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DISSERTATION

Funnel control for systems with known vector relative degree

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Declaration of Authorship

I, Lê Huy Hoàng , declare that this thesis titled, "Funnel control for systems with known vector relative degree" and the work presented in it are my own. I confirm that:

- This work was done wholly while in candidature for a research degree at Universität Hamburg.
- Where any part of this thesis has previously been submitted for a degree or any other qualification at Universität Hamburg or any other institution, this has been clearly stated.
- Where I have consulted the published work of others, this is always clearly acknowledged.
- Where I have quoted from the work of others, the source is always indicated. With the exception of such quotations, this thesis is entirely mine.
- I have acknowledged all main sources of references.
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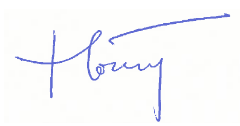
Hamburg, December 2018.



Lê Huy Hoàng

Abstract

The present dissertation examines funnel control which is a type of output feedback control for system with known vector relative degree. The goal of funnel control is to construct a simple controller for tracking the errors, which are the gap between system outputs $y(t)$ and an arbitrary given smooth enough reference signals $y_{\text{ref}}(t)$, within a prespecified performance funnel, and therefore, all involved quantities are bounded. This study addresses a number of questions concerned funnel control problem.

Firstly, it provides a *normal form* for linear differential-algebraic systems. A new definition of vector relative degree is proposed and based on that a *normal form* for linear differential-algebraic systems is introduced. The advantage of this *normal form* is not only placed on its simplicity but also the ability to design a new funnel controller tracking the error, $e(t) = y(t) - y_{\text{ref}}(t)$, within a prespecified performance funnel.

Secondly, this research proposes a feasible funnel control law for system with higher strict relative degree. We consider tracking control for non-linear multi-input, multi-output systems which have arbitrary strict relative degree and input-to-state stable internal dynamics. To this end, we introduce a new controller which involves the first $r-1$ derivatives of the tracking error, where r is the strict relative degree of the system. We derive an explicit bound for the resulting input and discuss the influence of the controller parameters. We further present some simulations where our funnel controller is applied to a mechanical system with higher relative degree and a two-input, two-output robot manipulator. The controller is also compared with other approaches.

Moreover, we contribute a feasible funnel controller for non-linear multi-input, multi-output systems with input-to-state stable internal dynamics and known vector relative degree $r = (r_1, r_2, \ldots, r_m)$. To address the funnel control problem, a new funnel controller involving the first $r_i - 1$ of the derivatives of each i-th element of the tracking error $e(t)$, is proposed. Furthermore, simulations are presented to demonstrate the work of this funnel controller in a number of examples.

As an application, we consider an overhead crane model whose control variables are the velocities of trolley and rope length respectively. The position of the load is considered as the output of the system. The objective is to design a closed-loop tracking controller which also takes into account the transient behaviour. Unfortunately, this system is shown that has no well-defined vector relative degree. Therefore, this does not allow us to apply established methods directly for adaptive control to achieve the objective. To overcome this problem, a dynamic state feedback is proposed, which results in a system with strict relative degree four, and then we can apply a funnel controller to this feedback system. Consequently, we made a simulation to show that our approach can be used to move loads from one to another given position in the situation where there are several obstacles which have to be circumnavigated.

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Contents

Nomenclature

$\mathbb{N}$	the set of natural numbers
$\mathbb{N}_0$	$:=\mathbb{N}\cup\{0\}$
$\mathbb{Z}$	the set of integers
$\mathbb{R}_{\geq 0}$	$:=[0,\infty)$
$\mathbb{C}_+,\mathbb{C}_-$	the open set of complex numbers with positive, negative real part, resp.
$\mathbb{R}^{n\times m}$	the set of real $n\times m$ matrices
$\mathrm{GL}_n(\mathbb{R})$	$:=\{\ A\in\mathbb{R}^{n\times n}\ \mid A$ invertible $\}$
$\mathbb{R}[s]$	the ring of polynomials with real coefficients
$\mathbb{R}[s]^{n\times m}$	the set of $n\times m$ matrices with entries in $\mathbb{R}[s]$
$\mathbb{R}(s)$	the quotient field of $\mathbb{R}[s]$
$\mathbb{R}(s)^{n\times m}$	the set of $n\times m$ matrices with entries in $\mathbb{R}(s)$
$\sigma(A)$	the spectrum of a matrix $A\in\mathbb{R}^{n\times n}$
$\|x\|$	the Euclidean norm of $x\in\mathbb{R}^n$
$\|A\|$	$:=\sup\{\ \|Ax\|\ \mid x\in\mathbb{R}^n,\|x\|=1\ \}$
$\mathscr{L}^\infty_{\mathrm{loc}}(I\to\mathbb{R}^n)$	the set of locally essentially bounded functions $f:I\to\mathbb{R}^n$, $I\subseteq\mathbb{R}$ an interval
$\mathscr{L}^1_{\mathrm{loc}}(I\to\mathbb{R}^n)$	the set of locally Lebesgue integrable functions $f:I\to\mathbb{R}^n$, $I\subseteq\mathbb{R}$ an interval
$\mathscr{L}^\infty(I\to\mathbb{R}^n)$	the set of essentially bounded functions $f:I\to\mathbb{R}^n$ with norm
$\|f\|_\infty$	$:=\operatorname{ess\,sup}_{t\in I}\|f(t)\|$
$\dot{f}\,(f^{(i)})$	the (ith) weak derivative of $f\in\mathscr{L}^1_{\mathrm{loc}}(I\to\mathbb{R}^n)$
$\mathscr{W}^{k,\infty}(I\to\mathbb{R}^n)$	the set of k-times weakly differentiable functions $f:I\to\mathbb{R}^n$ such that $f,\dots,f^{(k)}\in\mathscr{L}^\infty(I\to\mathbb{R}^n)$
$\mathscr{W}^{k,1}_{\mathrm{loc}}(I\to\mathbb{R}^n)$	$:=\left\{\ f\in\mathscr{L}^1_{\mathrm{loc}}(I\to\mathbb{R}^n)\ \middle|\ f^{(i)}\in\mathscr{L}^1_{\mathrm{loc}}(I\to\mathbb{R}^n)\text{ for }i=1,\dots,k\ \right\},k\in\mathbb{N}_0\cup\infty$
$\mathscr{C}^k(V\to\mathbb{R}^n)$	the set of k-times continuously differentiable functions $f:V\to\mathbb{R}^n$, $V\subseteq\mathbb{R}^m$; $\mathscr{C}(V\to\mathbb{R}^n)=\mathscr{C}^0(V\to\mathbb{R}^n)$
$f\|_W$	restriction of the function $f:V\to\mathbb{R}^n$ to $W\subseteq V$

Chapter 1

Introduction

This dissertation focuses on funnel control law for system with known vector relative degree. Funnel control belongs to the non-identification control style which is of adaptive control. The introduction shall start with a concise history of funnel control before drawing readers attention to the research motivation and major research questions.

1.1 Adaptive control

Adaptive control is the method in which a controller is structured to adapt a closed-loop system with parameters of varied or initially uncertain. Furthermore, the system class, determined by given assumptions, is the only priori information that is needed to create the controller. Consequently, in adaptive control the coefficients of the system are completely unidentified while, only few structural details are available, for example: vector relative degree, minimum phase, input-to-state stable internal dynamics. Therefore, it is the main task of this control type to build a common controller that are capable to apply to all single system belonging to a given class. The controller u, therefore, could be fully implemented by using error e, which is the gap between the system output y and a reference signal y_{ref}, see Figure 1.1.

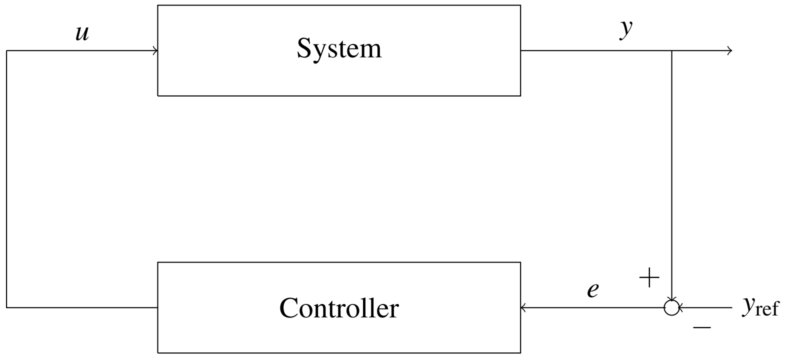

FIGURE 1.1: Closed-loop control systems.

1.1.1 Tracking without identification

Fundamental ideas in adaptive control were introduced in 1950s by trying to design an autopilot in aircraft. In 1970s, the model reference adaptive control and the self-turning regulator were successful both in theory and application. By using stability theory, these results work on well with systems which are fully known its parameters. At that time, using information form systems to design feedback controller or control with identification was the main stream of adaptive control, see Aström [3], and Narenda [70], for example. On the other hand, until

1980's, control without identification or non-identifier-based adaptive control was started by different researchers from 1983 to 1985, such for instance, Mareels [64], Martensson [65], Morse [67], Nussbaum [72], and Willems and Byrnes [87]. Based on this approach, feedback control laws were designed without using parameter from the systems. It means that the feedback controller does not depend on any estimation of the controlled system. Furthermore, these works created a new field to investigate within adaptive control, see the survey Ilchmann [39].

To illustrate the main idea of non-identification control , let us consider the simple class of systems

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \quad x(0) = x^0 \in \mathbb{R}^n \\ y(t) &= Cx(t), \end{aligned} \tag{1.1.1}$$

in which $x \in \mathbb{R}^n$ is the *state* vector, $u \in \mathbb{R}^m$ is the *input* vector, $y \in \mathbb{R}^m$ is the *output* vector, and the coefficients matrices $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{m\times n}$.

- If $m = 1$, then the system (1.1.1) is called the single-input, single-output system .
- If $m > 1$, then the system (1.1.1) is called the multi-input, multi-output system .

Definition 1.1.1. The function $G(s) := C(sI - A)^{-1}B$ is called the *transfer function* of the system (1.1.1).

We recall definition of poles and zeros of transfer function in [38, Def.2.1.1].

Definition 1.1.2 ([38, Def.2.1.1])**.** Let $G(s) \in \mathbb{R}(s)^{m\times m}$ be a rational matrix with *Smith-McMillan form*

$$U^{-1}(s)G(s)V^{-1}(s) = \operatorname{diag}\left(\frac{\varepsilon_1(s)}{\psi_1(s)}, \dots, \frac{\varepsilon_r(s)}{\psi_r(s)}, 0, \dots, 0\right) \in \mathbb{R}(s)^{m\times m},$$

where $U(s), V(s) \in \mathbb{R}[s]^{m\times m}$ are unimodular, $\operatorname{rank}_{\mathbb{R}(s)} G(s) = r$, $\psi_i(s) \not\equiv 0$, $\varepsilon_i(s), \psi_i(s) \in \mathbb{R}[s]$ are monic, coprime and satisfy $\varepsilon_i(s)|\varepsilon_{i+1}(s)$, $\psi_{i+1}(s)|\psi_i(s)$ for all $i = 1, \dots, r-1$. Set

$$\varepsilon(s) = \prod_{i=1}^{r} \varepsilon_i(s) \quad \psi(s) = \prod_{i=1}^{r} \psi_i(s).$$

s_0 is called a *(transmission) zero* of $G(s)$, if $\varepsilon(s_0) = 0$ and a *pole* of $G(s)$, if $\psi(s_0) = 0$.

The idea of non-identification is that we completely unknow each coefficient of A, B, C and only some structural assumptions of the systems (1.1.1) should be imposed. The following assumptions will typically be imposed

- *Strict relative degree r and sign of high-frequency gain matrix*
  System (1.1.1) has strict relative degree $r \in \mathbb{N}$, i.e, $CA^iB = 0$, for $i = 0 \dots, r-2$ and $\Gamma := CA^{r-1}B$, which is called high-frequency gain matrix, is invertible. Furthermore, Γ is either positive or negative definite (Γ is not required to be symmetric).
- *Minimum phase*
  $\det \begin{bmatrix} \lambda I - A & B \\ C & 0 \end{bmatrix} \neq 0$ for all $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda \geq 0$.

We note that a state space system (1.1.1) is normally called *minimum phase*, if it is stabilizable, detectable, and transfer function has no zeros in $\overline{\mathbb{C}}_+$. More clearly, we recall a proposition from [38, Prop.2.1.2] to change this condition to algebraic condition.

Proposition 1.1.3 ([38, Prop.2.1.2]). *The coefficients matrices $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{m\times n}$ from system* (1.1.1) *satisfies*

$$\det \begin{bmatrix} \lambda I - A & B \\ C & 0 \end{bmatrix} \neq 0 \text{ for all } \lambda \in \overline{\mathbb{C}}_+$$

if, and only if, the following three conditions are satisfied

(i) $\operatorname{rank} \begin{bmatrix} \lambda I - A & B \end{bmatrix} = n$ *for all* $\lambda \in \overline{\mathbb{C}}_+$, *i.e.* (A,B) *is stabilizable by state feedback,*

(ii) $\operatorname{rank} \begin{bmatrix} \lambda I - A \\ C \end{bmatrix} = n$ *for all* $\lambda \in \overline{\mathbb{C}}_+$, *i.e. (A, C) is detectable,*

(iii) $G(\lambda)$ *has no zeros in* $\overline{\mathbb{C}}_+$.

Remark 1.1.4.

- Suppose that the system (1.1.1) has strict relative degree r. Let us calculate its output derivatives,

$$\dot{y}(t) = C\dot{x}(t) = C(Ax(t) + Bu(t)) = CAx(t) + \underbrace{CB}_{=0} u(t) = CAx(t),$$
$$\ddot{y}(t) = CA\dot{x}(t) = CA(Ax(t) + Bu(t)) = CA^2x(t) + \underbrace{CAB}_{=0} u(t) = CA^2x(t),$$
$$\vdots$$
$$y^{(r)}(t) = CA^{r-1}\dot{x}(t) = CA^{r-1}(Ax(t) + Bu(t)) = CA^r x(t) + \underbrace{CA^{r-1}B}_{\in \mathrm{GL}_m(\mathbb{R})} u(t).$$

  Therefore, the strict relative degree, in the case of single-input, single-output systems, is exactly the number of times that the output $y(t)$ has to derivative in order to get appearing explicit of the input $u(t)$.
  On the other hand, we also have

$$G(s) = C(sI - A)^{-1}B = \sum_{k=0}^{\infty} \frac{1}{s^{k+1}} CA^k B = \sum_{k=r-1}^{\infty} \frac{1}{s^{k+1}} CA^k B = \frac{1}{s^r} CA^{r-1}B + \dots$$

  since $CA^iB = 0$, for $i = 0, \dots, r-2$. Hence, the strict relative degree, in the single-input, single-output systems, is the difference between the numerator and the denominator degree of the transfer function.
  Furthermore,

$$\lim_{s\to\infty} s^r G(s) = \lim_{s\to\infty} s^r \sum_{k=r-1}^{\infty} \frac{1}{s^{k+1}} CA^k B = CA^{r-1}B = \Gamma \in \mathrm{GL}_m(\mathbb{R}).$$

- In the case of nonlinear systems, the condition of *minimum phase* is substituted by the condition of *asymptotic stable zero dynamics* which will be discussed in following chapters. However, we note that asymptotic stable zero dynamics leads to minimum phase but not vice versa, for more detail see [55].

In control without identification, we need tracking output error, $e(t) = y(t) - y_{\text{ref}}(t)$, which is the difference between system output and a given reference signal. Using the estimation of output error $e(t)$, a feedback controller is constructed to work on arbitrary system from a given system class, and every reference signal from a chosen class of function. Furthermore, all other related signals of the closed loop system should be bounded and defined on $[0,\infty)$, and the tracking error $e(t)$ need to be addressed at least one of the following targets which were mentioned in survey [45].

(a) *Asymptotic tracking*: $\lim_{t\to\infty} e(t) = 0$.

(b) λ*-tracking*: $\limsup_{t\to\infty} \|e(t)\| \le \lambda$, for any prescribed $\lambda > 0$.

(c) *Prespecified transient behaviour*: $\|e(t)\| \le \dfrac{1}{\varphi(t)}$ for all $t > 0$, and for some suitable prescribed function φ.

In the initial researches of this type of feedback controller, the aim of tracking error is stabilization, i.e, the set of possible reference signal $\mathscr{Y}_{\text{ref}} = \{0\}$. Byrnes and Willems in [21] introduced an adaptive controller to stabilize any system in the class of single-input, single-output linear systems with minimum phase, relative degree one, and having negative high frequency gain $\Gamma = CB < 0$.

$$u(t) = k(t)y(t), \qquad \dot{k}(t) = \|y(t)\|^2, \quad k(0) = k^0 \tag{1.1.2}$$

The application of the controller (1.1.2) to the system (1.1.1) with $m = 1$, and $r = 1$ leads to a closed loop system. This closed loop system with arbitrary initial data $x^0 \in \mathbb{R}^n$, $k^0 \in \mathbb{R}$ yields a unique solution (x,k) satisfying

$$\lim_{t\to\infty} x(t) = 0 \text{ and } \lim_{t\to\infty} k(t) = k^\infty \in \mathbb{R}.$$

The remarkable advantage of the controller (1.1.2) is very simple and it combined the monotone gain function $k(\cdot)$ with the high gain property which is imposed to the system class. This type of controller was introduced in series of papers which were appeared around the middle of 1980s, such as papers [67, 87, 21, 64]. Some generalized versions of this kind of controller have been applied to several different classes of system, for instant, multivariable system [43], unknown sign of high frequency gain [72], nonlinear systems [77], discontinuous feedback strategies within the framework of differential inclusions [78, 79], infinite dimensional systems [62], transient behaviour [66], tracking including an internal model [35, 61]. After three decades, there are also various surveys were written to review the results in this field such as [38, 39, 45], and [63] for the case of infinite dimension. However, the canonical high gain adaptive controller (1.1.2) and some its modifications could be faced to many disadvantages, from both practical and theoretical perspectives. First of all, it is only available for system with relative degree one. Moreover, the gain function $k(t)$ is monotonically non-decreasing. This may lead to the situation that the gain function grows dramatically when perturbations are appeared, and after that although the perturbations may be not present anymore, the gain function is still kept very large and continuous increasing and the transient behaviour is not addressed. Furthermore, some small noises in measurement of the output can bring unexpected huge gain, and this kind of controller is also designed only for linear systems. Some simple modification of the controller (1.1.2) was proposed to overcome the disadvantages but it makes the control being complicated. To address some later disadvantage of high-gain control which is so call λ-tracking is proposed by made the control objective be slightly weaker. We substitute the asymptotic tracking in high-gain control by requirement that the error $e(t)$ closed to the strip having prespecified width $\lambda > 0$, $\lim_{t\to\infty} \text{dist}(\|e(t)\|, [0,\lambda]) = 0$. To begin with linear system, the control strategy for λ-tracking is

$$\begin{aligned} u(t) &= -k(t)e(t), \\ \dot{k}(t) &= \max\{\|e(t)\| - \lambda, 0\}, \quad k(0) = k^0. \end{aligned} \tag{1.1.3}$$

In contrast to (1.1.2), the gain function in (1.1.3) is constant on any interval where the error stays in the strip having width $\lambda > 0$. The new controller allows us to enlarge nonlinear system classes which can be applied because 0 is not compulsory to be an equilibrium point of the system. We can also apply to the larger set of reference signal $\mathscr{Y}_{\text{ref}}$, for example $\mathscr{Y}_{\text{ref}} = \mathscr{W}^{1,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R})$. In [1, 40], these results have been extended to the multi-input multi-output nonlinear system of the following form,

$$\begin{aligned} \dot{y} &= f(t,y,z) + g(t,y,z)u(t), \\ \dot{z} &= h(t,y,z), \end{aligned} \tag{1.1.4}$$

where for $n, m \in \mathbb{N}$ with $n > m$ and functions

$$\begin{aligned} f &: \mathbb{R}_{\geq 0} \times \mathbb{R}^m \times \mathbb{R}^{n-m} \to \mathbb{R}^m, \\ g &: \mathbb{R}_{\geq 0} \times \mathbb{R}^m \times \mathbb{R}^{n-m} \to \mathbb{R}^{m \times m}, \\ h &: \mathbb{R}_{\geq 0} \times \mathbb{R}^m \times \mathbb{R}^{n-m} \to \mathbb{R}^{n-m}, \end{aligned}$$

are deemed to be *Carathéodory function*[1] with equilibrium point $(y_e, z_e, 0) \in \mathbb{R}^m \times \mathbb{R}^{n-m} \times \mathbb{R}^m$, i.e, $f(t, y_e, z_e) = 0$ and $h(t, y_e, z_e) = 0$ for all t. The function g is supposed to be uniformly bounded and bounded away from zero, and dynamics $\dot{z} = h(t, y_e, z)$ is uniformly asymptotically stable. However, the assumption, which was imposed on the functions f, g has improved from globally Lipschitz at (y_e, z_e) in [1] to uniformly polynomial boundedness[2] in [40]. Furthermore, the controller (1.1.3) has been modified with $s \geq 1$ which is an upper bound of the polynomial degree of f and g as follows

$$\begin{aligned} u(t) &= -k(t)\|e(t)\|^{s-1}e(t), \\ \dot{k}(t) &= \begin{cases} (\|e(t)\| - \lambda)^s, & \|e(t)\| \geq \lambda, \\ 0, & \|e(t)\| < \lambda, \end{cases} \quad k(0) = k^0. \end{aligned} \tag{1.1.5}$$

With the control law (1.1.5), the system (1.1.4) can be driven to get the aim of λ-tracking. One of the advantages of system class (1.1.4) is it can be extended to infinite dimension case. To demonstrate this, a multi-input multi output system are considered in [47] which are of the form

$$\begin{aligned} \dot{y}(t) &= f(p(t), (Ty)(t)) + g(p(t), (Ty)(t), u(t)), \\ y|_{[-h,0]} &= y^0 \in \mathscr{C}([-h,0] \to \mathbb{R}^m), \end{aligned} \tag{1.1.6}$$

where $h \geq 0$ is the "memory" of the system, p may be thought of as a (bounded) disturbance term and T is nonlinear causal operator, g and f are assumed to be continuous. We remark here that diverse phenomena are incorporated within the class including, for example, diffusion processes, delays (both point and distributed) and hysteretic effects, see [47, Subsec. 2.4,2.5], [46, Appendix A] and references therein. $\mathscr{K}$ is denoted as the class of continuous, strictly increasing function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ satisfying $\alpha(0) = 0$, and

$$\mathscr{K}_\infty := \{\ \alpha \in \mathscr{K} \mid \alpha \text{ is unbounded } \}.$$

Furthermore, denote

$$\mathscr{J} := \{\ \alpha \in \mathscr{K} \mid \text{for each } \delta \geq 0, \text{ there exists } \Delta \geq 0 : \ \alpha(\delta\tau) \leq \Delta\alpha(\tau) \text{ for all } \tau \geq 0\ \},$$

[1] A function $\alpha : \mathbb{R}_{\geq 0} \times \mathbb{R}^p \to \mathbb{R}$ is called a *Carathéodory function* if $\alpha(\cdot, x)$ is measurable on $\mathbb{R}_{\geq 0}$ for each $x \in \mathbb{R}^p$, and $\alpha(t, \cdot)$ is continuous on $\mathbb{R}^p$ for all $t \in \mathbb{R}_{\geq 0}$

[2] A function $f : \mathbb{R}_{\geq 0} \times \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^q$ is called *uniformly polynomially bounded* if for some polynomials $p(\cdot) \in \mathbb{R}[s]$, we have $\|f(t,y,z)\| \leq p\left(\left\|\begin{pmatrix} y \\ z \end{pmatrix}\right\|\right)$ for all $(t,y,z) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^m \times \mathbb{R}^n$

and $\mathcal{J}_\infty = \mathcal{J} \cap \mathcal{K}_\infty$. Apart from some technical condition, the essential hypotheses are as follows.

(a) $f : \mathbb{R}^p \times \mathbb{R}^q \to \mathbb{R}^m$ is continuous, and for every compact set $C \subset \mathbb{R}^p$, there exist $\alpha_f \in \mathcal{J}$ and constant $c_f \geq 0$ such that

$$\|f(p,\omega)\| \leq c_f\Big(1+\alpha_f(\|\omega\|)\Big) \text{ for all } (p,\omega) \in C \times \mathbb{R}^q.$$

(b) $g : \mathbb{R}^p \times \mathbb{R}^q \times \mathbb{R}^m \to \mathbb{R}^m$ is continuous, and for every compact set $C \subset \mathbb{R}^p$, there exists a positive definite, symmetric $G \in \mathbb{R}^{m\times m}$ such that

$$u^\top Gg(p,\omega,u) \geq \|u\|^2 \quad \forall (p,\omega,u) \in C \times \mathbb{R}^q \times \mathbb{R}^m,$$

this assumption replaces the "positive high frequency gain" assumption in the case of linear systems.

(c) A weak bounded-input, bounded-output assumption on the operator T (which is a counterpart of the minimum phase assumption in the case of linear systems). And there exist $\alpha_T \in \mathcal{J}$ and constsnt $c_T \geq 0$ such tha, for all $y \in \mathcal{C}([-h,\infty) \to \mathbb{R}^m)$,

$$\|(Ty)(t)\| \leq c_T\Big(1+\max_{s\in[0,t]} \alpha_T(\|y(s)\|)\Big) \text{ for almost all } t \in \mathbb{R}_{\geq 0}.$$

The feedback and gain adaptation takes the form

$$\begin{aligned} u(t) &= -k(t)\varphi(e(t)), \\ \dot{k}(t) &= \psi_\lambda(e(t)), \quad k(0) = k^0, \end{aligned}$$

where the function φ and ψ_λ are determined as follows. Choose $\alpha \in \mathcal{J}_\infty$ with a property

$$\liminf_{s\to\infty} \frac{\alpha(s)}{s+\alpha_f(\alpha_T(s))} \neq 0.$$

For $\lambda > 0$, choose $\psi_\lambda : \mathbb{R}_{\geq 0} \to \mathbb{R}_{ge0}$ to be a continuous function with properties

$$\liminf_{s\to\infty} \frac{s\psi_\lambda(s)}{\alpha(s)} \neq 0 \text{ and } \psi_\lambda^{-1}(0) := \{\, s \mid \psi_\lambda(s) = 0 \,\} = [0,\lambda].$$

And

$$\varphi(e) = \begin{cases} \alpha(\|e\|)\|e\|^{-1}e, & \text{if } e \neq 0, \\ 0, & \text{if } e = 0. \end{cases}$$

Note that (1.1.5) is a particular case of this general structure. The concept of λ-tracking is implicit in [66], albeit it is a somewhat different context to that considered here. The concept as described above was introduced for linear systems (1.1.1) in [44], for infinite dimensional linear systems in [42], for the nonlinear class (1.1.4) in [1, 40], for the nonlinear class (1.1.6) in [47], and for systems modelled by differential inclusions in [77]. However, there are several questions that we need to find the answer such that: Could the disadvantages of monotonically non-decreasing gain function be eliminated and presspecified transient behaviour, and how to expand the results to system with higher relative degree (larger than one)? The answers could be partly addressed by the notion of funnel control which has been first introduced in [48].

1.1.2 Funnel control

The concept of funnel control has been developed at the first time by Ilchmann et al. [48] for systems with relative degree one, see also the survey [45] and the references therein. It is an adaptive controller since the gain is adapted to the actual needed value by a time-varying (non-dynamic) adaptation scheme[3]. Note that no exact tracking is pursued, but a tracking error with prescribed transient behavior. Controllers of high-gain type have various advantages when it comes to "real world" applications because it only require the assumption of the system structure, see [28]. In particular, the funnel controller is proved to be the appropriate tool for tracking problems in various applications, such as temperature control of chemical reactor models [54], control of industrial servo-systems [34, 52] and rigid, revolute joint robotic manipulators [33], speed control of wind turbine systems [30, 32], current control for synchronous machines [31], DC-link power flow control [82], voltage and current control of electrical circuits [16], oxygenation control during artificial ventilation therapy [75] and control of peak inspiratory pressure [76]. At the beginning, we can get rid of the disadvantages of the monotonically increasing gain function by following observation. Loosely speaking, the high-gain property ensures that the output $y(t)$ or the error $e(t)$ is decaying if the gain is sufficiently large. If k is a time varying function, we may "tune" its values $k(t)$ to be large only when required: k need not be a monotonically increasing function. Based on this idea and try to solve the issue of transient behaviour, we introduce the concept of a performance funnel. Let φ be a function of the following class

$$\Phi_r := \left\{ \varphi \in \mathscr{C}^r(\mathbb{R}_{\geq 0} \to \mathbb{R}) \;\middle|\; \begin{array}{r} \varphi, \dot{\varphi}, \ldots, \varphi^{(r)} \text{ are bounded,} \\ \varphi(\tau) > 0 \text{ for all } \tau > 0, \\ \text{and } \liminf_{\tau \to \infty} \varphi(\tau) > 0 \end{array} \right\}. \tag{1.1.7}$$

With $\varphi \in \Phi_r$, we associate the set

$$\mathscr{F}_\varphi := \{ \, (t,e) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^m \mid \varphi(t)\|e\| < 1 \, \}, \tag{1.1.8}$$

which we refer to as the performance funnel by Figure 1.2. This terminology arises from the

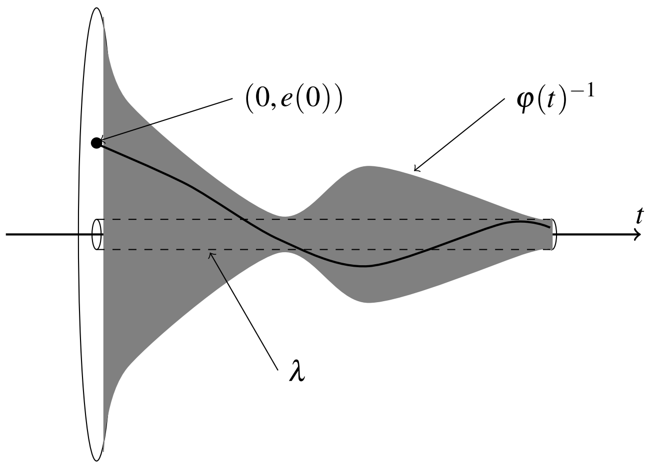

FIGURE 1.2: Error evolution in a funnel $\mathscr{F}_\varphi$ with boundary $\varphi(t)^{-1}$ for $t > 0$.

fact that, if a control structure can be devised which ensures that the tracking error evolves

[3]Note that often only controllers with dynamic gain adaptation are viewed as adaptive controllers of high-gain type.

within $\mathscr{F}_\varphi$, then we have guaranteed transient behaviour in the sense that

$$\|e(t)\| < \frac{1}{\varphi(t)}, \quad \forall t > 0,$$

and, moreover, if φ is chosen so that $\varphi(t) \geq \frac{1}{\lambda}$ for all t sufficiently large, then λ-tracking is achieved. The funnel boundary is given by the reciprocal of φ, see Figure 1.2. It is explicitly allowed that $\varphi(0) = 0$, meaning that no restriction on the initial value is imposed since $\varphi(0)\|e(0)\| < 1$; the funnel boundary $1/\varphi$ has a pole at $t = 0$ in this case. An important property of the class Φ_r is that the boundary of each performance funnel $\mathscr{F}_\varphi$ with $\varphi \in \Phi_r$ is bounded away from zero, i.e., because of boundedness of φ there exists $\lambda > 0$ such that $1/\varphi(t) \geq \lambda$ for all $t > 0$. The funnel boundary is not necessarily monotonically decreasing, while in most situations it is convenient to choose a monotone funnel. However, there are situations where widening the funnel over some later time interval might be beneficial, e.g., when the reference trajectory changes strongly or the system is perturbed by some calibration so that a large tracking error would enforce a large input action. Therefore, a variety of different funnel boundaries are possible, see e.g. [41, Sec. 3.2].
To ensure error evolution within the funnel, the controller (1.1.3) is replaced by

$$\begin{aligned} u(t) &= -k(t)e(t), \\ k(t) &= \frac{1}{1 - \varphi(t)\|e(t)\|}. \end{aligned} \tag{1.1.9}$$

The underlying idea of the "funnel controller" (1.1.9) comes in very natural way, if error $e(t)$ approaches the funnel boundary, then the gain $k(t)$ increases. Therefore, the contact with the boundary of the error $e(t)$ is precluded this feature in conjunction with the high gain property imposed on system class. Furthermore, all signals included the gain function, and input function are remained bounded and $\|e(\cdot)\|$ is bounded away from the funnel boundary. We also see that high gain values may only come into play when an output error grows closely to the funnel boundary in order to maintain the error evolution stay in the funnel. This remark lead us to choose the gain function in (1.1.9) and this simple structure has taken some advantages. First, $k(t)$ is not determined by a dynamical system (differential equation). On besides of this, the controller is also time varying proportional output feedback of striking simplicity. This structure was introduced for the class of nonlinear systems (1.1.6) in [48], modifications to mollify controller behaviour near the funnel boundary are contained in [51]. However, there still remains a question about tracking error for system with relative degree larger than one, see [38, 45, 68].

1.1.3 Funnel control for system with higher strict relative degree

In the case of systems with higher relative degree, high-gain adaptive control problem in general and funnel control problem in particular, becomes much more difficult. For instance, if we apply a conventional controller $u(t) = -ky(t)$ to a very simple system with relative degree two, take, for example, $\ddot{y}(t) = u(t)$; then we get the closed loop system

$$\ddot{y}(t) = -ky(t).$$

This system is not asymptotically stable for arbitrary gain k. In order to find an efficiently adaptive control law for system with higher strict relative degree, a filter or observer is frequently used to obtain approximations of the output derivatives and try to combine with a high gain controller. Mareels [64] is one of the first try to achieve stabilization for linear systems of higher relative degree, however, it is unsuccessful due to a counterexample to the main result presented in [37]. Later on, some results which relate to single-input, single-output systems are distributed by several researchers. Hoagg and Bernstein focused on the problem of adaptive stabilization of linear systems with higher relative degree in [37, 36]. Bullinger and Allgöwer introduced in [17] a high gain observer which can be cooperated with an adaptive controller to ensure tracking with prescribed asymptotic accuracy $\lambda > 0$ (λ-tracking). However, the unable to achieve the transient behavior of the tracking error is a big disadvantage of this type of controller. A "back-stepping" procedure is used by Ye [90, 91] to attain adaptive λ-tracking with non-decreasing gain. In [90], a linear minimum phase systems with nonlinear perturbation is considered, and the class of allowable nonlinearities is smaller than system class defined by (1.1.6). In [91], Ye also used a piecewise constant adaptive switching strategy to stabilize the systems of maximum relative degree in parametric strict feedback form.

To cope with the obstacle of higher relative degree, Ilchmann et al. [49, 50] developed a funnel controller by using the "back-stepping" procedure in conjunction with a pre-compensator. This controller achieves tracking with prescribed transient behaviour for a large class of systems governed by nonlinear (functional) differential equations,

$$\begin{aligned} y^{(r)}(t) &= \sum_{i=0}^{r-1} R_i y^{(i)}(t) + g(p(t),(Ty)(t)) + \Gamma u(t), \\ y|_{[-h,0]} &= y^0 \in \mathscr{C}^{r-1}([-h,0] \to \mathbb{R}^m), \end{aligned}$$

where g is a continuous function, $p(t)$ is a bounded disturbance, and T is a causal operator with a bounded-input bounded-output property, the matrix Γ is assumed to be positive (negative) definite. The works [49, 50] introduce a funnel controller based on a filter and "back-stepping" construction for systems with higher relative degree. First consider a filter with

$$\begin{aligned} \dot{\xi}_i(t) &= -\xi_i(t) + \xi_{i+1}(t), \quad i = 1,\dots,r-2, \\ \dot{\xi}_{r-1}(t) &= -\xi_{r-1}(t) + u(t). \end{aligned}$$

Introduce the projections

$$\pi_i : \mathbb{R}^{(r-1)m} \to \mathbb{R}^{im},\ \xi = (\xi_1,\dots,\xi_{r-1}) \mapsto (\xi_1,\dots,\xi_i)$$

for $i = 1,\dots,r-1$ and functions

$$\begin{aligned} \gamma_1(k,e) &= k \cdot e \\ \gamma_i(k,e,\pi_{i-1}\xi) &:= \gamma_{i-1}(k,e,\pi_{i-2}\xi) + \|D\gamma_{i-1}(k,e,\pi_{i-2}\xi)\|^2 k^4 \cdot (1 + \|\pi_{i-1}\xi\|^2) \\ &\quad \cdot (\xi_{i-1} + \gamma_{i-1}(k,e,\pi_{i-2}\xi)) \end{aligned}$$

The controller in [50] takes the form

$$\begin{aligned} u(t) &= -\gamma_r(k(t),e(t),\xi(t)), \\ k(t) &= \frac{1}{1-\varphi(t)^2\|e(t)\|^2}. \end{aligned}$$

We stress that in [50] T may only depend on y and Γ is assumed to be constant. The above presented controller works provided that $\Gamma \in \mathbb{R}^{m\times m}$ is positive definite. However, this approach can be modified such that it also works for systems in which it is not known whether Γ is positive or negative definite. In this case, the function γ_1 has to be modified by $\gamma_1(k,e) = \nu(k) \cdot e$, where $\nu : \mathbb{R}_{\geq 0} \to \mathbb{R}$ is smooth and satisfies the "Nussbaum property" [50].

$$\limsup_{k\to\infty} \frac{1}{k}\int_0^k \nu(k)dk = \infty,$$
$$\liminf_{k\to\infty} \frac{1}{k}\int_0^k \nu(k)dk = -\infty.$$

Unfortunately, the "back-stepping" procedure is quite impractical even in the known sign case, especially since it involves high powers of a gain function which typically takes very large values, cf. [29, Sec.4.4.3]. In the following we demonstrate input function for the cases of relative degree two and three.

$r = 2$: Here the controller takes the form

$$u = -ke - (\|e\|^2 + k^2) \cdot k^4(1 + \|\xi\|^2)(\xi + ke),$$

where we omit the argument t. This feedback law is dynamic and the gain occurs with $k(t)^7$. The presence of such a large power of the funnel gain $k(t)$ is problematic in practice; the controller produces inputs which might be impractical, cf. [29, Sec.4.4.3].

$r = 3$: Here the controller reads, for $m = 1$,

$$\begin{aligned} u = &-ke - k^4(e^2+k^2)(1+\xi_1^2)(\xi_1+ke) - \Big\{\big[e+(1+\xi_1^2) \\ &\cdot \big[2k^5(\xi_1+ke)+4k^3(e^2+k^2)(\xi_1+ke)+k^4(e^2+k^2)e\big]\big]^2 \\ &+ \big[k+k^4(1+\xi_1^2)\big[2e(\xi_1+ke)+k(e^2+k^2)\big]\big]^2 \\ &+ \big[k^4(e^2+k^2)\big[2\xi_1(\xi_1+ke)+(1+\xi_1^2)\big]\big]^2\Big\}k^4(1+\xi_1^2+\xi_2^2) \\ &\cdot \big[\xi_2+ke+k^4(e^2+k^2)(1+\xi_1^2)(\xi_1+ke)\big]. \end{aligned} \tag{1.1.10}$$

An expansion of the above product gives that this controller contains the 25th power (!) of the funnel gain $k(t)$, and the problems depicted for $r = 3$ are present here a fortiori.

An alternative approach to system with higher relative degree is *derivative feedback*, i.e., the system output $y(\cdot)$ and its higher derivatives are used. In the model of standard position control problem, the output $y(\cdot)$ and its derivatives are normally availbles, see [34, 33] and reference therein. The first attempt by this approach is using type of a proportional-derivative (PD) funnel controller given in [34], (see also the modification in [28]), for systems with relative degree two. Avoiding the "back-stepping" procedure, the authors construct a simple funnel control for single-input, single-output systems as follow

$$\begin{aligned} u(t) &= -k_0(t)^2 e(t) - k_1(t)\dot{e}(t), \\ k_i(t) &= \frac{\varphi_i(t)}{1-\varphi_i(t)|e^{(i)}(t)|}, \quad i = 0, 1. \end{aligned} \tag{1.1.11}$$

The funnel functions φ_0 for the error and φ_1 for the derivative of the error have to satisfy $\varphi_0 \in \Phi_2$, $\varphi_1 \in \Phi_1$, and they have to fulfill the compatibility condition

$$\forall t > 0\ \exists \delta > 0 :\ 1/\varphi_1(t) \geq \delta - \frac{d}{dt}(1/\varphi_0(t)) \qquad \forall\, t > 0. \tag{1.1.12}$$

Further versions of this type of controller for multi-input, multi-output system with strict relative degree two both in linear and nonlinear case were developed by Hackl in [28, 29, 33]. This controller is simple and its practicability has been verified experimentally. However, there is no straightforward extension to systems with relative degree larger than two. The only available generalization of this approach to systems with higher relative degree is the bang-bang funnel controller introduced by Liberzon and Trenn [60]. The authors consider single-input, single-output systems described by a nonlinear differential equation

$$\begin{aligned}\dot{x} &= F(x) + G(x)u, \quad x(0) = x^0 \in \mathbb{R}^n,\\ y &= H(x),\end{aligned}$$

with known relative degree r and positive high frequency gain. The bang-bang funnel controller was introduced as

$$u(t) = \begin{cases} U^-, & \text{if } q(t) = \text{ true}\\ U^+, & \text{if } q(t) = \text{ false}\end{cases} \tag{1.1.13}$$

where $q : \mathbb{R}_{\geq 0} \to \{\text{true, false}\}$ is the output of the switching logic $\mathscr{S}$ which maps the error signal to the switching signal q. The switching logic $\mathscr{S} : e \mapsto q$ is defined basing on r blocks $\mathfrak{B}_0, \mathfrak{B}_1, \ldots, \mathfrak{B}_{r-1}$,

$$\begin{aligned}\mathscr{S}(e) &= \mathfrak{B}_{r-1}(e^{(r-1)}, q_{r-1}, \psi_{r-1}),\\ (q_i, \psi_i) &= \mathfrak{B}_{i-1}(e^{(i-1)}, q_{i-1}, \psi_{i-1}), \quad i = r-1, \ldots, 2,\\ (q_i, \psi_i) &= \mathfrak{B}_0(e).\end{aligned} \tag{1.1.14}$$

Each block $\mathfrak{B}_i$ ensures $e^{(i)}$ remaining inside the funnel

$$\mathscr{F}_i := \left\{ (t, e^{(i)}) \in \mathbb{R}_{\geq 0} \times \mathbb{R} \;\middle|\; \varphi_i^-(t) < e^{(i)} < 1 \right\},$$

and

$$\begin{aligned}&\mathfrak{B}_0 : e \mapsto (q_1, \psi_1) \text{ with}\\ &\quad q_1(t) = \mathfrak{S}(e(t), \varphi_0^+(t) - \varepsilon_0^+, \varphi_0^-(t) + \varepsilon_0^-, q_1(t-)),\\ &\quad q_1(0-) = q_1^0 \in \{\text{true, false}\},\\ &\quad \psi_1(t) = \begin{cases}\dot{\varphi}_0^+(t), & \text{if } q_1(t) = \text{ true},\\ \dot{\varphi}_0^-(t), & \text{if } q_1(t) = \text{ false},\end{cases}\end{aligned}$$

$$\begin{aligned}&\mathfrak{B}_i : (e^{(i), q_i, \psi_i}) \mapsto (q_{i+1}, \psi_{i+1}),\ i = 1, \ldots, r-2 \text{ with}\\ &q_{i+1}(t) = \begin{cases}\mathfrak{S}(e^{(i)}(t), \psi_i^+(t) - \varepsilon_i^+, \varphi_i^-(t) + \varepsilon_i^-, q_{i+1}(t-)), \text{ if } q_i(t) = \text{ true},\\ \mathfrak{S}(e^{(i)}(t), \varphi_i^+(t) - \varepsilon_i^+, \psi_i^-(t) + \varepsilon_i^-, q_{i+1}(t-)), \text{ if } q_i(t) = \text{ false},\end{cases}\\ &\psi_i^+(t) = \min\{\psi_i(t), -\lambda_i^-\}\\ &\psi_i^-(t) = \max\{\psi_i(t), \lambda_i^+\}\\ &q_{i+1}(0-) = q_{i+1}^0 \in \{\text{true, false}\},\\ &\psi_{i+1}(t) = \begin{cases}\dot{\psi}_i(t+), & \text{if } q_i(t) = \text{ true} \wedge q_{i+1}(t) = \text{ true},\\ \dot{\varphi}_i^-(t), & \text{if } q_i(t) = \text{ true} \wedge q_{i+1}(t) = \text{ false},\\ \dot{\varphi}_i^+(t), & \text{if } q_i(t) = \text{ false} \wedge q_{i+1}(t) = \text{ true},\\ \dot{\psi}_i(t+), & \text{if } q_i(t) = \text{ false} \wedge q_{i+1}(t) = \text{ false},\end{cases}\end{aligned}$$

$\mathfrak{B}_{r-1} : (e^{(r-1),q_{r-1},\psi_{r-1}}) \mapsto q$ given as above with $i = r-1$, and $q = q_r$,

$$q(0-) = q^0 \in \{\text{true, false}\},$$

where $\mathfrak{S}(e,\bar{e},\underline{e},q_{\text{old}}) := [e \geq \bar{e} \vee (e > \underline{e} \wedge q_{\text{old}})]$ for $e,\bar{e},\underline{e} \in \mathbb{R}$, and $q_{\text{old}} \in \{\text{true, false}\}$. It is easy to see that the most limitation of this controller is only available to single-input, single-output systems and the involved compatibility conditions on the funnel boundaries, the safety distances and the settling times are quite complicated, see [60, Sec.IV].

In the conference paper [23], Chowdhury and Khalil defined a virtual (weighted) output which can be applied to single-input, single-output systems with higher relative degree to obtain the systems having relative degree one with respect to this virtual output. And, therefore, the conventional funnel controller from [48] is fiesible for the obtained systems. This funnel controller is also shown that (ignoring the additional use of a high-gain observer) for sufficiently small weighting parameter in the virtual output, the original tracking error evolves in a prescribed performance funnel. However, tuning of the weighting parameter has to be done a posteriori and hence depends on the system parameters and the chosen reference trajectory. As a consequence, this approach breaks the model-free propery of funnel control. Moreover, the controller is not robust, since the small perturbations of the reference signal may cause the tracking error to leave the performance funnel.

In [5], a "Prescribed Performance Controller" for systems with higher strict relative degree has been introduced by Bechlioulis and Rovithakis (and in [85] the influence of disturbances is discussed), however trivial internal dynamics are assumed. Moreover, the performance bounds are limited to class of smooth, strict decreasing functions. In addition, the proposed controller is not "simple" since some strategies in selection of control elements are required.

Therefore, a simple strategy in funnel control for system with higher relative degree is still an open problem. In the present dissertation we consider output trajectory tracking for nonlinear systems by funnel control. We assume knowledge of the vector relative degree of the system and that the internal dynamics are, in a certain sense, input-to-state stable, resembling the concept introduced by Sontag [83].

1.2 Contribution of dissertation

We briefly highlight the main contribution of this thesis.

In Chapter 2, we consider a general class of linear differential-algebraic systems

$$\begin{aligned} E\dot{x}(t) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t) \end{aligned}$$

where $E,A \in \mathbb{R}^{l\times n}$, $B \in \mathbb{R}^{l\times m}$, $C \in \mathbb{R}^{p\times n}$.

In Section 2.2, we study regular systems which means that $l = n$ and $\det(sE - A) \in \mathbb{R}[s] \setminus \{0\}$. We introduce some important fundamental concepts of system theory such that transfer function and its properties in Subsection 2.2.1, vector relative degree and its properties in Subsection 2.2.2. Furthermore, *normal form*, which is a prime concept in adaptive control, is introduced in Subsection 2.2.3. We also recall some important results from [9, 10, 69].

In Section 2.3, we present several results for general linear differential algebraic systems. At the beginning of this section, in Subsection 2.3.1, we recall some elementary results from [7]. In the proximal section of Section 2.3, we introduce a novel extension for the definition of vector relative degree for general linear differential-algebraic systems. This new notion does not only help us to open width the system class which have vector relative degree, but also indicates a way to obtain a *normal form* of general linear differential-algebraic systems.

Finally, in Subsection 2.3.3, we present a very new result in *normal form* of general linear differential-algebraic systems. This *normal form* plays an important role to construct a new efficiently funnel control law for differential-algebraic systems.

In Chapter 3, we present our new result in funnel control for non-linear system with known strict relative degree which has been published in [13]. We introduced a novel funnel controller for non-linear systems described by functional differential systems of the form

$$\begin{aligned} y^{(r)}(t) &= f\big(d(t), T(y,\dot{y},\ldots,y^{(r-1)})(t)\big) + \Gamma\big(d(t), T(y,\dot{y},\ldots,y^{(r-1)})(t)\big)\, u(t), \\ y|_{[-h,0]} &= y^0 \in \mathscr{C}^{r-1}([-h,0] \to \mathbb{R}^m), \end{aligned}$$

where $h > 0$ is the "memory" of the system, $r \in \mathbb{N}$ is the strict relative degree.
In Section 3.1, we indicate some necessary assumptions which were imposed on the system class as following.

(P1): The measurable "disturbance" satisfies $d \in \mathscr{L}^\infty(\mathbb{R}_{\geq 0} \to \mathbb{R}^p)$, $p \in \mathbb{N}$.

(P2): $f \in \mathscr{C}(\mathbb{R}^p \times \mathbb{R}^q \to \mathbb{R}^m)$, $q \in \mathbb{N}$.

(P3): the "high-frequency gain matrix function" $\Gamma \in \mathscr{C}(\mathbb{R}^p \times \mathbb{R}^q \to \mathbb{R}^{m\times m})$ takes values in the set of positive (negative) definite matrices.

(P4): $T : \mathscr{C}([-h,\infty) \to \mathbb{R}^{rm}) \to \mathscr{L}^\infty_{\text{loc}}(\mathbb{R}_{\geq 0} \to \mathbb{R}^q)$ is an operator with the following properties

a) T maps bounded trajectories to bounded trajectories.

b) T is causal.

c) T is locally Lipschitz continuous.

Moreover, minimum-phase linear time-invariant systems are shown being apart of this class of systems.
In Section 3.2, we introduce a new funnel control law for non-linear system with known strict relative degree. A theoretically comparison between our proposal and some others approach, which was in [34, 49, 50], is presented.
In the proximal section, we prove the feasibility of funnel strategy in application of considered system class; in particular we show that our proposed funnel controller achieves the control objective described in Subsection 1.1.2. Additionally we derive an explicit bound on the input generated by the controller and discuss the influence of the design parameters.
Finally, in Section 3.3, the performance of the funnel controller is illustrated by means of several examples in mechanics control problems from "mass on car system" to "robotic manipulator", where also our approach is compared to the feedback strategies in [34, 49, 50, 60].

In Chapter 4, we introduce and prove the feasibility of new funnel control for non-linear differential algebraic equations with known generalized vector relative degree of the form,

$$
\begin{aligned}
\begin{pmatrix} y_1^{(r_1)}(t) \\ y_2^{(r_2)}(t) \\ \vdots \\ y_p^{(r_p)}(t) \end{pmatrix} &= f_1\Big(y_1(t),\cdots,y_1^{(r_1-1)}(t),\cdots,y_p^{(r_p-1)}(t),y_{p+1}(t),\cdots,y_m(t)\Big) \\
&\quad + f_2\Big(d_1(t),(Ty)(t)\Big) + \Gamma_I\Big(d_2(t),(Ty)(t)\Big)u_I(t), \\
0 &= f_3\Big(y_1(t),\cdots,y_1^{(r_1-1)}(t),\cdots,y_p^{(r_p-1)}(t),y_{p+1}(t),\cdots,y_m(t)\Big) \\
&\quad + f_4\Big(d_3(t),(Ty)(t)\Big) + \Gamma_{II}\Big(d_4(t),(Ty)(t)\Big)u_I(t) \\
&\quad + f_5\Big(d_5(t),(Ty)(t)\Big)u_{II}(t), \\
y|_{[-h,0]} &= y^0 = (y_1^0,y_2^0,\ldots,y_m^0), \\
y_i^0 &\in \mathscr{C}^{r_i-1}([-h,0]\to\mathbb{R}),\ i=1,\ldots,p, \\
y_i^0 &\in \mathscr{C}([-h,0]\to\mathbb{R}),\ i=p+1,\ldots,m,
\end{aligned}
$$

where $r=(r_1,\ldots,r_p,0,\ldots,0)\in\mathbb{N}^{1\times m}$ is called the *generalized vector relative degree* of the systems. Denote $|r|=\sum_{i=1}^{p} r_i$, $u_I(t)=(u_1(t),\ldots,u_p(t))^\top$, $u_{II}(t)=(u_{p+1}(t),\ldots,u_m(t))^\top$. The functions $u=(u_I,u_{II})^\top:\mathbb{R}\to\mathbb{R}^p\times\mathbb{R}^{m-p}$ and $y:[-h,\infty)\to\mathbb{R}^m$, where $h>0$ is the "memory" of the system, are called *input* and *output* of the system, respectively.
In Section 4.1, we in turn present some assumptions which are imposed on system class including class of operator introduced in [11].

(i) the gain $\Gamma_I\in\mathscr{C}^1(\mathbb{R}^q\times\mathbb{R}^s\to\mathbb{R}^{p\times p})$ takes values in the set of positive (negative) definite matrices, $\Gamma_{II}\in\mathscr{C}^1(\mathbb{R}^q\times\mathbb{R}^s\to\mathbb{R}^{(m-p)\times p})$.

(ii) the disturbances $d_1,d_2\in\mathscr{L}^\infty(\mathbb{R}_{\geq 0}\to\mathbb{R}^q)$, and $d_3,d_4,d_5\in\mathscr{C}^1(\mathbb{R}_{\geq 0}\to\mathbb{R}^q)$ are bounded.

(iii) $f_1\in\mathscr{C}^1(\mathbb{R}^{|r|+m-p}\to\mathbb{R}^p)$, $f_2\in\mathscr{C}^1(\mathbb{R}^q\times\mathbb{R}^k\to\mathbb{R}^p)$, $f_3\in\mathscr{C}^1(\mathbb{R}^{|r|+m-p}\to\mathbb{R}^{m-p})$, $f_4\in\mathscr{C}^1(\mathbb{R}^q\times\mathbb{R}^k\to\mathbb{R}^{m-p})$, and $f_3'\cdot\begin{bmatrix}0\\ I_{m-p}\end{bmatrix}$ is bounded.

(iv) $f_5\in\mathscr{C}^1(\mathbb{R}^q\times\mathbb{R}^k\to\mathbb{R})$, and $\exists\alpha>0,\forall(d,v)\in\mathbb{R}^q\times\mathbb{R}^k:\ f_5(d,v)\geq\alpha$.

(v) Operator $T:\mathscr{C}([-h,\infty)\to\mathbb{R}^m)\to\mathscr{C}^1(\mathbb{R}_{\geq 0}\to\mathbb{R}^k)$ satisfies

  (a) T is a causal operator.

  (b) T is locally Lipschitz continuous.

  (c) T maps bounded trajectories to bounded trajectories.

  (d) $\exists z\in\mathscr{C}(\mathbb{R}^m\times\mathbb{R}^k\to\mathbb{R}^k)\ \exists\tilde{T}:[-h,\infty)\to\mathbb{R}^m)\to\mathscr{C}(\mathbb{R}_{\geq 0}\to\mathbb{R}^k)$ with properties $(a)-(c)\ \forall v\in\mathscr{C}([-h,\infty)\to\mathbb{R}^m)\ \forall t\geq 0$:

$$\frac{d}{dt}(Tv)(t)=z(v(t),(\tilde{T}v)(t)).$$

In Subsection 4.1.1, 4.1.2, we consider some important subclasses of the system class such as the systems with positive vector relative degree, and linear general differential-algebraic systems.

In the follow section, we introduce a new funnel controller with tracking for each element of outputs for non-linear system with known generalized vector relative degree and the feasibility of this funnel strategy. Additionally, we show that our new proposed funnel controller achieves the control objective described in Subsection 1.1.2. Furthermore, in Subsection 4.2.2, we demonstrate the performance of the funnel controller in simulation of several examples.

In Chapter 5, we study an application of funnel control to overhead crane case which was introduced in our proceeding paper [14]. The model of overhead crane was presented by Otto and Seifried in [73] with motion equation described as follows

$$\begin{bmatrix} \tau_s & 0 & 0 \\ 0 & \tau_l & 0 \\ \cos\varphi & 0 & l \end{bmatrix} \begin{pmatrix} \ddot{s} \\ \ddot{l} \\ \ddot{\varphi} \end{pmatrix} + \begin{pmatrix} \dot{s} \\ \dot{l} \\ 2\dot{\varphi}\dot{l} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -g\sin\varphi \end{pmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} u_s \\ u_l \end{pmatrix},$$

where τ_s, τ_l are time constants of trolley and winch actuator, respectively and g is the gravitational constant. The quantities s, l, φ are denoted to trolley position, length of the rope, and the swing angle, respectively. The system inputs are served by the reference velocity of the trolley u_s and the reference velocity of the rope u_l. We note that the rope velocity $\dot{l}$, and trolley velocity $\dot{s}$ are then rheonomic constraints on the system. And the position of the load $(s + l\sin\varphi, l\cos\varphi)$ plays the role of system output.

The control objective is to design a closed-loop tracking controller which also takes into account the transient behaviour. Unfortunately, this system is not well defined vector relative degree with respect to considered input, output vectors. Therefore, we use an dynamic feedback introduced in [56, 80] to achieve a new system which have strict relative degree. Computer simulations are shown in Section 5.2 to illustrate that our approach can be used to move loads from one to another given position in the situation where there are several obstacles which have to be circumnavigated.

To summary, some parts of present dissertation, which are published or submitted for publication, will be indicated in the following table

Chapter or Section	**Contained in**
Section 2.3	New
Chapter 3	Berger, Lê, Reis [13, 12]
Chapter 4	New
Chapter 5	Berger, Lê, Reis [14]

Chapter 2

Linear differential-algebraic systems

In this chapter, we study linear differential-algebraic multi-input multi-output systems. By investigating *zero dynamics* of the system, we suppose a new approach to generalize the definition of *vector relative degree*. Moreover, we base on this definition to introduce a novel *normal form* of general linear differential-algebraic system.

2.1 Preliminaries

We consider linear differential-algebraic systems,

$$\begin{aligned} E\dot{x}(t) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t) \end{aligned} \tag{2.1.1}$$

where $E, A \in \mathbb{R}^{l\times n}$, $B \in \mathbb{R}^{l\times m}$, $C \in \mathbb{R}^{p\times n}$. We denote $\Sigma_{l,n,m,p}$ being the class of these systems and write $[E,A,B,C] \in \Sigma_{l,n,m,p}$. We want to stress that this systems is not required to be *regular*[1]. The functions $u : \mathbb{R} \to \mathbb{R}^m$, $x : \mathbb{R} \to \mathbb{R}^n$, and $y : \mathbb{R} \to \mathbb{R}^p$ are called *input, (generalized) state variable*, and *output* of the system, respectively. We now denote what is so called the *behaviour* of system (2.1.1),

$$\mathfrak{B}_{[E,A,B,C]} := \left\{ (x,u,y) \in \mathscr{L}^1_{\text{loc}}(\mathbb{R} \to \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^p) \;\middle|\; \begin{array}{c} Ex \in \mathscr{W}^{1,1}_{\text{loc}}(\mathbb{R} \to \mathbb{R}^l) \text{ and } (x,u,y) \\ \text{fulfills (2.1.1) for almost all } t \in \mathbb{R} \end{array} \right\}.$$

$(x,u,y) \in \mathfrak{B}_{[E,A,B,C]}$ is called a *(weak) solution* of (2.1.1). Note that $Ex \in \mathscr{W}^{1,1}_{\text{loc}}(\mathbb{R} \to \mathbb{R}^l)$ is continuous but $x(t)$ might not be continuous.
We recall some basic notions in systems theory such that *system equivalent*, and *zero dynamics*.

Definition 2.1.1. Two systems $[E_i, A_i, B_i, C_i] \in \Sigma_{l,n,m,p}$, $i = 1,2$, are called *system equivalent* if, and only if,

$$\exists W \in \mathrm{GL}_l(\mathbb{R}), T \in \mathrm{GL}_n(\mathbb{R}) : \begin{bmatrix} sE_1 - A_1 & B_1 \\ C_1 & 0 \end{bmatrix} = \begin{bmatrix} W & 0 \\ 0 & I_p \end{bmatrix} \begin{bmatrix} sE_2 - A_2 & B_2 \\ C_2 & 0 \end{bmatrix} \begin{bmatrix} T & 0 \\ 0 & I_m \end{bmatrix},$$

and denoted by

$$[E_1, A_1, B_1, C_1] \overset{W,T}{\sim} [E_2, A_2, B_2, C_2].$$

System equivalence is an equivalence relation on $\Sigma_{l,n,m,p}$, see [59, 7, 15] and references therein. We discuss following an important concept which is so called *zero dynamics*, see [56, Sec.4.3]. Loosely speaking, *zero dynamics* is resulting in a trivial output.

Definition 2.1.2 ([7, Def.3.1,Def.3.9])**.**

[1] A matrix pencil $(sE - A)$ is said to be *regular* if $l = n$, and $\det(sE - A) \not\equiv 0$.

- The zero dynamics of the systems (2.1.1) is the set

$$\mathscr{ZD}_{[E,A,B,C]} := \left\{ (x,u,y) \in \mathfrak{B}_{[E,A,B,C]} \;\middle|\; y \overset{\text{a.e}}{=} 0 \right\}.$$

- The zero dynamics $\mathscr{ZD}_{[E,A,B,C]}$ is called autonomous if

$$\forall \omega \in \mathscr{ZD}_{[E,A,B,C]}, \forall I \subseteq \mathbb{R} \text{ open interval: if } \omega|_I \overset{\text{a.e}}{=} 0, \text{ then } \omega \overset{\text{a.e}}{=} 0.$$

- The zero dynamics $\mathscr{ZD}_{[E,A,B,C]}$ is called asymptotically stable if

$$\forall (x,u,y) \in \mathscr{ZD}_{[E,A,B,C]} : \lim_{t\to\infty} \operatorname*{esssup}_{[t,\infty)} \|(x,u)\| = 0.$$

Furthermore, we introduce a notion of (E,A,B)-invariant subspace, given in [2, 7, 74], playing as a tool to investigate zero dynamics properties. This concept is a generalization of controller invariance subspace or (A,B)-invariant subspace which were introduced in [4, 89, 88, 86].

Definition 2.1.3. Suppose $(E,A,B) \in \mathbb{R}^{l\times n} \times \mathbb{R}^{l\times n} \times \mathbb{R}^{n\times m}$, and $\mathscr{V}$ is a linear subspace of $\mathbb{R}^n$. If $A\mathscr{V} \subseteq E\mathscr{V} + \operatorname{im} B$, then $\mathscr{V}$ is called (E,A,B)-invariant subspace.

Given a system $[E,A,B,C] \in \Sigma_{l,n,m,p}$, the maximal (E,A,B)-invariant subspace included in $\ker C$ is denoted by $\max(E,A,B;\ker C)$.

Proposition 2.1.4 ([7, Prop.3.5])**.** *Let $[E,A,B,C] \in \Sigma_{l,n,m,p}$. Then the following statements are equivalent.*

(i) $\mathscr{ZD}_{[E,A,B,C]}$ *is autonomous.*

(ii) $\operatorname{rank}_{\mathbb{R}(s)} \begin{bmatrix} sE-A & -B \\ -C & 0 \end{bmatrix} = n+m.$

(iii) *(a)* $\operatorname{rank} B = m$,
(b) $\ker E \cap \max(E,A,B;\ker C) = \{0\}$,
(c) $\operatorname{im} B \cap E \max(E,A,B;\ker C) = \{0\}$.

Proposition 2.1.5 ([7, Thm.3.6])**.** *Consider $[E,A,B,C] \in \Sigma_{l,n,m,p}$ and suppose that the zero dynamics $\mathscr{ZD}_{[E,A,B,C]}$ is autonomous. Let $V \in \mathbb{R}^{n\times k}$ be such that $\operatorname{im} V = \max(E,A,B;\ker C)$ and $\operatorname{rank} V = k$. Then there exist $W \in \mathbb{R}^{n\times(n-k)}$ and $S \in \mathrm{GL}_l(\mathbb{R})$ such that $[V,W] \in GL_n(\mathbb{R})$ and*

$$[E,A,B,C] \overset{S,[V,W]}{\sim} [\widetilde{E},\widetilde{A},\widetilde{B},\widetilde{C}],$$

where

$$\widetilde{E} = \begin{bmatrix} I_k & E_2 \\ 0 & E_4 \\ 0 & E_6 \end{bmatrix}, \widetilde{A} = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \\ 0 & A_6 \end{bmatrix}, \widetilde{B} = \begin{bmatrix} 0 \\ I_m \\ 0 \end{bmatrix}, \widetilde{C} = \begin{bmatrix} 0 & C_2 \end{bmatrix} \tag{2.1.2}$$

such that

$$\max\left(\begin{bmatrix} E_4 \\ E_6 \end{bmatrix}, \begin{bmatrix} A_4 \\ A_6 \end{bmatrix}, \begin{bmatrix} I_m \\ 0 \end{bmatrix}; \ker C_2 \right) = \{0\}, \tag{2.1.3}$$

and $A_1 \in \mathbb{R}^{k\times k}$, $E_2, A_2 \in \mathbb{R}^{k\times(n-k)}$, $A_3 \in \mathbb{R}^{m\times k}$, $E_4, A_4 \in \mathbb{R}^{m\times(n-k)}$, $E_6, A_6 \in \mathbb{R}^{(l-k-m)\times(n-k)}$, $C_2 \in \mathbb{R}^{p\times(n-k)}$.

For uniqueness we have: If $[E,A,B,C], [\widehat{E},\widehat{A},\widehat{B},\widehat{C}] \in \Sigma_{l,n,m,p}$ *are in the form* (2.1.2) *such that* (2.1.3) *holds, and*

$$[E,A,B,C] \overset{S,T}{\sim} [\widehat{E},\widehat{A},\widehat{B},\widehat{C}] \text{ for some } S \in GL_l(\mathbb{R}), T \in GL_n(\mathbb{R}), \tag{2.1.4}$$

then

$$S = \begin{bmatrix} S_1 & 0 & S_3 \\ 0 & I_m & S_6 \\ 0 & 0 & S_9 \end{bmatrix}, \quad T = \begin{bmatrix} S_1^{-1} & T_2 \\ 0 & T_4 \end{bmatrix},$$

where $S_1 \in GL_k(\mathbb{R})$*,* $S_9 \in GL_{l-k-m}(\mathbb{R})$*,* $T_4 \in GL_{n-k}(\mathbb{R})$ *and* S_3, S_6, T_2 *are appropriate sizes. In particular the dimensions of the matrices in* (2.1.2) *are unique and* A_1 *is unique up to similarity, i.e.,* $\sigma(A_1)$ *is unique.*

In the next section, we will highlight some important results of system theory considered for the special DAE case: the matrix pencil $sE - A$ is regular. This case was studied by Berger, Reis and Ilchmann in [6, 10, 9], but a normal form base on vector relative degree for the whole case is still an open problem.

2.2 Regular systems

First of all, we need to study an important case of system pencil, that is *regular pencil*. The pencil matrix $sE - A$ is called *r*egular if $l = n$ and $\det(sE - A) \in \mathbb{R}[s] \setminus \{0\}$. Regularity of the systems pencil ensures that the differential-algebraic equation $E\dot{x} = Ax$ is solvable and its solution is unique for each consistent initial value, as shown in [59, Sec.2.1]. Therefore, we consider the following system.

$$\begin{aligned} E\dot{x}(t) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t), \end{aligned} \tag{2.2.1}$$

where $A, E \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$.

2.2.1 Transfer function

In this subsection, we declare *transfer function* which is a primary tool in control theory. Some elementary concepts of system theory are defined via transfer function.

Definition 2.2.1. Let $[E,A,B,C] \in \Sigma_{n,n,m,p}$ be a regular system. Then the transfer function of $[E,A,B,C]$ is defined by

$$G(s) = C(sE - A)^{-1}B \in \mathbb{R}(s)^{p\times m}.$$

Remark 2.2.2. Transfer function of the system (2.2.1) is a rational matrix function and invariant under system equivalence. For more detail, we introduce *Weierstrass form* of system pencil which was mentioned in some books and papers such as [9, 10, 27], [59, Sec.2.1].

Proposition 2.2.3 (Weierstrass form)**.** *For any regular matrix pencil* $sE - A \in \mathbb{R}[s]^{n\times n}$*, there exists* $W, T \in \mathrm{GL}_n(\mathbb{R})$ *such that*

$$sE - A = W \begin{bmatrix} sI_{n_s} - A_s & 0 \\ 0 & sN - I_{n_f} \end{bmatrix} T \tag{2.2.2}$$

for some $A_s \in \mathbb{R}^{n_s\times n_s}$*, and nilpotent* $N \in \mathbb{R}^{n_f\times n_f}$*. The dimensions* $n_s, n_f \in \mathbb{N}_0$ *are unique, and the matrices* A_s*, and* N *are unique up to similarity.*

The *index of nilpotency* of a nilpotent matrix $N \in \mathbb{R}^{k\times k}$ is defined to be the smallest $\nu \in \mathbb{N}_0$ such that $N^\nu = 0$. It can be shown that the index of nilpotency ν of N in (2.2.2) is uniquely defined by the regular pencil $sE - A$. Hence, we say that ν is the *index* of the pencil $sE - A$. In particular, if the nilpotent block is not appear (i.e $n_f = 0$), then the index of matrix pencil is zero.

Corollary 2.2.4. *Let $[E,A,B,C] \in \Sigma_{n,n,m,p}$. Then there exits $W,T \in \mathrm{GL}_n(\mathbb{R})$ such that*

$$[E,A,B,C] \overset{W,T}{\sim} \left[\begin{bmatrix} I_{n_s} & 0 \\ 0 & N \end{bmatrix}, \begin{bmatrix} A_s & 0 \\ 0 & I_{n_f} \end{bmatrix}, \begin{bmatrix} B_s \\ B_f \end{bmatrix}, \begin{bmatrix} C_s & C_f \end{bmatrix} \right] \tag{2.2.3}$$

for some $B_s \in \mathbb{R}^{n_s\times m}, B_f \in \mathbb{R}^{n_f\times m}, C_s \in \mathbb{R}^{p\times n_s}, C_f \in \mathbb{R}^{p\times n_f}, A_s \in \mathbb{R}^{n_s\times n_s}$, and nilpotent $N \in \mathbb{R}^{n_f\times n_f}$.

The form (2.2.3) is interpreted, in terms of the system (2.2.1), as follows

$$(x,u,y) \in \mathfrak{B}_{[E,A,B,C]} \text{ if, and only if } \begin{pmatrix} x_s(\cdot) \\ x_f(\cdot) \end{pmatrix} := Tx(\cdot)$$

solves the decoupled systems

$$\begin{aligned} &\dot{x}_s(t) = A_s x_s(t) + B_s u(t), \quad N\dot{x}_f(t) = x_f(t) + B_f u(t), \\ &y_s(t) = C_s x_s(t), \qquad\qquad\quad y_f(t) = C_f x_f(t), \\ &\qquad\qquad y(t) = y_s(t) + y_f(t), \end{aligned} \tag{2.2.4}$$

If $(x,u,y) \in \mathfrak{B}_{[E,A,B,C]}$ and in addition $u \in \mathscr{C}^{\nu-1}(\mathbb{R} \to \mathbb{R}^m)$, then by repeated multiplication of $N\dot{x}_f(t) = x_f(t) + B_f u(t)$ by N from the left, differentiation, and using the identity $N^\nu = 0$, it is easy to see that the solution satisfies

$$x_f(\cdot) = -\sum_{k=0}^{\nu-1} N^k B_f u^{(k)}(\cdot) \tag{2.2.5}$$

We also derive frequency domain result for $[E,A,B,C] \in \Sigma_{n,n,m,p}$ and its transfer function.

- A rational matrix function $G(s) \in \mathbb{R}(s)^{p\times m}$ is called *proper* if $\lim_{\lambda\to\infty} G(\lambda) = D$ for some $D \in \mathbb{R}^{p\times m}$,
- A rational matrix function $G(s) \in \mathbb{R}(s)^{p\times m}$ is called *strictly proper* if $\lim_{\lambda\to\infty} G(\lambda) = 0$.

Since the transfer function is invariant under system equivalence, we have

$$\begin{aligned} G(s) = C(sE-A)^{-1}B &= C_f(sN - I_{n_f})^{-1}B_f + C_s(sI_{n_s} - A_s)^{-1}B_s \\ &= \underbrace{-\sum_{i=0}^{\nu-1} C_f N^i B_f s^i}_{=:P(s)} + \underbrace{\sum_{i=1}^{\infty} C_s A_s^{i-1} B_s s^{-i}}_{=:G_{sp}(s)} \end{aligned}$$

where $P(s)$ is polynomial matrix, and $G_{sp}(s)$ is strictly proper rational matrix.

We now introduce the definition of transmission zeros and poles of a transfer function which were mentioned in [9, Sec.1.2] or [57, Sec.6.5].

Definition 2.2.5 ([9, Sec.1.2], [57, Sec.6.5])**.** Let $G(s) \in \mathbb{R}(s)^{m\times m}$ with *Smith-McMillan form*

$$U^{-1}(s)G(s)V^{-1}(s) = \operatorname{diag}\left(\frac{\varepsilon_1(s)}{\psi_1(s)}, \ldots, \frac{\varepsilon_r(s)}{\psi_r(s)}, 0, \ldots, 0 \right) \in \mathbb{R}(s)^{m\times m},$$

where $U(s), V(s) \in \mathbb{R}[s]^{m\times m}$ are unimodular, i.e, invertible over $\mathbb{R}[s]^{m\times m}$, $\operatorname{rank}_{\mathbb{R}(s)} G(s) = r$, $\varepsilon_i(s), \psi_i(s) \in \mathbb{R}[s]$ are monic, coprime and satisfy $\varepsilon_i(s)|\varepsilon_{i+1}(s)$, $\psi_{i+1}(s)|\psi_i(s)$ for all $i = 1,\dots,r-1$. A complex number s_0 is called a *transmission zero* of $G(s)$ if $\varepsilon_r(s_0) = 0$ and a *pole* of $G(s)$ if $\psi_1(s_0) = 0$.

Because the transfer function is invariant under system equivalence, the transmission zeros and poles are also invariant.

2.2.2 Vector relative degree

In this subsection, *vector relative degree*, another fundamental concept in systems theory, is introduced. *R*elative degree, roughly speaking, is a number of times the components of the output have to be differentiated to get input explicitly. Futhermore, the *vector relative degree* $r = (r_1, r_2, \dots, r_p) \in \mathbb{N}^p$ for multi input, multi output non-linear differential equation is a vector in which each component r_i is precisely the number of times one has to differentiate the i-th output $y_i(t)$ at $t = t^0$ in order to have at least one component of the input vector $u(t^0)$ explicitly appearing, see Isidori [56, Sec. 4.1, Sec. 5.1]. However, it is sophisticated to use this way for defining *vector relative degree* to differential-algebraic systems. Therefore, we need an alternative approach to define *vector relative degree* for differential-algebraic systems. Since Remark 1.1.4, it is well-known that for single input, single output linear differential system, the *relative degree* r is exactly equal to the difference between numerator and denominator degree of the transfer function. By this approach, the value of the *relative degree* can be negative. For more detail, let us begin with regular case of multi input, multi output linear differential-algebraic systems and introduce a definition of *vector relative degree* presented by Berger [7].

Definition 2.2.6 ([7, Def. B.1])**.**

- Transfer function matrix $G(s) \in \mathbb{R}(s)^{p\times m}$ has *vector relative degree* $(r_1, \cdots, r_p) \in \mathbb{Z}^{1\times p}$, if there exists a matrix $\Gamma \in \mathbb{R}^{p\times m}$ with $\operatorname{rank}\Gamma = p$ such that

$$\lim_{s\to\infty} \operatorname{diag}(s^{r_1}, \dots, s^{r_p}) G(s) = \Gamma.$$

  For convenience, we also say that system $[E, A, B, C] \in \Sigma_{n,n,m,p}$ has vector relative degree $(r_1, \dots, r_p)$ if its transfer function $G(s)$ has vector relative degree $(r_1, \dots, r_p)$.

- The matrix Γ is called *high-frequency gain matrix*.

- A vector relative degree $(r_1, \dots, r_p)$ of system $[E, A, B, C]$ is called an *ordered vector relative degree* if $r_1 \geq r_2 \geq \cdots \geq r_p$.

- If $r_1 = \cdots = r_p = r$, then we say that system $[E, A, B, C] \in \Sigma_{n,n,m,p}$ or $G(s)$ has *strict relative degree* r.

Remarks 2.2.7.

(i) If $[E, A, B, C]$ has a vector relative degree, then the vector relative degree is unique. Indeed, suppose $[E, A, B, C]$ has two vector relative degree $(r_1, \dots, r_p)$ and $(\rho_1, \dots, \rho_p)$, i.e,

$$\lim_{s\to\infty} \operatorname{diag}(s^{r_1}, \dots, s^{r_p}) G(s) = \Gamma \in \mathbb{R}^{p\times m} \text{ and } \operatorname{rank}\Gamma = p,$$

and

$$\lim_{s\to\infty} \operatorname{diag}(s^{\rho_1}, \dots, s^{\rho_p}) G(s) = \tilde{\Gamma} \in \mathbb{R}^{p\times m} \text{ and } \operatorname{rank}\tilde{\Gamma} = p.$$

Denote $F(s) := \operatorname{diag}(s^{r_1},\dots,s^{r_p})G(s)$, then $F(s)$ is proper. Moreover,

$$\begin{aligned}\lim_{s\to\infty}\operatorname{diag}(s^{\rho_1-r_1},\dots,s^{\rho_p-r_p})F(s) &= \lim_{s\to\infty}\operatorname{diag}(s^{\rho_1-r_1},\dots,s^{\rho_p-r_p})\operatorname{diag}(s^{r_1},\dots,s^{r_p})G(s)\\ &= \lim_{s\to\infty}\operatorname{diag}(s^{\rho_1},\dots,s^{\rho_p})G(s) = \tilde{\Gamma}\in\mathbb{R}^{p\times m}.\end{aligned}$$

Since $\lim_{s\to\infty}F(s) = \Gamma\in\mathbb{R}^{p\times m}$ and Γ is full row-rank, we have $\rho_i - r_i \le 0$ for all $i = 1,\dots,p$. Switching the role between r and ρ, we also obtain $r_i - \rho_i \le 0$ for all $i = 1,\dots,p$. Therefore, $r_i = \rho_i$ for all $i = 1,\dots,p$.

(ii) Vector relative degree of a system $[E,A,B,C]$ does not necessarily exist. For example, consider system

$$E = \begin{bmatrix}1 & 1\\ 0 & 0\end{bmatrix};\quad A = \begin{bmatrix}1 & 1\\ 1 & 2\end{bmatrix}; B = I_2; C = I_2.$$

We have

$$G(s) = C(sE-A)^{-1}B = \begin{bmatrix}\dfrac{2}{s-1} & 1\\ -\dfrac{1}{s-1} & -1\end{bmatrix}.$$

Then

$$\Gamma = \lim_{s\to\infty}\operatorname{diag}(s^0,s^0)G(s) = \begin{bmatrix}0 & 1\\ 0 & -1\end{bmatrix}.$$

Clearly that $\operatorname{rank}\Gamma = 1 < 2$, then system does not have vector relative degree.

(iii) Consider a multi-input multi-output linear system of the form

$$\begin{aligned}\dot{x} &= Ax + Bu,\\ y &= Cx,\end{aligned}$$

where $A\in\mathbb{R}^{n\times n}$, $B = [B_1, B_2 \dots, B_m]$ with $B_i\in\mathbb{R}^{n\times 1}$, and $C = \begin{bmatrix}C_1\\ C_2\\ \vdots\\ C_m\end{bmatrix}$ with $C_i\in\mathbb{R}^{1\times n}$, $i = 1,\dots,m$. Using definition of vector relative degree in [56, Sec.5.1] or [69, Def.2.1], the system $[I,A,B,C]$ has vector relative degree $(r_1,r_2,\dots,r_m)\in\mathbb{N}^{1\times m}$, if and only if

(a) $C_iA^kB_j = 0$ for all $i = 1,\dots,m$, $k = 0,\dots,r_i-2$, $j = 1,\dots,m$,

(b) and the matrix

$$\Gamma = \begin{bmatrix}C_1A^{r_1-1}B_1 & C_1A^{r_1-1}B_2 & \cdots & C_1A^{r_1-1}B_m\\ C_2A^{r_2-1}B_1 & C_2A^{r_2-1}B_2 & \cdots & C_2A^{r_2-1}B_m\\ \vdots & \vdots & \ddots & \vdots\\ C_mA^{r_m-1}B_1 & C_mA^{r_m-1}B_2 & \cdots & C_mA^{r_m-1}B_m\end{bmatrix}$$

is non-singular.

It is easy to see that

$$\lim_{s\to\infty}\operatorname{diag}(s^{r_1},s^{r_2},\dots,s^{r_m})C(sI-A)^{-1}B = \lim_{s\to\infty}\operatorname{diag}(s^{r_1},s^{r_2},\dots,s^{r_m})C\left(\sum_{k=0}^{\infty}\frac{A^k}{s^{k+1}}\right)B.$$

Therefore, two conditions $(a),(b)$ are equivalent to condition

$$\lim_{s\to\infty} \operatorname{diag}(s^{r_1}, s^{r_2}, \ldots, s^{r_m}) C \left(\sum_{k=0}^{\infty} \frac{A^k}{s^{k+1}} \right) B$$
$$= \begin{bmatrix} C_1 A^{r_1-1} B_1 & C_1 A^{r_1-1} B_2 & \cdots & C_1 A^{r_1-1} B_m \\ C_2 A^{r_2-1} B_1 & C_2 A^{r_2-1} B_2 & \cdots & C_2 A^{r_2-1} B_m \\ \vdots & \vdots & \ddots & \vdots \\ C_m A^{r_m-1} B_1 & C_m A^{r_m-1} B_2 & \cdots & C_m A^{r_m-1} B_m \end{bmatrix} \in \mathrm{GL}_m(\mathbb{R}).$$

It means that Definition 2.2.6 is consistent with the definition of vector relative degree given by [56, Sec.5.1], or [69, Def.2.1] for linear multi-input, multi-output differential systems.

(iv) Similar to [56, Sec.5.2], we also conclude that vector relative degree of a system is invariant under static output feedback or static state feedback.

(v) If $G(s)$ has strict relative degree $r \le 0$, then

$$r = -\max\left\{ i \in \{0, \ldots, \nu - 1\} \;\middle|\; \operatorname{rank} C_f N^i B_f = p \right\},$$

where C_f, N, B_f are quoted as in (2.2.3). Indeed, since r is the strict relative degree of $G(s)$, then

$$\lim_{s\to\infty} s^r G(s) = \Gamma \in \mathbb{R}^{p\times m}, \text{ and } \operatorname{rank}\Gamma = p.$$

Using notation in (2.2.3), we have

$$\mathbb{R}^{p\times m} \ni \Gamma = \lim_{s\to\infty} s^r \Big(C_f (sN - I_{n_f})^{-1} B_f + C_s (sI_{n_s} - A_s)^{-1} B_s \Big)$$
$$= \lim_{s\to\infty} s^r \left(-\sum_{i=0}^{\nu-1} C_f N^i B_f s^i + \sum_{i=1}^{\infty} C_s A_s^{i-1} B_s s^{-i} \right).$$

Because $r \le 0$, and $\sum_{i=1}^{\infty} C_s A_s^{i-1} B_s s^{-i}$ is strictly proper, we have

$$\Gamma = \lim_{s\to\infty} s^r \left(-\sum_{i=0}^{\nu-1} C_f N^i B_f s^i \right) = \lim_{s\to\infty} \left(-\sum_{i=0}^{\nu-1} C_f N^i B_f s^{i+r} \right).$$

Then

$$r = -\max\left\{ i \in \{0, \ldots, \nu - 1\} \;\middle|\; \operatorname{rank} C_f N^i B_f = p \right\},$$

and $\Gamma = -C_f N^{-r} B_f$.

(vi) Similar to [69, Lem.2.3], we can conclude that if $G(s)$ has vector relative degree $r = (r_1, \ldots, r_p) \in \mathbb{Z}^{1\times p}$. Then there exists a permutation matrix $P \in \mathbb{R}^{p\times p}$ such that the system $[E, A, B, PC]$ has ordered vector relative degree $rP = (\tilde{r}_1, \ldots, \tilde{r}_p)$, (i.e $\tilde{r}_1 \ge \cdots \ge \tilde{r}_p$).

Consider linear multi-input, multi-output differential-algebraic system (2.2.1) in special case with number of input equal to number of output, i.e, $p = m$. Then we have the following proposition.

Proposition 2.2.8. *Let $[E, A, B, C] \in \Sigma_{n,n,m,m}$ be such that transfer function $G(s)$ has vector relative degree $r = (r_1, \ldots, r_m)$. Then* *(i)* $G(s)$ *is invertible over* $\mathbb{R}(s)$*. In particular,* $\operatorname{rank} B = \operatorname{rank} C = m$*.*

(ii) $-r = (-r_1, \ldots, -r_m)$ *is vector relative degree of* $G^{-\top}(s)$*.*

(iii) $\begin{bmatrix} sE-A & -B \\ -C & 0 \end{bmatrix} \in \mathbb{R}(s)^{(n+m)\times(n+m)}$ *is invertible.*

(iv) $\mathscr{Z}\mathscr{D}_{[E,A,B,C]}$ *are autonomous.*

Proof.

(i) Suppose that $G(s) \in \mathbb{R}(s)^{m\times m}$ has vector relative degree $(r_1, \ldots, r_m) \in \mathbb{Z}^{1\times m}$. Since

$$\lim_{s\to\infty} \operatorname{diag}(s^{r_1}, \ldots, s^{r_m}) G(s) = \Gamma \in \mathrm{GL}_m(\mathbb{R}),$$

then $\operatorname{diag}(s^{r_1}, \ldots, s^{r_m}) G(s)$ is invertible over $\mathbb{R}(s)$. Therefore, $G(s)$ is also invertible over $\mathbb{R}(s)$. In particular, $G(s) = C(sE-A)^{-1}B \in \mathbb{R}(s)^{m\times m}$ is full-rank which implies $\operatorname{rank} B = \operatorname{rank} C = m$.

(ii) We have

$$\lim_{s\to\infty} \operatorname{diag}(s^{-r_1}, \ldots, s^{-r_m}) G^{-\top}(s) = \lim_{s\to\infty} \Big(\operatorname{diag}(s^{r_1}, \ldots, s^{r_m}) G(s) \Big)^{-\top} = \Gamma^{-\top} \in \mathrm{GL}_m(\mathbb{R}).$$

This shows that $-r = (-r_1, \ldots, -r_m)$ is vector relative degree of $G^{-\top}(s)$.

(iii) Since

$$\begin{bmatrix} -(sE-A)^{-1}BG^{-1}(s)C(sE-A)^{-1} + (sE-A)^{-1} & (sE-A)^{-1}BG^{-1}(s) \\ -G^{-1}(s)C(sE-A)^{-1} & -G^{-1}(s) \end{bmatrix} \begin{bmatrix} sE-A & -B \\ -C & 0 \end{bmatrix} = I_{n+m},$$

it follows that, $\begin{bmatrix} sE-A & -B \\ -C & 0 \end{bmatrix} \in \mathbb{R}(s)^{(n+m)\times(n+m)}$ is invertible.

(iv) This follows from (iii) and Proposition 2.1.4.

□

Remark 2.2.9. If $[E,A,B,C] \in \Sigma_{n,n,m,m}$ and $L(s) \in \mathbb{R}^{(n+m)\times(n+m)}$ is the inverse of $\begin{bmatrix} sE-A & -B \\ -C & 0 \end{bmatrix}$, then

$$\begin{bmatrix} I_n & (sE-A)^{-1}B \\ 0 & I_m \end{bmatrix} = L(s) \begin{bmatrix} sE-A & -B \\ -C & 0 \end{bmatrix} \begin{bmatrix} I_n & (sE-A)^{-1}B \\ 0 & I_m \end{bmatrix} = L(s) \begin{bmatrix} sE-A & 0 \\ -C & -C(sE-A)^{-1}B \end{bmatrix}.$$

Hence, the inverse of the transfer function can be represented by

$$G^{-1}(s) = -\begin{bmatrix} 0 & I_m \end{bmatrix} L(s) \begin{bmatrix} 0 \\ I_m \end{bmatrix}$$

Therefore, in this case, it follows from Proposition 2.2.8(ii) that

$$\lim_{s\to\infty} \operatorname{diag}(s^{r_1},\dots,s^{r_m})G(s) = \Gamma \in \mathrm{GL}_m(\mathbb{R}) \iff \lim_{s\to\infty} G^{-1}(s)\operatorname{diag}(s^{-r_1},\dots,s^{-r_m}) = \Gamma^{-1} \in \mathrm{GL}_m(\mathbb{R}).$$

2.2.3 Normal form

The *Byrnes-Isidori normal form* (or shortly *normal form*) was first introduced in [22] for nonlinear and linear single-input, single-output systems. In system theory, the *normal form* plays an important role in designing local and global feedback stabilization of nonlinear systems as in [18, 19, 20], adaptive observers as in [71], and adaptive controllers for linear systems as in [44, 53]. Later on, a *normal form* was developed for linear multi-input, multi-output systems as in [56, 69, 10, 9]. We recall in next subsection *normal form* of the systems based on vector relative degree in some special cases.

Systems with positive vector relative degree

We consider a linear system of type (2.2.1) with positive vector relative degree $r = (r_1, r_2, \dots, r_m) \in \mathbb{N}^{1\times m}$. Let us begin with a linear multi-input, multi-output system described by a differential equation

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t), \end{aligned} \tag{2.2.6}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$ and $C \in \mathbb{R}^{m\times n}$. It is clear that $G(s) = C(sI-A)^{-1}B \in \mathbb{R}(s)^{m\times m}$, which is the transfer function of the system (2.2.6), is strictly proper. Therefore, if system (2.2.6) has vector relative degree $r = (r_1, r_2, \dots, r_m)$, then $r_i > 0$ for all $i = 1, 2 \dots, m$. In other words, system (2.2.6) also belong to class of system which have positive vector relative degree. Following proposition will show the *normal form* for system (2.2.6)

Proposition 2.2.10 ([69, Thm 2.4])**.** *Consider a linear system of the form* (2.2.6) *with vector relative degree* $r = (r_1, r_2, \dots, r_m) \in \mathbb{N}^{1\times m}$, $|r| = \sum_{i=1}^{m} r_i$. *If a trajectory* (x,u,y) *belongs to system behaviour, then there exists* $T \in \mathrm{GL}_n(\mathbb{R})$ *such that*

$$Tx = \left(y_1, \dots, y_1^{(r_1-1)}, \dots, y_m, \dots, y_m^{(r_m-1)}, \eta^\top\right)^\top \in \mathscr{W}^{1,1}_{\mathrm{loc}}(\mathbb{R}\to\mathbb{R}^n)$$

fulfills

$$\begin{aligned} \begin{pmatrix} y_1^{(r_1)}(t) \\ y_2^{(r_2)}(t) \\ \vdots \\ y_m^{(r_m)}(t) \end{pmatrix} &= \begin{pmatrix} \sum_{j=1}^{m}\sum_{h=1}^{r_j} R^1_{jh} y_j^{(h-1)}(t) \\ \sum_{j=1}^{m}\sum_{h=1}^{r_j} R^2_{jh} y_j^{(h-1)}(t) \\ \vdots \\ \sum_{j=1}^{m}\sum_{h=1}^{r_j} R^m_{jh} y_j^{(h-1)}(t) \end{pmatrix} + S\eta(t) + \Gamma u(t), \\ \dot{\eta}(t) &= \sum_{i=1}^{m} P_i y_i(t) + Q\eta(t), \end{aligned} \tag{2.2.7}$$

where $R^i_{jh} \in \mathbb{R}$ *with* $i, j \in \{1, \dots, m\}$, $h \in \{1, \dots, r_i\}$, $S \in \mathbb{R}^{m\times(n-|r|)}$, $P_i \in \mathbb{R}^{n-|r|}$, $Q \in \mathbb{R}^{(n-|r|)\times(n-|r|)}$. *And*

$$\Gamma = \lim_{s\to\infty} \operatorname{diag}(s^{r_1},\dots,s^{r_m})G(s),$$

as in Remarks 2.2.7(iii). Moreover, the zero dynamics of (2.2.6) *are asymptotically stable if, and only if, Q is Hurwitz.*

The *normal form* for linear systems of the form (2.2.6) was considered in [69]. Moreover, a *normal form* for the case of nonlinear differential systems was also mentioned in [56, Sec.5.1] and [80, Subsec.3.3.1]. We introduce a new *normal form*, which can be inferred straightforward from combining results in Proposition 2.2.10 and [9, Thm.2.1], for differential-algebraic systems with positive vector relative degree.

$$\begin{aligned} E\dot{x}(t) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t), \end{aligned} \tag{2.2.8}$$

where $A, E \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{m\times n}$.

Proposition 2.2.11. *Suppose* $[E,A,B,C] \in \Sigma_{n,n,m,m}$ *has vector relative degree* $r = (r_1, r_2, \dots, r_m) \in \mathbb{N}^{1\times m}$, $|r| = \sum_{i=1}^{m} r_i$. *If* ν *denotes the index of* $sE - A$, *then a trajectory satisfies*

$$(x,u,y_1,\dots,y_m) \in \mathfrak{B}_{[E,A,B,C]} \cap \Big(\mathscr{W}^{1,1}_{\mathrm{loc}}(\mathbb{R}\to\mathbb{R}^n) \times \mathscr{W}^{\nu-1,1}_{\mathrm{loc}}(\mathbb{R}\to\mathbb{R}^m) \times \\ \times \mathscr{W}^{r_1,1}_{\mathrm{loc}}(\mathbb{R}\to\mathbb{R}) \times \cdots \times \mathscr{W}^{r_m,1}_{\mathrm{loc}}(\mathbb{R}\to\mathbb{R})\Big)$$

if and only if there exist $T \in \mathrm{GL}_n(\mathbb{R})$ *such that*

$$Tx = \Big(y_1, \dots, y_1^{(r_1-1)}, \dots, y_m, \dots, y_m^{(r_m-1)}, \eta^\top, x_c^\top, x_{\bar{c}}^\top\Big)^\top \in \mathscr{W}^{1,1}_{\mathrm{loc}}(\mathbb{R}\to\mathbb{R}^n)$$

fulfills, for almost all $t \in \mathbb{R}$,

$$\begin{aligned} \begin{pmatrix} y_1^{(r_1)}(t) \\ y_2^{(r_2)}(t) \\ \vdots \\ y_m^{(r_m)}(t) \end{pmatrix} &= \begin{pmatrix} \sum_{j=1}^{m}\sum_{h=1}^{r_j} R^1_{jh} y_j^{(h-1)}(t) \\ \sum_{j=1}^{m}\sum_{h=1}^{r_j} R^2_{jh} y_j^{(h-1)}(t) \\ \vdots \\ \sum_{j=1}^{m}\sum_{h=1}^{r_j} R^m_{jh} y_j^{(h-1)}(t) \end{pmatrix} + S\eta(t) + \Gamma u(t), \\ \dot{\eta}(t) &= \sum_{i=1}^{m} P_i y_i(t) + Q\eta(t), \\ x_c(t) &= -\sum_{i=0}^{\nu-1} N_c^i B_c u^{(i)}(t), \\ x_{\bar{c}}(t) &= 0, \end{aligned} \tag{2.2.9}$$

where $n_c, n_{\bar{c}} \in \mathbb{N}_0$, $\mu = n - n_c - n_{\bar{c}} - |r|$ *and* $R^i_{jh} \in \mathbb{R}$ *with* $i, j \in \{1,\dots,m\}$, $h \in \{1,\dots,r_i\}$, $S \in \mathbb{R}^{m\times\mu}$, $P_i \in \mathbb{R}^{\mu}$, $Q \in \mathbb{R}^{\mu\times\mu}$, $B_c \in \mathbb{R}^{n_c\times m}$. $N_c \in \mathbb{R}^{n_c\times n_c}$ *is nilpotent with index* ν, *and* $\mathrm{rank}\begin{bmatrix} N_c & B_c \end{bmatrix} = n_c$. *And*

$$\Gamma = \lim_{s\to\infty} \mathrm{diag}(s^{r_1}, \dots, s^{r_m}) C (sE - A)^{-1} B.$$

Moreover, the zero dynamics of system (2.2.8) *are asymptotically stable if, and only if, Q is Hurwitz.*

Proof. Since positive vector relative degree implies that transfer function $G(s) = C(sE - A)^{-1}B$ is strictly proper, we can apply [9, Lemma 7.1] to obtain the existence of $W_1, T_1 \in \mathrm{GL}_n(\mathbb{R})$ such that

$$[E,A,B,C] \overset{W_1,T_1}{\sim} \left[\begin{bmatrix} I_{n_s} & 0 & 0 \\ 0 & N_c & N_{c\bar{c}} \\ 0 & 0 & N_{\bar{c}} \end{bmatrix}, \begin{bmatrix} A_s & 0 & 0 \\ 0 & I_{n_c} & 0 \\ 0 & 0 & I_{n_{\bar{c}}} \end{bmatrix}, \begin{bmatrix} B_s \\ B_c \\ 0 \end{bmatrix}, \begin{bmatrix} C_s & 0 & C_{\bar{c}} \end{bmatrix} \right]$$

for some $A_s \in \mathbb{R}^{n_s \times n_s}$, $B_s \in \mathbb{R}^{n_s \times m}$, $C_s \in \mathbb{R}^{m \times n_s}$, $N_c \in \mathbb{R}^{n_c \times n_c}$, $N_{c\bar{c}} \in \mathbb{R}^{n_c \times n_{\bar{c}}}$, $N_{\bar{c}} \in \mathbb{R}^{n_{\bar{c}} \times n_{\bar{c}}}$, $B_c \in \mathbb{R}^{n_c \times m}$ and $C_{\bar{c}} \in \mathbb{R}^{m \times n_{\bar{c}}}$, where $N_c, N_{\bar{c}}$ are nilpotent and $\operatorname{rank}[N_c, B_c] = n_c$. The dimensions $n_s, n_c, n_{\bar{c}} \in \mathbb{N}_0$ are unique, and the matrices $A_s, N_c, N_{\bar{c}}$ are unique up to similarity.
Then with $T_1 x = (x_1, x_c, x_{\bar{c}}) \in \mathscr{W}^{1,1}_{\mathrm{loc}}(\mathbb{R} \to \mathbb{R}^n)$, we have

$$\begin{aligned} \dot{x}_1(t) &= A_s x_1(t) + B_s u(t), \\ \frac{d}{dt}(N_c x_c(t) + N_{c\bar{c}} x_{\bar{c}}(t)) &= x_c(t) + B_c u(t), \\ \frac{d}{dt}(N_{c\bar{c}} x_{\bar{c}}(t)) &= x_{\bar{c}}, \\ y(t) &= C_s x_1(t) + C_{\bar{c}} x_{\bar{c}}. \end{aligned}$$

Since, $N_{\bar{c}}$ is nilpotent, third equation from above implies $x_{\bar{c}}(t) = 0$. Therefore, we have equations

$$\begin{aligned} \dot{x}_1(t) &= A_s x_1(t) + B_s u(t), \\ \frac{d}{dt}(N_c x_c(t)) &= x_c(t) + B_c u(t), \\ x_{\bar{c}}(t) &= 0, \\ y(t) &= C_s x_1(t). \end{aligned}$$

Moreover, since the transfer function is invariant under system equivalence, we have

$$\begin{aligned} G(s) = C(sE-A)^{-1}B &= \begin{bmatrix} C_s & 0 & C_{\bar{c}} \end{bmatrix} \begin{bmatrix} sI_{n_s} - A_s & 0 & 0 \\ 0 & sN_c - I_{n_c} & sN_{c\bar{c}} \\ 0 & 0 & sN_{\bar{c}} - I_{n_{\bar{c}}} \end{bmatrix}^{-1} \begin{bmatrix} B_s \\ B_c \\ 0 \end{bmatrix} \\ &= C_s (sI_{n_s} - A_s)^{-1} B_s. \end{aligned}$$

Apply Proposition 2.2.10, there exists $T_2 \in \mathrm{GL}_{n_s}(\mathbb{R})$ such that (2.2.7) holds. We now set

$$W = W_1 \begin{bmatrix} T_2^{-1} & 0 \\ 0 & I \end{bmatrix}; \quad T = \begin{bmatrix} T_2 & 0 \\ 0 & I \end{bmatrix} T_1.$$

Therefore, $Tx = \left(y_1, \ldots, y_1^{(r_1-1)}, \ldots, y_m, \ldots, y_m^{(r_m-1)}, \eta^\top, x_c^\top, x_{\bar{c}}^\top \right)^\top \in \mathscr{W}^{1,1}_{\mathrm{loc}}(\mathbb{R} \to \mathbb{R}^n)$ fulfills (2.2.9) . □

The systems with positive strict relative degree which have been investigated in [9] is a special case of system class (2.2.8).

Systems with proper inverse transfer function

We consider a linear system (2.2.1) with *proper inverse transfer function*, i.e. its transfer function $G(s)$ has proper inverse .

$$\begin{aligned} E\dot{x}(t) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t), \end{aligned} \tag{2.2.10}$$

where $A, E \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{m\times n}$. This kind of system class was introduced in [10].

Remark 2.2.12. We note that the class of system which has vector relative degree $(r_1 \ldots, r_m)$ and $r_i \le 0$ for all $i = 1, \ldots, m$ is a subset of the class of system with proper inverse transfer function. Infact, using Proposition 2.2.8, if $G(s)$ has vector relative degree $(r_1, \ldots, r_m)$, and $r_i \le 0$ for all $i = 1, \ldots, m$, then $G^{-\top}(s)$ has non-negative vector relative degree. This implies that $G^{-\top}(s)$ is proper matrix function which lead to $G(s)$ has proper inverse. Hence, we conclude that the system which has vector relative degree $(r_1, \ldots, r_m)$, and $r_i \le 0$ for all $i = 1, \ldots, m$ belongs to the set of systems with proper inverse transfer function. However, the converse is, in general, not true. To clarify this, we consider a counterexample with

$$E = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}; \; A = \begin{bmatrix} -1 & -1 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}; \; B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}; \; C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

It is easy to calculate that

$$G(s) = C(sE - A)^{-1}B = \frac{1}{2} \begin{bmatrix} -s & -s \\ s+1 & s-1 \end{bmatrix},$$

then the inverse of transfer function is

$$G^{-1}(s) = \begin{bmatrix} 1 - \frac{1}{s} & 1 \\ -1 - \frac{1}{s} & -1 \end{bmatrix} \overset{s\to\infty}{\longrightarrow} \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \in \mathbb{R}^{2\times 2}.$$

Hence, $G(s)$ is invertible and its inverse is proper. However, $G(s)$ does not have vector relative degree because

$$\lim_{s\to\infty} \operatorname{diag}(s^{-1}, s^{-1}) G(s) = \frac{1}{2} \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix} \notin \mathrm{GL}_2(\mathbb{R}).$$

Proposition 2.2.13 ([10, Rem.2.6]). *Suppose $[E, A, B, C] \in \Sigma_{n,n,m,m}$ has proper inverse transfer function. If ν denotes the index of $sE - A$, then a trajectory satisfies*

$$(x, u, y) \in \mathfrak{B}_{[E,A,B,C]} \cap \left(\mathscr{W}^{1,1}_{\mathrm{loc}}(\mathbb{R} \to \mathbb{R}^n) \times \mathscr{W}^{1,1}_{\mathrm{loc}}(\mathbb{R} \to \mathbb{R}^m) \times \mathscr{W}^{\nu-1,1}_{\mathrm{loc}}(\mathbb{R} \to \mathbb{R}^m) \right)$$

if and only if there exist $T \in \mathrm{GL}_n(\mathbb{R})$ such that $Tx = \left(y^\top, \eta^\top, x_c^\top, x_{\bar{c}}^\top \right)^\top \in \mathscr{W}^{1,1}_{\mathrm{loc}}(\mathbb{R} \to \mathbb{R}^n)$ fulfills

$$\begin{aligned} 0 &= A_{11} y(t) + A_{12} \eta(t) + u(t), \\ \dot{\eta}(t) &= A_{21} y(t) + Q\eta(t), \\ x_c(t) &= \sum_{i=0}^{\nu-1} N_c^i E_c y^{(i+1)}(t), \\ x_{\bar{c}}(t) &= 0, \end{aligned} \tag{2.2.11}$$

where $n_c, n_{\bar{c}}, \mu \in \mathbb{N}_0$, *and* $A_{11} \in \mathbb{R}^{m\times m}$, $A_{12} \in \mathbb{R}^{m\times \mu}$, $A_{21} \in \mathbb{R}^{\mu\times m}$, $Q \in \mathbb{R}^{\mu\times\mu}$, $E_c \in \mathbb{R}^{n_c\times m}$, $N_c \in \mathbb{R}^{n_c\times n_c}$ *is nilpotent with index* ν, *and* rank $\begin{bmatrix} N_c & E_c \end{bmatrix} = n_c$.
Moreover, the zero dynamics of (2.2.1) *are asymptotically stable if, and only if,* Q *is Hurwitz.*

Remark 2.2.14. From counterexample in Remark 2.2.12 and the result of Proposition 2.2.13, it seems that knowing exactly the values of negative part in vector relative degree of a linear differential-algebraic systems is not necessary to contribute a *normal form*. On the other hand, the approach using transfer function to define vector relative degree is only applied for regular systems. Therefore, we have to find another way to define vector relative degree not only for regular systems but also for arbitrary linear differential-algebraic systems. The Remark 2.2.9 bring to us a new approach which will be discussed in following section to obtain a *normal form* of linear differential-algebraic systems.

2.3 Linear differential-algebraic systems

In the first part of this section, we recall some important results which were introduced in [7]. After that, we introduce a generalized definition of vector relative degree that can enlarge considered system class. Based on this new definition, we introduce a novel *normal form* of general linear differential-algebraic systems. This *normal form* is not only having simple structure but it also help us to create an efficient funnel controller for class of systems. First of all, we reconsider a general linear differential-algebraic systems as concerning at the beginning of this chapter,

$$\begin{aligned} E\dot{x}(t) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t) \end{aligned}$$

where $E, A \in \mathbb{R}^{l\times n}$, $B \in \mathbb{R}^{l\times m}$, $C \in \mathbb{R}^{p\times n}$.

2.3.1 Inversion form

The goal of this subsection is construction a *inversion form* of linear differential-algebraic systems. In order to solve this problem, we need to recall the definition of left- and right-invertibility of a systems that was introduced in [7, Sec.4].

Definition 2.3.1 ([7, Sec.4])**.** $[E,A,B,C] \in \Sigma_{l,n,m,p}$ is called

(i) *left-invertible* if

$$\forall (x,u,y) \in \mathfrak{B}_{[E,A,B,C]} : [y \overset{a.e}{=} 0 \wedge Ex(0) = 0] \Longrightarrow u \overset{a.e}{=} 0. \tag{2.3.1}$$

(ii) *right-invertible* if

$$\forall y \in \mathscr{C}^\infty(\mathbb{R} \to \mathbb{R}^p),\ \exists (x,u) \in \mathscr{L}^1_{\text{loc}}(\mathbb{R} \to \mathbb{R}^n \times \mathbb{R}^m) :\ (x,u,y) \in \mathfrak{B}_{[E,A,B,C]}.$$

(iii) *invertible* if $[E,A,B,C]$ are both *left-invertible* and *right-invertible*.

Proposition 2.3.2 ([7, Lem.4.3])**.** *Let* $[E,A,B,C] \in \Sigma_{l,n,m,p}$ *with autonomous zero dynamics. Then* $[E,A,B,C]$ *is left-invertible.*

We are now in the position to introduce an important result, which is so called *inversion form* of system $[E,A,B,C] \in \Sigma_{l,n,m,p}$, see [7, Thm. 4.4].

Theorem 2.3.3 ([7, Thm.4.4]). *Let $[E,A,B,C] \in \Sigma_{l,n,m,p}$ with autonomous zero dynamics and* $\operatorname{rank} C = p$. *Then, there exist $W \in \mathrm{GL}_l(\mathbb{R})$, $T \in \mathrm{GL}_n(\mathbb{R})$ such that*

$$[E,A,B,C] \overset{W,T}{\sim} [\widehat{E},\widehat{A},\widehat{B},\widehat{C}]$$

where

$$\widehat{E} = \begin{bmatrix} I_\mu & 0 & 0 \\ 0 & E_{22} & E_{23} \\ 0 & E_{32} & N \\ 0 & E_{42} & E_{43} \end{bmatrix}, \widehat{A} = \begin{bmatrix} Q & A_{12} & 0 \\ A_{21} & A_{22} & 0 \\ 0 & 0 & I_{n_3} \\ 0 & A_{42} & 0 \end{bmatrix}, \widehat{B} = \begin{bmatrix} 0 \\ I_m \\ 0 \\ 0 \end{bmatrix}, \widehat{C} = [0, I_p, 0], \tag{2.3.2}$$

$\mu = \dim \max(E,A,B;\ker C)$ and $N \in \mathbb{R}^{n_3 \times n_3}$, $n_3 = n - \mu - p$, is nilpotent with $N^\nu = 0$, and $N^{\nu-1} \neq 0$, $\nu \in \mathbb{N}$, $E_{22}, A_{22} \in \mathbb{R}^{m \times p}$, and all other matrices are of appropriate sizes.
For the uniqueness, we have

(i) If $[E,A,B,C], [\widetilde{E},\widetilde{A},\widetilde{B},\widetilde{C}] \in \Sigma_{l,n,m,p}$ are both in form (2.3.2) and

$$[E,A,B,C] \overset{W,T}{\sim} [\widetilde{E},\widetilde{A},\widetilde{B},\widetilde{C}] \text{ for some } W \in \mathrm{GL}_l(\mathbb{R}), T \in \mathrm{GL}_n(\mathbb{R})$$

then there exist $W_{11} \in \mathrm{GL}_\mu(R)$, $W_{33} \in \mathrm{GL}_{n_3}(R)$ such that

$$W = \begin{bmatrix} W_{11} & 0 & 0 & 0 \\ 0 & I_m & 0 & 0 \\ 0 & 0 & W_{33} & 0 \\ 0 & 0 & 0 & I_{(l+p)-(n+m)} \end{bmatrix} \quad T = \begin{bmatrix} W_{11}^{-1} & 0 & 0 \\ 0 & I_p & 0 \\ 0 & 0 & W_{33}^{-1} \end{bmatrix}$$

(ii) The matrices N, Q are unique up to similarity, so in particular the spectrum of Q, and the nilpotent index of N are unique .

Remarks 2.3.4.

(i) Let $[E,A,B,C] \in \Sigma_{l,n,m,p}$ with autonomous zero dynamics, and $\operatorname{rank} C = p$. Using inversion form (2.3.2), for any

$$(x,u,y) \in \mathfrak{B}_{[E,A,B,C]} \cap (\mathscr{W}^{1,1}_{\mathrm{loc}}(\mathbb{R} \to \mathbb{R}^n) \times \mathscr{W}^{1,1}_{\mathrm{loc}}(\mathbb{R} \to \mathbb{R}^m) \times \mathscr{W}^{\nu+1,1}_{\mathrm{loc}}(\mathbb{R} \to \mathbb{R}^p),$$

and $Tx = (\eta^\top, y^\top, x_3^\top)^\top \in \mathscr{W}^{1,1}_{\mathrm{loc}}(\mathbb{R} \to \mathbb{R}^{\mu+p+n_3})$, then (Tx,u,y) solves

$$\begin{aligned} \dot\eta(t) &= Q\eta(t) + A_{12}y(t), \\ 0 &= -\sum_{i=0}^{\nu-1} E_{23}N^i E_{32} y^{(i+2)}(t) - E_{22}\dot y(t) + A_{22}y(t) + A_{21}\eta(t) + u(t), \\ x_3(t) &= \sum_{i=0}^{\nu-1} N^i E_{32} y^{(i+1)}(t), \\ 0 &= -E_{42}\dot y(t) - \sum_{i=0}^{\nu-1} E_{43}N^i E_{32} y^{(i+2)}(t) + A_{42}y(t). \end{aligned}$$

(ii) According to [7, Cor. 4.6], let $[E,A,B,C] \in \Sigma_{l,n,m,p}$ with autonomous zero dynamics, and $\operatorname{rank} C = p$, then the zero dynamics $\mathscr{ZD}_{[E,A,B,C]}$ are asymptotically stable if and only if $\sigma(Q) \subseteq \mathbb{C}_-$.

(iii) Suppose $[E,A,B,C] \in \Sigma_{l,n,m,p}$ with autonomous zero dynamics, then $[E,A,B,C]$ is invertible if and only if $\operatorname{rank} C = p$, $E_{42} = 0$, $A_{42} = 0$ and $E_{43}N^jE_{32} = 0$ for $j = 0,\dots,\nu-1$, see [7, Prop. 4.7].

(iv) Because of Proposition 2.1.4, the autonomous zero dynamics of $[E,A,B,C]$ implies a full column rank of $\begin{bmatrix} sE-A & -B \\ -C & 0 \end{bmatrix}$ over $\mathbb{R}[s]$. This infers the existence of a left inverse of $\begin{bmatrix} sE-A & -B \\ -C & 0 \end{bmatrix}$ over $\mathbb{R}(s)$.

Proposition 2.3.5 ([7, Lem.A.1])**.** *Let $[E,A,B,C] \in \Sigma_{l,n,m,p}$ has autonomous zero dynamics, and be right invertible. Then, with the notation from Theorem 2.3.3, $L(s) \in \mathbb{R}(s)^{(n+m)\times(l+p)}$ is a left inverse of $\begin{bmatrix} sE-A & -B \\ -C & 0 \end{bmatrix}$ if and only if,*

$$L(s) = \begin{bmatrix} T & 0 \\ 0 & I_m \end{bmatrix} \begin{bmatrix} (sI_\mu - Q)^{-1} & 0 & 0 & X_{14}(s) & X_{15}(s) \\ 0 & 0 & 0 & X_{24}(s) & I_p \\ 0 & 0 & (sN - I_{n_3})^{-1} & X_{34}(s) & X_{35}(s) \\ X_{41}(s) & I_m & X_{43}(s) & X_{44}(s) & X_{45}(s) \end{bmatrix} \begin{bmatrix} W & 0 \\ 0 & I_p \end{bmatrix},$$

where $\left[X_{14}(s)^\top, X_{24}(s)^\top, X_{34}(s)^\top, X_{44}(s)^\top\right]^\top \in \mathbb{R}(s)^{(n+m)\times(l+p-n-m)}$, and

$$\begin{aligned} X_{15}(s) &= (sI-Q)^{-1}A_{12}, \quad X_{35}(s) = -s(sN-I)^{-1}E_{32}, \\ X_{41}(s) &= A_{21}(sI-Q)^{-1}, \quad X_{43}(s) = -sE_{23}(sN-I)^{-1}, \\ X_{45}(s) &= -(sE_{22}-A_{22}) + A_{21}(sI-Q)^{-1}A_{12} + s^2E_{23}(sN-I_{n_3})^{-1}E_{32}, \end{aligned}$$

and $L(s)$ is partitioned according to the block structure of (2.3.2).
If $L_1(s), L_2(s) \in \mathbb{R}(s)^{(n+m)\times(l+p)}$ are two left inverse of matrix $\begin{bmatrix} sE-A & -B \\ -C & 0 \end{bmatrix}$, then

$$\begin{bmatrix} 0 & I_m \end{bmatrix} L_1(s) \begin{bmatrix} 0 \\ I_p \end{bmatrix} = \begin{bmatrix} 0 & I_m \end{bmatrix} L_2(s) \begin{bmatrix} 0 \\ I_p \end{bmatrix}.$$

Remarks 2.3.6.

(i) Suppose $[E,A,B,C] \in \Sigma_{l,n,m,p}$ has autonomous zero dynamics, and be right invertible. Denote

$$\mathbb{R}(s)^{m\times p} \ni H(s) = -\begin{bmatrix} 0 & I_m \end{bmatrix} L(s) \begin{bmatrix} 0 \\ I_p \end{bmatrix}, \tag{2.3.3}$$

with $L(s) \in \mathbb{R}(s)^{(n+m)\times(l+p)}$ is a left inverse of $\begin{bmatrix} sE-A & -B \\ -C & 0 \end{bmatrix}$. By Proposition 2.3.5, $H(s)$ is independent of the choice of $L(s)$. Moreover,

$$H(s) = sE_{22} - A_{22} - A_{21}(sI-Q)^{-1}A_{12} - s^2E_{23}(sN-I_{n_3})^{-1}E_{32}. \tag{2.3.4}$$

(ii) Suppose $[E,A,B,C] \in \Sigma_{l,n,m,p}$ has autonomous zero dynamics, and be right invertible. Recall the Remark 2.2.9, if $l = n$, $p = m$, then $H(s) = G(s)^{-1}$. Therefore, in the case of regular system $[E,A,B,C] \in \Sigma_{n,n,m,m}$, $H(s)$ is exactly the inverse of transfer function $G(s) = C(sE-A)^{-1}B$. We also note that $G(s)$ exists only if $sE-A$ is a regular pencil.

2.3.2 Generalized vector relative degree

One of the limitation in defining vector relative degree via transfer function is only available for the case of *regular systems*. In order to get rid of this limitation, we introduce, in this subsection, a novel definition of vector relative degree for linear differential-algebraic systems and suppose it to name *generalized vector relative degree*. Let us begin with recalling the notion of column degree of a rational matrix function given in [26, Sec.2.4][2].

Definition 2.3.7. The degree of a rational function, which is expressed by a quotient of polynomials $r(s) := \dfrac{p(s)}{q(s)}$ with $p(x), q(x) \in \mathbb{R}[s]$, $q(s) \not\equiv 0$, is

$$\deg r(s) := \deg p(s) - \deg q(s).$$

Definition 2.3.8.

- We define the degree of rational vector $f(s) = (f_1(s), f_2(s), \ldots, f_m(s))^\top \in \mathbb{R}(s)^m$ by $\deg f(s) = \max\limits_{1 \le i \le m} \{\deg f_i(s)\}$.
- Given $H(s) = \begin{bmatrix} H_1(s) & \cdots & H_p(s) \end{bmatrix}$ with $H_i(s) \in \mathbb{R}(s)^m$, we denote the degree of i-th column rational vector $H_i(s)$ of $H(s)$ by $\deg H_i(s)$.

Remark 2.3.9. Refer to the Remark 2.2.9 and Remarks 2.3.6, in the case of regular system $[E,A,B,C] \in \Sigma_{n,n,m,m}$ having vector relative degree $r = (r_1, \ldots, r_m)$, it was shown that

$$\lim_{s\to\infty} \operatorname{diag}(s^{r_1}, \ldots, s^{r_m}) G(s) = \Gamma \in \mathrm{GL}_m(\mathbb{R})$$
$$\iff \lim_{s\to\infty} H(s) \operatorname{diag}(s^{-r_1}, \ldots, s^{-r_m}) = \Gamma^{-1} \in \mathrm{GL}_m(\mathbb{R}),$$

with $G(s) = C(sE - A)^{-1}B$ being the transfer function of $[E,A,B,C]$, and $H(s)$ as in (2.3.3). This brings to us a new idea by using $H(s)$ instead of transfer function $G(s)$ in Definition 2.2.6 for vector relative degree. We note that being regular of system pencil $sE - A$ is not requirement in using $H(s)$.

Now we consider $[E,A,B,C] \in \Sigma_{l,n,m,p}$ is not regular. According to Remarks 2.3.6, there are two assumptions imposed on system $[E,A,B,C]$ to ensure the existence of $H(s)$.

(a) $[E,A,B,C]$ has autonomous zero dynamics.

(b) $[E,A,B,C]$ is right invertible.

Definition 2.3.10. Let $[E,A,B,C] \in \Sigma_{l,n,m,p}$ be right invertible and has autonomous zero dynamics. $H(s)$ is determined as in (2.3.3). Set $r_i = \max\{\deg H_i(s), 0\}$,
$q := \#\{r_i > 0,\ i = 1, \ldots, p\}$, and

$$\hat{\Gamma} := \lim_{s\to\infty} H(s) \operatorname{diag}(s^{-r_1}, \ldots, s^{-r_p}) \in \mathbb{R}^{m\times p}. \tag{2.3.5}$$

We say that $r = (r_1, \ldots, r_p) \in N^{1\times p}$ is a *generalized vector relative degree* of $[E,A,B,C]$, if $\operatorname{rank} \hat{\Gamma}_q = q$ for $\hat{\Gamma}_q \in \mathbb{R}^{m\times q}$ which is obtained from $\hat{\Gamma}$ by deleting all the columns respective to $r_i = 0$.

Remarks 2.3.11.

(i) Refer to Proposition 2.3.5, $H(s)$ is unique. Therefore, $\hat{\Gamma}$ in (2.3.5) is unique which implies *generalized vector relative degree* of $[E,A,B,C]$ being well defined.

[2] In [26, Sec.2.4], the definition is given for polynomial matrix but the same for rational matrix.

(ii) If $[E,A,B,C]$ has a generalized vector relative degree, then the generalized vector relative degree is unique. This property is obtained by the uniqueness determining of $r_i = \max\{\deg H_i(s), 0\}$.

(iii) Generalized vector relative degree of a system does not necessarily exist. For instance, we consider an example with

$$E = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, A = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, C = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

For this system, we have

$$H(s) = \begin{bmatrix} s-1 & s+1 \\ s-1 & s-2 \end{bmatrix}.$$

Moreover,

$$\hat{\Gamma} = \lim_{s\to\infty} H(s)\,\mathrm{diag}(s^{-1}, s^{-1}) = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \text{ and } \hat{\Gamma}_q = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

Since $\operatorname{rank}\hat{\Gamma}_q = 1 < 2$, which is the number of positive column degree of $H(s)$. Hence, generalized vector relative degree of this system does not exist.

(iv) If $[E,A,B,C] \in \Sigma_{n,n,m,m}$, and $sE-A$ is regular with proper inverse transfer function then $[E,A,B,C]$ has generalized vector relative degree $(0,\dots,0)$. Indeed, if $[E,A,B,C] \in \Sigma_{n,n,m,m}$ with proper inverse transfer function then the inversion $G^{-1}(s)$ of its transfer function $G(s) = C(sE-A)^{-1}B$ exists. Therefore, we have

$$\begin{bmatrix} -(sE-A)^{-1}BG^{-1}(s)C(sE-A)^{-1} + (sE-A)^{-1} & (sE-A)^{-1}BG^{-1}(s) \\ -G^{-1}(s)C(sE-A)^{-1} & -G^{-1}(s) \end{bmatrix} \begin{bmatrix} sE-A & -B \\ -C & 0 \end{bmatrix} = I_{n+m}.$$

Hence, $\begin{bmatrix} sE-A & -B \\ -C & 0 \end{bmatrix} \in \mathbb{R}(s)^{(n+m)\times(n+m)}$ is invertible. Denote $L(s)$ is the unique inverse of $\begin{bmatrix} sE-A & -B \\ -C & 0 \end{bmatrix}$. We have

$$\begin{aligned} \begin{bmatrix} I_n & (sE-A)^{-1}B \\ 0 & I_m \end{bmatrix} &= L(s) \begin{bmatrix} sE-A & -B \\ -C & 0 \end{bmatrix} \begin{bmatrix} I_n & (sE-A)^{-1}B \\ 0 & I_m \end{bmatrix} \\ &= L(s) \begin{bmatrix} sE-A & 0 \\ -C & -C(sE-A)^{-1}B \end{bmatrix}. \end{aligned}$$

This implies

$$H(s) = -\begin{bmatrix} 0 & I_m \end{bmatrix} L(s) \begin{bmatrix} 0 \\ I_m \end{bmatrix} = G^{-1}(s).$$

We obtain,

$$\hat{\Gamma} = \lim_{s\to\infty} H(s)\,\mathrm{diag}(s^0, \dots, s^0) \in \mathbb{R}^{m\times m},$$

since $H(s) = G^{-1}(s)$ is proper. Moreover, this system satisfies condition $\operatorname{rank}\hat{\Gamma}_q = q$ because $q = 0$. Hence, $[E,A,B,C] \in \Sigma_{n,n,m,m}$ with proper inverse transfer function has generalized vector relative degree $(0,\dots,0)$.

(v) In regular case, it is obvious that if $[E,A,B,C]$ has a vector relative degree in conventional way $r=(r_1,\dots,r_p)$, then it also has generalized vector relative degree. For more detail, if $r=(r_1,\dots,r_p)$ is vector relative degree of $[E,A,B,C]$, then $\tilde{r}=(\tilde{r}_1,\dots,\tilde{r}_p)$, where $\tilde{r}_i=\max\{r_i,0\}$ for $i=1,\dots,p$, is generalized vector relative degree of $[E,A,B,C]$. However, the existence of having generalized vector relative degree of a system does not imply the existence of having vector relative degree. To verify this, we consider the following example with

$$E=\begin{bmatrix}1&0&0&0&0\\0&1&0&1&0\\0&-1&0&0&0\\0&0&0&0&1\\0&1&0&0&0\end{bmatrix};\ A=\begin{bmatrix}-1&1&-2&0&0\\3&5&0&0&0\\0&0&0&0&0\\0&0&0&1&0\\0&0&0&0&1\end{bmatrix}$$

$$B=\begin{bmatrix}0&0\\1&0\\0&1\\0&0\\0&0\end{bmatrix};\ C=\begin{bmatrix}0&1&0&0&0\\0&0&1&0&0\end{bmatrix}.$$

Then

$$G(s)=C(sE-A)^{-1}B=\begin{bmatrix}0&-\frac{1}{s}\\\frac{s+1}{6}&\frac{s^4+s^3+s^2-4s-8}{6s}\end{bmatrix}.$$

We have

$$\Gamma:=\lim_{s\to\infty}\operatorname{diag}(s,s^{-3})G(s)=\begin{bmatrix}0&-1\\0&\frac{1}{6}\end{bmatrix},\text{ and }\operatorname{rank}\Gamma=1<2.$$

This implies that the system does not have vector relative degree in the sense of Definition 2.2.6. On the other hand, we also have

$$H(s)=\begin{bmatrix}s^3+s-5-\frac{3}{s+1}&\frac{6}{s+1}\\-s&0\end{bmatrix}.$$

And we learn from that

$$\hat{\Gamma}:=\lim_{s\to\infty}H(s)\operatorname{diag}(s^{-3},1)=\begin{bmatrix}1&0\\0&0\end{bmatrix},\text{ and }\hat{\Gamma}_q=\begin{bmatrix}1\\0\end{bmatrix}.$$

Therefore, using the Definition 2.3.10, this system have generalized vector relative degree $r=(3,0)$ since $\operatorname{rank}\hat{\Gamma}_q=1$ which is equal to the number of positive column degree of $H(s)$.

(vi) In [7], Berger consider linear differential-algebraic system $[E,A,B,C]\in\Sigma_{l,n,m,m}$ which satisfy following properties

- $[E,A,B,C]$ has autonomous zero dynamics,
- $[E,A,B,C]$ is right-invertible,
- the matrix

$$\Gamma=-\lim_{s\to\infty}s^{-1}\begin{bmatrix}0&I_m\end{bmatrix}L(s)\begin{bmatrix}0\\I_m\end{bmatrix}\in\mathbb{R}^{m\times m}$$

exists and satisfies $\Gamma = \Gamma^\top \geq 0$, where $L(s) \in \mathbb{R}(s)^{(n+m)\times(l+m)}$ is a left inverse of $\begin{bmatrix} sE-A & -B \\ -C & 0 \end{bmatrix}$.

This type of system class can be viewed as a unified framework for two different cases: proper inverse transfer function and positive strict relative degree one. However, it is more general since $sE-A$ is not required regular. This type system class also does not serve as a subclass of linear differential-algebraic systems with generalized vector raltive degree since Γ is reqired semi positive definite and symmetry. Let us clarify this by reconsidering the example introduced in (iii) of this remark. This system was shown that it has not generalized vector relative degree. However,

$$\Gamma = -\lim_{s\to\infty} s^{-1} \begin{bmatrix} 0 & I_m \end{bmatrix} L(s) \begin{bmatrix} 0 \\ I_m \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

which satisfies $\Gamma = \Gamma^\top \geq 0$.
On the other hand, another example with

$$E = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, A = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, C = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix},$$

this system has

$$H(s) = \begin{bmatrix} s-1 & 2s+1 \\ s-1 & s-2 \end{bmatrix}.$$

It is easy to see that

$$\hat{\Gamma} = \lim_{s\to\infty} H(s)\,\mathrm{diag}(s^{-1}, s^{-1}) = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}, \text{and} \hat{\Gamma}_q = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}.$$

Hence, this system has generalized vector relative degree $(1,1)$ since $\mathrm{rank}\,\hat{\Gamma}_q = 2$ which is the number of positive column degree of $H(s)$. However, the matrix

$$\Gamma = -\lim_{s\to\infty} s^{-1} \begin{bmatrix} 0 & I_m \end{bmatrix} L(s) \begin{bmatrix} 0 \\ I_m \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$$

is not symmetric. It means that this system does not belong to system class introduced in [7]. We conclude that although there are some common elements, the set of systems which have generalized relative degree $(r_1, \ldots, r_m)$, $r_i \in \{0,1\}$ is really different from the set of systems which is considered in [7].

Definition 2.3.12. If $[E,A,B,C]$ has generalized vector relative degree $r = (r_1, \ldots, r_p) \in \mathbb{N}_0^{1\times p}$ satisfying $r_1 \geq \cdots \geq r_p$, then $r = (r_1, \ldots, r_p)$ is called *ordered generalized vector relative degree*.

Lemma 2.3.13. *If $[E,A,B,C]$ has generalized vector relative degree $r = (r_1, \ldots, r_p) \in \mathbb{N}_0^{1\times p}$. Then there exists a permutation matrix $P \in \mathbb{R}^{p\times p}$ such that the system $[E,A,B,PC]$ has ordered generalized vector relative degree $rP = (\tilde{r}_1, \ldots, \tilde{r}_p)$, (i.e $\tilde{r}_1 \geq \cdots \geq \tilde{r}_p$). Moreover, we have $\tilde{r}_1 \geq \tilde{r}_2 \geq \cdots \geq \tilde{r}_q > 0 = \cdots = 0$, and $\widetilde{\hat{\Gamma}}_q = \widetilde{\hat{\Gamma}} \begin{bmatrix} I_q \\ 0 \end{bmatrix}$.*

Proof. Let $\sigma : \{1,\dots,p\} \to \{1,\dots,p\}$ be a permutation such that $r_{\sigma(1)} \geq \cdots \geq r_{\sigma(p)}$. Moreover, set

$$P := \begin{bmatrix} e_{(p)}^{\sigma(1)} \\ \vdots \\ e_{(p)}^{\sigma(p)} \end{bmatrix} \in \mathrm{GL}_p(\mathbb{R}),$$

with $e_{(p)}^{j} := \big(\underbrace{0,\dots,0}_{j-1},1,\underbrace{0,\dots,0}_{p-j}\big)$.

Then $P^\top = P^{-1}$ and $H_P(s) = H(s)P^\top$ where $H_P(s) = -\begin{bmatrix}0 & I_m\end{bmatrix} L_P(s)\begin{bmatrix}0\\ I_p\end{bmatrix}$ and $L_P(s)$ is left inverse of matrix $\begin{bmatrix} sE-A & -B \\ -PC & 0\end{bmatrix}$. This implies that

$$H_P(s)\operatorname{diag}(s^{-r_{\sigma(1)}},\dots,s^{-r_{\sigma(p)}}) = H(s)\operatorname{diag}(s^{-r_1},\dots,s^{-r_p})P^\top.$$

And therefore, $[E,A,B,PC]$ has generalized vector relative degree $rP = (r_{\sigma(1)},\dots,r_{\sigma(p)})$ with $r_{\sigma(1)} \geq \cdots \geq r_{\sigma(p)}$. □

Remark 2.3.14. If $[E,A,B,C]$ has an ordered generalized vector relative degree $r = (r_1,\dots,r_q,0\cdots,0) \in \mathbb{N}_0^{1\times p}$, $r_1 \geq \cdots \geq r_q > 0$. Apply the *inversion form* in Theorem 2.3.3 to $[E,A,B,C]$, we get the following

$$\begin{aligned}\hat\Gamma &= \lim_{s\to\infty} H(s)\operatorname{diag}(s^{-r_1},\dots,s^{-r_q},1,\dots,1)\\ &= \lim_{s\to\infty}[(sE_{22}-A_{22}) - A_{21}(sI-Q)^{-1}A_{12} - s^2E_{23}(sN-I_{n_3})^{-1}E_{32}]\operatorname{diag}(s^{-r_1},\dots,s^{-r_q},1,\dots,1)\\ &= \lim_{s\to\infty}[sE_{22}-A_{22} + \sum_{k=0}^{\nu-1} s^{k+2}E_{23}N^kE_{32}]\operatorname{diag}(s^{-r_1},\dots,s^{-r_q},1,\dots,1).\end{aligned}$$

Proposition 2.3.15. *The generalized vector relative degree is invariant under static output feedback.*

Proof. Without lost of generality, we suppose that $[E,A,B,C]$ has odered generalized vector relative degree $r = (r_1,\dots,r_q,0\cdots,0) \in \mathbb{N}_0^{1\times p}$, $r_1 \geq \cdots \geq r_q > 0$. Consider a static output feedback $u(t) = Ky(t) + v(t)$ with $K \in \mathbb{R}^{m\times p}$. Apply this feedback to system (2.1.1), we have

$$\begin{aligned} E\dot x(t) &= (A+BKC)x(t) + Bv(t),\\ y(t) &= Cx(t).\end{aligned}$$

We note that

$$\begin{bmatrix} sE-(A+BKC) & -B \\ -C & 0\end{bmatrix} = \begin{bmatrix} sE-A & -B \\ -C & 0\end{bmatrix}\begin{bmatrix} I_n & 0 \\ -KC & I_m\end{bmatrix}.$$

Hence, $\operatorname{rank}_{\mathbb{R}(s)}\begin{bmatrix} sE-(A+BKC) & -B \\ -C & 0\end{bmatrix} = n+m$ since $\operatorname{rank}_{\mathbb{R}(s)}\begin{bmatrix} sE-A & -B \\ -C & 0\end{bmatrix} = n+m$. In conjunction with Proposition 2.1.4, we conclude that $[E,A+BKC,B,C] \in \Sigma_{l,n,m,p}$ has autonomous zero dynamics, and is right invertible if $[E,A,B,C] \in \Sigma_{l,n,m,p}$ has autonomous zero dynamics, and is right invertible. Denote $L(s), L_K(s) \in \mathbb{R}(s)^{(n+m)\times(l+p)}$ are left inverses of $\begin{bmatrix} sE-A & -B \\ -C & 0 \end{bmatrix}$ and $\begin{bmatrix} sE-A-BKC & -B \\ -C & 0 \end{bmatrix}$, respectively. Then

$$L_K(s) = \begin{bmatrix} I_n & 0 \\ -KC & I_m \end{bmatrix} L(s).$$

Using the notation from Proposition 2.3.5, we have

$$\begin{aligned} H_K(s) &= -\begin{bmatrix} 0 & I_m \end{bmatrix} L_K(s) \begin{bmatrix} 0 \\ I_p \end{bmatrix} \\ &= -\begin{bmatrix} 0 & I_m \end{bmatrix} \begin{bmatrix} I_n & 0 \\ -KC & I_m \end{bmatrix} L(s) \begin{bmatrix} 0 \\ I_p \end{bmatrix} \\ &= -\begin{bmatrix} -KC & I_m \end{bmatrix} L(s) \begin{bmatrix} 0 \\ I_p \end{bmatrix} \\ &= -\begin{bmatrix} 0 & -K & 0 & I_m \end{bmatrix} L(s) \begin{bmatrix} 0 \\ I_p \end{bmatrix} \\ &= K - X_{45}(s) = H(s) + K. \end{aligned}$$

Therefore,

$$\lim_{s\to\infty} H_K(s) \operatorname{diag}(s^{-r_1}, \ldots, s^{-r_q}, 1, \ldots, 1) = \lim_{s\to\infty} (H(s)+K) \operatorname{diag}(s^{-r_1}, \ldots, s^{-r_q}, 1, \ldots, 1).$$

Denote $\hat{\Gamma}_K = \lim_{s\to\infty} H_K(s) \operatorname{diag}(s^{-r_1}, \ldots, s^{-r_q}, 1, \ldots, 1)$, then we can show that $\hat{\Gamma}_K \begin{bmatrix} I_q \\ 0 \end{bmatrix} = \hat{\Gamma}_q$. This implies $\operatorname{rank} \hat{\Gamma}_K \begin{bmatrix} I_q \\ 0 \end{bmatrix} = \operatorname{rank} \hat{\Gamma}_q = q$. In conclusion, $[E, A+BKC, B, C]$ has also generalized vector relative degree $r = (r_1, \ldots, r_q, 0 \ldots, 0)$. In the other words, generalized vector relative degree is invariant under static output feedback. □

2.3.3 Normal form

In this subsection, our goal is to contribute a novel normal form for general linear differential-algebraic systems with generalized vector relative degree of the form (2.1.1). Without lost of generality, we suppose that $[E, A, B, C]$ has odered generalized vector relative degree $r = (r_1, \ldots, r_q, 0 \cdots, 0) \in \mathbb{N}_0^{1\times p}$, $r_1 \geq \cdots \geq r_q > 0$. Refer to Remark 2.3.14, we have

$$\begin{aligned} \hat{\Gamma} &= \lim_{s\to\infty} H(s) \operatorname{diag}(s^{-r_1}, \ldots, s^{-r_q}, 1, \ldots, 1) \\ &= \lim_{s\to\infty} [sE_{22} - A_{22} + \sum_{k=0}^{\nu-1} s^{k+2} E_{23} N^k E_{32}] \operatorname{diag}(s^{-r_1}, \ldots, s^{-r_q}, 1, \ldots, 1) \\ &= \begin{bmatrix} \hat{\Gamma}_{11} & \hat{\Gamma}_{12} \\ \hat{\Gamma}_{21} & \hat{\Gamma}_{22} \end{bmatrix} \in \mathbb{R}^{m\times p}, \end{aligned}$$

where $\hat{\Gamma}_{11} \in \mathbb{R}^{q\times q}$, $\hat{\Gamma}_{12} \in \mathbb{R}^{q\times(p-q)}$, $\hat{\Gamma}_{21} \in \mathbb{R}^{(m-q)\times q}$, $\hat{\Gamma}_{22} \in \mathbb{R}^{(m-q)\times(p-q)}$. According to the definition of generalized vector relative degree, we note that

$$\operatorname{rank} \begin{bmatrix} \hat{\Gamma}_{11} \\ \hat{\Gamma}_{21} \end{bmatrix} = q, \text{ since } \operatorname{rank} \hat{\Gamma} \begin{bmatrix} I_q \\ 0 \end{bmatrix} = q.$$

We suppose that $r_1 \geq r_2 \geq \cdots \geq r_h > 1 = \cdots = r_q > 0$. For $j = 1,\ldots,p$, denote $E_{32}^{(j)}$, $E_{22}^{(j)}$, and $A_{22}^{(j)}$ are the j-th column of E_{32}, E_{22}, and $-A_{22}$, respecttively. Then

$$\begin{aligned}\begin{bmatrix}\hat{\Gamma}_{11} & \hat{\Gamma}_{12}\\ \hat{\Gamma}_{21} & \hat{\Gamma}_{22}\end{bmatrix} &= \lim_{s\to\infty}[sE_{22} - A_{22} + \sum_{k=0}^{\nu-1} s^{k+2}E_{23}N^kE_{32}]\,\mathrm{diag}(s^{-r_1},\ldots,s^{-r_q},1,\ldots,1)\\ &= \begin{bmatrix}E_{23}N^{r_1-2}E_{32}^{(1)} & \cdots & E_{23}N^{r_h-2}E_{32}^{(h)} & E_{22}^{(h+1)} & \cdots & E_{22}^{(q)} & A_{22}^{(q+1)} & \cdots & A_{22}^{(p)}\end{bmatrix}.\end{aligned}$$

Therefore,

$$\hat{\Gamma}_q = \begin{bmatrix}\hat{\Gamma}_{11}\\ \hat{\Gamma}_{21}\end{bmatrix} = \begin{bmatrix}E_{23}N^{r_1-2}E_{32}^{(1)} & \cdots & E_{23}N^{r_h-2}E_{32}^{(h)} & E_{22}^{(h+1)} & \cdots & E_{22}^{(q)}\end{bmatrix} \in \mathbb{R}^{m\times q}. \quad (2.3.6)$$

Since $[E,A,B,C]$ has generalized vector relative degree, $\hat{\Gamma}_q$ is full column rank. Hence, there exist q rows of matrix $\hat{\Gamma}_q$ being linearly independent. For that reason, we can change the rows of the matrix $\mathrm{rank}\,\hat{\Gamma}_q$ such that $\hat{\Gamma}_{11} \in \mathrm{GL}_q(\mathbb{R})$. This action is obtained by changing the rows of the matrices $A_{21}, A_{22}, E_{22}, E_{23}$. By renumbered the input channels, the structure of the *inversion form* of system $[E,A,B,C]$ is still preserved the same as in (2.3.2). Therefore, without loss of generality, $\hat{\Gamma}_{11} \in \mathbb{R}^{q\times q}$ is supposed to be invertible. Consequently, we further construct a matrix

$$\Gamma = \begin{bmatrix}\Gamma_{11} & 0\\ \Gamma_{21} & I_{m-q}\end{bmatrix} \in \mathbb{R}^{m\times m}, \quad (2.3.7)$$

where $\Gamma_{11} = \hat{\Gamma}_{11}^{-1} \in \mathrm{GL}_q(\mathbb{R})$, $\Gamma_{21} = -\hat{\Gamma}_{21}\hat{\Gamma}_{11}^{-1} \in \mathbb{R}^{(m-q)\times q}$. Therefore,

$$\Gamma\hat{\Gamma}_q = \begin{bmatrix}I_q\\ 0\end{bmatrix}.$$

On the other hand, using the notation from Theorem 2.3.3, for any

$$(x,u,y) \in \mathfrak{B}_{[E,A,B,C]} \cap (\mathscr{W}^{1,1}_{\mathrm{loc}}(\mathbb{R}\to\mathbb{R}^n) \times \mathscr{W}^{1,1}_{\mathrm{loc}}(\mathbb{R}\to\mathbb{R}^m) \times \mathscr{W}^{\nu+1,1}_{\mathrm{loc}}(\mathbb{R}\to\mathbb{R}^p),$$

and $Tx = (\eta^\top, y^\top, x_3^\top)^\top \in \mathscr{W}^{1,1}_{\mathrm{loc}}(\mathbb{R}\to\mathbb{R}^{\mu+p+n_3})$, (Tx,u,y) solves

$$\begin{aligned}\dot{\eta}(t) &= Q\eta(t) + A_{12}y(t)\\ 0 &= -\sum_{i=0}^{\nu-1} E_{23}N^iE_{32}y^{(i+2)}(t) - E_{22}\dot{y}(t) + A_{22}y(t) + A_{21}\eta(t) + u(t)\\ x_3(t) &= \sum_{i=0}^{\nu-1} N^iE_{32}y^{(i+1)}(t)\\ 0 &= 0.\end{aligned} \quad (2.3.8)$$

Multiply both side of the second equation in (2.3.8) with non-singular matrix Γ in (2.3.7), we obtain

$$0 = -\sum_{i=0}^{\nu-1} \Gamma E_{23}N^iE_{32}y^{(i+2)} - \Gamma E_{22}\dot{y} + \Gamma A_{22}y + \Gamma A_{21}\eta + \Gamma u. \quad (2.3.9)$$

Apply (2.3.6)-(2.3.9), we obtain

$$0=\begin{bmatrix}-1&0&\cdots&0&0&\cdots&0\\0&0&\cdots&0&0&\cdots&0\\&&\ddots&&&&\\0&0&\cdots&0&0&\cdots&0\\0&0&\cdots&0&0&\cdots&0\\&&&&&\ddots&\\0&0&\cdots&0&0&\cdots&0\end{bmatrix}\begin{pmatrix}y_1^{(r_1)}\\y_2^{(r_1)}\\\vdots\\y_q^{(r_1)}\\y_{q+1}^{(r_1)}\\\vdots\\y_p^{(r_1)}\end{pmatrix}+\begin{bmatrix}R^1_{1,r_1}&0&\cdots&0&0&\cdots&0\\R^2_{1,r_1}&0&\cdots&0&0&\cdots&0\\&&\ddots&&&&\\R^q_{1,r_1}&0&\cdots&0&0&\cdots&0\\R^{q+1}_{1,r_1}&0&\cdots&0&0&\cdots&0\\&&&&&\ddots&\\R^m_{1,r_1}&0&\cdots&0&0&\cdots&0\end{bmatrix}\begin{pmatrix}y_1^{(r_1-1)}\\y_2^{(r_1-1)}\\\vdots\\y_q^{(r_1-1)}\\y_{q+1}^{(r_1-1)}\\\vdots\\y_p^{(r_1-1)}\end{pmatrix}+\cdots+$$

$$+\begin{bmatrix}R^1_{1,r_2+1}&0&\cdots&0&0&\cdots&0\\R^2_{1,r_2+1}&-1&\cdots&0&0&\cdots&0\\&&\ddots&&&&\\R^q_{1,r_2+1}&0&\cdots&0&0&\cdots&0\\R^{q+1}_{1,r_2+1}&0&\cdots&0&0&\cdots&0\\&&&&&\ddots&\\R^m_{1,r_2+1}&0&\cdots&0&0&\cdots&0\end{bmatrix}\begin{pmatrix}y_1^{(r_2)}\\y_2^{(r_2)}\\\vdots\\y_q^{(r_2)}\\y_{q+1}^{(r_2)}\\\vdots\\y_p^{(r_2)}\end{pmatrix}+\begin{bmatrix}R^1_{1,r_2}&R^1_{2,r_2}&\cdots&0&0&\cdots&0\\R^2_{1,r_2}&R^2_{2,r_2}&\cdots&0&0&\cdots&0\\&&\ddots&&&&\\R^q_{1,r_2}&R^q_{2,r_2}&\cdots&0&0&\cdots&0\\R^{q+1}_{1,r_2}&R^{q+1}_{2,r_2}&\cdots&0&0&\cdots&0\\&&&&&\ddots&\\R^m_{1,r_2}&R^m_{2,r_2}&\cdots&0&0&\cdots&0\end{bmatrix}\begin{pmatrix}y_1^{(r_2-1)}\\y_2^{(r_2-1)}\\\vdots\\y_q^{(r_2-1)}\\y_{q+1}^{(r_2-1)}\\\vdots\\y_p^{(r_2-1)}\end{pmatrix}$$

$$+\cdots+$$

$$+\begin{bmatrix}R^1_{1,2}&\cdots&R^1_{q,2}&0&\cdots&0\\&\ddots&&&&\\R^q_{1,2}&\cdots&R^q_{q,2}&0&\cdots&0\\R^{q+1}_{1,2}&\cdots&R^{q+1}_{q,2}&0&\cdots&0\\&&&&\ddots&\\R^m_{1,2}&\cdots&R^m_{q,2}&0&\cdots&0\end{bmatrix}\begin{pmatrix}\dot y_1\\\vdots\\\dot y_q\\\dot y_{q+1}\\\vdots\\\dot y_p\end{pmatrix}+\begin{bmatrix}R^1_{1,1}&\cdots&R^1_{q,1}&R^1_{q+1,1}&\cdots&R^1_{p,1}\\&\ddots&&&&\\R^q_{1,1}&\cdots&R^q_{q,1}&R^q_{q+1,1}&\cdots&R^q_{p,1}\\R^{q+1}_{1,1}&\cdots&R^{q+1}_{q,1}&R^{q+1}_{q+1,1}&\cdots&R^{q+1}_{p,1}\\&&&&\ddots&\\R^m_{1,1}&\cdots&R^m_{q,1}&R^m_{q+1,1}&\cdots&R^m_{p,1}\end{bmatrix}\begin{pmatrix}y_1\\\vdots\\y_q\\y_{q+1}\\\vdots\\y_p\end{pmatrix}$$

$$+\begin{bmatrix}\Gamma_{11}&0\\\Gamma_{21}&I_{m-q}\end{bmatrix}A_{21}x_1+\begin{bmatrix}\Gamma_{11}&0\\\Gamma_{21}&I_{m-q}\end{bmatrix}u.$$

As a consequence, the second equation of (2.3.8) can be written as follow

$$\begin{aligned}y_i^{(r_i)}(t)&=\sum_{j=1}^{q}\sum_{h=1}^{r_j}R^i_{jh}y_j^{(h-1)}(t)+\sum_{j=q+1}^{p}R^i_{j1}y_j(t)+\Gamma_iA_{21}\eta(t)+\Gamma_iu(t)\quad i=1,\dots,q\\0&=\sum_{j=1}^{q}\sum_{h=1}^{r_j}R^i_{jh}y_j^{(h-1)}(t)+\sum_{j=q+1}^{p}R^i_{j1}y_j(t)+\Gamma_iA_{21}\eta(t)+\Gamma_iu(t)\quad i=q+1,\dots,m\end{aligned}$$

where Γ_i is i-th row of matrix $\Gamma \in \mathbb{R}^{m\times m}$ from (2.3.7), $R^i_{jh} \in \mathbb{R}$. Then the system (2.3.8) can be transformed to

$$\begin{aligned}
\dot{\eta}(t) &= Q\eta(t) + A_{12}y(t)\\
y_i^{(r_i)}(t) &= \sum_{j=1}^{q}\sum_{h=1}^{r_j} R^i_{jh} y_j^{(h-1)}(t) + \sum_{j=q+1}^{p} R^i_{j1} y_j(t) + \Gamma_i A_{21}\eta(t) + \Gamma_i u(t), \quad i = 1,\dots,q,\\
0 &= \sum_{j=1}^{q}\sum_{h=1}^{r_j} R^i_{jh} y_j^{(h-1)}(t) + \sum_{j=q+1}^{p} R^i_{j1} y_j(t) + \Gamma_i A_{21}\eta(t) + \Gamma_i u(t), \quad i = q+1,\dots,m,\\
x_3(t) &= \sum_{i=0}^{\nu-1} N^i E_{32} y^{(i+1)}(t).
\end{aligned}$$

We conclude this subsection by a theorem.

Theorem 2.3.16. *Suppose* $[E,A,B,C] \in \Sigma_{l,n,m,p}$ *has generalized vector relative degree* $r = (r_1,\dots,r_q,0,\dots,0) \in \mathbb{N}_0^{1\times p}$, $r_i > 0$ *for all* $i = 1,\dots,q$. *If* ν *denotes the nilpotent index of matrix* N *in inversion form* (2.3.2)*, then a trajectory satisfies*

$$(x,u,y) \in \mathfrak{B}_{[E,A,B,C]} \cap \left(\mathscr{W}^{1,1}_{\text{loc}}(\mathbb{R}\to\mathbb{R}^n) \times \mathscr{W}^{1,1}_{\text{loc}}(\mathbb{R}\to\mathbb{R}^m) \times \mathscr{W}^{\nu+1,1}_{\text{loc}}(\mathbb{R}\to\mathbb{R}^m)\right)$$

if and only if there exist $T \in \mathrm{GL}_n(\mathbb{R})$ *such that* $Tx = (\eta^\top, y_1,\dots,y_q,y_{q+1},\dots,y_p,x_3^\top)^\top \in \mathscr{W}^{1,1}_{\text{loc}}(\mathbb{R}\to\mathbb{R}^\mu) \times \mathscr{W}^{\tilde{r}_1,1}_{\text{loc}}(\mathbb{R}\to\mathbb{R}) \times\cdots\times \mathscr{W}^{\tilde{r}_q,1}_{\text{loc}}(\mathbb{R}\to\mathbb{R}) \times \mathscr{W}^{\nu,1}_{\text{loc}}(\mathbb{R}\to\mathbb{R}) \cdots \times \mathscr{W}^{\nu,1}_{\text{loc}}(\mathbb{R}\to$

$\mathbb{R}) \times \mathscr{W}^{1,1}_{\mathrm{loc}}(\mathbb{R} \to \mathbb{R}^{n_3})$ *fulfills*

$$
\begin{aligned}
\dot\eta(t) &= Q\eta(t) + A_{12}y(t),\\
\begin{pmatrix} y_1^{(r_1)}(t)\\ y_2^{(r_2)}(t)\\ \vdots\\ y_q^{(r_q)}(t)\end{pmatrix} &= \begin{pmatrix} \sum\limits_{j=1}^{q}\sum\limits_{h=1}^{r_j} R^1_{jh} y_j^{(h-1)}(t) + \sum\limits_{j=q+1}^{p} R^1_{j1} y_j(t)\\ \sum\limits_{j=1}^{q}\sum\limits_{h=1}^{r_j} R^2_{jh} y_j^{(h-1)}(t) + \sum\limits_{j=q+1}^{p} R^2_{j1} y_j(t)\\ \vdots \\ \sum\limits_{j=1}^{q}\sum\limits_{h=1}^{r_j} R^q_{jh} y_j^{(h-1)}(t) + \sum\limits_{j=q+1}^{p} R^q_{j1} y_j(t)\end{pmatrix} + [\Gamma_{11}\ 0]A_{21}\eta(t)\\
&\qquad + [\Gamma_{11}\ 0]\begin{pmatrix} u_1(t)\\ u_2(t)\\ \vdots\\ u_m(t)\end{pmatrix},\\
0 &= \begin{pmatrix} \sum\limits_{j=1}^{q}\sum\limits_{h=1}^{r_j} R^{q+1}_{jh} y_j^{(h-1)}(t) + \sum\limits_{j=q+1}^{p} R^{q+1}_{j1} y_j(t)\\ \sum\limits_{j=1}^{q}\sum\limits_{h=1}^{r_j} R^{q+2}_{jh} y_j^{(h-1)}(t) + \sum\limits_{j=q+1}^{p} R^{q+2}_{j1} y_j(t)\\ \vdots \\ \sum\limits_{j=1}^{q}\sum\limits_{h=1}^{r_j} R^m_{jh} y_j^{(h-1)}(t) + \sum\limits_{j=q+1}^{p} R^m_{j1} y_j(t)\end{pmatrix} + [\Gamma_{21}\ I_{m-q}]A_{21}\eta(t)\\
&\qquad + [\Gamma_{21}\ I_{m-q}]\begin{pmatrix} u_1(t)\\ u_2(t)\\ \vdots\\ u_m(t)\end{pmatrix},\\
x_3(t) &= \sum_{i=0}^{\nu-1} N^i E_{32} y^{(i+1)}(t),
\end{aligned}
\tag{2.3.10}
$$

where Γ_{11}, Γ_{21} *from* (2.3.7)*,* $Q, A_{12}, A_{21}, N, E_{32}$ *from* (2.3.2)*,* $\tilde r_i = \max\{r_i, \nu\}$*,* $i = 1, \ldots, q$*.*

Remarks 2.3.17.

(i) We also note that

$$
\begin{aligned}
\hat A_{12} &= \begin{bmatrix}\Gamma_{11} & 0\end{bmatrix} A_{22}\begin{bmatrix}0\\ I_{p-q}\end{bmatrix} = \begin{bmatrix} R^1_{q+1,1} & \cdots & R^1_{p,1}\\ & \ddots & \\ R^q_{q+1,1} & \cdots & R^q_{p,1}\end{bmatrix} \in \mathbb{R}^{q\times(p-q)}\\
\hat A_{22} &= \begin{bmatrix}\Gamma_{21} & I_{m-q}\end{bmatrix} A_{22}\begin{bmatrix}0\\ I_{p-q}\end{bmatrix} = \begin{bmatrix} R^{q+1}_{q+1,1} & \cdots & R^{q+1}_{p,1}\\ & \ddots & \\ R^m_{q+1,1} & \cdots & R^m_{p,1}\end{bmatrix} \in \mathbb{R}^{(m-q)\times(p-q)}
\end{aligned}
\tag{2.3.11}
$$

(ii) We want to show the relation between the matrix Γ_{11} from (2.3.7) and high-gain matrix in the case of regular system. Consider a regular system $[E,A,B,C] \in \Sigma_{n,n,m,m}$ having vector relative degree $r = (r_1, \ldots, r_m)$. Without of loss generality, suppose that $r_1 \geq \cdots \geq r_m$. The transfer function is $G(s) = C(sE-A)^{-1}B$, and the high-gain matrix of $[E,A,B,C]$ is

$$\mathrm{GL}_m(\mathbb{R}) \ni \Gamma = \lim_{s\to\infty} \mathrm{diag}(s^{r_1}, \ldots, s^{r_m}) G(s).$$

Arcoording to Remark 2.2.9, we have $H(s) = G^{-1}(s)$, and

$$\lim_{s\to\infty} H(s)\,\mathrm{diag}(s^{-r_1},\ldots,s^{-r_m}) = \Gamma^{-1}.$$

On the other hand, from Remarks 2.3.11(v), $[E,A,B,C]$ has generalized vector relative degree $(r_1,\ldots,r_q,0\ldots,0)$ with $r_1 \geq \cdots \geq r_q > 0$, and

$$\lim_{s\to\infty} H(s)\,\mathrm{diag}(s^{-r_1},\ldots,s^{-r_q},1,\ldots,1) = \hat{\Gamma} \in \mathbb{R}^{m\times m},$$

Therefore,

$$\Gamma^{-1}\begin{bmatrix} I_q \\ 0 \end{bmatrix} = \hat{\Gamma}\begin{bmatrix} I_q \\ 0 \end{bmatrix} = \begin{bmatrix} \hat{\Gamma}_{11} \\ \hat{\Gamma}_{21} \end{bmatrix} \in \mathbb{R}^{m\times q}.$$

In conjunction with $\Gamma_{11} = \hat{\Gamma}_{11}^{-1} \in \mathrm{GL}_q(\mathbb{R})$ from (2.3.7), we conclude that

$$\Gamma_{11} = \begin{bmatrix} I_q & 0 \end{bmatrix} \Gamma \begin{bmatrix} I_q \\ 0 \end{bmatrix}.$$

We would like to stress that in high-gain adaptive control, it is usualy to assume that high-gain matrix Γ being positive (negative) definite. Hence, in the case of linear differential-algebraic systems, we can suppose Γ_{11} being positive (negative) definite instead of using high-gain matrix Γ.

(iii) By using left inverse of matrix $\begin{bmatrix} sE-A & -B \\ -C & 0 \end{bmatrix}$, we avoid to use the transfer function in defining the *generalized vector relative degree*. This approach is really efficient in finding out a nice *normal form* (2.3.10) for general linear differential-algebraic system. Based on this *normal form*, we conclude that only q first components of inputs impacting on q first outputs via differential part of the systems.

(iv) Consider regular system $[E,A,B,C] \in \Sigma_{n,n,m,m}$ with genneralized vector relative degree. If $E = I_n$, then $q = m$, and the third equation in normal form (2.3.10) vanishes. Moreover, we denote $n_3 = n - m - \mu = |r| - m$, and set

$$x_3 = (\dot{y}_1,\ldots,y_1^{(r_1-1)},\ldots,\dot{y}_m,\ldots,y_m^{(r_m-1)})^\top \in \mathbb{R}^{n_3},$$

$$N = \mathrm{diag}(N_1,\ldots,N_m) \in \mathbb{R}^{n_3\times n_3} \text{ with } N_i = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \\ & \ddots & \ddots & & \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix} \in \mathbb{R}^{(r_i-1)\times(r_i-1)},$$

$$E_{32} = \mathrm{diag}(E_{32(1)},\ldots,E_{32(m)}) \in \mathbb{R}^{n_3\times m} \text{ with } E_{32(i)} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \in \mathbb{R}^{(r_i-1)\times 1}.$$

We note that N is a nilpotent matrix with index of nilpotency $\nu = \max_{1\le i\le m}(r_i - 1)$. Now, we obtain

$$
\begin{aligned}
\dot{\eta}(t) &= Q\eta(t) + A_{12}y(t),\\
\begin{pmatrix} y_1^{(r_1)}(t)\\ y_2^{(r_2)}(t)\\ \vdots\\ y_m^{(r_m)}(t)\end{pmatrix} &= \begin{pmatrix} \sum_{j=1}^{m}\sum_{h=1}^{r_j} R_{jh}^{1}y_j^{(h-1)}(t)\\ \sum_{j=1}^{m}\sum_{h=1}^{r_j} R_{jh}^{2}y_j^{(h-1)}(t)\\ \vdots\\ \sum_{j=1}^{m}\sum_{h=1}^{r_j} R_{jh}^{m}y_j^{(h-1)}(t)\end{pmatrix} + \Gamma A_{21}\eta(t) + \Gamma\begin{pmatrix} u_1(t)\\ u_2(t)\\ \vdots\\ u_m(t)\end{pmatrix},\\
x_3(t) &= \sum_{i=0}^{\nu-1} N^i E_{32} y^{(i+1)}(t).
\end{aligned}
$$

This form is consistent with the *normal form* of linear differential systems which was introduced by Mueller in [69] and also recalled in Corollary 2.2.10. On the other hand, if $[E,A,B,C] \in \Sigma_{n,n,m,m}$ with proper inverse transfer function, then $[E,A,B,C]$ has generalized vector relative degree $(0,\dots,0)$, according to Remark 2.3.11. This implies the second equation in in (2.3.10) will be disappeared, and $q = 0$.

$$
\begin{aligned}
\dot{\eta}(t) &= Q\eta(t) + A_{12}y(t),\\
0 &= \begin{pmatrix} \sum_{j=1}^{m} R_{j1}^{1}y_j(t)\\ \sum_{j=1}^{m} R_{j1}^{2}y_j(t)\\ \vdots\\ \sum_{j=1}^{m} R_{j1}^{m}y_j(t)\end{pmatrix} + A_{21}\eta(t) + \begin{pmatrix} u_1(t)\\ u_2(t)\\ \vdots\\ u_m(t)\end{pmatrix},\\
x_3(t) &= \sum_{i=0}^{\nu-1} N^i E_{32} y^{(i+1)}(t).
\end{aligned}
$$

We rewrite this system in the short form

$$
\begin{aligned}
\dot{\eta}(t) &= Q\eta(t) + A_{12}y(t),\\
0 &= \hat{A}_{22}y(t) + A_{21}\eta(t) + u(t),\\
x_3(t) &= \sum_{i=0}^{\nu-1} N^i E_{32} y^{(i+1)}(t).
\end{aligned}
$$

This form is precisely the *normal form* of system with proper inverse transfer function presented in Proposition 2.2.13 which was cited from [10]. In conclusion, the *normal form* obtained in (2.3.10) is generalized, and consistent with results which was proposed in [69, 10].

Chapter 3

Funnel control for systems with known strict relative degree

In this chapter, we study funnel control for nonlinear multi-input, multi-output systems having strict relative degree r and input-to-state stable internal dynamics. We propose a new simple funnel controller which require the involvement of the first $r-1$ derivatives of the output error. We further discus the feasible of this controller and the influence of the controller parameters. We finally show the application of our controller to some mechanical systems by simulation, and compare our result with some other approaches.

3.1 System class

We consider a class of non-linear systems described by functional differential equations of the form

$$\begin{aligned} y^{(r)}(t) &= f\big(d(t), T(y,\dot{y},\dots,y^{(r-1)})(t)\big) + \Gamma\big(d(t), T(y,\dot{y},\dots,y^{(r-1)})(t)\big)\, u(t), \\ y|_{[-h,0]} &= y^0 \in \mathscr{C}^{r-1}([-h,0] \to \mathbb{R}^m), \end{aligned} \tag{3.1.1}$$

where $h > 0$ is the "memory" of the system, $r \in \mathbb{N}$ is the strict relative degree, and

(P1): the measurable "disturbance" satisfies $d \in \mathscr{L}^\infty(\mathbb{R}_{\geq 0} \to \mathbb{R}^p)$, $p \in \mathbb{N}$;

(P2): $f \in \mathscr{C}(\mathbb{R}^p \times \mathbb{R}^q \to \mathbb{R}^m)$, $q \in \mathbb{N}$,

(P3): the "high-frequency gain matrix function" $\Gamma \in \mathscr{C}(\mathbb{R}^p \times \mathbb{R}^q \to \mathbb{R}^{m\times m})$ takes values in the set of positive (negative) definite matrices[1];

(P4): $T : \mathscr{C}([-h,\infty) \to \mathbb{R}^{rm}) \to \mathscr{L}^\infty_{\text{loc}}(\mathbb{R}_{\geq 0} \to \mathbb{R}^q)$ is an operator with the following properties:

a) T maps bounded trajectories to bounded trajectories , i.e, for all $c_1 > 0$, there exists $c_2 > 0$ such that for all $\zeta \in \mathscr{C}([-h,\infty) \to \mathbb{R}^{rm})$,

$$\sup_{t\in[-h,\infty)} \|\zeta(t)\| \leq c_1 \Rightarrow \sup_{t\in[0,\infty)} \|T(\zeta)(t)\| \leq c_2,$$

b) T is causal , i.e, for all $t \geq 0$ and all $\zeta, \xi \in \mathscr{C}([-h,\infty) \to \mathbb{R}^{rm})$,

$$\zeta|_{[-h,t)} = \xi|_{[-h,t)} \Rightarrow T(\zeta)|_{[0,t)} \overset{\text{a.a.}}{=} T(\xi)|_{[0,t)},$$

where "a.a." stands for "almost all".

[1] One may wonder why Γ is not assumed to be uniformly bounded away from zero. The reason is that in the closed-loop system this is established anyway due to the boundedness of the involved signals.

c) T is locally Lipschitz continuous in the following sense: for all $t \geq 0$ there exist $\tau, \delta, c > 0$ such that for all $\zeta, \Delta\zeta \in \mathscr{C}([-h,\infty) \to \mathbb{R}^{rm})$ with $\Delta\zeta|_{[-h,t)} = 0$ and $\|\Delta\zeta|_{[t,t+\tau]}\|_\infty < \delta$ we have

$$\left\| \left(T(\zeta+\Delta\zeta) - T(\zeta) \right)\Big|_{[t,t+\tau]} \right\|_\infty \leq c\|\Delta\zeta|_{[t,t+\tau]}\|_\infty.$$

Some cases of systems with the form (3.1.1) have been studied in, for instance, [34, 50, 48, 46]. Based on these studies, the class of systems (3.1.1) contains linear and nonlinear systems which have strict relative degree and input-to-state stable internal dynamics (zero dynamics in the linear case). The operator T and its properties help us to open wide considered systems to infinite-dimensional linear systems, systems with hysteretic effects, nonlinear delay elements, or any combination of them. We note that the operator T usually present to the solution operator of the differential equation which depicts the internal dynamics of the system and property (P4a) describes the input-to-state stability of the internal dynamics.

Remark 3.1.1. One important class of linear differential-algebraic systems that relates to systems class (3.1.1) are regular systems with positive strict relative degree r and asymptotically stable zero dynamics which was introduced in [9], and also mentioned in Subsection 2.2.3

$$\begin{aligned} E\dot{x}(t) &= Ax(t) + Bu(t), \quad x(0) = x^0 \in \mathbb{R}^n \\ y(t) &= Cx(t), \end{aligned} \tag{3.1.2}$$

where $E, A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{m\times n}$ and $\Gamma = \lim_{s\to\infty} s^r C(sE-A)^{-1}B \in \mathbb{R}^{m\times m}$ is positive (negative) definite. Similar to (2.2.9), systems of this type can be transformed into *normal form*, also see [9],

$$\begin{aligned} y^{(r)}(t) &= \sum_{i=1}^{r} R_i y^{(i-1)}(t) + S\eta(t) + \Gamma u(t), && y(0) = Cx^0, \\ \dot{\eta}(t) &= Py(t) + Q\eta(t), && \eta(0) = \eta^0 \in \mathbb{R}^\mu, \\ x_c(t) &= -\sum_{i=0}^{\nu-1} N_c^i B_c u^{(i)}(t), && \\ x_{\bar{c}}(t) &= 0, && \end{aligned}$$

where $n_c, n_{\bar{c}} \in \mathbb{N}_0$, $\mu = n - n_c - n_{\bar{c}} - rm$, $R_i \in \mathbb{R}^{m\times m}$ for $i = 1, 2, \ldots, r$, $S \in \mathbb{R}^{m\times\mu}$, $P \in \mathbb{R}^{\mu\times m}$, $B_c \in \mathbb{R}^{n_c\times m}$, $N_c \in \mathbb{R}^{n_c\times n_c}$ is nilpotent with index ν, and rank $\begin{bmatrix} N_c & B_c \end{bmatrix} = n_c$. $Q \in \mathbb{R}^{\mu\times\mu}$ is a Hurwitz matrix, i.e., all eigenvalues of Q have negative real part. We begin with two first equations in above *normal form*

$$\begin{aligned} y^{(r)}(t) &= \sum_{i=1}^{r} R_i y^{(i-1)}(t) + S\eta(t) + \Gamma u(t), && y(0) = Cx^0 \\ \dot{\eta}(t) &= Py(t) + Q\eta(t), && \eta(0) = \eta^0 \in \mathbb{R}^\mu \end{aligned}$$

This subsystem belongs to type (3.1.1) with $\Gamma = \lim_{s\to\infty} s^r C(sE-A)^{-1}B$ and

$$\begin{aligned} f\big(d(t), T(y,\dot{y},\ldots,y^{(r-1)})(t)\big) &= T(y,\dot{y},\ldots,y^{(r-1)})(t) \\ &= \sum_{i=1}^{r} R_i y^{(i-1)}(t) + Se^{Qt}\eta^0 + \int_0^t Se^{Q(t-\tau)}Py(\tau)d\tau. \end{aligned}$$

T is clearly causal, locally Lipschitz, and the Hurwitz property of Q implies that T has the bounded-input-bounded-output property (P4a). Note that T is parameterized by $\eta^0 \in \mathbb{R}^\mu$. Although, system (3.1.2) is not apart of system class (3.1.1), we still shown in the following sections that a feasible funnel controller of system class (3.1.1) also applied to system of the form (3.1.2). And we also note that for the third equation

$$x_c(t) = -\sum_{i=0}^{\nu-1} N_c^i B_c u^{(i)}(t),$$

the input function $u(t)$ is required smooth enough, $u \in \mathscr{W}_{\text{loc}}^{\nu-1,1}(\mathbb{R} \to \mathbb{R}^m)$. This implies the required smoothness of input function when we apply funnel controller for linear differential-algebraic system with positive strict relative degree (3.1.2).
In [13], we also indicate that *minimum-phase linear time-invariant* systems is a subclass of systems (3.1.1). Finally, we want to emphasize that systems with equation

$$y^{(r)}(t) = f\big(d_1(t), T_1(y,\dot{y},\ldots,y^{(r-1)})(t)\big) + \Gamma\big(d_2(t), T_2(y,\dot{y},\ldots,y^{(r-1)})(t)\big)\, u(t),$$

where d_i is as in (P1) and T_i is as in (P4) for $i = 1,2$ are included in the class (3.1.1). This can be achieved by setting $d := (d_1, d_2)$, $T := (T_1, T_2)$ and a suitable adjustment of f and Γ.

3.2 Funnel controller with derivative feedback

For systems of type (3.1.1), we propose the following funnel controller:

$$\boxed{\begin{array}{rcccl} e_0(t) &=& e(t) &=& y(t) - y_{\text{ref}}(t),\\ e_1(t) &=& \dot{e}_0(t) &+& k_0(t)\cdot e_0(t),\\ e_2(t) &=& \dot{e}_1(t) &+& k_1(t)\cdot e_1(t),\\ &\vdots&&&\\ e_{r-1}(t) &=& \dot{e}_{r-2}(t) &+& k_{r-2}(t)\cdot e_{r-2}(t),\\ \multicolumn{5}{l}{k_i(t) = \dfrac{1}{1-\varphi_i^2(t)\|e_i(t)\|^2}, \quad i = 0,\ldots,r-1,}\\ \multicolumn{5}{l}{u(t) = \begin{cases} -k_{r-1}(t)\cdot e_{r-1}(t), & \text{if } \Gamma \text{ is pointwise positive definite},\\ k_{r-1}(t)\cdot e_{r-1}(t), & \text{if } \Gamma \text{ is pointwise negative definite},\end{cases}} \end{array}} \tag{3.2.1}$$

where the reference signal and funnel functions have the following properties:

$$\boxed{\begin{array}{l} y_{\text{ref}} \in \mathscr{W}^{r,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R}^m),\\ \varphi_0 \in \Phi_r,\ \ \varphi_1 \in \Phi_{r-1},\ \ldots,\ \ \varphi_{r-1} \in \Phi_1. \end{array}} \tag{3.2.2}$$

At the beginning, we need to solve a basic question about the existence of solutions of the initial value problem which is obtained by apply funnel controller (3.2.1) to a system (3.1.1). Some care must be exercised with the existence of a solution since k_i introduces a pole on the right hand side of the closed-loop differential equation. We like to stress that a *solution* of closed-loop system (3.1.1), (3.2.1) on $[-h,\omega)$ is a function $y \in \mathscr{C}^{r-1}([-h,\omega) \to \mathbb{R}^m)$, $\omega \in (0,\infty]$, with $y|_{[-h,0]} = y^0$ such that $y^{(r-1)}|_{[0,\omega)}$ is absolutely continuous and satisfies the differential equation in (3.1.1) with u defined in (3.2.1) for almost all $t \in [0,\omega)$. A solution y is called *maximal*, if it has no right extension that is also a solution . Existence of solutions of functional differential equations has been investigated in [48] for instance.

Remark 3.2.1 (Funnel control for systems with $r \in \{1,2,3\}$)**.** In the following we determine the funnel controllers explicitly for the cases $r = 1,2,3$. We assume for convenience that the high-frequency gain matrix function Γ is pointwise positive definite.

$r = 1$: The control law (3.2.1) reduces to the "classical" funnel controller

$$\begin{aligned} u(t) &= -k(t)e(t), \\ k(t) &= \frac{1}{1-\varphi^2(t)\|e(t)\|^2}. \end{aligned}$$

Moreover, our assumptions on the reference signal and the funnel function φ reduce to those made in [48].

$r = 2$: We obtain the controller

$$\begin{aligned} u(t) &= -k_1(t)(\dot e(t)+k_0(t)e(t)), \\ k_0(t) &= \frac{1}{1-\varphi_0^2(t)\|e(t)\|^2}, \\ k_1(t) &= \frac{1}{1-\varphi_1^2(t)\|\dot e(t)+k_0(t)e(t)\|^2}. \end{aligned}$$

$r = 3$: Here the controller (3.2.1) takes the form

$$\begin{aligned} u(t) = &-k_2(t)\cdot\big[\ddot e(t)+2k_0(t)^2(\varphi_0^2(t)e^\top(t)\dot e(t)+\varphi_0(t)\dot\varphi_0(t)\|e(t)\|^2)e(t) \\ &+k_0(t)\dot e(t)+k_1(t)(\dot e(t)+k_0(t)e(t))\big], \\ k_0(t) =& \frac{1}{1-\varphi_0^2(t)\|e(t)\|^2}, \\ k_1(t) =& \frac{1}{1-\varphi_1^2(t)\|\dot e(t)+k_0(t)e(t)\|^2}, \\ k_2(t) =& \frac{1}{1-\varphi_2^2(t)\|\ddot e(t)+2k_0(t)^2(\varphi_0^2(t)e^\top(t)\dot e(t)+\varphi_0(t)\dot\varphi_0(t)\|e(t)\|^2)e(t)} \\ &\overline{+k_0(t)\dot e(t)+k_1(t)(\dot e(t)+k_0(t)e(t))\|^2}. \end{aligned}$$

Remark 3.2.2 (The intuition behind the funnel controller (3.2.1))**.** We like to repeat the underlying of "classical" funnel controller for systems with relative degree one and input-to-state stable internal dynamics in [48]. It is based on an idea that if the error $e(t)$ approaches the funnel boundary, then the gain function $k(t)$ takes a large value. This property incorporate with high-gain property imposed to system class, keeps $e(t)$ bounded away from the funnel boundary.

From this basic idea, we describe the working of the controller (3.2.1) through an example of single-input, single-output system which is depicted in Figure 3.1. We suppose that the error $e = e_0$ approaches the upper funnel boundary $\varphi_0(t)^{-1}$ at time $t_0 > 0$. Then $k_0(t_0)$, and consequently $k_0(t_0)\cdot e(t_0)$ are very large. Since $e_1 = \dot e + k_0 \cdot e$ evolves in the performance funnel $\mathscr{F}_{\varphi_1}$, we may infer that $\dot e(t_0) = e_1(t_0) - k_0(t_0)\cdot e(t_0)$ takes a large negative value. It means that e will be decreased extremely. That is, whenever the error e approaches the funnel boundary $\varphi_0(t)^{-1}$, the controller ensures a repelling effect.

This argumentation can be repeated for the functions $e_1,\dots,e_{r-2}$. Finally, since e_{r-1} includes the first $r-1$ derivatives of e, the system with artificial output e_{r-1} has relative degree one, and the classical high gain property applies to e_{r-1}.

We now show feasibility of the funnel controller (3.2.1).

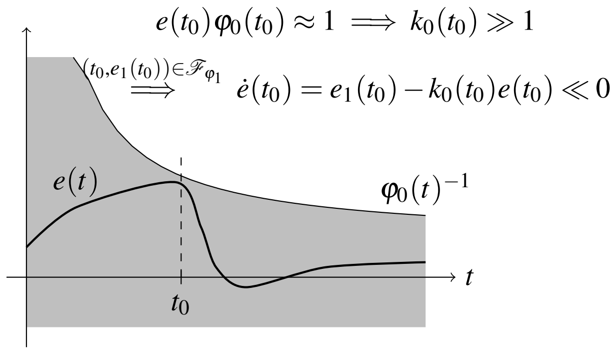

FIGURE 3.1: Error in the performance funnel $\mathscr{F}_{\varphi_0}$

Theorem 3.2.3. *Consider a system* (3.1.1) *with strict relative degree $r \in \mathbb{N}$ and properties (P1)-(P4). For Φ_i as defined in* (1.1.7)*, given $\varphi_i \in \Phi_{r-i}$, $i = 0,\dots,r-1$. Let $y_{\text{ref}} \in \mathscr{W}^{r,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R}^m)$ be a reference signal, and $y|_{[-h,0]} = y^0 \in \mathscr{C}^{r-1}([-h,0] \to \mathbb{R}^m)$ an initial value such that $e_0,\dots,e_{r-1}$ as defined in* (3.2.1) *fulfill*

$$\varphi_i(0)\|e_i(0)\| < 1 \text{ for } i = 0,\dots,r-1. \tag{3.2.3}$$

Then the application of the funnel controller (3.2.1) *to* (3.1.1) *yields an initial value problem, which has a solution, and every maximal solution $y : [-h,\omega) \to \mathbb{R}^m$, $\omega \in (0,\infty]$, has the following properties*[2]*:*

(i) The solution is global (i.e., $\omega = \infty$).

(ii) The input $u : \mathbb{R}_{\geq 0} \to \mathbb{R}^m$, the gain functions $k_0,\dots,k_{r-1} : \mathbb{R}_{\geq 0} \to \mathbb{R}$ and $y,\dots,y^{(r-1)} : \mathbb{R}_{\geq 0} \to \mathbb{R}^m$ are bounded.

(iii) The functions $e_0,\dots,e_{r-1} : \mathbb{R}_{\geq 0} \to \mathbb{R}^m$ evolve in their respective performance funnels and are uniformly bounded away from the funnel boundaries in the following sense:

$$\forall\, i = 0,\dots,r-1\ \exists\, \varepsilon_i > 0\ \forall\, t > 0 : \|e_i(t)\| \leq \varphi_i(t)^{-1} - \varepsilon_i. \tag{3.2.4}$$

In particular, the error $e(t) = y(t) - y_{\text{ref}}(t)$ evolves in the funnel $\mathscr{F}_{\varphi_0}$ as in (1.1.8) *and stays uniformly away from its boundary.*

Proof. We may, without loss of generality, assume that the high-frequency gain matrix function Γ of system (3.1.1) is pointwisely positive definite. We proceed in several steps.
Step 1: We show that a maximal solution $y : [-h,\omega) \to \mathbb{R}^m$, $\omega \in (0,\infty]$, of (3.1.1), (3.2.1) exists. We aim at reformulating (3.1.1), (3.2.1) as an initial value problem

$$\begin{aligned} \dot{x}(t) &= F\big(t, x(t), T(x)(t)\big), \\ x|_{[-h,0]} &= \Big(y^0, \dot{y}^0, \dots, (\tfrac{d}{dt})^{r-1} y^0\Big)\Big|_{[-h,0]}, \end{aligned} \tag{3.2.5}$$

where $x = \big(y, \dot{y}, \dots, y^{(r-1)}\big)$ and F is some suitable continuous function.
Step 1a: Define, for $i = 0,\dots,r-1$, the sets

$$\mathscr{D}_i := \left\{\ (t, e_0, \dots, e_i) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^m \times \cdots \times \mathbb{R}^m\ \middle|\ (t, e_j) \in \mathscr{F}_{\varphi_j}, j = 0,\dots,i\ \right\},$$

[2]Note that maximal solutions are not unique in general.

where $\mathscr{F}_{\varphi_j}$ is as in (1.1.8), and the functions $K_i : \mathscr{D}_i \to \mathbb{R}^m$ recursively by

$$\begin{aligned} K_0(t,e_0) &:= \frac{e_0}{1-\varphi_0^2(t)\|e_0\|^2}, \\ K_i(t,e_0,\ldots,e_i) &:= \frac{e_i}{1-\varphi_i^2(t)\|e_i\|^2} + \frac{\partial K_{i-1}}{\partial t}(t,e_0,\ldots,e_{i-1}) \\ &\quad + \sum_{j=0}^{i-1} \frac{\partial K_{i-1}}{\partial e_j}(t,e_0,\ldots,e_{i-1}) \left(e_{j+1} - \frac{e_j}{1-\varphi_j^2(t)\|e_j\|^2} \right). \end{aligned}$$

Choose some interval $I \subseteq \mathbb{R}_{\geq 0}$ with $0 \in I$ and let $(e_0,\ldots,e_{r-1}) : I \to \mathbb{R}^{rm}$ be such that, for all $t \in I$, $\big(t, e_0(t),\ldots,e_{r-1}(t)\big) \in \mathscr{D}_{r-1}$ and $(e_0,\ldots,e_{r-1})$ satisfies the relations in (3.2.1). Then $e = e_0$ satisfies, on the interval I,

$$e^{(i)} = e_i - \sum_{j=0}^{i-1} \left(\frac{d}{dt}\right)^{i-j-1} (k_j e_j) \quad \text{for all } i = 1,\ldots,r-1. \tag{3.2.6}$$

Step 1b: We show by induction that for all $i = 0,\ldots,r-1$ we have

$$\forall\, t \in I: \ \sum_{j=0}^{i} \left(\frac{d}{dt}\right)^{i-j} \Big(k_j(t)e_j(t)\Big) = K_i\Big(t, e_0(t),\ldots,e_i(t)\Big). \tag{3.2.7}$$

Equation (3.2.7) is obviously true for $i = 0$. Assume that $i \in \{1,\ldots,r-1\}$ and the statement holds for $i-1$. Then

$$\begin{aligned} \sum_{j=0}^{i} \left(\frac{d}{dt}\right)^{i-j} \Big(k_j(t)e_j(t)\Big) &= k_i(t)e_i(t) + \frac{d}{dt}\left(\sum_{j=0}^{i-1} \left(\frac{d}{dt}\right)^{i-j-1} \Big(k_j(t)e_j(t)\Big) \right) \\ &= k_i(t)e_i(t) + \frac{d}{dt} K_{i-1}\Big(t, e_0(t),\ldots,e_{i-1}(t)\Big) \\ &= K_i\Big(t, e_0(t),\ldots,e_i(t)\Big). \end{aligned}$$

Step 1c: Define

$$\tilde{K}_0 : \mathbb{R}_{\geq 0} \times \mathbb{R}^m \to \mathbb{R}^m,\ (t,y) \mapsto y - y_{\text{ref}}(t)$$

and the set

$$\tilde{\mathscr{D}}_0 := \left\{ \ (t,y) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^m \ \middle| \ \big(t, \tilde{K}_0(t,y)\big) \in \mathscr{D}_0 \ \right\}.$$

Furthermore, recursively define for $i = 1,\ldots,r-1$ the maps

$$\begin{aligned} &\tilde{K}_i : \tilde{\mathscr{D}}_{i-1} \times \mathbb{R}^m \to \mathbb{R}^m, \\ &\quad (t,y_0,\ldots,y_i) \mapsto y_i - y_{\text{ref}}^{(i)}(t) + K_{i-1}\big(t, \tilde{K}_0(t,y_0),\ldots,\tilde{K}_{i-1}(t,y_0,\ldots,y_{i-1})\big) \end{aligned}$$

and the sets

$$\tilde{\mathscr{D}}_i := \left\{ \ (t,y_0,\ldots,y_i) \in \tilde{\mathscr{D}}_{i-1} \times \mathbb{R}^m \ \middle| \ \big(t, \tilde{K}_0(t,y_0),\ldots,\tilde{K}_i(t,y_0,\ldots,y_i)\big) \in \mathscr{D}_i \ \right\}.$$

It now follows from a simple induction, invoking (3.2.6) and (3.2.7) that, for all $t \in I$ and all $i = 0,\ldots,r-1$,

$$\begin{aligned} e_i(t) &= y^{(i)}(t) - y_{\text{ref}}^{(i)}(t) + K_{i-1}\big(t, e_0(t),\ldots,e_{i-1}(t)\big) \\ &= \tilde{K}_i\big(t, y(t), \dot{y}(t),\ldots,y^{(i)}(t)\big). \end{aligned}$$

Therefore, the feedback u in (3.2.1) reads

$$u(t) = \frac{-\tilde{K}_{r-1}\big(t, y(t), \dots, y^{(r-1)}(t)\big)}{1 - \varphi_{r-1}^2(t)\|\tilde{K}_{r-1}\big(t, y(t), \dots, y^{(r-1)}(t)\big)\|^2}, \quad t \in I.$$

Step 1d: Define

$$\begin{aligned} &F : \tilde{\mathscr{D}}_{r-1} \times \mathbb{R}^q \to \mathbb{R}^{rm}, \\ &(t, y_0, y_1, \dots, y_{r-1}, \eta) \mapsto \left(y_1, \dots, y_{r-1}, f\big(d(t), \eta\big) - \frac{\Gamma(d(t), \eta)\tilde{K}_{r-1}(t, y_0, \dots, y_{r-1})}{1 - \varphi_{r-1}^2(t)\|\tilde{K}_{r-1}(t, y_0, \dots, y_{r-1})\|^2} \right). \end{aligned}$$

Then the initial value problem (3.1.1), (3.2.1) is equivalent to (3.2.5). In particular, $\big(0, x(0)\big) \in \tilde{\mathscr{D}}_{r-1}$ and F is measurable in t, continuous in $\big(y_0, y_1, \dots, y_{r-1}, \eta\big)$ and locally essentially bounded. Hence an application of [46, Theorem B.1][3] yields existence of solutions to (3.2.5) and every solution can be extended to a maximal solution. Furthermore, for a maximal solution $x = \left(y, \dot{y}, \dots, y^{(r-1)}\right) : [-h, \omega) \to \mathbb{R}^{rm}$, $\omega \in (0, \infty]$, of (3.2.5), the closure of the graph of this solution is not a compact subset of $\tilde{\mathscr{D}}_{r-1}$. As a consequence, for $(e_0, \dots, e_{r-1}) : [0, \omega) \to \mathbb{R}^{rm}$ defined by

$$e_i(t) := \tilde{K}_i\big(t, y(t), \dot{y}(t), \dots, y^{(i)}(t)\big), \quad t \in [0, \omega),$$

it follows that the closure of the graph of $(e_0, \dots, e_{r-1})$ is not a compact subset of $\mathscr{D}_{r-1}$.

Step 2: We show that $k_0, \dots, k_{r-1}$ as in (3.2.1) are bounded on $[0, \omega)$. For all $i \in \{0, \dots, r-1\}$, set $\psi_i(t) := \varphi_i(t)^{-1}$ for $t \in (0, \omega)$, let $\tau_i \in (0, \omega)$ be arbitrary but fixed and set $\lambda_i := \inf_{t \in (0, \omega)} \psi_i(t) > 0$. Since $\dot{\varphi}_i$ is bounded and $\liminf_{t \to \infty} \varphi_i(t) > 0$ we find that $\frac{d}{dt} \psi_i|_{[\tau_i, \infty)}$ is bounded and hence there exists a Lipschitz bound $L_i > 0$ of $\psi_i|_{[\tau_i, \infty)}$.

Step 2a: We show that k_i is bounded for $i \in \{0, \dots, r-2\}$. Choose $\varepsilon_i > 0$ small enough so that

$$\begin{aligned} \varepsilon_i &\le \min\left\{ \frac{\lambda_i}{2}, \inf_{t \in (0, \tau_i]} \big(\psi_i(t) - \|e_i(t)\|\big) \right\} \\ \text{and} \quad L_i &\le \frac{\lambda_i^2}{4\varepsilon_i} - \sup_{t \in [\tau_i, \infty)} |\psi_{i+1}(t)|. \end{aligned} \tag{3.2.8}$$

Using a standard procedure in funnel control, see e.g. [41], we show that for all $t \in (0, \omega)$ holds $\psi_i(t) - \|e_i(t)\| \ge \varepsilon_i$. By definition of ε_i this holds on $(0, \tau_i]$. Seeking a contradiction suppose that there exists some $t_{i1} \in [\tau_i, \omega)$ with $\psi_i(t_{i1}) - \|e_i(t_{i1})\| < \varepsilon_i$. Set

$$t_{i0} = \max\{\, t \in [\tau_i, t_{i1}) \mid \psi_i(t) - \|e_i(t)\| = \varepsilon_i \,\}.$$

Then, for all $t \in [t_{i0}, t_{i1}]$, we have that

$$\begin{aligned} \psi_i(t) - \|e_i(t)\| &\le \varepsilon_i, \\ \|e_i(t)\| &\ge \psi_i(t) - \varepsilon_i \ge \frac{\lambda_i}{2}, \\ k_i(t) = \frac{1}{1 - \varphi_i^2(t)\|e_i(t)\|^2} &\ge \frac{\psi_i(t)}{2\varepsilon_i} \ge \frac{\lambda_i}{2\varepsilon_i}. \end{aligned}$$

[3] In [46] a domain $\mathscr{D} \subseteq \mathbb{R}_{\ge 0} \times \mathbb{R}$ is considered, but the generalization to the higher dimensional case is straightforward.

Therefore, we find that by (3.2.1)

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|e_i(t)\|^2 &= e_i^\top(t)\left(e_{i+1}(t)-k_i(t)e_i(t)\right)\\ &= -k_i(t)\|e_i(t)\|^2+e_i^\top(t)e_{i+1}(t)\\ &\le \left(-\frac{\lambda_i^2}{4\varepsilon_i}+\sup_{t\in[\tau_i,\infty)}|\psi_{i+1}(t)|\right)\|e_i(t)\|\\ &\overset{(3.2.8)}{\le} -L_i\|e_i(t)\|\end{aligned}$$

for all $t\in[t_{i0},t_{i1}]$. Then

$$\begin{aligned}\|e_i(t_{i1})\|-\|e_i(t_{i0})\| &= \int_{t_{i0}}^{t_{i1}}\frac{1}{2}\|e_i(t)\|^{-1}\frac{d}{dt}\|e_i(t)\|^2\,\mathrm{d}t\\ &\le -L_i(t_{i1}-t_{i0})\\ &\le -|\psi_i(t_{i1})-\psi_i(t_{i0})|\\ &\le \psi_i(t_{i1})-\psi_i(t_{i0}),\end{aligned}$$

and thus we obtain $\varepsilon_i=\psi_i(t_{i0})-\|e_i(t_{i0})\|\le\psi_i(t_{i1})-\|e_i(t_{i1})\|<\varepsilon_i$, a contradiction.
Step 2b: We show that k_{r-1} is bounded. By (3.2.6) and Step 1 we have, invoking $x=(y,\dot y,\dots,y^{(r-1)})$,

$$\begin{aligned}\dot e_{r-1}(t) = f\big(d(t),T(x)(t)\big)-k_{r-1}(t)\Gamma\big(d(t),T(x)(t)\big)e_{r-1}(t)\\ -y_{\mathrm{ref}}^{(r)}(t)+\sum_{i=0}^{r-2}\left(\frac{d}{dt}\right)^{r-i-1}[k_i(t)e_i(t)].\end{aligned}$$

In the following we will prove by induction that there exist constants $M_{i,j},N_{i,j},K_{i,j}>0$ such that, for all $t\in[0,\omega)$,

$$\left\|\left(\frac{d}{dt}\right)^j[k_i(t)e_i(t)]\right\|\le M_{i,j},\quad \left\|\left(\frac{d}{dt}\right)^j e_i(t)\right\|\le N_{i,j},\quad \left|\left(\frac{d}{dt}\right)^j k_i(t)\right|\le K_{i,j},\qquad (3.2.9)$$

for $i=0,\dots,r-2$, $j=0,\dots,r-1-i$. First, we may infer from Step 2a that $k_0,\dots,k_{r-2}$ are bounded. Furthermore, $e_0,\dots,e_{r-1}$ are bounded since they evolve in the respective performance funnels, cf. (3.2.1). Therefore, (3.2.9) is true whenever $j=0$. We prove (3.2.9) for $i=r-2$ and $j=1$: We find that

$$\begin{aligned}\dot e_{r-2}(t) &= e_{r-1}(t)-k_{r-2}(t)e_{r-2}(t),\\ \dot k_{r-2}(t) &= 2k_{r-2}^2(t)\big(\varphi_{r-2}^2(t)e_{r-2}^\top(t)\dot e_{r-2}(t)+\varphi_{r-2}(t)\dot\varphi_{r-2}(t)\|e_{r-2}(t)\|^2\big),\\ \frac{d}{dt}[k_{r-2}(t)e_{r-2}(t)] &= \dot k_{r-2}(t)e_{r-2}(t)+k_{r-2}(t)\dot e_{r-2}(t),\end{aligned}$$

and all of these signals are bounded since $k_{r-2},\varphi_{r-2},\dot\varphi_{r-2},e_{r-2},e_{r-1}$ are bounded. Now let $p\in\{0,\dots,r-3\}$ and $q\in\{0,\dots,r-1-p\}$ and assume that (3.2.9) is true for all $i=p+1,\dots,r-2$ and all $j=0,\dots,r-1-i$ as well as for $i=p$ and all $j=0,\dots,q-1$. We show that it is true for $i = p$ and $j = q$:

$$\begin{aligned}
\left(\frac{d}{dt}\right)^q e_p(t) &= \left(\frac{d}{dt}\right)^{q-1} [e_{p+1}(t) - k_p(t)e_p(t)] \\
&= \left(\frac{d}{dt}\right)^{q-1} e_{p+1}(t) - \left(\frac{d}{dt}\right)^{q-1} [k_p(t)e_p(t)], \\
\left(\frac{d}{dt}\right)^q k_p(t) &= \left(\frac{d}{dt}\right)^{q-1} \Big(2k_p^2(t)\big(\varphi_p^2(t)e_p^\top(t)\dot{e}_p(t) + \varphi_p(t)\dot{\varphi}_p(t)\|e_p(t)\|^2\big)\Big), \\
\left(\frac{d}{dt}\right)^q [k_p(t)e_p(t)] &= \left(\frac{d}{dt}\right)^{q-1} \big(\dot{k}_p(t)e_p(t) + k_p(t)\dot{e}_p(t)\big).
\end{aligned}$$

Then, successive application of the product rule and using the induction hypothesis as well as the fact that $\varphi_p, \dot{\varphi}_p, \ldots, \varphi_p^{(r-p)}$ are bounded, yields that the above terms are bounded. Therefore, the proof of (3.2.9) is complete.

By (3.2.9) and (3.2.6) it follows that $e^{(i)}$ is bounded on $[0,\omega)$ and hence, invoking boundedness of $y_{\text{ref}}, \ldots, y_{\text{ref}}^{(r-1)}$, also $y^{(i)}$ is bounded on $[0,\omega)$ for all $i = 0, \ldots, r-1$. By the bounded-input, bounded-output property (P4a) of the operator T it follows that $T(x)$ is bounded, where $x = \left(y, \dot{y}, \ldots, y^{(r-1)}\right)$. We denote $M_T := \|T(x)|_{[0,\omega)}\|_\infty$. Since f is continuous and d is bounded, we may further infer that $f(d(\cdot), T(x)(\cdot))$ is bounded on $[0,\omega)$, i.e., there exists $M_F > 0$ such that

$$\text{for almost all } t \in [0,\omega): \quad \|f\big(d(t), T(x)(t)\big)\| \le M_F.$$

Define the compact set

$$\mathscr{M} := \left\{ (\delta, \eta, e) \in \mathbb{R}^p \times \mathbb{R}^q \times \mathbb{R}^m \;\middle|\; \begin{array}{l} \|\delta\| \le \|d|_{[0,\omega)}\|_\infty \\ \|\eta\| \le M_T \\ \|e\| = 1. \end{array} \right\},$$

then, since Γ is pointwise positive definite and the map

$$\mathscr{M} \ni (\delta, \eta, e) \mapsto e^\top \Gamma(\delta, \eta) e \in \mathbb{R}_{>0}$$

is continuous, it follows that there exists $\gamma > 0$ such that

$$\forall\, (\delta, \eta, e) \in \mathscr{M}: \quad e^\top \Gamma(\delta, \eta) e \ge \gamma.$$

Therefore, we have

$$\begin{aligned}
&e_{r-1}(t)^\top \Gamma\big(d(t), T(x)(t)\big) e_{r-1}(t) \\
&= \left(\frac{e_{r-1}(t)^\top}{\|e_{r-1}(t)\|} \Gamma\big(d(t), T(x)(t)\big) \frac{e_{r-1}(t)}{\|e_{r-1}(t)\|} \right) \|e_{r-1}(t)\|^2 \\
&\ge \gamma \|e_{r-1}(t)\|^2
\end{aligned}$$

for all $t \in [0,\omega)$. Now, choose $\varepsilon_{r-1} > 0$ small enough so that

$$\varepsilon_{r-1} \le \min\left\{ \frac{\lambda_{r-1}}{2}, \inf_{t \in (0, \tau_{r-1}]} \big(\psi_{r-1}(t) - \|e_{r-1}(t)\|\big) \right\}$$

and

$$L_{r-1} \le \frac{\lambda_{r-1}^2}{4\varepsilon_{r-1}}\gamma - M_F - \sup_{t\in[0,\omega)} \|y_{\text{ref}}^{(r)}(t)\| - \sum_{i=0}^{r-2} M_{i,r-1-i}. \tag{3.2.10}$$

We show that

$$\forall t \in (0,\omega):\ \psi_{r-1}(t) - \|e_{r-1}(t)\| \ge \varepsilon_{r-1}.$$

By definition of ε_{r-1} this holds on $(0,\tau_{r-1}]$. Seeking a contradiction suppose that

$$\exists t_{r-1,1} \in [\tau_{r-1},\omega):\ \psi_{r-1}(t_{r-1,1}) - \|e_{r-1}(t_{r-1,1})\| < \varepsilon_{r-1}.$$

Define

$$t_{r-1,0} = \max\{\ t \in [\tau_{r-1}, t_{r-1,1})\ \mid\ \psi_{r-1}(t) - \|e_{r-1}(t)\| = \varepsilon_{r-1}\ \},$$

then, for all $t \in [t_{r-1,0}, t_{r-1,1}]$, we have that

$$\begin{aligned} \psi_{r-1}(t) - \|e_{r-1}(t)\| &\le \varepsilon_{r-1}, \\ \|e_{r-1}(t)\| \ge \psi_{r-1}(t) - \varepsilon_{r-1} &\ge \frac{\lambda_{r-1}}{2}, \\ k_{r-1}(t) = \frac{1}{1-\varphi_{r-1}^2(t)\|e_{r-1}(t)\|^2} \ge \frac{\psi_{r-1}(t)}{2\varepsilon_{r-1}} &\ge \frac{\lambda_{r-1}}{2\varepsilon_{r-1}}. \end{aligned}$$

We obtain, for all $t \in [t_{r-1,0}, t_{r-1,1}]$, that

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|e_{r-1}(t)\|^2 = e_{r-1}^\top(t)\dot e_{r-1}(t) \\ &= e_{r-1}^\top(t)\Big(f\big(d(t),T(x)(t)\big) - k_{r-1}(t)\Gamma\big(d(t),T(x)(t)\big)e_{r-1}(t) \\ &\quad - y_{\text{ref}}^{(r)}(t) + \sum_{i=0}^{r-2}\left(\frac{d}{dt}\right)^{r-1-i}[k_i(t)e_i(t)]\Big) \\ &\le \left(M_F - \frac{\lambda_{r-1}^2}{4\varepsilon_{r-1}}\gamma + \sup_{t\in(0,\omega)}\|y_{\text{ref}}^{(r)}(t)\| + \sum_{i=0}^{r-2} M_{i,r-1-i}\right)\|e_{r-1}(t)\| \\ &\overset{(3.2.10)}{\le} -L_{r-1}\|e_{r-1}(t)\|, \end{aligned}$$

and therefore,

$$\begin{aligned} \|e_{r-1}(t_{r-1,1})\| - \|e_{r-1}(t_{r-1,0})\| &= \int_{t_{r-1,0}}^{t_{r-1,1}} \frac{1}{2}\|e_{r-1}(t)\|^{-1}\frac{d}{dt}\|e_{r-1}(t)\|^2\,\mathrm{d}t \\ &\le -L_{r-1}(t_{r-1,1}-t_{r-1,0}) \\ &\le -|\psi_{r-1}(t_{r-1,1}) - \psi_{r-1}(t_{r-1,0})| \\ &\le \psi_{r-1}(t_{r-1,1}) - \psi_{r-1}(t_{r-1,0}), \end{aligned}$$

and thus we obtain

$$\varepsilon_{r-1} = \psi_{r-1}(t_{r-1,0}) - \|e_{r-1}(t_{r-1,0})\| \le \psi_{r-1}(t_{r-1,1}) - \|e_{r-1}(t_{r-1,1})\| < \varepsilon_{r-1},$$

a contradiction.

Step 3: We show that $\omega = \infty$. Assume that $\omega < \infty$. Then, since e_i, k_i, $i = 0,\dots,r-1$ are bounded by Step 2, it follows that the closure of the graph of $(e_0, e_1, \dots, e_{r-1})$ is a compact subset of $\mathscr{D}_{r-1}$, a contradiction. Hence $\omega = \infty$ which shows (i). Statements (ii) and (iii) are then immediate consequences of Step 2. □

Remark 3.2.4. Note that it follows from Theorem 3.2.3 that the funnel controller (3.2.1) solves the *Prescribed Performance Control Problem* as formulated for the system class in [85]. Furthermore, the funnel controller (3.2.1) is of much lower complexity than the controller proposed in [85].

In the following we derive explicit formulas for the ε_i appearing in (3.2.4) and bounds for the input u and the derivatives $e^{(i)}$ of the tracking error. We use the notation and assumptions from Theorem 3.2.3. For simplicity we assume that we have "finite" funnel boundaries, i.e., $\varphi_i(0) > 0$ for $i = 0,\dots,r-1$.

For all $i \in \{0,\dots,r-1\}$, set $\psi_i(t) := \varphi_i(t)^{-1}$ for all $t \geq 0$ and $\lambda_i := \inf_{t\geq 0} \psi_i(t) > 0$. Since $\dot{\varphi}_i$ is bounded and $\liminf_{t\to\infty} \varphi_i(t) > 0$ we find that $\dot{\psi}_i$ is bounded and hence there exists a Lipschitz bound $L_i > 0$ of ψ_i. For $i = 0,\dots,r-2$ set

$$\varepsilon_i := \min\left\{ \frac{\lambda_i^2}{4(L_i + \|\psi_{i+1}\|_\infty)}, \frac{\lambda_i}{2}, \psi_i(0) - \|e_i(0)\| \right\}.$$

Then ε_i satisfies (3.2.8) and hence $\psi_i(t) - \|e_i(t)\| \geq \varepsilon_i$ for all $t > 0$ and $i = 0,\dots,r-2$ as shown in the proof of Theorem 3.2.3.

For $i = r-1$ we first need to define the following constants in an iterative way. Set $N_{i,0} := \|\psi_i\|_\infty$ for $i = 0,\dots,r-1$ and

$$K_{i,0} := \frac{N_{i,0}}{\varepsilon_i}, \qquad M_{i,0} := N_{i,0} \cdot K_{i,0}$$

for $i = 0,\dots,r-2$. Therefore, (3.2.9) holds for $i = 0,\dots,r-2$ and $j = 0$ since

$$\begin{aligned} k_i(t) = \frac{1}{(1-\varphi_i(t)\|e_i(t)\|)(1+\varphi_i(t)\|e_i(t)\|)} \leq \frac{1}{1-\varphi_i(t)\|e_i(t)\|} &= \frac{\psi_i(t)}{\psi_i(t) - \|e_i(t)\|} \\ &\leq \frac{\psi_i(t)}{\varepsilon_i}, \qquad t \geq 0. \end{aligned}$$

Define, for $i = 0,\dots,r-2$ and $j = 0,\dots,r-i-1$

$$\begin{aligned}
N_{i,j} &:= N_{i+1,j-1} + M_{i,j-1}, \\
L_{i,0} &:= N_{i,0}^2, \\
L_{i,j} &:= 2\sum_{l=0}^{j-1} \binom{j-1}{l} N_{i,l} \cdot N_{i,j-l}, \\
\Phi_{i,0} &:= \|\varphi_i\|_\infty^2, \\
\Phi_{i,j} &:= 2\sum_{l=0}^{j-1} \binom{j-1}{l} \|\varphi_i^{(l)}\|_\infty \cdot \|\varphi_i^{(j-l)}\|_\infty, \\
\Sigma_{i,j} &:= \frac{1}{2}\left(\Phi_{i,0} \cdot L_{i,j+1} + \Phi_{i,1} \cdot L_{i,j} + \Phi_{i,j} \cdot L_{i,1} + L_{i,0} \cdot \Phi_{i,j+1}\right) \\
&\quad + \sum_{l_1=1}^{j-1} \binom{j}{l_1} \left(\Phi_{i,l_1} \sum_{l_2=0}^{j-l_1} \binom{j-l_1}{l_2} N_{i,l_2} \cdot N_{i,j-l_1-l_2} + L_{i,j-l_1} \sum_{l_2=0}^{l_1} \binom{l_1}{l_2} \|\varphi_i^{(l_2)}\|_\infty \cdot \|\varphi_i^{(l_1-l_2)}\|_\infty \right), \\
K_{i,j} &:= K_{i,0}^2 \cdot \Sigma_{i,j-1} + \sum_{l_1=1}^{j-1} \binom{j-1}{l_1} \Sigma_{i,j-l_1-1} \left(\sum_{l_2=0}^{l_1-1} \binom{l_1-1}{l_2} K_{i,l_2+1} \cdot K_{i,l_1-l_2-1} \right), \\
M_{i,j} &:= \sum_{l=0}^{j} \binom{j}{l} K_{i,l} \cdot N_{i,j-l}.
\end{aligned}$$

Then cumbersome but straightforward calculations show that the above defined constants $N_{i,j}$, $K_{i,j}$, $M_{i,j}$ satisfy (3.2.9). Set

$$\hat{K}_{-1} := 0, \quad \hat{K}_i := \sum_{j=0}^{i} M_{j,i-j} \quad \text{for } i = 0, \ldots, r-2.$$

Using the notation from the proof of Theorem 3.2.3 we see that any maximal solution $y : [-h, \infty) \to \mathbb{R}^m$ of (3.1.1), (3.2.1) satisfies

$$y^{(i)}(t) = e_i(t) + y_{\text{ref}}^{(i)}(t) - K_{i-1}\big(t, e_0(t), \ldots, e_{i-1}(t)\big), \quad t \geq 0.$$

Therefore, using (3.2.7), it follows that

$$\|e^{(i)}(t)\| \leq \psi_i(t) + \hat{K}_{i-1}, \quad t \geq 0,$$

and

$$\|y^{(i)}\|_\infty \leq \|\psi_i\|_\infty + \|y_{\text{ref}}^{(i)}\|_\infty + \hat{K}_{i-1}$$

for $i = 0, \ldots, r-1$. Define the compact set

$$\mathscr{B} := \left\{ \zeta \in \mathscr{C}([-h, \infty) \to \mathbb{R}^m)^r \;\middle|\; \|\zeta_i\|_\infty \leq \|\psi_{i-1}\|_\infty + \|y_{\text{ref}}^{(i-1)}\|_\infty + \hat{K}_{i-2},\ i = 1, \ldots, r \right\},$$

and

$$M_1 := \sup_{\zeta \in \mathscr{B}} \|T(\zeta)\|_\infty.$$

With this we may set

$$M_F := \sup\{\, \|f(\delta, z)\| \mid \|z\| \leq M_1, \wedge \|\delta\| \leq \|d\|_\infty \,\}.$$

Furthermore, let the set $\mathscr{M}$ be as in Step 2b of the proof of Theorem 3.2.3 and set

$$\gamma := \min_{(\delta, \eta, e) \in \mathscr{M}} e^\top \Gamma(\delta, \eta) e > 0.$$

Now we are in the position to define

$$\varepsilon_{r-1} := \min \left\{ \frac{\gamma \cdot \lambda_{r-1}^2}{4\left(L_{r-1} + M_F + \|y_{\text{ref}}^{(r)}\|_\infty + \sum_{i=0}^{r-2} M_{i,r-i-1}\right)}, \frac{\lambda_{r-1}}{2}, \psi_{r-1}(0) - \|e_{r-1}(0)\| \right\},$$

and

$$u_{\text{bd}} := \frac{\|\psi_{r-1}\|_\infty^2}{\varepsilon_{r-1}},$$

which is an upper bound for the input u in the closed-loop system as can be concluded from the proof of Theorem 3.2.3. Then ε_{r-1} satisfies (3.2.10) and $\varepsilon_{r-1} \leq \frac{\lambda_{r-1}}{2}$ and hence $\psi_{r-1}(t) - \|e_{r-1}(t)\| \geq \varepsilon_{r-1}$ for all $t > 0$. We may now also extend the definitions of the constants $K_{i,0}$, $M_{i,0}$ to $i = r-1$; in particular, $K_{r-1,0} := \frac{N_{r-1,0}}{\varepsilon_{r-1}}$ is a bound for k_{r-1}. We summarize our findings in the following result.

Proposition 3.2.5. *Use the notation and assumptions from Theorem 3.2.3 and assume that $\varphi_i(0) > 0$ for $i = 0, \ldots, r-1$. Then the following statements are true for any maximal solution $y : [-h, \infty) \to \mathbb{R}^m$ of* (3.2.1),(3.1.1)*:*

1. (3.2.4) *holds with $\varepsilon_0, \ldots, \varepsilon_{r-1}$ as defined above,*

2. $k_i(t) \le K_{i,0}$ *and* $\|e^{(i)}(t)\| \le \varphi_i(t)^{-1} + \hat{K}_{i-1}$ *for all* $t \ge 0$ *and all* $i = 0,\dots,r-1$,

3. $\|u\|_\infty \le u_{\text{bd}}$.

Remark 3.2.6.

(i) Proposition 3.2.5 may be exploited for the design of suitable funnel functions $\varphi_0,\dots,\varphi_{r-1}$ in the presence of control constraints in the following way: If a bound $\hat{u}$ is given so that the desired control $u(\cdot)$ (of the form as in (3.2.1)) must satisfy $\|u(t)\| \le \hat{u}$ for all $t \ge 0$, then, if possible, $\varphi_0,\dots,\varphi_{r-1}$ must be chosen such that $u_{\text{bd}} \le \hat{u}$. Of course, there is a minimum feasibility requirement on the control depending on the system parameters, i.e., a lower bound for u_{bd}. For instance, if $r = 1$ and we choose $\varphi_0 = \psi_0^{-1}$ to be constant and assume that $\|e(0)\| \le \dfrac{\psi_0}{2}$, then $L_0 = 0$,

$$\varepsilon_0 = \frac{\gamma \psi_0^2}{4 \max\left\{\frac{\psi_0}{2}, M_F + \|\dot{y}_{\text{ref}}\|_\infty\right\}},$$

$$M_F = M_F(\psi_0) = \sup\left\{ \|f(\delta,z)\| \;\middle|\; \|\delta\| \le \|d\|_\infty \wedge \|z\| \le \sup_{\|\zeta\|_\infty \le \psi_0 + \|y_{\text{ref}}\|_\infty} \|T(\zeta)\|_\infty \right\}$$

and hence

$$u_{\text{bd}} = \frac{\psi_0^2}{\varepsilon_0} = \frac{4\max\left\{\frac{\psi_0}{2}, M_F(\psi_0) + \|\dot{y}_{\text{ref}}\|_\infty\right\}}{\gamma} \underset{\psi_0 \to 0}{\searrow} \frac{4\left(M_F^* + \|\dot{y}_{\text{ref}}\|_\infty\right)}{\gamma},$$

where

$$M_F^* := \sup\left\{ \|f(\delta,z)\| \;\middle|\; \|z\| \le \sup_{\|\zeta\|_\infty \le \|y_{\text{ref}}\|_\infty} \|T(\zeta)\|_\infty \wedge \|\delta\| \le \|d\|_\infty \right\}.$$

Obviously, $\dfrac{\psi_0}{2}$ is monotonically increasing in ψ_0 and $M_F(\psi_0)$ is monotonically non-increasing in ψ_0, thus in the choice of ψ_0 there is trade-off between these two quantities.

(ii) In recent result, Berger[8] has succesful improved the estimate for the bound of tracking errors derivatives in Proposition 3.2.5. Indeed, refer to [8, Sec.4], $\psi_i : \mathbb{R}_{\ge 0} \to \mathbb{R}_{>0}$ is continuously differentiable and $\dot{\psi}_i$ is bounded for $i = 0,\dots,r-1$. Set $\varepsilon_i(t)$ is the solution of following initial value problems

$$\begin{aligned} \dot{\varepsilon}_i(t) &= \dot{\psi}_i(t) - \psi_{i+1}(t) + \frac{\psi_i(t)(\psi_i(t) - \varepsilon_i(t))}{2\varepsilon_i(t)}, \\ \varepsilon_i(0) &= \psi_i(0) - \|e_i(0)\|, \end{aligned} \tag{3.2.11}$$

for $i = 0,\dots,r-2$. In [8, Sec.4], Berger shows that (3.2.11) has a unique global solution $\varepsilon_i : \mathbb{R}_{\ge 0} \to \mathbb{R}$ satisfying

$$\varepsilon_{i,\min} \le \varepsilon_i(t) \le \psi_i(t) - \varepsilon_{i,\max} \text{ for all } t \ge 0,$$

with

$$\begin{aligned}
\lambda_i &:= \inf_{t\geq 0} \psi_i(t) > 0, \quad i = 0,\dots,r-2,\\
\kappa_i &:= \|\psi_{i+1} - \dot\psi_i\|_\infty, \quad i = 0,\dots,r-2,\\
\varepsilon_{i,\min} &:= \min\left\{\frac{\lambda_i^2}{2\kappa_i + \|\psi_i\|_\infty}, \psi_i(0) - \|e_i(0)\|\right\} > 0,\\
\varepsilon_{i,\max} &:= \min\left\{\frac{\lambda_{i+1}\lambda_i}{\|\psi_i\|_\infty}, \frac{\lambda_i}{2}, \|e_i(0)\|\right\} \leq 0.
\end{aligned}$$

Furthermore,

$$\begin{aligned}
\|e_i(t)\| &\leq \psi_i(t) - \varepsilon_i(t),\\
k_i(t) = \frac{1}{1-\varphi_i^2(t)\|e_i(t)\|^2} &\leq \frac{\psi_i(t)}{\varepsilon_i(t)},
\end{aligned} \tag{3.2.12}$$

for $i = 0,\dots,r-2$, $t \geq 0$.
Therefore, the estimate of $\|e^{(i)}(t)\|$ for $i = 0,\dots,r-2$ in Proposition 3.2.5 can be improved by time varying function $\varepsilon_i(t)$; and still ensuring that $\varepsilon_i(\cdot)$ can be calculated a priori. For instance, if $r \geq 4$, we have a priori estimation of $\ddot e(t)$. Now we calculate $\ddot e(t)$ from (3.2.1).

$$\begin{aligned}
\ddot e(t) &= e_2(t) - (k_0(t) + k_1(t))e_1(t) + k_0(t)^2 e(t)\\
&\quad - k_0(t)^2\Big[2\varphi_0(t)\dot\varphi_0(t)\|e(t)\|^2 + \varphi_0(t)^2 e(t)^\top (e_1(t) - k_0(t)e(t))\Big]\, e(t).
\end{aligned}$$

In conjunction with (3.2.12), we have a priori estimation of $\ddot e(t)$

$$\begin{aligned}
\|\ddot e(t)\| &\leq \Big(\psi_2(t) - \varepsilon_2(t)\Big)\\
&\quad + \left[\frac{\psi_0(t)}{\varepsilon_0(t)} + \frac{\psi_1(t)}{\varepsilon_1(t)} + 2\Big(\frac{\psi_0(t)}{\varepsilon_0(t)}\Big)^2 \varphi_0(t)^2\Big(\psi_0(t)-\varepsilon_0(t)\Big)^2\right]\Big(\psi_1(t)-\varepsilon_1(t)\Big)\\
&\quad + \left[\Big(\frac{\psi_0(t)}{\varepsilon_0(t)}\Big)^2 + 2\Big(\frac{\psi_0(t)}{\varepsilon_0(t)}\Big)^3 \varphi_0(t)^2\Big(\psi_0(t)-\varepsilon_0(t)\Big)^2\right]\Big(\psi_0(t)-\varepsilon_0(t)\Big)\\
&\quad + 2\Big(\frac{\psi_0(t)}{\varepsilon_0(t)}\Big)^2 |\dot\varphi_0(t)|\Big(\psi_0(t)-\varepsilon_0(t)\Big)^2.
\end{aligned} \tag{3.2.13}$$

for all $t \geq 0$.
We note that ε_i depends only on ψ_i, ψ_{i+1} and $\|e_i(0)\|$. For the typical funnel functions, it is supposed that $\lim_{t\to\infty} \psi_i(t) = \lambda_i > 0$. Hence, for very large times t, we may have $\psi_i(t) \approx \lambda_i$, $\psi_{i+1}(t) \approx \lambda_{i+1}$, and $\dot\psi_i(t) \approx 0$. Using (3.2.11), we can approximate $\varepsilon_i(t)$ by the solution of differential equation

$$\dot x(t) = -\lambda_{i+1} + \frac{\lambda_i(\lambda_i - x(t))}{2x(t)}.$$

This implies that $\varepsilon_i(t) \approx \dfrac{\lambda_i^2}{2\lambda_{i+1} + \lambda_i}$. As a result, we have

$$\frac{\psi_i(t)}{\varepsilon_i(t)} \approx \frac{2\lambda_{i+1}}{\lambda_i} + 1.$$

Therefore, for very large times t, and $\lambda_{i+1} \le \beta\lambda_i$ with $\beta > 0$, we can choose priori a constant $\delta > 0$ such that $\dfrac{\psi_i(t)}{\varepsilon_i(t)} < \delta$ for $i = 0,\dots,r-2$. As a consequence, if functions θ_i are given such that $\|e^{(i)}(t)\| \le \theta_i(t)$ must be satisfied for all $t \ge 0$ and all $i = 0,\dots,r-1$, then it always posible to choose $\psi_0,\dots,\psi_{r-1}$ sufficiently small to achieve this.

Remark 3.2.7. We recall the linear differential-algebraic system with positive strict relative degree $r \in \mathbb{N}$ mentioned in Remark 3.1.1.

$$\begin{aligned} E\dot{x}(t) &= Ax(t) + Bu(t), \quad x(0) = x^0 \in \mathbb{R}^n \\ y(t) &= Cx(t), \end{aligned}$$

where $E,A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{m\times n}$ and $\Gamma = \lim_{s\to\infty} s^r C(sE-A)^{-1}B \in \mathbb{R}^{m\times m}$ is positive (negative) definite. We will show that this kind of systems can be applied funnel controller (3.2.1) with the condiction (3.2.2) is substituted by

$$\boxed{\begin{aligned} & y_{\text{ref}} \in \mathscr{W}^{n,\infty}(\mathbb{R}_{\ge 0} \to \mathbb{R}^m), \\ & \varphi_0 \in \Phi_n,\ \ \varphi_1 \in \Phi_{n-1},\ \dots,\ \ \varphi_{r-1} \in \Phi_{n-r+1}. \end{aligned}} \tag{3.2.14}$$

Proposition 3.2.8. *Let $[E,A,B,C] \in \Sigma_{n,n,m,m}$ be regular, has positive strict relative degree $r \in \mathbb{N}$, positive (negative) definite high frequency gain matrix $\Gamma = \lim_{s\to\infty} s^r C(sE-A)^{-1}B$, and asymptotically stable zero dynamics. For funnel functions φ_i, $i = 0,\dots,r-1$, reference signal y_{ref} as in* (3.2.14)*, and any consistent initial value $x^0 \in \mathbb{R}^n$ such that $e_0,\dots,e_{r-1}$ as defined in* (3.2.1) *fulfill*

$$\varphi_i(0)\|e_i(0)\| < 1 \text{ for } i = 0,\dots,r-1.$$

Then the application of the funnel controller (3.2.1) *to* (3.1.2) *yields an initial value problem, which has a solution, and every maximal solution $x : [0,\omega) \to \mathbb{R}^n$, $\omega \in (0,\infty]$, has the following properties:*

(i) The solution is global (i.e., $\omega = \infty$).

(ii) The input $u : \mathbb{R}_{\ge 0} \to \mathbb{R}^m$, the gain functions $k_0,\dots,k_{r-1} : \mathbb{R}_{\ge 0} \to \mathbb{R}$ and $x : \mathbb{R}_{\ge 0} \to \mathbb{R}^n$ are bounded.

(iii) The functions $e_0,\dots,e_{r-1} : \mathbb{R}_{\ge 0} \to \mathbb{R}^m$ evolve in their respective performance funnels and are uniformly bounded away from the funnel boundaries in the following sense:

$$\forall\, i = 0,\dots,r-1\ \exists\, \varepsilon_i > 0\ \forall\, t > 0 : \|e_i(t)\| \le \varphi_i(t)^{-1} - \varepsilon_i.$$

In particular, the error $e(t) = Cx(t) - y_{\text{ref}}(t)$ evolves in the funnel $\mathscr{F}_{\varphi_0}$ as in (1.1.8) *and stays uniformly away from its boundary.*

Proof. Without loss of generality, we may consider $[E,A,B,C]$ in the form

$$\begin{aligned} y^{(r)}(t) &= \sum_{i=1}^{r} R_i y^{(i-1)}(t) + S\eta(t) + \Gamma u(t), && y(0) = Cx^0, \\ \dot{\eta}(t) &= Py(t) + Q\eta(t), && \eta(0) = \eta^0 \in \mathbb{R}^\mu, \\ x_c(t) &= -\sum_{i=0}^{\nu-1} N_c^i B_c u^{(i)}(t), && \\ x_{\bar{c}}(t) &= 0, && \end{aligned}$$

where ν is the index of $sE-A$, $n_c, n_{\bar{c}} \in \mathbb{N}_0$, $\mu = n - n_c - n_{\bar{c}} - rm$, $R_i \in \mathbb{R}^{m\times m}$ for $i = 1,2,\dots,r$, $S \in \mathbb{R}^{m\times\mu}$, $P \in \mathbb{R}^{\mu\times m}$, $B_c \in \mathbb{R}^{n_c\times m}$, $N_c \in \mathbb{R}^{n_c\times n_c}$ is nilpotent with index ν, and rank $\begin{bmatrix} N_c & B_c \end{bmatrix} = n_c$. $Q \in \mathbb{R}^{\mu\times\mu}$ is a Hurwitz matrix.
Ignoring the last two algebraic equation of that form, the claim in $(i),(iii)$, and the boundedness of u, $k_0,\dots,k_{r-1}$ follow directly from Theorem 3.2.3. It remains to show that x is a bounded function. Indeed, we have $N_c \in \mathbb{R}^{n_c\times n_c}$ is nilpotent with index ν, then $\nu \le n_c$. Hence, $\nu + r \le n$. Since $\varphi_i \in \Phi_{n-i}$, it is obtained that $u(\cdot)$ at least $(\nu-1)$ times continuously differentiable and all of these derivatives are bounded functions. Moreover,

$$\begin{aligned} x_c(t) &= -\sum_{i=0}^{\nu-1} N_c^i B_c u^{(i)}(t), \\ x_{\bar{c}}(t) &= 0. \end{aligned}$$

Therefore, x_c and $x_{\bar{c}}$ are bounded function. This implies the boundedness of x. □

3.3 Applications

3.3.1 Mass on car system

We consider an example of a mass-spring system mounted on a car from [81] to show the work of (3.2.1) in simulation. The mass $m_2[kg]$ moves on a ramp which is inclined by the angle $\alpha[rad]$ and mounted on a car with mass $m_1[kg]$, for which it is possible to control the force $u = F[N]$ acting on it, see Figure 3.2. The equations of motion for the system are given by

$$\begin{bmatrix} m_1+m_2 & m_2\cos\alpha \\ m_2\cos\alpha & m_2 \end{bmatrix} \begin{pmatrix} \ddot{x}(t) \\ \ddot{s}(t) \end{pmatrix} + \begin{pmatrix} 0 \\ ks(t)+d\dot{s}(t) \end{pmatrix} = \begin{pmatrix} u(t) \\ 0 \end{pmatrix}, \tag{3.3.1}$$

where $x[m]$ is the horizontal car position and $s[m]$ the relative position of the mass on the ramp. The constants $k[N/m]$, $d[Ns/m]$ are the coefficients of the spring and damper, resp. The output of the system is given by the horizontal position of the mass on the ramp,

$$y(t) = x(t) + s(t)\cos\alpha.$$

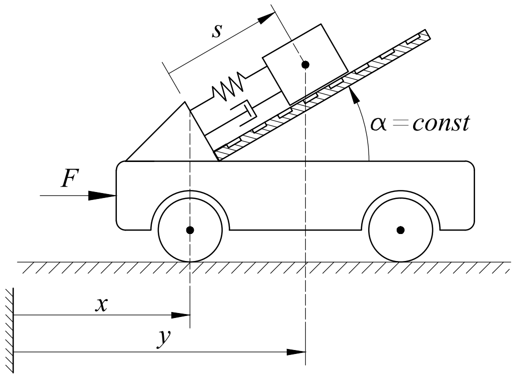

FIGURE 3.2: Mass on car system.

The reference trajectory is $y_{\text{ref}}(t) = \cos t[m]$. System (3.3.1) can be reformulated such that it belongs to the class (3.1.1), see [81], with a relative degree r depending on the angle $\alpha[rad]$ and the damping $d[Ns/m]$. We consider two cases.

Case 1: If $0 < \alpha < \frac{\pi}{2}$, see Figure 3.2, then system (3.3.1) has relative degree $r = 2$ and the high-frequency gain matrix reads $\Gamma = \frac{\sin^2\alpha}{m_1 + m_2 \sin^2\alpha} > 0$; for the simulation, we choose the parameters $m_1 = 4[kg], m_2 = 1[kg], k = 2[N/m], d = 1[Ns/m]$, the initial values $x(0) = 0$, $\dot{x}(0) = 0$, $s(0) = 0$, $\dot{s}(0) = 0$ and $\alpha = \frac{\pi}{4}$. For the controller (3.2.1) we choose the funnel functions

$$\varphi_0(t) = (5e^{-2t} + 0.1)^{-1}, \qquad \varphi_1(t) = (10e^{-2t} + 0.5)^{-1},$$

and obviously the initial errors lie within the respective funnel boundaries, i.e., (3.2.3) is satisfied, thus Theorem 3.2.3 yields that funnel control is feasible. We compare the controller (3.2.1) with the proportional-derivative funnel controller (1.1.11) proposed in [34], which has been explained in Subsection 1.1.3, and choose the same funnel functions φ_0, φ_1 for it. These functions satisfy the compatibility condition (1.1.12) and hence the controller (1.1.11) may be applied to (3.3.1) by [34].

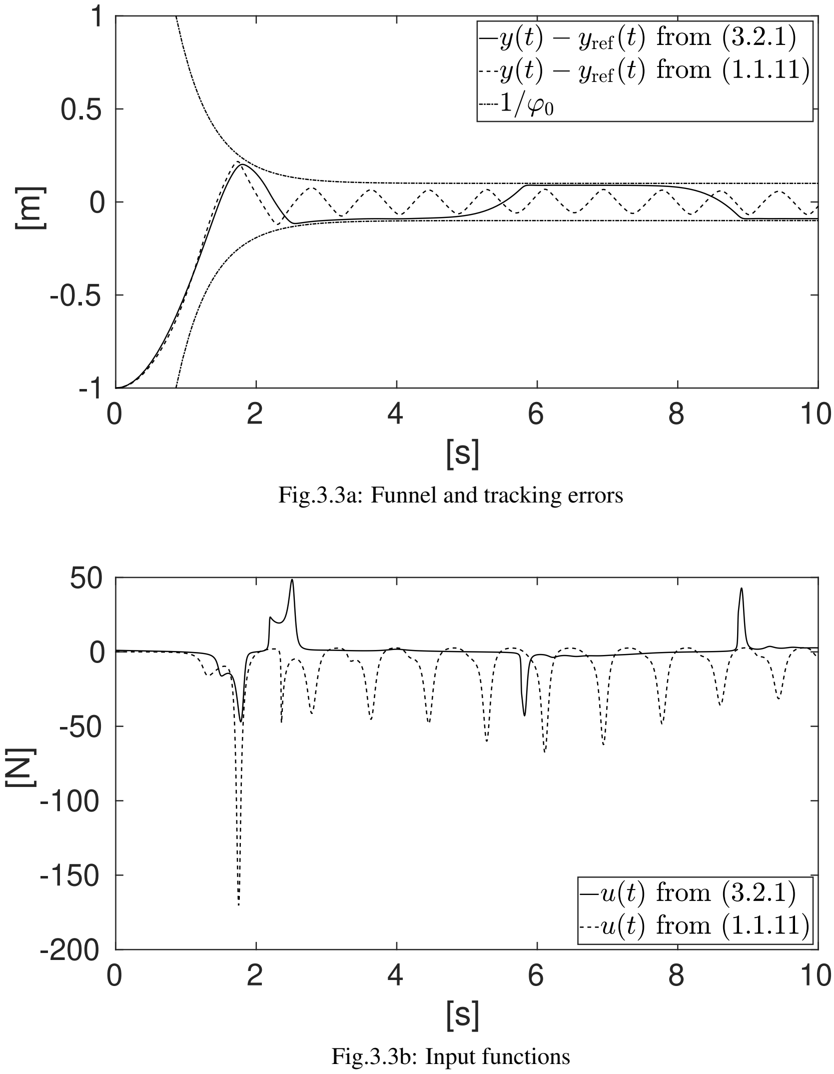

Fig.3.3a: Funnel and tracking errors

Fig.3.3b: Input functions

FIGURE 3.3: Simulation of the controllers (3.2.1) and (1.1.11) for the mass on car system (3.3.1) with $\alpha = \dfrac{\pi}{4}$.

The simulation of the controllers (3.2.1) and (1.1.11) applied to (3.3.1) over the time interval $[0, 10]$ has been performed in MATLAB (solver: `ode45`, rel. tol.: 10^{-14}, abs. tol.:

10^{-10}) and is depicted in Figure 3.3. Figure 3.3a shows the tracking errors corresponding to the two different controllers applied to the system, while Figure 3.3b shows the respective input functions generated by them. It can be seen that our proposed funnel controller (3.2.1) requires less input action than the controller (1.1.11), both in magnitude and over time. For instance, in the time interval $[3, 5.5]$ there is no input action generated by (3.2.1), but several (large) oscillations generated by (1.1.11). It seems that the controller (3.2.1) better exploits the inherent system properties and thus requires less input action than the controller proposed in [34].

Case 2: If $\alpha = 0$ and $d \neq 0$, see Figure 3.4, then system (3.3.1) has relative degree $r = 3$ and high-frequency gain matrix $\Gamma = \frac{d}{m_1 m_2} > 0$. For the simulation, we choose the parameters $m_1 = 4[kg], m_2 = 1[kg], k = 2[N/m], d = 1[Ns/m]$ and the initial values $x(0) = 0$, $\dot{x}(0) = 0$, $s(0) = 0$, $\dot{s}(0) = 0$.

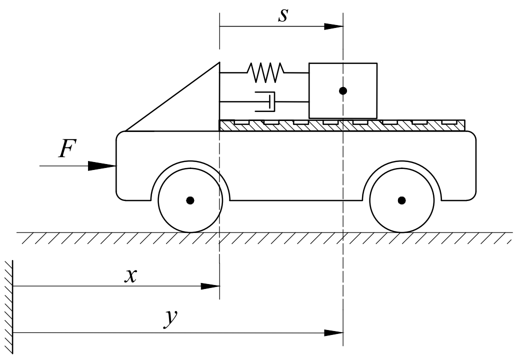

FIGURE 3.4: Mass on car system with $\alpha = 0$.

For the illustration of the controller (3.2.1) we choose the funnel functions

$$\varphi_0(t) = (5e^{-2t} + 2)^{-1}, \quad \varphi_1(t) = \varphi_2(t) = (ae^{-t} + b)^{-1}$$

with the three sets of parameter values

$$\begin{aligned} \text{C1}: &\quad a = 1.4,\ b = 0.05, \\ \text{C2}: &\quad a = 5,\ b = 0.05, \\ \text{C3}: &\quad a = 1.4,\ b = 0.5; \end{aligned}$$

the initial errors lie within the respective funnel boundaries, i.e., conditions (3.2.3) are satisfied, thus Theorem 3.2.3 yields that funnel control is feasible.

The simulation of the controller (3.2.1) with the different parameter sets C1–C3 applied to the relative degree 3 system (3.3.1) with $\alpha = 0$ over the time interval $[0, 10]$ has been performed in MATLAB (solver: `ode45`, rel. tol.: 10^{-14}, abs. tol.: 10^{-10}) and is depicted in Figure 3.5. Figure 3.5a shows the tracking errors corresponding to the different controllers applied to the system, while Figure 3.5b shows the respective input functions generated by them. The difference in the performance of the controllers is discussed in the next subsection.

We did not provide the comparison of the controller (3.2.1) with the backstepping funnel controller (1.1.10) proposed in [50] here. A simulation of (1.1.10) for the system (3.3.1) with funnel function $\varphi = \varphi_0$ is not feasible due to numerical issues, cf. the explanation in Subsection 1.1.3.

Now, we discuss the influence of the design parameters of the funnel controller (3.2.1). Of particular interest is the influence of the choice of the funnel functions φ_i in (3.2.1) on the

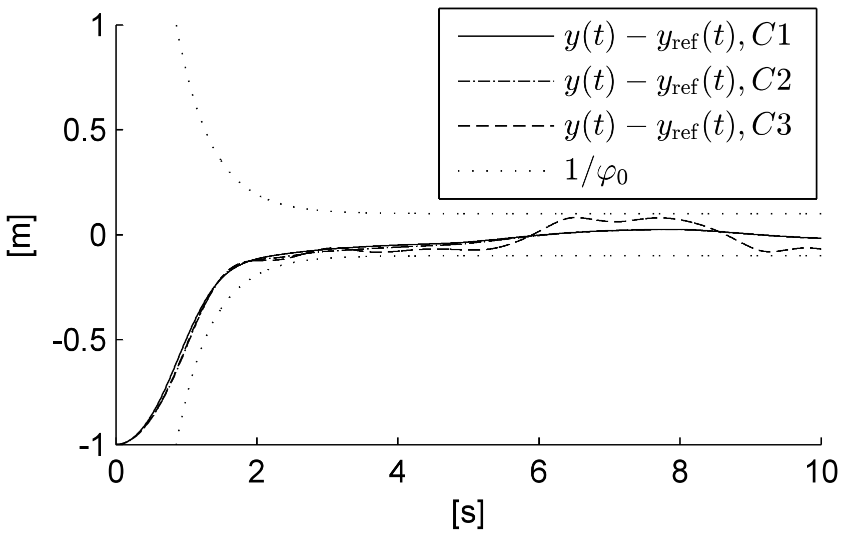

Fig.3.5a: Funnel and tracking errors

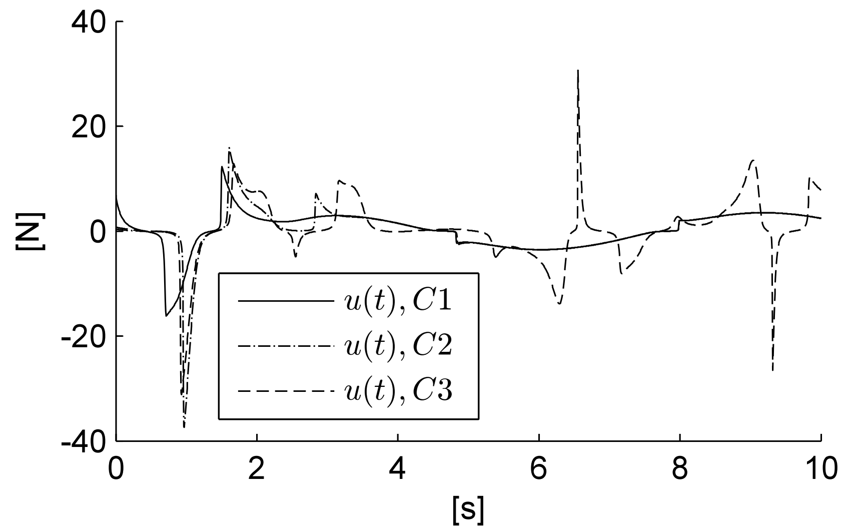

Fig.3.5b: Input functions

FIGURE 3.5: Simulation of the controllers (3.2.1) for the mass on car system (3.3.1) with $\alpha = 0$ and different sets of funnel functions φ_1, φ_2.

controller performance, that means the maximal absolute value of the input u and its oscillation behavior. We assume that the choice of φ_0 is done by the designer based on specific objectives for the transient behavior of the tracking error such as desired tracking accuracy, and the choice of φ_i is free apart from the initial conditions (3.2.3) for $i = 1, \ldots, r-1$. In principle, based on the explicit formula for u_{bd} derived in Proposition 3.2.5, a minimization of this bound over all possible funnel functions could be performed. This is a highly complicated venture left for future research. However, as a rule of thumb, we may conclude that the performance funnels $\mathscr{F}_{\varphi_i}$ corresponding to φ_i should be chosen as tight as possible, i.e., starting as close to $\|e_i(0)\|$ as possible and then decaying to a small value.

In order to illustrate this we consider Case 2 of the mass on car system (3.3.1) and discuss the resulting controller performance for the choices of parameter values C1–C3. The case C1 represents an "optimal" choice of the parameters as far as the experiments show. It can be

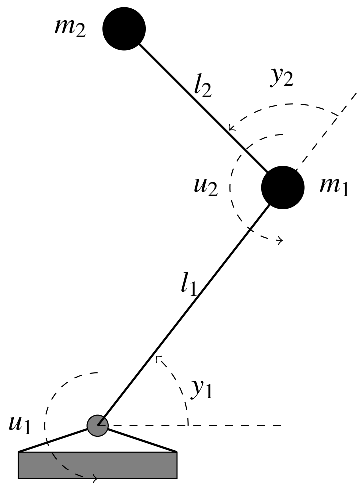

FIGURE 3.6: Planar rigid revolute joint robotic manipulator.

see in Figure 3.3b that increasing the value of a as in C2 results in a peaking behavior of the input u for small t, while increasing the value of b as in C3 leads to possible peaks at later time instants, but smaller maximal input values than in C2 in general. Furthermore, the distance of the tracking error to the funnel boundary seems to depend on the parameter b; in case C3 (for larger b), the error gets closer to the boundary than in cases C1 and C2. These observations have been confirmed in several other experiments.

In order to improve the performance of the controller and reduce unnecessary large control actions one may use alternative gain functions in (3.2.1) as discussed e.g. in [51]. For instance, using the future distance to the future funnel boundary instead of the vertical distance to the funnel boundary as in (3.2.1) may increase the ability of the controller (3.2.1) to avoid large control values. For instance, using the future distance to the future funnel boundary instead of the vertical distance to the funnel boundary as in (3.2.1) may increase the ability of the controller (3.2.1) to avoid large control values.

3.3.2 Robotic manipulator

We show the application of the funnel controller (3.2.1) for a nonlinear multi-input, multi-output system by considering an example of a robotic manipulator from [33], see also [58, p. 77], as depicted Figure 3.6. The robotic manipulator is planar, rigid, with revolute joints and has two degrees of freedom. The two joints are actuated by $u_1[Nm]$ and $u_2[Nm]$. We assume that the links are massless, have lengths $l_1[m]$ and $l_2[m]$, resp., and point masses $m_1[kg]$ and $m_2[kg]$ are attached to their ends. The two outputs are the joint angles $y_1[rad]$ and $y_2[rad]$ and the equations of motion are given by (see also [84, pp.259])

$$M(y(t))\ddot{y}(t) + C(y(t),\dot{y}(t))\dot{y}(t) + \mathbf{g}(y(t)) = u(t) \tag{3.3.2}$$

with initial value $(y(0),\dot{y}(0)) = (0,0)$, inertia matrix

$$M : \mathbb{R}^2 \to \mathbb{R}^{2\times 2},$$
$$(y_1,y_2) \mapsto M(y_1,y_2) := \begin{bmatrix} m_1 l_1^2 + m_2(l_1^2 + l_2^2 + 2l_1 l_2 \cos(y_2)) & m_2(l_2^2 + l_1 l_2 \cos(y_2)) \\ m_2(l_2^2 + l_1 l_2 \cos(y_2)) & m_2 l_2^2 \end{bmatrix}$$

centrifugal and Coriolis force matrix

$$\begin{aligned}&C:\mathbb{R}^2\times\mathbb{R}^2\to\mathbb{R}^{2\times 2},\\&(y_1,y_2,v_1,v_2)\mapsto C(y_1,y_2,v_1,v_2):=\begin{bmatrix}-2m_2l_1l_2\sin(y_2)v_1 & -m_2l_1l_2\sin(y_2)v_2\\ -m_2l_1l_2\sin(y_2)v_1 & 0\end{bmatrix},\end{aligned}$$

and gravity vector

$$\begin{aligned}&\mathbf{g}:\mathbb{R}^2\to\mathbb{R}^2,\\&(y_1,y_2)\mapsto \mathbf{g}(y_1,y_2):=g\begin{bmatrix}m_1l_1\cos(y_1)+m_2(l_1\cos(y_1)+l_2\cos(y_1+y_2))\\ m_2l_2\cos(y_1+y_2)\end{bmatrix},\end{aligned}$$

where $g=9.81[m/s^2]$ is the acceleration of gravity. If we multiply system (3.3.2) with $M(y(t))^{-1}$, which is pointwise positive definite, from the left we see that the resulting system belongs to the class (3.1.1) with $r=m=2$.

For the simulation, we choose the parameters $m_1=m_2=1[kg]$, $l_1=l_2=1[m]$ and the reference trajectories $y_{\mathrm{ref},1}(t)=\sin t[rad]$ and $y_{\mathrm{ref},2}(t)=\sin 2t[rad]$. For the controller (3.2.1) we choose the funnel functions

$$\varphi_0(t)=(e^{-2t}+0.1)^{-1},\quad \varphi_1(t)=(3e^{-2t}+0.1)^{-1}.$$

The initial errors lie within the respective funnel boundaries, i.e., conditions (3.2.3) are satisfied, thus Theorem 3.2.3 yields that funnel control is feasible. We compare the controller (3.2.1) with the funnel controller proposed in [33], that is (already fixing the gain scaling functions)

$$\begin{aligned}u(t)&=-M(y(t))\big(K_0(t)^2e(t)+K_0(t)K_1(t)\dot e(t)\big),\\K_i(t)&=\operatorname{diag}\left(\frac{1}{1-\varphi_i(t)|e_1^{(i)}(t)|},\frac{1}{1-\varphi_i(t)|e_2^{(i)}(t)|}\right),\quad i=0,1\end{aligned}\tag{3.3.3}$$

and we choose the same funnel functions φ_0,φ_1 for it. The controller (3.3.3) is a modification of (1.1.11), first introduced in [28] for single-input, single-output systems and tailored to multi-input,multi-output systems with mass matrix in [33]. We remark that there is a typo in the controller formula [33, (8)], the sign of the input u must be the opposite.

The simulation of the controllers (3.2.1) and (3.3.3) applied to (3.3.2) over the time interval [0,10] has been performed in MATLAB (solver: *ode45*, rel. tol: 10^{-14}, abs. tol: 10^{-10}) and is depicted in Figure 3.7 (tracking error components) and Figure 3.8 (input components). It can be seen that the funnel controller (3.2.1) outperforms the controller (3.3.3) as it generates a smaller maximal control action and does not "oscillate" as (3.3.3) does e.g. in the interval $[4,6]$. Moreover, we stress that the controller (3.3.3) requires knowledge of the mass matrix $M(\cdot)$ of the system (3.3.2) and is specifically constructed for systems with strict relative degree two. On the other hand, knowledge of $M(\cdot)$ is not necessary for the control strategy (3.2.1).

3.3.3 Comparison with the bang-bang funnel controller

We finally compare the funnel controller (3.2.1) with the bang-bang funnel controller (1.1.13) developed in [60] by using the same academic example also presented in [60]: a nonlinear

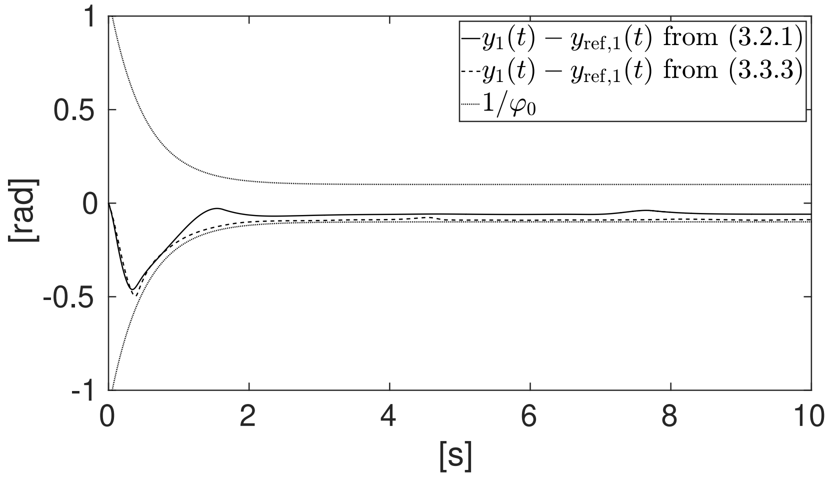

Fig. 3.7a: Funnel and first tracking error components

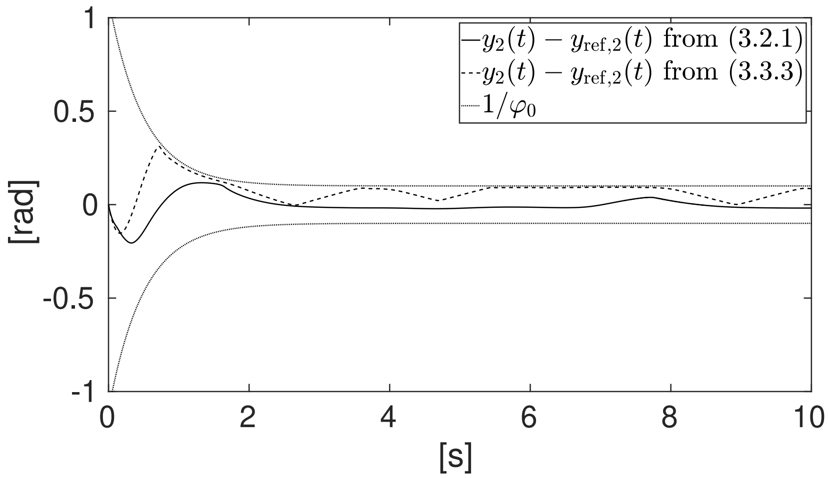

Fig. 3.7b: Funnel and second tracking error components

FIGURE 3.7: Funnel and tracking errors for the controllers (3.2.1) and (3.3.3) applied to (3.3.2).

single input, single output system with relative degree 4

$$\begin{aligned} y^{(4)}(t) &= z(t)\, y^{(3)}(t)^2 + e^{z(t)}\, u(t), \\ \dot{z}(t) &= z(t)\big(a - z(t)\big)\big(z(t)+b\big) - cy(t) \end{aligned} \tag{3.3.4}$$

with initial values

$$z(0) = 0, \quad y^{(i)}(0) = y_{\mathrm{ref}}^{(i)}(0),\ i = 0,\dots,3,$$

where we choose the reference signal $y_{\mathrm{ref}}(t) = 5\sin t$. For the simulation we choose the parameters

$$a = 0.09, \quad b = 0.05, \quad c = 0.008.$$

For the controller (3.2.1) we choose the constant funnel functions

$$\varphi_0(t) = 1, \quad \varphi_1(t) = 10, \quad \varphi_2(t) = 10, \quad \varphi_3(t) = 10.$$

The funnel φ_0 for the tracking error is the same as in [60], but apart from that we have chosen $\varphi_1, \dots, \varphi_3$ so that the corresponding performance funnels are tighter than in [60]; this is

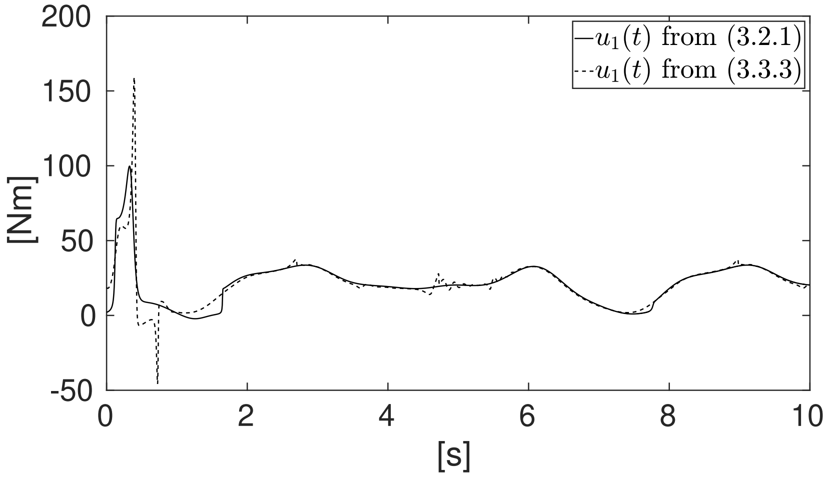

Fig. 3.8a: First input components

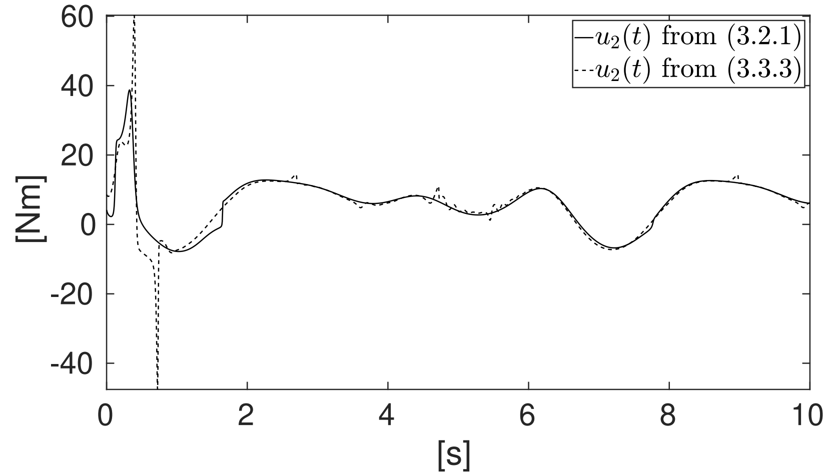

Fig. 3.8b: Second input components

FIGURE 3.8: Input functions for the controllers (3.2.1) and (3.3.3) applied to (3.3.2).

allowed in our framework, but in [60] several complicated compatibility assumptions require the funnel boundaries to be large enough. We also stress that the controller design (3.2.1) is quite different from the bang-bang funnel controller in [60].

The simulation of the controller (3.2.1) applied to (3.3.4) over the time interval [0,10] has been performed in MATLAB (solver: *ode15s*, rel. tol: 10^{-14}, abs. tol: 10^{-10}), see Figure 3.9. It can be seen that the funnel controller (3.2.1) generates a maximal control action of approximately 5, while for the bang-bang funnel controller in [60] the value is around 254. Obviously, the controller (3.2.1) achieves a better performance than the controller proposed in [60].

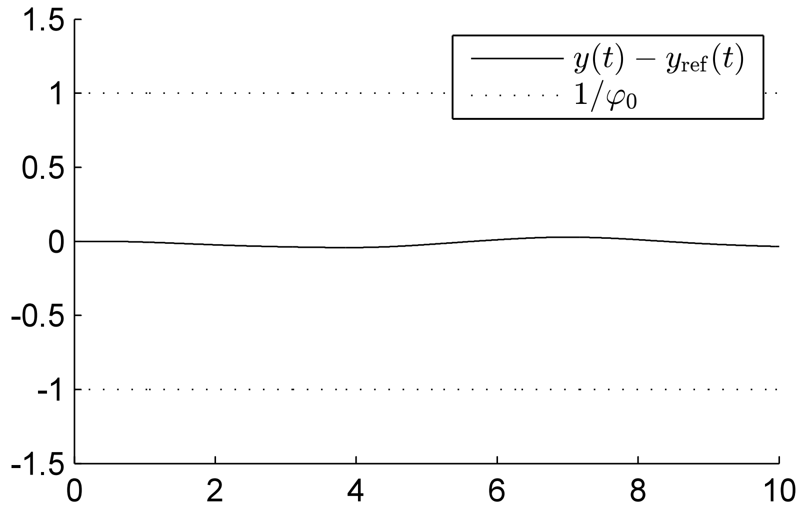

Fig. 3.9a: Funnel and tracking error

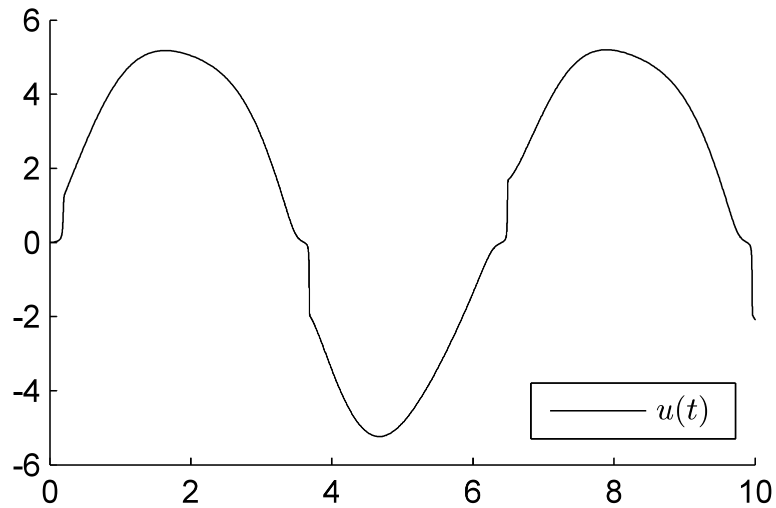

Fig. 3.9b: Input function

FIGURE 3.9: Simulation of the controller (3.2.1) for the system (3.3.4).

Chapter 4

Funnel control for systems with known generalized vector relative degree

In this chapter, we continue to develop the study of funnel control in Chapter 3 to non-linear functional differential-algebraic systems with known generalized vector relative degree $r = (r_1, \ldots, r_m)$ and input-to-state stable internal dynamics. We present a simple funnel controller which require the involvement of the first $r_i - 1$ derivatives of the each element in output errors. We further discus the feasible of this controller, and its modifications applying to some related system classes. We finally show the application of our controller to some systems by simulation.

4.1 System class

We consider a non-linear functional differential-algebraic systems of the form,

$$\begin{aligned}
\begin{pmatrix} y_1^{(r_1)}(t) \\ y_2^{(r_2)}(t) \\ \vdots \\ y_p^{(r_p)}(t) \end{pmatrix} &= f_1\Big(y_1(t), \cdots, y_1^{(r_1-1)}(t), \cdots, y_p^{(r_p-1)}(t), y_{p+1}(t), \cdots, y_m(t)\Big) \\
&\quad + f_2\Big(d_1(t), (Ty)(t)\Big) + \Gamma_I\Big(d_2(t), (Ty)(t)\Big) u_I(t), \\
0 &= f_3\Big(y_1(t), \cdots, y_1^{(r_1-1)}(t), \cdots, y_p^{(r_p-1)}(t), y_{p+1}(t), \cdots, y_m(t)\Big) \\
&\quad + f_4\Big(d_3(t), (Ty)(t)\Big) + \Gamma_{II}\Big(d_4(t), (Ty)(t)\Big) u_I(t) \\
&\quad + f_5\Big(d_5(t), (Ty)(t)\Big) u_{II}(t), \\
y|_{[-h,0]} &= y^0 = (y_1^0, y_2^0, \ldots, y_m^0), \\
y_i^0 &\in \mathscr{C}^{r_i-1}([-h,0] \to \mathbb{R}), \quad i = 1, \ldots, p, \\
y_i^0 &\in \mathscr{C}([-h,0] \to \mathbb{R}), \quad i = p+1, \ldots, m,
\end{aligned} \tag{4.1.1}$$

We now call $r = (r_1, \ldots, r_p, 0, \ldots, 0) \in \mathbb{N}_0^{1\times m}$, $r_i > 0$, $i = 1, \ldots, p$ to be the generalized vector relative degree of the systems. Denote $|r| = \sum_{i=1}^{p} r_i$, $u_I(t) = (u_1(t), \ldots, u_p(t))^\top$, $u_{II}(t) = (u_{p+1}(t), \ldots, u_m(t))^\top$.

First of all, we recall the operator class $\mathscr{T}_{m,k}$, which was introduced by [11], for operator T.

Definition 4.1.1. For $t \geq 0$, $\omega \in \mathscr{C}([-h,t] \to \mathbb{R}^m)$, $\tau > t$ and $\delta > 0$, define the following set of extensions of ω:

$$\mathscr{C}(\omega;t,\tau,\delta) := \left\{ v \in \mathscr{C}([-h,\tau] \to \mathbb{R}^m) \;\middle|\; \begin{array}{l} v|_{[-h,t]} = \omega \wedge \forall s \in [t,\tau]: \\ \|v(s) - \omega(t)\| \leq \delta \end{array} \right\}.$$

An operator $T : \mathscr{C}([-h,\infty) \to \mathbb{R}^m) \to \mathscr{C}^1(\mathbb{R}_{\geq 0} \to \mathbb{R}^k)$ is said to be of class $\mathscr{T}_{m,k}$ if and only if

(i) T is a causal operator , i.e, for all $t \geq 0$ and all $\zeta, \xi \in \mathscr{C}([-h,\infty) \to \mathbb{R}^m)$,

$$\zeta|_{[-h,t)} = \xi|_{[-h,t)} \Rightarrow T(\zeta)|_{[0,t)} \overset{a.a}{=} T(\xi)|_{[0,t)},$$

where "a.a" stands for "almost all".

(ii) T is locally Lipschitz continuous in the following sense: $\forall t \geq 0$, $\forall \omega \in \mathscr{C}([-h,t] \to \mathbb{R}^m)$, $\exists \tau > t$, $\exists \delta > 0$, $\exists c_0 > 0$ $\forall u, v \in \mathscr{C}(\omega;t,\tau,\delta)$ such that

$$\max_{s\in[t,\tau]} \|(Tu)(s) - (Tv)(s)\| \leq c_0 \max_{s\in[t,\tau]} \|u(s) - v(s)\|.$$

(iii) T maps bounded trajectories to bounded trajectories , i.e, $\forall c_1 > 0$ $\exists c_2 > 0$, $\forall v \in \mathscr{C}([-h,\infty) \to \mathbb{R}^m)$

$$\sup_{t\in[-h,\infty)} \|v(t)\| \leq c_1 \Rightarrow \sup_{t\in[0,\infty)} \|(Tv)(t)\| \leq c_2.$$

(iv) $\exists z \in \mathscr{C}(\mathbb{R}^m \times \mathbb{R}^k \to \mathbb{R}^k)$ $\exists \tilde{T} : \mathscr{C}([-h,\infty) \to \mathbb{R}^m) \to \mathscr{C}(\mathbb{R}_{\geq 0} \to \mathbb{R}^k)$ with all above properties, $\forall v \in \mathscr{C}([-h,\infty) \to \mathbb{R}^m)$ $\forall t \geq 0$:

$$\frac{d}{dt}(Tv)(t) = z(v(t), (\tilde{T}v)(t)).$$

Definition 4.1.2 (System class $\Sigma_{m,p,k,s}$)**.** The functional differential-algebraic equation (4.1.1) is said to define a system of class $\Sigma_{m,p,k,s}$ if, and only if,

(i) the gain $\Gamma_I \in \mathscr{C}^1(\mathbb{R}^s \times \mathbb{R}^k \to \mathbb{R}^{p\times p})$ takes values in the set of positive (negative) definite matrices, $\Gamma_{II} \in \mathscr{C}^1(\mathbb{R}^s \times \mathbb{R}^k \to \mathbb{R}^{(m-p)\times p})$.

(ii) the disturbances $d_1, d_2 \in \mathscr{L}^\infty(\mathbb{R}_{\geq 0} \to \mathbb{R}^s)$, and $d_3, d_4, d_5 \in \mathscr{C}^1(\mathbb{R}_{\geq 0} \to \mathbb{R}^s)$ are bounded.

(iii) $f_1 \in \mathscr{C}^1(\mathbb{R}^{|r|+m-p} \to \mathbb{R}^p)$, $f_2 \in \mathscr{C}^1(\mathbb{R}^s \times \mathbb{R}^k \to \mathbb{R}^p)$, $f_3 \in \mathscr{C}^1(\mathbb{R}^{|r|+m-p} \to \mathbb{R}^{m-p})$, $f_4 \in \mathscr{C}^1(\mathbb{R}^s \times \mathbb{R}^k \to \mathbb{R}^{m-p})$, and $f_3' \cdot \begin{bmatrix} 0 \\ I_{m-p} \end{bmatrix}$ is bounded.

(iv) $f_5 \in \mathscr{C}^1(\mathbb{R}^s \times \mathbb{R}^k \to \mathbb{R})$, and $\exists \alpha > 0, \forall (d,v) \in \mathbb{R}^s \times \mathbb{R}^k :\ f_5(d,v) \geq \alpha$.

(v) $T \in \mathscr{T}_{m,k}$.

In the next subsections, we will derive some class of systems that can be consistent with system class (4.1.1) satisfying definition 4.1.2.

4.1.1 Systems with positive vector relative degree

One of the most important subclass of $\Sigma_{m,p,k,s}$ is a class of non-linear systems described by functional ordinary differential equations. We note that from Remark 2.3.11, generalized vector relative degree and vector relative degree are exactly the same in case of ordinary differential systems. Moreover, if the vector relative degree of these systems exists, then its components are positive. Therefore, we consider the class of systems which have positive vector relative degree $r = (r_1, r_2, \ldots, r_m)$, $r_i \in \mathbb{N}$ and $r_i > 0$ for all $i = 1, \ldots, m$. Since the system does not contain the part which is respective to generalized relative degree $r_i = 0$, we consider a slightly extended class described as follows

$$\begin{aligned}
\begin{pmatrix} y_1^{(r_1)}(t) \\ y_2^{(r_2)}(t) \\ \vdots \\ y_m^{(r_m)}(t) \end{pmatrix} &= f\big(d(t), T(y_1, \ldots, y_1^{(r_1-1)}, \ldots, y_m, \ldots, y_m^{(r_m-1)})(t)\big) \\
&\quad + \Gamma\big(d(t), T(y_1, \ldots, y_1^{(r_1-1)}, \ldots, y_m, \ldots, y_m^{(r_m-1)})(t)\big)\, u(t), \\
y_i^0 &\in \mathscr{C}^{r_i-1}([-h, 0] \to \mathbb{R}), \quad i = 1, \ldots, m.
\end{aligned} \tag{4.1.2}$$

In this case, operator T only need to satisfy properties $(i)-(iii)$ in definition (4.1.1), and property (iv) can be eliminated because of disappearance of algebraic constraint in equation. Therefore, the set of available operator T can be slightly larger, which implies a larger considered system class.

Remark 4.1.3. We recall the class of linear regular differential-algebraic systems with positive vector relative degree which were introduced in Subsection 2.2.3.

$$\begin{aligned}
E\dot{x}(t) &= Ax(t) + Bu(t), \\
y(t) &= Cx(t),
\end{aligned}$$

where $A, E \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{m\times n}$. Suppose system has vector relative degree $r = (r_1, r_2, \ldots, r_m) \in \mathbb{N}^{1\times m}$, $r_i > 0$ for all $i = 1, 2, \ldots, m$, $|r| = \sum_{i=1}^{m} r_i$, and ν denotes the index of $sE - A$. We have already known in Subsection 2.2.3 that systems of this type can be transformed into *normal form*,

$$\begin{aligned}
\begin{pmatrix} y_1^{(r_1)}(t) \\ y_2^{(r_2)}(t) \\ \vdots \\ y_m^{(r_m)}(t) \end{pmatrix} &= \begin{pmatrix} \sum_{j=1}^{m}\sum_{h=1}^{r_j} R_{jh}^1 y_j^{(h-1)}(t) \\ \sum_{j=1}^{m}\sum_{h=1}^{r_j} R_{jh}^2 y_j^{(h-1)}(t) \\ \vdots \\ \sum_{j=1}^{m}\sum_{h=1}^{r_j} R_{jh}^m y_j^{(h-1)}(t) \end{pmatrix} + S\eta(t) + \Gamma u(t), \\
\dot{\eta}(t) &= \sum_{i=1}^{m} P_i y_i(t) + Q\eta(t), \\
x_c(t) &= -\sum_{i=0}^{\nu-1} N_c^i B_c u^{(i)}(t), \\
x_{\bar{c}}(t) &= 0,
\end{aligned}$$

where $n_c, n_{\bar{c}} \in \mathbb{N}_0$, $\mu = n - n_c - n_{\bar{c}} - |r|$ and $R_{jh}^i \in \mathbb{R}$ with $i, j \in \{1, \ldots, m\}$, $h \in \{1, \ldots, r_i\}$, $S \in \mathbb{R}^{m\times\mu}$, $P_i \in \mathbb{R}^{\mu}$, $Q \in \mathbb{R}^{\mu\times\mu}$, $B_c \in \mathbb{R}^{n_c\times m}$, $N_c \in \mathbb{R}^{n_c\times n_c}$ is nilpotent with index ν, and $\operatorname{rank}\begin{bmatrix} N_c & B_c \end{bmatrix} = n_c$. $\Gamma = \lim_{s\to\infty} \operatorname{diag}(s^{r_1}, \ldots, s^{r_m}) C(sE - A)^{-1} B$ is the high-gain matrix. We note that the zero dynamics of these systems are asymptotically stable if, and only if, Q is Hurwitz. Ignore the third, and fourth equations, we focus on the first and second equations of this form

$$\begin{pmatrix} y_1^{(r_1)}(t) \\ y_2^{(r_2)}(t) \\ \vdots \\ y_m^{(r_m)}(t) \end{pmatrix} = \begin{pmatrix} \sum\limits_{j=1}^{m}\sum\limits_{h=1}^{r_j} R_{jh}^1 y_j^{(h-1)}(t) \\ \sum\limits_{j=1}^{m}\sum\limits_{h=1}^{r_j} R_{jh}^2 y_j^{(h-1)}(t) \\ \vdots \\ \sum\limits_{j=1}^{m}\sum\limits_{h=1}^{r_j} R_{jh}^m y_j^{(h-1)}(t) \end{pmatrix} + S\eta(t) + \Gamma u(t),$$

$$\dot{\eta}(t) = \sum_{i=1}^{m} P_i y_i(t) + Q\eta(t).$$

It is easy to see that this subsystem is a member of type (4.1.2) with

$$f\big(d(t), T(y_1,\dots,y_1^{(r_1-1)},\dots,y_m,\dots,y_m^{(r_m-1)})(t)\big) = T(y_1,\dots,y_1^{(r_1-1)},\dots,y_m,\dots,y_m^{(r_m-1)})(t)$$

$$:= \begin{pmatrix} \sum\limits_{j=1}^{m}\sum\limits_{h=1}^{r_j} R_{jh}^1 y_j^{(h-1)}(t) \\ \sum\limits_{j=1}^{m}\sum\limits_{h=1}^{r_j} R_{jh}^2 y_j^{(h-1)}(t) \\ \vdots \\ \sum\limits_{j=1}^{m}\sum\limits_{h=1}^{r_j} R_{jh}^m y_j^{(h-1)}(t) \end{pmatrix} + Se^{Qt}\eta^0 + \int_0^t Se^{Q(t-\tau)} \sum_{i=1}^{m} P_i y_i(\tau) d\tau.$$

T is clearly causal, locally Lipschitz, and the Hurwitz property of Q implies that T has the bounded-input-bounded-output property. Note that in this case, the part of equation respecting to zero elements in general vector relative degree are disappeared. Therefore, the property (iv) in definition 4.1.1 is not necessary required for operator T. In conclusion, a funnel controller applied to system class 4.1.2 also work for linear regular differential-algebraic systems with positive vector relative degree, and asymptotically stable zero dynamics. However, we would like to stress that for the third equation

$$x_c(t) = -\sum_{i=0}^{\nu-1} N_c^i B_c u^{(i)}(t),$$

the input function $u(t)$ is required smooth enough, $u \in \mathscr{W}_{\text{loc}}^{\nu-1,1}(\mathbb{R} \to \mathbb{R}^m)$. This implies the required smoothness of input function when we apply funnel controller for linear differential-algebraic system with positive vector relative degree.

4.1.2 Linear differential-algebraic systems with generalized vector relative degree

In this subsection, another important class of system related to system class (4.1.1) will be considered. First of all, we recall some results for the general linear differential-algebraic systems which was examined in Section 2.3. In this case, we consider the invariant system with the same number of input and output as follow description,

$$\begin{aligned} E\dot{x}(t) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t) \end{aligned} \tag{4.1.3}$$

where $A, E \in \mathbb{R}^{l\times n}$, $B \in \mathbb{R}^{l\times m}$, $C \in \mathbb{R}^{m\times n}$. Suppose that (4.1.3) is right invertible, have autonomous, asymptotically stable zero dynamics, and have generalized vector relative degree $r = (r_1, \cdots, r_p, 0, \cdots, 0) \in \mathbb{N}^{1\times m}$. Refer to Theorem 2.3.16, the *normal form* of the system (4.1.3) can be obtained as follows

$$
\begin{aligned}
\dot{\eta}(t) &= Q\eta(t) + A_{12}y(t),\\
\begin{pmatrix} y_1^{(r_1)}(t) \\ y_2^{(r_2)}(t) \\ \vdots \\ y_p^{(r_p)}(t) \end{pmatrix} &= \begin{pmatrix} \sum\limits_{j=1}^{p}\sum\limits_{h=1}^{r_j} R_{jh}^1 y_j^{(h-1)}(t) + \sum\limits_{j=p+1}^{m} R_{j1}^1 y_j(t) \\ \sum\limits_{j=1}^{p}\sum\limits_{h=1}^{r_j} R_{jh}^2 y_j^{(h-1)}(t) + \sum\limits_{j=p+1}^{m} R_{j1}^2 y_j(t) \\ \vdots \\ \sum\limits_{j=1}^{p}\sum\limits_{h=1}^{r_j} R_{jh}^p y_j^{(h-1)}(t) + \sum\limits_{j=p+1}^{m} R_{j1}^p y_j(t) \end{pmatrix} + [\Gamma_{11}\ 0]A_{21}\eta(t)\\
&\qquad + [\Gamma_{11}\ 0]\begin{pmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_m(t) \end{pmatrix},\\
0 &= \begin{pmatrix} \sum\limits_{j=1}^{p}\sum\limits_{h=1}^{r_j} R_{jh}^{p+1} y_j^{(h-1)}(t) + \sum\limits_{j=p+1}^{m} R_{j1}^{p+1} y_j(t) \\ \sum\limits_{j=1}^{p}\sum\limits_{h=1}^{r_j} R_{jh}^{p+2} y_j^{(h-1)}(t) + \sum\limits_{j=p+1}^{m} R_{j1}^{p+2} y_j(t) \\ \vdots \\ \sum\limits_{j=1}^{p}\sum\limits_{h=1}^{r_j} R_{jh}^{m} y_j^{(h-1)}(t) + \sum\limits_{j=p+1}^{m} R_{j1}^{m} y_j(t) \end{pmatrix} + [\Gamma_{21}\ I_{m-p}]\,\eta(t)\\
&\qquad + [\Gamma_{21}\ I_{m-p}]\begin{pmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_m(t) \end{pmatrix},\\
x_3(t) &= \sum_{i=0}^{\nu-1} N^i E_{32} y^{(i+1)}(t).
\end{aligned}
\tag{4.1.4}
$$

where, $Q \in \mathbb{R}^{\mu\times\mu}$ is Hurwitz, $\mu = \dim\max(E, A, B; \ker C)$, $n_3 = n - \mu - m$, $N \in \mathbb{R}^{n_3\times n_3}$ is nilpotent with index $\nu \in \mathbb{N}$, i.e. $N^\nu = 0$ and $N^{\nu-1} \neq 0$, $R_{jh}^i \in \mathbb{R}$ for $i = 1, \ldots, p$, $j = 1, \ldots, p$, $h = 1, \ldots, r_j$, and $R_{j1}^i \in \mathbb{R}$ for $i = p+1, \ldots, m$, $j = p+1, \ldots, m$, and Γ_{11}, Γ_{21} from (2.3.7), E_{32}, A_{21}, A_{12} from (2.3.2) are matrices with suitable size.

We first limit the consideration in subsystem made by three first equations in (4.1.4).

$$
\begin{aligned}
\dot{\eta}(t) &= Q\eta(t) + A_{12}y(t),\\
\begin{pmatrix} y_1^{(r_1)}(t) \\ y_2^{(r_2)}(t) \\ \vdots \\ y_p^{(r_p)}(t) \end{pmatrix} &= \begin{pmatrix} \sum\limits_{j=1}^{p}\sum\limits_{h=1}^{r_j} R_{jh}^1 y_j^{(h-1)}(t) + \sum\limits_{j=p+1}^{m} R_{j1}^1 y_j(t) \\ \sum\limits_{j=1}^{p}\sum\limits_{h=1}^{r_j} R_{jh}^2 y_j^{(h-1)}(t) + \sum\limits_{j=p+1}^{m} R_{j1}^2 y_j(t) \\ \vdots \\ \sum\limits_{j=1}^{p}\sum\limits_{h=1}^{r_j} R_{jh}^p y_j^{(h-1)}(t) + \sum\limits_{j=p+1}^{m} R_{j1}^p y_j(t) \end{pmatrix} + [\Gamma_{11}\ 0]A_{21}\eta(t)\\
&\qquad + [\Gamma_{11}\ 0]\begin{pmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_m(t) \end{pmatrix},\\
0 &= \begin{pmatrix} \sum\limits_{j=1}^{p}\sum\limits_{h=1}^{r_j} R_{jh}^{p+1} y_j^{(h-1)}(t) + \sum\limits_{j=p+1}^{m} R_{j1}^{p+1} y_j(t) \\ \sum\limits_{j=1}^{p}\sum\limits_{h=1}^{r_j} R_{jh}^{p+2} y_j^{(h-1)}(t) + \sum\limits_{j=p+1}^{m} R_{j1}^{p+2} y_j(t) \\ \vdots \\ \sum\limits_{j=1}^{p}\sum\limits_{h=1}^{r_j} R_{jh}^{m} y_j^{(h-1)}(t) + \sum\limits_{j=p+1}^{m} R_{j1}^{m} y_j(t) \end{pmatrix} + [\Gamma_{21}\ I_{m-p}]A_{21}\eta(t)\\
&\qquad + [\Gamma_{21}\ I_{m-p}]\begin{pmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_m(t) \end{pmatrix},
\end{aligned}
$$

It easy to see that this subsystem should be a system of type (4.1.1) with

$$(Ty)(t) := \eta(t) = e^{Qt}\eta^0 + \int_0^t e^{Q(t-\tau)}A_{12}y(\tau)d\tau.$$

T is clearly causal, locally Lipschitz, and the Hurwitz property of Q implies that T has the bounded-input, bounded-output property, and

$$\frac{d}{dt}(Ty)(t) = Qe^{Qt}\eta^0 + A_{12}y(t) + Q\int_0^t e^{Q(t-\tau)}A_{12}y(\tau)d\tau.$$

Define,

$$(\tilde{T}y)(t) := Qe^{Qt}\eta^0 + Q\int_0^t e^{Q(t-\tau)}A_{12}y(\tau)d\tau,$$

then $\tilde{T}$ is also causal, locally Lipschitz, and maps bounded trajectories to bounded trajectories. We now set

$$z(y(t),(\tilde{T}y)(t)) := A_{12}y(t) + (\tilde{T}y)(t).$$

Then, $\frac{d}{dt}(Ty)(t) = z(y(t),(\tilde{T}y)(t))$ which implies the operator T satisfying all properties (i)-(iv) in Definition 4.1.1. Furthermore, all other functions can be specify as follows

$$f_1\Big(y_1(t),\cdots,y_1^{(r_1-1)}(t),\cdots,y_p^{(r_p-1)}(t),y_{p+1}(t),\cdots,y_m(t)\Big)$$
$$= \begin{pmatrix} \sum_{j=1}^{p}\sum_{h=1}^{r_j} R_{jh}^1 y_j^{(h-1)}(t) + \sum_{j=p+1}^{m} R_{j1}^1 y_j(t) \\ \sum_{j=1}^{p}\sum_{h=1}^{r_j} R_{jh}^2 y_j^{(h-1)}(t) + \sum_{j=p+1}^{m} R_{j1}^2 y_j(t) \\ \vdots \\ \sum_{j=1}^{p}\sum_{h=1}^{r_j} R_{jh}^p y_j^{(h-1)}(t) + \sum_{j=p+1}^{m} R_{j1}^p y_j(t) \end{pmatrix},$$

$$f_2\Big(d_1(t),(Ty)(t)\Big) = [\Gamma_{11}\ 0]A_{21}(Ty)(t),$$
$$\Gamma_I\Big(d_2(t),(Ty)(t)\Big) = \Gamma_{11},$$

$$f_3\Big(y_1(t),\cdots,y_1^{(r_1-1)}(t),\cdots,y_p^{(r_p-1)}(t),y_{p+1}(t),\cdots,y_m(t)\Big)$$
$$= \begin{pmatrix} \sum_{j=1}^{p}\sum_{h=1}^{r_j} R_{jh}^{p+1} y_j^{(h-1)}(t) + \sum_{j=p+1}^{m} R_{j1}^{p+1} y_j(t) \\ \sum_{j=1}^{p}\sum_{h=1}^{r_j} R_{jh}^{p+2} y_j^{(h-1)}(t) + \sum_{j=p+1}^{m} R_{j1}^{p+2} y_j(t) \\ \vdots \\ \sum_{j=1}^{p}\sum_{h=1}^{r_j} R_{jh}^m y_j^{(h-1)}(t) + \sum_{j=p+1}^{m} R_{j1}^m y_j(t) \end{pmatrix},$$

$$f_4\Big(d_1(t),(Ty)(t)\Big) = [\Gamma_{21}\ I_{m-p}]A_{21}(Ty)(t),$$
$$\Gamma_{II}\Big(d_4(t),(Ty)(t)\Big) = \Gamma_{21},$$
$$f_5\Big(d_5(t),(Ty)(t)\Big) = 1.$$

And the function f_3 satisfies condition (iii) in Definition (4.1.2) since

$$f_3' \cdot \begin{bmatrix} 0 \\ I_{m-p} \end{bmatrix} = \begin{bmatrix} R_{p+1,1}^{p+1} & \cdots & R_{m,1}^{p+1} \\ & \ddots & \\ R_{p+1,1}^m & \cdots & R_{p,1}^m \end{bmatrix} \overset{(2.3.11)}{=} \hat{A}_{22} \in \mathbb{R}^{(m-p)\times(m-p)}.$$

Although, the system (4.1.3) does not really belong to a subclass of system class type as (4.1.1) since the forth equation in (4.1.4) was not included. However, we want to emphasize that in equation

$$x_3(t) = \sum_{i=0}^{\nu-1} N^i E_{32} y^{(i+1)}(t),$$

the output function $y(t)$ is required smooth enough for $x_3(t)$ being well defined. Therefore, funnel controller for class systems (4.1.1) can also be applied to (4.1.3) with some minor regulation. That why in the next section, we will introduce a funnel controller not only for system class (4.1.1) but also can be applied to class of systems (4.1.2) as well as class of systems (4.1.3).

4.2 Funnel controller

More than one decade since the concept of funnel control has been introduced by Ilchmann, Ryan, and Sangwin in [48], plenty of papers in funnel control of the both sides theoretical and application were published by several authors such that [45, 34, 7], and the references therein. During several years, giving a feasible funnel controller for systems which have vector relative degree is still an open problem. In [13], we have already proposed a new funnel feedback control for systems with known strict relative degree. In that paper, feedback strategy is proved working efficiency for a wide multi-input, multi-output systems with arbitrary strict relative degree in simulation comparison with "back-stepping" funnel controller given by [49, 50] or bang bang funnel controller supposed by [60]. In particular, for multi-input, multi-output systems with strict relative degree two, funnel controller introduced by [13] is better than PD controller given by [33] for simulation to robotic manipulators.

On the other hand, funnel control problem is considered not only to the system described by ordinary differential equations, but also to the system described by differential-algebraic equations. A series of papers were introduced to solve problem of funnel control for a certain class of systems. Firstly, a funnel controller is successful created for regular systems with has proper inverse transfer function published in [10]. In the same year, in book chapter [9], the authors have also given funnel strategy for regular systems with positive strict relative degree approached via "back-stepping" funnel control. Moreover, by using the approach via systems with autonomous zero dynamics, Berger in [7] has proposed a funnel controller for general linear differential algebraic system with the input does affect at most the first derivative of the output. This result is also extended to non-linear functional differential-algebraic systems in [11]. However, a feasible funnel feedback controller for general differential-algebraic systems with known vector relative degree is still a challenge. This section address that problem by introducing a funnel controller for systems class (4.1.1). Since system (4.1.1) have generalized vector relative degree, we will build a funnel controller with tracking for each element of outputs.

$$
\boxed{\begin{array}{l}
\text{For } i = 1,\dots,p, \\
\quad \begin{array}{rclcl}
e_{i0}(t) &=& e_i(t) &=& y_i(t) - y_{\mathrm{ref},i}(t), \\
e_{i1}(t) &=& \dot e_{i0}(t) &+& k_{i0}(t)e_{i0}(t), \\
e_{i2}(t) &=& \dot e_{i1}(t) &+& k_{i1}(t)e_{i1}(t), \\
&\vdots&&& \\
e_{i,r_i-1}(t) &=& \dot e_{i,r_i-2}(t) &+& k_{i,r_i-2}(t)e_{i,r_i-2}(t),
\end{array} \\
k_{iq}(t) = \dfrac{1}{1-\varphi_{iq}^2(t)|e_{iq}(t)|^2}, \quad q = 0,\dots,r_i-2. \\
\text{For } i = p+1,\dots,m, \\
\quad e_i(t) = y_i(t) - y_{\mathrm{ref},i}(t). \\
\text{Set} \\
e_I(t) = (e_{1,r_1-1}(t),\dots,e_{p,r_p-1}(t))^\top, \qquad e_{II}(t) = (e_{p+1}(t),\dots,e_m(t))^\top, \\
k_I(t) = \dfrac{1}{1-\varphi_I^2(t)\|e_I(t)\|^2}, \qquad k_{II}(t) = \dfrac{\hat k}{1-\varphi_{II}^2(t)\|e_{II}(t)\|^2}, \\
\text{then} \\
u(t) = \begin{pmatrix} u_I(t) \\ u_{II}(t) \end{pmatrix} = \begin{cases} \begin{pmatrix} -k_I(t)e_I(t) \\ -k_{II}(t)e_{II}(t) \end{pmatrix} & \text{if } \Gamma_I \text{ is pointwise positive definite,} \\[2ex] \begin{pmatrix} k_I(t)e_I(t) \\ -k_{II}(t)e_{II}(t) \end{pmatrix}, & \text{if } \Gamma_I \text{ is pointwise negative definite,} \end{cases}
\end{array}}
\tag{4.2.1}
$$

where the reference signal and funnel functions have the following properties:

$$
\boxed{\begin{array}{c}
y_{\mathrm{ref}} = (y_{\mathrm{ref},1},\dots,y_{\mathrm{ref},m}), \quad y_{\mathrm{ref},i} \in \mathscr{W}^{r_i,\infty}(\mathbb{R}_{\ge 0} \to \mathbb{R}), \\
\varphi_I, \varphi_{II} \in \Phi_1, \varphi_{i0} \in \Phi_{r_i}, \varphi_{i1} \in \Phi_{r_i-1}, \dots, \varphi_{i,r_i-2} \in \Phi_2, i = 1,\dots,p,
\end{array}}
\tag{4.2.2}
$$

and $\hat k$ satisfies a following condition

$$
\hat k > \alpha^{-1} \sup_{Y \in \mathbb{R}^{|r|+m-p}} \left\| f_3'(Y) \begin{bmatrix} 0 \\ I_{m-p} \end{bmatrix} \right\|. \tag{4.2.3}
$$

Remarks 4.2.1.

- Similar to the case of systems with strict relative degree in previous chapter, we need to consider the existence of solution of the initial value problem resulting from the application of the funnel controller (4.2.1) to a system (4.1.1). A solution of (4.1.1), (4.2.1) on $[-h,\omega)$ is a function $y(t) = (y_1(t),\dots,y_m(t))$, $y_i \in \mathscr{C}^{r_i-1}([-h,\omega) \to \mathbb{R})$, with $i = 1,\dots,p$, and $y_i \in \mathscr{C}([-h,\omega] \to \mathbb{R})$, with $i = p+1,\dots,m$ for some $\omega \in (0,\infty]$, and $y|_{[-h,0]} = y^0$ such that $y_i^{r_i-1}|_{[0,\omega)}$ with $i = 1,\dots,p$, and $y_i|_{[0,\omega)}$ with $i = p+1,\dots,m$ are absolutely continuous and satisfies the differential-algebraic equation in (4.1.1) with $u(t)$ defined in (4.2.1) for almost all $t \in [0,\infty)$. $y(t)$ is called *maximal*, if it has no right extension that is also a solution.

- In conjunction with (iii) in Definition 4.1.2, the condition (4.2.3) is essential for the solvability of the closed loop system (4.1.1), (4.2.1). For more detail, the condition (4.2.3) guarantees the invertibility of $\alpha\hat k I_{m-p} - f_3'(Y)\begin{bmatrix} 0 \\ I_{m-p} \end{bmatrix}$. This property is crucial for the explicit solution of the algebraic constrain in the closed loop system (4.1.1), (4.2.1) since it ensures the index-1 property of the system.

- In the case of system subclass (4.1.2), it can be seen that the system does not contain the part which is respective to generalized relative degree $r_i = 0$. This implies $\Gamma = \Gamma_I$ and the funnel controller, therefore, (4.2.1) does not contain second part, $-k_{II}(t)e_{II}(t)$. Hence, the funnel controller for these systems can be simplified as follow

$$
\boxed{\begin{array}{l}
\text{For } i = 1, \dots, m, \\
\quad \begin{array}{rclcl}
e_{i0}(t) & = & e_i(t) & = & y_i(t) - y_{\mathrm{ref},i}(t), \\
e_{i1}(t) & = & \dot{e}_{i0}(t) & + & k_{i0}(t)e_{i0}(t), \\
e_{i2}(t) & = & \dot{e}_{i1}(t) & + & k_{i1}(t)e_{i1}(t), \\
 & \vdots & & & \\
e_{i,r_i-1}(t) & = & \dot{e}_{i,r_i-2}(t) & + & k_{i,r_i-2}(t)e_{i,r_i-2}(t),
\end{array} \\
k_{iq}(t) = \dfrac{1}{1-\varphi_{iq}^2(t)|e_{iq}(t)|^2}, \quad q = 0, \dots, r_i - 2. \\
\text{Set} \\
\bar{e}(t) = (e_{1,r_1-1}(t), \dots, e_{m,r_m-1}(t))^\top, \\
\bar{k}(t) = \dfrac{1}{1-\varphi^2(t)\|\bar{e}(t)\|^2}, \\
\text{then} \\
u(t) = \begin{cases} -\bar{k}(t)\cdot\bar{e}(t), & \text{if } \Gamma \text{ is pointwise positive definite,} \\ \bar{k}(t)\cdot\bar{e}(t), & \text{if } \Gamma \text{ is pointwise negative definite.} \end{cases}
\end{array}}
\tag{4.2.4}
$$

- In the case of linear differential-algebraic system having proper inverse transfer function of type (2.2.10). Using normal form (2.2.11), we see that the system does not contain the part which relate to generalized relative degree $r_i > 0$. This implies $p = 0$, and the funnel controller, therefore, (4.2.1) does not contain the first part $\pm k_I(t)e_I(t)$. Hence, the funnel controller for these systems can be simplified as follow

$$
\begin{aligned}
e(t) &= y(t) - y_{\mathrm{ref}}(t), \\
k(t) &= \frac{\hat{k}}{1-\varphi^2(t)\|e(t)\|^2}, \\
u(t) &= -k(t)e(t).
\end{aligned}
$$

This controller become the funnel controller introduced in [10] for this type of system class. In [7], Berger has successful applied this funnel controller for the system class which was mentioned in Remarks 2.3.11.

Remark 4.2.2. Let (4.1.1) be a system of $\Sigma_{m,p,k,s}$, the reference signal and funnel functions be as in (4.2.2). By causality of operator $T \in \mathscr{T}_{m,k}$ there exists $j : \mathbb{R}^m \to \mathbb{R}^k$ such that

$$\forall v \in \mathscr{C}([-h,\infty) \to \mathbb{R}^m) : (Tv)(0) = j(v(0)).$$

Hence, an initial value

$$
\begin{aligned}
&y^0 = (y_1^0, y_2^0, \dots, y_m^0), \\
&y_i^0 \in \mathscr{C}^{r_i-1}([-h,0] \to \mathbb{R}),\ i = 1, \dots, p, \\
&y_i^0 \in \mathscr{C}([-h,0] \to \mathbb{R}),\ i = p+1, \dots, m,
\end{aligned}
\tag{4.2.5}
$$

is called *consistent* for the closed loop system (4.1.1), (4.2.1), if

$$f_3\left(y_1^0(0),\ldots,\left(\frac{d}{dt}\right)^{r_1-1}y_1^0(0),\ldots,\left(\frac{d}{dt}\right)^{r_p-1}y_p^0(0)\right)+f_4\Big(d_3(0),j(y^0(0))\Big)$$
$$+\Gamma_{II}\Big(d_4(0),j(y^0(0))\Big)u_I(0)+f_5\Big(d_5(0),j(y^0(0))\Big)u_{II}(0)=0,$$

where $u_I(0),u_{II}(0)$ are defined by (4.2.1) with a note that $e_i(0)=y_i^0(0)-y_{\mathrm{ref},i}(0)$.

4.2.1 Feasible funnel controller

We show the feasibility of the funnel controller (4.2.1) for the class $\Sigma_{m,p,k,s}$ with generalized vector relative degree as defined in Definition 4.1.2.

Theorem 4.2.3. *Consider system* (4.1.1) *belonging to* $\Sigma_{m,p,k,s}$. *Let* $\varphi_I,\varphi_{II},\varphi_{iq}$, $i=1,\ldots,p$, $q=0,\ldots,r_i-2$ *from* (4.2.2) *defining performance funnel. Then for any reference signal* y_{ref} *satisfying* (4.2.2), *any consistent initial value* $y|_{[-h,0]}=y^0$ *as in* (4.2.5) *such that* e_I,e_{II},e_{iq}, $i=1,\ldots,p$, $q=0,\ldots,r_i-2$ *defined in* (4.2.1) *fulfill*

$$\begin{aligned}\varphi_I(0)\|e_I(0)\|&<1,\\ \varphi_{II}(0)\|e_{II}(0)\|&<1,\\ \varphi_{iq}(0)|e_{iq}(0)|&<1,\quad i=1,\ldots,p,\ q=0,\ldots,r_i-2,\end{aligned}\tag{4.2.6}$$

and $\hat{k}>0$ *be such that* (4.2.3) *is satisfied. Then the application of the funnel controller* (4.2.1) *to system* (4.1.1) *yields a closed-loop initial value problem that has a solution and every solution can be extended to a global solution. Furthermore, for every global solution* $y(\cdot)$,

i) The input $u:\mathbb{R}_{\geq 0}\to\mathbb{R}^m$, *the gain functions* $k_I,k_{II},k_{iq}:\mathbb{R}_{\geq 0}\to\mathbb{R}$, $i=1,\ldots,p$, $q=0,\ldots,r_i-2$ *are bounded.*

ii) The functions $e_I:\mathbb{R}_{\geq 0}\to\mathbb{R}^p$, $e_{II}:\mathbb{R}_{\geq 0}\to\mathbb{R}^{m-p}$ *and* $e_{iq}:\mathbb{R}_{\geq 0}\to\mathbb{R}$, $i=1,\ldots,p$, $q=0,\ldots,r_i-2$ *evolve in their respective performance funnels, i.e.,*

$$\begin{aligned}&(t,e_I)\in\mathscr{F}_{\varphi_I};\ (t,e_{II})\in\mathscr{F}_{\varphi_{II}};\ (t,e_{iq})\in\mathscr{F}_{\varphi_{iq}};\\ &\text{for all } i=1,\ldots,p,\ q=0,\ldots,r_i-2 \text{ and } t\geq 0.\end{aligned}$$

Furthermore, the signals $e_I(\cdot),e_{II}(\cdot),e_{iq}(\cdot)$ *are uniformly bounded away from the funnel boundaries in the following sense:*

$$\begin{aligned}\exists\,\varepsilon_I>0\ \forall t>0:\|e_I(t)\|&\leq\varphi_I(t)^{-1}-\varepsilon_I,\\ \exists\,\varepsilon_{II}>0\ \forall t>0:\|e_{II}(t)\|&\leq\varphi_{II}(t)^{-1}-\varepsilon_{II},\\ \forall i=1,\ldots,p,\ q=0,\ldots,r_i-2,\exists\,\varepsilon_{iq}>0\ \forall t>0:|e_{iq}(t)|&\leq\varphi_{iq}(t)^{-1}-\varepsilon_{iq}.\end{aligned}\tag{4.2.7}$$

In particular, each error component $e_i(t)=y_i-y_{\mathrm{ref},i}(t)$ *evolves in the funnel* $\mathscr{F}_{\varphi_{i0}}$, *with* $i=1,\ldots,p$, *or* $\mathscr{F}_{\varphi_{II}}$, *with* $i=p+1,\ldots,m$, *and stays uniformly away from its boundary.*

Proof. Without loss of generality, we may assume that the matrix function Γ_I of system (4.1.1) is point-wisely positive definite. We prove this theorem by several steps.
Step 1: We show that a maximal solution $y:[-h,\omega)\to\mathbb{R}^m$, $\omega\in(0,\infty]$, of closed-loop system

(4.1.1), (4.2.1) exists,

$$
\begin{aligned}
\begin{pmatrix} y_1^{(r_1)}(t) \\ \vdots \\ y_p^{(r_p)}(t) \end{pmatrix} &= f_1(y_1(t),\ldots,y_1^{(r_1-1)}(t),\ldots,y_p^{(r_p-1)}(t),y_{p+1}(t),\ldots,y_m(t)) \\
&\qquad + f_2(d_1(t),(Ty)(t)) - \Gamma_I(d_2(t),(Ty)(t))k_I(t)e_I(t), \\
0 &= f_3(y_1(t),\ldots,y_1^{(r_1-1)}(t),\ldots,y_p^{(r_p-1)}(t),y_{p+1}(t),\ldots,y_m(t)) \\
&\qquad + f_4(d_3(t),(Ty)(t)) - \Gamma_{II}(d_4(t),(Ty)(t))k_I(t)e_I(t) \\
&\qquad - f_5(d_5(t),(Ty)(t))k_{II}(t)e_{II}(t), \\
y|_{[-h,0]} &= y^0 = (y_1^0,y_2^0,\ldots,y_m^0), \\
y_i^0 &\in \mathscr{C}^{r_i-1}([-h,0]\to\mathbb{R}), \quad i=1,\ldots,p, \\
y_i^0 &\in \mathscr{C}([-h,0]\to\mathbb{R}), \quad i=p+1,\ldots,m.
\end{aligned}
\tag{4.2.8}
$$

Step 1a: Define, for $i=1,\ldots,p$, and $q=0,\ldots,r_i-2$, the sets

$$
\mathscr{D}_{iq} := \left\{ (t,e_{i0},\ldots,e_{iq}) \in \mathbb{R}_{\geq 0}\times\mathbb{R}\times\cdots\times\mathbb{R} \;\middle|\; (t,e_{ij}) \in \mathscr{F}_{\varphi_{ij}}, j=0,\ldots,q \right\},
$$

where $\mathscr{F}_{\varphi_{ij}}$ is as in (4.2.2), and the functions $K_{iq}:\mathscr{D}_{iq}\to\mathbb{R}$ recursively by

$$
\begin{aligned}
K_{i0}(t,e_{i0}) &:= \frac{e_{i0}}{1-\varphi_{i0}^2(t)|e_{i0}|^2}, \\
K_{iq}(t,e_{i0},\ldots,e_{iq}) &:= \frac{e_{iq}}{1-\varphi_{iq}^2(t)|e_{iq}|^2} + \frac{\partial K_{i,q-1}}{\partial t}(t,e_{i0},\ldots,e_{i,q-1}) \\
&\qquad + \sum_{j=0}^{q-1} \frac{\partial K_{i,q-1}}{\partial e_{ij}}(t,e_{i0},\ldots,e_{i,q-1}) \left(e_{i,j+1} - \frac{e_{ij}}{1-\varphi_{ij}^2(t)|e_{ij}|^2} \right).
\end{aligned}
$$

Now set

$$
\begin{aligned}
\mathscr{D}_I &:= \left\{ (t,e_{10},\ldots,e_{1,r_1-1},\ldots,e_{p,r_p-1}) \in \mathbb{R}_{\geq 0}\times\mathbb{R}^{|r|} \;\middle|\; \begin{array}{r} (t,e_{i0},\ldots,e_{i,r_i-2}) \in \mathscr{D}_{i,r_i-2}, \\ (t,e_I) \in \mathscr{F}_{\varphi_I}, \end{array} \right\}, \\
\mathscr{D}_{II} &:= \left\{ (t,e_{p+1},\ldots,e_m) \in \mathbb{R}_{\geq 0}\times\mathbb{R}^{m-p} \;\middle|\; (t,e_{II}) \in \mathscr{F}_{\varphi_I} \right\}, \\
\mathscr{D} &:= \left\{ (t,e_{10},\ldots,e_{1,r_1-1},\ldots,e_{p,r_p-1},e_{p+1},\ldots,e_m) \;\middle|\; \begin{array}{r} (t,e_{10},\ldots,e_{1,r_1-1},\ldots,e_{p,r_p-1}) \in \mathscr{D}_I, \\ (t,e_{p+1},\ldots,e_m) \in \mathscr{D}_{II}, \end{array} \right\}.
\end{aligned}
$$

Choose some interval $I\subseteq\mathbb{R}_{\geq 0}$ with $0\in I$ and let $(e_{10},\ldots,e_{1,r_1-1},\ldots,e_{p,r_p-1}):I\to\mathbb{R}^{|r|}$ be such that, for all $t\in I$, $\big(t,e_{10}(t),\ldots,e_{1,r_1-1}(t),\ldots,e_{p,r_p-1}(t)\big)\in\mathscr{D}_I$, $(t,e_{p+1}(t),\ldots,e_m(t))\in\mathscr{D}_{II}$ and $(e_{i0},\ldots,e_{i,r_i-1})$, $i=1,\ldots,p$ satisfies the relations in (4.2.1). Then $e_i=e_{i0}$ satisfies, on the interval I,

$$
e_i^{(q)} = e_{iq} - \sum_{j=0}^{q-1} \left(\frac{d}{dt}\right)^{q-1-j} k_{ij}e_{ij}, \quad q=1,\ldots,r_i-1. \tag{4.2.9}
$$

Step 1b: We show by induction that for all $i=1,\ldots,p$, and $q=0,\ldots,r_i-2$ we have

$$
\forall\, t\in I:\ \sum_{j=0}^{q} \left(\frac{d}{dt}\right)^{q-j} \Big(k_{ij}(t)e_{ij}(t)\Big) = K_{iq}\Big(t,e_{i0}(t),\ldots,e_{iq}(t)\Big). \tag{4.2.10}
$$

Equation (4.2.10) is obviously true for $q=0$. Assume that $q \in \{1,\dots,r_i-2\}$ and the statement holds for $q-1$. Then

$$\begin{aligned}\sum_{j=0}^{q}\left(\frac{d}{dt}\right)^{q-j}\Big(k_{ij}(t)e_{ij}(t)\Big) &= k_{iq}(t)e_{iq}(t)+\frac{d}{dt}\left(\sum_{j=0}^{q-1}\left(\frac{d}{dt}\right)^{q-j-1}\Big(k_{ij}(t)e_{ij}(t)\Big)\right)\\ &= k_{iq}(t)e_{iq}(t)+\frac{d}{dt}K_{i,q-1}\Big(t,e_{i0}(t),\dots,e_{i,q-1}(t)\Big)\\ &= K_{iq}\Big(t,e_{i0}(t),\dots,e_{iq}(t)\Big).\end{aligned}$$

Therefore, invoking (4.2.9), we have

$$e_i^{(q)} = e_{iq} - K_{i,q-1}\Big(t,e_{i0}(t),\dots,e_{i,q-1}(t)\Big),\quad q=1,\dots,r_i-1. \tag{4.2.11}$$

In particular,

$$e_i^{(r_i-1)} = e_{i,r_i-1} - K_{i,r_i-2}\Big(t,e_{i0}(t),\dots,e_{i,r_i-2}(t)\Big).$$

Then

$$e_i^{(r_i)} = \dot e_{i,r_i-1} - \frac{d}{dt}K_{i,r_i-2}\Big(t,e_{i0}(t),\dots,e_{i,r_i-2}(t)\Big).$$

Step 1c: Define, for $i=1,\dots,p$,

$$\tilde K_{i0}:\mathbb{R}_{\ge 0}\times\mathbb{R}\to\mathbb{R},\ (t,y_{i0})\mapsto y_{i0}-y_{\mathrm{ref},i}(t)$$

and the set

$$\tilde{\mathscr{D}}_{i0} := \left\{\ (t,y_i)\in\mathbb{R}_{\ge 0}\times\mathbb{R}\ \middle|\ \big(t,\tilde K_{i0}(t,y_i)\big)\in\mathscr{D}_{i0}\ \right\}.$$

Furthermore, recursively define for $q=1,\dots,r_i-1$ the maps

$$\begin{aligned}&\tilde K_{iq}:\tilde{\mathscr{D}}_{i,q-1}\times\mathbb{R}\to\mathbb{R},\\ &\quad (t,y_{i0},\dots,y_{iq})\mapsto y_{iq}-y_{\mathrm{ref},i}^{(q)}(t)+K_{i,q-1}\big(t,\tilde K_{i0}(t,y_{i0}),\dots,\tilde K_{i,q-1}(t,y_{i0},\dots,y_{i,q-1})\big).\end{aligned}$$

We also define the sets, for $q=1,\dots,r_i-1$,

$$\tilde{\mathscr{D}}_{iq} := \left\{\ (t,y_{i0},\dots,y_{iq})\in\tilde{\mathscr{D}}_{i,q-1}\times\mathbb{R}\ \middle|\ \big(t,\tilde K_{i0}(t,y_{i0}),\dots,\tilde K_{iq}(t,y_{i0},\dots,y_{iq})\big)\in\mathscr{D}_{iq}\ \right\},$$

and

$$\tilde{\mathscr{D}}_I := \left\{\ (t,y_{10},\dots,y_{1,r_1-1},\dots,y_{p,r_p-1})\ \middle|\ \begin{array}{r}(t,y_{i0},\dots,y_{i,r_i-1})\in\tilde{\mathscr{D}}_{i,r_i-2}\times\mathbb{R},\\ \big(t,\tilde K_{10},\dots,\tilde K_{1,r_1-1},\dots,\tilde K_{p,r_p-1}\big)\in\mathscr{D}_I.\end{array}\right\}$$

Recall (4.2.11), we have, for all $t\in I$, $i=1,\dots,p$, and $q=0,\dots,r_i-1$,

$$e_{iq}(t)=\tilde K_{iq}(t,y_i(t),\dots,y_i^{(q)}(t)).$$

Moreover,

$$\begin{aligned}e_I(t) &= \Big(\tilde K_{1,r_1-1}(t,y_1(t),\dots,y_1^{(r_1-1)}(t)),\dots,\tilde K_{p,r_p-1}(t,y_p(t),\dots,y_p^{(r_p-1)}(t))\Big)^\top,\\ &:= \tilde K_I(t,y_1(t),\dots,y_1^{(r_1-1)}(t)\dots,y_p^{(r_p-1)}(t)).\end{aligned}$$

Define,

$$\tilde{\mathscr{D}}_{II} := \left\{ (t, y_{p+1}, \dots, y_m) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^{m-p} \;\middle|\; (t, y_{p+1} - y_{\text{ref},p+1}, \dots, y_m - y_{\text{ref},m}) \in \mathscr{D}_{II} \right\},$$

and denote,

$$\begin{aligned} X_I &= \left(y_1, \dots, y_1^{(r_1-1)} \dots, y_p^{(r_p-1)}\right)^\top, \\ X_{II} &= (y_{p+1}, \dots, y_m)^\top, \\ X_{\text{ref},II} &= (y_{\text{ref},p+1}, \dots, y_{\text{ref},m})^\top. \end{aligned}$$

Then

$$e_{II}(t) = X_{II}(t) - X_{\text{ref},II}(t).$$

The feedback u in (4.2.1) reads

$$u(t) = \begin{pmatrix} \dfrac{-\tilde{K}_I(t, X_I(t))}{1 - \varphi_I(t)^2 \|\tilde{K}_I(t, X_I(t))\|^2} \\ \dfrac{-\hat{k} \cdot (X_{II}(t) - X_{\text{ref},II}(t))}{1 - \varphi_{II}(t)^2 \|X_{II}(t) - X_{\text{ref},II}(t)\|^2} \end{pmatrix}, \qquad t \in I.$$

Step 1d: Now, we set

$$\begin{aligned} H_i &= \begin{pmatrix} 1 & 0 & \dots & 0 \end{pmatrix} \in \mathbb{R}^{1 \times r_i}, \quad \text{for } i = 1, \dots, p, \\ H &= \operatorname{diag}(H_1, \dots, H_p) \in \mathbb{R}^{p \times |r|}, \\ S &= \begin{bmatrix} H & 0 \\ 0 & I_{m-p} \end{bmatrix} \in \mathbb{R}^{m \times |r| + m - p}, \end{aligned} \tag{4.2.12}$$

then,

$$S \begin{pmatrix} X_I \\ X_{II} \end{pmatrix} = y.$$

We define an operator $T_1 : \mathscr{C}([-h, \infty) \to \mathbb{R}^{|r|+m-p}) \to \mathscr{C}^1(\mathbb{R}_{\geq 0} \to \mathbb{R}^k)$ such that

$$T_1 \begin{pmatrix} X_I \\ X_{II} \end{pmatrix} (t) = T \left(S \begin{pmatrix} X_I \\ X_{II} \end{pmatrix} \right) (t).$$

T_1 is causal, localy Lipschitz and maps bounded trajectories to bounded trajectories since the properties of operator T. Set

$$\tilde{\mathscr{D}} := \left\{ (t, X_I, X_{II}) \;\middle|\; (t, X_I) \in \tilde{\mathscr{D}}_I \text{ and } (t, X_{II}) \in \tilde{\mathscr{D}}_{II} \right\}.$$

We rewrite f_1, f_2, and Γ_I from system (4.1.1) in vector form

$$f_1 = \begin{pmatrix} f_1^1 \\ \vdots \\ f_1^p \end{pmatrix} \text{ with } f_1^i : \mathscr{C}^1(\mathbb{R}^{|r|+m-p} \to \mathbb{R}) \qquad i = 1, \dots, p,$$

$$f_2 = \begin{pmatrix} f_2^1 \\ \vdots \\ f_2^p \end{pmatrix} \text{ with } f_2^i : \mathscr{C}^1(\mathbb{R}^s \times \mathbb{R}^k \to \mathbb{R}) \qquad i = 1, \dots, p,$$

$$\Gamma_I = \begin{pmatrix} \Gamma_I^1 \\ \vdots \\ \Gamma_I^p \end{pmatrix} \text{ with } \Gamma_I^i : \mathscr{C}^1(\mathbb{R}^s \times \mathbb{R}^k \to \mathbb{R}^{1\times p}) \qquad i = 1, \dots, p.$$

We now define functions

$$\begin{aligned} F_I : \tilde{\mathscr{D}} \times \mathbb{R}^k \to \mathbb{R}^{|r|}, \quad & (t, y_{10}, \dots, y_{1,r_1-1}, \dots, y_{p,r_p-1}, y_{p+1}, \dots, y_m, \eta) \\ \mapsto \Bigg(& y_{11}, \dots, f_1^1(y_{10}, \dots, y_{1,r_1-1}, \dots, y_{p,r_p-1}, y_{p+1}, \dots, y_m) + f_2^1(d_1(t), \eta) \\ & - \frac{\Gamma_I^1(d_2(t), \eta)\tilde{K}_I(t, y_{10}, \dots, y_{1,r_1-1}, \dots, y_{p,r_p-1})}{1 - \varphi_I(t)^2 \|\tilde{K}_I(t, y_{10}, \dots, y_{1,r_1-1}, \dots, y_{p,r_p-1})\|^2}, \dots \\ & \dots, f_1^p(y_{10}, \dots, y_{1,r_1-1}, \dots, y_{p,r_p-1}, y_{p+1}, \dots, y_m) + f_2^p(d_1(t), \eta) \\ & - \frac{\Gamma_I^p(d_2(t), \eta)\tilde{K}_I(t, y_{10}, \dots, y_{1,r_1-1}, \dots, y_{p,r_p-1})}{1 - \varphi_I(t)^2 \|\tilde{K}_I(t, y_{10}, \dots, y_{1,r_1-1}, \dots, y_{p,r_p-1})\|^2} \Bigg). \end{aligned}$$

$$\begin{aligned} F_{II} : \tilde{\mathscr{D}} \times \mathbb{R}^k \to \mathbb{R}^{m-p}, \quad & (t, y_{10}, \dots, y_{1,r_1-1}, \dots, y_{p,r_p-1}, y_{p+1}, \dots, y_m, \eta) \\ \mapsto \Bigg(& f_3(y_{10}, \dots, y_{1,r_1-1}, \dots, y_{p,r_p-1}, y_{p+1}, \dots, y_m) + f_4(d_3(t), \eta) \\ & - \frac{\Gamma_{II}(d_4(t), \eta)\tilde{K}_I(t, y_{10}, \dots, y_{1,r_1-1}, \dots, y_{p,r_p-1})}{1 - \varphi_I(t)^2 \|\tilde{K}_I(t, y_{10}, \dots, y_{1,r_1-1}, \dots, y_{p,r_p-1})\|^2} \\ & - f_5(d_5(t), \eta) \frac{\hat{k} \cdot ((y_{p+1}, \dots, y_m) - (y_{\text{ref},p+1}(t), \dots, y_{\text{ref},m}(t)))}{1 - \varphi_{II}(t)^2 \|(y_{p+1}, \dots, y_m) - (y_{\text{ref},p+1}(t), \dots, y_{\text{ref},m}(t))\|^2} \Bigg). \end{aligned}$$

Then the closed-loop system (4.2.8) can be rewritten as

$$\begin{aligned} \dot{X}_I(t) &= F_I\left(t, \begin{pmatrix} X_I(t) \\ X_{II}(t) \end{pmatrix}, T_1 \begin{pmatrix} X_I \\ X_{II} \end{pmatrix}(t)\right), \\ 0 &= F_{II}\left(t, \begin{pmatrix} X_I(t) \\ X_{II}(t) \end{pmatrix}, T_1 \begin{pmatrix} X_I \\ X_{II} \end{pmatrix}(t)\right). \end{aligned} \tag{4.2.13}$$

The equation (4.2.13) is an index-1 differential-algebraic equation. We need to differentiate the algebraic constrain to obtain an ordinary differential equation in all system variable, the solution of which satisfies the algebraic constraint.

Step 1e: We note that $\frac{d}{dt}(Ty)(t) = z(y(t), (\tilde{T}y)(t))$, then define an operator $T_2 : \mathscr{C}([-h, \infty) \to$

$\mathbb{R}^{|r|+m-p}) \to \mathscr{C}(\mathbb{R}_{\geq 0} \to \mathbb{R}^k)$ such that

$$T_2 \begin{pmatrix} X_I \\ X_{II} \end{pmatrix} (t) = \tilde{T}\left(S \begin{pmatrix} X_I \\ X_{II} \end{pmatrix} \right)(t),$$

where S as in (4.2.12). T_2 is causal, localy Lipschitz and maps bounded trajectories to bounded trajectories since the properties of operator $\tilde{T}$. Furthermore,

$$u_I(t) = \frac{-\tilde{K}_I(t, X_I(t))}{1 - \varphi_I(t)^2 \|\tilde{K}_I(t, X_I(t))\|^2},$$
$$u_{II}(t) = \frac{-\hat{k} \cdot (X_{II}(t) - X_{\text{ref},II}(t))}{1 - \varphi_{II}(t)^2 \|X_{II}(t) - X_{\text{ref},II}(t)\|^2}$$

Then

$$\begin{aligned} \frac{d}{dt} u_I(t) = \frac{-1}{(1 - \varphi_I(t)^2 \|\tilde{K}_I(t, X_I(t))\|^2)^2} \Bigg[\left(\frac{\partial \tilde{K}_I(t, X_I(t))}{\partial t} + \frac{\partial \tilde{K}_I(t, X_I(t))}{\partial X_I} \dot{X}_I(t) \right) \times \\ \times (1 - \varphi_I(t)^2 \|\tilde{K}_I(t, X_I(t))\|^2) + 2\tilde{K}_I(t, X_I(t)) \Bigg(\varphi_I(t) \dot{\varphi}_I(t) \|\tilde{K}_I(t, X_I(t))\|^2 + \\ + \varphi_I(t)^2 \tilde{K}_I(t, X_I(t))^\top \left(\frac{\partial \tilde{K}_I(t, X_I(t))}{\partial t} + \frac{\partial \tilde{K}_I(t, X_I(t))}{\partial X_I} \dot{X}_I(t) \right) \Bigg) \Bigg], \end{aligned} \tag{4.2.14}$$

$$\begin{aligned} \frac{d}{dt} u_{II}(t) = \frac{-\hat{k}}{(1 - \varphi_{II}(t)^2 \|X_{II}(t) - X_{\text{ref},II}(t)\|^2)^2} \Bigg[(\dot{X}_{II}(t) - \dot{X}_{\text{ref},II}(t))(1 - \varphi_{II}(t)^2 \times \\ \times \|X_{II}(t) - X_{\text{ref},II}(t)\|^2) + 2(X_{II}(t) - X_{\text{ref},II}(t)) \Big(\varphi_{II}(t) \dot{\varphi}_{II}(t) \|X_{II}(t) - X_{\text{ref},II}(t)\|^2 + \\ + \varphi_{II}(t)^2 (X_{II}(t) - X_{\text{ref},II}(t))^\top (\dot{X}_{II}(t) - \dot{X}_{\text{ref},II}(t)) \Big) \Bigg]. \end{aligned} \tag{4.2.15}$$

By differentiation of the second equation in (4.2.13), and using (4.2.14),(4.2.15), we obtain

$$
\begin{aligned}
0 = {} & \frac{\partial f_3(X_I(t),X_{II}(t))}{\partial X_I}\dot{X}_I(t) + \frac{\partial f_3(X_I(t),X_{II}(t))}{\partial X_{II}}\dot{X}_{II}(t) \\
& + f_4'\left(d_3(t), T_1\begin{pmatrix} X_I \\ X_{II}\end{pmatrix}(t)\right)\left(\genfrac{}{}{0pt}{}{\dot{d}_3(t)}{z\left(S\begin{pmatrix} X_I(t) \\ X_{II}(t)\end{pmatrix}, T_2\begin{pmatrix} X_I \\ X_{II}\end{pmatrix}(t)\right)}\right) \\
& - \Gamma_{II}'\left(d_4(t), T_1\begin{pmatrix} X_I \\ X_{II}\end{pmatrix}(t)\right)\left(\genfrac{}{}{0pt}{}{\dot{d}_4(t)}{z\left(S\begin{pmatrix} X_I(t) \\ X_{II}(t)\end{pmatrix}, T_2\begin{pmatrix} X_I \\ X_{II}\end{pmatrix}(t)\right)}\right)\frac{\tilde{K}_I(t,X_I(t))}{1-\varphi_I(t)^2\|\tilde{K}_I(t,X_I(t))\|^2} \\
& - \frac{\Gamma_{II}\left(d_4(t), T_1\begin{pmatrix} X_I \\ X_{II}\end{pmatrix}(t)\right)}{(1-\varphi_I(t)^2\|\tilde{K}_I(t,X_I(t))\|^2)^2}\Bigg[\left(\frac{\partial \tilde{K}_I(t,X_I(t))}{\partial t} + \frac{\partial \tilde{K}_I(t,X_I(t))}{\partial X_I}\dot{X}_I(t)\right)\times \\
& \times (1-\varphi_I(t)^2\|\tilde{K}_I(t,X_I(t))\|^2) + 2\tilde{K}_I(t,X_I(t))\Bigg(\varphi_I(t)\dot{\varphi}_I(t)\|\tilde{K}_I(t,X_I(t))\|^2 + \\
& + \varphi_I(t)^2\tilde{K}_I(t,X_I(t))^\top\left(\frac{\partial \tilde{K}_I(t,X_I(t))}{\partial t} + \frac{\partial \tilde{K}_I(t,X_I(t))}{\partial X_I}\dot{X}_I(t)\right)\Bigg)\Bigg] \\
& - f_5'\left(d_5(t), T_1\begin{pmatrix} X_I \\ X_{II}\end{pmatrix}(t)\right)\left(\genfrac{}{}{0pt}{}{\dot{d}_5(t)}{z\left(S\begin{pmatrix} X_I(t) \\ X_{II}(t)\end{pmatrix}, T_2\begin{pmatrix} X_I \\ X_{II}\end{pmatrix}(t)\right)}\right)\times \\
& \times \frac{\hat{k}\cdot(X_{II}(t)-X_{\mathrm{ref},II}(t))}{1-\varphi_{II}(t)^2\|X_{II}(t)-X_{\mathrm{ref},II}(t)\|^2} - \frac{\hat{k}f_5\left(d_5(t), T_1\begin{pmatrix} X_I \\ X_{II}\end{pmatrix}(t)\right)}{(1-\varphi_{II}(t)^2\|X_{II}(t)-X_{\mathrm{ref},II}(t)\|^2)^2}\times \\
& \times\Bigg[(\dot{X}_{II}(t)-\dot{X}_{\mathrm{ref},II}(t))(1-\varphi_{II}(t)^2\|X_{II}(t)-X_{\mathrm{ref},II}(t)\|^2) + 2(X_{II}(t)-X_{\mathrm{ref},II}(t))\times \\
& \times\Bigg(\varphi_{II}(t)\dot{\varphi}_{II}(t)\|X_{II}(t)-X_{\mathrm{ref},II}(t)\|^2 + \varphi_{II}(t)^2(X_{II}(t)-X_{\mathrm{ref},II}(t))^\top\times \\
& \times(\dot{X}_{II}(t)-\dot{X}_{\mathrm{ref},II}(t))\Bigg)\Bigg]. \qquad (4.2.16)
\end{aligned}
$$

Then put $\dot{X}_{II}(t)$ on the the left side, and using $\dot{X}_I(t) = F_I\left(t, \begin{pmatrix} X_I(t) \\ X_{II}(t) \end{pmatrix}, T_1\begin{pmatrix} X_I \\ X_{II} \end{pmatrix}(t)\right)$, we have

$$
\begin{aligned}
&-\left[\frac{\partial f_3(X_I(t),X_{II}(t))}{\partial X_{II}} - \frac{\hat{k} f_5\left(d_5(t), T_1\begin{pmatrix} X_I \\ X_{II} \end{pmatrix}(t)\right)}{1-\varphi_{II}(t)^2\|X_{II}(t)-X_{\mathrm{ref},II}(t)\|^2}\times\right.\\
&\qquad\left.\times\left(I_{m-p} + \frac{2\varphi_{II}(t)^2(X_{II}(t)-X_{\mathrm{ref},II}(t))(X_{II}(t)-X_{\mathrm{ref},II}(t))^\top}{1-\varphi_{II}(t)^2\|X_{II}(t)-X_{\mathrm{ref},II}(t)\|^2}\right)\right]\dot{X}_{II}(t)\\
&\qquad= \frac{\partial f_3(X_I(t),X_{II}(t))}{\partial X_I} F_I\left(t, \begin{pmatrix} X_I(t) \\ X_{II}(t) \end{pmatrix}, T_1\begin{pmatrix} X_I \\ X_{II} \end{pmatrix}(t)\right)\\
&\qquad + f_4'\left(d_3(t), T_1\begin{pmatrix} X_I \\ X_{II} \end{pmatrix}(t)\right)\begin{pmatrix} \dot{d}_3(t) \\ z\left(S\begin{pmatrix} X_I(t) \\ X_{II}(t) \end{pmatrix}, T_2\begin{pmatrix} X_I \\ X_{II} \end{pmatrix}(t)\right) \end{pmatrix}\\
&-\Gamma_{II}'\left(d_4(t), T_1\begin{pmatrix} X_I \\ X_{II} \end{pmatrix}(t)\right)\begin{pmatrix} \dot{d}_4(t) \\ z\left(S\begin{pmatrix} X_I(t) \\ X_{II}(t) \end{pmatrix}, T_2\begin{pmatrix} X_I \\ X_{II} \end{pmatrix}(t)\right) \end{pmatrix}\frac{\tilde{K}_I(t,X_I(t))}{1-\varphi_I(t)^2\|\tilde{K}_I(t,X_I(t))\|^2}\\
&-\frac{\Gamma_{II}\left(d_4(t), T_1\begin{pmatrix} X_I \\ X_{II} \end{pmatrix}(t)\right)}{(1-\varphi_I(t)^2\|\tilde{K}_I(t,X_I(t))\|^2)^2}\left[\left(\frac{\partial \tilde{K}_I(t,X_I(t))}{\partial t} + \frac{\partial \tilde{K}_I(t,X_I(t))}{\partial X_I}F_I\left(t, \begin{pmatrix} X_I(t) \\ X_{II}(t) \end{pmatrix}, T_1\begin{pmatrix} X_I \\ X_{II} \end{pmatrix}(t)\right)\right)\right.\\
&(1-\varphi_I(t)^2\|\tilde{K}_I(t,X_I(t))\|^2) + 2\tilde{K}_I(t,X_I(t))\Bigg(\varphi_I(t)\dot{\varphi}_I(t)\|\tilde{K}_I(t,X_I(t))\|^2 + \varphi_I(t)^2\tilde{K}_I(t,X_I(t))^\top\\
&\qquad\left.\left(\frac{\partial \tilde{K}_I(t,X_I(t))}{\partial t} + \frac{\partial \tilde{K}_I(t,X_I(t))}{\partial X_I}F_I\left(t, \begin{pmatrix} X_I(t) \\ X_{II}(t) \end{pmatrix}, T_1\begin{pmatrix} X_I \\ X_{II} \end{pmatrix}(t)\right)\right)\Bigg)\right]\\
&-f_5'\left(d_5(t), T_1\begin{pmatrix} X_I \\ X_{II} \end{pmatrix}(t)\right)\begin{pmatrix} \dot{d}_5(t) \\ z\left(S\begin{pmatrix} X_I(t) \\ X_{II}(t) \end{pmatrix}, T_2\begin{pmatrix} X_I \\ X_{II} \end{pmatrix}(t)\right) \end{pmatrix}\frac{\hat{k}\cdot(X_{II}(t)-X_{\mathrm{ref},II}(t))}{1-\varphi_{II}(t)^2\|X_{II}(t)-X_{\mathrm{ref},II}(t)\|^2}\\
&\quad-\frac{\hat{k} f_5\left(d_5(t), T_1\begin{pmatrix} X_I \\ X_{II} \end{pmatrix}(t)\right)}{(1-\varphi_{II}(t)^2\|X_{II}(t)-X_{\mathrm{ref},II}(t)\|^2)^2}\Bigg[-\dot{X}_{\mathrm{ref},II}(t)(1-\varphi_{II}(t)^2\|X_{II}(t)-X_{\mathrm{ref},II}(t)\|^2)+\\
&\qquad + 2(X_{II}(t)-X_{\mathrm{ref},II}(t))\Big(\varphi_{II}(t)\dot{\varphi}_{II}(t)\|X_{II}(t)-X_{\mathrm{ref},II}(t)\|^2+\\
&\qquad\qquad + \varphi_{II}(t)^2(X_{II}(t)-X_{\mathrm{ref},II}(t))^\top(-\dot{X}_{\mathrm{ref},II}(t))\Big)\Bigg]. \qquad (4.2.17)
\end{aligned}
$$

In order to get ordinary differential equation, we need to prove the matrix

$$
\begin{aligned}
\mathscr{M}(t,X_I(t),X_{II}(t)) := \frac{\partial f_3(X_I(t),X_{II}(t))}{\partial X_{II}} - \frac{\hat{k} f_5\left(d_5(t), T_1\begin{pmatrix} X_I \\ X_{II} \end{pmatrix}(t)\right)}{1-\varphi_{II}(t)^2\|X_{II}(t)-X_{\mathrm{ref},II}(t)\|^2}\times\\
\times\left(I_{m-p} + \frac{2\varphi_{II}(t)^2(X_{II}(t)-X_{\mathrm{ref},II}(t))(X_{II}(t)-X_{\mathrm{ref},II}(t))^\top}{1-\varphi_{II}(t)^2\|X_{II}(t)-X_{\mathrm{ref},II}(t)\|^2}\right) \qquad (4.2.18)
\end{aligned}
$$

is invertible for all $t \in I$.
Since, matrix

$$\mathscr{G}(t, X_{II}(t)) := \frac{2\varphi_{II}(t)^2 (X_{II}(t) - X_{\text{ref},II}(t))(X_{II}(t) - X_{\text{ref},II}(t))^\top}{1 - \varphi_{II}(t)^2 \|X_{II}(t) - X_{\text{ref},II}(t)\|^2}$$

is symmetric, and semi positive definite, then the matrix $I_{m-p} + \mathscr{G}(t, X_{II}(t))$ is invertible for all $t \in I$, and $\left\| \left(I_{m-p} + \mathscr{G}(t, X_{II}(t)) \right)^{-1} \right\| \leq 1$. Therefore, according to (4.2.3) and Definition 4.1.2(iv), we have

$$\left\| (1 - \varphi_{II}(t)^2 \|X_{II}(t) - X_{\text{ref},II}(t)\|^2) \hat{k}^{-1} \left[f_5 \left(d_5(t), T_1 \begin{pmatrix} X_I \\ X_{II} \end{pmatrix} (t) \right) \right]^{-1} \times \right.$$
$$\left. \times (I_{m-p} + \mathscr{G}(t, X_{II}(t)))^{-1} \frac{\partial f_3(X_I(t), X_{II}(t))}{\partial X_{II}} \right\| \leq \hat{k}^{-1} \alpha^{-1} \left\| \frac{\partial f_3(X_I(t), X_{II}(t))}{\partial X_{II}} \right\| < 1, \ \forall t \in I.$$

Hence,

$$(1 - \varphi_{II}(t)^2 \|X_{II}(t) - X_{\text{ref},II}(t)\|^2) \hat{k}^{-1} \left[f_5 \left(d_5(t), T_1 \begin{pmatrix} X_I \\ X_{II} \end{pmatrix} (t) \right) \right]^{-1} \times$$
$$\times \left(I_{m-p} + \mathscr{G}(t, X_{II}(t)) \right)^{-1} \frac{\partial f_3(X_I(t), X_{II}(t))}{\partial X_{II}} - I_{m-p}$$

is invertible. Then the matrix

$$\mathscr{M}(t, X_I(t), X_{II}(t)) = \frac{\partial f_3(X_I(t), X_{II}(t))}{\partial X_{II}} - \frac{\hat{k} f_5 \left(d_5(t), T_1 \begin{pmatrix} X_I \\ X_{II} \end{pmatrix} (t) \right)}{1 - \varphi_{II}(t)^2 \|X_{II}(t) - X_{\text{ref},II}(t)\|^2} \times$$
$$\times \left(I_{m-p} + \mathscr{G}(t, X_{II}(t)) \right)$$

is invertible.

We define a function

$$
\begin{aligned}
&\tilde{F}_{II}: \tilde{\mathscr{D}} \times \mathbb{R}^k \times \mathbb{R}^k \to \mathbb{R}^{m-p}, (t, X_I, X_{II}, \eta, \mu) \\
&\mapsto [\mathscr{M}(t, X_I, X_{II})]^{-1} \Bigg\{ \frac{\partial f_3(X_I, X_{II})}{\partial X_I} F_I\left(t, \begin{pmatrix} x_I \\ X_{II} \end{pmatrix}, \eta\right) + f_4'(d_3(t), \eta) \begin{pmatrix} \dot{d}_3(t) \\ z\left(S\begin{pmatrix} X_I \\ X_{II}\end{pmatrix}, \mu\right)\end{pmatrix} \\
&\qquad - \Gamma_{II}'(d_4(t), \eta) \begin{pmatrix} \dot{d}_4(t) \\ z\left(S\begin{pmatrix} X_I \\ X_{II}\end{pmatrix}, \mu\right)\end{pmatrix} \frac{\tilde{K}_I(t, X_I)}{1 - \varphi_I(t)^2 \|\tilde{K}_I(t, X_I)\|^2} \\
&\qquad - \frac{\Gamma_{II}(d_4(t), \eta)}{(1 - \varphi_I(t)^2\|\tilde{K}_I(t, X_I)\|^2)^2} \Bigg[\left(\frac{\partial \tilde{K}_I(t, X_I)}{\partial t} + \frac{\partial \tilde{K}_I(t, X_I)}{\partial X_I} F_I\left(t, \begin{pmatrix} x_I \\ X_{II}\end{pmatrix}, \eta\right)\right) \times \\
&\qquad \times (1 - \varphi_I(t)^2 \|\tilde{K}_I(t, X_I)\|^2) + 2\tilde{K}_I(t, X_I) \Bigg(\varphi_I(t)\dot{\varphi}_I(t)\|\tilde{K}_I(t, X_I)\|^2 + \\
&\qquad + \varphi_I(t)^2 \tilde{K}_I(t, X_I)^\top \left(\frac{\partial \tilde{K}_I(t, X_I)}{\partial t} + \frac{\partial \tilde{K}_I(t, X_I)}{\partial X_I} F_I\left(t, \begin{pmatrix} x_I \\ X_{II}\end{pmatrix}, \eta\right)\right)\Bigg)\Bigg] \\
&\qquad - f_5'(d_5(t), \eta) \begin{pmatrix} \dot{d}_5(t) \\ z\left(S\begin{pmatrix} X_I \\ X_{II}\end{pmatrix}, \mu\right)\end{pmatrix} \frac{\hat{k} \cdot (X_{II} - X_{\mathrm{ref},II}(t))}{1 - \varphi_{II}(t)^2 \|X_{II} - X_{\mathrm{ref},II}(t)\|^2} \\
&\qquad - \frac{\hat{k} f_5(d_5(t), \eta)}{(1 - \varphi_{II}(t)^2\|X_{II} - X_{\mathrm{ref},II}(t)\|^2)^2} \Bigg[-\dot{X}_{\mathrm{ref},II}(t)(1 - \varphi_{II}(t)^2 \|X_{II} - X_{\mathrm{ref},II}(t)\|^2) \\
&+ 2(X_{II} - X_{\mathrm{ref},II}(t)) \Big(\varphi_{II}(t)\dot{\varphi}_{II}(t)\|X_{II} - X_{\mathrm{ref},II}(t)\|^2 + \varphi_{II}(t)^2 (X_{II} - X_{\mathrm{ref},II}(t))^\top (-\dot{X}_{\mathrm{ref},II}(t))\Big)\Bigg]\Bigg\}.
\end{aligned}
$$

According to (4.2.17), it is obtained

$$
\dot{X}_{II}(t) = \tilde{F}_{II}\left(t, \begin{pmatrix} X_I(t) \\ X_{II}(t)\end{pmatrix}, T_1\begin{pmatrix} X_I \\ X_{II}\end{pmatrix}(t), T_2 \begin{pmatrix} X_I \\ X_{II}\end{pmatrix}(t)\right).
$$

In conjunction with first equation in (4.2.13), we have an ordinary differential equation

$$
\begin{aligned}
\dot{X}_I(t) &= F_I\left(t, \begin{pmatrix} X_I(t) \\ X_{II}(t)\end{pmatrix}, T_1\begin{pmatrix} X_I \\ X_{II}\end{pmatrix}(t)\right), \\
\dot{X}_{II}(t) &= \tilde{F}_{II}\left(t, \begin{pmatrix} X_I(t) \\ X_{II}(t)\end{pmatrix}, T_1\begin{pmatrix} X_I \\ X_{II}\end{pmatrix}(t), T_2 \begin{pmatrix} X_I \\ X_{II}\end{pmatrix}(t)\right),
\end{aligned} \tag{4.2.19}
$$

with initial value

$$
\begin{aligned}
X_I|_{[-h,0]} &= \left(y_1^0, \ldots, \left(\frac{d}{dt}\right)^{r_1 - 1} y_1^0, \ldots, \left(\frac{d}{dt}\right)^{r_p - 1} y_p^0\right), \\
X_{II}|_{[-h,0]} &= (y_{p+1}^0, \ldots, y_m^0).
\end{aligned}
$$

Step 1f: Consider the initial value problem (4.2.19), we have $(0, X_I(0), X_{II}(0)) \in \tilde{\mathscr{D}}$, F_I is measurable in t, continuous in (X_I, X_{II}, η), and locally essentially bounded, and $\tilde{F}_{II}$ is measurable in t, continuous in (X_I, X_{II}, η, μ), and locally essentially bounded. Therefore, apply [46, Theorem B.1][1], we obtain the existence of solutions to (4.2.19), and every solution can

[1] In [46] a domain $\mathscr{D} \subseteq \mathbb{R}_{\geq 0} \times \mathbb{R}$ is considered, but the generalization to the higher dimensional case is straightforward.

be extended to a maximal solution. Furthermore, for a maximal solution $(X_I, X_{II}) : [-h,\omega) \to \mathbb{R}^{|r|+m-p}$, $\omega \in (0,\infty]$, of (4.2.19), the closure of the graph of this solution is not a compact subset of $\tilde{\mathscr{D}}$.

The solution $(X_I(t), X_{II}(t))$ of (4.2.19) in particular satisfies (4.2.16). Integration gives, for all $t \in [0,\omega)$,

$$0 = F_{II}\left(t, \begin{pmatrix} X_I(t) \\ X_{II}(t) \end{pmatrix}, T_1 \begin{pmatrix} X_I \\ X_{II} \end{pmatrix}(t)\right) - F_{II}\left(0, \begin{pmatrix} X_I(0) \\ X_{II}(0) \end{pmatrix}, T_1 \begin{pmatrix} X_I \\ X_{II} \end{pmatrix}(0)\right).$$

It follows from the choice of y^0 which implies the value of $(X_I, X_{II})|_{[-h,0]}$ that $(X_I(t), X_{II}(t))$ satisfies the second equation in (4.2.13). And therefore, $(X_I(t), X_{II}(t))$ satisfies (4.2.13) for all $t \in [-h,\omega)$. This leads to a maximal solution $(X_I, X_{II}) : [-h,\omega) \to \mathbb{R}^{|r|+m-p}$, $\omega \in (0,\infty]$, of (4.2.13), and the closure of the graph of this solution is not a compact subset of $\tilde{\mathscr{D}}$.

Consequently, $(e_{10}, \ldots, e_{1,r_1-1}, \ldots, e_{p,r_p-1}, e_{p+1}, \ldots, e_m) : [0,\omega) \to \mathbb{R}^{|r|+m-p}$ defined by

$$\begin{aligned} e_{iq}(t) &= \tilde{K}_{iq}(t, y_i(t), \ldots, y_i^{(q)}(t)), \text{ for } i = 1,\ldots,p \text{ and } q = 0,\ldots,r_i - 1, \\ e_i(t) &= y_i(t) - y_{\text{ref},i}(t), \text{ for } i = p+1,\ldots,m, \end{aligned}$$

where $t \in [0,\omega)$. It follows that the closure of the graph of $(e_{10}, \ldots, e_{1,r_1-1}, \ldots, e_{p,r_p-1}, e_{p+1}, \ldots, e_m)$ is not a compact subset of $\mathscr{D}$.

Step 2:We show that $k_I(\cdot)$, $k_{II}(\cdot)$, $k_{iq}(\cdot)$, for $i = 1,\ldots,p$, $\quad q = 0,\ldots,r_i-2$, as in (4.2.1) are bounded on $[0,\omega)$.

Step 2a: We show that $k_{iq}(\cdot)$ for $i = 1,\ldots,p$, $q = 0,\ldots,r_i-2$ as in (4.2.1) are bounded on $[0,\omega)$. This step is similar to *Step 2a* of the proof of Theorem 3.2.3.

Step 2b: In the following we will prove by induction that there exist constants $\tilde{M}_{iq}^j, \tilde{N}_{iq}^j, \tilde{K}_{iq}^j > 0$ such that, for all $t \in [0,\omega)$,

$$\left|\left(\frac{d}{dt}\right)^j [k_{iq}(t)e_{iq}(t)]\right| \le \tilde{M}_{iq}^j, \quad \left|\left(\frac{d}{dt}\right)^j e_{iq}(t)\right| \le \tilde{N}_{iq}^j, \quad \left|\left(\frac{d}{dt}\right)^j k_{iq}(t)\right| \le \tilde{K}_{iq}^j, \qquad (4.2.20)$$

for $i = 1,\ldots,p$, $\quad q = 0,\ldots,r_i-2$, and $j = 0,\ldots,r_i-1-q$.
First, we may infer from *Step 3a* that $k_{iq}(\cdot)$, for $i = 1,\ldots,p$, $\quad q = 0,\ldots,r_i-2$, are bounded. Furthermore, e_{iq} are bounded since they evolve in the respective performance funnels, cf.(4.2.1). Therefore, for each $i = 1,\ldots,p$, (4.2.20) is true whenever $j = 0$. We prove (4.2.20) for $q = r_i-2$ and $j = 1$. We find that

$$\begin{aligned} \dot{e}_{i,r_i-2}(t) &= e_{i,r_i-1}(t) - k_{i,r_i-2}(t)e_{i,r_i-2}(t), \\ \dot{k}_{i,r_i-2}(t) &= 2k_{i,r_i-2}^2(t)\left(\varphi_{i,r_i-2}^2(t)e_{i,r_i-2}(t)\dot{e}_{i,r_i-2}(t) + \varphi_{i,r_i-2}(t)\dot{\varphi}_{i,r_i-2}(t)|e_{i,r_i-2}(t)|^2\right), \\ \frac{d}{dt}[k_{i,r_i-2}(t)e_{i,r_i-2}(t)] &= \dot{k}_{i,r_i-2}(t)e_{i,r_i-2}(t) + k_{i,r_i-2}(t)\dot{e}_{i,r_i-2}(t). \end{aligned}$$

Therefore, $\dot{e}_{i,r_i-2}(t)$, $\dot{k}_{i,r_i-2}(t)$, and $\frac{d}{dt}[k_{i,r_i-2}(t)e_{i,r_i-2}(t)]$ are bounded since k_{i,r_i-2}, φ_{i,r_i-2}, $\dot{\varphi}_{i,r_i-2}$, e_{i,r_i-2}, and e_{i,r_i-1} are bounded. Now let $s \in \{0,\ldots,r_i-3\}$ and $l \in \{0,\ldots,r_i-1-s\}$ and assume that (4.2.20) is true for all $q = s+1,\ldots,r_i-2$ and all $j = 0,\ldots,r_i-1-q$ as well as for $q = s$ and all $j = 0, \dots, l-1$. We show that it is true for $q = s$ and $j = l$:

$$\begin{aligned}
\left(\frac{d}{dt}\right)^{l} e_{is}(t) &= \left(\frac{d}{dt}\right)^{l-1} [e_{i,s+1}(t) - k_{is}(t)e_{is}(t)] \\
&= \left(\frac{d}{dt}\right)^{l-1} e_{i,s+1}(t) - \left(\frac{d}{dt}\right)^{l-1} [k_{is}(t)e_{is}(t)], \\
\left(\frac{d}{dt}\right)^{l} k_{is}(t) &= \left(\frac{d}{dt}\right)^{l-1} \Big(2k_{is}^2(t)\big(\varphi_{is}^2(t)e_{is}(t)\dot{e}_{is}(t) + \varphi_{is}(t)\dot{\varphi}_{is}(t)|e_{is}(t)|^2\big)\Big), \\
\left(\frac{d}{dt}\right)^{l} [k_{is}(t)e_{is}(t)] &= \left(\frac{d}{dt}\right)^{l-1} \big(\dot{k}_{is}(t)e_{is}(t) + k_{is}(t)\dot{e}_{is}(t)\big).
\end{aligned}$$

Then, successive application of the product rule and using the induction hypothesis as wells as the fact that $\varphi_{is}, \dot{\varphi}_{is}, \dots, \varphi_{is}^{(r_i-s)}$ are bounded, yields that the above terms are bounded. Therefore, the proof of (4.2.20) is complete.
By (4.2.20) and (4.2.9) it follows that $e_i^{(q)}$ is bounded on $[0,\omega)$ for all $i = 1, \dots, p, \quad q = 0, \dots, r_i - 1$.
Step 2c: We show that $k_I(\cdot)$ is bounded.
Recall from *Step 1a*, we have, for $i = 1, \dots, p$,

$$e_i^{(r_i)}(t) = \dot{e}_{i,r_i-1}(t) - \sum_{j=0}^{r_i-2} \left(\frac{d}{dt}\right)^{r_i-1-j} k_{ij}(t)e_{ij}(t).$$

Then,

$$\begin{aligned}
\dot{e}_I(t) = {} & f_1(y_1(t), \dots, y_1^{(r_1-1)}(t), \dots, y_p^{(r_p-1)}(t), y_{p+1}(t), \dots, y_m(t)) + f_2(d_1(t), (Ty)(t)) \\
& + \begin{pmatrix} \sum_{j=0}^{r_1-2} \left(\frac{d}{dt}\right)^{r_1-1-j} k_{1j}(t)e_{1j}(t) \\ \sum_{j=0}^{r_2-2} \left(\frac{d}{dt}\right)^{r_2-1-j} k_{2j}(t)e_{2j}(t) \\ \vdots \\ \sum_{j=0}^{r_p-2} \left(\frac{d}{dt}\right)^{r_p-1-j} k_{pj}(t)e_{pj}(t) \end{pmatrix} - \begin{pmatrix} y_{\text{ref},1}^{(r_1)}(t) \\ \vdots \\ y_{\text{ref},p}^{(r_p)}(t) \end{pmatrix} - \Gamma_I(d_2(t), (Ty)(t))k_I(t)e_I(t).
\end{aligned}$$

Now we set

$$\begin{aligned}
\hat{F}_I(t, (Ty)(t)) := {} & \begin{pmatrix} \sum_{j=0}^{r_1-2} \left(\frac{d}{dt}\right)^{r_1-1-j} k_{1j}(t)e_{1j}(t) \\ \sum_{j=0}^{r_2-2} \left(\frac{d}{dt}\right)^{r_2-1-j} k_{2j}(t)e_{2j}(t) \\ \vdots \\ \sum_{j=0}^{r_p-2} \left(\frac{d}{dt}\right)^{r_p-1-j} k_{pj}(t)e_{pj}(t) \end{pmatrix} - \begin{pmatrix} y_{\text{ref},1}^{(r_1)}(t) \\ \vdots \\ y_{\text{ref},p}^{(r_p)}(t) \end{pmatrix} + f_2(d_1(t), (Ty)(t)) \\
& + f_1(y_1(t), \dots, y_1^{(r_1-1)}(t), \dots, y_p^{(r_p-1)}(t), y_{p+1}(t), \dots, y_m(t)). \qquad (4.2.21)
\end{aligned}$$

By the bounded-input, bounded-output property (iii) of operator T, it follows that $T(y)$ is bounded, and we denote $M_T := \|T(y)|_{[0,\omega)}\|_\infty$. By *Step 2b*, property (iii) of operator T, continuous property f_1, and boundedness of d_1, we conclude that $\hat{F}_I(t, (Ty)(t))$ is bounded on $[0,\omega)$, i.e, there exists $M_{\hat{F}_I} > 0$ such that

$$\text{for almost all } t \in [0,\omega): \quad \|\hat{F}_I(t,(Ty)(t))\| \le M_{\hat{F}_I}.$$

Recall the definition of $e_I(\cdot)$, $e_{II}(\cdot)$, and combine with (4.2.21), we have

$$\begin{aligned} \dot{e}_I(t) &= \hat{F}_I(t,(Ty)(t)) - \Gamma_I(d_2(t),(Ty)(t))k_I(t)e_I(t) \\ k_I(t) &= \frac{1}{1-\varphi_I^2(t)\|e_I(t)\|^2}. \end{aligned} \tag{4.2.22}$$

We now define a compact set

$$\Omega = \left\{ (\delta,\eta,e_I) \in \mathbb{R}^p \times \mathbb{R}^q \times \mathbb{R}^m \;\middle|\; \begin{array}{l} \|\delta\| \le \|d_2|_{[0,\omega)}\|_\infty \\ \|\eta\| \le M_T \\ \|e_I\| = 1. \end{array} \right\},$$

then, since Γ_I is pointwise positive definite and the map

$$\Omega \ni (\delta,\eta,e_I) \mapsto e_I^\top \Gamma(\delta,\eta)e_I \in \mathbb{R}_{>0}$$

is continuous, it follows that there exists $\gamma > 0$ such that

$$\forall\,(\delta,\eta,e_I) \in \Omega: \quad e_I^\top \Gamma(\delta,\eta)e_I \ge \gamma.$$

Therefore, we have

$$e_I(t)^\top \Gamma_I(d_2(t),(Ty)(t))e_I(t) \ge \gamma\|e_I(t)\|^2$$

for all $t \in [0,\omega)$. Choose $\varepsilon_I > 0$ small enough so that

$$\begin{aligned} \varepsilon_I &\le \min\left\{ \frac{\lambda_I}{2}, \inf_{t\in(0,T_I]} (\psi_I(t) - \|e_I(t)\|) \right\} \\ \text{and} \quad L_I &\le \frac{\lambda_I^2}{4\varepsilon_I}\gamma - M_{\hat{F}_I}, \end{aligned} \tag{4.2.23}$$

We show that

$$\forall\, t \in (0,\omega): \; \psi_I(t) - \|e_I(t)\| \ge \varepsilon_I. \tag{4.2.24}$$

By definition of ε_I this holds on $(0,T_I]$. Seeking a contradiction suppose that

$$\exists t_{I,1} \in [T,\omega): \; \psi_I(t_{I,1}) - \|e_I(t_{I,1})\| < \varepsilon_I.$$

Set $t_{I,0} = \max\{t \in [T_I,t_{I,1}) \mid \psi_I(t) - \|e_I(t)\| = \varepsilon_I\}$. Then, for all $t \in [t_{I,0},t_{I,1}]$, we have that

$$\begin{aligned} \psi_I(t) - \|e_I(t)\| &\le \varepsilon_I, \\ \|e_I(t)\| \ge \psi_I(t) - \varepsilon_I &\ge \frac{\lambda_I}{2}, \\ k_I(t) = \frac{1}{1-\varphi_I^2(t)\|e_I(t)\|^2} &\ge \frac{\lambda_I}{2\varepsilon_I}. \end{aligned}$$

Now we have, for all $t \in [t_{I,0}, t_{I,1}]$,

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|e_I(t)\|^2 = e_I^\top(t)\dot{e}_I(t) &= e_I^\top(t)\left[\hat{F}_I(t,(Ty)(t)) - \Gamma_I(d_2(t),(Ty)(t))k_I(t)e_I(t)\right] \\
&\le \left(M_{\hat{F}_I} - \frac{\lambda_I^2}{4\varepsilon_I}\gamma\right)\|e_I(t)\| \\
&\overset{(4.2.23)}{\le} -L_I\|e_I(t)\|.
\end{aligned}
$$

Then

$$
\begin{aligned}
\|e_I(t_{I,1})\| - \|e_I(t_{I,0})\| &= \int_{t_{I,0}}^{t_{I,1}} \frac{1}{2}\|e_I(t)\|^{-1}\frac{d}{dt}\|e_I(t)\|^2\,dt \\
&\le -L_I(t_{I,1}-t_{I,0}) \\
&\le -|\psi(t_{I,1}) - \psi(t_{I,0})| \\
&\le \psi(t_{I,1}) - \psi(t_{I,0}),
\end{aligned}
$$

and thus we obtain $\varepsilon_I = \psi_I(t_{I,0}) - \|e_I(t_{I,0})\| \le \psi_I(t_{I,1}) - \|e_I(t_{I,1})\| < \varepsilon_I$, a contradiction.
Step 2d: We show that $k_{II}(\cdot)$ is bounded. Seeking a contradiction, assume that $k_{II}(t) \to \infty$ for $t \to \omega$. Set

$$
\begin{aligned}
\hat{F}_{II}(t,(Ty)(t)) := f_3(y_1(t),\ldots,y_1^{(r_1-1)}(t),\ldots,y_p^{(r_p-1)}(t),y_{p+1}(t),\ldots,y_m(t)) \\
+ f_4(d_3(t),(Ty)(t)). \quad (4.2.25)
\end{aligned}
$$

By *Step 2b*, property (iii) of operator T, continuous property f_3, and boundedness of d_3, we conclude that $\hat{F}_{II}(\cdot,(Ty)(\cdot))$ is bounded on $[0,\omega)$, i.e, there exists $M_{\hat{F}_{II}} > 0$ such that

$$
\text{for almost all } t \in [0,\omega): \quad \|\hat{F}_{II}(t,(Ty)(t))\| \le M_{\hat{F}_{II}}.
$$

Recall the definition of $e_I(\cdot)$, $e_{II}(\cdot)$, and combine with (4.2.25), we have

$$
\begin{aligned}
0 &= \hat{F}_{II}(t,(Ty)(t)) - \Gamma_{II}(d_4(t),(Ty)(t))k_I(t)e_I(t) \\
&\qquad - f_5(d_5(t),(Ty)(t))k_{II}(t)e_{II}(t), \\
k_{II}(t) &= \frac{\hat{k}}{1-\varphi_{II}^2(t)\|e_{II}(t)\|^2}.
\end{aligned} \quad (4.2.26)
$$

We show that $e_{II}(t) \to 0$ for $t \to \omega$. Seeking a contradiction, assume that there exist $\kappa > 0$ and a sequence $(t_n) \subset \mathbb{R}_{\ge 0}$ with $t_n \nearrow \omega$ such that $\|e_{II}(t_n)\| \ge \kappa$ for all $n \in \mathbb{N}$. Then, from (4.2.26) we obtain, for all $t \ge 0$,

$$
\|\hat{F}_{II}(t,(Ty)(t))\| = \|\Gamma_{II}(d_4(t),(Ty)(t))k_I(t)e_I(t) + f_5(d_5(t),(Ty)(t))k_{II}(t)e_{II}(t)\|.
$$

Since, k_I is bounded by *Step 2c*, y, d_4 are bounded and Γ_{II} is continuous, there exists $\gamma_{II} > 0$ such that $\sup_{t\ge 0}\|\Gamma_{II}(d_4(t),(Ty)(t))k_I(t)e_I(t)\| \le \gamma_{II}$. Since $k_{II}(t) \to \infty$ for $t \to \omega$, $\|e_{II}(t_n)\| \ge \kappa$ and $f_5(d_5(t_n),(Ty)(t_n)) \ge \alpha$, we find that for $n \in \mathbb{N}$ large enough

$$
\|\hat{F}_{II}(t,(Ty)(t))\| \ge \alpha\kappa k_{II}(t_n) - \gamma_{II} \to \infty \text{ for } n \to \infty.
$$

This is contradiction to boundedness property of $\hat{F}_{II}(\cdot,(Ty)(\cdot))$. Hence, $e_{II}(t) \to 0$ for $t \to \omega$. Therefore, $\lim_{t\to\infty}\varphi_{II}(t)\|e_{II}(t)\|^2 = 0$ because $\varphi_{II}(\cdot)$ is bounded. This leads to $\lim_{t\to\infty}k_{II}(t) = \hat{k}$, a

contradiction. In conclusion, $k_{II}(\cdot)$ is bounded.
Step 3: We show that $\omega = \infty$. Seeking a contradiction, suppose that $\omega < \infty$. Then, since $e_{iq}, k_{i,q}$ for $i = 1, \ldots, p$, $q = 0, \ldots, r_i - 2$, e_I, e_{II}, k_I, k_{II} are bounded by *Step 2*, it follows that the closure of the graph of $(e_{10}, \ldots, e_{1,r-1-1}, \ldots, e_{p,r_p-1}, e_{p+1}, \ldots, e_m)$ is a compact subset of $\mathscr{D}$, a contradiction. □

We show that funnel controller (4.2.4) which is a simplification version of funnel controller (4.2.1) works on well in application to class of systems (4.1.2).

Corollary 4.2.4. *Consider a system* (4.1.2) *with vector relative degree* $r = (r_1, r_2, \ldots, r_m)^\top \in \mathbb{N}^m$. *For* Φ, Φ_i *as defined in* (1.1.7), *suppose reference signal* y_{ref}, *and* φ, φ_{iq}, $i = 1, \ldots, m$, $q = 0, \ldots, r_i - 2$ *as given in* (4.2.2). *Let* $y|_{[-h,0]} = y^0 = (y_1^0, \ldots, y_m^0)$, $y_i^0 \in \mathscr{C}^{r_i-1}([-h,0] \to \mathbb{R})$ *an initial value such that* $\bar{e}, e_{iq}$, $i = 1, \ldots, m$, $q = 0, \ldots, r_i - 2$ *as defined in* (4.2.4) *fulfill*

$$\begin{aligned} \varphi(0)\|\bar{e}(0)\| &< 1 \\ \varphi_{iq}(0)\|e_{iq}(0)\| &< 1 \text{ for } i = 1, \ldots, m,\ q = 0, \ldots, r_i - 2. \end{aligned} \tag{4.2.27}$$

Then the application of the funnel controller (4.2.4) *to* (4.1.2) *yields an initial value problem, which has a solution, and every maximal solution* $y : [-h, \omega) \to \mathbb{R}^m$, $\omega \in (0, \infty]$, *has the following properties:*

(i) The solution is global (i.e., $\omega = \infty$*).*

(ii) The input $u : \mathbb{R}_{\geq 0} \to \mathbb{R}^m$, *the gain functions* $\bar{k}, k_{iq} : \mathbb{R}_{\geq 0} \to \mathbb{R}$, $i = 1, \ldots, m$, $q = 0, \ldots, r_i - 2$ *and* $y_i, \ldots, y_i^{(r_i-1)} : \mathbb{R}_{\geq 0} \to \mathbb{R}$, $i = 1, \ldots, m$ *are bounded.*

(iii) The functions $\bar{e} : \mathbb{R}_{\geq 0} \to \mathbb{R}^m$, $e_{iq} : \mathbb{R}_{\geq 0} \to \mathbb{R}$, $i = 1, \ldots, m$, $q = 0, \ldots, r_i - 2$ *evolve in their respective performance funnels and are uniformly bounded away from the funnel boundaries in the following sense:*

$$\begin{aligned} \forall i = 1, \ldots, m,\ q = 0, \ldots, r_i - 2\ \exists \varepsilon_{iq} > 0\ \forall t > 0 : \|e_{iq}(t)\| \leq \varphi_{iq}(t)^{-1} - \varepsilon_{iq}, \\ \exists \bar{\varepsilon} > 0\ \forall t > 0 : \|\bar{e}(t)\| \leq \varphi(t)^{-1} - \bar{\varepsilon}, \end{aligned} \tag{4.2.28}$$

In particular, the error $e_i(t) = y_i(t) - y_{\text{ref},i}(t)$, $i = 1, \ldots, m$, *evolves in the funnel* $\mathscr{F}_{\varphi_{i0}}$ *as in* (1.1.8) *and stays uniformly away from its boundary.*

Proof. This corollary is achieved by applying directly theorem 4.2.3 with noting that the part which is respective to generalized relative degree $r_i = 0$ vanish. □

We now consider a linear differential-algebraic system with positive vector relative degree of type (2.2.8).

$$\begin{aligned} E\dot{x}(t) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t), \end{aligned}$$

where $A, E \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, $C \in \mathbb{R}^{m \times n}$.
We will prove that funnel controller (4.2.4) combined with regulation condition (4.2.29) can be also applied to this kind of system,

$$\boxed{\begin{aligned} &y_{\text{ref}} \in \mathscr{W}^{n,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R}^m), \\ &\quad \varphi \in \Phi_{n-|r|}, \varphi_{i0} \in \Phi_n, \varphi_{i1} \in \Phi_{n-1}, \ldots, \varphi_{i,r_i-2} \in \Phi_{n-r_i+2}, i = 1, \ldots, m, \end{aligned}} \tag{4.2.29}$$

with $|r| = \sum_{i=1}^{m} r_i$.

Proposition 4.2.5. *Let $[E,A,B,C] \in \Sigma_{n,n,m,m}$ be regular, has asymptotically stable zero dynamic, positive vector relative degree $r = (r_1,\dots,r_m)$, $r_i > 0$, $i = 1,\dots,m$, and positive (negative) definite high-gain matrix $\Gamma = \lim_{s\to\infty} \operatorname{diag}(s^{r_1},\dots,s^{r_m})C(sE-A)^{-1}B$. For funnel functions φ, φ_{iq}, $i = 1,\dots,m$, $q = 0,\dots,r_i-2$, reference signal y_{ref} as in* (4.2.29)*, and any consistent initial value $x^0 \in \mathbb{R}^n$ such that $\bar{e}, e_{iq}$, $i = 1,\dots,p$, $q = 0,\dots,r_i-2$ as defined in* (4.2.4) *fulfill*

$$\begin{aligned}\varphi(0)\|\bar{e}(0)\| &< 1,\\ \varphi_{iq}(0)|e_{iq}(0)| &< 1, \quad i = 1,\dots,m,\ q = 0,\dots,r_i-2.\end{aligned}$$

Then the application of the funnel controller (4.2.4) *to system* (2.2.8) *yields a closed-loop initial value problem that has a solution and every solution can be extended to a global solution. Furthermore, for every global solution $x(\cdot)$,*

i) The input $u : \mathbb{R}_{\geq 0} \to \mathbb{R}^m$, the gain functions $\bar{k}, k_{iq} : \mathbb{R}_{\geq 0} \to \mathbb{R}$, $i = 1,\dots,m$, $q = 0,\dots,r_i-2$ are bounded.

ii) The functions $\bar{e} : \mathbb{R}_{\geq 0} \to \mathbb{R}^m$, and $e_{iq} : \mathbb{R}_{\geq 0} \to \mathbb{R}$, $i = 1,\dots,m$, $q = 0,\dots,r_i-2$ evolve in their respective performance funnels, i.e.,

$$(t,\bar{e}) \in \mathscr{F}_\varphi;\ (t,e_{iq}) \in \mathscr{F}_{\varphi_{iq}} \text{ for all } i = 1,\dots,m,\ q = 0,\dots,r_i-2 \text{ and } t \geq 0.$$

Furthermore, the signals $\bar{e}(\cdot), e_{iq}(\cdot)$ are uniformly bounded away from the funnel boundaries in the following sense:

$$\begin{aligned}\forall i = 1,\dots,m,\ q = 0,\dots,r_i-2\ \exists \varepsilon_{iq} > 0\ \forall t > 0 : \|e_{iq}(t)\| &\leq \varphi_{iq}(t)^{-1} - \varepsilon_{iq},\\ \exists \varepsilon > 0\ \forall t > 0 : \|\bar{e}(t)\| &\leq \varphi(t)^{-1} - \varepsilon.\end{aligned}$$

In particular, each error component $e_i(t) = C_i x(t) - y_{\text{ref},i}(t)$, $i = 1,\dots,m$, evolves in the funnel $\mathscr{F}_{\varphi_{i0}}$ and stays uniformly away from its boundary.

Proof. Without limitation of generality, we may consider $[E,A,B,C]$ in the form

$$\begin{aligned}\begin{pmatrix} y_1^{(r_1)}(t)\\ y_2^{(r_2)}(t)\\ \vdots\\ y_m^{(r_m)}(t)\end{pmatrix} &= \begin{pmatrix} \sum\limits_{j=1}^{m}\sum\limits_{h=1}^{r_j} R_{jh}^1 y_j^{(h-1)}(t)\\ \sum\limits_{j=1}^{m}\sum\limits_{h=1}^{r_j} R_{jh}^2 y_j^{(h-1)}(t)\\ \vdots\\ \sum\limits_{j=1}^{m}\sum\limits_{h=1}^{r_j} R_{jh}^m y_j^{(h-1)}(t)\end{pmatrix} + S\eta(t) + \Gamma u(t),\\ \dot{\eta}(t) &= \sum_{i=1}^{m} P_i y_i(t) + Q\eta(t),\\ x_c(t) &= -\sum_{i=0}^{\nu-1} N_c^i B_c u^{(i)}(t),\\ x_{\bar{c}}(t) &= 0,\end{aligned}$$

where $n_c, n_{\bar{c}} \in \mathbb{N}_0$, $\mu = n - n_c - n_{\bar{c}} - |r|$ and $R_{jh}^i \in \mathbb{R}$ with $i,j \in \{1,\dots,m\}$, $h \in \{1,\dots,r_i\}$, $S \in \mathbb{R}^{m\times\mu}$, $P_i \in \mathbb{R}^{\mu}$, $B_c \in \mathbb{R}^{n_c\times m}$, $Q \in \mathbb{R}^{\mu\times\mu}$ is a Hurwitz matrix. $N_c \in \mathbb{R}^{n_c\times n_c}$ is nilpotent with index ν, and rank $\begin{bmatrix} N_c & B_c\end{bmatrix} = n_c$.

Ignoring the last two algebraic equation of that form, the claim in (ii), and the boundedness of u, $\hat{k}$, k_{iq}, $i = 1,\dots,m$, $q = 0,\dots,r_i-2$ follow directly from Corollary 4.2.4. It remains to show that x is a bounded function. Indeed, we have already known that $\nu + |r| \leq n$. Since $\varphi \in$

Φ_{n-q}, $\varphi_{iq} \in \Phi_{n-q}$, $q = 0, \ldots, r_i - 2$ it is obtained that $u(t)$ at least $(\nu - 1)$ times continuously differentiable and all of these derivatives are bounded functions. Therefore, x_c and $x_{\bar{c}}$ are bounded function. This implies the boundedness of x. □

We continue to consider a linear differential-algebraic sytem with generalized relative degree of type (4.1.3).

$$E\dot{x}(t) = Ax(t) + Bu(t),$$
$$y(t) = Cx(t)$$

where $A, E \in \mathbb{R}^{l\times n}$, $B \in \mathbb{R}^{l\times m}$, $C \in \mathbb{R}^{m\times n}$.

Proposition 4.2.6. *Let $[E,A,B,C] \in \Sigma_{l,n,m,m}$ be right invertible and has autonomous, asymptotically stable zero dynamics, and has a generalized vector relative degree $(r_1, \cdots, r_p, 0, \cdots, 0) \in \mathbb{N}_0^{1\times m}$, $r_i > 0$, for $i = 1, \ldots, p$. Suppose that the matrix Γ_{11} which is defined as in* (2.3.7) *is positive (negative) definite. Let $\varphi_{iq}, \varphi_I, \varphi_{II}$ as in* (4.2.2) *defining performance funnel, and reference signal $y_{\text{ref}} \in \mathscr{W}^{n,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R}^m)$. Then, the closed-loop system* (4.1.3),(4.2.1) *with any consistent initial value $x^0 \in \mathbb{R}^n$ has a solution and every solution can be extended to a global solution. Moreover, for every global solution $x(\cdot)$,*

i) the input function $u : \mathbb{R}_{\geq 0} \to \mathbb{R}^m$, the gain functions $k_{iq}, k_I, k_{II} : \mathbb{R}_{\geq 0} \to \mathbb{R}$, $i = 1, \ldots, p$, $q = 0, \ldots, r_i - 2$, the state function $x : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$ are bounded.

ii) the functions $e_{iq} : \mathbb{R}_{\geq 0} \to \mathbb{R}$, $i = 1, \ldots, p$, $q = 0, \ldots, r_i - 2$ and $e_I : \mathbb{R}_{\geq 0} \to \mathbb{R}^p$, $e_{II} : \mathbb{R}_{\geq 0} \to \mathbb{R}^{m-p}$ evolve in their respective performance funnels, i.e.,

$(t, e_I) \in \mathscr{F}_{\varphi_I}$; $(t, e_{II}) \in \mathscr{F}_{\varphi_{II}}$; $(t, e_{iq}) \in \mathscr{F}_{\varphi_{iq}}$ for all $i = 1, \ldots, p$, $q = 0, \ldots, r_i - 2$ and $t \geq 0$.

Furthermore, the signals $e_I(\cdot), e_{II}(\cdot), e_{iq}(\cdot)$ are uniformly bounded away from the funnel boundaries in the following sense:

$$\begin{aligned} \forall i = 1, \ldots, p,\ q = 0, \ldots, r_i - 2\ \exists \varepsilon_{iq} > 0\ \forall t > 0 : \|e_{iq}(t)\| &\leq \varphi_{iq}(t)^{-1} - \varepsilon_{iq}, \\ \exists \varepsilon_I > 0\ \forall t > 0 : \|e_I(t)\| &\leq \varphi_I(t)^{-1} - \varepsilon_I, \\ \exists \varepsilon_{II} > 0\ \forall t > 0 : \|e_{II}(t)\| &\leq \varphi_{II}(t)^{-1} - \varepsilon_{II}. \end{aligned} \tag{4.2.30}$$

In particular, each error component $e_i(t) = C_i x(t) - y_{\text{ref},i}(t)$ evolves in the funnel $\mathscr{F}_{\varphi_{i0}}$, with $i = 1, \ldots, p$, or $\mathscr{F}_{\varphi_{II}}$, with $i = p+1, \ldots, m$, and stays uniformly away from its boundary.

Proof. Without loss of generality, we may consider system (4.1.3) in the form

$$
\begin{aligned}
\dot\eta(t) &= Q\eta(t) + A_{12}y(t),\\
\begin{pmatrix} y_1^{(r_1)}(t)\\ y_2^{(r_2)}(t)\\ \vdots\\ y_p^{(r_p)}(t)\end{pmatrix} &= \begin{pmatrix} \sum\limits_{j=1}^{p}\sum\limits_{h=1}^{r_j} R_{jh}^{1} y_j^{(h-1)}(t) + \sum\limits_{j=p+1}^{m} R_{j1}^{1} y_j(t)\\ \sum\limits_{j=1}^{p}\sum\limits_{h=1}^{r_j} R_{jh}^{2} y_j^{(h-1)}(t) + \sum\limits_{j=p+1}^{m} R_{j1}^{2} y_j(t)\\ \vdots\\ \sum\limits_{j=1}^{p}\sum\limits_{h=1}^{r_j} R_{jh}^{p} y_j^{(h-1)}(t) + \sum\limits_{j=p+1}^{m} R_{j1}^{p} y_j(t)\end{pmatrix} + [\Gamma_{11}\ 0]A_{21}\eta(t)\\
&\quad + [\Gamma_{11}\ 0]\begin{pmatrix} u_1(t)\\ u_2(t)\\ \vdots\\ u_m(t)\end{pmatrix},\\
0 &= \begin{pmatrix} \sum\limits_{j=1}^{p}\sum\limits_{h=1}^{r_j} R_{jh}^{p+1} y_j^{(h-1)}(t) + \sum\limits_{j=p+1}^{m} R_{j1}^{p+1} y_j(t)\\ \sum\limits_{j=1}^{p}\sum\limits_{h=1}^{r_j} R_{jh}^{p+2} y_j^{(h-1)}(t) + \sum\limits_{j=p+1}^{m} R_{j1}^{p+2} y_j(t)\\ \vdots\\ \sum\limits_{j=1}^{p}\sum\limits_{h=1}^{r_j} R_{jh}^{m} y_j^{(h-1)}(t) + \sum\limits_{j=p+1}^{m} R_{j1}^{m} y_j(t)\end{pmatrix} + [\Gamma_{21}\ I_{m-p}]A_{21}\eta(t)\\
&\quad + [\Gamma_{21}\ I_{m-p}]\begin{pmatrix} u_1(t)\\ u_2(t)\\ \vdots\\ u_m(t)\end{pmatrix},\\
x_3(t) &= \sum_{i=0}^{\nu-1} N^i E_{32} y^{(i+1)}(t).
\end{aligned}
$$

where, $Q \in \mathbb{R}^{\mu\times\mu}$ is Hurwitz, $\mu = \dim\max(E,A,B;\ker C)$, $n_3 = n-\mu-m$, $N \in \mathbb{R}^{n_3\times n_3}$ is nilpotent with index $\nu \in \mathbb{N}$, i.e. $N^\nu = 0$ and $N^{\nu-1} \neq 0$, $R_{jh}^i \in \mathbb{R}$ for $i = 1,\dots,p$, $j = 1,\dots,p$, $h = 1,\dots,r_j$, and $R_{j1}^i \in \mathbb{R}$ for $i = p+1,\dots,m$, $j = p+1,\dots,m$, and E_{32}, A_{21}, A_{12} are matrices with suitable size. We recall the Subsection 4.1.2 which states that the subsystem made by three first equations of (4.1.4) belongs to the class of system (4.1.1). We note that

$$
(Ty)(t) := \eta(t) = e^{Qt}\eta^0 + \int_0^t e^{Q(t-\tau)}A_{12}y(\tau)d\tau.
$$

T is clearly causal, locally Lipschitz, and the Hurwitz property of Q implies that T has the bounded-input-bounded-output property, and

$$
\frac{d}{dt}(Ty)(t) = Qe^{Qt}\eta^0 + Q\int_0^t e^{Q(t-\tau)}A_{12}y(\tau)d\tau = z(y(t),(\tilde{T}y)(t)).
$$

Hence, operator T satisfies all properties (i)-(iv) in Definition 4.1.1. Furthermore, all other functions can be specify as follows

$$f_1\Big(y_1(t),\cdots,y_1^{(r_1-1)}(t),\cdots,y_p^{(r_p-1)}(t),y_{p+1}(t),\cdots,y_m(t)\Big)$$
$$=\begin{pmatrix} \sum\limits_{j=1}^{p}\sum\limits_{h=1}^{r_j} R_{jh}^1 y_j^{(h-1)}(t)+\sum\limits_{j=p+1}^{m} R_{j1}^1 y_j(t)\\ \sum\limits_{j=1}^{p}\sum\limits_{h=1}^{r_j} R_{jh}^2 y_j^{(h-1)}(t)+\sum\limits_{j=p+1}^{m} R_{j1}^2 y_j(t)\\ \vdots\\ \sum\limits_{j=1}^{p}\sum\limits_{h=1}^{r_j} R_{jh}^p y_j^{(h-1)}(t)+\sum\limits_{j=p+1}^{m} R_{j1}^p y_j(t)\end{pmatrix},$$

$$f_2\Big(d_1(t),(Ty)(t)\Big)=[\Gamma_{11}\ 0]A_{21}(Ty)(t),$$
$$\Gamma_I\Big(d_2(t),(Ty)(t)\Big)=\Gamma_{11},$$

$$f_3\Big(y_1(t),\cdots,y_1^{(r_1-1)}(t),\cdots,y_p^{(r_p-1)}(t),y_{p+1}(t),\cdots,y_m(t)\Big)$$
$$=\begin{pmatrix} \sum\limits_{j=1}^{p}\sum\limits_{h=1}^{r_j} R_{jh}^{p+1} y_j^{(h-1)}(t)+\sum\limits_{j=p+1}^{m} R_{j1}^{p+1} y_j(t)\\ \sum\limits_{j=1}^{p}\sum\limits_{h=1}^{r_j} R_{jh}^{p+2} y_j^{(h-1)}(t)+\sum\limits_{j=p+1}^{m} R_{j1}^{p+2} y_j(t)\\ \vdots\\ \sum\limits_{j=1}^{p}\sum\limits_{h=1}^{r_j} R_{jh}^m y_j^{(h-1)}(t)+\sum\limits_{j=p+1}^{m} R_{j1}^m y_j(t)\end{pmatrix},$$

$$f_4\Big(d_1(t),(Ty)(t)\Big)=[\Gamma_{21}\ I_{m-p}]A_{21}(Ty)(t),$$
$$\Gamma_{II}\Big(d_4(t),(Ty)(t)\Big)=\Gamma_{21},$$
$$f_5\Big(d_5(t),(Ty)(t)\Big)=1.$$

We, therefore, apply Theorem 4.2.3 to get most of statement results of Proposition 4.2.6. What remains to prove is that the state functions are bounded. Or more precisely, we need to show the boundedness of x_3 in (4.1.4). Indeed, we abuse the definition of $\begin{pmatrix} X_I(t)\\ X_{II}(t)\end{pmatrix}$, and $F_I,\tilde{F}_{II}$ from *Step 1c,d,e* in the proof of Theorem 4.2.3. We have $\begin{pmatrix} X_I(t)\\ X_{II}(t)\end{pmatrix}$ solves the initial value problem (4.2.19),

$$
\begin{aligned}
\dot{X}_I(t) &= F_I\left(t, \begin{pmatrix} X_I(t) \\ X_{II}(t) \end{pmatrix}, T_1\begin{pmatrix} X_I \\ X_{II} \end{pmatrix}(t)\right),\\
\dot{X}_{II}(t) &= \tilde{F}_{II}\left(t, \begin{pmatrix} X_I(t) \\ X_{II}(t) \end{pmatrix}, T_1\begin{pmatrix} X_I \\ X_{II} \end{pmatrix}(t), T_2\begin{pmatrix} X_I \\ X_{II} \end{pmatrix}(t)\right),\\
X_I(0) &= \left(y_1^0, \dots, \left(\frac{d}{dt}\right)^{r_1-1} y_1^0, \dots, \left(\frac{d}{dt}\right)^{r_p-1} y_p^0\right)(0),\\
X_{II}(0) &= (y_{p+1}^0, \dots, y_m^0)(0).
\end{aligned}
$$

We note that by *Step 2* in proof of Theorem 4.2.3, $\begin{pmatrix} X_I \\ X_{II} \end{pmatrix}$ is bounded. Therefore, $\begin{pmatrix} \dot{X}_I \\ \dot{X}_{II} \end{pmatrix}$ is bounded since F_I, $\tilde{F}_{II}$ are continuously differentiable. Then again, we obtain that $\frac{d}{dt}\left(T_1\begin{pmatrix} X_I \\ X_{II} \end{pmatrix}\right)$, $\frac{d}{dt}\left(T_2\begin{pmatrix} X_I \\ X_{II} \end{pmatrix}\right)$ are bounded. And the boundedness of $\begin{pmatrix} \ddot{X}_I \\ \ddot{X}_{II} \end{pmatrix}$ is also gotten by differentiating (4.2.19). Iteratively, we have that

$$
\begin{aligned}
\forall j = 0, \dots, \nu+1 : &\left(\exists c_0, \dots, c_j > 0 \forall t \geq 0 : \left\| \begin{pmatrix} X_I \\ X_{II} \end{pmatrix}(t) \right\| \leq c_0, \dots, \left\| \begin{pmatrix} X_I^{(j)} \\ X_{II}^{(j)} \end{pmatrix}(t) \right\| \leq c_j \right)\\
\Rightarrow &\left(\exists C_1, C_2 > 0 \forall t \geq 0 : \left\| \left(T_1\begin{pmatrix} X_I \\ X_{II} \end{pmatrix}\right)^{(j)}(t) \right\| \leq C_1, \left\| \left(T_1\begin{pmatrix} X_I \\ X_{II} \end{pmatrix}\right)^{(j)}(t) \right\| \leq C_2 \right),
\end{aligned}
$$

and successive differentiation of (4.2.19) yields that $\begin{pmatrix} X_I \\ X_{II} \end{pmatrix}, \begin{pmatrix} \dot{X}_I \\ \dot{X}_{II} \end{pmatrix}, \dots, \begin{pmatrix} X_I^{(\nu+1)} \\ X_{II}^{(\nu+1)} \end{pmatrix}$ are bounded. As a consequense, $y, \dot{y}, \dots, y^{(\nu+1)}$ are bounded. Hence,

$$
x_3(t) = \sum_{i=0}^{\nu-1} N^i E_{32} y^{(i+1)}(t)
$$

is also bounded. In conclusion, x is bounded. □

4.2.2 Applications

The following simulations aim to illustrate the theoretical results of the applying funnel controller (4.2.1) to system which belong to class (4.1.1).

Firstly, we consider an academic example of nonlinear multi-input, multi-output system described as follow

$$
\begin{aligned}
\ddot{y}_1(t) &= -\sin y_1(t) + y_1(t)\dot{y}_1(t) + y_2(t)^2 + y_1(t)(Ty)(t)\\
&\qquad + (y_1(t)^2 + y_2(t)^4 + 1)u_1(t),\\
0 &= y_1(t)^3 + y_1(t)\dot{y}_1(t)^2 + y_2(t) + y_2(t)(Ty)(t)\\
&\qquad + (y_1(t) + (Ty)(t))y_2(t)u_1(t) + u_2(t).
\end{aligned}
\tag{4.2.31}
$$

where operator

$$
(Ty)(t) = (T(y_1, y_2))(t) := e^{-2t}\eta^0 + \int_0^t e^{-2(t-s)}\Big(2y_1(s) - y_2(s)\Big) ds, \quad t \geq 0,
$$

for any fix $\eta^0 \in \mathbb{R}$. Operator $T \in \mathscr{T}_{2,1}$ because T satisfies all properties in definition 4.1.1. Obviously, the equation (4.2.31) belongs to system class (4.1.1) with

$$\begin{aligned}
f_1 &= -\sin y_1(t) + y_1(t)\dot{y}_1(t) + y_2(t)^2,\\
f_2 &= y_1(t)(Ty)(t),\\
f_3 &= y_1(t)^3 + y_1(t)\dot{y}_1(t)^2 + y_2(t),\\
f_4 &= y_2(t)(Ty)(t),\\
f_5 &= 1,\\
\Gamma_I &= y_1(t)^2 + y_2(t)^4 + 1,\\
\Gamma_{II} &= (y_1(t) + (Ty)(t))y_2(t),
\end{aligned}$$

since $\begin{bmatrix} \frac{\partial f_3}{\partial y_1} & \frac{\partial f_3}{\partial \dot{y}_1} & \frac{\partial f_3}{\partial y_2} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = 1$ is bounded. Furthermore, system (4.2.31) has generalized vector relative degree $r = (2,0)$. For simulation, we choose reference signal $y_{\text{ref}} = (\cos 2t, \sin t)^\top$, and initial value $y^0 = (0,0)^\top$, and $\eta^0 = 0$. We can set $\hat{k} = 2$ in funnel controller (4.2.1) since $\begin{bmatrix} \frac{\partial f_3}{\partial y_1} & \frac{\partial f_3}{\partial \dot{y}_1} & \frac{\partial f_3}{\partial y_2} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = 1$ and $f_5 = 1$. Because there is not any constraint condition imposed on funnel boundary, we may choose the same funnel function φ for all values.

$$\begin{aligned}
\varphi : \mathbb{R}_{\geq 0} &\to \mathbb{R}_{\geq 0},\\
t &\mapsto \frac{1}{2} t e^{-t} + 2\arctan t.
\end{aligned}$$

This simulation has been performed in MATLAB (solver: ode15s, relative tolerance: 10^{-14}, absolute tolerance: 10^{-5}) over the time interval [0,10], see Figure 4.1.

Figure 4.1a shows the output errors stay in the funnel and Figure 4.1b shows the input components that drive tracking errors.

We continue consider another example of linear general differential-algebraic systems. We note that this kind of system does not belong to class (4.1.1), however it still can be applied funnel controller (4.2.1) according to Proposition 4.2.6. We recall example in Remark 2.3.11 which do not have vector relative degree but have generalized vector relative degree. For more detail, we have

$$\begin{aligned}
E\dot{x}(t) &= Ax(t) + Bu(t)\\
y(t) &= Cx(t)
\end{aligned} \tag{4.2.32}$$

where

$$E = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}; \; A = \begin{bmatrix} -1 & 1 & -2 & 0 & 0 \\ 3 & 5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}; \; C = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}.$$

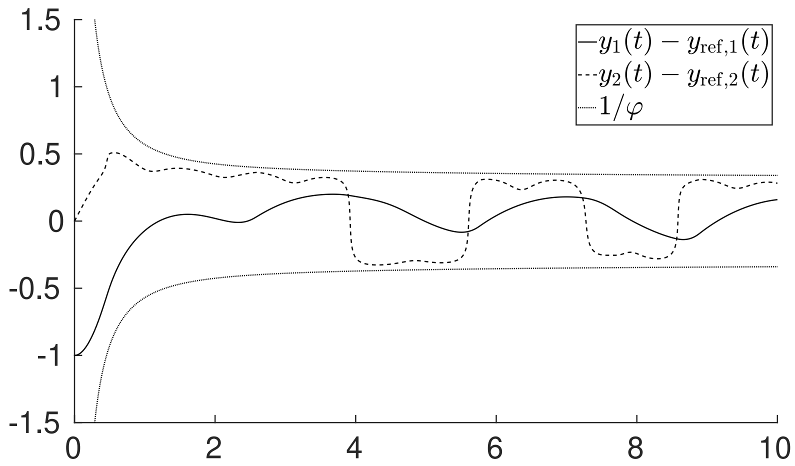

Fig. 4.1a: Funnel and tracking errors

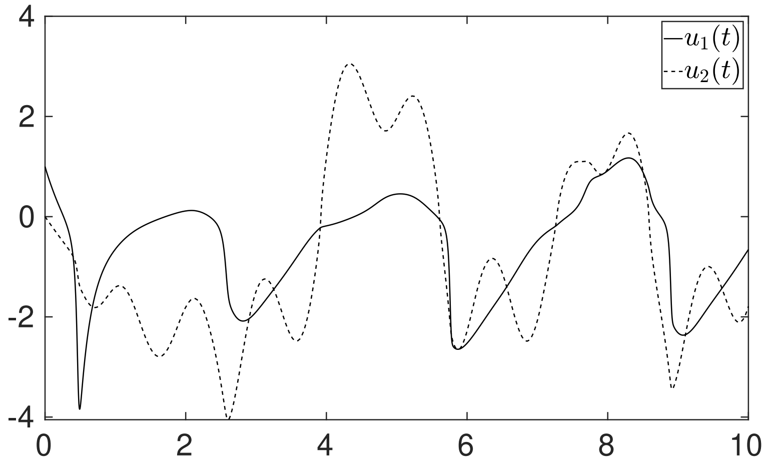

Fig. 4.1b: Input functions

FIGURE 4.1: Simulation of the controller (4.2.1) for the system (4.2.31).

We have already shown in Remark 2.3.11 that the system (4.2.32) is 2-input, 2 output systems and has generalized vector relative degree $r = (3,0)$. Interpretation this system to *normal form* (4.1.4), the system has matrix $Q = -1$ which means that its zero dynamics is asymptotically stable and the matrix $A_{22} = \begin{bmatrix} 5 & 0 \\ 0 & 0 \end{bmatrix}$. Therefore, we can set $\hat{k} = 6$ in funnel controller (4.2.1). We may choose, in simulation, reference signal $y_{\mathrm{ref}} = (\sin t, \cos 2t)^\top$, and initial value $x^0 = (0,1,1,0,0)^\top$. Since there is not any constraint condition imposed on funnel boundary, we may choose the same funnel function φ for all values.

$$\begin{aligned} \varphi : \mathbb{R}_{\geq 0} &\to \mathbb{R}_{\geq 0}, \\ t &\mapsto \frac{1}{2} t e^{-t} + 2 \arctan t. \end{aligned}$$

This simulation has been performed in MATLAB (solver: ode15s, relative tolerance: 10^{-14}, absolute tolerance: 10^{-5}) over the time interval [0,10], see Figure 4.2.

For more detail, Figure 4.2a shows the output errors stay in the funnel and Figure 4.2b shows the input components that drive tracking errors.

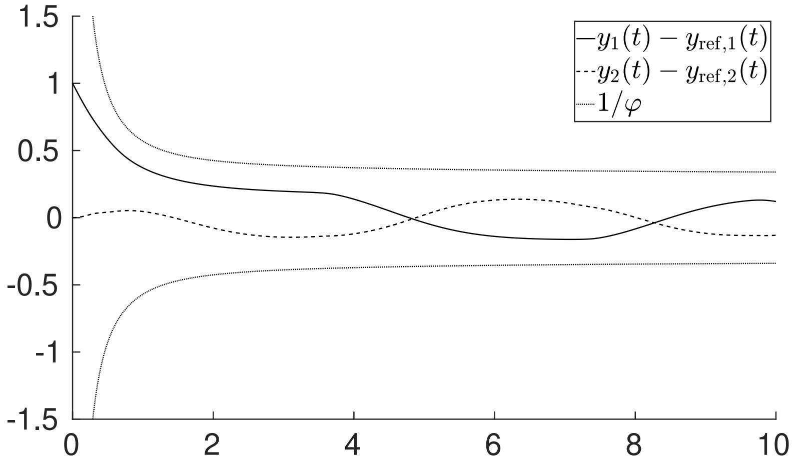

Fig. 4.2a: Funnel and tracking errors

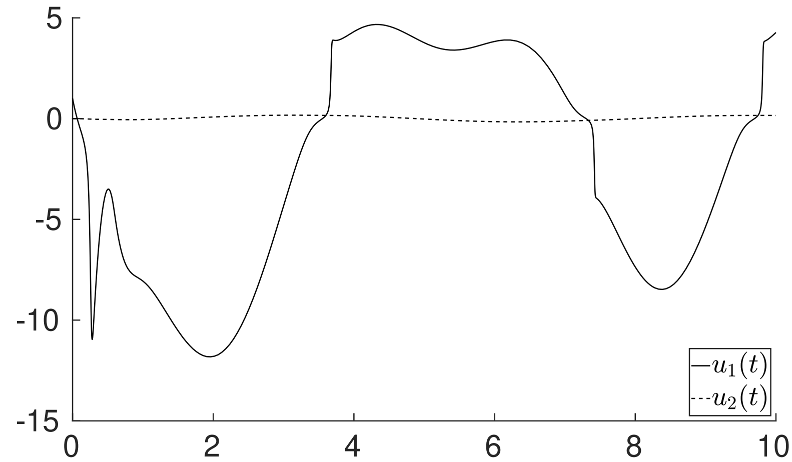

Fig. 4.2b: Input functions

FIGURE 4.2: Simulation of the controller (4.2.1) for the system (4.2.32).

Chapter 5

Application: Adaptive control of an overhead crane

In this chapter, we study an application of funnel control to an overhead crane model which has been introduced by Otto and Seifried in [73] and Fliess et al. in [24, 25]. The objective is to design a closed-loop tracking controller that also takes into account the transient behaviour. Unfortunately, its vector relative degree is not available with respect to considered input, output vectors. That means the funnel controller projected in chapters 3, and 4 is not subjected to direct application. The dynamic state feedback is then ultilized to acquire a new system with strict relative degree four. This allows us to propose a new output feedback controller based on funnel control that can perform the objective. Computer simulations are displayed, demonstrating that our approach can be employed to move loads from one to another given position in situations of having several circumnavigated obstacles.

5.1 Overhead crane model

The overhead crane model is an overhead gantry crane with a trolley moving along a horizontal axis. A suspended load is attached to the trolley by four ropes, which are assumed to be rigid and massless. Moreover, the winches on the trolley are synchronized which help controlling the length of the ropes so that the attached load does not rotate around itself. Therefore, this model can be represented by a point mass connected to the trolley by a single rope as shown in Figure 5.1.

Normally, the control inputs are the external force applied to the trolley and the hoisting torque. However, force and torque controlled actuators are difficult to realize and often pose robustness issues due to drive train friction. Hence, the velocity-controlled is considered which the system control inputs u_s (in $\mathrm{m\,s^{-1}}$), and u_l (in $\mathrm{m\,s^{-1}}$) are set as reference velocities of the trolley and rope length respectively. The equations of motion of the overhead crane model are

$$\begin{aligned} \tau_s \ddot{s} + \dot{s} &= u_s \\ \tau_l \ddot{l} + \dot{l} &= u_l \\ \cos(\varphi)\ddot{s} + l\ddot{\varphi} + 2\dot{\varphi}\dot{l} &= -g\sin(\varphi), \end{aligned} \tag{5.1.1}$$

where s (in m) is the trolley position, l (in m) is the rope length, and φ (in rad) is the swing angle. We note that the rope velocity $\dot{l}$ (in $\mathrm{m\,s^{-1}}$), and trolley velocity $\dot{s}$ (in $\mathrm{m\,s^{-1}}$) are then rheonomic constraints on the system. And τ_s (in s), τ_l (in s) are time constants of trolley and winch actuator, resp., and g (in $\mathrm{m/s^2}$) is the gravitational constant. The constants do not depend on the trolley or load mass. As output of the model we choose the position of the load

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} s + l\sin\varphi \\ l\cos\varphi \end{pmatrix}.$$

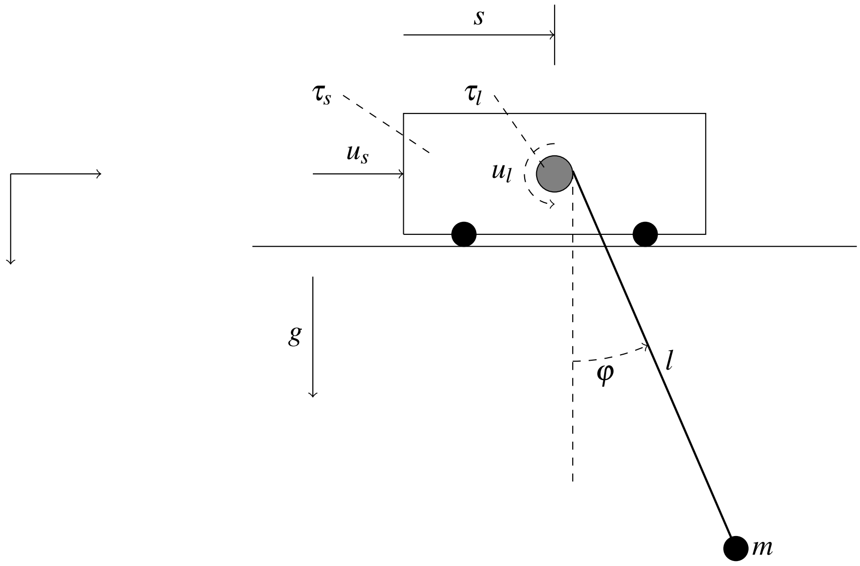

FIGURE 5.1: Crane model.

Before study crane model for more detail, we recall basic notations by Isidori[56, Sec.1.2]. Consider the multivariable non-linear systems which are described in state space form by equations of the following kind

$$\begin{aligned} \dot{x} &= f(x) + \sum_{i=1}^{m} g_i(x)u_i, \\ y_1 &= h_1(x), \\ &\vdots \\ y_m &= h_m(x), \end{aligned} \tag{5.1.2}$$

where $f(x)$, $g_1(x), \ldots, g_m(x)$, and $h_1(x), \ldots, h_m(x)$ are smooth functions, defined on an open set X of $\mathbb{R}^n$. We denote the *derivative of h_i along f* at x by

$$L_f h_i(x) = \sum_{j=1}^{n} \frac{\partial h_i}{\partial x_j} f_j(x).$$

If h_i is being differentiated k times along f, the function $L_f^k h_i(x)$ satisfies the recursion

$$L_f^k h_i(x) = \sum_{j=1}^{n} \frac{\partial (L_f^{k-1} h_i)}{\partial x_j} f_j(x)$$

with note that $L_f^0 h_i(x) = h_i(x)$. Moreover, we also denote

$$L_{g_j} L_f^k h_i(x) = \sum_{p=1}^{n} \frac{\partial (L_f^k h_i)}{\partial x_p} g_p(x).$$

For sake of better overview, we reintroduce the definition of vector relative degree in [56, Sec.5.1].

Definition 5.1.1 ([56, Sec.5.1])**.** A multivariable nonlinear system of the form (5.1.2) has a (vector) relative degree $(r_1, \ldots, r_m)$ at a point x^0 if

(i) $L_{g_j}(L_f^k h_i(x)) = 0$ for all $1 \le j \le m$, $0 \le k < r_i - 1$, $1 \le i \le m$, and for all x in a neighborhood of x^0,

(ii) the matrix

$$\Gamma(x) = \begin{bmatrix} L_{g_1}(L_f^{r_1-1}h_1(x)) & \cdots & L_{g_m}(L_f^{r_1-1}h_1(x)) \\ \vdots & \ddots & \vdots \\ L_{g_1}(L_f^{r_m-1}h_m(x)) & \cdots & L_{g_m}(L_f^{r_m-1}h_m(x)) \end{bmatrix}$$

is nonsingular at $x = x^0$.

Remark 5.1.2. In order to compare this definition with other definition on vector relative degree which has been introduced in previous chapters, we calculate the vector relative degree of a linear system

$$\begin{aligned} \dot{x} &= Ax + Bu, \\ y &= Cx, \end{aligned}$$

where $A \in \mathbb{R}^{n\times n}$, $B = [B_1, B_2 \ldots, B_m]$ with $B_i \in \mathbb{R}^{n\times 1}$, and $C = \begin{bmatrix} C_1 \\ C_2 \\ \vdots \\ C_m \end{bmatrix}$ with $C_i \in \mathbb{R}^{1\times n}$, $i = 1, \ldots, m$.

In this case, since $f(x) = Ax$, $g_j(x) = B_j$, $h_i(x) = C_i x$, we obtain

$$L_f^k h_i(x) = C_i A^k x, \qquad \text{for } i = 1, \ldots, m,$$

and therefore

$$L_{g_j} L_f^k h_i(x) = C_i A^k B_j, \qquad \text{for } i = 1, \ldots, m, \quad j = 1, \ldots, m.$$

Thus, the vector relative degree $(r_1, \ldots, r_m)$ is characterized by the conditions

(a) $C_i A^k B_j = 0$ for all $1 \le j \le m$, $0 \le k < r_i - 1$, $1 \le i \le m$,

(b) and the matrix

$$\Gamma = \begin{bmatrix} C_1 A^{r_1-1} B_1 & C_1 A^{r_1-1} B_2 & \cdots & C_1 A^{r_1-1} B_m \\ C_2 A^{r_2-1} B_1 & C_2 A^{r_2-1} B_2 & \cdots & C_2 A^{r_2-1} B_m \\ \vdots & \vdots & \ddots & \vdots \\ C_m A^{r_m-1} B_1 & C_m A^{r_m-1} B_2 & \cdots & C_m A^{r_m-1} B_m \end{bmatrix}$$

is non-singular.

In conjunction with Remark 2.2.7(iii), it is shown that Definition 2.2.6 and Definition 5.1.1 are consistent.

To analyze the properties of the crane system, we transform (5.1.1) into the form (5.1.2) by denoting $x_1 := s$, $x_2 := \dot{s}$, $x_3 := l$, $x_4 := \dot{l}$, $x_5 := \varphi$, $x_6 := \dot{\varphi}$, and $u_1 := u_s$, $u_2 := u_l$.

$$\begin{aligned} \dot{x} &= f(x) + g_1(x)u_1 + g_2(x)u_2, \\ \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} &= \begin{pmatrix} x_1 + x_3 \sin x_5 \\ x_3 \cos x_5 \end{pmatrix}, \end{aligned} \tag{5.1.3}$$

where

$$x := \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix}, f(x) := \begin{pmatrix} x_2 \\ -\frac{x_2}{\tau_s} \\ x_4 \\ -\frac{x_4}{\tau_l} \\ x_6 \\ -\frac{2x_4x_6 + g\sin x_5}{x_3} + \frac{x_2\cos x_5}{x_3\tau_s} \end{pmatrix}, g_1(x) := \begin{pmatrix} 0 \\ \frac{1}{\tau_s} \\ 0 \\ 0 \\ 0 \\ -\frac{\cos x_5}{x_3\tau_s} \end{pmatrix}, g_2(x) := \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{\tau_l} \\ 0 \\ 0 \end{pmatrix}.$$

Since $y_1 = x_1 + x_3 \sin x_5$, and $y_2 = x_3 \cos x_5$, we have

$$L_f y_1 = x_2 + x_4 \sin x_5 + x_3 x_6 \cos x_5,$$
$$L_f y_2 = x_4 \cos x_5 - x_3 x_6 \sin x_5.$$

Then we obtain

$$\begin{array}{lllll} L_{g_1} y_1 & = & L_{g_2} y_1 & = & 0, \\ L_{g_1} y_2 & = & L_{g_2} y_2 & = & 0, \end{array}$$

and

$$L_{g_1}(L_f y_1) = \frac{\sin^2 x_5}{\tau_s},$$
$$L_{g_2}(L_f y_1) = \frac{\sin x_5}{\tau_l},$$
$$L_{g_1}(L_f y_2) = \frac{\sin x_5 \cos x_5}{\tau_s},$$
$$L_{g_2}(L_f y_2) = \frac{\cos x_5}{\tau_l}.$$

We have matrix

$$\Gamma(x) = \begin{bmatrix} L_{g_1}(L_f y_1) & L_{g_2}(L_f y_1) \\ L_{g_1}(L_f y_2) & L_{g_2}(L_f y_2) \end{bmatrix} = \begin{bmatrix} \frac{\sin^2 x_5}{\tau_s} & \frac{\sin x_5}{\tau_l} \\ \frac{\sin x_5 \cos x_5}{\tau_s} & \frac{\cos x_5}{\tau_l} \end{bmatrix}.$$

Obviously, the rank of $\Gamma(x)$ is constant one, whence the crane model system has no strict (vector) relative degree at any point of the state space. It means that we can not directly apply funnel controller (3.2.1) in Chapter 3 to this system. Therefore, we need to modify - by means of control laws - this system to get a new one having vector relative degree. We will rather use a feedback structure which incorporates an additional set of state variables, namely a *dynamic state feedback*, see Isidori[56, Sec.5.4], or Seifried[80, Subsec.3.3.5]. We introduce a technique which is presented in [56, Sec.5.4], namely *dynamic extension algorithm* to build a *dynamic state feedback*. To demonstrate this algorithm, we use the notation of the matrix $\Gamma(x)$ from the Definition 5.1.1,

$$\Gamma(x) = \begin{bmatrix} L_{g_1}(L_f^{r_1-1} h_1(x)) & \cdots & L_{g_m}(L_f^{r_1-1} h_1(x)) \\ \vdots & \ddots & \vdots \\ L_{g_1}(L_f^{r_m-1} h_m(x)) & \cdots & L_{g_m}(L_f^{r_m-1} h_m(x)) \end{bmatrix},$$

with the symbol r_i denoted the number satisfying

$$L_{g_j}(L_f^{r_i-1}h_i(x)) \neq 0, \text{ and } L_{g_j}(L_f^k h_i(x)) = 0$$

for $1 \leq j \leq m$, $0 \leq k < r_i - 1$, $1 \leq i \leq m$, and for all x on a neighborhood U of x^0. Suppose that $\Gamma(x)$ has constant rank on U, and rank $\Gamma(x) < m$ which means that system (5.1.2) has no vector relative degree.

Dynamic extension algorithm. Denote $\Gamma_i(x)$, $i = 1,\ldots,m$ being the $i-th$ row of matrix $\Gamma(x)$. Since $\Gamma(x)$ is singular, and has constant rank on U, it is possible to find $p \in \{1,\ldots,m\}$, and $p-1$ smooth functions $c_1(x),\ldots,c_{p-1}(x)$ defined on U, and i_0, j_0 such that $c_{i_0}(x)$ is not identically zero,

$$\Gamma_p(x) = \sum_{i=1}^{p-1} c_i(x)\Gamma_i(x),$$

and

$$\Gamma_{i_0 j_0}(x^0) = L_{g_{j_0}}(L_f^{r_{i_0}-1}h_{i_0}(x^0)) \neq 0.$$

Then, we define the dynamic feedback

$$\begin{aligned} u_j &= v_j \text{ for } j \neq j_0 \\ u_{j_0} &= \frac{1}{\Gamma_{i_0 j_0}}\Big(p(x) + q(x)\xi - \sum_{j=1, j\neq j_0}^{m} \Gamma_{i_0 j}(x)v_j\Big), \\ \dot{\xi} &= v_{j_0}, \end{aligned}$$

where $p(x)$ and $q(x)$ are arbitrary functions satisfying $p(x^0) = 0$ and $q(x^0) = 1$.
Apply to system (5.1.2), we obtain an extended system

$$\begin{aligned} \dot{x} &= f(x) + \sum_{j=1, j\neq j_0}^{m} g_j(x)v_j + \frac{g_{j_0}(x)}{\Gamma_{i_0 j_0}}\Big(p(x) + q(x)\xi - \sum_{j=1, j\neq j_0}^{m} \Gamma_{i_0 j}(x)v_j\Big), \\ \dot{\xi} &= v_{j_0}, \\ y_1 &= h_1(x), \\ &\vdots \\ y_m &= h_m(x). \end{aligned}$$

We repeat the procedure for new system until achieving vector relative degree. The system with dynamic state feedback is depicted in Figure 5.2.

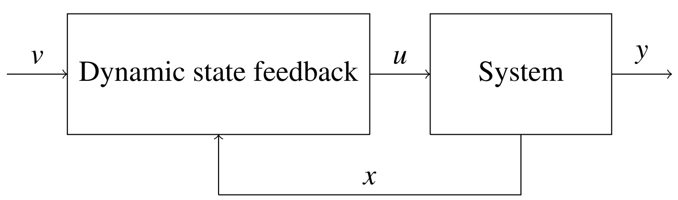

FIGURE 5.2: Extended system with dynamic state feedback.

The *dynamic extension algorithm* is applied to a system in which the high gain matrix $\Gamma(x)$ is singular, or in the other words, the rank of $\Gamma(x)$ is less than m which equals to the number of output or the number of input. Therefore, by using a dynamic state feedback, we can get an extended system in which the rank of the high gain matrix is possibly larger.

Hence, a system having vector relative degree may be achieved after a finite number of iterations. That is the reason why *dynamic extension algorithm* could be possible applied to extend the multi-input, multi-output non-linear system (5.1.3) until the extended system has a well-defined vector relative degree. By iteration approach, in each step of the algorithm, system inputs or combination of those inputs are delayed by using integrators until the gain matrix of obtained system has full rank. This technique usually add some new state variables to the original system. Moreover, by [56, Remark.5.4.6], any two regularizing dynamic extension have necessarily the same dimension and only differ by change of coordinates and regular static feedback. For the system (5.1.3), we use above procedure to get the following *dynamic state feedback*

$$\begin{aligned}
\dot{x}_7 &= x_8,\\
\dot{x}_8 &= v_2,\\
u_1 &= v_1,\\
u_2 &= x_4 + x_3 x_6^2 \tau_l + \frac{(x_2 - v_1)\tau_l \sin x_5}{\tau_s} + \frac{(x_7 - g\sin^2 x_5)\tau_l}{\cos x_5}.
\end{aligned} \tag{5.1.4}$$

Hence, we have a new system

$$\begin{aligned}
\dot{\tilde{x}} &= \tilde{f}(x) + \tilde{g}_1(x)v_1 + \tilde{g}_2(x)v_2,\\
\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} &= \begin{pmatrix} x_1 + x_3 \sin x_5 \\ x_3 \cos x_5 \end{pmatrix},
\end{aligned} \tag{5.1.5}$$

where

$$\tilde{x} := \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \end{pmatrix}, \tilde{f}(\tilde{x}) := \begin{pmatrix} x_2 \\ -\dfrac{x_2}{\tau_s} \\ x_4 \\ x_3 x_6^2 + \dfrac{x_2 \sin x_5}{\tau_s} + \dfrac{x_7 - g\sin^2 x_5}{\cos x_5} \\ x_6 \\ -\dfrac{2x_4 x_6 + g\sin x_5}{x_3} + \dfrac{x_2 \cos x_5}{x_3 \tau_s} \\ x_8 \\ 0 \end{pmatrix}, \tilde{g}_1(\tilde{x}) := \begin{pmatrix} 0 \\ \dfrac{1}{\tau_s} \\ 0 \\ -\dfrac{\sin x_5}{\tau_s} \\ 0 \\ -\dfrac{\cos x_5}{x_3 \tau_s} \\ 0 \\ 0 \end{pmatrix}, \tilde{g}_2(\tilde{x}) := \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

Since $y_1 = x_1 + x_3 \sin x_5$, and $y_2 = x_3 \cos x_5$, we have

$$\begin{aligned}
L_{\tilde{f}} y_1 &= x_2 + x_4 \sin x_5 + x_3 x_6 \cos x_5,\\
L_{\tilde{f}} y_2 &= x_4 \cos x_5 - x_3 x_6 \sin x_5,\\
L^2_{\tilde{f}} y_1 &= (x_7 - g)\tan x_5,\\
L^2_{\tilde{f}} y_2 &= x_7,\\
L^3_{\tilde{f}} y_1 &= x_8 \tan x_5 + x_6(1 + \tan^2 x_5)(x_7 - g),\\
L^3_{\tilde{f}} y_2 &= x_8.
\end{aligned}$$

Then, we obtain

$$\begin{array}{ccccccccccccc} L_{\tilde{g}_1}y_1 &=& L_{\tilde{g}_2}y_1 &=& L_{\tilde{g}_1}(L_{\tilde{f}}y_1) &=& L_{\tilde{g}_2}(L_{\tilde{f}}y_1) &=& L_{\tilde{g}_1}(L^2_{\tilde{f}}y_1) &=& L_{\tilde{g}_2}(L^2_{\tilde{f}}y_1) &=& 0, \\ L_{\tilde{g}_1}y_2 &=& L_{\tilde{g}_2}y_2 &=& L_{\tilde{g}_1}(L_{\tilde{f}}y_2) &=& L_{\tilde{g}_2}(L_{\tilde{f}}y_2) &=& L_{\tilde{g}_1}(L^2_{\tilde{f}}y_2) &=& L_{\tilde{g}_2}(L^2_{\tilde{f}}y_2) &=& 0, \end{array}$$

and

$$\begin{aligned} L_{\tilde{g}_1}(L^3_{\tilde{f}}y_1) &= \frac{g-x_7}{x_3\tau_s\cos x_5}, \\ L_{\tilde{g}_2}(L^3_{\tilde{f}}y_1) &= \frac{\sin x_5}{\cos x_5}, \\ L_{\tilde{g}_1}(L^3_{\tilde{f}}y_2) &= 0, \\ L_{\tilde{g}_2}(L^3_{\tilde{f}}y_2) &= 1, \end{aligned}$$

Obviously, the matrix

$$\Gamma(\tilde{x}) = \begin{bmatrix} L_{\tilde{g}_1}(L^3_{\tilde{f}}y_1) & L_{\tilde{g}_2}(L^3_{\tilde{f}}y_1) \\ L_{\tilde{g}_1}(L^3_{\tilde{f}}y_2) & L_{\tilde{g}_2}(L^3_{\tilde{f}}y_2) \end{bmatrix} = \begin{bmatrix} \dfrac{g-x_7}{x_3\tau_s\cos x_5} & \dfrac{\sin x_5}{\cos x_5} \\ 0 & 1 \end{bmatrix} \tag{5.1.6}$$

is nonsingular at any point of state space which satisfies $x_7 \neq g$. Therefore, based on Definition 5.1.1, the extended system (5.1.5) has strict relative degree $r = 4$ at any point of state space which satisfies $x_7 \neq g$.

Remarks 5.1.3. If $x_3 = 0$, or $\cos x_5 = 0$, then $y_2 = 0$. This is unrealistic position where the load is at the same horizontal level as the trolley. Therefore, in designing an adaptive controller for system (5.1.5), the reference trajectory of the load and tracking area must be chosen such that the load always moves under the trolley to avoid the case $x_3 = 0$, $\cos x_5 = 0$, see Figure 5.3.

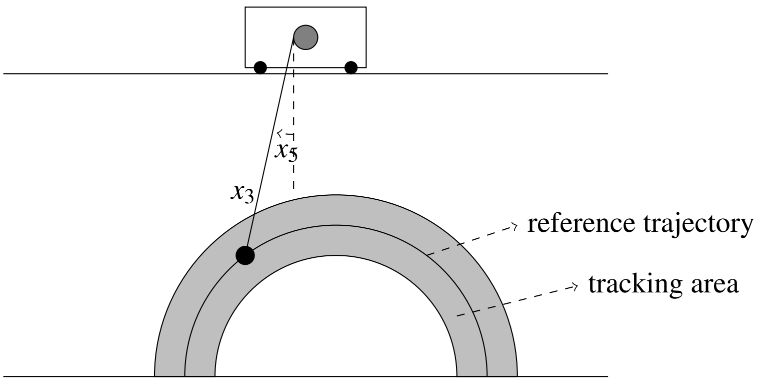

FIGURE 5.3: Crane model and reference trajectory

We set an output feedback controller based on funnel control for the extended systems (5.1.5)

$$\boxed{\begin{array}{l} \begin{array}{rclcl} e_0(t) &=& e(t) &=& y(t) - y_{\mathrm{ref}}(t), \\ e_1(t) &=& \dot{e}_0(t) &+& k_0(t) \cdot e_0(t), \\ e_2(t) &=& \dot{e}_1(t) &+& k_1(t) \cdot e_1(t), \\ e_3(t) &=& \dot{e}_2(t) &+& k_2(t) \cdot e_2(t), \end{array} \\ k_i(t) = \dfrac{1}{1 - \varphi_i^2(t)\|e_i(t)\|^2}, \quad i = 0,1,2, \\ k_3(t) = \dfrac{\hat{k}}{1 - \varphi_3^2(t)\|e_3(t)\|^2}, \\ w(t) = -k_3(t) \cdot e_3(t), \\ v(t) = \Gamma(\tilde{x})^{-1} \cdot w(t) \end{array}} \tag{5.1.7}$$

with $\hat{k}$ is a positive constant, and $\Gamma(\tilde{x})$ from (5.1.6).

Remark 5.1.4.

(i) Since the Remark 5.1.3, we need to choose reference signal and funnel function such that $y_2 \neq 0$. We note that $\|e(t)\| \leq \varphi_0(t)^{-1}$. Hence, we can choose the reference signal and funnel functions such that

$$y_{\mathrm{ref},2}(t) - \varphi_0(t)^{-1} > 0 \text{ for all } t \geq 0. \tag{5.1.8}$$

The condition (5.1.8) ensures that $y_2 > 0$.

(ii) A crucial point of funnel controller (5.1.7) is that $\Gamma(\tilde{x})^{-1}$ could not exist when $x_7 = \ddot{y}_2 = g$. On the other hand, refer to Remark 3.2.6 (ii), the bound of $\ddot{e}(t)$ can be calculated a priori and can be adjusted by using the funnel functions $\varphi_0, \varphi_1, \varphi_2$, and φ_3. Indeed, set $\psi_i(t) := \varphi_i(t)^{-1}$, the estimation (3.2.13) shows that

$$\begin{aligned} \|\ddot{e}(t)\| \leq & \Big(\psi_2(t) - \varepsilon_2(t)\Big) \\ & + \left[\frac{\psi_0(t)}{\varepsilon_0(t)} + \frac{\psi_1(t)}{\varepsilon_1(t)} + 2\Big(\frac{\psi_0(t)}{\varepsilon_0(t)}\Big)^2 \varphi_0(t)^2 \Big(\psi_0(t) - \varepsilon_0(t)\Big)^2\right] \Big(\psi_1(t) - \varepsilon_1(t)\Big) \\ & + \left[\Big(\frac{\psi_0(t)}{\varepsilon_0(t)}\Big)^2 + 2\Big(\frac{\psi_0(t)}{\varepsilon_0(t)}\Big)^3 \varphi_0(t)^2 \Big(\psi_0(t) - \varepsilon_0(t)\Big)^2\right] \Big(\psi_0(t) - \varepsilon_0(t)\Big) \\ & + 2\Big(\frac{\psi_0(t)}{\varepsilon_0(t)}\Big)^2 |\dot{\varphi}_0(t)| \Big(\psi_0(t) - \varepsilon_0(t)\Big)^2. \end{aligned}$$

with $\varepsilon_i(t)$ is the solution of following initial value problems

$$\begin{aligned} \dot{\varepsilon}_i(t) &= \dot{\psi}_i(t) - \psi_{i+1}(t) + \frac{\psi_i(t)(\psi_i(t) - \varepsilon_i(t))}{2\varepsilon_i(t)}, \\ \varepsilon_i(0) &= \psi_i(0) - \|e_i(0)\|, \end{aligned} \tag{5.1.9}$$

for $i = 0,1,2$. We note that (5.1.9) is proved to have a unique global solution $\varepsilon_i : \mathbb{R}_{\geq 0} \to \mathbb{R}$ satisfying

$$\begin{aligned} \varepsilon_{i,\min} \leq \varepsilon_i(t) \leq \psi_i(t) - \varepsilon_{i,\max} & \text{ for all } t \geq 0, \\ \|e_i(t)\| \leq \psi_i(t) - \varepsilon_i(t) & \text{ for all } t \geq 0, \end{aligned}$$

with

$$\begin{aligned}
\lambda_i &:= \inf_{t\geq 0} \psi_i(t) > 0, \quad i = 0,1,2,\\
\kappa_i &:= \|\psi_{i+1} - \dot{\psi}_i\|_\infty, \quad i = 0,1,2,\\
\varepsilon_{i,\min} &:= \min\left\{ \frac{\lambda_i^2}{2\kappa_i + \|\psi_i\|_\infty}, \psi_i(0) - \|e_i(0)\| \right\} > 0,\\
\varepsilon_{i,\max} &:= \min\left\{ \frac{\lambda_{i+1}\lambda_i}{\|\psi_i\|_\infty}, \frac{\lambda_i}{2}, \|e_i(0)\| \right\} \leq 0,
\end{aligned}$$

by [8, Lem. 4.1, Lem. 4.3]. Consequently, we can choose the reference signal and funnel functions such that

$$\begin{aligned}
&\ddot{y}_{\text{ref},2}(t) + \Big(\psi_2(t) - \varepsilon_2(t)\Big)\\
&\quad + \left[\frac{\psi_0(t)}{\varepsilon_0(t)} + \frac{\psi_1(t)}{\varepsilon_1(t)} + 2\Big(\frac{\psi_0(t)}{\varepsilon_0(t)}\Big)^2 \varphi_0(t)^2\Big(\psi_0(t) - \varepsilon_0(t)\Big)^2\right]\Big(\psi_1(t) - \varepsilon_1(t)\Big)\\
&\quad + \left[\Big(\frac{\psi_0(t)}{\varepsilon_0(t)}\Big)^2 + 2\Big(\frac{\psi_0(t)}{\varepsilon_0(t)}\Big)^3 \varphi_0(t)^2\Big(\psi_0(t) - \varepsilon_0(t)\Big)^2\right]\Big(\psi_0(t) - \varepsilon_0(t)\Big)\\
&\quad + 2\Big(\frac{\psi_0(t)}{\varepsilon_0(t)}\Big)^2 |\dot{\varphi}_0(t)|\Big(\psi_0(t) - \varepsilon_0(t)\Big)^2 \leq g_0 < g, \quad (5.1.10)
\end{aligned}$$

for all $t \geq 0$, and g_0 is a constant. The condition (5.1.10) ensures that $\ddot{y}_2(t) < g$. Therefore, this can get rid of the singularity of $\Gamma(\tilde{x})$. Hence, system (5.1.5) has strict relative degree $r = 4$.

Now, with the suitable chosen of y_{ref}, ψ_0, $\psi_1(t)$, $\psi_2(t)$, $\psi_3(t)$, and initial value $\tilde{x}^0$ satisfying conditions (5.1.8), and (5.1.10), the final feedback controller, consisting of the dynamic state feedback (5.1.4) and the output feedback controller (5.1.7) is depicted in Figure 5.4.

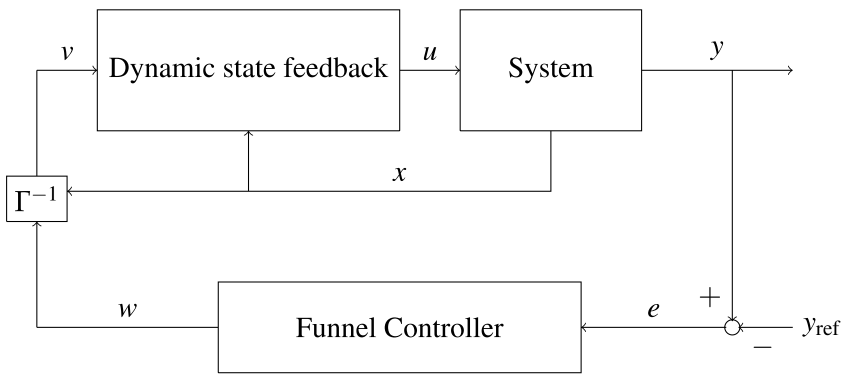

FIGURE 5.4: Combination of funnel controller and dynamic extension.

Theorem 5.1.5. *Consider the crane systems* (5.1.3)*, and let* $y_{\text{ref}} \in \mathscr{W}^{4,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R}^2)$*,* $\varphi_i \in \Phi_{4-i}$ *for* $i = 0,1,2,3$*, and initial value* $\tilde{x}^0$ *satisfy conditions* (5.1.8)*,* (5.1.10)*, and* e_i *as defined in* (5.1.7) *satisfy*

$$\varphi_i(0)\|e_i(0)\| < 1, \textit{ for } i = 0,1,2,3.$$

Then the combination of dynamic state feedback (5.1.4) *and output feedback controller* (5.1.7) *applied to the crane systems* (5.1.3) *yields a closed-loop system which has a maximal solution* $x : [0,\omega) \to \mathbb{R}^6$ *with the properties,* *(i)* $\omega = \infty$,

(ii) all involved signals $x(\cdot)$, $u(\cdot)$, $k_i(\cdot)$ *for* $i = 0,1,2,3$ *are bounded,*

(iii) the errors evolve uniformly within the respective performance funnel in the sense

(a) for $i = 0,1,2$, *and* $\varepsilon_i(t)$ *as in* (5.1.9), $\|e_i(t)\| \le \varphi_i(t)^{-1} - \varepsilon_i(t)$ *for all* $t \ge 0$,

(b) $\exists \varepsilon_3 > 0$, $\forall t \ge 0$: $\|e_3(t)\| \le \varphi_3(t)^{-1} - \varepsilon_3$.

Proof. First we consider clsoed-loop system which is obtained by application output feedback controller (5.1.7) to the extended system (5.1.5). Since $y_1 = x_1 + x_3 \sin x_5$, and $y_2 = x_3 \cos x_5$, we calculate the derivatives of the outputs of the extended system (5.1.5)

$$\begin{aligned}
\dot y_1 &= x_2 + x_4 \sin x_5 + x_3 x_6 \cos x_5,\\
\dot y_2 &= x_4 \cos x_5 - x_3 x_6 \sin x_5,\\
\ddot y_1 &= (x_7 - g)\tan x_5,\\
\ddot y_2 &= x_7,\\
y_1^{(3)} &= x_8 \tan x_5 + x_6(1+\tan^2 x_5)(x_7 - g),\\
y_2^{(3)} &= x_8,\\
y_1^{(4)} &= 2x_6(x_7-g)(1+\tan^2 x_5)\tan x_5 + 2(1+\tan^2 x_5)x_6 x_8\\
&\quad + (1+\tan^2 x_5)(x_7-g)\left[-\frac{2x_4x_6 + g\sin x_5}{x_3} + \frac{x_2\cos x_5}{x_3\tau_s}\right] - \frac{x_7-g}{x_3\tau_s\cos x_5}v_1 + v_2\tan x_5,\\
y_2^{(4)} &= v_2,
\end{aligned}$$

Hence, the extended system (5.1.5) can be written in the form of input-output relation.

$$\begin{aligned}
y_1^{(4)} &= 2\frac{\ddot y_1}{\ddot y_2 - g}\left(y_1^{(3)} - y_2^{(3)}\frac{\ddot y_1}{\ddot y_2 - g}\right) + 2\left(y_1^{(3)} - y_2^{(3)}\frac{\ddot y_1}{\ddot y_2 - g}\right)\times\\
&\quad\times\left[\frac{y_2^{(3)}}{\ddot y_2 - g} - \frac{\ddot y_2}{y_2} - \frac{\ddot y_1}{\ddot y_2 - g}\frac{1}{\left(1+\left(\frac{\ddot y_1}{\ddot y_2 - g}\right)^2\right)(\ddot y_2 - g)}\left(y_1^{(3)} - y_2^{(3)}\frac{\ddot y_1}{\ddot y_2 - g}\right)\right]\\
&\quad - g\frac{\ddot y_1}{y_2} + \frac{1}{y_2\tau_s}\Big(\dot y_1(\ddot y_2 - g) - \dot y_2\ddot y_1\Big) - \frac{1}{\tau_s}\left(y_1^{(3)} - y_2^{(3)}\frac{\ddot y_1}{\ddot y_2 - g}\right)\\
&\quad - \frac{\ddot y_2 - g}{y_2\tau_s}v_1 + \frac{\ddot y_1}{\ddot y_2 - g}v_2,\\
y_2^{(4)} &= v_2,
\end{aligned}$$

Set

$$p(y,\dot{y},\ddot{y},y^{(3)}) := 2\frac{\ddot{y}_1}{\ddot{y}_2-g}\left(y_1^{(3)}-y_2^{(3)}\frac{\ddot{y}_1}{\ddot{y}_2-g}\right)+2\left(y_1^{(3)}-y_2^{(3)}\frac{\ddot{y}_1}{\ddot{y}_2-g}\right)\times$$
$$\times\left[\frac{y_2^{(3)}}{\ddot{y}_2-g}-\frac{\dot{y}_2}{y_2}-\frac{\ddot{y}_1}{\ddot{y}_2-g}\frac{1}{\left(1+\left(\frac{\ddot{y}_1}{\ddot{y}_2-g}\right)^2\right)(\ddot{y}_2-g)}\left(y_1^{(3)}-y_2^{(3)}\frac{\ddot{y}_1}{\ddot{y}_2-g}\right)\right]$$
$$-g\frac{\ddot{y}_1}{y_2}+\frac{1}{y_2\tau_s}\Big(\dot{y}_1(\ddot{y}_2-g)-\dot{y}_2\ddot{y}_1\Big)-\frac{1}{\tau_s}\left(y_1^{(3)}-y_2^{(3)}\frac{\ddot{y}_1}{\ddot{y}_2-g}\right).$$

We note that $\frac{\ddot{y}_1}{\ddot{y}_2-g}$ is well-defined since $\ddot{y}_1=(\ddot{y}_2-g)\tan x_5$, and $\tan x_5$ are determined. Furthermore, $\frac{1}{\left(1+\left(\frac{\ddot{y}_1}{\ddot{y}_2-g}\right)^2\right)(\ddot{y}_2-g)}\left(y_1^{(3)}-y_2^{(3)}\frac{\ddot{y}_1}{\ddot{y}_2-g}\right)$ is also well-defined since $y_1^{(3)}-y_2^{(3)}\tan x_5 = x_6(1+\tan^2 x_5)(\ddot{y}_2-g)$, and $x_6,\tan x_5$ are determined. As a consequence, the function $p(y,\dot{y},\ddot{y},y^{(3)})$ is well-defined. Hence, we have an equivalence of system (5.1.5)

$$\begin{pmatrix} y_1^{(4)} \\ y_2^{(4)} \end{pmatrix} = \begin{pmatrix} p(y,\dot{y},\ddot{y},y^{(3)}) \\ 0 \end{pmatrix} + \underbrace{\begin{bmatrix} \frac{g-\ddot{y}_2}{y_2\tau_s} & \frac{\ddot{y}_1}{\ddot{y}_2-g} \\ 0 & 1 \end{bmatrix}}_{\Gamma(\tilde{x})} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}.$$

Now, set

$$w_1 = \frac{g-\ddot{y}_2}{y_2\tau_s}v_1+\frac{\ddot{y}_1}{\ddot{y}_2-g}v_2,$$
$$w_2 = v_2,$$

then we have system

$$\begin{pmatrix} y_1^{(4)} \\ y_2^{(4)} \end{pmatrix} = \begin{pmatrix} p(y,\dot{y},\ddot{y},y^{(3)}) \\ 0 \end{pmatrix} + \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}. \tag{5.1.11}$$

The system (5.1.11) belongs to the system class (3.1.1) with strict relative degree $r=4$, and "high-frequency gain" being indentity matrix. Therefore, the funnel controller

$$\begin{aligned}
e_0(t) &= e(t) = y(t)-y_{\mathrm{ref}}(t),\\
e_1(t) &= \dot{e}_0(t) + k_0(t)\cdot e_0(t),\\
e_2(t) &= \dot{e}_1(t) + k_1(t)\cdot e_1(t),\\
e_3(t) &= \dot{e}_2(t) + k_2(t)\cdot e_2(t),\\
k_i(t) &= \frac{1}{1-\varphi_i^2(t)\|e_i(t)\|^2}, \quad i=0,1,2,\\
k_3(t) &= \frac{\hat{k}}{1-\varphi_3^2(t)\|e_3(t)\|^2},\\
w(t) &= -k_3(t)\cdot e_3(t),
\end{aligned} \tag{5.1.12}$$

with constant $\hat{k}>0$, can be applied to system (5.1.11) by Theorem 3.2.3 in the case $r=4$. We note that in the proof of Theorem 3.2.3, $\hat{k}=1$ is considered, but the case of a constant $\hat{k}>0$ is straightforward. Therefore, the closed-loop system (5.1.11), (5.1.12) has a maximal solution $y : [0, \omega) \to \mathbb{R}^2$ with the properties,

(i) $\omega = \infty$,

(ii) all involved signals $y(\cdot)$, $\dot{y}(\cdot)$, $\ddot{y}(\cdot)$, $y^{(3)}(\cdot)$, $w(\cdot)$, $k_i(\cdot)$ for $i = 0, 1, 2, 3$ are bounded.

Moreover, in conjunction with results from [8, Lem. 4.1, Lem. 4.3], the errors evolve uniformly within the respective performance funnel in the sense

(a) for $i = 0, 1, 2$, and $\varepsilon_i(t)$ as in (5.1.9), $\|e_i(t)\| \le \varphi_i(t)^{-1} - \varepsilon_i(t)$ for all $t \ge 0$,

(b) $\exists \varepsilon_3 > 0$, $\forall t \ge 0$: $\|e_3(t)\| \le \varphi_3(t)^{-1} - \varepsilon_3$.

On the other hand, $y_{\text{ref}}(t)$, $\psi_0(t)$, $\psi_1(t)$, $\psi_2(t)$, $\psi_3(t)$, and initial value $\tilde{x}^0$ satisfying condition (5.1.10) implies $\ddot{y}_2(t) \le g_0 < g$ by Remark 5.1.4(ii). This leads to the boundedness of

$$\begin{aligned} v_1 &= \frac{y_2 \tau_s}{g - \ddot{y}_2} w_1 + \frac{\ddot{y}_1 y_2 \tau_s}{(g - \ddot{y}_2)^2} w_2, \\ v_2 &= w_2. \end{aligned}$$

Moreover, we have

$$\begin{aligned} x_1 &= y_1 - y_2 \frac{\ddot{y}_1}{\ddot{y}_2 - g}, \\ x_2 &= \dot{y}_1 - \dot{y}_2 \frac{\ddot{y}_1}{\ddot{y}_2 - g} - \frac{y_2}{\ddot{y}_2 - g} \left(y_1^{(3)} - y_2^{(3)} \frac{\ddot{y}_1}{\ddot{y}_2 - g} \right), \\ x_3 &= y_2 \sqrt{1 + \left(\frac{\ddot{y}_1}{\ddot{y}_2 - g} \right)^2}, \\ x_4 &= \dot{y}_2 \sqrt{1 + \left(\frac{\ddot{y}_1}{\ddot{y}_2 - g} \right)^2} - y_2 \frac{1}{(\ddot{y}_2 - g) \sqrt{1 + \left(\frac{\ddot{y}_1}{\ddot{y}_2 - g} \right)^2}} \left(y_1^{(3)} - y_2^{(3)} \frac{\ddot{y}_1}{\ddot{y}_2 - g} \right) \frac{\ddot{y}_1}{\ddot{y}_2 - g}, \\ x_5 &= \arctan \frac{\ddot{y}_1}{\ddot{y}_2 - g}, \\ x_6 &= \frac{1}{\left(1 + \left(\frac{\ddot{y}_1}{\ddot{y}_2 - g} \right)^2 \right) (\ddot{y}_2 - g)} \left(y_1^{(3)} - y_2^{(3)} \frac{\ddot{y}_1}{\ddot{y}_2 - g} \right), \\ x_7 &= \ddot{y}_2, \\ x_8 &= y_2^{(3)}. \end{aligned}$$

Then, because of the boundedness of $y(\cdot)$, $\dot{y}(\cdot)$, $\ddot{y}(\cdot)$, $y^{(3)}(\cdot)$, and $\ddot{y}_2(t) \le g_0 < g$, we have the boundedness of $x_i(\cdot)$, for $i = 1, \ldots, 8$.
Finally, we also have

$$\begin{aligned} u_1 &= v_1, \\ u_2 &= x_4 + x_3 x_6^2 \tau_l + \frac{(x_2 - v_1) \tau_l \sin x_5}{\tau_s} + \frac{(x_7 - g \sin^2 x_5) \tau_l}{\cos x_5}. \end{aligned}$$

Since $v_1(\cdot)$, $x_i(\cdot)$, for $i = 1, \ldots, 8$ are bounded, then $u_1(\cdot)$, and $u_2(\cdot)$ are bounded. In conclusion, we get all the statements $(i) - (iii)$. □

5.2 Simulations

For the simulation, we have the constant parameters

$$\tau_s = 0.03[s], \quad \tau_l = 0.02[s], \quad g = 9.81[m/s^2].$$

We choose funnel functions for $e_0(t)$ $e_1(t)$, $e_2(t)$, $e_3(t)$ being

$$\begin{aligned}\varphi_0(t) &= 10,\\ \varphi_1(t) &= 5,\\ \varphi_2(t) &= (4e^{-2t} + 0.1)^{-1},\\ \varphi_3(t) &= (20e^{-2t} + 0.5)^{-1},\end{aligned}$$

respectively. Furthermore, we may choose the reference signals

$$\begin{aligned}y_{\text{ref},1}(t) &= 3(t - \sin(t))[m],\\ y_{\text{ref},2}(t) &= 9 + 3\cos(t)[m],\end{aligned}$$

for $t \in [0, 2\pi]$, and an initial values $\tilde{x}^0 = (0,0,12,0,0,0,-3,0)$. We note that this initial value is realizable since $x_3(0) = l(0) = 12[m]$ is the length of the rope, and $x_1(0) = s(0) = 0[m]$, $x_5(0) = \varphi(0) = 0[rad]$ at the beginning position of the load. Moreover, $x_7(0) = -3$ since $\ddot{y}_{\text{ref},2}(t) = -3\cos t$ and $\ddot{y}_2(t) = x_7(t)$. With this initial values, the reference signals, and the funnel functions, the conditions (5.1.8), (5.1.10) are satisfied and the system (5.1.5) have strict relative 4. The simulation of the controller (5.1.7) with $\hat{k} = 100$ applied to (5.1.5) over the time interval $[0, 2\pi]$ has been performed in MATLAB (solver: *ode45*, rel. tol: 10^{-14}, abs. tol: 10^{-10}). In conclusion, we have succeeded in using combination of output feedback controller (5.1.7) and dynamics feedback (5.1.4) to drive overhead crane model (5.1.3), see Figure 5.5.

For the chosen reference signals, $y^{(3)}_{\text{ref},1}(t) = 3\cos t$, then $y^{(3)}_{\text{ref},1}(0) = 3$. On the other hand, $y^{(3)}_1(t) = x_8 \tan x_5 + x_6(1 + \tan^2 x_5)(x_7 - g)$. Thus, $y^{(3)}_1(0) = 0$ since $\tilde{x}^0 = (0,0,12,0,0,0,-3,0)$. Therefore, there is a gap between $y^{(3)}_{\text{ref},1}(0)$ and $y^{(3)}_1(0)$. Thus, u_s takes certain values in its initial phase for reducing the gap and forcing the load to follow reference trajectory, see Figure 5.5c.

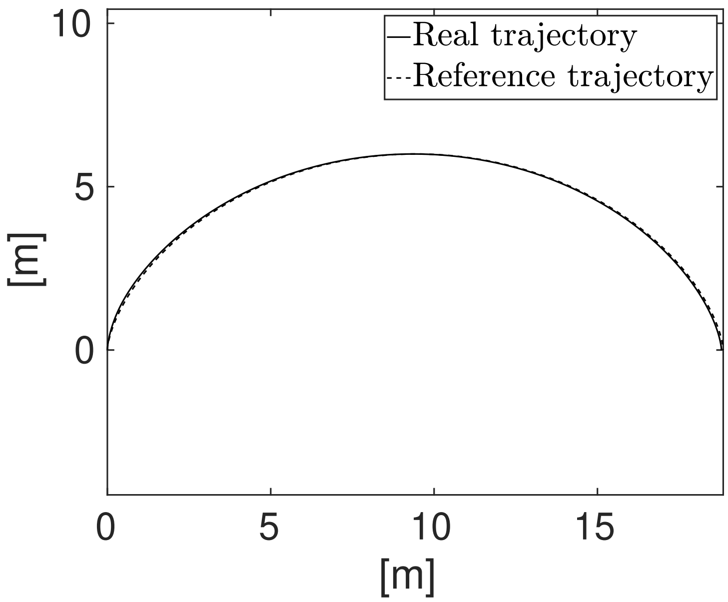

Fig. 5.5a: Load trajectory

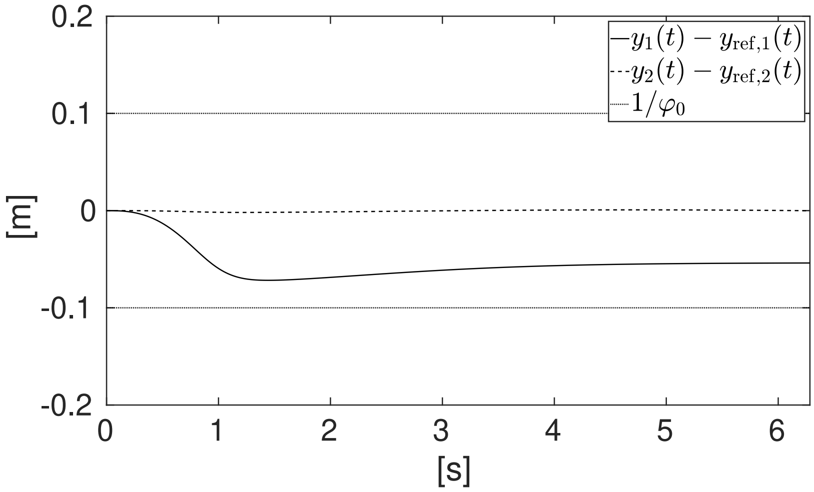

Fig. 5.5b: Funnel and tracking errors

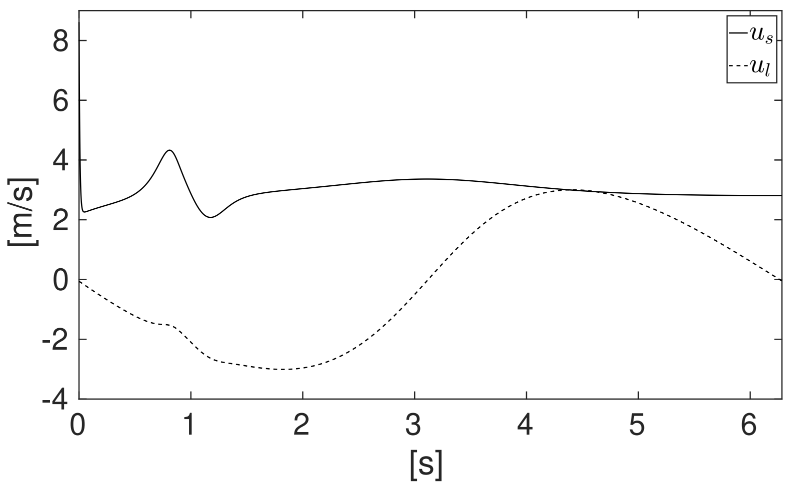

Fig. 5.5c: Input functions

FIGURE 5.5: Simulation of the controller (5.1.7) and (5.1.4) for the system (5.1.3)

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Bibliography

[1] F. Allgöwer, J. Ashman, and A. Ilchmann. “High-gain adaptive λ-tracking for nonlinear systems”. In: *Automatica* 33.5 (1997), pp. 881–888.

[2] J D. Aplevich. “Minimal representations of implicit linear systems”. In: *Automatica* 21.3 (1985), pp. 259–269.

[3] K. J. Åstrom. “Adaptive feedback control”. In: *Proceedings of the IEEE* 75.2 (1987), pp. 185–217.

[4] G. Basile and G. Marro. “Controlled and conditioned invariant subspaces in linear system theory”. In: *Journal of Optimization Theory and Applications* 3.5 (1969), pp. 306–315.

[5] C. P. Bechlioulis and G. A. Rovithakis. “A low-complexity global approximation-free control scheme with prescribed performance for unknown pure feedback systems”. In: *Automatica* 50.4 (2014), pp. 1217–1226.

[6] T. Berger. *On differential-algebraic control systems*. Ilmenau: Univ.-Verlag Ilmenau, 2013.

[7] T. Berger. “Zero dynamics and funnel control of general linear differential-algebraic systems”. In: *ESAIM: Control, Optimisation and Calculus of Variations* 22.2 (2016), pp. 371–403.

[8] T. Berger. “Tracking with prescribed performance for linear non-minimum phase systems”. In: *Universität Hamburg, Hamburger Beiträge zur Angewandten Mathematik* 0 (2018), p. 2018.

[9] T. Berger, A. Ilchmann, and T. Reis. “Normal forms, high-gain, and funnel control for linear differential-algebraic systems”. In: *Control and Optimization with Differential-Algebraic Constraints* 23 (2012), pp. 127–164.

[10] T. Berger, A. Ilchmann, and T. Reis. “Zero dynamics and funnel control of linear differential-algebraic systems”. In: *Mathematics of Control, Signals, and Systems* (2012), pp. 1–45.

[11] T. Berger, A. Ilchmann, and T. Reis. “Funnel control for nonlinear functional differential-algebraic systems”. In: *Proceedings of the 21st International Symposium on Mathematical Theory of Networks and Systems*. Groningen, 2014, pp. 46–53.

[12] T. Berger, H. H. Lê, and T. Reis. “Funnel control for nonlinear systems with higher relative degree”. In: *Proceedings in Applied Mathematics and Mechanics 18*. Special Issue: 89th Annual Meeting of the GAMM, Munich. 2018.

[13] T. Berger, H. H. Lê, and T. Reis. “Funnel control for nonlinear systems with known strict relative degree”. In: *Automatica* 87 (2018), pp. 345–357.

[14] T. Berger, H. H. Lê, and T. Reis. “Funnel control for overhead crane model”. In: *Proceedings in Applied Mathematics and Mechanics 18*. Special Issue: 89th Annual Meeting of the GAMM, Munich. 2018.

[15] T. Berger and T. Reis. "Controllability of linear differential-algebraic systemsa survey". In: *Surveys in differential-algebraic equations I*. Cham: Springer, 2013, pp. 1–61.

[16] T. Berger and T. Reis. "Zero dynamics and funnel control for linear electrical circuits". In: *Journal of the Franklin Institute* 351.11 (2014), pp. 5099–5132.

[17] E. Bullinger and F. Allgöwer. "Adaptive λ-tracking for nonlinear higher relative degree systems". In: *Automatica* 41.7 (2005), pp. 1191–1200.

[18] C. I. Byrnes and A. Isidori. "Global feedback stabilization of nonlinear systems". In: *1985, 24th IEEE Conference on Decision and Control*. IEEE. Fort Lauderdale, FL, USA, 1985, pp. 1031–1037.

[19] C. I. Byrnes and A. Isidori. "Local stabilization of minimum-phase nonlinear systems". In: *Systems & Control Letters* 11.1 (1988), pp. 9–17.

[20] C. I. Byrnes and A. Isidori. "New results and examples in nonlinear feedback stabilization". In: *Systems & Control Letters* 12.5 (1989), pp. 437–442.

[21] C. I. Byrnes and J. C. Willems. "Adaptive stabilization of multivariable linear systems". In: *1984, 23rd IEEE Conference on Decision and Control*. Vol. 23. IEEE. Las Vegas, Nevada, USA, 1984, pp. 1574–1577.

[22] C.I. Byrnes and A. Isidori. "A frequency domain philosophy for nonlinear systems, with applications to stabilization and to adaptive control". In: *1984, 23rd IEEE Conference on Decision and Control*. Vol. 23. IEEE. Las Vegas, Nevada, USA, 1984, pp. 1569–1573.

[23] D. Chowdhury and H. K. Khalil. "Funnel control of higher relative degree systems". In: *2017 American Control Conference*. IEEE. Seattle, WA, USA, 2017, pp. 598–603.

[24] M. Fliess, J. Lévine, and P. Rouchon. "A simplified approach of crane control via a generalized state-space model". In: *Proceedings of the 30th IEEE Conference on Decision and Control, 1991*. IEEE. Brighton, UK, 1991, pp. 736–741.

[25] M. Fliess, J. Lévine, P. Martin, and P. Rouchon. "Flatness and defect of non-linear systems: introductory theory and examples". In: *International journal of control* 61.6 (1995), pp. 1327–1361.

[26] P. A. Fuhrmann and U. Helmke. *The mathematics of networks of linear systems*. New York: Springer, 2015.

[27] F. R. Gantmacher. *The Theory of Matrices Vol. I and II*. New York: Chelsea, 1960.

[28] C. M. Hackl. "High-gain adaptive position control". In: *International Journal of Control* 84.10 (2011), pp. 1695–1716.

[29] C. M. Hackl. "Contributions to high-gain adaptive control in mechatronics". PhD thesis. München: Technische Universität München, 2012.

[30] C. M. Hackl. "Funnel control for wind turbine systems". In: *2014 IEEE Conference on Control Applications*. IEEE. Juan Les Antibes, France, 2014, pp. 1377–1382.

[31] C. M. Hackl. "Current PI-funnel control with anti-windup for synchronous machines". In: *2015 IEEE 54th Annual Conference on Decision and Control*. IEEE. Osaka, Japan, 2015, pp. 1997–2004.

[32] C. M. Hackl. "Speed funnel control with disturbance observer for wind turbine systems with elastic shaft". In: *2015 IEEE 54th Annual Conference on Decision and Control*. IEEE. Osaka, Japan, 2015, pp. 2005–2012.

[33] C. M. Hackl and R. M. Kennel. “Position funnel control for rigid revolute joint robotic manipulators with known inertia matrix”. In: *2012 20th Mediterranean Conference on Control & Automation*. IEEE. Barcelona, Spain, 2012, pp. 615–620.

[34] C. M. Hackl, N. Hopfe, A. Ilchmann, M. Mueller, and S. Trenn. “Funnel control for systems with relative degree two”. In: *SIAM Journal on Control and Optimization* 51.2 (2013), pp. 965–995.

[35] U. Helmke, D. Prätzel-Wolters, and S. Schmid. “Adaptive tracking for scalar minimum phase systems”. In: *Control of Uncertain Systems*. Ed. by D. Hinrichsen and B. Mårtensson. Boston: Birkhäuser, 1990, pp. 101–117.

[36] J. B Hoagg and D. S. Bernstein. “Direct adaptive command following and disturbance rejection for minimum phase systems with unknown relative degree”. In: *International Journal of Adaptive Control and Signal Processing* 21.1 (2007), pp. 49–75.

[37] J. B. Hoagg and D. S. Bernstein. “Direct adaptive dynamic compensation for minimum phase systems with unknown relative degree”. In: *IEEE Transactions on Automatic Control* 52.4 (2007), pp. 610–621.

[38] A. Ilchmann. “Non-identifier-based adaptive control of dynamical systems: A survey”. In: *IMA Journal of Mathematical Control and Information* 8.4 (1991), pp. 321–366.

[39] A. Ilchmann. *Non-identifier-based high-gain adaptive control*. London: Springer, 1993.

[40] A. Ilchmann. “Adaptive λ-tracking for polynomial minimum phase systems”. In: *Dynamics and Stability of Systems* 13.4 (1998), pp. 341–371.

[41] A. Ilchmann. “Decentralized tracking of interconnected systems”. In: *Mathematical System Theory - Festschrift in Honor of Uwe Helmke on the Occasion of his Sixtieth Birthday*. Ed. by K. Hüper and J. Trumpf. CreateSpace, 2013, pp. 229–245.

[42] A. Ilchmann and H. Logemann. “Adaptive λ-tracking for a class of infinite-dimensional systems”. In: *Systems & Control letters* 34.1 (1998), pp. 11–21.

[43] A. Ilchmann, D. H. Owens, and D. Prätzel-Wolters. “High-gain robust adaptive controllers for multivariable systems”. In: *Systems & Control Letters* 8.5 (1987), pp. 397–404.

[44] A. Ilchmann and E. P. Ryan. “Universal λ-tracking for nonlinearly-perturbed systems in the presence of noise”. In: *Automatica* 30.2 (1994), pp. 337–346.

[45] A. Ilchmann and E. P. Ryan. “High-gain control without identification: a survey”. In: *GAMM-Mitteilungen* 31.1 (2008), pp. 115–125.

[46] A. Ilchmann and E. P. Ryan. “Performance funnels and tracking control”. In: *International Journal of Control* 82.10 (2009), pp. 1828–1840.

[47] A. Ilchmann, E. P. Ryan, and C. J. Sangwin. “Systems of controlled functional differential equations and adaptive tracking”. In: *SIAM Journal on Control and Optimization* 40.6 (2002), pp. 1746–1764.

[48] A. Ilchmann, E. P. Ryan, and C. J. Sangwin. “Tracking with prescribed transient behaviour”. In: *ESAIM: Control, Optimisation and Calculus of Variations* 7 (2002), pp. 471–493.

[49] A. Ilchmann, E. P. Ryan, and P. Townsend. “Tracking control with prescribed transient behaviour for systems of known relative degree”. In: *Systems & Control Letters* 55.5 (2006), pp. 396–406.

[50] A. Ilchmann, E. P Ryan, and P. Townsend. “Tracking with prescribed transient behavior for nonlinear systems of known relative degree”. In: *SIAM Journal on Control and Optimization* 46.1 (2007), pp. 210–230.

[51] A. Ilchmann, E. P. Ryan, and S. Trenn. “Tracking control: Performance funnels and prescribed transient behaviour”. In: *Systems & Control Letters* 54.7 (2005), pp. 655–670.

[52] A. Ilchmann and H. Schuster. “PI-funnel control for two mass systems”. In: *IEEE Transactions on Automatic Control* 54.4 (2009), pp. 918–923.

[53] A. Ilchmann and S. Townley. “Simple adaptive stabilization of high-gain stabilizable systems”. In: *Systems & control letters* 20.3 (1993), pp. 189–198.

[54] A. Ilchmann and S. Trenn. “Input constrained funnel control with applications to chemical reactor models”. In: *Systems & control letters* 53.5 (2004), pp. 361–375.

[55] A. Ilchmann and F. Wirth. “On minimum phase”. In: *at–Automatisierungstechnik* 61.12 (2013), pp. 805–817.

[56] A. Isidori. *Nonlinear control systems*. 3rd. Berlin: Springer, 1995.

[57] T. Kailath. *Linear systems*. Prentice-Hall Englewood Cliffs, NJ, 1980.

[58] R. Kelly, V. S. Davila, and A. Loría. *Control of robot manipulators in joint space*. London: Springer, 2005.

[59] P. Kunkel and V. Mehrmann. *Differential-algebraic equations: analysis and numerical solution*. Zürich: European Mathematical Society, 2006.

[60] D. Liberzon and S. Trenn. “The bang-bang funnel controller for uncertain nonlinear systems with arbitrary relative degree”. In: *IEEE Transactions on Automatic Control* 58.12 (2013), pp. 3126–3141.

[61] H. Logemann and A. Ilchmann. “An adaptive servomechanism for a class of infinite-dimensional systems”. In: *SIAM Journal on Control and Optimization* 32.4 (1994), pp. 917–936.

[62] H. Logemann and B. Martensson. “Adaptive stabilization of infinite-dimensional systems”. In: *IEEE Transactions on Automatic Control* 37.12 (1992), pp. 1869–1883.

[63] H. Logemann and S. Townley. “Adaptive control of infinite-dimensional systems without parameter estimation: an overview”. In: *IMA journal of mathematical control and information* 14.2 (1997), pp. 175–206.

[64] I. Mareels. “A simple selftuning controller for stably invertible systems”. In: *Systems & Control Letters* 4.1 (1984), pp. 5–16.

[65] B. Mårtensson. “The order of any stabilizing regulator is sufficient a priori information for adaptive stabilization”. In: *Systems & Control Letters* 6.2 (1985), pp. 87–91.

[66] D. E. Miller and E. J. Davison. “An adaptive controller which provides an arbitrarily good transient and steady-state response”. In: *IEEE Transactions on Automatic Control* 36.1 (1991), pp. 68–81.

[67] A. S. Morse. “Recent problems in parameter adaptive control”. In: *Outils et Modeles Mathématiques pour l’Automatique, l’Analyse de Systemes et le Traitment du Signal* 3 (1983), pp. 733–740.

[68] A. S. Morse. “Overcoming the obstacle of high relative degree”. In: *European Journal of Control* 2.1 (1996), pp. 29–35.

[69] M. Mueller. “Normal form for linear systems with respect to its vector relative degree”. In: *Linear algebra and its applications* 430.4 (2009), pp. 1292–1312.

[70] K. S. Narendra. “The maturing of adaptive control”. In: *Foundations of Adaptive Control*. Ed. by P. V. Kokotovic. New York: Springer, 1991, pp. 1–36.

[71] S. Nicosia and A. Tornambè. "High-gain observers in the state and parameter estimation of robots having elastic joints". In: *Systems & Control Letters* 13.4 (1989), pp. 331–337.

[72] R. D Nussbaum. "Some remarks on a conjecture in parameter adaptive control". In: *Systems & Control Letters* 3.5 (1983), pp. 243–246.

[73] S. Otto and R. Seifried. "Real-time trajectory control of an overhead crane using servo-constraints". In: *Multibody System Dynamics* (2017), pp. 1–17.

[74] K. Özçaldiran. "A geometric characterization of the reachable and the controllable subspaces of descriptor systems". In: *Circuits, systems, and signal processing* 5.1 (1986), pp. 37–48.

[75] A. Pomprapa, S. R. Alfocea, C. Göbel, B. J. E. Misgeld, and S. Leonhardt. "Funnel control for oxygenation during artificial ventilation therapy". In: *IFAC Proceedings Volumes* 47.3 (2014), pp. 6575–6580.

[76] A. Pomprapa, S. Weyer, S. Leonhardt, M. Walter, and B. Misgeld. "Periodic funnel-based control for peak inspiratory pressure". In: *2015 IEEE 54th Annual Conference on Decision and Control*. IEEE. 2015, pp. 5617–5622.

[77] E. P. Ryan. "A nonlinear universal servomechanism". In: *IEEE Transactions on Automatic Control* 39.4 (1994), pp. 753–761.

[78] E.P. Ryan. "Adaptive stabilization of a class of uncertain nonlinear systems: A differential inclusion approach". In: *Systems & control letters* 10.2 (1988), pp. 95–101.

[79] E.P. Ryan. "Universal $W^{1,\infty}$-tracking for a class of nonlinear systems". In: *Systems & control letters* 18.3 (1992), pp. 201–210.

[80] R. Seifried. *Dynamics of underactuated multibody systems: modeling, control and optimal design*. Vol. 205. Springer Science & Business Media, 2013.

[81] R. Seifried and W. Blajer. "Analysis of servo-constraint problems for underactuated multibody systems". In: *Mechanical Sciences* 4.1 (2013), pp. 113–129.

[82] A. Senfelds and A. Paugurs. "Electrical drive DC link power flow control with adaptive approach". In: *2014 55th International Scientific Conference on Power and Electrical Engineering of Riga Technical University*. IEEE. Riga, Latvia, 2014, pp. 30–33.

[83] E. D Sontag. "Smooth stabilization implies coprime factorization". In: *IEEE Transactions on Automatic Control* 34.4 (1989), pp. 435–443.

[84] M. W. Spong, S. Hutchinson, and M. Vidyasagar. *Robot modeling and control*. Vol. 3. New York: John Wiley & Sons, 2006.

[85] A. Theodorakopoulos and G. A Rovithakis. "Low-complexity prescribed performance control of uncertain MIMO feedback linearizable systems". In: *IEEE Transactions on Automatic Control* 61.7 (2016), pp. 1946–1952.

[86] H. Trentelman, A. A. Stoorvogel, and M. Hautus. *Control theory for linear systems*. London: Springer, 2001.

[87] J. C Willems and C. I. Byrnes. "Global adaptive stabilization in the absence of information on the sign of the high frequency gain". In: *Analysis and Optimization of Systems, Proc. of the 6th INRIA Conference, Nice, France*. Ed. by A. Bensoussan and J. L. Lions. 62. Berlin: Springer, 1984, pp. 49–57.

[88] W. M. Wonham. *Linear Multivariable Control: a Geometric Approach*. New York: Springer, 1985.

[89] W. M. Wonham and A S. Morse. “Decoupling and pole assignment in linear multivariable systems: a geometric approach”. In: *SIAM Journal on Control* 8.1 (1970), pp. 1–18.

[90] X. Ye. “Universal λ-tracking for nonlinearly-perturbed systems without restrictions on the relative degree”. In: *Automatica* 35.1 (1999), pp. 109–119.

[91] X. Ye. “Switching adaptive output-feedback control of nonlinearly parametrized systems”. In: *Automatica* 41.6 (2005), pp. 983–989.