# Integrability in $\mathcal{N} = 1$ Gauge Theories

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## Abstract

In this thesis we investigate two questions that shed new light on fundamental properties of supersymmetric gauge theories.

Our first topic deals with the question, which implications results for $\mathcal{N} = 4$ Super-Yang-Mills have on theories with less symmetry. Specifically we conjecture that the dilatation operator in the $SU(2,1|1)$ sector of any $\mathcal{N} = 1$ superconformal gauge theory can be found from the one for $\mathcal{N} = 4$ Super-Yang-Mills by a redefinition of the coupling constant. This implies its integrability. We prove this conjecture perturbatively up to three loops for the vacuum of this sector and discuss generalizations to the whole sector.

Our second investigation concerns the protected spectrum of $\mathcal{N} = 2$ superconformal QCD. It is much richer than naively expected. In particular it contains states with arbitrarily large spin, which has been shown by means of the superconformal index. However their form was as of yet unknown. We present an algorithm that explicitly constructs these states in terms of the fundamental fields of the theory.



## Zusammenfassung

In dieser Arbeit befassen wir uns mit zwei Fragen, welche neues Licht auf fundamentale Eigenschaften supersymmetrischer Eichtheorien werfen.

Im ersten Thema beschäftigen wir uns mit der Frage, welche Implikationen Resulate für $\mathcal{N} = 4$ Super-Yang-Mills auf weniger symmetrische Theorien haben. Insbesondere stellen wir die Vermutung auf, dass der Dilatationsoperator im $SU(2,1|1)$ Sektor einer jeden $\mathcal{N} = 1$ superkonformen $\mathcal{N} = 1$ Eichtheorie durch eine Umdefinierung der Kopplungskonstante aus dem Dilatationsoperator von $\mathcal{N} = 4$ Super-Yang-Mills erhalten werden kann. Daraus folgt, dass er integrabel ist. Wir zeigen durch störungstheoretische Rechnungen bis zu drei Schleifen, dass diese Vermutung zumindest für das Vakuum dieses Sektors zutrifft und diskutieren anschließend Erweiterungen auf den gesamten Sektor.

In unserer zweiten Untersuchung betrachten wir das geschützte Spektrum von $\mathcal{N} = 2$ superkonformer QCD. Dieses ist viel umfangreicher als ursprünglich erwartet. Insbesondere enthält es Zustände mit beliebig hohem Spin, was mithilfe des superkonformen Indexes gezeigt wurde. Ihre Gestalt war bisher allerdings nicht bekannt. Wir präsentieren einen Algorithmus, der ihre explizite Darstellung in Abhänigkeit der fundamentalen Felder der Theorie konstruiert.

**This thesis is based on the publications:**

- J.P. Cartensen, E. Pomoni, Integrability in $\mathcal{N} = 1$ gauge theories, *in progress*
- M. Sprenger, J.P. Cartensen, E. Pomoni, New protected states in $\mathcal{N} = 2$ superconformal QCD, *in progress*

# Contents

# Chapter 1

# Introduction

## 1.1 Motivation

Gauge theories are arguably the most important framework to describe real world phenomena in modern physics. Historically the concept of gauge symmetry was first discovered in the theory of electromagnetism and since then gauge theories have found applications throughout many areas of physics, most notably in the form of Yang-Mills theories [1] in the Standard Model of particle physics, which describes the fundamental interactions at high energies and small scales. Despite remarkable success in calculating measurable quantities there are still many open questions about the true nature of gauge theories as exemplified by the fact that for general compact simple gauge groups the very existence of a non-trivial gauge theory is one of the unsolved Millennium Prize Problems as defined by the Clay Mathematics Institute [2].

However recent decades have seen a plethora of major developments in our understanding of these theories. A particularly exciting branch is the discovery of hidden global symmetries that do not reveal themselves at the level of the classical Lagrangian. In the most favorable cases a theory has as many symmetries as degrees of freedom. This is loosely speaking the definition of an *integrable model*. Due to this large amount of symmetry the theory is highly constrained and many problems can be brought into the form of integral or even algebraic equations, which can be solved exactly, at least in principle. This is in stark contrast to the usual methods of perturbation theory, where only the first few of orders in a series expansion in the coupling constant can be computed. It might seem unexpected that a system as complicated as a four-dimensional gauge theory can be this constrained but indeed it has been found that integrability emerges in very diverse contexts in these theories. One example is the high-energy (Regge) limit of scattering amplitudes in ordinary quantum chromodynamics (QCD) [3–6]. Another seemingly unrelated example is the scaling behavior of gauge invariant local operators in the maximally supersymmetric gauge theory in four dimensions called $\mathcal{N} = 4$ Super-Yang-Mills (SYM) in the planar limit [7–10]. This scaling behavior is measured by the dilatation operator as defined in eq. (1.6). Both of these are related by the fact that they turn out to be described by integrable spin chains. A great review on many topics pertaining to integrability in these contexts is [11].

On the one hand the first example shows that the emergence of integrability does not necessarily require a large amount of supersymmetry, but instead might be a somewhat universal feature in many different gauge theories. The second example on the other hand only deals with a very particular, highly idealized theory. It is then a natural and interesting question to ask, whether and how these restrictions can be relaxed in order to accommodate more realistic theories. In other words:

> **Question 1**: Is it possible to start with results for the dilatation operator in $\mathcal{N} = 4$ SYM and infer statements for the dilatation operator in theories with less (super-)symmetry?

Some inroads have been made on the quest to answer this question. In [12–16] QCD with gauge group $SU(3)$ and *pure* $\mathcal{N} = 1, 2, 4$ theories (i.e. without chiral multiplets or hypermultiplets) with gauge group $SU(N)$ were analyzed in a unified fashion on the light-cone. It was found that at least in these four theories integrability of the dilatation operator seems to be sensitive neither to conformal symmetry nor to supersymmetry and only depends on the planar limit up to two loops. In [17] a one loop analysis of large $N$ QCD identified the sector of purely gluonic operators constructed with self-dual field strengths and an arbitrary number of derivatives to be integrable.

A very different approach was taken in [18]. Rather than looking at a specific theory, a general argument for the integrability of a sector present in any $\mathcal{N} = 2$ superconformal gauge theory was devised. The details of this are explained in the following sections. Most of this thesis is devoted to the investigation, whether a similar argument can be made for $\mathcal{N} = 1$ superconformal gauge theories.

In order to infer properties of theories with little symmetry from more symmetric ones it is advantageous to have a controlled way to produce one from the other. One such method is the orbifolding of a theory. The importance of this technique in the context of string and gauge theories was realised in [19]. This procedure breaks the symmetry of a given theory in a prescribed way and often the orbifolded theory behaves similarly as its mother theory [20, 21]. Important classes of theories that can be obtained in this way are quiver gauge theories that are orbifolds of $\mathcal{N} = 4$ SYM [22, 23], which get their name from the fact that their field content is succinctly

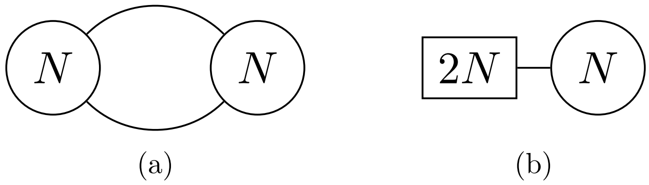



**Figure 1.1:** Examples of $\mathcal{N}=2$ quiver theories. (a) shows the $\mathbb{Z}_2$ orbifold of $\mathcal{N}=4$ SYM with gauge group $SU(2N)$. Each blob stands for an $SU(N)$ gauge group and the corresponding $\mathcal{N}=2$ vector multiplet, while the lines correspond to hypermultiplets in the bifundamental representation. (b) corresponds to $\mathcal{N}=2$ superconformal QCD, which has gauge group $SU(N)$ and $N_F=2N$ fundamental hypermultiplets. This is obtained as the result of ungauging one of the gauge groups in (a).

summarized in quiver diagrams. Figure 1.1 (a) shows the example of the $\mathbb{Z}_2$ orbifold of $\mathcal{N}=4$ SYM. In fact $\mathbb{Z}_M$ orbifolds of $\mathcal{N}=4$ SYM are $\mathcal{N}=2$ superconformal gauge theories and provided the testing ground for the analysis in [18] mentioned above. These theories can in turn be orbifolded to obtain $\mathcal{N}=1$ superconformal gauge theories, for example by $\mathbb{Z}_M\times\mathbb{Z}_k$ orbifolds, which we will use as our main examples.

In this way one obtains theories, where all fields are either in the adjoint or in the bifundamental representation of the gauge groups. A further step that can be taken to get more realistic theories is to ungauge one or more of the gauge groups. This introduces a flavor symmetry and the corresponding fields will then transform in the fundamental representation of one of the gauge groups. Thus this is a way to introduce fundamental matter into the theory. A particularly important example of this is shown in figure 1.1 (b), where the ungauging of one of the gauge groups of the $\mathbb{Z}_2$ orbifold of $\mathcal{N}=4$ SYM leads to $\mathcal{N}=2$ superconformal QCD ($\mathcal{N}=2$ SCQCD), which has $N_F=2N$ fundamental hypermultiplets.

Our second line of inquiry deals with this theory. Specifically it deals with the so-called *protected spectrum* in the theory. These are states that do not receive quantum corrections (their symmetry protects them from it). Another way to phrase this is that they are annihilated by some subset of the symmetry generators. This implies that they organize in so-called short superconformal multiplets, that have fewer members than generic long ones. A comprehensive treatment of superconformal representation theory is given in [24].

These types of states are an interesting object of study, because many of their properties can be computed exactly. In [25] it was shown that $\mathcal{N}=2$ superconformal QCD has more protected states than is naively expected. In particular these new protected states can have arbitrarily large spin.

This is the first interacting 4d superconformal field theory, where this has been observed and this fact is thus by itself highly notable. Soon after, a similar effect was observed in certain 3d superconformal theories [26–28]. It also has important physical consequences. One comes from the AdS/CFT correspondence [29, 30]. Namely it implies that the low energy limit of the dual string theory of $\mathcal{N}=2$ SCQCD cannot be a normal supergravity theory but instead it must be a higher spin theory. This is in contrast to $\mathcal{N}=4$ SYM and to the interpolating theory, both of which have a gravity dual, whose low energy limit is a supergravity theory. Secondly protected states usually serve as vacua in spin chains, so the existence of such a tower of new vacua has profound implications, which are worth to be investigated.

The first objective however is to understand the origin of these states. In [25] it was observed that this can be understood by the relation between $\mathcal{N}=2$ SCQCD and the $\mathbb{Z}_2$ orbifold of $\mathcal{N}=4$ SYM. Namely there is a theory that interpolates between those two, aptly called the *interpolating theory*. It assigns two different coupling constants $g, \check{g}$ to the two gauge groups. If they are equal $g=\check{g}$, one recovers the orbifold theory. In the limit where one of the couplings goes to zero $\check{g}\to 0$ the corresponding vector multiplet decouples from the theory and one finds $\mathcal{N}=2$ SCQCD. The new protected states must come from states that are unprotected in the interpolating theory and become protected only in the limit $\check{g}\to 0$. The natural question to ask is

**Question 2**: What are these new protected states in $\mathcal{N}=2$ SCQCD?

This question is specified in section 1.5 and answered in chapter 9.

## 1.2 Statement of problem 1 and results

The result of the analysis in [18] is that any $\mathcal{N}=2$ superconformal gauge theory possesses an integrable sector, called the $SU(2,1|2)$ sector, which consists of gauge invariant local operators that are products of fields from the $\mathcal{N}=2$ vector multiplet with an arbitrary number of covariant derivatives acting on them. This conclusion was reached by realizing that the dilatation operator restricted to this sector is the same as the dilatation operator in $\mathcal{N}=4$ SYM up to a redefinition of the coupling constant. Since this redefinition has no effect on the integrability, the result follows. We investigate, whether this can be extended to the analogous sector in $\mathcal{N}=1$ theories. This sector is called the $SU(2,1|1)$ sector and is composed of gauge invariant local operators that are products of fields from the $\mathcal{N}=1$ vector multiplet and covariant derivatives acting on them. A more precise definition of the sector will be given in section 4.1.

In order to understand our main result, it is first necessary to introduce the spin chain picture for gauge invariant local single trace operators. A comprehensive review of this and many other topics related to integrability can be found in the review collection [11]. The idea is that the elementary fields in an operator are mapped to the different sites of the spin chain. Since the trace is cyclic, the spin chains will be closed. A fundamental consequence is that in this picture the dilatation operator acting on the operators is mapped to the Hamiltonian acting on the spin chain. In particular one can define the lowest energy states with respect to this Hamiltonian as the vacuum of the spin chain and higher energy states will be excitations. In our case the operator corresponding to the vacuum is a trace over a local product of the supersymmetric field strengths $W_+$, whose precise definition will be given later

$$\mathcal{O}_{vac} = \mathrm{tr}\,\left(W_+ \ldots W_+\right)\,, \tag{1.1}$$

Excitations are then given by acting with background gauge covariant derivatives $\boldsymbol{\nabla}_+, \boldsymbol{\nabla}_{+\dot{\alpha}}$ on the field strength factors. A single such derivative is also called a *magnon*. The Hamiltonian induces scattering processes of these magnons on the spin chain and an interesting observable is the corresponding S-matrix. If the spin chain is integrable, the S-matrix factorizes according to the Yang-Baxter equation [31–33]. This equation implies that the scattering of any number of magnons can be reduced to the scattering of only 2 magnons.

Our main result is that *for the vacuum* of this sector in any $\mathcal{N} = 1$ superconformal gauge theory the dilatation operator in the planar limit is indeed identical to the one in $\mathcal{N} = 4$ SYM up to a coupling redefinition

$$D_{\mathcal{N}=1}(g) = D_{\mathcal{N}=4}(f(g)) \tag{1.2}$$

up to three loops. If this result also holds for up to two excitations instead of just the vacuum, this would establish equality of the S-matrix of the two theories up to said coupling redefinition. Since $\mathcal{N} = 4$ SYM is known to be integrable [34], this would imply integrability for the whole sector. We will present the current status of these efforts.

It is worth mentioning that the redefinition of the coupling constant for $\mathcal{N} = 2$ theories is universal in the sense that it does not only hold for the dilatation operator but also for a host of other observables, namely Wilson loops, the Bremsstrahlung function and the entanglement entropy [35, 36].

The following sections give an outline of the argument and introduce the necessary formalism. It relies on the following features

- $\mathcal{N} = 1$ supersymmetry,
- gauge invariance and the background field formalism,

- the choice of the sector.

The first two of these will be discussed in section 1.3 and chapter 2 and the last one is treated in section 4.1. We also demand

- planarity,
- conformality.

Planarity is demanded because this is the regime, in which $D_{\mathcal{N}=4}$ is known to be integrable. It requires further investigation, whether conformality can be relaxed as indicated by the treatment in [16].

## 1.3 Outline of the argument

Our argument rests on two main pillars:

*Supersymmetric perturbation theory* [37] ensures that supersymmetry is manifestly preserved at all stages of the calculation. This is particularly convenient for calculations in the $SU(2,1|1)$ sector, since all fields are related by supersymmetry transformations and can thus be expressed by a single superfield $V$ and (superspace) derivatives thereof. In particular this implies that in superspace language there is only a single wave function renormalization $Z_V$.

*The background field formalism* [38, 39] in turn keeps background gauge invariance manifest. After a suitable splitting of $V$ into quantum and background fields $V_Q$ and $V_B$ respectively, it ensures that the renormalization of the coupling constant and the background vector superfield are related by

$$Z_g(g)\sqrt{Z_{V_B}(g)} = 1\,. \tag{1.3}$$

A further consequence of the background field formalism is that quantum fields and ghosts only appear in loops. This implies that the renormalization factors coming from the vertices always cancel against the ones from the propagators. For a review see [40, section 3.2].

The upshot is that the renormalization of vertices, which are already present at tree level, is governed by a single counterterm $Z_g(g)$. In principle new vertices can be produced in the effective action during the process of renormalization, which require different counterterms. If one can show however that these new vertices don't contribute to the dilatation operator in the $SU(2,1|1)$ sector, this implies the existence of a unique function $f(g)$ in eq. (1.2). Our goal is to show just that.

Composite operators will generically mix under renormalization

$$\mathcal{O}_i^{ren} = Z_i^j \mathcal{O}_j^{bare} \,. \tag{1.4}$$

It is standard textbook material (see e.g. [41, 42]) that the anomalous dimensions and therefore the dilatation operator are given by

$$\gamma \equiv \mu \frac{\mathrm{d}}{\mathrm{d}\mu} \ln Z \,, \tag{1.5}$$

read as a matrix equation. Here $\mu$ is the energy scale introduced by the renormalization scheme. We will use dimensional reduction [43] in $d = 4 - 2\epsilon$ dimensions, where the scale enters through $\boldsymbol{g} \to \boldsymbol{g}\mu^\epsilon$ in order to keep the marginal couplings $\boldsymbol{g} = (g_1, \ldots, g_n)$ dimensionless. The explicit form of the dilatation operator is then given by

$$D = \mu \frac{\mathrm{d}}{\mathrm{d}\mu} \ln Z(\boldsymbol{g}\mu^\epsilon, \epsilon) = \lim_{\epsilon \to 0} \left( \epsilon\, x \cdot \frac{\partial}{\partial x} \ln Z(\boldsymbol{g}x, \epsilon) \right)_{x=1} \,. \tag{1.6}$$

A direct consequence of the presence of the factor of $\epsilon$ and the limit $\epsilon \to 0$ is that only the first order pole in $\epsilon$ from $Z$ can contribute. All higher order poles must cancel in the logarithm to get a finite result for the anomalous dimensions. The derivative with respect to $x$ produces a factor of $2L$, where $L$ is the loop number.

The task is thus to compute the renormalization of operators in the $SU(2,1|1)$ sector and show that the new vertices will never contribute. Since our argument relies on the presence of both supersymmetry and background gauge invariance we are naturally led to use a formalism called *background covariant supersymmetric Feynman rules* [44, 45], which preserves both symmetries for most of the calculation. A consequence of this preservation of symmetry is that far fewer terms appear in the calculation and that many cancellations are automatic. The resulting terms are generally less divergent. As was realized in [44] this method still needs to split up space-time covariant derivatives into ordinary derivatives and background connections, when performing the momentum integration. This breaks gauge invariance. This problem was solved in [46] with a technique that maintains gauge invariance all the way. We will explain both of these methods in chapter 2.

## 1.4 New vs. old vertices

The background field formalism implies that the renormalization of all terms in the expansion of [1]

$$S_{\text{gauge}} = \frac{1}{2g^2}\int \mathrm{d}^6 z\, W^\alpha(x,\theta)W_\alpha(x,\theta) = -\frac{1}{2g^2}\int \mathrm{d}^8 z\, (e^{-V}\boldsymbol{\nabla}^\alpha e^V)\bar{\boldsymbol{\nabla}}^2(e^{-V}\boldsymbol{\nabla}_\alpha e^V) \tag{1.7}$$

is determined by the same factor $Z_g$. Here $\mathrm{d}^8z = \mathrm{d}^4x\,\mathrm{d}^2\theta\mathrm{d}^2\bar{\theta}$ and $\mathrm{d}^6z = \mathrm{d}^4x\,\mathrm{d}^2\theta$ are the measures of full and chiral superspace, respectively, $W_\alpha$ is the full superfield strength and $V$ is the quantum vector superfield. The interaction terms in this expansion are the terms that we call old vertices. Some of these are quantum-background interactions. A useful representation of these is given in terms of the background field strength

$$\boldsymbol{W}_\alpha = \frac{1}{2}i[\bar{\boldsymbol{\nabla}}^{\dot{\alpha}}, \{\bar{\boldsymbol{\nabla}}_{\dot{\alpha}}, \boldsymbol{\nabla}_\alpha\}] = W_\alpha|_{V=0}\,, \tag{1.8}$$

which comes from the complete field strength upon setting the quantum field $V = 0$. This yields

$$S_{\text{gauge},\boldsymbol{W}} = \frac{1}{g^2}\int \mathrm{d}^8 z\, \boldsymbol{W}^\alpha(e^{-V}\boldsymbol{\nabla}_\alpha e^V) \tag{1.9}$$

We see that their structure is highly restricted.

In the presence of local operators it is possible that new effective vertices are produced in perturbation theory, which renormalize differently. These are called new vertices. They must be Lorentz scalars and the background field formalism restricts them to be gauge invariant, but other than that they are unrestricted. One example of a possible new term in the effective action is

$$S_{\text{new},W^4} = \int \mathrm{d}^8 z\, W^\alpha W_\alpha \bar{W}^{\dot{\alpha}}\bar{W}_{\dot{\alpha}}\,. \tag{1.10}$$

Figure 1.2 shows examples of both old and new vertices. Diagram 1.2 (a) can be identified with the structure

$$\boldsymbol{W}^\alpha V(\boldsymbol{\nabla}_\alpha V) \subset S_{\text{gauge},\boldsymbol{W}}\,, \tag{1.11}$$

which is part of the second order expansion of eq. (1.9). Gauge invariance requires

[1] We will go into more detail about superfields and supersymmetric actions, when we properly introduce superspace in chapter 2.

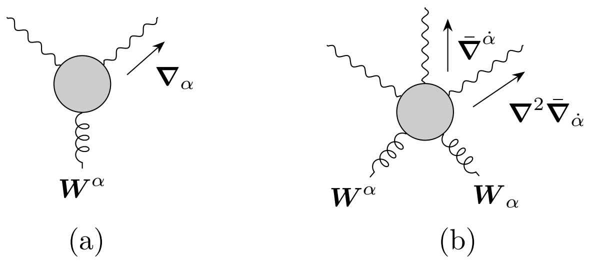



**Figure 1.2:** (a) Example of a vertex that is already present at tree level, (b) Example of a new vertex

the presence of all the other terms as well. Diagram 1.2 (b) on the other hand is part of the expansion of eq. (1.10)

$$\frac{1}{2}\int \mathrm{d}^8 z\, \boldsymbol{W}^\alpha \boldsymbol{W}_\alpha V \left\{ \bar{\boldsymbol{\nabla}}^{\dot\alpha} V, \boldsymbol{\nabla}^2 \bar{\boldsymbol{\nabla}}_{\dot\alpha} V \right\} \subset S_{\mathrm{new},W^4} \tag{1.12}$$

and as such constitutes a new vertex.

Looking at eq. (1.9) it is immediately clear that any vertex, which contains two or more background field strengths must be a new vertex. In fact all of the new vertices that we will encounter in our calculations will have this feature. Our goal is to show that these do not contribute to anomalous dimensions in our sector. These calculations will occupy most of this thesis.

## 1.5 Statement of problem 2

Let us now turn back to the protected spectrum of $\mathcal{N}=2$ SCQCD. Let $\phi$ be the scalar from the $\mathcal{N}=2$ vector multiplet and $Q_{\mathcal{I}i}$ the scalars from the hypermultiplets. We use indices $\mathcal{I},\mathcal{J}=\pm$ for the $SU(2)_R$ symmetry, $i,j=1,...N_f$ for the flavor group $U(N_f)$ and $a,b=1,...N$ for the color group $SU(N)$. We define the mesonic operators

$$\mathcal{M}^{\mathcal{I}a}_{\mathcal{J}\ b} = \frac{1}{\sqrt{2}} Q_{\mathcal{J}}{}^{a}{}_{i} \bar{Q}^{\mathcal{I}}{}_{b}^{\ i} \tag{1.13}$$

$$\mathcal{M}^{a}_{1\ b} = \mathcal{M}_{\mathcal{I}}{}^{\mathcal{I}} \tag{1.14}$$

$$\mathcal{M}_{3\mathcal{J}}{}^{\mathcal{I}} = \mathcal{M}_{\mathcal{I}}{}^{\mathcal{J}} - \frac{1}{2}\mathcal{M}_{\mathcal{K}}{}^{\mathcal{K}}\delta_{\mathcal{I}}{}^{\mathcal{J}} \tag{1.15}$$

The naive expectation for the protected spectrum in $\mathcal{N}=2$ SCQCD is then given by

$$\operatorname{tr}\phi^{k+2}, \qquad \operatorname{tr}\left(T\phi^k\right) \quad \text{and} \quad \mathcal{M}_3 \tag{1.16}$$

with $k\geq 0$ and $T=\phi\bar\phi-\mathcal{M}_1$. This guess comes from the knowledge that this is the correct protected spectrum for $\mathcal{N}=4$ SYM. However in [25] the existence of a tower of higher-spin protected states in $\mathcal{N} = 2$ SCQCD was shown. The tool that was used, is called the superconformal index [47], see also the reviews [48, 49]. This index is invariant under exactly marginal transformations and can thus be computed in the free field limit.

The index vanishes on long multiplets and it counts equivalence classes of short multiplets (i.e. protected states) in the following sense. Suppose there are three short multiplets $S_1, S_2, S_3$ and they recombine to some long multiplets $L_1, L_2$ as follows

$$L_1 = S_1 \oplus S_2\,, \tag{1.17}$$

$$L_2 = S_2 \oplus S_3\,. \tag{1.18}$$

Since the index vanishes on long multiplets we find

$$\mathcal{I}(S_1) = -\mathcal{I}(S_2) = \mathcal{I}(S_3) \tag{1.19}$$

and we say that $S_1$ and $S_3$ are in the same equivalence class. In [25] it was found that the superconformal index of $\mathcal{N} = 2$ SCQCD does not match the result from only taking into account eq. (1.16). From this mismatch they were able to extract constraints on the quantum numbers of the lowest lying new protected states. Since the index doesn't separate between the equivalence classes, their quantum numbers cannot be completely fixed in this way. We will briefly review this procedure in chapter 9.

The next natural step is to fix the quantum numbers of the states completely and to explicitly construct their representations in terms of the fundamental fields of the theory. We did just that by using the complete one loop dilatation operator of $\mathcal{N} = 2$ SCQCD, which was derived in [50]. The strategy for this is outlined in chapter 9 and the main results are collected in section 9.3. This is joint work with Martin Sprenger.

# Chapter 2

# Supersymmetric perturbation theory

In this section we explain the formalism that lies at the basis of our calculations. We start by an introduction to standard $\mathcal{N}=1$ superspace, which is the natural language for $\mathcal{N}=1$ supersymmetric theories. We then describe the basics of supersymmetric perturbation theory and finally we introduce the covariant formalism.

## 2.1 $\mathcal{N}=1$ superspace

Supersymmetry, first described by Haag, Łopuszański and Sohnius [51], is a nontrivial extension of Poincaré symmetry by new fermionic symmetry generators $\mathcal{Q}_\alpha, \bar{\mathcal{Q}}_{\dot{\alpha}}$. It thus relates bosons and fermions. The fermionic generators imply that in contrast to conventional symmetries it is not described by a Lie algebra but rather by a *super* Lie algebra, thereby evading the Coleman-Mandula theorem [52]. The new non-vanishing commutation relations are

$$\left\{\hat{\mathcal{Q}}_\alpha, \hat{\bar{\mathcal{Q}}}_{\dot{\alpha}}\right\} = \hat{P}_{\alpha\dot{\alpha}}\,, \tag{2.1}$$

$$\left[\hat{J}_{\alpha\beta}, \hat{\mathcal{Q}}_\gamma\right] = \frac{1}{2} i C_{\gamma(\alpha}\,\hat{\mathcal{Q}}_{\beta)} \tag{2.2}$$

where $\hat{P}$ and $\hat{J}$ are the generators of the Poincaré algebra and $C_{\alpha\beta} = i\epsilon_{\alpha\beta}$ is the antisymmetric symbol. Our notation follows the conventions of [53]: The simplest non-trivial representation of the (universal cover of the) Lorentz group $SL(2,\mathbb{C})$ is the Weyl spinor representation $(\frac{1}{2}, 0)$ and its complex conjugate $(0, \frac{1}{2})$, which we label by indices $\alpha$ and $\dot{\alpha}$ respectively. The combined index $a = (\alpha\dot{\alpha})$ is thus an index of the $(\frac{1}{2}, \frac{1}{2})$ (i.e. vector) representation of the Lorentz group.

A theory is best described in a language that is tailored to its properties. Just as relativistic theories have a natural description in Minkowski space, supersymmetry takes its most natural shape in superspace. The idea is to extend usual spacetime by fermionic directions $\theta, \bar{\theta}$ in order to give a geometric interpretation to the supersymmetry generators $\mathcal{Q}$, $\bar{\mathcal{Q}}$. A point in superspace is thus labeled by

$$z^A = \left(x^{\alpha\dot{\alpha}}, \theta^\alpha, \bar{\theta}^{\dot{\alpha}}\right) \tag{2.3}$$

and the action of the supersymmetry generators in terms of differential operators on this space can be shown to be

$$\mathcal{Q}_\alpha = i\partial_\alpha + \frac{1}{2}\bar{\theta}^{\dot{\alpha}}\partial_{\alpha\dot{\alpha}} \quad \bar{\mathcal{Q}}_{\dot{\alpha}} = i\bar{\partial}_{\dot{\alpha}} + \frac{1}{2}\theta^\alpha\partial_{\alpha\dot{\alpha}}\,. \tag{2.4}$$

Fields defined on this space, which transform covariantly under these transformations are called *superfields*. The simplest example is the scalar superfield, which is invariant under these transformations

$$A(x', \theta', \bar{\theta}') = A(x, \theta, \bar{\theta})\,. \tag{2.5}$$

As in other nontrivial geometries it is possible to construct covariant derivatives, which are invariant under the action of $\mathcal{Q}_\alpha$ and $\bar{\mathcal{Q}}_{\dot{\alpha}}$ (and covariant under the Poincaré generators). These are

$$D_\alpha = \partial_\alpha + \frac{1}{2}\bar{\theta}^{\dot{\alpha}}i\partial_{\alpha\dot{\alpha}} \quad \bar{D}_{\dot{\alpha}} = \bar{\partial}_{\dot{\alpha}} + \frac{1}{2}\theta^\alpha i\partial_{\alpha\dot{\alpha}}\,. \tag{2.6}$$

Their commutation relations are given by

$$\left\{D_\alpha, \bar{D}_\alpha\right\} = i\partial_{\alpha\dot{\alpha}}\,. \tag{2.7}$$

Any superfield can be expanded in the fermionic coordinates. Since they are fermionic, this expansion will terminate. The coefficients of this expansion are fields on ordinary spacetime, called component fields. Eq. (2.5) relates these component fields. This is, what is meant, when people say, supersymmetry relates fermions and bosons. The fields that are physically most important, are even more constrained. The two examples we will deal with are the *chiral* superfield $\Phi$ and the *real (or vector)* superfield $V$.

The chiral superfield is constrained by $\bar{D}_{\dot{\alpha}}\Phi = 0$. Upon the coordinate transformation $y_{\alpha\bar{\alpha}} = x_{\alpha\bar{\alpha}} + \frac{1}{2}i\theta_\alpha\theta_{\bar{\alpha}}$ it becomes independent of $\bar{\theta}$ and its component expansion is simply given by

$$\Phi(y, \theta) = \phi(y) + \theta^\alpha\psi_\alpha(y) + \theta^2 F(y)\,, \tag{2.8}$$

where $\phi$ is a complex scalar, $\psi$ is a Weyl spinor and $F$ is a complex auxiliary field.

The real superfield fulfills $V = V^\dagger$, which gives

$$\begin{aligned} V(z) =& C(x) + \theta^\alpha \chi_\alpha(x) + \bar{\theta}^{\dot{\alpha}} \bar{\chi}_{\dot{\alpha}}(x) - \theta^2 M(x) - \bar{\theta}^2 \bar{M}(x) + \theta^\alpha \bar{\theta}^{\dot{\alpha}} A_{\alpha\dot{\alpha}}(x) \\ & - \bar{\theta}^2 \theta^\alpha \lambda_\alpha(x) - \theta^2 \bar{\theta}^{\dot{\alpha}} \bar{\lambda}_{\dot{\alpha}}(x) + \theta^2 \bar{\theta}^2 D(x) \,. \end{aligned} \tag{2.9}$$

Since there is a real vector field $A_{\alpha\dot{\alpha}}$, this is seen as a supersymmetric version of the gauge field and indeed it posseses a huge amount of gauge freedom (see eq. (2.23)). By a proper gauge choice, called *Wess-Zumino gauge* [54], it takes the form

$$V(z) = \theta^\alpha \bar{\theta}^{\dot{\alpha}} A_{\alpha\dot{\alpha}}(x) - \bar{\theta}^2 \theta^\alpha \lambda_\alpha(x) - \theta^2 \bar{\theta}^{\dot{\alpha}} \bar{\lambda}_{\dot{\alpha}}(x) + \theta^2 \bar{\theta}^2 D(x) \,, \tag{2.10}$$

with the real vector field $A_{\alpha\dot{\alpha}}$, the Weyl spinor $\lambda_\alpha$ and the real auxiliary field $D$.

As in the non-supersymmetric case $V$ does not transform covariantly under gauge transformations. Instead we define the gauge covariant quantity

$$W_\alpha = i \bar{D}^2 e^{-V} D_\alpha e^V \,, \tag{2.11}$$

which is called the supersymmetric field strength, because it contains the self dual field strength $f_{\alpha\beta}$ of the component vector field $A_{\alpha\dot{\alpha}}$. Similarly $\bar{W}_{\dot{\alpha}}$ contains the anti-self dual part $f_{\dot{\alpha}\dot{\beta}}$. Note that this is a chiral field $\bar{D}_{\dot{\alpha}} W_\beta = 0$.

The chiral part of the classical action for the vector field is then given by

$$S_{\text{gauge}} = \frac{1}{2g^2} \int \mathrm{d}^6 z \; \mathrm{tr} \, (W^\alpha W_\alpha) \tag{2.12}$$

$$= -\frac{1}{2g^2} \int \mathrm{d}^8 z \; (e^{-V} D^\alpha e^V) \bar{D}^2 (e^{-V} D_\alpha e^V) \,. \tag{2.13}$$

Here $\mathrm{d}^8 z = \mathrm{d}^4 x \, \mathrm{d}^2\theta \mathrm{d}^2\bar{\theta}$ and $\mathrm{d}^6 z = \mathrm{d}^4 x \, \mathrm{d}^2\theta$ are the measures of full and chiral superspace respectively. When this action is written in components, it reduces to

$$S_{\text{gauge}} = \frac{1}{2g^2} \int \mathrm{d}^4 x \, \left( -\frac{1}{2} f^{\alpha\beta} f_{\alpha\beta} + \bar{\lambda}^{\dot{\alpha}} i \nabla^\alpha_{\dot{\alpha}} \lambda_\alpha + D^2 \right) \,. \tag{2.14}$$

These are just the standard terms for a vector field and a Weyl fermion interacting with it. The auxiliary field $D$ has no dynamics.

Renormalizable gauge matter interactions take the form

$$S_{\text{K}} = \frac{1}{g^2} \int \mathrm{d}^8 z \; \mathrm{tr} \, (\bar{\Phi}_j \left( e^V \right)^i_j \Phi^i) \,, \tag{2.15}$$

where $V^i_j = V^a (T^a)^i_j$ and the form of the $T^a$ depends on the representation of the chiral fields. For example for fields in the adjoint representation this can be written as

$$S_{\mathrm{K}} = \frac{1}{g^2} \int \mathrm{d}^8 z \, \mathrm{tr}\,(e^{-V} \bar{\Phi} e^{V} \Phi)\,, \tag{2.16}$$

where now $V = V^a T^a$ with the $T^a$ in the fundamental representation. In components eq. (2.15) reads

$$S_{\mathrm{K}} = \int \mathrm{d}^4 x \; \bar{Q}_i \Box Q^i + \bar{\psi}_i^{\dot{\alpha}} i \nabla^{\alpha}_{\dot{\alpha}} \psi^i_{\alpha} + i \bar{Q}_i (\lambda^{\alpha})^i_j \psi^j_{\alpha} - i \bar{\psi}_i^{\dot{\alpha}} (\bar{\lambda}_{\dot{\alpha}})^i_j Q^j + \bar{Q}_i (D)^i_j Q^j + F^i \bar{F}_i\,. \tag{2.17}$$

Again we find the standard kinetic terms for the component fields, minimally coupled to the gauge field. however in this term there are also Yukawa interactions between the components of the two multiplets. Finally there can be a superpotential for the chiral field

$$S_{\mathrm{W}} = \int \mathrm{d}^6 z \, \mathcal{W}(\Phi)\,, \tag{2.18}$$

where $\mathcal{W}$ is some holomorphic function. In renormalizable theories it has to terminate after the third order

$$\mathcal{W}(\Phi) = m^2 \Phi^2 + \lambda \Phi^3\,. \tag{2.19}$$

The name $\lambda$ for the coupling is an unfortunate convention. It should not be confused with the fermion in the vector multiplet. In components this reads

$$S_{\mathrm{W}} = \int \mathrm{d}^4 x \; m \left( \frac{1}{2} \psi^{\alpha} \psi_{\alpha} + QF \right) + \lambda \left( \frac{1}{2} Q \psi^{\alpha} \psi_{\alpha} + \frac{1}{2} Q^2 F \right)\,. \tag{2.20}$$

After integration of the auxiliary field $F$ we can also write this as

$$\begin{aligned} S_{\mathrm{W}} = \int \mathrm{d}^4 x \; \Big( & - m^2 \bar{Q} Q + m(\psi^2 + \bar{\psi}^2) - \frac{1}{2} m \lambda \left( Q \bar{Q}^2 + \bar{Q} Q^2 \right) \\ & - \frac{1}{4} \lambda^2 Q^2 \bar{Q}^2 + \lambda \left( Q \psi^2 + \bar{Q} \bar{\psi}^2 \right) \Big)\,, \end{aligned} \tag{2.21}$$

which leads to mass terms for the fields, a cubic scalar coupling, a quartic scalar coupling and Yukawa terms.

## 2.2 Feynman rules for superfields

Starting from the Lagrangians in the last section it is possible to derive Feynman rules for superfields in just the same way as one does for ordinary fields. Since this material is covered in the vast literature (see e.g. [53, 55–57]), we will not review the full details here.

The classical action for the vector field

$$S_{\text{gauge}} = \frac{1}{2g^2}\int \mathrm{d}^6z \ \mathrm{tr}\,(W^\alpha W_\alpha) = -\frac{1}{2g^2}\int \mathrm{d}^8z \ (e^{-V}D^\alpha e^V)\bar{D}^2(e^{-V}D_\alpha e^V) \tag{2.22}$$

is invariant under the gauge transformation

$$e^V \to e^{i\bar{\Lambda}}e^V e^{-i\Lambda} \quad \text{for} \quad \bar{D}_{\dot{\alpha}}\Lambda = 0\,. \tag{2.23}$$

As in non-supersymmetric QFT one has to introduce a gauge fixing function, which in non-abelian theories leads to the introduction of two chiral Faddeev-Popov ghosts $c, c'$. A suitable gauge fixing function is given by

$$F = \bar{D}^2 V\,. \tag{2.24}$$

We will skip the details of the Faddeev-Popov procedure, which is analogous to the non-supersymmetric case and just cite the result. The extra term from gauge fixing is

$$S_{\text{GF}} = -\frac{1}{\alpha g^2}\int \mathrm{d}^8z \ \mathrm{tr}\,(D^2V)(\bar{D}^2V) \tag{2.25}$$

and the Faddeev-Popov term is

$$S_{FP} = i\int \mathrm{d}^8z \ \mathrm{tr}\,(c'-\bar{c}')L_{\frac{1}{2}V}\left[(c+\bar{c})\coth L_{\frac{1}{2}V}(c-\bar{c})\right]\,, \tag{2.26}$$

where $L_XY = [X,Y]$. In Feynman gauge ($\alpha = 1$) we find

$$S_{\text{gauge}} + S_{\text{GF}} = \int \mathrm{d}^8z \ \mathrm{tr}\left(-\frac{1}{2}V\Box_0 V + \frac{1}{2}\left[V, D^\alpha V\right](\bar{D}^2 D_\alpha V) + \ldots\right)\,, \tag{2.27}$$

with $\Box_0 = \frac{1}{2}\partial^{\alpha\dot{\alpha}}\partial_{\alpha\dot{\alpha}}$.

In order to derive the Feynman rules one starts with the generating functional with source terms

$$\int \mathrm{d}^4x\mathrm{d}^2\theta\mathrm{d}^2\bar{\theta}\ JV + \int \mathrm{d}^4x\mathrm{d}^2\theta\ j\Phi + \int \mathrm{d}^4x\mathrm{d}^2\bar{\theta}\ \bar{j}\bar{\Phi}\,, \tag{2.28}$$

where $J$ is real, $j$ is chiral and $\bar{j}$ is antichiral. We then take functional derivatives

$$\frac{\delta J(x,\theta,\bar{\theta})}{\delta J(x',\theta',\bar{\theta}')} = \delta^4(x-x')\delta^4(\theta-\theta')\,, \tag{2.29}$$

$$\frac{\delta j(x,\theta)}{\delta j(x',\theta')} = \bar{D}^2\delta^4(x-x')\delta^4(\theta-\theta')\,, \tag{2.30}$$

$$\frac{\delta \bar{j}(x, \bar{\theta})}{\delta \bar{j}(x', \bar{\theta}')} = D^2 \delta^4(x - x') \delta^4(\theta - \theta') \,. \tag{2.31}$$

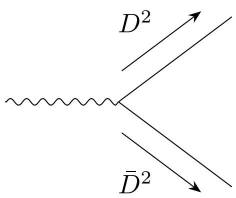
Note that the presence of $\bar{D}^2$ and $D^2$ implies that whenever there is a chiral line leaving a vertex, there is an extra factor of $\bar{D}^2$ acting on that line and similarly for antichiral lines. For purely chiral vertices one of those factors is used to complete the chiral superspace integral $\int \mathrm{d}^6 z\ \bar{D}^2 = \int \mathrm{d}^8 z$ .

In total we find the Feynman rules for the propagators in the massless case

$$\langle VV \rangle = -\frac{1}{p^2} \delta^4(\theta - \theta') \,, \qquad \langle \bar{\Phi} \Phi \rangle = \frac{1}{p^2} \delta^4(\theta - \theta') \,, \tag{2.32}$$

and the vertices can be read from the action with the additional rule for chiral fields described above. As an example the $\bar{\Phi} V \Phi$ vertex looks like

$$\langle \bar{\Phi} V^2 \Phi \rangle = \tag{2.33}$$




,

where the arrows indicate the direction, in which the derivatives act. Each vertex comes with a superspace integral.

One can see that a supersymmetric Feynman graph has the same structure as a normal Feynman graph but is supplemented by superspace derivatives acting on the propagators. This extra structure encodes the supersymmetry. In order to compute supersymmetric Feynman graphs, one must know how to manipulate these derivatives. This procedure is called the $D$-algebra. Let us outline its fundamental features.

The derivatives act on the $\delta^4(\theta_1 - \theta_2)$ functions that come with the propagators. One can move the derivatives along the lines by means of the transfer rule

$$D_{1\alpha} \delta^4(\theta_1 - \theta_2) = -\delta^4(\theta_1 - \theta_2) \overleftarrow{D_{2\alpha}} \tag{2.34}$$

and at the vertices one can integrate by parts to shift a derivative from one line to another. Whenever the derivatives are collected one the same line one can use $\left\{ D_\alpha, \bar{D}_\alpha \right\} = i \partial_{\alpha\dot{\alpha}}$ and $D^3 = \bar{D}^3 = 0$ to reduce the number of superspace derivatives and convert them to ordinary spacetime derivatives. One can show that

$$\delta^4(\theta) = \theta^2 \bar{\theta}^2 \,, \tag{2.35}$$

and writing $\delta_{12} = \delta^4(\theta_1 - \theta_2)$ one then finds

$$\delta_{12}\delta_{12} = \delta_{12}D^\alpha\delta_{12} = \delta_{12}D^2\delta_{12} = \delta_{12}D^2\bar{D}^{\dot{\alpha}}\delta_{12} = 0\,, \tag{2.36}$$

and only

$$\delta_{12}D^2\bar{D}^2\delta_{12} = \delta_{12}\,. \tag{2.37}$$

If there are no derivatives acting on a product of $\delta$-functions, we have the ordinary rule

$$\int \mathrm{d}^4\theta\ \delta^4(\theta' - \theta)\delta^4(\theta - \theta'') = \delta^4(\theta' - \theta'')\,. \tag{2.38}$$

The goal is now to free the lines of derivatives by the manipulations described above, so that the $\theta$ integrals can be performed. This can be done successively for all lines in a loop except until one reaches the situation where only two $\theta$ coordinates and two $\delta$-functions are left. Eqs. (2.36) and (2.37) imply that we need a factor of $D^2\bar{D}^2$ in the loop in order to get a non-vanishing result.

The end result of these manipulations is an ordinary Feynman graph without superspace derivatives (they were all converted to spacetime derivatives or used in the fermionic integration), which is local in $\theta$, i.e. all vertices in the graph are evaluated at the same $\theta$ coordinate.

This is the simplest version of supersymmetric Feynman rules. It keeps supersymmetry manifest instead of splitting up the computation into components. This leads to fewer diagrams than ordinary perturbation theory because many cancellations between component fields are automatic, when they are packaged in superfields.

Despite these computational advantages there are some shortcomings. When one deals with gauge theories one has to deal with the vector field $V$, which has scaling dimension 0. Due to this Feynman graphs with external $V$ fields have a notoriously high degree of divergence. This is only cured, when all contributions are added and gauge invariance is restored. These problems could be avoided, if we used a formalism that doesn't break gauge invariance in intermediate steps and instead is solely based on gauge invariant quantities. In the next section we will introduce such a formalism.

## 2.3 Covariant D-algebra

In the last section we discussed ordinary supersymmetric perturbation theory. One of the shortcomings this technique is that it doesn't keep explicit gauge invariance. In non-supersymmetric theories there is the well-known background field formalism, which splits the fields into quantum and background parts and only fixes the gauge symmetry of the quantum part. Background gauge invariance is maintained throughout the calculation. This formalism exists with minor modifications also in supersymmetric theories. While the splitting in non-supersymmetric theories is usually linear this is not suitable here, because the non-abelian gauge symmetry is highly nonlinear in $V$. Inspiration for the correct splitting can be found by looking at the transformation properties eq. (2.23)

$$e^V \to e^{i\bar{\Lambda}} e^V e^{-i\Lambda} \quad \text{for} \quad \bar{D}_{\dot{\alpha}} \Lambda = 0\,. \tag{2.39}$$

It turns out that a convenient splitting is given by

$$e^V = e^{\mathbf{\Omega}} e^{V_Q} e^{\bar{\mathbf{\Omega}}}\,, \tag{2.40}$$

where $\mathbf{\Omega}$ is called the background prepotential and $V_Q$ is the quantum vector superfield. We also define the *background gauge covariant derivatives*

$$\boldsymbol{\nabla}_\alpha = e^{-\mathbf{\Omega}} D_\alpha e^{\mathbf{\Omega}} \quad \text{and} \quad \bar{\boldsymbol{\nabla}}_{\dot{\alpha}} = e^{\bar{\mathbf{\Omega}}} \bar{D}_{\dot{\alpha}} e^{-\bar{\mathbf{\Omega}}}\,. \tag{2.41}$$

Then the classical action eq. (2.22) takes the form

$$S_{\text{gauge}} = -\frac{1}{2g^2} \int \mathrm{d}^8 z \, (e^{-V_Q} \boldsymbol{\nabla}^\alpha e^{V_Q}) \bar{\boldsymbol{\nabla}}^2 (e^{-V_Q} \boldsymbol{\nabla}_\alpha e^{V_Q})\,. \tag{2.42}$$

From now on we will drop the index and $V = V_Q$ will always denote the quantum field. Since our goal is to only gauge fix the quantum part and keep explicit background gauge symmetry, we have to choose a background covariant gauge fixing function. A good candidate is given by the covariantization of eq. (2.24)

$$F = \bar{\boldsymbol{\nabla}}^2 V\,, \tag{2.43}$$

which leads to the new term in the action

$$S_{\text{GF}} = -\frac{1}{\alpha g^2} \int \mathrm{d}^8 z \, \mathrm{tr}\, (\boldsymbol{\nabla}^2 V)(\bar{\boldsymbol{\nabla}}^2 V). \tag{2.44}$$

The Faddeev-Popov ghosts will now be background covariantly chiral $\bar{\boldsymbol{\nabla}}_{\dot{\alpha}} c = 0$ rather than chiral. Since our gauge fixing function explicitly depends on the background, it requires the introduction of another background covariantly chiral ghost $b$, called Nielsen-Kallosh ghost [58, 59], which introduces a term

$$S_{\text{NK}} = \int \mathrm{d}^8 z \, \mathrm{tr}\, \bar{b} b\,. \tag{2.45}$$

It has no interaction with the quantum fields. The quadratic part of the gauge fixed action can be brought in the form

$$(S_{\text{gauge}} + S_{\text{GF}})|_{V^2} = \int \mathrm{d}^8 z\, V(\Box - i\boldsymbol{W}^\alpha \boldsymbol{\nabla}_\alpha - i\bar{\boldsymbol{W}}^{\dot\alpha}\bar{\boldsymbol{\nabla}}_{\dot\alpha})V\,, \tag{2.46}$$

where $\Box = \frac{1}{2}\boldsymbol{\nabla}^{\alpha\dot\alpha}\boldsymbol{\nabla}_{\alpha\dot\alpha}$ is the background covariant d'Alembertian and $\boldsymbol{W}_\alpha$ is the background covariant field strength defined by $[\bar{\boldsymbol{\nabla}}_{\dot\alpha}, \boldsymbol{\nabla}_{\beta\dot\beta}] = C_{\dot\alpha\dot\beta}\boldsymbol{W}_\beta$.

For covariantly chiral fields in some background one can covariantize the identity

$$\bar{D}^2 D^2 \Phi = \Box_0 \Phi \quad \rightarrow \quad \bar{\boldsymbol{\nabla}}^2 \boldsymbol{\nabla}^2 \Phi = \Box_+ \Phi \tag{2.47}$$

to find the kinetic operator

$$\Box_+ = \Box - \frac{i}{2}(\boldsymbol{\nabla}^\alpha \boldsymbol{W}_\alpha) - i\boldsymbol{W}^\alpha \boldsymbol{\nabla}_\alpha \tag{2.48}$$

and similarly for covariantly antichiral fields. See appendix A for more details.

Now that we have this background gauge covariant formulation at hand, the strategy employed in [44, 45] is to perform the $D$-algebra directly with the gauge covariant derivatives

$$(\boldsymbol{\nabla}_\alpha, \bar{\boldsymbol{\nabla}}_{\dot\alpha}, \boldsymbol{\nabla}_{\alpha\dot\alpha}) \tag{2.49}$$

rather then splitting them up into ordinary superspace derivatives $(D_\alpha, \bar{D}_{\dot\alpha}, \partial_{\alpha\dot\alpha})$ and the superspace background connections $(\boldsymbol{\Gamma}_\alpha, \boldsymbol{\Gamma}_{\dot\alpha}, \boldsymbol{\Gamma}_{\alpha\dot\alpha})$ , which are the supersymmetric versions of $A_\mu$. By working directly with the covariant derivatives, manifest gauge invariance is maintained. This procedure is called the *covariant D-algebra*.

This improves the powercounting, because neither $\boldsymbol{\Omega}$ nor the background connections $\boldsymbol{\Gamma}_\alpha$ and $\boldsymbol{\Gamma}_{\dot\alpha}$ (with mass dimension 1/2) will ever appear explicitly as external fields. When this formalism was originally introduced in [44, 60] it was still necessary to split $\boldsymbol{\nabla}_{\alpha\dot\alpha}$ at a later stage of the calculation, thereby breaking gauge invariance. This was solved in [46], see section 2.4, so that gauge invariance can be kept all the way and the only quantity that appears as external field is the gauge covariant background field strength $\boldsymbol{W}_\alpha$ with mass dimension 3/2. This is a big improvement over having the field $V$ with dimension 0 as an external field.

The fundamental commutation relations of the covariant derivatives read

$$\{\boldsymbol{\nabla}_\alpha, \bar{\boldsymbol{\nabla}}_{\dot\alpha}\} = i\boldsymbol{\nabla}_{\alpha\dot\alpha}\,, \tag{2.50}$$

$$[\boldsymbol{\nabla}_\alpha, \boldsymbol{\nabla}_{\beta\dot\beta}] = C_{\alpha\beta}\bar{\boldsymbol{W}}_{\dot\beta}, \qquad [\bar{\boldsymbol{\nabla}}_{\dot\alpha}, \boldsymbol{\nabla}_{\beta\dot\beta}] = C_{\dot\alpha\dot\beta}\boldsymbol{W}_\beta\,, \tag{2.51}$$

$$\boldsymbol{\nabla}^\alpha \boldsymbol{W}_\alpha = -\bar{\boldsymbol{\nabla}}^{\dot\alpha}\bar{\boldsymbol{W}}_{\dot\alpha}\,. \tag{2.52}$$

These relations are more complicated than the simple $\left\{D_\alpha, \bar{D}_{\dot\alpha}\right\} = i\partial_{\alpha\dot\alpha}$, but they turn out to be very useful in actual calculations. There is a plethora of relations that follow from these definitions. Appendix A is devoted to their derivation. Two of the most important ones are

$$\begin{aligned}\left[\boldsymbol{\nabla}_\alpha, \bar{\boldsymbol{\nabla}}^2\right] &= i\boldsymbol{W}_\alpha - i\boldsymbol{\nabla}_{\alpha\dot\alpha}\bar{\boldsymbol{\nabla}}^{\dot\alpha}\,,\\ \left[\bar{\boldsymbol{\nabla}}_{\dot\alpha}, \boldsymbol{\nabla}^2\right] &= i\bar{\boldsymbol{W}}_{\dot\alpha} - i\boldsymbol{\nabla}_{\alpha\dot\alpha}\boldsymbol{\nabla}^{\alpha}\,,\end{aligned} \tag{2.53}$$

which will be used ubiquitously throughout this thesis.

Another feature of this calculation is that the D-algebra is performed *before* expanding the kinetic operators, i.e. in the presence of the background. Let us see, how this works: The kinetic operators for the vector and (anti-)chiral superfields in the presence of the background are given by

$$\begin{aligned}\hat{\Box} &= \Box - i\boldsymbol{W}^\alpha\boldsymbol{\nabla}_\alpha - i\bar{\boldsymbol{W}}^{\dot\alpha}\bar{\boldsymbol{\nabla}}_{\dot\alpha}\,,\\ \Box_+ &= \Box - \frac{i}{2}(\boldsymbol{\nabla}^\alpha\boldsymbol{W}_\alpha) - i\boldsymbol{W}^\alpha\boldsymbol{\nabla}_\alpha\,,\\ \Box_- &= \Box - \frac{i}{2}(\bar{\boldsymbol{\nabla}}^{\dot\alpha}\bar{\boldsymbol{W}}_{\dot\alpha}) - i\bar{\boldsymbol{W}}^{\dot\alpha}\bar{\boldsymbol{\nabla}}_{\dot\alpha}\,.\end{aligned} \tag{2.54}$$

Rather than expanding the propagators in terms of the background field strengths and then do the D-algebra we make use of the commutation relations (see appendix A)

$$\begin{aligned}\left[\boldsymbol{\nabla}_\alpha, \hat{\Box}\right] &= \frac{1}{2}(\boldsymbol{\nabla}_{\alpha\dot\alpha}\bar{\boldsymbol{W}}^{\dot\alpha}) - i(\boldsymbol{\nabla}_\alpha\boldsymbol{W}^\beta)\boldsymbol{\nabla}_\beta\,,\\ \left[\bar{\boldsymbol{\nabla}}_{\dot\alpha}, \hat{\Box}\right] &= \frac{1}{2}(\boldsymbol{\nabla}_{\alpha\dot\alpha}\boldsymbol{W}^{\alpha}) - i(\bar{\boldsymbol{\nabla}}_{\dot\alpha}\bar{\boldsymbol{W}}^{\dot\beta})\bar{\boldsymbol{\nabla}}_{\dot\beta}\,,\end{aligned} \tag{2.55}$$

and

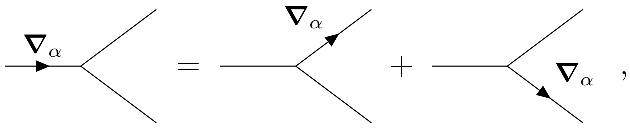
$$\begin{aligned}\left[\boldsymbol{\nabla}_\alpha, \Box_+\right] &= (\boldsymbol{\nabla}^2\boldsymbol{W}_\alpha) + \bar{\boldsymbol{W}}^{\dot\alpha}\boldsymbol{\nabla}_{\alpha\dot\alpha} + i(\boldsymbol{\nabla}_\alpha\boldsymbol{W}_\beta)\boldsymbol{\nabla}^\beta\,,\\ \left[\bar{\boldsymbol{\nabla}}_{\dot\alpha}, \Box_-\right] &= (\bar{\boldsymbol{\nabla}}^2\bar{\boldsymbol{W}}_{\dot\alpha}) + \boldsymbol{W}^{\alpha}\boldsymbol{\nabla}_{\alpha\dot\alpha} + i(\bar{\boldsymbol{\nabla}}_{\dot\alpha}\bar{\boldsymbol{W}}_{\dot\beta})\bar{\boldsymbol{\nabla}}^{\dot\beta}\,,\\ \left[\boldsymbol{\nabla}_\alpha, \Box_-\right] &= \left[\bar{\boldsymbol{\nabla}}_{\dot\alpha}, \Box_+\right] = 0\,.\end{aligned} \tag{2.56}$$

These rules are used in conjunction with the general rule for computing commutators of inverse operators

$$\begin{aligned}\left[A, B^{-1}\right] &= AB^{-1} - B^{-1}A = B^{-1}BAB^{-1} - B^{-1}ABB^{-1}\\ &= -B^{-1}\left[A, B\right]B^{-1}\,,\end{aligned} \tag{2.57}$$

to find the commutation relations for the propagators $\hat{\Box}^{-1}$ and $\Box_\pm^{-1}$. They are significantly more complicated than $[D_\alpha, \Box_0] = 0$, which is used in ordinary D-algebra but they have the advantage that the external background fields that they create have much higher mass dimensions, which will lead to fewer divergent terms!

At the vertices one can use the Leibniz rule to transfer the derivatives from one line to the others. Pictorially this is given by



(2.58)

where the arrows indicate the direction in which the derivative is acting. The goal is to collect multiple factors of covariant superspace derivatives in one place, use the commutation relations to reduce them to spacetime derivatives and external fields and to use the fact that as in ordinary D-algebra a factor of $\boldsymbol{\nabla}^2\bar{\boldsymbol{\nabla}}^2$ in every loop is necessary to get a non-vanishing result.

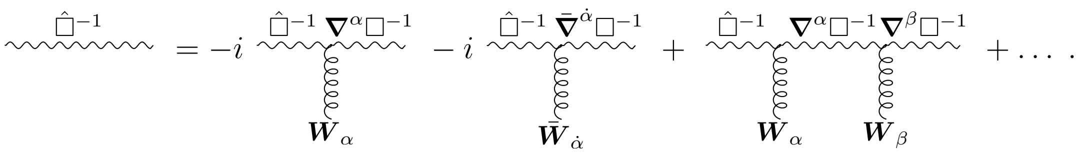
At the end one still has to expand the propagators in terms of the background fields. Writing $\hat{\Box} = \Box + \boldsymbol{A}$ with $\boldsymbol{A} = i\boldsymbol{W}_\alpha\boldsymbol{\nabla}^\alpha + i\bar{\boldsymbol{W}}_{\dot\alpha}\bar{\boldsymbol{\nabla}}^{\dot\alpha}$ its inverse is given by the recursive expression

$$\hat{\Box}^{-1} = \Box^{-1} - \hat{\Box}^{-1}\boldsymbol{A}\Box^{-1}\,, \tag{2.59}$$

which can be used to expand to any given order in the background fields. Eq. (2.59) can be proved by writing

$$\begin{aligned}
\hat{\Box}(\Box^{-1} - \hat{\Box}^{-1}\boldsymbol{A}\Box^{-1}) &= \hat{\Box}\Box^{-1} - \hat{\Box}\,\hat{\Box}^{-1}\boldsymbol{A}\Box^{-1} \\
&= (\Box + \boldsymbol{A})\Box^{-1} - \hat{\Box}\,\hat{\Box}^{-1}\boldsymbol{A}\Box^{-1} \\
&= \Box\,\Box^{-1} + \boldsymbol{A}\Box^{-1} - \boldsymbol{A}\Box^{-1} \\
&= \mathrm{id}\,. 
\end{aligned} \tag{2.60}$$

Graphically eq. (2.59) is represented by



(2.61)

The expansions for $\Box_\pm^{-1}$ follow analogously. As we will see in section 4.1 the terms involving $\bar{\boldsymbol{W}}$ will not contribute in our sector and thus the expansions will simplify drastically.

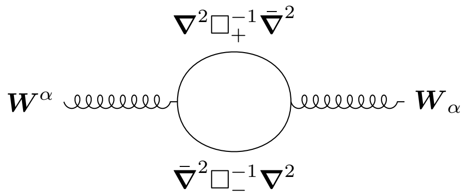



**Figure 2.1:** Illustration of how superspace derivatives appear on chiral lines on the example of the one loop self-energy of $\boldsymbol{W}$

Note that the operator $\Box\frac{1}{2}\boldsymbol{\nabla}^{\alpha\dot\alpha}\boldsymbol{\nabla}_{\alpha\dot\alpha}$ still contains the connection $\boldsymbol{\Gamma}_{\alpha\dot\alpha}$.

$$\Box = \Box_0 - \frac{i}{2}\partial^{\alpha\dot\alpha}\boldsymbol{\Gamma}_{\alpha\dot\alpha} - i\boldsymbol{\Gamma}_{\alpha\dot\alpha}\partial^{\alpha\dot\alpha} - \frac{1}{2}\boldsymbol{\Gamma}^{\alpha\dot\alpha}\boldsymbol{\Gamma}_{\alpha\dot\alpha} \stackrel{\text{def}}{=} \Box_0 + \boldsymbol{B}\,. \tag{2.62}$$

Naively it is necessary to expand $\Box^{-1}$ in a similar fashion as $\hat\Box^{-1}$ in terms of $\boldsymbol{\Gamma}_{\alpha\dot\alpha}$, however in the next section we will see how to avoid this.

As explained for ordinary supersymmetric Feynman rules below eq. (2.29) taking functional derivatives introduces a factor of $(\boldsymbol{\nabla}^2)\bar{\boldsymbol{\nabla}}^2$ acting on (anti-)chiral lines. This implies in particular that the (anti-)chiral kinetic operators always come in the combination

$$\boldsymbol{\nabla}^2\Box_+^{-1}\bar{\boldsymbol{\nabla}}^2 \qquad \text{and} \qquad \bar{\boldsymbol{\nabla}}^2\Box_-^{-1}\boldsymbol{\nabla}^2\,, \tag{2.63}$$

which will have important implications later. Figure 2.1 illustrates this with an example of a contribution to the one loop self-energy of the background field strength.

We summarize the strategy of computation as follows

1. draw all possible quantum vacuum graphs with $\hat\Box^{-1}$ propagators (possibly with $\boldsymbol{W}$ emerging from the vertices but *not* from the propagators)

2. Bring all spinor derivatives to one place and convert them to spacetime derivatives by using their commutation relations with the operators $\hat\Box$ and $\Box_\pm$ and using integration by parts at the vertices. This will produce factors of $\boldsymbol{W}$ and $\bar{\boldsymbol{W}}$

3. Expand

$$\begin{aligned}\hat\Box^{-1} = &\Box^{-1} + i\,\Box^{-1}(\boldsymbol{W}^\alpha\boldsymbol{\nabla}_\alpha + \bar{\boldsymbol{W}}^{\dot\alpha}\bar{\boldsymbol{\nabla}}_{\dot\alpha})\Box^{-1}\\ &- \Box^{-1}(\boldsymbol{W}^\alpha\boldsymbol{\nabla}_\alpha + \bar{\boldsymbol{W}}^{\dot\alpha}\bar{\boldsymbol{\nabla}}_{\dot\alpha})\Box^{-1}(\boldsymbol{W}^\alpha\boldsymbol{\nabla}_\alpha + \bar{\boldsymbol{W}}^{\dot\alpha}\bar{\boldsymbol{\nabla}}_{\dot\alpha})\Box^{-1}\\ &+ \mathcal{O}(\boldsymbol{W}^3)\end{aligned} \tag{2.64}$$

and similarly for $\Box_\pm^{-1}$.

4. Use

$$\boldsymbol{\nabla}^{\alpha}\boldsymbol{\nabla}^{\beta}\boldsymbol{\nabla}^{\gamma} = \bar{\boldsymbol{\nabla}}^{\dot{\alpha}}\bar{\boldsymbol{\nabla}}^{\dot{\beta}}\bar{\boldsymbol{\nabla}}^{\dot{\gamma}} = 0 \,, \tag{2.65}$$

$$\delta(\theta_1 - \theta_2) P(\boldsymbol{\nabla}_{\alpha}, \bar{\boldsymbol{\nabla}}_{\dot{\alpha}}) \delta(\theta_1 - \theta_2) = 0 \tag{2.66}$$

for any polynomial $P$ in two variables of degree smaller 2 in either variable and

$$\delta(\theta_1 - \theta_2) \boldsymbol{\nabla}^2 \bar{\boldsymbol{\nabla}}^2 \delta(\theta_1 - \theta_2) = \delta(\theta_1 - \theta_2) \tag{2.67}$$

to complete the D-algebra.

Steps 2 and 3 don't have to be in this order. It might sometimes be convenient to first expand the propagators and then commute the derivatives.

## 2.4 Diagrams with covariant spacetime derivatives

Many of the diagrams we are interested in contain background covariant spacetime derivatives. A standard way of treating them is to split them up into an ordinary derivative and the connection and compute both contributions separately. This is inconvenient, because the individual contributions will not be gauge invariant. It would be advantageous to find a way of computing these diagrams by working with covariant derivatives directly. At the one loop level this was first achieved by the *Schwinger-DeWitt technique* [38, 61], which has been successfully applied and generalized, see e.g. [62–66]. For the discussion of higher loops we will follow the method introduced in [46], whose essential features we will outline below. For an alternative approach, see e.g. [67]. The main ingredients are the use of the so-called *parallel displacement propagator* [38] and the *covariant Taylor expansion* [65].

After following the steps of the last section one is left with a diagram, where the propagators are given by $\Box^{-1}$, where $\Box = \frac{1}{2}\boldsymbol{\nabla}^{\alpha\dot{\alpha}}\boldsymbol{\nabla}_{\alpha\dot{\alpha}}$, and there are external background fields at the vertices and there might be covariant derivatives $\boldsymbol{\nabla}_{\beta\dot{\beta}}$ acting on the propagators. First we consider the propagators. We write the Green's function in terms of the heat kernel as

$$D_{\Box}(x, x') = i \int_0^{\infty} \mathrm{d}s K(x, x'|s) \,, \tag{2.68}$$

where

$$K(x, x'|s) = e^{is(\Box + i\delta)} \delta^d(x - x') \mathbb{1} \,, \qquad \delta \searrow 0 \,. \tag{2.69}$$

We will usually omit the $\delta$, which is needed for the causal ordering in the Feynman propagator. The Fourier integral representation of $\delta^d(x-x')\mathbb{1}$ is

$$\delta^d(x-x')\mathbb{1} = \int \frac{\mathrm{d}^d k}{(2\pi)^d}\, e^{ik(x-x')} I(x,x')\,, \tag{2.70}$$

where $I(x,x')$ is the parallel displacement propagator. It has the following defining properties: It tranforms covariantly under gauge transformations, it satisfies the boundary condition $I(x,x)=\mathbb{1}$ and it satisfies $(x-x')^a \boldsymbol{\nabla}_a I(x,x') = 0$. From these properties one can show that

$$\boldsymbol{\nabla}_{(a_1}\ldots\boldsymbol{\nabla}_{a_n)}\, I(x,x')\big|_{x=x'} = 0\,, \tag{2.71}$$

$$I(x,x')I(x',x) = \mathbb{1}\,, \tag{2.72}$$

$$(I(x,x'))^\dagger = I(x',x)\,. \tag{2.73}$$

The Green's function has the explicit expression

$$\begin{aligned} D_\Box(x,x') &= i\int_0^\infty \mathrm{d}s \int \frac{\mathrm{d}^d k}{(2\pi)^d}\, e^{is\Box} e^{ik(x-x')} I(x,x') \\ &= i\int_0^\infty \mathrm{d}s \int \frac{\mathrm{d}^d k}{(2\pi)^d}\, e^{ik(x-x')} e^{\frac{is}{2}(\boldsymbol{\nabla}^a + ik^a)^2} I(x,x') \\ &= i\int_0^\infty \mathrm{d}s \int \frac{\mathrm{d}^d k}{(2\pi)^d}\, e^{-isk^2} e^{ik(x-x')} e^{is\Box - sk^a\boldsymbol{\nabla}_a} I(x,x')\,. \end{aligned} \tag{2.74}$$

When there are derivatives acting on the Green's function one finds

$$\begin{aligned} &\boldsymbol{\nabla}_{a_1}\ldots\boldsymbol{\nabla}_{a_n} D_\Box(x,x') \\ &\quad = i\int_0^\infty \mathrm{d}s \int \frac{\mathrm{d}^d k}{(2\pi)^d}\, e^{-isk^2} e^{ik(x-x')} (\boldsymbol{\nabla}+ik)_{a_1}\ldots(\boldsymbol{\nabla}+ik)_{a_n} e^{is\Box - sk^a\boldsymbol{\nabla}_a} I(x,x')\,. \end{aligned} \tag{2.75}$$

In order to evaluate the action of covariant derivatives we introduce the covariant Taylor expansion. For a function $\phi(x)$ transforming in a certain representation of the gauge group we can write down its covariant Taylor series as

$$\phi(x) = I(x,x')\sum_{n=0}^\infty \frac{1}{n!}(x-x')^{a_1}\ldots(x-x')^{a_n}\boldsymbol{\nabla}_{a_1}\ldots\boldsymbol{\nabla}_{a_n}\, \phi(y)\big|_{y=x'}\,. \tag{2.76}$$

In particular this can be applied to $\boldsymbol{\nabla}_b I(x,x')$ considered as a function of $x$:

$$\boldsymbol{\nabla}_b I(x,x') = I(x,x')\sum_{n=0}^\infty \frac{1}{n!}(x-x')^{a_1}\ldots(x-x')^{a_n}\boldsymbol{\nabla}_{a_1}\ldots\boldsymbol{\nabla}_{a_n}\boldsymbol{\nabla}_b\, I(y,x')\big|_{y=x'}\,. \tag{2.77}$$

Using eq. (2.71) and $[\boldsymbol{\nabla}_a, \boldsymbol{\nabla}_b] = -i\boldsymbol{F}_{ab}$, with the background field strength $F_{ab}$, it can be shown that [64]

$$\boldsymbol{\nabla}_b I(x,x') = iI(x,x') \sum_{n=1}^{\infty} \frac{n}{(n+1)!} (x-x')^{a_1} \dots (x-x')^{a_n} \boldsymbol{\nabla}'_{a_1} \dots \boldsymbol{\nabla}'_{a_{n-1}} \boldsymbol{F}_{a_n b}(x') \,,$$
$$\boldsymbol{\nabla}_b I(x,x') = -i \sum_{n=1}^{\infty} \frac{(-1)^n}{(n+1)!} (x-x')^{a_1} \dots (x-x')^{a_n} (\boldsymbol{\nabla}_{a_1} \dots \boldsymbol{\nabla}_{a_{n-1}} \boldsymbol{F}_{a_n b}(x)) I(x,x') \,. \tag{2.78}$$

Note that this means that we can express covariant derivatives acting on $I(x,x')$ in terms of undifferentiated factors of $I(x,x')$ and derivatives of $\boldsymbol{F}_{ab}$. Manifest covariance is thus preserved. Since the mass dimension of $\boldsymbol{F}$ is $[\boldsymbol{F}]_\Delta = 2$ this also improves powercounting. We will also use

$$\begin{aligned}
\int \frac{\mathrm{d}^d k}{(2\pi)^d}\, & k_{b_1} \dots k_{b_m} (x-x')^{a_1} \dots (x-x')^{a_n} e^{-isk^2} e^{ik(x-x')} \\
&= (-i)^n \int \frac{\mathrm{d}^d k}{(2\pi)^d}\, k_{b_1} \dots k_{b_m} e^{-isk^2} \frac{\partial^n}{\partial k_{a_1} \dots \partial k_{a_n}} \left( e^{ik(x-x')} \right) \\
&= i^n \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{\partial^n}{\partial k_{a_1} \dots \partial k_{a_n}} \left( k_{b_1} \dots k_{b_m} e^{-isk^2} \right) e^{ik(x-x')} \\
&= i^n \int \frac{\mathrm{d}^d k}{(2\pi)^d}\, K^{a_1 \dots a_n}{}_{;b_1 \dots b_m}(s,k)\; e^{ik(x-x')} \,,
\end{aligned} \tag{2.79}$$

where we introduced the notation

$$K^{a_1 \dots a_n}_{b_1 \dots b_m}(s,k) = \frac{\partial^n}{\partial k_{a_1} \dots \partial k_{a_n}} \left( k_{b_1} \dots k_{b_m} e^{-isk^2} \right) . \tag{2.80}$$

This way one can evaluate eq. (2.75) up to an arbitrary order in the derivative expansion and the coordinates of the vertices appear only through explicit background field dependence and factors of

$$e^{ik(x-x')} I(x,x') \,. \tag{2.81}$$

We have now freed all of the propagators from explicit derivatives acting on them. Let us now explore how to treat the explicit background fields at the vertices. More details can be found in [46, Sec. 3]. The goal is to shift all of the background field dependence to a single vertex by using the covariant Taylor expansion. Consider some background field $\mathcal{V}(x)$ at the vertex at position $x$, that is connected to some other vertex $x'$ by a propagator. Using eq. (2.76) we can write this as

$$\mathcal{V}(x) = I(x,x') \sum_{n=0}^{\infty} \frac{1}{n!} (x-x')^{a_1} \dots (x-x')^{a_n} \boldsymbol{\nabla}_{a_1} \dots \boldsymbol{\nabla}_{a_n} \left. \mathcal{V}(y) \right|_{y=x'} \,. \tag{2.82}$$

We have thus shifted the background field dependence from one vertex to another at the cost of introducing an auxiliary factor of $I(x, x')$. This procedure can be repeated to successively shift the background fields. We end up with a diagram consisting of a background field dependence at a single vertex $x_0$, a collection of propagators and some auxiliary parallel displacement propagators coming from shifting the external background fields. The structure of the parallel displacement propagators is now given by

$$\mathrm{tr}\,\left(I(x_0, x)\chi(x, x_0)\right)\,, \tag{2.83}$$

where $x$ is some vertex that is connected to $x_0$ by a propagator and $\chi(x, x_0)$ is a product of parallel displacement propagators and also contains the background fields at $x_0$. The dependence on the other coordinates is suppressed. Using the covariant Taylor expansion this is

$$I(x_0, x)\chi(x, x_0) = \sum_{n=0}^{\infty} \frac{1}{n!}(x - x_0)^{a_1} \ldots (x - x_0)^{a_n} \boldsymbol{\nabla}_{a_1} \ldots \boldsymbol{\nabla}_{a_n}\, \chi(y, x_0)\big|_{y=x_0}\,. \tag{2.84}$$

We thus effectively delete the factor of $I(x_0, x)$ and introduce new background fields at $x_0$. The coordinate dependence can again be traded for derivatives with respect to the corresponding edge momentum. This procedure can also be repeated until there are only two vertices left. As argued in [46] it follows then from gauge invariance that the product of all parallel displacement propagators reduces to the identity. The complete dependence on the coordinates is then given by factors of

$$e^{ik(x_i - x_j)}\,, \tag{2.85}$$

which after switching to momentum space reduces to momentum conserving $\delta$-functions. The edge momenta only come as products of $K^{a_1 \ldots a_n}_{b_1 \ldots b_m}(s, k)$ and thus the momentum integrals are Gaussian and can be simply evaluated.

Let us summarize the main features of this formalism. Using the covariant Taylor expansion one shifts the entire external background field dependence to one vertex. This produces more derivatives on the external fields and propagators with higher powers. Then one successively deletes the vertices until only two are left. This again produces more background fields at the designated vertex. At the end one can perform the momentum integrals, because they reduce to Gaussian ones. Due to the trivial momentum structure only logarithmically divergent graphs will contribute.

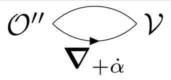
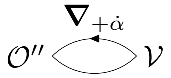
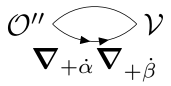
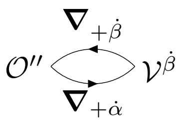
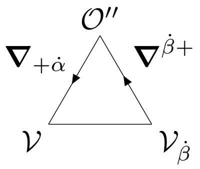
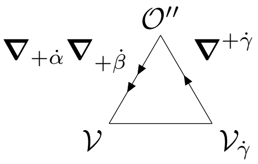
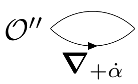
| Diagram | Mass Dimensions | Superficial UV divergence |
|---|---|---|
|   | $[\mathcal{V}]_\Delta = [\mathcal{O}']_\Delta - 1$ | $\frac{1}{2}\mathcal{I}_1\mathcal{O}''(x)(\boldsymbol{\nabla}_{+\dot\alpha}\mathcal{V}(x))$ |
|   | $[\mathcal{V}]_\Delta = [\mathcal{O}']_\Delta - 1$ | $-\frac{1}{2}\mathcal{I}_1\mathcal{O}''(x)(\boldsymbol{\nabla}_{+\dot\alpha}\mathcal{V}(x))$ |
|   | $[\mathcal{V}]_\Delta = [\mathcal{O}']_\Delta - 2$ | $\mathcal{I}_1\mathcal{O}''(x)\Big(\frac{1}{6}\left\{\boldsymbol{\nabla}_{+\dot\alpha},\boldsymbol{\nabla}_{+\dot\beta}\right\} + \frac{i}{2}\left\{\boldsymbol{\nabla}_{+},\boldsymbol{W}_{+}\right\}C_{\dot\alpha\dot\beta}\Big)\mathcal{V}(x)$ |
|   | $[\mathcal{V}]_\Delta = [\mathcal{O}']_\Delta - 2$ | $\mathcal{I}_1\mathcal{O}''(x)\Big(-\frac{1}{12}\left\{\boldsymbol{\nabla}_{+\dot\alpha},\boldsymbol{\nabla}_{+\dot\beta}\right\}\mathcal{V}^{\dot\beta}(x) + \frac{i}{4}\left\{\boldsymbol{\nabla}_{+},\boldsymbol{W}_{+}\right\}\mathcal{V}_{\dot\alpha}(x) - \frac{i}{4}\mathcal{V}_{\dot\alpha}(x)\left\{\boldsymbol{\nabla}_{+},\boldsymbol{W}_{+}\right\}\Big)$ |
|   | $[\mathcal{V}]_\Delta + [\mathcal{V}_{\dot\beta}]_\Delta = [\mathcal{O}']_\Delta$ | $-\frac{1}{2}\mathcal{I}_1\mathcal{O}''(x)\mathcal{V}(x)\mathcal{V}_{\dot\alpha}(x)$ |
|   | $[\mathcal{V}]_\Delta + [\mathcal{V}_{\dot\gamma}]_\Delta = [\mathcal{O}']_\Delta - 1$ | $-\mathcal{I}_1\mathcal{O}''(x)\Big(\frac{1}{3}(\boldsymbol{\nabla}_{+\dot\alpha}\mathcal{V})\mathcal{V}_{\dot\beta} + \frac{1}{6}\mathcal{V}(\boldsymbol{\nabla}_{+\dot\alpha}\mathcal{V}_{\dot\beta}) + (\dot\alpha\leftrightarrow\dot\beta)\Big)$ |

**Table 2.1:** Examples of diagrams with covariant derivatives and their UV divergences. $\mathcal{O}'$ is the part of the operator that takes part in the loop calculation, while $\mathcal{O}''$ is the rest of the operator, which is external to the diagram and $\mathcal{I}_1 = 1/\epsilon$ is the superficial UV divergence of the ordinary scalar two-point function.

## 2.5 Examples of diagrams with covariant derivatives

In this section we will discuss in detail some easy examples in order to clarify the procedure outlined in section 2.4. We use the schematic notation $\mathcal{O}'$ for the part of the operator that takes part in the loop calculation, while $\mathcal{O}''$ is the rest of the operator, which is external to the diagram. We discuss two examples in detail and collect more results in table 2.1. The simplest example to consider is an integral of the form

**I**  

$$= \mathcal{O}''(x)(\boldsymbol{\nabla}_{+\dot\alpha}D_\Box(x,x'))\mathcal{V}(x')D_\Box(x',x)\,, \tag{2.86}$$

where the explicit background field dependence after performing the D-algebra is given by $\mathcal{O}''(x)$ and $\mathcal{V}(x')$ and the mass dimensions fulfill the relation $[\mathcal{V}]_\Delta = [\mathcal{O}']_\Delta - 1$. As always we concentrate on terms with superficial degree of divergence $\omega \geq 0$ (see eq. (3.1)), which means that the background fields that are produced during the expansion can have at most dimension 1. Let us first rewrite eq. (2.86) by means of eqs. (2.74) and (2.75) as

$$
\begin{aligned}
-\int_0^\infty \mathrm{d}s \int_0^\infty \mathrm{d}s' \int \frac{\mathrm{d}^d k}{(2\pi)^d} \int \frac{\mathrm{d}^d k'}{(2\pi)^d}\, e^{-isk^2} e^{ik(x-x')} \mathcal{O}''(x) \left(e^{is\Box - sk^a \nabla_a} (\boldsymbol{\nabla} + ik)_{+\dot\alpha} I(x,x')\right) \\
\mathcal{V}(x') e^{-is'k'^2} e^{ik'(x'-x)} e^{is'\Box' - s'k'^a \nabla'_a} I(x',x)\,.
\end{aligned}
$$

Due to eq. (2.78) the term involving $\boldsymbol{\nabla}_{+\dot\alpha} I(x,x')$ will have $\omega < 0$. The same goes for the exponentiated kinetic operator acting on the parallel displacement propagators. We are left with

$$
\begin{aligned}
-i \int_0^\infty \mathrm{d}s \int_0^\infty \mathrm{d}s' \int \frac{\mathrm{d}^d k}{(2\pi)^d} \int \frac{\mathrm{d}^d k'}{(2\pi)^d}\, e^{-isk^2 - is'k'^2} e^{i(k-k')(x-x')} \\
\mathcal{O}''(x) k_{+\dot\alpha} I(x,x') \mathcal{V}(x') I(x',x)\,. \qquad (2.87)
\end{aligned}
$$

Using the covariant Taylor expansion the background field dependence can now be shifted to the vertex at $x$. The expansion again has to terminate after the first order

$$
\begin{aligned}
-i \int_0^\infty \mathrm{d}s \int_0^\infty \mathrm{d}s' \int \frac{\mathrm{d}^d k}{(2\pi)^d} \int \frac{\mathrm{d}^d k'}{(2\pi)^d}\, e^{-isk^2 - is'k'^2} e^{i(k-k')(x-x')} \mathcal{O}''(x) k_{+\dot\alpha} I(x,x') \\
I(x',x) \left(1 + (x'-x)^a \boldsymbol{\nabla}_{a,y}\right) \mathcal{V}(y)\big|_{y=x} I(x',x)\,. \qquad (2.88)
\end{aligned}
$$

Since there are no derivatives acting on the parallel displacement propagators anymore their product will reduce to the identity. The coordinate dependence can also be rephrased by eq. (2.79), which leads us to

$$
\begin{aligned}
-\, i \int_0^\infty \mathrm{d}s \int_0^\infty \mathrm{d}s' \int \frac{\mathrm{d}^d k}{(2\pi)^d} \int \frac{\mathrm{d}^d k'}{(2\pi)^d}\, k_{+\dot\alpha} e^{-isk^2} e^{i(k-k')(x-x')} \mathcal{O}''(x) \\
\left(e^{-is'k'^2} + i \frac{\partial}{\partial k_a} \left(e^{-is'k'^2}\right) \boldsymbol{\nabla}_{a,y}\right) \mathcal{V}(y)\big|_{y=x}\,.
\end{aligned}
$$

We can now switch to the momentum representation by integrating over the positions of the vertices, which will lead to momentum conserving $\delta$-functions.

$$-i\int \mathrm{d}^dx \int \mathrm{d}^dx' \int_0^\infty \mathrm{d}s \int_0^\infty \mathrm{d}s' \int \frac{\mathrm{d}^dk}{(2\pi)^d} \int \frac{\mathrm{d}^dk'}{(2\pi)^d}\, k_{+\dot\alpha} e^{-isk^2} e^{i(k-k')(x-x')}\mathcal{O}''(x)$$
$$\left(e^{-is'k'^2} + i\frac{\partial}{\partial k'_a}\left(e^{-is'k'^2}\right)\boldsymbol{\nabla}_{a,y}\right)\mathcal{V}(y)|_{y=x}$$
$$= -i\int \mathrm{d}^dx \int_0^\infty \mathrm{d}s \int_0^\infty \mathrm{d}s' \int \frac{\mathrm{d}^dk}{(2\pi)^d}\, k_{+\dot\alpha} e^{-isk^2}\mathcal{O}''(x)$$
$$\left(e^{-is'k^2} + i\frac{\partial}{\partial k_a}\left(e^{-is'k^2}\right)\boldsymbol{\nabla}_{a,y}\right)\mathcal{V}(y)|_{y=x}\,. \quad (2.89)$$

Since the integral over $k$ is Gaussian, only the even terms can contribute. This reduces the integral to

$$= -i\int \mathrm{d}^dx \int_0^\infty \mathrm{d}s \int_0^\infty \mathrm{d}s'\; s' \int \frac{\mathrm{d}^dk}{(2\pi)^d}\, k_{+\dot\alpha}k^a e^{-i(s+s')k^2}\mathcal{O}''(x)(\boldsymbol{\nabla}_a\mathcal{V}(x))\,. \quad (2.90)$$

Performing the integrals over $s$ and $s'$ we see that this reduces to the ordinary momentum space representation

$$= \int \mathrm{d}^dx \int \frac{\mathrm{d}^dk}{(2\pi)^d}\, \frac{k_{+\dot\alpha}k^a}{(k^2)^3}\mathcal{O}''(x)(\boldsymbol{\nabla}_a\mathcal{V}(x)) = \frac{1}{2}\int \mathrm{d}^dx \int \frac{\mathrm{d}^dk}{(2\pi)^d}\, \frac{1}{(k^2)^2}\mathcal{O}''(x)(\boldsymbol{\nabla}_{+\dot\alpha}\mathcal{V}(x))\,. \quad (2.91)$$

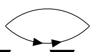
In the last step we used that under symmetric integration

$$k_{+\dot\alpha}k^a \to \frac{2}{d}k^2\delta^a_{+\dot\alpha}\,. \quad (2.92)$$

Eq. (2.91) is the standard one loop two point integral evaluated at external momentum $p = 0$. This introduces a new IR divergence, which has to be regulated. One way to do this is to introduce a non-vanishing external momentum. This will not change the UV divergence, because the superficial UV divergence of any logarithmically divergent graph is independent of external momenta. The superficial UV divergence of this graph is then given by

$$\frac{1}{2}\mathcal{I}_1\mathcal{O}''(\boldsymbol{\nabla}_{+\dot\alpha}\mathcal{V})\,. \quad (2.93)$$

As a second example let us consider

**III** $\mathcal{O}''$  $\mathcal{V}$ $= \mathcal{O}''(x)(\boldsymbol{\nabla}_{+\dot\alpha}\boldsymbol{\nabla}_{+\dot\beta}D_\Box(x,x'))\mathcal{V}(x')D_\Box(x',x)\,,$ (2.94)

$\boldsymbol{\nabla}_{+\dot\alpha}\boldsymbol{\nabla}_{+\dot\beta}$

where now $[\mathcal{V}]_\Delta = [\mathcal{O}']_\Delta - 2$. The new feature that arises here is that fields of mass dimension 2 can be produced, which means that there can now be field strength terms from the action of derivatives on $I(x,x')$. From eq. (2.78) one can show that

$$(\boldsymbol{\nabla}+ik)_{+\dot\alpha}(\boldsymbol{\nabla}+ik)_{+\dot\beta}I(x,x') \\ = \left(-k_{+\dot\alpha}k_{+\dot\beta}+\frac{i}{2}\boldsymbol{F}_{+\dot\alpha,+\dot\beta}-\frac{1}{2}\left(k_{+\dot\alpha}(x-x')^a\boldsymbol{F}_{a,+\dot\beta}+(\dot\alpha\leftrightarrow\dot\beta)\right)+\dots\right)I(x,x')\,, \tag{2.95}$$

where the dots stand for higher dimensional background fields. Let us concentrate on the last term. Since the background field strength $\boldsymbol{F}$ has dimension 2 no more background fields will be produced. This implies that no more explicit factors of $(x-x')$ or $k$ are produced either. The integral will necessarily take the form

$$\int\frac{\mathrm{d}^dk}{(2\pi)^d}\,\frac{\partial}{\partial k_a}\left(k_{+\dot\alpha}e^{-isk^2}\right)\propto\delta^a_{+\dot\alpha}\,. \tag{2.96}$$

This means, that we get contributions with external structure $\boldsymbol{F}_{+\dot\alpha,+\dot\beta}+(\dot\alpha\leftrightarrow\dot\beta)=0$, which vanish due to antisymmetry of the field strength. The first term in eq. (2.95) can give an non-vanishing contribution only, if the momenta are properly contracted. The only term that can achieve this is the second order expansion of the background field

$$\mathcal{V}(x')\propto\frac{1}{2}I(x',x)(x'-x)^{a_1}(x'-x)^{a_2}\boldsymbol{\nabla}_{a_1,y}\boldsymbol{\nabla}_{a_2,y}\,\mathcal{V}(y)|_{y=x}\,. \tag{2.97}$$

The calculation is analogous to the one of the previous example, only instead of eq. (2.92) one needs

$$k_{+\dot\alpha}k_{+\dot\beta}k^{a_1}k^{a_2}\to\frac{4}{d(d+2)}(k^2)^2\left(\delta^{a_1}_{+\dot\alpha}\delta^{a_2}_{+\dot\beta}+\delta^{a_2}_{+\dot\alpha}\delta^{a_1}_{+\dot\beta}\right)\,. \tag{2.98}$$

The result is

$$\frac{1}{6}\int\mathrm{d}^dx\int\frac{\mathrm{d}^dk}{(2\pi)^d}\,\frac{1}{(k^2)^2}\mathcal{O}''(x)\left(\left\{\boldsymbol{\nabla}_{+\dot\alpha},\boldsymbol{\nabla}_{+\dot\beta}\right\}\mathcal{V}(x)\right)\,. \tag{2.99}$$

The second term in eq. (2.95) is even easier to evaluate, because there are no further background fields produced. It immediately reduces to

$$\frac{i}{2}\int\mathrm{d}^dx\int\frac{\mathrm{d}^dk}{(2\pi)^d}\,\frac{1}{(k^2)^2}\mathcal{O}''(x)\left(\boldsymbol{F}_{+\dot\alpha,+\dot\beta}\mathcal{V}(x)\right)\,. \tag{2.100}$$

Putting the terms together and using eq. (A.19) the divergent part is

$$\mathcal{I}_1\mathcal{O}''\left(\frac{1}{6}\left\{\boldsymbol{\nabla}_{+\dot\alpha},\boldsymbol{\nabla}_{+\dot\beta}\right\}+\frac{i}{2}\left\{\boldsymbol{\nabla}_{+},\boldsymbol{W}_{+}\right\}C_{\dot\alpha\dot\beta}\right)\mathcal{V}. \tag{2.101}$$

# Chapter 3

# Renormalization

The dilatation operator depends on the first order pole in $\epsilon$ of the counterterm $Z$. This in turn is calculated by extracting the superficial UV divergence from a given Feynman diagram. That's the divergence that is left after subtracting all subdivergences. In this chapter we will introduce the necessary machinery. All results will be collected in appendix B.

## 3.1 The R-operation

The main tool of renormalization goes under the name *R-operation* [68–70]. This is a method to systematically render a Feynman integral finite by successively subtracting UV subdivergences. In order to understand the method let us first introduce the necessary vocabulary.

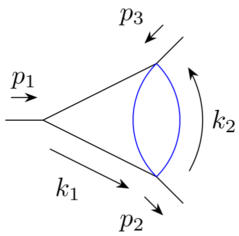
Given a graph $G$ its *superficial degree of (UV) divergence* is defined by

$$\omega(G) = d\,L + N - D\,, \tag{3.1}$$

where $d$ is the space-time dimension, $L$ is the number of loops and $N$ and $D$ are the powers of loop momenta in the numerator and denominator, respectively. A Feynman diagram $G$ is called *superficially divergent*, if $\omega(G) \geq 0$, otherwise it is called *superficially convergent*. Note that this formula counts the power of loop momenta. If all loop momenta were rescaled by a factor $x$ the leading power in $x$ in the limit $x \to \infty$ of the Feynman integral would be $x^{\omega(G)}$. As an example the standard one loop scalar two point function has $\omega = d - 4$, as can be seen from

$$\int \frac{\mathrm{d}^d k}{(2\pi)^d}\, \frac{1}{k^2(k-p)^2} \overset{k \to xk}{\to} x^d \int \frac{\mathrm{d}^d k}{(2\pi)^d}\, \frac{1}{(xk)^2(xk-p)^2} \overset{x \to \infty}{\to} x^{d-4} \int \frac{\mathrm{d}^d k}{(2\pi)^d}\, \frac{1}{k^4} \tag{3.2}$$

and thus it has a UV divergence for $d \geq 4$. Next consider the two loop graph




$$= \int \frac{\mathrm{d}^d k_1}{(2\pi)^d} \int \frac{\mathrm{d}^d k_2}{(2\pi)^d} \frac{1}{k_1^2 k_2^2 (k_2 - k_1 + p_2)^2 (k_1 + p_3 - p_1)^2} \,. \tag{3.3}$$

Simultaneous rescaling of $k_1$ and $k_2$ leads to $\omega(G) = 2d - 8 = 0$ in four dimensions but only rescaling $k_2$ also leads to $\omega(\gamma) = d - 4 = 0$. This signals a divergence of the subgraph $\gamma \subset G$, which is indicated in blue. A one-particle irreducible (1PI) subgraph $\gamma$ with $\omega(\gamma) \geq 0$ is called a *UV subgraph*. Subgraphs that can diverge independently of each other are called *UV disjoint*. A set of UV disjoint subgraphs is sometimes called a *spinney* and finally the set of all spinneys is called a wood and indicated by $W(G)$. Note that $W(G)$ also contains $G$ itself. The reduced wood is given by $\bar{W}(G) = W(G) \backslash \{G\}$.

We are now in a position to define the R-operation. It is given by the so-called *forest formula*

$$RG = \sum_{S \in W(G)} \Delta(S) * G/S \,, \tag{3.4}$$

where $\Delta(S)$ is the counterterm operation to be defined below and $G/S$ is the reduced diagram, constructed from $G$ by contracting to points all subgraphs that are contained in $S$. The $*$-operation is simply given by a product for logarithmic divergences. For higher divergences it acts as an insertion operator. It inserts the momenta coming from the counterterm into the reduced diagram. We also introduce the $\bar{R}$-operation, which subtracts the subdivergences but does not subtract the superficial divergence

$$\bar{R}G = \sum_{S \in \bar{W}(G)} \Delta(S) * G/S \,. \tag{3.5}$$

For a spinney $S = \{\gamma_1, \ldots, \gamma_k\}$ consisting of multiple UV divergent subgraphs the counterterm factorizes

$$\Delta(S) = \prod_{i=1}^{k} \Delta(\gamma_i) \,. \tag{3.6}$$

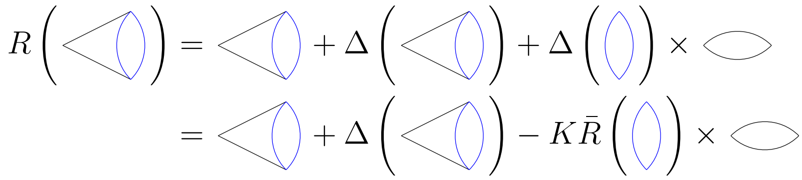
These graphs only have to be weakly disconnected, i.e. they can share a common vertex. On these individual subgraphs there is ambiguity in the counterterm operation, which reflects the choice of different renormalization schemes. We use the *minimal* *subtraction* scheme ($MS$-scheme) [71], where the counterterm operation is given by

$$\Delta(G) = -K\bar{R}G\,, \tag{3.7}$$

and the pole operator $K$, as the name suggests, extracts the poles of a function in $\epsilon$. The superficial divergence of a graph is then given by

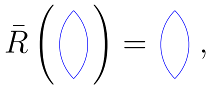
$$K\bar{R}G\,. \tag{3.8}$$

Let us see eq. (3.4) in action for the simple two loop graph from above.



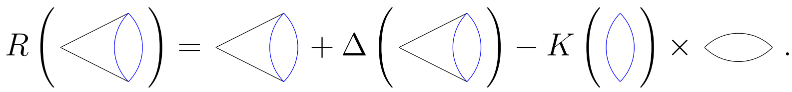
(3.9)

where



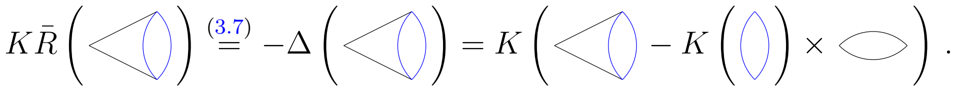
(3.10)

because the one loop graph has no subdivergences that could be subtracted. We end up with



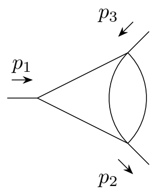
(3.11)

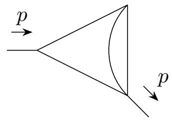
The quantity that we are most interested in, is the superficial divergence of this graph. Since the right hand side is renormalized, it is completely finite. This precisely defines the counterterm of the two loop graph itself. This implies that

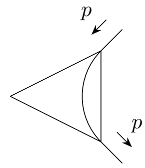


(3.12)

In order to facilitate the computation of the counterterms it is important to note they will always be polynomials in the external momenta of order $\omega(G)$. In particular most of our graphs have $\omega(G) = 0$ in $d = 4$, which implies that their overall divergence is *independent of the external momenta*. This can be used for a technique called *infrared rearrangement*: One can rearrange the external momentum structure in such a way that the evaluation of the diagrams becomes simple. The only thing to be careful about is that the momentum rearrangement should be done in an infrared safe way, which means that it does not introduce new IR divergences. Look again at our favorite example.

$$K\bar{R}\left(\right.$$ 

$$\left.\right) = K\bar{R}\left(\right.$$ 

$$\left.\right) \neq K\bar{R}\left(\right.$$ 

$$\left.\right) . \tag{3.13}$$

We find that the counterterms for the first two graphs are identical. Only the third counterterm is different because that particular momentum rearrangement introduces an infrared divergence due to the presence of a $1/k^4$ propagator. This can be remedied by a proper generalization of the R-operation, called the R*-operation [72, 73] (see also [74] for a modern approach), which also subtracts IR divergences.

In practice we can always reduce the graphs to propagator type diagrams, i.e. diagrams that only have one external momentum scale $p^2$ and then set $p^2 = 1$. Results for the unrenormalized graphs are readily found in the literature, see e.g. [75] for some of the graphs dicussed here. They can often be expressed through the functions

$$G(\alpha, \beta) = \frac{\Gamma(\frac{d}{2} - \alpha)\Gamma(\frac{d}{2} - \beta)\Gamma(\alpha + \beta - \frac{d}{2})}{(4\pi)^{\frac{d}{2}}\Gamma(\alpha)\Gamma(\beta)\Gamma(D - \alpha - \beta)} , \tag{3.14}$$

$$G_1(\alpha, \beta) = \frac{1}{2}(-G(\alpha, \beta - 1) + G(\alpha - 1, \beta) + G(\alpha, \beta)) , \tag{3.15}$$

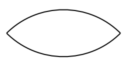
$$G_2(\alpha, \beta) = \frac{1}{2}(-G(\alpha, \beta - 1) - G(\alpha - 1, \beta) + G(\alpha, \beta)) . \tag{3.16}$$

By $I$ we denote the unrenormalized integrals themselves, evaluated at $p^2 = 1$ and

$$\mathcal{I} = K\bar{R}I . \tag{3.17}$$

Let us give some examples to illustrate how these computations are done. A complete list of results can be found in appendix B.

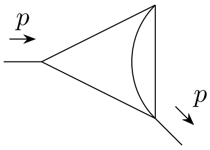
The result for the one loop self-energy graph in this notation is

$$I_1 =$$ 
$$= \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{1}{k^2(k-p)^2}\bigg|_{p^2=1} = G(1,1) \tag{3.18}$$

and the divergence can easily be computed by performing a series expansion in $\epsilon$

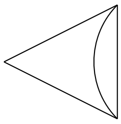
$$\mathcal{I}_1 = K(I_1) = \frac{1}{(4\pi)^2}\left(\frac{1}{\epsilon}\right) . \tag{3.19}$$

Let us now turn back to the two loop diagram. We are free to choose the particularly simple momentum assignment

$I_2 =$  $\Big|_{p^2=1} = \int \frac{\mathrm{d}^d k_1}{(2\pi)^d} \int \frac{\mathrm{d}^d k_2}{(2\pi)^d} \frac{1}{k_1^2 k_2^2 (k_2 - k_1 + p)^2 (k_1 - p)^2}\Big|_{p^2=1}$

$$= G(1,1)G(3-\tfrac{d}{2},1)\,, \tag{3.20}$$

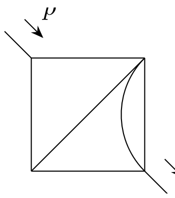
where the lsat line was taken from [75]. To reduce cluttering we will usually not indicate the external momenta explicitly. We will thus just write

$I_2 =$  $= G(1,1)G(3-\tfrac{d}{2},1)\,, \quad (3.21)$

Using this result and eq. (3.12) we can find

$$\mathcal{I}_2 = K\left(I_2 - \mathcal{I}_1 I_1\right) = \frac{1}{(4\pi)^4}\left(-\frac{1}{2\epsilon^2} + \frac{1}{2\epsilon}\right)\,. \tag{3.22}$$

At higher loops one generally has more subdivergences to deal with. Let us take a look at the graph

$I_3 =$  $\Big|_{p^2=1}\,. \quad (3.23)$

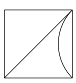
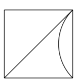
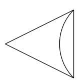
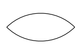
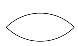
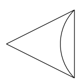
We have again made use of infrared rearrangement to get the simplest possible momentum structure. The explicit integral is then given by

$$I_3 = \int \frac{\mathrm{d}^d k_1}{(2\pi)^d} \int \frac{\mathrm{d}^d k_2}{(2\pi)^d} \int \frac{\mathrm{d}^d k_3}{(2\pi)^d} \frac{1}{k_1^2 k_2^2 k_3^2 (k_1 - p)^2 (k_2 - k_1)^2 (p + k_3 - k_2)^2}\Big|_{p^2=1}\,. \tag{3.24}$$

Again from [75] we find

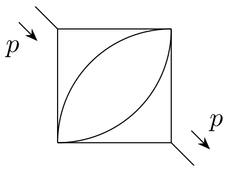
$$I_3 = G(1,1)G(3-\tfrac{d}{2},1)G(5-d,1)\,. \tag{3.25}$$

The forest formula can now be applied to yield

$\mathcal{I}_3 = K\bar{R}\Big($  $\Big) = K\Big($  $+ \Delta\Big($  $\Big)$  $+ \Delta\Big($  $\Big)$  $\Big)$

$$= K\left(I_3 - \mathcal{I}_2 I_1 - \mathcal{I}_1 I_2\right) = \frac{1}{(4\pi)^6}\left(\frac{1}{6\epsilon^3} - \frac{1}{2\epsilon^2} + \frac{2}{3\epsilon}\right) . \tag{3.26}$$

As a final scalar example consider the graph

$$I_{3b} =$$


$$\Bigg|_{p^2=1} . \tag{3.27}$$

With this simple momentum assignement the explicit integral expression is

$$I_{3b} = \int \frac{\mathrm{d}^d k_1}{(2\pi)^d} \int \frac{\mathrm{d}^d k_2}{(2\pi)^d} \int \frac{\mathrm{d}^d k_3}{(2\pi)^d} \left.\frac{1}{k_1^2 k_2^2 k_3^2 (k_1-p)^2 (k_2-p)^2 (k_3+k_2-k_1)^2}\right|_{p^2=1} \tag{3.28}$$

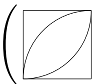
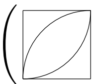
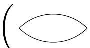
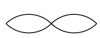
We computed this integral in Mincer [76, 77], using its O4 topology, hence we do not have an analytic expression. We find

$$I_{3b} = N^3\left(\frac{1}{3\epsilon^3} + \frac{7}{3\epsilon^2} + \frac{31}{2\epsilon} + \text{finite}\right) . \tag{3.29}$$

Note the conversion factor

$$N = (4\pi)^{-2} \exp\left(-\epsilon\gamma + \epsilon\ln(4\pi) - \epsilon^2\frac{\zeta(2)}{2}\right) \tag{3.30}$$

from Mincer's $\overline{MS}$-scheme to our $MS$ conventions. The result of applying the rofest formula is then

$$\mathcal{I}_{3b} = K\bar{R}\Bigg($$

$$\Bigg) = K\Bigg($$

$$+ \Delta\Big($$

$$\Big)$$

$$\Bigg)$$

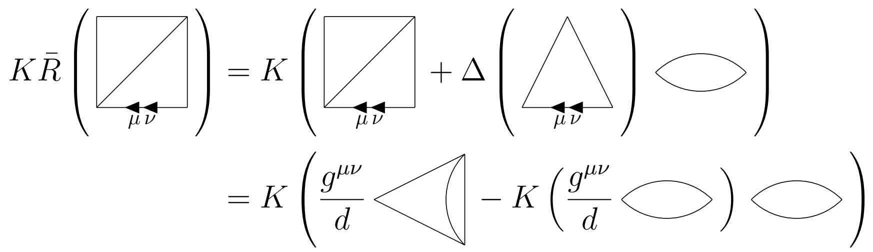
$$= K\left(I_{3b} - \mathcal{I}_1 I_{11}\right) = \frac{1}{(4\pi)^6}\left(\frac{1}{3\epsilon^3} - \frac{2}{3\epsilon^2} + \frac{1}{3\epsilon}\right) . \tag{3.31}$$

Excitations introduce tensor structures, which require a new class of counterterms. A useful modern reference for the renormalization of tensor diagrams is [74]. We should mention that the literature on these kinds of counterterms is very scarce.

One might naively expect, that one can extract the counterterm of a tensor graph from its tensor reduced version. This is wrong, because the $K$ operation does not commute with tensor contraction. The simplest case of tensor contraction is done by multiplying with

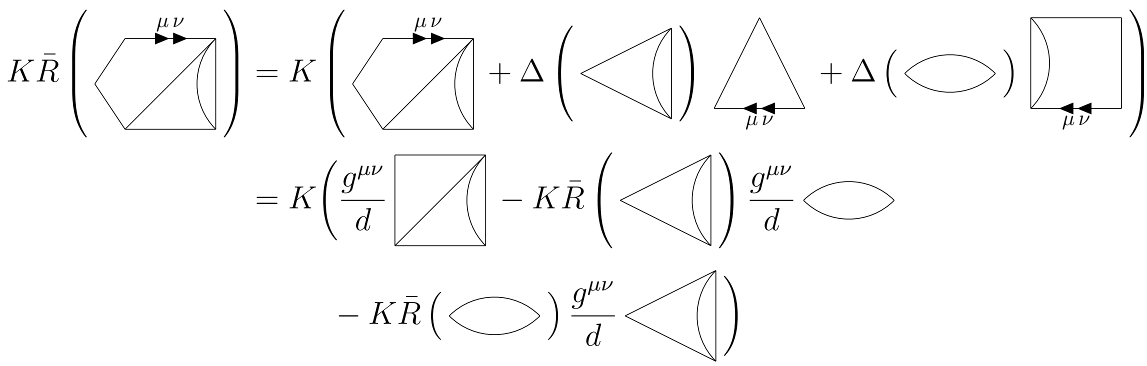
$$P^{\mu\nu} = \frac{g^{\mu\nu}}{d} , \tag{3.32}$$

and the presence of the factor $d$ is the reason, why this operation in general does not commute with $K$. The correct procedure is to write down the nested structure of the forest formula explicitly and then perform the tensor reduction from the innermost layer. Let us again go through a few examples. As a two loop example consider



$$= K\left(I_{3b} - \mathcal{I}_1 I_{11}\right) = \frac{1}{(4\pi)^4}\left(\frac{1}{3\epsilon^3} - \frac{2}{3\epsilon^2} + \frac{1}{3\epsilon}\right) g^{\mu\nu}\,. \tag{3.33}$$

An example at three loops is



$$= K\left(\frac{g^{\mu\nu}}{d}\left(I_3 - \mathcal{I}_2 I_1 - \mathcal{I}_1 I_2\right)\right) = \frac{1}{(4\pi)^4}\left(\frac{1}{24\epsilon^3} - \frac{5}{48\epsilon^2} + \frac{11}{96\epsilon}\right) g^{\mu\nu}\,. \tag{3.34}$$

A few comments are in order. First note that for the computation of the superficial divergence we do not need the explicit analytic expressions for the unrenormalized tensor diagrams. Instead the tensor reduced versions are enough. Also note that all of these graphs are logarithmically divergent and their counterterms are independent of the momenta. The only available tensor structure for them is thus $g^{\mu\nu}$. We list all our results in appendix B.

## 3.2 Contributing diagrams

We are only interested in the first order pole in $\epsilon$ of the superficial UV divergence of a given diagram, since this is the relevant part for the anomalous dimensions of the local operators as explained below eq. (1.6).

An important result is the *Weinberg-Dyson convergence theorem* [78], which was

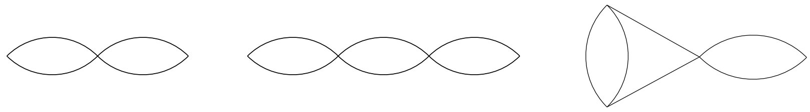


**Figure 3.1:** Some examples of one-vertex-reducible graphs.

first proven in [79]. An elementary proof based on the $R$-operation can be found in [80]. It states that a Feynman diagram $G$ is absolutely (UV) convergent, if

1. $\omega(G) < 0$
2. $\forall \gamma \subset G : \omega(\gamma) < 0$.

A corollary of this theorem is that after replacing all subdiagrams with their renormalized values, a superficially convergent Feynman diagram $G$ is absolutely convergent. This will give us one of the main tools to decide, whether a particular graph will contribute: Since we are only interested in the superficial UV divergence, we only have to look at diagrams with $\omega \geq 0$.

Consider a graph that can be separated into two graphs by deleting a single vertex. Such graphs are called *one-vertex-reducible* (1VR), while the converse is called *one-vertex-irreducible* (1VI). Some examples of 1VR graphs are given in figure 3.1. Due to eq. (3.6) the counterterm factorizes to the counterterms of the subgraphs. It should be mentioned that in general this doesn't hold for tensor graphs due to contraction anomalies. In [74] it is shown however that it still holds for the overall UV divergence, which is relevant to us. A consequence is that the lowest order pole of the overall UV divergence of a graph with $C$ 1VI components starts is $\epsilon^{-C}$.

Since we are only interested in first order poles in $\epsilon$ we can discard all 1VR diagrams. As we will see, these types of diagrams are produced during the D-algebra by canceling propagators.

Discarding these diagrams on the spot saves a lot of effort in the computation of the first order pole, since often they occur quite early during the D-algebra. Too late did it become clear to the author, that this strategy deprives one of the very valuable cross check of the cancellation of higher order poles. A future analysis should include a reinsertion of these contributions and carefully check the cancellation.

# Chapter 4

# The $SU(2,1|1)$ sector

## 4.1 Field content and closedness

This sector comprises the local single trace operators that are composed out of elementary fields from the vector multiplet with derivatives acting on them. Also the index of the first copy of $SU(2)$ in the Lorentz group is chosen to be $\alpha = +$, thereby picking the highest weight states. This choice avoids mixing with chiral matter fields.

The motivation for chosing this sector comes from comparing general $\mathcal{N} = 1$ SCFTs to $\mathcal{N} = 4$ SYM. The one sector that is expected to behave similarly and can be compared to $\mathcal{N} = 4$ is the one that is connected by supersymmetry to the gauge field. The same reasoning has been employed in [18] for the case of $\mathcal{N} = 2$ theories.

In the background covariant superspace language the gauge covariant building blocks of this sector are the superfield strength $W_+$ and the derivatives $\boldsymbol{\nabla}_+$ and $\boldsymbol{\nabla}_{+\dot\alpha}$

$$\boldsymbol{\nabla}_+ , \quad \boldsymbol{\nabla}_{+\dot\alpha} , \quad W_+ \, . \tag{4.1}$$

In the conventions of [81] (see table 1 there, see also [82] for details on the representation theory) the scaling dimensions $\Delta$, the $U(1)_r$ charges and the $SU(2)_\alpha$ charge $j$ of the superfields are

$$\begin{gathered}
[\boldsymbol{W}_+]_{(\Delta,r,j)} = \left(\tfrac{3}{2}, 1, \tfrac{1}{2}\right), \quad [\bar{\boldsymbol{W}}_{\dot\alpha}]_{(\Delta,r,j)} = \left(\tfrac{3}{2}, -1, 0\right), \\
[\boldsymbol{\nabla}_+]_{(\Delta,r,j)} = \left(\tfrac{1}{2}, -1, \tfrac{1}{2}\right), \quad [\bar{\boldsymbol{\nabla}}_{\dot\alpha}]_{(\Delta,r,j)} = \left(\tfrac{1}{2}, 1, 0\right), \\
[\boldsymbol{\nabla}_{+\dot\alpha}]_{(\Delta,r,j)} = \left(1, 0, \tfrac{1}{2}\right), \\
[Q]_{(\Delta,r,j)} = \left(1, 1 - \tfrac{N_C}{N_F}, 0\right), \quad [\bar{Q}]_{(\Delta,r,j)} = \left(1, \tfrac{N_C}{N_F} - 1, 0\right) \; ,
\end{gathered} \tag{4.2}$$

where $Q$ are the chiral matter superfields. It is easy to check that the relation

$$\Delta = \tfrac{r}{2} + 2j \quad \text{at} \quad g = 0 \tag{4.3}$$

is fulfilled by all fields including the covariant derivatives, *except for* $Q$ and $\bar{Q}$ and $\bar{\boldsymbol{W}}_{\dot{\alpha}}$. In contrast these violate the relation by at least 2/3 (we are in the conformal window $\frac{3}{2} \leq \frac{N_F}{N_C} \leq 3$):

$$\Delta \geq \tfrac{r}{2} + 2j + \tfrac{2}{3} \quad \text{at} \quad g = 0. \tag{4.4}$$

It is worth emphasizing that these relations are not BPS-conditions. They are modified when $g \neq 0$, but at small coupling they will not be modified enough to account for a change of 2/3 in $\Delta$. Thus they cannot mix. Since the 't Hooft coupling expansion is believed to converge [83], this statement is also true for any finite value of the coupling constant in the planar limit.

At first glance it might be worrying that $\bar{\boldsymbol{\nabla}}_{\dot{\alpha}}$ fulfills relation (4.3) and thus can mix with the sector, however since there are no anti-chiral fields it can act on directly, it will always appear in one of the combinations

$$\left\{\boldsymbol{\nabla}_+, \bar{\boldsymbol{\nabla}}_{\dot{\alpha}}\right\} = i\boldsymbol{\nabla}_{+\dot{\alpha}} \text{ or } \left[\bar{\boldsymbol{\nabla}}_{\dot{\alpha}}, \boldsymbol{\nabla}_{+\dot{\beta}}\right] = C_{\dot{\alpha}\dot{\beta}}\boldsymbol{W}_+ \tag{4.5}$$

and thus reduces to fields and derivatives that are indeed present in the sector.

## 4.2 Simplifications in our sector

The fact that this sector is closed under renormalization to all loops has a couple of important consequences for the computation. Without doing any explicit calculation we can restrict all of the external fields to be in our sector. In particular the expansion of the kinetic operators from eq. (2.54) simplifies to

$$\begin{aligned} \hat{\Box} &= \Box + i\boldsymbol{W}_+\boldsymbol{\nabla}^+ \,, \\ \Box_+ &= \Box + i\boldsymbol{W}_+\boldsymbol{\nabla}^+ = \hat{\Box}\,, \\ \Box_- &= \Box\,. \end{aligned} \tag{4.6}$$

From here one easily see that $\hat{\Box}\boldsymbol{\nabla}^2 = \Box_-\boldsymbol{\nabla}^2$. As mentioned at the end of section 2.3 the operator $\Box_-^{-1}$ will always be sandwiched like $\bar{\boldsymbol{\nabla}}^2\Box_-^{-1}\boldsymbol{\nabla}^2$ in the beginning of the calculation. The computation in eq. (A.58) shows that

$$\bar{\boldsymbol{\nabla}}^2\Box_-^{-1}\boldsymbol{\nabla}^2 = \bar{\boldsymbol{\nabla}}^2\hat{\Box}^{-1}\boldsymbol{\nabla}^2\,, \tag{4.7}$$

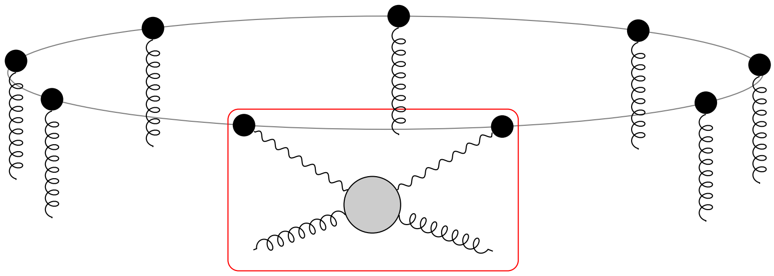


**Figure 4.1:** Example of a closed spin chain with nine sites. Gluon lines indicate the tree level without interactions between different sites. The interaction in the red rectangle will be the focus of our perturbative calculations.

where the equality holds up to terms that leave the sector. There is thus effectively only one kinetic operator $\hat{\Box}$ and its commutation relations from eq. (A.35) simplify to

$$\begin{aligned} \left[\boldsymbol{\nabla}_\alpha, \hat{\Box}\right] &= i\delta_\alpha^+ (\boldsymbol{\nabla}_+ \boldsymbol{W}_+)\boldsymbol{\nabla}^+ \,, \\ \left[\boldsymbol{\nabla}_{\alpha\dot{\alpha}}, \hat{\Box}\right] &= i\delta_\alpha^+ \left((\boldsymbol{\nabla}_{+\dot{\alpha}} \boldsymbol{W}_+)\boldsymbol{\nabla}^+ + (\boldsymbol{\nabla}_+ \boldsymbol{W}_+)\boldsymbol{\nabla}_{\dot{\alpha}}^+\right) \,, \\ \left[\bar{\boldsymbol{\nabla}}_{\dot{\alpha}}, \hat{\Box}\right] &= \left[\boldsymbol{\nabla}^2, \hat{\Box}\right] = \left[\bar{\boldsymbol{\nabla}}^2, \hat{\Box}\right] = 0 \,. \end{aligned} \tag{4.8}$$

More details can be found in appendix A.5.

## 4.3 Spin chain picture and graphical notation

Throughout the computation we will use a spin chain picture, which maps these single trace operators to closed spin chains, where each site represents a field $W_+$, possibly with derivatives acting on it. The vacuum of this spin chain is

$$\mathrm{tr}\,(W_+ \ldots W_+) \tag{4.9}$$

and the derivatives $\boldsymbol{\nabla}_{+\dot{\alpha}}$ and $\boldsymbol{\nabla}_+$ are treated as excitations. See figure 4.1 for an example of a spin chain with nine sites. The red triangle indicates the interaction that arises due to contributions from loops. The explicit expansion of $W_+$ in terms of the background fields is

$$\begin{aligned} W_+ &= \frac{1}{2} i \left[\bar{\boldsymbol{\nabla}}^{\dot{\alpha}}, \left\{\bar{\boldsymbol{\nabla}}_{\dot{\alpha}}, e^{-V} \boldsymbol{\nabla}_+ e^V\right\}\right] = i\bar{\boldsymbol{\nabla}}^2 \left(\boldsymbol{\nabla}_+ + \sum_{k=1}^{\infty} \frac{1}{n!} [\ldots[\boldsymbol{\nabla}_+, \underbrace{V], \ldots], V]}_{k}\right) \\ &= \boldsymbol{W}_+ + i(\bar{\boldsymbol{\nabla}}^2 \boldsymbol{\nabla}_+ V) + \ldots \,. \end{aligned} \tag{4.10}$$

The first term is the tree level result and the next term will lead to perturbative corrections. Higher orders will contribute in general, however at a given loop order they are of shorter range and lead to very similar but simpler diagrams. One can

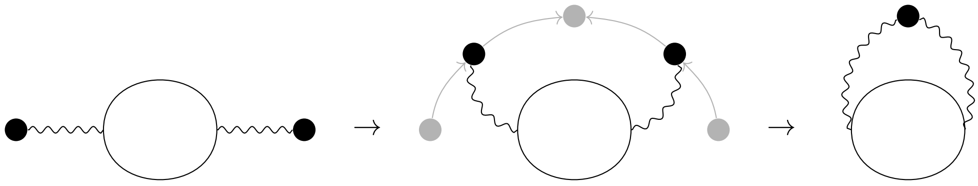


**Figure 4.2:** Ordinary Feynman diagrams are constructed by closing the operator above the diagram.

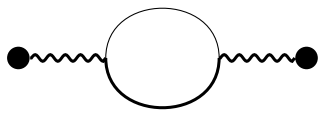


**Figure 4.3:** In the planar limit external fields can only come from the thick lines.

easily transfer the results from the longest range diagrams to the shorter ones and hence we will not consider them further.

We give a brief explanation of our rather unconventional notation. See the leftmost picture in figure 4.2 for a typical example of a diagram. The different sites of the composite operator are denoted by big black dots. In an ordinary diagram notation these points would coincide. The purpose of this notation is to declutter the diagrams because a big part of the calculation happens graphically directly on them. A drawback is that the external momentum structure is only obvious, when we go back to the ordinary notation at the end of the computation. The diagrams are to be read in such a way that the operator sits above the diagram as is shown in figure 4.2.

We will only look at theories with gauge group $SU(N)$ in the *planar limit* [84]

$$N \to \infty\,. \tag{4.11}$$

For theories with matter in some representations the usual 't Hooft limit has to be generalized to the *Veneziano limit* [85], which also takes appropriate limits of the number $N_R$ of fields in the representation $R$. The upshot is that in this limit only planar diagrams contribute. This implies that in our diagrammatic notation the external fields can only come from the lower lines. Figure 4.3 shows an example of this, where only the thick lines can contribute external fields.

# Chapter 5

# One Loop

Our argument compares general Lagrangian $\mathcal{N}=1$ SCFTs to $\mathcal{N}=4$ SYM. For this reason it is not necessary to compute all diagrams that contribute to the dilatation operator, we will instead focus on those diagrams that can potentially be different. This kind of strategy was first employed in the context of the calculation of Wilson loops in $\mathcal{N}=2$ superconformal QCD in [86]. Other notable papers that rely on this strategy are [18, 87, 88] and [89]. In the latter the term *difference method* for this argument was coined. Since any diagram that only contains gauge fields and ghosts will automatically be identical to $\mathcal{N}=4$ we do not have to consider them further. Also since our sector does not contain chiral matter fields, they cannot appear as external legs but only in closed loops. Since the one loop self-energy correction vanishes due to conformality this implies that the first differences can appear at two loops. In fact a more sophisticated argument in [90] shows that also at two loops there are no possible diagram that can lead to a difference to $\mathcal{N}=4$ SYM. However in order to get used to the formalism it is helpful to see, how it works for simple one-loop and two-loop examples. To help navigate the calculations we indicate changes in the D-algebra in blue.

## 5.1 The vacuum

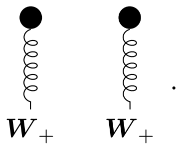
The simplest example is the vacuum. We focus on the interaction that takes place in the red rectangle in figure 4.1. At one loop these must always be interactions between two neighboring sites, which we indicate by

$$\overset{W_+}{\bullet} \qquad \overset{W_+}{\bullet} \;. \tag{5.1}$$

The first term

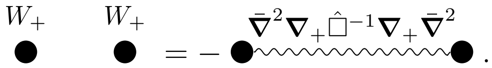
$$W_+ = \frac{1}{2} i \left[ \bar{\boldsymbol{\nabla}}^{\dot{\alpha}}, \left\{ \bar{\boldsymbol{\nabla}}_{\dot{\alpha}}, e^{-V} \boldsymbol{\nabla}_+ e^V \right\} \right] = \boldsymbol{W}_+ + i(\bar{\boldsymbol{\nabla}}^2 \boldsymbol{\nabla}_+ V) + \dots \,. \tag{5.2}$$

gives us the tree level result




(5.3)

The first loop contribution comes, when both factors of $W_+$ are expanded to first order. There is thus a factor of $i(\bar{\boldsymbol{\nabla}}^2 \boldsymbol{\nabla}_+ V)$ at each of the two sites. Wick contraction of the two factors of $V$ yields the propagator $\hat{\square}^{-1}$ from eq. (2.54), which we denote by a photon line. The derivatives act on the propagator. We end up with




(5.4)

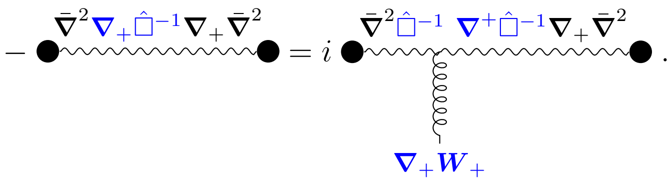
We will start by commuting the factor of $\boldsymbol{\nabla}_+$ from the left site of the operator through the diagram using the commutation relation eq. (A.61)

$$\left[ \boldsymbol{\nabla}_+, \hat{\square} \right] = i(\boldsymbol{\nabla}_+ \boldsymbol{W}_+) \boldsymbol{\nabla}^+ \,, \tag{5.5}$$

which implies

$$\left[ \boldsymbol{\nabla}_+, \hat{\square}^{-1} \right] = -\hat{\square}^{-1} \left[ \boldsymbol{\nabla}_+, \hat{\square} \right] \hat{\square}^{-1} = -i \hat{\square}^{-1} (\boldsymbol{\nabla}_+ \boldsymbol{W}_+) \boldsymbol{\nabla}^+ \hat{\square}^{-1} \,. \tag{5.6}$$

When the $\boldsymbol{\nabla}_+$ commutes through the propagator, it hits the other $\boldsymbol{\nabla}_+$ and we find $\boldsymbol{\nabla}_+ \boldsymbol{\nabla}_+ = 0$. This leads to




(5.7)

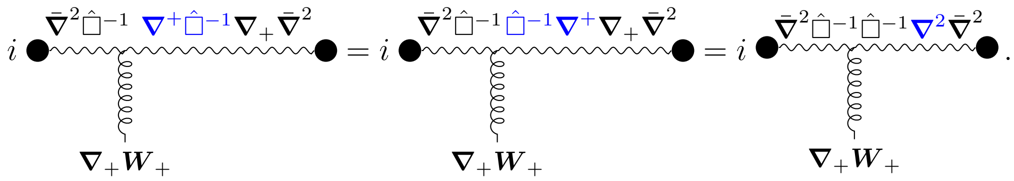
Since the commutation relation for $\boldsymbol{\nabla}^+$ is

$$\left[ \boldsymbol{\nabla}^+, \hat{\square} \right] = 0 \,, \tag{5.8}$$

we can immediately commute it to the right side of the diagram, where it combines with $\boldsymbol{\nabla}_+$ due to

$$\boldsymbol{\nabla}^2 = \frac{1}{2}\boldsymbol{\nabla}^\alpha\boldsymbol{\nabla}_\alpha = \boldsymbol{\nabla}^+\boldsymbol{\nabla}_+ \,. \tag{5.9}$$

We find

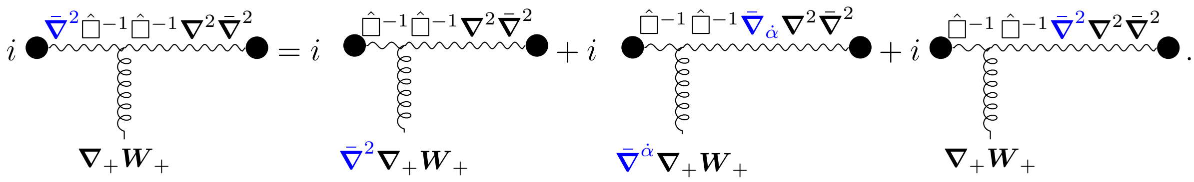


(5.10)

Following this we can use the commutation relation eq. (A.61)

$$\left[\bar{\boldsymbol{\nabla}}_{\dot\alpha}, \hat{\Box}\right] = 0 \tag{5.11}$$

to commute the $\bar{\boldsymbol{\nabla}}^2$ through the diagram. Graphically this yields



(5.12)

In the last two diagrams we use the commutation relation

$$\{\boldsymbol{\nabla}_\alpha, \bar{\boldsymbol{\nabla}}_{\dot\alpha}\} = i\boldsymbol{\nabla}_{\alpha\dot\alpha} \tag{5.13}$$

to reduce the number of $\boldsymbol{\nabla}_\alpha$. As always in supersymmetric perturbation theory one needs a factor of $\boldsymbol{\nabla}^2\bar{\boldsymbol{\nabla}}^2$ for every loop to get a non-vanishing result (see e.g. [53]). The only place where extra factors $\boldsymbol{\nabla}_\alpha$ can come from, is the expansion of the propagators $\hat{\Box}^{-1}$ in terms of the $\boldsymbol{W}_\alpha$.

$$\hat{\Box}^{-1} = \Box^{-1} - i\,\Box^{-1}\boldsymbol{W}_+\boldsymbol{\nabla}^+\Box^{-1} + \dots \,. \tag{5.14}$$

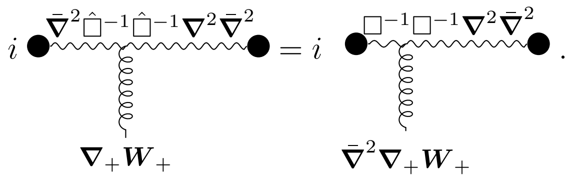
This expansion will necessarily lead to diagrams with negative superficial degree of divergence $\omega < 0$ and will thus not contribute as discussed in section 3.2. This can be seen by the fact that the external fields will have a higher dimension than the operator itself

$$[\boldsymbol{W}_+(\bar{\boldsymbol{\nabla}}^{\dot\alpha}\boldsymbol{\nabla}_+\boldsymbol{W}_+)]_\Delta > [\boldsymbol{W}_+(\boldsymbol{\nabla}_+\boldsymbol{W}_+)]_\Delta > [\boldsymbol{W}_+\boldsymbol{W}_+]_\Delta \,, \tag{5.15}$$

which does not leave enough momentum factors in the diagram to make it UV-divergent. Thus the only diagram that has enough $\boldsymbol{\nabla}^2$ is the first one and we have

$$\hat{\Box}^{-1} = \Box^{-1} \, , \tag{5.16}$$

where the equality is understood up to terms that lead to finite diagrams. No new superspace derivatives are produced after this point and we arrive at

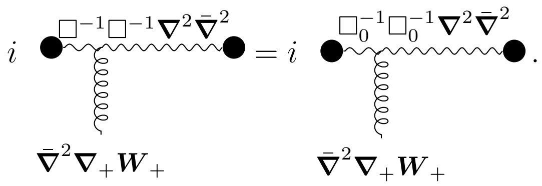


(5.17)

In principle we have to expand $\Box^{-1}$ in terms of the background connection $\boldsymbol{\Gamma}_{\alpha\dot{\alpha}}$, but by the same argument as before only the ordinary d'Alembertian gives an infinite contribution. Thus we have

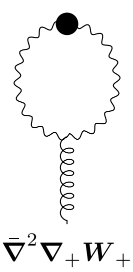
$$\Box^{-1} = \Box_0^{-1} \, , \tag{5.18}$$

and we arrive at



(5.19)

As in figure 4.2 we can now bring this into the form of a standard Feynman diagram. The result is

$i$ 

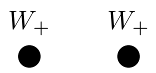
$$= i(\bar{\boldsymbol{\nabla}}^2\boldsymbol{\nabla}_+\boldsymbol{W}_+)\mathcal{I}_1 \, , \tag{5.20}$$

where the superficial divergence $\mathcal{I}_1 = 1/((4\pi)^2\epsilon)$ was introduced in chapter 3. The external field structure of this diagram is

$$\bar{\boldsymbol{\nabla}}^2\boldsymbol{\nabla}_+\boldsymbol{W}_+ \equiv \frac{1}{2}\left\{\bar{\boldsymbol{\nabla}}^{\dot{\alpha}}, \left[\bar{\boldsymbol{\nabla}}_{\dot{\alpha}}, \{\boldsymbol{\nabla}_+, \boldsymbol{W}_+\}\right]\right\} = \frac{1}{2}\left\{\bar{\boldsymbol{\nabla}}^{\dot{\alpha}}, \left[\left\{\bar{\boldsymbol{\nabla}}_{\dot{\alpha}}, \boldsymbol{\nabla}_+\right\}, \boldsymbol{W}_+\right]\right\}$$

$$= \frac{1}{2}\left\{\left[\bar{\boldsymbol{\nabla}}^{\dot\alpha},\left\{\bar{\boldsymbol{\nabla}}_{\dot\alpha},\boldsymbol{\nabla}_+\right\}\right],\boldsymbol{W}_+\right\} = -i\left\{\boldsymbol{W}_+,\boldsymbol{W}_+\right\}\ , \tag{5.21}$$

where we repeatedly used chirality of $\boldsymbol{W}_+$. Hence we find



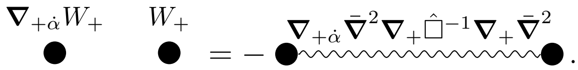

$$= \{\boldsymbol{W}_+,\boldsymbol{W}_+\}\,\mathcal{I}_1 + \text{finite}\ . \tag{5.22}$$

Note that the anticommutator is exactly the structure we expect from the dilatation operator acting on two fermions at one loop. In our formalism this comes as a natural consequence of the action of derivatives.

## 5.2 Excitations

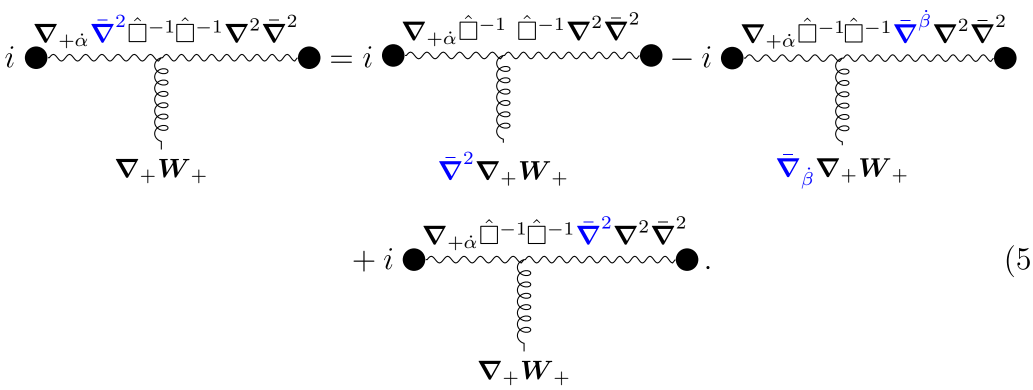
### 5.2.1 $D((\boldsymbol{\nabla}_{+\dot\alpha}W_+)W_+)$

The next step is to compute excitations above this vacuum. Consider the case where there is one derivative acting on one of the fields



(5.23)

The first steps are identical to the previous computation. There are again three terms

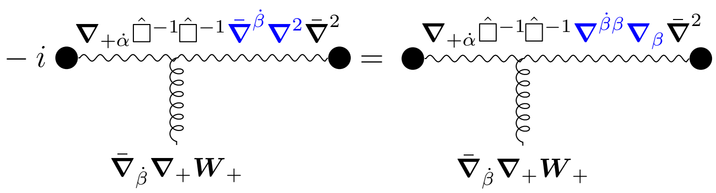


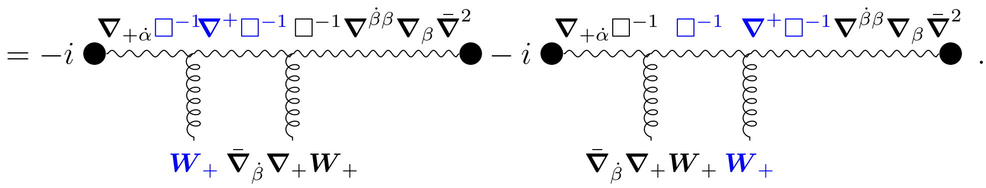
(5.24)

As before the third term still vanishes. The second term however can be expanded as in

$$\hat{\Box}^{-1} = \Box^{-1} - i\,\Box^{-1}\boldsymbol{W}_+\boldsymbol{\nabla}^+\Box^{-1} + \ldots \qquad (\textit{in our sector}) \tag{5.25}$$

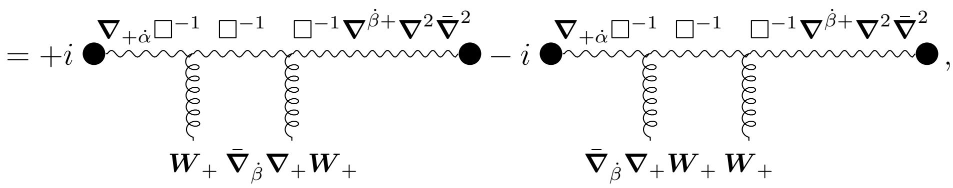
and gives the contribution



$$. \tag{5.26}$$

These terms give

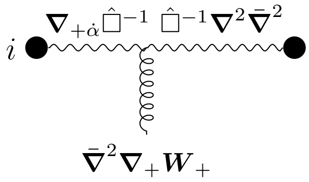


$$, \tag{5.27}$$

which according to table 2.1 results in

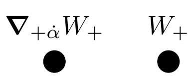
$$= -\frac{1}{2}\,\mathcal{I}_1\left[(\boldsymbol{\nabla}_{+\dot\alpha}\boldsymbol{W}_+), \boldsymbol{W}_+\right] . \tag{5.28}$$

Meanwhile the first term evaluates to



$$= \frac{1}{2}\mathcal{I}_1\left[\boldsymbol{\nabla}_{+\dot\alpha}, \{\boldsymbol{W}_+, \boldsymbol{W}_+\}\right] = \mathcal{I}_1\left\{(\boldsymbol{\nabla}_{+\dot\alpha}\boldsymbol{W}_+), \boldsymbol{W}_+\right\} . \tag{5.29}$$

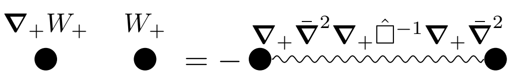
We thus find



$$\begin{aligned} &= \left(\{(\boldsymbol{\nabla}_{+\dot\alpha}\boldsymbol{W}_+), \boldsymbol{W}_+\} - \frac{1}{2}\left[(\boldsymbol{\nabla}_{+\dot\alpha}\boldsymbol{W}_+), \boldsymbol{W}_+\right]\right)\mathcal{I}_1 \\ &= \left(\frac{1}{2}(\boldsymbol{\nabla}_{+\dot\alpha}\boldsymbol{W}_+)\boldsymbol{W}_+ + \frac{3}{2}\boldsymbol{W}_+(\boldsymbol{\nabla}_{+\dot\alpha}\boldsymbol{W}_+)\right)\mathcal{I}_1 . \end{aligned} \tag{5.30}$$

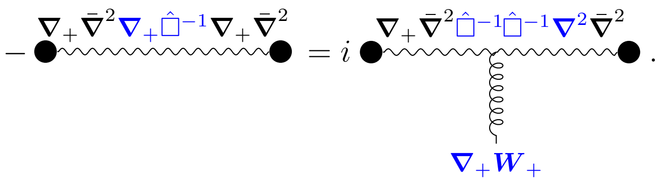
### 5.2.2 $D((\boldsymbol{\nabla}_+ W_+)W_+)$

The first steps are similar to the computation above. First use the expansion of $W_+$ to write the lowest order diagram



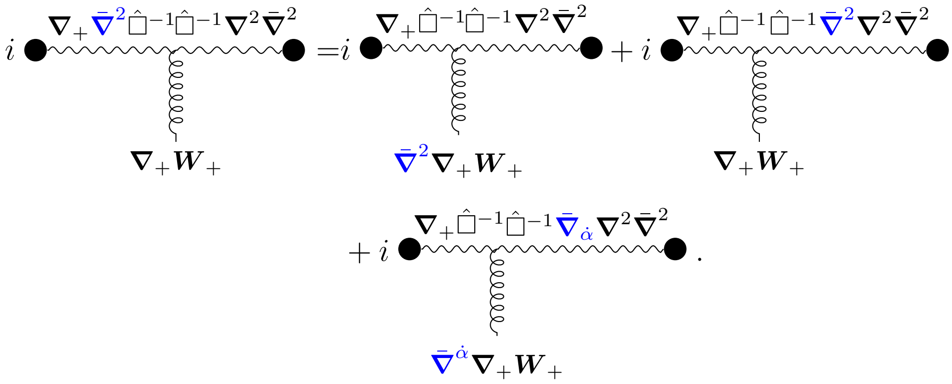
$$\tag{5.31}$$

Then we commute the undotted derivative to the right



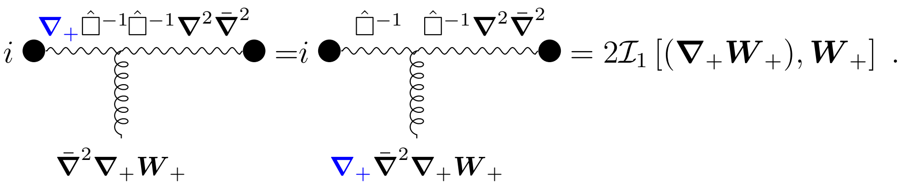

(5.32)

Commuting the dotted derivative again produces three terms



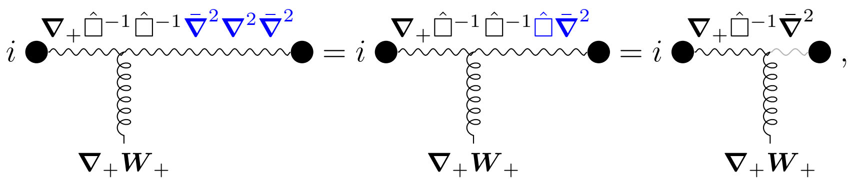

(5.33)

This is where the differences start. Contrary to the last section all of these terms will contribute. The first term gives



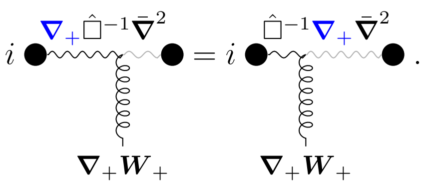

(5.34)

By means of eqs. (A.45) and (A.56) the second term becomes



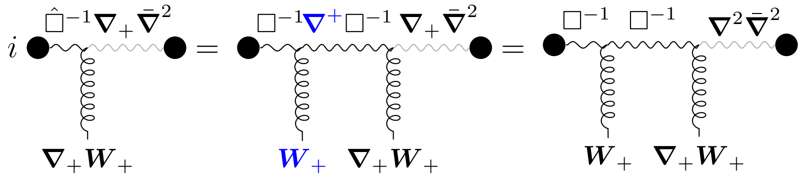

(5.35)

where the gray color indicates that the propagator belonging to that line has been canceled. After this simplification we can again commute the undotted derivative through the diagram. The commutator will only produce finite terms.



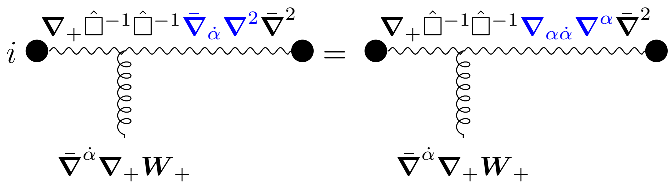

(5.36)

Finally we can expand the propagator to arrive at



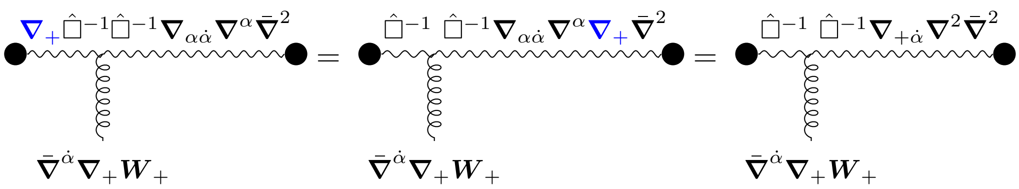

$$= \mathcal{I}_1 \boldsymbol{W}_+(\boldsymbol{\nabla}_+\boldsymbol{W}_+)\ . \tag{5.37}$$

In the third term we first use eq. (A.11) to reduce the number of spinor derivatives




$$. \tag{5.38}$$

Then we commute the derivative to the right. The only term that gives a divergent contribution is the one, where it actually combines with the other undotted derivative



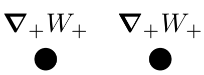

$$\begin{aligned} &= -\frac{i}{2}\mathcal{I}_1 \boldsymbol{\nabla}_{+\dot\alpha}\boldsymbol{\nabla}_+{}^{\dot\alpha}\boldsymbol{W}_+ \\ &\overset{(A.22)}{=} -\frac{1}{2}\mathcal{I}_1\left[(\boldsymbol{\nabla}_+\boldsymbol{W}_+),\boldsymbol{W}_+\right]\ . \end{aligned} \tag{5.39}$$

Since it commutes through an even number of spinor derivatives, no minus sign is produced. The final diagram was computed in section 2.5. Hence in total we find

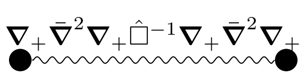

$$\begin{aligned} &= \mathcal{I}_1\left(\frac{3}{2}\left[(\boldsymbol{\nabla}_+\boldsymbol{W}_+),\boldsymbol{W}_+\right] + \boldsymbol{W}_+(\boldsymbol{\nabla}_+\boldsymbol{W}_+)\right) \\ &= \mathcal{I}_1\left(\left[(\boldsymbol{\nabla}_+\boldsymbol{W}_+),\boldsymbol{W}_+\right] + \frac{1}{2}\left\{(\boldsymbol{\nabla}_+\boldsymbol{W}_+),\boldsymbol{W}_+\right\}\right) \\ &= \mathcal{I}_1\left(\frac{3}{2}(\boldsymbol{\nabla}_+\boldsymbol{W}_+)\boldsymbol{W}_+ - \frac{1}{2}\boldsymbol{W}_+(\boldsymbol{\nabla}_+\boldsymbol{W}_+)\right)\ . \end{aligned} \tag{5.40}$$

### 5.2.3 $D((\boldsymbol{\nabla}_+W_+)(\boldsymbol{\nabla}_+W_+))$

We also have to consider the cases, when there are two excitations. We start with

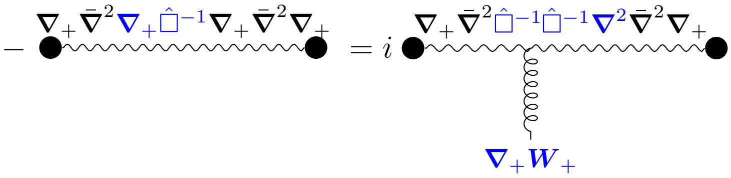



$$= -$$



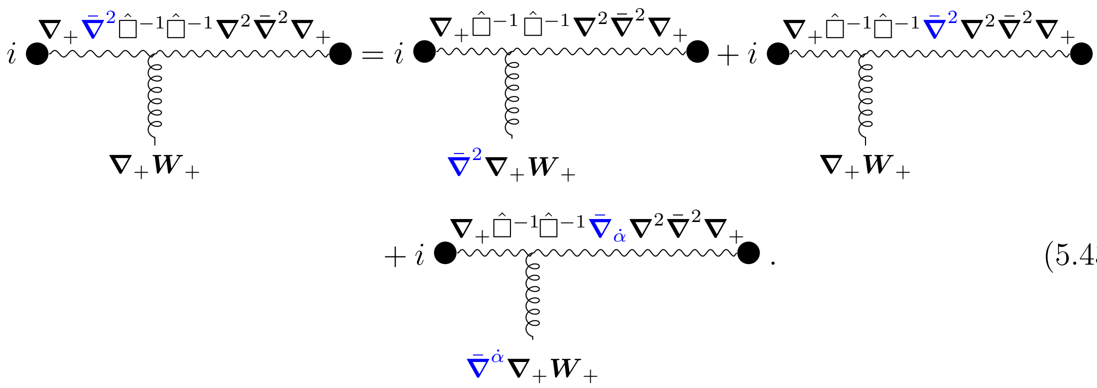

$$. \tag{5.41}$$

As before we start by commuting the undotted derivative



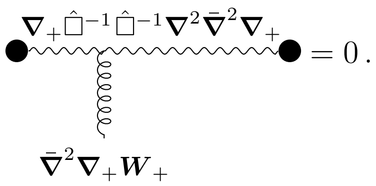

(5.42)

and then commute the dotted derivatives, thereby producing three terms



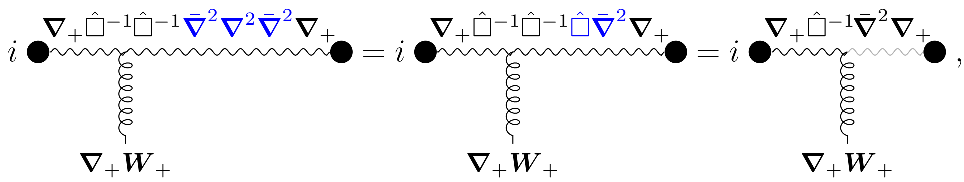

(5.43)

In the first term we can reduce the number of spinor derivatives by means of the commutation relations, hence it will not contribute.



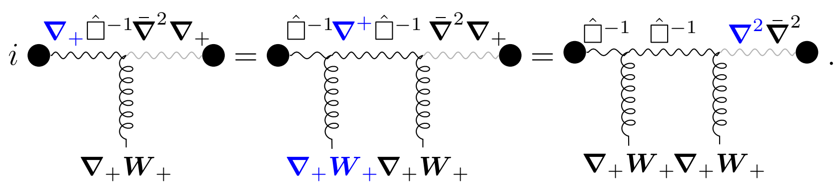

(5.44)

By means of eqs. (A.45) and (A.56) the second term becomes



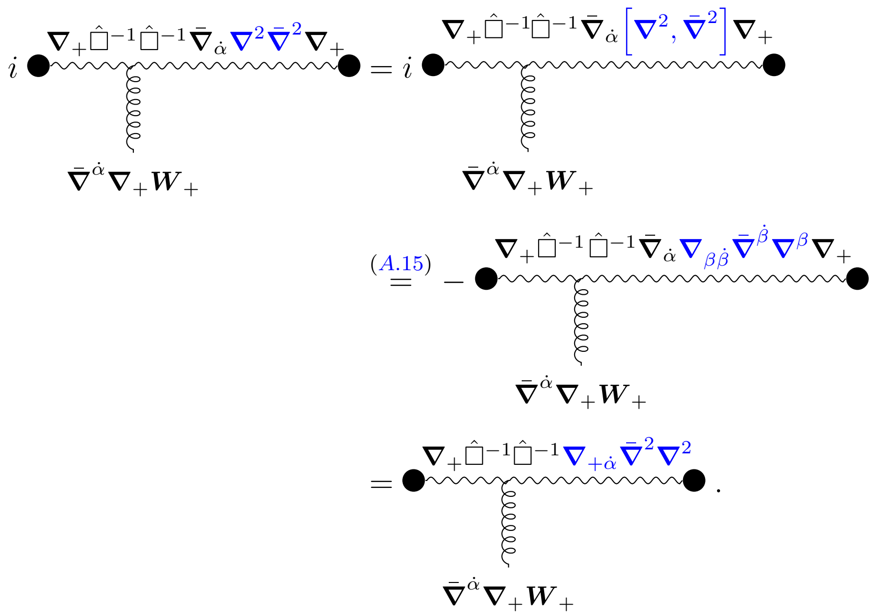

(5.45)

where the gray color indicates that the propagator belonging to that line has been canceled. We can now commute the left derivative through the diagram



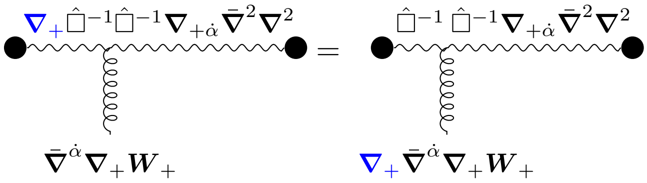

(5.46)

The third term is



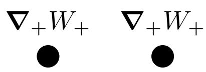

(5.47)

Finally we can commute the leftmost derivative through the diagram. The only contribution is




$$
\begin{aligned}
&= -\frac{1}{2}\mathcal{I}_1\boldsymbol{\nabla}_{+\dot{\alpha}}\boldsymbol{\nabla}_+\bar{\boldsymbol{\nabla}}^{\dot{\alpha}}\boldsymbol{\nabla}_+\boldsymbol{W}_+ = -\frac{i}{2}\mathcal{I}_1\boldsymbol{\nabla}_{+\dot{\alpha}}\boldsymbol{\nabla}_+{}^{\dot{\alpha}}\boldsymbol{\nabla}_+\boldsymbol{W}_+ \\
&= -\frac{1}{2}\mathcal{I}_1\left[(\boldsymbol{\nabla}_+\boldsymbol{W}_+), (\boldsymbol{\nabla}_+\boldsymbol{W}_+)\right] = 0\,. \qquad (5.48)
\end{aligned}
$$

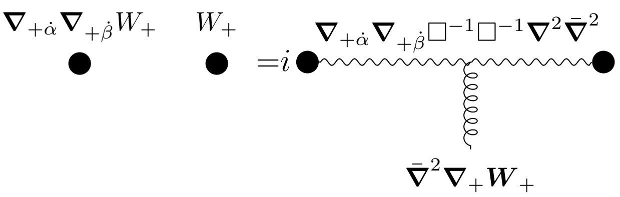
Hence we find

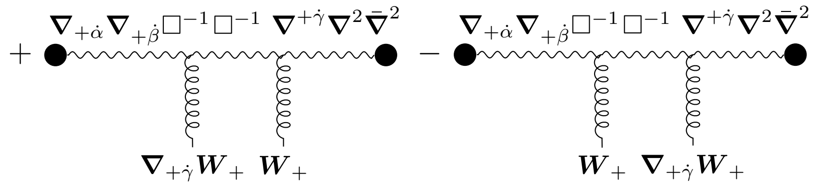



$$= (\boldsymbol{\nabla}_+\boldsymbol{W}_+)(\boldsymbol{\nabla}_+\boldsymbol{W}_+)\mathcal{I}_1\,. \qquad (5.49)$$

### 5.2.4 $D((\boldsymbol{\nabla}_{+\dot{\alpha}}\boldsymbol{\nabla}_{+\dot{\beta}}W_+)W_+)$

Analogous computations lead to



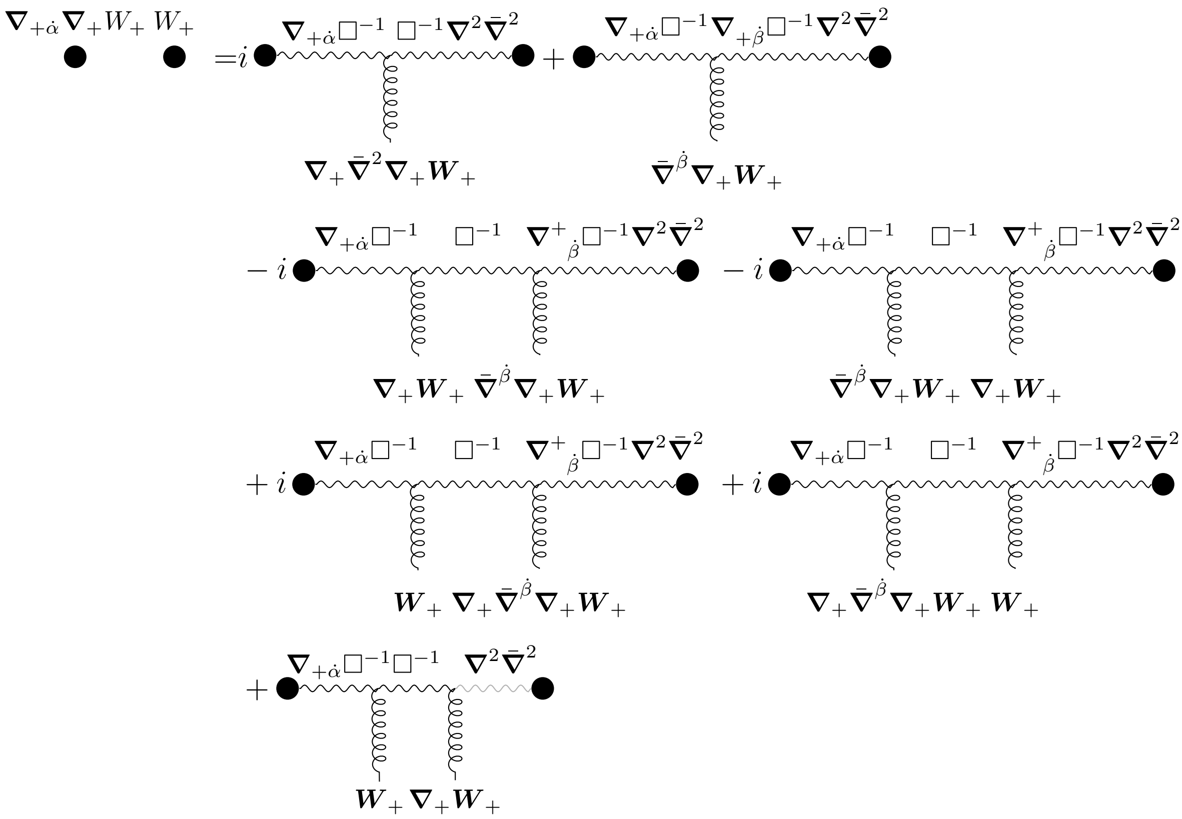



$$
\begin{aligned}
=&\mathcal{I}_1\Big(-\frac{1}{2}(\big[\nabla_{+\dot\alpha},\nabla_{+\dot\beta}\big]W_+)W_+ + \frac{1}{2}W_+(\big\{\nabla_{+\dot\alpha},\nabla_{+\dot\beta}\big\}W_+)\\
&+\frac{5}{6}(\nabla_{+\dot\beta}W_+)(\nabla_{+\dot\alpha}W_+)+\frac{5}{6}(\nabla_{+\dot\alpha}W_+)(\nabla_{+\dot\beta}W_+)\Big),
\end{aligned}
\tag{5.50}
$$

where the results are again taken from table 2.1.

### 5.2.5 $D((\nabla_{+\dot\alpha}\nabla_+W_+)W_+)$

Finally we consider the most complicated one loop diagram




$$
\begin{aligned}
=&\mathcal{I}_1\Big(\frac{1}{3}(\nabla_{+\dot\alpha}\nabla_+W_+)W_+ + (\nabla_+W_+)(\nabla_{+\dot\alpha}W_+)\\
&+\frac{5}{12}(\nabla_{+\dot\alpha}W_+)(\nabla_+W_+)-\frac{4}{3}W_+(\nabla_{+\dot\alpha}\nabla_+W_+)\Big).
\end{aligned}
\tag{5.51}
$$

## 5.3 Results

In this section we collect the results for the one loop dilatation operator. Remember that there is an extra factor of $2L = 2$ in the dilatation operator. We consistently omit a factor of $g^2/(4\pi)^2$. From the results in the previous sections we find

$$\begin{aligned}
D^{(1)}(W_+W_+) &= 4\boldsymbol{W}_+\boldsymbol{W}_+\,, && (5.52)\\
D^{(1)}((\boldsymbol{\nabla}_{+\dot\alpha}W_+)W_+) &= (\boldsymbol{\nabla}_{+\dot\alpha}\boldsymbol{W}_+)\boldsymbol{W}_+ + 3\boldsymbol{W}_+(\boldsymbol{\nabla}_{+\dot\alpha}\boldsymbol{W}_+)\,, && (5.53)\\
D^{(1)}((\boldsymbol{\nabla}_{+}W_+)W_+) &= 3(\boldsymbol{\nabla}_{+}\boldsymbol{W}_+)\boldsymbol{W}_+ - \boldsymbol{W}_+(\boldsymbol{\nabla}_{+}\boldsymbol{W}_+)\,, && (5.54)\\
D^{(1)}((\boldsymbol{\nabla}_{+}W_+)(\boldsymbol{\nabla}_{+}W_+)) &= 2(\boldsymbol{\nabla}_{+}\boldsymbol{W}_+)(\boldsymbol{\nabla}_{+}\boldsymbol{W}_+)\,, && (5.55)\\
D^{(1)}((\boldsymbol{\nabla}_{+\dot\alpha}\boldsymbol{\nabla}_{+\dot\beta}W_+)W_+) &= -(\left[\boldsymbol{\nabla}_{+\dot\alpha},\boldsymbol{\nabla}_{+\dot\beta}\right]\boldsymbol{W}_+)\boldsymbol{W}_+ + \boldsymbol{W}_+(\left\{\boldsymbol{\nabla}_{+\dot\alpha},\boldsymbol{\nabla}_{+\dot\beta}\right\}\boldsymbol{W}_+)\\
&\quad + \frac{5}{3}(\boldsymbol{\nabla}_{+\dot\alpha}\boldsymbol{W}_+)(\boldsymbol{\nabla}_{+\dot\beta}\boldsymbol{W}_+) + \frac{5}{3}(\boldsymbol{\nabla}_{+\dot\beta}\boldsymbol{W}_+)(\boldsymbol{\nabla}_{+\dot\alpha}\boldsymbol{W}_+)\,, && (5.56)\\
D^{(1)}((\boldsymbol{\nabla}_{+\dot\alpha}\boldsymbol{\nabla}_{+}W_+)W_+) &= \frac{2}{3}(\boldsymbol{\nabla}_{+\dot\alpha}\boldsymbol{\nabla}_{+}\boldsymbol{W}_+)\boldsymbol{W}_+ + 2(\boldsymbol{\nabla}_{+}\boldsymbol{W}_+)(\boldsymbol{\nabla}_{+\dot\alpha}\boldsymbol{W}_+)\\
&\quad + \frac{5}{6}(\boldsymbol{\nabla}_{+\dot\alpha}\boldsymbol{W}_+)(\boldsymbol{\nabla}_{+}\boldsymbol{W}_+) - \frac{8}{3}\boldsymbol{W}_+(\boldsymbol{\nabla}_{+\dot\alpha}\boldsymbol{\nabla}_{+}\boldsymbol{W}_+)\,. && (5.57)
\end{aligned}$$

Remember that these are the same in any theory superconformal field theory. We thus find

$$D^{(1)}_{\mathcal{N}=4} - D^{(1)}_{\mathcal{N}=1} = 0\,. \qquad (5.58)$$

At one loop order the difference is trivial, i.e. the dilatation operator in any superconformal gauge theory in the $SU(2,1|1)$ sector is identical to $\mathcal{N}=4$ SYM.

# Chapter 6

# Two Loops

## 6.1 The vacuum

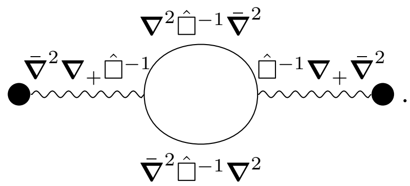
At two loops we encounter the first diagram that can *naively* lead to a difference between general $\mathcal{N} = 1$ SCFTs and $\mathcal{N} = 4$ SYM. Specifically there can be a chiral loop. The corresponding diagram is the one we already used for illustratory purposes in figures 4.2 and 4.3.

Due to the analysis in [90] also this diagram will not contribute to a difference and will find

$$\mathcal{Z}^{(2)}\Big|_{\text{diff}} = 0\,, \tag{6.1}$$

for any superconformal gauge theory. This is due to the very restricted field content and matter representations. We will still go through the computation, because many of the features of the more complicated three loop computation appear already here in a simpler way. In everything that follows we calculate the contribution coming from a single chiral field.

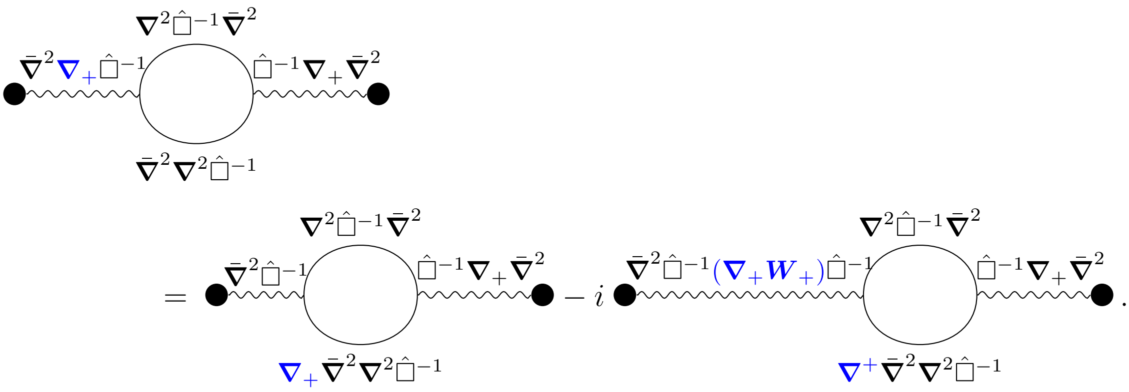
The derivative structure at the beginning of the calculation is given by[1]




. (6.2)

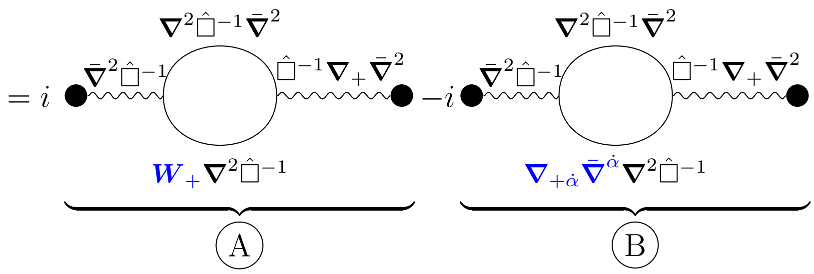
We start by commuting the factor of $\boldsymbol{\nabla}_+$ from the left site of the operator through the diagram, making us of eqs. (2.58) and (A.61) and the fact that $\boldsymbol{\nabla}^3 = 0$. We find

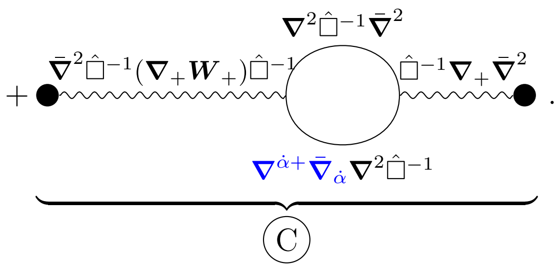
[1] Actually there are two distinct D-algebra structures on this diagram, however they are just mirrors of each other and hence we consider only one of them explicitly.




(6.3)

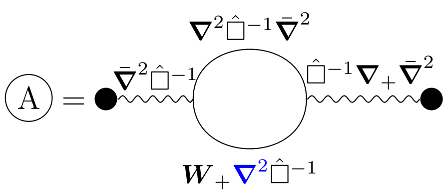
Application of eq. (A.11) yields

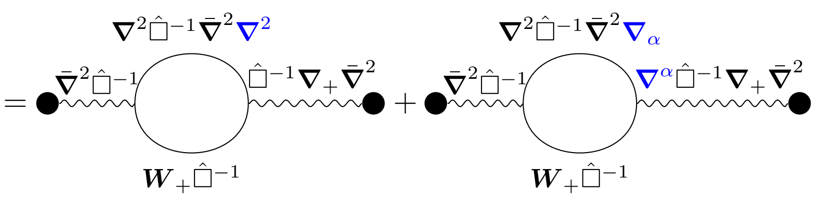


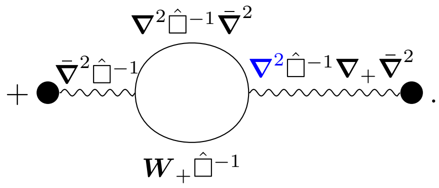




(6.4)

Note that due to eq. (A.61) factors of $\nabla^2$ can be moved freely across propagators. Application of eq. (2.58) on the right vertex allows us to move the factor of $\nabla^2$ away from the lower line. First consider



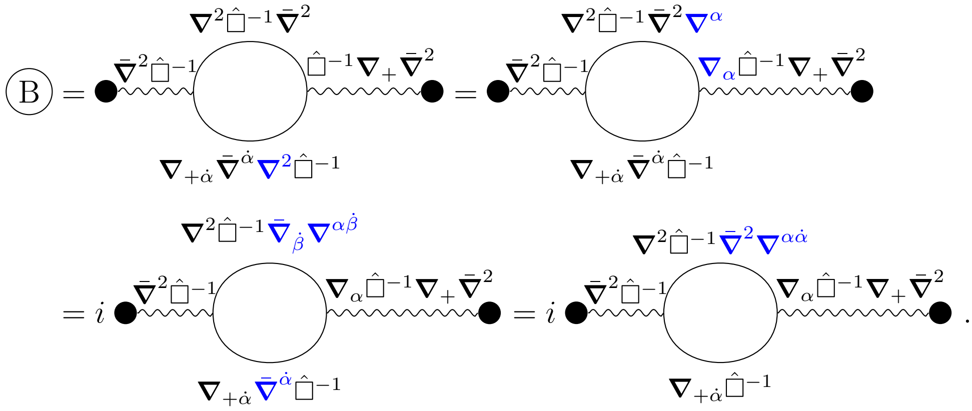



(6.5)

In the first and second term one can use the commutation relations to reduce the number of dotted derivatives. Since a factor of $\bar{\nabla}^2$ in the inner loop is needed, these terms will vanish. In the third term the factor $\nabla^2$ on the right line can again be moved across the propagator, which yields $\nabla^3 = 0$. In total we find

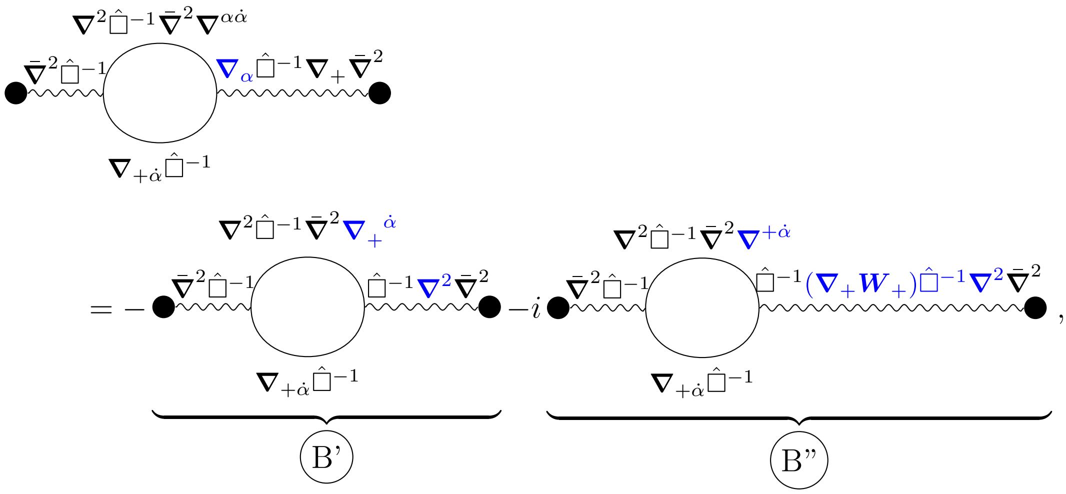
$$\text{(A)} = 0\,. \tag{6.6}$$

Following the same arguments as above we see that the only contribution in (B) that does not immediately vanish is the one, where the factor $\nabla^2$ splits onto two different lines.



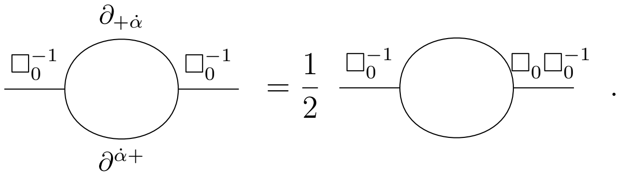

(6.7)

The factor of $\nabla_\alpha$ can now be commuted through the propagator on the right leg, again producing two contributions



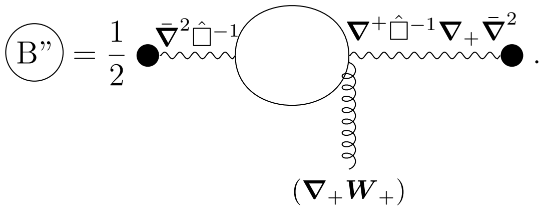

(6.8)

which we will consider in turn. The *new vertices* only come from (B'), which we will calculate at the end of this section. First focus on diagram (B"). The only non-vanishing contribution is the one where the derivatives are commuted onto the external leg. At this point we can complete the D-algebra in the inner loop. We use the relationship



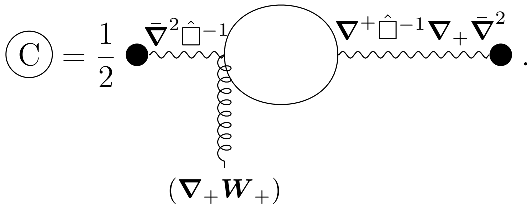

(6.9)

between ordinary Feynman diagrams and commute the derivatives in an appropriate way to make the subdiagram structure obvious. We find




(6.10)

A completely analogous computation, which we will not repeat here, shows that



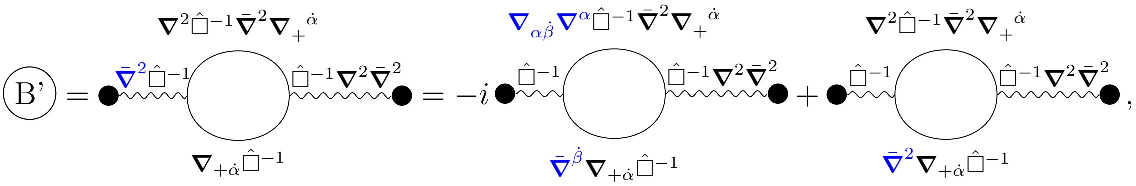

(6.11)

Both of these vertices have the structure $(\boldsymbol\nabla_+ \boldsymbol W_+)\boldsymbol\nabla^+$, which is precisely the vertex from the commutation relation of $\boldsymbol\nabla_+$ with $\hat\Box^{-1}$ that stems from the tree level action.

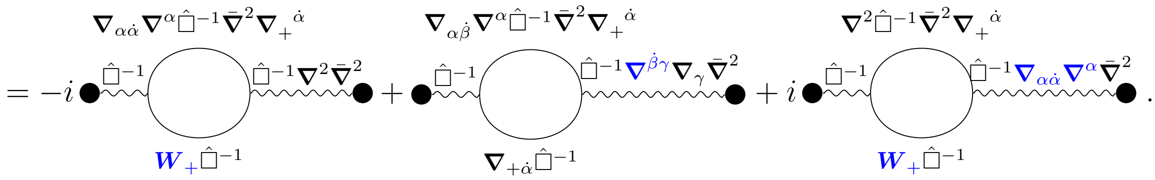
Finally consider diagram (B'). The strategy for the computation of this diagram is again to commute the leftmost factor $\bar{\boldsymbol\nabla}^2$ all the way to the right. This time however there are potentially new vertices produced by

$$\left[\bar{\boldsymbol\nabla}^{\dot\beta}, \boldsymbol\nabla_{+\dot\alpha}\right] = \delta^{\dot\beta}_{\dot\alpha} \boldsymbol W_+ \, . \tag{6.12}$$

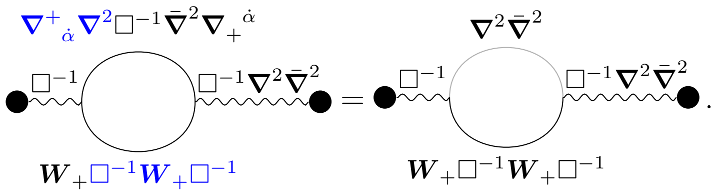
The two terms that arise are




which by means of eq. (A.9) evaluate to



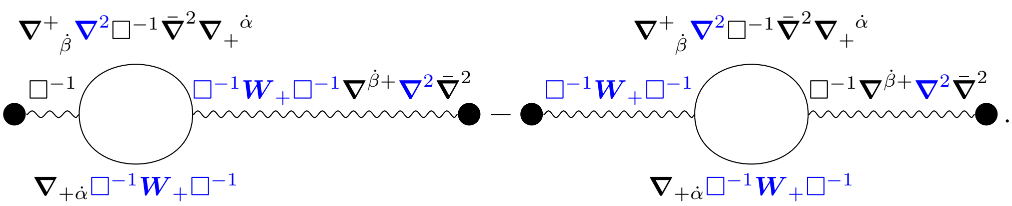

At this point we have to expand the propagators in terms of the background field strengths in order to have enough factors of $\nabla^2$ to get a non-vanishing result. In the first term it is necessarily the propagator from the chiral loop that has to be expanded, because otherwise the chiral loop does not have a factor of $\nabla^2$. We find




(6.13)

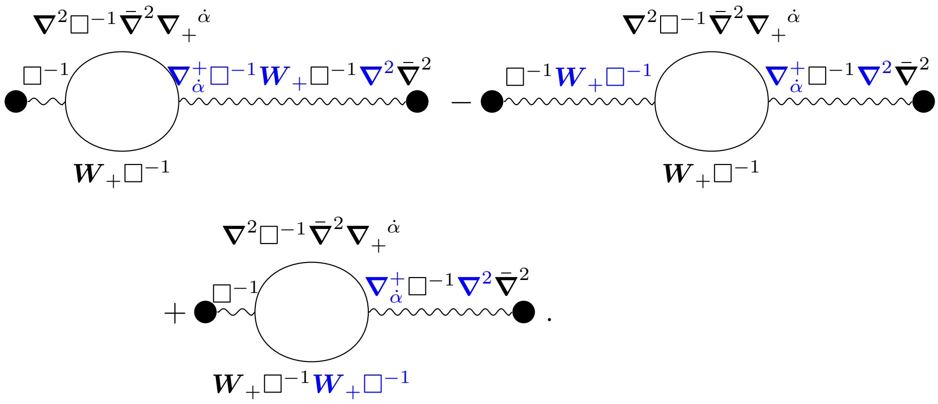
Due to $\nabla^+{}_{\dot\alpha}\nabla_+{}^{\dot\alpha} = \Box$ the upper propagator is deleted and thus this a 1VR graph. According to section 3.2 its overall UV divergence does not have a first order pole in $\epsilon$ and will not contribute.

The second term has the expansion




(6.14)

Note the relative minus sign between the two terms. This comes from the fact that in the second diagram the factor of $\nabla^+$ from the expansion of $\hat{\Box}^{-1}$ has to be commuted through the vertex containing the factor of $W_+$. It is easy to see that their difference is finite and we will not consider it further. Finally the third term gives



. (6.15)

The difference of the first two terms is again finite and the third term evaluates to

$$\frac{1}{2}\left(\mathcal{I}_2 - \mathcal{I}_2 - \mathcal{I}_{11}\right) , \tag{6.16}$$

with $\mathcal{I}_{11} = -\mathcal{I}_1^2$. Thus the overall UV divergence does not have a first order pole.

## 6.2 Results

According to the treatment of the last section we find that the only contributions to the dilatation operator that contain a chiral field at two loops come from the diagrams B” and C. These are given by

$$\text{B”} + \text{C} = \mathcal{I}_2 \left\{\boldsymbol{W}_+, \boldsymbol{W}_+\right\} . \tag{6.17}$$

For the dilatation operator we have to multiply this result by $2L = 4$ and extract the first order pole. We again omit a factor of $g^4/(4\pi)^4$, which gives us

$$D^{(2)}_{\text{chiral}}(W_+W_+) = 4\boldsymbol{W}_+\boldsymbol{W}_+ . \tag{6.18}$$

Note again, that in a superconformal gauge theory the field content is very restricted. In fact one finds that the combinatorical factor coming from the chiral loop must always be identical to the one in $\mathcal{N} = 4$ SYM. This is a consequence of the vanishing of the one loop $\beta$ function: The vector and and ghost contributions to the one loop self-energy correction of the vector field are identical, hence the contributions of the chiral fields also have to be the same. This leaves us with

$$D^{(2)}_{\mathcal{N}=4} - D^{(2)}_{\mathcal{N}=1} = 0 \tag{6.19}$$

in the $SU(2,1|1)$ sector. The first non-vanishing contributions are found at three loop order, which we will calculate in the next chapter.

# Chapter 7

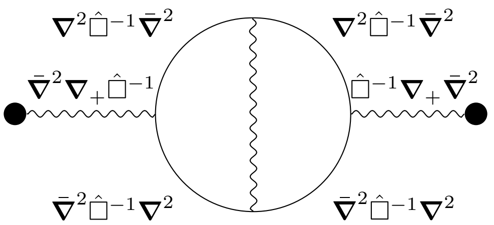
# Three Loops

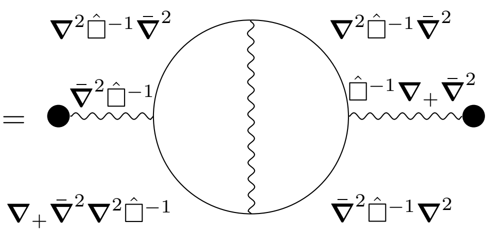
We trust that after consultation of the one and two loop computations the reader has gained some familiarity with the structure of the computations. In order to keep the three loop computation reasonably concise we will stop highlighting changes in blue and often perform multiple steps at once.

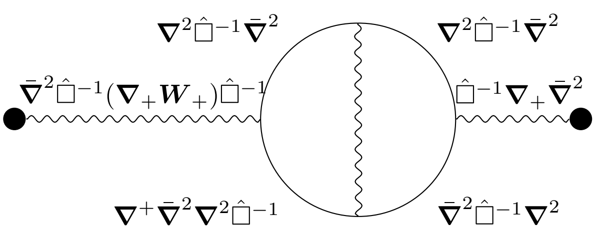
## 7.1 Topology 1

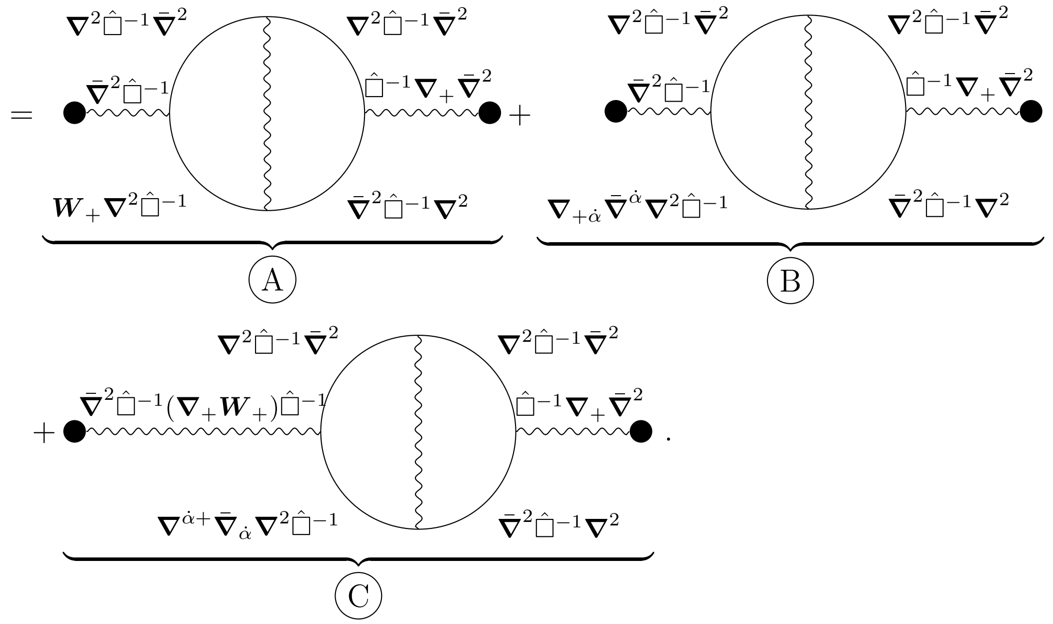
### 7.1.1 The vacuum

We begin by considering the topology given by a chiral loop with a vector superfield propagating inside the loop. First, as always, we commute the factor $\boldsymbol{\nabla}_+$ from the left



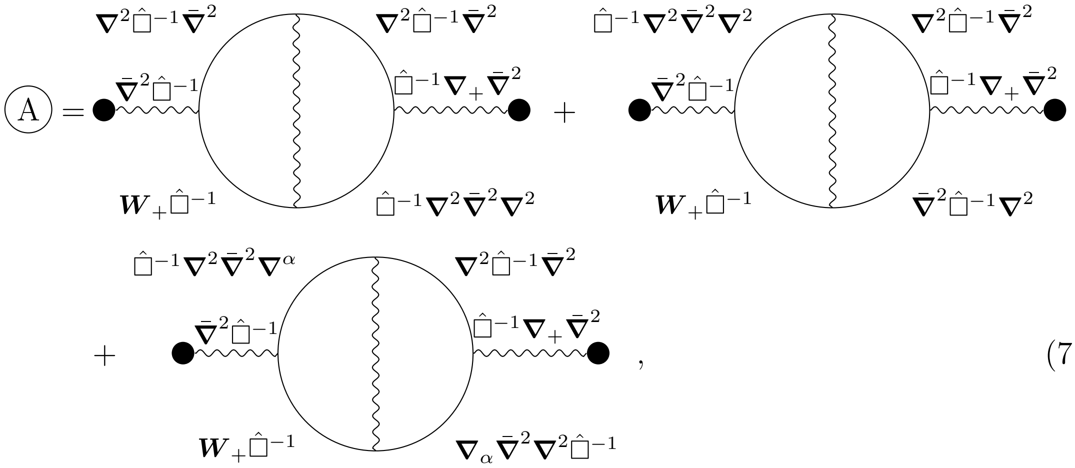

$=$ 

$- i$ 



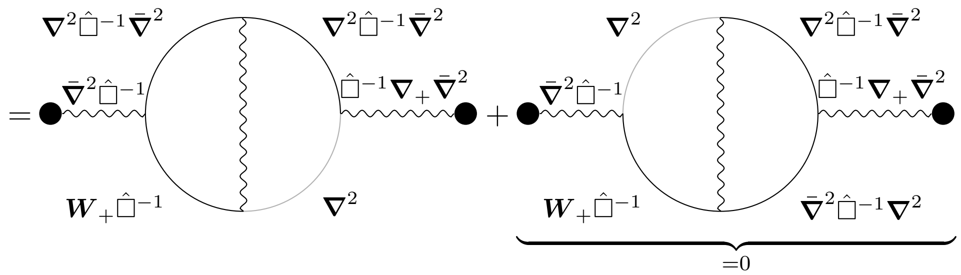

(7.1)

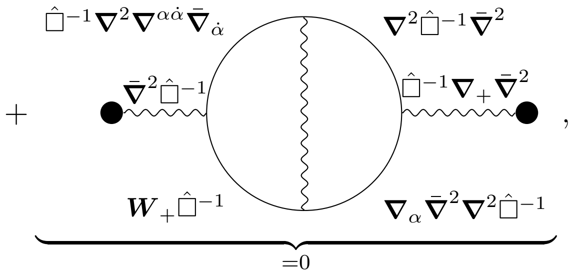
In the first term we can integrate by parts the factor of $\nabla^2$ away from the lower left line. The derivatives cannot go on the upper right line because $\nabla^3 = 0$. We find




(7.2)

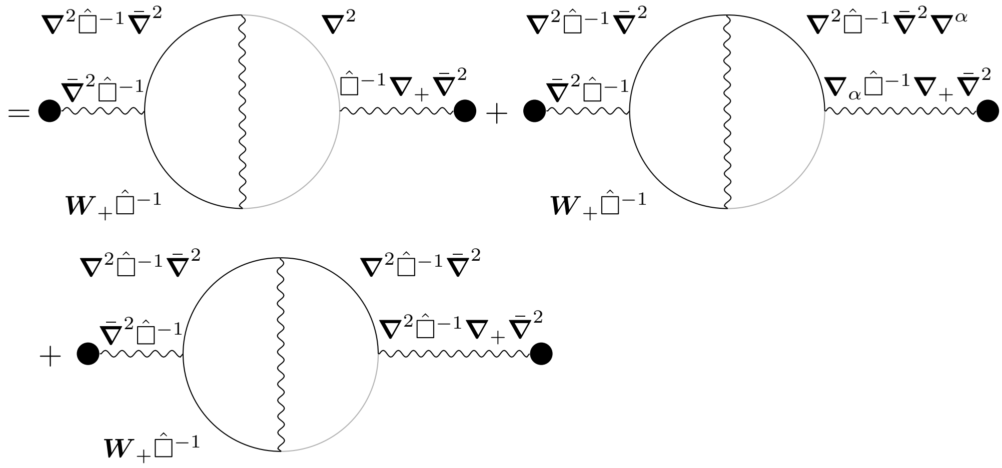
which, upon using the commutation relations, takes the form





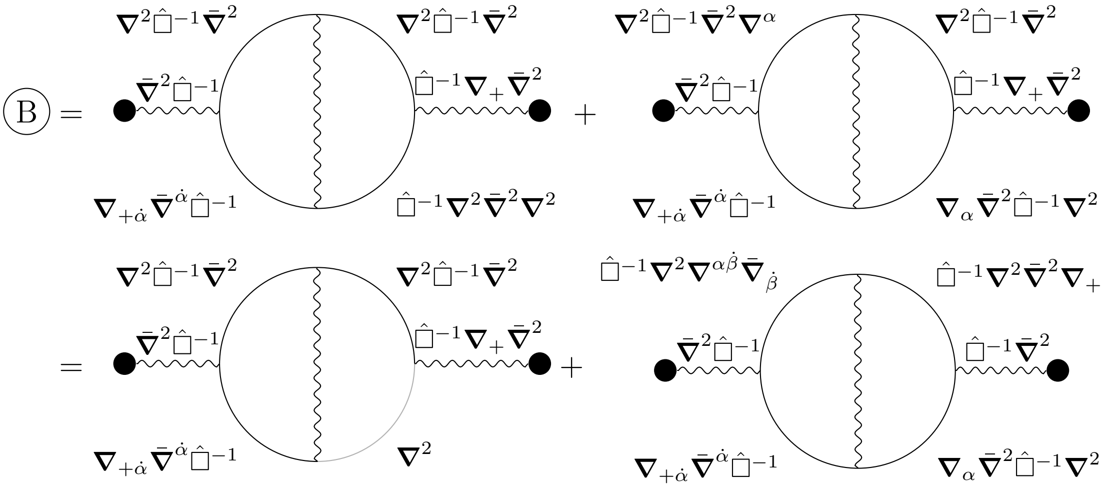

, (7.3)

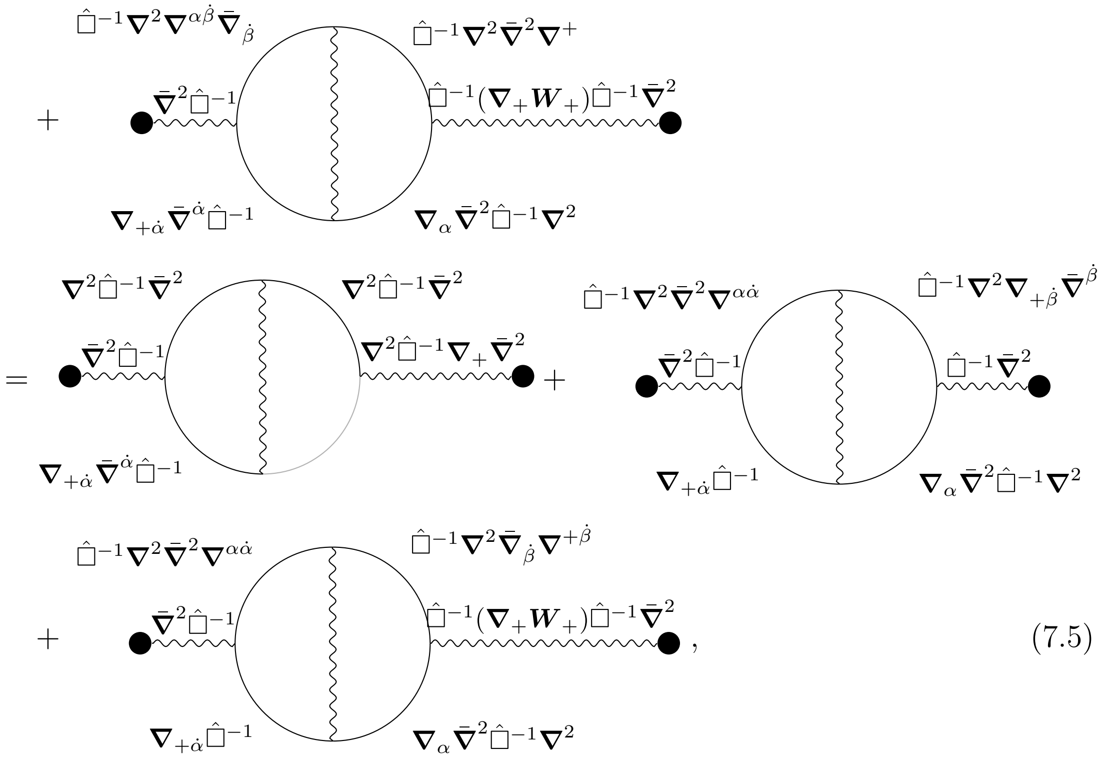
where the second and third term vanish immediately by a lack of $\bar{\boldsymbol{\nabla}}^2$ in the left loop. In the first term we integrate by parts the factor $\boldsymbol{\nabla}^2$ at the right vertex, which yields




$=0$ , (7.4)

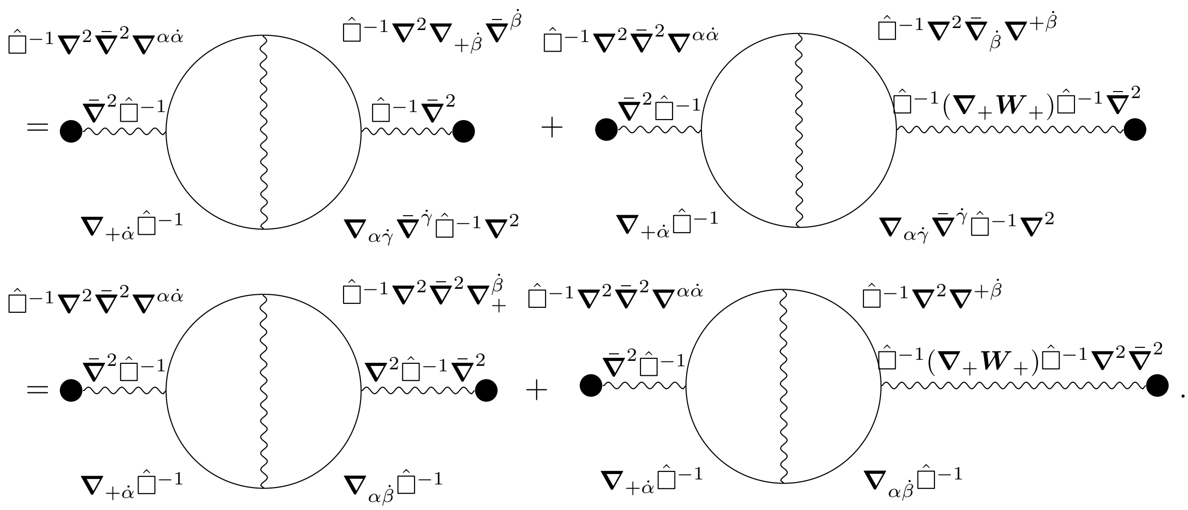
where again the first and second term vanish due to a lack of $\bar{\boldsymbol{\nabla}}^2$, while the last term vanishes because of $\left[\boldsymbol{\nabla}^2, \hat{\Box}\right] = 0$ in our sector. We follow the same procedure for the second term. The non-vanishing contributions in this case are





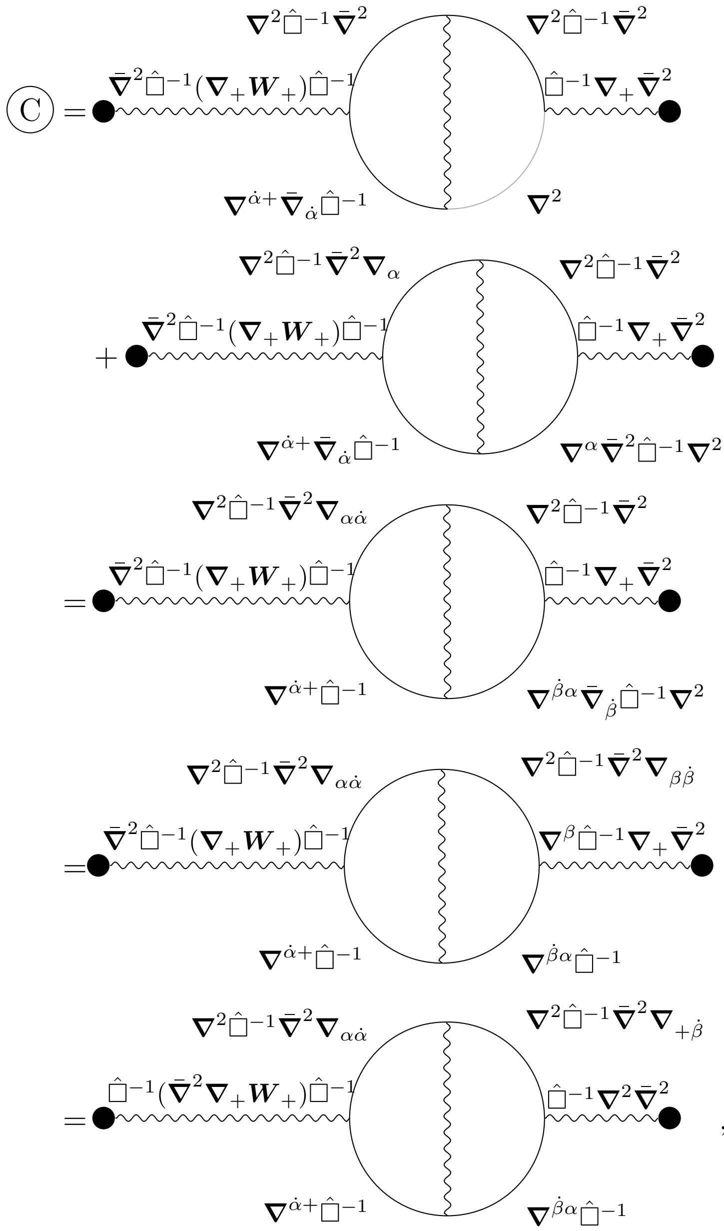
(7.5)

where the first term vanishes again due to $\left[\boldsymbol{\nabla}^2, \hat{\Box}\right] = 0$. This yields



(7.6)

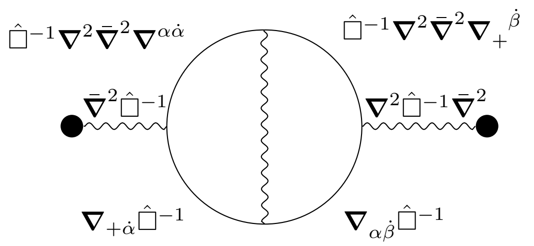
The first diagram is the one that yields new vertices. Below we will show that these contributions do not have a first order pole and thus have no effect on the dilatation operator. The second term gives a contribution that is equivalent to the one from diagram (C), which evaluates to




, (7.7)

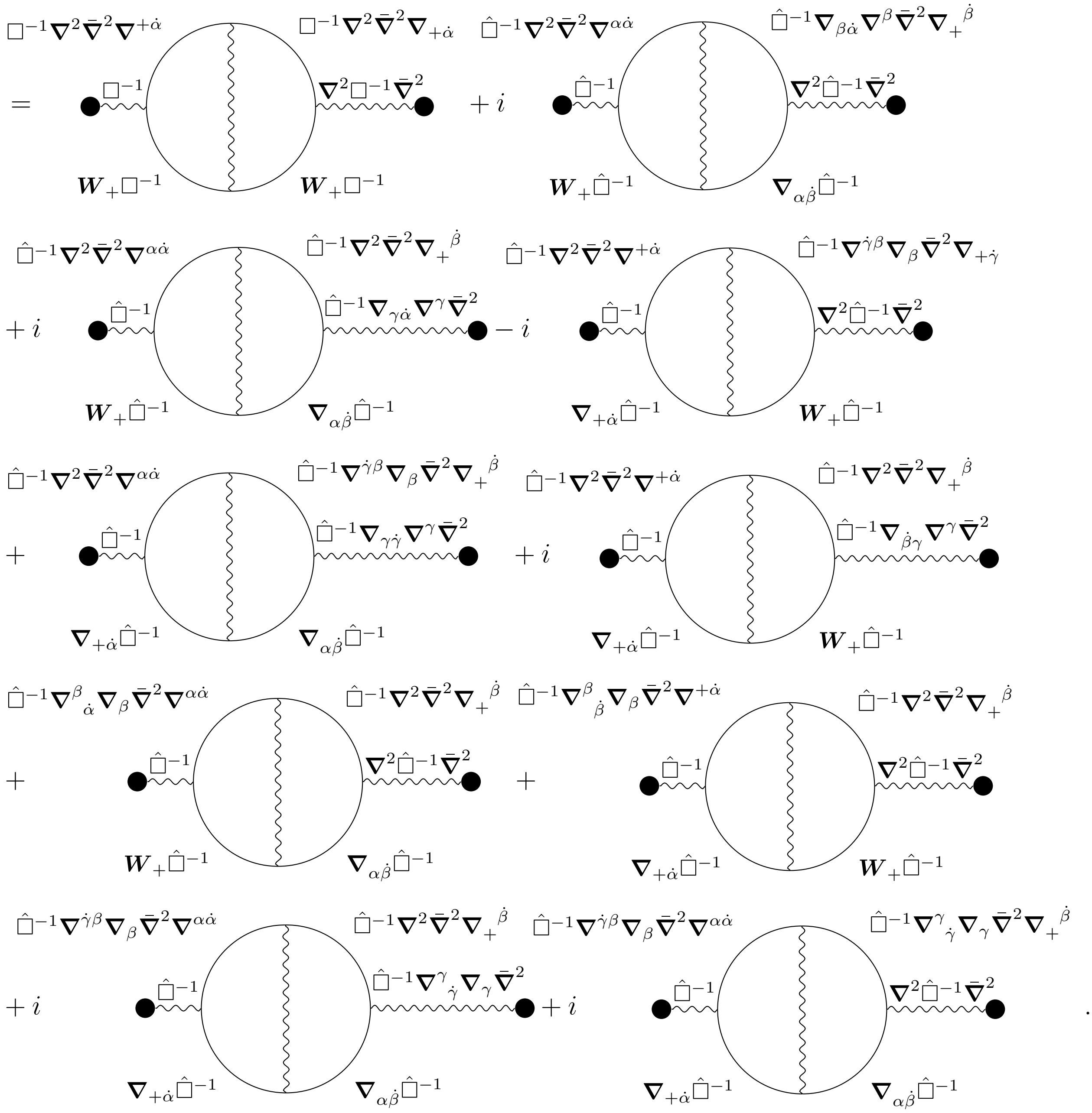
which has the structure of a renormalized tree level vertex.

As mentioned above, the only diagrams that can contribute new vertices are produced from



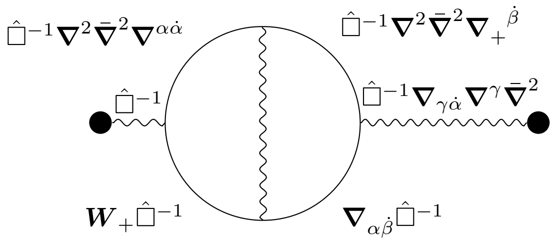

. (7.8)

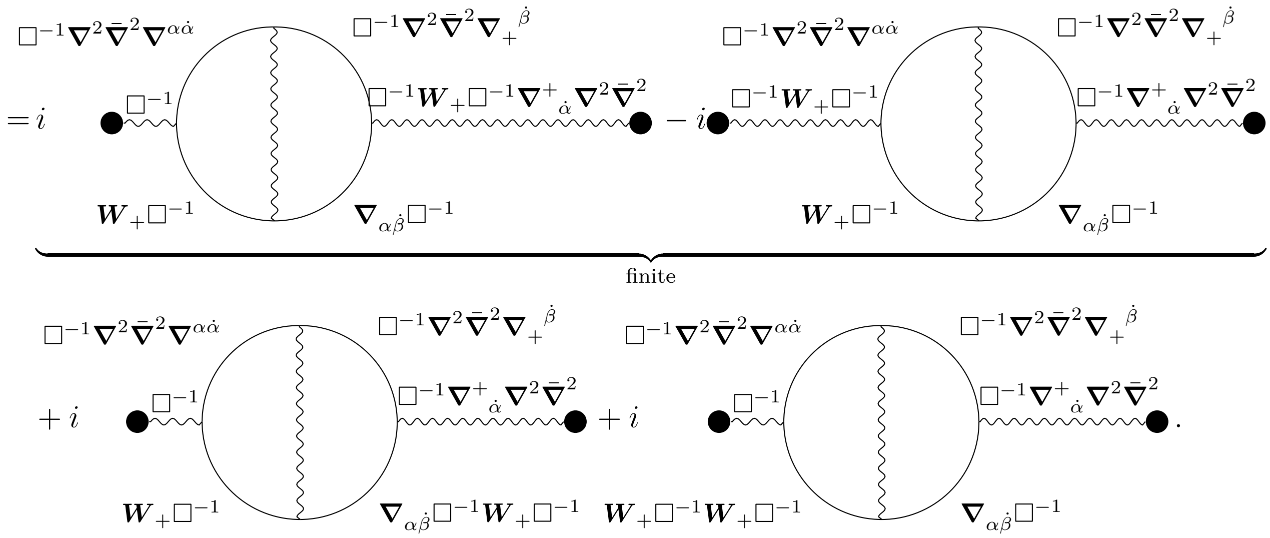
By commuting the leftmost factor of $\bar{\boldsymbol{\nabla}}^2$ through the diagram one finds




(7.9)

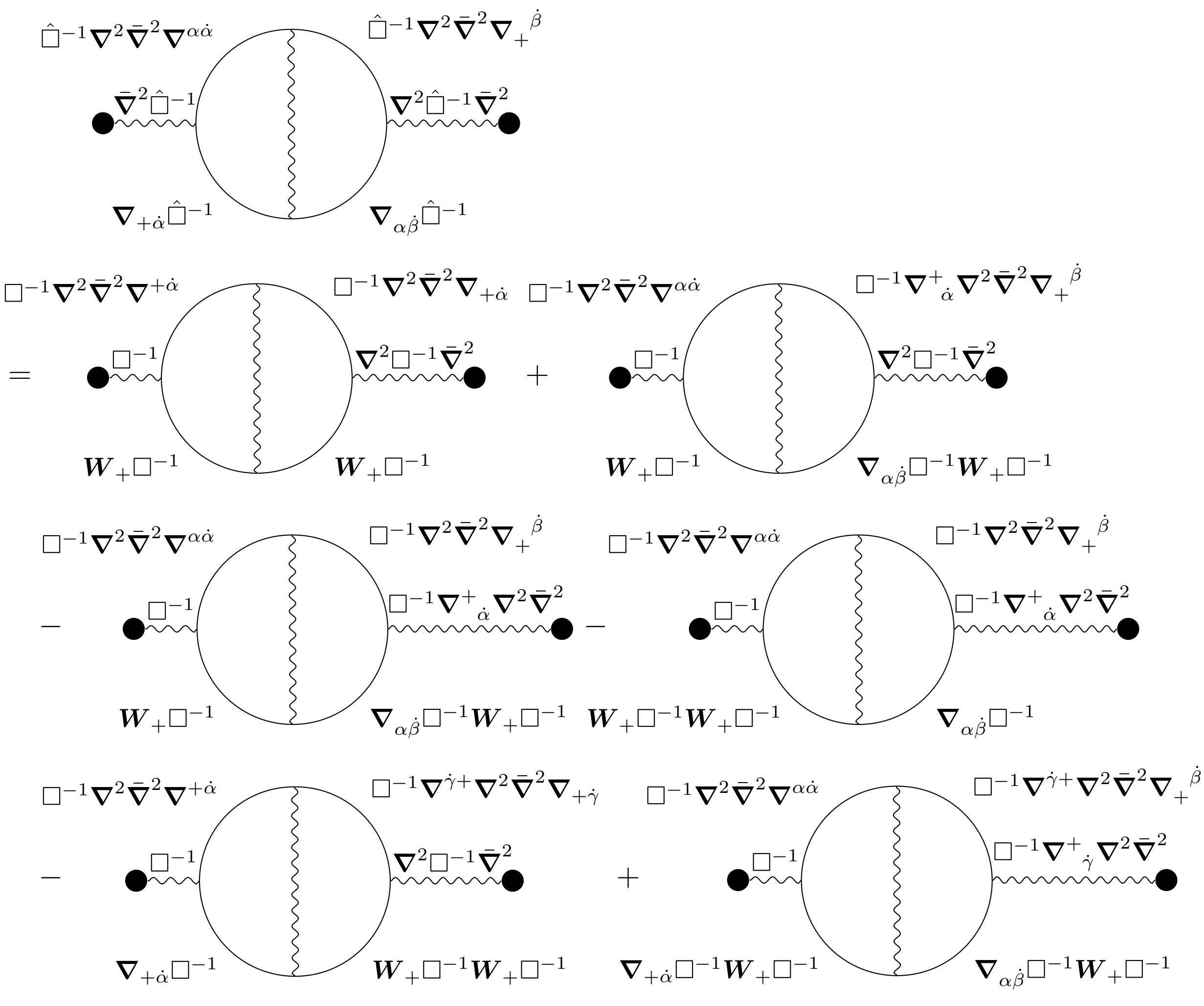
Now one has to expand the $\hat{\square}^{-1}$ propagators. As an example we look at the third term

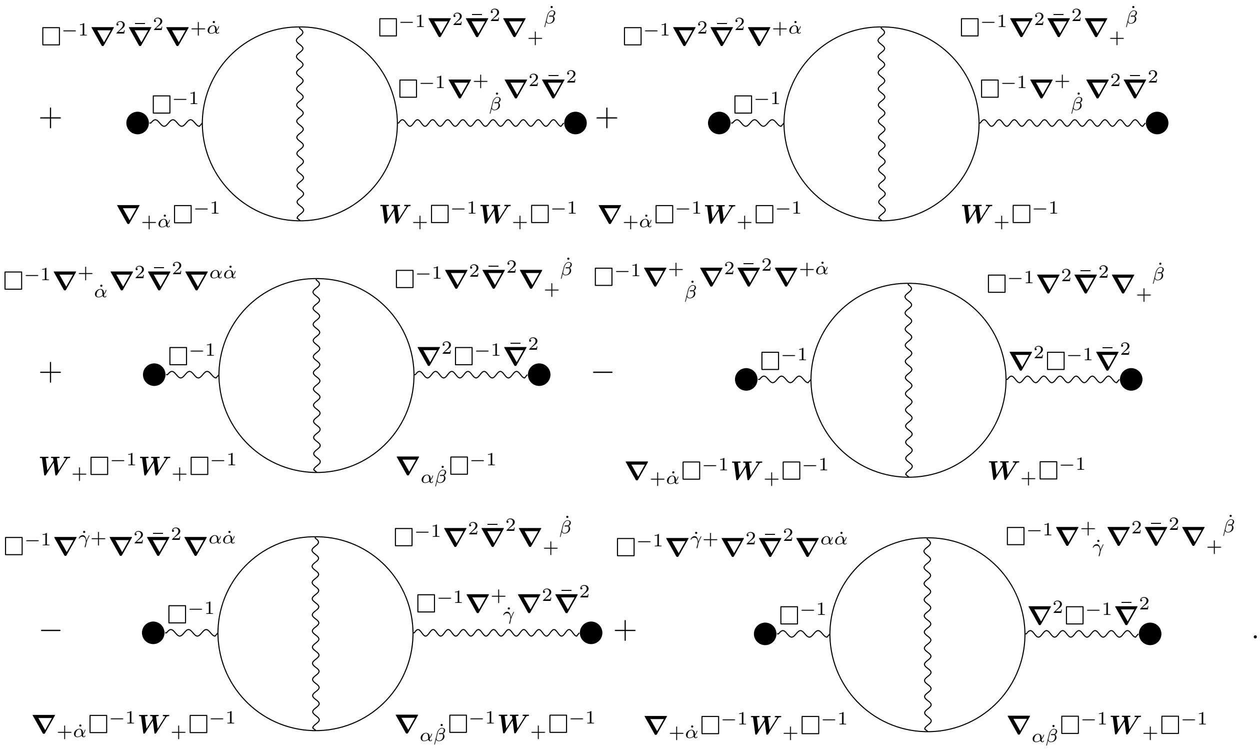





(7.10)

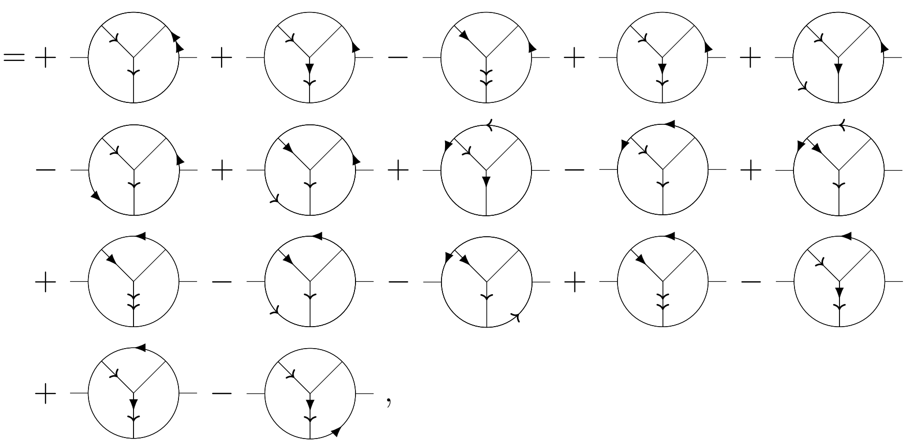
Similar cancellations take place in the other terms. The result is






(7.11)

All of these graphs reduce to the so-called Benz topology. The momentum contractions can be reduced to usual scalar products of momenta. In usual Feynman diagram notation the result is given by




(7.12)

where the two different kinds of arrows denote scalar products of the corresponding momenta. These graphs can in turn be reduced to the basis of appendix B. For this we use a reduction procedure similar to the one implemented in Mincer [76, 77]. Using the explicit results from that section it is easy to see that this evaluates to

$$\frac{1}{2}\left(\mathcal{I}_1\mathcal{I}_2 - \mathcal{I}_3 + \mathcal{I}_{32t}\right) + \frac{3}{2}\mathcal{I}_{3b} \tag{7.13}$$

which does not have a first order pole and will thus not contribute.

### 7.1.2 Results

We thus find that the first order pole contribution to the counterterm from this topology is given purely by eq. (7.7) and the last diagram in eq. (7.6). They give identical results and upon reduction to our basis, they evaluate to

$$\mathcal{Z}^{(3)}_{\text{contr,Top1}} = -\frac{1}{16}\mathcal{I}_{3n} + \frac{1}{2}\mathcal{I}_{3t} + \frac{1}{2}\mathcal{I}_{3bb} - \frac{37}{16}\mathcal{I}_{3}\,. \tag{7.14}$$

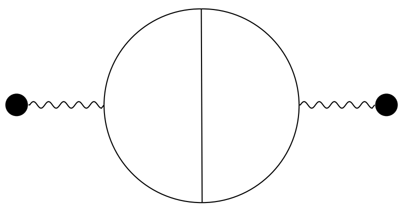
The contribution to the dilatation operator from this topology is thus given by

$$D^{(3)}_{\text{Top1}}(W_+W_+) = \left(6\zeta(3) - \frac{647}{576}\right)(\boldsymbol{W}_+\boldsymbol{W}_+)\,. \tag{7.15}$$

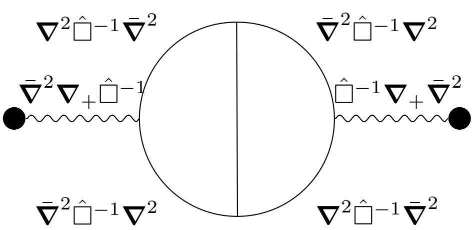
## 7.2 Topology 2

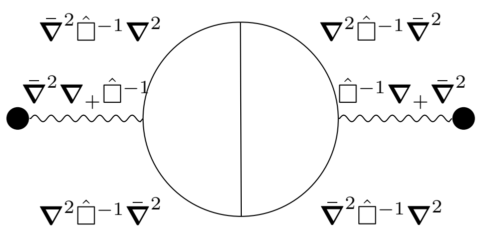
### 7.2.1 The Vacuum

The next topology to consider is the one with a chiral loop and a chiral field propagating inside, namely the one given by


,

with the appropriate derivatives acting on the propagators. In this case we have to consider both D-algebra structures separately, because upon closing the diagram there is no symmetry relating them. The computation is fairly similar to the last one. The starting point are the two diagrams

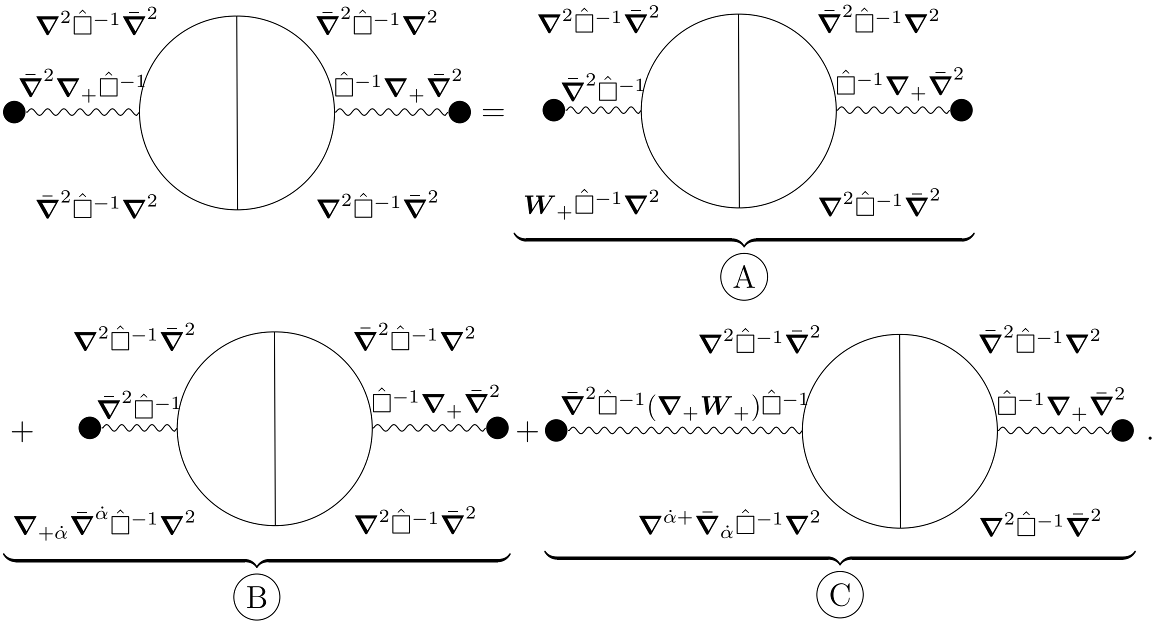


and


.

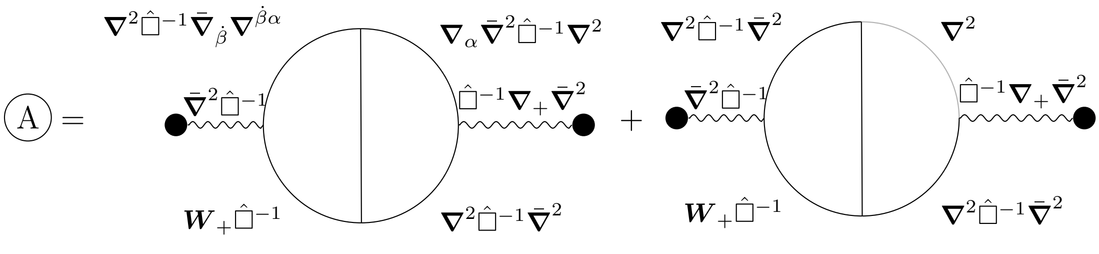
(7.16)

Their computation is quite similar to the previous toplogy and thus we will skip some of the steps. The first one evaluates to



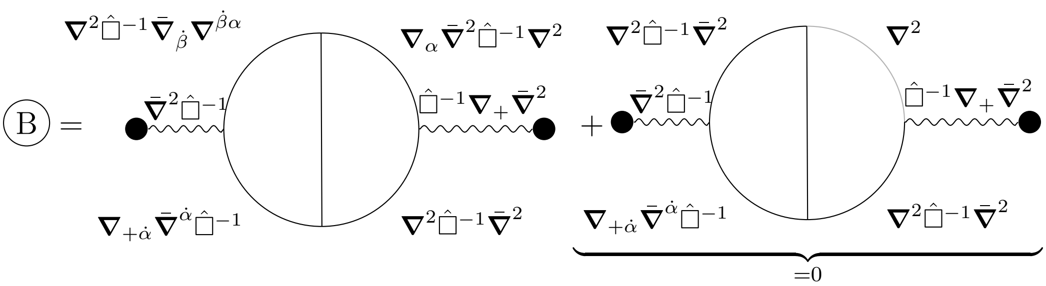

(7.17)

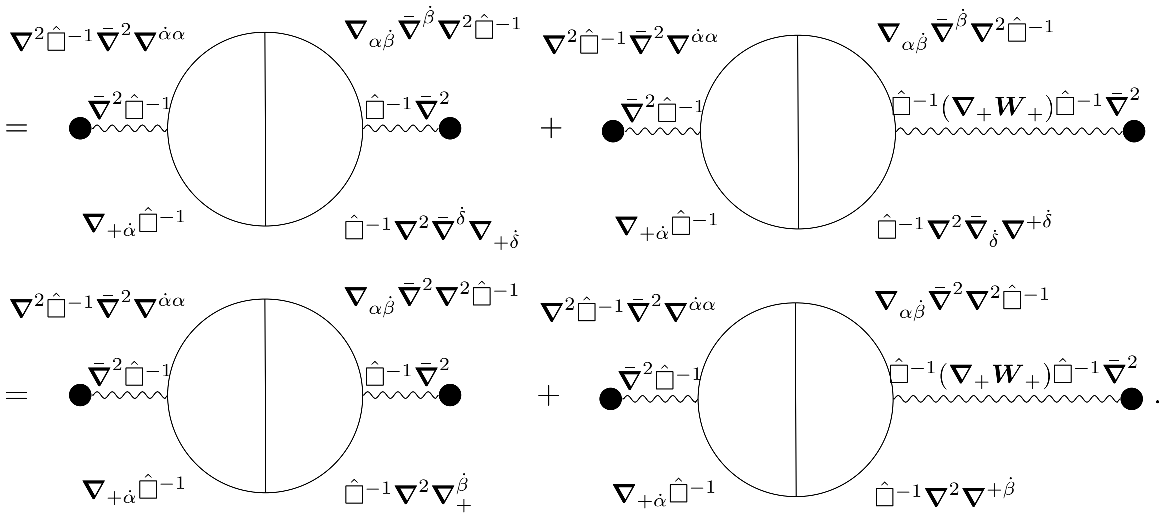
The first contribution again vanishes




$= 0$ (7.18)

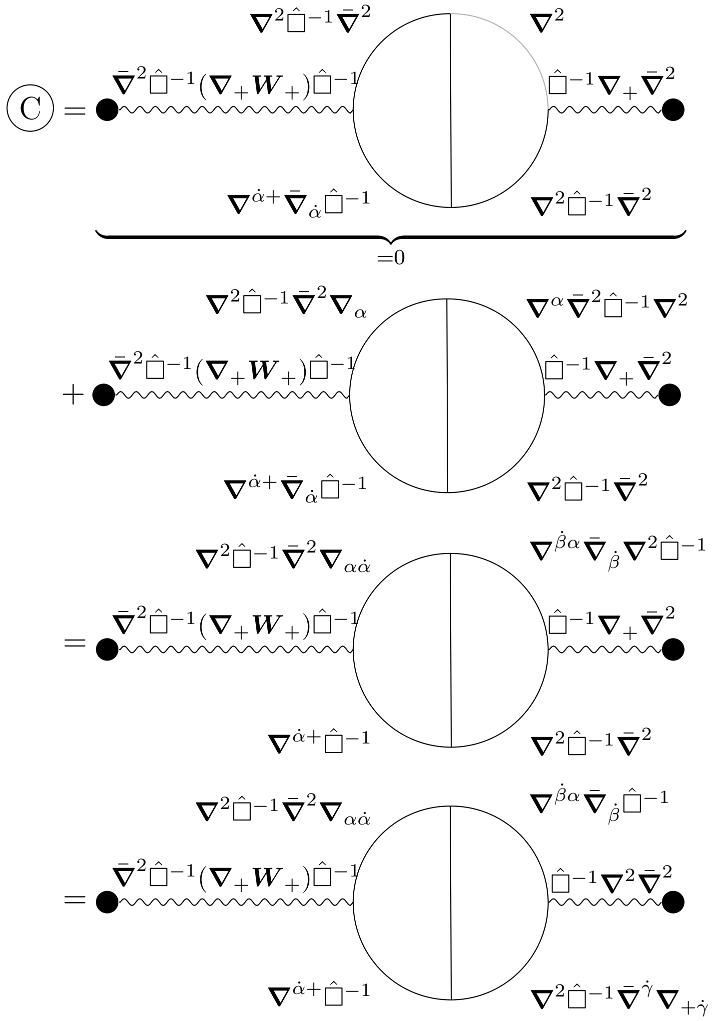
due to the lack of $\bar{\nabla}^2$ and $\left[\nabla^2, \hat{\Box}\right] = 0$. The next one is

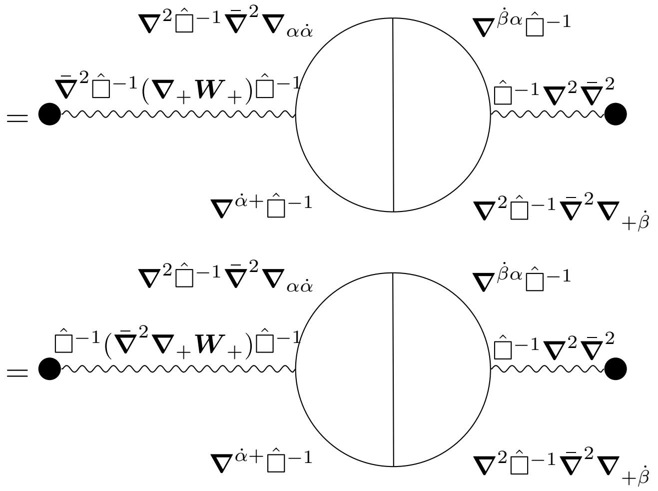





(7.19)

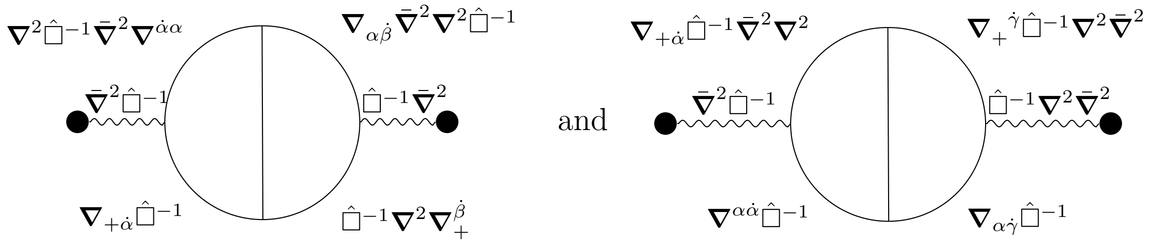
The first diagram in the last equation is the one that can potentially produce new vertices. Finally contribution C only produces old vertices.





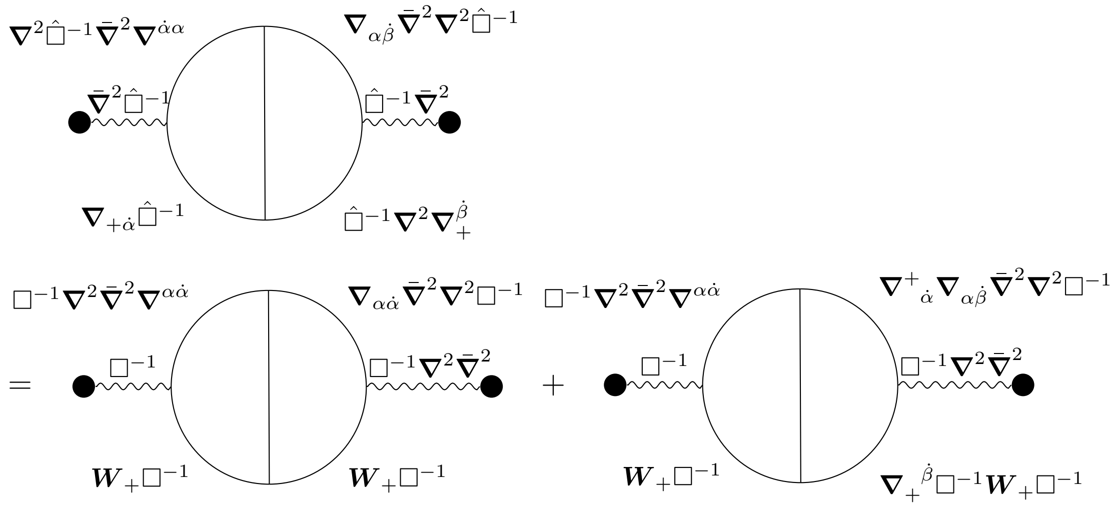

. (7.20)

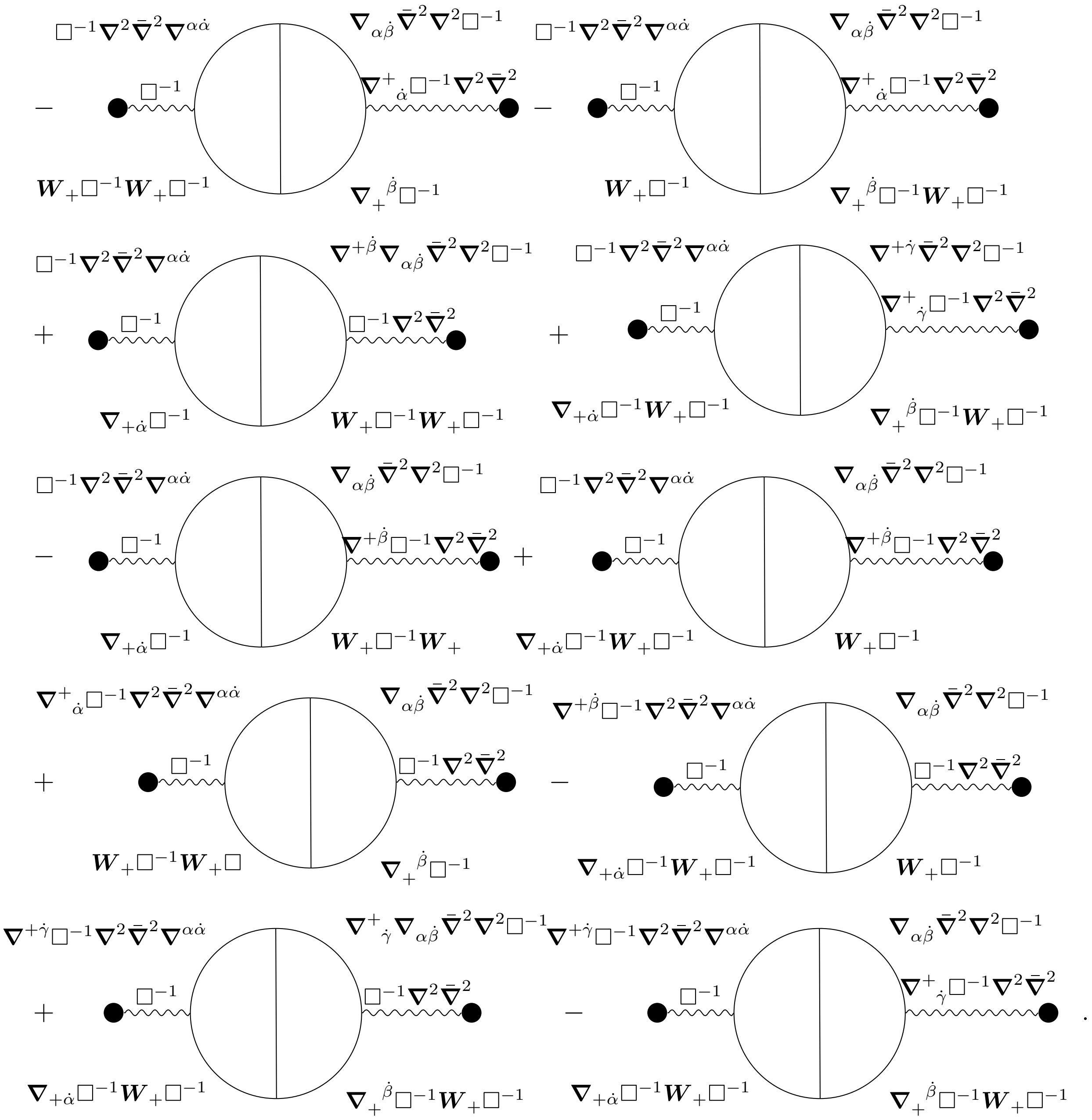
Since the calculation of the other D-algebra structure is virtually identical, we will skip it. The same diagram C also appears in that topology and the two diagrams, which contain all the potentially dangerous contributions from these two topologies, are




. (7.21)

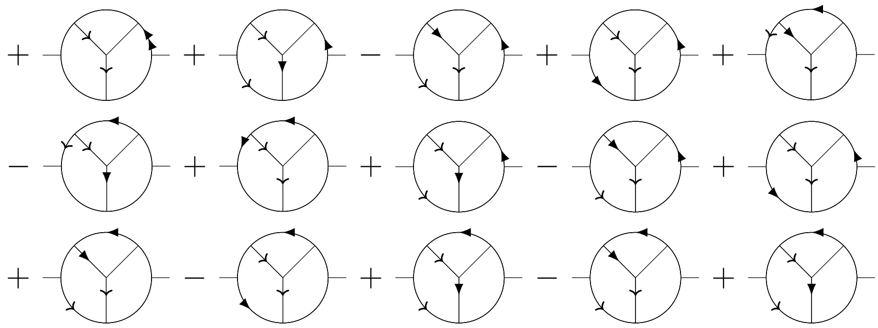
After finishing the D-algebra the first diagram yields

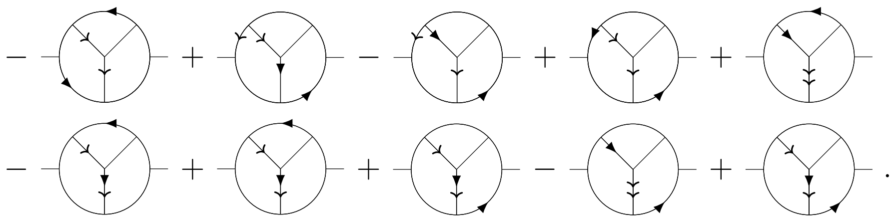





(7.22)

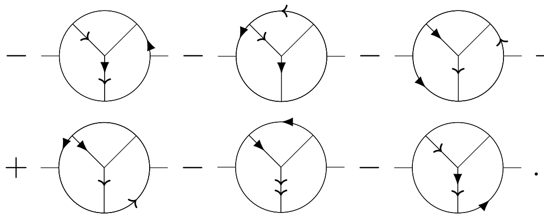
and analogously for the second topology. Reducing this to ordinary Feynman diagrams leads to





(7.23)

Similarly the other D-algebra structure yields



(7.24)

which together evaluates to

$$= -\mathcal{I}_3 - \mathcal{I}_{32t} + \frac{3}{2}\mathcal{I}_{3b} + \frac{1}{2}\mathcal{I}_{3bb}\,. \tag{7.25}$$

This has no first order pole and thus does not contribute to the dilatation operator.

### 7.2.2 Results

In complete analogy to topology 1 the total first order pole contribution to the dilatation operator of this topology again comes only from eq. (7.20) and the last diagram in eq. (7.19). Note that the momentum contractions are different from topology 2. They evaluate to

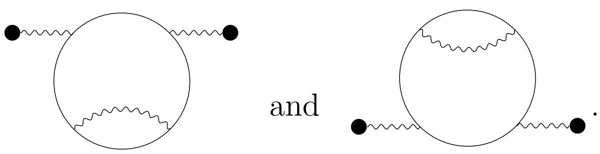
$$\mathcal{Z}^{(3)}_{\text{contr,Top2}} = \frac{5}{16}\mathcal{I}_3 - \frac{1}{2}\mathcal{I}_{3bb} + \frac{17}{8}\mathcal{I}_{3n}\,. \tag{7.26}$$

The contribution to the dilatation operator from this topology is thus given by

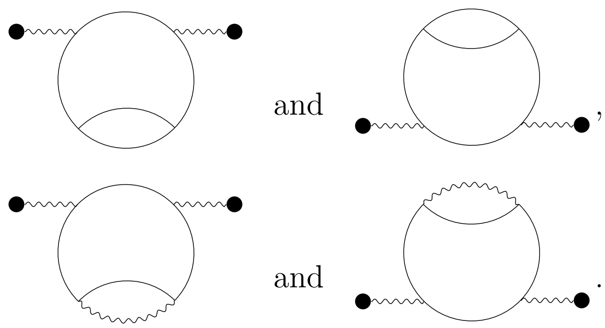
$$D^{(3)}_{\text{Top2}}(W_+W_+) = -\frac{9}{64}(\boldsymbol{W}_+\boldsymbol{W}_+)\,. \tag{7.27}$$

## 7.3 Topology 3

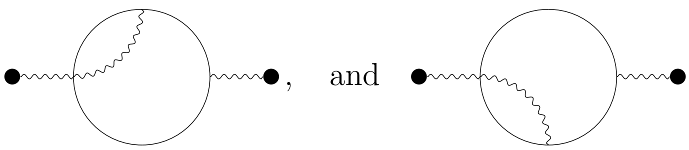
Next we consider the two topologies where there is a chiral field propagating inside the chiral loop. We have two different D-algebra structures



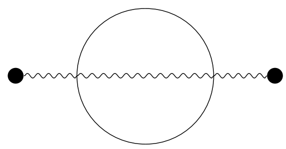
In our notation it looks like they are identical, however remember that the operator sites are closed above the diagram, hence the symmetry is broken and these are indeed two different diagrams. There are two more pairs of diagrams like these, which are treated in the same way and hence we will not consider them further. They are




Importantly the results for the topologies

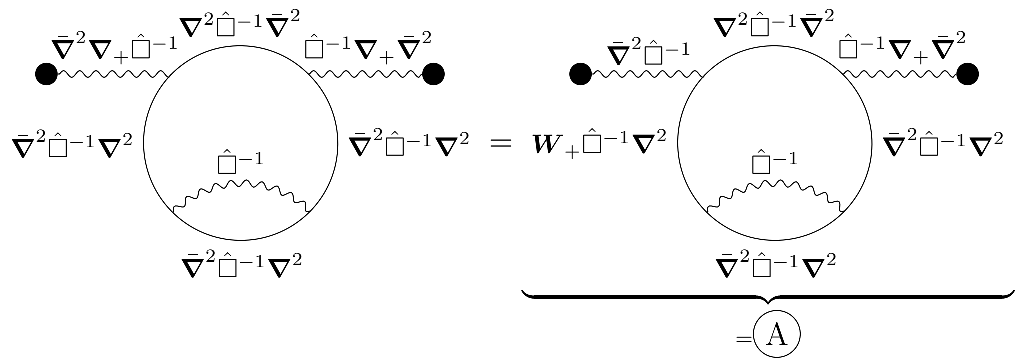


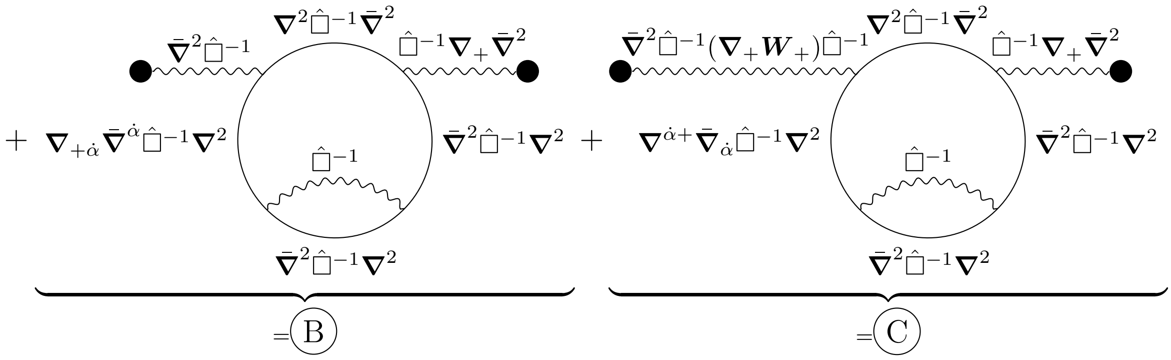

can also be directly inferred from these, because the computation is virtually identical. The topology



(7.28)

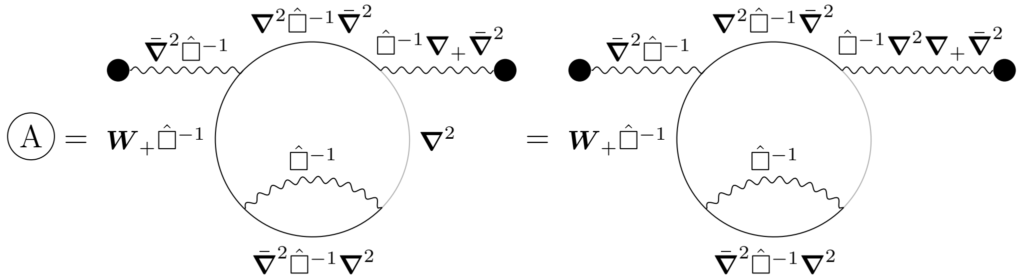
is even simpler and will be discussed very briefly at the end of this section. All these topologies turn out to be relatively simple to evaluate. We find





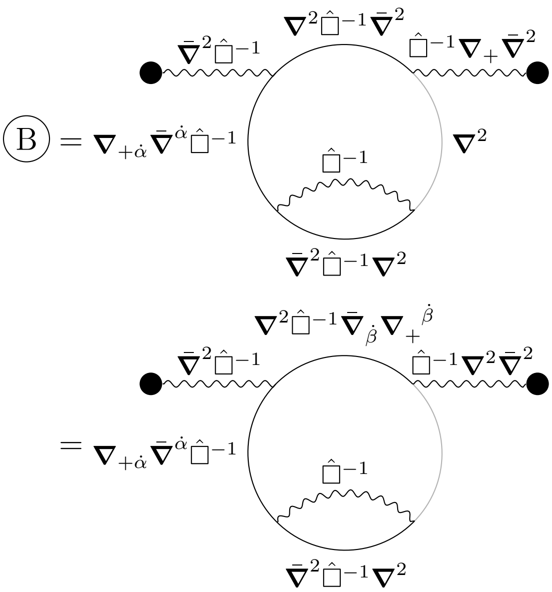

(7.29)

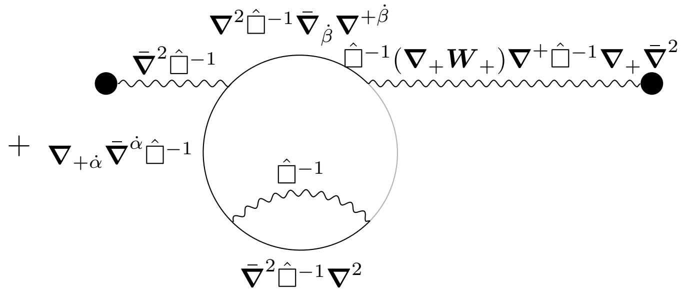
The first contribution vanishes due to $\boldsymbol\nabla^3 = 0$




$$= 0\,. \tag{7.30}$$

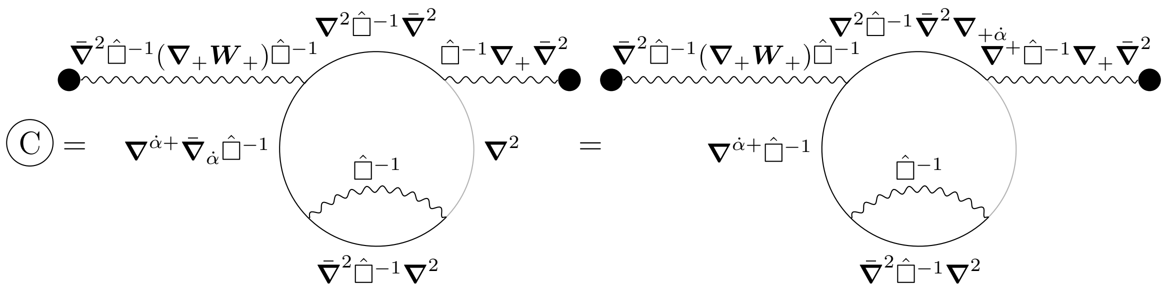
The second contribution is





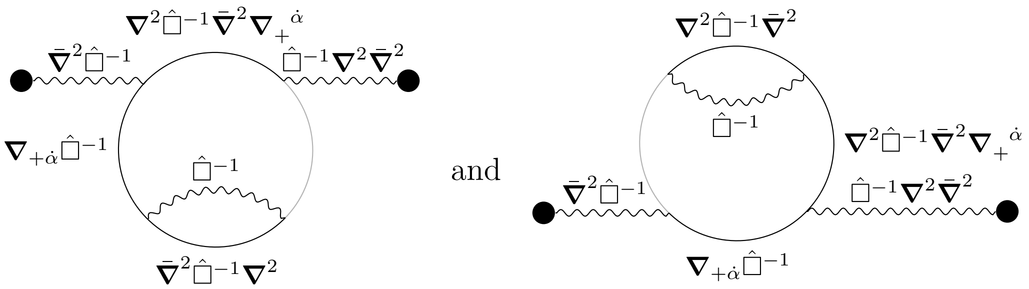

, (7.31)

where the first diagram is the only one that produces a new vertex and the last contribution is



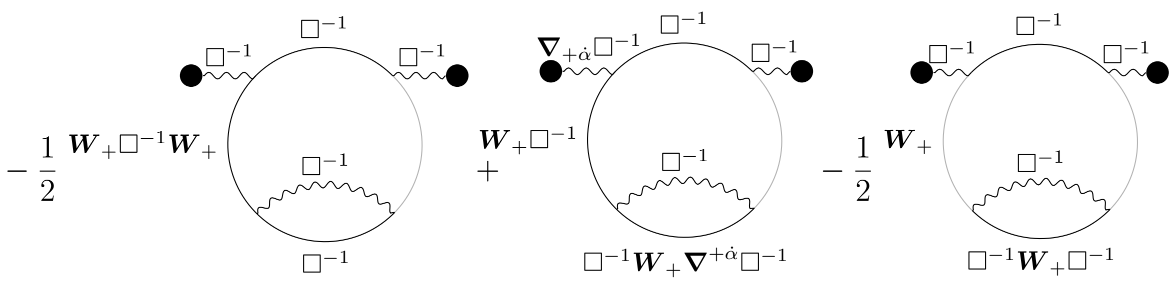

. (7.32)

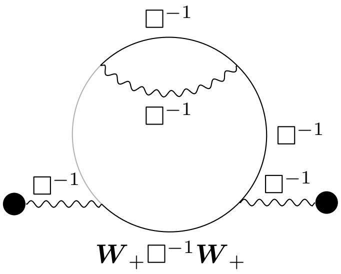
An analogous calculation for the other topology shows that the only graphs with new vertices come from the two diagrams

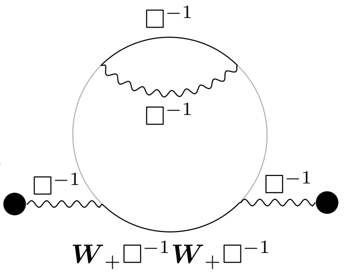


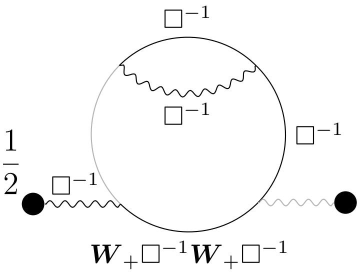

. (7.33)

After expansion in terms of the background fields and reduction of the scalar products we find



$+\frac{1}{2}$ 

$+\frac{1}{2}$ 

$-\frac{1}{2}$ 


$$= -\mathcal{I}_{32t} - \frac{1}{2}\mathcal{I}_3\,, \tag{7.34}$$

which has no first order pole.

### 7.3.1 Results

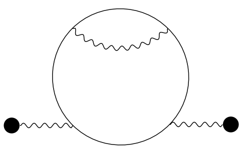
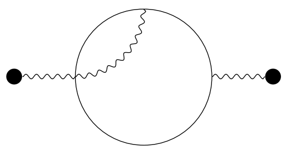
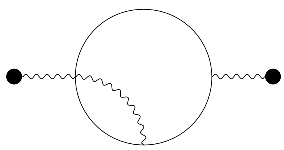
As in the previous topologies the only contributions to the dilatation operator come from eq. (7.32) and the last diagram in eq. (7.31) and they give

$$\mathcal{Z}^{(3)}_{\text{contr,Top3}} = \frac{1}{2}\mathcal{I}_3 - \frac{1}{2}\mathcal{I}_{3n}\,. \tag{7.35}$$

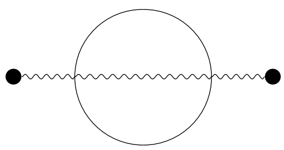
The contribution to the dilatation operator from this topology is thus given by

$$D^{(3)}_{\text{Top3}}(W_+W_+) = \frac{25}{8}(\boldsymbol{W}_+\boldsymbol{W}_+)\,. \tag{7.36}$$

The same result also holds for

 ,  , and  .

The topology

Top 4 =  (7.37)

is even easier. The D-algebra in this case is trivial and we immediately find that the only contribution is

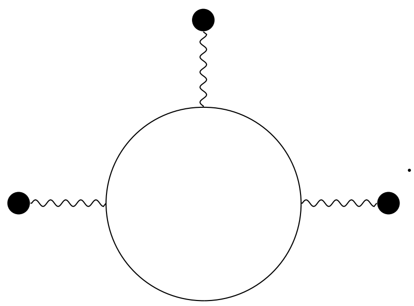
$$\mathcal{Z}^{(3)}_{\text{contr,Top4}} = \mathcal{I}_{3n}\,, \tag{7.38}$$

which yields

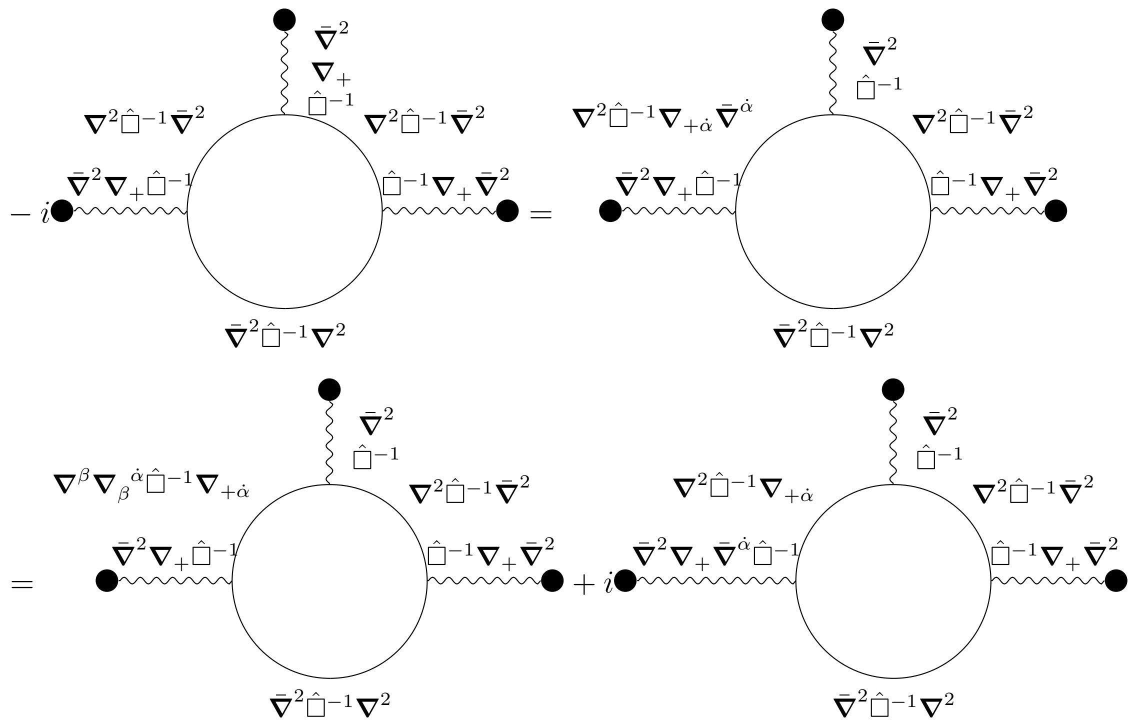
$$D^{(3)}_{\text{Top4}}(W_+W_+) = \frac{9}{4}(\boldsymbol{W}_+\boldsymbol{W}_+)\,. \tag{7.39}$$

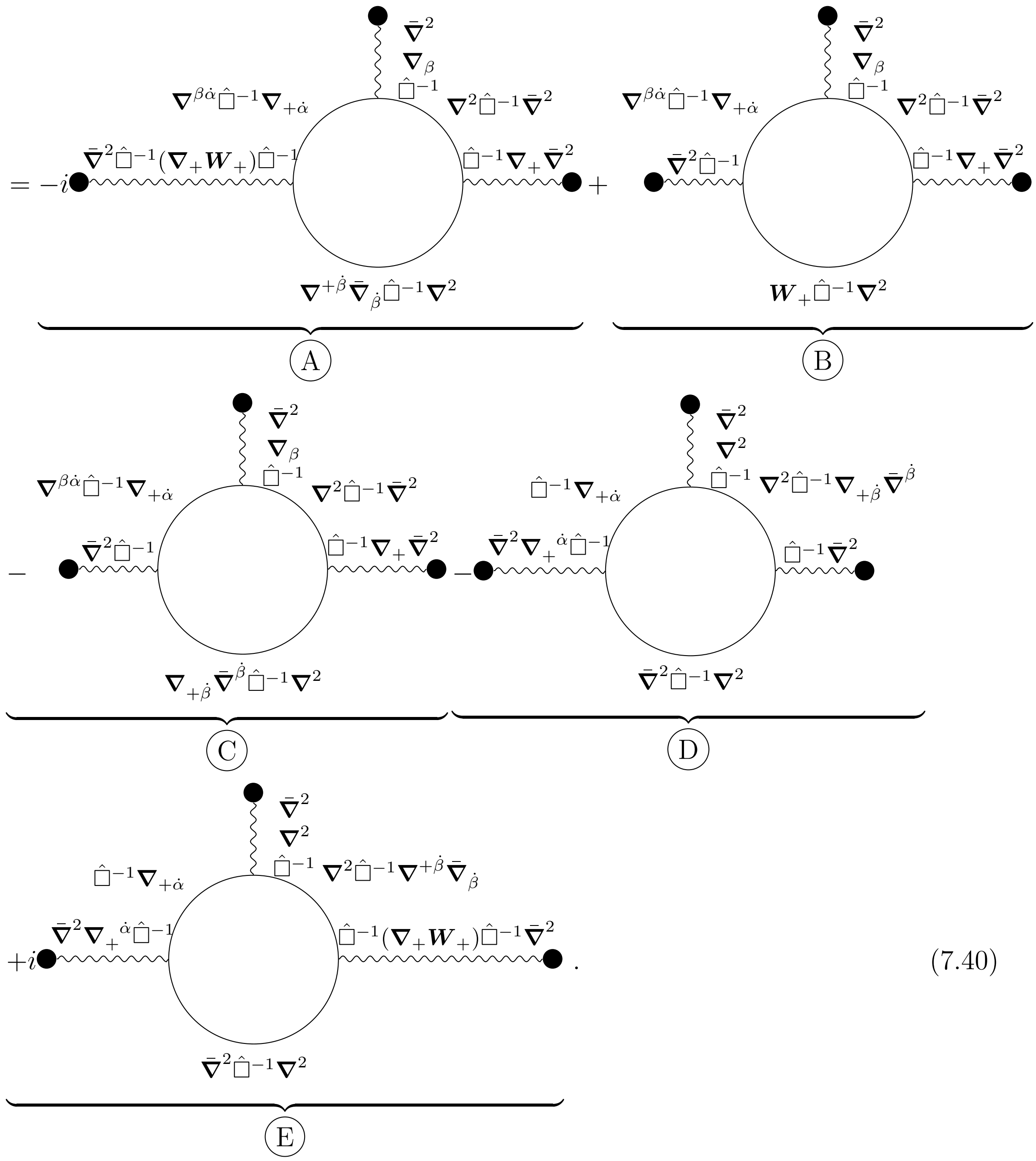
## 7.4 Other topologies

There are other topologies at three loops, which contain chiral loops. All of these have in common, that there is no extra line propagating *inside* the loop. We are not aware of any superconformal gauge theory, where they can contribute to the difference to $\mathcal{N} = 4$ SYM. It is nevertheless instructive to look at examples of such topologies. We will consider the most intricate such topology, which is



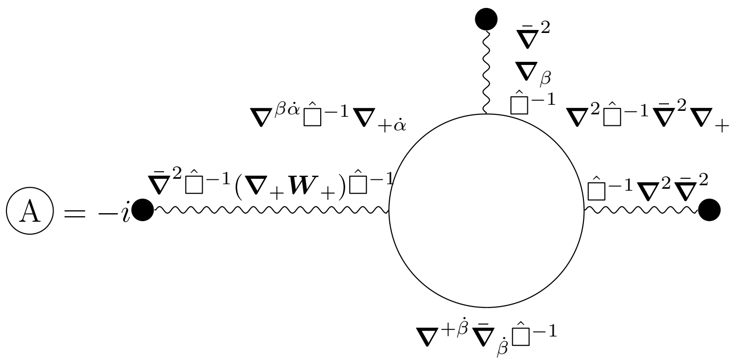
It is also interesting because it is the only topology in our calculation that has an interaction range of three sites instead of two. In this topology it turns out that the most efficient way is to start at the site in the middle and commute the factor of $\boldsymbol{\nabla}_+$ onto the chiral loop. We find

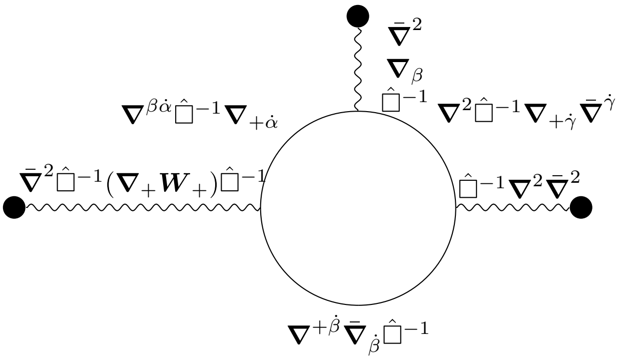


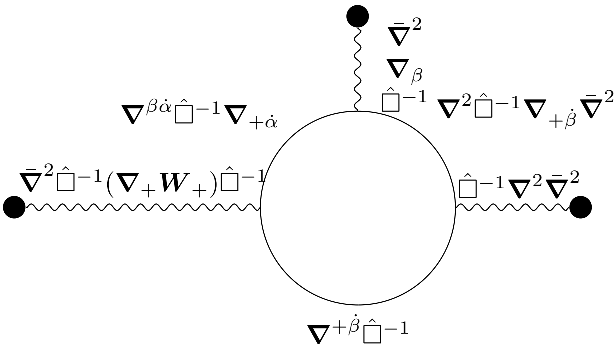



(7.40)

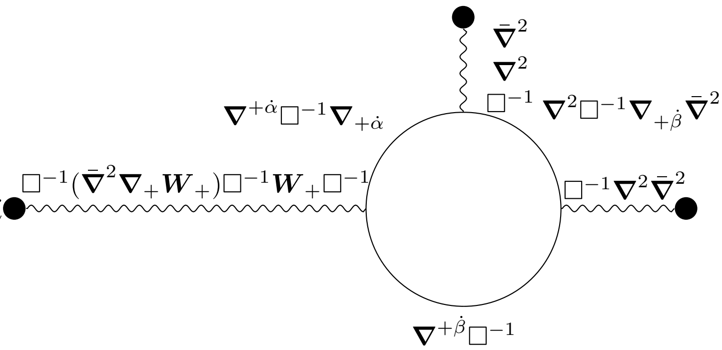
Again we consider these contributions in turn. The first one gives

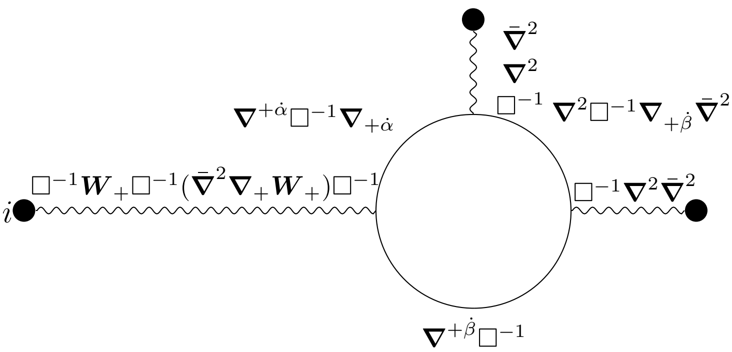


$=$ 

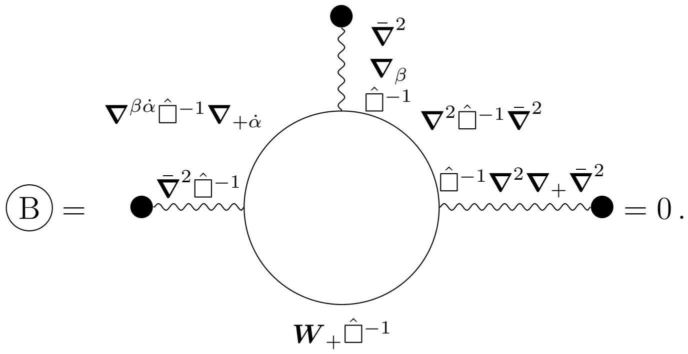

$= -$  ,


and after expansion of the propagators this is

$= i$ 

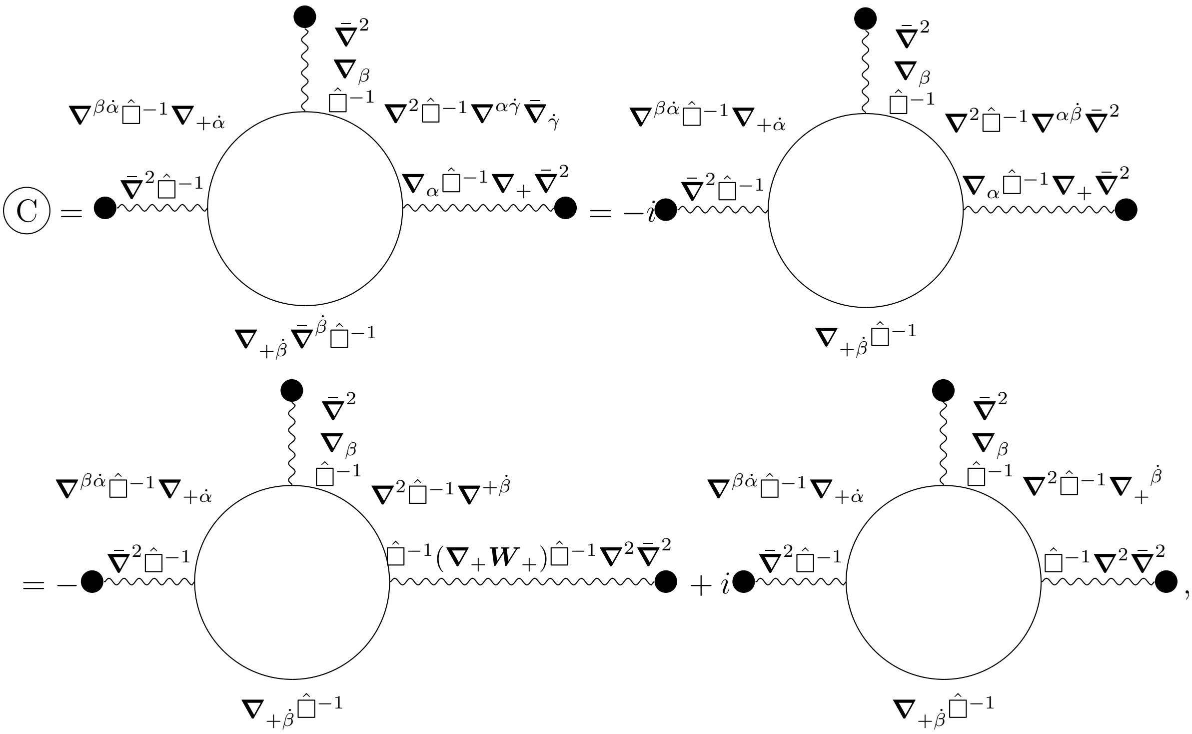

$- i$  . (7.41)

The second contribution is particularly simple due to $\boldsymbol{\nabla}^3 = 0$.

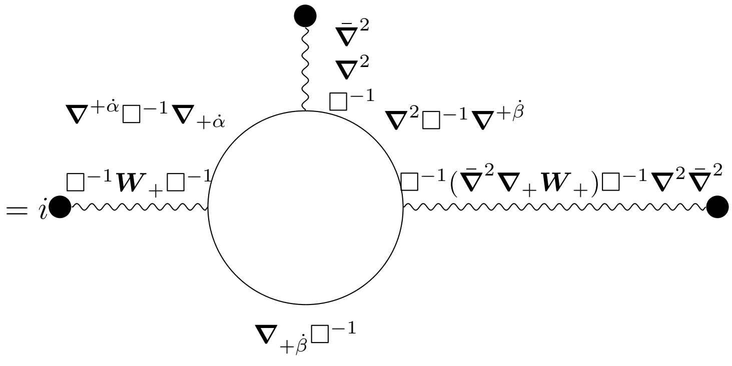


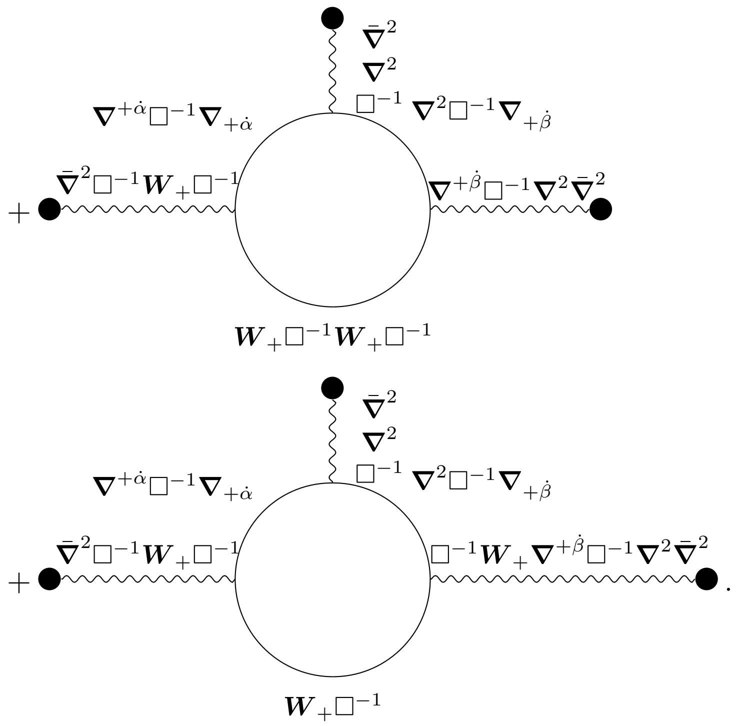

(7.42)

For C we find



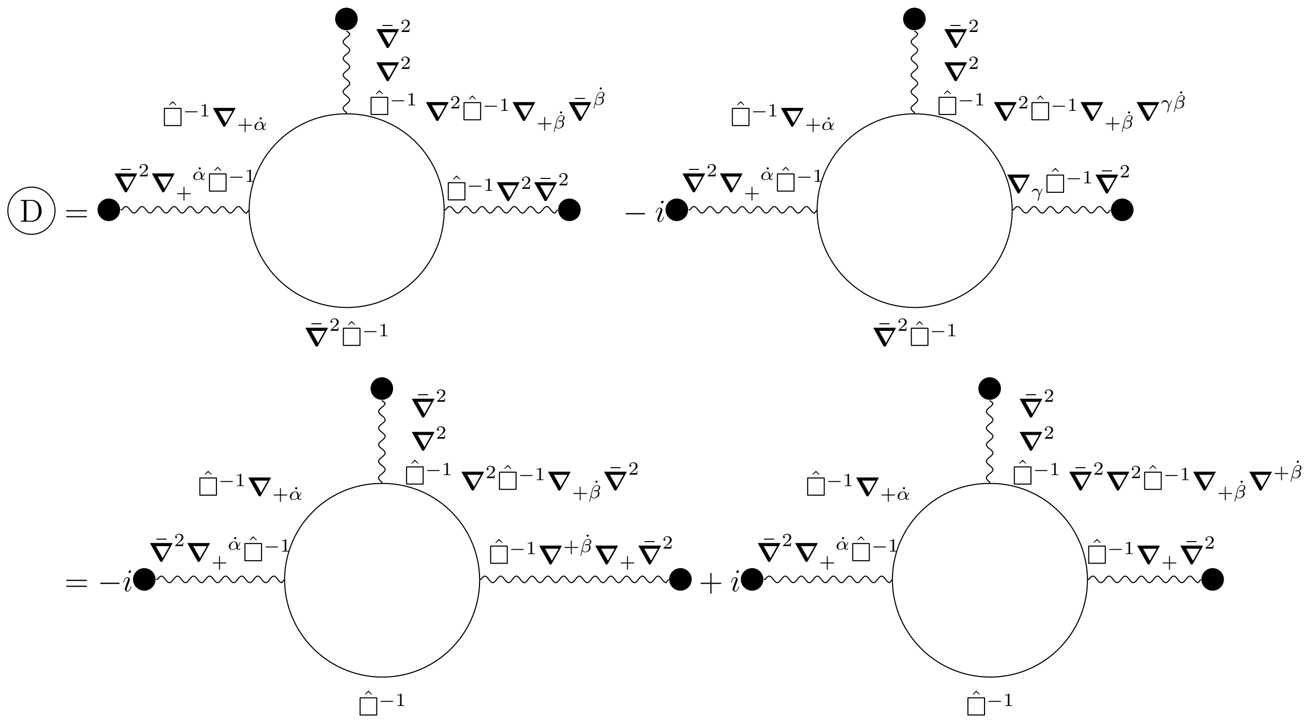

which after expansion of the propagators evaluates to

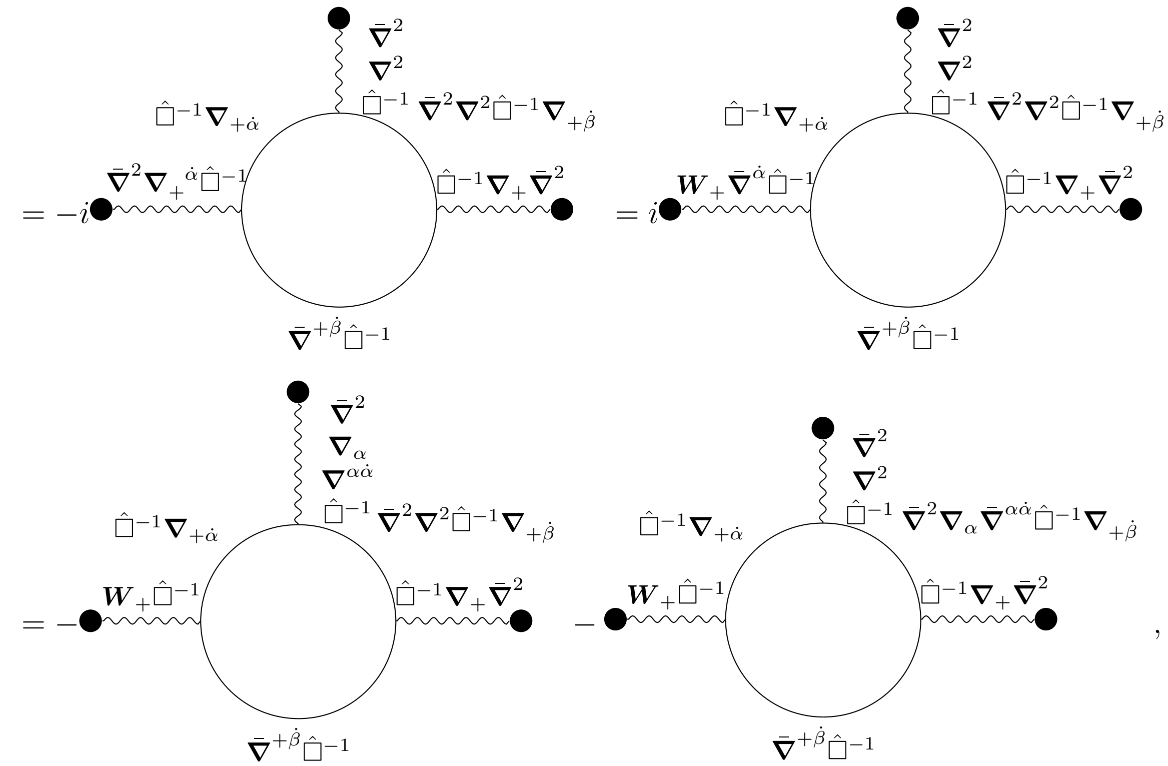




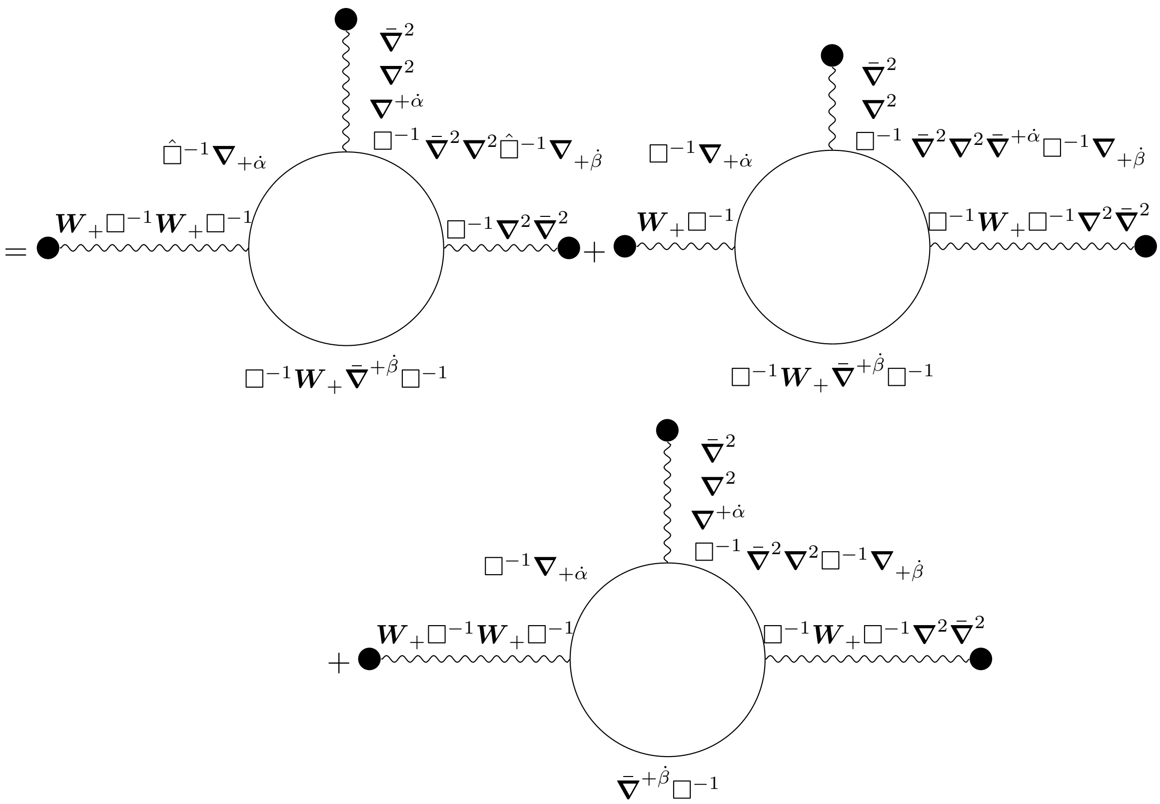

(7.43)

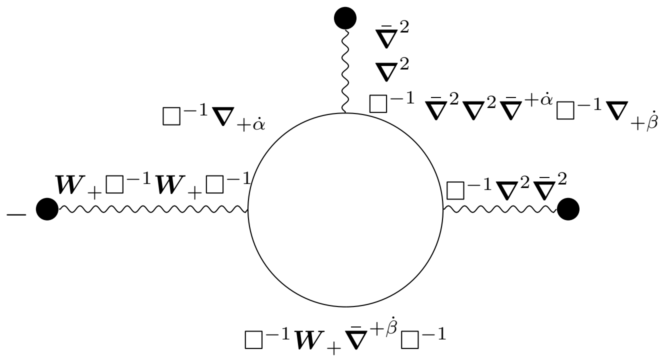
The fourth contribution is





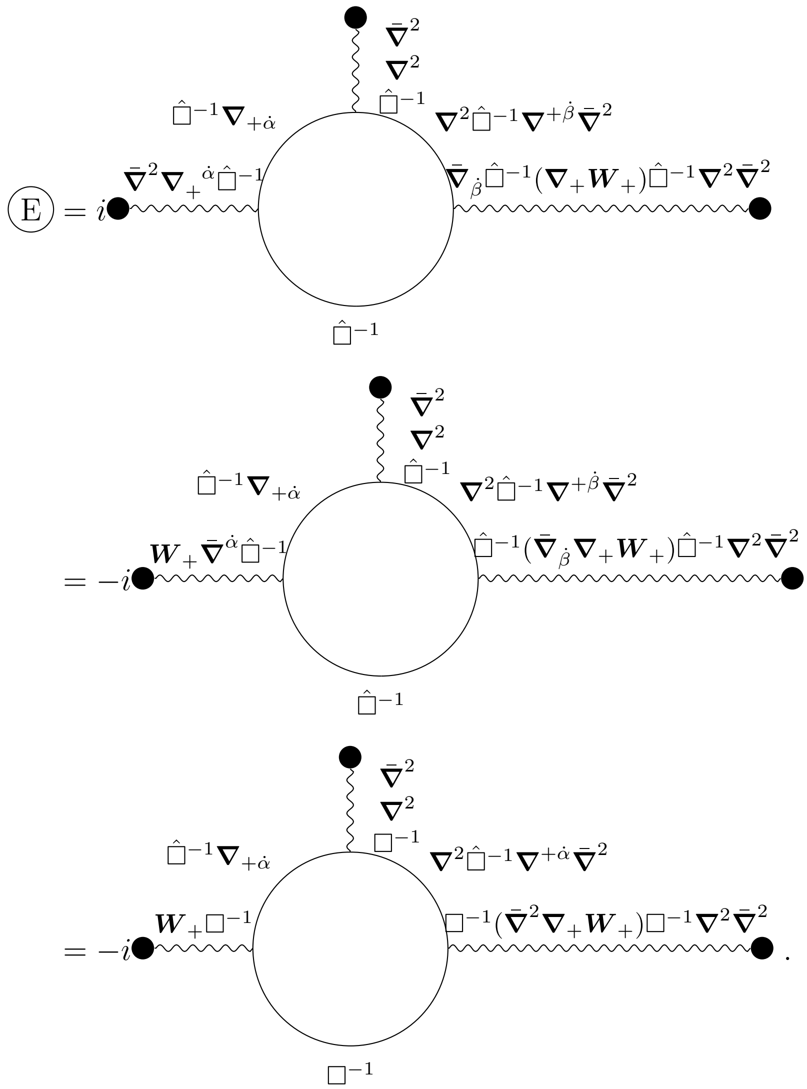
and after expansion





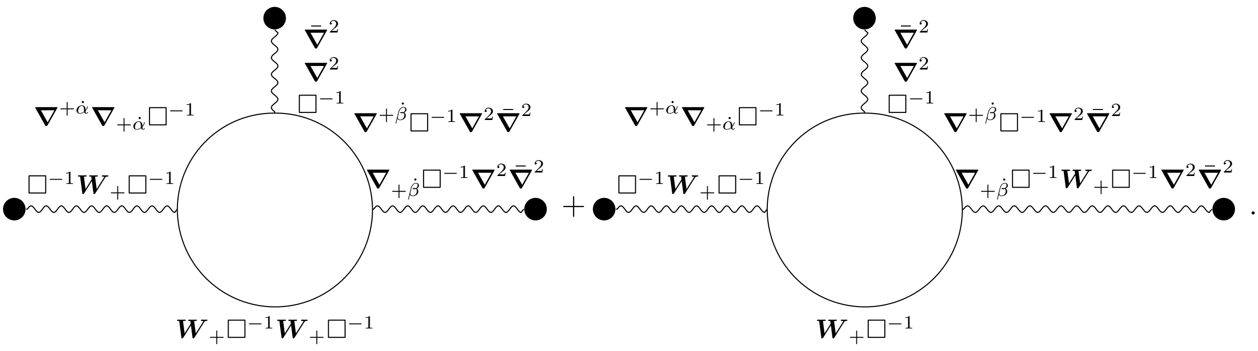

. (7.44)

Finally the last contribution is



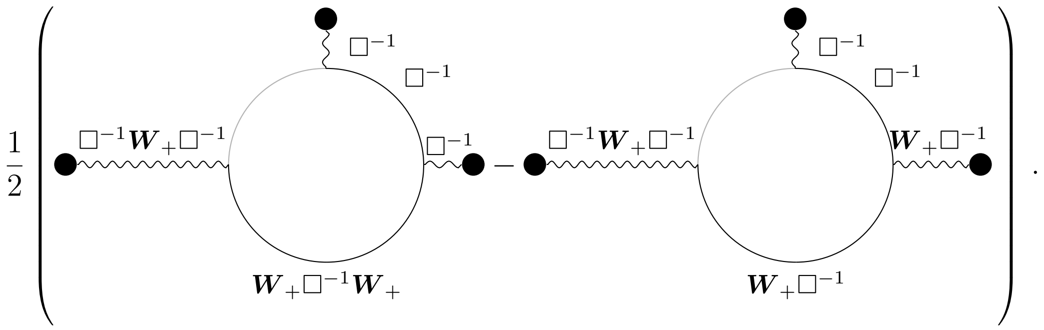

. (7.45)

Going through the same procedure as for the other topologies one finds that the only diagrams, in which new vertices could contribute, are




(7.46)

After reduction to our basis the dangerous diagrams take the form




(7.47)

These clearly cancel.

Let us reiterate that we do not know any $\mathcal{N} = 1$ superconformal gauge theory, in which this topology could contribute.

# Chapter 8

# Class $\mathcal{S}_k$

It turns out that the issue of extending our calculation to the whole $SU(2,1|1)$ sector is subtle. For the argument outlined in section 1 it is necessary to keep manifest gauge invariance. This can be achieved by using the manifestly covariant formalism from section 2.4. However since this formalism relies on shifting the external fields to a single vertex it obscures the external structure of divergent subgraphs thereby making it difficult idenfify the corresponding counterterms. This issue is as of yet unresolved. As a first step towards the resolution of this problem in this section we present partial results for a particular set of theories, called class $\mathcal{S}_k$, which were introduced in [91] and further studied in [92–100]. These allow for a simplified version of our argument. At least up to three loops we can formulate the argument without referring to the cancellation of new vertices. In principle we could even use a non-covariant formalism, however as argued in section 2 this leads to a stark increase in the amount of diagrams that have to be computed. In the following sections we will present those features of theories of class $\mathcal{S}_k$ that are important for our computation and than give an overview over the next steps of the calculation.

## 8.1 Theories of class $\mathcal{S}_k$

These theories were introduced in [91]. In general they are given by twisted compactifications of six-dimensional $(1,0)$ SCFTs on a Riemann surface. In this section we are only interested in the subclass of these theories, which admit a Lagrangian description. In particular we consider the set of quiver gauge theories obtained as $\mathbb{Z}_k$ orbifolds of elliptic $\mathcal{N}=2$ quivers. They are conformal by inheritance arguments [20, 21, 23]. They are illustrated by the diagram in figure 8.1. Each blob stands for an $SU(N)$ gauge group with a vector superfield $V_{(i,c)}$ and the field representations under these gauge groups are given in table 8.1.

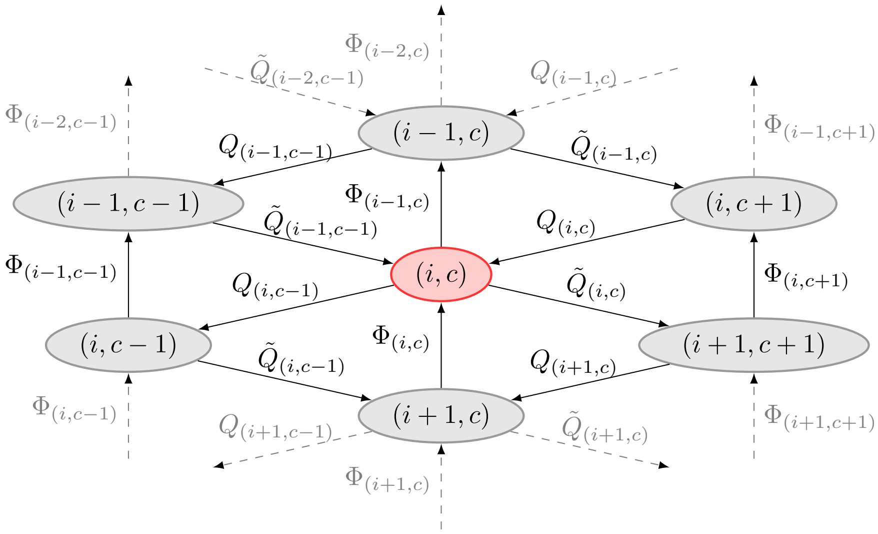



**Figure 8.1:** Quiver diagram for a theory of class $\mathcal{S}_k$ with $i \in \{1, \dots, k\}$ and $c \in \{0, \dots, M\}$. The nodes stand for gauge groups and their corresponding vector multiplets and lines stand for chiral fields in the bifundamental representation of the gauge groups they connect. We consider the sector made out of fields from the vectormultiplet corresponding to the gauge group $SU(N)_{(i,c)}$ denoted by the red blob. The figure is adapted from [94].

The superpotential is given by

$$\mathcal{L}_W = \sum_{i=1}^{k}\sum_{c=1}^{M} \mathrm{tr}\left(g_{(i,c)} Q_{(i,c-1)} \Phi_{(i,c)} \tilde{Q}_{(i,c-1)} - g_{(i+1,c)} \tilde{Q}_{(i,c)} \Phi_{(i,c)} Q_{(i+1,c)}\right) . \tag{8.1}$$

In this sum cyclicity of the indices $i$ and $c$ is understood $(k+1, c) = (1, c)$, $(i, 0) = (i, c)$. Notice that this can simply be read from the diagrams by following the triangular paths. The sign is determined by the orientation of the path. In this section we will use the rescaled fields $V_{(i,c)} \to g_{(i,c)} V_{(i,c)}$ in order to make the coupling dependence explicit and to get the conventional prefactors for the vertices. The gauge matter interactions are then given by the Kähler potential

$$\begin{aligned}\mathcal{L}_K = \sum_{i=1}^{k}\sum_{c=1}^{M} \mathrm{tr}\Big( & e^{-g_{(i+1,c)}V_{(i+1,c)}} \bar{\Phi}_{(i,c)} e^{g_{(i,c)}V_{(i,c)}} \Phi_{(i,c)} \\ & + e^{-g_{(i,c)}V_{(i,c)}} \bar{Q}_{(i,c-1)} e^{g_{(i,c-1)}V_{(i,c-1)}} Q_{(i,c-1)} \\ & + e^{-g_{(i,c-1)}V_{(i,c-1)}} \bar{\tilde{Q}}_{(i,c-1)} e^{g_{(i+1,c)}V_{(i+1,c)}} \tilde{Q}_{(i,c-1)} \Big)\end{aligned} \tag{8.2}$$

and together with

$$\mathcal{L}_{\text{gauge}} = \sum_{i=1}^{k}\sum_{c=1}^{M} W^{\alpha}_{(i,c)} W_{\alpha(i,c)} \tag{8.3}$$

| | $SU(N)_{(i,c-1)}$ | $SU(N)_{(i,c)}$ | $SU(N)_{(i+1,c)}$ |
|---|---|---|---|
| $V_{(i,c)}$ | $\mathbf{1}$ | adj. | $\mathbf{1}$ |
| $\Phi_{(i,c)}$ | $\mathbf{1}$ | $\square$ | $\overline{\square}$ |
| $Q_{(i,c-1)}$ | $\square$ | $\overline{\square}$ | $\mathbf{1}$ |
| $\tilde{Q}_{(i,c-1)}$ | $\overline{\square}$ | $\mathbf{1}$ | $\square$ |

**Table 8.1:** Representations of the fields under the various gauge groups. It is understood that $SU(N)_{(k+1,c)} = SU(N)_{(1,c)}$. The table is adapted from [94].

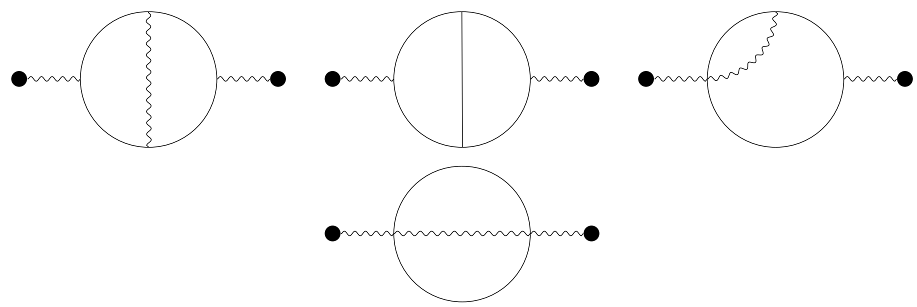


**Figure 8.2:** Topoplogies that contribute to the difference between theories of class $\mathcal{S}_k$ and $\mathcal{N} = 4$ SYM.

the complete classical Lagrangian is given by

$$\mathcal{L}_{\mathcal{S}_k} = \mathcal{L}_{\text{gauge}} + \mathcal{L}_K + \mathcal{L}_W \, . \tag{8.4}$$

We will again make use of the difference method, only looking at diagrams that can potentially be different from $\mathcal{N} = 4$ SYM. As observed in [18, 90] for the case of $\mathcal{N} = 2$ theories the only differences come from the presence of other gauge couplings. Say we consider the $SU(2,1|1)$ sector that is made out of fields from the vector multiplet of the gauge group $SU(N)_{(i,c)}$, then the only fields that can contribute to diagrams up to three loops are the solid ones in figure 8.1. The shaded lines start contributing at higher orders. As can be easily seen from the interaction terms in eqs. (8.1) and (8.2) the diagrams that lead to differences either contain at least of the six neighboring vector superfields $V_{(i-1,c-1)}, V_{(i,c-1)}, V_{(i+1,c)}, V_{(i+1,c+1)}, V_{(i,c+1)}$ or $V_{(i-1,c)}$ or at least one of the four chiral fields $\Phi_{(i-1,c)}, \Phi_{(i,c)}, \Phi_{(i-1,c-1)}$ or $\Phi_{(i,c+1)}$. All of these diagrams have the form of topologies 1, 2 and the simplified version of 3 in section 7. These are shown in figure 8.2. In particular there are no contributions from one loop self-energy corrections of the chiral fields inside the loop as in the original version of topology 3, because these cancel. As an illustration consider all the diagrams that have the field $\Phi_{(i,c)}$ at the left vertex. These are listed in figure 8.3. Note that all of them come with a prefactor $g^2_{(i,c)} g^2_{(i+1,c)}$ except for the last one, which has a prefactor of $g^2_{(i,c)} g^2_{(i,c)}$. This prefactor is identical to $\mathcal{N} = 4$ SYM and thus this diagram will not contribute

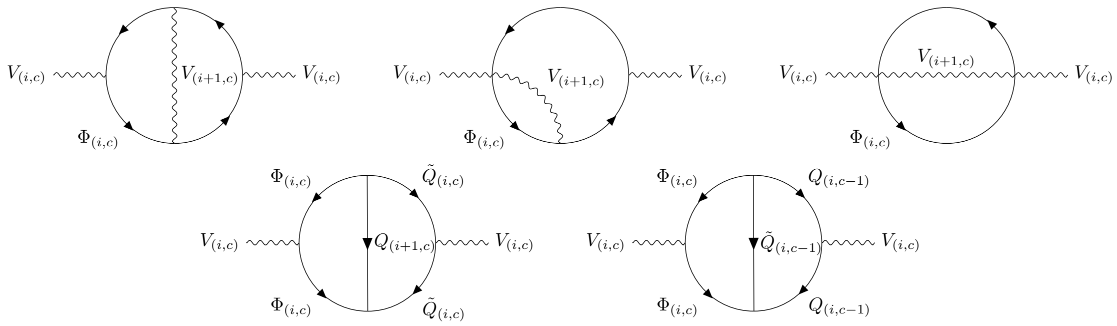
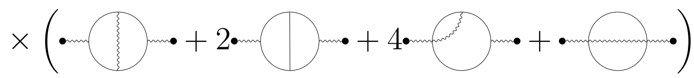


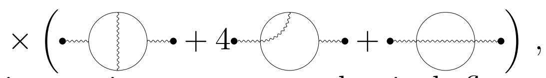

**Figure 8.3:** All two loop subgraphs, where the chiral field that connects to the left vertex is $\Phi_{(i,c)}$. The first four topologies have a prefactor $g^2_{(i,c)}g^2_{(i+1,c)}$, while the last one has a prefactor of $g^2_{(i,c)}g^2_{(i,c)}$.

to the difference. Analogous computations work for all the other chiral fields. After collecting the diagrams according to the coupling constants one finds for the difference

$$\mathcal{Z}^{(3)}\Big|_{\text{diff}} = g^4_{(i,c)}\left(4g^2_{(i,c)} - g^2_{(i+1,c)} - g^2_{(i-1,c)} - g^2_{(i,c+1)} - g^2_{(i,c-1)}\right)$$



$$+\, g^4_{(i,c)}\left(2g^2_{(i,c)} - g^2_{(i-1,c-1)} - g^2_{(i+1,c+1)}\right)$$



(8.5)

where the diagrams in this equation are meant to be single flavor. We can now collect the results from sections 7.1.2, 7.2.2 and 7.3.1. In total we find

$$\begin{aligned}\mathcal{D}^{(3)}_{\text{diff}} = &g^4_{(i,c)}\left(4g^2_{(i,c)} - g^2_{(i+1,c)} - g^2_{(i-1,c)} - g^2_{(i,c+1)} - g^2_{(i,c-1)}\right)\\ &\times\left(\mathcal{D}^{(3)}_{\text{diff,Top1}} + 2\mathcal{D}^{(3)}_{\text{diff,Top2}} + 4\mathcal{D}^{(3)}_{\text{diff,Top3}} + \mathcal{D}^{(3)}_{\text{diff,Top4}}\right)\\ &+ g^4_{(i,c)}\left(2g^2_{(i,c)} - g^2_{(i-1,c-1)} - g^2_{(i+1,c+1)}\right)\\ &\times\left(\mathcal{D}^{(3)}_{\text{diff,Top1}} + 4\mathcal{D}^{(3)}_{\text{diff,Top3}} + \mathcal{D}^{(3)}_{\text{diff,Top4}}\right)\,,\end{aligned} \tag{8.6}$$

which upon defining

$$\begin{aligned}A &= -2g^2_{(i,c)} + g^2_{(i+1,c)} + g^2_{(i-1,c)} + g^2_{(i,c+1)} + g^2_{(i,c-1)} - g^2_{(i-1,c-1)} - g^2_{(i+1,c+1)}\,,\\ B &= 6g^2_{(i,c)} - g^2_{(i+1,c)} - g^2_{(i-1,c)} - g^2_{(i,c+1)} - g^2_{(i,c-1)} - g^2_{(i-1,c-1)} - g^2_{(i+1,c+1)}\,,\end{aligned} \tag{8.7}$$

takes the explicit form

$$\mathcal{D}^{(3)}_{\text{diff}}(W_+W_+) = g^4_{(i,c)}\left(12\zeta(3)B + \frac{9}{32}A\right)(\boldsymbol{W}_+\boldsymbol{W}_+)\,. \tag{8.8}$$

This can now related to the one loop result from eq. (5.52)

$$D^{(1)}_{\mathcal{N}=4}(g^2_{(i,c)})(W_+W_+) = 4g^2_{(i,c)}(\boldsymbol{W}_+\boldsymbol{W}_+)\,. \tag{8.9}$$

From here we find

$$D_{\mathcal{N}=1}(\boldsymbol{g}^2)(W_+W_+) = D_{\mathcal{N}=4}(f(\boldsymbol{g}^2))(\boldsymbol{W}_+\boldsymbol{W}_+) \tag{8.10}$$

with

$$f(\boldsymbol{g}^2) = g^2_{(i,c)} + g^4_{(i,c)}\left(A\frac{9}{128} + 3B\zeta(3)\right) . \tag{8.11}$$

Note that in contrast to the $\mathcal{N} = 2$ case there is a term with subleading transcendentality.

An important consistency check is that for $k = 1$ they should reduce to the result for an $\mathcal{N} = 2$ quiver. Setting $k = 1$ implies $g_{(i,c)} = g_{(j,c)} = g_c$. Indeed upon this substitution one finds

$$\begin{aligned} A &= 0 \\ B &= 2(2g_c^2 - g_{c+1}^2 - g_{c-1}^2) \end{aligned} \tag{8.12}$$

and thus

$$f(\boldsymbol{g})_{\mathcal{N}=2} = g_c^2 + 6g_c^2(2g_c^2 - g_{c+1}^2 - g_{c-1}^2) , \tag{8.13}$$

which precisely matches the result from [35].

Since we omitted diagrams that only contribute higher order poles in our calculations we have unfortunately deprived ourselves of the chance to cross check the cancellation of higher order poles. This should be remedied in future work.

## 8.2 Excitations

The argument in the preceeding section has a fairly straightforward generalization, when derivatives are added. The topologies that contribute will be precisely the same, only that there are extra derivatives acting on the propagators connected to the operator insertion. It follows from powercounting that more terms can contribute. Another feature is that similarly to the computations in section 5.2 the D-algebra produces terms, where the dimensions of the external fields are smaller than the dimensions of the fields of the operator participating in the interaction. In order to treat these in a covariant fashion one has to use the formalism from section 2.4. In order for this to be feasibly done at three loops, it has to be automated. The end result of this computation are logarithmically divergent vacuum graphs with vector indices. These are the central object of interest in [74], where they go by the name LTVGs or *logarithmic tensor vacuum graphs*. This paper also introduces a method how to properly extract their overall UV divergences, which is the quantity needed for the computation of the dilation operator. The UV divergences for some of the graphs that occur in our calculation are collected in appendix B. Since our calculation is not finished, neither is the list of counterterms.

As an illustration of what a result would look like, let us consider the action of the dilatation operator on $(\boldsymbol{\nabla}_{+\dot{\alpha}}\boldsymbol{W}_+)\boldsymbol{W}_+$. Since our calculation preserves manifest gauge invariance it follows that the only external field structures are $(\boldsymbol{\nabla}_{+\dot{\alpha}}\boldsymbol{W}_+)\boldsymbol{\nabla}_+$ and $\boldsymbol{\nabla}_+(\boldsymbol{W}_{+\dot{\alpha}}\boldsymbol{W}_+)$.

$$\begin{aligned} D^{(3)}_{\mathcal{N}=4}(g^2_{(i,c)})((\boldsymbol{\nabla}_{+\dot{\alpha}}\boldsymbol{W}_+)\boldsymbol{W}_+) - D^{(3)}_{\mathcal{N}=1}(\boldsymbol{g}^2)(\boldsymbol{\nabla}_{+\dot{\alpha}}\boldsymbol{W}_+)\boldsymbol{W}_+) \\ = h_1(\boldsymbol{g})(\boldsymbol{\nabla}_{+\dot{\alpha}}\boldsymbol{W}_+)\boldsymbol{W}_+) + h_2(\boldsymbol{g})(\boldsymbol{\nabla}_+(\boldsymbol{W}_{+\dot{\alpha}}\boldsymbol{W}_+))\,. \end{aligned} \tag{8.14}$$

Compare this to the one loop dilatation operator, which is inferred from eq. (5.30)

$$D^{(1)}_{\mathcal{N}=4}\left(g^2_{(i,c)}\right)((\boldsymbol{\nabla}_{+\dot{\alpha}}\boldsymbol{W}_+)\boldsymbol{W}_+) = g^2_{(i,c)}\left(((\boldsymbol{\nabla}_{+\dot{\alpha}}\boldsymbol{W}_+)\boldsymbol{W}_+) + 3(\boldsymbol{W}_+(\boldsymbol{\nabla}_{+\dot{\alpha}}\boldsymbol{W}_+))\right)\,. \tag{8.15}$$

In order for the difference argument to work the right-hand sides of these equations have to match up to the redefinition $g^2_{(i,c)} \to f(\boldsymbol{g}^{\mathbf{2}})$. This implies that

$$h_2(\boldsymbol{g}) = 3h_1(\boldsymbol{g}) = f(\boldsymbol{g})\,. \tag{8.16}$$

This is by no means trivial and would provide a very strong check of the argument. As mentioned above this calculation is currently work in progress.

Similar statements with checks along the same lines can be made for higher excitations. As mentioned in the introduction, it is enough to look at diagrams with up to two excitations, because this would establish equality of the S-matrix on the spin chain of the two theories up to the coupling redefinition. Since integrability in $\mathcal{N} = 4$ SYM implies a factorization of the S-matrix, higher excitations can be obtained from the Yang-Baxter equation.

# Chapter 9

# Extra protected states in $\mathcal{N} = 2$ SCQCD

In this section we describe the calculation of the new protected states. In order to do this we review some notions of superconformal representation theory, the superconformal index and the emergence of equivalence classes of short multiplets for the $\mathcal{N} = 2$ superconformal algebra. We follow the conventions of [24]. We will then briefly review the Sieve algorithm from [25] used to get the constraints on the quantum numbers of the new protected states and finally we describe our strategy to get their explicit forms.

## 9.1 Equivalence classes of short multiplets

Let us start by briefly discussing some properties of the representation theory of the $\mathcal{N} = 2$ superconformal algebra. For more details on the topics mentioned here the reader should consult the classic reference [24]. The bosonic symmetries of the $\mathcal{N} = 2$ superconformal group are the Lorentz group with spin indices $(j, \bar{j})$ and the so-called R-symmetry group $SU(2)_R \times U(1)_r$, for which we use the quantum numbers $R$ and $r$, respectively. A superconformal multiplet is formed by acting with the supercharges and the raising operator for the $SU(2)_R$ group on the superconformal primaries. A general such multiplet is denoted by $\mathcal{A}^{\Delta}_{R,r(j,\bar{j})}$, where $\Delta$ is the conformal dimension of the superconformal primary.

It is possible to put constraints on these quantum numbers by demanding that the action of some of the generators on the superconformal primary vanish. These are called shortening conditions and the resulting multiplets have fewer members than the general unconstrained $\mathcal{A}$ multiplet. They are denoted by $\mathcal{B}$, $\bar{\mathcal{B}}$, $\hat{\mathcal{B}}$, $\mathcal{C}$, $\bar{\mathcal{C}}$, $\hat{\mathcal{C}}$, $\mathcal{D}$, $\bar{\mathcal{D}}$, $\mathcal{E}$, $\bar{\mathcal{E}}$ depending on the specific shortening condition. Table 9.1 gives a complete overview over the multiplets and their shortening conditions. When constraints like these

| Shortening Conditions | | | | Multiplet |
|---|---|---|---|---|
| $\mathcal{B}_1$ | $\mathcal{Q}^1_\alpha\|R,r\rangle^{h.w.}=0$ | $j=0$ | $\Delta=2R+r$ | $\mathcal{B}_{R,r(0,\bar{j})}$ |
| $\bar{\mathcal{B}}_2$ | $\bar{\mathcal{Q}}_{2\dot{\alpha}}\|R,r\rangle^{h.w.}=0$ | $\bar{j}=0$ | $\Delta=2R-r$ | $\bar{\mathcal{B}}_{R,r(j,0)}$ |
| $\mathcal{E}$ | $\mathcal{B}_1\cap\mathcal{B}_2$ | $R=0$ | $\Delta=r$ | $\mathcal{E}_{r(0,\bar{j})}$ |
| $\bar{\mathcal{E}}$ | $\bar{\mathcal{B}}_1\cap\bar{\mathcal{B}}_2$ | $R=0$ | $\Delta=-r$ | $\bar{\mathcal{E}}_{r(j,0)}$ |
| $\hat{\mathcal{B}}$ | $\mathcal{B}_1\cap\bar{B}_2$ | $r=0,\ j,\bar{j}=0$ | $\Delta=2R$ | $\hat{\mathcal{B}}_R$ |
| $\mathcal{C}_1$ | $\epsilon^{\alpha\beta}\mathcal{Q}^1_\beta\|R,r\rangle^{h.w.}_\alpha=0$<br>$(\mathcal{Q}^1)^2\|R,r\rangle^{h.w.}=0$ for $j=0$ | | $\Delta=2+2j+2R+r$<br>$\Delta=2+2R+r$ | $\mathcal{C}_{R,r(j,\bar{j})}$<br>$\mathcal{C}_{R,r(0,\bar{j})}$ |
| $\bar{\mathcal{C}}_2$ | $\epsilon^{\dot{\alpha}\dot{\beta}}\bar{\mathcal{Q}}_{2\dot{\beta}}\|R,r\rangle^{h.w.}_{\dot{\alpha}}=0$<br>$(\bar{\mathcal{Q}}_2)^2\|R,r\rangle^{h.w.}=0$ for $\bar{j}=0$ | | $\Delta=2+2\bar{j}+2R-r$<br>$\Delta=2+2R-r$ | $\bar{\mathcal{C}}_{R,r(j,\bar{j})}$<br>$\bar{\mathcal{C}}_{R,r(j,0)}$ |
| $\mathcal{F}$ | $\mathcal{C}_1\cap\mathcal{C}_2$ | $R=0$ | $\Delta=2+2j+r$ | $\mathcal{C}_{0,r(j,\bar{j})}$ |
| $\bar{\mathcal{F}}$ | $\bar{\mathcal{C}}_1\cap\bar{\mathcal{C}}_2$ | $R=0$ | $\Delta=2+2\bar{j}-r$ | $\bar{\mathcal{C}}_{0,r(j,\bar{j})}$ |
| $\hat{\mathcal{C}}$ | $\mathcal{C}_1\cap\bar{\mathcal{C}}_2$ | $r=\bar{j}-j$ | $\Delta=2+2R+j+\bar{j}$ | $\hat{\mathcal{C}}_{R(j,\bar{j})}$ |
| $\hat{\mathcal{F}}$ | $\mathcal{C}_1\cap\mathcal{C}_2\cap\bar{\mathcal{C}}_1\cap\bar{\mathcal{C}}_2$ | $R=0,r=\bar{j}-j$ | $\Delta=2+j+\bar{j}$ | $\hat{\mathcal{C}}_{0(j,\bar{j})}$ |
| $\mathcal{D}$ | $\mathcal{B}_1\cap\bar{\mathcal{C}}_2$ | $r=\bar{j}+1$ | $\Delta=1+2R+\bar{j}$ | $\mathcal{D}_{R(0,\bar{j})}$ |
| $\bar{\mathcal{D}}$ | $\bar{\mathcal{B}}_2\cap\mathcal{C}_1$ | $-r=j+1$ | $\Delta=1+2R+j$ | $\bar{\mathcal{D}}_{R(j,0)}$ |
| $\mathcal{G}$ | $\mathcal{E}\cap\bar{\mathcal{C}}_2$ | $r=\bar{j}+1,R=0$ | $\Delta=r=1+\bar{j}$ | $\mathcal{D}_{0(0,\bar{j})}$ |
| $\bar{\mathcal{G}}$ | $\bar{\mathcal{E}}\cap\mathcal{C}_1$ | $-r=j+1,R=0$ | $\Delta=-r=1+j$ | $\bar{\mathcal{D}}_{0(j,0)}$ |

**Table 9.1:** Shortening conditions and short multiplets for the $\mathcal{N}=2$ superconformal algebra [24]. Table taken from [25].

are imposed on the general $\mathcal{A}^\Delta_{R,r(j,\bar{j})}$ multiplet it was shown in [24] that the long multiplet can be written as the sum of shorter multiplets. These relations are called recombination rules and the important ones for us are

$$\mathcal{A}^{2R+r+2j+2}_{R,r(j,\bar{j})}\simeq\mathcal{C}_{R,r(j,\bar{j})}\oplus\mathcal{C}_{R+\frac{1}{2},r+\frac{1}{2}(j-\frac{1}{2},\bar{j})}\,, \tag{9.1}$$

$$\mathcal{A}^{2R-r+2\bar{j}+2}_{R,r(j,\bar{j})}\simeq\bar{\mathcal{C}}_{R,r(j,\bar{j})}\oplus\bar{\mathcal{C}}_{R+\frac{1}{2},r-\frac{1}{2}(j,\bar{j}-\frac{1}{2})}\,, \tag{9.2}$$

$$\mathcal{A}^{2R+j+\bar{j}+2}_{R,j-\bar{j}(j,\bar{j})}\simeq\hat{\mathcal{C}}_{R(j,\bar{j})}\oplus\hat{\mathcal{C}}_{R+\frac{1}{2}(j-\frac{1}{2},\bar{j})}\oplus\hat{\mathcal{C}}_{R+\frac{1}{2}(j,\bar{j}-\frac{1}{2})}\oplus\hat{\mathcal{C}}_{R+1(j-\frac{1}{2},\bar{j}-\frac{1}{2})}\,. \tag{9.3}$$

The second ingredient for our argument is called the superconformal index introduced in [47], see also the reviews [48, 49]. We are sepcifically interested in the so-called *left superconformal index* which takes the form (see [25], eq.(5.3))

$$\mathcal{I}^L(t,y,v)=\mathrm{Tr}(-1)^F t^{2(\Delta+j)}y^{2\bar{j}}v^{r-R}\,, \tag{9.4}$$

where the trace is over the Hilbert space of the theory. An important property is that this object only receives contributions from states with

$$\Delta-2j-2R-r=0\,. \tag{9.5}$$

In particular this implies that the index vanishes on long multiplets. From eq. (9.1) one can then see that

$$\mathcal{I}^{\mathrm{L}}[\mathcal{C}_{R,r(j,\bar{j})}] + \mathcal{I}^{\mathrm{L}}[\mathcal{C}_{R+\frac{1}{2},r+\frac{1}{2}(j-\frac{1}{2},\bar{j})}] = 0\,. \tag{9.6}$$

One can see that the index introduces equivalence classes of short multiplets. These can be labeled by the invariants $\tilde{R} \equiv R + j$, $\tilde{r} \equiv r + j$, $\bar{j}$ and the overall sign. The corresponding class is denoted by $[\tilde{R}, \tilde{r}, \bar{j}]^{\mathrm{L}}_{+}$. A similar analysis can be done for the $\hat{\mathcal{C}}$ multiplets with the result that the invariant quantum numbers for the left index are $\hat{R} \equiv j + \bar{j}$, $\bar{j}$ and the overall sign. Moreover $r$ is fixed by $r = \bar{j} - j$. This equivalence class is denoted by $[\hat{R}, \bar{j}]^{\mathrm{L}}_{+}$. The explicit formulas for the indices of these equivalence classes are

$$\mathcal{I}^{\mathrm{L}}_{[\tilde{R},\tilde{r},\bar{j}]^{\mathrm{L}}_{+}} = (-1)^{2\bar{j}+1} t^{6+4\tilde{R}+2\tilde{r}} v^{-2+\tilde{r}-\tilde{R}} \frac{(1-t^2 v)(t-\frac{v}{y})(t-vy)}{(1-t^3 y)(1-\frac{t^3}{y})} (y^{2\bar{j}} + \ldots + y^{-2\bar{j}})\,, \tag{9.7}$$

$$\begin{aligned}\mathcal{I}^{\mathrm{L}}_{[\hat{R},\bar{j}]^{\mathrm{L}}_{+}} &= (-1)^{2\bar{j}} \frac{t^{6-2\bar{j}+4\hat{R}} v^{-1+2\bar{j}-\hat{R}}(1-t^2 v)}{(1-t^3 y)(1-t^3/y)} \\ &\quad \times \left(t(y^{2\bar{j}+1} + \ldots + y^{-(2\bar{j}+1)}) - v(y^{2\bar{j}} + \ldots + y^{-2\bar{j}})\right). \end{aligned} \tag{9.8}$$

## 9.2 Review of the Sieve algorithm

To find the quantum numbers of the extra protected states, we employ the Sieve algorithm as described in [25]. Let us restate the simplest example of this algorithm here. We begin by expanding the difference between the true and the naive index in $t$,

$$\mathcal{I}_{\mathrm{QCD}} - \mathcal{I}_{\mathrm{naive}} = -\frac{t^{13}}{v}\left(y + \frac{1}{y}\right) + \ldots \tag{9.9}$$

This discrepancy can be compared with the $t$-expansion of a general $\mathcal{C}$-multiplet $\left[\tilde{R}, \tilde{r}, \bar{j}\right]^{L}_{\pm}$,

$$\mathcal{I}^{L}_{\left[\tilde{R},\tilde{r},\bar{j}\right]} = -t^{6+4\tilde{R}+2\tilde{r}} v^{\tilde{r}-\tilde{R}} \left(y^{2\bar{j}} + \cdots + y^{-2\bar{j}}\right) + \ldots, \tag{9.10}$$

from which we find

$$\tilde{R} = \frac{3}{2}, \quad \tilde{r} = \frac{1}{2}, \quad \bar{j} = \frac{1}{2}. \tag{9.11}$$

Furthermore, using the definition of the left superconformal index from eq. (9.4), we can read off some of the quantum numbers of the lowest extra protected state, namely

$$\Delta + j = \frac{13}{2}, \quad \bar{j} = \frac{1}{2}, \quad r - R = -1. \tag{9.12}$$

Note that from the relation $\tilde{r} = \bar{j}$ we see that this is a $\hat{\mathcal{C}}$-multiplet.

After having determined the lowest state, we subtract its full index to find the next-lowest discrepancy,

$$\mathcal{I}_{\text{QCD}} - \mathcal{I}_{\text{naive}} - \mathcal{I}_{\hat{\mathcal{C}}\left[\frac{3}{2},\frac{1}{2}\right]^L_+} = -t^{18} + \ldots , \tag{9.13}$$

from which we extract the constraints

$$\Delta + j = \frac{18}{2}, \quad \bar{j} = 0, \quad r - R = 0 , \tag{9.14}$$

and the state has to be fermionic. As before, we determine the multiplet to be of type $\mathcal{C}$ with quantum numbers $[2, 2, 0]^L_+$. Continuing in this way, we can successively find the constraints for all extra protected states, with the lowest ones given as in [25]:

- $\hat{\mathcal{C}}$-multiplets: $\left[2, \frac{1}{2}\right]^L_+, [4, 1]^L_+, \left[4, \frac{3}{2}\right]^L_+, \ldots$
- $\mathcal{C}$-multiplets: $[2, 2, 0]^L_+, [2, 3, 0]^L_+, [2, 4, 0]^L_+, \ldots$

Note that the superconformal index does not uniquely fix the multiplet in which the extra proteced state appears. Indeed, the index vanishes on long multiplets. Therefore, the sum of the superconformal indices which can recombine into a long multiplet also vanishes. The index therefore fixes the extra protected multiplet only up to this subtletly. However, so far, all states we obtained were immediately contained in the multiplet the index suggests.

## 9.3 Results

In this section we describe the procedure for the determination of the extra states for the first state before listing our current results. As reviewed above, the quantum numbers of the first extra protected state have to obey the constraints

$$\Delta + j = \frac{13}{2}, \quad \bar{j} = \frac{1}{2}, \quad r - R = -1 , \tag{9.15}$$

and the state has to be fermionic. It immediately follows from these constraints that the length of the operator is bounded from above by $L = 6$, since the lowest possible dimension of a field is that of a scalar with dimension 1. The simplest way of finding the wanted state then is to write down all possible operators of a given length consistent with eq. (9.15) and act with the spin-chain Hamiltonian of [50] on an arbitrary linear combination of these states. To simplify this procedure, we start from a handful of states and act with the Hamiltonian repeatedly on the list of so-generated operators until no new states are created. While this does not guarantee that we find the full basis of states, it will certainly generate a closed subsector for us.

| | Constraints | | | | Extra state | | | | | Highest weight | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | $\Delta + j$ | $\bar{j}$ | r-R | Length | $\Delta$ | $j$ | $\bar{j}$ | $R$ | $r$ | $\Delta$ | $j$ | $\bar{j}$ | $R$ | $r$ | $R_{(j,\bar{j})}$ |
| $\hat{\mathcal{C}}\left[2,\frac{1}{2}\right]$ | $\frac{13}{2}$ | $\frac{1}{2}$ | $-1$ | 4 | $\frac{11}{2}$ | 1 | $\frac{1}{2}$ | $\frac{3}{2}$ | $\frac{1}{2}$ | $\frac{10}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | 1 | 0 | $1_{(\frac{1}{2},\frac{1}{2})}$ |
| $\hat{\mathcal{C}}\left[4,\frac{3}{2}\right]$ | $\frac{19}{2}$ | $\frac{3}{2}$ | $-1$ | 4 | $\frac{15}{2}$ | $\frac{4}{2}$ | $\frac{3}{2}$ | $\frac{3}{2}$ | $\frac{1}{2}$ | $\frac{14}{2}$ | $\frac{3}{2}$ | $\frac{3}{2}$ | 1 | 0 | $1_{(\frac{3}{2},\frac{3}{2})}$ |
| $\hat{\mathcal{C}}\left[4,1\right]$ | $\frac{20}{2}$ | 1 | $-2$ | 6 | $\frac{17}{2}$ | $\frac{3}{2}$ | $\frac{2}{2}$ | $\frac{5}{2}$ | $\frac{1}{2}$ | $\frac{16}{2}$ | $\frac{2}{2}$ | $\frac{2}{2}$ | 2 | 0 | $2_{(1,1)}$ |
| $\hat{\mathcal{C}}\left[6,\frac{5}{2}\right]$ | $\frac{25}{2}$ | $\frac{5}{2}$ | $-1$ | 4 | $\frac{19}{2}$ | $\frac{6}{2}$ | $\frac{5}{2}$ | $\frac{3}{2}$ | $\frac{1}{2}$ | $\frac{18}{2}$ | $\frac{5}{2}$ | $\frac{5}{2}$ | 1 | 0 | $1_{(\frac{5}{2},\frac{5}{2})}$ |
| $\hat{\mathcal{C}}\left[6,2\right]$ | $\frac{26}{2}$ | 2 | $-2$ | 6 | | | | | | $\frac{20}{2}$ | 2 | 2 | 2 | 0 | $2_{(2,2)}$ |
| $\hat{\mathcal{C}}\left[6,\frac{3}{2}\right]$ | $\frac{27}{2}$ | $\frac{3}{2}$ | $-3$ | 8 | | | | | | $\frac{22}{2}$ | $\frac{3}{2}$ | $\frac{3}{2}$ | 3 | 0 | $3_{(\frac{3}{2},\frac{3}{2})}$ |
| $\hat{\mathcal{C}}\left[8,\frac{7}{2}\right]$ | $\frac{31}{2}$ | $\frac{7}{2}$ | $-1$ | 4 | | | | | | $\frac{22}{2}$ | $\frac{7}{2}$ | $\frac{7}{2}$ | 1 | 0 | $1_{(\frac{7}{2},\frac{7}{2})}$ |
| $\mathcal{C}\left[2,2,0\right]$ | $\frac{18}{2}$ | 0 | 0 | 6 | $\frac{15}{2}$ | $\frac{3}{2}$ | 0 | $\frac{3}{2}$ | $\frac{3}{2}$ | $\frac{14}{2}$ | 1 | 0 | $\frac{2}{2}$ | 1 | $1_{(1,0)}$ |
| $\mathcal{C}\left[2,3,0\right]$ | $\frac{20}{2}$ | 0 | $+1$ | 7 | $\frac{17}{2}$ | $\frac{3}{2}$ | 0 | $\frac{3}{2}$ | $\frac{5}{2}$ | $\frac{16}{2}$ | 1 | 0 | 1 | 2 | $1_{(1,0)}$ |
| $\mathcal{C}\left[2,4,0\right]$ | $\frac{22}{2}$ | 0 | $+2$ | 8 | $\frac{19}{2}$ | $\frac{3}{2}$ | 0 | $\frac{3}{2}$ | $\frac{7}{2}$ | $\frac{18}{2}$ | 1 | 0 | 1 | 3 | $1_{(1,0)}$ |
| $\mathcal{C}\left[2,5,0\right]$ | $\frac{24}{2}$ | 0 | $+3$ | 9 | $\frac{21}{2}$ | $\frac{3}{2}$ | 0 | $\frac{3}{2}$ | $\frac{9}{2}$ | $\frac{20}{2}$ | 1 | 0 | 1 | 4 | $1_{(1,0)}$ |

**Table 9.2:** Table of currently known extra protected states.

For the constraints listed in eq. (9.15) this procedure successfully finds the wanted state at length $L = 4$ with quantum numbers

$$\Delta = \frac{11}{2}, \quad j = 1, \quad \bar{j} = \frac{1}{2}, \quad R = \frac{3}{2}, \quad r = \frac{1}{2} \tag{9.16}$$

and the state is fermionic. The explicit form is quite ugly and will be listed at the end of this section. To learn more about the multiplet in which this operator lives, we determine the highest weight by acting with the conformal supercharges $\mathcal{S}, \bar{\mathcal{S}}$ and find a state with quantum numbers

$$\Delta = \frac{10}{2}, \quad j = \frac{1}{2}, \quad \bar{j} = \frac{1}{2}, \quad R = 1, \quad r = 0. \tag{9.17}$$

By acting on the highest weight state with the appropriate combination of supercharges $\mathcal{Q}, \bar{\mathcal{Q}}$ we have established that it satisfies the appropriate semi-shortening conditions for a $\hat{\mathcal{C}}$ multiplet.

We repeat the procedure explained above to determine more of the extra protected states. We list the results in table 9.2. We write out the quantum numbers only for the states that we have explicitly determined (i.e. we have only determined the highest weight states for some of the higher extra multiplets). We see some nice patterns emerging. Especially the structure for the $\hat{\mathcal{C}}$-multiplets seems to be clear. It is apparently given by $1_{(n/2,n/2)}$ for odd $n$.

Finally let us give the explicit forms of the lowest new protected state and the corresponding highest weight state. They come from the multiplet $1_{(\frac{1}{2},\frac{1}{2})}$ and the former takes the form

$$
\begin{aligned}
&\mathcal{O}_{1_{(\frac{1}{2},\frac{1}{2})}} \\
=&-\frac{1}{2}\mathrm{tr}\,(\bar\phi(D_{++\dot+}\lambda_{+1})Q_1\bar Q_1)+\frac{1}{2}\mathrm{tr}\,((D_{++\dot+}\lambda_{+1})\bar\phi Q_1\bar Q_1)-\frac{7}{2}\mathrm{tr}\,((D_{++\dot+}\bar Q_1)\bar\phi\lambda_{+1}Q_1)\\
&+\frac{7}{2}\mathrm{tr}\,(Q_1(D_{++\dot+}\bar Q_1)\lambda_{+1}\bar\phi)-\frac{7}{2}\mathrm{tr}\,(\bar\phi(D_{++\dot+}Q_1)\bar Q_1\lambda_{+1})-\frac{7}{2}\mathrm{tr}\,(\bar\phi\lambda_{+1}(D_{++\dot+}Q_1)\bar Q_1)\\
&-2\mathrm{tr}\,(\bar\phi\lambda_{+1}\lambda_{+1}\bar\lambda_{\dot+1})+2\mathrm{tr}\,(\bar\lambda_{\dot+1}\lambda_{+1}\lambda_{+1}\bar\phi)-\mathrm{tr}\,((D_{++\dot+}\bar\phi)Q_1\bar Q_1\lambda_{+1})\\
&+\mathrm{tr}\,((D_{++\dot+}\bar\phi)\lambda_{+1}Q_1\bar Q_1)+6\mathrm{tr}\,((D_{++\dot+}\bar Q_1)\psi_+\bar Q_1Q_1)-\frac{7}{2}\mathrm{tr}\,(\psi_+(D_{++\dot+}\bar Q_1)Q_1\bar Q_1)\\
&-\frac{13}{2}\mathrm{tr}\,(\tilde\psi_+Q_1(D_{++\dot+}\bar Q_1)Q_1)+2\mathrm{tr}\,(Q_1(D_{++\dot+}\tilde\psi_+)Q_1\bar Q_1)-\frac{7}{2}\mathrm{tr}\,((D_{++\dot+}Q_1)\tilde\psi_+Q_1\bar Q_1)\\
&+6\mathrm{tr}\,(Q_1\tilde\psi_+(D_{++\dot+}Q_1)\bar Q_1)-\frac{13}{2}\mathrm{tr}\,(\psi_+\bar Q_1(D_{++\dot+}Q_1)\bar Q_1)+2\mathrm{tr}\,((D_{++\dot+}\psi_+)\bar Q_1Q_1\bar Q_1)\\
&-\frac{1}{2}\mathrm{tr}\,(F_{++}\bar\lambda_{\dot+1}Q_1\bar Q_1)+\frac{1}{2}\mathrm{tr}\,(\bar\lambda_{\dot+1}F_{++}Q_1\bar Q_1)-\frac{7}{2}\mathrm{tr}\,(\tilde\psi_+\bar\lambda_{\dot+1}\lambda_{+1}Q_1)\\
&+\frac{7}{2}\mathrm{tr}\,(Q_1\tilde\psi_+\lambda_{+1}\bar\lambda_{\dot+1})+\frac{7}{2}\mathrm{tr}\,(\bar\lambda_{\dot+1}\lambda_{+1}\psi_+\bar Q_1)+\frac{7}{2}\mathrm{tr}\,(\bar\lambda_{\dot+1}\psi_+\bar Q_1\lambda_{+1})\,,
\end{aligned}
\tag{9.18}
$$

while the latter is given by

$$
\begin{aligned}
&\mathcal{O}_{\mathrm{h.w.}1_{(\frac{1}{2},\frac{1}{2})}} \\
=&-\frac{7}{2}\mathrm{tr}\,(\bar\phi\phi Q_1(D_{++\dot+}\bar Q_1))-\frac{7}{2}\mathrm{tr}\,(\phi\bar\phi Q_1(D_{++\dot+}\bar Q_1))+\frac{7}{2}\mathrm{tr}\,(\bar\phi\phi(D_{++\dot+}Q_1)\bar Q_1)\\
&+\frac{7}{2}\mathrm{tr}\,(\phi\bar\phi(D_{++\dot+}Q_1)\bar Q_1)+\mathrm{tr}\,(\bar\phi(D_{++\dot+}\phi)Q_1\bar Q_1)-\mathrm{tr}\,((D_{++\dot+}\phi)\bar\phi Q_1\bar Q_1)\\
&+2\mathrm{tr}\,(\lambda_{+1}\phi\bar\phi\bar\lambda_{\dot+1})-2\mathrm{tr}\,(\lambda_{+1}\phi\bar\lambda_{\dot+1}\bar\phi)-2\mathrm{tr}\,(\phi\lambda_{+1}\bar\phi\bar\lambda_{\dot+1})\\
&+2\mathrm{tr}\,(\phi\lambda_{+1}\bar\lambda_{\dot+1}\bar\phi)+\frac{7}{2}\mathrm{tr}\,(\bar\phi\lambda_{+1}\bar{\tilde\psi}_{\dot+}\bar Q_1)+\frac{7}{2}\mathrm{tr}\,(\lambda_{+1}\bar\phi\bar{\tilde\psi}_{\dot+}\bar Q_1)\\
&-\frac{7}{2}\mathrm{tr}\,(\bar\phi\lambda_{+1}Q_1\bar\psi_{\dot+})-\frac{7}{2}\mathrm{tr}\,(\lambda_{+1}\bar\phi Q_1\bar\psi_{\dot+})-\mathrm{tr}\,((D_{++\dot+}\bar\phi)\phi Q_1\bar Q_1)\\
&+\mathrm{tr}\,(\phi(D_{++\dot+}\bar\phi)Q_1\bar Q_1)-6\mathrm{tr}\,((D_{++\dot+}\bar Q_1)Q_2\bar Q_1Q_1)+\frac{7}{2}\mathrm{tr}\,(Q_2(D_{++\dot+}\bar Q_1)Q_1\bar Q_1)\\
&+\frac{13}{2}\mathrm{tr}\,(\bar Q_2Q_1(D_{++\dot+}\bar Q_1)Q_1)-4\mathrm{tr}\,(Q_1(D_{++\dot+}\bar Q_2)Q_1\bar Q_1)+\frac{7}{2}\mathrm{tr}\,((D_{++\dot+}Q_1)\bar Q_2Q_1\bar Q_1)\\
&-6\mathrm{tr}\,(\bar Q_2(D_{++\dot+}Q_1)\bar Q_1,Q_1)+\frac{13}{2}\mathrm{tr}\,(\bar Q_1(D_{++\dot+}Q_1)\bar Q_1Q_2)-4\mathrm{tr}\,((D_{++\dot+}Q_2)\bar Q_1Q_1\bar Q_1)\\
&-\frac{7}{2}\mathrm{tr}\,(\bar\lambda_{\dot+1}\phi Q_1\tilde\psi_+)-\frac{7}{2}\mathrm{tr}\,(\phi\bar\lambda_{\dot+1}Q_1\tilde\psi_+)-\frac{7}{2}\mathrm{tr}\,(\bar{\tilde\psi}_{\dot+}\tilde\psi_+Q_1\bar Q_1)\\
&-6\mathrm{tr}\,(\tilde\psi_+\bar{\tilde\psi}_{\dot+}\bar Q_1Q_1)+\frac{13}{2}\mathrm{tr}\,(\bar Q_1\psi_+\bar Q_1\bar{\tilde\psi}_{\dot+})-\mathrm{tr}\,(\bar\lambda_{\dot+1}\lambda_{+2}Q_1\bar Q_1)\\
&-\mathrm{tr}\,(\lambda_{+2}\bar\lambda_{\dot+1}Q_1\bar Q_1)-\frac{7}{2}\mathrm{tr}\,(\bar\lambda_{\dot+1}\lambda_{+1}Q_2\bar Q_1)+\frac{7}{2}\mathrm{tr}\,(\lambda_{+1}\bar\lambda_{\dot+1}Q_2\bar Q_1)
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{7}{2}\mathrm{tr}\,(\bar{\lambda}_{\dot{+}1}\phi\psi_+\bar{Q}_1)+\frac{7}{2}\mathrm{tr}\,(\phi\bar{\lambda}_{\dot{+}1}\psi_+\bar{Q}_1)+\mathrm{tr}\,(\bar{\lambda}_{\dot{+}2}\lambda_{+1}Q_1\bar{Q}_1)\\
&+\mathrm{tr}\,(\lambda_{+1}\bar{\lambda}_{\dot{+}2}Q_1\bar{Q}_1)+6\mathrm{tr}\,(\bar{\psi}_{\dot{+}}\psi_+\bar{Q}_1Q_1)+\frac{7}{2}\mathrm{tr}\,(\psi_+\bar{\psi}_{\dot{+}}Q_1\bar{Q}_1)\\
&+\frac{7}{2}\mathrm{tr}\,(\bar{\lambda}_{\dot{+}1}\lambda_{+1}Q_1\bar{Q}_2)-\frac{7}{2}\mathrm{tr}\,(\lambda_{+1}\bar{\lambda}_{\dot{+}1}Q_1\bar{Q}_2)-\frac{13}{2}\mathrm{tr}\,(\bar{\psi}_{\dot{+}}Q_1\tilde{\psi}_+Q_1)\,.
\end{aligned}
\tag{9.19}
$$

All higher states have too many terms to write them down here. These forms are not very informative and it would be good to understand the origin of their structure.

# Chapter 10

# Conclusion and outlook

The broad motivation for our work is to investigate, how the better understood properties of highly symmetric and idealized theories can shed light on properties of less symmetric and more realistic theories. Our two subjects of focus were the dilatation operator in $\mathcal{N} = 1$ superconformal gauge theories and the protected spectrum of $\mathcal{N} = 2$ superconformal QCD.

First we extended the perturbative argument of [18] to the vacuum of the $SU(2,1|1)$ sector in $\mathcal{N} = 1$ superconformal gauge theories. We have shown up to three loops that the dilatation operator acting on the vacuum of this sector can be obtained by a coupling redefinition of the one from $\mathcal{N} = 4$ SYM. In doing so we made extensive use of the closedness of the sector, the planar limit, background gauge invariance and conformal symmetry. We explicitly calculated this redefinition for theories of class $\mathcal{S}_k$ and checked that it equals the result for the corresponding $\mathcal{N} = 2$ quiver theories for $k = 1$.

One urgent question for further work is, if this result generalizes to the whole sector. We presented the current status of this investigation. At least for three loops this calculation seems attainable. However any reasonable treatment of this requires the automation of the procedure in [46]. Adding only one excitation will already provide a strong non-trivial check, whether this result generalizes. Adding two derivatives would ultimately establish equality of the S-matrix in the two theories (up to the redefinition of the coupling constant) and thereby prove integrability.

Another line of investigation is, whether our argument can be generalized to an all-loop argument. Indeed this was one of the driving motivations for using the covariant formalism, because it considerably improves powercounting and might lead to general powercounting theorems, which historically have been essential for extracting all-loop arguments from supersymmetric perturbation theory.

The unexpected universality of the coupling redefinition in $\mathcal{N} = 2$ theories that was observed in [36] hints at a deeper physical significance. It poses the question, if the same holds in $\mathcal{N} = 1$ theories. It should be investigated, whether our coupling redefinition also holds for other observables like Wilson loops, the Bremsstrahlung function and entanglement entropy.

Finally it would be interesting to see, if a similar argument can also be devised for theories without supersymmetry, which would then be very close to QCD. One reason to be optimistic is the presence of integrability in scattering amplitudes in the Regge limit and one loop integrability of the dilatation operator [17]. A perturbative argument could obviously not rely on supersymmetry, however the covariant formalism in [46] also works for non-supersymmetric theories, providing some hope that this is possible.

Our second subject deals with the new protected states in $\mathcal{N} = 2$ SCQCD that were found in [25]. As a starting point we used the constraints on their quantum numbers derived there. In order to explicitly construct them, we made a general ansatz consistent with these restrictions and then found the states, that vanish under the action of the Hamiltonian from [50].

The appearance of these higher-spin protected states has some profound physical consequences that beg to be better understood. Via the AdS/CFT correspondence they imply that the low energy limit of the string dual of this theory contains higher-spin states. One step towards a more general understanding of this phenomenon is the recent discovery of new protected spin 2 operators in a broader class of theories. [101]. Also the spin chain picture of the dilatation operator in this theory seems to be much more intricate than in other theories, because these new protected states serve as candidates for possible vacua. The implications of this should be investigated.

On a more direct level the explicit forms of these states are quite intricate and it is as of yet unclear, if there is a different formulation, which makes the structure more tractable.

It is important to note that explicit multi loop calculations for dilatation operators in the past have almost universally been restricted to chiral multiplets, e.g. [75], whereas by necessity we consider the vector multiplet. These calculations tend to be harder, because the D-algebra is much more involved. In order to facilitate these calculations we were led to use background covariant supersymmetric Feynman rules from [44] and [46] and modern concepts from renormalization, developed in [74] and we have applied them in way that to our knowledge has not been done before. We hope that our exposition of these topics might inspire other people to learn these powerful tools and put them to good use.

# Appendix A

# Covariant derivatives and kinetic operators

## A.1 General properties

Derivatives obey the (graded) Leibniz rule:

$$(\partial_A XY) = (\partial_A X)Y + (-)^{|AX|}(\partial_A Y)\,, \tag{A.1}$$

where $(-)^{|AX|}$ is $-1$ if both $A$ and $X$ are anticommuting and $+1$ otherwise. This property is implemented in a clean way by considering the graded commutator

$$[A, B\} = AB - (-)^{|AB|}BA\,. \tag{A.2}$$

Derivatives are then considered as *acting via the graded commutator*:

$$(\partial_A X) \equiv [\partial_A, X\}\,. \tag{A.3}$$

Together with the following general rules for the commutator

$$\begin{aligned} [a, bc] &= [a, b]c + b[a, c] \\ &= \{a, b\}c - b\{a, c\}\,, \\ \{a, bc\} &= \{a, b\}c - b[a, c] \\ &= [a, b]c + b\{a, c\}\,, \end{aligned} \tag{A.4}$$

this consistently implements the graded Leibniz rule. We take (A.3) as the definition for how any kind of derivative acts in general.

In order to define background covariant derivatives we first introduce the antisymmetric symbol $C_{\alpha\beta}$ with the normalization

$$C_{\alpha\beta}C^{\gamma\delta} = \delta_\alpha^\gamma \delta_\beta^\delta - \delta_\beta^\gamma \delta_\alpha^\delta \,, \tag{A.5}$$

which implies

$$C_{\alpha\beta}C^{\alpha\beta} = 2 \,. \tag{A.6}$$

A specific representation is given by

$$\begin{aligned} C^{+-} &= C_{-+} = i \,, \\ C^{-+} &= C_{+-} = -i \,. \end{aligned} \tag{A.7}$$

Covariant derivatives fulfill

$$\{\boldsymbol{\nabla}_\alpha, \bar{\boldsymbol{\nabla}}_{\dot\alpha}\} = i \boldsymbol{\nabla}_{\alpha\dot\alpha} \,, \tag{A.8}$$

$$[\boldsymbol{\nabla}_\alpha, \boldsymbol{\nabla}_{\beta\dot\beta}] = C_{\alpha\beta} \bar{\boldsymbol{W}}_{\dot\beta}, \qquad [\bar{\boldsymbol{\nabla}}_{\dot\alpha}, \boldsymbol{\nabla}_{\beta\dot\beta}] = C_{\dot\alpha\dot\beta} \boldsymbol{W}_\beta \,, \tag{A.9}$$

$$\boldsymbol{\nabla}^\alpha \boldsymbol{W}_\alpha = -\bar{\boldsymbol{\nabla}}^{\dot\alpha} \bar{\boldsymbol{W}}_{\dot\alpha} \,. \tag{A.10}$$

Two of the most important relations that follows from these definitions are

$$\begin{aligned} \left[\boldsymbol{\nabla}_\alpha, \bar{\boldsymbol{\nabla}}^2\right] &= i\boldsymbol{W}_\alpha - i\boldsymbol{\nabla}_{\alpha\dot\alpha}\bar{\boldsymbol{\nabla}}^{\dot\alpha} \,, \\ \left[\bar{\boldsymbol{\nabla}}_{\dot\alpha}, \boldsymbol{\nabla}^2\right] &= i\bar{\boldsymbol{W}}_{\dot\alpha} - i\boldsymbol{\nabla}_{\alpha\dot\alpha}\boldsymbol{\nabla}^{\alpha} \,, \end{aligned} \tag{A.11}$$

which we use as an example of how to prove these identities. The honest way to prove it is to let these operators act on something (which we call $A$) using the explicit nested structure by which derivatives act. We start with

$$2\bar{\boldsymbol{\nabla}}^2\boldsymbol{\nabla}_\alpha(A) \equiv \left[\bar{\boldsymbol{\nabla}}^{\dot\alpha}, \left\{\bar{\boldsymbol{\nabla}}_{\dot\alpha}, [\boldsymbol{\nabla}_\alpha, A]\right\}\right]$$

Using the general commutator rules (A.4) we can disentangle these nested commutators:

$$\begin{aligned} &= [\bar{\boldsymbol{\nabla}}^{\dot\alpha}, \{\bar{\boldsymbol{\nabla}}_{\dot\alpha}, \boldsymbol{\nabla}_\alpha\}]A + 2\{\bar{\boldsymbol{\nabla}}_{\dot\alpha}, \boldsymbol{\nabla}_\alpha\}[\bar{\boldsymbol{\nabla}}^{\dot\alpha}, A] + \boldsymbol{\nabla}_\alpha\{\bar{\boldsymbol{\nabla}}^{\dot\alpha}, [\bar{\boldsymbol{\nabla}}_{\dot\alpha}, A]\} \\ &\quad - A[\bar{\boldsymbol{\nabla}}^{\dot\alpha}, \{\bar{\boldsymbol{\nabla}}_{\dot\alpha}, \boldsymbol{\nabla}_\alpha\}] - 2[\bar{\boldsymbol{\nabla}}^{\dot\alpha}, A]\{\bar{\boldsymbol{\nabla}}_{\dot\alpha}, \boldsymbol{\nabla}_\alpha\} - \{\bar{\boldsymbol{\nabla}}^{\dot\alpha}, [\bar{\boldsymbol{\nabla}}_{\dot\alpha}, A]\}\boldsymbol{\nabla}_\alpha \,. \end{aligned}$$

Now they can be put back together as commutators again:

$$\begin{aligned} &= [[\bar{\boldsymbol{\nabla}}^{\dot\alpha}, \{\bar{\boldsymbol{\nabla}}_{\dot\alpha}, \boldsymbol{\nabla}_\alpha\}], A] + 2[\{\bar{\boldsymbol{\nabla}}_{\dot\alpha}, \boldsymbol{\nabla}_\alpha\}, [\bar{\boldsymbol{\nabla}}^{\dot\alpha}, A]] + [\boldsymbol{\nabla}_\alpha, \{\bar{\boldsymbol{\nabla}}^{\dot\alpha}, [\bar{\boldsymbol{\nabla}}_{\dot\alpha}, A]\}] \\ &= -2i[\boldsymbol{W}_\alpha, A] + 2[\{\bar{\boldsymbol{\nabla}}_{\dot\alpha}, \boldsymbol{\nabla}_\alpha\}, [\bar{\boldsymbol{\nabla}}^{\dot\alpha}, A]] + [\boldsymbol{\nabla}_\alpha, \{\bar{\boldsymbol{\nabla}}^{\dot\alpha}, [\bar{\boldsymbol{\nabla}}_{\dot\alpha}, A]\}] \,. \end{aligned} \tag{A.12}$$

It follows that

$$\begin{aligned}[\boldsymbol{\nabla}_\alpha, \bar{\boldsymbol{\nabla}}^2](A) &= i[\boldsymbol{W}_\alpha, A] - i[\boldsymbol{\nabla}_{\alpha\dot{\alpha}}, [\bar{\boldsymbol{\nabla}}^{\dot{\alpha}}, A]] \\ &= i\boldsymbol{W}_\alpha(A) - i\boldsymbol{\nabla}_{\alpha\dot{\alpha}}\bar{\boldsymbol{\nabla}}^{\dot{\alpha}}(A)\,, \end{aligned} \tag{A.13}$$

which is what we wanted to prove. Now that we have gained trust in the method let us do the same calculation without the explicit nested structure of the derivatives.

$$\begin{aligned}\left[\boldsymbol{\nabla}_\alpha, \bar{\boldsymbol{\nabla}}^2\right] &\overset{(A.4)}{=} \frac{1}{2}\left(\left\{\boldsymbol{\nabla}_\alpha, \bar{\boldsymbol{\nabla}}^{\dot{\beta}}\right\}\bar{\boldsymbol{\nabla}}_{\dot{\beta}} - \bar{\boldsymbol{\nabla}}^{\dot{\beta}}\left\{\boldsymbol{\nabla}_\alpha, \bar{\boldsymbol{\nabla}}_{\dot{\beta}}\right\}\right) = -\frac{1}{2}\left\{\left\{\boldsymbol{\nabla}_\alpha, \bar{\boldsymbol{\nabla}}_{\dot{\beta}}\right\}, \bar{\boldsymbol{\nabla}}^{\dot{\beta}}\right\} \\ &= -\frac{1}{2}\left[\left\{\boldsymbol{\nabla}_\alpha, \bar{\boldsymbol{\nabla}}_{\dot{\beta}}\right\}, \bar{\boldsymbol{\nabla}}^{\dot{\beta}}\right] - \left\{\boldsymbol{\nabla}_\alpha, \bar{\boldsymbol{\nabla}}_{\dot{\beta}}\right\}\bar{\boldsymbol{\nabla}}^{\dot{\beta}} \\ &= i\boldsymbol{W}_\alpha - i\boldsymbol{\nabla}_{\alpha\dot{\alpha}}\bar{\boldsymbol{\nabla}}^{\dot{\alpha}}\,. \end{aligned} \tag{A.14}$$

We see that we can just use normal commutator rules and end up with the same result. This implies

$$\begin{aligned}\left[\boldsymbol{\nabla}^2, \bar{\boldsymbol{\nabla}}^2\right] &= \frac{1}{2}\left[\boldsymbol{\nabla}^\alpha, \left[\boldsymbol{\nabla}_\alpha, \bar{\boldsymbol{\nabla}}^2\right]\right] \overset{A.11}{=} \frac{1}{2}\left[\boldsymbol{\nabla}^\alpha, i\boldsymbol{W}_\alpha - i\boldsymbol{\nabla}_{\alpha\dot{\alpha}}\bar{\boldsymbol{\nabla}}^{\dot{\alpha}}\right] \\ &= \frac{i}{2}(\boldsymbol{\nabla}^\alpha\boldsymbol{W}_\alpha) - i\boldsymbol{W}_\alpha\boldsymbol{\nabla}^\alpha - \frac{i}{2}\underbrace{[\boldsymbol{\nabla}^\alpha, \boldsymbol{\nabla}_{\alpha\dot{\alpha}}]}_{-2\boldsymbol{W}_{\dot{\alpha}}}\bar{\boldsymbol{\nabla}}^{\dot{\alpha}} - \frac{i}{2}\boldsymbol{\nabla}_{\alpha\dot{\alpha}}\underbrace{\left[\boldsymbol{\nabla}^\alpha, \bar{\boldsymbol{\nabla}}^{\dot{\alpha}}\right]}_{i\boldsymbol{\nabla}^{\alpha\dot{\alpha}} - 2\bar{\boldsymbol{\nabla}}^{\dot{\alpha}}\boldsymbol{\nabla}^\alpha} \\ &= \Box + \frac{i}{2}(\boldsymbol{\nabla}^\alpha\boldsymbol{W}_\alpha) - i\boldsymbol{W}_\alpha\boldsymbol{\nabla}^\alpha + i\bar{\boldsymbol{W}}_{\dot{\alpha}}\bar{\boldsymbol{\nabla}}^{\dot{\alpha}} + i\boldsymbol{\nabla}_{\alpha\dot{\alpha}}\boldsymbol{\nabla}^\alpha\bar{\boldsymbol{\nabla}}^{\dot{\alpha}}\,. \end{aligned} \tag{A.15}$$

We also find

$$\begin{aligned}\left[\boldsymbol{\nabla}^2, \boldsymbol{\nabla}_{\alpha\dot{\alpha}}\right] &= \frac{1}{2}\left[\boldsymbol{\nabla}^\beta, [\boldsymbol{\nabla}_\beta, \boldsymbol{\nabla}_{\alpha\dot{\alpha}}]\right] = \frac{1}{2}\left[\boldsymbol{\nabla}_\alpha, \bar{\boldsymbol{W}}_{\dot{\alpha}}\right] \\ &= \frac{1}{2}\left((\boldsymbol{\nabla}_\alpha\bar{\boldsymbol{W}}_{\dot{\alpha}}) - 2\bar{\boldsymbol{W}}_{\dot{\alpha}}\boldsymbol{\nabla}_\alpha\right) \\ &= -\bar{\boldsymbol{W}}_{\dot{\alpha}}\boldsymbol{\nabla}_\alpha\,, \end{aligned} \tag{A.16}$$

and

$$\left[\bar{\boldsymbol{\nabla}}^2, \boldsymbol{\nabla}_{\alpha\dot{\alpha}}\right] = -\boldsymbol{W}_\alpha\bar{\boldsymbol{\nabla}}_{\dot{\alpha}}\,. \tag{A.17}$$

Another important result is

$$\begin{aligned}\left[\boldsymbol{\nabla}_{\alpha\dot{\alpha}}, \boldsymbol{\nabla}_{\beta\dot{\beta}}\right] &\overset{(A.8)}{=} -\frac{i}{2}\left([\boldsymbol{\nabla}_{\alpha\dot{\alpha}}, \{\boldsymbol{\nabla}_\beta, \bar{\boldsymbol{\nabla}}_{\dot{\beta}}\}] - \left[(\alpha\dot{\alpha}) \leftrightarrow (\beta\dot{\beta})\right]\right) \\ &= -\frac{i}{2}\left((\{\boldsymbol{\nabla}_\beta, [\boldsymbol{\nabla}_{\alpha\dot{\alpha}}, \bar{\boldsymbol{\nabla}}_{\dot{\beta}}]\} + \{\bar{\boldsymbol{\nabla}}_{\dot{\beta}}, [\boldsymbol{\nabla}_{\alpha\dot{\alpha}}, \boldsymbol{\nabla}_\beta]\}) - \left[(\alpha\dot{\alpha}) \leftrightarrow (\beta\dot{\beta})\right]\right) \\ &\overset{(A.9)}{=} -\frac{i}{2}\left((\{\boldsymbol{\nabla}_\beta, \boldsymbol{W}_\alpha\}C_{\dot{\alpha}\dot{\beta}} + \{\bar{\boldsymbol{\nabla}}_{\dot{\beta}}, \bar{\boldsymbol{W}}_{\dot{\alpha}}\}C_{\alpha\beta}) - \left[(\alpha\dot{\alpha}) \leftrightarrow (\beta\dot{\beta})\right]\right)\,. \end{aligned} \tag{A.18}$$

We thus found the explicit form of the field strength

$$\begin{aligned} \boldsymbol{F}_{\alpha\dot\alpha,\beta\dot\beta} &= i\left[\boldsymbol{\nabla}_{\alpha\dot\alpha},\boldsymbol{\nabla}_{\beta\dot\beta}\right] \\ &= \frac{1}{2}\left(\{\boldsymbol{\nabla}_{(\alpha},\boldsymbol{W}_{\beta)}\}C_{\dot\alpha\dot\beta}+\{\bar{\boldsymbol{\nabla}}_{(\dot\alpha},\bar{\boldsymbol{W}}_{\dot\beta)}\}C_{\alpha\beta}\right). \end{aligned} \tag{A.19}$$

In particular we have

$$[\boldsymbol{\nabla}_{+\dot\alpha},\boldsymbol{\nabla}_{+\dot\beta}] = -i\{\boldsymbol{\nabla}_+,\boldsymbol{W}_+\}\,C_{\dot\alpha\dot\beta}\,. \tag{A.20}$$

Letting two contracted derivatives act on $\boldsymbol{W}_+$ we find

$$\begin{aligned} C^{\dot\alpha\dot\beta}\left[\boldsymbol{\nabla}_{+\dot\alpha},\left[\boldsymbol{\nabla}_{+\dot\beta},\boldsymbol{W}_+\right]\right] &= C^{\dot\alpha\dot\beta}\left[\boldsymbol{\nabla}_{+\dot\alpha},\boldsymbol{\nabla}_{+\dot\beta}\boldsymbol{W}_+-\boldsymbol{W}_+\boldsymbol{\nabla}_{+\dot\beta}\right] \\ &= C^{\dot\alpha\dot\beta}\left(\left[\left[\boldsymbol{\nabla}_{+\dot\alpha},\boldsymbol{\nabla}_{+\dot\beta}\right],\boldsymbol{W}_+\right]+\left[\boldsymbol{\nabla}_{+\dot\beta},\left[\boldsymbol{\nabla}_{+\dot\alpha},\boldsymbol{W}_+\right]\right]\right) \\ &= C^{\dot\alpha\dot\beta}\left(\left[\left[\boldsymbol{\nabla}_{+\dot\alpha},\boldsymbol{\nabla}_{+\dot\beta}\right],\boldsymbol{W}_+\right]-\left[\boldsymbol{\nabla}_{+\dot\alpha},\left[\boldsymbol{\nabla}_{+\dot\beta},\boldsymbol{W}_+\right]\right]\right), \end{aligned} \tag{A.21}$$

which yields

$$C^{\dot\alpha\dot\beta}\left[\boldsymbol{\nabla}_{+\dot\alpha},\left[\boldsymbol{\nabla}_{+\dot\beta},\boldsymbol{W}_+\right]\right] = \frac{1}{2}C^{\dot\alpha\dot\beta}\left(\left[\left[\boldsymbol{\nabla}_{+\dot\alpha},\boldsymbol{\nabla}_{+\dot\beta}\right],\boldsymbol{W}_+\right]\right) \overset{(A.20)}{=} -i\left[\{\boldsymbol{\nabla}_+,\boldsymbol{W}_+\},\boldsymbol{W}_+\right]. \tag{A.22}$$

The covariant d'Alembertian is defined as

$$\Box = \frac{1}{2}\boldsymbol{\nabla}^{\alpha\dot\alpha}\boldsymbol{\nabla}_{\alpha\dot\alpha} \tag{A.23}$$

and we can compute its commutation relations with the derivatives

$$\begin{aligned} [\boldsymbol{\nabla}_\alpha,\Box] &= \frac{1}{2}\left(\boldsymbol{\nabla}^{\beta\dot\beta}\left[\boldsymbol{\nabla}_\alpha,\boldsymbol{\nabla}_{\beta\dot\beta}\right]+\left[\boldsymbol{\nabla}_\alpha,\boldsymbol{\nabla}_{\beta\dot\beta}\right]\boldsymbol{\nabla}^{\beta\dot\beta}\right) \\ &\overset{(A.9)}{=} \frac{1}{2}\left(\boldsymbol{\nabla}^{\beta\dot\beta}C_{\alpha\beta}\bar{\boldsymbol{W}}_{\dot\beta}+C_{\alpha\beta}\bar{\boldsymbol{W}}_{\dot\beta}\boldsymbol{\nabla}^{\beta\dot\beta}\right) \\ &= \frac{1}{2}\left(\boldsymbol{\nabla}_{\alpha\dot\alpha}\bar{\boldsymbol{W}}^{\dot\alpha}+\bar{\boldsymbol{W}}^{\dot\alpha}\boldsymbol{\nabla}_{\alpha\dot\alpha}\right) \end{aligned} \tag{A.24}$$

and similarly

$$\left[\bar{\boldsymbol{\nabla}}_{\dot\alpha},\Box\right] = \frac{1}{2}\left(\boldsymbol{\nabla}_{\alpha\dot\alpha}\boldsymbol{W}^\alpha+\boldsymbol{W}^\alpha\boldsymbol{\nabla}_{\alpha\dot\alpha}\right). \tag{A.25}$$

From eqs. (A.24) and (A.25) it follows that

$$\left[\boldsymbol{\nabla}^2,\Box\right] = \frac{1}{2}(\boldsymbol{\nabla}^\alpha\left[\boldsymbol{\nabla}_\alpha,\Box\right]+\left[\boldsymbol{\nabla}^\alpha,\Box\right]\boldsymbol{\nabla}_\alpha) = \frac{1}{2}\left[\boldsymbol{\nabla}^\alpha,\left[\boldsymbol{\nabla}_\alpha,\Box\right]\right]$$

$$
\begin{aligned}
&= \frac{1}{2}\{\boldsymbol{\nabla}^\alpha, [\boldsymbol{\nabla}_\alpha, \Box]\} - [\boldsymbol{\nabla}_\alpha, \Box]\,\boldsymbol{\nabla}^\alpha \\
&= \frac{1}{4}\left\{\boldsymbol{\nabla}^\alpha, \left[\boldsymbol{\nabla}_{\alpha\dot\alpha}, \bar{\boldsymbol{W}}^{\dot\alpha}\right]\right\} - \frac{1}{2}\bar{\boldsymbol{W}}^{\dot\alpha}\,[\boldsymbol{\nabla}^\alpha, \boldsymbol{\nabla}_{\alpha\dot\alpha}] - [\boldsymbol{\nabla}_\alpha, \Box]\,\boldsymbol{\nabla}^\alpha \\
&= \frac{1}{2}\left\{\bar{\boldsymbol{W}}^{\dot\alpha}, \bar{\boldsymbol{W}}_{\dot\alpha}\right\} - \left(\frac{1}{2}\left[\boldsymbol{\nabla}_{\alpha\dot\alpha}, \bar{\boldsymbol{W}}^{\dot\alpha}\right] + \bar{\boldsymbol{W}}^{\dot\alpha}\boldsymbol{\nabla}_{\alpha\dot\alpha}\right)\boldsymbol{\nabla}^\alpha\,,
\end{aligned}
\tag{A.26}
$$

and analogously

$$
\left[\bar{\boldsymbol{\nabla}}^2, \Box\right] = \frac{1}{2}\{\boldsymbol{W}^\alpha, \boldsymbol{W}_\alpha\} - \left(\frac{1}{2}[\boldsymbol{\nabla}_{\alpha\dot\alpha}, \boldsymbol{W}^\alpha] + \boldsymbol{W}^\alpha\boldsymbol{\nabla}_{\alpha\dot\alpha}\right)\bar{\boldsymbol{\nabla}}^{\dot\alpha}\,. \tag{A.27}
$$

Using $F_{\alpha\dot\alpha,\beta\dot\beta}$ from eq. (A.19) one can also show that

$$
[\boldsymbol{\nabla}_{+\dot\alpha}, \Box] = -\frac{i}{2}\left[\left(\boldsymbol{\nabla}^{\beta\dot\beta}F_{+\dot\alpha,\beta\dot\beta}\right) + 2F_{+\dot\alpha,\beta\dot\beta}\boldsymbol{\nabla}^{\beta\dot\beta}\right]\,. \tag{A.28}
$$

## A.2 The vector kinetic operator

Using the results of the previous section we can investigate the properties of the kinetic operator for the vector field

$$
\hat\Box = \Box - i\boldsymbol{W}^\alpha\boldsymbol{\nabla}_\alpha - i\bar{\boldsymbol{W}}^{\dot\alpha}\bar{\boldsymbol{\nabla}}_{\dot\alpha} \tag{A.29}
$$

and compute its commutation relations with the covariant derivatives

$$
\begin{aligned}
\left[\boldsymbol{\nabla}_\alpha, \hat\Box\right] &= [\boldsymbol{\nabla}_\alpha, \Box] - i\left[\boldsymbol{\nabla}_\alpha, \boldsymbol{W}^\beta\boldsymbol{\nabla}_\beta\right] - i\left[\boldsymbol{\nabla}_\alpha, \bar{\boldsymbol{W}}^{\dot\alpha}\bar{\boldsymbol{\nabla}}_{\dot\alpha}\right] \\
&= \frac{1}{2}\left[\boldsymbol{\nabla}_{\alpha\dot\alpha}, \bar{\boldsymbol{W}}^{\dot\alpha}\right] + \bar{\boldsymbol{W}}^{\dot\alpha}\boldsymbol{\nabla}_{\alpha\dot\alpha} - i\left\{\boldsymbol{\nabla}_\alpha, \boldsymbol{W}^\beta\right\}\boldsymbol{\nabla}_\beta + i\boldsymbol{W}^\beta\underbrace{\{\boldsymbol{\nabla}_\alpha, \boldsymbol{\nabla}_\beta\}}_{=0} \\
&\quad - i\underbrace{\left\{\boldsymbol{\nabla}_\alpha, \bar{\boldsymbol{W}}^{\dot\alpha}\right\}}_{=0}\bar{\boldsymbol{\nabla}}_{\dot\alpha} + i\bar{\boldsymbol{W}}^{\dot\alpha}\underbrace{\left\{\boldsymbol{\nabla}_\alpha, \bar{\boldsymbol{\nabla}}_{\dot\alpha}\right\}}_{=i\boldsymbol{\nabla}_{\alpha\dot\alpha}} \\
&= \frac{1}{2}\left[\boldsymbol{\nabla}_{\alpha\dot\alpha}, \bar{\boldsymbol{W}}^{\dot\alpha}\right] - i\left\{\boldsymbol{\nabla}_\alpha, \boldsymbol{W}^\beta\right\}\boldsymbol{\nabla}_\beta \\
&= \frac{1}{2}(\boldsymbol{\nabla}_{\alpha\dot\alpha}\bar{\boldsymbol{W}}^{\dot\alpha}) - i(\boldsymbol{\nabla}_\alpha\boldsymbol{W}^\beta)\boldsymbol{\nabla}_\beta
\end{aligned}
\tag{A.30}
$$

and analogously

$$
\left[\bar{\boldsymbol{\nabla}}_{\dot\alpha}, \hat\Box\right] = \frac{1}{2}(\boldsymbol{\nabla}_{\alpha\dot\alpha}\boldsymbol{W}^\alpha) - i(\bar{\boldsymbol{\nabla}}_{\dot\alpha}\bar{\boldsymbol{W}}^{\dot\beta})\bar{\boldsymbol{\nabla}}_{\dot\beta}\,. \tag{A.31}
$$

From here one can show

$$
\left[\boldsymbol{\nabla}^2, \hat\Box\right] = \frac{1}{2}\left(\boldsymbol{\nabla}^\alpha\left[\boldsymbol{\nabla}_\alpha, \hat\Box\right] + \left[\boldsymbol{\nabla}^\alpha, \hat\Box\right]\boldsymbol{\nabla}_\alpha\right) = \frac{1}{2}\left[\boldsymbol{\nabla}^\alpha, \left[\boldsymbol{\nabla}_\alpha, \hat\Box\right]\right]
$$

$$
\begin{aligned}
&= \frac{1}{4}\underbrace{\left\{\boldsymbol{\nabla}^{\alpha},\left[\boldsymbol{\nabla}_{\alpha\dot{\alpha}},\bar{\boldsymbol{W}}^{\dot{\alpha}}\right]\right\}}_{=0} - \frac{i}{2}\left(\underbrace{\left[\boldsymbol{\nabla}^{\alpha},\left\{\boldsymbol{\nabla}_{\alpha},\boldsymbol{W}^{\beta}\right\}\right]}_{=2\boldsymbol{\nabla}^2\boldsymbol{W}^{\beta}}\boldsymbol{\nabla}_{\beta} + \left\{\boldsymbol{\nabla}_{\alpha},\boldsymbol{W}^{\beta}\right\}\underbrace{\left\{\boldsymbol{\nabla}^{\alpha},\boldsymbol{\nabla}_{\beta}\right\}}_{=0}\right) \\
&\quad - \frac{1}{2}\underbrace{\left[\boldsymbol{\nabla}_{\alpha\dot{\alpha}},\bar{\boldsymbol{W}}^{\dot{\alpha}}\right]}_{=i(\boldsymbol{\nabla}^2\boldsymbol{W}_{\alpha})}\boldsymbol{\nabla}^{\alpha} + i\left\{\boldsymbol{\nabla}_{\alpha},\boldsymbol{W}^{\beta}\right\}\underbrace{\boldsymbol{\nabla}_{\beta}\boldsymbol{\nabla}^{\alpha}}_{=-\delta^{\alpha}_{\beta}\boldsymbol{\nabla}^2} \\
&= -\frac{i}{2}(\boldsymbol{\nabla}^2\boldsymbol{W}^{\alpha})\boldsymbol{\nabla}_{\alpha} - i(\boldsymbol{\nabla}_{\alpha}\boldsymbol{W}^{\alpha})\boldsymbol{\nabla}^2\,, \qquad \text{(A.32)}
\end{aligned}
$$

and

$$
\left[\bar{\boldsymbol{\nabla}}^2,\hat{\Box}\right] = -\frac{i}{2}(\bar{\boldsymbol{\nabla}}^2\bar{\boldsymbol{W}}^{\dot{\alpha}})\bar{\boldsymbol{\nabla}}_{\dot{\alpha}} - i(\bar{\boldsymbol{\nabla}}_{\dot{\alpha}}\bar{\boldsymbol{W}}^{\dot{\alpha}})\bar{\boldsymbol{\nabla}}^2\,. \qquad \text{(A.33)}
$$

We also find

$$
\begin{aligned}
\left[\boldsymbol{\nabla}_{\alpha\dot{\alpha}},\hat{\Box}\right] &= -\frac{i}{2}(\boldsymbol{\nabla}_{\alpha}\boldsymbol{\nabla}_{\beta\dot{\alpha}}\boldsymbol{W}^{\beta}) - \frac{i}{2}(\bar{\boldsymbol{\nabla}}_{\dot{\alpha}}\boldsymbol{\nabla}_{\alpha\dot{\beta}}\bar{\boldsymbol{W}}^{\dot{\beta}}) + i(\boldsymbol{\nabla}_{\alpha\dot{\alpha}}\bar{\boldsymbol{W}}_{\dot{\beta}})\bar{\boldsymbol{\nabla}}^{\dot{\beta}} + i(\boldsymbol{\nabla}_{\alpha\dot{\alpha}}\boldsymbol{W}_{\beta})\boldsymbol{\nabla}^{\beta} \\
&\quad - i(\bar{\boldsymbol{\nabla}}_{\dot{\alpha}}\bar{\boldsymbol{W}}^{\dot{\beta}})\boldsymbol{\nabla}_{\alpha\dot{\beta}} - i(\boldsymbol{\nabla}_{\alpha}\boldsymbol{W}^{\beta})\boldsymbol{\nabla}_{\beta\dot{\alpha}}\,. \qquad \text{(A.34)}
\end{aligned}
$$

For reference we collect all of them here

$$
\begin{aligned}
\left[\boldsymbol{\nabla}_{\alpha},\hat{\Box}\right] &= \frac{1}{2}(\boldsymbol{\nabla}_{\alpha\dot{\alpha}}\bar{\boldsymbol{W}}^{\dot{\alpha}}) - i(\boldsymbol{\nabla}_{\alpha}\boldsymbol{W}^{\beta})\boldsymbol{\nabla}_{\beta}\,, \\
\left[\bar{\boldsymbol{\nabla}}_{\dot{\alpha}},\hat{\Box}\right] &= \frac{1}{2}(\boldsymbol{\nabla}_{\alpha\dot{\alpha}}\boldsymbol{W}^{\alpha}) - i(\bar{\boldsymbol{\nabla}}_{\dot{\alpha}}\bar{\boldsymbol{W}}^{\dot{\beta}})\bar{\boldsymbol{\nabla}}_{\dot{\beta}}\,, \\
\left[\boldsymbol{\nabla}^2,\hat{\Box}\right] &= -\frac{i}{2}(\boldsymbol{\nabla}^2\boldsymbol{W}^{\alpha})\boldsymbol{\nabla}_{\alpha} - i(\boldsymbol{\nabla}_{\alpha}\boldsymbol{W}^{\alpha})\boldsymbol{\nabla}^2\,, \\
\left[\bar{\boldsymbol{\nabla}}^2,\hat{\Box}\right] &= -\frac{i}{2}(\bar{\boldsymbol{\nabla}}^2\bar{\boldsymbol{W}}^{\dot{\alpha}})\bar{\boldsymbol{\nabla}}_{\dot{\alpha}} - i(\bar{\boldsymbol{\nabla}}_{\dot{\alpha}}\bar{\boldsymbol{W}}^{\dot{\alpha}})\bar{\boldsymbol{\nabla}}^2\,, \\
\left[\boldsymbol{\nabla}_{\alpha\dot{\alpha}},\hat{\Box}\right] &= -\frac{i}{2}(\boldsymbol{\nabla}_{\alpha}\boldsymbol{\nabla}_{\beta\dot{\alpha}}\boldsymbol{W}^{\beta}) - \frac{i}{2}(\bar{\boldsymbol{\nabla}}_{\dot{\alpha}}\boldsymbol{\nabla}_{\alpha\dot{\beta}}\bar{\boldsymbol{W}}^{\dot{\beta}}) + i(\boldsymbol{\nabla}_{\alpha\dot{\alpha}}\bar{\boldsymbol{W}}_{\dot{\beta}})\bar{\boldsymbol{\nabla}}^{\dot{\beta}} + i(\boldsymbol{\nabla}_{\alpha\dot{\alpha}}\boldsymbol{W}_{\beta})\boldsymbol{\nabla}^{\beta} \\
&\quad - i(\bar{\boldsymbol{\nabla}}_{\dot{\alpha}}\bar{\boldsymbol{W}}^{\dot{\beta}})\boldsymbol{\nabla}_{\alpha\dot{\beta}} - i(\boldsymbol{\nabla}_{\alpha}\boldsymbol{W}^{\beta})\boldsymbol{\nabla}_{\beta\dot{\alpha}}\,.
\end{aligned}
\qquad \text{(A.35)}
$$

## A.3 The (Anti-)chiral kinetic operators

The (anti-)chiral kinetic operators are

$$
\Box_{+}\Phi = \bar{\boldsymbol{\nabla}}^2\boldsymbol{\nabla}^2\Phi \text{ and } \Box_{-}\bar{\Phi} = \boldsymbol{\nabla}^2\bar{\boldsymbol{\nabla}}^2\bar{\Phi}\,. \qquad \text{(A.36)}
$$

Making use of the commutation relations and the fact that $\Phi$ is chiral one can show that the explicit form is

$$
\Box_{+}\Phi = \bar{\boldsymbol{\nabla}}^2\boldsymbol{\nabla}^2\Phi = \left[\bar{\boldsymbol{\nabla}}^2,\boldsymbol{\nabla}^2\right]\Phi = \frac{1}{2}\left[\boldsymbol{\nabla}^{\alpha},\left[\bar{\boldsymbol{\nabla}}^2,\boldsymbol{\nabla}_{\alpha}\right]\right]\Phi
$$

$$= (-\frac{i}{2}\{\boldsymbol{\nabla}^\alpha, \boldsymbol{W}_\alpha\} - i\boldsymbol{W}^\alpha\boldsymbol{\nabla}_\alpha + \frac{i}{2}([\boldsymbol{\nabla}^\alpha, \boldsymbol{\nabla}_{\alpha\dot\alpha}]\underbrace{\bar{\boldsymbol{\nabla}}^{\dot\alpha}}_{\propto 0} + \boldsymbol{\nabla}_{\alpha\dot\alpha}\underbrace{\left[\boldsymbol{\nabla}^\alpha, \bar{\boldsymbol{\nabla}}^{\dot\alpha}\right]}_{\propto -\left\{\boldsymbol{\nabla}^\alpha, \bar{\boldsymbol{\nabla}}^{\dot\alpha}\right\}}))\Phi$$

$$= (\Box - \frac{i}{2}\{\boldsymbol{\nabla}^\alpha, \boldsymbol{W}_\alpha\} - i\boldsymbol{W}^\alpha\boldsymbol{\nabla}_\alpha)\Phi\,. \tag{A.37}$$

Thus

$$\Box_+ = \Box - \frac{i}{2}(\boldsymbol{\nabla}^\alpha\boldsymbol{W}_\alpha) - i\boldsymbol{W}^\alpha\boldsymbol{\nabla}_\alpha \tag{A.38}$$

and similarly

$$\Box_- = \Box - \frac{i}{2}(\bar{\boldsymbol{\nabla}}^{\dot\alpha}\bar{\boldsymbol{W}}_{\dot\alpha}) - i\bar{\boldsymbol{W}}^{\dot\alpha}\bar{\boldsymbol{\nabla}}_{\dot\alpha}\,. \tag{A.39}$$

The following commutation relations hold

$$\begin{aligned}
[\boldsymbol{\nabla}_\alpha, \Box_+] &= (\boldsymbol{\nabla}^2\boldsymbol{W}_\alpha) + \bar{\boldsymbol{W}}^{\dot\alpha}\boldsymbol{\nabla}_{\alpha\dot\alpha} + i(\boldsymbol{\nabla}_\alpha\boldsymbol{W}_\beta)\boldsymbol{\nabla}^\beta\,,\\
\left[\bar{\boldsymbol{\nabla}}_{\dot\alpha}, \Box_-\right] &= (\bar{\boldsymbol{\nabla}}^2\bar{\boldsymbol{W}}_{\dot\alpha}) + \boldsymbol{W}^\alpha\boldsymbol{\nabla}_{\alpha\dot\alpha} + i(\bar{\boldsymbol{\nabla}}_{\dot\alpha}\bar{\boldsymbol{W}}_{\dot\beta})\bar{\boldsymbol{\nabla}}^{\dot\beta}\,,\\
[\boldsymbol{\nabla}_\alpha, \Box_-] &= \left[\bar{\boldsymbol{\nabla}}_{\dot\alpha}, \Box_+\right] = 0\,.
\end{aligned} \tag{A.40}$$

From the definition it is easy to see that

$$\boldsymbol{\nabla}^2\Box_+\bar{\boldsymbol{\nabla}}^2 = \boldsymbol{\nabla}^2\bar{\boldsymbol{\nabla}}^2\boldsymbol{\nabla}^2\bar{\boldsymbol{\nabla}}^2 = \Box_-\boldsymbol{\nabla}^2\bar{\boldsymbol{\nabla}}^2 \tag{A.41}$$

and

$$\bar{\boldsymbol{\nabla}}^2\Box_-\boldsymbol{\nabla}^2 = \bar{\boldsymbol{\nabla}}^2\boldsymbol{\nabla}^2\bar{\boldsymbol{\nabla}}^2\boldsymbol{\nabla}^2 = \Box_+\bar{\boldsymbol{\nabla}}^2\boldsymbol{\nabla}^2\,, \tag{A.42}$$

which in turn implies

$$\boldsymbol{\nabla}^2\Box_+^{-1}\bar{\boldsymbol{\nabla}}^2 = \Box_-^{-1}\Box_-\boldsymbol{\nabla}^2\underbrace{\Box_+^{-1}\bar{\boldsymbol{\nabla}}^2}_{\text{chiral}} = \Box_-^{-1}\boldsymbol{\nabla}^2\Box_+\Box_+^{-1}\bar{\boldsymbol{\nabla}}^2 = \Box_-^{-1}\boldsymbol{\nabla}^2\bar{\boldsymbol{\nabla}}^2 \tag{A.43}$$

and

$$\bar{\boldsymbol{\nabla}}^2\Box_-^{-1}\boldsymbol{\nabla}^2 = \Box_+^{-1}\bar{\boldsymbol{\nabla}}^2\boldsymbol{\nabla}^2\,. \tag{A.44}$$

This can also be used in $D$-algebra manipulations.

$$\boldsymbol{\nabla}^2\bar{\boldsymbol{\nabla}}^2\boldsymbol{\nabla}^2 = \Box_-\boldsymbol{\nabla}^2\,, \tag{A.45}$$

$$\bar{\boldsymbol{\nabla}}^2\boldsymbol{\nabla}^2\bar{\boldsymbol{\nabla}}^2 = \Box_+\bar{\boldsymbol{\nabla}}^2\,. \tag{A.46}$$

By a similar calculation one shows that

$$\boldsymbol{\nabla}^2\bar{\boldsymbol{\nabla}}^2\boldsymbol{\nabla}^\alpha = (\Box_- + i\boldsymbol{W}_\beta\boldsymbol{\nabla}^\beta + i\boldsymbol{\nabla}_{\beta\dot{\alpha}}\bar{\boldsymbol{\nabla}}^{\dot{\alpha}}\boldsymbol{\nabla}^\beta)\boldsymbol{\nabla}^\alpha\,, \tag{A.47}$$

$$\bar{\boldsymbol{\nabla}}^2\boldsymbol{\nabla}^2\bar{\boldsymbol{\nabla}}^{\dot{\alpha}} = (\Box_+ + i\bar{\boldsymbol{W}}_{\dot{\beta}}\bar{\boldsymbol{\nabla}}^{\dot{\beta}} + i\boldsymbol{\nabla}_{\alpha\dot{\beta}}\boldsymbol{\nabla}^\alpha\bar{\boldsymbol{\nabla}}^{\dot{\beta}})\bar{\boldsymbol{\nabla}}^{\dot{\alpha}}\,. \tag{A.48}$$

## A.4 Explicit forms of the background covariant derivatives

The explicit form of the background covariant derivatives in terms of the background prepotential is

$$\boldsymbol{\nabla}_\alpha = e^{-\boldsymbol{\Omega}}D_\alpha e^{\boldsymbol{\Omega}} = D_\alpha + e^{-\boldsymbol{\Omega}}(D_\alpha e^{\boldsymbol{\Omega}}) = D_\alpha + (D_\alpha\boldsymbol{\Omega})\,, \tag{A.49}$$

$$\bar{\boldsymbol{\nabla}}_{\dot{\alpha}} = e^{\bar{\boldsymbol{\Omega}}}\bar{D}_{\dot{\alpha}}e^{-\bar{\boldsymbol{\Omega}}} = \cdots = \bar{D}_{\dot{\alpha}} - (\bar{D}_{\dot{\alpha}}\bar{\boldsymbol{\Omega}})\,. \tag{A.50}$$

From $\boldsymbol{\nabla}_A = D_A - i\boldsymbol{\Gamma}_A$ we see

$$\boldsymbol{\Gamma}_\alpha = i(D_\alpha\boldsymbol{\Omega})\,, \tag{A.51}$$

$$\boldsymbol{\Gamma}_{\dot{\alpha}} = -i(\bar{D}_{\dot{\alpha}}\bar{\boldsymbol{\Omega}})\,. \tag{A.52}$$

Furthermore

$$\boldsymbol{\nabla}_{\alpha\dot{\alpha}} = \partial_{\alpha\dot{\alpha}} - i\{\bar{D}_{\dot{\alpha}}, \boldsymbol{\Gamma}_\alpha\} - i\{D_\alpha, \boldsymbol{\Gamma}_{\dot{\alpha}}\} - \{\boldsymbol{\Gamma}_\alpha, \boldsymbol{\Gamma}_{\dot{\alpha}}\}\,, \tag{A.53}$$

and hence

$$\boldsymbol{\Gamma}_{\alpha\dot{\alpha}} = \{\bar{D}_{\dot{\alpha}}, \boldsymbol{\Gamma}_\alpha\} + \{D_\alpha, \boldsymbol{\Gamma}_{\dot{\alpha}}\} + i\{\boldsymbol{\Gamma}_\alpha, \boldsymbol{\Gamma}_{\dot{\alpha}}\}\,. \tag{A.54}$$

The covariant d'Alembertian is found to be

$$\begin{aligned}\Box &= \frac{1}{2}\partial^{\alpha\dot{\alpha}}\partial_{\alpha\dot{\alpha}} - \frac{i}{2}\partial^{\alpha\dot{\alpha}}\boldsymbol{\Gamma}_{\alpha\dot{\alpha}} - i\boldsymbol{\Gamma}_{\alpha\dot{\alpha}}\partial^{\alpha\dot{\alpha}} - \frac{1}{2}\boldsymbol{\Gamma}^{\alpha\dot{\alpha}}\boldsymbol{\Gamma}_{\alpha\dot{\alpha}} \\ &= \Box_0 - \frac{i}{2}\partial^{\alpha\dot{\alpha}}\boldsymbol{\Gamma}_{\alpha\dot{\alpha}} - i\boldsymbol{\Gamma}_{\alpha\dot{\alpha}}\partial^{\alpha\dot{\alpha}} - \frac{1}{2}\boldsymbol{\Gamma}^{\alpha\dot{\alpha}}\boldsymbol{\Gamma}_{\alpha\dot{\alpha}}\end{aligned} \tag{A.55}$$

## A.5 Simplifications in our sector

The general rules above simplify significantly in our sector. The fact that the only undotted external indices are lowered + renders many commutation relations and intricate relations between the different operators trivial. In this section we collect the results. From the definitions of the kinetic operators we see that in our sector

$$\begin{aligned} \Box_+ &= \hat{\Box}\,, \\ \Box_- &= \hat{\Box} - i\boldsymbol{W}_+\boldsymbol{\nabla}^+\,. \end{aligned} \tag{A.56}$$

From here we can see that

$$\Box_+\boldsymbol{\nabla}^2 = \Box_-\boldsymbol{\nabla}^2\,. \tag{A.57}$$

Since in the beginning of the calculation the $\Box_-^{-1}$ always appears in the combination $\bar{\boldsymbol{\nabla}}^2\Box_-^{-1}\boldsymbol{\nabla}^2$ we can use the above relation to write

$$\begin{aligned} \bar{\boldsymbol{\nabla}}^2\Box_-^{-1}\boldsymbol{\nabla}^2 &= \bar{\boldsymbol{\nabla}}^2\Box_-^{-1}\Box_+^{-1}\Box_+\boldsymbol{\nabla}^2 = \bar{\boldsymbol{\nabla}}^2\Box_-^{-1}\Box_+^{-1}\Box_-\boldsymbol{\nabla}^2 \\ &= \bar{\boldsymbol{\nabla}}^2\Box_+^{-1}\boldsymbol{\nabla}^2 + \bar{\boldsymbol{\nabla}}^2\left[\Box_-^{-1},\Box_+^{-1}\right]\Box_-\boldsymbol{\nabla}^2 \\ &= \bar{\boldsymbol{\nabla}}^2\Box_+^{-1}\boldsymbol{\nabla}^2 + \frac{i}{2}\bar{\boldsymbol{\nabla}}^2\Box_+^{-1}\Box_-^{-1}(\boldsymbol{\nabla}_{+\dot{\alpha}}\boldsymbol{W}_+)\boldsymbol{\nabla}^{\dot{\alpha}+}\underbrace{\boldsymbol{\nabla}^+\Box_-^{-1}\boldsymbol{\nabla}^2}_{=0} \\ &= \bar{\boldsymbol{\nabla}}^2\hat{\Box}^{-1}\boldsymbol{\nabla}^2\,, \end{aligned} \tag{A.58}$$

where we used the relation

$$\left[\Box_-^{-1},\Box_+^{-1}\right] = -\Box_+^{-1}\left[\Box_-^{-1},\Box_+\right]\Box_+^{-1} = \Box_+^{-1}\Box_-^{-1}\left[\Box_-,\Box_+\right]\Box_-^{-1}\Box_+^{-1} \tag{A.59}$$

and

$$\begin{aligned} \left[\Box_-,\Box_+\right] &= \left[\Box,\Box + i\boldsymbol{W}_+\boldsymbol{\nabla}^+\right] = \frac{i}{2}\left[\boldsymbol{\nabla}^{\alpha\dot{\alpha}}\boldsymbol{\nabla}_{\alpha\dot{\alpha}},\boldsymbol{W}_+\boldsymbol{\nabla}^+\right] \\ &= \frac{i}{2}\left(\left\{\boldsymbol{\nabla}^{\alpha\dot{\alpha}},\left[\boldsymbol{\nabla}_{\alpha\dot{\alpha}},\boldsymbol{W}_+\right]\right\}\boldsymbol{\nabla}^+ + \boldsymbol{W}_+\left\{\boldsymbol{\nabla}^{\alpha\dot{\alpha}},\left[\boldsymbol{\nabla}_{\alpha\dot{\alpha}},\boldsymbol{\nabla}^+\right]\right\}\right) \\ &= \frac{i}{2}(\boldsymbol{\nabla}_{+\dot{\alpha}}\boldsymbol{W}_+)\boldsymbol{\nabla}^{\dot{\alpha}+}\boldsymbol{\nabla}^+\,, \end{aligned} \tag{A.60}$$

where the last equation holds up to terms that leave the sector. Eq. (A.58) implies that effectively there is only the propagator $\hat{\Box}^{-1}$ in our sector. The eqs. (A.35) simplify to

$$\begin{aligned} \left[\boldsymbol{\nabla}_\alpha,\hat{\Box}\right] &= i\delta_\alpha^+(\boldsymbol{\nabla}_+\boldsymbol{W}_+)\boldsymbol{\nabla}^+\,, \\ \left[\boldsymbol{\nabla}_{\alpha\dot{\alpha}},\hat{\Box}\right] &= i\delta_\alpha^+\left((\boldsymbol{\nabla}_{+\dot{\alpha}}\boldsymbol{W}_+)\boldsymbol{\nabla}^+ + (\boldsymbol{\nabla}_+\boldsymbol{W}_+)\boldsymbol{\nabla}_{\dot{\alpha}}^+\right)\,, \\ \left[\bar{\boldsymbol{\nabla}}_{\dot{\alpha}},\hat{\Box}\right] &= \left[\boldsymbol{\nabla}^2,\hat{\Box}\right] = \left[\bar{\boldsymbol{\nabla}}^2,\hat{\Box}\right] = 0\,. \end{aligned} \tag{A.61}$$

We sometimes also need the commutation relations for $\Box$, which reduce to

$$\begin{aligned}
[\boldsymbol{\nabla}_\alpha, \Box] &= 0\,, \\
\left[\bar{\boldsymbol{\nabla}}_{\dot\alpha}, \Box\right] &= -\boldsymbol{W}_+ \boldsymbol{\nabla}^+_{\dot\alpha}\,, \\
\left[\boldsymbol{\nabla}^2, \Box\right] &= 0\,, \\
\left[\bar{\boldsymbol{\nabla}}^2, \Box\right] &= -\boldsymbol{W}_+ \boldsymbol{\nabla}^{\dot\alpha +} \bar{\boldsymbol{\nabla}}_{\dot\alpha}\,, \\
[\boldsymbol{\nabla}_{\alpha\dot\alpha}, \Box] &= i\delta^+_\alpha (\boldsymbol{\nabla}_+ \boldsymbol{W}_+) \boldsymbol{\nabla}^+_{\dot\alpha}\,.
\end{aligned} \tag{A.62}$$

# Appendix B

# Counterterms

In this appendix we collect results on the diagrams that appear in our calculations. In particular we are interested in their superficial UV-divergences.

Following the discussion of section 3.1 we use infrared rearrangement to convert the graphs to propagator type diagrams, then set their external momentum to $p^2 = 1$. In these graphs we always assume a momentum flow that does not introduce new IR divergences.

Results for the unrenormalized graphs are readily found in the literature, see e.g. [75] for some of the graphs discussed here.[1] We list them explicitly below. They can often be expressed through the functions

$$G(\alpha,\beta) = \frac{\Gamma(\frac{d}{2}-\alpha)\Gamma(\frac{d}{2}-\beta)\Gamma(\alpha+\beta-\frac{d}{2})}{(4\pi)^{\frac{d}{2}}\Gamma(\alpha)\Gamma(\beta)\Gamma(D-\alpha-\beta)}\,, \tag{B.1}$$

$$G_1(\alpha,\beta) = \frac{1}{2}(-G(\alpha,\beta-1)+G(\alpha-1,\beta)+G(\alpha,\beta))\,, \tag{B.2}$$

$$G_2(\alpha,\beta) = \frac{1}{2}(-G(\alpha,\beta-1)-G(\alpha-1,\beta)+G(\alpha,\beta))\,. \tag{B.3}$$

By $I$ we denote the unrenormalized integrals themselves, evaluated at $p^2 = 1$ and

$$\mathcal{I} = K\bar{R}I \tag{B.4}$$

stands for the superficial UV divergence.

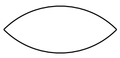
In those cases, where we don't list the closed form formulas for the unrenormalized scalar graphs, we have used their $\epsilon$-expansion as produced by Mincer [76, 77]. For example the graph $I_{3b}$ can be computed with the $O4$ topology in Mincer. The results in Mincer are in the $\overline{MS}$-scheme. In order to translate them to our conventions one

[1] There is a typo in the formula for $I_{32t}$ in [75].

has to multiply the result by a factor

$$N = (4\pi)^{-2} \exp\left(-\epsilon\gamma + \epsilon \ln(4\pi) - \epsilon^2 \frac{\zeta(2)}{2}\right) \tag{B.5}$$

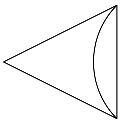
for every loop. We list our results here and give some examples of the computations in section 3.1

$I_1 =$ 

$$= \int \frac{\mathrm{d}^d k_1}{(2\pi)^d} \frac{1}{k_1^2 (k_1 - p)^2}\bigg|_{p^2=1}$$

$$= G(1,1)$$

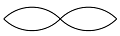
$$\mathcal{I}_1 = \frac{1}{(4\pi)^2}\left(\frac{1}{\epsilon}\right) \tag{B.6}$$

$I_2 =$ 

$$= \int \frac{\mathrm{d}^d k_1}{(2\pi)^d} \int \frac{\mathrm{d}^d k_2}{(2\pi)^d} \frac{1}{k_1^2 k_2^2 (k_2 - k_1 + p)^2 (k_1 - p)^2}\bigg|_{p^2=1}$$

$$= G(1,1)G(3 - \tfrac{d}{2}, 1)$$

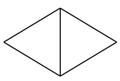
$$\mathcal{I}_2 = \frac{1}{(4\pi)^4}\left(-\frac{1}{2\epsilon^2} + \frac{1}{2\epsilon}\right) \tag{B.7}$$

$I_{11} =$ 

$$= \int \frac{\mathrm{d}^d k_1}{(2\pi)^d} \int \frac{\mathrm{d}^d k_2}{(2\pi)^d} \frac{1}{k_1^2 k_2^2 (k_1 - p)^2 (k_2 - p)^2}\bigg|_{p^2=1}$$

$$= G(1,1)^2$$

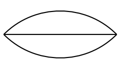
$$\mathcal{I}_{11} = -\mathcal{I}_1^2 \tag{B.8}$$

$I_{2t} =$ 

$$= \int \frac{\mathrm{d}^d k_1}{(2\pi)^d} \int \frac{\mathrm{d}^d k_2}{(2\pi)^d} \frac{1}{k_1^2 k_2^2 (k_1 - p)^2 (k_2 - p)^2 (k_1 - k_2)^2}\bigg|_{p^2=1}$$

$$= \frac{2}{d-4} G(1,1)(G(1,2) + G(3 - \tfrac{d}{2}, 2))$$

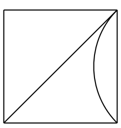
$$\mathcal{I}_{2t} = 0 \tag{B.9}$$

$I_{2n} =$ 

$$= \int \frac{\mathrm{d}^d k_1}{(2\pi)^d} \int \frac{\mathrm{d}^d k_2}{(2\pi)^d} \frac{1}{k_1^2 k_2^2 (p + k_2 - k_1)^2}\bigg|_{p^2=1}$$

$$= \frac{1}{(4\pi)^{d/2}} \frac{\Gamma(2 - \frac{d}{2})\Gamma^3(\frac{d}{2} - 1)}{\Gamma(3(\frac{d}{2} - 1))}$$

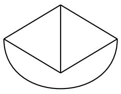
$$\mathcal{I}_{2n} = \frac{1}{(4\pi)^4}\left(-\frac{1}{4\epsilon}\right) \tag{B.10}$$

$I_3 =$ 

$$= \int \frac{\mathrm{d}^d k_1}{(2\pi)^d} \int \frac{\mathrm{d}^d k_2}{(2\pi)^d} \int \frac{\mathrm{d}^d k_3}{(2\pi)^d} \frac{1}{k_1^2 k_2^2 k_3^2 (k_1 - p)^2 (k_2 - k_1)^2 (p + k_3 - k_2)^2}\bigg|_{p^2=1}$$

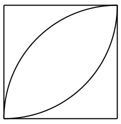
$$= G(1,1)G(3 - \tfrac{d}{2}, 1)G(5 - d, 1)$$

$$\mathcal{I}_3 = \frac{1}{(4\pi)^6}\left(\frac{1}{6\epsilon^3} - \frac{1}{2\epsilon^2} + \frac{2}{3\epsilon}\right)$$

$$I_{3t} =$$  $$= \int \frac{\mathrm{d}^d k_1}{(2\pi)^d} \int \frac{\mathrm{d}^d k_2}{(2\pi)^d} \int \frac{\mathrm{d}^d k_3}{(2\pi)^d} \frac{1}{k_1^2 k_2^2 k_3^2 (k_1 - k_2)^2 (k_1 + k_3 - p)^2 (k_2 + k_3 - p)^2}\bigg|_{p^2=1}$$

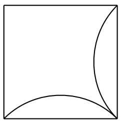
$$= I_{2t} G(5-d, 1)$$

$$\mathcal{I}_{3t} = \frac{1}{(4\pi)^6} \frac{2\zeta(3)}{\epsilon} \tag{B.11}$$

$$I_{3b} =$$  $$= \int \frac{\mathrm{d}^d k_1}{(2\pi)^d} \int \frac{\mathrm{d}^d k_2}{(2\pi)^d} \int \frac{\mathrm{d}^d k_3}{(2\pi)^d} \frac{1}{k_1^2 k_2^2 k_3^2 (k_1 - p)^2 (k_2 - p)^2 (k_3 + k_2 - k_1)^2}\bigg|_{p^2=1}$$

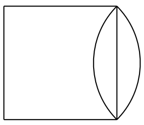
$$= N^3 \left(\frac{1}{3\epsilon^3} + \frac{7}{3\epsilon^2} + \frac{31}{2\epsilon} + \text{finite}\right)$$

$$\mathcal{I}_{3b} = \frac{1}{(4\pi)^6} \left(\frac{1}{3\epsilon^3} - \frac{2}{3\epsilon^2} + \frac{1}{3\epsilon}\right) \tag{B.12}$$

$$I_{3bb} =$$  $$= \int \frac{\mathrm{d}^d k_1}{(2\pi)^d} \int \frac{\mathrm{d}^d k_2}{(2\pi)^d} \int \frac{\mathrm{d}^d k_3}{(2\pi)^d} \frac{1}{k_1^2 k_2^2 k_3^2 (k_1 - p)^2 (k_2 - k_1)^2 (k_3 - k_1 + p)^2}\bigg|_{p^2=1}$$

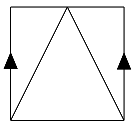
$$= G(1,1)^2 G(3 - \tfrac{d}{2}, 3 - \tfrac{d}{2})$$

$$\mathcal{I}_{3bb} = \frac{1}{(4\pi)^6} \left(\frac{1}{3\epsilon^3} - \frac{1}{3\epsilon^2} - \frac{1}{3\epsilon}\right) \tag{B.13}$$

$$I_{3n} =$$  $$= \int \frac{\mathrm{d}^d k_1}{(2\pi)^d} \int \frac{\mathrm{d}^d k_2}{(2\pi)^d} \int \frac{\mathrm{d}^d k_3}{(2\pi)^d} \frac{1}{k_1^2 k_2^2 k_3^2 ((k_1 - p)^2)^2 (k_2 + k_3 - k_1 + p)^2}\bigg|_{p^2=1}$$

$$= N^3 \left(-\frac{1}{12\epsilon^2} - \frac{7}{8\epsilon} + \text{finite}\right)$$

$$\mathcal{I}_{3n} = \frac{1}{(4\pi)^6} \left(\frac{1}{6\epsilon^2} - \frac{3}{8\epsilon}\right) \tag{B.14}$$

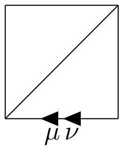
$$I_{32t} =$$  $$= \int \frac{\mathrm{d}^d k_1}{(2\pi)^d} \int \frac{\mathrm{d}^d k_2}{(2\pi)^d} \int \frac{\mathrm{d}^d k_3}{(2\pi)^d} \frac{-k_1 \cdot k_2}{k_1^2 k_2^2 k_3^2 (k_1 + p)^2 (k_2 + p)^2 (k_3 - k_1)^2 (k_3 - k_2)^2}\bigg|_{p^2=1}$$

$$= G_1(2,1) G_1(4 - \tfrac{d}{2}, 1) G_2(6-d, 1)$$

$$\mathcal{I}_{32t} = \frac{1}{(4\pi)^6} \left(-\frac{1}{3\epsilon}\right) \tag{B.15}$$

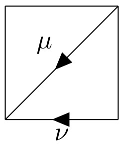
Here and below it is understood that a line with an arrow has another factor of the momentum of that line in the numerator. In $I_{32t}$ the two extra momenta are contracted. Note that there is only one quadratically divergent graph $I_{2n}$. Its superficial UV divergence is a second order polynomial in its external momentum $p^2$. This graph is needed to compute the superficial divergence of $I_{3n}$. We checked our results against the literature, in particular the ancillary files of [102]. A very useful reference for surveying the literature on Feynman diagrams is [103].

We also list our results for the tensor counterterms, on which there is barely any literature besides [74].

$$I_{2a}^{\mu\nu} =$$


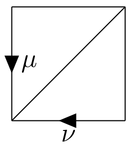

$$= \int \frac{\mathrm{d}^d k_1}{(2\pi)^d} \int \frac{\mathrm{d}^d k_2}{(2\pi)^d} \left. \frac{k_2^\mu k_2^\nu}{k_1^2 k_2^2 (k_1-p)^2 (k_2-p)^2 (k_2-k_1)^2} \right|_{p^2=1}$$

$$\mathcal{I}_{2a}^{\mu\nu} = \frac{1}{(4\pi)^4} \left( -\frac{1}{8\epsilon^2} + \frac{3}{16\epsilon} \right) g^{\mu\nu} \tag{B.16}$$

$$I_{2b}^{\mu\nu} =$$


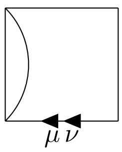

$$= \int \frac{\mathrm{d}^d k_1}{(2\pi)^d} \int \frac{\mathrm{d}^d k_2}{(2\pi)^d} \left. \frac{-(k_2-k_1)^\mu k_2^\nu}{k_1^2 k_2^2 (k_1-p)^2 (k_2-p)^2 (k_2-k_1)^2} \right|_{p^2=1}$$

$$\mathcal{I}_{2b}^{\mu\nu} = \frac{1}{(4\pi)^4} \left( -\frac{1}{8\epsilon^2} + \frac{1}{16\epsilon} \right) g^{\mu\nu} \tag{B.17}$$

$$I_{2c}^{\mu\nu} =$$


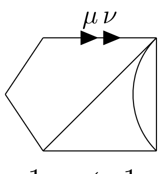

$$= \int \frac{\mathrm{d}^d k_1}{(2\pi)^d} \int \frac{\mathrm{d}^d k_2}{(2\pi)^d} \left. \frac{-k_1^\mu k_2^\nu}{k_1^2 k_2^2 (k_1-p)^2 (k_2-p)^2 (k_2-k_1)^2} \right|_{p^2=1}$$

$$\mathcal{I}_{2c}^{\mu\nu} = \frac{1}{(4\pi)^4} \left( -\frac{1}{8\epsilon} \right) g^{\mu\nu} \tag{B.18}$$

$$I_{2d}^{\mu\nu} =$$


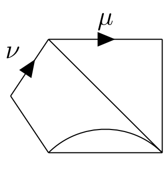

$$= \int \frac{\mathrm{d}^d k_1}{(2\pi)^d} \int \frac{\mathrm{d}^d k_2}{(2\pi)^d} \left. \frac{k_1^\mu k_1^\nu}{(k_1^2)^2 k_2^2 (k_1-p)^2 (k_2-k_1)^2} \right|_{p^2=1}$$

$$\mathcal{I}_{2d}^{\mu\nu} = \frac{1}{(4\pi)^4} \left( -\frac{1}{8\epsilon^2} + \frac{1}{16\epsilon} \right) g^{\mu\nu} \tag{B.19}$$

$$I_3^{\mu\nu 1} =$$



$$= \int \frac{\mathrm{d}^d k_1}{(2\pi)^d} \int \frac{\mathrm{d}^d k_2}{(2\pi)^d} \int \frac{\mathrm{d}^d k_3}{(2\pi)^d} \left. \frac{k_1^\mu k_1^\nu}{(k_1^2)^2 k_2^2 k_3^2 (k_1+p)^2 (k_2-k_1)^2 (k_3-k_2)^2} \right|_{p^2=1}$$

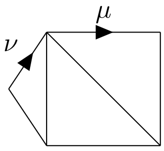
$$\mathcal{I}_3^{\mu\nu 1} = \frac{1}{(4\pi)^4} \left( \frac{1}{24\epsilon^3} - \frac{5}{48\epsilon^2} + \frac{11}{96\epsilon} \right) g^{\mu\nu} \tag{B.20}$$

$$I_3^{\mu\nu 2} =$$



$$= \int \frac{\mathrm{d}^d k_1}{(2\pi)^d} \int \frac{\mathrm{d}^d k_2}{(2\pi)^d} \int \frac{\mathrm{d}^d k_3}{(2\pi)^d} \left. \frac{k_2^\mu k_1^\nu}{k_1^2 k_2^2 k_3^2 (k_1-p)^2 (k_2-p)^2 (k_1-k_2)^2 (k_3-k_1+p)^2} \right|_{p^2=1}$$

$$\mathcal{I}_3^{\mu\nu 2} = \frac{1}{(4\pi)^4} \left( \frac{1}{48\epsilon} - \frac{1}{24\epsilon^2} \right) g^{\mu\nu} \tag{B.21}$$

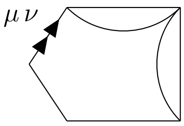
$$I_3^{\mu\nu 3} =$$




$$= \int \frac{\mathrm{d}^d k_1}{(2\pi)^d} \int \frac{\mathrm{d}^d k_2}{(2\pi)^d} \int \frac{\mathrm{d}^d k_3}{(2\pi)^d} \left. \frac{k_2^\mu k_1^\nu}{k_1^2 k_2^2 k_3^2 (k_1-p)^2 (k_2-p)^2 (k_1-k_3-p)^2 (k_3-k_2+p)^2} \right|_{p^2=1}$$

$$\mathcal{I}_3^{\mu\nu 3} = \frac{1}{(4\pi)^4} \left( \frac{1}{12\epsilon} \right) g^{\mu\nu} \tag{B.22}$$

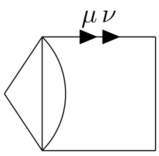
$$I_{3bb}^{\mu\nu 1} =$$




$$= \int \frac{\mathrm{d}^d k_1}{(2\pi)^d} \int \frac{\mathrm{d}^d k_2}{(2\pi)^d} \int \frac{\mathrm{d}^d k_3}{(2\pi)^d} \left. \frac{k_1^\mu k_1^\nu}{(k_1^2)^2 k_2^2 k_3^2 (k_1-p)^2 (k_2-k_1)^2 (k_3-k_1)^2} \right|_{p^2=1}$$

$$\mathcal{I}_{3bb}^{\mu\nu 1} = \frac{1}{(4\pi)^4} \left( \frac{1}{12\epsilon^3} - \frac{1}{24\epsilon^2} - \frac{5}{48\epsilon} \right) g^{\mu\nu} \tag{B.23}$$

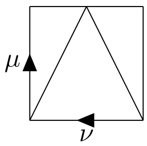
$$I_{3b}^{\mu\nu 1} =$$




$$= \int \frac{\mathrm{d}^d k_1}{(2\pi)^d} \int \frac{\mathrm{d}^d k_2}{(2\pi)^d} \int \frac{\mathrm{d}^d k_3}{(2\pi)^d} \left. \frac{k_2^\mu k_2^\nu}{k_1^2 (k_2^2)^2 k_3^2 (k_1-p)^2 (k_2-p)^2 (k_1+k_3-k_2)^2} \right|_{p^2=1}$$

$$\mathcal{I}_{3b}^{\mu\nu 1} = \frac{1}{(4\pi)^4} \left( \frac{1}{12\epsilon^3} - \frac{1}{8\epsilon^2} + \frac{5}{96\epsilon} \right) g^{\mu\nu} \tag{B.24}$$

$$I_{32t}^{\mu\nu 1} =$$




$$= \int \frac{\mathrm{d}^d k_1}{(2\pi)^d} \int \frac{\mathrm{d}^d k_2}{(2\pi)^d} \int \frac{\mathrm{d}^d k_3}{(2\pi)^d} \left. \frac{k_1^\mu k_3^\nu}{k_1^2 k_2^2 k_3^2 (k_1+p)^2 (k_2+p)^2 (k_1-k_3)^2 (k_2-k_3)^2} \right|_{p^2=1}$$

$$\mathcal{I}_{32t}^{\mu\nu 1} = \frac{1}{(4\pi)^4} \left( \frac{7}{48\epsilon} - \frac{1}{12\epsilon^2} \right) g^{\mu\nu} \, . \tag{B.25}$$

There are many more counterterms that can be computed this way. For the three loop counterterms we are not aware, that they have been published anywhere.

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**Eidesstattliche Erklärung**

Hiermit versichere ich an Eides statt, die vorliegende Dissertationsschrift selbst verfasst und keine anderen als die angegebenen Hilfsmittel und Quellen benutzt zu haben.

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