Exploring the minimal 4d Superconformal theories

Dissertation
zur Erlangung des Doktorgrades
an der Fakultät für Mathematik, Informatik und Naturwissenschaften
Fachbereich Physik
der Universität Hamburg

vorgelegt von
Frau Dipl.-Phys. Martina Cornagliotto

Hamburg
2018

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Datum der Disputation:	19.10.2018
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Abstract

The conformal bootstrap program is based on the idea that the symmetries of a conformal field theory could fix its dynamics completely. An even more constraining setup arises when supersymmetry is present. In this thesis, we consider a set of supersymmetric theories in two and four dimensions and, using the modern approach to superconformal bootstrap, we explore and constrain their conformal data both analytically and numerically. We start by reviewing the essential aspects of conformal field theories in dimension higher than two with a particular focus on the recent progress achieved by the numerical bootstrap. We then discuss the introduction of supersymmetry highlighting the evidence for the existence of non-Lagrangian theories. Motivated by this need for alternative methods to approach such theories we review the superconformal bootstrap setup. Such setup is then applied to various examples. First, we consider a $\mathcal{N}=2$ theory in two dimensions initiating the superconformal bootstrap for long multiplets, that exploits all constraints from superprimaries and their descendants. To this end, we work out the Casimir equations for four-point correlators of long multiplets of the two-dimensional global $\mathcal{N}=2$ superconformal algebra. After constructing the full set of conformal blocks we discuss two different applications. The first one concerns two-dimensional (2,0) theories. The numerical bootstrap analysis we perform serves a twofold purpose, as a feasibility study of our long multiplet bootstrap and also as an exploration of (2,0) theories. A second line of applications is directed towards four-dimensional $\mathcal{N}=3$ SCFTs. In this context, our results imply a new bound $c \geqslant \frac{13}{24}$ for the central charge of such models, which we argue cannot be saturated by an interacting SCFT. Afterwards, we consider another four-dimensional theory, which is arguably the minimal four-dimensional theory with $\mathcal{N}=2$ supersymmetry, the (A_1, A_2) Argyres-Douglas theory. We study the four-point function of its single Coulomb branch chiral ring generator and put numerical bounds on the low-lying spectrum of the theory. Of particular interest is an infinite family of semi-short multiplets labeled by the spin ℓ. Although the conformal dimensions of these multiplets are protected, their three-point functions are not. Using the numerical bootstrap we impose rigorous upper and lower bounds on their values for spins up to $\ell = 20$. Through a recently obtained inversion formula, we also estimate them for sufficiently large ℓ, and the comparison of both approaches shows consistent results. We also give a rigorous numerical range for the OPE coefficient of the next operator in the chiral ring, and estimates for the dimension of the first R-symmetry neutral non-protected multiplet for small spin.

Kurzfassung

Das Programm des konformen Bootstraps basiert auf der Idee, dass die Symmetrien einer konformen Feldtheorie ihre Dynamik komplett bestimmen können. In der Gegenwart von Supersymmetrie ergibt sich eine noch eingeschränktere Situation. In dieser Arbeit betrachten wir eine Gruppe supersymmetrischer Theorien in zwei und vier Dimensionen und nutzen den modernen Ansatz des superkonformen Bootstraps, um ihre konformen Daten sowohl analytisch als auch numerisch zu erforschen und einzuschränken. Wir beginnen mit einer Wiederholung der wichtigsten Aspekte konformer Feldtheorien in Dimensionen höher als zwei, wobei wir ein besonderes Augenmerk auf neuere Fortschritte des numerischen Bootstraps legen. Danach beschreiben wir die Einführung von Supersymmetrie und gehen dabei insbesondere auf Hinweise auf Theorien, die nicht durch Lagrangefunktionen beschrieben werden können, ein. Motiviert durch die Notwendigkeit alternativer Methoden, um diese Theorien zu beschreiben, geben wir einen Überblick über den superkonformen Bootstrap und wenden ihn anschließend auf verschiedene Beispiele an. Zunächst betrachten wir eine $\mathcal{N} = 2$ Theorie in zwei Dimensionen, wobei wir den superkonformen Bootstrap für lange Multiplets begründen. Dieser nutzt alle Zwangsbedingungen der superkonformen Primärfelder und ihrer Nachkommen. Dafür leiten wir die Casimirgleichungen für Vierpunktfunktionen der langen Multiplets der globalen zweidimensionalen $\mathcal{N} = 2$ superkonformen Algebra her. Nachdem wir einen vollständigen Satz konformer Blocks konstruiert haben, betrachten wir zwei verschiedene Anwendungen. Die erste sind zweidimenionale (2,0) Theorien. Unsere Analyse mithilfe des numerischen Bootstraps dient einem doppelten Zweck, erstens als Machbarkeitsstudie unseres Bootstraps für lange Multiplets und zweitens als Methode zur Erforschung von (2,0) Theorien. Eine zweite Anwendung findet sich in vierdimensionalen $\mathcal{N} = 3$ superkonformen Feldtheorien. Unsere Ergebnisse implizieren eine neue untere Schranke $c \geqslant \frac{13}{24}$ für die zentrale Ladung in diesen Modellen, von der wir argumentieren, dass sie in interagierenden superkonformen Feldtheorien niemals erreicht werden kann. Danach betrachten wir eine weitere vierdimensionale Feldtheorie namens (A_1, A_2) Argyres-Douglas-Theorie, welche die minimale vierdimensionale Feldtheorie mit $\mathcal{N} = 2$ Supersymmetrie ist. Wir untersuchen die Vierpunktfunktion des einzigen Generators des chiralen Rings auf dem Coulomb-Zweig und erhalten numerische Schranken für das Niedrigenergiespektrum dieser Theorie. Von besonderem Interesse ist eine unendliche Familie halbkurzer Multiplets, welche durch ihren Spin ℓ charakterisiert werden. Obwohl ihre konformen Dimensionen beschützt sind, gilt dies nicht für ihre Dreipunktfunktionen. Unter Anwendung des numerischen Bootstraps finden wir rigorose untere und obere Schranken für ihre Werte für Spins bis $\ell = 20$. Mithilfe einer kürzlich hergeleiteten Inversionsformel erhalten wir außerdem Näherungswerte für ausreichend große Werte von ℓ. Ein Vergleich der beiden Methoden zeigt, dass ihre Ergebnisse kompatibel sind. Weiterhin geben wir

ein rigoroses numerisches Intervall für den OPE Koeffizienten des nächsten Operators im chiralen Ring an sowie Näherungen für die Dimension des ersten nichtbeschützten Multiplets bei kleinem Spin, welches neutral unter der R-Symmetrie ist.

Contents

Chapter 1

Introduction

In the twentieth century, Quantum Field Theories (QFTs) have emerged as a general framework to describe a wide variety of physical phenomena. On the one hand, it has become clear that the interaction among fundamental particles are well described by the Standard Model of fundamental interactions, which is based on an non abelian gauge theories with gauge group $SU(3) \times SU(2) \times U(1)$. On the other hand, quantum field theories have proven a very effective tool in the description of low energy physics such as statistical mechanics and condensed matter.

The textbook approach to QFTs is based on the presence of a Lagrangian. Given an action, we have a well defined way to quantize it (either canonically or with the path integral) and extract predictions on physical observables. Nevertheless, despite some extremely rare examples our understanding of QFTs is mostly perturbative, i.e. it is valid only when a small physical parameter is available and our results can be expanded for small values of such parameter.

In the last few years, it turned out this approach may not be the best way to understand the intricate nature of QFTs. Already in the seventies, when Kenneth Wilson introduced the renormalization group (RG) flow [1], people realized that our Lagrangian formulation of the Standard Model could be an effective description. This means that our current picture may give very accurate predictions at the energy scale that is presently accessible to the experiment, but it may well break down at higher energies. In other words, it would result from the RG flow of a UV complete theory, namely a theory which describes physical phenomena at arbitrary high energies.

Furthermore, there has been a growing amount of evidence that many interesting features of the dynamics of the theory could be accessed by studying its symmetries without any reference to an underlying Lagrangian. Even more strikingly, people discovered that a wide range of QFTs (arguably most of them) do not admit a Lagrangian description at all. For such theories the conventional approach would be useless. The idea of exploring the dynamics of a QFT by studying its symmetries, which goes under the name of bootstrap, was partially explored in the sixties and then almost abandoned for a long time. In this thesis, we focus on the application of this idea to Conformal Field Theories (CFTs), a particular class of QFTs which enjoy scale invariance.

The study of CFTs has the privilege of being more constraining than ordinary QFTs still remaining physically relevant. Indeed, we can think of a UV-complete QFT as a RG flow between CFTs. Thus, a deep knowledge of CFTs may be extremely relevant for understanding the space of QFTs as well as various aspects of their non perturbative dynamics. Furthermore, many physical systems that are relevant for statistical mechanics become scale invariant at the critical point, i.e. the temperature for which the system undergoes a second order phase transition. Such systems are usually characterized by critical exponents, namely the exponents that appear in power law behaviour of physical observables when they approach the critical point. A precise comprehension of CFTs may give very accurate predictions on such quantities.

Compared to ordinary QFTs, CFTs enjoy a larger symmetry group, including dilatations as well as special conformal transformations, a combination of translations and inversions. This larger amount of symmetry made the application of the bootstrap to these theories particularly successful. In particular, for the case of two spacetime dimensions, when the conformal algebra becomes infinite dimensional, the conformal bootstrap was developed already in the seventies [2–4]. On the other hand, the case of higher dimensions was curiously not explored until very recently [5].

By now we have a large wealth of results on strongly coupled theories, that would otherwise be hard to study by conventional field theory techniques, even including models that are lacking a Lagrangian description. The bootstrap approach, by relying only on symmetries, combined with a few spectral assumptions, allows to obtain complete non-perturbative answers, without reference to any type of perturbative description. The three-dimensional Ising model represents a striking example, where the most accurate determination of the critical exponents comes from the numerical bootstrap [6–11].

In a parallel line of development, analytic approaches to the bootstrap have also been explored, and recent progress has given access to the spectrum of CFTs at large spin by means of the lightcone limit [12, 13]. These two methods were combined in [11, 14], where knowledge of operator dimensions and operator product expansion (OPE) coefficients, obtained numerically for the Ising model, was used to derive analytic approximations for the CFT data at large spin. Remarkably, the analytic results obtained matched the numerical data down to spin two. The success of the large spin expansions down to spin two was recently explained in [15], where it was shown that operators of spin greater than one must organize in families analytic in spin.

Superconformal bootstrap

In the landscape of all the possible conformal field theories a particular role is played by superconformal ones. The latter feature an additional invariance which connects bosonic and fermionic degrees of freedom and is known as supersymmetry. Despite their little phenomenological interest the study of superconformal field theory (SCFT) is particularly useful for several reasons. First of all, the presence of supersymmetry produces a large variety of strongly coupled interacting CFTs which are usually hard to obtain in the non-supersymmetric case. This poses the interesting question of classifying all the possible superconformal theories. This task has been the focus of a large amount of work in the last few years and it is still an open problem under many aspects.

Besides the goal of classifying SCFTs, it is also interesting to explore their dynamics. In four dimensions the maximally supersymmetric theory, $\mathcal{N} = 4$ Super Yang-Mills (SYM), is so constrained that it may soon become the first example of exactly solved interacting QFT in 4d. However, lowering the amount of supersymmetry the structure of the resulting theories becomes at the same time richer and more obscure. On the one hand, exact results can still be achieved using supersymmetric localization. On the other hand, such techniques are applicable only to Lagrangian theories and for a restricted class of observables. This is especially limiting if one considers that for $\mathcal{N} = 2$ supersymmetry non Lagrangian theories form an important subset of all the known $\mathcal{N} = 2$ SCFTs. And even more limiting for $\mathcal{N} = 3$, where the totality of known theories does not admit a Lagrangian description.

For these reasons, the bootstrap approach is an optimal tool for the exploration of SCFTs. A tremendous amount of work has been done studying SCFTs in various dimensions and with various amounts of supersymmetry [16–51]. It has led to non-perturbative results in known theories ranging from two-dimensional $\mathcal{N} = (2, 2)$ [34], to six-dimensional $\mathcal{N} = (2, 0)$ [32] SCFTs. Furthermore, the bootstrap line of thinking helped uncover a solvable subsector in four-dimensional superconformal theories [52].[1] More precisely, the results of [52] imply that any $4d$ $\mathcal{N} \geqslant 2$ SCFT contains a closed subsector isomorphic to a $2d$ chiral algebra. In this thesis we will mostly focus on four spacetime dimensions with $\mathcal{N} = 2$ and $\mathcal{N} = 3$ supersymmetry. However in order to obtain results for $\mathcal{N} = 3$ supersymmetry in Chapter 3 we will also explore the long multiplet bootstrap for $\mathcal{N} = 2$ supersymmetry in two dimensions.

[1] See also [53] and [26, 54] for similar results in six and three dimensions.

By virtue of exploring the space of SCFTs relying only on symmetries, and with minimum assumptions about the operator content of the theories, the bootstrap program also provides a way to discover new SCFTs. Although there have been few surprises so far, a puzzling result was obtained in the supersymmetric bootstrap of four-dimensional $\mathcal{N}=1$ SCFT. Namely the presence of a "kink" in the dimension bounds of the leading long operator (*i.e.* obeying no shortening conditions) appearing in the operator product expansion (OPE) between a chiral and an antichiral operator [17–19]. Unlike the Ising model case, where the kink appeared exactly at the location of a known theory, there is no currently known theory which lives at the $\mathcal{N}=1$ kink.[2] The long operator whose dimension is given by the position of the "kink" is one of the natural objects to study in order to shed light on this "minimal" $\mathcal{N}=1$ SCFT, similarly to what was done for the three-dimensional Ising model. Very recently the superconformal primary of said long multiplet was considered in [19], but the complete set of constraints arising from the full supermultiplet remains unexplored. The only other existing bootstrap analysis that went beyond the usual half-BPS multiplets is [20], but as in [19], the authors restrict to correlations of the superconformal primary.

Most of the study of SCFTs has been limited to the analysis of four-point functions of half-BPS operators. In this case there are no nilpotent invariants, and the correlation function of the superconformal primary completely determines that of its superdescendants. Moreover since the only superconformal invariants are the supersymmetrizations of the conformal and R-symmetry cross-ratios, the crossing equations for the superconformal primary four-point function capture all of the constraints, and there is no need to consider those arising from four-point functions involving superdescendants. The same is still true for the four-point functions of two chiral operators with two long multiplets that were studied in [31]. However, things change once we consider four-point functions that involve at most one half-BPS multiplet while the other fields satisfy fewer or no shortening conditions at all. This is the topic of the long multiplet bootstrap.

Long multiplet bootstrap

For a complete superconformal bootstrap analysis one should certainly consider all four-point functions, including those in which all fields belong to long multiplets of the superconformal algebra. Such four-point functions can depend on nilpotent superconformal

[2] While the presence of a "kink" is not enough to guarantee the existence of a fully consistent SCFT, it provides hints it might correspond to a new $\mathcal{N}=1$ SCFT. The four-dimensional bounds were extended to SCFTs in $2\leqslant d\leqslant 4$ with four-supercharges ($\mathcal{N}=1$ in four dimensions) [31], and the "kink" persisted in lower dimensions as well. (Although in fractional dimensions unitarity is not preserved [55, 56], the violations are probably mild as the results appear reasonable.)

invariants, and information is lost when restricting the external operators to the superconformal primaries. For the case of four generic long multiplets this might mean, as was the case in [19, 41, 44] for four-dimensional $\mathcal{N} = 1$ long multiplets, that correlation functions of superprimaries can (only) be decomposed into bosonic conformal blocks with *independent* coefficients. While supersymmetry relates the various operators in the exchanged multiplet, and in particular their conformal dimensions, it does not constrain the coefficients of the bosonic block decomposition. In other words, correlation functions of superprimaries in long multiplets possess no "superblock" decomposition. The only way the number of free parameters in these block decompositions may be reduced is through permutation symmetry in the case of identical fields [41], or by additional shortening conditions, such as for conserved currents [19, 20, 39, 41].

In order to fully exploit consequences of supersymmetry in the study of long multiplets, in Chapter 3 we will be working with the *full* four-point functions in superspace, *i.e.* we consider not only superprimaries as the external operators, but also superdescendants. While our explicit analysis below will focus on two-dimensional SCFTs the key lessons we learn are more general. We show that, even if there is no "superblock" decomposition (other than the one into bosonic blocks) when one restricts to external superconformal primaries, some of the OPE coefficients of external superdescendants can be fixed in terms of those of the primary. This means that the number of free parameters in the block decomposition of the full four-point function is reduced as compared to the decomposition in terms of bosonic blocks. Moreover, the constraints coming from the full set of crossing equations in superspace are stronger than those of just the superprimary. This is not too surprising since our approach effectively includes mixed correlators with respect to the bosonic conformal symmetry even if we analyze correlation functions of four identical supermultiplets of the superconformal algebra. The combination of a non-trivial superblock decomposition and the constraints from crossing symmetry of superdescendants explains why our long multiplet bootstrap is significantly more powerful than a conventional analysis of crossing symmetry for superprimaries in long multiplets. Recently the aforementioned $\mathcal{N} = 1$ kink was studied by considering simultaneously chiral operators and the *superconformal primary* of long multiplets as external states in the correlation functions [19]. Even though in this system the blocks corresponding the long four-point function were simply bosonic blocks, stronger results on the kink were obtained. It seems natural to expect an improvement if one adds the (more computationally expensive) *whole* long supermultiplet, and all the crossing symmetry constraints.

In order to illustrate the workings of our long multiplet bootstrap we shall consider models with a two-dimensional $\mathcal{N} = 2$ (global) superconformal symmetry. Our first goal is to construct the relevant superblocks for four-point functions of long multiplets. We will do so under some technical assumptions on the R-charges of the involved multiplets. The superblocks for the various types of exchanged operators, are obtained in superspace by solving both the quadratic and cubic super Casimir equations. The equations provided by higher Casimirs bring no new information in this case. We obtain a coupled system of six second-order differential equations and construct its solutions in terms of hypergeometric functions. Our analysis serves as a first step towards the computation of long superblocks in higher dimensions for theories with four supercharges, by solving the super Casimir equation in an arbitrary number of dimensions, as done in [31] with half-BPS operators. For this reason we focus only on the global superconformal algebra in two dimensions, and do not make use of the full super Virasoro algebra.

Two-dimensional $\mathcal{N} = (2, 0)$ SCFTs

Once the relevant superblocks for the $\mathcal{N} = 2$ superconformal algebra are constructed we can run the numerical bootstrap program for long multiplets. We do so in the context of two-dimensional $\mathcal{N} = (2, 0)$ SCFTs, putting together the holomorphic blocks we compute with anti-holomorphic global $sl(2)$ blocks. This serves a two-fold purpose, as a feasibility test of bootstrapping long multiplets, and also as an exploration of $\mathcal{N} = (2, 0)$ theories which are interesting in their own right. By focusing on the four-point function of four identical uncharged long multiplets, Bose symmetry fixes all OPE coefficients of external superdescendants in terms of those of the external superprimary. However the crossing equations for external superdescendants still provide non-trivial constraints on the CFT data. Indeed if one were to consider the four-point function of external superconformal primaries alone, one would not find any improvement over the bosonic conformal bootstrap, since there would be no superblocks as discussed above. We exemplify how the bounds obtained in this way are stronger than the pure bosonic bootstrap and how our bounds are saturated by known supersymmetric minimal models at a point.

Four-dimensional $\mathcal{N} = 3$ SCFTs

In a different direction, the blocks we compute in Chapter 3 are precisely the ones relevant for the study of the chiral algebras associated to the recently discovered four-dimensional $\mathcal{N} = 3$ SCFTs [57], further explored in [35, 57–67]. Here we take a purely field-theoretic approach to these theories, using the fact, shown in [52], that any four-dimensional theory with $\mathcal{N} \geqslant 2$ supersymmetry has a subsector isomorphic to a two-dimensional chiral algebra. The chiral algebras of $\mathcal{N} = 3$ SCFTs have precisely $\mathcal{N} = 2$ supersymmetry [64]. In the study of four-dimensional four-point functions of half-BPS $\mathcal{N} = 3$ operators, as done in [35], the relevant two-dimensional blocks are those of external half-BPS (two-dimensional $\mathcal{N} = 2$ chiral) operators, which were computed in [40]. However, if one wants to consider the four-dimensional stress-tensor multiplet, which in two dimensions gives rise to the $\mathcal{N} = 2$ stress tensor multiplet, one needs exactly the long blocks obtained in Chapter 3. In the spirit of the bootstrap our assumptions will be minimal, obtaining constraints valid for any *local* and *interacting* $\mathcal{N} = 3$ SCFT.

Therefore we study the four-point function of the stress-tensor multiplet, as it is the only non-trivial multiplet we are guaranteed to have in a local $\mathcal{N} = 3$ SCFT. We obtain an infinite set of OPE coefficients, between two stress-tensor multiplets and a set of protected operators, valid for any *local, interacting* $\mathcal{N} = 3$ SCFT, depending only on the central charge. This is a necessary first step of any numerical study of the full-blown system of crossing equations for four-dimensional $\mathcal{N} = 3$ stress-tensor multiplets.

Moreover, positivity of these OPE coefficients, as required by unitarity of the four-dimensional $\mathcal{N} = 3$ theory, is not automatic. Imposing unitarity yields the following analytic bound on the c anomaly coefficient

$$c_{4d} \geqslant \frac{13}{24}\,, \tag{1.1}$$

valid for any *local, interacting* $\mathcal{N} = 3$ SCFT. Unlike similar analytic bounds obtained on various central charges, for both $\mathcal{N} = 2$ and $\mathcal{N} = 4$ SCFTs [22, 48, 52, 68, 69], we argue this bound corresponds to a strict inequality and cannot be saturated by an interacting unitary $\mathcal{N} = 3$ SCFT.

(A_1, A_2) Argyres-Douglas theory

In Chapter 4 we focus on another four-dimensional theory, which may be argued to be the "simplest" four-dimensional $\mathcal{N} = 2$ SCFT: the (A_1, A_2) (or H_0) Argyres-Douglas theory [70, 71]. "Simplest" in this case means that it has the lowest possible c-anomaly coefficient among interacting SCFTs [68], and the lowest a-anomaly coefficient among the known ones. The (A_1, A_2) SCFT can be realized by going to a special point on the Coulomb branch of an $\mathcal{N} = 2$ supersymmetric gauge theory, with gauge group SU(3), where electric and magnetic particles become simultaneously massless [70, 71]. It is an isolated $\mathcal{N} = 2$ SCFT, with no exactly marginal deformations, and thus no weak-coupling description. As such, despite being known for a very long time, little is known about the spectrum of this theory. Known data includes the scaling dimension, Δ_ϕ, of the single generator of the Coulomb branch chiral ring, whose vev parametrizes the Coulomb branch, and the a- and c-anomaly coefficients [59]:

$$\Delta_\phi = \frac{6}{5}\,, \qquad c = \frac{11}{30}\,, \qquad a = \frac{43}{120}\,. \tag{1.2}$$

The full superconformal index [72–74] was recently computed using an $\mathcal{N}=1$ Lagrangian that flows to the (A_1, A_2) SCFT in the IR [75]. The chiral algebra of this theory is conjectured to be the Yang-Lee minimal model [76, 77], which gives access to the spectrum of a particular class of short operators, dubbed "Schur" operators. However, the chiral algebra is insensitive to the Coulomb branch data of the theory, and even though the dimensions of the operators parameterizing the Coulomb branch chiral ring are known, not much is known about the values of the corresponding three-point functions.[3]

The relatively low values of its central charge and of the dimension of its Coulomb branch chiral ring generator make the (A_1, A_2) Argyres-Douglas theory amenable to numerical bootstrap techniques. In fact, one could argue that this is the $\mathcal{N}=2$ SCFT with the best chance to be "solved" numerically. We approach this theory based on the existing Coulomb branch data, by considering four-point functions of $\mathcal{N}=2$ chiral operators, whose superconformal primaries are identified with the elements of the Coulomb branch chiral ring.[4] While the values of c and Δ_ϕ in (1.2) are not selected by the numerical bootstrap, thanks to supersymmetry they are exactly known and thus we can use them as input in our analysis. We note however, that nothing is known about the spectrum of non-supersymmetry preserving relevant deformations of the (A_1, A_2) theory, and this type of information was essential to corner the $3d$ Ising model to a small "island" [8].

The results we find are encouraging, and provide the first estimates for unprotected quantities in this theory. We start by obtaining a lower bound on the central charge valid for any $\mathcal{N}=2$ theory with a Coulomb branch chiral ring operator of dimension $\Delta_\phi = \frac{6}{5}$. This bound appears to be converging to a value *close* to $c = \frac{11}{30}$, however the numerics are not conclusive enough. If the bound on c converges to $\frac{11}{30}$, then there is a unique solution to the crossing equations at $\Delta_\phi = \frac{6}{5}$ that corresponds to the (A_1, A_2) theory. If the numerical bound falls short of $\frac{11}{30}$, we present evidence, in the form of valid bounds on OPE coefficients and estimates on operator dimensions, that the various solutions around $c \sim \frac{11}{30}$ do not look so different, as far as certain observables are concerned. While the results we obtain are not at the level of the precision numerics of the $3d$ Ising model, we are able to provide estimates for the CFT data of this theory. For example, we constrain the OPE coefficient of the square of the Coulomb branch chiral ring generator (after unit normalizing its two-point function) to lie in the interval

$$2.1418 \leqslant \lambda^2_{\mathcal{E}_{\frac{12}{5}}} \leqslant 2.1672\,. \tag{1.3}$$

[3]See [78] for a recent computation of the two-point function (in normalizations where the OPE coefficients are one) of a Coulomb branch chiral ring operator, for theories with a single chiral ring generator, in the limit of large $U(1)_r$ charge.

[4]Another natural operator to consider in the correlation functions would be the $\mathcal{N}=2$ stress-tensor multiplet, however, the superconformal blocks for this multiplet are not known, and we leave this for future work.

While this is a true bound, due to slow convergence it is still far from being optimal, and will improve as more of the constraints of the crossing equations are taken into account. In section 4.1.2 we present estimates for the optimal range, based on conservative extrapolations of the bounds. Similarly, we constrain the OPE coefficients of a family of semi-short multiplets, appearing in the self-OPE of $\mathcal{N}=2$ chiral operators, to lie in a narrow range, quoted in (4.3) for $\ell=2,4$, and in figure 4.6 for even spins up to $\ell=20$.

We also provide in (4.5) the first estimate of the dimension of the lowest-lying unprotected scalar appearing in the OPE of the $\mathcal{N}=2$ chiral operator with its conjugate. This operator corresponds to a long multiplet that is a singlet under $\mathrm{SU}(2)_R$ symmetry, and neutral under $U(1)_r$, and we find it is relevant. These estimates are obtained from the extremal functionals [79] that gave rise to the aforementioned OPE coefficient bounds. From these extremal functionals we also obtain rough estimates for the dimensions of the lowest-twist long operator for higher values of the spin, shown in figure 4.7. Surprisingly, for spin greater than zero these operators are very close to being double-twist operators, *i.e.* , $\Delta = 2\Delta_\phi + \ell$.

Finally, we make use of the inversion formula of [15] to obtain large-spin estimates of the CFT data. As our numerical results are much further away from convergence than [11], we refrain from using them as input in the inversion formula. As such the only input we provide is the identity and stress-tensor supermultiplet exchange (with the appropriate central charge). Interestingly, we find that this input already provides a reasonable estimate of the numerically-bounded quantities for small spin.

A hybrid approach, combining both the numerical bootstrap and the inversion formula seems to be the most promising way to proceed, perhaps along the lines of the one suggested in [11]. The results in this thesis are a first step in this direction, and give us hope that a large amount of CFT data can be bootstrapped for the (A_1, A_2) theory.

Chapter 2

The Superconformal Bootstrap Program

2.1 Introduction to Conformal field theory

The ambitious goal of the bootstrap program is to provide a complete non perturbative solution of quantum field theory by the only means of symmetries and physical constraints. In this section we will give a brief introduction to conformal field theories trying to emphasize how quantum field theories are highly constrained by symmetries and consistency conditions.

2.1.1 Conformal group in d dimesions

The conformal group is the subgroup of coordinate transformations $x'^{\mu}(x)$ which leaves the metric invariant up to an overall scale $\delta g_{\mu\nu} = \omega(x) g_{\mu\nu}$. We can find the conformal group by considering the infinitesimal coordinate transformation

$$x'^{\mu} = x^{\mu} + \epsilon^{\mu}(x), \tag{2.1}$$

and imposing that

$$\delta g_{\mu\nu} = \partial_{\mu}\epsilon_{\nu} + \partial_{\nu}\epsilon_{\mu} = c(x)\delta_{\mu\nu} \tag{2.2}$$

where $c(x)$ is a scalar function. Contracting both sides with $\delta^{\mu\nu}$ gives $c(x) = \frac{2}{d}\partial\epsilon(x)$. Furthermore, taking additional derivatives of equation 2.2, one can easily show that only four class of solutions are allowed in $d > 2$ dimensions. They read

$$\begin{aligned}
&\epsilon^\mu = \text{constant} && \text{infinitesimal translation, } c(x) = 0\ , \\
&\epsilon^\mu = x^\nu \omega_{[\nu\mu]} && \text{infinitesimal rotation, } c(x) = 0\ , \\
&\epsilon^\mu = \lambda x^\mu && \text{scale transformation, } c(x) = 2\lambda\ , \\
&\epsilon^\mu = 2(a\cdot x)x^\mu - x^2 a^\mu && \text{Special Conformal Transformations, } c(x) = a\cdot x,
\end{aligned} \tag{2.3}$$

with a^μ an arbitrary vector.

Integrating to finite transformations we find the Poincaré group

$$\begin{aligned}
x' &= x + a \\
x' &= \Lambda x\ ,
\end{aligned} \tag{2.4}$$

the dilatations

$$x' = \lambda x \tag{2.5}$$

and the special conformal transformation

$$x'^\mu = \frac{(x^\mu - a^\mu x^2)}{1 - 2(a\cdot x) + a^2 x^2}, \tag{2.6}$$

A general conformal transformation $x \to x'$ will be a composition of translations, rotations, scale transformations and special conformal transformations (SCTs).

From now on we will focus on the $d > 2$ case. In $d = 2$ conformal symmetry presents some special features which deserve a different derivation. We briefly comment on the $d = 2$ case in section 2.1.3, mostly outlining the relevant references on the subject.

2.1.2 The conformal algebra

Starting from the infinitesimal conformal transformations, it is a straightforward exercise to write the generators of the conformal algebra in differential form

$$\begin{aligned}
&P_\mu = -i\partial_\mu \ \to\ \text{translations,} \\
&M_{\mu\nu} = i(x_\mu\partial_\nu - x_\nu\partial_\mu) \ \to\ \text{rotations,} \\
&D = -ix^\mu\partial_\mu \ \to\ \text{dilatations,} \\
&K_\mu = -i(2x_\mu x^\nu\partial_\nu - x^2\partial_\mu) \ \to\ \text{SCTs}\ .
\end{aligned} \tag{2.7}$$

From these expressions we can easily derive the commutation relations of the algebra

$$[M_{\mu\nu}, M_{\rho\sigma}] = i(\delta_{\nu\rho}M_{\mu\sigma} + \delta_{\mu\sigma}M_{\nu\rho} - \delta_{\mu\rho}M_{\nu\sigma} - \delta_{\nu\sigma}M_{\mu\rho}), \tag{2.8}$$

$$[M_{\mu\nu}, P_\rho] = i(\delta_{\rho\nu}P_\mu - \delta_{\rho\mu}P_\nu), \tag{2.9}$$

$$[M_{\mu\nu}, K_\rho] = -i(\delta_{\rho\mu}K_\nu - \delta_{\rho\nu}K_\mu), \tag{2.10}$$

$$[D, P_\mu] = iP_\mu, \tag{2.11}$$

$$[D, K_\mu] = -iK_\mu, \tag{2.12}$$

$$[K_\mu, P_\nu] = 2i(\delta_{\mu\nu}D - M_{\mu\nu}) \ . \tag{2.13}$$

In the first three commutation relations we recognise the algebra of Euclidean rotations $SO(d)$ generated by $M_{\mu\nu}$ and we can see that P_μ, K_μ transform as vectors. The last three equations are more interesting. Equations (2.11) and (2.12) say that P_μ and K_μ can be thought of as raising and lowering operators for D.

Rewriting the conformal generators like

$$\begin{aligned} L_{\mu\nu} &= M_{\mu\nu}, \\ L_{-1,0} &= D, \\ L_{0,\mu} &= \frac{1}{2}(P_\mu + K_\mu), \\ L_{-1,\mu} &= \frac{1}{2}(P_\mu - K_\mu), \end{aligned} \tag{2.14}$$

with $L_{ab} = -L_{ba}$ and $a, b \in \{-1, 0, 1, \ldots, d\}$, it is easy to show that L_{ab} satisfy the commutation relations of $\mathrm{SO}(d{+}1, 1)$. The fact that the d-dimensional conformal algebra is $\mathrm{SO}(d+1, 1)$ suggests that it might be good to think about its action in terms of $\mathbb{R}^{d+1,1}$ instead of $\mathbb{R}^d$. This is the idea behind the embedding space formalism [80–85], which provides a simple and powerful way to understand the constraints of conformal invariance.

2.1.3 The special case of $d = 2$

The case of $d = 2$ has been object of great attention in the past and is nowadays textbook material [86]. Here we limit ourself to the illustration of the main reason why the two-dimensional case deeply differ from the higher-dimensional counterpart and then we focus on the latter. For $d = 2$ equation 2.2 admits additional solutions since it is equivalent to the Cauchy-Riemann conditions

$$\partial_1\epsilon_1 = \partial_2\epsilon_2, \qquad \partial_1\epsilon_2 = -\partial_2\epsilon_1 \tag{2.15}$$

which are solved for any analytic function on the complex plane. In particular if we write $\epsilon(x)$ in the complex coordinates

$$x = x_1 + ix_2 \ , \qquad \bar{x} = x_1 - ix_2 . \ , \qquad ds^2 = dx d\bar{x} \ , \tag{2.16}$$

the two-dimensional conformal transformations coincide with the coordinate transformations $x \to f(x)$ (and $\bar{x} \to \bar{f}(\bar{x})$) for any analytic function f. This generates of course an infinite dimensional algebra since it has as many degrees of freedom as those of an arbitrary analytic function in the complex plane. Indeed, a careful analysis shows that the conformal algebra factorizes in a holomorphic and anti-holomorphic sector and both of these sectors become infinite dimensional. To see this one can rearrange the usual bosonic generators $D, P^\mu, K^\mu, M^{\mu\nu}$ as

$$L_0 = \frac{iD - M_{12}}{\sqrt{2}} \qquad \bar{L}_0 = \frac{-iD - M_{12}}{\sqrt{2}} \tag{2.17}$$

$$L_1 = \frac{P_1 - iP_2}{\sqrt{2}} \qquad \bar{L}_1 = \frac{P_1 + iP_2}{\sqrt{2}} \tag{2.18}$$

$$L_{-1} = \frac{K_1 - iK_2}{\sqrt{2}} \qquad \bar{L}_{-1} = \frac{K_1 + iK_2}{\sqrt{2}} \tag{2.19}$$

satisfying the Gliozzi algebra

$$[L_m, L_n] = (m-n)L_{m+n} \qquad m, n = -1, 0, 1 \tag{2.20}$$

and then notice that this algebra can be extended to the full infinite dimensional Virasoro algebra

$$[L_m, L_n] = (m-n)L_{m+n} + \delta_{m,-n} m(m^2-1)\frac{c}{12} \tag{2.21}$$

where c is the central charge. Although in Chapter 3 we will be concerned with two-dimensional theories, we will never use the full Virasoro algebra, but we will rather use a supersymmetric extension of (2.20), which we will introduce in section 3.1.2.

2.1.4 Irreducible representations of the conformal algebra

Given the d-dimensional conformal algebra introduced in the previous section we are interested in finding its irreducible representations. To do that, we first need to identify the Cartan subalgebra. For simplicity we restrict to the four dimensional case in Euclidean signature. As we observed in the previous section, the conformal group is SO(5, 1) generated by $P_\mu, K_\mu, M_{\mu\nu}$ and D with $\mu = 1, 2, 3, 4$. A generic state in our theory $|\psi\rangle$ will be eigenstate of the dilation operator D and two of the $\mathfrak{so}(4)$ generators J_1, J_2, which are usually called spin. The set of generators $\{D, J_1, J_2\}$ spans the Cartan subalgebra in 4 dimensions[1]. The Cartan generators can be simultaneously diagonalized and we can characterize ther eigenstate as $|\Delta, j_1, j_2\rangle$ with

$$\begin{aligned} D\,|\Delta, j_1, j_2\rangle &= \Delta\,|\Delta, j_1, j_2\rangle \\ J_1\,|\Delta, j_1, j_2\rangle &= j_1\,|\Delta, j_1, j_2\rangle\,, \\ J_2\,|\Delta, j_1, j_2\rangle &= j_2\,|\Delta, j_1, j_2\rangle\,. \end{aligned} \tag{2.22}$$

Looking at the commutation relation (2.12) we see that K_μ act on the states as a lowering operator for the dimension. Since we are interested in physically sensible theories, dimensions have to be bounded from below. Thus, it must exist a state such that

$$K_\mu\,|\Delta_0, j_1, j_2\rangle = 0. \tag{2.23}$$

Such a state is called lowest-weight state. Given the lowest-weight state we can construct states of higher dimension by acting with momentum generators, which act like raising operators for the dimension of the state.

$$\begin{array}{c|c} \text{state} & \text{dimension} \\ \hline \dots & \dots \\ P_{\mu_1}P_{\mu_2}\,|\Delta_0, j_1, j_2\rangle & \Delta = \Delta_0 + 2 \\ \uparrow & \\ P_{\mu_1}\,|\Delta_0, j_1, j_2\rangle & \Delta = \Delta_0 + 1 \\ \uparrow & \\ |\Delta_0, j_1, j_2\rangle & \Delta = \Delta_0 \end{array} \tag{2.24}$$

Thus, a generic representation of the conformal algebra in four dimensions could be written as

$$\mathcal{A}^{\Delta_0}_{J_1,J_2} = \text{Span}\Big\{ P_{\mu_1}\dots P_{\mu_n}\,|\Delta_0, j_1, j_2\rangle \Big\} \tag{2.25}$$

for $n \geq 0$. The allowed values for the lowest-weight quantum numbers are constrained by the requirement of unitarity. The latter imposes that all the operators have positive norm, including descendants. The norm of a descendant is related to the commutator $[K^\mu, P^\nu]$ which contains the Cartan generators. Therefore positivity of the norm leads to a constraint of the kind $\Delta_0 \geq f(j_1, j_2)$ for some linear combination f. A careful

[1]In the generic d-dimensional case one would have the dilatation operator and a set of $[\frac{d}{2}]$ spins

computation of such conditions yields the bounds

$$\begin{aligned}
\Delta_0 \geq &\quad j_1 + j_2 + 2 && \text{if} \quad j_1 \neq 0 \ \text{ and } \ j_2 \neq 0 \\
\Delta_0 \geq &\quad j_1 + 1 && \text{if} \quad j_1 \neq 0 \ \text{ and } \ j_2 = 0 \\
\Delta_0 \geq &\quad j_2 + 1 && \text{if} \quad j_1 = 0 \ \text{ and } \ j_2 \neq 0 \\
\Delta_0 \geq &\quad 1 && \text{if} \quad j_1 = j_2 = 0.
\end{aligned} \tag{2.26}$$

If a lowest-weight state of dimension Δ_0 satisfy the condition then this automatically implies that all the descendants strictly satisfy the inequality. We are interested in finding a lowest-weight state such that the unitarity bound is saturated. Such a state would generate a null state in the theory, *i.e.* a descendant state with zero norm which is removed. The corresponding representation is often called short multiplet. If both spins are different from zero we have:

$$\Delta_0 = j_1 + j_2 + 2 \tag{2.27}$$

An important example is the case $j_1 = j_2 = 1/2$, when the unitarity conditions are saturated for $\Delta_0 = 3$ and the null state is

$$||P^{\alpha\dot{\alpha}} \left| J_{\alpha\dot{\alpha}} \right\rangle || = 0. \tag{2.28}$$

where $|J_{\alpha\dot{\alpha}}\rangle \equiv |3, 1/2, 1/2\rangle$. This means that in a generic conformal field theory a vector of exact dimension three is automatically a conserved current. Conversely, conserved currents always belong to short representations of the conformal algebra. Furthermore, in the limit $\Delta \to j_1 + j_2 + 2$ we can write the generic representation of the algebra $\mathcal{A}^{j_1+j_2+2}_{j_1,j_2}$ as the direct sum of an irreducible short representation $\mathcal{C}_{j_1,j_2}$ and a generic (long) representation with exactly the same quantum numbers of the null state

$$\mathcal{A}^{j_1+j_2+2}_{j_1,j_2} = \mathcal{C}_{j_1,j_2} \oplus \mathcal{A}^{j_1+j_2+3}_{j_1-1/2,j_2-1/2} \quad . \tag{2.29}$$

When both spins $j_1 = j_2 = 0$ the unitarity bound is saturated for $\Delta_0 = 1$ and we have a null state at level two represented by

$$||P^{\alpha\dot{\alpha}} P_{\alpha\dot{\alpha}} \left|\Phi\right\rangle || = 0. \tag{2.30}$$

where $|\Phi\rangle \equiv |1, 0, 0\rangle$. This shortening condition has the clear physical interpretation of the equation of motion for a free scalar field in four dimensions. In the limit $\Delta_0 \to 1$ the long representation $\mathcal{A}^1_{0,0}$ decomposes as

$$\mathcal{A}^1_{0,0} = \mathcal{B} \oplus \mathcal{A}^{\Delta=3}_{0,0}. \tag{2.31}$$

Here $\mathcal{B}$ denotes the corresponding short multiplet. Following the same path we denote with $\mathcal{B}^R_{J_2}$ the case $j_1 = 0, j_2 \neq 0$ and $\mathcal{B}^L_{J_1}$ the case $j_1 \neq 0, j_2 = 0$.

2.1.5 State-operator correspondence

The textbook quantization of a quantum field theory is performed by slicing the space-time with equal time surfaces and taking time-ordered correlators of local operators. Nevertheless, for conformal field theories on Euclidean space an alternative route is available. This can be understood by noticing that a conformal transformation maps the cylinder $S^{d-1} \times \mathbb{R}$ to $\mathbb{R}^d$. The map is easily constructed by taking polar coordinates on $\mathbb{R}^d$ and mapping the radial coordinate r to the time coordinate t (the direction along $\mathbb{R}$) on the cylinder

$$e^t = r \tag{2.32}$$

We notice that the origin in $\mathbb{R}^d$ is mapped to $t = -\infty$ on the cylinder and that a constant time quantization on the cylinder is mapped to a radial quantization on $\mathbb{R}^d$. It is then clear that, in radial quantization, the role of the Hamiltonian is played by the dilatation operator, which evolves operators along a radial direction. The constant radius surfaces are spheres S^{d-1} and they have an associated Hilbert space $\mathcal{H}$ on which we can act by inserting operators on the surface of the sphere. Correspondingly, correlation functions of local operators are taken to be radially ordered.

Consider now an eigenstate of the dilatation operator, such as those introduced in the previous section

$$D\,|\Delta\rangle = \Delta\,|\Delta\rangle \tag{2.33}$$

Using radial quantization one can rigorously prove the intuitive fact that such states are in one to one correspondence with local operators in the origin

$$\mathcal{O}(0) \qquad \longleftrightarrow \qquad \mathcal{O}(0)|0\rangle \equiv |\mathcal{O}\rangle. \tag{2.34}$$

This property goes under the name of state operator correspondence. The action on conformal primary operators are

$$[K_\mu, \mathcal{O}(0)] = 0 \qquad \longleftrightarrow \qquad K_\mu|\mathcal{O}\rangle = 0, \tag{2.35}$$

$$[D, \mathcal{O}(0)] = \Delta\mathcal{O}(0) \qquad \longleftrightarrow \qquad D|\mathcal{O}\rangle = \Delta|\mathcal{O}\rangle, \tag{2.36}$$

$$[M_{\mu\nu}, \mathcal{O}(0)] = \mathcal{S}_{\mu\nu}\mathcal{O}(0) \qquad \longleftrightarrow \qquad M_{\mu\nu}|\mathcal{O}\rangle = \mathcal{S}_{\mu\nu}|\mathcal{O}\rangle, \tag{2.37}$$

while conformal descendant operators are defined by acting with derivatives at the origin, for example

$$\partial_\mu \mathcal{O}(x)|_{x=0}|0\rangle \;=\; [P_\mu, \mathcal{O}(0)]|0\rangle \;=\; P_\mu|\mathcal{O}\rangle. \tag{2.38}$$

The operator $\mathcal{O}(x)$, away from the origin creates an infinite linear combination of descendants,

$$\mathcal{O}(x)|0\rangle \;=\; e^{x\cdot P}\mathcal{O}(0)e^{-x\cdot P}|0\rangle \;=\; e^{x\cdot P}|\mathcal{O}\rangle \;=\; \sum_{n=0}^{\infty}\frac{1}{n!}(x\cdot P)^n|\mathcal{O}\rangle. \tag{2.39}$$

Using radial quantization we can also prove that all operators in unitary CFTs are linear combinations of primaries and descendants. We start by the CFT defined on the cylinder $S_{d-1}\times\mathbb{R}$ and we compactify the time direction. As we mentioned above, in the cylinder picture the Hamiltonian is the dilatation operator, therefore the partition function on the torus $S^{d-1}\times S^1_\beta$ is given by

$$\mathcal{Z}_{S^{d-1}\times S^1_\beta} = \mathrm{Tr}(e^{-\beta D}) < \infty. \tag{2.40}$$

where in the last step we assumed it is finite. This means that $e^{-\beta D}$ is trace-class, and hence diagonalizable with a discrete spectrum (by the spectral theorem).[2] It follows that D is also diagonalizable, with real eigenvalues because D is Hermitian. Now consider a local operator $\mathcal{O}$, and assume for simplicity it is an eigenvector of dilatation with dimension Δ. By finiteness of the partition function, there are a finite number of primary operators $\mathcal{O}_p$ with dimension less than or equal to Δ. Using the inner product, we may subtract off the projections of $\mathcal{O}$ onto the conformal multiplets of $\mathcal{O}_p$ to get $\mathcal{O}'$. Now suppose (for a contradiction) that $\mathcal{O}'\neq 0$. Acting on it with K_μ's, we must eventually get zero (again by finiteness of the partition function), which means there is a new primary with dimension below Δ, a contradiction. Thus $\mathcal{O}' = 0$, and $\mathcal{O}$ is a linear combination of states in the multiplets $\mathcal{O}_p$.

2.1.6 Correlation functions and conformal invariants

Conformal symmetry imposes severe restrictions on correlators. In particular, using confomal transformation we can fix almost completely 2- and 3-point functions for scalar primary fields. Consider first a two-point function $\langle\mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\rangle$ of two scalar primaries $\mathcal{O}_1$ and $\mathcal{O}_2$. Invariance under translations imposes that the 2-point function

[2] Assuming $e^{-\beta D}$ is trace-class may be too strong for some applications. Boundedness of $e^{-\beta D}$ suffices for D to be diagonalizable (with a possibly continuous spectrum). An interesting example is Liouville theory, which has a divergent partition function and continuous spectrum, but still has many properties of a sensible CFT, like an OPE.

must depend only on $x_{12} = x_1 - x_2$, and in order to satisfy scale invariance it must have the form

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\rangle = \frac{C_{12}}{|x_{12}|^{\Delta_1+\Delta_2}}. \tag{2.41}$$

where C_{12} is a constant determined by the normalization of the fields and Δ_1 and Δ_2 are the scaling dimensions of $\mathcal{O}_1$ and $\mathcal{O}_2$ respectively. For primary scalars in a CFT, the correlators must satisfy a stronger Ward identity,

$$\begin{aligned}\langle \mathcal{O}_1(x_1)\ldots\mathcal{O}_n(x_n)\rangle &= \langle (U\mathcal{O}_1(x_1)U^{-1})\cdots(U\mathcal{O}_n(x_n)U^{-1})\rangle \\ &= \Omega(x_1')^{\Delta_1}\cdots\Omega(x_n')^{\Delta_n}\langle \mathcal{O}_1(x_n')\cdots\mathcal{O}_n(x_n')\rangle. \end{aligned} \tag{2.42}$$

This requires that either $\Delta_1 = \Delta_2$ or $C_{12} = 0$. In other words,

$$\langle O_1(x_1)\mathcal{O}_2(x_2)\rangle = \frac{C\delta_{\Delta_1\Delta_2}}{x_{12}^{2\Delta_1}}, \tag{2.43}$$

Conformal invariance is also powerful enough to fix a three-point function of primary scalars, up to an overall coefficient. Indeed, it is not hard to check that the famous formula [87]

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\mathcal{O}_3(x_3)\rangle = \frac{\lambda_{123}}{x_{12}^{\Delta_1+\Delta_2-\Delta_3}x_{23}^{\Delta_2+\Delta_3-\Delta_1}x_{31}^{\Delta_3+\Delta_1-\Delta_2}}, \tag{2.44}$$

with λ_{123} constant, satisfies the Ward identity (2.42).

Even though conformal invariance is not enough to fix the 4-point functions, some considerations may be done. Using conformal transformations we can identify independent invariants on which N-point functions might depend. Ordinary translation invariance tells us that an N-point function depends not on N independent coordinates x_i, but rather only on the differences $x_i - x_j$. If we consider for simplicity scalar objects, then rotational invariance tells us that for d large enough, there is only dependence on the $N(N-1)/2$ distances $r_{ij} = |x_i - x_j|$. Now, imposing scale invariance allows dependence only on the ratios r_{ij}/r_{kl}. Finally, since under the special conformal transformation, we have

$$|x_1' - x_2'|^2 = \frac{|x_1 - x_2|^2}{(1 + 2b\cdot x_1 + b^2x_1^2)(1 + 2b\cdot x_2 + b^2x_2^2)}, \tag{2.45}$$

only cross ratios of the form

$$\frac{r_{ij}r_{kl}}{r_{ik}r_{jl}}, \tag{2.46}$$

are invariant under the full conformal group. The number of independent cross-ratios of the form (2.46), formed from N coordinates, is $N(N-3)/2$. For a four-point functions the number of conformal invariant cross ratios is 2 and they can be written as

$$u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}, \qquad v = \frac{x_{23}^2 x_{14}^2}{x_{13}^2 x_{24}^2}. \tag{2.47}$$

To understand the geometrical interpretation of such cross ratios we can use conformal transformations to fix some of the degrees of freedom in the four-point function

- move x_4 to infinity, using special conformal transformations.
- move x_1 to zero using translations.
- move x_3 to $(1, 0, \ldots, 0)$ using rotations and dilatations.
- move x_2 to $(x, y, 0, \ldots, 0)$ using rotations that leave x_3 fixed.

This procedure leaves exactly two undetermined quantities x and y, giving two independent conformal invariants. Evaluating u and v for this special configuration of points gives

$$u = z\bar{z}, \qquad v = (1-z)(1-\bar{z}), \tag{2.48}$$

where $z \equiv x + iy$.

Conformal invariance is not enough to fix four points functions because they have a non trivial dependence on the cross ratios. For a scalar ϕ with dimension Δ_ϕ, the general formula

$$\langle \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4) \rangle \quad = \quad \frac{g(u,v)}{x_{12}^{2\Delta_\phi} x_{34}^{2\Delta_\phi}} \tag{2.49}$$

satisfies the Ward identity (2.42) for any function $g(u, v)$.

Since the left-hand side of (2.49) is manifestly invariant under permutations of the points x_i, we can use this invarince to fix some consistency conditions on $g(u, v)$,

$$g(u,v) = g(u/v, 1/v) \qquad \text{(from swapping } 1 \leftrightarrow 2 \text{ or } 3 \leftrightarrow 4\text{)}, \tag{2.50}$$

$$g(u,v) = \left(\frac{u}{v}\right)^{\Delta_\phi} g(v,u) \qquad \text{(from swapping } 1 \leftrightarrow 3 \text{ or } 2 \leftrightarrow 4\text{)}. \tag{2.51}$$

We will see shortly that $g(u, v)$ is actually determined in terms of the dimensions Δ_i and three-point coefficients f_{ijk} of the theory. As we show in the rest of this thesis, (2.51) will lead to powerful constraints on the Δ_i, λ_{ijk}.

2.1.7 The Operator Product Expansion

Let us consider two scalar operators $\mathcal{O}_i(x)\mathcal{O}_j(0)$ inside a sphere. In radial quantization the path integral over the interior of the sphere yield some state on the boundary. Since, as we saw previously, every state in a CFT is a linear combination of primaries and descendants, we can decompose this state as

$$\mathcal{O}_i(x)\mathcal{O}_j(0)|0\rangle = \sum_k C_{ijk}(x,P)\mathcal{O}_k(0)|0\rangle, \tag{2.52}$$

where k runs over primary operators and $C_{ijk}(x,P)$ encodes the contributions of all conformal descendatns of the primary $\mathcal{O}_k$. As long as all other operators are outside the sphere with radius $|x|$, we can use Eq. (2.52) in the path integral. Using the state-operator correspondence, we can write

$$\mathcal{O}_i(x_1)\mathcal{O}_j(x_2) = \sum_k C_{ijk}(x_{12},\partial_2)\mathcal{O}_k(x_2). \tag{2.53}$$

This is called Operator Product Expansion (OPE). We should notice that equation (2.53) is valid inside any correlation function where the other operators $\mathcal{O}_n(x_n)$ have $|x_{2n}| \geq |x_{12}|$. This means that we can do the OPE between two operators whenever it's possible to draw any sphere that separates the two operators from all the others. In other words we could have performed radial quantization around a different point x_3, giving

$$\mathcal{O}_i(x_1)\mathcal{O}_j(x_2) = \sum_k C'_{ijk}(x_{13},x_{23},\partial_3)\mathcal{O}_k(x_3), \tag{2.54}$$

where $C'_{ijk}(x_{13},x_{23},\partial_3)$ is some other differential operator. In the previous discussion we focused on scalar operators but the considerations made are still valid for operator with spin. In this case, collectively denoting the indices of the SO(d) inner product under which the operators transforms as a,b,c, the OPE looks like

$$\mathcal{O}_i^a(x_1)\mathcal{O}_j^b(x_2) = \sum_k C_{ijkc}^{ab}(x_{12},\partial_2)\mathcal{O}_k^c(x_2). \tag{2.55}$$

As we commented before, the OPE can be used inside correlation functions. We can use this property to see how conformal invariance strongly restricts the form of the OPE. For simplicity, suppose again $\mathcal{O}_i$, $\mathcal{O}_j$, and $\mathcal{O}_k$ are scalars. Consider equation (2.53) and take the correlation function on both sides with a third operator $\mathcal{O}_k(x_3)$ (taking

$|x_{23}| \geq |x_{12}|$, so that the OPE is valid),

$$\langle \mathcal{O}_i(x_1)\mathcal{O}_j(x_2)\mathcal{O}_k(x_3)\rangle = \sum_{k'} C_{ijk'}(x_{12}, \partial_2)\langle \mathcal{O}_{k'}(x_2)\mathcal{O}_k(x_3)\rangle. \tag{2.56}$$

On the left-hand side, we have the three-point function which is fixed by conformal invariance, and is given by (2.44). On the right hand side we have a two-points function which is given by $\langle \mathcal{O}_k(x_2)\mathcal{O}_{k'}(x_3)\rangle = \delta_{kk'} x_{23}^{-2\Delta_k}$, picking an orthonormal basis so that the coefficient $C_{12} = 1$ when two operators are orthogonal. The sum then collapses to a single term, giving

$$\frac{\lambda_{ijk}}{x_{12}^{\Delta_i+\Delta_j-\Delta_k} x_{23}^{\Delta_j+\Delta_k-\Delta_i} x_{31}^{\Delta_k+\Delta_i-\Delta_j}} = C_{ijk}(x_{12}, \partial_2) x_{23}^{-2\Delta_k}. \tag{2.57}$$

This determines C_{ijk} to be proportional to λ_{ijk}, times a differential operator that depends only on the dimensions (Δ_i). The operator can be obtained by matching the small $|x_{12}|/|x_{23}|$ expansion of both sides of (2.57).

2.1.8 Conformal data

As we saw in (2.56) it is possible to use the OPE to reduce a three-point function to a sum of two-point functions. In general, we can reduce any n-point function to a sum of $n-1$-point functions using the OPE,

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\cdots\mathcal{O}_n(x_n)\rangle = \sum_k C_{12k}(x_{12}, \partial_2)\langle \mathcal{O}_k(x_2)\cdots\mathcal{O}_n(x_n)\rangle. \tag{2.58}$$

Iterating this procedure, we can reduce everything to a sum of one-point functions, which are fixed by scale invariance,

$$\langle \mathcal{O}(x)\rangle = \begin{cases} 1 & \text{if } \mathcal{O} \text{ is the unit operator,} \\ 0 & \text{otherwise.} \end{cases} \tag{2.59}$$

In principles we are able to compute correlation function using the OPE. It is easy to see from the previous equations that all these correlators are determined by dimensions Δ_i, spins, and OPE coefficients λ_{ijk} of all the operators of the theory. These set of data are called CFT data. The knowledge of this data is the final goal of the conformal bootstrap method, which will be reviewed in the next section.

2.2 Conformal bootstrap

2.2.1 Conformal blocks from the OPE

As we said in the previous section, we can use OPE to compute correlators. Applying it to four-point function of identical scalars, fixed in (2.49) by Ward identities to be Recall that Ward identities imply

$$\langle\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle = \frac{g(u,v)}{x_{12}^{\Delta_\phi}x_{34}^{\Delta_\phi}}, \tag{2.60}$$

with the cross-ratios u, v are given by (2.47). The OPE can be written as

$$\phi(x_1)\phi(x_2) = \sum_{\mathcal{O}}\lambda_{\phi\phi\mathcal{O}}C_a(x_{12},\partial_2)\mathcal{O}^a(x_2), \tag{2.61}$$

where the operator $\mathcal{O}^a$, appearing the OPE of two scalars must transform in a spin-ℓs symmetric traceless representation of SO(d).

We can now pair up the operators (12) (34) and perform the OPE between them inside the correlation functions[3]

$$\begin{aligned}
&\langle\overbrace{\phi(x_1)\phi(x_2)}\overbrace{\phi(x_3)\phi(x_4)}\rangle\\
&\quad=\sum_{\mathcal{O},\mathcal{O}'}f_{\phi\phi\mathcal{O}}f_{\phi\phi\mathcal{O}'}C_a(x_{12},\partial_2)C_b(x_{34},\partial_4)\langle\mathcal{O}^a(x_2)\mathcal{O}'^b(x_4)\rangle\\
&\quad=\sum_{\mathcal{O}}f^2_{\phi\phi\mathcal{O}}C_a(x_{12},\partial_2)C_b(x_{34},\partial_4)\frac{I^{ab}(x_{24})}{x_{24}^{2\Delta_\mathcal{O}}}\\
&\quad=\frac{1}{x_{12}^{\Delta_\phi}x_{34}^{\Delta_\phi}}\sum_{\mathcal{O}}f^2_{\phi\phi\mathcal{O}}g_{\Delta_\mathcal{O},\ell_\mathcal{O}}(x_i),
\end{aligned} \tag{2.62}$$

where

$$g_{\Delta,\ell}(x_i) \equiv x_{12}^{\Delta_\phi}x_{34}^{\Delta_\phi}C_a(x_{12},\partial_2)C_b(x_{34},\partial_4)\frac{I^{ab}(x_{24})}{x_{24}^{2\Delta}}. \tag{2.63}$$

In (2.62), we have chosen an orthonormal basis of operators and used that

$$\langle\mathcal{O}^a(x)\mathcal{O}'^b(0)\rangle = \delta_{\mathcal{O}\mathcal{O}'}\frac{I^{ab}(x)}{x^{2\Delta_\mathcal{O}}}, \tag{2.64}$$

where $I^{ab}(x)$ is a tensor which encode the vectorial structure resulting from a two-points function of vector operators.

[3]This can be done whenever we can draw any sphere separating x_1, x_2 from x_3, x_4.

The functions $g_{\Delta,\ell}(x_i)$ are called *conformal blocks*, and are only functions of the cross-ratios. Even if it is not obvious from this computations, we can write the conformal block decomposition as

$$g(u,v) \quad = \quad \sum_{\mathcal{O}} f^2_{\phi\phi\mathcal{O}} g_{\Delta_{\mathcal{O}},\ell_{\mathcal{O}}}(u,v). \tag{2.65}$$

A conformal block represents the contribution of a single conformal multiplet to a four-point function. In other words it resums the contribution of all the descendant operators, leaving only a sum over primaries.

2.2.2 Conformal blocks ad eigenfunction of the Casimir

Conformal blocks can be computed in a simple and elegant way, for a formal derivation see Dolan & Osborn [88]. Recall that the conformal algebra is isomorphic to SO$(d+1,1)$, with generators L_{ab} given by (2.14). The quadratic Casimir of the algebra is given by $C = -\frac{1}{2}L^{ab}L_{ab}$ and it acts on every state in an irreducible representation with the same eigenvalue. Namely

$$\begin{aligned} C|\mathcal{O}\rangle \quad &= \quad \lambda_{\Delta,\ell}|\mathcal{O}\rangle, \\ \lambda_{\Delta,\ell} \quad &\equiv \quad \Delta(\Delta-d)+\ell(\ell+d-2). \end{aligned} \tag{2.66}$$

Consider $\mathcal{L}_{ab,i}$, the differential operator giving the action of L_{ab} on the operator $\phi(x_i)$. The action of L_{ab} can be rewritten as

$$\begin{aligned} (\mathcal{L}_{ab,1}+\mathcal{L}_{ab,2})\phi(x_1)\phi(x_2)|0\rangle \quad &= \quad ([L_{ab},\phi(x_1)]\phi(x_2)+\phi(x_1)[L_{ab},\phi(x_2)])\,|0\rangle \\ &= \quad L_{ab}\phi(x_1)\phi(x_2)|0\rangle. \end{aligned} \tag{2.67}$$

Thus we can write the Casimir C in differential form,

$$\begin{aligned} C\phi(x_1)\phi(x_2)|0\rangle \quad &= \quad \mathcal{D}_{1,2}\phi(x_1)\phi(x_2)|0\rangle, \\ \text{where} \qquad \mathcal{D}_{1,2} \quad &\equiv \quad -\frac{1}{2}(\mathcal{L}_1^{ab}+\mathcal{L}_2^{ab})(\mathcal{L}_{ab,1}+\mathcal{L}_{ab,2}). \end{aligned} \tag{2.68}$$

Acting with the Casimir on the four-point function we find that $g_{\Delta,\ell}$ must satisfy the differential equation

$$\mathcal{D}g_{\Delta,\ell}(u,v) \quad = \quad \lambda_{\Delta,\ell}g_{\Delta,\ell}(u,v), \tag{2.69}$$

where the second-order differential operator $\mathcal{D}$ is given by

$$\begin{aligned}\mathcal{D} &= 2(z^2(1-z)\partial_z^2 - z^2\partial_z) + 2(\bar{z}^2(1-\bar{z})\partial_{\bar{z}}^2 - \bar{z}^2\partial_{\bar{z}}) \\ &\quad +2(d-2)\frac{z\bar{z}}{z-\bar{z}}((1-z)\partial_z - (1-\bar{z})\partial_{\bar{z}}).\end{aligned} \tag{2.70}$$

From equation (2.69) appears clear that conformal blocks are eigenfunctions of the Casimir of the theory. It is also manifestly clear from this derivation that conformal blocks are functions of the conformal cross-ratios u, v. In even dimensions, the Casimir equation can be solved analytically. For example, in 2d and 4d [88, 89],

$$g^{(2d)}_{\Delta,\ell}(u,v) = k_{\Delta+\ell}(z)k_{\Delta-\ell}(\bar{z}) + k_{\Delta-\ell}(z)k_{\Delta+\ell}(\bar{z}), \tag{2.71}$$

$$g^{(4d)}_{\Delta,\ell}(u,v) = \frac{z\bar{z}}{z-\bar{z}}\left(k_{\Delta+\ell}(z)k_{\Delta-\ell-2}(\bar{z}) - k_{\Delta-\ell-2}(z)k_{\Delta+\ell}(\bar{z})\right), \tag{2.72}$$

$$k_\beta(x) \equiv x^{\beta/2}{}_2F_1\left(\frac{\beta}{2}, \frac{\beta}{2}, \beta, x\right). \tag{2.73}$$

In odd dimensions, no explicit formula in terms of elementary functions is known. However the blocks can still be computed in a series expansion using the Casimir equation or alternative techniques like recursion relations.

2.2.3 Crossing Symmetry and the Conformal Bootstrap

So far, using conformal symmetry and basic principles of quantum field theories led us to some beautiful results. We defined theories abstractly, organizing operator into irreducible representations of the conformal algebra, as primary or descendants, we used symmetries to fix the form of correlation functions, with two and three points functions fixed up to constants. We also reviewed the OPE and saw how it can be used to determine n-point functions as sums of $(n-1)$-point functions. In principle, using the OPE, all correlation functions can be written in terms of CFT data Δ_i, f_{ijk}. But what if someone hands us a random set of data, does that necessarily define a consistent CFT? The answer is no, a random set of CFT data does not always define a CFT. It has to be a "good" one. Performing the OPE in different order and between different pairs of operators, we would naively say that we get different expression in terms of CFT data. But since these results are expression of the same state at the end of the day they should all agree. This means the OPE should be associative,

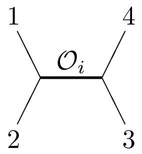
$$(\mathcal{O}_1\mathcal{O}_2)\mathcal{O}_3 = \mathcal{O}_1(\mathcal{O}_2\mathcal{O}_3), \tag{2.74}$$

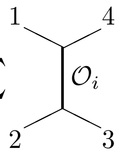
or more explicitly,

$$C_{12i}(x_{12},\partial_2)C_{i3j}(x_{23},\partial_3)\mathcal{O}_j(x_3) \quad = \quad C_{23i}(x_{23},\partial_3)C_{1ij}(x_{13},\partial_3)\mathcal{O}_j(x_3). \tag{2.75}$$

Where we have suppressed spin indices for simplicity, but the same holds with vector fields.

Equivalently, if we now insert a forth operator $\mathcal{O}_4(x_4)$ and take the correlator we find what is called *crossing symmetry equation*

$$\sum_i$$


$$=\quad\sum_i$$


. (2.76)

The left-hand side is the conformal block expansion of $\langle\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3\mathcal{O}_4\rangle$ in the $12 \leftrightarrow 34$ channel and the right-hand side is the same expansion but in the $14 \leftrightarrow 23$ channel.

The crossing equation (2.76) is a very powerful constraint on the CFT data. Using its implications to solve the theory is the goal of the conformal bootstrap. Solving this constraint can result in very complicated relations. Here we are interested in providing a simple illustrative presentation of the powerful results that can be accomplished by studying such relations. For this reason, from now on we will focus on the simplest possible case: a four-point function of identical scalars $\langle\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle$.

2.2.4 Conformal bound

In general the crossing equation (2.76) is very hard to solve. In fact, even thought it has been known for decades very little progress was made in solving it for CFTs in $d \geq 3$, and beyond special cases in $d = 2$ until ten years ago. In 2008 a breakthrough paper by Rattazzi, Rychkov, Tonni, and Vichi [5] changed the way of looking at the problem. Instead of trying to solve the crossing equation exactly, they constrained the space of solutions by studying the crossing equation geometrically. Thanks to their method we can make rigorous statements about some CFT data without necessarily computing them all. We rewrite the crossing equation as

$$\sum_{\mathcal{O}} \lambda^2_{\phi\phi\mathcal{O}} \underbrace{\left(v^{\Delta_\phi} g_{\Delta,\ell}(u,v) - u^{\Delta_\phi} g_{\Delta,\ell}(v,u)\right)}_{F^{\Delta_\phi}_{\Delta,\ell}(u,v)} \quad = \quad 0, \tag{2.77}$$

and identify the functions $F^{\Delta_\phi}_{\Delta,\ell}(u,v)$ with vectors $\vec{F}^{\Delta_\phi}_{\Delta,\ell}$ in the infinite-dimensional vector space of functions of u and v. The crossing equation (2.77) takes the form

$$\sum_{\Delta,\ell} p_{\Delta,\ell}\vec{F}^{\Delta_\phi}_{\Delta,\ell} \quad = \quad 0, \qquad p_{\Delta,\ell} = \lambda^2_{\phi\phi\mathcal{O}} =\geq 0, \tag{2.78}$$

where Δ, ℓ run over dimensions and spins of operators in the $\phi \times \phi$ OPE and we used the fact (dictated by unitarity) that $\lambda^2_{\phi\phi\mathcal{O}} \geq 0$. Equation (2.78) is saying that we have the vectors $\vec{F}_{\Delta,\ell}$ sum to zero, with positive coefficients. This is possible only for certain vectors. The way to distinguish whether equation (2.78) makes sense or not is to search for a *separating plane* α. If there exists a plane α such that all the vectors $\vec{F}^{\Delta_\phi}_{\Delta,\ell}$ lie on one side of it then the $\vec{F}^{\Delta_\phi}_{\Delta,\ell}$ cannot satisfy crossing, for any choice of coefficients $p_{\Delta,\ell}$. This suggests the following procedure for bounding CFT data. Suppose we want to look for a bound on the operator dimension. First we should make an hypothesis on which dimensions and spins Δ, ℓ appear in the $\phi \times \phi$ OPE. Then, we need to search for a linear functional α acting on all $\vec{F}^{\Delta_\phi}_{\Delta,\ell}$ satisfying the condition

$$\alpha(\vec{F}^{\Delta_\phi}_{\Delta,\ell}) \geq 0, \tag{2.79}$$

If α exists, the hypothesis is wrong[4]. Therefore we need to change the input and start the search again. Recursively using the algorithm we will find a bound on the dimension. A slight modification of this algorithm also lead to bounds on the OPE coefficients [90].

2.2.5 Numerical Techniques

The hard part in solving the algorithm described in the previous section is the middle step: finding a functional α such that

$$\alpha(\vec{F}^{\Delta_\phi}_{\Delta,\ell}) \geq 0, \qquad \text{for all } \Delta, \ell \text{ satisfying our hypothesis.} \tag{2.80}$$

The first difficulty we have is to deal with the the space of possible α since in principle it is infinite-dimensional. A clever way to fix it is to restrict our search to a finite-dimensional subspace instead of searching in the inifinite-dimensional space of all possible functionals. Of course, if we do not find any α in this subspace that satisfies the positivity constraints, we do not conclude anything about the spectrum: either no functional exists, or we just weren't searching a big enough subspace. But if we do find α in our subspace, we can immediately rule out our hypothesis about the spectrum. For numerical computations,

[4] We see this by applying α to both sides of (2.78) and finding a contradiction.

a good choice is sometimes a linear combinations of derivatives around the crossing-symmetric point $z = \bar{z} = \frac{1}{2}$,

$$\alpha(F) = \sum_{m+n\leq\Lambda} a_{mn}\partial_z^m \partial_{\bar{z}}^n F(z,\bar{z})|_{z=\bar{z}=\frac{1}{2}}, \tag{2.81}$$

where Λ is some cutoff. The functional α is now parameterized by a finite number of coefficients a_{mn}. Searching over these coefficients is something a computer can handle.[5] The second difficulty is handle the infinite number of positivity constraints (2.80) — one for each Δ, ℓ satisfying our hypothesis. To solve the inequalities (2.80), we need to encode them with a finite amount of data. For the spin we can restrict $\ell \leq \ell_{\max}$ for some large cutoff $\ell_{\max}$, find α and then go back and check afterwards that it satisfies $\alpha(F^{\Delta_\phi}_{\Delta,\ell}) \geq 0$ for $\ell > \ell_{\max}$. As for the continuous infinity of Δ's, in the literature one can find three different techniques. In the original paper on CFT bounds [5] we see that it is possible to restrict a finite set of linear inequalities for a_{mn} by discretizing Δ with a small spacing and impose a cutoff $\Delta_{\max}$. In [7, 91] they use a version of the simplex algorithm that is customized to handle continuously varying constraints. Finally, in [8, 9, 17, 92] they approximate the constraints (2.80) as positivity conditions on polynomials and use *semidefinite programming*, this last one is the approach we will take.

2.3 Four dimensional $\mathcal{N} = 2$ Superconformal field theory

In the previous sections we have presented the bootstrap program. Now we will introduce the superbootstrap as a supersymmetric generalization of the previous setup. We will therefore follow the same path of the previous sections introducing the supersymmetric formalism.

First of all we need to define the superconformal algebra. Then, we will construct its irreducible representations and analyze the shortening conditions. Finally, we will discuss some new features that come into the game when supersymmetry is present.

2.3.1 The four-dimensional $\mathcal{N} = 2$ superconformal algebra

In four spacetime dimensions one can introduce up to 16 supercharges, corresponding to $\mathcal{N} = 4$ supersymmetry. In the following we will be concerned only with the case of $\mathcal{N} = 2$ supersymmetry, where the addition of 8 supercharges promotes the conformal group

[5]Finding the optimal space of functionals is still an open problem. This subspace is not always obviously the best choice. New stronger results could be found with different choice of subspaces.

$SO(4,2)^6$ to its supersymmetric extension $SU(2,2|2)$. The maximal bosonic subgroup is just the conformal group $SO(4,2) \sim SU(2,2)$ times the R-symmetry group $SU(2)_R \times U(1)_r$. Let us first redefine the commutation relation of the conformal algebra in spinorial basis with spinor indices $\alpha, \dot{\alpha} = 1,2$. The two-form $M^{\mu\nu}$ decomposes into its self-dual ($\mathcal{M}_\alpha{}^\beta$) and anti self-dual ($\mathcal{M}^{\dot{\alpha}}{}_{\dot{\beta}}$) components of spin $(1,0)$ and $(0,1)$ respectively. The momentum and the special conformal generators can be written as bispinors $\mathcal{P}_{\alpha\dot{\alpha}}$ and $\mathcal{K}_{\alpha\dot{\alpha}}$ of spin $(1/2,1/2)$. The commutation relations in this basis read

$$\begin{aligned}
[\mathcal{M}_\alpha{}^\beta, \mathcal{M}_\gamma{}^\delta] &= \delta_\gamma{}^\beta \mathcal{M}_\alpha{}^\delta - \delta_\alpha{}^\delta \mathcal{M}_\gamma{}^\beta \ , \\
[\mathcal{M}^{\dot{\alpha}}{}_{\dot{\beta}}, \mathcal{M}^{\dot{\gamma}}{}_{\dot{\delta}}] &= \delta^{\dot{\alpha}}{}_{\dot{\delta}} \mathcal{M}^{\dot{\gamma}}{}_{\dot{\beta}} - \delta^{\dot{\gamma}}{}_{\dot{\beta}} \mathcal{M}^{\dot{\alpha}}{}_{\dot{\delta}} \ , \\
[\mathcal{M}_\alpha{}^\beta, \mathcal{P}_{\gamma\dot{\gamma}}] &= \delta_\gamma{}^\beta \mathcal{P}_{\alpha\dot{\gamma}} - \tfrac{1}{2}\delta_\alpha{}^\beta \mathcal{P}_{\gamma\dot{\gamma}} \ , \\
[\mathcal{M}^{\dot{\alpha}}{}_{\dot{\beta}}, \mathcal{P}_{\gamma\dot{\gamma}}] &= \delta^{\dot{\alpha}}{}_{\dot{\gamma}} \mathcal{P}_{\gamma\dot{\beta}} - \tfrac{1}{2}\delta^{\dot{\alpha}}{}_{\dot{\beta}} \mathcal{P}_{\gamma\dot{\gamma}} \ , \\
[\mathcal{M}_\alpha{}^\beta, \mathcal{K}^{\dot{\gamma}\gamma}] &= -\,\delta_\alpha{}^\gamma \mathcal{K}^{\dot{\gamma}\beta} + \tfrac{1}{2}\delta_\alpha{}^\beta \mathcal{K}^{\dot{\gamma}\gamma} \ , \\
[\mathcal{M}^{\dot{\alpha}}{}_{\dot{\beta}}, \mathcal{K}^{\dot{\gamma}\gamma}] &= -\,\delta^{\dot{\gamma}}{}_{\dot{\beta}} \mathcal{K}^{\dot{\alpha}\gamma} + \tfrac{1}{2}\delta^{\dot{\alpha}}{}_{\dot{\beta}} \mathcal{K}^{\dot{\gamma}\gamma} \ , \\
[\mathcal{D}, \mathcal{P}_{\alpha\dot{\alpha}}] &= \mathcal{P}_{\alpha\dot{\alpha}} \ , \\
[\mathcal{D}, \mathcal{K}^{\dot{\alpha}\alpha}] &= -\,\mathcal{K}^{\dot{\alpha}\alpha} \ , \\
[\mathcal{K}^{\dot{\alpha}\alpha}, \mathcal{P}_{\beta\dot{\beta}}] &= \delta_\beta{}^\alpha \delta^{\dot{\alpha}}{}_{\dot{\beta}} \mathcal{D} + \delta_\beta{}^\alpha \mathcal{M}^{\dot{\alpha}}{}_{\dot{\beta}} + \delta^{\dot{\alpha}}{}_{\dot{\beta}} \mathcal{M}_\beta{}^\alpha \ .
\end{aligned} \tag{2.82}$$

Additionally we have to define the R -symmetry algebra $\mathfrak{su}(2)_R \oplus \mathfrak{u}(1)_r$. For the $\mathfrak{su}(2)_R$ sector we can use creation and annihilation generators $\mathcal{R}^\pm$ and a Cartan generator $\mathcal{R}$, respecting the usual commutation relations

$$[\mathcal{R}^+, \mathcal{R}^-] = 2\mathcal{R} \ , \qquad [\mathcal{R}, \mathcal{R}^\pm] = \pm\mathcal{R}^\pm \ . \tag{2.83}$$

In Lorentz signature they also obey hermiticity conditions $(\mathcal{R}^+)^\dagger = \mathcal{R}^-$, $\mathcal{R}^\dagger = \mathcal{R}$. To close the bosonic sector we need to consider also the generator of the Abelian symmetry $\mathrm{U}(1)_r$, which we denote as r. These four generators can be arranged in the convenient basis $\mathcal{R}^\mathcal{I}{}_\mathcal{J}$, with

$$\mathcal{R}^1{}_2 = \mathcal{R}^+ \ , \qquad \mathcal{R}^2{}_1 = \mathcal{R}^- \ , \qquad \mathcal{R}^1{}_1 = \frac{1}{2}r + \mathcal{R} \ , \qquad \mathcal{R}^2{}_2 = \frac{1}{2}r - \mathcal{R} \ , \tag{2.84}$$

which obey the commutation relations

$$[\mathcal{R}^\mathcal{I}{}_\mathcal{J}, \mathcal{R}^\mathcal{K}{}_\mathcal{L}] = \delta^\mathcal{K}{}_\mathcal{J} \mathcal{R}^\mathcal{I}{}_\mathcal{L} - \delta^\mathcal{I}{}_\mathcal{L} \mathcal{R}^\mathcal{K}{}_\mathcal{J} \ . \tag{2.85}$$

Moreover, there are sixteen fermionic generators related to the Poincaré and conformal symmetry. The eight Poincaré supercharges are denoted by $\{\mathcal{Q}^\mathcal{I}_\alpha,\ \widetilde{\mathcal{Q}}_{\mathcal{I}\dot{\alpha}}\}$, whereas the

[6] Compared to the previous section, where we mostly worked with Euclidean signature, here we switch to Lorentzian one.

eight conformal supercharges are $\{\mathcal{S}^\alpha_\mathcal{J},\, \widetilde{\mathcal{S}}^{\mathcal{J}\dot\alpha}\}$. Their nonvanishing commutators read

$$\begin{aligned}
\{\mathcal{Q}^\mathcal{I}_\alpha,\, \widetilde{\mathcal{Q}}_{\mathcal{J}\dot\alpha}\} &= \delta^\mathcal{I}{}_\mathcal{J}\mathcal{P}_{\alpha\dot\alpha}\ ,\\
\{\widetilde{\mathcal{S}}^{\mathcal{I}\dot\alpha},\, \mathcal{S}_\mathcal{J}{}^\alpha\} &= \delta^\mathcal{I}{}_\mathcal{J}\mathcal{K}^{\dot\alpha\alpha}\ ,\\
\{\mathcal{Q}^\mathcal{I}_\alpha,\, \mathcal{S}_\mathcal{J}{}^\beta\} &= \tfrac{1}{2}\delta^\mathcal{I}{}_\mathcal{J}\delta_\alpha{}^\beta\mathcal{D} + \delta^\mathcal{I}{}_\mathcal{J}\mathcal{M}_\alpha{}^\beta - \delta_\alpha{}^\beta\mathcal{R}^\mathcal{I}{}_\mathcal{J}\ ,\\
\{\widetilde{\mathcal{S}}^{\mathcal{I}\dot\alpha},\, \widetilde{\mathcal{Q}}_{\mathcal{J}\dot\beta}\} &= \tfrac{1}{2}\delta^\mathcal{I}{}_\mathcal{J}\delta^{\dot\alpha}{}_{\dot\beta}\mathcal{D} + \delta^\mathcal{I}{}_\mathcal{J}\mathcal{M}^{\dot\alpha}{}_{\dot\beta} + \delta^{\dot\alpha}{}_{\dot\beta}\mathcal{R}^\mathcal{I}{}_\mathcal{J}\ .
\end{aligned} \tag{2.86}$$

The last step is to write the commutators of the supercharges with the bosonic symmetry generators:

$$[\mathcal{M}_\alpha{}^\beta, \mathcal{Q}^\mathcal{I}_\gamma] = \delta_\gamma{}^\beta\mathcal{Q}^\mathcal{I}_\alpha - \tfrac{1}{2}\delta_\alpha{}^\beta\mathcal{Q}^\mathcal{I}_\gamma\ , \qquad [\mathcal{M}^{\dot\alpha}{}_{\dot\beta}, \widetilde{\mathcal{Q}}_{\mathcal{I}\dot\delta}] = \delta^{\dot\alpha}{}_{\dot\delta}\widetilde{\mathcal{Q}}_{\mathcal{I}\dot\beta} - \tfrac{1}{2}\delta^{\dot\alpha}{}_{\dot\beta}\widetilde{\mathcal{Q}}_{\mathcal{I}\dot\delta}\ , \tag{2.87}$$

$$[\mathcal{M}_\alpha{}^\beta, \mathcal{S}_\mathcal{I}{}^\gamma] = -\delta_\alpha{}^\gamma\mathcal{S}_\mathcal{I}{}^\beta + \tfrac{1}{2}\delta_\alpha{}^\beta\mathcal{S}_\mathcal{I}{}^\gamma\ , \qquad [\mathcal{M}^{\dot\alpha}{}_{\dot\beta}, \widetilde{\mathcal{S}}^{\mathcal{I}\dot\gamma}] = -\delta^{\dot\gamma}{}_{\dot\beta}\widetilde{\mathcal{S}}^{\mathcal{I}\dot\alpha} + \tfrac{1}{2}\delta^{\dot\alpha}{}_{\dot\beta}\widetilde{\mathcal{S}}^{\mathcal{I}\dot\gamma}\ , \tag{2.88}$$

$$[\mathcal{D}, \mathcal{Q}^\mathcal{I}_\alpha] = \tfrac{1}{2}\mathcal{Q}^\mathcal{I}_\alpha\ , \qquad [\mathcal{D}, \widetilde{\mathcal{Q}}_{\mathcal{I}\dot\alpha}] = \tfrac{1}{2}\widetilde{\mathcal{Q}}_{\mathcal{I}\dot\alpha}\ , \tag{2.89}$$

$$[\mathcal{D}, \mathcal{S}_\mathcal{I}{}^\alpha] = -\tfrac{1}{2}\mathcal{S}_\mathcal{I}{}^\alpha\ , \qquad [\mathcal{D}, \widetilde{\mathcal{S}}^{\mathcal{I}\dot\alpha}] = -\tfrac{1}{2}\widetilde{\mathcal{S}}^{\mathcal{I}\dot\alpha}\ , \tag{2.90}$$

$$[\mathcal{R}^\mathcal{I}{}_\mathcal{J}, \mathcal{Q}^\mathcal{K}_\alpha] = \delta^\mathcal{K}_\mathcal{J}\mathcal{Q}^\mathcal{I}_\alpha - \frac{1}{4}\delta^\mathcal{I}_\mathcal{J}\mathcal{Q}^\mathcal{K}_\alpha\ , \qquad [\mathcal{R}^\mathcal{I}{}_\mathcal{J}, \widetilde{\mathcal{Q}}_{\mathcal{K}\dot\alpha}] = -\delta^\mathcal{I}_\mathcal{K}\widetilde{\mathcal{Q}}_{\mathcal{J}\dot\alpha} + \frac{1}{4}\delta^\mathcal{I}_\mathcal{J}\widetilde{\mathcal{Q}}_{\mathcal{K}\dot\alpha}\ , \tag{2.91}$$

$$[\mathcal{K}^{\dot\alpha\alpha}, \mathcal{Q}^\mathcal{I}_\beta] = \delta_\beta{}^\alpha\widetilde{\mathcal{S}}^{\mathcal{I}\dot\alpha}\ , \qquad [\mathcal{K}^{\dot\alpha\alpha}, \widetilde{\mathcal{Q}}_{\mathcal{I}\dot\beta}] = \delta_{\dot\beta}{}^{\dot\alpha}\mathcal{S}_\mathcal{I}{}^\alpha\ , \tag{2.92}$$

$$[\mathcal{P}_{\alpha\dot\alpha}, \mathcal{S}_\mathcal{I}{}^\beta] = -\delta_\alpha{}^\beta\widetilde{\mathcal{Q}}_{\mathcal{I}\dot\alpha}\ , \qquad [\mathcal{P}_{\alpha\dot\alpha}, \widetilde{\mathcal{S}}^{\mathcal{I}\dot\beta}] = -\delta_{\dot\alpha}{}^{\dot\beta}\mathcal{Q}^\mathcal{I}_\alpha\ . \tag{2.93}$$

This concludes the description of the $\mathcal{N}=2$ superconformal algebra. We now review its irreducible representations.

2.3.2 Irreducible representation of the $\mathcal{N}=2$ superconformal algebra

In this section we review the classification of unitary irreducible representations of $\mathfrak{su}(2,2|2)$ (*cf.* [72, 93, 94]). The main idea is similar to the construction of unitary irreducible representation of the conformal algebra. Compared to what we did in section 2.1.4 the bosonic Cartan subalgebra of $\mathfrak{su}(2,2|2)$ includes two additional generators, corresponding to the R-symmetry algebra $SU(2)_R \times U(1)_r$. Therefore a lowest weight state, also called the *superconformal primary* of the representation, is labelled by quantum numbers (Δ, j_1, j_2, r, R). In this case the supercharges $\mathcal{Q}^\mathcal{I}_\alpha$, $\widetilde{\mathcal{Q}}_{\mathcal{J}\dot\alpha}$ and the superconformal generators $\widetilde{\mathcal{S}}^{\mathcal{I}\dot\alpha}, \mathcal{S}_\mathcal{J}{}^\alpha$ act as creation and annihilation operators respectively. Therefore, a superprimary is defined by the condition

$$\mathcal{S}_\mathcal{I}{}^\alpha\,|\Delta, j_1, j_2, r, R\rangle = \widetilde{\mathcal{S}}^{\mathcal{I}\dot\alpha}\,|\Delta, j_1, j_2, r, R\rangle = 0. \tag{2.94}$$

Notice that this automatically implies that $\mathcal{K}^{\dot\alpha\alpha}$ annihilates the superprimary, making it also a conformal primary. Of course this condition needs to be supplemented by the usual highest weight conditions for the bosonic subgroup $SU(2)_{j_1} \times SU(2)_{j_2} \times SU(2)_R$. This can be done in the usual way by introducing creation annihilation basis for $SU(2)$ generators. A *generic representation* – also called a *long representation* – is obtained by the action of the eight Poincaré supercharges as well as the momentum generators and $SU(2)_R$ lowering operators on the highest weight state. Therefore the space of states is spanned by a set of generators of the form

$$\prod_{i,j,\alpha,\dot{\alpha}} (\mathcal{Q}^{\mathcal{I}}_{\alpha})^{n_{\mathcal{I}\alpha}} (\widetilde{\mathcal{Q}}_{\mathcal{K}\dot{\alpha}})^{\tilde{n}_{\mathcal{I}\dot{\alpha}}} |\Delta, j_1, j_2, r, R\rangle \qquad n_{\mathcal{I}\alpha}, \tilde{n}_{\mathcal{I}\dot{\alpha}} = 0, 1, \tag{2.95}$$

together with those generated by the action of $\mathcal{P}_{\alpha\dot{\alpha}}$ and of the $SU(2)$ lowering operators. Using commutation relations and defining property of the state in (2.95) one can show that all such states, either are annihilated by $\mathcal{K}^{\dot{\alpha}\alpha}$ or, when acted upon by $\mathcal{K}^{\dot{\alpha}\alpha}$ they generate a state which is a combination of a conformal primary and a conformal descendant. Therefore a long representation of $su(2,2|2)$ contains a number of conformal primaries equal to all the operators generated by (2.95) and the $SU(2)$ lowering operators. The number of such conformal primary states is what we will call the dimension of the representation. Of course the actual dimension of the representation is infinite since any conformal primary is accompanied by a whole tower of conformal descendants generated by the action of $\mathcal{P}_{\alpha\dot{\alpha}}$, however this nomenclature is very common in the literature and very useful to attribute a size to the multiplet. The dimension of a long representation $\mathcal{A}^{\Delta}_{R,r(j_1,j_2)}$ is given by

$$\dim \mathcal{A}^{\Delta}_{R,r(j_1,j_2)} = 256(2R+1)(2j_1+1)(2j_2+1). \tag{2.96}$$

Short representations occur when a superconformal descendant state becomes null due to a conspiracy of quantum numbers. The unitarity bounds for a superconformal primary operator are given by

$$\begin{aligned} &\Delta \geqslant \Delta_i \ , && j_i \neq 0 \ , \\ &\Delta = \Delta_i - 2 \quad \text{or} \quad \Delta \geqslant \Delta_i \ , && j_i = 0 \ , \end{aligned} \tag{2.97}$$

where we have defined

$$\Delta_1 := 2 + 2j_1 + 2R + r \ , \qquad \Delta_2 := 2 + 2j_2 + 2R - r \ . \tag{2.98}$$

Short representations occur when one or more of these bounds are saturated. The different ways in which this can happen correspond to different combinations of Poincaré supercharges that will annihilate the superconformal primary state in the representation.

There are two types of shortening conditions denoted by $\mathcal{B}$ and $\mathcal{C}$ (sometimes the $\mathcal{C}$

Shortening	Quantum Number Relations			Notation
$\varnothing$	$\Delta \geqslant \max(\Delta_1, \Delta_2)$			$\mathcal{A}^{\Delta}_{R,r(j_1,j_2)}$
$\mathcal{B}^1$	$\Delta = 2R + r$	$j_1 = 0$		$\mathcal{B}_{R,r(0,j_2)}$
$\bar{\mathcal{B}}_2$	$\Delta = 2R - r$	$j_2 = 0$		$\bar{\mathcal{B}}_{R,r(j_1,0)}$
$\mathcal{B}^1 \cap \mathcal{B}^2$	$\Delta = r$	$R = 0$		$\mathcal{E}_{r(0,j_2)}$
$\bar{\mathcal{B}}_1 \cap \bar{\mathcal{B}}_2$	$\Delta = -r$	$R = 0$		$\bar{\mathcal{E}}_{r(j_1,0)}$
$\mathcal{B}^1 \cap \bar{\mathcal{B}}_2$	$\Delta = 2R$	$j_1 = j_2 = r = 0$		$\hat{\mathcal{B}}_R$
$\mathcal{C}^1$	$\Delta = 2 + 2j_1 + 2R + r$			$\mathcal{C}_{R,r(j_1,j_2)}$
$\bar{\mathcal{C}}_2$	$\Delta = 2 + 2j_2 + 2R - r$			$\bar{\mathcal{C}}_{R,r(j_1,j_2)}$
$\mathcal{C}^1 \cap \mathcal{C}^2$	$\Delta = 2 + 2j_1 + r$	$R = 0$		$\mathcal{C}_{0,r(j_1,j_2)}$
$\bar{\mathcal{C}}_1 \cap \bar{\mathcal{C}}_2$	$\Delta = 2 + 2j_2 - r$	$R = 0$		$\bar{\mathcal{C}}_{0,r(j_1,j_2)}$
$\mathcal{C}^1 \cap \bar{\mathcal{C}}_2$	$\Delta = 2 + 2R + j_1 + j_2$	$r = j_2 - j_1$		$\hat{\mathcal{C}}_{R(j_1,j_2)}$
$\mathcal{B}^1 \cap \bar{\mathcal{C}}_2$	$\Delta = 1 + 2R + j_2$	$r = j_2 + 1$		$\mathcal{D}_{R(0,j_2)}$
$\bar{\mathcal{B}}_2 \cap \mathcal{C}^1$	$\Delta = 1 + 2R + j_1$	$-r = j_1 + 1$		$\bar{\mathcal{D}}_{R(j_1,0)}$
$\mathcal{B}^1 \cap \mathcal{B}^2 \cap \bar{\mathcal{C}}_2$	$\Delta = r = 1 + j_2$	$r = j_2 + 1$	$R = 0$	$\mathcal{D}_{0(0,j_2)}$
$\mathcal{C}^1 \cap \bar{\mathcal{B}}_1 \cap \bar{\mathcal{B}}_2$	$\Delta = -r = 1 + j_1$	$-r = j_1 + 1$	$R = 0$	$\bar{\mathcal{D}}_{0(j_1,0)}$

Table 2.1: Summary of unitary irreducible representations of the $\mathcal{N} = 2$ superconformal algebra.

type goes under the name of semishortening), each of which correspond to four different bounds depending on the R charge and chirality of the supercharges:

$$\mathcal{B}^{\mathcal{I}} \quad : \qquad \mathcal{Q}^{\mathcal{I}}_{\alpha}|\psi\rangle = 0 \ , \quad \alpha = 1,2 \ , \tag{2.99}$$

$$\bar{\mathcal{B}}_{\mathcal{I}} \quad : \qquad \widetilde{\mathcal{Q}}_{\mathcal{I}\dot{\alpha}}|\psi\rangle = 0 \ , \quad \dot{\alpha} = 1,2 \ , \tag{2.100}$$

$$\mathcal{C}^{\mathcal{I}} \quad : \qquad \begin{cases} \epsilon^{\alpha\beta}\mathcal{Q}^{\mathcal{I}}_{\alpha}|\psi\rangle_{\beta} = 0 \ , & j_1 \neq 0 \ , \\ \epsilon^{\alpha\beta}\mathcal{Q}^{\mathcal{I}}_{\alpha}\mathcal{Q}^{\mathcal{I}}_{\beta}|\psi\rangle = 0 \ , & j_1 = 0 \ , \end{cases} \tag{2.101}$$

$$\bar{\mathcal{C}}_{\mathcal{I}} \quad : \qquad \begin{cases} \epsilon^{\dot{\alpha}\dot{\beta}}\widetilde{\mathcal{Q}}_{\mathcal{I}\dot{\alpha}}|\psi\rangle_{\dot{\beta}} = 0 \ , & j_2 \neq 0 \ , \\ \epsilon^{\dot{\alpha}\dot{\beta}}\widetilde{\mathcal{Q}}_{\mathcal{I}\dot{\alpha}}\widetilde{\mathcal{Q}}_{\mathcal{I}\dot{\beta}}|\psi\rangle = 0 \ , & j_2 = 0 \ . \end{cases} \tag{2.102}$$

These shortening conditions may be combined in different ways as long as the generated bounds are compatible. We summarize all these possibilities in Table 2.1, where we also list the relations that must be satisfied by the quantum numbers of the superconformal primary in such a representation. In this thesis we follow the notation of [94], although it is worth mentioning that other notations may be found in the literature [72, 95]

In the limit where the dimension of a long representation approaches a unitarity bound, it becomes decomposable into a collection of short representations. This fact is often referred to as the existence of *recombination rules* for short representations into a long representation at the unitarity bound. The generic recombination rules are as follows,

$$\begin{aligned}
\mathcal{A}^{\Delta\to 2R+r+2+2j_1}_{R,r(j_1,j_2)} &\simeq \mathcal{C}_{R,r(j_1,j_2)} \oplus \mathcal{C}_{R+\frac{1}{2},r+\frac{1}{2}(j_1-\frac{1}{2},j_2)}\ , \\
\mathcal{A}^{\Delta\to 2R-r+2+2j_2}_{R,r(j_1,j_2)} &\simeq \bar{\mathcal{C}}_{R,r(j_1,j_2)} \oplus \bar{\mathcal{C}}_{R+\frac{1}{2},r-\frac{1}{2}(j_1,j_2-\frac{1}{2})}\ , \\
\mathcal{A}^{\Delta\to 2R+j_1+j_2+2}_{R,j_1-j_2(j_1,j_2)} &\simeq \hat{\mathcal{C}}_{R(j_1,j_2)} \oplus \hat{\mathcal{C}}_{R+\frac{1}{2}(j_1-\frac{1}{2},j_2)} \oplus \hat{\mathcal{C}}_{R+\frac{1}{2}(j_1,j_2-\frac{1}{2})} \oplus \hat{\mathcal{C}}_{R+1(j_1-\frac{1}{2},j_2-\frac{1}{2})}\ .
\end{aligned} \tag{2.103}$$

In special cases the quantum numbers of the long multiplet at threshold are such that some Lorentz quantum numbers in (2.103) would be negative and unphysical. In these cases the following exceptional recombination rules apply,

$$\begin{aligned}
\mathcal{A}^{2R+r+2}_{R,r(0,j_2)} &\simeq \mathcal{C}_{R,r(0,j_2)} \oplus \mathcal{B}_{R+1,r+\frac{1}{2}(0,j_2)}\ , \\
\mathcal{A}^{2R-r+2}_{R,r(j_1,0)} &\simeq \bar{\mathcal{C}}_{R,r(j_1,0)} \oplus \bar{\mathcal{B}}_{R+1,r-\frac{1}{2}(j_1,0)}\ , \\
\mathcal{A}^{2R+j_2+2}_{R,-j_2(0,j_2)} &\simeq \hat{\mathcal{C}}_{R(0,j_2)} \oplus \mathcal{D}_{R+1(0,j_2)} \oplus \hat{\mathcal{C}}_{R+\frac{1}{2}(0,j_2-\frac{1}{2})} \oplus \mathcal{D}_{R+\frac{3}{2}(0,j_2-\frac{1}{2})}\ , \\
\mathcal{A}^{2R+j_1+2}_{R,j_1(j_1,0)} &\simeq \hat{\mathcal{C}}_{R(j_1,0)} \oplus \hat{\mathcal{C}}_{R+\frac{1}{2}(j_1-\frac{1}{2},0)} \oplus \bar{\mathcal{D}}_{R+1(j_1,0)} \oplus \bar{\mathcal{D}}_{R+\frac{3}{2}(j_1-\frac{1}{2},0)}\ , \\
\mathcal{A}^{2R+2}_{R,0(0,0)} &\simeq \hat{\mathcal{C}}_{R(0,0)} \oplus \mathcal{D}_{R+1(0,0)} \oplus \bar{\mathcal{D}}_{R+1(0,0)} \oplus \hat{\mathcal{B}}_{R+2}\ .
\end{aligned} \tag{2.104}$$

Following the bootstrap philosophy we would like to avoid any reference to the Lagrangian formulation of $\mathcal{N}=2$ theories and recognize among the listed multiplets some of the well-known operators of $\mathcal{N}=2$ superconformal field theories. Indeed, the short representations are closely related to various nice features of theories with $\mathcal{N}=2$ supersymmetry. Here we focus on three classes of short representations

- $\mathcal{E}_r$: Half-BPS multiplets "of Coulomb type". These obey two $\mathcal{B}$-type shortening conditions of the same chirality: $\mathcal{B}^1 \cap \mathcal{B}^2$. In other terms, they are $\mathcal{N}=2$ chiral multiplets, annihilated by the action of *all* left-handed supercharges.

- $\hat{\mathcal{B}}_R$: Half-BPS multiplets "of Higgs type". These obey two $\mathcal{B}$-type shortening conditions of opposite chirality: $\mathcal{B}^1 \cap \bar{\mathcal{B}}_2$. These types of operators are sometimes called "Grassmann-analytic".

- $\hat{\mathcal{C}}_{0(j_1,j_2)}$: The stress tensor multiplet (the special case $j_1 = j_2 = 0$) and its higher spin generalizations. These obey the maximal set of semi-shortening conditions: $\mathcal{C}^1 \cap \mathcal{C}^2 \cap \bar{\mathcal{C}}_1 \cap \bar{\mathcal{C}}_2$.

The CFT data associated to these representations encodes some of the most basic physical information about an $\mathcal{N}=2$ SCFT. We now look at each in more detail, starting from the third and most universal class, which contains the stress tensor multiplet.

Stress tensor multiplet

The maximally semi-short multiplets $\hat{\mathcal{C}}_{0(j_1,j_2)}$ contain conserved tensors of spin $2+j_1+j_2$. It is a well-known fact [96, 97]that higher spin conserved currents are not allowed in an interacting CFT, and we can always assume their absence if we are interesting in non-trivial theories.

The only allowed case is $\hat{\mathcal{C}}_{0(0,0)}$, including a conserved tensor of spin two, which can be identified as the stress tensor of the theory. By definition, a *local* $\mathcal{N}=2$ SCFT will contain exactly one $\hat{\mathcal{C}}_{0(0,0)}$ multiplet. The superconformal primary of $\hat{\mathcal{C}}_{0(0,0)}$ is a scalar operator of dimension two that is invariant under all R-symmetry transformations. The multiplet also contains the conserved currents for $SU(2)_R \times U(1)_r$ symmetries. An analysis in $\mathcal{N}=2$ superspace [98] reveals that the three-point function of $\hat{\mathcal{C}}_{0(0,0)}$ multiplets involves two independent structures, whose coefficients can be parametrized in terms of the a and c anomalies. The latter are two important numbers which can be thought of as the four-dimensional generalization of the two-dimensional central charge. They are related to conformal anomalies since, for a CFT on a four-dimensional curved manifold, the expectation value of the trace of the stress tensor is parametrized by a linear combination of two Weyl invariants with coefficients a and c. For $\mathcal{N}=2$ theories a bound on the ratio of these coefficients was found in [99]

$$\frac{1}{2} \leqslant \frac{a}{c} \leqslant \frac{5}{4} \,. \tag{2.105}$$

The lower bound is saturated by the free hypermultiplet theory, and the upper bound by the free vector multiplet theory. By a generalization of the analysis of [100], one should be able to argue that these are the only $\mathcal{N}=2$ SCFTs saturating the bounds. For $\mathcal{N} \geq 3$ the superconformal algebra fixes $c=a$.

In this thesis we will not study the four-point function of the full stress tensor multiplet, but only a protected piece of this four-points functions. This allow us to derive a bound on the $c=a$ anomaly coefficient for $\mathcal{N}=3$ superconformal field theories.

Coulomb and Higgs branches

The field content of Lagrangian $\mathcal{N}=2$ field theories is organized in two types of multiplets. The vector multiplet contains a complex scalar, two Weyl fermions and a gauge boson all transforming in the adjoint representation of the gauge group (this is the on-shell content of the multiplet). On the other hand, an on-shell hypermultiplet contains two complex scalars, transforming in conjugate representations of the gauge group, and two Weyl fermions of opposite chirality. Scalar fields can acquire a vacuum expectation value (vev) without breaking Lorentz invariance and therefore one can consider the possible vacua of the theory, *i.e.* the possible vev of scalar operators in the two multiplets. For $\mathcal{N} = 2$ theory this study is particularly interesting since one has a manifold of vacua with a very interesting geometry, known as moduli space of vacua.

The moduli space of vacua can be divided into different branches. When only the scalars in the vector multiplets acquires a vev, one refers to the *Coulomb branch* of the theory, since one ends up with a collection of $U(1)$ massless gauge bosons. On the other hand for scalars in the hypermultiplets we have the *Higgs branch.* When both scalars in the vector and hypermultiplets have a non-vanishing vev we talk of *mixed branch.*

From a bootstrap perspective we are interested in extending the notion of Coulomb and Higgs branches to non-Lagrangian theories. This can be done by looking at the local operators parametrizing the two branches in a Lagrangian theory and find the multiplet they belong to in the classification of irreps in section 2.3.2. A careful analysis shows that the Coulomb branch is parametrized by vev's of chiral operators of the kind $\text{Tr}\phi^k$ with ϕ the complex scalar in the vector multiplet. These operators are neutral under $SU(2)_R$ and they have charge k under $U(1)_r$. Furthermore, they are annihilated by half of the supercharges and they can be unambiguously identified with superprimaries of a $\mathcal{E}_{r(0,0)}$ multiplet. In a similar way one can show that the operators parametrizing the Higgs branch of the theory are also 1/2 BPS, but they are neutral under $U(1)_r$ and charged under $SU(2)_R$. Therefore we can immediately identify the superprimaries of the $\hat{\mathcal{B}}_R$ multiplets as the generators of the Higgs branch.

Before looking in more details to the data associated to these two supermultiplets let us point out that a satisfactory understanding of the phenomenon of spontaneous conformal symmetry breaking has not yet been developed in the language of CFT operator algebras. In the bootstrap philosophy we should expect that local data contain information on the phases of the theory where conformal symmetry is spontaneously broken. Nevertheless, no method to extract this information is presently known. Even the basic question of whether a given CFT possesses nontrivial vacua remains out of reach. Since all known examples of vacuum manifolds in CFTs occur in supersymmetric theories, one might speculate that supersymmetry is a necessary condition for spontaneous conformal symmetry breaking.

We are now ready to examine the two BPS multiplets associated to such spontaneous symmetry breaking.

Coulomb branch data

We will refer to the data associated to $\mathcal{E}_r$ multiplets as *Coulomb branch data.* It is important to point out that most of our knowledge on the characterization of the Coulomb branch comes from the analysis of Lagrangian theories. It is therefore interesting to understand which features remain true when no Lagrangian formulation is available. One important example is the following. In the cohomology of the left-handed Poincaré supercharges, one finds a commutative ring of operators known as the *Coulomb branch chiral ring*, the elements of which can be identified with the superconformal primaries of $\mathcal{E}_r$ multiplets. For Lagrangian theories these operators are of the kind $\mathrm{Tr}\phi^r$ and one can show that moduli space of any $\mathcal{N} = 2$ SCFT is parameterized by assigning independent vevs to each of the Coulomb branch chiral ring generators. This statement is assumed to be true also for non-Lagrangian theories although no proof is available.

Another feature of the Coulomb branch chiral ring that was believed to be true for any $\mathcal{N} = 2$ theory is that such ring is freely generated, *i.e.* given a set of generators $\mathcal{E}_r$, they form a ring

$$\mathcal{E}_r\mathcal{E}_s = \sum_t c_{rst}\mathcal{E}_t, \tag{2.106}$$

and they do not satisfy any non-trivial relation. Such statement was recently disproved in a couple of papers, where the authors found counterexamples even in the realm of Lagrangian theories [101–103] see also [] for a general study of rank 1 SCFTs with non-freely generated chiral rings. Clearly, this piece of information is accessible to bootstrap techniques since it can be translated into a statement about the OPE coefficients of the $\mathcal{E}_r$ multiplets. For instance, a simple consequence of a freely generated Coulomb branch is that no $\mathcal{E}_r$ superconformal primary can square to zero in the chiral ring, so an $\mathcal{E}_{2r}$ operator must appear with nonzero coefficient in the OPE of the $\mathcal{E}_r$ with itself, which has been explored numerically [21, 27].

The number of generators of the Coulomb branch chiral ring is usually referred to as the *rank* of the theory. The set $\{r_1, \ldots r_{\mathrm{rank}}\}$ of $U(1)_r$ charges of these chiral ring generators is one of the most basic invariants of an $\mathcal{N} = 2$ SCFT. Unitarity implies $r \geqslant 1$, with $r = 1$ only in the case of the free vector multiplet, so we will always assume $r > 1$. In Lagrangian SCFTs, the r_i are all integers, but there are several non-Lagrangian models that possess $\mathcal{E}_r$ multiplets with interesting fractional values of r, and we will study in detail one such theory in chapter 4. For the familiar example of $SU(N)$ Lagrangian gauge theories the rank is simply $N - 1$.

In [104], Shapere and Tachikawa (ST) proved a remarkable formula that relates the a and c central charges to the generating r-charges $\{r_1, \dots r_{\text{rank}}\}$,

$$2a - c = \frac{1}{4}\sum_{i=1}^{\text{rank}}(2r_i - 1) \ . \tag{2.107}$$

The ST sum rule holds in all known theories which are not obtained by gauging a discrete symmetry, and it is tempting to conjecture that it is a general property of all $\mathcal{N}=2$ SCFTs, athough their derivation assumes the SCFT to be realized at a point on the moduli space a Lagrangian theory. The result can then be extended to all SCFTs connected to that class of theories by generalized S-dualities. Such dualities, in general, relate Lagrangian to non-Lagrangian theories and therefore they are a very useful tool for analyzing some general property of non-Lagrangian theories. For rank 1 theories obtained by gauging a discrete symmetry see [63] for a proposed correction to 2.107. Taking the ST sum rule as true also for rank 0, a theory with zero rank has $a/c = 1/2$, which is the lower bound in (2.105). As remarked above, there are strong reasons to believe that the only SCFT saturating this bound is the free hypermultiplet theory. An interacting SCFT of zero rank would be rather exotic, but we do not know how to rule it out with present methods. The existence of an interacting rank 0 SCFT would have dramatic consequences for the systematic classification of rank 1 SCFTs [63, 105].

A particularly interesting multiplet in the class we are analyzing is the $\mathcal{E}_2$ multiplet. Its top component, obtained by acting with four right-moving supercharges on the superconformal primary,[7] $\mathcal{O}_4 \sim \tilde{Q}^4\mathcal{E}_2$ is a scalar operator singlet under$SU(2)_R \times U(1)_r$ of exact dimension four. This means that the SCFT can be deformed by an exactly marginal operator preserving the full $\mathcal{N}=2$ supersymmetry. Furthermore, the converse is also true: any $\mathcal{N}=2$ supersymmetric exactly marginal operator $\mathcal{O}_4$ must be the top component of an $\mathcal{E}_2$ multiplet. Therefore, the number of $\mathcal{E}_2$ multiplets is equal to the (complex) dimension of the conformal manifold of the theory. For Lagrangian theories any exactly marginal operator $\mathcal{O}_4 \sim \text{Tr}(F^2 + i\tilde{F}^2)$ (where F is the Yang-Mills field strength) is dual to a complexified gauge coupling τ_i.

Higgs branch data

Although in this thesis we will be mostly concerned with Coulomb branch and stress tensor physics, for completeness we highlight some features of the Higgs branch. The $\hat{\mathcal{B}}_R$ superconformal primaries, which are also $SU(2)_R$ highest weights, form the *Higgs branch chiral ring*. In all known examples this ring is generated by a finite set of generators

[7] In an abuse of notation, we are denoting the superconformal primary with the same symbol $\mathcal{E}_2$ that represents the whole multiplet.

obeying polynomial relations. As for the Coulomb branch, we expect the algebraic variety defined by this ring to coincide with the Higgs branch of vacua.

A distinguished role is played by the $\hat{\mathcal{B}}_1$ multiplet as it encodes the information about the continuous global symmetries of the theory. Indeed, the multiplet contains a conserved current,

$$J_{\alpha\dot{\alpha}} = \epsilon^{\mathcal{J}\mathcal{K}} \mathcal{Q}^{\mathcal{I}}_{\alpha} \widetilde{\mathcal{Q}}_{\mathcal{J}\dot{\alpha}} \phi_{\mathcal{I}\mathcal{K}} \ , \tag{2.108}$$

where $\phi_{\mathcal{I}\mathcal{J}}$ is the operator of lowest dimension in the $\hat{\mathcal{B}}_1$ multiplet. It is an $SU(2)_R$ triplet and is often referred to as the *moment map* operator. The current $J_{\alpha\dot{\alpha}}$ generates a continuous global symmetry, and vice versa, if the theory enjoys a continuous global symmetry, it follows from Noether theorem that the CFT contains an associated conserved current $J_{\alpha\dot{\alpha}}$, and one can show that in an interacting $\mathcal{N}=2$ SCFT such a current must necessarily belong to a $\hat{\mathcal{B}}_1$ multiplet.

2.3.3 Chiral algebras of $\mathcal{N}=2$ SCFTs

Finally, some of the CFT data $\mathcal{N}=2$ SCFTs can be studied by determining the chiral algebra associated to the 4d SCFT. It was shown in [52] that any $\mathcal{N}\geqslant 2$ SCFT in four dimensions admits a subsector isomorphic to a two-dimensional chiral algebra. Here we only briefly describe the construction and refer to [52] for all the details. The chiral algebra is obtained by restricting operators to lie on a plane, on which we put coordinates $(z,\bar{z})$, and passing to the cohomology of a certain nilpotent supercharge $\mathbb{Q}$, that is a linear combination of a Poincaré and a conformal supercharge. The anti-holomorphic dependence is $\mathbb{Q}$ exact, and cohomology classes of local operators correspond to meromorphic operators on the two-dimensional plane on which we restricted the operators to lie. We call operators in the cohomology of said supercharge "Schur operators", since they correspond precisely to the class of operators contributing to the Schur limit of the superconformal index [72–74]. The stress tensor multiplet (denoted by $\hat{\mathcal{C}}_{0,(0,0)}$ in the notation of [94]) of an $\mathcal{N}=2$ SCFT contains one Schur operator, giving rise, in the cohomology, to a two-dimensional operator acting as the meromorphic stress tensor $T(z)$.[8] Therefore, the global $sl2$ symmetry on the plane is enhanced to the full Virasoro algebra, with the two-dimensional central charge determined in terms of the four-dimensional c anomaly coefficient,

$$c_{2d} = -12 c_{4d} \,. \tag{2.109}$$

Similarly, global symmetries of the four-dimensional theory give rise to affine Kac-Moody current algebras, with level determined from the four-dimensional flavor current central

[8]Note that in this section we are using z instead of x for the holomorphic coordinate.

charge

$$k_{2d} = -\frac{k_{4d}}{2} \,. \tag{2.110}$$

The two-dimensional affine current $J(z)$ arises from a Schur operator in the four dimensional $\hat{\mathcal{B}}_1$ multiplet, that also contains the conserved flavor symmetry current. More generally, each $\mathcal{N} = 2$ superconformal multiplet contributes at most one (non-trivial) operator to the cohomology, giving rise in two dimensions to global $sl2$ primaries.

2.4 Non Lagragian Theories

To help us to guide our thinking for the non-Lagragian theories, we will first review some aspects of Lagrangian $\mathcal{N} = 2$ field theories.

2.4.1 A recap on Lagrangian theories

To construct an $\mathcal{N} = 2$ four-dimensional Lagrangian we need to start from the building blocks which are vector multiplets, transforming in the adjoint representation of a gauge group G, and hypermultiplets (the *matter content*), transforming in some representation $\mathfrak{R}$ of G. For the theory to be microscopically well-defined, the gauge group should contain no abelian factors, so we can take G to be semi-simple,

$$G = G_1 \times G_2 \times \cdots G_n \ . \tag{2.111}$$

To each simple factor G_i is associated a complexified gauge coupling $\tau_i \in \mathbb{C}$, Im $\tau_i > 0$. For each choice of $(G, \mathfrak{R}, \{\tau_i\})$ there is a unique, classically conformally invariant $\mathcal{N} = 2$ Lagrangian. Conformal invariance and $\mathcal{N} = 2$ supersymmetry at quantum level require the matter content to be chosen so that the one loop beta functions for the gauge couplings vanish. All the possible pairs $(G, \mathfrak{R})$ from which arise a $\mathcal{N} = 2$ SCFTs can be classified by solving a combinatorial problem, whose complete solution has been described in [106]. Two of the simplest and significant examples are $\mathcal{N} = 2$ superconformal QCD, which has gauge group $G = SU(N_c)$ and $N_f = 2N_c$ hypermultiplets in the fundamental representation, and $\mathcal{N} = 4$ super Yang-Mills theory (which can be regarded as an $\mathcal{N} = 2$ SCFT), for which G is any simple group and the hypermultiplets transform in the adjoint representation. Deforming a given CFT by exactly marginal operators one can realize a space of theories which is called conformal manifold. Usually one refers to the conformal manifold of an $\mathcal{N} = 2$ SCFT as the submanifold of the full conformal manifold where $\mathcal{N} = 2$ supersymmetry is preserved. For Lagrangian theories, this submanifold coincides with the space of gauge couplings $\{\tau_i\}$, up to the discrete identifications induced by generalized S-dualities.

2.4.2 Isolated SCFTs

In the last years there has been growing evidence that Lagrangian SCFTs represent only a small subsector of all SCFTs. A wealth of strongly coupled $\mathcal{N} = 2$ SCFTs with no marginal deformations are known to exist – by virtue of being isolated, they cannot have a conventional Lagrangian description. However, recently $\mathcal{N} = 1$ RG flows, between $\mathcal{N} = 1$ Lagrangian theories in the UV and some of the non-Lagrangian theories in the IR, have been found. There are several arguments in favour of the existence of such theories and, although no perturbative approach is applicable many of their features can be extracted by alternative approaches. One example is that of generalized S-dualitiy [107], which establishes a correspondence between a Lagrangian theory with infinite marginal coupling and a weakly gauged description which involves one or more isolated SCFTs and a set of vector multiplets. In this dual description the gauging procedure is described in what we may call a quasi-Lagrangian fashion: the isolated SCFT is treated as a non-Lagrangian black box with a certain flavor symmetry, which is allowed to talk to the vector multiplets through minimal coupling of the conserved flavor current of the isolated SCFT to the gauge field.

The web of generalized S-dualities for large classes of theories can be elegantly described through the class $\mathcal{S}$ constructions of [108, 109]. These theories arise from twisted compactifications of the six-dimensional $(2, 0)$ theories on a punctured Riemann surface, with additional discrete data specified at each puncture. The marginal deformations of the four-dimensional theory correspond to the moduli of the Riemann surface, and weakly gauged theories arise if the Riemann surface degenerates. In this picture the isolated theories correspond to three-punctured spheres which have no continuous moduli. They do, however, depend on the discrete data at the three punctures as well as on a choice of $\mathfrak{g} \in \{A_n, D_n, E_n\}$ for the six-dimensional ancestor theory. In this way several infinite classes of isolated theories can be constructed. A few of these theories turn out to be equal to theories of free hypermultiplets, but most cases do not have a known Lagrangian description.

In order to describe the currently known landscape of $\mathcal{N} = 2$ SCFTs, then, it is clearly not sufficient to only consider Lagrangians with hypermultiplets and vector multiplets. We can certainly accommodate any theory in a framework which takes as fundamental the spectrum and algebra of local operators. This is the basic starting point for the bootstrap approach that we take in this thesis. Before entering the technical details of such approach, let us describe in more detail two non-Lagrangian theories which are of interest for the following chapters of this thesis.

2.4.3 The (A_1, A_2) Argyres-Douglas theory

One particular class of theories which does not have a Lagrangian description are the Argyres-Douglas theories [70, 71] which were first obtained by going to a special point on the Coulomb branch of an $\mathcal{N} = 2$ theory in which several BPS particles, with mutually non-local charges, become massless simultaneously. Among the various Argyres-Douglas models a particular class appears to be the "simplest", that is, the (A_1, A_{2n}) theories obtained in [110]. They are rank n theories, *i.e.* , their Coulomb branches have complex dimension n, and have trivial Higgs branches [111].

In this thesis we focus on the $n = 1$ case of the (A_1, A_{2n}) Argyres-Douglas family, which is of rank one and thus the simplest in this class. In fact, among all interacting rank one SCFTs obtained through the systematic classification of [61–63, 105, 112], it corresponds to the theory with the smallest a-anomaly coefficient,which provides a measure of degrees of freedom in CFT [113]. This theory was originally obtained on the Coulomb branch of a pure SU(3) gauge theory, or alternatively from an SU(2) gauge theory with a single hypermultiplet [70, 71]. There is no standard nomenclature for this model, and in this work we follow the (A_1, A_2) naming convention based on its BPS quiver [114]. To emphasize its original construction it was also named $AD_{N_f=0}(\mathrm{SU}(3))$ and $AD_{N_f=1}(\mathrm{SU}(2))$ in [115]. Finally, it can also be realized in F-theory, on a single D3-brane probing a codimension one singularity of type H_0 where the dilaton is constant [116, 117]. For this reason the theory is often referred to as the H_0 theory.

The (A_1, A_2) theory is an intrinsically interacting isolated fixed point with no marginal coupling: it does not have a conformal manifold nor a weak-coupling expansion. Recently, there has been progress in obtaining RG flows from $\mathcal{N} = 1$ Lagrangian theories that end on Argyres-Douglas SCFTs in the IR [75, 118–121], and in particular the (A_1, A_2) theory can be obtained starting from a deformation of SU(2) $\mathcal{N} = 2$ superconformal QCD. This allows for the computation of some information about the theory, such as the superconformal index. The values of the a- and c-anomaly coefficients are known, first obtained through a holographic computation in [59], and the dimension of the single generator of the Coulomb branch chiral ring is also known.

The chiral algebra of the (A_1, A_2) theory is conjectured to be the $\mathcal{M}_{2,5}$ minimal model [76, 77], also known as the Yang-Lee edge singularity. The first indication of this conjecture comes from the central charge. The basic chiral algebra dictionary states that $4d$ and $2d$ central charges are related by $c_{2d} = -12c_{4d}$; for the (A_1, A_2) theory this gives

$c_{2d} = -\frac{22}{5}$, which is indeed the correct value for the Yang-Lee model. Thanks to the interplay between $2d$ and $4d$ descriptions one can actually prove that $c_{4d} \geqslant \frac{11}{30}$ for any interacting $\mathcal{N} = 2$ SCFTs [68].[9] This bound is saturated by the (A_1, A_2) theory, which in some sense sits at the origin of the $\mathcal{N} = 2$ theory space, as all other interacting SCFTs must have higher values of the c-central charge. Another entry of the chiral algebra dictionary states that the Schur limit of the superconformal index [72–74] should match the $2d$ vacuum character. For the Yang-Lee minimal model the vacuum character seems to match the expression for the Schur index proposed in [122], while the character of the non-vacuum module has been matched to the index in the presence of a surface defect [123]. Using the Yang-Lee model we can compute three-point functions of Schur operators, *i.e.* , the operators captured by the chiral algebra, modulo ambiguities when lifting operators from the $2d$ chiral algebra to representations of the four-dimensional superconformal algebra. A conjectured prescription on how to lift these ambiguities for the (A_1, A_2) Argyres-Douglas theory has been put forward in [124]. Coulomb branch chiral ring operators, however, are not captured by the chiral algebra.

The features described above suggest that the (A_1, A_2) theory might be the simplest $\mathcal{N} = 2$ interacting SCFT. Despite this, apart from the aforementioned quantities not much is known about the CFT data of this theory. The fact that the theory has a Coulomb branch operator of relatively low dimension, $r_0 = \frac{6}{5}$, and a very low c central charge, makes it well suited for the bootstrap program. Hence, in chapter 4 we will see how is possible to use modern bootstrap tools in order to access *non-protected* dynamical data. First we will study the two operators that are guaranteed to be present: the stress tensor, and the $\mathcal{N} = 2$ chiral operator that parametrizes the Coulomb branch. Since the superconformal blocks of the former remain elusive we focus on the latter. A preliminary analysis of chiral correlators was already started in [21, 27], however the main goal of those papers was the exploration of the landscape of $\mathcal{N} = 2$ SCFTs through their Coulomb branch data. In this thesis we will instead focus exclusively on the (A_1, A_2) theory, and attempt to "zoom in" on it by studying an $\mathcal{N} = 2$ chiral operator of fixed dimension $r_0 = \frac{6}{5}$.[10]

[9] Similar bounds can be obtained for $\mathcal{N} = 3$ [49] and $\mathcal{N} = 4$ [22, 48] theories, and also for $\mathcal{N} = 2$ theories with flavor symmetries [52, 69, 77].

[10] There are other known SCFTs with a Coulomb branch chiral ring operator of dimension $r_0 = \frac{6}{5}$, in particular higher rank theories whose lowest dimensional Coulomb branch generator has this dimension are obtained in F-theory by probing a singularity of type H_0 with N D3-branes [116, 117]. However, these theories have larger values of the c-anomaly coefficient [59], and by fixing the central charge we can focus on the (A_1, A_2) theory.

2.4.4 Four dimensional $\mathcal{N} = 3$ Superconformal field theory

The study of $\mathcal{N} = 3$ SCFT does not follow the same path of the two siblings cases $\mathcal{N} = 2$ and $\mathcal{N} = 4$, due to CPT invariance in fact the Lagrangian formulation of any $\mathcal{N} = 3$ theory becomes automatically $\mathcal{N} = 4$. Given the growing evidence for the existence of non-Lagrangian superconformal field theories, however, numerous papers have recently revisited the status of $\mathcal{N} = 3$ SCFTs. In [58] several of their properties have been studied. In particular it was found that the a and c anomaly coefficients are always the same and that pure $\mathcal{N} = 3$ theories (*i.e.* theories whose symmetry does not enhance to $\mathcal{N} = 4$) have no marginal deformations and therefore they are always isolated. In contrast with the most familiar $\mathcal{N} = 2$ theories, pure $\mathcal{N} = 3$ SCFTs cannot have a flavor symmetry that is not an R-symmetry. Moreover, since the only possible free multiplet of an $\mathcal{N} = 3$ SCFT is a vector multiplet, the low energy theory at a generic point on the moduli space must involve vector multiplets, and the types of short multiplets whose expectations values can parametrize such branches were analyzed in [58].

When we decompose a $\mathcal{N} = 3$ vector multiplet in $\mathcal{N} = 2$ multiplets, it must contains both an $\mathcal{N} = 2$ vector and hyper multiplet. This implies that the theories possess both $\mathcal{N} = 2$ Higgs and Coulomb branches that are rotated by $\mathcal{N} = 3$. The first evidence for $\mathcal{N} = 3$ theories were presented in [57]. They were found by studying N D3-branes in the presence of an S-fold plane, which is a generalization of the standard orientifold construction that also includes the S-duality group. In [59] the classification of different variants of $\mathcal{N} = 3$ preserving S-folds was done and this lead to additional $\mathcal{N} = 3$ SCFTs to be discovered. Another generalization was presented in [60] where, in addition to the S-duality group also T-duality was included in the orientifold construction. This background is known as a U-fold, and the study of M5-branes on this background leads to $\mathcal{N} = 3$ theories associated with the exceptional $(2, 0)$ theories. By studying Coulomb branch geometries of rank one $\mathcal{N} = 2$ SCFTs (*i.e.* , with a one complex dimensional Coulomb branch) [61, 62, 105, 112] has recovered the known $\mathcal{N} = 3$ SCFTs, and also led to new ones [61, 63].

Some of these theories are obtained by starting from $\mathcal{N} = 4$ SYM with gauge group U(1) or SU(2) and gauging discrete symmetries, while others correspond to genuine $\mathcal{N} = 3$ SCFTs not obtained by discrete gauging. However, as emphasized in [59, 63] the local dynamics of the theory on $\mathbb{R}^4$ does not change even when the theory is obtained by a discrete gauging, only the spectrum of local and non-local operators are different. In particular, the central charges and correlation functions remain the same.

Some of the theories constructed in [59], labeled by the number N of D3-branes and by integers k, ℓ associated to the S-fold, have enhanced $\mathcal{N} = 4$ supersymmetry, or arise as discretely gauged versions of $\mathcal{N} = 4$. In the class of these theories the non-trivial $\mathcal{N} = 3$ SCFT with the smallest central charge are labeled by $N = 1$ and $\ell = k = 3$, with central charge given by $\frac{15}{12}$, which corresponds to a rank one theory with Coulomb branch parameter of scaling dimension three. As shown in [58] Coulomb branch operators of $\mathcal{N} = 3$ theories must have integer dimensions. But since theories with a Coulomb branch generator of dimension one or two enhance to $\mathcal{N} = 4$, it follows that dimension three is the smallest dimension a Coulomb branch can have in genuine $\mathcal{N} = 3$ theory. This theory could indeed correspond to the "minimal" $\mathcal{N} = 3$ SCFT. Higher rank versions of this minimal theory can be obtained by increasing the number of D3-branes. More generally, the rank N theories with $k = \ell$, are not obtained from others by discrete gauging, and have an N dimensional Coulomb branch.

As said before $\mathcal{N} = 3$ SCFTs are hard to study by standard field theory approach because they have no relevant or marginal deformations. Recent progress in understanding $\mathcal{N} = 3$ theories includes [64–67]. In this thesis we study the four-point function of the stress-tensor multiplet for any interacting $\mathcal{N} = 3$ superconformal field theory and we find a new analytic bound on the c anomaly coefficient. For completeness, in this review we also include a list of the irreducible representations of the $\mathcal{N} = 3$ superconformal algebra, which are crucial for the bootstrap approach.

2.4.5 Unitary representations of the $\mathcal{N} = 3$ superconformal algebra

Let us summarize now the unitary representations of the four-dimensional $\mathcal{N} = 3$ superconformal algebra [58, 72, 93, 95, 125, 126]. We list the possible representations in table 2.2, following the naming conventions of [35].

In the first column we listed the name of the representation. The second column list the quantum numbers of the superconformal primary, denoted by $(j_1, j_2)^{\Delta}_{[R_1,R_2],r}$, where (j_1, j_2) are the spins, as before Δ is the conformal dimension, (R_1, R_2) are the Dynkin labels of $SU(3)_R$ and r is the $U(1)_r$ R-charge. The stress tensor of an $\mathcal{N} = 3$ SCFT belongs in the $\hat{B}_[1, 1]$ multiplet and it will be the subject of section 3.3. To gain intuition on the various multiplets listed in table 2.2 we will always decompose them in the $\mathcal{N} = 2$ ones reviewed above.

2.5 Superconformal bootstrap

The application of the bootstrap to superconformal field theories follows the same philosophy of the ordinary conformal bootstrap, although it uses supersymmetry to further

Name	Superconformal primary	Conditions
$\mathcal{A}^{\Delta}_{[R_1,R_2],r,(j_1,j_2)}$	$(j_1,j_2)^{\Delta}_{[R_1,R_2],r}$	$\Delta > 2+2j_1+\frac{2}{3}(2R_1+R_2)-\frac{r}{6}$ $\Delta > 2+\bar{2}\jmath+\frac{2}{3}(R_1+2R_2)+\frac{r}{6}$
$\mathcal{B}_{[R_1,R_2],r,j_2}$	$(0,j_2)^{\frac{2}{3}(2R_1+R_2)-\frac{r}{6}}_{[R_1,R_2],r}$	$-6j_2+2(R_1-R_2)-6>r$
$\bar{\mathcal{B}}_{[R_1,R_2],r,j}$	$(j_1,0)^{\frac{2}{3}(R_1+2R_2)+\frac{r}{6}}_{[R_1,R_2],r}$	$6j_1+2(R_1-R_2)+6<r$
$\hat{\mathcal{B}}_{[R_1,R_2]}$	$(0,0)^{R_1+R_2}_{[R_1,R_2],2(R_1-R_2)}$	
$\mathcal{C}_{[R_1,R_2],r,(j_1,j_1)}$	$(j_1 j_2)^{2+2j_1+\frac{2}{3}(2R_1+R_2)-\frac{r}{6}}_{[R_1,R_2],r}$	$6(j_1-j_2)+2(R_1-R_2)>r$
$\bar{\mathcal{C}}_{[R_1,R_2],r,(j_1,j_2)}$	$(j_1,j_2)^{2+2j_2+\frac{2}{3}(R_1+2R_2)+\frac{r}{6}}_{[R_1,R_2],r}$	$6(j_1-j_2)+2(R_1-R_2)<r$
$\hat{\mathcal{C}}_{[R_1,R_2],(j_1,j_2)}$	$(j_1,j_2)^{2+j_1+j_2+R_1+R_2}_{[R_1,R_2],6(j_1-j_2)+2(R_1-R_2)}$	
$\mathcal{D}_{[R_1,R_2],j_2}$	$(0,j_2)^{1+j_2+R_1+R_2}_{[R_1,R_2],2(R_1-R_2)-6-6j_2}$	
$\bar{\mathcal{D}}_{[R_1,R_2],j_1,}$	$(j_1,0)^{1+j_1,+R_1+R_2}_{[R_1,R_2],2(R_1-R_2)+6+6j_1,}$	

TABLE 2.2: Unitary representations of $\mathcal{N}=3$. The second column shows the charges of the superconformal primary in the representation, while the tird one lists the conditions that the charges have to obey. The A_2, respectively $\bar{A}_2$ shortening cases are obtained by putting respectively $j_1=0$ and $j_2=0$. This changes the null states drastically, but not the labels.

constraint the space of allowed conformal data. The idea is to consider four point functions of local operators and use the crossing equation to constrain the dimensions and OPE coefficients of the superprimary operators. While most of the research on the subject in the first years has focused on four point functions of protected multiplets, in Chapter 3 of this thesis we will describe also the application to long multiplets, thus exploiting the full power of superconformal symmetry. On the other hand, in Chapter 4 we will move back to short multiplets and apply the superconformal bootstrap to a specific theory. In this section we just want to describe the general setup and, to do that, we focus on the four-point functions of a particular class of short multiplets, which we will use in Chapter 4. We postpone the discussion of the long multiplet bootstrap to chapter 3.

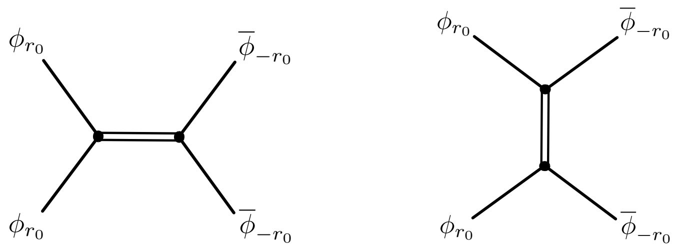
2.5.1 OPE decomposition and crossing symmetry

The operators we consider in this section are the Coulomb branch operators $\mathcal{E}_r$ and $\bar{\mathcal{E}}_r$, which we introduced in section 2.3.2. We denote the superconformal primary of the chiral (anti-chiral) multiplets $\mathcal{E}_r$ ($\bar{\mathcal{E}}_r$) by ϕ_r ($\bar{\phi}_{-r}$), where r is the $U(1)_r$ charge of the superconformal primary, with unitarity requiring $r \geqslant 1$ ($-r \geqslant 1$). The dimensions of the superconformal primaries ϕ_r ($\bar{\phi}_{-r}$) are fixed in terms of their $U(1)_r$ charges by $\Delta_\phi = r$ ($\Delta_{\bar{\phi}} = -r$).

The numerical bootstrap program applied to chiral correlators was considered in [21, 27] for the case of two identical operators, and their conjugates, and in [27] for two distinct operators, and their conjugates. Here we review the setup for two identical operators $\mathcal{E}_r$, and conjugates. We will be investigating the four-point function

$$\left\langle \phi_{r_0}(x_1)\bar{\phi}_{-r_0}(x_2)\phi_{r_0}(x_3)\bar{\phi}_{-r_0}(x_4)\right\rangle \,. \tag{2.112}$$

We should determine what operators can be exchanged in each channel and find the corresponding superconformal blocks. After obtaining the superconformal blocks in all channels we have to work out the constraints imposed by crossing symmetry. Of all the short and semi-short multiplets appearing in the partial wave expansion, the only coefficient we are able to fix is that of the stress tensor, which must appear in the $\phi_{r_0} \times \bar{\phi}_{-r_0}$ OPE.



(A) Chiral OPE channel. (B) Nonchiral OPE channel.

FIGURE 2.1: The two inequivalent OPE channels for the $\mathcal{E}_r$ four-point function.

There are two qualitatively different OPE channels to consider depending on whether we take the non-chiral OPE $\phi_{r_0}(x_1) \times \bar{\phi}_{-r_0}(x_2)$ or the chiral OPE $\phi_{r_0}(x_1) \times \phi_{r_0}(x_2)$ (see Fig. 2.1). We now describe the various selection rules for superconformal representations appearing in these two channels, as well as the corresponding superconformal blocks.

2.5.1.1 Non-chiral channel

We begin with the selection rules for the non-chiral OPE. The problem simplifies due to the fact that an operator $\mathcal{O}(x_3)$ can participate in a non-zero three-point function $\langle \phi_{r_0}(x_1)\bar{\phi}_{-r_0}(x_2)\mathcal{O}(x_3)\rangle$ only if the superconformal primary of the multiplet to which it belongs also participates in such a non-vanishing three-point function (this result was derived in [21]). Selection rules for the $U(1)_r$ and $SU(2)_R$ impose that any such operator $\mathcal{O}(x_3)$ is an $SU(2)_R$ singlet and have $r_{\mathcal{O}} = 0$. Furthermore, they must have $j_1 = j_2 =: j$. Taken together, these conditions imply the following selection rule

$$\phi_r \times \bar{\phi}_{-r} \sim \mathbf{1} + \hat{\mathcal{C}}_{0(j,j)} + \mathcal{A}^{\Delta>2j+2}_{0,0(j,j)} \,. \tag{2.113}$$

Here the $\hat{\mathcal{C}}_{0(j,j)}$ multiplets include conserved currents of spin $2j+2$, which for $j>0$ are absent in interacting theories [96, 97] and thus we will set them to zero. The multiplet $\hat{\mathcal{C}}_{0(0,0)}$ corresponds to the superconformal multiplet that contains the stress tensor. By an abuse of notation we will often replace the subscript (j,j) by ℓ, with $\ell = 2j$. The superconformal block decomposition in this channel can be written as

$$\langle \phi_r(x_1)\bar{\phi}_{-r}(x_2)\phi_r(x_3)\bar{\phi}_{-r}(x_4)\rangle = \frac{1}{x_{12}^{2\Delta_\phi}x_{34}^{2\Delta_\phi}} \sum_{\mathcal{O}_{\Delta,\ell}} |\lambda_{\phi\bar{\phi}\mathcal{O}}|^2 \mathcal{G}_{\Delta,\ell}(z,\bar{z})\,, \tag{2.114}$$

where the superblocks $\mathcal{G}_{\Delta,\ell}(z,\bar{z})$, capturing the supersymmetric multiplets being exchanged in (2.113), were computed in [40],

$$\mathcal{G}_{\Delta,\ell}(z,\bar{z}) = \frac{1}{z\bar{z}} g^{2,2}_{\Delta+2,\ell}(z,\bar{z})\,. \tag{2.115}$$

The function $g^{\Delta_{12},\Delta_{34}}_{\Delta,\ell}(z,\bar{z})$ is the standard bosonic block for the decomposition of a correlation function with four distinct operators, defined in (B.1). Although not immediately obvious, the bosonic block with shifted arguments in (2.115) can be written as a finite sum of $g^{0,0}_{\Delta,\ell}(z,\bar{z})$ blocks, as expected from supersymmetry. The block reduces to 1 for the identity exchange, *i.e.* , $\Delta = \ell = 0$.

The stress-tensor multiplet $\hat{\mathcal{C}}_{0(0,0)}$ corresponds to $\Delta = 2$, $\ell = 0$ in (2.115), and its OPE coefficient can be fixed using the Ward identities (see for example [21]):

$$\left|\lambda_{\phi\bar{\phi}\mathcal{O}_{\Delta=2,\ell=0}}\right|^2 = \frac{\Delta_\phi^2}{6c}\,, \tag{2.116}$$

while long multiplets $\mathcal{A}^{\Delta>\ell+2}_{0,0,\ell}$ contribute as (2.115) with $\Delta > \ell+2$.

When writing the crossing equations it will be useful to have the block expansion with a slightly different ordering

$$\langle \bar{\phi}_{-r}(x_1)\phi_r(x_2)\phi_r(x_3)\bar{\phi}_{-r}(x_4)\rangle = \frac{1}{x_{12}^{2\Delta_\phi}x_{34}^{2\Delta_\phi}} \sum_{\mathcal{O}_{\Delta,\ell}} (-1)^\ell |\lambda_{\phi\bar{\phi}\mathcal{O}}|^2 \tilde{\mathcal{G}}_{\Delta,\ell}(z,\bar{z})\,, \tag{2.117}$$

where the function $\tilde{\mathcal{G}}_{\Delta,\ell}(z,\bar{z})(z,\bar{z})$ is defined as

$$\tilde{\mathcal{G}}_{\Delta,\ell}(z,\bar{z}) = \frac{1}{z\bar{z}} g^{2,-2}_{\Delta+2,\ell}(z,\bar{z})\,. \tag{2.118}$$

2.5.1.2 Chiral channel

We now turn to the chiral OPE. In this case only $SU(2)_R$ singlets with $r_{\mathcal{O}} = 2r_0$ and $j_1 = j_2 =: j$ are allowed, and the spin $\ell := 2j$ is required to be even because we are considering the OPE of two identical scalars. The complete selection rules for this channel are worked out in [21] and their result read

$$\phi_r \times \phi_r \sim \mathcal{E}_{2r} + \mathcal{C}_{0,2r-1(j-1,j)} + \mathcal{B}_{1,2r-1(0,0)} + \mathcal{C}_{\frac{1}{2},2r-\frac{3}{2}(j-\frac{1}{2},j)} + \mathcal{A}^{\Delta>2+2r+2j}_{0,2r-2(j,j)}\,, \quad (2.119)$$

where we already imposed Bose symmetry, and we assumed ϕ_r to be above the unitarity bound, *i.e.* , $r > 1$. If one considers different operators, or if $r = 1$, additional multiplets are allowed to appear (see *e.g.* [27]). Chirality of ϕ_r requires each supermultiplet contributes with a single conformal family, and therefore the superblock decomposition contains only bosonic blocks:

$$\langle \phi_r(x_1)\phi_r(x_2)\bar{\phi}_{-r}(x_3)\bar{\phi}_{-r}(x_4)\rangle = \sum_{\Delta,\ell} |\lambda_{\phi\phi\mathcal{O}_{\Delta,\ell}}|^2 g^{0,0}_{\Delta,\ell}(z,\bar{z})\,. \quad (2.120)$$

Since we are considering the OPE between two identical ϕ_r multiplets, Bose symmetry requires the above sum to include only even ℓ. The precise contribution from each of the multiplets appearing in (2.119) is the following

$$\begin{array}{lcll}
\mathcal{A}_{0,2r-2(j,j)} & : & g^{0,0}_{\Delta,\ell}\,, & \Delta > 2+2r+\ell\,,\ \ell \text{ even}\,, \\
\mathcal{C}_{\frac{1}{2},2r-\frac{3}{2}(j-\frac{1}{2},j)} & : & g^{0,0}_{\Delta=2r+\ell+2,\ell}\,, & \ell \geqslant 2\,,\ \ell \text{ even}\,, \\
\mathcal{B}_{1,2r-1(0,0)} & : & g^{0,0}_{\Delta=2r+2,\ell=0}\,, & \\
\mathcal{C}_{0,2r-1(j-1,j)} & : & g^{0,0}_{\Delta=2r+\ell,\ell}\,, & \ell \geqslant 2\,,\ \ell \text{ even}\,, \\
\mathcal{E}_{2r} & : & g^{0,0}_{\Delta=2r,\ell=0}\,, &
\end{array} \quad (2.121)$$

where $\ell = 2j$ is even (see [27] for the contribution in the case of different operators). While the short multiplets being exchanged in this channel have their dimensions fixed by supersymmetry, their OPE coefficients are not known. In fact, it is not even guaranteed all these multiplets are present as their physical meaning is not as clear as the short multiplets exchanged in the non-chiral channel. The $\mathcal{E}_{2r}$ multiplet corresponds to an operator in the Coulomb branch chiral ring and therefore must be present, although the value of its OPE coefficient is not known. Note that the contribution of the two short operators $\mathcal{B}_{1,2r-1(0,0)}$ and $\mathcal{C}_{\frac{1}{2},2r-\frac{3}{2}(j-\frac{1}{2},j)}$ is identical to that of a long multiplet saturating the unitarity bound $\Delta = 2+2r+\ell$, as follows directly from the decomposition of the long multiplet when hitting the unitarity bound [94]. On the other hand, the contribution of the short multiplets $\mathcal{E}_{2r}$ and $\mathcal{C}_{0,2r-1(j-1,j)}$ is isolated from the continuous spectrum of long operators by a gap; this will be relevant for the numerical analysis of Chapter 4.

2.5.1.3 Crossing symmetry

Starting from the previous expressions for the chiral and antichiral OPE it is rather easy to write down the crossing equations

$$(z\bar{z})^{\Delta_\phi} \sum_{\mathcal{O}\in\phi\phi} |\lambda_{\phi\phi\mathcal{O}}|^2 g^{0,0}_{\Delta_\mathcal{O},\ell_\mathcal{O}}(1-z,1-\bar{z}) \\ = ((1-z)(1-\bar{z}))^{\Delta_\phi} \sum_{\mathcal{O}\in\phi\bar{\phi}} |\lambda_{\phi\bar{\phi}\mathcal{O}}|^2 (-1)^\ell \tilde{\mathcal{G}}_{\Delta_\mathcal{O},\ell_\mathcal{O}}(z,\bar{z}) \,, \tag{2.122a}$$

$$((1-z)(1-\bar{z}))^{\Delta_\phi} \sum_{\mathcal{O}\in\phi\bar{\phi}} |\lambda_{\phi\bar{\phi}\mathcal{O}}|^2 \mathcal{G}_{\Delta_\mathcal{O},\ell_\mathcal{O}}(z,\bar{z}) \\ = (z\bar{z})^{\Delta_\phi} \sum_{\mathcal{O}\in\phi\bar{\phi}} |\lambda_{\phi\bar{\phi}\mathcal{O}}|^2 \mathcal{G}_{\Delta_\mathcal{O},\ell_\mathcal{O}}(1-z,1-\bar{z}) \,. \tag{2.122b}$$

The full system of equations comprises (2.122) together with equation (2.122a) with $z \to 1-z$ and $\bar{z} \to 1-\bar{z}$. These are collected in a form suitable for the numerical implementation in (B.3).

2.5.1.4 Numerical bootstrap

In this short section we give details of the numerical implementation that will be necessary to understand the results of Chapter 4. This follows the same idea described in section 2.2.5. Schematically, the final form of the crossing equations given in (B.3) is

$$|\lambda_{\mathcal{O}^\star}|^2 \vec{V}_{\mathcal{O}^\star} + \sum_{\mathcal{O}} |\lambda_\mathcal{O}|^2 \vec{V}_\mathcal{O} + \vec{V}_{\text{fixed}} = 0 \,. \tag{2.123}$$

Here $\mathcal{O}^\star$ is a superconformal multiplet whose OPE coefficient we would like to bound numerically. The term $\vec{V}_{\text{fixed}}$ encodes the contribution of the identity, or of the identity and stress tensor if we fix the central charge c, and is given in (B.5). OPE coefficient bounds are obtained using the SDPB solver of [9] to solve the following optimization problem

$$\begin{aligned} &\vec{\Psi}\cdot\vec{V}_\mathcal{O} \geqslant 0 \,, \qquad \forall\, \mathcal{O} \in \{\text{trial spectrum}\}\,, \\ &\vec{\Psi}\cdot\vec{V}_{\mathcal{O}^\star} = \pm 1 \,, \\ &\text{Maximize}\left(\vec{\Psi}\cdot\vec{V}_{\text{fixed}}\right) \,, \end{aligned} \tag{2.124}$$

where the minus sign in the second line can be consistently imposed at the same time as the first line, *only* when the contribution of $\mathcal{O}^\star$ is isolated from the contribution of the remaining $\mathcal{O} \in \{\text{trial spectrum}\}$ [17]. As is standard in the bootstrap literature, we truncate the infinite-dimensional functional as

$$\vec{\Psi} = \sum_{m,n}^{m+n\leqslant\Lambda} \vec{\Psi}_{m,n} \partial_z^m \partial_{\bar{z}}^n \Big|_{z=\bar{z}=\frac{1}{2}} . \tag{2.125}$$

The result of the extremization problem (2.124) provides a bound on the OPE coefficient of $\mathcal{O}^\star$ as

$$\pm|\lambda_{\mathcal{O}^\star}|^2 \leqslant -\text{Max}\left(\Psi \cdot \vec{V}_{\text{fixed}}\right) . \tag{2.126}$$

When the bound is saturated, there is a unique solution to the (truncated) crossing equations [16, 79], with different extremization problems possibly leading to different solutions.[11] At finite Λ, this corresponds to an approximate solution to the full crossing system, with the spectrum encoded in the extremal functional [79].

[11] See, however, [127] for subtitles that arise when considering systems of mixed correlators.

Chapter 3

Long Multiplet Bootstrap

3.1 Two-dimensional $\mathcal{N} = 2$ global long superconformal blocks

In this section we obtain the two-dimensional $\mathcal{N} = 2$ global superconformal blocks for the four-point functions of long multiplets. Since the two-dimensional conformal algebra factorizes into left and right movers, we only consider the holomorphic part. Anti-holomorphic blocks will be added in the next section. As a warm-up we review the case of the $\mathcal{N} = 1$ blocks [40], which captures some of the main features of the $\mathcal{N} = 2$ case, while being computationally less involved. The procedure is as follows: We start by writing the form of the correlation function of four arbitrary operators as required by superconformal symmetry. It will include a general function of the independent superconformal invariants, which amounts to (two) five, for the ($\mathcal{N} = 1$) $\mathcal{N} = 2$ case. The superconformal blocks are then obtained by solving the eigenvalue problem associated to the quadratic and cubic Casimirs. In the $\mathcal{N} = 2$ case this produces a system of six coupled differential equations for the quadratic Casimir. In order to solve this system we start from a physically motivated Ansatz in terms of the expected bosonic block decomposition of the superconformal block.

3.1.1 Warm-up example: the $\mathcal{N} = 1$ superconformal blocks in two dimensions

We start by revisiting the computation of global superconformal blocks in $\mathcal{N} = 1$ SCFTs [40], highlighting the main features that are relevant for the $\mathcal{N} = 2$ computation.[1] Recall

[1] As in [40] we focus on four-point functions that are single-valued on their own, without adding the anti-holomorphic dependence, which reduces the number of invariants and is enough for the illustrative purposes in this section. We thank C. Behan for pointing this out.

the global $\mathcal{N}=1$ superconformal algebra is described, along with the global conformal generators ($L_{\pm 1}\,,\ L_0$), by the fermionic generators G_r ($r=\pm 1/2$) with the following commutation relations

$$\{G_r, G_s\} = 2L_{r+s} \quad \text{and} \quad [L_n, G_{\pm\frac{1}{2}}] = \left(\frac{n}{2} \mp \frac{1}{2}\right) G_{\pm\frac{1}{2}+n}\,. \tag{3.1}$$

Introducing a single fermionic variable θ we can write a generic superfield Φ, which we label by the holomorphic dimension of its superprimary h, as a function of (x,θ), where x is the usual holomorphic coordinate. The generators can be represented by the differential operators

$$\begin{aligned} &\mathcal{L}_{-1} = -\partial_x\,, && \mathcal{L}_0 = -x\partial_x - \frac{1}{2}\theta\partial_\theta - h\,, && \mathcal{L}_1 = -x^2\partial_x - x\theta\partial_\theta - 2xh\,, \\ &\mathcal{G}_{-\frac{1}{2}} = \partial_\theta - \theta\partial_x\,, && \mathcal{G}_{+\frac{1}{2}} = x\partial_\theta - \theta x\partial_x - 2h\theta\,. \end{aligned} \tag{3.2}$$

To study the four-point function

$$\langle \Phi(x_1,\theta_1)\Phi(x_2,\theta_2)\Phi(x_3,\theta_3)\Phi(x_4,\theta_4)\rangle\,, \tag{3.3}$$

we need to introduce the four-point superconformal invariants on which this correlator can depend. Defining the superconformal distance as $\mathbf{z}_{ij} = x_i - x_j - \theta_i\theta_j$ it is easy to see the two four-point invariants of the theory are

$$I_1 = \frac{\mathbf{z}_{12}\mathbf{z}_{34}}{\mathbf{z}_{14}\mathbf{z}_{23}} \to \frac{x_1 - x_2 - \theta_1\theta_2}{x_2} \quad \text{and} \quad I_2 = \frac{\mathbf{z}_{13}\mathbf{z}_{24}}{\mathbf{z}_{14}\mathbf{z}_{23}} \to \frac{x_1}{x_2}\,, \tag{3.4}$$

where the arrows mean we used a superconformal transformation to set $\mathbf{z}_3 = 0$ and $\mathbf{z}_4 \to \infty$. After taking this limit, the four-point function can be written as an arbitrary function of the two invariants

$$G(x_1,x_2,\theta_1,\theta_2) = \frac{1}{(x_1-x_2)^{2h_\phi}}\left[g_0(z) + \frac{\theta_1\theta_2}{x_2}g_\theta(z)\right]\,, \tag{3.5}$$

where h_ϕ is the dimension of the superprimary of Φ, $z = 1 - \frac{x_1}{x_2}$ is the bosonic cross ratio, and of course we used that the Taylor expansion of the function on the fermionic cross ratio truncates. Let us take a step back to interpret the two functions which appeared: g_0 is the piece that survives after taking all fermionic coordinates to zero and thus the four-point function of the superconformal primary of Φ. On the other hand, g_θ corresponds (up to factors of the bosonic cross ratio) to the correlation function of the two superconformal primaries at points three and four, and two (global) superdescendants at points one and two.

These functions admit a decomposition in blocks, corresponding to the exchange of a given superconformal multiplet. As in [40], we obtain these blocks by acting with the quadratic Casimir, and solving the corresponding eigenfunction equation, in terms of the eigenvalue of the quadratic Casimir on the exchanged supermultiplet $\mathfrak{c}_2$. The superconformal Casimir $\mathbf{C}^{(d)}$ is given by

$$\mathbf{C}^{(2)} = L_0^2 - \frac{1}{2}\left(L_1 L_{-1} + L_{-1} L_1\right) + \frac{1}{4}\left(G_{+\frac{1}{2}} G_{-\frac{1}{2}} - G_{-\frac{1}{2}} G_{+\frac{1}{2}}\right) . \tag{3.6}$$

Applying the differential form of the Casimir on the four-point function (3.5) we obtain a coupled system of two differential equations

$$\begin{aligned} z^2\left((1-z)\partial_z^2 - \partial_z\right) g_0 + \frac{1}{2} z g_\theta &= \mathfrak{c}_2 g_0(z) , \\ \left[z^2(1-z)\partial_z^2 + z(2-3z)\partial_z - z + \frac{1}{2}\right] g_\theta + \frac{1}{2} z\left((1-z)\partial_z^2 - \partial_z\right) g_0 &= \mathfrak{c}_2 g_\theta(z) , \end{aligned} \tag{3.7}$$

where $\mathfrak{c}_2 = h(h-\frac{1}{2})$ is the eigenvalue of quadratic Casimir on the superconformal multiplet being exchanged, with h denoting the scaling dimension of its superconformal primary. Solving these equations lead us to two sets of solutions with physical boundary conditions. The first one, obtained in [40], reads

$$\begin{aligned} g_0(z) &= g_h^{0,0}\left(\frac{z}{z-1}\right) , \\ g_\theta(z) &= \frac{h}{z} g_h^{0,0}\left(\frac{z}{z-1}\right) , \end{aligned} \tag{3.8}$$

and it corresponds to the case in which the superprimary itself (of weight h) is exchanged in the OPE. Note that the argument of the usual $sl2$ block

$$g_h^{h_{12},h_{34}}(z) = z^h {}_2F_1\left(h - h_{12}, h + h_{34}, 2h, z\right) , \tag{3.9}$$

is $\frac{z}{z-1}$ since this combination corresponds to the standard bosonic cross-ratio of $\frac{x_{12}x_{34}}{x_{13}x_{24}}$.

However, there is a second solution, which has the physical interpretation of a superconformal descendant being exchanged

$$\begin{aligned} g_0(z) &= g_{h+\frac{1}{2}}^{0,0}\left(\frac{z}{z-1}\right) , \\ g_\theta(z) &= \frac{1-2h}{2z} g_{h+\frac{1}{2}}^{0,0}\left(\frac{z}{z-1}\right) , \end{aligned} \tag{3.10}$$

where we recall that h corresponds to the dimension of the superconformal primary, which is what figures in the Casimir eigenvalue.

Notice that if one restricts to the correlation function of superconformal primaries by setting the fermionic coordinates to zero in eq. (3.5), one finds from eqs. (3.8) and (3.10) that the "superblock" is a sum of bosonic blocks with arbitrary coefficients. In fact the operator being exchanged in eq. (3.10) can even be a descendant of an operator which itself does not appear in the OPE decomposition, implying there is not even a constraint on the spectrum. However if one considers the whole supermultiplet as the external field, then one gets superblocks, in the sense that the coefficients of the block decomposition of external superdescendants are fixed in terms of those of external superprimaries. In practice, by considering the whole superfield as the external operator we are considering a mixed system in which supersymmetry was already used to reduce it to the set of independent of correlators. Exactly the same will happen for the $\mathcal{N}=2$ superblocks computed in the remaining of this section. As we will see in section 3.2, for the $\mathcal{N}=2$ case the crossing equations of external superdescendants provide non-trivial constraints and are essential in obtaining bounds that are stronger than the pure bosonic bootstrap.

3.1.2 $\mathcal{N}=2$ long multiplet four-point function

We now apply a similar strategy to the case of $\mathcal{N}=2$ supersymmetry. Although there are new features with respect to the much simpler $\mathcal{N}=1$ case, some of the main points are the same, even if obscured by the cumbersome technical details. The global part of the two-dimensional $\mathcal{N}=2$ superconformal algebra has four fermionic generators G_r, $\overline{G}_r$ ($r=\pm 1/2$), alongside the standard Virasoro generators L_m ($m=-1,0,1$) and the additional U(1) R-symmetry current algebra generator J_0. The commutation relations are given by

$$[L_m, L_n] = (m-n)L_{m+n}\,, \qquad \{G_r, G_s\} = \{\overline{G}_r, \overline{G}_s\} = 0\,, \tag{3.11}$$

$$[L_m, J_n] = -nJ_{m+n}\,, \qquad \{G_r, \overline{G}_s\} = L_{r+s} + \frac{1}{2}(r-s)J_{r+s}\,, \tag{3.12}$$

$$[L_m, G_r] = \left(\frac{m}{2}-r\right)G_{r+m}\,, \qquad [J_m, G_r] = G_{m+r}\,, \tag{3.13}$$

$$[L_m, \overline{G}_r] = \left(\frac{m}{2}-r\right)\overline{G}_{r+m}\,, \qquad [J_m, \overline{G}_r] = -\overline{G}_{m+r}\,. \tag{3.14}$$

Introducing θ and $\bar{\theta}$ as the two fermionic directions, following the steps of the previous subsection, we start by writing the differential action of the generators as (see for example

[128])

$$
\begin{aligned}
\mathcal{L}_{-1} &= -\partial_x \,, \\
\mathcal{L}_0 &= -x\partial_x - \frac{1}{2}\theta\partial_\theta - \frac{1}{2}\bar{\theta}\partial_{\bar{\theta}} - h \,, \\
\mathcal{L}_1 &= -x^2\partial_x - x\theta\partial_\theta - x\bar{\theta}\partial_{\bar{\theta}} - 2xh + q\theta\bar{\theta} \,, \\
\mathcal{G}_{+\frac{1}{2}} &= \frac{1}{\sqrt{2}}\big(x\partial_\theta + \theta\bar{\theta}\partial_\theta - x\bar{\theta}\partial_x - (2h+q)\bar{\theta}\big) \,, \\
\overline{\mathcal{G}}_{+\frac{1}{2}} &= \frac{1}{\sqrt{2}}\big(x\partial_{\bar{\theta}} - \theta\bar{\theta}\partial_{\bar{\theta}} - x\theta\partial_x - (2h-q)\theta\big) \,, \\
\mathcal{G}_{-\frac{1}{2}} &= \frac{1}{\sqrt{2}}(\partial_\theta - \bar{\theta}\partial_x) \,, \\
\overline{\mathcal{G}}_{-\frac{1}{2}} &= \frac{1}{\sqrt{2}}(\partial_{\bar{\theta}} - \bar{\theta}\partial_x) \,, \\
\mathcal{J}_0 &= -\theta\partial_\theta + \bar{\theta}\partial_{\bar{\theta}} - q \,,
\end{aligned}
\tag{3.15}
$$

where h and q are the conformal weight and the R-charge respectively, of the superconformal primary of the superfield.

Superconformal invariants

The form of the long multiplet four-point function is fixed by superconformal invariance up to an arbitrary function of all four-point superconformal invariants. Defining the supersymmetric distance

$$
Z_{ij} = x_i - x_j - \theta_i\bar{\theta}_j - \bar{\theta}_i\theta_j \,, \quad \text{with } \theta_{ij} = \theta_i - \theta_j \,, \tag{3.16}
$$

there are five such invariants, most naturally written as[2]

$$
\begin{aligned}
U_1 &= \frac{Z_{13}Z_{24}}{Z_{23}Z_{14}} \,, & U_4 &= \frac{\theta_{12}\bar{\theta}_{12}}{Z_{12}} + \frac{\theta_{24}\bar{\theta}_{24}}{Z_{24}} - \frac{\theta_{14}\bar{\theta}_{14}}{Z_{14}} \,, \\
U_2 &= \frac{\theta_{13}\bar{\theta}_{13}}{Z_{13}} + \frac{\theta_{34}\bar{\theta}_{34}}{Z_{34}} - \frac{\theta_{14}\bar{\theta}_{14}}{Z_{14}} \,, & U_5 &= \frac{Z_{12}Z_{34}}{Z_{23}Z_{14}} \,, \\
U_3 &= \frac{\theta_{23}\bar{\theta}_{23}}{Z_{23}} + \frac{\theta_{34}\bar{\theta}_{34}}{Z_{34}} - \frac{\theta_{24}\bar{\theta}_{24}}{Z_{24}} \,.
\end{aligned}
\tag{3.17}
$$

With foresight we define a new basis of invariants I_a as

$$
\begin{aligned}
I_0 &= 1 - U_1 \,, & I_1 &= -U_5 - (1-U_1) \,, \\
I_2 &= U_4(1-U_1) + U_2U_1 \,, & I_3 &= U_3 \,, & I_4 &= U_2 \,.
\end{aligned}
\tag{3.18}
$$

[2]Such invariants also appeared in [128] although only four were considered independent there. However, as will become clear later, all five invariants we write are independent and required for the four-point function expansion.

Naturally, all these invariants should be nilpotent at some power, with the exception of the one that corresponds to the supersymmetrization of the bosonic conformal invariant. Indeed we find they obey the following identities

$$\begin{aligned} &I_1^2 = -2I_3I_4(1-I_0)\,, && I_2^2 = 2I_3I_4(1-I_0)\,, && I_3^2 = 0\,, \\ &I_1I_2 = I_1I_3 = I_1I_4 = 0\,, && I_2I_3 = I_2I_4 = 0\,, && I_4^2 = 0\,, \end{aligned} \tag{3.19}$$

with the non-nilpotent invariant being I_0. The four-point function then has a finite Taylor expansion in the nilpotent invariants, with each term being a function of the super-symmetrization of the bosonic cross ratio I_0.

The four-point function

We write a generic long $\mathcal{N} = 2$ superconformal multiplet as

$$\Phi(x,\theta,\bar\theta) = \phi(x) + \theta\psi(x) + \bar\theta\chi(x) + \theta\bar\theta T(x)\,, \tag{3.20}$$

and we label the multiplet by the quantum numbers of its superconformal primary $\phi(x)$, namely the R-charge, q, and holomorphic dimension h. In our conventions then $\psi(x)$ ($\chi(x)$) has dimension and charge $h+\frac{1}{2}$ and $q+1$ ($h+\frac{1}{2}$ and $q-1$), while $T(x)$ has charge q and dimension $h+1$. Notice also that $T(x)$ is, in general, not a conformal primary, since it is not annihilated by the special conformal transformations. The superdescendant of dimension $h+1$ that is a conformal primary corresponds to the combination $P = -T - \frac{q}{2h}\partial\phi(x)$.[3]

We now write the most general form of the four-point function, as required by superconformal invariance. As usual we write the correlation function as a prefactor carrying the appropriate conformal weights, times a function of superconformal invariants

$$\langle\Phi(x_1,\theta_1,\bar\theta_1)\Phi(x_2,\theta_2,\bar\theta_2)\Phi(x_3,\theta_3,\bar\theta_3)\Phi(x_4,\theta_4,\bar\theta_4)\rangle = \frac{1+\frac{q_1\theta_{12}\bar\theta_{12}}{Z_{12}}}{Z_{12}^{2h}}\frac{1+\frac{q_3\theta_{34}\bar\theta_{34}}{Z_{34}}}{Z_{34}^{2h'}}F(I_a) \tag{3.21}$$

where for simplicity we took $h_1 = h_2 = h$, $h_3 = h_4 = h'$, $q_2 = -q_1$ and $q_4 = -q_3$ for the conformal dimensions and charges of the superprimaries. Given the properties of the nilpotent invariants (3.19) the function $F(I_a)$ can be expanded as

$$F(I_a) = f_0(I_0) + I_1f_1(I_0) + I_2f_2(I_0) + I_3f_3(I_0) + I_4(1-I_0)f_4(I_0) + I_3I_4(1-I_0)f_5(I_0)\,. \tag{3.22}$$

[3]Note that since we are working only with the global part of the conformal algebra, by conformal primary we do not mean a Virasoro primary but rather what is sometimes called a quasi-primary.

Furthermore we can use a superconformal transformation to set $x_4 = \infty$, $x_3 = 0$, and the fermionic variables of the last two fields to zero yielding

$$\begin{aligned} F(z,\theta_\alpha,\bar\theta_\alpha) =& f_0(z) + \frac{f_1(z)(\theta_1\bar\theta_2 - \theta_2\bar\theta_1)}{x_2} + \frac{f_2(z)(\theta_1\bar\theta_2 + \theta_2\bar\theta_1)}{x_2} + \frac{f_3(z)\theta_2\bar\theta_2}{x_2} + \frac{f_4(z)\theta_1\bar\theta_1}{x_2} \\ & - \frac{f_5(z)\theta_1\theta_2\bar\theta_1\bar\theta_2}{x_2^2}, \end{aligned} \tag{3.23}$$

with $\alpha = 1, 2$ and $z = 1 - \frac{x_1}{x_2}$.[4] The natural form of equation (3.23) is what motivated the choice of invariants I_a in eq. (3.18). Our goal is to obtain how each superconformal multiplet appearing in the double OPE of Φ contributes to each of these functions $f_i(z)$. Following the warm-up example the next step is to write the Casimir operators and act with them on the correlation function.

Quadratic and cubic Casimirs

In the case under investigation it turns out that the quadratic Casimir $\mathbf{C}^{(2)}$ is not enough to completely fix the form of the superconformal blocks, and we must also use the cubic one $\mathbf{C}^{(3)}$. This can readily be seen by looking at the eigenvalue of the quadratic Casimir on a superconformal multiplet whose superprimary has dimension h_{ex} and charge q_{ex},

$$\mathfrak{c}_2 = h_{\text{ex}}^2 - \tfrac{q_{\text{ex}}^2}{4}\,, \tag{3.24}$$

which does not distinguish the sign of the R-charge. Superconformal multiplets with opposite charges are distinguished by the cubic Casimir, whose eigenvalue is

$$\mathfrak{c}_3 = -q_{\text{ex}}\mathfrak{c}_2\,. \tag{3.25}$$

The quadratic and cubic Casimirs are given by [129]

$$\mathbf{C}^{(2)} = L_0^2 - \frac{1}{4}J_0^2 - \frac{1}{2}\{L_1, L_{-1}\} + \frac{1}{2}[\overline{G}_{+\frac{1}{2}}, G_{-\frac{1}{2}}] + \frac{1}{2}[G_{+\frac{1}{2}}, \overline{G}_{-\frac{1}{2}}]\,, \tag{3.26}$$

$$\begin{aligned} \mathbf{C}^{(3)} =& (L_0^2 - \frac{1}{4}J_0^2 - \frac{1}{2}L_{-1}L_1)J_0 + G_{-\frac{1}{2}}\overline{G}_{+\frac{1}{2}}(1 - L_0 - \frac{3}{2}J_0) \\ & - \overline{G}_{-\frac{1}{2}}G_{+\frac{1}{2}}(1 - L_0 + \frac{3}{2}J_0) - L_{-1}\overline{G}_{+\frac{1}{2}}G_{+\frac{1}{2}} + L_1 G_{-\frac{1}{2}}\overline{G}_{-\frac{1}{2}}\,. \end{aligned} \tag{3.27}$$

Acting with the quadratic Casimir on the four point function, through the differential action (3.15) of the generators, yields a system of six coupled differential equations for the six functions $f_i(z)$ in eq. (3.23). These are rather long and thus we collect them in appendix A.0.1; the next step is find a solution for this system of coupled

[4] Notice again that the standard two-dimensional cross-ratio is related to z by $\mathbf{z} = \frac{z}{z-1} = \frac{x_{12}x_{34}}{x_{13}x_{24}}$.

differential equations, and then to constrain said solution further by demanding it is also an eigenfunction of the cubic Casimir equation.

3.1.3 Long superconformal blocks

The easiest, and physically more transparent, way to solve the system of Casimir equations given in appendix A.0.1 is to give an Ansatz in terms of the expected bosonic block decomposition of superblocks.[5] Instead of given an Ansatz for the functions f_i, it is more convenient to "change basis" from the f_i to functions $\hat{f}_i$ that match the individual four-point functions of each external superconformal descendant (but conformal primary) field. This change of basis reads

$$
\begin{aligned}
&f_0(z) = \hat{f}_0(z)\,,\\
&f_2(z) = -\frac{2q_1\hat{f}_0(z)+\hat{f}_1(z)+\hat{f}_2(z)}{2z}\,,\\
&f_1(z) = \frac{4h\hat{f}_0(z)+\hat{f}_1(z)-\hat{f}_2(z)}{2z}\,,\\
&f_3(z) = \frac{2h\hat{f}_3(z)-q_1(z-1)z\hat{f}_0{}'(z)}{2hz}\,,\\
&f_4(z) = \frac{q_1\hat{f}_0{}'(z)}{2h}+\frac{\hat{f}_4(z)}{z}\,,\\
&f_5(z) = \frac{1}{4h^2z^2}\Big(2h(2h-1)\left(4h^2-q_1^2\right)\hat{f}_0(z)-q_1^2z^2(\hat{f}_0{}'(z)-(1-z)\hat{f}_0{}''(z))-\\
&-4h^2\big((q_1-2h)\hat{f}_1(z)-(q_1+2h)\hat{f}_2(z)-\hat{f}_5\big)\\
&+2hq_1z(\hat{f}_3(z)-\hat{f}_3{}'(z))+2hq_1z(1-z)\hat{f}_4{}'(z)\Big)\,.
\end{aligned}
\tag{3.28}
$$

This was obtained by expanding the superfields (3.20) on the left-hand-side of the four-point function (3.21), and obtaining the combinations of f_i that captures the correlation function of each of the conformal primaries appearing in eq. (3.20).[6]

Each of the functions $\hat{f}_i$ then has the interpretation as corresponding the correlators listed in eq. (3.29), and admits a decomposition in regular bosonic blocks. Recall that the Casimir equations depend on the quantum numbers of the superprimary of the multiplet being exchanged: $q_{\rm ex}$ and $h_{\rm ex}$. As such the most generic contribution of a given multiplet may be decomposed into a sum of bosonic blocks with dimensions that

[5]Note that by giving an Ansatz as a sum of bosonic blocks we are already fixing the boundary conditions and don't have to worry about removing shadow-block solutions.

[6]Note that T in eq. (3.20) is not the conformal primary combination as discussed below that equation.

are determined by the dimensions of the various fields in the multiplet,

$$
\begin{aligned}
\hat{f}_0\Big|_{h_{\rm ex}} &= a_0 g^{0,0}_{h_{\rm ex}} + b_0 g^{0,0}_{h_{\rm ex}+1} + c_0 g^{0,0}_{h_{\rm ex}+\frac{1}{2}} \to \langle\phi_1\phi_2\phi_3\phi_4\rangle\,,\\
\hat{f}_1\Big|_{h_{\rm ex}} &= a_1 g^{0,0}_{h_{\rm ex}} + b_1 g^{0,0}_{h_{\rm ex}+1} + c_1 g^{0,0}_{h_{\rm ex}+\frac{1}{2}} \to -\langle\chi_1\psi_2\phi_3\phi_4\rangle\,,\\
\hat{f}_2\Big|_{h_{\rm ex}} &= a_2 g^{0,0}_{h_{\rm ex}} + b_2 g^{0,0}_{h_{\rm ex}+1} + c_2 g^{0,0}_{h_{\rm ex}+\frac{1}{2}} \to \langle\psi_1\chi_2\phi_3\phi_4\rangle\,,\\
\hat{f}_3\Big|_{h_{\rm ex}} &= a_3 g^{-1,0}_{h_{\rm ex}} + b_3 g^{-1,0}_{h_{\rm ex}+1} + c_3 g^{-1,0}_{h_{\rm ex}+\frac{1}{2}} \to \langle\phi_1 P_2\phi_3\phi_4\rangle\,,\\
\hat{f}_4\Big|_{h_{\rm ex}} &= a_4 g^{1,0}_{h_{\rm ex}} + b_4 g^{1,0}_{h_{\rm ex}+1} + c_4 g^{1,0}_{h_{\rm ex}+\frac{1}{2}} \to \langle P_1\phi_2\phi_3\phi_4\rangle\,,\\
\hat{f}_5\Big|_{h_{\rm ex}} &= a_5 g^{0,0}_{h_{\rm ex}} + b_5 g^{0,0}_{h_{\rm ex}+1} + c_5 g^{0,0}_{h_{\rm ex}+\frac{1}{2}} \to -\langle P_1 P_2\phi_3\phi_4\rangle\,.
\end{aligned}
\tag{3.29}
$$

Where again $g^{h_{12},h_{34}}_{h_{\rm ex}}$ is the standard $sl2$ conformal block, (3.9), with argument $\frac{z}{z-1}$ (see footnote 4). Next we note that since we are considering the OPE channel between two oppositely charged fields, by U(1) R-charge conservation only uncharged operators can appear. Then we have two possibilities

- The superconformal primary itself is uncharged ($q_{\rm ex} = 0$), which means both the primary (the exchange with coefficient a_i in eq. (3.29)) and its dimension $h_{\rm ex} + 1$ superdescendant can appear (b_i in eq. (3.29)), but not its dimension $h_{{\rm ex}+\frac{1}{2}}$ and thus $c_i = 0$,
- The superconformal primary has charge $q_{\rm ex} = \pm 1$, which means only one of its two dimension $h_{\rm ex} + \frac{1}{2}$ can appear (the exchange with coefficient c_i in eq. (3.29)), and thus $a_i = b_i = 0$.

This is in accord with the study of $\mathcal{N} = 2$ three-point functions of [130]. For the exchange of a given supermultiplet labeled by $q_{\rm ex}$ and $h_{\rm ex}$ the various coefficients in eq. (3.29) are constrained by the Casimir equations.

Uncharged supermultiplet exchange

First we consider the solutions where the superconformal primary has zero charge, in which case $c_i = 0$ in eq. (3.29). Plugging the Ansatz (3.29) in the quadratic Casimir equations for $\hat{f}_i$, obtained from the ones in appendix A.0.1, we find the following solution

$$\begin{aligned}
a_2 &= 4a_0\left(h-\frac{h_{\rm ex}}{2}\right)+a_1\,,\\
a_3 &= a_1-\frac{a_0\left(-2h^2+hh_{\rm ex}-hq_1+\frac{1}{2}h_{\rm ex}q_1\right)}{h}\,,\\
a_4 &= a_1-\frac{a_0\left(-2h^2+hh_{\rm ex}-hq_1+\frac{1}{2}h_{\rm ex}q_1\right)}{h}\,,\\
a_5 &= \frac{a_0\left(2h^2-h(2h_{\rm ex}-1)+\frac{1}{2}(h_{\rm ex}-1)h_{\rm ex}\right)(2h+q_1)^2}{2h^2}+\frac{a_1q_1(2h-(h_{\rm ex}-1))}{h}\,,
\end{aligned} \tag{3.30}$$

$$\begin{aligned}
b_2 &= 4b_0\left(h+\frac{h_{\rm ex}}{2}\right)+b_1\,,\\
b_3 &= -\frac{b_1(h_{\rm ex}+1)}{h_{\rm ex}}-\frac{b_0(h_{\rm ex}+1)\left(h+\frac{h_{\rm ex}}{2}\right)(2h+q_1)}{hh_{\rm ex}}\,,\\
b_4 &= -\frac{b_1(h_{\rm ex}+1)}{h_{\rm ex}}-\frac{b_0(h_{\rm ex}+1)\left(h+\frac{h_{\rm ex}}{2}\right)(2h+q_1)}{hh_{\rm ex}}\,,\\
b_5 &= \frac{b_0\left(2h^2+h(2h_{\rm ex}+1)+\frac{1}{2}h_{\rm ex}(h_{\rm ex}+1)\right)(2h+q_1)^2}{2h^2}+\frac{b_1q_1(2h+h_{\rm ex}+1)}{h}\,,
\end{aligned} \tag{3.31}$$

where one of the unfixed a_i, and one of the unfixed b_i correspond to normalizations, thus leaving one arbitrary parameter in each of the above solutions. This solution automatically solves the constraints coming from the cubic and quartic Casimirs, and thus no more parameters can be fixed in general. Until now we were considering arbitrary fields of pairwise equal charges, however the parameters can be further constrained if the operators are assumed to be conjugates of each other. If, in addition, we consider the case $\Phi_1 = \Phi_2$, for which we need to consider uncharged operators ($q_1 = -q_2 = 0$), we notice that, for example, a_1 and a_2 correspond to the same three-point functions up to a permutation of the first two fields. Therefore imposing Bose symmetry fixes two more parameters as

$$a_1 = -a_0(2h-h_{\rm ex}))\,,\qquad \text{and} \qquad b_1 = -b_0(2h+h_{\rm ex})\,. \tag{3.32}$$

In this case then, the contribution of a given operator to the OPE of the descendants of the external superfield is fixed in terms of that of the external primary operator.

Charged supermultiplet exchange

Finally, we turn to the exchange of a multiplet whose superconformal primary has charge $q_{\rm ex} = \pm 1$. By solving the quadratic Casimir equation we clearly cannot distinguish the sign of $q_{\rm ex}$ and thus we must also consider the cubic Casimir. Unlike the quadratic case, the eigenvalue $\mathfrak{c}_3 = -q_{\rm ex}\mathfrak{c}_2$ depends on the sign of $q_{\rm ex}$, and this allows to fix completely all of the c_i, in terms of the charge of the exchanged supermultiplet $q_{\text{ex}} = \pm 1$, to be

$$
\begin{aligned}
c_1 &= -c_0(2h + q_1)\,, \\
c_2 &= -c_0(q_1 - 2h)\,, \\
c_3 &= \frac{-q_{\text{ex}}(c_0(2h_{\text{ex}} + 1)(2h + q_1 q_{\text{ex}}))}{4h}\,, \\
c_4 &= -\frac{-q_{\text{ex}}(c_0(2h_{\text{ex}} + 1)(2h - q_1 q_{\text{ex}}))}{4h}\,, \\
c_3 &= \frac{c_0(2h_{\text{ex}} + 1)(2h - q_1)}{4h}\,, \\
c_5 &= -\frac{c_0\left(-64h^4 - 32h^3 + 4h^2\left(4h_{\text{ex}}^2 - 1\right) + 16h^2 q_1^2 + 8hq_1^2 - \left(4h_{\text{ex}}^2 - 1\right) q_1^2\right)}{16h^2}\,,
\end{aligned}
\tag{3.33}
$$

where c_0 is a normalization. This solution automatically satisfies the quartic Casimir equation. The final system of Casimir equations are collected in eq. (A.6).

3.1.4 Decomposition of the $\mathcal{N} = 2$ stress-tensor four-point function

In order to prepare for our analysis of four-dimensional $\mathcal{N} = 3$ theories in section 3.3, and as a consistency check for the superblocks we constructed, we want to decompose the $\mathcal{N} = 2$ stress-tensor four-point function in terms of our blocks (3.29). The stress-tensor multiplet of an $\mathcal{N} = 2$ superconformal field theory in two dimensions is composed of the U(1) current, $J(x)$, two fermionic supercurrents, $G(x)$ and $\bar{G}(x)$, and the stress tensor itself $T(x)$. These four currents can be naturally organized in a long supermultiplet

$$
\mathcal{T}(x, \theta, \bar{\theta}) = J(x) + \theta G(x) + \bar{\theta}\bar{G}(x) + \theta\bar{\theta}T(x)\,, \tag{3.34}
$$

whose superprimary has dimension one and charge zero. Therefore, the four point function

$$
\langle \mathcal{T}(x_1, \theta_1, \bar{\theta}_1)\mathcal{T}(x_2, \theta_2, \bar{\theta}_2)\mathcal{T}(x_3, \theta_3, \bar{\theta}_3)\mathcal{T}(x_4, \theta_4, \bar{\theta}_4)\rangle\,, \tag{3.35}
$$

corresponds precisely to the type we have studied in this section, and it admits a decomposition in the blocks we just computed. Thus, after fixing the following list of correlators, for example by using Ward identities, we can decompose them in blocks of eq. (3.29) as[7]

$$
\begin{aligned}
\langle J(x_1)J(x_2)J(x_3)J(x_4)\rangle &= \sum_{h_{\text{ex}}} \lambda_{ex}^2 \hat{f}_0(h_{\text{ex}})\,, \\
\langle \bar{G}(x_1)G(x_2)J(x_3)J(x_4)\rangle &= -\sum_{h_{\text{ex}}} \lambda_{ex}^2 \hat{f}_1(h_{\text{ex}})\,, \\
\langle G(x_1)\bar{G}(x_2)J(x_3)J(x_4)\rangle &= \sum_{h_{\text{ex}}} \lambda_{ex}^2 \hat{f}_2(h_{\text{ex}})\,, \\
\langle T(x_1)T(x_2)J(x_3)J(x_4)\rangle &= -\sum_{h_{\text{ex}}} \lambda_{ex}^2 \hat{f}_5(h_{\text{ex}})\,,
\end{aligned}
\tag{3.36}
$$

while the contributions of $\hat{f}_3$ and $\hat{f}_4$ have to amount to zero, since the corresponding correlator vanishes. Note that we assumed the four $J(x)$ currents to be equal, and thus Bose symmetry requires the $sl2$ block decomposition of the four identical currents to be in terms of exchanged operators with even holomorphic dimension. This constraints the dimension of the superprimary of the exchanged operators, h_{ex}, for the three types of solutions, as can be read from eq. (3.29). In turn, this implies that for uncharged exchanges either the superconformal primary (a in eq. (3.29)) or its descendent (b in eq. (3.29)) appear, but not both at the same time. Moreover, Bose symmetry fixes the unfixed coefficients in eqs. (3.30) and (3.31) according to eq. (3.32). Therefore we label the OPE coefficient $\lambda_{ex}(X)$ with $X = a, b, c$ according to which of the solutions in eqs. (3.30), (3.31) and (3.33) is turned on, and for the corresponding one we normalize $X_0 = 1$. Doing so we find

$$
\begin{aligned}
&\lambda_{\text{ex}}^2(a) = 1\,,\ h_{\text{ex}} = 0\,, \\
&\lambda_{\text{ex}}^2(a) = \frac{\sqrt{\pi}2^{1-2h_{\text{ex}}}\left(c(h_{\text{ex}}-1)\Gamma(h_{\text{ex}}+3) + 12((h_{\text{ex}}-2)h_{\text{ex}}-2)\Gamma(h_{\text{ex}})\right)}{ch_{\text{ex}}\Gamma\left(h_{\text{ex}}+\frac{1}{2}\right)}\,,\ h_{\text{ex}} = 2k\,, \\
&\lambda_{\text{ex}}^2(b) = \frac{\sqrt{\pi}2^{-2h_{\text{ex}}-1}\left(c(h_{\text{ex}}-2)(h_{\text{ex}}-1)h_{\text{ex}}(h_{\text{ex}}+1) + 12(h_{\text{ex}}(h_{\text{ex}}+2)-2)\right)\Gamma\left(h_{\text{ex}}\right)}{ch_{\text{ex}}\Gamma\left(h_{\text{ex}}+\frac{3}{2}\right)}\,, \\
&\hspace{20em} h_{\text{ex}} = 2k-1\,, \\
&\lambda_{\text{ex}}^2(c, q_{\text{ex}}=1) = \frac{\sqrt{\pi}2^{-2h_{\text{ex}}-3}\left(4h_{\text{ex}}^2-9\right)\left(4ch_{\text{ex}}^2-c-48\right)\Gamma\left(h_{\text{ex}}-\frac{1}{2}\right)}{c\left(2h_{\text{ex}}+1\right)\Gamma\left(h_{\text{ex}}\right)}\,,\ h_{\text{ex}} = \frac{4k-1}{2}\,, \\
&\lambda_{\text{ex}}^2(c, q_{\text{ex}}=-1) = \frac{\sqrt{\pi}2^{-2h_{\text{ex}}-3}\left(4h_{\text{ex}}^2-9\right)\left(4ch_{\text{ex}}^2-c-48\right)\Gamma\left(h_{\text{ex}}-\frac{1}{2}\right)}{c\left(2h_{\text{ex}}+1\right)\Gamma\left(h_{\text{ex}}\right)}\,,\ h_{\text{ex}} = \frac{4k-1}{2}\,.
\end{aligned}
\tag{3.37}
$$

where the first term corresponds to the identity contribution, and below k is a positive integer. All these OPE coefficients are positive for unitarity theories, as they correspond, in our normalizations, to OPE coefficient squared between two identical currents and a generic operator. For negative central charges, *i.e.* , non-unitary theories, of course this

[7] The minus signs may appear strange but they just follow from the way $\hat{f}_i$ was defined, and so are to be combined with the corresponding a_i, b_i and c_i in eq. (3.29).

is no longer the case and this will be crucial to constrain the space of four-dimensional SCFTs following [52] in section 3.3.

3.2 Bootstrapping two-dimensional $(2,0)$ theories

As a first application of our bootstrap program for long operators we shall consider two-dimensional theories with $\mathcal{N}=(2,0)$ supersymmetry. The superblocks we constructed in the previous section suffice to analyze constraints from crossing symmetry of uncharged fields. In addition, we shall assume that our four external fields are identical and scalar ($h=\bar{h}$). These two assumptions would be easy to drop, but they simplify things a bit. In particular, for *identical* uncharged scalar operators the contribution from any given supermultiplet is determined by a single OPE coefficient, as shown in section 3.1.3.

Below we shall briefly review the history and status of $\mathcal{N}=(2,0)$ theories before we work out the crossing symmetry constraints. We then combine the decomposition into our left moving superblocks with a standard decomposition into right moving bosonic blocks to prepare for a numerical analysis. The results on bounds for central charges and conformal weights are summarized in the final subsection. Let us note that any $\mathcal{N}=(2,2)$ theory is an $\mathcal{N}=(2,0)$ one so that we cannot remove these solutions to the crossing equations from our analysis. In particular, we will see how we can recover models with more supersymmetry within the smaller system of crossing equations of $\mathcal{N}=(2,0)$ theories. One clear example is the $k=2$ minimal model with $\mathcal{N}=(2,2)$. While this model seemed to appear *inside* the region that is allowed by crossing symmetry of chiral operators, for the truncation of this system considered in [30], the central charge bounds in our long multiplet bootstrap are such that the $k=2$ minimal model actually saturates them.

3.2.1 The landscape of two-dimensional $\mathcal{N}=(2,0)$ theories

The study of two-dimensional models with (2,0) supersymmetry goes back more than two decades. Originally the main motivation came from heterotic string theory which relies on worldsheet models in which left movers are acted upon by an $\mathcal{N}=2$ superconformal algebra while right movers carry an action of the Virasoro algebra only. Extending the simplest realizations of this setup, which involves free fields, to non-trivial curved backgrounds turned out much more difficult than in the case of (2,2) supersymmetry. This has two reasons. On the one hand, the reduced amount of supersymmetry provides less control over the infrared fixed points of renormalization group flows in potential two-dimensional gauge theory realizations. On the other hand, exact worldsheet constructions need to adapt to the fact that left- and right movers are not identical, an issue that could be overcome in a few cases which we will describe below. But even with some exactly solvable models around, it remains an open question how typical they are within the landscape of two-dimensional (2,0) theories. More recent developments provided a new view onto this landscape. In fact, a large family of (2,0) theories are expected to emerge when one wraps $M5$ branes on a 4-manifold $\mathcal{M}_4$ [131]. Thereby the rich geometry of 4-manifolds becomes part of the landscape of two-dimensional (2,0) theories.

Realization as infrared fixed points of two-dimensional gauge theories were initiated by Witten in [132]. In this paper, some gauged linear sigma models for (2,2) theories are deformed by terms that break the right moving supersymmetry. These could be shown to flow to conformal field theories in the infrared [133]. The framework was extended to a larger class of gauged linear sigma models in [134] and arguments for the existence of infrared fixed points were given in [135]. More recently, realizations of (2,0) theories that are based on two-dimensional *non-abelian* gauge theories were pioneered in work of Gadde et al. [136]. Within this extended setup, interesting new non-perturbative triality relations emerged. Controlling the infrared behavior of these theories, however, remains a tricky issue, even with the use of modern technology [137].

Soon after the early work in the context of gauged linear sigma models, the first families of exact solutions were constructed in [138], following earlier ideas in [139], see also [140–142]. In all existing constructions, the left moving $\mathcal{N} = 2$ sector is realized as a gauged (coset) WZNW model, following the work of Kazama and Suzuki [143]. The simplest realization was found in [138]. These authors suggested to start from a WZNW model with the appropriate number of free fermions added, as in the Kazama-Suzuki construction. Then they gauged the subgroup used by Kazama and Suzuki, allowing for an asymmetry between the left and right moving sector. Such asymmetric gaugings are severely constrained by anomaly cancellation conditions and hence the construction of [138] only gives rise to a scarce list of models. If we require $c_L < 3$, one obtains the $(2, 0)$ minimal models of [138] in which the left moving $\mathcal{N} = 2$ superconformal algebra or central charge $c_L = 3k/(k+2)$ is combined with a right moving SU(2) current algebra at level k. In this case, consistency requires $k = 2(Q^2 - 1)$ so that the lowest allowed left moving central charge is $c_L = 9/4$.

Quite recently, Gadde and Putrov constructed another infinite family of $(2, 0)$ theories with $c_L = c_R = 3k/(k + 2)$, this time for any value of $k = 2, 3, \ldots$ [144]. Once again, the left moving chiral symmetry is the usual $\mathcal{N} = 2$ superconformal algebra, while on the right their models preserve a subalgebra of the SU(2) current algebra given by

$$\mathcal{W}_{\mathcal{R}} = \mathrm{SU}(2)_k/\mathrm{U}(1)_{2k} \times \mathrm{U}(1)_{k(k+2)} = \mathrm{PF}_k \times \mathrm{U}_{k(k+2)} \ , \tag{3.38}$$

i.e. , a product of the parafermionic chiral algebra and a U(1) current algebra. Let us recall that the sector of parafermions are labeled by pairs (l, α) with $l = 0, 1, \ldots, k$ and $\alpha = -k+1, \ldots, k$ such that $l+\alpha$ is even. The two pairs (l, α) and $(k-l, k-\alpha)$ correspond to the same sector. The conformal weight of the primary fields in these sectors satisfies

$$h^{\mathrm{PF}_k}_{(l,\alpha)} = \frac{l(l+2)}{4(k+2)} - \frac{\alpha^2}{4k} \ \ mod \ \ 1 \ .$$

The U(1) current algebra $\mathrm{U}(1)_K$, on the other hand, possesses K sectors with primaries of conformal weight

$$h^{\mathrm{U}_K}_m = m^2/2K \ .$$

Working with a smaller chiral algebra for right movers, as compared to the affine current algebra that was used in [138], allows for additional freedom so that now there is a modular invariant for any value k of the level. It takes the form

$$Z_k(q, \bar{q}) = \sum_{l=0}^{k} \sum_{\alpha \in \mathbb{Z}_{2k}} \sum_{s \in \mathbb{Z}_{k+2}} \chi^{\mathrm{SMM}_k}_{(l,2s-\alpha)}(\bar{q}) \chi^{\mathrm{PF}_k}_{(l,\alpha)}(q) \chi^{\mathrm{U}_{k(k+2)}}_{ks+\alpha}(q) \ . \tag{3.39}$$

In order to complete the description of these models we also recall that Neveu-Schwarz sector representations (l, m) of the $\mathcal{N} = 2$ superconformal algebra come with $l = 0, \ldots, k$ and $m = -2k+1, \ldots, 2k$ subject to the selection rule $l+m$ even, and field identification $(l, m) \cong (k-l, 2k-m)$. The conformal weight and charge of the corresponding primaries obey

$$\bar{h}^{\mathrm{SMM}_k}_{(l,m)} = \frac{l(l+2) - m^2}{4(k+2)} \ mod \ 1 \quad , \quad \bar{q}^{\mathrm{SMM}_k}_{(l,m)} = \frac{m}{k+2} \ mod \ 2 \ .$$

The first non-trivial example of the modular invariant (3.39) appears for $k = 2$ at $c_L = 3/2 = c_R$. It consists of 12 sectors and its modular invariant reads

$$Z(q, \bar{q}) = \sum_{l=0}^{2} \sum_{m=-3}^{4} \chi^{\mathrm{SMM}_2}_{(l,m)}(\bar{q}) \, \chi^{\mathrm{Ising}}_l(q) \chi^{\mathrm{U}_8}_m(q) \ . \tag{3.40}$$

Here, Ising stands for the Ising model whose three sectors are labeled by $l = 0, 1, 2$ and SMM_2 denotes the $\mathcal{N}$=2 supersymmetric minimal model with central charge $c = c_l = 3/2$. Only the six NS sector representation of the corresponding superconformal algebra appear in the modular invariant. These are labeled by (l, m) with $l + m$ even and $l = 0, 1, 2$, $m = -3, -2, \ldots, 3, 4$. The pairs (l, m) and $(2-l, 4-m)$ denote the same representation.

Of course, for each value of the central charge $c_L = c_R = 3k/(k+2)$ one can also construct a minimal model in which the $(2,0)$ supersymmetry happens to be extended to $(2,2)$. In particular, for $c_L = c_R = 3/2$ we have at least two $(2,0)$ models, one heterotic theory with modular invariant (3.40) and the usual diagonal supersymmetric minimal model. In our numerical analysis below the point $c_L = c_R = 3/2$ will appear at the boundary of the allowed region. We will access this point by studying the four-point function of an uncharged scalar field Φ of weight $h = \bar{h} = 1/2$. In both theories, the heterotic and the (2,2) minimal model, this field Φ involves the same primary $\bar{\varphi}_{(l,m)} = \bar{\varphi}_{(2,0)}$ of the left moving superconformal algebra. While it is combined with a right moving primary $\varphi_{(2,0)}$ from the same sector in the $(2,2)$ minimal model, the right moving contributions in the heterotic theory (3.40) are built as a product of the $l = 2$ field ε in the Ising model and the identity of the U_8 theory, *i.e.* , $\Phi^{\text{het}} = \bar{\varphi}_{(2,0)} \cdot \varepsilon$. Using *e.g.* , a free fermion representation of the $SU(2)_2$ current algebra it is not difficult to see that the field $\Phi = \bar{\varphi}_{(2,0)} \cdot \varphi_{(2,0)}$ and Φ^{het} possess identical four point functions. Hence we will not be able to distinguish between them in our bootstrap analysis below.

Let us also point out that in both models, the field Φ does not appear in the OPE of Φ with itself. In case of the heterotic theory (3.40), this may be seen from the well known fusion rule $\varepsilon \times \varepsilon \sim id$ of the $c = 1/2$ Virasoro algebra. The fusion rules of the $\mathcal{N} = 2$ super-Virasoro algebra, which can be found in [145], imply that $\varphi_{(2,0)} \times \varphi_{(2,0)} \sim id$ for $k = 2$. Hence, the field Φ of the (2,2) minimal model can not appear in its own self-OPE, as we had claimed. Our conclusion results from a low level truncation in the fusion rules of the $\mathcal{N} = 2$ super-Virasoro algebra.[8] For values $k \geq 3$, the corresponding fields Φ with left moving quantum numbers $(l, m) = (2, 0)$ do appear in their self-OPE, both for the heterotic and the (2,2) minimal models.

3.2.2 The (2,0) crossing equations

The goal of this subsection is to derive the $(2,0)$ crossing symmetry equations (3.46) for six functions $\hat{g}_i = \hat{g}_i(I_0, \bar{z}), i = 1, \ldots, 6$ of two variables, I_0 and $\bar{z}$. In order to do so, we combine the theory of left moving superblocks from the previous section with the well-known theory of bosonic blocks for the right movers. Blocks of the latter depend on a single cross ratio $\bar{z}$. So, let us consider the four-point function of a two-dimensional uncharged superfield, $\Phi(x, \theta, \bar{\theta}, \bar{x})$, with equal holomorphic and anti-holomorphic dimensions $h = \bar{h}$. Here, x ($\bar{x}$) denotes the (anti-)holomorphic bosonic coordinate while θ and $\bar{\theta}$ are both left-moving (holomorphic) fermionic variables. The four-point function can

[8] Unfortunately, this truncation was omitted in [31].

then be written as

$$\langle\Phi(x_1,\theta_1,\bar\theta_1,\bar x_1)\Phi(x_2,\theta_2,\bar\theta_2,\bar x_2)\Phi(x_3,\theta_3,\bar\theta_3,\bar x_3)\Phi(x_4,\theta_4,\bar\theta_4,\bar x_4)\rangle = \frac{1}{Z_{12}^{2h}}\frac{1}{Z_{34}^{2h}}\frac{1}{\bar x_{12}^{2\bar h}\bar x_{34}^{2\bar h}}$$
$$\times\Big(g_0(I_0,\bar z)+I_1 g_1(I_0,\bar z)+I_2 g_2(I_0,\bar z)+I_3 g_3(I_0,\bar z) \qquad (3.41)$$
$$+I_4(1-I_0)g_4(I_0,\bar z)+I_3I_4(1-I_0)g_5(I_0,\bar z)\Big)\,.$$

This representation of the four-point function follows the one we have used in eqs. (3.21) and (3.22), only that now the left moving coordinates are accompanied by right moving bosonic variables $\bar x_i$. Consequently, the functions $f_i(I_0)$ in eq. (3.21) are replaced by functions $g_i(I_0,\bar z)$, containing an additional dependence on the usual cross-ratio

$$\bar z = \frac{\bar x_{12}\bar x_{34}}{\bar x_{13}\bar x_{24}}\,, \qquad \bar x_{ij} = \bar x_i - \bar x_j\,. \qquad (3.42)$$

All other notations are as in the previous section. Following the usual logic we obtain the crossing equation by comparing the correlation function (3.41) with the one in the crossed channel in which $(x_1,\theta_1,\bar\theta_1,\bar x_1)$ and $(x_3,\theta_3,\bar\theta_3,\bar x_3)$ are exchanged. This leads to the equation

$$g_0(I_0,\bar z)+I_1g_1(I_0,\bar z)+I_2g_2(I_0,\bar z)+I_3g_3(I_0)+I_4g_4(I_0,\bar z)+I_3I_4(1-I_0)g_5(I_0,\bar z)=$$
$$(I_0+I_1)^{2h}\left(\frac{\bar z}{\bar z-1}\right)^{2\bar h}\Bigg(g_0(I_0^t,1-\bar z)+\frac{-I_1}{I_0+I_1}g_1(I_0^t,1-\bar z)$$
$$+\frac{2I_4(1-I_0)-I_2}{I_0+I_1}g_2(I_0^t,1-\bar z)+\frac{I_4(1-I_0)+I_3-I_2}{I_0}g_3(I_0^t,1-\bar z)$$
$$-I_4g_4(I_0^t,1-\bar z)+\frac{I_3I_4(1-I_0)}{I_0(I_0+I_1)}g_5(I_0^t,1-\bar z)\Bigg)\,.$$
$$(3.43)$$

Upon swapping $(x_1,\theta_1,\bar\theta_1)$ with $(x_3,\theta_3,\bar\theta_3)$ the invariants I_i become I_i^t. The latter may be expressed in terms of I_i as

$$I_0^t=\frac{1+I_1}{I_0+I_1}\,,\quad I_1^t=\frac{-I_1}{I_0+I_1}\,,\quad I_2^t=\frac{2I_4(1-I_0)-I_2}{I_0+I_1}\,,\quad I_3^t=\frac{I_4(1-I_0)+I_3-I_2}{I_0}\,,$$
$$I_4^t=-I_4\,,\qquad I_3^tI_4^t(1-I_0^t)=\frac{I_3I_4(1-I_0)}{I_0(I_0+I_1)}\,. \qquad (3.44)$$

Next we Taylor expand equation (3.43) in the nilpotent invariants ($I_{i\neq0}$) with the end result collected in eq. (A.8) as it is rather long. By comparing the coefficients of the six different nilpotent structures we obtain a system of six crossing equations for the six functions $g_i = g_i(I_0,\bar z), i = 0,\dots 5$, of the two variables I_0 and $\bar z$.

Finally we want each function of the two cross-ratios I_0 and $\bar{z}$ to admit a block decomposition that can be interpreted as the exchange of a given representation in the correlation function of the various operators that make up the external superfield $\Phi(x_1, \theta_1, \bar{\theta}_1, \bar{x}_1)$. This is achieved as in eq. (3.28) by going to a "primary basis" which can be decomposed in terms of the $\hat{f}_i$ blocks we have determined, *i.e.* , we rewrite the crossing equations in terms of $\hat{g}_i(I_0, \bar{z})$, where the $\hat{g}_i$ are related to the g_i in the same way the $\hat{f}_i$ are related to f_i. In addition, we express the variable I_0 in terms of a new variable

$$\mathbf{z} = \frac{I_0}{I_0 - 1}\,, \tag{3.45}$$

that reduces to the standard cross ratio z upon setting all the fermionic variables to zero. With these notations, the six crossing equations can be written in the form

$$\begin{aligned}
0 =& (1-z)^{2h}\hat{g}_0(\mathbf{z},\bar{z}) - z^{2h}\hat{g}_0(1-\mathbf{z},1-\bar{z})\,,\\
0 =& (1-z)^{2h+1}\hat{g}_3(\mathbf{z},\bar{z}) - z^{2h+1}\hat{g}_3(1-\mathbf{z},1-\bar{z})\,,\\
0 =& -2(z-1)z\left((z-1)z^{2h}\hat{g}_0(\mathbf{z},\bar{z})^{(1,0)}(1-\mathbf{z},1-\bar{z}) + z(1-z)^{2h}\hat{g}_0(\mathbf{z},\bar{z})^{(1,0)}(\mathbf{z},\bar{z})\right)\,,\\
&+ 2z^{2h+1}\hat{g}_1(1-\mathbf{z},1-\bar{z}) - 2(1-z)^{2h+1}\hat{g}_1(\mathbf{z},\bar{z})\,,\\
0 =& (1-z)^{2h+1}\hat{g}_3(\mathbf{z},\bar{z}) + z^{2h+1}\hat{g}_3(1-\mathbf{z},1-\bar{z})\,,\\
0 =& (1-z)^{2h+1}\hat{g}_4(\mathbf{z},\bar{z}) + z^{2h+1}\hat{g}_4(1-\mathbf{z},1-\bar{z})\,,\\
0 =& 2h(z-1)z^{2h+1}\hat{g}_0(\mathbf{z},\bar{z})^{(1,0)}(1-\mathbf{z},1-\bar{z}) + (z-1)z^{2h+2}\hat{g}_1(\mathbf{z},\bar{z})^{(1,0)}(1-\mathbf{z},1-\bar{z})\\
&+ z(1-z)^{2(h+1)}\hat{g}_1(\mathbf{z},\bar{z})^{(1,0)}(\mathbf{z},\bar{z}) + 2hz(1-z)^{2h+1}\hat{g}_0(\mathbf{z},\bar{z})^{(1,0)}(\mathbf{z},\bar{z})\\
&+ 2h^2(2z-1)z^{2h}\hat{g}_0(1-\mathbf{z},1-\bar{z}) + 2h^2(2z-1)(1-z)^{2h}\hat{g}_0(\mathbf{z},\bar{z})\\
&+ z^{2h+1}(2h(z+2)+z)\hat{g}_1(1-\mathbf{z},1-\bar{z}) + z^{2h+2}\hat{g}_5(1-\mathbf{z},1-\bar{z})\\
&- (1-z)^{2(h+1)}\hat{g}_5(\mathbf{z},\bar{z}) + (1-z)^{2h+1}(2h(z-3)+z-1)\hat{g}_1(\mathbf{z},\bar{z})\,.
\end{aligned} \tag{3.46}$$

Note that we have written the equations in a way such that they have an obvious symmetry under $\mathbf{z} \to 1 - \mathbf{z}$ and $\bar{z} \to 1 - \bar{z}$. This will prove to be convenient for the numerical implementation.

3.2.3 Block expansions

Each of the functions $\hat{g}_i$ in the crossing equation (3.46) admits a decomposition into superblocks in the left moving variable $\mathbf{z}$ and regular bosonic blocks depending on $\bar{z}$

$$\hat{g}_i(\mathbf{z}, \bar{z}) = \sum_{h_{\text{ex}}, q_{\text{ex}}, \bar{h}_{\text{ex}}} \lambda^2_{h_{\text{ex}}, q_{\text{ex}}, \bar{h}_{\text{ex}}} \hat{f}_i(\mathbf{z}) g_{\bar{h}_{\text{ex}}}(\bar{z}) \,. \tag{3.47}$$

We recall that on the supersymmetric (left) side, $h_{\text{ex}}, q_{\text{ex}}$ are the quantum numbers of the superconformal primary in a given supermultiplet, even if the operator appearing in the OPE is not the superprimary itself. Unitarity requires that the summation in eq. (3.47) is restricted by $h_{\text{ex}} \geqslant \frac{q_{\text{ex}}}{2}$ and $\bar{h}_{\text{ex}} \geqslant 0$. Here, the superblocks are given by eq. (3.29), with the coefficients fixed by eqs. (3.30), (3.31) and (3.33). Recall that since we are considering identical external fields, Bose symmetry fixes all coefficients as given in eq. (3.32), up to a normalization. We normalize them by setting $X_0 = 1$, with $X = a, b, c$ depending on which of the solutions we consider. The bosonic blocks, on the other hand, possess the standard expression

$$g_{\bar{h}_{\text{ex}}}(\bar{z}) = \bar{z}^{\bar{h}} {}_2F_1\left(\bar{h}_{\text{ex}}, \bar{h}_{\text{ex}}, 2\bar{h}_{\text{ex}}, \bar{z}\right) \,. \tag{3.48}$$

With our normalizations, the squares $\lambda^2_{h_{\text{ex}}, q_{\text{ex}}, \bar{h}_{\text{ex}}}$ of the OPE coefficients are the same that would appear in the four-point functions of the superconformal primary of $\Phi(x, \theta, \bar{\theta}, \bar{x})$, *i.e.* , when we set all fermionic variables in eq. (3.41) to zero. Hence, they are positive numbers.

On the supersymmetric side, we found in section 3.1.3, that there could be the following types of operators exchanged

- The superconformal primary (of dimension h_{ex}) of an uncharged ($q_{\text{ex}} = 0$) superconformal multiplet is exchanged – the solution given by the a_i in eq. (3.30),
- The superconformal descendant of dimension $h_{\text{ex}} + 1$ of an uncharged ($q_{\text{ex}} = 0$) superconformal multiplet whose superconformal primary has dimension h_{ex} – the solution given by b_i in eq. (3.31),
- The uncharged superconformal descendant of dimension $h_{\text{ex}} + \frac{1}{2}$ of a charged superconformal multiplet whose superconformal primary has dimension h_{ex} and charge $q_{\text{ex}} = \pm 1$ – the solution given by c_i in eq. (3.33) with $q_{\text{ex}} = \pm 1$.

Now we want to see which pairings of the above quantum numbers with the anti-holomorphic dimension $\bar{h}$ can appear in the OPE of identical uncharged scalars. Defining

$$\Delta = h + \bar{h} \,, \qquad \ell = h - \bar{h} \,, \tag{3.49}$$

we want to obtain the range of $\Delta_{\rm ex}$ and $\ell_{\rm ex}$ for operators that can appear in the self-OPE of the external superfield. Note that in two-dimensions, since the conformal group factorizes, parity does not exchange states in the same representation. In particular ℓ can be both positive and negative, and since we focus on $\mathcal{N}=(2,0)$ theories (which clearly have no symmetry between $\mathbf{z}$ and $\bar{z}$, as visible in eq. (3.47)), we must consider both signs of ℓ independently.[9] However ℓ should still be half-integer for single-valuedness of correlation functions. This means that the sum (3.47) will have a discrete parameter ℓ_{ex}, and a continuous one $\Delta_{\rm ex}$ satisfying

$$\begin{aligned}
&\Delta_{ex} \geqslant |\ell_{\rm ex}|\,, \quad \text{for } q_{\rm ex}=0\,, \quad \Delta_{ex} \geqslant \ell_{\rm ex}\,, \quad \text{for } |q_{\rm ex}| \leqslant 2\ell_{\rm ex}\,, \\
&\Delta_{ex} \geqslant |q_{\rm ex}| - \ell_{\rm ex}\,, \quad \text{for } |q_{\rm ex}| \geqslant 2\ell_{\rm ex}\,.
\end{aligned} \tag{3.50}$$

Furthermore, Bose symmetry constrains the spin of the operators, appearing in the OPE of the superconformal primary of $\Phi(x,\theta,\bar{\theta},\bar{x})$, to be even, putting constraints on the spin of the superconformal primary of multiplet $\ell_{\rm ex}$.

Of the multiplets appearing in the OPE three are noteworthy. One corresponds to the identity operator, which has $\Delta_{ex}=\ell_{ex}=q_{\rm ex}=0$ and comes from the a_i solution in eq. (3.30). The other two correspond to the holomorphic and anti-holomorphic stress tensors, which are given respectively by a b_i solution in eq. (3.31) with $\Delta_{\rm ex}=\ell_{\rm ex}=1$, $q_{\rm ex}=0$; and by an a_i solution with $\Delta_{ex}=-\ell_{ex}=2$ and $q_{\rm ex}=0$.

3.2.4 Numerical implementation

To analyze the crossing equations (3.46) we proceed numerically, as pioneered in [5], using the SDPB solver of [9]. We follow the, by now standard, procedure to obtain numerical bounds (see, *e.g.* , [148, 149] for reviews). In the block decomposition (3.47) we approximate the superblocks by polynomials in the exchanged dimension $\Delta_{\rm ex}$, as first implemented in [17], and truncate the infinite sum over the spins from $-L_{max} \leqslant \ell \leqslant L_{max}$.[10]

By searching for six-dimensional linear functionals

$$\vec{\Phi} = \sum_{n,m=0}^{n+m\leqslant\Lambda} \vec{\Phi}_{m,n} \partial_{\mathbf{z}}^m \partial_{\bar{z}}^n \big|_{\mathbf{z}=\bar{z}=\frac{1}{2}}\,, \tag{3.51}$$

[9]For the bosonic case, when putting together holomorphic blocks to make the whole conformal block, one usually symmetrizes in $z \leftrightarrow \bar{z}$, and therefore can restrict the OPE decompositions to positive spin (see *e.g.* , [146]). Parity odd blocks, anti-symmetric under this exchange, were considered in [147].

[10]Note that due to the explicit $\Delta_{\rm ex}$ factors in the crossing equations, and derivatives of blocks, one must be careful to consistently approximate all terms in the crossing equation to a polynomial of the same degree in $\Delta_{\rm ex}$.

whose action on the crossing equations is subject to a given set of conditions, we can rule out assumptions on the spectrum of operators $\{\Delta_{\text{ex}}, \ell_{\text{ex}}, q_{\text{ex}}\}$ appearing in the OPE, and on their OPE coefficients. The cutoff Λ implies we are effectively studying a Taylor series expansion of the crossing equations, truncated by Λ. Therefore for each Λ, we obtain valid bounds, that will get stronger as we increase the number of terms kept in the Taylor expansion. Each of the equations (3.46) has a definite symmetry under $\mathbf{z} \to 1 - \mathbf{z}$ and $\bar{z} \to 1 - \bar{z}$, according to which only even or odd $m + n$ derivatives in eq. (3.51) will be non-trivial. However, unlike the typical bootstrap setups, the equations have no symmetry in $z \leftrightarrow \bar{z}$ and we cannot restrict to derivatives with $m < n$.

3.2.5 Numerical results for $\mathcal{N} = (2, 0)$ theories

3.2.5.1 Central charge bounds

In exploring the space of $\mathcal{N} = (2, 0)$ SCFTs the first question one wants to answer concerns the range of allowed central charges. Here we explore what values are allowed for both left and right central charges, while allowing the other one to be arbitrary, and compare the numerical bounds with the known landscape of theories described in section 3.2.1. A peculiarity of two dimensions, as already discussed in [7], is that one cannot find a lower bound on the central charge without imposing a small gap in the spectrum of scalar operators. Therefore to obtain central charge bounds we require that all scalar superprimaries appearing in the OPE of our external field have dimension larger than a certain value, which we denote by $h_{\text{gap}} = \bar{h}_{\text{gap}}$. The bounds are then obtained for various different values of $h_{\text{gap}} = \bar{h}_{\text{gap}}$.[11]

Right central charge

We start by obtaining a lower bound on c_R (the central charge of the non-supersymmetric side), displayed on figure 3.1, overlapped with the bound obtained from the purely bosonic crossing equations. The bounds in figure 3.1 obtained for the full set of crossing equations (3.46) (colored dots and lines) assume various different values of h_{gap}, while for the crossing equations of just the superconformal primary (the first equation in (3.46)) we picked a single illustrative value of h_{gap} (dashed black line).[12] In order to

[11]That it is necessary to impose a gap is expected from the fact that the unitarity bounds in two dimensions do not have a gap between the dimension of the identity operator (0) and that of the first generic operator. This implies that the optimization problem we try to solve, by minimizing the value of the functional on the identity, while remaining positive on all other blocks, is only possible if the continuum is isolated from the identity by a gap imposed by hand. Why the gap must be at least of the order of $h_{\text{gap}} \sim 1.7h$, as empirically observed in the numerical results, is not clear to us.

[12]The first equation in the list (3.46) is exactly the equation for a bosonic theory, and bounds for various gaps were obtained in.

obtain a non-trivial central charge bound we found we needed to impose a gap in the scalar superprimary spectrum of $h_{\text{gap}} = \bar{h}_{\text{gap}} \sim 1.7h$, where $h = \bar{h}$ is the dimension of the external superprimary. The size of the minimum gap appears to be similar to the one needed in for the bosonic case. The bounds are shown only for $\Lambda = 20$, to avoid cluttering, which is enough for them to have approximately converged in the scaled used. (The rate of convergence is exemplified for the left central charge bound c_L in figure 3.3.) Finally, note that the bounds start with $h = \bar{h}$ slightly above zero, as at the point $h = \bar{h} = 0$ the external field becomes shorter (it becomes the identity) and the blocks we computed are not valid.

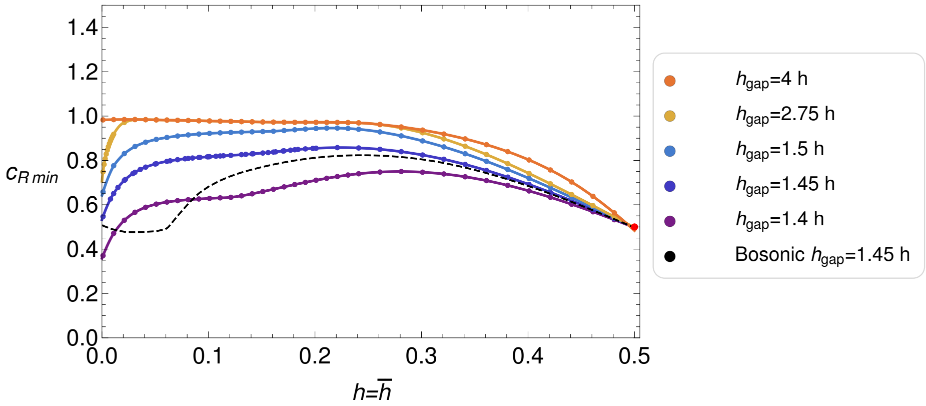


FIGURE 3.1: Lower bound on the allowed right central charge c_R (non-supersymmetric side) of $\mathcal{N} = (2,0)$ SCFTs as a function of the external dimension, $h = \bar{h}$, after imposing different gaps on the spectrum of superprimary scalar operators $h_{\text{gap}} = \bar{h}_{\text{gap}}$. The lines with dots correspond to the full set of crossing equations. The dashed black line corresponds to the bound obtained from the crossing equations of the superconformal primary alone, which matches with the bosonic bootstrap bounds, and is obtained for a single $h_{\text{gap}} = 1.45h$. The red dot marks the central charge and external dimension of the known $(2,2)$ and $(2,0)$ minimal models described below eq. (3.40) (see text for discussion). The bounds were obtained for $\Lambda = 20$ which, with the shown scale is enough to have obtained a converged plot as exemplified in figure 3.3.

The bounds for the full set of crossing equations are much stronger than the purely bosonic ones, in particular the minimum corresponding to the central charge of the two-dimensional Ising model is absent. This exemplifies the amount of constraints lost if one were to consider only the correlation function of the superconformal primary, for which the "superblocks" are just bosonic conformal blocks. Even though Bose symmetry fixes the four-point function of external superdescendants in terms of that of the external superprimary one, the crossing equations for external descendants provide non-trivial constraints, further reducing the space of allowed CFTs. This is stark contrast with the case of half-BPS operators, such as the two-dimensional chiral operators considered in [31], where the only invariants were the supersymmetrization of the bosonic cross-ratios u and v.

We see the bounds exhibit a strong dependence on the gap imposed, with the exception of a neighborhood of $h = \bar{h} = \frac{1}{2}$, where all bounds appear to give the same value approaching $c_R = \frac{1}{2}$. This leads to the natural question of whether there is a physical theory with $h = \bar{h} = \frac{1}{2}$ saturating our bounds. Looking at the landscape of known physical $\mathcal{N} = (2,0)$ SCFTs, briefly described in section 3.2.1, we see that the uncharged scalar operators of most of the $\mathcal{N} = (2,0)$ models there described, and also of the $\mathcal{N} = (2,2)$ minimal models, have the property that said scalar appears in its self-OPE. Therefore by imposing a gap $h_{\text{gap}} \geqslant 1.7h$ we exclude all these theories by hand. The exceptions are the $\mathcal{N} = (2,2)$ minimal model with central charge $c_L = c_R = \frac{3}{2}$, and the heterotic model described around eq. (3.40). Both these theories have an uncharged scalar of dimension $h = \bar{h} = \frac{1}{2}$ and thus should be allowed in our setup. To understand how these theories should appear in our plots we must first point out that by c_R we mean the central charge coefficient read off from the exchange of a superprimary operator on the left, and $sl2$ primary on the right, with $h_{\text{ex}} = q_{\text{ex}} = 0$ and $\bar{h}_{\text{ex}} = 2$. It could happen that there is more than one such operator. For example, if the theory actually has $\mathcal{N} = 2$ supersymmetry on the right side, we expect there to be two such operators: the anti-holomorphic stress tensor, and the Sugawara stress tensor, made out of the U(1) current. As such we are not guaranteed to be bounding the OPE coefficient of the anti-holomorphic stress tensor.

Let us start by describing the $\mathcal{N} = (2,2)$ minimal model for which the $h = \bar{h} = \frac{1}{2}$ operator of charge zero does not appear in its self-OPE, and thus should appear inside our allowed region.[13] As pointed out above we are not guaranteed to be bounding c_R, which for this model should be $\frac{3}{2}$. In this case we are obtaining a sum of OPE coefficient squared, namely that of the stress tensor and of the Sugawara stress tensor. Computing this OPE coefficient we find it should give rise to an apparent central charge of $c_R = \frac{1}{2}$, and thus this solution appears to saturate our bounds, and is indicated by a red dot in figure 3.1. Next we turn to the heterotic model described around eq. (3.40). In this case there is no supersymmetry on the right side, and the coefficient we are bounding corresponds exactly to c_R. Again in this case $c_R = \frac{1}{2}$, which is indicated by the same red dot in figure 3.1, and this solution too appears to saturate our bounds. As explained in section the correlation function of these two solutions are the same, and this is not in conflict the fact that there is a unique solution to the crossing equations for theories that sit on the numerical exclusion curves [16, 79].

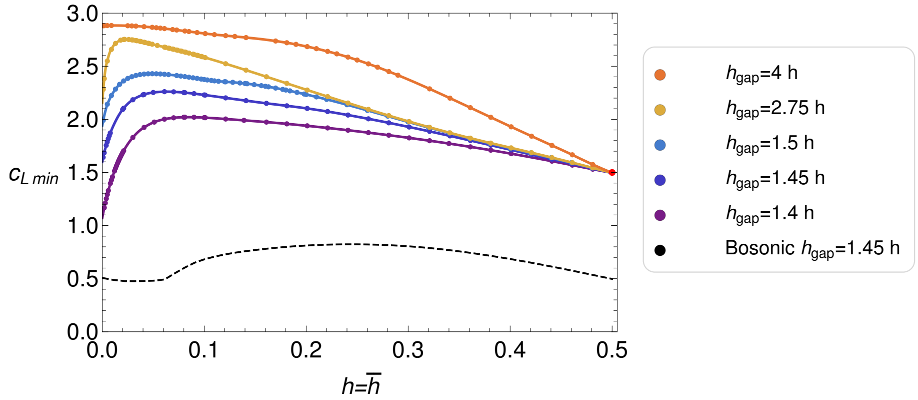


FIGURE 3.2: Lower bound on the allowed left central charge c_L (supersymmetric side) of $\mathcal{N} = (2, 0)$ SCFTs as a function of the external dimension, $h = \bar{h}$, after imposing different gaps on the spectrum of superprimary scalar operators $h_{\rm gap} = \bar{h}_{\rm gap}$. The lines with dots correspond to the full set of crossing equations. The dashed black line corresponds to the bound obtained from the crossing equations of the superconformal primary alone, which matches with the bosonic bootstrap bounds, and is shown for a single value of $h_{\rm gap}$. The red dot marks the central charge and external dimension of the known (2, 2) and (2, 0) minimal models discussed below eq. (3.40) which have identical four-point functions for this external operator. The bounds were obtained for $\Lambda = 20$, and the rate of convergence of the numerical bounds is shown in figure 3.3 for $h = \bar{h} = 0.5$.

Left central charge

Next we turn to c_L (the central charge of the $\mathcal{N} = 2$ side) shown in figure 3.2, where we obtain, as expected, a much stronger bound than for the non-supersymmetric side. We obtain c_L from the OPE coefficient of the exchange of the holomorphic stress tensor, which is a global superdescendant of the U(1) current. Thus, unlike in the c_R case, the stress tensor is distinguished from the Sugawara stress tensor: the latter is part of a global superprimary, while the former is always a superdescendant. As before to obtain a non-trivial c_L bound we must impose a gap at least of the order of $1.7h$. The plot is obtained at fixed $\Lambda = 20$ and, once again, we mark the position of the $\mathcal{N} = (2, 2)$ minimal model and the heterotic model described in section 3.2.1 as a red dot. Again, while the bounds display a large dependence on the gap imposed, for external dimension $h = \bar{h} = \frac{1}{2}$ all gaps give the same bound, around $\frac{3}{2}$. The dependence of the bounds on the cutoff Λ is shown for this value of the external dimension in figure 3.3, where we see for $\Lambda = 20$ the bounds have almost stabilized to a value close to $\frac{3}{2}$. This is precisely the central charge of the $\mathcal{N} = (2, 2)$ minimal model and the heterotic model, which appear to also saturate both the c_R and c_L bounds. Recall from section 3.2.1 that the

[13]This minimal model also has chiral operators ($h = \frac{|q|}{2}$) of charge $\pm\frac{1}{4}$ and $\pm\frac{1}{2}$, which appear inside the dimension bounds of Figure 5 of [31] (the $k = 2$ minimal model), and at least for the Λ considered there, not saturating them.

four-point function of both this models is equal, corresponding to the unique solution obtained when a bound is saturated.

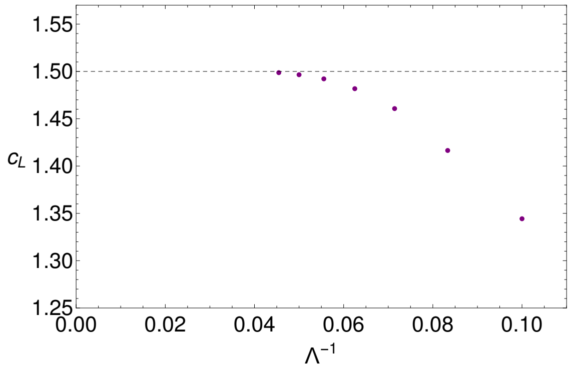


FIGURE 3.3: Lower bound on the allowed left central charge c_L (supersymmetric side) of $\mathcal{N} = (2, 0)$ SCFTs for $h = \bar{h} = 0.5$ and $h_{\text{gap}} = \bar{h}_{\text{gap}} = 0.8$, as a function of the inverse of the number of derivatives (Λ^{-1}) to exemplify the convergence of our numerical results with Λ.

Finally, if we impose that imposing $c_L = c_R = c$ we seem to find a bound on c identical to that of figure 3.2, for the cases of $h_{\text{gap}} = \bar{h}_{\text{gap}}$ we tested. This follows from a technical subtlety, namely, the functional is normalized to one on the sum of the holomorphic and anti-holomorphic stress tensor blocks, but this allows it to be zero on one of them, and one on the other. As such we are obtaining a bound on the minimum of both OPE coefficients, which are inversely proportional to the central charge, and thus what we obtain is the maximum of the c_L and c_R bounds, explaining the observed feature.

3.2.5.2 Dimension bounds

Lastly, we turn to bounding the dimensions of the first long scalar operators, whose global superconformal primaries appear in this OPE (this corresponds to the a_i solution in eq. (3.30). The upper bound on the dimension of the superconformal primary is shown in figure 3.4 for various values of the cutoff Λ. The orange line in the plot corresponds to the solution of generalized free field theory, *i.e.* , the four-point function given by a sum of products of two-point functions $h_{\text{ex}} = 2h$.

Note that since we are only using *global* superconformal blocks and not Virasoro superblocks we should expect super Virasoro descendants to appear in the OPE independently of their superconformal primaries. In particular the following descendant of the identity $\left(J_{-1}J_{-1} - \frac{2}{3}L_{-2}\right)\bar{L}_{-2}|0\rangle$ corresponds to a scalar operator of dimension $h_{\text{ex}} = \bar{h}_{\text{ex}} = 2$, which is a global superprimary, and therefore should appear in the OPE channel we are studying in figure 3.4. For $h \lesssim 0.5$ the numerical results demand an

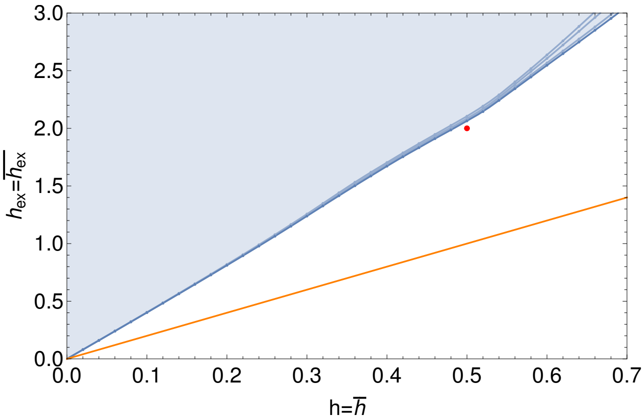


Figure 3.4: Upper bound on the dimension of the first uncharged scalar long superconformal primary that appears in the OPE, as a function of the dimension of the external operator $h = \bar{h}$ for $\Lambda = 16, 18, \ldots, 22$ derivatives. The red dot marks the dimension of the known $(2,2)$ and $(2,0)$ minimal models discussed below eq. (3.40). The orange line corresponds to the generalized free field theory solution $h_{\rm ex} = 2h$.

operator of a smaller dimension to be present, while for $h \gtrsim 0.5$ the numerical bounds allow for solutions without such $h_{\rm ex} = \bar{h}_{\rm ex} = 2$ operators.

The four-point function of the $\mathcal{N} = (2,2)$ and the $\mathcal{N} = (2,0)$ models discussed above only has one super Virasoro multiplet being exchanged, that of the vacuum. Therefore we expect the dimension of the first global superconformal primary to be exactly $h_{\rm ex} = \bar{h}_{\rm ex} = 2$ in both cases, marked by the red dot in figure 3.4. The numerical upper bound on the dimension is converging slower than the central charge bounds (figure 3.3), and it is not clear whether it is converging to the red dot, although it seems plausible.

The remaining $\mathcal{N} = (2,2)$ minimal models, and $\mathcal{N} = (2,0)$ theories described in subsection 3.2.1 share the property that the external field appears in its own OPE, *i.e.* , their solution corresponds to $h_{\rm ex} = \bar{h}_{\rm ex} = h = \bar{h}$, for $h < \frac{1}{2}$. This means they are deep inside the allowed region in figure 3.4, below the generalized free field theory solution (orange line in the plot). This leaves open the question of whether there exist new theories saturating the numerical bounds for $h < 0.5$, or if the solution to crossing symmetry of this particular correlator cannot be part of a full-fledged SCFT.

One could hope that by allowing the external field to appear in its own OPE, the remaining minimal models would also saturate the numerical bounds. However the next scalar in the minimal models also sits well inside the numerical bound of figure 3.4, and since we cannot force the external scalar to be exchanged, only to allow for its presence, we end up with the same result as in figure 3.4. We could keep repeating the procedure, allowing for both the external scalar, and the first operator exchanged after it in the known solutions. However, preliminary explorations suggest the resulting bound would be very weak.

3.3 Consequences for four-dimensional physics

Finally, we discuss the implications of the blocks we computed in section 3.1 for four-dimensional $\mathcal{N}=3$ SCFTs. As briefly reviewed in Chapter 2 any $\mathcal{N}\geq 2$ SCFT contains a protected sub sector isomorphic to a 2d chiral algebra. The construction described uses up all of the supersymmetry of a pure $\mathcal{N}=2$ theory, and the two-dimensional chiral algebra has no supersymmetry left. However, if the four-dimensional theory has more supersymmetry, then the chiral algebra will also be supersymmetric. This follows immediately from the fact that the extra supercharges, enhancing the supersymmetry beyond $\mathcal{N}=2$, commute with $\mathbb{Q}$ and thus relate different representatives of $\mathcal{N}=2$ multiplets in cohomology. This is the case of theories with $\mathcal{N}=4$ supersymmetry, for which the chiral algebra will necessarily contain the "small" $\mathcal{N}=4$ super algebra, as discussed in detail in [52]. If the theory has instead $\mathcal{N}=3$ supersymmetry one will end up precisely with a $\mathcal{N}=2$ two-dimensional chiral algebra as first discussed in [64], with the full list of $\mathcal{N}=3$ supermultiplets containing Schur operators given in [35].

3.3.1 Four-dimensional $\mathcal{N}=3$ SCFTs

As reviewed in Section 2.4 the first examples of pure $\mathcal{N}=3$ SCFTs (*i.e.* , theories which do not have $\mathcal{N}=4$ supersymmetry) were recently constructed using a generalization of orientifolds in string theory, called S-folds, in [57].[14] Several properties of pure $\mathcal{N}=3$ SCFTs can be obtained from representation theory alone, which had been studied long ago in [72, 93, 125], but only recently was the case of $\mathcal{N}=3$ explored in detail [58], shortly before the first $\mathcal{N}=3$ theories were constructed. Similarly to the $\mathcal{N}=4$ case, the a and c conformal anomalies of $\mathcal{N}=3$ SCFTs have to be equal, and pure $\mathcal{N}=3$ theories cannot have any flavor symmetry which is not an R-symmetry. They are also isolated theories, in the sense that pure $\mathcal{N}=3$ theories have no exactly marginal deformations.[15] Despite having no exactly marginal deformations, thus making them hard to study by the traditional field-theoretic approaches, various examples of non-trivial, pure, $\mathcal{N}=3$ SCFTs have been constructed by now [57, 59, 60] using string-theoretic technology. These theories were also recovered, and new ones obtained, by the systematic study

[14]Already in [150] a truncation of type IIB supergravity, whose CFT dual would correspond to a four-dimensional $\mathcal{N}=3$ SCFT, had been considered.

[15]The only $\mathcal{N}=3$ superconformal multiplet which could accommodate supersymmetric exactly marginal deformations also contains extra supersymmetry currents, enhancing $\mathcal{N}=3$ to $\mathcal{N}=4$ [58, 126].

of $\mathcal{N} = 2$ SCFTs with a one complex dimensional Coulomb branch in the work of [61, 62, 105, 112].

Nevertheless, we still seem far from having a complete classification of $\mathcal{N} = 3$ SCFTs. One can hope that the situation is more tractable than the $\mathcal{N} = 2$ case, due to the extra supersymmetry, yet richer than $\mathcal{N} = 4$ where we might already have the complete classification. Some of the known $\mathcal{N} = 3$ theories are obtained from $\mathcal{N} = 4$ SYM by gauging a discrete subgroup which, as pointed out in [59, 63], does not change the correlation functions nor the central charges of the theory, changing only the spectrum of local and non-local operators. Among all non-trivial (*i.e.* , that do not come from discrete gauging) pure $\mathcal{N} = 3$ theories known to date, the one with the smallest central charge, and thus in a sense the simplest theory, has $a = c = \frac{15}{12}$.[16] One could wonder if this indeed corresponds to the "minimal" theory, or if there is a theory with lower central charge, perhaps not obtainable from S-fold constructions (and their generalizations). Thus we shall try to address these questions by field theoretic methods, and refrain from making any assumptions about the theories, apart from that it is a local and interacting $\mathcal{N} = 3$ SCFT.

3.3.2 Chiral algebra constraints on four-dimensional $\mathcal{N} = 3$ SCFTs

We take a bootstrap approach, bypassing the need for any perturbative description and making only use of the fact that any local $\mathcal{N} = 3$ SCFTs will have a stress tensor. The existence of the stress-tensor operator, together with all other operators that sit in the same $\mathcal{N} = 3$ superconformal multiplet, is the *minimal* assumption one can make about local $\mathcal{N} = 3$ theories.[17] Therefore the constraints we obtain in this section are valid for *any* $\mathcal{N} = 3$ SCFT and do not rely on any string-theoretic construction. We also do not assume any information about the Coulomb branch of the theory. A downside of making the minimal set of assumptions about the theory is that we cannot impose it only has $\mathcal{N} = 3$ supersymmetry. By simply considering the $\mathcal{N} = 3$ stress tensor four-point function we cannot distinguish between non-trivial $\mathcal{N} = 3$ SCFTs, and theories which are either $\mathcal{N} = 4$ theories or $\mathcal{N} = 3$ theories obtained from $\mathcal{N} = 4$ ones by gauging discrete symmetries.

To be able to rule out $\mathcal{N} = 4$ solutions one would have to impose that the multiplets containing the additional supercurrents, enhancing the symmetry from $\mathcal{N} = 3$ to $\mathcal{N} = 4$, are absent. However such multiplets are not exchanged in the most universal OPEs such as the stress tensor self-OPE. This limitation can be overcome if one wants to construct

[16] In the notation of [59] this corresponds to $N = 1$ and $\ell = k = 3$.

[17] In the notation of [95, 126] the stress tensor multiplet is denoted by $B_1\bar{B}_1(0,0)^2_{[1,1],0}$ and by $\hat{\mathcal{B}}_{[1,1]}$ in [35].

the explicit chiral algebra of an $\mathcal{N} = 3$ SCFT, as done in [35, 64], but that requires making assumptions about the *complete* list of generators of the chiral algebra, and thus is well suited to studying *specific* known $\mathcal{N} = 3$ theories, but not to exploring the allowed space of $\mathcal{N} = 3$ SCFTs.[18]

The stress tensor multiplet

Unsurprisingly, the operators in the stress-tensor multiplet of four-dimensional $\mathcal{N} = 3$ SCFTs give rise in cohomology to a two-dimensional $\mathcal{N} = 2$ stress-tensor multiplet. This corresponds to a long multiplet in two dimensions, $\mathcal{T}(z, \theta, \bar{\theta}) = J(z) + G(z)\theta + \bar{G}(z)\bar{\theta} + \theta\bar{\theta}T(z)$, therefore requiring precisely the blocks computed in section 3.1.

The four-dimensional origin of each of the global conformal primaries in the superfield $\mathcal{T}(z, \theta, \bar{\theta})$ becomes more transparent if we view the $\mathcal{N} = 3$ theory as an $\mathcal{N} = 2$ one. When viewed as an $\mathcal{N} = 2$ theory the $U(3)_R$ R-symmetry group of $\mathcal{N} = 3$ theories decomposes as $U(1)_F \times U(2)_R$, where the first factor is the R-symmetry of the $\mathcal{N} = 2$ superconformal algebra, while the second factor corresponds to a global symmetry from the $\mathcal{N} = 2$ point of view, *i.e.* , it commutes with the $\mathcal{N} = 2$ superconformal algebra. Decomposing the $\mathcal{N} = 3$ stress tensor multiplet in $\mathcal{N} = 2$ representations one finds

- the $U(1)_F$ flavor current multiplet ($\hat{\mathcal{B}}_1$ in the notation of [94]),
- the stress tensor multiplet ($\hat{\mathcal{C}}_{0,(0,0)}$), and
- two supercurrent multiplets, containing extra currents enhancing $\mathcal{N} = 2$ to $\mathcal{N} = 3$ ($\mathcal{D}_{1/2,(0,0)}$ and $\bar{\mathcal{D}}_{1/2,(0,0)}$).

As described above the $U(1)_F$ flavor symmetry gives rise in cohomology to a U(1) AKM current algebra, whose generator is given precisely by the dimension one superconformal primary of $\mathcal{T}(z, \theta, \bar{\theta})$: $J(z)$. The $\mathcal{N} = 2$ stress tensor multiplet gives rise to the two-dimensional stress tensor $T(z)$, while the extra supercurrents furnish $G(z)$ and $\bar{G}(z)$ [52]. All of these two-dimensional global conformal primaries are related by the action of four of the extra supercharges (two Poincaré supercharges and their conjugates) which appear in the $\mathcal{N} = 3$ in addition to those of the $\mathcal{N} = 2$ subalgebra, and which commute with $\mathbb{Q}$.

[18] An attempt to reach a compromise between the two options was explored in [35] by constructing a candidate subalgebra of a large class of known $\mathcal{N} = 3$ SCFTs.

Four-dimensional OPE coefficients from the chiral algebra

We decomposed the four-point function of the two-dimensional $\mathcal{N}=2$ stress-tensor multiplet $\mathcal{T}(z,\theta,\bar{\theta})$ in superblocks in section 3.1.4. Interpreting this decomposition in the context of the two-dimensional chiral algebra, each two-dimensional global superconformal primary operator arises as the representative of a four-dimensional superconformal multiplet. Thus, the two-dimensional OPE coefficients obtained in this way amount to the computation of an infinite number of four-dimensional OPE coefficients. Furthermore, even though the two-dimensional chiral algebra is not unitary, implying the sign of the two-dimensional OPE coefficients have a priori no constraint. By re-interpreting these OPE coefficients in a four-dimensional language we can impose unitarity of the four-dimensional theory and constrain their sign. This constrains which chiral algebras can arise from four-dimensional $\mathcal{N}=3$ SCFTs.

Although the selection rules for the four-dimensional OPE of two $\mathcal{N}=3$ stress-tensor multiplets remain elusive, and obtaining them is a project in itself, we can leverage knowledge of selection rules for $\mathcal{N}=2$ SCFTs to interpret the computed two-dimensional OPE coefficients in terms of four-dimensional ones. The superconformal primary of the two-dimensional stress tensor multiplet is the aforementioned AKM current. In four-dimensional language it arises from an $\mathcal{N}=2$ $\hat{\mathcal{B}}_1$ multiplet, whose OPE selection rules were obtained in [151]

$$\hat{\mathcal{B}}_1 \times \hat{\mathcal{B}}_1 = \mathcal{I} + \hat{\mathcal{B}}_1 + \hat{\mathcal{B}}_2 + \sum_{\ell=0}^{\infty} \hat{\mathcal{C}}_{0,\ell} + \sum_{\ell=0}^{\infty} \hat{\mathcal{C}}_{1,\ell}\,. \tag{3.52}$$

Here we only listed multiplets containing Schur operators, and thus relevant for our computation. Of these multiplets the $\hat{\mathcal{C}}_{0,\ell}$ with spin $\ell > 0$ contain conserved currents of spin larger than two, which are expected to be absent in interacting theories [96, 97]. As such we set their OPE coefficients to zero by hand, thereby restricting only to *interacting* theories. We point out even though we are interpreting these OPE coefficients in terms of four-dimensional $\mathcal{N}=2$ representations, by decomposing the full correlation in two-dimensional superblocks, the four-dimensional $\mathcal{N}=2$ multiplets were organized in $\mathcal{N}=3$ representations. In other words, the superblock decomposition allows us to identify which two-dimensional multiplets are global superconformal primaries, and which are global superdescendants, thereby identifying which $\mathcal{N}=3$ multiplet each $\mathcal{N}=2$ multiplet belongs to. Recall that the OPE coefficients $a_{h_{\text{ex}}}$, $b_{h_{\text{ex}}}$, $c_{h_{\text{ex}},q_{\text{ex}}=\pm 1}$ in eq. (3.29) correspond to a global superprimary, the $G_{-1/2}\overline{G}_{-1/2}$ descendant, and $G_{-1/2}/\overline{G}_{-1/2}$ descendants, respectively. Therefore it is straightforward to identify which $\mathcal{N}=3$ multiplet is being exchanged by making use of the decomposition of $\mathcal{N}=3$ in

$\mathcal{N}=2$ of [35]. The relevant decompositions are

$$\begin{aligned} &\hat{\mathcal{C}}_{[1,1],(\frac{\ell}{2},\frac{\ell}{2})} \to \hat{\mathcal{C}}_{1,(\frac{\ell}{2},\frac{\ell}{2})} \oplus \hat{\mathcal{C}}_{1,(\frac{\ell+1}{2},\frac{\ell+1}{2})}\,, \qquad && \hat{\mathcal{B}}_{[1,1]} \to \hat{\mathcal{B}}_1 \oplus \hat{\mathcal{C}}_{0,(0,0)}\,, \\ &\hat{\mathcal{C}}_{[2,0],(\frac{\ell}{2},\frac{\ell+1}{2})} \to \hat{\mathcal{C}}_{1,(\frac{\ell+1}{2},\frac{\ell+1}{2})}\,, && \hat{\mathcal{B}}_{[2,2]} \to \hat{\mathcal{B}}_2 \oplus \hat{\mathcal{C}}_{1,(0,0)}\,, \end{aligned} \tag{3.53}$$

where we followed the labeling of $\mathcal{N}=3$ multiplets of [35], and restricted the decompositions to the types of Schur multiplets exchanged in eq. (3.52)

All in all, we obtain from the two-dimensional OPE coefficients in eq. (3.37), the following four-dimensional OPE coefficients

$$\lambda^2_{\hat{\mathcal{B}}_{[1,1]}\ \text{desc.}} = -\frac{2}{c_{2d}}\,, \tag{3.54}$$

$$\lambda^2_{\hat{\mathcal{B}}_{[2,2]}\ \text{prim.}} = 2-\frac{2}{c_{2d}}\,, \tag{3.55}$$

$$\begin{aligned}\lambda^2_{\hat{\mathcal{C}}_{[1,1],\ell}\ \text{prim.}} &= \frac{3\sqrt{\pi}2^{-2\ell-3}(\ell(\ell+4)+1)\Gamma(\ell+3)}{c_{2d}(\ell+3)\Gamma\left(\ell+\frac{7}{2}\right)} \\ &+\frac{\sqrt{\pi}2^{-2\ell-5}(\ell+2)(\ell+4)(\ell+5)\Gamma(\ell+3)}{\Gamma\left(\ell+\frac{7}{2}\right)}\,, \quad \ell \text{ odd}\,,\end{aligned} \tag{3.56}$$

$$\begin{aligned}\lambda^2_{\hat{\mathcal{C}}_{[1,1],\ell-1}\ \text{desc.}} &= \frac{3\sqrt{\pi}2^{-2\ell-3}(\ell(\ell+6)+6)\Gamma(\ell+2)}{c_{2d}(\ell+2)\Gamma\left(\ell+\frac{7}{2}\right)} \\ &+\frac{\sqrt{\pi}2^{-2\ell-5}\ell(\ell+1)\Gamma(\ell+4)}{(\ell+2)\Gamma\left(\ell+\frac{7}{2}\right)}\,, \quad \ell \text{ odd}\,,\end{aligned} \tag{3.57}$$

$$\begin{aligned}|\lambda_{\hat{\mathcal{C}}_{[2,0],\left(\frac{\ell-1}{2},\frac{\ell}{2}\right)}\ \text{desc.}}|^2 &= \frac{3\sqrt{\pi}2^{-2\ell-3}(\ell+1)(\ell+4)\Gamma(\ell+2)}{c_{2d}(\ell+3)\Gamma\left(\ell+\frac{5}{2}\right)} \\ &+\frac{\sqrt{\pi}2^{-2\ell-5}(\ell+1)\Gamma(\ell+5)}{(\ell+3)\Gamma\left(\ell+\frac{5}{2}\right)}\,, \quad \ell \text{ odd}\,,\end{aligned} \tag{3.58}$$

where the OPE coefficients in eqs. (3.55) and (3.56) correspond to the exchange of a two-dimensional global superprimary, eqs. (3.54) and (3.57) to a $G_{-1/2}\overline{G}_{-1/2}$ descendant, and eq. (3.58) to a $G_{-1/2}/\overline{G}_{-1/2}$. We point out that we only know the OPE coefficient of the *Schur* operator that is exchanged in the $\mathcal{T}\mathcal{T}$ OPE, which is not enough to obtain all four-dimensional OPE coefficients appearing in the full three-point function of two stress tensors and the multiplet in question. The above, nonetheless, provides the subset of the selection rules of $\mathcal{N}=3$ stress tensor multiplet OPE that is captured by the chiral algebra.

A new $\mathcal{N}=3$ unitarity bound

Unitarity requires all of the above coefficients to be positive, implying lower bounds on the value of $c_{4d} = -\frac{c_{2d}}{12}$, with the strongest one coming from the OPE coefficient

$\lambda^2_{\hat{\mathcal{C}}_{[1,1],\ell=0}\text{ desc.}}$ $(= b_{h_{\text{ex}}=3})$, *i.e.* , the exchange of a dimension four descendant (a $\hat{\mathcal{C}}_{1,\ell=1}$ multiplet) of an uncharged global superprimary of dimension three (a $\hat{\mathcal{C}}_{1,\ell=0}$ multiplet). This yields the following unitarity bound

$$c_{4d} \geqslant \frac{13}{24}\,, \tag{3.59}$$

valid for any local *interacting* $\mathcal{N} = 3$ SCFT. Unlike the previous unitarity bounds obtained from chiral algebra correlators [22, 48, 52, 68, 69] the inequality (3.59) is not saturated by any known theory, and in fact we will argue that the bound is a strict inequality. Similar bounds, relying only on the existence of a stress tensor, and the absence of higher spin currents, for theories with $\mathcal{N} = 2$ and $\mathcal{N} = 4$ supersymmetry ($c_{4d} \geqslant \frac{11}{30}$ and $c_{4d} \geqslant \frac{3}{4}$ respectively [22, 48, 68]), are saturated by the interacting theories with lowest central charge known in each case: the simplest Argyres-Douglas point [70, 71] for the former, and by $\mathcal{N} = 4$ super Yang-Mills (SYM) with gauge group SU(2) for the latter, but we claim this cannot happen for (3.59). Moreover there is no known SCFT whose central charge is close to saturating it. In particular, there is no known theory with central charge in between this value and that of $\mathcal{N} = 4$ SYM with gauge group SU(2). These values are below those values that were seen in the systematic classification of theories with a one-dimensional Coulomb branch of [61, 62, 105, 112]. Moreover, making use of eq. (5.1) of [62] one can obtain, under certain assumptions, including that the Coulomb branch is freely generated, that for a rank one $\mathcal{N} = 3$ SCFT $c_{4d} \geqslant \frac{3}{4}$.[19] A theory close to saturating (3.59) would then seem to be an interacting rank zero SCFT (*i.e.* , with no Coulomb branch) or with non-freely generated Coulomb branches.[20]

Reconstructing $4d$ operators from the chiral algebra

We should emphasize that it is still not clear what is the full set of conditions a two-dimensional chiral algebra must satisfy such that it arises from a consistent four-dimensional SCFT. In the case at hand, however, we will give an argument as to why an $\mathcal{N} = 2$ chiral algebra with $c_{2d} = -13/2$ cannot correspond to an *interacting* four dimensional $\mathcal{N} = 3$ SCFT.

Let us suppose that there exists an *interacting* four-dimensional SCFT for some given value of c_{4d}. Then we can construct in the chiral algebra the operators that are exchanged in the $\mathcal{T}\mathcal{T}$ OPE. In our discussion here we will focus on uncharged dimension

[19] We thank Mario Martone for discussions on this point.

[20] In [152] six-dimensional theories were found that could have rank zero, although this was not the only possibility there, we thank I. García-Extebarria for bringing this reference to our attention.

three global superprimaries, since the bound (3.59) arises from the exchange of a superdescendant of such an operator. From four-dimensional selection rules we know the global superprimaries of the operators being exchanged have to belong in $\hat{\mathcal{C}}_{[1,1],\ell=0}$ or $\hat{\mathcal{C}}_{[0,0],\ell=1}$ representations, and we impose the latter to be absent to focus on interacting theories. When passing to the cohomology of [52], Schur operators from different $4d$ multiplets can give rise to global supermultiplets of the 2-dimensional $\mathcal{N} = 2$ algebra that look identical. In particular they may contain two-dimensional superprimaries of the same weight and $U(1)_F$ (and also $U(1)_r$) charges. One such example is given by the $\mathcal{N} = 2$ multiplets $\hat{\mathcal{C}}_{0,\ell}$ and $\hat{\mathcal{C}}_{1,\ell-1}$ in a four-dimensional theory.

For the arguments we outline below it will be crucial to distinguish between $4d$ multiplets that give rise to identical superconformal multiplets in cohomology. The ambiguities that can appear were discussed in [35], and for theories with a single chiral algebra generator a conjectural prescription on how to lift them was put forward in [124]. Since such prescription does not apply to the case at hand we simply exploit that cohomology inherits a bit more structure from the reduction process than the spectrum of charges and weights. Namely, it also induces an indefinite quadratic form. Orthogonal $4d$ multiplets remain so in cohomology, but their superprimaries may give rise to states of negative norm. In this way, it may be possible to distinguish between two multiplets with identical spectra of weights and charges. This is the case for the $\mathcal{N} = 2$ multiplets relevant here (see (3.53)) $\hat{\mathcal{C}}_{1,0}$ and $\hat{\mathcal{C}}_{0,1}$ [52] which indeed reduce to identical superprimaries, but with norms of opposite signs. When we reduce the stress tensor operator product expansion of the four-dimensional theory we obtain the superdescendant of a $2d$ uncharged superconformal primary of dimension $h = 3$ which has negative norm with respect to the induced quadratic form.

Let us now look at the two-dimensional side of the story. For central charges around the value $c_{2d} = -\frac{13}{2}$, the subspace of uncharged dimension $h = 3$ superprimaries is 2-dimensional and its quadratic form is indefinite, *i.e.* , it possesses one positive and one negative eigenvalue. Let us stress that both eigenvalues are non-zero. Given any choice of an orthonormal basis $\mathcal{O}_1$ and $\mathcal{O}_2$ of this space, we can reach any other choice by an $\mathrm{SO}(1,1)$ transformation. Let us denote the unique parameter of $\mathrm{SO}(1,1)$ by b and the corresponding basis vectors by $\mathcal{O}_1(b)$ and $\mathcal{O}_2(b)$. Without loss of generality we can assume that the vectors $\mathcal{O}_1(b)$ are those with negative norm while the norm of $\mathcal{O}_2(b)$ is positive. According to our previous discussion, we must show the operator product expansion of the stress tensor in the 2-dimensional theory contains the global superdescendant of $\mathcal{O}_1(b)$ and is orthogonal to the global superdescendant of $\mathcal{O}_2(b)$ for some choice of the parameter b. This is indeed possible for all values of the central charge $c_{2d} > -\frac{13}{2}$. In fact, one can show that the 3-point functions

$$\langle \mathcal{T}(w_1)\mathcal{T}(w_2)G\overline{G}\mathcal{O}_2(b;z)\rangle = 0 \quad \text{for some} \quad b = b(c_{2d}) \ ,$$

where $G\overline{G}$ means we are referring to the dimension four superdescendant of $\mathcal{O}_2(b;z)$. The negative norm field $\mathcal{O}_1$, whose descendant appears in the operator product expansion, is given by

$$\mathcal{O}_1(b(c_{2d});z) = 2\left(6(\bar{G}G)(z) - (JT)(z) + J(z)''\right) - \frac{3}{2}T(z)' \,. \tag{3.60}$$

Here $(AB)(z)$ means the normal-ordered product of $A(z)$ and $B(z)$. The dimension four superdescendant of $\mathcal{O}_1(b,z)$ corresponds to the operator exchanged in the $\mathcal{T}\mathcal{T}$ OPE. When c_{2d} approaches the value $c_{2d} = -\frac{13}{2}$, however, the boost parameter b tends to infinity and the field $\mathcal{O}_1(b(c_{2d} = -\frac{13}{2});z)$ has vanishing 2-point function. This means that the stress tensor OPE at $c_{2d} = -\frac{13}{2}$ is inconsistent with the cohomological reduction from a four-dimensional interacting $\mathcal{N} = 3$ theory with central charge $c_{4d} = \frac{13}{24}$. Hence we conclude that such a theory cannot exist. Let us stress, though, that our argument relies on one additional assumption, namely that the quadratic form in cohomology coincides with the usual Shapovalov form in the vacuum sector of the $\mathcal{N} = 2$ Virasoro algebra. This is not guaranteed, much as it is not guaranteed that the global $\mathcal{N} = 2$ superconformal symmetry that acts on cohomology is enhanced to a super Virasoro symmetry. On the other hand, such an enhancement is seen in many explicit examples and it seems natural to expect that it extends to the relevant quadratic form.

An immediate question that arises is whether our arguments could be refined to obtain a bound stronger than (3.59). In particular there is no known $\mathcal{N} = 3$ theory whose chiral algebra is generated only by the stress tensor multiplet $\mathcal{T}$ (the $\mathcal{N} = 2$ and $\mathcal{N} = 4$ SCFTs with smallest central charge have as chiral algebras the (super) Virasoro vacuum module), could such a theory exist? A necessary condition for this to happen would be the existence of a null state in the chiral algebra involving a power of the stress tensor as discussed in [153].

Chapter 4

Bootstrapping the (A_1, A_2) Argyres-Douglas theory

In this chapter we apply bootstrap techniques in order to constrain the CFT data of the (A_1, A_2) Argyres-Douglas theory, which as reviewed before is the minimal of the Argyres-Douglas models. After studying the four-point function of its single Coulomb branch chiral ring generator and putting numerical bounds on the low-lying spectrum of the theory, we will focus on a particularly interesting infinite family of semi-short multiplets labeled by the spin ℓ. Although the conformal dimensions of these multiplets are protected, their three-point functions are not. Using the numerical bootstrap we impose rigorous upper and lower bounds on their values for spins up to $\ell = 20$. By the means of a recently obtained inversion formula [15] , we also estimate them for sufficiently large ℓ, and the comparison of both approaches shows consistent results. Finally, we will give a rigorous numerical range for the OPE coefficient of the next operator in the chiral ring, and estimates for the dimension of the first R-symmetry neutral non-protected multiplet for small spin.

4.1 Numerical results

In this section we attempt to zoom in on the (A_1, A_2) Argyres-Douglas theory reviewed in the previous section, following up on the numerical analysis of the Coulomb branch presented in [21, 27], where the landscape of theories with one or two Coulomb branch chiral ring operators was explored.

An interesting question that was left open in [27] was whether the (A_1, A_2) theory saturates the numerical lower bounds on the central charge c. While it has been established analytically that this theory has the lowest possible central charge among interacting $\mathcal{N}=2$ SCFTs [68], this does not preclude solutions to the crossing equations (2.122) for $r_0 = \frac{6}{5}$ and values of c smaller than $\frac{11}{30}$. The central charge bounds from [21, 27] were obtained for $\Lambda \leqslant 22$, and there was no particularly clear trend that would allow for an extrapolation to $\Lambda \to \infty$.

In this work we present improved numerical results, with extrapolations consistent with, but not definite proof of, the saturation of the c-bound by the (A_1, A_2) SCFT. Moreover, our results seem to imply that, even if there is more than one crossing symmetric four-point function for $r_0 = \frac{6}{5}$ and $c = \frac{11}{30}$, these solutions do not differ by much as far as some observables are concerned, and can be used as an approximation to the low-lying spectrum of the (A_1, A_2) theory. In particular, we are able to obtain the first predictions for unprotected OPE coefficients in the form of *true* upper and lower bounds for OPE coefficients, together with conservative extrapolations for $\Lambda \to \infty$. In addition, we also estimate the value of the lowest-twist unprotected long multiplets appearing in the non-chiral OPE. In this section we focus on the lowest spin operators, but numerical results for larger spins are presented in section 4.2, where we compare them to estimates arising from the Lorentzian inversion formula of [15] adapted to the supersymmetric case.

4.1.1 Central charge bound

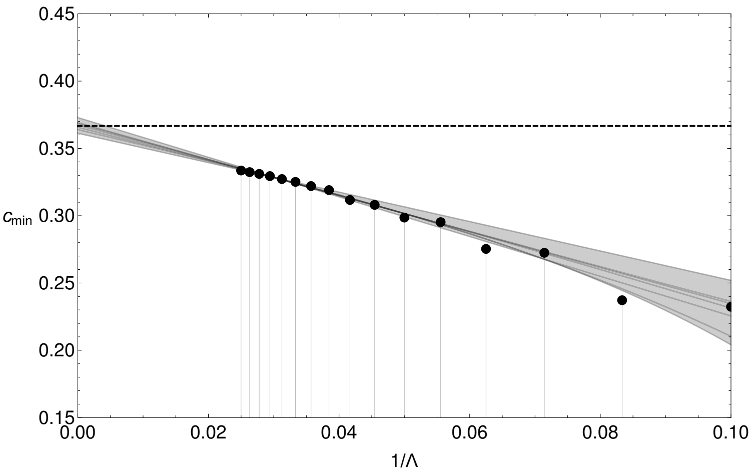


FIGURE 4.1: Numerical lower bound (black dots) on the central charge of theories with an $\mathcal{N}=2$ chiral operator of dimension $r_0 = \frac{6}{5}$ as a function of the inverse cutoff Λ. The lines correspond to various extrapolations to infinitely many derivatives, and the horizontal dashed line marks $c = \frac{11}{30}$ – the central charge of the (A_1, A_2) SCFT.

Our first task is to obtain numerical lower bounds on the central charge $c_{min}(\Lambda)$, for fixed $r_0 = \frac{6}{5}$, as a function of the cutoff Λ. The resulting bound $c_{min}(\Lambda)$ is shown in figure 4.1, together with various different extrapolations to $\Lambda \to \infty$. While the results are consistent with the bound converging to $c_{min} = \frac{11}{30}$ (the dashed line in figure 4.1), they are still not conclusive enough. In what follows we will be agnostic about the $\Lambda \to \infty$ fate of the c-bound, and concentrate on a region around $c \sim \frac{11}{30}$ in an attempt to estimate the CFT data of the (A_1, A_2) theory.

4.1.2 OPE coefficient bounds

We now concentrate on OPE-coefficient bounds for the different short multiplets appearing in the chiral channel, for varying $c \geqslant c_{min}(\Lambda)$, and external dimension fixed to $r_0 = \frac{6}{5}$. In particular, we obtain an upper bound for the OPE coefficient of the $\mathcal{B}_{1,\frac{7}{5}(0,0)}$ multiplet, and both *lower and upper* bounds (see discussion in subsection 2.5.1.4) for the coefficients of the $\mathcal{E}_{\frac{12}{5}}$ and $\mathcal{C}_{0,\frac{7}{5}(\frac{\ell}{2}-1,\frac{\ell}{2})}$ multiplets.

For fixed Λ, there is a unique solution to the truncated crossing equations (2.122) at $c = c_{min}(\Lambda)$ [16, 79], and indeed we will see below that upper and lower bounds (when available) coincide. As already discussed, it is plausible that $c_{min}(\Lambda) \to \frac{11}{30}$ as $\Lambda \to \infty$, and so in this limit the meeting point of upper and lower bounds would be at $c = \frac{11}{30} \simeq 0.367$.

An important subtlety in all the plots that follow is that we cannot fix the central charge exactly: each time we quote a value of c, the corresponding plot captures values *less or equal* than the given number. This follows from the fact that we allow for a continuum of long multiplets with dimensions consistent with the unitarity bounds; for the non-chiral channel this means long multiplets with $\Delta > \ell + 2$ (2.113). However, as is clear from the superconformal blocks (2.115), the contribution of a long multiplet at the unitarity bound mimics the contribution of a conserved current $\hat{\mathcal{C}}_{0,\ell}$. This has two important consequences. First, we cannot restrict ourselves to interacting theories, because it is not possible to set to zero the OPE coefficient of the conserved currents of spin greater than two ($\hat{\mathcal{C}}_{0,\ell\geqslant 1}$), without imposing a gap on the spectrum of all long multiplets. Second, even if we fix the central charge according to (2.116), a long multiplet at the unitarity bound with an arbitrary (positive) coefficient, will increase the value of the $\lambda_{\phi\bar{\phi}\mathcal{O}_{\Delta=2,\ell=0}}$ coefficient, which means that we are really allowing for all central charges smaller than the fixed value. This implies that a given bound can only get weaker as c is increased, and explains the flatness of some of the bounds presented below.

OPE coefficient bound for $\mathcal{B}_{1,\frac{7}{5}(0,0)}$

Let us first consider the OPE coefficient squared of $\mathcal{B}_{1,\frac{7}{5}(0,0)}$. A numerical upper bound, as a function of the central charge, and for fixed external dimension $r_0 = \frac{6}{5}$, is shown

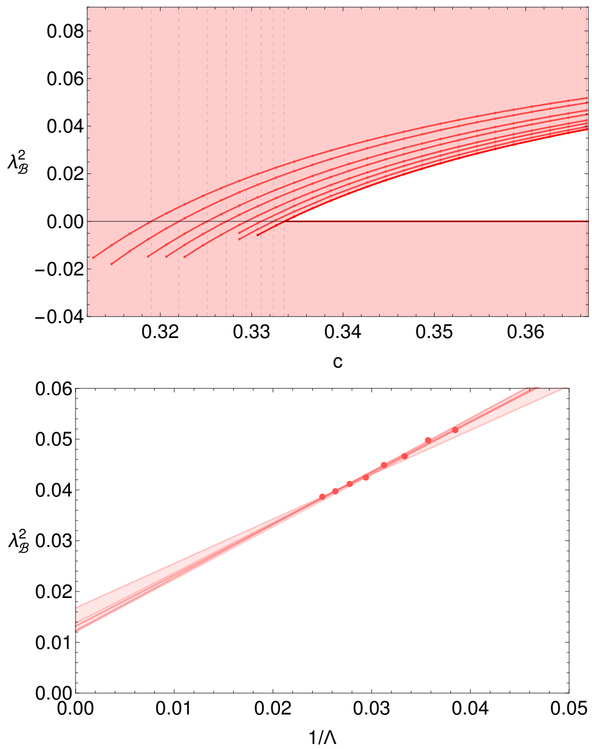


FIGURE 4.2: Numerical upper bound on the OPE coefficient squared of the operator $\mathcal{B}_{1,\frac{7}{5}(0,0)}$ appearing in the chiral channel for $\Lambda = 26, \ldots, 40$, and external dimension $r_0 = \frac{6}{5}$. Left: Upper bound on the OPE coefficient for different values of the central charge, with the strongest bound corresponding to $\Lambda = 40$; the dashed lines mark the minimum central charge as extracted from figure 4.1 for each cutoff Λ, and the solid line marks $c = \frac{11}{30}$. Right: Bound on the OPE coefficient for $c = \frac{11}{30}$ as a function of the inverse cutoff Λ, together with various extrapolations to infinitely many derivatives.

on the left-hand side of figure 4.2 for various values of the cutoff Λ. For each value of Λ, the upper bound vanishes for $c = c_{min}(\Lambda)$ (marked by the dashed vertical lines in the figure), and becomes negative for $c < c_{min}(\Lambda)$, implying there is no unitary solution to the crossing equations. This is consistent with what was found for $\Lambda = 12$ in [27], and suggests this operator is responsible for the existence of the central charge bound. Since such a multiplet is associated with the mixed branch, and the (A_1, A_2) theory has no mixed branch, it would be natural to expect its absence to be a feature of the four-point function of the (A_1, A_2) theory. However, as can be seen on the right-hand side of figure 4.2, the numerical results appear to leave room for solutions to crossing with a small value of this OPE coefficient, as it is not clear if the upper bound will converge to zero as $\Lambda \to \infty$. If there is more than one solution, it is plausible that the one corresponding to the (A_1, A_2) theory is one in which $\mathcal{B}_{1,\frac{7}{5}(0,0)}$ has zero OPE coefficient. We should point out though, that the absence of a mixed branch does not guarantee that the aforementioned multiplet is absent, as it is possible that such a multiplet is present, but one cannot give it a vev and thus no mixed branch exists [105, 115].

OPE coefficient bound for $\mathcal{E}_{\frac{12}{5}}$

Turning to the OPE coefficient of the Coulomb branch chiral ring operator $\mathcal{E}_{\frac{12}{5}}$, we can now place upper and lower bounds as a function of the central charge. We present the results on the left-hand side of figure 4.3 for several values of Λ.

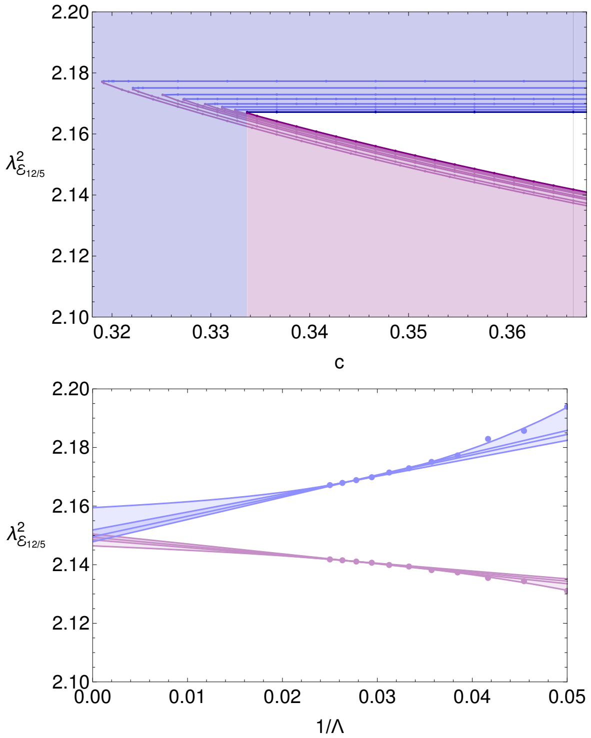


FIGURE 4.3: Numerical upper and lower bounds on the OPE coefficient squared of the chiral operator $\mathcal{E}_{\frac{12}{5}}$ for increasing number of derivatives and external dimension $r_0 = \frac{6}{5}$. Left: Bounds on the OPE coefficient for different values of the central charge, with cutoffs $\Lambda = 26, \ldots, 40$, the vertical line marks $c = \frac{11}{30}$. Right: Various extrapolations of the lower and upper bounds at $c = \frac{11}{30}$ for infinite Λ.

As already discussed, the plots in this section allow for all central charges $c \geqslant c_{\text{fixed}}$, since a gap in the spectrum of spin zero long multiplets is not imposed. This explains the flatness of the upper bound: solutions to crossing saturating it can effectively have central charges equal to $c_{min}(\Lambda)$.[1] The lower bound, however, must be saturated by theories with central charge equal to the fixed value. At $c_{min}(\Lambda)$ the upper and lower bounds coincide, fixing a unique value of the OPE coefficient, and as c is increased a wider range of values, and distinct solutions to crossing, are allowed. We show the

[1]A natural solution would be to impose small gaps in the spectrum of long multiplets, this removes the conserved currents of spin greater than two and fixes the central charge. However, we have no intuition on the size of these gaps, not even for spin zero, as there is no understanding of the number of non-supersymmetry preserving relevant deformations. We experimented imposing that the spectrum of long multiplets obeys $\Delta \geqslant 2 + \epsilon + \ell$ for various small values of ϵ, and although the upper bound gets stronger than that of figure 4.3, it varies smoothly with ϵ and thus there is no justification to pick any specific value. The lower bound, on the other hand, shows a much smaller dependence on ϵ.

allowed range, for $c = \frac{11}{30}$, as a function of $1/\Lambda$ on the right-hand side of figure 4.3. The lines correspond to different extrapolations through (subsets) of the data points, and the shaded region aims to give an idea of where the bounds are converging to. If $c_{min}(\Lambda \to \infty) = \frac{11}{30}$, then the upper and lower bound should converge to the same value, which is not ruled out by the extrapolations. In any case, our results indicate that the OPE coefficient of $\mathcal{E}_{\frac{12}{5}}$ is constrained to a narrow range.

We have thus obtained the following *rigorous* bounds for the value of this OPE coefficient in the (A_1, A_2) theory:

$$2.1418 \leqslant \lambda^2_{\mathcal{E}_{\frac{12}{5}}} \leqslant 2.1672\,, \qquad \text{for } \Lambda = 40\,. \tag{4.1}$$

Furthermore, the most conservative of the extrapolations presented in figure 4.3 gives

$$2.146 \lesssim \lambda^2_{\mathcal{E}_{\frac{12}{5}}} \lesssim 2.159\,, \qquad \text{extrapolated for } \Lambda \to \infty\,. \tag{4.2}$$

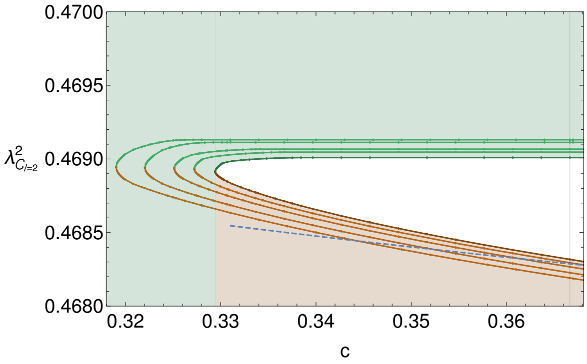
OPE coefficient bounds for $\mathcal{C}_{0,\frac{7}{5}(0,1)}$ and $\mathcal{C}_{0,\frac{7}{5}(1,2)}$

Let us now focus on the $\mathcal{C}_{0,\frac{7}{5}(\frac{\ell}{2}-1,\frac{\ell}{2})}$ family of multiplets. Like in the $\mathcal{E}_{\frac{12}{5}}$ case, upper and lower bounds are possible thanks to the gap that separates these $\mathcal{C}$-type multiplets from the continuum of long operators. The bounds for $\ell = 2, 4$, as a function of c, are shown in figures 4.4a and 4.4b respectively, while bounds for higher values of ℓ can be found in figure 4.6, for fixed $c = \frac{11}{30}$. The dashed lines in figure 4.4 are estimates of the OPE coefficient valid for sufficiently large ℓ, that will be discussed in detail in section 4.2.1. Similarly to the $\mathcal{E}_{\frac{12}{5}}$ multiplet, the OPE coefficients of these multiplets in the (A_1, A_2) theory are constrained to lie in a narrow range:

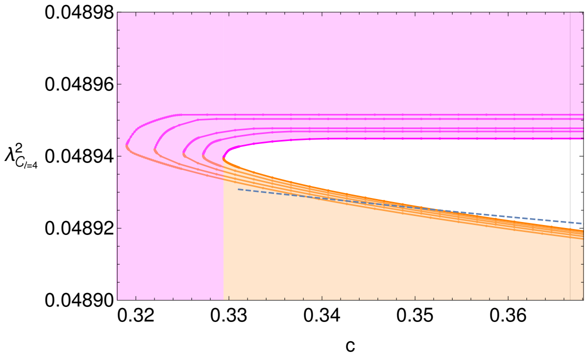
$$0.46831 \leqslant \lambda^2_{\mathcal{C}_{0,\frac{7}{5},(0,1)}} \leqslant 0.46901\,, \quad 0.048919 \leqslant \lambda^2_{\mathcal{C}_{0,\frac{7}{5},(1,2)}} \leqslant 0.048945\,, \quad \text{for } \Lambda = 34\,. \tag{4.3}$$

The upper bounds in figure 4.4 now show a mild dependence on the central charge, and so we can compare the extrapolations of the upper and lower bounds at $c = \frac{11}{30}$ with the extrapolation of the value of the OPE coefficient for the unique solution at $c_{min}(\Lambda)$. Like before, the extrapolations (not shown) do not rule out that $c_{min} \to \frac{11}{30}$ as $\Lambda \to \infty$. As visible in figure 4.4, the value of the OPE coefficient at $c_{min}(\Lambda)$ (the meeting point) shows a very mild dependence on Λ, unlike the OPE coefficient of $\mathcal{E}_{\frac{12}{5}}$, we can therefore obtain the following estimates

$$\begin{aligned} 0.4687 \lesssim \lambda^2_{\mathcal{C}_{0,\frac{7}{5}(0,1)}}(c = c_{min}(\Lambda)) \lesssim 0.4688\,, \\ 0.04892 \lesssim \lambda^2_{\mathcal{C}_{0,\frac{7}{5}(1,2)}}(c = c_{min}(\Lambda)) \lesssim 0.04894\,, \end{aligned} \qquad \text{extrapolated for } \Lambda \to \infty\,. \tag{4.4}$$



(A) $\mathcal{C}_{0,\frac{7}{5}(0,1)}$



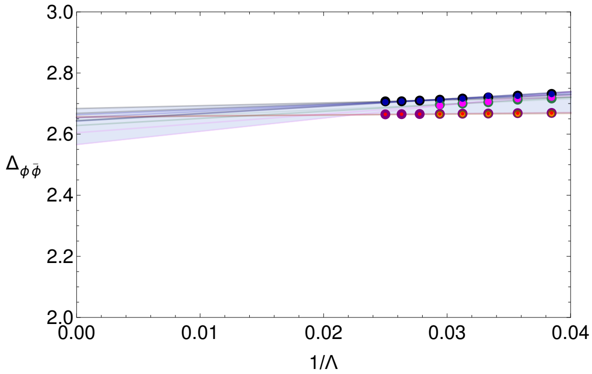
(B) $\mathcal{C}_{0,\frac{7}{5}(1,2)}$

FIGURE 4.4: Numerical upper and lower bounds on the OPE coefficient squared of the chiral channel multiplet $\mathcal{C}_{0,\frac{7}{5}(\frac{\ell}{2}-1,\frac{\ell}{2})}$, for $\ell = 2, 4$, and external dimension $r_0 = \frac{6}{5}$. The bounds were obtained for cutoffs $\Lambda = 26, \ldots, 34$ and the vertical line marks $c = \frac{11}{30}$. The dashed line corresponds to the value obtained from the Lorentzian inversion formula of [15] applied to the chiral channel and using as input only the exchange of the identity and stress-tensor superblocks in the non-chiral channel, and thus valid for sufficiently large ℓ (see section 4.2.1 for more details).

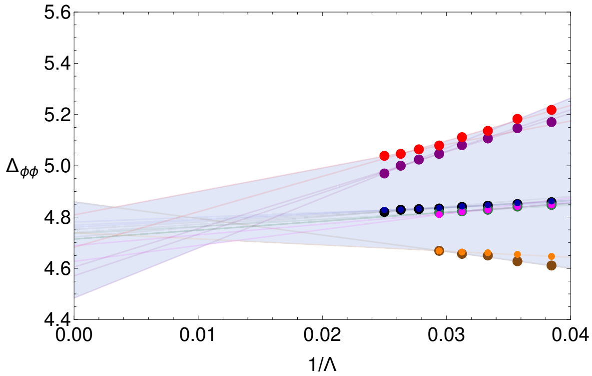
4.1.3 Dimensions of unprotected operators

Finally, we estimate dimensions of unprotected long operators. In [21, 27] numerical upper bounds on the dimensions of the first long in the non-chiral and chiral channels were obtained, for various values of c and r_0. The best bound obtained in [21] for the dimension of the first scalar long operator in the non-chiral channel reads $\Delta_{\phi\bar{\phi}} \leqslant 2.68$, for $\Lambda = 18$, $r_0 = \frac{6}{5}$ and $c \leqslant \frac{11}{30}$. On the other hand, the bound obtained for the first scalar long operator in the chiral channel was very weak and converged too slowly without further assumptions (figure 2 of [27]). Removing the $\mathcal{B}_{1,\frac{7}{5}(0,0)}$ multiplet this bound improved to $\Delta_{\phi\phi} \leqslant 4.93$ for $\Lambda = 20$, $r_0 = \frac{6}{5}$, and did not appear to depend on c [27].

Here, instead, we extract the dimensions of the first long $\mathcal{A}^{\Delta_{\phi\bar{\phi}}}_{0,0,\ell}$ and $\mathcal{A}^{\Delta_{\phi\phi}}_{0,\frac{2}{5},\ell}$ multiplets in the approximate solutions to crossing saturating the various bounds presented above.



(A) Dimension of first long multiplet in the non-chiral channel arising from different bounds.



(B) Dimension of first long multiplet in the chiral channel arising from different bounds.

Figure 4.5: Numerical estimates for the first scalar long operator in the non-chiral (a) and chiral (b) channels obtained from the functionals of figures 4.1, 4.2, 4.3, and 4.4. The data points are color-coded according to the bound the extremal functional was extracted from, and in the cases where the bounds are plotted as a function of c, the functional for $c = \frac{11}{30}$ was used. The lines give an estimate of the extrapolation to infinitely many derivatives.

The results for $\ell = 0$, and various values of the cutoff Λ, are given in figure 4.5a for $\Delta_{\phi\bar{\phi}}$, and in figure 4.5b for $\Delta_{\phi\phi}$. The dimensions were extracted from the extremal functionals [79] of figures 4.2–4.4 for $c = \frac{11}{30}$, and from figure 4.1, and are color coded according to the bound they came from. Note that these are the dimensions of the first long operator present in each of the extremal solutions, and are not rigorous upper bounds, which would require a different extremization problem. The lines in the figures show various extrapolations of the dimensions to $\Lambda \to \infty$.

The estimates of the dimensions of the lowest-lying long multiplet in the non-chiral channel appear all consistent with each other, even at finite Λ. This implies that, even if the various extremization problems are solved by different solutions to the crossing equations, these solutions do not differ by much as far as $\Delta_{\phi\bar{\phi}}$ is concerned, and we can take the spread of the values as an estimate for the uncertainty in the value of this dimension. Then, conservative extrapolations for $\Lambda \to \infty$ of the values coming from the various functionals give

$$2.56 \lesssim \Delta_{\phi\bar{\phi}} \lesssim 2.68\,, \qquad \text{from the extrapolations as } \Lambda \to \infty\,, \tag{4.5}$$

for $\Delta_{\phi\bar{\phi}}$ in the (A_1, A_2) theory. Similarly, the dimensions of the leading twist non-chiral operators with spin $\ell > 0$, obtained from the various extremal functionals of figures 4.1–4.4, are shown in figure 4.7 for $\Lambda = 34$. We will comment on these results in section 4.2.2.

Less coherent are the results for $\Delta_{\phi\phi}$, the values extracted from the extremal functionals of figures 4.1–4.4 look very different for finite Λ, and the extrapolations are not conclusive. This is shown in figure 4.5b. Since the dimensions obtained are so disparate, it is not clear we can get any meaningful estimate for this operator in the (A_1, A_2) theory.

4.2 Inverting the OPEs

In recent years, starting from [12, 13], there has been much progress in understanding the large spin spectrum of CFTs by studying analytically the crossing equations in a Lorentzian limit. In $d > 2$, by studying the four-point function $\langle \phi_1(x_1)\phi_1(x_2)\phi_2(x_3)\phi_2(x_4)\rangle$ in the lightcone limit, the authors of [12, 13] found that in the $t-$channel there must be exchanged, in a distributional sense, double-twist operators, *i.e.* , operators whose dimensions approach $\Delta_1 + \Delta_2 + \ell$ for $\ell \to \infty$, and whose OPE coefficients tend to the values of generalized free field theory. Corrections to these dimensions and OPE coefficients, or rather weighted averages of these quantities, in a large spin expansion can be obtained in terms of the leading twist operators exchanged in the $s-$channel.[2] Assuming the existence of individual operators close to the average values, this procedure was set up to systematically compute the OPE coefficients and dimensions of the double-twist operators in an asymptotic expansion in the inverse spin [155]. The lightcone limit of the crossing equations has been used to constrain the large spin spectrum of various CFTs with different global symmetries and supersymmetries [12–14, 32, 156–169]. Remarkably, this has resulted in predictions for OPE coefficients and anomalous dimensions of operators that match the numerical results down to spin two [11, 14].

Recently, the work of [15] has explained this agreement, by showing that the spectrum organizes in analytic families.[3] There, a "Lorentzian" inversion formula for the $s-$channel OPE of a given correlator was obtained, with the crucial feature that the

[2]Some of the steps taken in the derivations [12, 13] rely on intuitive assumptions, some of which have started to be put on a firm footing in [154].

[3]We thank Marco Meineri for many discussions on [15].

result of the inversion is a function that is analytic in spin (a function valid only for spin greater than one). This established, for sufficiently large spin, the existence of each individual double-twist operator. The inversion formula explained the organization of the spectrum, and allows one to compute individual OPE coefficients and anomalous dimensions, avoiding the asymptotic expansions, and obtaining the coefficients instead of averages.

Motivated by the success of [11], we take the first steps towards a systematic analysis of the (A_1, A_2) theory for large spin, considering both crossing equations (2.122). We apply the inversion formula obtained in [15] to invert the chiral (2.120) and non-chiral (2.114) OPEs. The block decomposition of the former happens to be simply a decomposition in bosonic blocks (2.121), and thus the inversion formula directly applies. The latter has a decomposition in superblocks, but as we shall see, the formula can still be applied, although we must work as if we had a correlator of unequal external operators. The only required modifications will be on the spin down to which the formula holds, and on what the crossed-channel decompositions are. Since the numerical results in supersymmetric theories are not yet at the level of accuracy of the $3d$ Ising model, we refrain from using numerical data as input to the analysis, and instead compare the large-spin estimates coming from the inversion formula with the numerical results. The only input we give is the exchange of the identity and stress-tensor supermultiplet in the non-chiral OPE, and thus find results that are good estimates for sufficiently large spin. We find a reasonable agreement between the OPE coefficients of the leading-twist short operators exchanged in the chiral channel, and the analytical estimate, already for low spin, see figure 4.6. Our analysis also shows that the anomalous dimensions of the double-twist operators in the non-chiral channel, arising from the stress tensor exchange, are small, and we confirm this by matching to numerical estimates of these dimensions obtained from the bounds of section 4.1 (see figure 4.7).

We start with a brief summary of the results of [15] relevant for our purposes and refer the reader to that reference for further details. Starting from the four-point function of unequal scalar operators,

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\mathcal{O}_3(x_3)\mathcal{O}_4(x_4)\rangle = \frac{1}{|x_{12}|^{\Delta_1+\Delta_2}|x_{34}|^{\Delta_3+\Delta_4}} \left|\frac{x_{14}}{x_{24}}\right|^{\Delta_{21}} \left|\frac{x_{14}}{x_{13}}\right|^{\Delta_{34}} \mathcal{G}(z,\bar{z})\,, \quad (4.6)$$

the main result is a "Lorentzian" inversion formula for the $s-$channel decomposition of $\mathcal{G}(z,\bar{z})$ in conformal blocks,

$$\mathcal{G}(z,\bar{z}) = \sum_{\Delta,\ell} \lambda^2_{\Delta,\ell} g^{\Delta_{12},\Delta_{34}}_{\Delta,\ell}(z,\bar{z})\,. \quad (4.7)$$

The OPE coefficients $\lambda^2_{\Delta,\ell}$ (for $\ell > 1$) of the above decomposition (4.7) are then encoded in the residues of a function $c(\ell,\Delta)$ that is analytic in spin, contrasting with the one that can be obtained from a Euclidean inversion of the OPE. The condition $\ell > 1$ arises during the contour manipulations needed to go from the Euclidean inversion of the OPE, valid only for integer ℓ, to the "Lorentzian" formula of [15]. This condition requires looking at the $t-$ and $u-$ channels to bound the growth of $\mathcal{G}(z,\bar{z})$ in a particular region, and it is valid for any unitary CFT. The function $c(\ell,\Delta)$ receives contributions from the $t-$ and $u-$channels, with the even and odd spin operators defining two independent trajectories, as

$$c(\ell,\Delta) = c^t(\ell,\Delta) + (-1)^\ell c^u(\ell,\Delta)\,, \tag{4.8}$$

where c^t and c^u are defined in (3.20) of [15].

The poles of $c(\ell,\Delta)$ in Δ, at fixed ℓ, encode the dimensions of the operators in the theory, with the residues giving the OPE coefficients.[4] As described in section 3.2 of [15], if one is interested only in getting the poles and residues, the inversion formula can be written as

$$\begin{aligned} c^t(\ell,\Delta)\big|_{\text{poles}} &= \int_0^1 \frac{\delta z}{2z} z^{\frac{\ell-\Delta}{2}} \left(\sum_{m=0}^{\infty} z^m \sum_{k=-m}^{m} B^{(m,k)}_{\ell,\Delta} C^t(z,\ell+\Delta+2k) \right)\,, \\ C^t(z,\beta) &= \int_z^1 \delta\bar{z} \frac{(1-\bar{z})^{\frac{\Delta_{21}+\Delta_{34}}{2}}}{\bar{z}^2} \kappa_\beta k_\beta^{\Delta_{12},\Delta_{34}}(\bar{z}) \mathrm{dDisc}\left[\mathcal{G}(z,\bar{z})\right]\,, \end{aligned} \tag{4.9}$$

and similarly for $c^u(\ell,\Delta)$. Here dDisc denotes the double-discontinuity of the function, $k_\beta^{\Delta_{12},\Delta_{34}}(\bar{z})$ is defined in equation (B.1), and

$$\kappa_\beta = \frac{\Gamma\left(\frac{\beta-\Delta_{21}}{2}\right)\Gamma\left(\frac{\Delta_{21}+\beta}{2}\right)\Gamma\left(\frac{\beta-\Delta_{34}}{2}\right)\Gamma\left(\frac{\Delta_{34}+\beta}{2}\right)}{2\pi^2\Gamma(\beta-1)\Gamma(\beta)}\,. \tag{4.10}$$

The $z \to 0$ limit of the block in (4.9) gave the collinear block $k_\beta^{\Delta_{12},\Delta_{34}}(\bar{z})$ which does not take into account all descendants, these are instead taken into account by the functions $B^{(m,k)}_{\ell,\Delta}$, as discussed in [15]. Since we shall focus only on leading twist operators we do not need to subtract descendants and thus do not need these functions, apart from $B^{(0,0)}_{\ell,\Delta} = 1$. A term $z^{\frac{\tau(\Delta+\ell)}{2}}$ in the bracketed term in (4.9) implies there exists a pole at $\Delta - \ell = \tau(\Delta+\ell)$, with its residue, taken at fixed ℓ, providing the OPE coefficient; see [15] for more details.

[4] In some cases the residues need to be corrected as discussed in (3.9) of [15], but for the computations carried out in this section this correction is not needed.

4.2.1 Inverting the chiral OPE

The Lorentzian inversion formula obtained in [15] can be directly applied to invert the $s-$channel OPE of the correlator (2.120),

$$\langle \phi(x_1)\phi(x_2)\bar{\phi}(x_3)\bar{\phi}(x_4)\rangle = \frac{1}{x_{12}^{2\Delta_\phi} x_{34}^{2\Delta_\phi}} \sum_{\mathcal{O}_{\Delta,\ell}} |\lambda_{\phi\phi\mathcal{O}_{\Delta,\ell}}|^2 g_{\Delta,\ell}^{0,0}(u,v)\,, \tag{4.11}$$

as it is exactly of the form (4.6), with $\mathcal{G}(z,\bar{z})$ admitting a decomposition in bosonic blocks, with $\Delta_{12} = \Delta_{34} = 0$. We can thus apply (4.8) directly, with the $t-$ and $u-$channel decompositions as dictated by crossing symmetry of (4.11). Since the operators at point one and two are identical, the $u-$ and $t-$ channels give identical contributions, and thus only even spins appear in the $s-$channel OPE of (4.11), precisely in agreement with Bose symmetry.

We now want to make use of the generating functional (4.9), to obtain the dimensions and OPE coefficients of the $s-$channel operators (at least for large enough spin) by providing information about the $t-$channel decomposition. For large spin (that is large β) the leading contributions in (4.9) come from the $\bar{z} \to 1$ limit of the integrand, with the leading contribution corresponding to the lowest twist operators exchanged in the $t-$channel. The $t-$channel decompositions are given by the non-chiral OPE, as follows from (2.122a). From (2.113) we see that, after the identity, the leading contribution comes from the superconformal multiplet of the stress tensor $\hat{\mathcal{C}}_{0,0}$, and, since we are interested on interacting theories, there is no other contribution with the same twist. The next contributions will arise from long multiplets, for which we only currently have the numerical estimates for their dimensions obtained in section 4.1.3. At large spin the contributions of one of these operators of twist τ behaves as $\ell^{-\tau}$ [12, 15].[5] The leading twist operators have been estimated numerically from the various extremal functionals obtained in section 4.1. The leading spin zero operator could have twist as low as $\tau \sim 2.5$, while the higher spin operators (figure 4.7) all appear to have twists close to $2r_0 = 2.4$, with small corrections depending on the spin. This is to be compared with the contribution of the stress tensor with exactly $\tau = 2$. For sufficiently large spin the contributions of long multiplets are subleading, so in what follows we shall consider only the stress-tensor and identity exchanges in the $t-$ and $u-$channels.

The identity and stress tensor contribute to (4.9) according to the crossing equation (2.122a), with the identity contributing as $|\lambda_{\phi\bar{\phi}1}|^2 \tilde{\mathcal{G}}_{0,0}(u,v) = 1$. The stress-tensor superblock is given by (2.118) with $\Delta = 2$, $\ell = 0$, and with OPE coefficient given by

[5] Here we are using the bosonic results of [12, 15], while we have a superblock contribution at twist τ. However, decomposing the superblock in bosonic blocks we find a finite number of bosonic blocks with twist τ together with a finite number of higher twist, and so the presence of the superblock will only modify the coefficient of the leading behavior for large ℓ, which is unimportant for our point here.

(2.116), and we find

$$C^t(z,\beta) \supset \int_0^1 \frac{\delta\bar{z}}{\bar{z}^2}\kappa_\beta k_\beta^{\Delta_{12},\Delta_{34}}(\bar{z})\mathrm{dDisc}\left[\frac{(z\bar{z})^{r_0}}{((1-z)(1-\bar{z}))^{r_0}}\left(1+|\lambda_{\phi\bar{\phi}\hat{\mathcal{C}}_{0,0}}|^2\tilde{\mathcal{G}}_{2,0}(1-z,1-\bar{z})\right)\right]. \tag{4.12}$$

From the identity contribution, which is the leading one for large spin, we recover the existence of double-twist operators $[\phi\phi]_{m,\ell}$ (see for example section 4.2 of [15]), namely operators with dimensions approaching

$$\Delta_{[\phi\phi]_{m,\ell}} \underset{\ell\gg 1}{\longrightarrow} 2r_0+2m+\ell\,, \qquad \ell \text{ even}\,, \tag{4.13}$$

and with OPE coefficients approaching those of generalized free field theory,

$$\lambda^2_{\text{gfft}} = \frac{\left((-1)^\ell+1\right)\left((r_0-1)_m\right)^2\left((r_0)_{m+\ell}\right)^2}{m!\ell!\,(m+2r_0-3)_m\,(\ell+2)_m\,(m+\ell+2r_0-2)_m\,(2m+\ell+2r_0-1)_\ell}\,, \tag{4.14}$$

at large spin. In (4.14) $(a)_b$ denotes the Pochhammer symbol.

To compute the leading correction to these dimensions and OPE coefficients at large spin we take into account the contribution of the stress-tensor multiplet to the OPE. To do so we take the $z\to 0$ limit of (4.12); as pointed in [15], the correct procedure should be to subtract a known sum, such that the limit $z\to 0$ commutes with the infinite sum over $t-$channel primaries. However, when anomalous dimensions are small this procedure gives small corrections to the naive one of taking a series expansion in z and extracting anomalous dimensions from the terms proportional to $\log z$ (the generating function should have $z^{\gamma/2}\approx 1+\frac{1}{2}\gamma\log z+\ldots$) and corrections to OPE coefficients from the terms without $\log z$. For the case considered below the situation is even better as the anomalous dimensions of the operators we are interested in vanish. Taking the small z limit, the first observation is that anomalous dimensions, *i.e.* , log-terms, only come with a power of $z^{\Delta_\phi+2}$, and thus only the operators $[\phi\phi]_{m\geqslant 2,\ell}$ acquire an anomalous dimension. This is consistent with the fact that from the block decomposition (2.121) we identify the double-twist operators with $m=0,1$ as short multiplets, $\mathcal{C}_{0,2r_0-1(\frac{\ell}{2}-1,\frac{\ell}{2})}$ and $\mathcal{C}_{\frac{1}{2},2r_0-\frac{3}{2}(\frac{\ell}{2}-\frac{1}{2},\frac{\ell}{2})}$ respectively, whose dimensions are protected.[6]

We can now compute corrections to the OPE coefficient of the $\mathcal{C}_{0,2r_0-1,(\frac{\ell}{2}-1,\frac{\ell}{2})}$ operators, for $r_0=\frac{6}{5}$ and $c=\frac{11}{30}$, from (4.12). The result is plotted in figure 4.6, where we performed the integral in (4.12) numerically (after taking the leading z term), together with the

[6]We assume $\ell\geqslant 2$ here since the inversion formula is not guaranteed to converge for $\ell=0$.

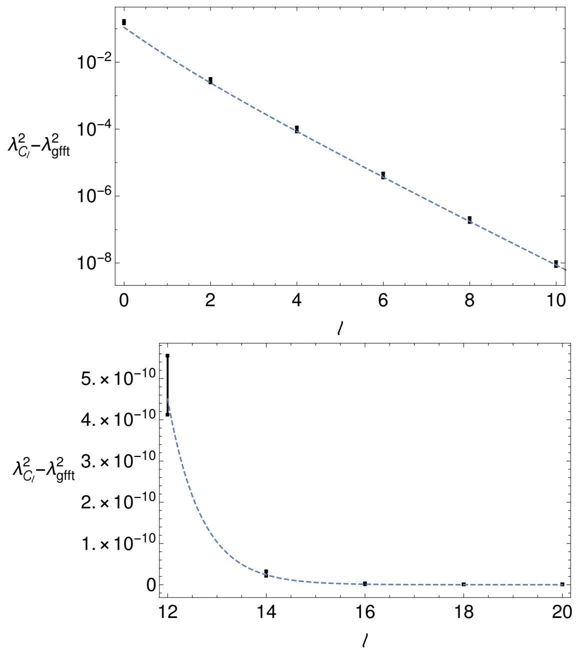


FIGURE 4.6: Comparison between the numerical bounds on the OPE coefficient squared of the leading twist operators in the chiral channel and the results from the inversion formula (4.12), for $c = \frac{11}{30}$ and external dimension $r_0 = \frac{6}{5}$. The black boxes mark the numerically allowed range for the squared OPE coefficients of the $\mathcal{C}_{0,\frac{7}{5},(\frac{\ell}{2}-1,\frac{\ell}{2})}$ operators for different values of ℓ (the $\ell = 0$ operator should be interpreted as $\mathcal{E}_{\frac{12}{5}}$) and with $\Lambda = 36$. The dashed line shows the result of equation (4.12), where we considered only the contribution of the identity and stress-tensor operators in the non-chiral channel, and thus is an approximate result for sufficiently large spin. The formula (4.12) is not guaranteed to be valid for $\ell = 0$, and the results here are just shown as an illustration.

numerical upper and lower bonds on the OPE coefficients, obtained in section 4.1.2.[7] The results for $\ell = 0$ (where the multiplet becomes an $\mathcal{E}_{2r_0}$) are also shown even though the formula is only guaranteed to be valid for $\ell \geqslant 2$.[8]

We point out that the only input was the leading $t-$ and $u-$channels contributions and thus the resulting OPE coefficients are an approximation for sufficient large spin. Indeed, the neglected contribution of the long multiplets should behave like $\ell^{-\tau}$ for large spin, with $\tau \sim 2.4$, while the stress tensor contributes as $\tau = 2$. Nevertheless, we see in figure 4.6 that starting from $\ell = 4$ the analytical result is already inside the numerically allowed range for the OPE coefficient. This is shown clearly in figure 4.4b where the result of (4.12) for $\ell = 4$ is shown as a dashed blue line, together with the numerically allowed range. For $\ell = 2$, however, the result of (4.12) (blue dashed line in

[7]Note that by the usual lightcone methods [12, 13, 155], we could obtain an asymptotic expansion in $\frac{1}{\ell}$ of the correction to the generalized free field theory OPE coefficients (4.14) arising from the stress tensor exchange. By considering the contributions of the stress tensor superblock to (4.9) we are effectively re-summing the lightcone expansion to all orders.

[8]The formula could only be valid for $\ell = 0$ if, for some reason, the growth of the four-point function of the (A_1, A_2) theory, in the limit relevant for the dropping of arcs of integration along the derivation of the inversion formula, was better than the generic growth expected in any CFT and derived in [15].

figure 4.4a) is clearly insufficient, as it is outside the numerically allowed region. Note that the numerical results are not optimal yet, *i.e.* , while they provide true bounds they have not yet converged, and the optimal bounds will be more restrictive. Thus, the fact that the $\ell = 4$ estimate was inside the numerical bounds should not be taken to mean the subleading contributions are negligible for such a low spin. What is in fact surprising is that the estimates from (4.12) are so close to the numerically obtained ranges for such low values of the spin. These results leave us optimistic that better estimates can be obtained by providing a few of the subleading contributions, as was done in [11] for the $3d$ Ising model. The computation used to obtain figure 4.6 could be easily extended to obtain estimates for the OPE coefficients of the $\mathcal{C}_{\frac{1}{2},2r_0-\frac{3}{2}(\frac{\ell}{2}-\frac{1}{2},\frac{\ell}{2})}$ multiplets, and also the dimensions and OPE coefficients of the remaining operators in (2.119). One particularly interesting multiplet would be $\mathcal{B}_{1,\frac{7}{5}(0,0)}$ since, as discussed before, we expect it to be absent in the (A_1, A_2) theory. However, this corresponds to a spin zero contribution and thus convergence of the inversion formula is not guaranteed.

4.2.2 Inverting the non-chiral OPE

Next we turn to the non-chiral channel, where we have a decomposition in superconformal blocks, and so we must obtain a supersymmetric version of the inversion formula of [15]. We consider the inversion of the $s-$channel OPE of (2.114), with the superblocks given by (2.115),

$$\langle \phi(x_1)\bar{\phi}(x_2)\phi(x_3)\bar{\phi}(x_4)\rangle = \frac{(z\bar{z})^{-\frac{\mathcal{N}}{2}}}{x_{12}^{2\Delta_\phi}x_{34}^{2\Delta_\phi}}\left(\sum_{\mathcal{O}_{\Delta,\ell}}|\lambda_{\phi\bar{\phi}\mathcal{O}_{\Delta,\ell}}|^2 g^{\mathcal{N},\mathcal{N}}_{\Delta+\mathcal{N},\ell}(z,\bar{z})\right), \tag{4.15}$$

where we are interested in taking $\mathcal{N} = 2$, but the same equation is also valid for $\mathcal{N} = 1$, and so all that follows generalizes easily to that case. Fortunately, the fact that, up to the overall prefactor $(z\bar{z})^{-\mathcal{N}/2}$ in (4.15), the blocks relevant for the $s-$channel decomposition are identical to bosonic blocks of operators with unequal dimensions makes the task of obtaining an inversion formula very easy. We can use the results of [15] with small modifications: The Lorentzian inversion formula applies to the term between brackets in (4.15), and the fact that the pre-factor is not the correct one for operators of unequal dimension plays a small role in the derivation of [15]. The only time the prefactor is considered is when bounding the growth of the correlator, needed to show the inversion formula is valid for spin greater than one. The modified prefactor here seems to ameliorate the growth: we are inverting $(z\bar{z})^{\frac{\mathcal{N}}{2}}$ times a CFT correlator whose growth is bounded as discussed in [15]. The condition $\ell > 1$ on the inversion formula (4.9) came from the need to have ℓ large such that one could drop the arcs at infinity during the derivation of [15]. The prefactor's behavior in this limit means the inversion formula will be valid for all $\ell > 1-\mathcal{N}$, and the results we obtain for $\mathcal{N}=2$ should be valid for all spins. Apart from this, the prefactor will only play a role when representing the correlator by its $t-$ and $u-$channel OPEs. As such we apply (4.9) with

$$\mathcal{G}(z,\bar{z}) = \sum_{\mathcal{O}_{\Delta,\ell}} |\lambda_{\phi\bar{\phi}\mathcal{O}_{\Delta,\ell}}|^2 g^{\mathcal{N},\mathcal{N}}_{\Delta+\mathcal{N},\ell}(z,\bar{z})\,. \tag{4.16}$$

The $t-$ and $u-$channels of the correlator (4.15) are given by a non-chiral and chiral OPE respectively. Using the crossing equation (2.122b) we see that the $t-$channel expansion of $\mathcal{G}(z,\bar{z})$ is

$$\mathcal{G}(z,\bar{z}) = (z\bar{z})^{\frac{\mathcal{N}}{2}} \left(\frac{z\bar{z}}{(1-z)(1-\bar{z})}\right)^{r_0} \sum_{\Delta,\ell} |\lambda_{\phi\bar{\phi}\mathcal{O}_{\Delta,\ell}}|^2 \mathcal{G}_{\Delta,\ell}(1-z,1-\bar{z})\,, \tag{4.17}$$

with the superblock given by (2.115). While the $u-$channel is given by

$$\mathcal{G}(z,\bar{z}) = (z\bar{z})^{r_0+\frac{\mathcal{N}}{2}} \sum_{\Delta,\ell} |\lambda_{\phi\phi\mathcal{O}}|^2 g_{\Delta,\ell}\left(\frac{1}{z},\frac{1}{\bar{z}}\right)\,. \tag{4.18}$$

Once again, the leading contributions to the $s-$channel spectrum at large spin, *i.e.* , the leading contributions for $\bar{z}\to 1$ in (4.17), are from the $t-$channel identity and stress-tensor multiplet. The subleading contributions in the $t-$channel come from long multiplets with $\Delta > \ell+2$. On the other hand, the leading twist contribution in the $u-$channel arises from the $\mathcal{E}_{2r_0}$ and $\mathcal{C}_{0,2r_0-1,(\frac{\ell}{2}-1,\frac{\ell}{2})}$ multiplets, whose twists are all exactly $2r_0$, and so one should consider the infinite sum over ℓ. From a lightcone computation, *e.g.* , [170], we expect an individual chiral operator of twist τ_c to contribute to the anomalous dimensions of the non-chiral operators at large ℓ as $\frac{(-1)^\ell}{\ell^{\tau_c}}$. Similarly, a non-chiral operator of twist τ contributes to the same anomalous dimension at large ℓ as $\frac{1}{\ell^\tau}$. In the case at hand, $\tau=2$ for the stress-tensor multiplet and $\tau_c = 2.4$ for each of the infinite number of leading operators in the chiral channel. The contribution of an individual chiral operator in the $u-$channel is thus subleading for sufficiently large spin. This is similar to what happened in section 4.2.1, and while in this case the dimensions of the operators are protected, their OPE coefficients are not. Indeed, the value of these OPE coefficients remains elusive, and the best estimate we have to go on comes from the numerically obtained bounds for the operators with $\ell \leqslant 20$ presented in figure 4.6. An interesting possibility would be to attempt to combine the numerical ranges for low spin with the estimate for the large spin OPE coefficients obtained from (4.12). The numerical bounds on the OPE coefficients would turn into an estimate, in the form of an interval, for the anomalous dimension; we leave this exploration for future work. Here we apply the inversion formula (4.9) only to the exchange of the identity and stress-tensor multiplets

$$C^t(z,\beta) \supset \int_0^1 \frac{\delta\bar{z}}{\bar{z}^2}\kappa_\beta k_\beta^{1,1}(\bar{z})\mathrm{dDisc}\left[\frac{(z\bar{z})^{r_0+1}}{((1-z)(1-\bar{z}))^{r_0}}\left(1+|\lambda_{\phi\bar{\phi}\hat{\mathcal{C}}_{0,0}}|^2\mathcal{G}_{2,0}(1-z,1-\bar{z})\right)\right] \quad (4.19)$$

where one should recall that $\Delta_{12}=\Delta_{34}=\frac{\mathcal{N}}{2}=1$ when taking the double-discontinuity.

Like before, the exchange of the identity in (4.19) gives the existence of double-twist operators $\left[\phi\bar{\phi}\right]_{m,\ell}$, with dimensions

$$\Delta_{\left[\phi\bar{\phi}\right]_{m,\ell}} \underset{\ell\gg 1}{\longrightarrow} 2r_0+2m+\ell\,. \quad (4.20)$$

Computing the OPE coefficients from the identity exchange we find, for the leading twist operators,

$$|\lambda_{\phi\bar{\phi}\left[\phi\bar{\phi}\right]_{0,\ell}}|^2 \underset{\ell\gg 1}{\longrightarrow} \frac{4^{2-\ell}r_0(r_0)_{\ell-2}(2r_0+1)_{\ell-2}}{(1)_{\ell-2}(r_0+\ell-2)\left(r_0+\frac{1}{2}\right)_{\ell-2}}\,, \quad (4.21)$$

which are precisely the OPE coefficients of generalized free field theory, now decomposed in superblocks instead of bosonic blocks.

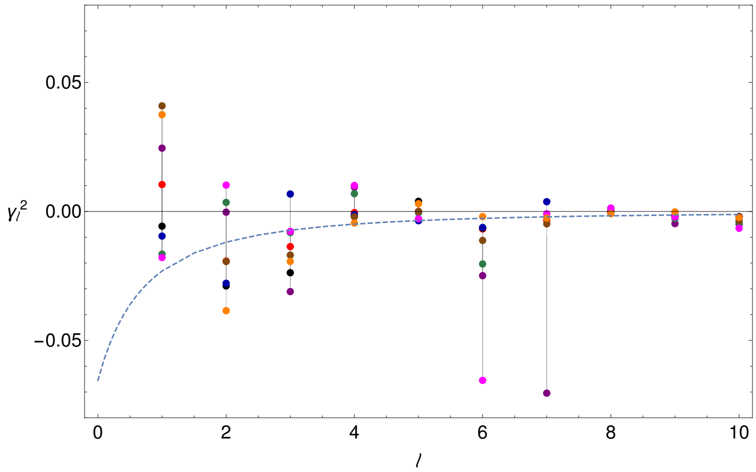


FIGURE 4.7: Anomalous dimension ($\gamma_\ell = \Delta_\ell - (2\Delta_\phi + \ell)$) of the first spin ℓ long multiplet in the non-chiral channel. The colored dots are the dimension estimates extracted from the extremal functionals of the various bounds (figures 4.1 and 4.2-4.4 for $c=\frac{11}{30}$) as indicated by their colors, with $\Lambda = 34$. The dashed line corresponds to the result from the inversion formula (4.19), for $c=\frac{11}{30}$ and external dimension $r_0=\frac{6}{5}$, taking into account only the exchange of the identity and stress tensor in the $t-$channel, and is thus an approximate result for sufficiently large spin.

The stress-tensor exchange provides corrections to these dimensions and OPE coefficients. As an illustration we computed its contribution to the anomalous dimensions of the leading twist operators $\left[\phi\bar{\phi}\right]_{0,\ell}$, $\gamma_\ell=\Delta_\ell-(2\Delta_\phi+\ell)$. From the numerical estimates (see figure 4.7) we see the anomalous dimensions starting at spin one are rather small, and so we simply take the zeroth order of the procedure outlined in [15] to commute the $z \to 0$ limit with the sum over primaries in (4.19). These results are also shown in figure 4.7 for $\ell \geqslant 1$ as a dashed blue line, together with estimates for these values arising from the various extremal functionals of section 4.1, color coded according to which bound they came from.[9] We are plotting the results starting from spin $\ell = 1$. The leading $\ell = 0$ operator is the stress tensor itself, which was not present in the generalized free field theory solution. As such the dimension of $\left[\phi\bar{\phi}\right]_{0,0}$ must come down from $2r_0 = 2.4$ to exactly 2. The value of the anomalous dimensions coming from (4.19) is still insufficient for this to happen, as clear from figure 4.7. For $\ell \geqslant 1$, however, the numerical estimates of leading twist operators' dimensions are very close the values of double-twist operators (4.20). Indeed, the maximum anomalous dimension in figure 4.7, ignoring the two outlying points, is of the order of $\gamma_1 \sim 0.04$, in a dimension that is close to $2r_0 + 1 = 3.4$. The anomalous dimensions obtained from (4.19) (dashed blue line in (4.7)) are close to the numerically obtained values starting from $\ell = 2$, despite the fact that our results are only valid for sufficiently large spin, as we have only considered the identity and stress tensor contributions in the $t-$channel, and completely disregarded any $u-$channel contribution. In particular, for spin $\ell \gtrsim 8$ the numerical estimates arising from the different extremization problems of section 4.1 are all cluttered, approaching the value (4.20), and close to the values coming from (4.19).

[9] We omitted two spin seven dimensions, as we could not accurately estimate them from the functionals. The two points that appear to be outlying in spin 6 and 7 correspond to cases where there were two zeros of the functional very close to one another, and we extracted the dimension of the first. We expect that higher derivative orders would fix both situations.

Chapter 5

Conclusions and outlook

In this thesis we have shown examples of the application of the superconformal bootstrap techniques to supersymmetric theories in two and four dimensions. The superconformal bootstrap program has been very successful in recent years, allowing to obtain non-perturbative results in theories that would be hard to access by other means. Such program has two ambitious goals: charting out the theory space and solving specific theories. To make progress in these directions it is necessary, on the one hand, to consider less supersymmetric multiplets inside the correlation functions and, on the other hand, to focus on some minimal models which could be more easily accessed by these techniques. In chapter 3 we initiated, in two dimensions, the bootstrap of long multiplets, using the whole superfield as the external operator. In chapter 4 we examined, using the numerical bootstrap, what we argued to be the simplest $\mathcal{N} = 2$ theory in four dimensions.

Long multiplet bootstrap

While long multiplets have been considered in the past, from the point of view of kinematics [20, 31, 39, 41, 44, 171], and recently through a numerical analysis of dynamical information [19], all previous work has been restricted to considering only the superconformal primary of long multiplets. Unlike the case of external chiral operators (or BPS operators in general) where the four-point function depends only on the supersymmetrization of the regular bosonic conformal and R-symmetry invariants, for more general external fields one starts finding nilpotent superconformal invariants. This implies that information is lost by restricting the four-point function to the superconformal primary, *i.e.* setting all fermionic coordinates to zero. Even in cases such as the one considered in section 3.2, where Bose symmetry fixes the correlation function involving external descendants from that of external primaries, the crossing symmetry constraints for correlators with external superdescendants were nontrivial. Upon setting all fermionic coordinates to zero the superblocks reduce to bosonic conformal blocks, and the crossing equations are simply those of a non-supersymmetric theory. Supersymmetry manifests itself in the constraints appearing when one considers also external superdescendents.

Although we treated the two-dimensional $\mathcal{N} = 1$ case only as a warm-up example, making manifest some of the features important to our discussion, we expect that also in the case of the $\mathcal{N} = (1,1)$ bootstrap, non-trivial constraints arise from considering the full four-point function. As was pointed out in [31] for a two-dimensional $\mathcal{N} = (1,1)$ SCFT, if we restrict the external operators to be the superconformal primaries, *i.e.* setting all fermionic coordinates to zero, the superconformal blocks reduce to a sum of bosonic blocks.[1] Once again the non-trivial constraints should come from considering the correlation functions of external superdescendants.

In two-dimensions, the blocks obtained in Chapter 3 are restricted to the OPE channel between opposite charged operators for brevity, but it would be straightforward to obtain results in the OPE channel between operators of different charge. The charged sector, and hence the full set of superblocks for any value of the external charges, is important of one wants to distinguish the (2,2) minimal models for the (2,0) heterotic theories (3.39) of Gadde and Putrov. Of course, studying the space of (2,2) theories is of independent interest. The numerical bootstrap approach to $\mathcal{N} = (2,2)$ case can be easily addressed simply by patching together the holomorphic and anti-holomorphic superblocks of section 3.1, extending the work done in [30] from chiral operators to long ones.

Interestingly, the blocks we have computed could be extremely useful also for the study of superconformal defects. One instance is the case of BPS lines in three dimensions, where the preserved superalgebra is exactly the same we considered [172, 173]. More in general tow dimensional superconformal algebras are always product of two one-dimensional superconformal algebra. Therefore all the results that are obtained for two-dimensional theories can be extended to the case of line defects [174].

Finally, one clear future direction would be to extend these results to higher dimensions, following what was done in [31] for correlators involving chiral operators. In particular they defined the superconformal algebra with four supercharges in an arbitrary number of dimensions, allowing to write the Casimir operator in $2 \leqslant d \leqslant 4$. Recall that this corresponds to theories with $\mathcal{N} = (2,2)$ in two dimensions and $\mathcal{N} = 1$ in four dimensions. By solving the Casimir equation in arbitrary d one gets, in one blow, the superblocks

[1]Similarly for the case of three-dimensional $\mathcal{N} = 1$, which has the same number of supercharges, the superblock turns into a regular bosonic block after setting all fermionic variables to zero [29].

involving chiral fields for all these theories. Our approach in this thesis provides the case of the two-dimensional $\mathcal{N} = (2, 2)$ long blocks, and the structure of the four-point function, *i.e.* the number of superconformal invariants will be the same in higher dimensions. Therefore one could write both the quadratic and cubic Casimirs in arbitrary dimensions and proceed along the lines of section 3.1. One technical difficulty we can foresee is the need to use spinning blocks, as even if one consider a scalar superconformal primary, among its superdescendants operators with spin will appear. Moreover, solving the Casimir equation is much easier if one can give an Ansatz for the superblock in terms of a sum of bosonic blocks. Such a procedure requires constructing conformal primaries out of superdescendants which can get cumbersome. In fact precisely the methods developed in this thesis have recently been used to obtain the full superblocks for the case of a chiral, and anti-chiral and two generic long scalar multiplets in [175]. In that case the computations were simplified by the fact that there were only two nilpotent invariants and there was no need to construct conformal primaries.

Alternatively, it would be of interest to extend the approach proposed in [176] to the case of superconformal groups. Quite generally, it leads to a reformulation of conformal Casimir equations as eigenvalue equations for certain Calogero-Sutherland Hamiltonians, in agreement with [177]. As was shown at the example of three-dimensional fermionic seed blocks in [176], the reformulation in terms of Calogero-Sutherland models is very universal and in particular works for spinning blocks as well as for scalars. Hence, one would expect that a universal set of Casimir equations for long multiplets of superconformal groups can be derived in any dimension. Moreover, by exploiting the integrability of Calogero-Sutherland Hamiltonians it should be possible to develop a systematic solution theory [177, 178], without the need for an Ansatz that decomposes superblocks in terms of bosonic ones.

Another clear future direction corresponds to obtaining the $\mathcal{N} = 1$ stress-tensor multiplet superblocks, which despite being an essential multiplet to consider in any bootstrap studies, remain unknown.[2] In this case however the superconformal primary has spin one. It could happen that the extra conditions arising from conservation make the Casimir differential equations in this case simpler to solve, otherwise it could simply be obtained from imposing conservation on the generic long blocks.

Application to $\mathcal{N} = 3$ theories in 4d

In a different direction, the holomorphic long blocks we computed, plus the same blocks relaxing the charge conditions we took for simplicity, together with the blocks involving

[2]although recently progress has been made in [179], once again only the superprimaries were considered

external chiral operators of [31, 40] are all the blocks that are required for the study of chiral algebras [52] associated to $\mathcal{N} = 3$ SCFTs. These blocks allowed us to obtain an infinite number of four-dimensional (sums of squared) OPE coefficients of $\mathcal{N} = 3$ theories, in terms of a single parameter, the central charge of the four-dimensional theory. These numbers correspond to the coupling between the Schur operators in the four-dimensional stress-tensor multiplet, and the Schur operators that appear in its self-OPE. They are universal, in the sense that no assumptions about specific $\mathcal{N} = 3$ theories were made, apart from the demand that the theory be interacting, and are a necessary ingredient in the superconformal bootstrap program of $\mathcal{N} = 3$ stress tensors.

Requiring unitarity of the four-dimensional theory provided a new analytic unitarity bound

$$c_{4d} \geqslant \frac{13}{24}\,, \tag{5.1}$$

valid for any interacting theory. Unlike similar bounds for $\mathcal{N} = 2$ and $\mathcal{N} = 4$ SCFTs, we have argued this bound *cannot* be saturated by any interacting unitary SCFT. Our arguments have provided a first non-trivial example of a chiral algebra that cannot appear as cohomology of a four-dimensional SCFT. Namely they provided an example of what can go wrong when we try to interpret a given chiral algebra as arising from a four-dimensional SCFT. Since there are also no known theories close to saturating (5.1) one might wonder if they could be ruled out by reasonings similar to the one used here, and whether its possible to obtain a stronger bound saturated by a physical $4d$ SCFT. We leave this question for future work, as it would require going deeper in the bigger question of what are the requirements for a two-dimensional chiral algebra to correspond to fully consistent four-dimensional SCFT. Similar reasoning might also help improve the bounds obtained in [52, 69]. Adding extra assumptions about specific theories by considering mixed systems of correlators, such as including chiral operators (arising from four-dimensional half-BPS multiplets) could provide new constraints on the space of theories, although one starts getting ambiguities in the four-dimensional interpretation of two-dimensional multiplets, as discussed in [35] for the simplest half-BPS correlator.

Finally, the blocks we have computed are a piece of the full four-dimensional superblocks of (non-chiral) Schur operators, obtained by performing the chiral algebra twist on the full blocks. An essential superblock for the $\mathcal{N} = 3$ superconformal bootstrap program corresponds to having stress-tensor multiplets as the external state. Although these blocks are still unknown, our analysis captures the chiral algebra subsector of these blocks, and in particular the statement that information is lost by setting all fermionic variables to zero (*i.e.* considering the correlation function of superconformal primaries) remains true for the whole system.

The (A_1, A_2) Argyres Douglas theory

In Chapter 4 we followed a different strategy and we focused on a single $\mathcal{N} = 2$ theory in four dimensions, the (A_1, A_2) Argyres Douglas theory. One could argue this is the simplest interacting theory in four dimensions with $\mathcal{N} = 2$ supersymmetry, namely the one with the lowest possible value of the central charge c, as it was analytically shown in [68]. In Chapter 4 we started by a numerical analysis of this theory improving the results of [27]. We provided evidence that the (A_1, A_2) theory saturates the c-bound $c = \frac{11}{30}$ and we considered various OPE coefficients providing upper and lower bounds on their values. These constrained the OPE coefficients to lie in very narrow ranges, and correspond to the first results on non-protected observables for this Argyres Douglas theory, we were also able to estimate the dimension of the lowest-twist unprotected long operator appearing in the non-chiral OPE.

In Section 4.2 we tackled the (A_1, A_2) theory from a different angle, i.e. using the Lorentzian inversion formula. Such formula was derived in [15] and it allows to extract individual OPE coefficients and scaling dimensions of the exchanged operators starting from the result of an arbitrary four-point function. A crucial feature of the inversion formula is that the CFT data, for a spin greater than a certain value, is analytic in spin thus operators organize in trajectors. This is guaranteed to be true in any CFT for spin greater than one [Caron-Huot]. In this thesis we performed the Lorentzian inversion of both the chiral and non-chiral OPEs, and while in the former case the inversion is identical to the non-supersymet ric one, and the CFT data is only guaranteed to be analytic for $\ell > 1$, in the latter supersymmetry implies analyticity for $\ell > 1 - \mathcal{N}$.[3] Furthermore, the large spin behavior of the CFT data in one channel is dominated by the low twist CFT data in the crossed channel. As such, by giving a minimal input, namely the leading twist CFT data, which corresponds to the exchange of the identity and stress tensor supermultiplets in the non-chiral channel, we obtained estimates for the chiral and non-chiral CFT data, valid for large spin.

All in all, we have seen that both in the chiral and non-chiral channels the estimates coming from applying the inversion formula, and providing only the leading twist operators (identity plus stress-tensor supermultiplet), come very close to the numerically obtained bounds/estimates. Surprisingly the estimates are not that far off for spin as low as zero in the chiral case and one in the non-chiral case. This leaves us optimistic

[3] The non-chiral inversion formula we used applies both in the $\mathcal{N} = 1$ and $\mathcal{N} = 2$ cases and so we left the result general.

that the spectrum of the (A_1, A_2) can be bootstrapped, similarly to the $3d$ Ising model. The numerical results for $\mathcal{N} = 2$ theories suffer from slow convergence and thus the estimates for OPE coefficients and anomalous dimensions we obtain are not yet with the precision of those of the $3d$ Ising model. By using this data as input to the inversion formulas they would in turn produce ranges for the various quantities appearing in the chiral and non-chiral OPEs. Finally, another direction corresponds to using the output of each inversion formula as input for the other to obtain better estimates. We leave these two directions for future work.

Appendix A

Casimir and crossing equations

This appendix collects some lengthy equations used to obtain the $\mathcal{N}=2$ superconformal blocks in section 3.1 and the crossing equations for $\mathcal{N}=(2,0)$ SCFTs in section 3.2.

A.0.1 Casimir equations

Quadratic Casimir differential equation

The application of the quadratic Casimir (obtained from eq. (3.26)) to the four-point function, through the differential action (3.15) of the generators yields a system of six coupled differential equations for the six functions $f_i(z)$ in eq. (3.23).

After some rearrangements we find that two of the six functions are completely determined in terms of the function f_0

$$
\begin{aligned}
f_1(z) &= \frac{z^3 f_0''(z) - z^2 f_0''(z) + z^2 f_0'(z) + \mathfrak{c}_2 f_0(z)}{z}\,,\\
f_5(z) &= \frac{z^2\left((z-1)\left((2\mathfrak{c}_2+2z-1)f_0''(z) + 2z f_1''(z)\right) + (2\mathfrak{c}_2+2z-1)f_0'(z)\right.}{z^2}\\
&\quad + \frac{\left.(6z-4)f_1'(z)\right) + 2\mathfrak{c}_2 f_0(z)(\mathfrak{c}_2+z-1)}{z^2}\,,
\end{aligned}
\tag{A.1}
$$

and that the differential equations involving f_0 is totally decoupled and can be written in terms of (minus) the usual bosonic Casimir

$$
\mathcal{C}_2 = z^2\left((z-1)\frac{\partial^2 f(z)}{\partial z^2} + \frac{\partial f(z)}{\partial z}\right)\,, \tag{A.2}
$$

as

$$
2\mathcal{D}(f_0)(\mathfrak{c}_2+4z-2) + \frac{\partial^2\left(2\mathcal{D}(f_0)(z-1)z^2\right)}{\partial z^2} + \frac{\partial(-2\mathcal{D}(f_0)z(5z-4))}{\partial z^1} = 0\,, \tag{A.3}
$$

where

$$\mathcal{D}(f_0) = 2\mathfrak{c}_2\mathcal{C}_2(f_0(z)) + (\mathfrak{c}_2 - 1)\mathfrak{c}_2 f_0(z) + \mathcal{C}_2(\mathcal{C}_2(f_0(z)))\,. \tag{A.4}$$

The other three functions are determined by the following equations

$$\begin{aligned}
&f_2(z) + z f_2'(z) + z^2\left(-f_3''(z)\right) - 2z f_3'(z) - \frac{\mathfrak{c}_2 f_3(z)}{z-1} = 0\,,\\
&f_2(z) + z f_2'(z) + (z-1)z^2 f_4''(z) + 2(2z-1)z f_4'(z) + f_4(z)(\mathfrak{c}_2 + 2z) = 0\,,\\
&-2(\mathfrak{c}_2 - 1)f_2(z) - z f_3'(z) + 2(z-1)^2 z^3 f_4^{(3)}(z) + 8(z-1)(2z-1)z^2 f_4''(z)\\
&+ z\left(2\mathfrak{c}_2 z - 2\mathfrak{c}_2 + 28z^2 - 27z + 3\right) f_4'(z) + z f_4(z)(2\mathfrak{c}_2 + 8z - 3) = 0\,.
\end{aligned} \tag{A.5}$$

Recall that the eigenvalue of the quadratic Casimir is $\mathfrak{c}_2 = h_{ex}^2 - \frac{q_{ex}^2}{4}$, where h_{ex} and q_{ex} are the charges of the superconformal primary of the supermultiplet being exchanged. This system is rather cumbersome to solve, and thus to solve it in section 3.1.3 we change "basis" from the functions $f_i(z)$ defined in eq. (3.23), to functions $\hat{f}_i$ (defined in eq. (3.28)) where one can more easily give an Ansatz in terms of a sum of bosonic blocks (3.29). The solution for the exchange of uncharged supermultiplets is collected in eqs. (3.30) and (3.31), according to whether a superconformal primary or descendant is exchanged.

Cubic and quadratic Casimir equations for the charged exchange

As clear from the quadratic Casimir eigenvalue the equations in appendix A.0.1 do not distinguish between the exchange of a superconformal multiplet with positive or negative charge, and thus we need also to consider the cubic Casimir (3.27). Considering these two equations suffices to fix all parameters in the Ansatz (3.29), giving the solution in eq. (3.33), and the quartic Casimir gives no new information. However some of the equations arising from the quartic Casimir appear in a simpler form, and using them we can easily simplify the system of Casimir equations, solving for all $\hat{f}_i(z)$ in terms of

$\hat{f}_0(z)$,[1]

$$\begin{aligned}
\hat{f}_1(z) &= \frac{z^2(2h+q_1)\left(\hat{f}_0'(z)+(z-1)\hat{f}_0''(z)\right)}{\mathfrak{c}_2}\,, \\
\hat{f}_2(z) &= -\frac{z^2(2h-q_1)\left(\hat{f}_0'(z)+(z-1)\hat{f}_0''(z)\right)}{\mathfrak{c}_2}\,, \\
\hat{f}_3(z) &= \frac{(z-1)z(q_1+2hq_{ex})\hat{f}_0'(z)}{2h}\,, \\
\hat{f}_4(z) &= \frac{z(2hq_{ex}-q_1)\hat{f}_0'(z)}{2h}\,, \\
\hat{f}_5(z) &= -\frac{z^2\left(\mathfrak{c}_2\left(4h^2+q_1^2\right)-8h^2\left(-4h^2+q_1^2+\mathfrak{c}_4\right)\right)\left(\hat{f}_0'(z)+(z-1)\hat{f}_0''(z)\right)}{4h^2\mathfrak{c}_2} \\
&\quad+\frac{2h(2h-1)\mathfrak{c}_2\left(4h^2-q_1^2\right)\hat{f}_0(z)}{4h^2\mathfrak{c}_2}\,,
\end{aligned} \tag{A.6}$$

where $\mathfrak{c}_4 = q_{ex}^2\mathfrak{c}_2$, and find a differential equation for $\hat{f}_0(z)$ only

$$\mathfrak{c}_2\hat{f}_0(z)+z^2\left(\hat{f}_0'(z)+(z-1)\hat{f}_0''(z)\right)=0\,. \tag{A.7}$$

We recognize this equation as the bosonic Casimir equation with eigenvalue $h(h-1)=\mathfrak{c}_2$, whose solution, for $q_{ex}=\pm1$, is simply given by the $sl(2)$ bosonic block with holomorphic dimension $h_{ex}+\frac{1}{2}$. Inserting this solution into eq. (A.6) gives immediately the result for the functions $\hat{f}_i$ given in eq. (3.33), and all other equations arising from the system of Casimirs are satisfied.

A.0.2 $\mathcal{N}=(2,0)$ crossing equations

Here we collect the Taylor expansion of the crossing equations (3.43) in the nilpotent invariants ($I_{i\neq0}$),

$$\begin{aligned}
&(\bar{z}-1)^{2\bar{h}}(g_0(I_0,\bar{z})+I_1g_1(I_0,\bar{z})+I_2g_2(I_0,\bar{z})+I_3g_3(I_0,\bar{z})+I_4g_4(I_0,\bar{z})-I_3I_4(1-I_0)g_5(I_0,\bar{z}))= \\
&I_0^{2h}\bar{z}^{2\bar{h}}\left(g_0(I_0^{-1},1-\bar{z})+\frac{I_1}{I_0}\left(2hg_0(I_0^{-1},1-\bar{z})+\left(1-\frac{1}{I_0}\right)g_0'(I_0^{-1},1-\bar{z})-g_1(I_0^{-1},1-\bar{z})\right)\right. \\
&-\frac{I_2}{I_0}\left(g_2(I_0^{-1},1-\bar{z})+g_3(I_0^{-1},1-\bar{z})\right)+\frac{I_3}{I_0}g_3(I_0^{-1},1-\bar{z})+\frac{I_4(1-I_0)}{I_0}\left(2g_2(I_0^{-1},1-\bar{z})\right. \\
&\left.+g_3(I_0^{-1},1-\bar{z})+g_4(I_0^{-1},1-\bar{z})\right)+\frac{I_3I_4(1-I_0)}{I_0^2}\left(2h(2h-1)g_0(I_0^{-1},1-\bar{z})+\left(1-\frac{2}{I_0}+\right.\right. \\
&\left.\left.\left.\frac{1}{I_0^2}\right)g_0''(I_0^{-1},1-\bar{z})-2(2h-1)g_1(I_0^{-1},1-\bar{z})-2\left(1-\frac{1}{I_0}\right)g_1'(I_0^{-1},1-\bar{z})+g_5(I_0^{-1},1-\bar{z})\right)\right)\,,
\end{aligned} \tag{A.8}$$

[1] Note that we always assume that the external fields are not chiral.

with the coefficient of each invariant giving rise to a crossing equation, as discussed in section 3.2.2, ultimately culminating in the crossing equation (3.46).

Appendix B

Blocks and Crossing

We write the bosonic blocks for the exchange of a conformal primary of dimension Δ and spin ℓ, in the four-point function of unequal scalar operators of dimensions $\Delta_{i=1,\dots,4}$, as [88]

$$\begin{aligned} g^{\Delta_{12},\Delta_{34}}_{\Delta,\ell}(z,\bar z) &= \frac{z\bar z}{z-\bar z}\left(k^{\Delta_{12},\Delta_{34}}_{\Delta+\ell}(z)k^{\Delta_{12},\Delta_{34}}_{\Delta-\ell-2}(\bar z) - z\leftrightarrow \bar z\right)\,, \\ k^{\Delta_{12},\Delta_{34}}_{\beta}(z) &= z^{\frac{\beta}{2}}\,{}_2F_1\left(\frac{\beta-\Delta_{12}}{2},\frac{\beta+\Delta_{34}}{2};\beta;z\right)\,, \end{aligned} \tag{B.1}$$

where $\Delta_{ij}=\Delta_i-\Delta_j$, and z and $\bar z$ are obtained from the standard conformally invariant cross-ratios

$$z\bar z = \frac{x_{12}^2x_{34}^2}{x_{13}^2x_{24}^2}\,, \qquad (1-z)(1-\bar z) = \frac{x_{14}^2x_{23}^2}{x_{13}^2x_{24}^2}\,. \tag{B.2}$$

The crossing equations (2.122) (see [21, 27] for a derivation) are written here in a form suitable for the numerical analysis of section 4.1

$$\sum_{\mathcal{O}\in\phi\bar\phi}|\lambda_{\phi\bar\phi\mathcal{O}}|^2\begin{bmatrix}(-1)^\ell\tilde{\mathcal{F}}_{\pm,\Delta,\ell}(z,\bar z)\\ \mathcal{F}_{-,\Delta,\ell}(z,\bar z)\end{bmatrix} + \sum_{\mathcal{O}\in\phi\phi}|\lambda_{\phi\phi\mathcal{O}}|^2\begin{bmatrix}\mp(-1)^\ell F_{\pm,\Delta,\ell}(z,\bar z)\\ 0\end{bmatrix} = 0\,, \tag{B.3}$$

where the first line encodes two separate crossing equations, differing by the signs indicated, and where we defined (recall that $\Delta_\phi=\Delta_{\bar\phi}=r_0$)

$$\begin{aligned} F_{\pm,\Delta,\ell}(z,\bar z) &\equiv ((1-z)(1-\bar z))^{r_0}\,g^{0,0}_{\Delta,\ell}(z,\bar z)\pm(z\bar z)^{r_0}g^{0,0}_{\Delta,\ell}(1-z,1-\bar z)\,, \\ \mathcal{F}_{\pm,\Delta,\ell}(z,\bar z) &\equiv ((1-z)(1-\bar z))^{r_0}\,\mathcal{G}_{\Delta,\ell}(z,\bar z)\pm(z\bar z)^{r_0}\mathcal{G}_{\Delta,\ell}(1-z,1-\bar z)\,, \\ \tilde{\mathcal{F}}_{\pm,\Delta,\ell}(z,\bar z) &\equiv ((1-z)(1-\bar z))^{r_0}\,\tilde{\mathcal{G}}_{\Delta,\ell}(z,\bar z)\pm(z\bar z)^{r_0}\tilde{\mathcal{G}}_{\Delta,\ell}(1-z,1-\bar z)\,, \end{aligned} \tag{B.4}$$

with the superblocks $\mathcal{G}_{\Delta,\ell}$ and $\tilde{\mathcal{G}}_{\Delta,\ell}$ given in (2.115) and (2.118). In (B.3) the stress tensor and the identity contribute as

$$\overrightarrow{V}_{\text{fixed}} = \begin{bmatrix} \tilde{\mathcal{F}}_{\pm,\Delta=0,\ell=0}(z,\bar{z}) \\ \mathcal{F}_{-,\Delta=0,\ell=0}(z,\bar{z}) \end{bmatrix} + \frac{r_0^2}{6c} \begin{bmatrix} \tilde{\mathcal{F}}_{\pm,\Delta=2,\ell=0}(z,\bar{z}) \\ \mathcal{F}_{-,\Delta=2,\ell=0}(z,\bar{z}) \end{bmatrix} . \tag{B.5}$$

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