Essays in Optimal Mechanism Design

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Chapter 1

Introduction

The supply side of many markets is controlled by a single firm. A monopoly position can be acquired via different channels: luck, technological advantage, law, predation, and many others. The textbook analysis of monopoly considers a firm that chooses an output and sells it a uniform market-clearing price. Since the firm does not internalize the negative effect of increased prices on consumer surplus, the optimal price is set above the marginal cost and thus creates a deadweight loss of social welfare (Varian, 2014, p. 465). However, this model is rather simplistic because the firm is allowed to produce arbitrarily large outputs and it can not charge different prices to different consumers.

Mechanism design theory provides a framework for analyzing much richer monopoly problems. The theory treats the market as a "communication system" where the monopolist is a "message center" (Hurwicz, 1960). The center announces a (direct) mechanism and asks every buyer to report her valuation. Upon collecting these reports, the mechanism determines an allocation and a vector of payments. The allocation specifies a quantity that the center should allocate to every buyer, and the vector of payments specifies for every buyer a total expected payment that she should pay to the center. Of course, real markets operate differently from such mechanisms because buyers and sellers do not exchange reports. Does this mean that economists should rather focus on studying arbitrarily sophisticated mappings from market behavior into market outcomes? Fortunately, the famous Revelation Principle (Myerson, 1979) states that there is no loss of generality if we restrict attention to direct truthful mechanisms because they could replicate any strategic behavior within indirect mechanisms.

Mechanism design theory offers elegant solutions to the problem of optimal auction design (Harris and Raviv, 1981; Myerson, 1981; Riley and Samuelson, 1981) and the problem of optimal nonlinear pricing (Maskin and Riley, 1984; Mussa and Rosen, 1978). The first problem considers a seller that trades a single object with a group of potential buyers, and the second problem considers a seller that engages in second-degree price discrimination by offering a schedule of prices for every quantity or quality. In a truthful mechanism, a buyer's information rent is a value function of her monetary payoff. Therefore, as stated in the famous Myerson's lemma (Myerson, 1981, p. 63), it is completely determined by the allocation rule. It is routine to show that the seller seeks to maximize the expectation of virtual valuations (or simply marginal revenues) of the buyer(s). The optimal mechanism features a trade-off between allocative efficiency and rent extraction and "shuts down" buyer types with negative virtual valuations.

Although Myerson's lemma is extremely useful, it relies on strong assumptions. For example, buyers may not communicate or exchange side-payments before an auction. However, if they conspired to suppress competition by placing at most one bid, they would secure a lower expected payment to the seller and thus larger information rents. Collusive behavior of buyers has become the subject of analysis in a small but growing field of literature within auction theory. Mailath and Zemsky (1991) consider second-price auctions and prove that there exists a mechanism of collusion that manages to suppress competition without compulsory participation or outside subsidies. Since the seller can no longer rely on competitive forces to push the selling price up, the presence of collusion results in a larger reserve price.

Chapter 2 of this thesis contributes to the literature on collusion in auctions by relaxing the assumption of full commitment power. We consider a second-price auction where the seller promises to keep the object if a reserve price is not met. But when it is not met, the seller has no incentive to keep the object because he could make a profit by running a new auction with a lower reserve price. So we assume that the seller can commit to his promise only for a limited time, captured a discounting factor. Our goal is to understand whether there exists a mechanism of collusion that maximizes the collusive surplus at every possible auction without relying on outside subsidies and coercion. In addition, we find sequentially optimal reserve prices in the presence of such collusion.

Our main result states that buyers can not collude efficiently if they expect to collude efficiently in the future. To get some intuition, suppose that the cartel somehow manages to collude efficiently in two consecutive auctions. Since collusion yields larger information rents, more buyer types should wait for the later auction to realize these extra rents. As the time between auctions goes to zero, almost all buyer types should wait for the later auction, which means that there are no present gains of collusion. However, it is not privately optimal to pursue this "social goal" unless such waiting is incentivized with a "tax" on premature bidding. To sum up, the cartel collects a "tax" but fails to offer much in return. A buyer could therefore reduce her expected payment by leaving the cartel.

The remaining chapters relax another assumption behind Myerson's lemma, namely expected payoff maximization. Chapter 3 extends the problem of optimal nonlinear pricing to intention-based social preferences in the spirit of Rabin (1993). We consider a seller who may (complete information) or may not (incomplete information) know a buyer's valuation. After the seller makes his offer, the buyer evaluates the seller's kindness by comparing the payoff that she believes the seller intended to give her against some benchmark of "fair payoff". Naturally, the buyer is willing to reward the seller if he is kind towards her and punish him otherwise. We assume that the buyer's beliefs about the seller's intentions are consistent with the optimal behavior, and the fair payoff is a convex combination of the lowest and the highest payoffs among the Pareto-efficient outcomes.

Our results say that under complete information, the optimal contract stipulates efficient quantity and a price below the buyer's valuation. Also, the price is increasing in the seller's cost of production. Under incomplete information, the optimal mechanism is characterized by no distortion at the top and downward distortion elsewhere. Compared to Maskin and Riley (1984), the size of distortion is smaller because the seller internalizes a psychological cost that he imposes on the buyer with his unfair mechanism. The optimal mechanism can be implemented by a schedule of price-quantity pairs that involve quantity discounts and lower prices than Maskin and Riley (1984).

Chapter 4 extends the problem of optimal auction design to reference-dependent preferences in the spirit of Koszegi and Rabin (2006). That is, every buyer holds an expectation about her payment to the seller and suffers a loss of utility when she has to pay more than expected. We assume that her reference point may be stochastic and in equilibrium it must be consistent with the strategies of all buyers. Our findings suggest that first-price auctions are revenue superior to second-price auctions, and the optimal reserve price is lower with loss-averse buyers than with loss-neutral buyers. Furthermore, the seller prefers negotiations with one buyer to an efficient auction with two buyers for a sufficiently large degree of loss aversion, and he always prefers public reserve prices to secret reserve prices.

Our contribution to the literature is twofold. In terms of methodology, this thesis shows that it is possible to adopt standard concepts and techniques to richer environments such as limited commitment and behavioral preferences. For instance, Chapter 2 uses the revelation principle and the payoff equivalence principle to characterize the ex-post efficient collusion for an arbitrary revelation policy. Chapter 3 exploits a shortcut from optimal control theory to derive a first-order condition on the optimal allocation under incomplete information. Chapter 4 introduces the concept of imputed values to represent the expected revenue from a first-price auction as the expectation of "psychological" virtual valuations.

In terms of predictions, our results differ significantly from the standard models. In contrast to Mailath and Zemsky (1991), Chapter 2 shows that for sufficiently weak commitment power, efficient collusion is not feasible and the sequentially optimal reserves are lower under an efficient collusion than under competition. Chapter 3 finds the degree of allocative inefficiency under second-degree price discrimination as well as optimal prices to be lower than in Maskin and Riley (1984). Finally, Chapter 4 shows that the optimal reserve is lower compared to Riley and Samuelson (1981) and may be decreasing in the number of buyers. Therefore, our messages may enhance the expected revenue in those real-world markets where our sets of assumptions better capture preferences of buyers.

Chapter 2

Collusion in Sequentially Optimal Auctions

Abstract

We study collusion among buyers when the seller can not commit to withhold the object from the market forever when it fails to sell. Our setting is characterized by second-price auctions, ex-ante symmetric buyers, and full-inclusive rings. We find that for sufficiently large reserve prices, any ex-post efficient mechanism of collusion must violate budget balance or voluntary participation. Yet, we show that a "second-best" allocation can always be implemented by a second-price preauction knockout. Furthermore, the collusion exerts downward pressure on the optimal reserve prices because it reduces demand for the object. As the time between auctions goes to zero, this effect reverses the classical result that the optimal reserve price is larger under collusion.

Keywords: Bidding Rings, Sequentially Optimal Auctions, Efficient Collusion.

2.1 Introduction

2.1.1 Motivation

The main focus of auction theory is optimal auction design. Myerson (1981) and Riley and Samuelson (1981) make use of mechanism design theory to show that an optimal auction allocates the object to the buyer with the highest virtual value and "shuts down" buyer types with negative virtual values. Their analysis relies on a number of strong assumptions such as independently distributed values, risk neutrality, competitive behavior, and many more. The assumption of *competitive behavior* is especially restrictive. From the empirical perspective, it is not consistent with a rich body of evidence that documents collusion between buyers. For example, 81% of the Sherman Act Section 1 criminal cases filed in the 1980s involved auctions markets (Hendricks and Porter, 1989).

From a theoretical perspective, this assumption has a strong effect on predictions. If we instead assume that buyers can also share information and exchange side-payments, then they are able to form a bidding ring that fully suppresses competition by placing at most one bid above the reserve price (Mailath and Zemsky, 1991). For any reserve price, the ring maximizes the collusive surplus, increases every buyer's payoff and balances the budget. Since competition is suppressed, the seller can not rely on the second-highest value to push the selling price above the reserve. Therefore, he chooses a larger reserve price compared to Riley and Samuelson (1981).

Although models of collusive behavior improve upon the standard framework, they suffer from all other strong assumptions that they inherit. One of them is *full commitment*: if the reserve price is not met, the seller keeps the object forever. However, it is not a sequentially optimal plan of actions because the seller could earn a profit by re-auctioning the object. In many real-world auction markets, for instance, objects that fail to sell at a given auction are sold at a later auction. Examples include auctions for wine and art as well as government auctions for oil tracts and real estate (McAfee and Vincent, 1997).

Why could there be a loss of generality when we study collusive behavior under the assumption of full commitment? First of all, under limited commitment a bidding ring may fail to maximize the collusive surplus without outside subsidies or forced participation. To see this, suppose that the time horizon is limited to two periods, and a bidding ring manages to suppress competition in period 2. To realize these extra rents from collusion in period 2, many buyer types should refrain from bidding in period 1. As the time between periods goes to zero, there is no cost of waiting, so all buyer types should refrain from bidding and the ring fails to obtain any surplus in period 1. But the ring must discourage its members from bidding in period 1, so it collects a “tax” on bidding by means of a reserve price or an entry fee. As a result, a buyer’s expected payment to the ring may be positive, so it may be optimal to leave the ring.

Second of all, the presence of collusion may reduce the optimal reserve price. As argued above, an efficient ring postpones the purchase of the object. Because of time discounting, the seller prefers to sell the object as early as possible. In order to encourage the ring to purchase the object earlier, the seller should set a lower reserve price. Therefore, the upward effect on the optimal reserve from Mailath and Zemsky (1991) becomes offset by a downward effect, and only a formal analysis can show which effect dominates.

2.1.2 Research Question and Results

This paper investigates the impact of limited commitment on the existence of efficient collusion and on the optimal reserve price. We propose a model where a seller auctions off a single object to a number of buyers using a second-price auction with a reserve price. Buyers are assumed to be ex-ante symmetric and risk-neutral. We confine attention to bidding rings that include all buyers. If all bids fail to meet the reserve in a given period, there is another second-price auction with a reserve price in the next period. The reserve prices must be sequentially optimal, that is, they maximize the seller’s expected profit from every auction off and on equilibrium path, conditional on the information about the values that the seller learns from the event of no-sale.

Our findings suggest that there exist such reserve prices that make it impossible to implement an efficient collusion with an ex-ante budget balanced and individually rational mechanism. Yet we construct a second-best allocation and show that it can be implemented for any reserve price with second-price preauction knockouts in the spirit of Graham and Marshall (1987). Furthermore, the presence of a ring, whether efficient or second-best, pushes the optimal reserve prices down because the ring buys the object later. As the time between auctions goes to zero, the initial reserve price with an efficient ring becomes smaller than the initial reserve price without a ring.

The rest of this paper is structured as follows. Section 2 reviews the literature on collusion in auctions and sequentially optimal mechanisms. Section 3 introduces our assumptions about auction environment and bidding rings. Section 4 considers the case with two periods and Section 5 considers the case with infinitely many periods. Section 6 discusses and extends the model and Section 7 concludes.

2.2 Literature Review

To the best of our knowledge, our paper is the first to study the impact of limited commitment on collusion in auctions. Yet, we draw many insights from the literature on collusion in auctions and the literature on optimal pricing without commitment. These two fields are very different so we shall review them separately.

2.2.1 Collusion in Auctions

The literature on collusion in auctions studies three problems that are closely related to our research question. The first problem, which is collusion in static auctions, extends the standard independent private values model of Riley and Samuelson (1981) to the possibility of bidding rings. Graham and Marshall (1987) consider a single-object second-price auction with a reserve price and ex-ante symmetric buyers. They introduce a mechanism of collusion, called the second-price Preauction Knockout, and show that it is truthful in dominant strategies, individually rational, ex-ante budget balanced, and ex-post efficient. In this mechanism, every participant receives a lump-sum transfer equal to her expected contribution to the ring's profit, and the right to bid in the auction is allocated using a second-price auction.

Mailath and Zemsky (1991) generalize this model to the case of ex-ante asymmetric buyers. Using Payoff Equivalence, they show that since competitive allocation is ex-post efficient, the expected payoff of any buyer in the ring is different from her payoff in the auction by at most a constant. Further, they construct a version of the Arrow-d'Aspremont-Gerard-Varet mechanism to prove that there exists an individually rational, incentive compatible, and ex-post budget balanced mechanism that implements ex-post efficient collusion. As shown by Krishna (2009), this existence result is an implication of a theorem by Krishna and Perry (1998).

The second problem, which is collusion in repeated auctions, extends the model of tacit collusion with incomplete information by Athey et al. (2004) to auction markets. Their model considers a cartel of price-competing firms that are privately informed about their costs. It uncovers a trade-off between efficiency and information rent which results in rigid prices and thus inefficient collusion. A similar result is obtained by Skrzypacz and Hopenhayn (2004) for the case of repeated auctions: if buyers can not communicate and the only source of information is the identity of winners, the optimal collusion is not efficient. However, if buyers are allowed to communicate prior to the auction, as studied by Aoyagi (2007), then efficient collusion can be sustained by using a redistribution scheme that transfers the winner's surplus to the loser in the form of a continuation payoff.

The third problem, which is optimal auctions in the presence of collusion, extends the model of optimal auction design by Myerson (1981) and Riley and Samuelson (1981) to the possibility of bidding rings. Graham and Marshall (1987) show that the optimal reserve price is increasing in the ring's size, and Mailath and Zemsky (1991) derive a similar result for the case of ex-ante asymmetric buyers. These findings are consistent with the exclusion principle, obtained by Che and Kim (2009), which states that the optimal prevention of less-than-full inclusive rings involves a positive probability that each member of the ring does not receive the object.

2.2.2 Optimal Pricing Without Commitment

The literature on optimal pricing without commitment is inspired by the seminal paper by Coase (1972) which conjectures that a seller of a durable product fails to exercise his monopoly power because he can not resist the temptation to sell at lower prices in future. Fudenberg et al. (1985) use dynamic programming techniques to characterize a strong-Markov equilibrium of the corresponding bargaining game, which (unfortunately) does not exist in general. However, Ausubel and Deneckere (1989) show that for a general class of demand functions, there exists a weak-Markov equilibrium, and every weak-Markov equilibrium satisfies the Coase conjecture. They also show that it is not optimal to randomize over prices (except in the initial period), and a similar result, obtained by Skreta (2006), states that there is no loss of generality if the strategy space of the seller is restricted to posted prices.

These models of bargaining can be generalized to the case of many buyers. Milgrom (1987) considers a series of sealed-bid auctions in continuous time where the reserve price must be optimal at each point in time. He finds that there is a unique symmetric stationary equilibrium where the reserve price is always equal to the seller's valuation. Going back to discrete time, McAfee and Vincent (1997) and Liu et al. (2017) consider a second-price auction with a reserve price when the seller can not permanently withhold the object from the market when the reserve is not met. These papers show that there always exists a symmetric stationary equilibrium, and every such equilibrium satisfies the Coase conjecture. Under a uniform distribution of values, there is a linear stationary equilibrium where the initial reserve price is positive, but falls arbitrarily fast as the commitment power vanishes. Again, there is no loss of generality when we restrict attention to second-price auctions with reserve prices: as shown by Skreta (2015), they remain optimal mechanisms under ex-ante symmetry of buyers even with limited commitment.

2.3 Model

2.3.1 Environment

Our auction environment is identical to that in Liu et al. (2017). There is one seller and $n \geq 2$ buyers, $N = \{1, 2, ..., n\}$. The seller owns an indivisible object that has no value to him, but has a value of X_i to each buyer $i \in N$. Each value X_i is independently and identically distributed on $[0, 1]$ according to the distribution function F with density f, such that $f(x) > 0$ for all $x \in [0, 1]$. The realization of X_i is only observed by buyer i, while the other players know its distribution F. Let Y denote the highest of $n-1$ values, $Y \equiv \max\{X_1, ..., X_{n-1}\}$, with the distribution function $G \equiv F^{n-1}$ and density $g = G'$.

In period $t = 1$, the seller conducts a second-price auction where he announces a reserve price $r_1 \in [0, 1]$. Each buyer $i \in N$ observes r_1 and can place a (sealed) bid $b_{i,1} \in [r_1, 1]$ or wait. If at least one bid is placed, the highest bidder receives the object and pays the competitive price, defined as

$$P_{i,1} \equiv \max_{j \neq i}\{b_{j,1}, r_1\}$$

and the game ends. Otherwise, the seller retains the object and the game proceeds to period $t = 2$, where he conducts a second-price auction with a reserve $r_2 \in [0, 1]$, as described above. To account for both finite and infinite time horizon, we separately consider two settings:

1. $T = \{1, 2\}$: the game ends after $t = 2$.
2. $T = \{1, 2, ..., \infty\}$: the game continues until the object is sold.

Each player is risk-neutral and discounts her payoff in period t by δ^{t-1}. The exogenous parameter $\delta \in (0, 1)$ is used to reflect the time for which the seller can commit to keep the object off the market. His commitment power is decreasing in δ, with $\delta \to 0$ being the standard case of full commitment.

For every $t \geq 2$, let $h_t = (r_1, ..., r_{t-1})$ denote the history of reserve prices when the object is not sold by period t, and let h_1 denote the history when the seller announces an initial reserve p_1; we use H_t to denote the set of all such histories. A pure strategy of the seller specifies a mapping from the set of histories into the set of reserve prices, $p_t : H_t \to \mathbb{R}_+$ for every period $t \in T$. A pure strategy of buyer $i \in N$ specifies a mapping from the set of histories, the set of valuations, and the set of current reserve prices into the set of bids, $\beta_t^i : H_t \times [0, 1] \times [0, 1] \to [0, 1]$ for every period $t \in T$, such that any bid below the current reserve r_t is treated as waiting.

In any period t, the seller can commit to the current reserve r_t, but he can not commit to future reserves. To capture the lack of commitment, we use the solution concept of perfect Bayesian equilibrium (Osborne and Rubinstein, 1994, p. 233). That is, we look for a profile of strategies and beliefs that satisfy *sequential rationality:* every player's equilibrium strategy is a best response to the equilibrium strategies of the other players in every information set, conditional on reaching this set; and *consistent beliefs:* every player's beliefs are derived from the equilibrium strategies using Bayes' rule, whenever possible.

2.3.2 Bidding Rings

The previous section defines our auction environment under the assumption of competitive behavior. However, we also allow buyers to collude by forming *a bidding ring.* There is a (full-inclusive) bidding ring in a given auction if all buyers agree to place at most one bid above the reserve price. If buyer i receives the object in period t, she pays *the collusive price*, defined as

$$\widehat{P}_{i,t} \equiv r_t$$

In order to coordinate their bids in the auction, the buyers must select the sole bidder and determine side-payments beforehand. Following Mailath and Zemsky (1991), we do not specify the rules by which these tasks are accomplished. Instead, we consider the class of direct mechanisms where the ring decides on the sole bidder and side-payments based on vectors of reports sent by its members.

A direct mechanism in period t consists of a selection rule $\mathbf{Q}_t : [0,1]^n \to [0,1]^n$ and a side-payment rule $\mathbf{S}_t : [0,1]^n \to \mathbb{R}^n$. For every vector of reports that buyers may send to the ring center, the selection rule specifies the probability that buyer $i \in N$ is allowed to bid in the auction[1], $Q_{i,t}(\mathbf{x})$, and the side-payment rule specifies i's payment to the ring, $S_{i,t}(\mathbf{x})$. For a given incentive compatible mechanism in period t, let $s_{i,t} : [0,1] \to \mathbb{R}$ denote the expected side-payment function of buyer i when her report to the ring is some $z_i \in [0,1]$,

$$s_{i,t}(z_i; r_t) = \int_{[0,1]^{n-1}} S_{i,t}(z_i, \mathbf{x_{-i}}; r_t) f_{-i}(\mathbf{x_{-i}}) \mathbf{dx_{-i}}$$

In our analysis, we evaluate mechanisms of collusion with standard properties:

1. *Incentive compatibility*: it is optimal for every buyer to report her true value to the ring if all other buyers also report their true values.
2. *Ex-ante budget balance*: the sum of side-payments is equal to zero in expectation.
3. *Individual rationality*: the equilibrium payoff of every buyer is larger than her payoff when she competes against a ring with $n-1$ buyers.

2.4 Two Periods

Our primary interest lies in the case of infinite time horizon, $T = \{1, 2, ..., \infty\}$. It involves a long and somewhat tedious analysis, so it is not easy to develop an intuition behind the equations. Fortunately, we can obtain qualitatively identical results in a much simpler case

[1]Of course, the probability of sale is at most one, $\sum_{i \in N} Q_{i,t}(\mathbf{x}) \leq 1$ for all $\mathbf{x}$.

of two periods, $T = \{1, 2\}$. Compared to Graham and Marshall (1987), we introduce an opportunity to auction-off the object at $t = 2$ in case it does not sell at $t = 1$. Our goal is to see whether such opportunity affects the existence and operation of a ring at $t = 1$ as well as the difference in optimal reserve prices with and without a ring. For the sake of exposition, we only consider anonymous mechanisms[2] and omit discussion of technical details such as concavity and absolute continuity.

Of course, ex-post efficient collusion is always feasible at $t = 2$ because this is a regular second-price auction with full commitment. But as we show in this section, the existence of efficient collusion at $t = 1$ critically depends on the presence of a ring at $t = 2$. This is why we separately consider two situations:

1. there is no collusion at $t = 2$.
2. there is ex-post efficient collusion at $t = 2$.

2.4.1 No Collusion at $t = 2$

Suppose that there were no bids in the first auction, so the game proceeds to the second auction. The seller announces a reserve price r_2, followed by *competition* between the buyers. For any $r_2 \in [0, 1]$, it is a weakly dominant strategy for every buyer to bid her true value. Thus the (unconditional) expected payment of a buyer with value $x \geq r_2$ is equal to

$$m_2(x; r_2) = r_2 G(r_2) + \int_{r_2}^{x} y dG(y)$$

Since the first auction failed, the seller learnt that each buyer's value is less than some cut-off, denoted c. Then the optimal reserve at $t = 2$, denoted $p_2(c)$, maximizes the (unconditional) expected profit,

$$\Pi_2(c) = \max_{r} n \times \int_{r}^{c} m_2(x; r) dF(x)$$

Let us go backwards to $t = 1$ and study the benchmark case of *competition*. Suppose that for any reserve price $r_1 \in [0, 1]$, there is a symmetric equilibrium characterized by a mapping from the set of reserve prices into the set of values, $\mu : [0, 1] \to [0, 1]$. It says that a buyer

[2] Roughly speaking, an anonymous mechanism does not condition a buyer's allocation and payment on her identity.

bids her value if it exceeds $\mu(r_1)$ and does not bid otherwise. The cut-off type $\mu(r_1)$ must be indifferent between bidding and waiting,

$$G(\mu)\mu - m_1(\mu; r_1) = \delta[G(\mu)\mu - m_2(\mu; p_2(\mu))] \tag{2.1}$$

where $m_1(x; r_1)$ is the first-period expected payment of a buyer with value $x \geq \mu(r_1)$,

$$m_1(x; r_1) = r_1 G(\mu(r_1)) + \int_{\mu(r_1)}^{x} y dG(y)$$

How should the buyers organize the efficient collusion? Under full commitment, collusion is (ex-post) efficient when a buyer with value x bids at $t = 1$ if and only if her value is the highest, $x \geq Y$, and her payoff from the auction is positive, $x - r_1 \geq 0$. Under limited commitment, however, the second condition becomes

$$x - r_1 \geq \delta(x - \max\{Y, r_2\})$$

because the buyer with the largest valuation could wait at $t = 1$ and obtain the payoff of $(x - \max\{Y, r_2\})$ at $t = 2$. Since the condition must hold for every $Y < x$, it also holds in expectation,

$$\frac{1}{G(x)} \int_0^x (x - r_1) dG(y) \geq \delta \frac{1}{G(x)} \int_0^x (x - \max\{y, r_2\}) dG(y) \tag{2.2}$$

or simply

$$G(x)x - \widehat{m}_1(x; r_1) \geq \delta[G(x)x - m_2(x; r_2)] \tag{2.3}$$

Let $\mu^e(r_1)$ denote the lowest value that satisfies Ineq. (2.3). Then the second-period reserve will be $p_2(\mu^e(r_1))$, and the efficient cut-off $\mu^e(r_1)$ is implicitly defined by

$$G(\mu^e)\mu^e - \widehat{m}_1(\mu^e; r_1) = \delta[G(\mu^e)\mu^e - m_2(\mu^e; p_2(\mu^e))] \tag{2.4}$$

Note that the expected payment of the cut-off type $\mu^e(r_1)$ is not affected by the presence of a ring because he can win only at the reserve price,

$$\begin{aligned} \widehat{m}_1(\mu(r_1); r_1) &= r_1 G(\mu(r_1)) \\ &= m_1(\mu(r_1); r_1) \end{aligned}$$

It implies that Eq. (2.1) and Eq. (2.4) are identical, so the efficient cut-off is the same as the competitive cut-off, $\mu^e(r_1) = \mu(r_1)$. Precisely, whether a ring operates or not, a buyer with value $x \geq \mu(r_1)$ wins the object at $t = 1$ if $Y < x$. Further, a buyer with value $x < \mu(r_1)$ does not win the object but obtains her second-stage equilibrium payoff if $Y < \mu(r_1)$. Since the allocation of the object is the same with and without a ring, any efficient mechanism of collusion can be described by the Payoff Equivalence principle.

Lemma 2.1 (Payoff Equivalence)**.** *The equilibrium payoff function with an efficient ring differs from the equilibrium payoff function without a ring by at most a constant,*

$$\widehat{U}(x; r_1) - U(x; r_1) \equiv k(r_1), \ \forall x \in [0, 1]$$

Proof. See Appendix. □

As in Graham and Marshall (1987), the efficient collusion can be implemented by a second-price preauction knockout (PAKT), defined as follows:

- each buyer $i \in N$ sends a report $z_i \in [0, 1]$ to the ring center;
- if buyer i's report z_i is higher than all other reports and the efficient cut-off, $z_i \geq \max_{j \neq i}\{z_j, \mu(r_1)\}$, she is advised to bid her report z_i and pay the difference of what she would pay without a ring and what she actually pays to the seller,

$$\tau_{i,1} = P_{i,1} - \widehat{P}_{i,1} \tag{2.5}$$

- otherwise, buyer i is advised not to bid;
- every buyer receives a lump-sum transfer of

$$k(r_1) = \int_{\mu(r_1)}^{1} \int_{\mu(r_1)}^{x} (y - r_1) dG(y) dF(x)$$

Proposition 2.2. *In the two-period model without collusion at $t = 2$, the PAKT is efficient, incentive compatible, individually rational, and ex-ante budget balanced for every $r_1 \in [0, 1]$.*

Proof. See Appendix. □

Finally, we compare the optimal reserve prices with and without a ring. When there is no ring at $t = 1$, the optimal reserve p_1 maximizes the expected profit, given by

$$\max_r \left[n \times \int_{\mu(r)}^1 m_1(x;r)dF(x) + \delta\Pi_2(\mu(r)) \right]$$

If p_1 is interior, it satisfies the first-order condition

$$n \times \left[\int_{\mu(p_1)}^1 \frac{\partial}{\partial r} m_1(x;p_1)dF(x) - \mu'(p_1)m_1(\mu(p_1);p_1)f(\mu(p_1)) \right] + \delta\Pi_2'(\mu(p_1))\mu'(p_1) = 0$$

But when there *is* a ring, the optimal reserve $\widehat{p}_1$ maximizes

$$\max_r \left[n \times \int_{\mu(r)}^1 \widehat{m}_1(x;r)dF(x) + \delta\Pi_2(\mu(r)) \right] \tag{2.6}$$

Proposition 2.3. *In the two-period model without collusion at $t = 2$, the optimal reserve at $t = 1$ is greater when there is a ring, $\widehat{p}_1 \geq p_1$.*

Proof. The first derivative of (2.6) is

$$n \times \left[\int_{\mu(r)}^1 \frac{\partial \widehat{m}_1(x;r)}{\partial r} dF(x) - \mu'(r)\widehat{m}_1(\mu(r);r)f(\mu(r)) \right] + \delta\Pi_2'(\mu(r))\mu'(r) \tag{2.7}$$

Evaluated at p_1, it simplifies to

$$-n \times \int_{\mu(p_1)}^1 \frac{\partial}{\partial r}(m_1(x;p_1) - \widehat{m}_1(x;p_1))dF(x)$$

Observe that the fact

$$\frac{\partial}{\partial r}(m_1(x;p_1) - \widehat{m}_1(x;p_1)) = -\int_{\mu(p_1)}^1 [G(x) - G(p_1) + \mu'(p_1)g(\mu(p_1))(\mu(p_1) - p_1)]dF(x) < 0$$

implies that (2.7) is positive at p_1. Since (2.6) is assumed to be concave, the proposition follows. □

Propositions 2.2 and 2.3 generalize the findings by Graham and Marshall (1987) to two periods. It is not surprising because in both models, the efficient collusion does not distort the allocation and does not affect the continuation payoff. In fact, it only reduces the expected payment at $t = 1$, thereby leading to similar results in both models.

2.4.2 Ex-Post Efficient Collusion at $t = 2$

Let us suppose again that the first auction failed, so the seller announces a second auction. For any reserve price $r_2 \in [0, 1]$, there exists an ex-post efficient, individually rational, incentive compatible and budget-balanced mechanism of collusion (Mailath and Zemsky, 1991). Then the (unconditional) expected payment of a buyer with value $x \geq r_2$ is

$$\widehat{m}_2(x; r_2) = r_2 G(x)$$

If the seller believes that each buyer's value is distributed on $[0, c]$, the optimal second-period reserve price, denoted $\widehat{p}_2(c)$, maximizes

$$\widehat{\Pi}_2(c) = \max_r \; n \times \int_r^c \widehat{m}_2(x; r) dF(x)$$

Now proceed backwards to $t = 1$. We say that collusion is *ex-post efficient* if a buyer with value x is nominated to bid at $t = 1$ if and only if her value is the largest, $x > Y$, and she obtains a larger payoff from winning at $t = 1$ than at $t = 2$,

$$x - r_1 \geq \delta(x - r_2)$$

Thus the lowest type who should bid at $t = 1$, denoted $\widehat{\mu}^e(r_1)$, must have the same payoff from both auctions,

$$G(\widehat{\mu}^e)\widehat{\mu}^e - \widehat{m}_1(\widehat{\mu}^e; r_1) = \delta[G(\widehat{\mu}^e)\widehat{\mu}^e - \widehat{m}_2(\widehat{\mu}^e; \widehat{p}_2(\widehat{\mu}^e))] \tag{2.8}$$

Note that the presence of collusion at $t = 2$ changes the definition of efficient collusion at $t = 1$, $\widehat{\mu}^e(r_1) \neq \mu^e(r_1)$. This is because the expected payment of the cut-off type $\widehat{\mu}^e(r_1)$ at $t = 2$ is collusive, $\widehat{m}_2(\widehat{\mu}^e; \widehat{p}_2(\widehat{\mu}^e))$, rather than competitive, $m_2(\mu^e; p_2(\mu^e))$.

Consider a mechanism that implements the cut-off $\widehat{\mu}^e(r_1)$ for some $r_1 \in [0, 1]$. It imposes a constraint on the side-payment function as follows.

Lemma 2.4. *The expected side-payment function of any efficient mechanism is given by*

$$s_1(x; r_1) = s_1(0; r_1) + \begin{cases} \displaystyle\int_{\widehat{\mu}^e(r_1)}^x (y - r_1) dG(y) + \delta s_2(\widehat{\mu}^e(r_1); \widehat{p}_2(\widehat{\mu}^e(r_1))) & \textit{if } x \geq \widehat{\mu}^e(r_1) \\ 0 & \textit{if } x < \widehat{\mu}^e(r_1) \end{cases} \tag{IC}$$

where for all $c \in [0,1]$,

$$s_2(c;\widehat{p}_2(c)) = \int_{\widehat{p}_2(c)}^{c} (y-\widehat{p}_2(c))dG(y) - \int_{\widehat{p}_2(c)}^{c}\int_{\widehat{p}_2(c)}^{x} (y-\widehat{p}_2(c))dG(y)dF(x)$$

Proof. See Appendix. □

If the cut-off type $\widehat{\mu}^e(r_1)$ leaves the ring to bid her value at $t = 1$, she wins the object when a ring with $n-1$ buyers does not bid, $Y < \widehat{\mu}^e(r_1)$. But if she stays in the ring, she is selected to bid under the same circumstances, $Y < \widehat{\mu}^e(r_1)$. Thus, it is rational for her to stay in the ring only if her expected side-payment is negative,

$$s_1(\widehat{\mu}^e(r_1); r_1) \leq 0 \tag{IR}$$

Since the mechanism is ex-ante budget-balanced, the expected side-payment is on average zero,

$$s_1(0;r_1) + \int_{\widehat{\mu}^e(r_1)}^{1} \left[\int_{\widehat{\mu}^e(r_1)}^{x} (y-r_1)dG(y) + \delta s_2(\widehat{\mu}^e(r_1);\widehat{p}_2(\widehat{\mu}^e(r_1)))\right] dF(x) = 0 \tag{BB}$$

Using Eq. (BB) and Eq. (IC), we can write Ineq. (IR) as,

$$\int_{\widehat{\mu}^e(r_1)}^{1} \left[\int_{\widehat{\mu}^e(r_1)}^{x} (y-r_1)dG(y) + \delta s_2(\widehat{\mu}^e(r_1);\widehat{p}_2(\widehat{\mu}^e(r_1)))\right] dF(x) \geq \delta s_2(\widehat{\mu}^e(r_1);\widehat{p}_2(\widehat{\mu}^e(r_1)))$$

Under full commitment, $\delta = 0$, the constraint is always satisfied because the left-hand side is positive and the right-hand side is zero. But for any $\delta > 0$, there is a reserve $\underline{r}_1 < 1$ such that no-one should bid at $t = 1$, that is $\widehat{\mu}^e(\underline{r}_1) = 1$. Then the constraint fails because the integral on the left-hand side collapses to zero while the right-hand side is positive. Intuitively, the cut-off type is better-off bidding for the object on her own because she would pay to the ring more than she would receive from it. Therefore, the main result of Mailath and Zemsky (1991) does not extend to limited commitment:

Proposition 2.5. *In the two-period model, there are no incentive-compatible, ex-ante budget balanced, and individually rational mechanisms that implement the ex-post efficient collusion in each period for all first-period reserve prices.*

2.5 Infinite Horizon

In this section, we analyze the main case of infinite time horizon, $T = \{1, 2, ...\}$. As shown in the previous section, the existence of efficient collusion may be affected by the presence of a ring in future. To verify that this result survives in the main case, we again separately consider two assumptions:

1. there is competition in every $t > 1$.
2. there is collusion in every $t > 1$.

2.5.1 Competition in every $t > 1$

Our first setting assumes that if all bids fail to meet the first-period reserve price $r_1 \in [0, 1]$, the buyers will compete in all auctions that may occur in future. First, we derive the expected payment function and the equilibrium payoff function when there is no ring at $t = 1$. Second, we derive the efficient cut-off rule and collusive surplus. Third, we characterize efficient mechanisms and construct an explicit example. Finally, we compare optimal reserve price with and without a ring at $t = 1$.

2.5.1.1 Competition at $t = 1$

Let us begin with the benchmark case where the buyers also compete in the initial period $t = 1$. Suppose that there exists a perfect Bayesian equilibrium with Markov strategies $\mu : [0, 1] \to [0, 1]$ and $p : [0, 1] \to [0, 1]$. The function $\mu(\cdot)$, with $\mu'(r) > 0$, is a mapping from the set of reserve prices into the set of values: in an auction with reserve r, a buyer with value x bids x if $x \geq \mu(r)$, and does not bid otherwise. The function $p(\cdot)$, with $p'(c) > 0$, is a mapping from the set of values into the set of reserve prices: when the seller believes that values are distributed on $[0, c]$, he announces a reserve $p(c)$.

In any auction with reserve $r \geq 0$, a buyer with value $x \geq \mu(r)$ wins the object with probability $G(x)$. Thus the equilibrium payoff function is

$$U(x; r) = \begin{cases} G(x)x - m(x; r) & \text{if } x \geq \mu(r) \\ \delta U(x; p(\mu(r))) & \text{if } x < \mu(r) \end{cases}$$

where $m(x;r)$ denotes the expected payment of a buyer with value $x \geq \mu(r)$,

$$m(x;r) = G(\mu(r))r + \int_{\mu(r)}^{x} ydG(y)$$

The Revelation Principle suggests that this game can be treated as a direct mechanism where every buyer is allowed to play her equilibrium strategy of any type. Perfect Bayesian equilibrium prescribes every buyer to play a best response, so it is optimal to play the equilibrium strategy of one's true type. It is thus an incentive compatible mechanism, whose equilibrium payoff function U must be absolutely continuous (Krishna, 2009, p. 83). In particular, the cut-off type $\mu(r)$ is defined by the indifference between bidding and waiting,

$$G(\mu)\mu - m(\mu;r) = \delta[G(\mu)\mu - m(\mu;p(\mu))] \tag{2.9}$$

To characterize sequentially optimal reserves, consider an auction where (the seller believes that) each buyer's value is distributed on $[0,c]$. Then the optimal reserve $p(c)$ satisfies the dynamic programming equation

$$\Pi(c) = \max_{r} \left[n \int_{\mu(r)}^{c} m(x;r)dF(x) + \delta\Pi(\mu(r)) \right]$$

Using the Envelope theorem,

$$\Pi'(c) = m(c;p(c))f(c)$$

The optimal reserve $p(c)$ must solve the first-order condition

$$\int_{\mu(p(c))}^{c} \frac{\partial}{\partial r} m(x;p(c))dF(x) = [m(\mu(p(c));p(c)) - \delta m(\mu(p(c));p(\mu(p(c))))]\mu'(p(c))f(\mu(p(c)))$$

The definition of the cut-off $\mu(\cdot)$ can be used to simplify the right-hand side as

$$\int_{\mu(p(c))}^{c} \frac{\partial}{\partial r} m(x;p(c))dF(x) = (1-\delta)G(\mu(p(c)))\mu(p(c)f(\mu(p(c)))\mu'(p(c)) \tag{2.10}$$

Eq. (2.9) and Eq. (2.10) can be found in McAfee and Vincent (1997) for the case of uniform distribution, $F(x) = x$. So our results generalize their setting to any distribution of values where a strong Markov equilibrium exists.

2.5.1.2 Efficient Collusion

What collusion at $t = 1$ should be considered efficient? Of course, a buyer with value x should be nominated to place a bid only if her value is the highest, $x > Y$. Besides, a buyer should not place a bid in case she can achieve a larger payoff in a future auction. Let $\mu^e(r_1)$ denote the lowest type who should place a bid for a given reserve $r_1 \in [0, 1]$. If the object does not sell at $t = 1$, every player infers that the values are distributed on $[0, \mu^e(r)]$. Our previous argument implies that $(p(c), \mu(r))$ are best responses in the continuation game, so the continuation payoff of a buyer with value x is $U(x; p(\mu^e(r)))$. It follows that the cut-off type $\mu^e(r)$ must obtain the same payoff from the auction at $t = 1$ and from the optimal auction at $t = 2$,

$$G(\mu^e)\mu^e - \widehat{m}(\mu^e; r_1) = \delta U(\mu^e; p(\mu^e)) \tag{2.11}$$

where $\widehat{m}(x; r_1)$ denotes the expected payment of a buyer with value $x \geq \mu^e(r)$,

$$\widehat{m}(x; r_1) = G(x)r_1$$

As in the two-period model, the expected payment of the cut-off type $\mu(r_1)$ is not affected by the presence of a ring,

$$\begin{aligned} \widehat{m}(\mu(r_1); r_1) &= G(\mu(r_1))r_1 \\ &= m(\mu(r_1); r_1) \end{aligned}$$

Substitute it into Eq. (2.11) to obtain

$$G(\mu^e)\mu^e - m(\mu; r_1) = \delta U(\mu^e; p(\mu^e))$$

which is the same equation that defines the competitive cut-off $\mu(r)$. Hence, the efficient collusion must replicate the competitive allocation, $\mu^e(r_1) = \mu(r_1), \ \forall r_1 \geq 0$.

Finally, note that membership in an efficient ring reduces the expected payment of all bidding types $x \geq \mu(r)$,

$$m(x; r_1) - \widehat{m}(x; r_1) = \begin{cases} \displaystyle\int_{\mu(r_1)}^{x} (y - r_1) dG(y) & \text{if } x \geq \mu(r_1) \\ 0 & \text{if } x < \mu(r_1) \end{cases}$$

Thus, we can define *the gains of collusion* as the expected value of such reduction,

$$
\begin{aligned}
k(r_1) &\equiv \int_{\mu(r_1)}^{1} [m(x;r_1) - \widehat{m}(x;r_1)]dF(x) \\
&= \int_{\mu(r_1)}^{1} \int_{\mu(r_1)}^{x} (y - r_1) dG(y) dF(x)
\end{aligned}
$$

2.5.1.3 Characterization of Efficient Mechanisms

How does an efficient ring select its sole bidder and divide the gains of collusion? Consider a mechanism of collusion that implements the efficient cut-off $\mu(r_1)$ in a perfect Bayesian equilibrium. Fix the equilibrium strategies of the other buyers ($j \neq i$) and the seller. Suppose that buyer i with value x reports some value $z \in [0,1]$, but plans to bid according to her true value in all future auctions. Then her expected payoff is

$$
\begin{cases}
G(z)x - \widehat{m}(z;r_1) - s_i(z;r_1) & \text{if } z \geq \mu(r_1) \\
-s_i(z;r_1) + \delta U(x; p(\mu(r_1))) & \text{if } z < \mu(r_1)
\end{cases}
$$

Incentive compatibility requires that it be optimal for every type to report her true value to the ring, $z = x$. Further, since buyer i's payoff is absolutely continuous in x, her equilibrium payoff $\widehat{U}_i$ is also absolutely continuous (Milgrom and Segal, 2002) and can be written as the integral of its derivative,

$$
\widehat{U}_i(x;r_1) = \widehat{U}_i(0;r_1) + \begin{cases}
G(\mu(r_1))\mu(r_1) - \widehat{m}(\mu(r_1);r_1) + \displaystyle\int_{\mu(r_1)}^{x} G(y)dy & \text{if } x \geq \mu(r_1) \\
\delta U(x; p(\mu(r_1))) & \text{if } x < \mu(r_1)
\end{cases} \tag{2.12}
$$

where we used the indifference $\delta U(\mu(r_1); p(\mu(r_1))) = G(\mu(r_1))\mu(r_1) - \widehat{m}(\mu(r_1);r_1)$.

Lemma 2.6 (Payoff Equivalence). *The equilibrium payoff of each buyer $i \in N$ with a ring differs from her equilibrium payoff without a ring by at most a constant,*

$$
\widehat{U}_i(x;r_1) - U(x;r_1) = -s_i(0;r_1), \ \forall x \in [0,1]
$$

Proof. Integrate by parts to re-write the equilibrium payoff under competition as

$$\begin{aligned} G(x)x - m(x;r_1) &= G(x)x - G(\mu(r_1))r_1 - \int_{\mu(r_1)}^{x} y dG(y) \\ &= G(x)x - G(\mu(r_1))r_1 - \left[G(x)x - G(\mu(r_1))\mu(r_1) - \int_{\mu(r_1)}^{x} G(y)dy \right] \\ &= G(\mu(r_1))\mu(r_1) - \widehat{m}(\mu(r_1);r_1) + \int_{\mu(r_1)}^{x} G(y)dy \end{aligned}$$

which can be summarized as

$$U(x;r_1) = \begin{cases} G(\mu(r_1))\mu(r_1) - \widehat{m}(\mu(r_1);r_1) + \int_{\mu(r_1)}^{x} G(y)dy & \text{if } x \geq \mu(r_1) \\ \delta U(x;p(\mu(r_1))) & \text{if } x < \mu(r_1) \end{cases} \tag{2.13}$$

The lemma follows from subtracting Eq. (2.13) from Eq. (2.12) and the fact $\widehat{U}_i(0;r_1) = -s_i(0;r_1)$. □

Payoff Equivalence can be used to show that

$$\begin{aligned} \widehat{U}_i(x;r_1) - U(x;r_1) &= -s_i(x;r_1) + [m(x;r_1) - \widehat{m}(x;r_1)] \\ &= -s_i(0;r_1) \end{aligned} \tag{2.14}$$

By re-arranging the terms, we obtain the expected side-payment function,

$$\begin{aligned} s_i(x;r_1) &= s_i(0;r_1) + [m(x;r_1) - \widehat{m}(x;r_1)] \\ &= s_i(0;r_1) + \int_{\mu(r_1)}^{x} (y - r_1)dG(y) \end{aligned} \tag{2.15}$$

Ex-ante budget balance requires that the side-payments sum up to zero in expectation,

$$\sum_{i \in N} \int_0^1 s_i(x;r_1)dF(x) = 0$$

Using Eq. (2.15), it can be expanded to

$$\sum_{i \in N} s_i(0;r_1) + n \int_{\mu(r_1)}^{1} \int_{\mu(r_1)}^{x} (y - r_1)dG(y)dF(x) = 0$$

If a buyer leaves the ring, she competes against a ring with $n-1$ buyers. That is, she wins if and only if her bid is larger than $\max\{Y, \mu(r_1)\}$. Since it is the same allocation as if there were no ring at all, she obtains her competitive payoff $U(x; r_1)$. Then *individual rationality* requires

$$\widehat{U}_i(x; r_1) \geq U(x; r_1), \ \forall i \in N$$

Using Eq. (2.14), it simply requires a negative side-payment for buyer type 0,

$$s_i(0; r_1) \leq 0, \ \forall i \in N$$

These findings are reminiscent of the characterization of efficient mechanisms in Mailath and Zemsky (1991) as follows. To ensure that the object is allocated efficiently, the ring center must introduce a side-payment rule that replicates the shape of expected payment under competition. Then the expected profit of the ring can be distributed among its members via individual transfers $-s_i(0; r_1)$. Of course, they must be positive so that every buyer prefers to stay in the ring.

2.5.1.4 Implementation of Efficient Collusion

It was shown in the two-period horizon that efficient collusion can be implemented with a second-price preauction knockout when there is competition in the second period. The next proposition states that the finding extends to the infinite horizon.

Proposition 2.7. *The second-price PAKT, where $\mu(r)$ is defined by Eq. (2.9), is efficient, incentive compatible, individually rational, and ex-ante budget balanced for all $r_1 \in [0,1]$.*

Proof. To verify incentive compatibility, suppose that all buyers $j \neq i$ report their true values. If buyer i with value x is selected to bid, the ring will ask her to pay

$$\tau_{i,1} = \begin{cases} Y - r_1 & \text{if } Y > \mu(r_1) \\ 0 & \text{if } Y \leq \mu(r_1) \end{cases}$$

Thus her expected payoff from reporting some value $z \in [0,1]$ is

$$k(r_1) + \begin{cases} G(z)x - \widehat{m}(z; r_1) - \int_{\mu(r)}^{z} (y - r_1) dG(y) & \text{if } z \geq \mu(r_1) \\ \delta U(x; p(\mu(r_1))) & \text{if } z < \mu(r_1) \end{cases}$$

Net of the constant $k(r_1)$, it is the same payoff she would obtain without a ring by bidding as if her value were z. Since it is a best response to bid as if buyer i's value is x, it is also a best response to report her true value to the ring.

To verify budget balance, note that a buyer's expected contribution to the ring is

$$E[\tau_{i,1}] = \int_{\mu(r_1)}^{1} \int_{\mu(r_1)}^{x} (y - r_1) dG(y) dF(x)$$

which is equal to the lump-sum transfer that she receives from the ring, $k(r_1)$.

And to verify individual rationality, note that the expected payment of a buyer with value $x < r_1$ is $s_i(0; r_1) = -k(r_1)$. Since $k(r_1) \geq 0$, we have $s_i(0; r_1) \leq 0$, $\forall i \in N$, as required. □

2.5.1.5 Optimal Reserve Price

What is an optimal response to the efficient collusion at $t = 1$? The seller's instantaneous profit from announcing some reserve $r_1 \in [0,1]$ is equal to

$$n \int_{\mu(r_1)}^{1} \widehat{m}(x; r_1) dF(x) \tag{2.16}$$

It is different from his instantaneous profit under competition in that the expected payment of each buyer with value $x \geq \mu(r)$ reduces from $m(x; r)$ to $\widehat{m}(x; r)$. To reflect the reduction explicitly, let us re-write Eq. (2.16) using our definition of $k(r_1)$,

$$n \left(\int_{\mu(r_1)}^{1} m(x; r_1) dF(x) - k(r_1) \right)$$

In the event of no sale, the seller be facing the same infinite horizon problem as under competition. Thus, his continuation profit does not differ from his competitive continuation profit, $\Pi(\mu(r))$. Overall, the optimal reserve price with a ring, denoted $\widehat{p}_1$, must solve

$$\max_r \left[n \left(\int_{\mu(r)}^1 m(x;r)dF(x) - k(r) \right) + \delta\Pi(\mu(r)) \right]$$

The gains of collusion $k(\cdot)$ are a strictly decreasing function because high reserve prices erode the value of competition $[m(x;r_1) - \widehat{m}(x;r_1)]$. This gives the seller an incentive to set a higher reserve, as stated in the next proposition.

Proposition 2.8. *The optimal reserve price at $t = 1$ is greater with a ring, $\widehat{p}_1 \geq p(1)$.*

Proof. The first derivative of the expected profit with a ring is

$$\frac{\partial}{\partial r} \left(n \int_{\mu(r)}^1 m(x;r)dF(x) + \delta\Pi(\mu(r)) \right) - nk'(r) \tag{2.17}$$

Evaluated at $p(1)$, the first term vanishes. In contrast, the second term is positive because

$$-k'(p(1)) = \int_{\mu(p(1))}^1 [G(x) - G(p(1)) + \mu'(p(1))g(\mu(p(1)))(\mu(p(1)) - p(1))]dF(x) \geq 0$$

These facts imply that Eq. (3.8) is positive at $p(1)$. Since the expected profit is assumed to be concave, the proposition follows. □

2.5.2 Collusion in every $t > 1$

In this section, we assume that a bidding ring operates not only at $t = 1$, but also in all future auctions, $t > 1$. First, we characterize individually rational and budget balanced mechanisms that implement a given cut-off rule. Second, we derive the ex-post efficient allocation and show that it can not be implemented without outside subsidies or compulsory participation. Third, we define a "second-best" allocation and show that it can be implemented with pre-auction knockouts. For both types of collusion, we also find the optimal reserve prices and compare them to the optimal reserve prices under competition.

2.5.2.1 Incentive Compatibility

Suppose that the ring implements some cut-off rule $\widehat{\mu} : [0,1] \to [0,1]$ as a perfect Bayesian equilibrium. That is, the ring buys the object in an auction with a reserve $r \in [0,1]$ if and only if its highest value exceeds $\widehat{\mu}(r)$. Let $\widehat{p} : [0,1] \to [0,1]$ denote the optimal reserve prices for a given distribution of values. In such equilibrium, buyer $i \in N$ wins the object in a given auction if and only if $x_i \geq \max_{j \neq i}\{x_j, r\}$. Also, if an auction with $r \geq 0$ fails to sell the object, each player infers that the values are distributed on $[0, \widehat{\mu}(r)]$.

Consider an auction with a reserve price $r \geq 0$. If buyer i with value x reports some $z \in [0,1]$ in the current auction and plans to report her true value x in any future auction, her payoff is

$$\begin{cases} G(z)x - \widehat{m}(z;r) - s_i(z;r) & \text{if } z \geq \widehat{\mu}(r) \\ -s_i(z;r) + \delta \widehat{U}_i(x; \widehat{p}(\widehat{\mu}(r))) & \text{if } z < \widehat{\mu}(r) \end{cases} \tag{2.18}$$

Incentive compatibility requires that it be optimal to report her true value, $z = x$. Since her payoff is maximized at x, we obtain the first-order condition

$$0 = \begin{cases} g(x)x - \widehat{m}'(x;r) - s_i'(x;r) & \text{if } x \geq \widehat{\mu}(r) \\ -s_i'(x;r) & \text{if } x < \widehat{\mu}(r) \end{cases}$$

By taking integrals on both sides,

$$s_i(x;r) = \begin{cases} s_i(\widehat{\mu}(r);r) + \displaystyle\int_{\widehat{\mu}(r)}^{x} (y-r)dG(y) & \text{if } x \geq \widehat{\mu}(r) \\ s_i(0;r) & \text{if } x < \widehat{\mu}(r) \end{cases} \tag{2.19}$$

Since buyer i's payoff in Eq. (2.18) is absolutely continuous in x for all $z \geq \widehat{\mu}(r)$, her equilibrium payoff $\widehat{U}_i$ is also absolutely continuous. In particular, the cut-off type is indifferent between reporting her true value and any lower value,

$$G(\widehat{\mu}(r))\widehat{\mu}(r) - \widehat{m}(\widehat{\mu}(r);r) - s_i(\widehat{\mu}(r);r) = -s_i(0;r) + \delta \widehat{U}_i(\widehat{\mu}(r); \widehat{p}(\widehat{\mu}(r)))$$

which pins down the expected side-payment of the cut-off type,

$$s_i(\widehat{\mu}(r);r) = s_i(0;r) + G(\widehat{\mu}(r))\widehat{\mu}(r) - \widehat{m}(\widehat{\mu}(r);r) - \delta \widehat{U}_i(\widehat{\mu}(r); \widehat{p}(\widehat{\mu}(r))) \tag{2.20}$$

Substitute Eq. (2.20) back into Eq. (2.19) to obtain the expected side-payment function,

$$s_i(x;r) = s_i(0;r) + \begin{cases} \int_{\widehat{\mu}(r)}^{x} (y-r)dG(y) + G(\widehat{\mu}(r))(\widehat{\mu}(r)-r) - \delta\widehat{U}_i(\widehat{\mu}(r);\widehat{p}(\widehat{\mu}(r))) & \text{if } x \geq \widehat{\mu}(r) \\ 0 & \text{if } x < \widehat{\mu}(r) \end{cases} \tag{2.21}$$

To get some intuition for the result, note that the function consists of three separable terms. The first term, $s_i(0;r)$, is constant over all types. It can be used by the ring center to distribute the gains of collusion among its members (e.g. as lump-sum transfers in preauction knockouts). The second term, $\int_{\widehat{\mu}(r)}^{x}(y-r)dG(y)$, is strictly increasing in x, but paid only by bidding types, $x \geq \widehat{\mu}(r)$. It suggests that a buyer's side-payment is increasing in her chance of receiving the object, which is used by the ring center to ensure the efficient self-selection of buyer types. The last term is the size of a jump discontinuity at the cut-off $\widehat{\mu}(r)$. It is analogous to a reserve price or an entry fee imposed by the ring center to achieve the desired level of participation $\widehat{\mu}(r)$.

2.5.2.2 Individual Rationality and Budget Balance

For every reserve price in every auction, a mechanism of collusion must be individually rational and budget balanced. Ex-ante budget balance requires that the sum of side-payments be zero in expectation,

$$\sum_{i \in N} \frac{1}{F(c)} \int_0^c s_i(x;r)dF(x) = 0$$

The function $s_i(x;r)$ is specified by Eq. (2.21), which implies

$$\sum_{i \in N} \left[s_i(0;r) + \frac{1}{F(c)} \int_{\widehat{\mu}(r)}^{c} \left(G(\widehat{\mu}(r))(\widehat{\mu}(r)-r) - \delta\widehat{U}_i(\widehat{\mu}(r);\widehat{p}(\widehat{\mu}(r))) + \int_{\widehat{\mu}(r)}^{x}(y-r)dG(y) \right) dF(x) \right] = 0 \tag{2.22}$$

Suppose that buyer i with value x leaves the ring and bids in the auction as if she were reporting her value to the ring. That is, she bids her value if $x \geq \widehat{\mu}(r)$ and does not bid otherwise. Then her payoff is

$$\begin{cases} G(x)x - \widehat{m}(x;r) - \int_{\widehat{\mu}(r)}^{x}(y-r)dG(y) & \text{if } x \geq \widehat{\mu}(r) \\ \delta\widehat{U}_i(x;\widehat{p}(\widehat{\mu}(r))) & \text{if } x < \widehat{\mu}(r) \end{cases}$$

Such deviation is not profitable if and only if the expected side-payment of non-bidding types and the cut-off type be negative,

$$s_i(\widehat{\mu}(r); r) \leq 0, \ \forall i \in N \tag{2.23}$$

$$s_i(0; r) \leq 0, \ \forall i \in N \tag{2.24}$$

Since it is not necessarily optimal to bid x if and only if $x \geq \widehat{\mu}(r)$, Ineq. (2.23) and Ineq. (2.24) are necessary for individual rationality, but not sufficient. However, they are enough for us to show the impossibility of the ex-post efficient collusion in the next section.

2.5.3 Efficient Collusion

2.5.3.1 Definition

It is *ex-post efficient* for the ring to buy the object in the auction that gives its members the largest sum of payoffs. Suppose that the ring is represented by its member with value x. If she bids in the current auction with reserve r_t, her payoff is $x - r_t$. But if she waits and bids in the next auction with reserve r_{t+1}, her payoff will be $\delta(x - r_{t+1})$. It is thus efficient to bid in the current auction only if

$$x - r_t \geq \delta(x - r_{t+1})$$

The lowest type who should place a bid is denoted $\mu^e(r)$ and defined by

$$\widehat{\mu}^e - r = \delta(\widehat{\mu}^e - \widehat{p}^e(\widehat{\mu}^e)) \tag{EF}$$

Note that Eq. (EF) implies

$$G(\widehat{\mu}^e(r))\widehat{\mu}^e(r) - \widehat{m}(\widehat{\mu}^e(r); r) - \delta \widehat{U}_i^e(\widehat{\mu}^e(r); \widehat{p}^e(\widehat{\mu}^e(r))) = \delta s_i^e(\widehat{\mu}^e(r); \widehat{p}^e(\widehat{\mu}^e(r)))$$

so the expected side-payment function in any efficient mechanism is

$$s_i^e(x; r) = s_i^e(0; r) + \begin{cases} \displaystyle\int_{\widehat{\mu}^e(r)}^{x} (y - r) dG(y) + \delta s_i^e(\widehat{\mu}^e(r); \widehat{p}^e(\widehat{\mu}^e(r))) & \text{if } x \geq \widehat{\mu}^e(r) \\ 0 & \text{if } x < \widehat{\mu}^e(r) \end{cases} \tag{2.25}$$

Under full commitment, $\delta = 0$, the term $\delta s_i^e(\widehat{\mu}^e(r); \widehat{p}^e(\widehat{\mu}^e(r)))$ is equal to zero. This is why the right to bid can be efficiently allocated by an auction without a reserve price. However, this term is in general not equal to zero under limited commitment, $\delta > 0$. It suggests that any efficient mechanism entails a feature that works as a reserve price or an entry fee, thereby creating a jump discontinuity at $\widehat{\mu}(r)$.

2.5.3.2 Existence

Is it possible to organize the ex-post efficient collusion using an individually rational and budget balanced mechanism? Let us consider the initial period $t = 1$ with some reserve $r_1 \in [0, 1]$. Our previous derivations suggest that budget balance requires

$$\sum_{i \in N} \left[s_i^e(0; r_1) + \int_{\widehat{\mu}^e(r_1)}^1 \left(\int_{\widehat{\mu}^e(r_1)}^x (y - r_1) dG(y) + \delta s_i^e(\widehat{\mu}^e(r_1); \widehat{p}^e(\widehat{\mu}^e(r_1))) \right) dF(x) \right] = 0 \quad (2.26)$$

and individual rationality requires

$$s_i^e(0; r_1) + \delta s_i^e(\widehat{\mu}^e(r_1); \widehat{p}^e(\widehat{\mu}^e(r_1))) \leq 0, \ \forall i \in N \quad (2.27)$$

Since Ineq. (2.27) holds for each $i \in N$, it also holds for the sum over N,

$$\sum_{i \in N} s_i^e(0; r_1) + \delta \sum_{i \in N} s_i^e(\widehat{\mu}^e(r_1); \widehat{p}^e(\widehat{\mu}^e(r_1))) \leq 0$$

The first sum is pinned down by Eq. (2.26), so we can write this inequality as

$$\sum_{i \in N} \int_{\widehat{\mu}^e(r_1)}^1 \left(\int_{\widehat{\mu}^e(r_1)}^x (y - r_1) dG(y) + \delta s_i^e(\widehat{\mu}^e(r_1); \widehat{p}^e(\widehat{\mu}^e(r_1))) \right) dF(x) \geq \delta \sum_{i \in N} s_i^e(\widehat{\mu}^e(r_1); \widehat{p}^e(\widehat{\mu}^e(r_1))) \quad (2.28)$$

The next lemma establishes that the right-hand side is strictly positive.

Lemma 2.9. *For all $c \in [0, 1]$,*

$$\sum_{i \in N} s_i^e(c; \widehat{p}^e(c)) > 0$$

Proof. One can also express the expected side-payment (2.25) as

$$s_i^e(x; r) = s_i^e(c; r) - \int_x^c (y - r) dG(y), \ \forall x \geq \widehat{\mu}^e(r_1)$$

Then budget balance can be written as

$$\sum_{i \in N} \left[s_i^e(0;r)F(\widehat{\mu}^e(r)) + \int_{\widehat{\mu}^e(r)}^{c} \left(s_i^e(c;r) - \int_x^c (y-r)dG(y) \right) dF(x) \right] = 0$$

Re-arrange the terms to obtain

$$(F(c) - F(\widehat{\mu}^e(r))) \sum_{i \in N} s_i^e(c;r) = \sum_{i \in N} \left[\int_{\widehat{\mu}^e(r)}^{c} \int_x^c (y-r)dG(y)dF(x) - s_i^e(0;r)F(\widehat{\mu}^e(r)) \right] \tag{2.29}$$

Consider the optimal reserve price, $r = \widehat{p}^e(c)$. As we will show in the next section, the seller never excludes all types from trade, $\widehat{\mu}^e(\widehat{p}^e(c)) < c$, for all $c \in [0,1]$. Thus, the double integral

$$\int_{\widehat{\mu}^e(\widehat{p}^e(c))}^{c} \int_x^c (y - \widehat{p}^e(c))dG(y)dF(x)$$

is strictly positive. Recall that Ineq. (2.24) requires $s_i^e(0;r) \leq 0$ for all $i \in N$. These facts imply that the right-hand side of Eq. (2.29) is strictly positive, which completes the proof. □

Consider the first-period reserve $\underline{r}_1 < 1$ such that it is efficient for the ring to abstain from bidding, $\widehat{\mu}^e(\underline{r}_1) = 1$. Then the left-hand side of Eq. (2.28) is equal to zero. But its right-hand side is strictly positive for all $\delta > 0$, as implied by the above lemma. Thus, Eq. (2.28) is violated, and the next proposition follows.

Proposition 2.10. *There is no incentive compatible, ex-ante budget balanced, and individually rational mechanism that implements the ex-post efficient cut-off rule $\widehat{\mu}^e(r)$ in every period for every first-period reserve price.*

The result extends Proposition 2.5 to the infinite horizon and also generalizes it to non-anonymous mechanisms, which leads to a slightly longer proof. Precisely, the infinite horizon requires Lemma 2.9 that verifies a strictly positive side-payment of the cut-off type in the next period; and non-anonymity requires a summation operator in the budget balance equation.

The impossibility of efficient collusion can be explained as follows. Consider a buyer who is supposed to bid, $x \geq \widehat{\mu}^e(r)$. Whether she reports her true value to ring or leaves the ring and bids her value, she wins the object with probability $G(x)$ and pays $\max\{Y, \widehat{\mu}^e(r_1)\}$.

However, membership in the ring also entails a lump-sum transfer of $-s_i^e(0; r_1) \geq 0$ and an entry fee of $\delta s_i^e(\widehat{\mu}^e(r_1); \widehat{p}^e(\widehat{\mu}^e(r_1))) > 0$. As r_1 increases, fewer types compete for the right to bid, so the ring's revenue and the lump-sup transfer decrease. For a sufficiently high r_1, the entry fee is larger than the lump-sum transfer, so it is optimal for a buyer $x \geq \widehat{\mu}^e(r)$ to leave the ring.

2.5.3.3 Optimal Reserve Prices

Suppose for a moment that the ring finds a way to operate efficiently by receiving subsidies or forcing participation[3]. How large is then the collusive price $\widehat{p}^e(c)$, compared to the competitive price $p(c)$? To answer this question, we begin by noting that the strategy $\widehat{p}^e(c)$ must satisfy the dynamic programming equation

$$\widehat{\Pi}^e(c) = \max_r \left[n \int_{\widehat{\mu}^e(r)}^{c} \widehat{m}(x; r) dF(x) + \delta \widehat{\Pi}^e(\widehat{\mu}^e(r)) \right]$$

It is useful to reformulate the problem in terms of bargaining theory. We can view the ring as a single buyer with value $\max\{X_1, ..., X_n\}$ who can buy the object for a price r,

$$\begin{aligned} n \int_{\widehat{\mu}^e(r)}^{c} \widehat{m}(x; r) dF(x) &= \int_{\widehat{\mu}^e(r)}^{c} nrG(x)f(x)dx \\ &= r[F^n(c) - F^n(\widehat{\mu}^e(r))] \end{aligned}$$

Another useful property is that the ring buys the object if and only if a single buyer would buy it,

$$\widehat{\mu}^e(r) = \frac{r - \delta \widehat{p}^e(\widehat{\mu}^e(r))}{1 - \delta}$$

These two properties suggest our problem is equivalent to bargaining with a single buyer whose value is distributed according to F^n. Although an equilibrium with Markov strategies does not exist for every distribution of values, it is known to exist and can be found analytically for the uniform distribution, $F(x) = x$. For the case of competition, a linear equilibrium is constructed by McAfee and Vincent (1997). For the case of ex-post efficient

[3] The latter option is not improbable given the criminal nature of collusion, as noted by Krishna (2009, p.166).

collusion, a linear equilibrium $\widehat{p}^e(c) = pc$ and $\widehat{\mu}^e(r) = \mu r$ is the solution to

$$(p\mu)^{-n} + \delta n(p\mu) = n + 1$$
$$\mu = \frac{1 - \delta(p\mu)}{1 - \delta}$$

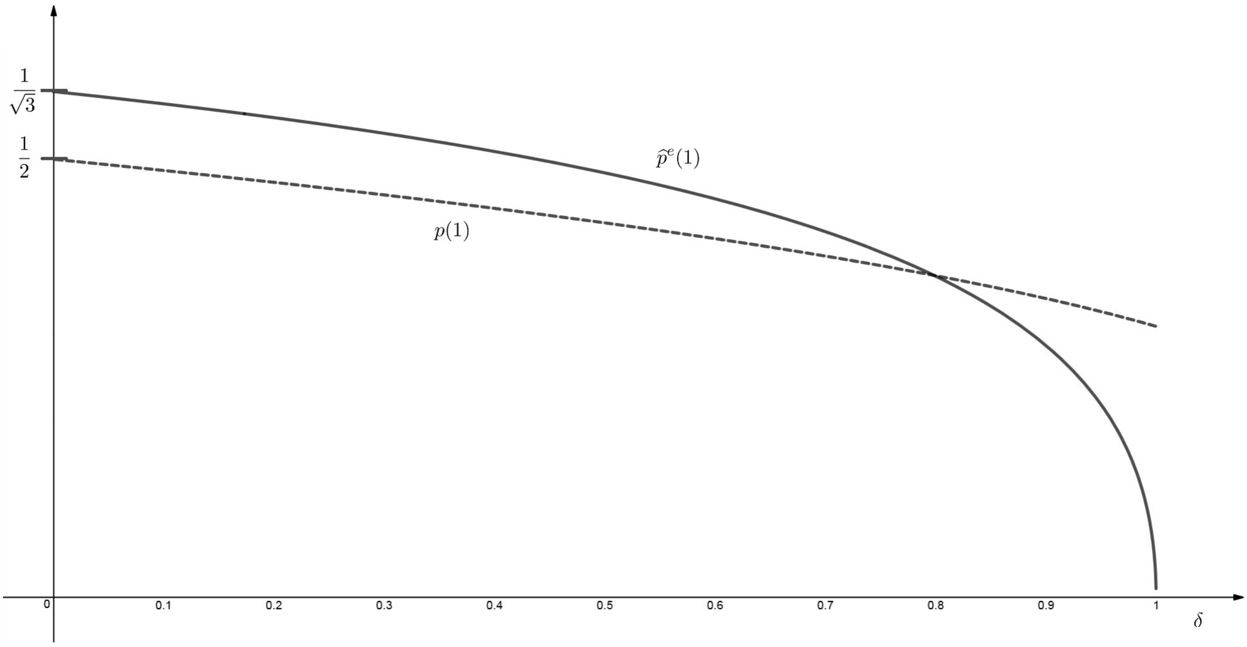

FIGURE 2.1: Optimal first-period reserve prices under efficient collusion *(solid)* and competition *(dotted)* with two buyers, $n = 2$, as functions of the discounting factor $\delta \in (0, 1)$.

Figure 2.1 compares optimal first-period prices in these equilibria for various values of the discounting factor. It suggests that the difference of $p(1)$ and $\widehat{p}^e(1)$ is decreasing in the discounting factor. Under full commitment, $\delta = 0$, and with two buyers, $n = 2$, the optimal reserve is larger with a ring,

$$\widehat{p}^e(1) = \frac{1}{\sqrt{3}} > \frac{1}{2} = p(1)$$

which is a standard example, found in Krishna (2009, p.166). The result is solely driven by the fact that the seller facing a ring can not rely on competitive forces to push the price above a reserve. However, limited commitment creates an opposite effect. An efficient ring buys the object later, $\widehat{\mu}^e(r) > \mu(r)$ for all $\delta > 0$. In other words, the presence of a ring *excludes* some types from an auction. The seller prefers to trade with these types early, which can only be done by setting a lower reserve price. Regardless of the number of buyers, the second effect comes to dominate the first effect as δ approaches one. Therefore, the conclusion of Graham and Marshall (1987) and Mailath and Zemsky (1991) fail to hold for sufficiently low commitment power.

2.5.4 Second-Best Collusion

In our analysis of ex-post efficient collusion, the source of tension between budget balance and individual rationality was a discontinuity of the expected side-payment function. It is then natural to find a "second-best" allocation that smooths the side-payment function and check whether it can be implemented.

2.5.4.1 Definition

Consider the "second-best" cut-off rule, denoted $\widehat{\mu}^*(r)$, which is implicitly defined by

$$\begin{aligned} G(\widehat{\mu}^*)\widehat{\mu}^* - \widehat{m}(\widehat{\mu}^*; r) &= \widehat{U}_i^*(\widehat{\mu}^*(r); \widehat{p}^*(\widehat{\mu}^*(r))) \\ &= \delta[G(\widehat{\mu}^*)\widehat{\mu}^* - m(\widehat{\mu}^*; \widehat{p}^*(\widehat{\mu}^*)) + k^*(\widehat{p}^*(\widehat{\mu}^*))] \end{aligned} \tag{2.30}$$

where for all $r \in [0, 1]$,

$$k^*(r) = \int_{\widehat{\mu}^*}^{c} \int_{\widehat{\mu}^*}^{x} (y - r) dG(y) dF(x)$$

Eq. (2.30) states that the cut-off type's expected payoff from an auction is equal to her continuation payoff. It is easy to see that the definition of the second-best cut-off $\widehat{\mu}^*$ differs from the definition of the competitive cut-off μ only by the term $\delta k^*(\widehat{p}^*(\widehat{\mu}^*))$ on the right-hand side. It reflects the presence of a ring in the next auction, which allows to reap the gains of collusion $k^*(\widehat{p}^*(\widehat{\mu}^*)) > 0$. Thus, the ring leads to a higher level of exclusion, $\widehat{\mu}^*(r) > \mu(r)$.

Note that Eq. (2.30) implies

$$G(\widehat{\mu}^*(r))\widehat{\mu}^*(r) - \widehat{m}(\widehat{\mu}^*(r); r) - \delta \widehat{U}_i^*(\widehat{\mu}^*(r); \widehat{p}^*(\widehat{\mu}^*(r))) = 0$$

so the expected side-payment function is continuous,

$$s_i^*(x; r) = s_i^*(0; r) + \begin{cases} \displaystyle\int_{\widehat{\mu}(r)}^{x} (y - r) dG(y) & \text{if } x \geq \widehat{\mu}^*(r) \\ 0 & \text{if } x < \widehat{\mu}^*(r) \end{cases}$$

It is routine to verify that the ring pays each buyer a transfer $-s_i^*(0; r)$ and asks every bidding type $x \geq \widehat{\mu}^*(r)$ to pay an amount that ensure revenue equivalence with and without collusion,

$$\begin{aligned}\widehat{m}(x;r)+s_i^*(x;r) &= G(x)r+s_i^*(0;r)+\int_{\widehat{\mu}(r)}^{x}(y-r)dG(y)\\ &= s_i^*(0;r)+G(\widehat{\mu}(r))r+\int_{\widehat{\mu}(r)}^{x}ydG(y)\\ &= m(x;r)+s_i^*(0;r)\end{aligned}$$

2.5.4.2 Implementation

Suppose that in every contingency throughout the game, the ring uses the pre-auction knockout where the cut-off is $\widehat{\mu}^*(r)$ and the lump-sum transfer is $k^*(r)$.

Proposition 2.11. *For any reserve price in every period, the PAKT is incentive compatible, individually rational, and ex-ante budget balanced.*

Proof. Consider an auction with some reserve $r \in [0,1]$. First, suppose that there is no ring in the current auction but there is a "second-best" ring in all future auctions. The lowest type who places a bid in the current auction is $\widehat{\mu}^*(r)$ because he is indifferent between bidding and waiting, as argued before.

Second, suppose that there is also a second-best ring in the current auction. To verify *incentive compatibility*, suppose that buyers $2,...,n$ report their true values. If buyer 1 with value x reports some $z \in [0,1]$, her payoff is

$$k^*(r)+\begin{cases} G(z)x-\widehat{m}(z;r)-\displaystyle\int_{\widehat{\mu}^*(r)}^{z}(y-r)dG(y) & \text{if } z \geq \widehat{\mu}^*(r)\\ \delta\widehat{U}^*(x;\widehat{p}(\widehat{\mu}^*(r))) & \text{if } z < \widehat{\mu}^*(r)\end{cases}$$

It is different from her payoff without a ring when she bids as if her value is z, by the constant $k^*(r)$. Since it is optimal to bid as if 1's value is x, it is optimal to report the true value.

To verify *ex-ante budget balance*, note that a buyer's expected contribution to the ring is equal to the lump-sum transfer from the ring,

$$\int_{\widehat{\mu}^*(r)}^{c}\int_{\widehat{\mu}^*(r)}^{x}(y-r)dG(y)dF(x)=k^*(r)$$

To verify *individual rationality*, consider a buyer who left the ring. She competes against $n-1$ buyers who bid Y if $Y > \widehat{\mu}^*(r)$ and do not bid otherwise. But it is the same situation if there were no ring at all. In this case, as shown above, the buyer's payoff is lower than her payoff in the ring by $k^*(r) > 0$. Thus, it is optimal to stay in the ring. □

What makes the second-best collusion "work"? In contrast to the ex-post efficient collusion, implementation of the second-best collusion does not require that the ring announce a reserve price or an entry fee. In fact, as stated in the last proposition, it is sufficient to sell the right to bid using a second-price auction without a reserve and distribute the expected revenue equally among the buyers. If a buyer left the ring, she would win the object with the same probability and pay the same price. However, she would lose a strictly positive lump-sum transfer, which is why it is optimal to participate.

2.5.4.3 Optimal Reserve Prices

What reserve prices should the seller announce when he faces the "second-best" collusion? Similar to the case of ex-post efficient collusion, the optimal price strategy $\widehat{p}^*(c)$ satisfies the dynamic programming equation,

$$\widehat{\Pi}^*(c) = \max_r \left[\int_{\widehat{\mu}^*(r)}^{c} \widehat{m}(x;r) dF(x) + \delta \widehat{\Pi}^*(\widehat{\mu}^*(r)) \right] \tag{2.31}$$

Again, the ring can be viewed as a single buyer because it always pays the reserve, $\widehat{m}(x;r) = G(x)r$. However, the ring follows a different cut-off rule in that it buys the object earlier than a single buyer would buy, $\widehat{\mu}^*(r) < \widehat{\mu}^e(r)$.

The first-order condition for Eq. (2.31) requires that $r = \widehat{p}^*(c)$ solve

$$\int_{\widehat{\mu}^*(r)}^{c} \frac{\partial \widehat{m}(x;r)}{\partial r} dF(x) = [\widehat{m}(\widehat{\mu}^*(r);r) - \delta \widehat{m}(\widehat{\mu}^*(r); \widehat{p}^*(\widehat{\mu}^*(r)))] \frac{d\widehat{\mu}^*(r)}{dr} f(\widehat{\mu}^*(r))$$

which can be transformed using Eq. (2.30) into

$$\int_{\widehat{\mu}^*(r)}^{c} \frac{\partial \widehat{m}(x;r)}{\partial r} dF(x) = [(1-\delta) G(\widehat{\mu}^*(r)) \widehat{\mu}^*(r) + \delta s(\widehat{\mu}^*(r); \widehat{p}^*(\widehat{\mu}^*(r))))] \frac{d\widehat{\mu}^*(r)}{dr} f(\widehat{\mu}^*(r))$$

Although one can not obtain an analytic solution from the first-oder condition, it is probable that the strategy $\widehat{p}^*(c)$ falls in between the cases of ex-post efficient collusion, $\widehat{p}^e(c)$, and competition, $p(c)$. On the one hand, it is similar to efficient collusion in that it reduces participation in the auction and also prevents the selling price from rising above the reserve. On the other hand, it is less extreme than efficient collusion in that the seller is able to sell at positive reserve prices even as the time between periods goes to zero. As a result, we were unable to say whether the optimal first-period reserve is larger with a second-best ring or with competition as $\delta \to 1$.

2.6 Discussion and Extensions

2.6.1 Transparency Policy

In our analysis, we assume an extreme *transparency policy*:

1. the ring center does not disclose any information to the buyers (i.e. their reports).
2. the ring center forgets (or commits to ignoring) any reports from the past.

As a result, neither a buyer's report to a mechanism nor the mechanism itself can not depend on past reports. In reality, however, nothing prevents the buyers from selecting the sole-bidder and exchanging side-payments in advance, instead of re-collecting reports after every failed auction. It is tempting to guess that one may construct a sophisticated transparency policy which mitigates the tension between efficiency, voluntary participation and budget balance. To prove the guess wrong, we follow the mechanism design approach and show that *the efficient collusion is impossible for any transparency policy.*

First, consider again our main setting with no disclosure, denoted *ND*. Fix a perfect Bayesian equilibrium $(\beta_1^{ND}(\cdot), ..., \beta_n^{ND}(\cdot))$ that implements the efficient cut-off $\widehat{\mu}^e(r)$. Suppose that buyer $i \in N$ with value x observes a first-period reserve $r_1 \in [0, 1]$ and commits to playing her equilibrium strategy of type z, that is $\beta_i^{ND}(z)$. Although buyer i is free to "mimic" any type $z \in [0, 1]$, her payoff is maximized by choosing $z = x$ because $\beta_i^{ND}(x)$ is her equilibrium strategy. Thus, we can treat the situation as a direct incentive-compatible mechanism[4] $(\widehat{\mathbf{Q}}^e, \mathbf{S}^{ND})$ with the efficient allocation rule and the expected payment function of $s_i^{ND}(x; r_1)$.

[4]The result is just an illustration of the Revelation Principle, see Myerson (1981).

Second, consider *any other* transparency policy, denoted D, that has a perfect Bayesian equilibrium $(\beta_1^D(\cdot), ..., \beta_n^D(\cdot))$ which also implements the efficient cut-off $\widehat{\mu}^e(r)$. Recall that when the seller faces a full-inclusive ring, his profit is fully defined by the ring's cut-off rule $\widehat{\mu}(r)$. Since policies ND and D involve the same cut-off $\widehat{\mu}^e(r)$, they thus lead to the same price strategy $\widehat{p}^e(r)$. Using the Revelation Principle again, the situation can be seen as a direct incentive-compatible mechanism $(\widehat{\mathbf{Q}}^e, \mathbf{S}^D)$ with the same allocation as ND and the expected payment function $s_i^D(x; r_1)$.

Since both mechanisms are incentive compatible and have the same allocation rule, Revenue Equivalence implies that the expected payment function of each buyer $i \in N$ under ND and D differs by at most a constant,

$$s_i^{ND}(x; r_1) - s_i^D(x; r_1) \equiv k_i, \ \forall x \in [0, 1]$$

Since every mechanism of collusion that may be used under either policy must be ex-ante budget balanced, the mechanisms $(\widehat{\mathbf{Q}}^e, \mathbf{S}^{ND})$ and $(\widehat{\mathbf{Q}}^e, \mathbf{S}^D)$ are also ex-ante budget balanced,

$$\sum_{i \in N} \int_0^1 s_i^{ND}(x; r_1) dF(x) = \sum_{i \in N} \int_0^1 s_i^D(x; r_1) dF(x) = 0$$

Proposition 2.12. *For any transparency policy, there are no ex-ante budget balanced and individually rational mechanisms that implement the ex-post efficient cut-off rule* $\widehat{\mu}^e(r)$ *in every period for every first-period reserve price.*

Proof. Combine the last two equations to obtain

$$\begin{aligned}\sum_{i \in N} k_i &= \sum_{i \in N} \int_0^1 k_i dF(x) \\ &= \sum_{i \in N} \int_0^1 \left[s_i^D(x; r_1) - s_i^{ND}(x; r_1) \right] dF(x) \\ &= 0\end{aligned}$$

from which it follows that

$$\sum_{i \in N} (s_i^D(x; r_1) - s_i^{ND}(x; r_1)) = \sum_{i \in N} k_i = 0, \ \forall x \in [0, 1]$$

Any buyer $i \in N$ with value $\widehat{\mu}^e(r_1)$ can leave the ring at $t = 1$ and bid her value in the auction. For any transparency policy, her belief about the distribution of values is *prior* because other buyers could not send reports prior to $t = 1$. Hence, the outside option at $t = 1$ does not depend on transparency policy, and individual rationality requires

$$\underbrace{U_i^D(\widehat{\mu}^e(r_1); r_1)}_{\text{collusive payoff of } \widehat{\mu}^e(r_1)} \geq \underbrace{G(\widehat{\mu}^e(r_1))\widehat{\mu}^e(r_1) - \widehat{m}(\widehat{\mu}^e(r_1); r_1)}_{\text{competitive payoff of } \widehat{\mu}^e(r_1)}, \ \forall i \in N$$

or simply

$$s_i^D(\widehat{\mu}^e(r_1); r_1) \leq 0, \ \forall i \in N$$

By taking a sum over N on both sides, we can write

$$\sum_{i \in N} s_i^D(\widehat{\mu}^e(r_1); r_1) \leq 0$$

Proposition 2.10 shows that there is a price $\underline{r}_1 < 1$ such that the no-disclosure policy fails the individual rationality constraint,

$$\sum_{i \in N} s_i^{ND}(\widehat{\mu}^e(\underline{r}_1); \underline{r}_1) > 0$$

which implies that *any transparency policy* also fails the constraint,

$$\sum_{i \in N} s_i^D(\widehat{\mu}^e(\underline{r}_1); r_1) = \sum_{i \in N} s_i^{ND}(\widehat{\mu}^e(\underline{r}_1); r_1) \ \Rightarrow \ \sum_{i \in N} s_i^D(\widehat{\mu}^e(\underline{r}_1); \underline{r}_1) > 0$$

□

The proposition and its proof are similar to Theorem 16 in Milgrom and Weber (1982), where the mechanism design approach is used to derive the effect of public information on the expected revenue in first-price auctions. When the values are not correlated, public information does not affect the expected revenue because it basically leads to a mechanism with the same allocation rule. Similarly, transparency policy does not affect the total expected side-payment because the allocation is always ex-post efficient and the budget is balanced in expected terms. Therefore, it is not possible to share information in such a way that every buyer stays in the ring in the initial period for a sufficiently large first-period reserve price.

2.6.2 Participation Constraints

Mailath and Zemsky (1991) propose two types of participation constraints:

1. if a buyer leaves the ring, it continues to operate with $n-1$ buyers;
2. if a buyer leaves the ring, it breaks down completely.

Under full commitment, these constraints are identical because efficient collusion and competition have the same allocation rule. But under limited commitment, they are not identical *in general*, so our assumption that the ring continues to operate with $n-1$ buyers is not without loss of generality.

There are at least two reasons why it may be satisfactory to ignore the second constraint. First, a real-world ring may fail to include all buyers even if the participation constraint is satisfied. This is because of many omitted factors that discourage buyers from colluding, such as costly communication and fear of prosecution. A ring can hardly be called successful or durable if it fails to operate when some of its members fall victim to these factors. But the second constraint entails that the ring breaks down unless it is full, so whenever it is satisfied, a more important property is violated.

Second, the two constraints are identical in the special but important case of second-best collusion. In the proof of Proposition 3.7, we show the competitive equilibrium implements the second-best cut-off. Thus, a buyer who leaves the ring faces the same problem whether there is a ring with $n-1$ buyers or full competition. It suggests that there is no loss of generality when we ignore the possibility of the ring breaking down.

2.6.3 Reputational Equilibria

Our analysis of optimal reserve prices restricts attention to equilibria with Markov strategies. Since they satisfy the Coase conjecture, the optimal first-period reserve becomes lower under efficient collusion as the time between periods goes to zero. However, these equilibria are generally not unique: Ausubel and Deneckere (1989) construct *reputational* equilibria where the initial price is very close to the optimal price under full commitment. But then the optimal price under collusion should be larger even for small time between periods, which poses a challenge to our claim.

We shall justify our restriction to stationary equilibria with an informal argument. Recall that the main price path of a reputational strategy is described by a very slow rate of sales. As shown in the proof of Proposition 2.10, a slow rate of sales "destroys" the ring's revenue because the latter only comes from bidding types. The cut-off type is asked to pay a positive entry fee to the ring but receives an arbitrarily small transfer in return. As a result, she is better-off leaving the ring and bidding for the object on her own. It follows that the main price path is rather a measure against collusion than a best response to it.

Of course, stationary equilibria are not immune to the tension between budget balance and individual rationality, either. However, their equilibrium path involves a very fast rate of sales where the ring secures a high expected revenue. The tension is present only in off-equilibrium reserve prices, that become increasingly suboptimal as the time between periods goes to zero. To sum up, reputational equilibria are much less "reasonable" because they suffer from the impossibility of efficient collusion not only in off-equilibrium, but also in on-equilibrium histories.

2.7 Conclusions

This paper has identified the impact of limited commitment on collusion among buyers. We have constructed a model where a seller can always run a second-price auction with a reserve price until he sells a single and indivisible object. It was crucial that the seller could only commit to an auction in the current period, but not in the future periods. Our goal was to understand if there exists a mechanism of collusion that implements efficient collusion and also satisfies individual rationality and budget balance constraints. Our findings suggest that such mechanism does not exist for all reserve prices because a bidding ring that discourages its members from inefficient participation may run a deficit. Furthermore, the best response against a ring that implements efficient collusion may involve lower reserve prices because buyers act more patiently when they collude.

Yet the model has a number of limitations that can be improved in further research. First, we only consider full-inclusive rings. Although it was shown to be the equilibrium size of the ring with commitment (see Graham and Marshall (1987)), some factors that are often sacrificed for the sake of tractability can prevent the formation of a full-inclusive ring. As noted earlier, the cost of coordination or the risk of being exposed may be prohibitively high to form a full-inclusive ring. But it is technically difficult to study rings with $k < n$ members because the buyers outside the ring do not generally bid in the same period as the buyers in the ring. The distribution of their values becomes a new state variable in the dynamic programming problem facing the seller; the result is a much less tractable model.

Second, we treat mechanisms of collusion as being proposed by an impartial ring center or simply exogenously given. As explained by Mailath and Zemsky (1991), this assumption is justified by the lack of a non-cooperative theory of coalition formation. Although the problem is also present in the static model, a lack of commitment makes it especially acute. While our model requires that every auction be sequentially optimal, it does not impose any requirement of optimality on the mechanisms of collusion. Simply put, a ring center can commit to any plan of mechanisms throughout the game. But if a bidding ring also becomes constrained by sequential optimality, it may be unable to implement even the second-best collusion because it is not efficient (and thus, not optimal). The problem may be overcome by incorporating a bargaining stage where the buyers must reach an agreement before they are asked to place bids.

2.8 Appendix

2.8.1 Proof of Lemma 2.1

Without a ring, a buyer with value x who bids as if her value were $z \in [0, 1]$, obtains a payoff

$$\begin{cases} G(z)x - m_1(z; r_1) & \text{if } z \geq \mu(r_1) \\ \delta U(x; p_2(\mu(r_1))) & \text{if } z < \mu(r_1) \end{cases}$$

With an efficient ring, a buyer with value x who reports some value $z \in [0, 1]$, obtains a payoff

$$\begin{cases} G(z)x - \widehat{m}_1(z; r_1) - s(z; r_1) & \text{if } z \geq \mu(r_1) \\ -s(z; r_1) + \delta U(x; p_2(\mu(r_1))) & \text{if } z < \mu(r_1) \end{cases}$$

The Envelope theorem implies that the shape of the equilibrium payoff is the same in both cases,

$$U'(x;r_1) = \widehat{U}'(x;r_1) = \begin{cases} G(x) & \text{if } x \geq \mu(r_1) \\ \delta U'(x;p_2(\mu(r_1))) & \text{if } x < \mu(r_1) \end{cases}$$

2.8.2 Proof of Lemma 2.4

If a buyer with value x reports $z \in [0,1]$, her payoff is

$$\begin{cases} G(z)x - \widehat{m}(z;r_1) - s_1(z;r_1) & \text{if } z \geq \mu(r_1) \\ -s_1(z;r_1) + \delta \widehat{U}_2(z,x;\widehat{p}_2(\widehat{\mu}^e(r_1))) & \text{if } z < \mu(r_1) \end{cases}$$

Incentive compatibility requires that the buyer report $z = x$. It implies the first-order condition

$$0 = \begin{cases} g(x)(x - r_1) - s_1'(x;r_1) & \text{if } x \geq \widehat{\mu}^e(r_1) \\ -s_1'(x;r_1) & \text{if } x < \widehat{\mu}^e(r_1) \end{cases}$$

Integrate on both sides and re-arrange the terms to obtain

$$s_1(x;r_1) = \begin{cases} s_1(\widehat{\mu}^e(r_1);r_1) + \int_{\widehat{\mu}^e(r_1)}^{x} (y - r_1) dG(y) & \text{if } x \geq \widehat{\mu}^e(r_1) \\ s_1(0;r_1) & \text{if } x < \widehat{\mu}^e(r_1) \end{cases} \tag{2.32}$$

Also, incentive compatibility implies that the equilibrium payoff of each buyer $i \in N$ is absolutely continuous. In particular, the cut-off type must be indifferent between reporting her true value and any lower value,

$$\begin{aligned} &G(\widehat{\mu}^e(r_1))\widehat{\mu}^e(r_1) - \widehat{m}_1(\widehat{\mu}^e(r_1);r_1) - s_1(\widehat{\mu}^e(r_1);r_1) = \\ &\quad = -s_1(0;r_1) + \delta[G(\widehat{\mu}^e(r_1))\widehat{\mu}^e(r_1) - \widehat{m}_2(\widehat{\mu}^e(r_1);\widehat{p}_2(\widehat{\mu}^e(r_1)) - s_2(\widehat{\mu}^e(r_1);\widehat{p}_2(\widehat{\mu}^e(r_1)))] \end{aligned}$$

Using Eq. (2.8), it can be re-written as

$$s_1(\widehat{\mu}^e(r_1);r_1) = s_1(0;r_1) + \delta s_2(\widehat{\mu}^e(r_1);\widehat{p}_2(\widehat{\mu}^e(r_1))) \tag{2.33}$$

Finally, substitute Eq. (2.33) into Eq. (2.32) to verify the lemma.

2.8.3 Proof of Proposition 2.2

To verify *incentive compatibility*, suppose that all buyers $j \neq i$ report their true values. If buyer i with value x who reports some value $z \in [0, 1]$ to the ring, her payoff is

$$k(r_1) + \begin{cases} G(z)x - \widehat{m}_1(z; r_1) - \int_{\widehat{\mu}^*(r)}^{z} (y - r_1) dG(y) & \text{if } z \geq \mu(r_1) \\ \delta U(x; p_2(\mu(r_1))) & \text{if } z < \mu(r_1) \end{cases}$$

Except for $k(r_1)$, it is the same payoff that buyer i would obtain without a ring by bidding as if her value were z. Since it is a best response to bid as if i's value were x, it is also a best response for i to report x.

To verify *ex-ante budget balance*, note that any buyer receives the transfer of $k(r_1)$ from the ring. It is equal to her expected contribution to the ring, given by

$$\int_{\mu(r_1)}^{1} \int_{\mu(r_1)}^{x} (y - r_1) dG(y) dF(x)$$

To verify *individual rationality*, note that a buyer's payoff in the ring is larger than her payoff outside the ring by a strictly positive constant, $k(r_1) > 0$.

Chapter 3

Fair Nonlinear Pricing

Abstract

We consider a problem of optimal nonlinear pricing where the buyer rewards the seller for charging fair prices and punishes him for unfair prices. As in Rabin (1993), the buyer's perception of fairness is determined by her belief about underlying intentions of the seller. Our findings suggest that under complete information, the buyer receives an offer to purchase the efficient quantity for a price below her valuation. Under incomplete information, the optimal truthful mechanism is characterized by no distortion at the top and downward distortion elsewhere. However, the distortion is not as large as in the standard model because the seller internalizes a psychological cost of paying an unfair price. As a result, he is motivated to reduce this cost by improving allocative efficiency. The optimal mechanism can be implemented by a schedule that stipulates lower per-unit prices, so our model seems to be more consistent with behavior of buyers and sellers in controlled experiments.

Keywords: Nonlinear pricing, Quantity Discounts, Intention-Based Social Preferences, Optimal Mechanism Design.

3.1 Introduction

Firms often charge lower prices to consumers who buy large quantities of their product (e.g. "three for the price of two"). Economic literature usually explains such nonlinear pricing within the framework of adverse selection. For example, Maskin and Riley (1984) consider a seller who faces a buyer with unobserved willingness to pay. The seller offers a schedule that stipulates a price for every quantity that the buyer may choose to purchase. Of course, the seller's profit would be maximized if the buyer always purchased the efficient quantity and earned no surplus from the trade. But the buyer could then purchase a subefficient quantity and earn a positive surplus, so the first-best solution is not incentive feasible. It follows that the second-best solution must trade-off allocative efficiency and information rent, which results in quantity discounts.

Although the model manages to capture the rationale behind quantity discounts, some predictions are not fully consistent with laboratory evidence. In a recent experiment, Hoppe and Schmitz (2015) divide subjects into two groups: sellers choose prices for two quantities, and buyers choose a quantity they wish to purchase. As predicted by the model, sellers tend to offer both quantities at such prices that buyers with higher willingness to pay purchase a larger quantity. However, buyers often reject offers, which is ruled out in the model because of individual rationality, and both quantities are traded at lower prices than predicted. To explain these inconsistencies, the authors extend the standard model to allow for inequity-averse buyers in the spirit of Fehr and Schmidt (1999). Using the logit-QRE approach to estimate the strength of aversion, they show that the extension improves the accuracy of predictions by only 1-2 percentage points.

In this paper, we modify the standard model of nonlinear pricing with the goal to improve its predictive power. It is assumed that the buyer's utility depends not only on her monetary payoff, but also on her psychological payoff, defined as the product of her kindness and the seller's kindness. As in Rabin (1993), an agent is considered kind if and only if she intends to give the other agent a payoff above some fair payoff, defined as a combination of payoffs from Pareto-efficient outcomes. Our analysis relies on the mechanism design approach to obtain the optimal schedule of prices in two information settings: the seller (i) observes the buyer's value, and (ii) does not observe her value.

Some of our findings are consistent with Maskin and Riley (1984): there is no quantity distortion under complete information, while there is a downward quantity distortion for inefficient types under incomplete information. However, the degree of distortion is smaller because the psychological cost is internalized by the seller. In order to reduce this cost, he chooses to offer a higher quantity so that the buyer earns a larger rent and the seller becomes kinder. The optimal menu of contracts can be implemented with a schedule that entails lower prices, which is why our predictions seem to fit the findings of Hoppe and Schmitz (2015) better.

The rest of this paper is organized as follows. Section 2 reviews the literature on intention-based social preferences and nonlinear pricing. Section 3 outlines the model and Section 4 derives the optimal contract under complete information. The optimal menu of contracts under incomplete information is derived in Section 5 for two buyer types and in Section 6 for a continuum of types. Section 7 discusses its implementation and quantity discounts and Section 8 concludes.

3.2 Literature Review

Economic theory traditionally assumes that agents seek to maximize their monetary payoff. The last decades, however, have seen the emergence of behavioral theories that challenge the traditional view on preferences. In particular, theories of *social preferences* assume that an economic agent may also care about the payoff of other agents besides their own. It is common to distinguish between outcome-based (Bolton and Ockenfels, 2000; Fehr and Schmidt, 1999) and intention-based (Dufwenberg and Kirchsteiger, 2004; Rabin, 1993) social preferences, depending on whether an agent focuses on consequences of, or motivation behind, feasible strategies.

Rabin (1993) relies on the concept of psychological games (Geanakoplos et al., 1989) to model a strategic situation where players wish to reward kind opponents and to hurt mean ones. It is assumed that a player is believed to be kind if and only if she intends to give the other player a payoff above the equitable payoff. The latter is defined as a normalized average of the lowest and the highest payoffs among Pareto-efficient outcomes. In equilibrium, every player's strategy maximizes the sum of her monetary payoff and her psychological payoff given her second-order beliefs, and her second-order beliefs are consistent with the strategies. Dufwenberg and Kirchsteiger (2004) extend the model to sequential games and propose the concept of sequential reciprocity equilibrium that specifies how players update their beliefs about each other's intentions over time.

The problem of optimal nonlinear pricing is a classical application of adverse selection theory, see Laffont and Martimort (2009). It is usually assumed that a monopolist can engage in second-degree price discrimination by offering a schedule of prices for each quality (Mussa and Rosen, 1978) or quantity (Maskin and Riley, 1984). By the Revelation Principle, it is sufficient to find the optimal truthful menu of contracts and propose its implementation by a schedule of nonlinear prices, see Wilson (1996). Recently, Bierbrauer and Netzer (2016) extend the mechanism design theory to intention-based social preferences of Rabin (1993). They show that under complete information about a weight of kindness, any Pareto-efficient social choice function can be weakly implemented.

The theoretical literature on fair pricing is surprisingly scarce. Rabin (1993) applies his model to a normal form game where a monopolist chooses a price and a consumer chooses a "reservation" price. His analysis suggests that the largest equilibrium price lies strictly below the consumer's valuation, and moreover, it is increasing in the monopolist's cost of production. Rotemberg (2011) considers a market where consumers believe that a firm is fair if they fail to reject the hypothesis that it is benevolent towards them. According to his results, the equilibrium price is rather rigid and more responsive to shocks in costs than to shocks in demand. For a summary of empirical studies on fair pricing, see Xia et al. (2004).

3.3 Model

There are two agents: a seller (he) and a buyer (she). The seller offers a contract (q, t) that the buyer can accept or reject. A contract stipulates a quantity q to be delivered by the seller and a monetary transfer t to be paid by the buyer. If the buyer accepts (q, t), her (monetary) *payoff* is equal to $U = xu(q) - t$, where x is her type[1], distributed according to some function F on $[0, w]$. We assume that $u' > 0$, $u'' < 0$ and $u(0) = 0$. Also, the seller's *profit* is equal to $\Pi = t - cq$, where $c > 0$ denotes his marginal cost of production. It is useful to define *social surplus* as the sum of the monetary payoff and the profit,

$$S(q, x) = xu(q) - cq$$

[1]Maskin and Riley (1984) consider a more general payoff function $U = \int_0^q p(y, x)dy - t$. Thus our payoff function can be treated as $p(q, x) = xu'(q)$.

Let $q^e(x)$ denote the socially efficient quantity, defined implicitly by $xu'(q^e) = c$.

In contrast to the standard model, the buyer's utility function depends on the seller's kindness towards her as well as her kindness towards the seller. First, *the seller's kindness* is given by the term $(U^S - U^F)$ where U^S denotes the payoff that the buyer believes that the seller believes that the buyer will earn; and U^F denotes the payoff that the buyer believes to be fair. Second, *the buyer's kindness* is given by the term $(\Pi - \Pi^F)$, where Π is the actual profit of the seller and Π^F is the fair profit to give to the seller. Finally, the buyer's *psychological payoff* is given by the product of both kindnesses, that is $(\Pi - \Pi^F) \times (U^S - U^F)$.

We assume that the utility function of the buyer is equal to the sum of her material payoff and her psychological payoff,

$$V = U + (\Pi - \Pi^F) \times (U^S - U^F)$$

Note that her psychological payoff is strictly increasing in Π if and only if the seller is kind, $(U^S - U^F) > 0$. It captures the idea that the buyer is willing to reward a kind seller by giving him more profit, and to punish an unkind seller by giving him less profit.

A contract is said to be *Pareto-efficient* if there is no other contract that yields a strictly larger profit and a strictly larger payoff. Let U^l (U^h) denote the lowest (highest) payoffs and Π^l (Π^h) denote the lowest (highest) profits in the set of Pareto-efficient contracts. Then the fair payoff is defined as

$$U^F = \theta U^h + (1 - \theta) U^l$$

and the fair profit is defined as

$$\Pi^F = \theta \Pi^h + (1 - \theta) \Pi^l$$

where $\theta \in [0, 1]$ is an exogenous parameter.

In what follows we separately consider two information environments. Under complete information, the seller observes the buyer's value and offers a single contract. We say that a contract (q^*, t^*) is *optimal* if the profit from any contract (q, t) is weakly lower than from (q^*, t^*), subject to

(i) consistency of second-order beliefs: if it is optimal to accept a contract, the buyer believes $U^S = U$, and if it is optimal to reject a contract, the buyer believes $U^S = 0$;

(ii) consistency of social norms: the fair payoff U^F and the fair profit Π^F follow from the set of contracts that are Pareto-efficient given the optimality of (q^*, t^*).

Under incomplete information, the seller does not observe the buyer's value and offers a menu of contracts. Similarly, a menu $(q^*(\cdot), t^*(\cdot))$ is said to be *optimal* if the expected profit from any menu $(q(\cdot), t(\cdot))$ is weakly lower than from $(q^*(\cdot), t^*(\cdot))$, subject to

(i) consistency of second-order beliefs: for any menu of contracts, if the buyer's optimal choice results in some payoff function $U(x)$, the buyer believes $U^S = E[U(X)]$;

(ii) consistency of social norms: the fair payoff U^F and the fair profit Π^F follow from the set of menus that are Pareto-efficient given the optimality of $(q^*(\cdot), t^*(\cdot))$.

3.4 Complete Information

In this section, we examine the case of complete information. That is, the seller observes the buyer's value $x \in [0, w]$ and offers her a single contract (q, t). We shall proceed by deriving a set of necessary conditions on the optimal contract, denoted by (q^*, t^*).

3.4.1 Optimization Problem

For a given fair payoff U^F, a contract (q, t) is *feasible* if the buyer prefers to accept (q, t) given that she believes that the seller intends to give her the payoff of $U^S = U$. Thus the buyer's utility from acceptance is

$$V = U + (\Pi - \Pi^F) \times (U - U^F)$$

and her utility from rejection is

$$R = 0 + (0 - \Pi^F) \times (U - U^F)$$

It is optimal to accept the contract if the difference of these utility levels is positive,

$$\begin{aligned} V - R &= U + \Pi \times (U - U^F) \\ &= xu(q) - t + (t - cq) \times [(xu(q) - t) - U^F] \geq 0 \end{aligned}$$

which can be re-arranged as

$$\underbrace{xu(q) - t}_{\text{monetary payoff}} \geq \underbrace{(t - cq) \times [U^F - (xu(q) - t)]}_{\text{psychological cost}} \tag{PC}$$

The participation constraints requires that the monetary payoff must compensate for the "psychological cost" of accepting an unfair contract[2]. Note that if $U^F = 0$, the set of feasible contracts is the same as in the standard model. But if $U^F > 0$, some contracts with sufficiently low payoffs are rejected.

The optimal contract (q^*, t^*) must be the most profitable feasible contract,

$$\max_{(q,t)} \Pi = t - cq, \text{ subject to (PC)}$$

Suppose that (q^*, t^*) satisfies Eq. (PC) with strict inequality. Then there is a contract (q^*, t') with $t' > t$ that satisfies Eq. (PC) with equality. Since (q^*, t') is more profitable and still feasible, (q^*, t^*) is not optimal, a contradiction. Hence, Eq. (PC) is binding in the optimum.

Let us consider the payoff variable $U = xu(q) - t$, so that $t = xu(q) - U$. Thus the optimization problem can be stated as

$$\max_{(q,U)} S(q,x) - U,$$
$$\text{subject to } \; U = (S(q,x) - U) \times (U^F - U)$$

The constraint implicitly defines U as a function of q. Differentiate both sides to obtain

$$\frac{dU}{dq} = \left(S_1(q,x) - \frac{dU}{dq}\right) \times (U^F - U) - \frac{dU}{dq} \times (S(q,x) - U) \tag{3.1}$$

The first-order condition for the optimization problem reads

$$S_1(q^*, x) = \frac{dU^*}{dq}$$

[2]Although there exist fair contracts, the optimal contract is unfair and so are local deviations from it.

Using Eq. (3.1), it can also be written as

$$\underbrace{S_1(q^*, x)}_{\text{efficiency}} = \underbrace{-S_1(q^*, x) \times (S(q^*, x) - U^*)}_{\text{psychological cost}}$$

It is clear the condition is satisfied at $S'(q^*, x) = 0 \Leftrightarrow q^* = q^e$. In order to understand why the optimal quantity is efficient, consider a deviation, $q \neq q^e$. If the deviation is profitable, it must reduce the buyer's payoff, $U(q) < U(q^e)$. However, the constraint suggests that the deviation increases her psychological cost, which must be compensated by a larger payoff, $U(q) > U(q^e)$, a contradiction. Thus, the seller can not profitably deviate from q^e.

3.4.2 Fair Payoff

To find the fair payoff U^F, first consider the contract (q^e, cq^e) where the allocation is efficient and the buyer reaps all surplus from trade. Since the profit of the seller is zero, Eq. (PC) implies that the contract is feasible. Since it also maximizes the buyer's payoff, the contract is Pareto-efficient. It follows that the largest payoff among Pareto-efficient contracts is

$$\begin{aligned} U^h &= xu(q^e) - cq^e \\ &= S(q^e, x) \end{aligned}$$

Second, consider the optimal contract (q^e, t^*). Since any other contract is less profitable, it is Pareto-efficient. Also, any other Pareto-efficient contract must yield a higher payoff to the buyer (otherwise it would be Pareto-dominated by the optimal contract). It implies that the lowest payoff among all Pareto-efficient contracts is

$$\begin{aligned} U^l &= xu(q^e(x)) - t^*(x) \\ &= U^* \end{aligned}$$

These two results lead to our first lemma:

Lemma 3.1. *If the optimal contract is (q^e, t^*), the fair payoff is given by*

$$U^F = \theta S(q^e, x) + (1 - \theta)U^*(x)$$

In economic terms, the lemma states that the fair payoff is a convex combination of the payoff under perfect competition, $S(q^e, x)$, and the payoff under first-degree price discrimination with intention-based social preferences, $U^*(x)$.

3.4.3 Optimal Contract

Using $q^*(x) = q^e(x)$ and $U^F = \theta S(q^e(x), x) + (1-\theta)U^*(x)$, the optimal payoff satisfies

$$\begin{aligned} U^*(x) &= (S(q^e(x), x) - U^*(x)) \times (U^F - U^*(x)) \\ &= (S(q^e(x), x) - U^*(x)) \times (\theta S(q^e(x), x) + (1-\theta)U^*(x) - U^*(x)) \\ &= \theta (S(q^e(x), x) - U^*(x))^2 \end{aligned}$$

One could argue that the profit from offering the optimal contract may be negative for low buyer types. Suppose that the seller shuts down all types on $[0, \mu]$, so μ denotes the lowest type whom the seller offers $(q^e(x), U^*(x))$. Then the profit from trading with type μ must be equal to zero,

$$\begin{aligned} 0 &= S(q^e(\mu), \mu) - U^*(\mu) \\ &= S(q^e(\mu), \mu) - \theta(S(q^e(\mu), \mu) - U^*(\mu))^2 \\ &= S(q^e(\mu), \mu) \end{aligned}$$

In economics terms, the profit from trading with μ is zero, so the buyer of type μ does not have any psychological payoff. Thus the social surplus from trading with type μ must be zero. Our assumptions imply that $S(q^e(0), 0) = 0 \Rightarrow \mu = 0$, so the seller trades with all types.

Proposition 3.2. *The optimal contract for buyer type x is $(q^e(x), U^*(x))$ where $U^*(x)$ solves*

$$U^*(x) = \theta(S(q^e(x), x) - U^*(x))^2 \ , \ U^*(x) \leq S(q^e(x), x)$$

It is easy to identify the effect of reciprocity on the optimal contract. In the standard model, the seller optimally produces the efficient quantity and reaps all social surplus, $(q^{FB}, U^{FB}) = (q^e, 0)$. Since the quantity is also efficient in our model, reciprocity only affects the *distribution* of social surplus. If $\theta = 0$, the buyer's payoff is zero, $U^*(x) = 0$, but for all $\theta > 0$, the buyer's payoff is positive, $U^*(x) > 0$. More precisely, the functional relationship between θ and $U^*(x)$ is implicitly governed by

$$\frac{dU^*(x)}{d\theta} = \frac{(S(q^e(x), x) - U^*(x))^2}{1 + \theta \times 2(S(q^e, x) - U^*(x))} > 0$$

which implies that the seller's profit $[S(q^e(x), x) - U^*(x)]$ is strictly decreasing in θ.

3.4.4 Implementation and Predictive Power

In the optimum, the buyer of type x consumes $q^e(x)$ units. By inverting $q^e(x)$, we can find the type $x^e(q)$ for whom it is efficient to purchase a given quantity q. Then the seller can implement his optimal contract by charging the price

$$\begin{aligned} T(q) \equiv t^*(x^e(q)) &= x^e(q)u(q) - U^*(x^e(q)) \\ &= x^e(q)u(q) - \theta(T(q) - cq)^2 \end{aligned}$$

Our analysis replicates both results of Rabin (1993). First, the optimal price $T(q)$ lies strictly below the buyer's valuation $x^e(q)u(q)$. Second, it is increasing in the cost of production,

$$\frac{\partial T(q)}{\partial c} = \frac{\theta \times 2(T(q) - cq) \times q}{1 + \theta \times 2(T(q) - cq)} > 0$$

Let us compare our predictions to the experimental results of Hoppe and Schmitz (2015). In a setting with complete information[3], their results can be broadly summarized as follows:

(EC.1) Sellers offer a single contract to each type;

(EC.2) No distortion at the top;

(EC.3) No distortion at the bottom;

(EC.4) Selling prices are lower than in the standard model;

(EC.5) Some offers are rejected.

[3]The authors call this setting SI-I.

The results (EC.1), (EC.2) and (EC.3) are consistent with the standard model while (EC.4) and (EC.5) are not. Fortunately, our model predicts the results (EC.1), (EC.2), (EC.3) and (EC.4) but fails to predict (EC.5). It is nevertheless wrong to say that our model can not provide an explanation for rejection of contracts. We saw that the buyer rejects "out-of-equilibrium" contracts that stipulate sufficiently low payoffs. In reality, the seller may not know how low is sufficiently low because the parameter θ may be unobserved. Hence, his contract could be rejected when he underestimates the size of θ.

3.5 Incomplete Information: Two Types

The main case of our analysis is a continuous distribution of types. It is however very helpful to begin with a simple case where the space of types is binary. Suppose that the buyer may be a high type, denoted $\overline{x}$, or a low type, denoted $\underline{x}$, with $\overline{x} > \underline{x} > 0$. In contrast to the previous section, the seller does not observe whether the buyer is $\underline{x}$ or $\overline{x}$, but he knows that the probability of facing the high type is equal to $f \in (0, 1)$.

3.5.1 Incentive and Participation Constraints

The seller offers a menu that consists of two contracts, one for each type. Consider a menu of contracts $\{(\underline{q}, \underline{t}), (\overline{q}, \overline{t})\}$ and fix a fair payoff U^F, a second-order belief U^S and fair profits $\overline{\Pi}^F$ and $\underline{\Pi}^F$. The menu is *incentive compatible* if every buyer type prefers her contract to the other contract,

$$\overline{x}u(\overline{q}) - \overline{t} + (\overline{t} - c\overline{q} - \overline{\Pi}^F) \times (U^S - U^F) \geq \overline{x}u(\underline{q}) - \underline{t} + (\underline{t} - c\underline{q} - \overline{\Pi}^F) \times (U^S - U^F)$$
$$\underline{x}u(\underline{q}) - \underline{t} + (\underline{t} - c\underline{q} - \underline{\Pi}^F) \times (U^S - U^F) \geq \underline{x}u(\overline{q}) - \overline{t} + (\overline{t} - c\overline{q} - \underline{\Pi}^F) \times (U^S - U^F)$$

and *individually rational* if every buyer type prefers her contract to the outside option,

$$\overline{x}u(\overline{q}) - \overline{t} + (\overline{t} - c\overline{q} - \overline{\Pi}^F) \times (U^S - U^F) \geq -\overline{\Pi}^F \times (U^S - U^F)$$
$$\underline{x}u(\underline{q}) - \underline{t} + (\underline{t} - c\underline{q} - \underline{\Pi}^F) \times (U^S - U^F) \geq -\underline{\Pi}^F \times (U^S - U^F)$$

As in the previous section, introduce the payoff variables $\overline{U} = \overline{x}u(\overline{q}) - \overline{t}$ and $\underline{U} = \underline{x}u(\underline{q}) - \underline{t}$. We can use them to simplify the incentive compatibility constraints as

$$\overline{U} + (\overline{S}(\overline{q}) - \overline{U})(U^S - U^F) \geq \underline{U} + u(\underline{q})\Delta x + (\underline{S}(\underline{q}) - \underline{U})(U^S - U^F) \tag{IC.1}$$

$$\underline{U} + (\underline{S}(\underline{q}) - \underline{U})(U^S - U^F) \geq \overline{U} - u(\overline{q})\Delta x + (\overline{S}(\overline{q}) - \overline{U})(U^S - U^F) \tag{IC.2}$$

and the individual rationality constraints as

$$\overline{U} + (\overline{S}(\overline{q}) - \overline{U})(U^S - U^F) \geq 0 \tag{IR.1}$$

$$\underline{U} + (\underline{S}(\underline{q}) - \underline{U})(U^S - U^F) \geq 0 \tag{IR.2}$$

It follows that the fair profits $\overline{\Pi}^F$ and $\underline{\Pi}^F$ are again irrelevant to the problem. Also, note that the seller does not know the buyer's type at the time of choosing a menu of contracts. Therefore, his intentions can not be contingent on the buyer's type, and the buyer believes that the seller intends to give her the *expected* payoff,

$$U^S = f\overline{U} + (1 - f)\underline{U}$$

3.5.2 Optimization Problem

The objective of the seller is to choose a menu of contracts that maximizes his expected profit subject to the incentive compatibility and individual rationality constraints,

$$\max_{\{(\underline{q},\underline{U}),(\overline{q},\overline{U})\}} f(\overline{S}(\overline{q}) - \overline{U}) + (1 - f)(\underline{S}(\underline{q}) - \underline{U}) \ ,$$
$$\text{subject to (IC.1)} - \text{(IR.2)}$$

Luckily, the constraint (IR.1) is not binding, as implied by (IC.1), (IR.2), and $\overline{x} > \underline{x}$,

$$\begin{aligned}\overline{U} + (\overline{S}(\overline{q}) - \overline{U})(f\overline{U} + (1 - f)\underline{U} - U^F) &\geq \underline{U} + u(\underline{q})\Delta x + (\underline{S}(\underline{q}) - \underline{U})(f\overline{U} + (1 - f)\underline{U} - U^F) \\ &> \underline{U} + (\underline{S}(\underline{q}) - \underline{U})(f\overline{U} + (1 - f)\underline{U} - U^F) \geq 0\end{aligned}$$

As in the standard model, it is also reasonable to guess that the constraint (IC.2) is not binding, either. However, the remaining constraints (IC.1) and (IR.2) must be binding in the optimum. If they were not, the seller could extract a slightly larger profit without violating the constraints.

Therefore, it is sufficient to consider a relaxed problem:

$$\max_{\{(\underline{q},\underline{U}),(\overline{q},\overline{U})\}} f(\overline{S}(\overline{q})-\overline{U})+(1-f)(\underline{S}(\underline{q})-\underline{U}), \text{ subject to}$$

$$\overline{U}=(\overline{S}(\overline{q})-\overline{U})(U^F-(f\overline{U}+(1-f)\underline{U}))+u(\underline{q})\Delta x, \tag{3.2}$$

$$\underline{U}=(\underline{S}(\underline{q})-\underline{U})(U^F-(f\overline{U}+(1-f)\underline{U})) \tag{3.3}$$

Multiply Eq. (3.2) by f and Eq. (3.3) by $1-f$ and sum them up to obtain

$$f\overline{U}+(1-f)\underline{U}=[f(\overline{S}(\overline{q})-\overline{U})+(1-f)(\underline{S}(\underline{q})-\underline{U})](U^F-(f\overline{U}+(1-f)\underline{U}))+fu(\underline{q})\Delta x \tag{3.4}$$

Note that Eq. (3.2) and Eq. (3.3) define $\underline{U}$ and $\overline{U}$ as functions of $\underline{q}$ and $\overline{q}$. Differentiate both sides of Eq. (3.4) with respect to $\overline{q}$,

$$\begin{aligned}f\frac{\partial\overline{U}}{\partial\overline{q}}+(1-f)\frac{\partial\underline{U}}{\partial\overline{q}}=(U^F-(f\overline{U}+(1-f)\underline{U}))\left[f\overline{S}'(\overline{q})-\left(f\frac{\partial\overline{U}}{\partial\overline{q}}+(1-f)\frac{\partial\underline{U}}{\partial\overline{q}}\right)\right]\\-\left(f\frac{\partial\overline{U}}{\partial\overline{q}}+(1-f)\frac{\partial\underline{U}}{\partial\overline{q}}\right)[f(\overline{S}(\overline{q})-\overline{U})+(1-f)(\underline{S}(\underline{q})-\underline{U})]\end{aligned} \tag{3.5}$$

and with respect to $\underline{q}$,

$$\begin{aligned}f\frac{\partial\overline{U}}{\partial\underline{q}}+(1-f)\frac{\partial\underline{U}}{\partial\underline{q}}=(U^F-(f\overline{U}+(1-f)\underline{U}))\left[(1-f)\underline{S}'(\underline{q})-\left(f\frac{\partial\overline{U}}{\partial\underline{q}}+(1-f)\frac{\partial\underline{U}}{\partial\underline{q}}\right)\right]\\-\left(f\frac{\partial\overline{U}}{\partial\underline{q}}+(1-f)\frac{\partial\underline{U}}{\partial\underline{q}}\right)[f(\overline{S}(\underline{q})-\overline{U})+(1-f)(\underline{S}(\underline{q})-\underline{U})]+fu'(\underline{q})\Delta x\end{aligned} \tag{3.6}$$

3.5.3 Solution

Suppose that both quantities are strictly positive in the optimum. Then their first-order effect on the expected profit must be equal to zero,

$$f\overline{S}'(\overline{q}^*)=\left(f\frac{\partial\overline{U}^*}{\partial\overline{q}}+(1-f)\frac{\partial\underline{U}^*}{\partial\overline{q}}\right)$$

$$(1-f)\underline{S}'(\underline{q}^*)=\left(f\frac{\partial\overline{U}^*}{\partial\underline{q}}+(1-f)\frac{\partial\underline{U}^*}{\partial\underline{q}}\right)$$

Using Eq. (3.5) and Eq. (3.6), the FOCs can be rewritten as

$$f\overline{S}'(\overline{q}^*) = -f\overline{S}'(\overline{q}^*)[f(\overline{S}(\overline{q}^*) - \overline{U}^*) + (1-f)(\underline{S}(\underline{q}^*) - \underline{U}^*)]$$
$$(1-f)\underline{S}'(\underline{q}^*) = -(1-f)\underline{S}'(\underline{q}^*)[f(\overline{S}(\overline{q}^*) - \overline{U}^*) + (1-f)(\underline{S}(\underline{q}^*) - \underline{U}^*)] + fu'(\underline{q}^*)\Delta x$$

or equivalently,

$$\overline{x}u'(\overline{q}^*) - c = 0$$
$$\underbrace{\left(\underline{x} - \frac{f}{1-f}\Delta x\right) u'(\underline{q}^*) - c}_{\text{rent extraction-efficiency trade-off}} = -\underbrace{(\underline{x}u'(\underline{q}^*) - c)[f(\overline{S}(\overline{q}^*) - \overline{U}^*) + (1-f)(\underline{S}(\underline{q}^*) - \underline{U}^*)]}_{\text{(marginal) psychological cost}} \quad (3.7)$$

One can see that there is no distortion at the top, $\overline{q}^* = \overline{q}^e$, and a downward distortion at the bottom, $\underline{q}^* < \underline{q}^e$. Furthermore, note that the left-hand side of Eq. (3.7) is equal to 0 at the optimal quantity for type $\underline{x}$ in the standard model, denoted $\underline{q}^{SB}$. In other words, the first derivative with respect to $\underline{q}$ is positive at $\underline{q}^{SB}$, so the distortion is smaller, $\underline{q}^* > \underline{q}^{SB}$.

It is routine to verify that the fair payoff is equal to a convex combination of the expected payoff under perfect competition and second-degree price discrimination,

$$U^F = \theta[f\overline{S}(\overline{q}^e) + (1-f)\underline{S}(\underline{q}^e)] + (1-\theta)[f\overline{U}^* + (1-f)\underline{U}^*]$$
$$= \theta[f\overline{S}(\overline{q}^e) + (1-f)\underline{S}(\underline{q}^e)] + (1-\theta)U^S$$

Thus the seller's unkindness in the optimum is equal to

$$U^F - U^S = \theta[f\overline{S}(\overline{q}^e) + (1-f)\underline{S}(\underline{q}^e)] + (1-\theta)U^S - U^S$$
$$= \theta[f\overline{S}(\overline{q}^e) + (1-f)\underline{S}(\underline{q}^e) - U^S]$$
$$= \theta[f(\overline{S}(\overline{q}^e) - \overline{U}^*) + (1-f)(\underline{S}(\underline{q}^e) - \underline{U}^*)]$$

and Eq. (3.2) and Eq. (3.3) can be written as

$$\overline{U}^* = \theta(\overline{S}(\overline{q}^e) - \overline{U}^*)[f(\overline{S}(\overline{q}^e) - \overline{U}^*) + (1-f)(\underline{S}(\underline{q}^e) - \underline{U}^*)] + u(\underline{q}^*)\Delta x$$
$$\underline{U}^* = \theta(\underline{S}(\underline{q}^*) - \underline{U}^*)[f(\overline{S}(\overline{q}^e) - \overline{U}^*) + (1-f)(\underline{S}(\underline{q}^e) - \underline{U}^*)]$$

For any $\theta > 0$, even the low type receives a strictly positive payoff. Also, the introduction of fairness widens the gap between the optimal payoffs because the high type suffers a larger psychological cost, and there is less distortion for the low type. Clearly, it means that the seller's profit is strictly lower than in the standard model.

3.5.4 Implementation and Predictive Power

How could the seller implement the optimal contracts $(\overline{q}^e, \overline{U}^*)$ and $(\underline{q}^*, \underline{U}^*)$? It is, in fact, sufficient to offer two quantities, namely $\overline{q}^e$ and $\underline{q}^*$, at prices $T(\overline{q}^e)$ and $T(\underline{q}^*)$, defined as

$$T(\overline{q}^e) \equiv \overline{t}^* = \overline{x}u(\overline{q}^e) - \overline{U}^*$$
$$T(\underline{q}^*) \equiv \underline{t}^* = \underline{x}u(\underline{q}^*) - \underline{U}^*$$

Let $(\overline{q}^{SB}, \overline{U}^{SB}) = (\overline{q}^e, u(\underline{q}^{SB})\Delta x)$ and $(\underline{q}^{SB}, \underline{U}^{SB}) = (\underline{q}^{SB}, 0)$ denote the solution from the standard model. The seller then charges $T^{SB}(\overline{q}^e) = \overline{x}u(\overline{q}^e) - \overline{U}^{SB}$ for $\overline{q}^e$ units and $T^{SB}(\underline{q}^{SB}) = \underline{x}u(\underline{q}^{SB})$ for $\underline{q}^{SB}$ units. We already argued that the high type receives a larger payoff in our model, $\overline{U}^* > \overline{U}^{SB}$, so the price of $\overline{q}^e$ units is lower, $T(\overline{q}^e) < T^{SB}(\overline{q}^e)$.

Naturally, the prices $T(\underline{q}^*)$ and $T^{SB}(\underline{q}^{SB})$ should not be compared because they involve different quantities, $\underline{q}^* > \underline{q}^{SB}$. Instead, we shall compare *per-unit* prices charged to the low type. Observe that the fact $u''(x) < 0$ implies

$$\begin{aligned}
\frac{d}{dq}\frac{u(q)}{q} &= \frac{d}{dq}\frac{1}{q}\int_0^q u'(y)dy \\
&= \frac{1}{q}u'(q) - \frac{1}{q^2}\int_0^q u'(y)dy \\
&= \frac{1}{q^2}\int_0^q \int_y^q u''(t)dtdy \\
&< 0
\end{aligned}$$

Since $\underline{q}^* > \underline{q}^{SB}$, we have $\underline{x}u(\underline{q}^*)/\underline{q}^* < \underline{x}u(\underline{q}^{SB})/\underline{q}^{SB}$. In addition, the low type receives a positive rent, $\underline{U}^* > 0$, so she faces a lower per-unit price, $T(\underline{q}^*)/\underline{q}^* < T^{SB}(\underline{q}^{SB})/\underline{q}^{SB}$.

Let us compare our predictions to the experimental findings of Hoppe and Schmitz (2015). In a setting with incomplete information[4], they can be summarized as follows:

(EI.1) Sellers offer a schedule of prices;

[4] The authors call this setting PI-I.

(EI.2) No distortion at the top;

(EI.3) Downward distortion at the bottom;

(EI.4) Selling prices are lower than in the standard model;

(EI.5) Some offers are rejected.

Recall that the standard model predicts findings (EI.1), (EI.2) and (EI.3), but fails to predict (EI.4) and (EI.5). In contrast, our model predicts findings (EI.1), (EI.2), (EI.3), and (EI.4). As in the case of complete information, we fail to predict (EI.5) because a buyer type never rejects her contract from the optimal menu. Nevertheless, she may reject her contract from a suboptimal menu if the payoff is sufficiently low. To give more credibility to our model, we shall argue once again that contracts get rejected because some sellers underestimate the size of θ.

3.6 Incomplete Information: Mechanism Design

In this section, we consider the main case of a continuous distribution of buyer types. The seller offers a menu of contracts $(q(\cdot), t(\cdot))$ where $(q(x), t(x))$ denotes the contract designed for type $x \in [0, w]$. Upon receiving a menu, the buyer can select a contract from the menu or reject all contracts. Our goal is to find the optimal menu of contracts $(q^*(\cdot), t^*(\cdot))$ using a number of necessary conditions. First, we derive the set of incentive-compatible and individually rational menus of contracts. Second, we derive the first-order condition for the optimization problem of the seller. Third, we find the fair payoff and fourth, we characterize the optimal mechanism.

3.6.1 Incentive and Participation Constraints

The seller is free to choose any arbitrary menu of contracts. Despite intention-based preferences, however, it is sufficient to study a small class of incentive-compatible menus. We say that a menu is *incentive compatible* if for all $x \in [0, w]$, it is optimal to choose the contact designed for type x,

$$V(x, x) \geq V(z, x), \forall z \in [0, w]$$

where $V(z, x)$ denotes the utility of the buyer with type x from the contract for type z.

Proposition 3.3 (*Revelation Principle*)**.** *For a given fair payoff $U^F \geq 0$, consider an arbitrary menu and fix an optimal choice of contracts from this menu. Then there exists an incentive compatible menu with the same allocation and the same transfers as in the original menu.*

Proof. See Appendix. □

The *incentive compatibility* constraint imposes a functional relationship between a menu's allocation and transfer rules. To find this relationship, note that

$$V(z, x) = xu(q(z)) - t(z) + (t(z) - cq(z) - \Pi^F(x)) \times (U^S - U^F)$$

where $U^F \geq 0$ and $U^S \geq 0$ can so far be treated as given because they are not affected by the buyer's choice. From the definition of incentive compatibility, we must have for any x and z that

$$V(x, x) \geq V(z, x) \text{ and } V(z, z) \geq V(x, z)$$

By adding up these inequalities, we find that

$$(x - z)(u(q(x)) - u(q(z))) \geq 0$$

It implies that the allocation $q(\cdot)$ is increasing and thus almost everywhere differentiable. Moreover, the payment $t(\cdot)$ is also a.e. differentiable with the same points of non-differentiability. But following Laffont and Martimort (2009, p. 135), we shall consider only differentiable menus of contracts.

Let us use $R(x)$ to denote the utility of buyer type x from rejecting all contracts,

$$R(x) = -\Pi^F(x) \times (U^S - U^F)$$

Then we can focus on that type's *surplus from trade*, defined as the difference of her utility from accepting the contract of some type z and rejecting all contracts,

$$V(z, x) - R(x) = xu(q(z)) - t(z) + (t(z) - cq(z)) \times (U^S - U^F)$$

It must be optimal to choose $z = x$, so the first-order condition reads

$$xu'(q(x))q'(x) - t'(x) + (t'(x) - cq'(x)) \times (U^S - U^F) = 0 \tag{3.8}$$

One can exploit the first-order condition to show that for any x and z

$$\begin{aligned} t(x) - t(z) &= \int_z^x [yu'(q(y))q'(y) + (t'(y) - cq'(y)) \times (U^S - U^F)]dy \\ &= xu(q(x)) - zu(q(z)) - \int_z^x u(q(y))dy + [(t(x) - cq(x)) - (t(z) - cq(z))](U^S - U^F) \end{aligned}$$

which can be re-arranged as

$$\begin{aligned} V(x,x) - R(x) &= xu(q(x)) - t(x) + (t(x) - cq(x))(U^S - U^F) \\ &= xu(q(z)) - t(z) + (t(z) - cq(z))(U^S - U^F) + \int_z^x u(q(y))dy - (x-z)u(q(z)) \\ &= V(z,x) - R(x) + \int_z^x u(q(y))dy - (x-z)u(q(z)) \end{aligned}$$

Since $q(\cdot)$ is increasing, the term $\int_z^x u(q(y))dy - (x-z)u(q(z))$ is positive. Thus the constraint $q'(x) \geq 0$ is not only necessary, but also sufficient for incentive compatibility.

Consider once again the payoff variable, $U(x) = xu(q(x)) - t(x)$. Using the first-order condition, its derivative is equal to

$$\begin{aligned} U'(x) &= u(q(x)) + xu'(q(x))q'(x) - t'(x) \\ &= u(q(x)) - (t'(x) - cq'(x)) \times (U^S - U^F) \\ &= u(q(x)) - [S_1(q(x),x)q'(x) + S_2(q(x),x) - U'(x)] \times (U^S - U^F) \end{aligned}$$

For a menu to be *individually rational*, every type must prefer her contract to the outside option,

$$V(x,x) - R(x) = U(x) + [S(q(x),x) - U(x)] \times (U^S - U^F) \geq 0, \ \forall x \in [0,w]$$

If a menu is incentive compatible and individually rational, the payoff of type x is $U(x)$. Thus the second-order belief U^S must be equal to the expected payoff,

$$U^S = \int_0^w U(y)dF(y)$$

3.6.2 Optimization Problem

By definition, the optimal menu $(q^*(\cdot), U^*(\cdot))$ maximizes the expected profit of the seller. For a given $U^F \geq 0$, it is thus a solution to the optimization problem

$$\max_{(q(x),U(x))} \int_0^w (S(q(x), x) - U(x))dF(x), \text{ subject to}$$

$$U'(x) = u(q(x)) + [S_1(q(x), x)q'(x) + S_2(q(x), x) - U'(x)] \left(U^F - \int_0^w U(y)dF(y)\right) \quad \text{(IC)}$$

$$U(x) - [S(q(x), x) - U(x)] \left(U^F - \int_0^w U(y)dF(y)\right) \geq 0 \quad \text{(PC)}$$

$$q'(x) \geq 0 \quad \text{(SOC)}$$

It is helpful to simplify the set of constraints. To begin with, integrate (IC) on $[0, x]$,

$$\begin{aligned} U(x) = U(0) + \int_0^x u(q(y))dy + [S(q(x), x) - U(x)] \left(U^F - \int_0^w U(y)dF(y)\right) \\ - [S(q(0), 0) - U(0)] \left(U^F - \int_0^w U(y)dF(y)\right) \end{aligned}$$

This implies that (PC) can be simplified to

$$\int_0^x u(q(y))dy + U(0) - [S(q(0), 0) - U(0)] \left(U^F - \int_0^w U(y)dF(y)\right) \geq 0 \tag{3.9}$$

Since $q'(x) \geq 0$, the left-hand side of Ineq. (3.9) is increasing in x. It follows that (PC) is satisfied if and only if Ineq. (3.9) holds for $x = 0$. Suppose for a moment that the optimal menu satisfies Ineq. (3.9) with strict inequality for $x = 0$. Consider another menu $(q^*(\cdot), \widehat{U}(\cdot, \epsilon))$ such that the left-hand side of Ineq. (3.9) is reduced by $\epsilon > 0$,

$$\begin{aligned} \widehat{U}(x, \epsilon) = \int_0^x u(q^*(y))dy + [S(q^*(x), x) - \widehat{U}(x, \epsilon)] \times \left(U^F - \int_0^w \widehat{U}(y, \epsilon)dF(y)\right) \\ + U^*(0) - (S(q^*(0), 0) - U^*(0)) \left(U^F - \int_0^w U^*(y)dF(y)\right) - \epsilon \end{aligned}$$

For sufficiently small $\epsilon > 0$, the menu $(q^*(x), \widehat{U}(x,\epsilon))$ satisfies (PC) and (IC). Next, differentiate the equation with respect to ϵ and evaluate it at $\epsilon = 0$,

$$\widehat{U}_\epsilon(x,0) = -\widehat{U}_\epsilon(x,0)\left(U^F - \int_0^w U^*(y)dF(y)\right) - [S(q^*(x),x) - U^*(x)]\int_0^w \widehat{U}_\epsilon(y,0)dF(y) - 1$$

It is easy to verify that $\widehat{U}_\epsilon(x,0) < 0$. This means that the seller can reduce the buyer's rent by reducing the constant in Ineq. (3.9). But the allocation remains the same, so the seller receives a larger share of social surplus. As a result, the menu $(q^*(\cdot), U^*(\cdot))$ is not optimal, a contradiction. It allows us to simplify (PC) to

$$U(0) = [S(q(0),0) - U(0)]\left(U^F - \int_0^w U(y)dF(y)\right)$$

3.6.3 First-Order Condition

For a moment, ignore (SOC) and consider a relaxed problem:

$$\begin{aligned} &\max_{(q(x),U(x))} \int_0^w (S(q(x),x) - U(x))dF(x)\ , \\ &\text{subject to } U(x) = \int_0^x u(q(y))dy + [S(q(x),x) - U(x)] \times \left(U^F - \int_0^w U(y)dF(y)\right) \end{aligned}$$

Let $q^*(x)$ denote an optimal allocation. Fix some function $\tau(x)$ and a number $\alpha \in \mathbb{R}$ to define a perturbed allocation $q(x,\alpha) \equiv q^*(x) + \alpha\tau(x)$. Also, let $U(x,\alpha)$ denote a payoff function that satisfies the constraint given $q(x,\alpha)$. Then the objective becomes a function of α alone,

$$\Pi(\alpha) \equiv \int_0^w (S(q(x,\alpha),x) - U(x,\alpha))dF(x)$$

Now take expectation on both sides of the constraint,

$$\begin{aligned} \int_0^w U(x,\alpha)dF(x) &= \int_0^w \int_0^x u(q(y,\alpha))dydF(x) \\ &\quad + \int_0^w [S(q(x,\alpha),x) - U(x,\alpha)]dF(x) \times \left(U^F - \int_0^w U(x,\alpha)dF(x)\right) \\ &= \int_0^w \frac{1-F(x)}{f(x)} u(q(x,\alpha))dF(x) + \Pi(\alpha)\left(U^F - \int_0^w U(x,\alpha)dF(x)\right) \end{aligned}$$

Each side of the equation is a function of α. By taking the derivative on each side,

$$\begin{aligned}\int_0^w U_\alpha(x,\alpha)dF(x) = \int_0^w \frac{1-F(x)}{f(x)} u'(q(x,\alpha))\tau(x)dF(x) \\ + \Pi'(\alpha)\left(U^F - \int_0^w U(x,\alpha)dF(x)\right) - \Pi(\alpha)\int_0^w U_\alpha(x,\alpha)dF(x)\end{aligned} \tag{3.10}$$

Since $q(x,0) = q^*(x)$ is an optimal control, the objective $\Pi(\alpha)$ is maximized at $\alpha = 0$. This implies the first-order condition

$$\begin{aligned}\Pi'(0) &= \int_0^w \left[\frac{\partial}{\partial q} S(q(x,0),x) \times q_\alpha(x,0) - U_\alpha(x,0)\right] dF(x) \\ &= \int_0^w \frac{\partial}{\partial q} S(q^*(x),x) \times \tau(x)dF(x) - \int_0^w U_\alpha(x,0)dF(x) \\ &= 0\end{aligned}$$

Now we can evaluate Eq. (3.10) at $\alpha = 0$ and plug it into the first-order condition,

$$\int_0^w \frac{\partial}{\partial q} S(q^*(x),x) \times \tau(x)dF(x) = \int_0^w \left(\frac{1-F(x)}{f(x)} u'(q^*(x)) - \Pi(0)\frac{\partial}{\partial q} S(q^*(x),x)\right)\tau(x)dF(x)$$

Since $q^*(x)$ is optimal for every function $\tau(x)$, we obtain the following condition:

$$\underbrace{\left(x - \frac{1-F(x)}{f(x)}\right) u'(q^*(x)) - c}_{\text{rent extraction-efficiency trade-off}} = \underbrace{-\left(xu'(q^*(x)) - c\right)\int_0^w \left(S(q^*(y),y) - U^*(y)\right)dF(y)}_{\text{(marginal) psychological cost}} \tag{3.11}$$

Similar to the standard model, the equation captures the first-order effect of a small change to the optimal allocation on the expected profit. Besides the familiar trade-off between allocative efficiency and information rent, we observe a new effect on the right-hand side. It states that a small change to the allocation results in a lower psychological cost, internalized by the seller through the constraint. As a result, a lower cost partially offsets an increase in information rent and incentives the seller to enhance allocative efficiency.

3.6.4 Second-Order Conditions

So far the second-order condition $q'(x) \geq 0$ has been neglected. While this constraint may be binding in general, there is a class of distributions for which it is slack. Let $\lambda(x)$ denote the hazard rate of F,

$$\lambda(x) = \frac{f(x)}{1 - F(x)}$$

Lemma 3.4. *If the hazard rate is increasing, the optimal allocation $q^*(x)$ is strictly increasing whenever $q^*(x) > 0$.*

Proof. Observe that we can re-write Eq. (4.5) as

$$G(q, x) \equiv u'(q) \left(x - \frac{1}{\lambda(x)} \right) - c + (xu'(q) - c)\Pi(0) = 0 \tag{3.12}$$

By the implicit function theorem,

$$\frac{dq^*(x)}{dx} = -\frac{G_x(q, x)}{G_q(q, x)} \tag{3.13}$$

where

$$G_x(q, x) = u'(q) \left(1 + \frac{\lambda'(x)}{\lambda^2(x)} + \Pi(0) \right)$$

and

$$G_q(q, x) = u''(q) \left(x - \frac{1}{\lambda(x)} + x\Pi(0) \right)$$

Since $\lambda'(x) \geq 0$, we have $G_x(q, x) > 0$. Also, Eq. (3.12) can be re-arranged as

$$u'(q) \left(x - \frac{1}{\lambda(x)} + x\Pi(0) \right) = c(1 + \Pi(0))$$

which implies $x - 1/\lambda(x) + x\Pi(0) > 0$, and so $G_q(q, x) < 0$. We conclude that the optimal allocation $q^*(x)$ is strictly increasing whenever $q^*(x) > 0$. □

It is not necessarily optimal to choose the interior solution $q^*(x)$. Instead, the seller could "shut down" buyer types on some interval $[0, \mu]$. In this case, his expected profit is

$$\begin{aligned} \Pi^*(\mu) &= \int_\mu^w (S(q^*(x), x) - U^*(x)) dF(x) \\ &= \int_\mu^w \left[u(q^*(x)) \left(x - \frac{1}{\lambda(x)} \right) - cq^*(x) \right] dF(x) - \Pi^*(\mu) \left(U^F - \int_\mu^w U^*(y) dF(y) \right) \end{aligned}$$

Lemma 3.5. *For a given $U^F \geq 0$, the optimal cut-off μ^* satisfies*

$$u(q^*(\mu^*)) \left(\mu^* - \frac{1}{\lambda(\mu^*)} \right) - cq^*(\mu^*) + \Pi^*(\mu^*)(\mu^* u(q^*(\mu^*)) - cq^*(\mu^*)) = 0$$

Proof. See Appendix. □

3.6.5 Fair Payoff

Given that $(q^*(\cdot), t^*(\cdot))$ is the optimal menu, what is the fair payoff? First, let us find the highest payoff among Pareto-efficient menus. Consider such menu that the allocation is ex-post efficient, $q(x) = q^e(x)$, and the profit is zero ex-post, $t(x) = cq^e(x)$. The latter property implies that the buyer's surplus from trade is equal to her payoff,

$$\begin{aligned} U(z, x) &= xu(q^e(z), x) - cq^e(z) \\ &= S(q^e(z), x) \end{aligned}$$

Since the allocation is efficient, $q^e(x) \in \arg\max S(q, x)$, the menu is incentive compatible. Then the buyer's surplus from trade is equal to the maximized social surplus, $S(q^e(x), x)$. It implies that the menu is also individually rational. Besides, any menu that yields a higher payoff must result in a negative profit. Thus the highest expected payoff from the set of Pareto-efficient menus is

$$U^h = \int_0^w S(q^e(y), y)dF(y)$$

Second, let us find the lowest payoff among Pareto-efficient menus. Consider the optimal menu $(q^*(\cdot), t^*(\cdot))$. Since any other menu yields a lower expected profit, it is Pareto-efficient. If there were a Pareto-efficient menu with a lower expected payoff, then it would have a higher expected profit, a contradiction. Thus, the lowest expected payoff from the set of Pareto-efficient menus is

$$U^l = \int_{\mu^*}^w U^*(y)dF(y)$$

These findings can be summarized in the following lemma:

Lemma 3.6. *If $(q^*(\cdot), t^*(\cdot))$ is the optimal menu, the fair payoff is equal to*

$$U^F = \theta \int_0^w S(q^e(y), y)dF(y) + (1 - \theta) \int_{\mu^*}^w U^*(y)dF(y)$$

Similar to the case of complete information, the fair payoff can be interpreted as a convex combination of the payoff under perfect competition and the payoff under the second-degree price discrimination with social preferences.

3.6.6 Effects of Reciprocity

Our results about the optimal menu can be summarized as follows:

Proposition 3.7. *Suppose that the hazard rate is increasing. Then the optimal mechanism is* $(0,0)$ *for all* $x < \mu^*$ *and* $(q^*(x), U^*(x))$ *for all* $x \geq \mu^*$*, which satisfies*

$$\begin{aligned} U^*(x) = &\int_{\mu^*}^{x} u(q^*(y))dy \\ &+ \theta \times (S(q^*(x), x) - U^*(x)) \left[\int_0^w (S(q^e(y), y)dF(y) - \int_{\mu^*}^{w} U^*(y))dF(y) \right] \end{aligned}$$

and

$$\left(x - \frac{1-F(x)}{f(x)} \right) u'(q^*(x)) - c = -(xu'(q^*(x)) - c) \int_{\mu^*}^{w} (S(q^*(y), y) - U^*(y))dF(y)$$

Recall that the optimal mechanism from the standard model, $(q^s(\cdot), U^s(\cdot))$, satisfies

$$q^s(x) \in \arg\max_q \left(S(q, x) - \frac{u(q)}{\lambda(x)} \right) \quad \text{and} \quad U^s(x) = \int_{\mu^s}^{x} u(q^s(y))dy$$

where the cut-off μ^s is given by

$$S(q^s(\mu^s), \mu^s) - \frac{u(q^s(\mu^s))}{\lambda(\mu^s)} = 0$$

Evaluate the first derivative of $S(q, x) - u(q)/\lambda(x)$ at $q^*(x)$,

$$\begin{aligned} \frac{\partial}{\partial q} \left(S(q^*(x), x) - \frac{u(q^*(x))}{\lambda(x)} \right) &= (xu'(q^*(x)) - c) - \frac{1-F(x)}{f(x)} u'(q^*(x)) \\ &= -(xu'(q^*(x)) - c) \int_{\mu^*}^{w} (S(q^*(y), y) - U^*(y))dF(y) \\ &< 0 \end{aligned}$$

But this derivative is equal to zero at $q^s(x)$, so the optimal allocation with social preferences is more efficient, $q^*(x) > q^s(x)$.

Next, Eq. (4.5) can be used to show the inequality

$$\begin{aligned} S(q^*(x), x) - \frac{u(q^*(x))}{\lambda(x)} + \Pi^*(x)(xu(q^*(x)) - cq^*(x)) &= \max_q \left[S(q, x) - \frac{u(q)}{\lambda(x)} + \Pi^*(x)(xu(q) - cq) \right] \\ &\geq \max_q \left[S(q, x) - \frac{u(q)}{\lambda(x)} \right] = S(q^s(x), x) - \frac{u(q^s(x))}{\lambda(x)} \end{aligned}$$

Thus the seller "shuts down" fewer buyer types, $\mu^* < \mu^s$. Furthermore, each type $x \in [0, w]$ earns a larger payoff,

$$\begin{aligned} U^*(x) &= \int_{\mu^*}^{x} u(q^*(y))dy + \theta(S(q^*(x), x) - U^*(x)) \times \left[\int_0^w (S(q^e(y), y)dF(y) - \int_{\mu^*}^{w} U^*(y))dF(y) \right] \\ &\geq \int_{\mu^*}^{x} u(q^*(y))dy \\ &> \int_{\mu^s}^{x} u(q^s(y))dy = U^s(x) \end{aligned}$$

3.7 Incomplete Information: Nonlinear Pricing

3.7.1 Implementation

We derived the optimal mechanism $(q^*(\cdot), U^*(\cdot))$, which is equivalent to the menu of contracts $(q^*(x), t^*(x))$ where $t^*(x) = xu(q^*(x)) - U^*(x)$. However, the seller must implement this menu by a schedule of prices $T(q)$. Under the assumption of Proposition 3.7, the function $q^*(x)$ is strictly increasing and therefore invertible. Let $x^*(q)$ denote its inverse, showing the type who buys a given quantity q. Then the optimal price is equal to

$$\begin{aligned} T(q) &\equiv t^*(x^*(q)) \\ &= x^*(q)u(q) - \int_{\mu^*}^{x^*(q)} u(q^*(y))dy - (T(q) - cq)(U^F - U^S) \end{aligned} \tag{3.14}$$

where the seller's unkindness, $(U^F - U^S)$, can also be written in terms of quantities,

$$\begin{aligned} U^F - U^S &= \theta \left[\int_0^w S(q^e(y), y) dF(y) - \int_{\mu^*}^w U^*(y) dF(y) \right] \\ &= \theta \left[\int_0^w (yu(q^e(y)) - cq^e(y)) dF(y) - \int_{\mu^*}^w (yu(q^*(y)) - t^*(y)) dF(y) \right] \\ &= \int_0^{q^e(w)} [(x^e(z)u(z) - cz) f(x^e(z)) - (x^*(z)u(z) - T(z)) f(x^*(z))] dz \end{aligned}$$

Let us verify that $T(q)$ does implement the optimal menu. Suppose that the buyer with value x faces the schedule $T(q)$. Then her optimal choice of quantity, q^*, solves

$$\max_q \left[xu(q) - T(q) - (T(q) - cq) \times (U^F - U^S) \right]$$

Thus it must satisfy the first-order condition

$$xu'(q^*) - T'(q^*) - (T'(q^*) - c) \times (U^F - U^S) = 0 \tag{3.15}$$

By differentiating Eq. (3.14) we obtain

$$T'(q) = x^*(q)u'(q) - (T'(q) - c)(U^F - U^S)$$

Evaluate the last equation at $q = q^*$ and plug it into Eq. (3.15),

$$xu'(q^*) = x^*(q^*)u'(q^*)$$

which is satisfied at $q^* = q^*(x)$. Thus it is optimal for the buyer with value $x \geq \mu^*$ to purchase $q^*(x)$ units.

3.7.2 Per-Unit Prices

Since the buyer pays $T(q)$ for q units, she faces the per-unit price of

$$p(q) \equiv \frac{T(q)}{q}$$

To derive $p(q)$, divide both sides of Eq. (3.14) by q,

$$\begin{aligned} p(q) &= \frac{1}{q}\left(x^*(q)u(q) - \int_{\mu^*}^{x^*(q)} u(q^*(y))dy\right) - (p(q) - c)(U^F - U^S) \\ &= \frac{1}{q}\int_0^q x^*(y)u'(y)dy - (p(q) - c)(U^F - U^S) \end{aligned} \tag{3.16}$$

In contrast, the per-unit price in the standard model, denoted $p^s(q)$, is given by

$$p^s(q) = \frac{1}{q}\int_0^q x^s(y)u'(y)dy$$

where $x^s(q)$ is the inverse of $q^s(x)$. But we showed that $q^s(x) \leq q^*(x)$, which implies $x^s(q) \geq x^*(q)$. Therefore, the introduction of social preferences reduces the per-unit price,

$$\begin{aligned} p(q) &= \frac{1}{q}\int_0^q x^*(y)u'(y)dy - \theta(p(q) - c)(U^F - U^S) \\ &< \frac{1}{q}\int_0^q x^*(y)u'(y)dy \\ &\leq \frac{1}{q}\int_0^q x^s(y)u'(y)dy \end{aligned}$$

Now differentiate both sides of Eq. (3.16) with respect to q,

$$p'(q) = \frac{1}{q^2}\int_0^q (x^*(q)u'(q) - x^*(y)u'(y))dy - p'(q)(U^F - U^S) \tag{3.17}$$

Is it optimal to offer quantity discounts, $p'(q) \leq 0$, or quantity premiums, $p'(q) \geq 0$? As shown in the next proposition, the answer is the same as in the standard model.

Proposition 3.8. *The per-unit price $p(q)$ is strictly decreasing.*

Proof. Let us use $x(q) = x^*(q)$ and $q(x) = q^*(x)$. Then we can write

$$\begin{aligned} x(q)u'(q) - x(y)u'(y) &= \int_y^q \frac{d}{dt}x(t)u'(t)dt \\ &= \int_y^q [x(t)u''(t) + x'(t)u'(t)]dt \\ &= \int_y^q [x(t)u''(t)q'(x(t)) + u'(t)]x'(t)dt \end{aligned}$$

Using Eq. (3.13), we obtain

$$
\begin{aligned}
x(q)u''(q)q'(x(q)) + u'(q) &= x(q)u''(q)\frac{u'(q)\left(1+\lambda'(x(q))/\lambda^2(x(q))+\Pi(0)\right)}{-u''(q)\left(x(q)-1/\lambda(x(q))+x(q)\Pi(0)\right)} + u'(q) \\
&= u'(q)\left(-x(q)\frac{1+\lambda'(x(q))/\lambda^2(x(q))+\Pi(0)}{x(q)-1/\lambda(x(q))+x(q)\Pi(0)}+1\right) \\
&= \frac{u'(q)}{x(q)-1/\lambda(x(q))+x(q)\Pi(0)}\left[x(q)-\frac{1}{\lambda(x(q))}-x(q)\left(1+\frac{\lambda'(x(q))}{\lambda^2(x(q))}\right)\right] \\
&= -\frac{u'(q)}{x(q)-1/\lambda(x(q))+x(q)\Pi(0)}\left[\frac{1}{(\lambda(x(q)))^2}\left(\lambda(x(q))+x(q)\lambda'(x(q))\right)\right]
\end{aligned}
$$

which is negative because $\lambda(x(q)) > 0$, $\lambda'(x(q)) \geq 0$, and $x(q) - 1/\lambda(x(q)) + x(q)\Pi(0) > 0$. It implies that the integral in Eq. (3.17) is also negative.

Finally, suppose that $p'(q) \geq 0$. Then the right-hand side of Eq. (3.17) is strictly negative while the left-hand side is positive, a contradiction. It implies that the per-unit price is strictly decreasing, $p'(q) < 0$. □

3.8 Conclusions

This paper has extended the model of nonlinear pricing by Maskin and Riley (1984) such that the buyer feels a loss of utility from being kind towards a seller whom she believes to be unkind. Our goal was to find the optimal schedule of prices under complete as well as incomplete information and to compare its structure against standard predictions and laboratory evidence. Although our analysis obtains a number of familiar properties such as a full separation of buyer types, no distortion at the top and downward distortion elsewhere, we have showed that the size of distortion is smaller because the seller internalizes the buyer's psychological cost. As a result, the buyer faces lower prices and weaker quantity discounts, so our model has helped to bridge the gap between theory and evidence.

It may prove useful to change some features of the model in further research. First, it is simplistic to specify psychological payoff as the product of two kindness functions. Instead, one could use a general function that is increasing in the seller's profit if he is kind and decreasing otherwise. In this case, the first-order condition of incentive compatibility should define a different relationship between a menu's allocation and transfer rules than Eq. (3.8). Second, it is arbitrary to assign psychological payoff to the buyer but not to the seller. If both agents care about each other's intentions, it is reasonable to expect a less efficient allocation or even complete shutdown. This is because a seller who faces an unkind buyer prefers to earn zero profit to sharing the surplus from trade with the buyer.

Third, real-world buyers may be heterogeneous also with respect to the parameter θ, which is why it could be unobserved by the seller. As mentioned earlier, it could be a reason why buyers reject some offers in the laboratory setting. Or, the seller might offer a menu of contracts to screen buyer types with respect to θ. Fourth, our setup can only be interpreted as a seller trading with a single buyer, not with a population of buyers. In the latter case, a buyer can not influence the total profit because the measure on her is zero. Compared to the standard model, the presence of psychological payoff would only affect the constraint of individual rationality and result in a different distribution of surplus.

3.9 Appendix

3.9.1 Proof of Proposition 4.6

Let us denote an optimal choice of type x under the original menu as $\beta(x)$. That is, the contract of type $\beta(x)$ maximizes the expected utility of buyer type x,

$$x \in \arg\max \Big[U(\beta(z), x) + (t(\beta(z)) - cq(\beta(z)))(U^S - U^F) \Big] \tag{3.18}$$

where U^S is equal to the buyer's expected payoff,

$$\begin{aligned} U^S &= \int_0^w U(\beta(x), x) dF(x) \\ &= \int_0^w [xu(q(\beta(x))) - t(\beta(x))] dF(x) \end{aligned}$$

Note that we got rid of the term $-\Pi^F(x)(U^S - U^F)$ because it is a constant with respect to z and thus does not affect incentives.

Now consider the menu $(q^c(x), t^c(x))$ that is designed to replicate the allocation, $q^c(x) = q(\beta(x))$, and the transfers, $t^c(x) = t(\beta(x))$, from the original menu. For a moment, suppose that it is incentive compatible. Then the expected payoff and thus the second-order belief

U^S are the same as in the original menu,

$$\begin{aligned} U^S &= \int_0^w [xu(q^c(x)) - t^c(x)]dF(x) \\ &= \int_0^w [xu(q(\beta(x))) - t(\beta(x))]dF(x) \\ &= \int_0^w U(\beta(x), x)dF(x) \end{aligned}$$

Observe that the expected utility from the contract of type z is then

$$U(\beta(z), x) + (t(\beta(z)) - cq(\beta(z)))(U^S - U^F)$$

It is the same function as (3.18), so it is also maximized at $z = x$. This means that the menu $(q^c(x), t^c(x))$ is incentive compatible, so it does replicate the original menu.

3.9.2 Proof of Lemma 3.5

The optimal cut-off μ^* solves the first-order condition

$$\begin{aligned} 0 &= \frac{d}{d\mu}\Pi^*(\mu^*) \\ &= \frac{d}{d\mu}\int_{\mu^*}^w S(q^*(x), x)dF(x) - \frac{d}{d\mu}\int_{\mu^*}^w U^*(x)dF(x) \\ &= -f(\mu^*)\left[u(q^*(\mu^*))\left(\mu^* - \frac{1}{\lambda(\mu^*)}\right) - cq^*(\mu^*)\right] + \int_{\mu^*}^w \frac{d}{d\mu}\left[u(q^*(x))\left(x - \frac{1}{\lambda(x)}\right) - cq^*(x)\right] dF(x) \\ &\quad - \left(U^F - \int_{\mu^*}^w U^*(y)dF(y)\right) dF(x)\frac{d}{d\mu}\Pi^*(\mu^*) + \Pi^*(\mu^*)\frac{d}{d\mu}\int_{\mu^*}^w U^*(y)dF(y) \end{aligned}$$

In particular, it implies that

$$\begin{aligned} \frac{d}{d\mu}\int_{\mu^*}^w U^*(y)dF(y) &= \frac{d}{d\mu}\int_{\mu^*}^w S(q^*(x), x)dF(x) \\ &= -f(\mu^*)(\mu^* u(q^*(\mu^*)) - cq^*(\mu^*)) + \int_{\mu^*}^w \frac{d}{dq}(xu(q^*(x)) - cq^*(x))\frac{dq^*(x)}{d\mu}dF(x) \end{aligned}$$

Therefore the first-order condition simplifies to

$$\begin{aligned} 0 = &- f(\mu^*)\left[u(q^*(\mu^*))\left(\mu^* - \frac{1}{\lambda(\mu^*)}\right) - cq^*(\mu^*) + \Pi^*(\mu^*)(\mu^* u(q^*(\mu^*)) - cq^*(\mu^*))\right] \\ &+ \int_{\mu^*}^w \frac{d}{dq}\left[u(q^*(x))\left(x - \frac{1}{\lambda(x)}\right) - cq^*(x) + (xu(q^*(x)) - cq^*(x))\Pi^*(\mu^*)\right]\frac{dq^*(x)}{d\mu}dF(x) \end{aligned}$$

But Eq. (4.5) implies that the integrand in the second line is equal to zero for every $x \geq \mu^*$. Thus the second line vanishes and the lemma follows.

Chapter 4

Optimal Auctions with Loss Averse Buyers

4.1 Introduction

Within the framework of independent private values, Riley and Samuelson (1981) show that any two auction formats that allocate the object efficiently are revenue equivalent, and the optimal reserve price must exclude all buyer types with negative virtual valuations. Both results follow from the fact that every buyer maximizes her expected monetary payoff by choosing the equilibrium probability of winning. However, the assumption of payoff maximization is sometimes fails not consistent with empirical findings, see DellaVigna (2007). But if the buyers do not seek to maximize their monetary payoff, the analysis of optimal auctions can not rely exclusively on the notions of revenue equivalence and virtual valuations. Therefore, auction theory may be sending a wrong message to people and organizations that search for an optimal mechanism to sell their assets.

This paper extends the problem of optimal auction design to reference-dependent preferences in the spirit of Koszegi and Rabin (2006). It is assumed that every buyer holds an expectation of her payment to the seller, and she suffers a psychological loss whenever she has to pay more than expected. In equilibrium, every buyer type maximizes the expectation of her monetary payoff net of psychological loss, and her expectations are stochastic and consistent with the equilibrium strategies. Our analysis is confined to the set of first-price and second-price auctions with reserve prices that can be made public or kept secret.

Our findings suggest that first-price auctions are revenue superior to second-price auctions for any reserve price. This is because the winner of a second-price auction may suffer a psychological loss in the event that she expected to win and pay a lower price, so her willingness-to-pay for the object is lower compared to a first-price auction. Furthermore, the optimal reserve price in a first-price auction is lower than in Riley and Samuelson (1981). It follows from the fact that loss aversion enhances the exclusionary effect of reserve prices, so a given level of exclusion can be achieved with a lower reserve price. These arguments imply that optimally structured negotiations with one buyer may be revenue superior to an efficient second-price auction with two buyers, and public reserve prices are always revenue superior to secret reserve prices.

The rest of this paper is structured as follows. Section 2 reviews the literature on optimal auction design and reference-dependent preferences and Section 3 outlines the model. Section 4 derives symmetric increasing equilibria in first-price and second-price auctions for a given reserve price. Section 5 finds the optimal auction format and the optimal reserve price in a first-price auction. Section 6 derives the value of competition and the value of public information, and Section 7 concludes.

4.2 Literature Review

The problem of optimal auction design with standard preferences is analyzed by Myerson (1981) and Riley and Samuelson (1981). According to the Revelation Principle, first-price and second-price auctions may be treated as mechanisms where every buyer chooses her equilibrium probability of winning. Thus the derivative of the equilibrium payoff function is simply the equilibrium probability of winning, as can be verified with the Envelope theorem or the first-order condition. It is routine to demonstrate that this implies the revenue equivalence of first-price and second-price auctions. Ultimately, an optimal auction should award the object to the buyer with the highest virtual valuation and exclude buyer types with negative virtual valuations.

These results have inspired many auxiliary projects. One of them investigates the revenue effect of increasing the number of buyers. Bulow and Klemperer (1996) draw some analogies between auction theory and monopoly theory to show that virtual valuations can be interpreted as marginal revenues. By writing the expected revenue from an auction as the expected maximum of all marginal revenues, they prove that an efficient auction with $n+1$ buyers is revenue superior to an optimal auction with n buyers. Another project deals with the revenue effect of keeping reserve prices secret. Riley and Samuelson (1981) argue that secret reserves are equivalent to public reserves in second-price auctions because buyers bid truthfully in both cases, and Elyakime et al. (1994) show that they are inferior to public reserves in first-price auctions because the seller loses his position of a "Stackelberg leader".

Influenced by the Prospect Theory (Kahneman and Tversky, 1979; Tversky and Kahneman, 1991), several recent papers study auctions with buyers that evaluate an outcome not only based on its monetary payoff, but also on its relation to some reference point. Rosenkranz and Schmitz (2007) assume that in case of winning, a buyer derives a psychological payoff equal to a weighted difference of her payment and her reference payment. Precisely, the reference payment is given by a convex combination of the reserve price and an exogenous parameter. They find that first-price and second-price auctions remain revenue equivalent, but the optimal reserve price is increasing in the number of buyers. Furthermore, the seller may prefer secret reserves over public reserves, and he may prefer an optimal auction with n buyers to an efficient auction with $n+1$ buyers.

It may seem restrictive to consider exogenous and deterministic reference points. Koszegi and Rabin (2006) propose a model of preferences with stochastic reference points. For every outcome and for every reference point, the utility function is equal to the sum of material payoff and gain-loss utility. In a personal equilibrium, the optimal choice must maximize the expectation of the utility function, and the reference point must be consistent with the optimal choice. Lange and Ratan (2010) incorporate these preferences into the context of auctions by assuming that every buyer suffers a loss of utility when she consumes less or pays more than expected, and she forms her expectations after placing her bid. Among other results, they show that first-price auctions are revenue superior to second-price auctions when the object is qualitatively different from money.

Besides, Eisenhuth (2010) extends the analysis of Myerson (1981) by using the assumptions of Lange and Ratan (2010) to prove the optimality of all-pay mechanisms. Ahmad (2015) extends the analysis of Rosenkranz and Schmitz (2007) to the case where the reference payment is equal to the expected price. Runco (2013) compares first-price to second-price auctions with private and common values when the reference point is proportional to the value of the object. Finally, Ehrhart and Ott (2014) study reference-dependent bidding in dynamic auctions.

4.3 Model

There is a single seller and n potential buyers. The seller owns an object that has zero value to him[1] and a value of X_i to each buyer $i = 1, ..., n$. We assume that each X_i is a random variable, distributed on $[0, w] \subset \mathbb{R}_+$ according to the distribution function F which admits a continuous density $f \equiv F'$ with full support. The realization of X_i is private information of buyer i in the sense that the other buyers and the seller only know the distribution of X_i. Let us denote the highest value among $n-1$ buyers as $Y \equiv \{X_1, ..., X_{n-1}\}$ with the distribution function $G \equiv F^{n-1}$ and continuous density $g = G'$.

The object can be sold by means of any auction from the set of first- and second-price auctions with a reserve price. Let $b_i \geq 0$ denote a sealed bid placed by buyer i and $p_i \equiv \max_{j \neq i}\{b_j, r\}$ denote the largest of all other bids, including the reserve price. Also, let $W(b_i, p_i)$ denote the price that buyer i has to pay in the event that she wins. That is, $W(b_i, p_i) = b_i$ in a first-price auction and $W(b_i, p_i) = p_i$ in a second-price auction.

It is assumed that each buyer $i = 1, ..., n$ holds a reference level of every bid in the auction. Let b_i^r denote buyer i's reference level of her bid and let p_i^r denote buyer i's reference level of the largest of all other bids, including the reserve price. If $b_i^r > p_i^r$, buyer i expects to win and pay the price $W(b_i, p_i^r)$. If $b_i^r < p_i^r$, buyer i expects to lose and pay nothing. Each buyer suffers a disutility in the event that she has to pay more than expected. Precisely, *the psychological loss* of buyer i in the event that she places the highest bid is given by

$$\begin{cases} \max\{W(b_i, p_i) - W(b_i, p_i^r), 0\} & \text{if } b_i^r > p_i^r \\ W(b_i, p_i) & \text{if } b_i^r < p_i^r \end{cases}$$

which means the following. In the event that buyer i expected to win, $b_i^r > p_i^r$, her psychological loss is equal to the difference of the price $W(b_i, p_i)$ and the reference price $W(b_i, p_i^r)$, whenever it is positive. But in the event that she expected to lose, $b_i^r < p_i^r$, her psychological loss is equal to the price $W(b_i, p_i)$ because she expected to pay zero.

[1]Later this assumption will be relaxed.

The utility function of each buyer $i = 1, ..., n$ is given by the difference of her monetary payoff and her discounted psychological loss,

$$
\begin{aligned}
u(b_i, p_i; x_i, b_i^r, p_i^r) &= (x_i - W(b_i, p_i))1\{b_i > p_i\} \\
&\quad -\theta \times \left(\max\{W(b_i, p_i) - W(b_i, p_i^r), 0\}1\{b_i^r > p_i^r\} + W(b_i, p_i)1\{b_i^r < p_i^r\}\right) 1\{b_i > p_i\}
\end{aligned}
$$

where $\theta \in [0, 1]$ denotes the degree of loss aversion.

A strategy of buyer i specifies a bid for every value, $\beta_i : [0, w] \to \mathbb{R}_+$. For a given vector of strategies $\boldsymbol{\beta_{-i}} = (\beta_1, ..., \beta_{i-1}, \beta_{i+1}, ..., \beta_n)$, the expected utility of buyer i is defined by

$$U(b_i, x_i) = E[u(b_i, \boldsymbol{\beta_{-i}(X_{-i})}; x_i, b_i^r, p_i^r)]$$

A profile of strategies $(\beta_1, ..., \beta_n)$ is said to be an equilibrium if for all $x_i \in [0, w]$ and for all $i \in N$,

$$U(\beta_i(x_i), x_i) \geq U(b_i, x_i), \;\; \forall b_i \geq 0$$

where

$$b_i^r = \beta_i(x_i) \text{ and } p_i^r \sim \max_{j \neq i}\{\beta_j(X_j), r\}$$

Thus, the equilibrium strategy of every buyer must be Bayesian rational given the equilibrium strategies of the other players and the distribution of her reference bids. Also, reference bids may be stochastic and their distribution must be consistent with the equilibrium strategies.

Our definition of equilibrium is very close to Lange and Ratan (2010). The only difference comes from the assumption that the probability that buyer i expects to win, that is $\Pr(b_i^r > p_i^r)$, is formed after buyer i learns her value, but before she places a bid. In other words, a buyer's expectations about the probability of winning are *lagged*[2].

4.4 Equilibrium Analysis

This section is devoted to the analysis of equilibria of first-price and second-price auctions for a given reserve price $r \geq 0$. For simplicity, we restrict our attention to symmetric and

[2] On the other hand, we assume that a buyer expects to pay $W(b_i, p_i^r)$ upon winning, so she forms her expectation about the price after placing her bid. It is routine to show that if the buyer expects to pay $W(b_i^r, p_i^r)$ instead, our propositions remain to be true but proofs become much more technical.

increasing equilibria. The procedure is broken down into three steps: (i) construct the expected utility function, (ii) derive the equilibrium from necessary conditions, and (iii) use an example to graph the equilibrium strategy.

4.4.1 First-Price Auctions

Suppose that there is a symmetric and increasing equilibrium $(\beta_I, ..., \beta_I)$ with $\beta_I(x) = 0$ if $x < \mu$ and $\beta_I(\mu) = r$ for some $\mu \in [0, w]$. Recall that the winner of a first-price auction has to pay her bid, $W_I(b, p) = b$.

First, consider a buyer with value $x < \mu$. Since she does not bid in equilibrium, she always loses and pays nothing. It follows that this buyer also expects to lose and pay nothing. If she deviates from her equilibrium strategy by placing a bid $b \geq r$ and wins, her psychological loss must be equal to the price, that is b. Thus her utility is

$$u_I(b, p; x, b^r, p^r) = (x - b)1\{b > p\} - \theta \times b1\{b > p\}$$

Her expected utility from placing a bid b for all $x < \mu$ is then

$$U_I(b, x) = \begin{cases} G(\beta_I^{-1}(b)) \times (x - b) - \theta G(\beta_I^{-1}(b)) \times b & \text{if } b \geq r \\ 0 & \text{if } b < r \end{cases}$$

Second, consider a buyer with value $x \geq \mu$. In equilibrium, she wins if and only if $\beta_I(x) > p$, so she feels a psychological loss when she expected to lose and pay nothing, $\beta_I(x) < p^r$,

$$u_I(b, p; x, \beta_I(x), p^r) = (x - b)1\{b > p\} - \theta \times b1\{\beta_I(x) < p^r\}1\{b > p\}$$

The probability of the event that she expects to lose and pay nothing is equal to $1 - G(x)$. Thus, her expected utility from placing a bid b for all $x \geq \mu$ is

$$U_I(b, x) = \begin{cases} G(\beta_I^{-1}(b)) \times (x - b) - \theta G(\beta_I^{-1}(b)) \times (1 - G(x))b & \text{if } b \geq r \\ 0 & \text{if } b < r \end{cases}$$

In equilibrium, every type must prefer her equilibrium bid to any other bid. In particular, the cut-off type μ must prefer bidding r to bidding 0. In fact, we shall make an educated guess that she is indifferent between $b = r$ and $b = 0$,

$$\begin{aligned} U_I(r, \mu) &= G(\mu)(\mu - r) - \theta G(\mu) \times r(1 - G(\mu)) \\ &= 0 = U_I(0, \mu) \end{aligned}$$

which simplifies to

$$\mu = r + \theta r(1 - G(\mu)) \tag{4.1}$$

Now consider the first derivative of U_I with respect to b for $x \geq \mu$,

$$\frac{\partial}{\partial b} U_I(b, x) = \frac{g(\beta_I^{-1}(b))}{\beta_I'(\beta_I^{-1}(b))} \times (x - b) - G(\beta_I^{-1}(b)) - \theta(1 - G(x)) \times \left[\frac{g(\beta_I^{-1}(b))}{\beta_I'(\beta_I^{-1}(b))} b + G(\beta_I^{-1}(b)) \right]$$

Since β_I is the equilibrium strategy, it is optimal for a buyer with value x to bid $b = \beta_I(x)$,

$$\beta_I(x)g(x) + G(x)\beta_I'(x) = g(x)x - \theta(1 - G(x)) \left[g(x)\beta_I(x) + G(x)\beta_I'(x) \right] \tag{4.2}$$

Using the boundary condition $\beta_I(\mu) = r$, Eq. (4.2) yields the following equilibrium strategy.

Proposition 4.1. *For a given reserve price $r \in [0, w]$, the profile of strategies $(\beta_I, ..., \beta_I)$ where*

$$\beta_I(x) = \begin{cases} r\dfrac{G(\mu)}{G(x)} + \dfrac{1}{G(x)} \displaystyle\int_\mu^x \frac{y}{1 + \theta(1 - G(y))} dG(y) & \text{if } x \geq \mu \\ 0 & \text{if } x < \mu \end{cases}$$

and the cut-off μ is defined by Eq. (4.1), is an equilibrium of the first-price auction.

Proof. For simplicity, let us define the function

$$v(x) = \frac{x}{1 + \theta(1 - G(x))}$$

so that the equilibrium bid of any $x \geq \mu$ can be re-written as

$$\begin{aligned} \beta_I(x) &= r\frac{G(\mu)}{G(x)} + \frac{1}{G(x)} \int_\mu^x v(y) dG(y) \\ &= v(x) - \frac{1}{G(x)} \int_\mu^x v'(y) G(y) dy \end{aligned}$$

where the last equality follows from integration by parts and observing that $v(\mu) = r$.

First, consider a buyer type whose equilibrium bid is strictly positive, $x \geq \mu$. If she places some bid $\beta_I(z) \geq r$, then her expected utility is

$$\begin{aligned} U_I(\beta_I(z), x) &= G(z)(x - \beta_I(z)) - \theta G(z)(1 - G(x))\beta_I(z) \\ &= G(z)x - G(z)\beta_I(z)(1 + \theta(1 - G(x))) \end{aligned}$$

Divide both sides by $(1 + \theta(1 - G(x)))$ to obtain

$$\begin{aligned} \frac{1}{1 + \theta(1 - G(x))} U_I(\beta_I(z), x) &= G(z)v(x) - G(z)\beta_I(z) \\ &= G(z)v(x) - G(z)v(z) + \int_\mu^z v'(y)G(y)dy \\ &= G(z)(v(x) - v(z)) + \int_\mu^z v'(y)G(y)dy \end{aligned}$$

Thus the effect of deviation from $\beta_I(x)$ to $\beta_I(z)$ amounts to

$$\begin{aligned} \frac{1}{1 + \theta(1 - G(x))} [U_I(\beta_I(x), x) - U_I(\beta_I(z), x)] &= G(z)(v(z) - v(x)) - \int_x^z v'(y)G(y)dy \\ &= \int_x^z (G(z) - G(y))v'(y)dy \end{aligned}$$

Since $v(\cdot)$ is increasing, the integral is positive. This implies that placing a bid other than $\beta_I(x)$ results in a loss,

$$U_I(\beta_I(x), x) - U_I(\beta_I(z), x) \geq 0, \forall \beta_I(z) \geq r$$

Our argument also implies that it can not be optimal to abstain from bidding,

$$\begin{aligned} U_I(\beta_I(x), x) &= (1 + \theta(1 - G(x))) \times \int_\mu^x v'(y)G(y)dy \\ &\geq 0 = U_I(0, x) \end{aligned}$$

Second, consider a buyer type who does not bid, $x < \mu$. If she bids an amount $\beta_I(z) \geq r$, her payoff is

$$U_I(\beta_I(z), x) = G(z) \times (x - \beta_I(z) - \theta\beta_I(z))$$

Since $x < \mu$ and $\beta_I(z) \geq r$, we have

$$\begin{aligned} U_I(\beta_I(z), x) &< G(z) \times (\mu - r - \theta r) \\ &< G(z) \times (\mu - r - \theta r(1 - G(\mu))) \\ &= 0 \\ &= U_I(\beta_I(x), x) \end{aligned}$$

Thus, it is optimal for every buyer type $x < \mu$ to abstain from bidding. □

Let us illustrate our findings with a standard example. Suppose that there are two buyers with uniformly distributed values on $[0, 1]$, that is $F(x) = x$. Figure 4.1 graphs the equilibrium strategy for $\theta = 0$, $\theta = 0.5$ and $\theta = 1$. When $\theta = 0$, the optimal bid is linear, $\beta_I(x) = x/2$. For larger values of θ, the optimal bid is lower, but also convex because higher types suffer a lower psychological loss.

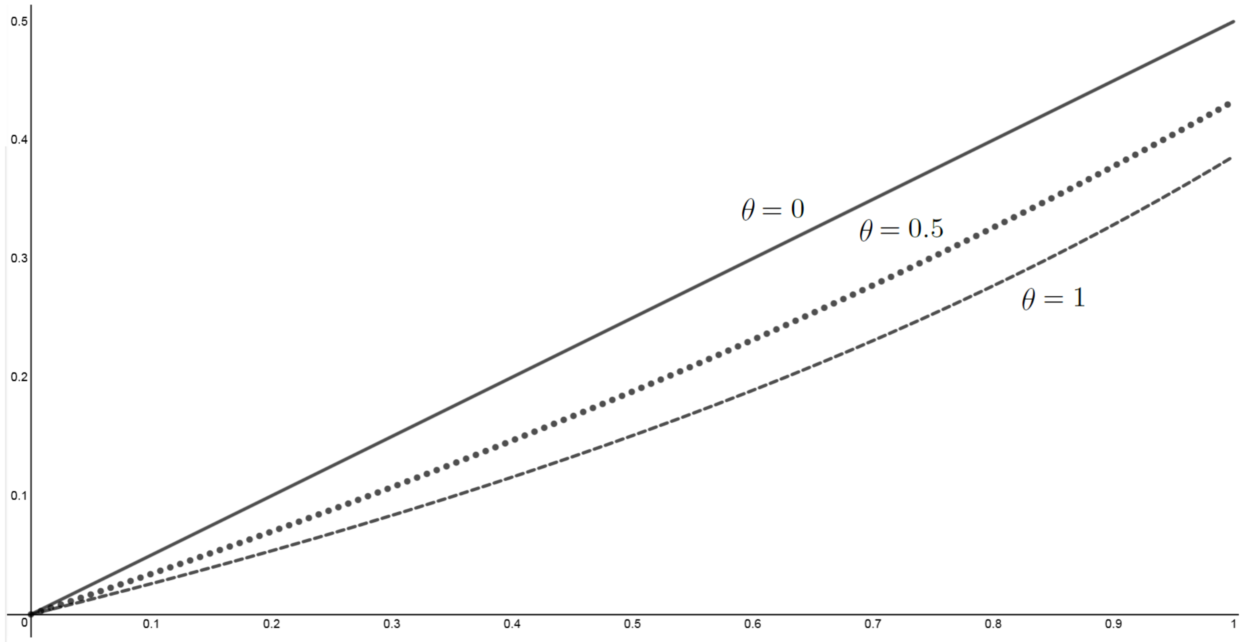

FIGURE 4.1: The equilibrium bid in a first-price auction with two buyers and uniformly distributed values for three degrees of loss aversion.

4.4.2 Second-Price Auctions

Similar to the case of first-price auctions, consider a symmetric and increasing equilibrium $(\beta_{II}, ..., \beta_{II})$ with $\beta_{II}(x) = 0$ if $x < \mu$ and $\beta_{II}(\mu) = r$. Recall that the winner of a second-price auction has to pay the largest of all other bids, $W_{II}(b, p) = p$.

First, consider a buyer with value $x < \mu$. In equilibrium, she never wins and thus always expects to pay nothing. It follows that her utility from bidding b is

$$u_{II}(b, p; x, b^r, p^r) = (x - p)1\{b > p\} - \theta \times p1\{b > p\}$$

and her expected utility from $b \geq r$ is then

$$U_{II}(b, x) = G(\beta_{II}^{-1}(b))x - \left(G(\mu)r + \int_{\mu}^{\beta_{II}^{-1}(b)} \beta_{II}(y)dG(y)\right) - \theta \times \left(G(\mu)r + \int_{\mu}^{\beta_{II}^{-1}(b)} \beta_{II}(y)dG(y)\right)$$

Second, consider a buyer with value $x \geq \mu$. She expects to win if $\beta_{II}(x) > p^r$ and to lose if $\beta_{II}(x) < p^r$, so her utility from bidding $b \geq r$ is

$$\begin{aligned} u_{II}(b, p; x, \beta_{II}(x), p^r) &= (x - p)1\{b > p\} \\ &\quad - \theta \times (\max\{p - p^r, 0\}1\{\beta_{II}(x) > p^r\} + p\{\beta_{II}(x) < p^r\})1\{b > p\} \end{aligned}$$

and her expected utility from $b \geq r$ is equal to

$$\begin{aligned} &U_{II}(b, x) = G(\beta_{II}^{-1}(b))x - G(\mu)r - \int_{\mu}^{\beta_{II}^{-1}(b)} \beta_{II}(y)dG(y) - \\ &\theta \left(G(\mu)(1 - G(x))r + \int_{\mu}^{\beta_{II}^{-1}(b)} \left[G(y)\beta_{II}(y) - G(\mu)r - \int_{\mu}^{y} \beta_{II}(y^r)dG(y^r) + (1 - G(x))\beta_{II}(y)\right] dG(y)\right) \end{aligned}$$

The first line denotes the expected monetary payoff. The second line denotes the expected psychological loss: in the event that she wins at the reserve r, whose probability is $G(\mu)$, she suffers the loss of r in the event that she expects to lose, whose probability is $(1 - G(x))$; in the event that she pays the second-highest bid $\beta_{II}(y)$, she suffers the loss of $\max\{\beta_{II}(y) - \max\{\beta_{II}(y^r), r\}, 0\}$ in the event that she expects to win and $\beta_{II}(y)$ in the event that she expects to lose, whose probability is $(1 - G(x))$.

Note that the equilibrium utility of the cut-off type μ is the same as in a first-price auction,

$$U_{II}(r, \mu) = G(\mu)(\mu - r) - \theta G(\mu) \times r(1 - G(\mu))$$

Thus, the cut-off μ is the same in first-price and second-price auctions.

For all $x \geq \mu$, the first derivative of U_{II} with respect b is

$$\begin{aligned}\frac{\partial}{\partial b}U_{II}(b,x) &= (x-b)\frac{g(\beta_{II}^{-1}(b))}{\beta_{II}'(\beta_{II}^{-1}(b))} \\ &- \theta\left(G(\beta_{II}^{-1}(b))b - \left[G(\mu)r + \int_{\mu}^{\beta_{II}^{-1}(b)} \beta_{II}(y^r)dG(y^r)\right] + (1-G(\beta_{II}^{-1}(b)))b\right)\frac{g(\beta_{II}^{-1}(b))}{\beta_{II}'(\beta_{II}^{-1}(b))}\end{aligned}$$

In equilibrium, it is optimal to bid $b = \beta_{II}(x)$, so the first-order condition is

$$x - \beta_{II}(x) - \theta\left(G(x)\beta_{II}(x) - \left[G(\mu)r + \int_{\mu}^{x} \beta_{II}(y^r)dG(y^r)\right] + (1-G(x))\beta_{II}(x)\right) = 0 \quad (4.3)$$

Differentiate both sides and re-arrange the terms to obtain

$$\beta_{II}'(x) + \theta\left[\beta_{II}'(x) - \beta_{II}(x)g(x)\right] = 1 \quad (4.4)$$

As shown in the next proposition, the equilibrium strategy can be found by solving this differential equation and using the boundary condition $\beta_{II}(\mu) = r$.

Proposition 4.2. *For a given reserve price $r \in [0, w]$, the profile of strategies $(\beta_{II}, ..., \beta_{II})$ where*

$$\beta_{II}(x) = \begin{cases} \exp\left(\frac{\theta}{1+\theta}(G(x)-G(\mu))\right)r + \frac{1}{1+\theta}\int_{\mu}^{x}\exp\left(\frac{\theta}{1+\theta}(G(x)-G(s))\right)ds & \text{if } x \geq \mu \\ 0 & \text{if } x < \mu \end{cases}$$

and the cut-off μ is given by Eq. (4.1) is an equilibrium of a second-price auction.

Proof. Introduce the function $v(x,y)$, defined as

$$v(x,y) = x - \theta\left(G(y)\beta_{II}(y) - \left[G(\mu)r + \int_{\mu}^{y} \beta_{II}(y^r)dG(y^r)\right] + (1-G(x))\beta_{II}(y)\right)$$

Since β_{II} is a solution to Eq. (4.4), it also solves Eq. (4.3),

$$\begin{aligned}\beta_{II}(x) &= x - \theta\left(G(x)\beta_{II}(x) - \left[G(\mu)r + \int_{\mu}^{x} \beta_{II}(y^r)dG(y^r)\right] + (1-G(x))\beta_{II}(x)\right) \\ &= v(x,x)\end{aligned}$$

First, consider a buyer with value $x \geq \mu$ who places some bid $\beta_{II}(z) \geq r$. Her expected utility is equal to

$$U_{II}(\beta_{II}(z), x) = G(\mu)\left(x - r - \theta(1 - G(x))r\right)$$
$$+ \int_{\mu}^{z} \left(x - \beta_{II}(y) - \theta \left[G(y)\beta_{II}(y) - \left(G(\mu)r + \int_{\mu}^{y} \beta_{II}(y^r) dG(y^r) \right) + (1 - G(x))\beta_{II}(y) \right] \right) dG(y)$$

Using the function $v(x, y)$, it can be re-written as

$$U_{II}(\beta_{II}(z), x) = G(\mu)\left(x - r - \theta(1 - G(x))r\right) + \int_{\mu}^{z} (v(x, y) - v(y, y)) dG(y)$$

It is routine to verify that the function $v(x, y)$ is strictly increasing in x. Thus, we have $v(x, y) - v(y, y) > 0$ for all $y < x$ and $v(x, y) - v(y, y) < 0$ for all $y > x$. If follows that the choice $z = x$ maximizes the integral, thereby maximizing the expected utility function. Also, there is no incentive to abstain from bidding because

$$\begin{aligned}
U_{II}(\beta_{II}(x), x) &= G(\mu)\left(x - r - \theta(1 - G(x))r\right) + \int_{\mu}^{x} (v(x, y) - v(y, y)) dG(y) \\
&> G(\mu)\left(\mu - r - \theta(1 - G(\mu))r\right) + \int_{\mu}^{x} (v(x, y) - v(y, y)) dG(y) \\
&= \int_{\mu}^{x} (v(x, y) - v(y, y)) dG(y) \\
&\geq 0 = U_{II}(0, x)
\end{aligned}$$

Second, consider any buyer type who does not bid, $x < \mu$. If she instead places a positive bid, $\beta_{II}(z) \geq r$, her expected utility is

$$\begin{aligned}
U_{II}(\beta_{II}(z), x) &= G(z)x - \left(G(\mu)r + \int_{\mu}^{z} \beta_{II}(y) dG(y) \right)(1 + \theta) \\
&< G(z) \times (\mu - r - \theta r) \\
&< G(z) \times (\mu - r - \theta r(1 - G(\mu))) \\
&= 0 = U_{II}(\beta_{II}(x), x)
\end{aligned}$$

where the first inequality follows from $x < \mu$ and $\beta_{II}(z) \geq r$. □

Figure 4.2 illustrates Proposition 4.2 for the case of $n = 2$ and $F(x) = x$ on $[0, 1]$. For $\theta = 0$, it is optimal to bid one's true value, $\beta_{II}(x) = x$. For larger values of θ, the optimal bid is lower and “almost” linear. This is because higher types suffer a lower loss when they expect to lose, but they suffer a higher loss when they expect to win.

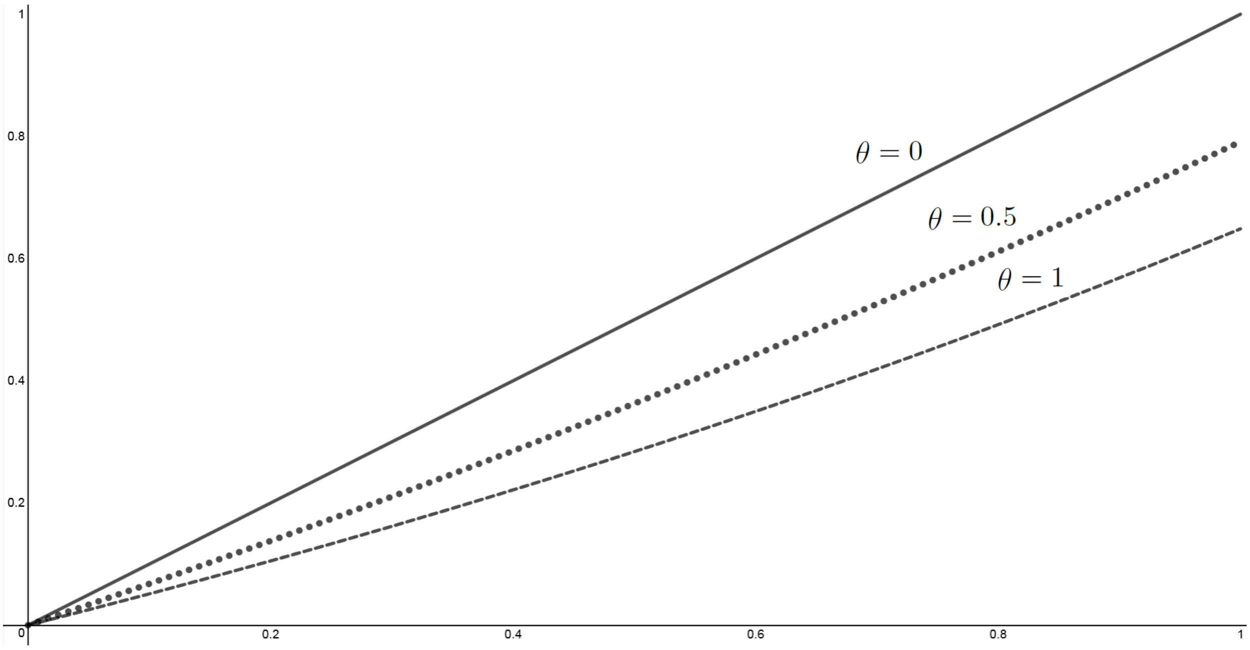

FIGURE 4.2: The equilibrium bid in a second-price auction with two buyers and uniformly distributed values for three degrees of loss aversion.

4.5 Optimal Auction

In this section, we use our results about the existence of symmetric increasing equilibria to find the optimal auction. First, we treat a buyer's problem as if she were purchasing some quantity of a good sold at a nonlinear unit price. With a new concept of imputed values, first-price auctions are shown to be superior to second-price auctions. Second, we propose a concept of psychological virtual values in order to find the optimal reserve price and study its properties.

4.5.1 Optimal Payment Rule

Fix some reserve price $r \geq 0$ and consider a first-price auction. Let $w_I(x)$ denote the *expected* price that a buyer with value x would pay upon winning. In a first-price auction, it is simply her bid,

$$w_I(x) = \beta_I(x)$$

Consider a buyer with value x who places some bid $\beta_I(z) \geq r$. Then we can write her expected utility as

$$U_I(\beta_I(z), x) = G(z)[x - w_I(z) - \theta(1 - G(x))w_I(z)]$$

and the first-order condition as

$$g(x)x - \left[g(x)w_I(x) + G(x)\frac{\partial}{\partial x}w_I(x)\right] - \theta(1 - G(x))\left[g(x)w_I(x) + G(x)\frac{\partial}{\partial x}w_I(x)\right] = 0 \quad (4.5)$$

It is clear that loss aversion affects the utility function by creating the new term $\theta(1 - G(x))w_I(z)$. Although it reflects non-monetary motivation, the term can be interpreted as a "value tax" on buying the object. That is, a buyer purchases $G(z)$ units and pays $w(z)(1 + \tau(x))$ for every unit where $\tau(x) \equiv \theta(1 - G(x))$ is the average value tax. It is interesting that the tax rate is regressive because $\tau(x)$ is decreasing in the volume of consumption $G(x)$.

In the presence of loss aversion, a buyer's value x no longer reflects her willingness-to-pay for the object. So let $v_I(x)$ denote the *imputed value* of a buyer with value x in a first-price auction. It is defined as the price that makes her indifferent between winning and losing,

$$x - v_I(x) - \theta(1 - G(x))v_I(x) = 0 \quad (4.6)$$

Re-arrange the terms to see that a buyer's imputed value is lower than her monetary value,

$$v_I(x) = x - \theta(1 - G(x))v_I(x) \quad (4.7)$$

Multiply both sides of Eq. (4.6) by $g(x)$ and subtract it from Eq. (4.5),

$$-\left[g(x)(w_I(x) - v_I(x)) + G(x)\frac{d}{dx}w_I(x)\right] - \theta(1 - G(x))\left[g(x)(w_I(x) - v_I(x)) + G(x)\frac{d}{dx}w_I(x)\right] = 0$$

Divide both sides by $G(x)$ and re-arrange the terms to obtain

$$\frac{d}{dx}w_I(x) = \frac{g(x)}{G(x)}(v_I(x) - w_I(x)) \quad (4.8)$$

Consider a second-price auction and let $w_{II}(x)$ denote the expected price that a buyer with value x pays conditional on winning,

$$w_{II}(x) = \frac{1}{G(x)}\left[G(\mu)r + \int_{\mu}^{x} \beta_{II}(y)dG(y)\right]$$

If a buyer with value x places some bid $\beta_{II}(z) \geq r$, her expected utility is

$$U_{II}(\beta_{II}(z), x) = G(z)\left[x - w_{II}(z) - \theta\left((1 - G(x))w_{II}(z) + \frac{1}{G(z)}\int_{\mu}^{z} G(y)(\beta_{II}(y) - w_{II}(y)))dG(y)\right)\right]$$

and the first-order condition is

$$\begin{aligned} & g(x)x - \left[g(x)w_{II}(x) + G(x)\frac{d}{dx}w_{II}(x)\right] \\ -\theta&\left((1 - G(x))\left[g(x)w_{II}(x) + G(x)\frac{d}{dx}w_{II}(x)\right] + G(x)(\beta_{II}(x) - w_{II}(x))g(x)\right) = 0 \end{aligned}$$

As we argued above, the term $\theta(1 - G(x))$ can be interpreted as a regressive value tax. But the utility function in a second-price auction has another term related to loss aversion, namely $G(y)(\beta_{II}(y) - w_{II}(y))$. In a similar fashion, it can be interpreted as a "quantity tax" because it is added to the price $w_{II}(z)$ for every "unit" purchased by a buyer.

Let $v_{II}(x, y)$ denote the *imputed value* of a buyer in a second-price auction when her (monetary) value is x and the largest of all other (monetary) values is y, defined as

$$x - v_{II}(x, y) - \theta\left((1 - G(x))v_{II}(x, y) + G(y)(v_{II}(x, y) - w_{II}(y))\right) = 0$$

or more conveniently,

$$v_{II}(x, y) = x - \theta\left((1 - G(x))v_{II}(x, y) + G(y)(v_{II}(x, y) - w_{II}(y))\right) \tag{4.9}$$

By a similar argument as before, we can write the first-order condition as

$$\frac{d}{dx}w_{II}(x) = \frac{g(x)}{G(x)}(v_{II}(x, x) - w_{II}(x)) \tag{4.10}$$

Proposition 4.3. *For any $\theta \geq 0$ and any $r \geq 0$, the expected revenue from a first-price auction is larger than the expected revenue from a second-price auction.*

Proof. Define the difference in expected prices as

$$\Delta(x) = w_I(x) - w_{II}(x)$$

Using Eq. (4.8) and Eq. (4.10), we obtain

$$\begin{aligned}\Delta'(x) &= \frac{d}{dx}w_I(x) - \frac{d}{dx}w_{II}(x)\\ &= \frac{g(x)}{G(x)}(v_I(x) - v_{II}(x,x) - \Delta(x))\end{aligned}$$

Subtract Eq. (4.7) from Eq. (4.9) to obtain

$$(v_I(x) - v_{II}(x,x))(1 + \theta(1 - G(x))) = \theta G(x)(v_{II}(x,x) - w_{II}(x))$$

Note that Eq. (4.9) implies $v_{II}(x,x) = \beta_{II}(x)$, so

$$\begin{aligned}v_{II}(x,x) - w_{II}(x) &= \beta_{II}(x) - \frac{1}{G(x)}\left[G(\mu)r + \int_\mu^x \beta_{II}(y)dG(y)\right]\\ &= \frac{1}{G(x)}\int_\mu^x G(y)\frac{d}{dy}\beta_{II}(y)dy\end{aligned}$$

where we used integration by parts. It implies $v_I(x) - v_{II}(x,x) \geq 0$ for $x = \mu$ and $v_I(x) - v_{II}(x,x) > 0$ for $x > \mu$.

Furthermore, we have $\Delta'(x) \geq 0$ whenever $\Delta(x) \leq 0$. Since $\Delta(\mu) = 0$, we have $\Delta'(x) \geq 0$ for all $x \geq \mu$. Finally, suppose that there is revenue equivalence, $\Delta(x) = 0$ and $\Delta'(x) = 0$ for all $x \geq \mu$. Then we must observe $v_I(x) - v_{II}(x,x) = 0$ for all $x \geq \mu$, a contradiction. Therefore, $\Delta'(x) > 0$ for some $x \geq \mu$. □

Our result is an application of the self-selection approach (Milgrom and Weber, 1982). Each buyer can be viewed as a population of buyers with values in $[0, w]$. Then a buyer with value x buys $G(x)$ units and pays a non-linear price $w(x)$ per unit. If she were to increase her consumption by $\epsilon \to 0$, her benefit would increase by $\epsilon(v(x) - w(x))$, where $v(x)$ is her willingness-to-pay. But it must not be profitable to consume more, so the price $w(x)$ must be sufficiently steep to offset the increase in benefit. As we showed, the willingness-to-pay is lower in a second-price auction because a buyer feels a loss in the event that she expected to pay a lower price. Thus its per-unit price should be less steep.

4.5.2 Optimal Reserve Price

We have seen that it is optimal for the seller to choose a first-price auction. But what reserve price should he announce? For a given $r \geq 0$, the expected payment of a buyer with value $x \geq \mu$ is equal to $G(x)w_I(x)$. Note that its derivative can be found by multiplying both sides of Eq. (4.8) and collecting the terms,

$$G(x)\frac{d}{dx}w_I(x) + g(x)w_I(x) = g(x)v_I(x)$$

Integrate on both sides and use the boundary condition $G(\mu)w_I(\mu) = G(\mu)r$ to obtain

$$\begin{aligned} G(x)w_I(x) &= G(\mu)r + \int_{\mu}^{x} v_I(y)dG(y) \\ &= G(x)v_I(x) - \int_{\mu}^{x} G(y)\frac{d}{dy}v_I(y)dy \end{aligned}$$

where the last equality follows from integration by parts.

Recall that the cut-off type μ is indifferent between winning and losing at price r. In other words, her imputed value is equal to the reserve price, $v_I(\mu) = r$. Thus the cut-off type can be written as $\mu = v_I^{-1}(r)$, so *the expected revenue* is equal to

$$\begin{aligned} \Pi_I(r) &= n \times \int_{v_I^{-1}(r)}^{w} G(x)w_I(x)dF(x) \\ &= n \times \int_{v_I^{-1}(r)}^{w} \left[G(x)v_I(x) - \int_{v_I^{-1}(r)}^{x} G(y)\frac{d}{dy}v_I(y)dy \right] dF(x) \\ &= n \times \int_{v_I^{-1}(r)}^{w} \left[v_I(x) - \frac{1-F(x)}{f(x)}\frac{d}{dx}v_I(x) \right] G(x)dF(x) \\ &= \int_{v_I^{-1}(r)}^{w} \left[v_I(x) - \frac{1-F(x)}{f(x)}\frac{d}{dx}v_I(x) \right] dF^n(x) \end{aligned}$$

where the third equality is obtained by changing the order of integration.

Let λ_H denote the hazard rate of a given distribution H,

$$\lambda_H(x) = \frac{h(x)}{1-H(x)}$$

Similar to the standard model of auctions, the term $[v_I(x) - v_I'(x)/\lambda_F(x)]$ can be referred to as a "psychological" virtual value of a buyer with monetary value x. Therefore, the expected revenue is equal to the expectation of the highest psychological virtual value among all buyers with imputed values above the reserve price.

Proposition 4.4. *If the hazard rate of F is increasing, the optimal reserve price in a first-price auction is given by $r^* = v_I(\mu^*)$, where μ^* is implicitly defined by*

$$\mu^* - \frac{1}{\lambda_F(\mu^*)} = \frac{\theta g(\mu^*)}{[\lambda_F(\mu^*)]^2} \times \left[1 + \theta g(\mu^*)\left(\frac{1}{\lambda_G(\mu^*)} - \frac{1}{\lambda_F(\mu^*)}\right)\right]^{-1} \tag{4.11}$$

Proof. Given the one-to-one and onto mapping from reserve prices into cut-offs, $r = v_I(\mu)$, we can instead search for the optimal cut-off,

$$\max_{\mu\in[0,w]} \Pi_I(\mu) = \int_\mu^w \left[v_I(x) - \frac{1-F(x)}{f(x)}\frac{d}{dx}v_I(x)\right] dF^n(x)$$

Using the Leibniz integral rule, the first derivative of Π_I is

$$\frac{d\Pi_I(\mu)}{d\mu} = -\frac{dF^n(\mu)}{d\mu}\left(v_I(\mu) - \frac{1-F(\mu)}{f(\mu)}\frac{dv_I(\mu)}{d\mu}\right)$$

Thus the first-order condition requires that the psychological virtual value of the cut-off type be zero,

$$v_I(\mu^*) - \frac{1-F(\mu^*)}{f(\mu^*)}\frac{dv_I(\mu^*)}{d\mu} = 0 \tag{4.12}$$

The last term can be found by totally differentiating Eq. (4.7),

$$\begin{aligned}\frac{dv_I(\mu)}{d\mu} &= 1 - \theta \times \left((1-G(\mu))\frac{dv_I(\mu)}{d\mu} - g(\mu)v_I(\mu)\right)\\ &= 1 - \theta \times g(\mu)\left(\frac{1}{\lambda_G(\mu)}\frac{dv_I(\mu)}{d\mu} - v_I(\mu)\right)\end{aligned}$$

Evaluate it at $\mu = \mu^*$ using the first-order condition,

$$\begin{aligned}\frac{dv_I(\mu^*)}{d\mu} &= 1 - \theta \times g(\mu^*)\left(\frac{1}{\lambda_G(\mu^*)}\frac{dv_I(\mu^*)}{d\mu} - \frac{1}{\lambda_F(\mu^*)}\frac{dv_I(\mu^*)}{d\mu}\right)\\ &= 1 - \theta \times g(\mu^*)\frac{dv_I(\mu^*)}{d\mu}\left(\frac{1}{\lambda_G(\mu^*)} - \frac{1}{\lambda_F(\mu^*)}\right)\end{aligned}$$

To verify the proposition, solve for $dv_I(\mu^*)/d\mu$, plug it back into Eq. (4.12) and re-arrange the terms. □

As in the case of loss neutrality, the optimal auction problem can be interpreted in terms of monopoly theory (Bulow and Roberts, 1989). Suppose that a monopolist faces a continuum of buyers, indexed by the set $X = [0, w]$ where the density of $x \in X$ is given by $f(x)$. If a buyer with index x buys one unit, she enjoys a gross surplus of x and pays a value tax of $\tau(x) = \theta(1 - G(x))$. When the monopolist charges some price $p \geq 0$, the demand function is

$$\begin{aligned} D(p) &= \Pr[x \geq (1 + \tau(x))p] \\ &= \Pr[p \leq v_I(x)] \\ &= 1 - F(v_I^{-1}(p)) \end{aligned}$$

Then the inverse demand function is given by $p(q) = v(F^{-1}(1 - q))$, and the revenue by

$$q \times p(q) = q \times v_I(F^{-1}(1 - q))$$

Differentiate the revenue to obtain the marginal revenue function,

$$\begin{aligned} \mathrm{MR}(p) &\equiv v_I(F^{-1}(1 - q)) - \frac{q \times v_I'(F^{-1}(1 - q))}{f(F^{-1}(1 - q))} \\ &= p - \frac{1 - F(v_I^{-1}(p))}{f(v_I^{-1}(p))} v_I'(v_I^{-1}(p)) \end{aligned}$$

where we used $p = v_I(F^{-1}(1 - q))$ and $q = 1 - F(v_I^{-1}(p)$. Thus, the optimal reserve price $r^* = v_I(\mu^*)$ can be thought of such monopoly price that the marginal revenue is zero,

$$\mathrm{MR}(v_I(\mu^*)) = v_I(\mu^*) - \frac{1 - F(\mu^*)}{f(\mu^*)} \frac{d}{dx} v_I(\mu^*) = 0$$

Let us compare our results to the standard model. As argued earlier, the seller maximizes the expected highest (psychological) virtual valuation. In the optimum, he chooses to "exclude" buyer types with negative virtual values, see Eq. (4.12). Because of loss aversion, a given reserve price has a stronger exclusionary effect, $\mu = v_I^{-1}(r) > r$. Conversely, the seller can achieve a target level of exclusion with a lower reserve price, $r = v_I(\mu) < \mu$. Thus it is reasonable to guess that the optimal reserve price is lower with loss-averse buyers. The next lemma confirms this intuition for the simple case of two buyers, $n = 2$.

Lemma 4.5. *With two buyers, the optimal reserve in a first-price auction is strictly decreasing in the degree of loss aversion.*

Proof. The fact $n = 2$ implies $G = F$, so Eq. (4.11) simplifies to

$$\begin{aligned} 0 &= \mu^* - \frac{1}{\lambda_F(\mu^*)} - \frac{\theta f(\mu^*)}{[\lambda_F(\mu^*)]^2} \\ &= \mu^* - \frac{1}{\lambda_F(\mu^*)}\left[1 + \frac{\theta f(\mu^*)}{\lambda_F(\mu^*)}\right] \\ &= \mu^* - \frac{1}{\lambda_F(\mu^*)}\left[1 + \theta(1 - F(\mu^*))\right] \end{aligned}$$

Divide both sides by $[1 + \theta(1 - F(\mu^*))]$ to obtain

$$v_I(\mu^*) - \frac{1}{\lambda_F(\mu^*)} = r^* - \frac{1}{\lambda_F(v_I^{-1}(r^*))} = 0$$

By the implicit function theorem,

$$\frac{dr^*}{d\theta} = -\frac{\lambda_F'(v_I^{-1}(r^*))}{[\lambda_F(v_I^{-1}(r^*))]^2} \times \frac{dv_I^{-1}(r^*)}{d\theta} \Big/ \frac{\partial}{\partial r^*}\left(r^* - \frac{1}{\lambda_F(v_I^{-1}(r^*))}\right)$$

The virtual valuation is assumed to be increasing in μ, so it is also increasing in r^* and the denominator is positive. Recall that the function $v_I^{-1}(r)$ specifies the type who is willing to pay r. As the degree of loss aversion increases, such cut-off type also increases, $dv_I^{-1}(r^*)/d\theta > 0$. This implies that the numerator is negative, and the proposition follows. □

Rosenkranz and Schmitz (2007) argue that the optimal reserve price under reference-dependent preferences is increasing in the number of buyers. In contrast, Lemma 4.5 leads to a different conclusion. For example, if there is only one buyer, $n = 1$, then imputed values do not differ from monetary values, $v(x) = x$. This implies that the optimal reserve price with one buyer is not affected by loss aversion. But if there are two buyers, $n = 2$, the optimal reserve is lower because of loss aversion. Therefore, the optimal reserve is lower with two buyers than with one buyer.

Let us again consider the case of uniform distribution, illustrated in Figure 4.3. It is well-known that the optimal reserve price is equal to 1/2 under standard preferences (Riley and Samuelson, 1981, p. 386). Under reference-dependent preferences, the expected revenue is given by the area below the "marginal revenue" curve on $[\mu^*, 1]$. Note that loss aversion leads to a higher level of exclusion, $\mu^* > 1/2$. The optimal reserve price, however, is less than 1/2 because it is equal to the imputed value of the cut-off type, $r^* = v_I(\mu^*)$, rather than her monetary value.

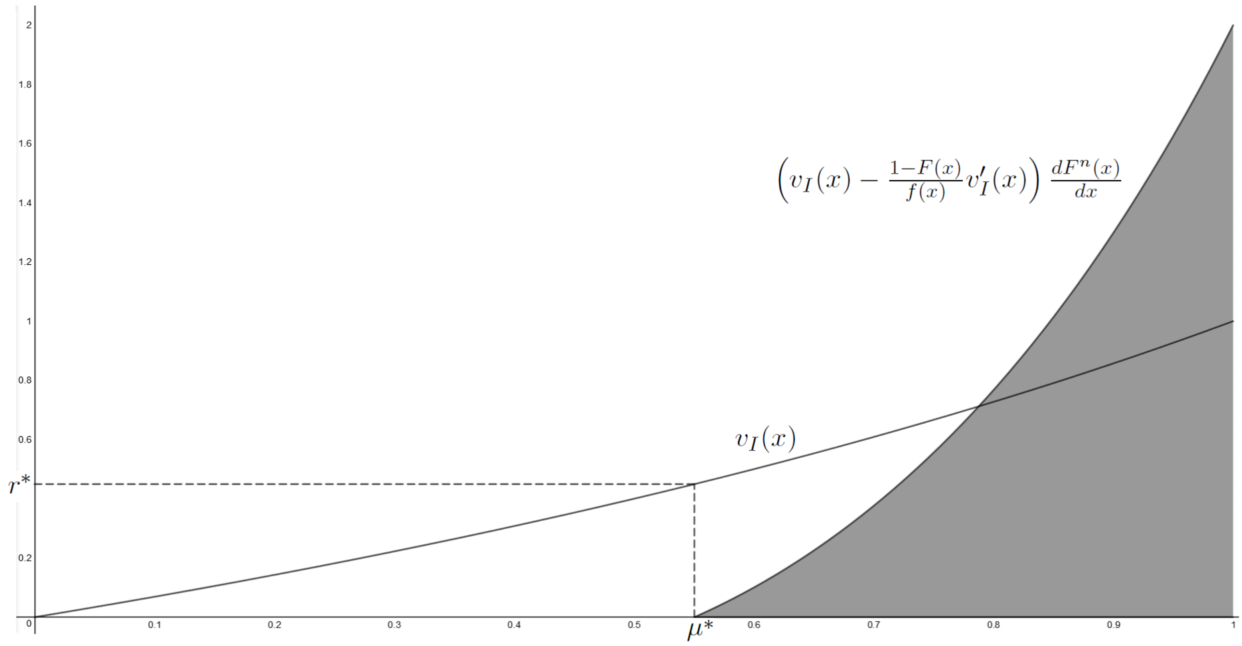

FIGURE 4.3: The expected revenue *(shaded area)* and the optimal reserve price r^* in a first-price auction with two buyers and uniformly distributed values for $\lambda = 0.5$.

4.6 Applications

In this section, the starting point of our analysis is a second-price auction with one buyer and the optimal reserve price that the seller announces publicly. We shall separately consider two questions of auction design:

1. *Value of Competition:* does the expected revenue increase when the seller "replaces" the reserve price with a second buyer?
2. *Value of Information:* does the expected revenue increase when the seller chooses a secret reserve price?

4.6.1 Value of Competition

Under standard preferences, Bulow and Klemperer (1996) show that an efficient second-price auction with $n+1$ buyers is more profitable than an optimal second-price auction with n buyers. However, Rosenkranz and Schmitz (2007) argue that the result fails to hold with reference-dependent preferences when n is sufficiently large. Below we strengthen their argument by proving that a second-price auction might be inferior to negotiations in the extreme case of $n = 1$.

First, suppose that the seller chooses not to invite a second buyer. Since $G(x) = 1$ for all x, buyer 1 with value x bids for the object if and only if $x \geq r$. As argued earlier, the optimal reserve price is thus the same as in Riley and Samuelson (1981), namely r^* at $\theta = 0$.

Let $p_N(x)$ denote the expected selling price in an optimal auction with buyer 1 when her value is x. Clearly, $p_N(x)$ is either the reserve price or zero,

$$p_N(x) = \begin{cases} r^*|_{\theta=0} & \text{if } x \geq r^*|_{\theta=0} \\ 0 & \text{if } x < r^*|_{\theta=0} \end{cases}$$

Second, suppose that the seller chooses to invite a second buyer. Similarly, let $p_A(x)$ denote the expected selling price in an efficient second-price auction with two buyers when the value of buyer 1 is x. To compute $p_A(x)$, fix some value of buyer 1, $X_1 = x$. If she wins, $x > Y$, the selling price is equal to the bid of buyer 2, $\beta_{II}(Y)$. But if she loses, $x < Y$, it is equal to the bid of buyer 1, $\beta_{II}(x)$. Therefore, we have

$$\begin{aligned} p_A(x) &= \int_0^x \beta_{II}(y)dF(y) + (1 - F(x))\beta_{II}(x) \\ &= F(x)w_{II}(x) + (1 - F(x))v_{II}(x, x) \\ &= F(x)v_{II}(x, x) - \int_0^x F(y)\frac{d}{dy}v_{II}(y, y)dy + \int_0^x \frac{d}{dy}v_{II}(y, y)dy - F(x)v_{II}(x, x) \\ &= \int_0^x (1 - F(y))\frac{d}{dy}v_{II}(y, y)dy \end{aligned}$$

Then the value of competition is equal to the expected difference of the selling prices,

$$\Delta^C = \int_0^w (p_A(x) - p_N(x))dF(x)$$

It is clear that the value of competition is strictly decreasing in the degree of loss aversion. This is because the revenue from negotiations is not affected by θ while the revenue from a second-price auction is strictly decreasing in θ. It is far from clear, however, whether there exists a sufficiently large degree of loss aversion such that negotiations become optimal. We shall now demonstrate by an example that it can be the case.

Consider the uniform distribution of values on $[0, 1]$, illustrated in Figure 4.4. The optimal reserve with one buyer is equal to $1/2$, so the revenue from negotiations is $1/4$. The revenue from a second-price auction is equal to $1/3$ with loss-neutral buyers, but goes down as θ

increases. In fact, it is optimal to negotiate with one buyer for $\theta > \underline{\theta}_{II} \approx 0.44$. Although a first-price auction yields a larger revenue that a second-price auction, it is also inferior to negotiations for $\theta > \underline{\theta}_I \approx 0.70$.

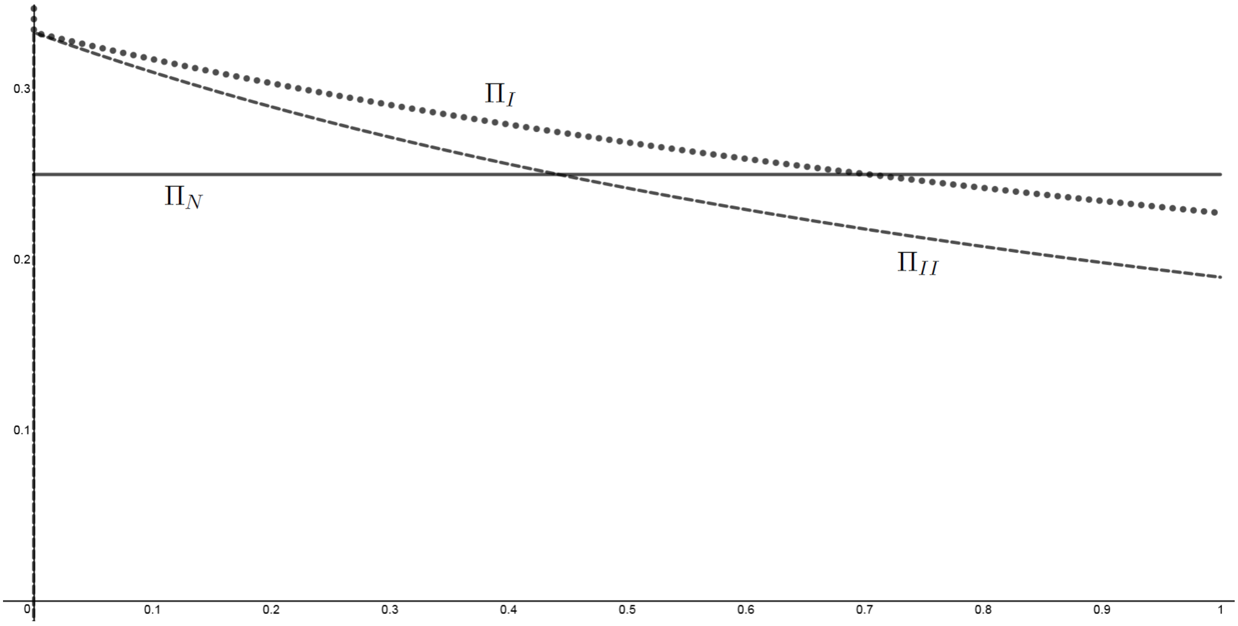

FIGURE 4.4: The expected revenue from optimal negotiations (*solid*), a first-price auction (*dotted*), and a second-price auction (*dashed*), as functions of θ.

4.6.2 Value of Information

In the model of Riley and Samuelson (1981), the seller is indifferent between secret and public reserve prices in a second-price auction because it is a weakly dominant strategy to bid one's value for any information policy. However, as shown by Rosenkranz and Schmitz (2007), the expected revenue under reference-dependent preferences may be larger when reserve prices are secret. In contrast to both findings, we shall prove below that secret reserve prices are strictly inferior to public reserve prices in our model.

So far the seller's value was assumed to be deterministic. Thus his strategy is just a reserve $r \geq 0$. Suppose that the seller keeps his reserve secret and some r^e is his equilibrium strategy. In equilibrium, the buyers know with certainty that the reserve is r^e, so they place the same bids as if r^e were public. Therefore, the equilibrium revenue of the seller can be replicated by choosing the public reserve r^e.

Following Elyakime et al. (1994), we shall instead assume that the seller's value is a random variable S. Precisely, S is distributed on $[0, w]$ according to the distribution function H with a continuous density $h \equiv H'$. The realization of S is only observed by the seller, and his strategy is a mapping from values into reserve prices, $p : [0, w] \to \mathbb{R}_+$.

4.6.2.1 Equilibrium Analysis

First, fix a strategy of the seller $p(\cdot)$, with $p'(\cdot) > 0$. The expected utility of the buyer with value x from placing some bid $b \geq 0$ is

$$\begin{aligned} U(b,x) = &\int_0^{p^{-1}(b)} (x - p(s))dH(s) \\ &- \theta \int_0^{p^{-1}(b)} \left(\int_0^s (p(s) - p(s^r))dH(s^r) + (1 - H(p^{-1}(\beta(x)))) \times p(s) \right) dH(s) \end{aligned}$$

The first line captures the expected monetary payoff: in the event that the buyer wins, $p(s) \leq b$, she has to pay $p(s)$. The second line captures the expected psychological loss: in the event that the buyer wins, her loss is $p(s) - p(s^r)$ in the event that she expected to pay a lower price, $p(s^r) < p(s)$; and her loss is $p(s)$ in the event that she expected to lose, whose probability is $(1 - H(p^{-1}(\beta(x))))$.

The first derivative with respect to the bid is

$$\frac{\partial}{\partial b} U(b,x) = \left[x - b - \theta \left(\int_0^{p^{-1}(b)} (b - p(s^r))dH(s^r) + (1 - H(p^{-1}(\beta(x)))) \times b \right) \right] \frac{h(p^{-1}(b))}{p'(p^{-1}(b))}$$

It must be optimal to bid $b = \beta(x)$, so the first-order condition reads

$$\beta(x) = x - \theta \left(\int_0^{p^{-1}(\beta(x))} (\beta(x) - p(s))dH(s) + (1 - H(p^{-1}(\beta(x))))\beta(x) \right) \tag{4.13}$$

Second, fix a strategy of the buyer, $\beta(\cdot)$ with $\beta'(\cdot) > 0$. The expected revenue of the seller with value s from setting some reserve $r \geq 0$ is

$$\begin{aligned} \Pi(r,s) &= sF(\beta^{-1}(r)) + r(1 - F(\beta^{-1}(r))) \\ &= sF(\beta^{-1}(r)) + \int_{\beta^{-1}(r)}^1 \left(\beta(x) - \frac{1 - F(x)}{f(x)} \beta'(x) \right) dF(x) \end{aligned}$$

The first derivative with respect to the reserve price is

$$\frac{\partial \Pi(r,s)}{\partial r} = \left[s - r + \frac{1 - F(\beta^{-1}(r))}{f(\beta^{-1}(r))} \beta'(\beta^{-1}(r)) \right] \frac{f(\beta^{-1}(r))}{\beta'(\beta^{-1}(r))}$$

In equilibrium, the revenue is maximized by choosing $r = p(s)$, so the first-order condition reads

$$p(s) = s + \frac{1 - F(\beta^{-1}(p(s)))}{f(\beta^{-1}(p(s)))} \beta'(\beta^{-1}(p(s))) \tag{4.14}$$

Proposition 4.6. *The profile of strategies (β, p) which simultaneously solves Eq.* (4.13) *and Eq.* (4.14) *is an equilibrium of a second-price auction with secret reserve prices.*

Proof. See Appendix. □

4.6.2.2 Revenue Ranking

Can the seller benefit from using secret reserve prices? First, suppose that the reserve prices are public. Consider the price strategy $\widehat{p}(s)$, defined by

$$\widehat{p}(s) = \begin{cases} \beta^{-1}(p(s)) & \text{if } s \leq \beta(w) \\ w & \text{if } s > \beta(w) \end{cases}$$

Then the probability of sale is the same under public and secret reserve prices. For convenience, let us denote it by

$$Q(x) \equiv H(p^{-1}(\beta(x)))$$

and let q denote its derivative, $q(x) \equiv dQ(x)/dx$.

Let $\widehat{w}(x)$ denote the expected price paid by the buyer with value x, conditional on winning,

$$\begin{aligned} \widehat{w}(x) &= \frac{1}{H(\widehat{p}^{-1}(x))} \int_0^{\widehat{p}^{-1}(x)} \widehat{p}(s) dH(s) \\ &= \frac{1}{Q(x)} \int_0^{p^{-1}(\beta(x))} \beta^{-1}(p(s)) dH(s) \end{aligned}$$

Suppose that the buyer with value x commits to bidding as if her value were some z for any reserve price. Then her expected utility is equal to

$$\widehat{U}(\beta(z), x) = Q(z) \times (x - \widehat{w}(z))$$

and the first-order condition is

$$q(x)(x - \widehat{w}(x)) - Q(x)\frac{d}{dx}\widehat{w}(x) = 0$$

which can be re-arranged as

$$\frac{d}{dx}\widehat{w}(x) = \frac{q(x)}{Q(x)} \times (x - \widehat{w}(x)) \tag{4.15}$$

Second, suppose that the reserve prices are secret. As before, let $w(x)$ denote the expected price paid by the buyer with value x, conditional on winning,

$$w(x) = \frac{1}{Q(x)} \int_0^{p^{-1}(\beta(x))} p(s)dH(s)$$

If the buyer with value x bids as if her value were some $z \in [0, w]$, her expected utility is

$$U(\beta(z), x) = Q(z) \times \left[x - w(z) - \theta\left((1 - Q(x))w(z) + \frac{1}{Q(z)}\int_0^z Q(t) \times (\beta(t) - w(t))dQ(t)\right)\right]$$

and the first-order condition reads

$$\begin{aligned} &q(x)(x - w(x)) - Q(x)\frac{d}{dx}w(x) \\ &- \theta\left[(1 - Q(x))\left(q(x)w(x) - Q(x)\frac{d}{dx}w(x)\right) + Q(x)(\beta(x) - w(x))q(x)\right] = 0 \end{aligned}$$

Let $v(x, s)$ denote the imputed value of the buyer with value x when the seller's value is s. Intuitively, it reflects the price at which the buyer is indifferent between winning and losing,

$$x - v(x, s) - \theta[(1 - Q(x))v(x, s) + Q(s)(v(x, s) - w(x))] = 0 \tag{4.16}$$

Multiply Eq. (4.16) by $q(x)$ and subtract it from the first-order condition to obtain

$$\frac{d}{dx}w(x) = \frac{q(x)}{Q(x)}(v(x, x) - w(x)) \tag{4.17}$$

Proposition 4.7. *For any $\theta > 0$, the expected revenue under public reserve prices is strictly larger than the expected revenue under secret reserve prices.*

Proof. Let $\Delta(x)$ define the effect of public information on the expected price,

$$\Delta(x) = \widehat{w}(x) - w(x)$$

By subtracting Eq. (4.17) from Eq. (4.15), we observe

$$\Delta'(x) = \frac{q(x)}{Q(x)}(x - v(x,x) - \Delta(x))$$

It is straightforward to see that the term $x - v(x,x)$ is strictly positive for all $x > 0$,

$$x - v(x,x) = \theta[(1 - Q(x))v(x,x) + Q(x)(v(x,x) - w(x))] > 0$$

Using a similar argument as in Proposition 4.3, we conclude that $\Delta'(x) \geq 0$ for all x and $\Delta'(x) > 0$ for some $x \geq 0$. □

To understand this result, let us interpret the problem in terms of industrial organization. The seller (the buyer) can be viewed as a population of sellers (buyers) with values in $[0, w]$. By choosing her bid, a buyer decides on the range of sellers that she is willing to trade with. Clearly, it is optimal to trade with a seller of type s as long as his price $p(s)$ is lower than the buyer's willingness-to-pay. But if the reserve is secret, a buyer feels a psychological loss from trading with a seller because she might have expected to pay a lower price or even not pay at all. Thus, her willingness-to-pay is lower, $v(x, s) < x$, which implies a lower demand and thus a lower (expected) revenue.

4.7 Conclusions

This paper has found the optimal auction among first-price and second-price auctions when buyers are loss averse in the monetary domain. We have assumed that reference points are formed before bidding, and they must be consistent with the equilibrium strategies. Our results suggest that it is optimal to choose a first-price auction and announce a reserve price below the optimal reserve with loss-neutral buyers. Furthermore, optimally structured negotiations with one buyer may be revenue superior to an efficient auction with two buyers, and public reserve prices are revenue superior to secret reserve prices.

However, a number of research questions remain to be addressed. First, loss aversion may affect the optimal choice of bids when the buyer is able to re-sell the object after winning[3]. In this case, the buyer would suffer a loss in the monetary domain when the she fails to re-sell the object but expected to re-sell it, thereby facing a different optimization problem in the initial auction. Second, loss aversion may affect the operation of bidding rings. Collusion reduces the number of serious bids and thus leads to lower uncertainty and lower psychological loss. By doing so, it should increase the gains of collusion as well as the selection process within the ring. Third, loss aversion may affect auctions where the seller can not fully commit to his choice of mechanism. If the buyer chooses to wait for the next auction, she will face a greater variance of selling prices and thus more psychological loss. Hence, she may be willing to accept a larger price in the present auction, possibly failing the Coase conjecture.

4.8 Appendix: Proof of Proposition 4.6

First, introduce the function $v(x,s)$, defined as

$$v(x,s) = x - \theta\left(\int_0^s (p(s) - p(s^r))dH(s^r) + (1 - H(p^{-1}(\beta(x))))p(s)\right)$$

Since β satisfies Eq. (4.13), it follows that $\beta(x) = v(x, p^{-1}(\beta(x)))$.

Suppose that the seller plays his equilibrium strategy and the buyer has value x, but places some bid $\beta(z) \geq 0$. Her expected utility is then

$$\begin{aligned} U(\beta(z),x) &= \int_0^{p^{-1}(\beta(z))} \left(x - p(s) - \theta\left(\int_0^s (p(s) - p(s^r))dH(s^r) + (1 - H(p^{-1}(\beta(x)))) \times p(s)\right)\right) dH(s) \\ &= \int_0^{p^{-1}(\beta(z))} (v(x,s) - p(s))dH(s) \end{aligned}$$

It is clear that that the function $v(x,s)$ is strictly increasing in x. Also, the integrand $(v(x,s) - p(s))$ is equal to zero at $s = p^{-1}(\beta(x))$,

$$\begin{aligned} v(x,p^{-1}(\beta(x))) - p(p^{-1}(\beta(x))) &= v(x,p^{-1}(\beta(x))) - \beta(x) \\ &= v(x,p^{-1}(\beta(x))) - v(x,p^{-1}(\beta(x))) = 0 \end{aligned}$$

[3] I am grateful to Andreas Lange for pointing out the impact of loss aversion in auctions with re-sale.

It implies that $v(x,s) - p(s) > 0$ for all $s < p^{-1}(\beta(x))$ and $v(x,s) - p(s) < 0$ for all $s > p^{-1}(\beta(x))$. Thus, the expected utility is maximized by ensuring that $p^{-1}(\beta(z)) = p^{-1}(\beta(x))$, or simply $z = x$. Since the value of the integral is positive, it is optimal to participate in the auction.

Second, introduce the function $\phi(\cdot)$, defined as

$$\phi(x) = \beta(x) - \frac{1 - F(x)}{f(x)}\beta'(x)$$

Using Eq. (4.14), the equilibrium reserve price satisfies $\phi(\beta^{-1}(p(s))) = s$, or simply $p(s) = \beta(\phi^{-1}(s))$. Suppose that the buyer plays her equilibrium strategy and the seller with value s chooses some reserve $p(t) \geq 0$. Then his expected revenue in excess of s is

$$\begin{aligned}\Pi(p(t), s) - s &= \int_{\beta^{-1}(p(t))}^{1} (\phi(x) - s)dF(x) \\ &= \int_{\phi^{-1}(t)}^{1} (\phi(x) - s)dF(x)\end{aligned}$$

The term $\phi(x) - s$ is equal to zero at $x = \phi^{-1}(s)$. Assuming that $\phi(x)$ is strictly increasing, we have $\phi(x) - s > 0$ for all $x > \phi^{-1}(s)$ and $\phi(x) - s < 0$ for all $x < \phi^{-1}(s)$. It follows that the revenue-maximizing choice is $t = s$.

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Appendix A

Summary

Chapter 2: *Collusion in Sequentially Optimal Auctions*

We study collusion among bidders when the seller can not commit to withhold the object from the market forever when it fails to sell. Our setting is characterized by second-price auctions, ex-ante symmetric buyers, and full-inclusive rings. We find that for sufficiently large reserve prices, any ex-post efficient mechanism of collusion must violate budget balance or voluntary participation. Yet, we show that a "second-best" allocation can always be implemented by a second-price preauction knockout. Furthermore, the collusion exerts downward pressure on the optimal reserve prices because it reduces demand for the object. As the time between auctions goes to zero, this effect reverses the classical result that the optimal reserve price is larger under collusion.

Wir betrachten geheime Absprachen zwischen Bietern, wobei der Verkäufer das Objekt nicht dem Markt dauerhaft vorenthalten kann, wenn es nicht verkauft wird. Unser Umfeld ist gekennzeichnet durch Zweitpreisauktionen, ex ante symmetrische Käufer und vollumfängliche Ringe. Wir stellen fest, dass für ausreichend große Reservationspreise jeglicher ex-post effiziente Kollusionsmechanismus die Budgetrestriktion oder die freiwillige Teilnahme verletzen muss. Wir zeigen, dass dennoch eine "zweitbeste" Allokation immer durch einen Knock-out in einer Zweitpreisvorauktion implementiert werden kann. Zudem übt die geheime Absprache Abwärtsdruck auf die optimalen Mindestpreise aus, weil sie die Nachfrage für das Objekt verringert. Da die Zeit zwischen Auktionen gegen Null geht, hebt dieser Effekt das klassische Ergebnis auf, welches besagt der optimale Mindestpreis sei höher bei geheimen Absprachen.

Chapter 3: *Fair Nonlinear Pricing*

We consider a problem of optimal nonlinear pricing where the buyer rewards the seller for charging fair prices and punishes him for unfair prices. As in Rabin (1993), the buyer's perception of fairness is determined by her belief about underlying intentions of the seller. Our findings suggest that under complete information, the buyer receives an offer to purchase the efficient quantity for a price below her valuation. Under incomplete information, the optimal truthful mechanism is characterized by no distortion at the top and downward distortion elsewhere. However, the distortion is not as large as in the standard model because the seller internalizes a psychological cost of paying an unfair price. As a result, he is motivated to reduce this cost by improving allocative efficiency. The optimal mechanism can be implemented by a schedule that stipulates lower per-unit prices, so our model seems to be more consistent with behavior of buyers and sellers in controlled experiments.

Wir betrachten ein Problem der optimalen nichtlinearen Preisbildung, bei welcher der Käufer den Verkäufer für eine faire Preisgestaltung belohnt und ihn für unfaire Preise bestraft. Wie bei Rabin (1993) ist die Wahrnehmung von Preisfairness aus Käufersicht davon bestimmt welche tieferen Absichten er von dem Verkäufer vermutet. Unsere Ergebnisse deuten darauf hin, dass dem Käufer bei vollständiger Information die effiziente Menge für einen Preis unter seiner Wertschätzung zum Kauf angeboten wird. Bei unvollständiger Information ist der optimale wahrheitsgemäße Mechanismus gekennzeichnet durch keine Verzerrung am oberen Ende und ist anderweitig nach unten verzerrt. Die Verzerrung ist jedoch nicht so groß wie in dem Standard-Modell, weil der Verkäufer die psychologischen Kosten dafür, einen unfairen Preis zu zahlen, internalisiert. Dies führt dazu, dass er motiviert ist, seine Kosten durch eine Verbesserung der Allokationseffizienz zu reduzieren. Der optimale Mechanismus kann mit einem Programmplan umgesetzt werden, in welchem niedrigere Stückpreise festlegt werden. Daher stimmt unser Modell mehr mit dem Verhalten von Käufern und Verkäufern in kontrollierten Experimenten überein.

Chapter 4: *Optimal Auctions with Loss Averse Buyers*

We extend the standard analysis of optimal auctions to allow for reference-dependent preferences: a buyer suffers a psychological disutility in the event that she has to pay more than her reference payment. The latter is assumed to be stochastic and consistent with the distribution of her payment in equilibrium. Our findings suggest that a first-price auction is strictly preferred to a second-price auction, and the optimal reserve price in a first-price auction is lower than that under standard preferences. Furthermore, the seller may prefer optimal negotiations with one buyer to an auction with two buyers for a sufficiently large degree of loss aversion; and he prefers public reserve prices to secret reserve prices for any degree of loss aversion.

Wir erweitern die Standardanalyse von optimalen Auktionen um referenzabhängige Präferenzen zu erlauben: Ein Käufer leidet unter einem psychologischen negativen Nutzen falls er mehr zahlen muss als seine Referenzzahlung. Letztere wird als stochastisch und vereinbar mit der Verteilung seiner Zahlung im Gleichgewicht angenommen. Unsere Ergebnisse zeigen, dass eine Erstpreisauktion gegenüber einer Zweitpreisauktion streng bevorzugt wird und dass der optimale Mindestpreis in einer Erstpreisauktion niedriger ist als unter Standardpräferenzen. Zudem kann es sein, dass der Verkäufer optimale Verhandlungen mit einem Käufer gegenüber einer Auktion mit zwei Käufern für ein ausreichend hohes Maß an Verlustaversion bevorzugt; und er bevorzugt öffentliche Mindestpreise gegenüber geheimen Mindestpreisen für jeglichen Grad an Verlustaversion.