta) \to M$, $h_t := h(\cdot, t)$ such that h_0 is given by the inclusion $K \hookrightarrow M$ and for every $t \in (-\delta, \delta)$, we have*
$$h_t(K \cap f_0^{-1}(P)) = h_t(K) \cap f_t^{-1}(P).$$

Part 1 is a consequence of Whitney condition (a).

Part 2 follows from the classical transversality theorem.

For part 3, consider the set $U \subset J^1(M,N)$ which is defined as follows: $\sigma \in J^1(M,N)$ with representative $f : M \to N$ and $\alpha(\sigma) =: x$, $\beta(\sigma) = f(x) =: y$ is contained in U iff one of the following conditions holds: $x \notin A$, $y \notin P$, or $\mathrm{d}f(T_xM) + T_yS = T_yN$. Then f is transverse to P along A iff $f \in M(U)$. Whitney condition (a) implies that $J^1(M,N) \setminus U$ is closed.

Parts 4 and 5 may be deduced from Thom's first isotopy lemma ([GWPL76, chapter 2, Theorem 5.2]).

Similar statements hold for a Whitney stratified subset Q of $J^k(M,N)$: If $j^k f \pitchfork Q$, we call f *k-jet-transverse* to Q. $\mathcal{T}' := \{ f \in C^\infty(M,N) \mid j^k f \pitchfork Q \}$ is a dense subset of $C^\infty(M,N)$ and $\mathcal{T}'_A := \{ f \in C^\infty(M,N) \mid j^k f \pitchfork Q \text{ along } A \}$ is open if $A \subset M$ and Q are closed. This follows from the above theorem together with [GG73, Theorem 4.9 and Proposition 3.4]. The other statements imply their jet analogues immediately.

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Eidesstattliche Eigenständigkeitserklärung

Hiermit erkläre ich an Eides statt, dass ich die vorliegende Dissertationsschrift selbst verfasst und keine anderen als die angegebenen Hilfsmittel und Quellen benutzt habe.

Hamburg, 8. September 2017

Mara Sommerfeld