Hopf-algebraic structures inspired by Kitaev models

Defects, comodule algebras and idempotents for non-semisimple Hopf algebras

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Summary

In this thesis we study algebraic structures that are important for models of commuting-projector Hamiltonians which realize topological phases of matter. The Kitaev model is such a model, where the projectors are defined using the structure of a semisimple Hopf algebra. In the first part of the thesis we construct a Kitaev model based on more general Hopf-algebraic data – semisimple bicomodule algebras – thereby implementing defects and boundaries. In the second part of the thesis we find generalizations of the idempotents used in the standard Kitaev model to non-semisimple Hopf algebras.

More precisely, in the first part of the thesis, we construct a Kitaev model, consisting of a Hamiltonian which is the sum of commuting local projectors, for surfaces with boundaries and defects of dimension 0 and 1. Specifically, we show that one can consider cell decompositions of surfaces whose 2-cells are labeled by semisimple Hopf algebras and 1-cells are labeled by semisimple bicomodule algebras. We introduce an algebra whose representations label the 0-cells and which reduces to the Drinfeld double of a Hopf algebra in the absence of defects. In this way we generalize the algebraic structure underlying the standard Kitaev model without defects or boundaries, where all 1-cells and 2-cells are labeled by a single Hopf algebra and where point defects are labeled by representations of its Drinfeld double. In the standard case, commuting local projectors are constructed using the Haar integral for semisimple Hopf algebras. A central insight we gain in this theis is that in the presence of defects and boundaries, the suitable generalization of the Haar integral is given by the unique symmetric separability idempotent for a semisimple (bi-)comodule algebra. This enables us to provide an explicit construction of a Kitaev model allowing for defects and boundaries.

In the second part of the thesis we obtain representation-theoretic results. We study the isotypic decomposition of the regular module of a not necessarily semisimple, finite-dimensional Hopf algebra over an algebraically closed field of characteristic zero. For a semisimple Hopf algebra, it is known that the idempotents realizing the isotypic decomposition can be explicitly expressed in terms of characters and the Haar integral. Here we investigate Hopf algebras with the Chevalley property, which are not necessarily semisimple. We find explicit expressions for idempotents in terms of Hopf-algebraic data, where we replace the Haar integral by the regular character of the dual Hopf algebra. For a large class of Hopf algebras we show that these form a complete set of orthogonal idempotents. Finally, we give an example which illustrates that the Chevalley property is crucial.

Zusammenfassung

In dieser Arbeit untersuchen wir algebraische Strukturen, die wichtig sind für Modelle von Hamilton-Operatoren mit kommutierenden Projektoren, welche topologische Phasen der Materie realisieren. Das Kitaev-Modell ist ein solches Modell, bei dem die Projektoren mithilfe der Struktur einer halbeinfachen Hopf-Algebra definiert werden. Im ersten Teil der Arbeit konstruieren wir ein Kitaev-Modell, das auf allgemeineren Hopf-algebraischen Daten basiert – halbeinfache Bikomodul-Algebren – und dabei Defekte und Ränder implementiert. Im zweiten Teil der Arbeit finden wir Verallgemeinerungen der im Standard-Kitaev-Modell verwendeten Idempotenten für nicht-halbeinfache Hopf-Algebren.

Genauer gesagt konstruieren wir im ersten Teil der Arbeit ein Kitaev-Modell, das aus einem Hamilton-Operator besteht, der eine Summe von kommutierenden lokalen Projektoren ist, für Flächen mit Rändern und Defekten von Kodimension 0 und 1. Insbesondere zeigen wir, dass man Zellzerlegungen von Flächen betrachten kann, deren 2-Zellen mit halbeinfachen Hopf-Algebren und deren 1-Zellen mit halbeinfachen Bikomodul-Algebren dekoriert sind. Wir führen eine Algebra ein, deren Darstellungen die 0-Zellen dekorieren und die sich im Spezialfall ohne Defekte auf das Drinfeld-Doppel einer Hopf-Algebra reduziert. Auf diese Weise verallgemeinern wir die algebraische Struktur, die dem Standard-Kitaev-Modell ohne Defekte oder Ränder zugrunde liegt, bei dem alle 1-Zellen und 2-Zellen mit einer einzigen Hopf-Algebra und Punktdefekte mit Darstellungen seines Drinfeld-Doppels dekoriert werden. Im Standardfall werden kommutierende lokale Projektoren mithilfe des Haar-Integrals für halbeinfache Hopf-Algebren konstruiert. Eine zentrale Einsicht dieser Arbeit ist, dass bei Vorhandensein von Defekten und Rändern die geeignete Verallgemeinerung des Haar-Integrals durch die eindeutige symmetrische Separabilitätsidempotente einer halbeinfachen (Bi-)Komodul-Algebra gegeben ist. Dies ermöglicht es uns, eine explizite Konstruktion eines Kitaev-Modells anzugeben, welches Defekte und Ränder zulässt.

Im zweiten Teil der Arbeit erlangen wir darstellungstheoretische Resultate. Wir untersuchen die isotypische Zerlegung des regulären Moduls einer nicht unbedingt halbeinfachen, endlichdimensionalen Hopf-Algebra über einem algebraisch abgeschlossenen Körper in Charakteristik 0. Für eine halbeinfache Hopf-Algebra ist bekannt, dass die Idempotenten, die die isotypische Zerlegung realisieren, explizit durch Charaktere und das Haar-Integral ausgedrückt werden können. Hier untersuchen wir Hopf-Algebren mit der Chevalley-Eigenschaft, die nicht unbedingt halbeinfach sind. Wir finden explizite Ausdrücke für Idempotente durch Hopf-algebraische Daten, wobei wir das Haar-Integral durch den regulären Charakter der dualen Hopf-Algebra ersetzen. Für eine große Klasse von Hopf-Algebren zeigen wir, dass diese einen vollständigen Satz orthogonaler Idempotente bilden. Abschließend geben wir ein Beispiel, das zeigt, dass die Chevalley-Eigenschaft von entscheidender Bedeutung ist.

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Contents

1 Introduction

The Kitaev model is a family of quantum systems on a two-dimensional lattice, or more generally a graph embedded in a surface, which provide fundamental examples of topological phases of matter. It has initially been proposed by Kitaev [Kit1] as a model for an error-correcting code, the so-called toric code, allowing for fault-tolerant quantum gates by braiding anyons, in the context of quantum computing. The algebraic input datum for the construction of such a Kitaev model is a finite-dimensional semisimple complex Hopf algebra [BMCA]; for the toric code it is the group algebra of the group with two elements. The ground states of the model are described by a three-dimensional topological field theory of Turaev-Viro type [BK2] and as such, mathematically, the Kitaev model provides a link between low-dimensional topology, Hopf algebras and tensor categories.

In this introductory chapter we will explain the concepts mentioned in the above paragraph in more detail.

1.1 The Kitaev model as a quantum many-body system

In terms of physics, the Kitaev model describes a quantum many-body system in two dimensions with local interactions. As such it is described by a Hilbert space that is the tensor product of local degrees of freedom, i.e. finite-dimensional Hilbert spaces associated with the edges of an underlying graph embedded in a surface Σ, and a Hamiltonian that is the sum of short-range interaction terms, i.e. Hamiltonians that each only act on a few local degrees of freedom in a small neighborhood. More precisely, such a Hamiltonian is called *local*, if there exists $n \in \mathbb{N}$, such that for any graph in Σ, the Hamiltonian considered on that graph has the property that every summand is the identity on all except for at most n tensor factors of the Hilbert space. We note at this point already that, in this thesis, instead of Hilbert spaces and Hamiltonians, we will consider vector spaces over an algebraically closed field $\Bbbk$ of characteristic zero and diagonalizable endomorphisms, respectively, i.e. we do not consider scalar products.

Specifically, for the Kitaev model, which depends on a finite-dimensional semisimple Hopf algebra H and a cell decomposition of a surface Σ with sets Σ^0, Σ^1 and Σ^2 of vertices, edges and plaquettes, respectively, the Hilbert space is the tensor product $\mathcal{H} = \bigotimes_{e \in \Sigma^1} H$ of copies of H for all edges $e \in \Sigma^1$. (If H is a semisimple complex $*$-Hopf algebra, then $\mathcal{H}$ has the structure of a Hilbert space [BMCA], but, since here we will only consider it as vector space, it will be enough to consider a semisimple Hopf algebra H over $\Bbbk$.) In particular, for the toric code, the degrees of freedom at each edge are described by a two-dimensional Hilbert space $H = \mathbb{C}\mathbb{Z}_2$, whose distinguished basis given by the two group elements of $\mathbb{Z}_2$ can be interpreted as the spin-up and spin-down states of a spin-$\frac{1}{2}$ system. The local terms of the Hamiltonian $h = \sum_{v \in \Sigma^0} (\mathrm{id} - A_v) + \sum_{p \in \Sigma^2} (\mathrm{id} - B_p)$ are given by vertex operators A_v, which act only on the tensor factors associated with the edges incident to a single vertex $v \in \Sigma^0$, and plaquette operators B_p, which act only on the tensor factors associated with the edges in the boundary of a single plaquette $p \in \Sigma^2$. Furthermore, these local terms are mutually commuting projectors making it particularly easy to diagonalize the Hamiltonian, leading to what is called an exactly solvable system with a frustration-free Hamiltonian. Here, a local Hamiltonian is called *frustration-free* if its lowest-eigenvalue eigenspace, i.e. the *ground-state space*, is contained in the ground-state space of each individual term of the Hamiltonian.

In the case of a group algebra $H = \mathbb{C}G$ for a finite group G, the Kitaev model has a lattice-gauge-theoretic interpretation with gauge group G as follows. The Hilbert space $\mathcal{H} = \bigotimes_{e \in \Sigma^1} \mathbb{C}G$ is, as a vector space, the space of functions on the set of assignments of group elements to each edge of the graph. Such an assignment can be interpreted as a discretized version of a G-connection on the underlying surface Σ, whose holonomy along a given edge of the graph is given by the group element assigned to that edge. The vertex operators implement gauge invariance at the individual vertices by averaging with respect to the Haar integral and the plaquette operators implement flatness of the connection at the individual plaquettes by projecting to the subspace of connections with trivial holonomy around the plaquette. The resulting ground-state space can then be interpreted as the space of gauge-invariant functions on the set of flat G-connections on the surface Σ. Even for a general semisimple Hopf algebra, the Kitaev model has been exhibited as an instance of a suitable notion of Hopf algebra gauge theory [Me, BR].

1.2 Topological phases of matter and topological field theories

The main theoretical relevance of the Kitaev model does not derive from any ability to describe particularly realistic physical systems, but rather from the fact that it provides a family of explicit, manageable representatives of a family of topological phases of matter.

Here, two-dimensional topological phases of matter are understood, not in a precise or exhaustive way, as equivalence classes of quantum many-body systems on a surface, such as the Kitaev model, which might differ in their microscopic description, but which have the same macroscopic properties at low energies, which includes the ground-state space as well as states with (finitely many) localized, gapped excitations – also called quasi-particles. This low-energy sector of a topological phase of matter has the following main characteristic properties.

Firstly, the ground-state space depends only on the topology of the underlying surface and its dimension is typically larger than one for topologically non-trivial surfaces. For example, the toric code considered on the torus has a four-dimensional ground-state space. This is known as *ground-state degeneracy.*

Furthermore, the ground-state space is *robust* against local perturbations: Any local observable, i.e. an operator that is the identity outside of a sufficiently small region, acts as (a multiple of) the identity on the ground-state space [CDHPRRS]. This means that the degrees of freedom on the ground-state space are non-local or, in other words, topological. For example, in the toric code, the four-dimensional ground-state space is identified with the first homology on the torus with coefficients in $\mathbb{Z}_2$, corresponding to the following four spin configurations: Either all spins are down, or all spins are down except along one of the two non-homotopic non-contractible loops around the torus, or all spins are down except along both of the non-contractible loops.

Finally, in a topological phase of matter, the localized gapped excitations are *anyons*. This means that the observables which exchange localized excitations of identical type with each other, by adiabatically moving them around in the surface, form a representation of the so-called surface braid group (which is the ordinary braid group when the underlying surface is a sphere) that is non-trivial, i.e. it does not factor through the symmetric group. (This is to be contrasted with identical bosons and identical fermions, whose exchange corresponds to the trivial representation and the sign representation of the symmetric group, respectively.) This means that the exchange of two identical anyons is not necessarily given by acting with a factor of 1 or -1, as for bosons and fermions, respectively, but rather it can be given by *any* phase or, more generally, any unitary operator.

A quantum system which satisfies the above properties is said to possess *topological order* [W], the theoretical study of which emerged from the experimental discovery of the fractional quantum Hall effect [TSG].

On the one hand, the above characteristic properties of topological phases lend themselves well to the implementation of a quantum computer, as we will explain in the next section 1.3. On the other hand, this characterization of (two-dimensional) topological phases, almost by definition, points to three-dimensional topological field theories as the low-energy effective theories of such quantum many-body systems. Topological field theories are a mathematical framework [At], that has been developed as an attempt to capture and study some of the structural properties of quantum field theories in a mathematically rigorous way. A three-dimensional oriented topological field theory is defined to be a symmetric monoidal functor $Z : \mathrm{cob}^{\mathrm{or}}_{2,3} \longrightarrow \mathrm{vect}(\Bbbk)$ from a symmetric monoidal category $\mathrm{cob}^{\mathrm{or}}_{2,3}$ of compact oriented surfaces and three-dimensional compact oriented cobordisms to a symmetric monoidal category $\mathrm{vect}(\Bbbk)$ of finite-dimensional vector spaces (or, alternatively, finite-dimensional Hilbert spaces) and linear maps. Other variants of topological field theories are based on categories of manifolds that are not necessarily oriented or that have other additional structures such as framings. Relevant for this thesis, as explained in Section 1.4, are also topological field theories with defects, which are defined on a cobordism category of stratified oriented manifolds.

Concretely, a topological field theory assigns to any compact oriented surface Σ a finite-dimensional vector space $Z(\Sigma)$ and to any compact oriented three-manifold M with boundary $\partial M = \overline{\Sigma_1} \sqcup \Sigma_2$ a linear map $Z(M) : Z(\Sigma_1) \to Z(\Sigma_2)$ between the vector spaces assigned to the components of the boundary, where $\overline{\Sigma_1}$ is the same manifold as Σ_1 but with opposite orientation. In particular, by considering as a cobordism the mapping cylinder on a surface Σ corresponding to a diffeomorphism of Σ, the topological field theory produces a linear automorphism of the associated vector space $Z(\Sigma)$ for each diffeomorphism of Σ. This induces a representation of the mapping class group of the surface Σ on the vector space $Z(\Sigma)$. The mapping class group of a surface Σ is defined to be the quotient of the group of diffeomorphisms of Σ by the subgroup of diffeomorphisms isotopic to the identity.

In order to capture not only vacuum states of a topological phase, but also states with localized gapped excitations, one needs the structure of an extended three-dimensional topological field theory, which is defined on a larger class of surfaces and cobordisms. In this case, there is a (finite) braided tensor category $\mathcal{C}$ that is assigned to the circle, and there are (finite-dimensional) vector spaces assigned to all (compact oriented) surfaces with boundary, where each boundary circle must be labeled by an object in $\mathcal{C}$. This assignment depends functorially on the objects attached to the circles and is compatible with gluing surfaces along boundaries. The idea is that such a vector space represents a ground-state space with localized gapped excitations at each boundary circle, with particle type prescribed by the corresponding object in $\mathcal{C}$. Furthermore, an extended topological field theory assigns linear maps to three-dimensional cobordisms with corners between surfaces with boundary, and this, in particular, yields representations of the mapping class groups of surfaces with boundary. For example, for the sphere with n boundary circles the mapping class group is the braid group on n strands. If one only has the vector spaces and the mapping class group representations on them, then one has the structure of a *modular functor*. In particular, the mapping class group of a surface with several boundary components includes diffeomorphisms which braid the boundary circles around each other. In this way, a modular functor captures the feature of topological phases that the exchange of anyonic excitations in the surface is given by a, generally non-trivial, unitary operator on the state space. To conclude, in this paragraph we have motivated the idea that a modular functor is the mathematical structure that describe the low-energy effective behavior of a two-dimensional topological phase of matter in the sense described above. For a more direct relation between the braided tensor category $\mathcal{C}$ that enters and the anyon model that is described by the modular functor, see also [Kit2].

It is well-established [BK2] that the ground-state space of the Kitaev model is described by a three-dimensional topological field theory. More precisely, for any semisimple Hopf algebra over $\Bbbk$ and compact oriented surface Σ, the ground-state space of the corresponding Kitaev model is canonically isomorphic to the vector space assigned to Σ by the topological field theory of Turaev-Viro type for the spherical fusion category H–mod of finite-dimensional H-modules. Such a three-dimensional topological field theory is part of a class of topological field theories that appear in many contexts and that have various constructions. Let us briefly mention these different realizations.

Firstly, if the semisimple Hopf algebra H is a group algebra for a finite group G, then the Dijkgraaf-Witten construction provides a gauge-theoretic approach [FQ, MNS]. Defects and boundaries have also been studied in this framework [FSV2].

Secondly, for any spherical fusion category there exists a state-sum construction, which dates back to work of Turaev and Viro [TV], who considered the representation category of a certain quantum group. Barrett and Westbury have later generalized the construction to spherical fusion categories [BW2]. Here, similarly to the Kitaev construction, one first constructs a larger vector space which depends on a choice of auxiliary combinatorial data on the surface, such as a triangulation, and then projects onto a subspace using maps assigned to three-dimensional manifolds. For an exposition see [BK1, TVi]. This construction has been extended to include defects in [CMS]. More recently, state-sum constructions based on non-semisimple categories have been considered [FSS2]. These constructions provide a useful counterpart to which we can compare our constructions in the framework of Kitaev models.

Finally, Levin-Wen string-net models also realize the class of topological field theories of Turaev-Viro type. Originally constructed as a family of microscopic models with a commuting-projector Hamiltonian in order to realize a large class of topological phases, they have been turned into a mathematically rigorous construction in [Kir]. Recently they have been extended to fully fledged topological field theories, at least in the case that is based on the group algebra of $\mathbb{Z}_2$, which corresponds to the toric code, in [BG]. Here, the vector spaces assigned to surfaces are not constructed as subspaces, but rather as quotients of larger vector spaces, where certain local relations on discs are taken into account.

It is well known that topological field theories of Turaev-Viro type can also be realized by the Reshetikhin-Turaev construction [BK1, TVi]. Here the relevant modular tensor category is the Drinfeld center of a spherical fusion category.

1.3 Quantum codes and error correction

The basic idea of quantum computing is to employ quantum-mechanical systems for the storage and processing of information. This means that information is encoded as a state of a (finite-dimensional) Hilbert space, say $\mathbb{C}^N$ with the standard Hermitian scalar product, and manipulated by unitary operators on this Hilbert space. If the Hilbert space is factorized into a tensor product $(\mathbb{C}^2)^{\otimes n}$ of two-dimensional Hilbert spaces $\mathbb{C}^2$, then these are referred to as *qubits*. More generally, one can consider *qu-d-its* $\mathbb{C}^d$ for any $d \in \mathbb{N}$.

In realistic physical systems, the stored information might be corrupted over time and unitary operators might not be perfectly realized. Moreover, since physical systems are never completely isolated from their environment, the effect of decoherence can compromise the very quantum-mechanical nature of the system. Therefore, it is crucial for the functioning of a quantum computer to be protected from such errors. Like in classical computing, one thus encodes the information with a redundancy and implements a mechanism to recover the original state from a corrupted state using this redundancy. This means that a *quantum code* is defined as a linear subspace $\mathcal{C} \subseteq \mathcal{H}$ of a finite-dimensional Hilbert space $\mathcal{H}$, which is also referred to as the *quantum medium*. The code subspace should be chosen such that a sufficiently small error occuring to a state encoding information will result in a state outside the code subspace that can be uniquely corrected back to the original state in the code space.

Here, an error is representated by a linear operator on the Hilbert space $\mathcal{H}$. The set of correctable errors is determined by the quantum code $\mathcal{C} \subseteq \mathcal{H}$ and it is a fundamental problem in the theory of quantum error correction to maximize both the set of correctable errors and the dimension of the code subspace, i.e. the amount of information that can be encoded, while minimizing the dimension of the total Hilbert space.

Often, a quantum code is specified by a set of commuting projectors, called the *stabilizer*, on the Hilbert space $\mathcal{H}$. In this case it is called a *stabilizer code* [G]. The code subspace $\mathcal{C} \subseteq \mathcal{H}$ is then the simultaneous $(+1)$-eigenspace of these projectors. An error that does not preserve the code subspace, i.e. that does not commute with all projectors of the stabilizer, can be detected by measuring the eigenvalues of the projectors. If the error is sufficiently small, it can also be corrected by acting with the stabilizer projectors.

More precisely, let $\mathcal{H} = (\mathbb{C}^2)^{\otimes n}$ be a tensor product of n qubits. A linear operator on $\mathcal{H}$ is called *k-local*, for $k \in \mathbb{N}$, if it is the identity on all but at most k tensor factors of $\mathcal{H}$. A quantum code $\mathcal{C} \subseteq \mathcal{H}$ is called a *k-code*, for $k \in \mathbb{N}$, if for any k-local operator $O : \mathcal{H} \to \mathcal{H}$, which for example describes a k-local error, the linear operator $\pi_{\mathcal{C}} \circ O|_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$ on the code space is (a scalar multiple of) the identity. Here, $\pi_{\mathcal{C}} : \mathcal{H} \to \mathcal{C}$ is the orthogonal projection onto the subspace $\mathcal{C} \subseteq \mathcal{H}$, which in the case of a stabilizer code is given by the composition of all the commuting projectors of the stabilizer. In this sense, errors that affect sufficiently few qubits at the same time can be corrected by the stabilizer projectors and one speaks of an *error-correcting code*.

The idea of topological quantum computing is to physically realize an error-correcting quantum code by a quantum many-body system in a topological phase, such as the Kitaev model. Here the quantum medium $\mathcal{H}$, as a tensor product of qu-d-its, for $d \in \mathbb{N}$, is realized as the state space $\mathcal{H} = \bigotimes_{e \in \Sigma^1} H$, where the local degrees of freedom are associated with the edges $e \in \Sigma^1$ of a graph embedded into a surface Σ and are given by a semisimple Hopf algebra H with dimension d. (For $H = \mathbb{C}\mathbb{Z}_2$, one considers a system of qubits.) The code space $\mathcal{C} \subseteq \mathcal{H}$ is realized as the ground-state space of the system. For this it is important that the ground-state space is degenerate, otherwise it would encode only a single qu-d-it. Since the Hamiltonian is a sum of commuting projectors, i.e. the ground-state space is given as the simultaneous $(+1)$-eigenspace of commuting projectors, the Kitaev model provides a stabilizer code. The dynamics described by the Hamiltonian thus tend to correct sufficiently small errors, since it costs energy for a state to violate the stabilizer conditions that define the ground-state space. This means that error correction does not have to implemented by some artificial procedure but rather is built in at the physical level. It has been rigorously shown [CDHPRRS], for the case of a group algebra $H = \mathbb{C}G$ for any finite group G, that the Kitaev model yields an error-correcting quantum code.

It is remarkable how closely the defining properties of an error-correcting quantum code mirror the characteristic properties of topological phases of matter: The topological stability of a degenerate ground-state space, on which sufficiently local operators can only act by (a multiple of) the identity, is precisely the condition that sufficiently local errors restricted to code subspace are correctable by the stabilizer projectors.

However, topological phases of matter not only lend themselves well to realizing quantum codes, but are also characterized by the existence of an interesting class of observables that braid localized anyonic excitations around each other, as explained in the previous section 1.2. On the other hand, in order to realize a quantum computer we need to implement not only the error-correcting code, the so-called *quantum memory*, but also linear operators on the quantum medium $\mathcal{H}$ preserving the code space $\mathcal{C} \subseteq \mathcal{H}$, so that the quantum computer cannot only store information but also actually perform computations on it. In order to obtain a quantum computer one must realize a *library*, i.e. a finite set, of unitary operators on the quantum code $\mathcal{C} \subseteq \mathcal{H}$, called *quantum gates*, which usually act on only one or two qu-d-its at the same time. The computations that can be performed, the so-called *quantum circuits*, are all the finite compositions of the quantum gates. The quantum computer is called *universal*, if the group generated by the quantum gates lies densely in the group of all unitary operators on the code space.

In topological quantum computing, the idea is to realize the unitary operators on the code space by the operators that correspond to braiding localized anyonic excitations around each other. The benefit of these operators is again that they are stable under local perturbations, since they only depend on the homotopy class of the path along which the anyon is moved. Mathematically more precisely, in terms of the underlying topological field theory or modular functor, these operators are given by the acting with the appropriate element of the mapping class group, which is an isotopy-invariant. It is known that in this way modular functors allow for universal quantum computation [FLW].

It turns out that the toric code based on the group algebra of $\mathbb{Z}_2$, as originally suggested by Kitaev, does not allow for universal quantum computation. However, it has been shown that it suffices to consider only slightly larger and more complicated groups such as the symmetric group S_3 to achieve universal quantum computation with a model based on anyons [Mo].

1.4 Defects and boundaries in topological field theories

It is natural to consider topological field theories not just on surfaces, but on surfaces with additional structure. In terms of physics, we want to allow for defects and boundaries; in mathematical terms, we consider the theories on a suitable class of decorated stratified manifolds called *defect surfaces* in the sense of [FSS2], but see also e.g. [CMS]. (For this thesis, models on oriented surfaces are relevant, whereas in [FSS2] surfaces with 2-framings have been considered.) Here, one considers manifolds with a collection of distinguished submanifolds that are labelled by certain additional data of a type that depends on the theory. For a boundary, for example, such a datum encodes a boundary condition.

The study of defects and boundaries in topological field theories has received increased attention in recent years. They are interesting for a variety of reasons. Firstly, considering a topological field theory with defects amounts to a unification of an entire family of topological field theories. Usually topological field theories come in families parametrized by a certain algebraic or category-theoretic input datum, such as spherical fusion categories in the case of Turaev-Viro theories or modular tensor categories for Reshetikhin-Turaev theories. In the framework of topological field theories with defects these input data are interpreted as the possible labels for the top-dimensional strata of the underlying manifolds. Defects of co-dimension 1 between such top-dimensional strata corresponding to various theories then allow to consider these theories within a single one. In this regard, Turaev-Viro theories are a natural subclass of three-dimensional topological field theories to consider, since boundaries and defects between such theories always exist in the sense of [FSV1], while in general this does not hold for every pair of three-dimensional topological field theories.

Secondly, defects and boundaries provide links with the (categorified) representation theory of the algebraic structures that label the top-dimensional manifolds. For example, surface defects in three-dimensional topological field theories of Turaev-Viro type are labelled by (semisimple) bimodule categories over the spherical fusion categories which describe the theories separated by the defect. In this way, many constructions in the categorified representation theory of such fusion categories obtain a geometric underpinning, as further demonstrated in [FSS2].

Furthermore, defects are related to symmetries of topological field theories. In fact, many symmetries of and dualities between theories can be interpreted as invertible defects. In this sense, defects can be seen as a framework that generalizes such features. Accordingly, there exist generalized orbifold constructions for topological field theories with defects [CRS].

For the present thesis, most crucially a further aspect of introducing defects and boundaries into the theory is relevant. It is known that defects leads to higher-dimensional vector spaces assigned to surfaces and more interesting mapping class group actions. For example, this has been demonstrated in [FS], see also [BJQ], for so-called permutation twist defects, which effectively increase the genus of the surface, i.e. the vector space assigned to a surface with such defects is isomorphic to the vector space assigned to a surface of higher genus without defects. In such theories the dimension of the vector spaces grows exponentially with the genus, as can be read off from the Verlinde formula.

This is particularly relevant for applications to topological quantum computing, since as explained in Section 1.3, the dimension of the ground-state space determines the amount of information that can be stored in the code and the action of the mapping class group on this vector space determines the variety of operations that can be performed on the code. A study in this direction, using boundaries in order to achieve higher computational power, is e.g. [LLW].

There have been already several approaches to include defects or boundaries in Kitaev models based on group algebras [BK, BMD, BSW, CCW]. In this thesis we follow an approach that deals with the more general case of semisimple Hopf algebras.

1.5 Projectors in the Kitaev model in terms of Hopf-algebraic structure

Let us review in more detail the construction of the Kitaev model, emphasizing the central role played by the Hopf-algebraic structure; see also [BMCA, BK2]. We shall also use this opportunity to fix our conventions regarding the notation. Throughout the thesis we fix an algebraically closed field $\Bbbk$ of characteristic zero. All vector spaces will be finite-dimensional over $\Bbbk$, including those underlying any algebras or modules over algebras.

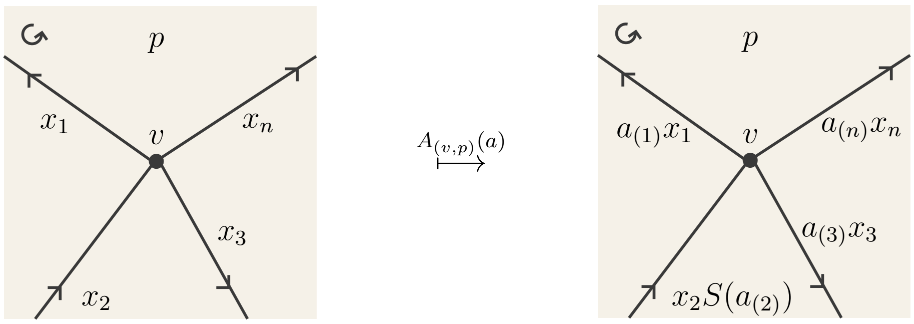
For the construction of the Kitaev model without defects or boundaries, one fixes a semisimple Hopf algebra H over $\Bbbk$. General references for Hopf algebras are e.g. [Mon, Ka]. This means that, in addition to being a finite-dimensional semisimple unital algebra over $\Bbbk$, H is equipped with a co-multiplication $\Delta : H \longrightarrow H \otimes H$ and a co-unit $\varepsilon : H \longrightarrow \Bbbk$, which are morphisms of algebras. The co-product is usually written in Sweedler notation as $\Delta(x) = x_{(1)} \otimes x_{(2)} \in H \otimes H$ for $x \in H$, where $x_{(1)} \otimes x_{(2)}$ is in general a sum of pure tensors, but the summation symbol is omitted. For an n-fold coproduct one writes $x_{(1)} \otimes \cdots \otimes x_{(n)}$, which is well-defined due to co-associativity. Furthermore, H has an involutive antipode $S : H \longrightarrow H$, which is an anti-algebra-morphism as well as an anti-coalgebra-morphism and can, hence, be seen as an isomorphism of Hopf algebras $S : H^{\mathrm{op,cop}} \longrightarrow H$. Here, $H^{\mathrm{op,cop}}$ has the opposite multiplication as well as the opposite co-multiplication compared to H. Semisimplicity of the Hopf algebra H implies that it possesses a distinguished idempotent $\ell \in H$, the *Haar integral*. This is the unique normalized two-sided integral of H, i.e. it satisfies $x\ell = \varepsilon(x)\ell = \ell x$ for all $x \in H$ and $\varepsilon(\ell) = 1$. These defining properties also imply that $\ell \in H$ is an idempotent and cocommutative. It provides the crucial algebraic ingredient entering in the construction of the commuting projectors of the Kitaev model, as explained below. Lastly, note that the dual H^* of a finite-dimensional semisimple Hopf algebra H is again a finite-dimensional semisimple Hopf algebra. Its multiplication is defined by dualising the co-multiplication of H so that for $f, g \in H^*$ we have $(f \cdot g)(x) := f(x_{(1)})g(x_{(2)})$ for all $x \in H$. Likewise, the co-multiplication is defined in terms of the multiplication of H as $(f_{(1)} \otimes f_{(2)})(x \otimes y) := f(xy)$ for all $f \in H^*$ and $x, y \in H$.

The Kitaev model depends not only on the semisimple Hopf algebra H, but also on a compact oriented surface Σ and a cell decomposition thereof, i.e. a CW complex structure. This means that Σ is decomposed into sets Σ^0, Σ^1 and Σ^2 of vertices, edges and plaquettes, respectively. The edges are assumed to have their own orientation. In other words, (Σ^0, Σ^1) is a directed graph embedded into the surface Σ such that its complement in Σ is a disjoint union of a set Σ^2 of discs. The starting point of the construction is then to consider the vector space $\mathcal{H} = \bigotimes_{e\in\Sigma^1} H$, which has the interpretation of the state space of a quantum system composed of local quantum systems with state spaces H associated with the edges of the cell decomposition – or in terms of quantum computing, $\mathcal{H} = \bigotimes_{e\in\Sigma^1} H$ is a quantum medium composed of qu-d-it spaces H, where $d = \dim(H)$, as explained in Section 1.3.

The construction of the Kitaev model proceeds in a natural way using only the Hopf-algebraic structure that is present as well as the combinatorial data contained in the cell decomposition.

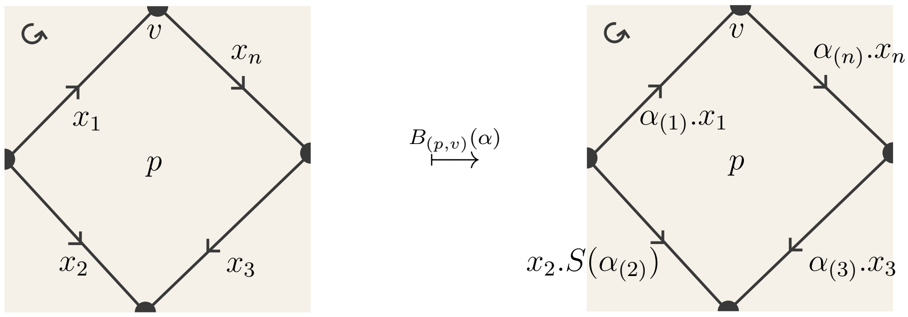
The crucial idea is that the vector space $\mathcal{H} = \bigotimes_{e\in\Sigma^1} H$ admits certain natural actions of the Hopf algebra H and the dual Hopf algebra H^*, which give rise to representations of the Drinfeld double $D(H)$.

More precisely, for each pair of a vertex $v \in \Sigma^0$ and an auxiliary plaquette $p \in \Sigma^2$ that is incident to v (and which we assume to lie on at most one side of any edge incident to v) – a so-called *site* (v, p) – there is an action $A_{(v,p)}$ of the Hopf algebra H on $\bigotimes_{e\in\Sigma^1} H$ by the n-fold coproduct of an element $a \in H$ acting in counter-clockwise order on the individual copies of H associated to the n edges incident to the given vertex v. For example, the action $A_{(v,p)}(a)$ looks like the following for $n = 4$:



where $x_i \in H$ is any element of the copy of H associated with the relevant edge. Here, the auxiliary plaquette $p \in \Sigma^2$ determines the edge at which the counter-clockwise order of the edges around v starts, which is given with respect to the orientation of the surface Σ, and the relative orientation of an edge with respect to the vertex v determines whether the Hopf algebra acts by left multiplication or by right multiplication via the antipode. In particular, this action does not depend on the choice of auxiliary plaquette p if one acts by a cocommutative element $a \in H$. Hence, it allows one to define, for each vertex $v \in \Sigma^0$, an idempotent endomorphism $A_v : \mathcal{H} \longrightarrow \mathcal{H}$ by acting with the unique Haar integral $\ell \in H$ of the semisimple Hopf algebra H. In terms of representation theory, this idempotent gives a projection to the subspace of invariants with respect to the H-action. This defines one part of the family of commuting projectors for the Hamiltonian of the Kitaev model, the so-called *vertex operators*.

For the remaining projectors, the so-called *plaquette operators*, one considers in the spirit of Poincaré duality an action $B_{(p,v)}$ of the dual Hopf algebra H^* on the copies of H associated to the edges in the boundary of a plaquette $p \in \Sigma^2$. For this, one analogously chooses an auxiliary vertex $v \in \Sigma^0$ in the boundary of the given plaquette p, assuming that the edges in the boundary incident to v are not loops, and orders the edges in the boundary in clockwise order with respect to the orientation of Σ. Depending on the relative orientation of an edge with respect to the plaquette $p \in \Sigma^2$, the dual Hopf algebra H^* acts on the copy of H associated to that edge via one of two natural actions which are intertwined by the involutive antipode of H, for details see [BMCA, BK2]. The action $B_{(p,v)}(\alpha)$ of an element $\alpha \in H^*$ thus looks like follows, when one edge is oriented counter-clockwise around the boundary of p and the others clockwise:



On the tensor factors associated with the edges not in the boundary of p The plaquette operator for the plaquette $p \in \Sigma^2$, an idempotent endomorphism $B_p : \mathcal{H} \longrightarrow \mathcal{H}$, is then defined by acting via this action with the unique Haar integral $\lambda \in H^*$ of the dual Hopf algebra H^*. Again in terms of representation theory, this gives a projection to the subspace of H^*-invariants.

The thus defined family of projectors $\{(A_v)_{v\in\Sigma^0}, (B_p)_{p\in\Sigma^2}\}$ has the important property that they commute pairwise, which leads to the construction of the frustration-free Hamiltonian $h = \sum_{v\in\Sigma^0}(\mathrm{id} - A_v) + \sum_{p\in\Sigma^2}(\mathrm{id} - B_p)$ or, in quantum information theoretic terms, a stabilizer code. Proving this property requires a careful analysis of how the actions of the Hopf algebra H and its dual H^* on the vector space $\mathcal{H}$ for a given site (v, p) interact, revealing the structure of the Drinfeld double $D(H)$. Indeed, it turns out that the H- and H^*-actions satisfy the so-called *straightening formula*, which is the defining commutation relation that H and H^* satisfy as subalgebras inside the the algebra $D(H)$, which has the underlying vector space $H^* \otimes H$, cf. [BMCA, Theorem 1]. Since in this way the structure of the Drinfeld double is an outcome of the construction, one can regard the Kitaev model as an independent motivation of the algebraic structure of the Drinfeld double.

Furthermore, representations of the Drinfeld double in the Kitaev model become important when studying point-like excitations. This is done by considering subspaces of the full state space $\mathcal{H}$, where at a few distinguished disjoint sites one does not project to the subspace of $D(H)$-invariants by vertex and plaquette operators, but rather leaves a larger subspace with a non-trivial residual $D(H)$-action for each distinguished site, see [BK2].

1.6 Summary of results

This thesis consists of two main parts, Chapters 2 and 3. While Chapter 2 is mainly a construction in mathematical physics, in Chapter 3 we obtain representation-theoretic results. The common theme of both chapters is that they investigate Hopf-algebraic structures which appear in the Kitaev model.

The main result of Chapter 2 is the construction of a Kitaev model, consisting of a commuting-projector Hamiltonian, for surfaces with defects and boundaries, using general Hopf-algebraic and representation-theoretic input data.

For this construction it is necessary to first realize the data labeling the defects, which are known for Turaev-Viro theory in a category-theoretic language, concretely in Hopf-algebraic and representation-theoretic terms. Specifically, topological field theories of Turaev-Viro type are parameterized by spherical fusion categories [BW2]. The data for defects separating two such theories are semisimple bimodule categories [KK, FSV1, FSS2]. The idea for obtaining the data for a Kitaev construction is to invoke Tannaka-Krein duality [D]. It states that a semisimple

Hopf algebra is equivalent to specifying a fusion category (the representation category of the Hopf algebra, admitting a canonical spherical structure) together with a monoidal fibre functor valued in finite-dimensional vector spaces (the forgetful functor assigning to a representation its underlying vector space). This recovers semisimple Hopf algebras as the input datum for the Kitaev models without defects, which we think of here as the labels for the two-dimensional strata of the defect surface.

We extend this idea and employ, for the bimodule categories labelling line defects on the surface in Turaev-Viro theory, the appropriate bimodule versions of fibre functors. By a bimodule version of Tannaka-Krein duality, which we explain in Subsubsection 2.1.1.1, this realizes these categories as the representation categories of bicomodule algebras over Hopf algebras. We thus identify bicomodule algebras as the labels for line defects and, as a special case, comodule algebras for boundaries.

Having established the algebraic data for line defects of the surface, we turn our attention to vertices where such line defects can join. They are labeled by objects in a category which serves as possible labels for generalized Wilson lines in a corresponding three-dimensional topological field theory, including boundary Wilson lines and Wilson lines at the intersection of surface defects. This category has been determined as a suitable generalization [FSS1, FSS2] of the Drinfeld center for a spherical fusion category, which labels bulk Wilson lines. Here, in Subsection 2.1.3, this category is realized as a representation category as follows: For a vertex at which line defects meet, the bicomodule algebras of the line defects and the algebras dual to the Hopf algebras attached to the adjacent two-dimensional strata naturally assemble into an algebra, which we introduce in Definition 5. This algebra, which we call *vertex algebra*, reduces in special cases to the Drinfeld double of the Hopf algebra, whose representations label point-like excitations in the Kitaev model without defects. The category of possible labels for such a vertex is then the category of modules over this algebra. We show in Theorem 8 that this category is equivalent to the category of generalized Wilson lines at the intersection of surface defects in a corresponding three-dimensional field theory [FSS2].

Furthermore, a choice of cell decomposition on the underlying surface enters the construction of the Kitaev model. In the standard Kitaev model without defects, every 1-cell (or *edge*) of the cell decomposition is labeled by a single Hopf algebra. In our setting this should be seen as the regular bicomodule algebra and we consider this label as the *transparent* defect. In our case, edges of the cell decomposition are either transparently labeled or they constitute a non-trivial defect and are labeled by an arbitrary bicomodule algebra.

Our construction proceeds in the following steps – mirroring the construction of the standard Kitaev model without defects, as in e.g. [BMCA, BK2]. We first define in Definition 9 a vector space with local degrees of freedom for each edge and each 0-cell (or *vertex*) of the cell decomposition. Then we show in Subsection 2.2.1 that this vector space admits, locally with respect to the cell decomposition, the structure of a bimodule over the algebras attached to the vertices. This is analogous to the representations of the Drinfeld double for each site in the standard Kitaev model without defects. In this case one then proceeds to use the Haar integral for any semisimple Hopf algebra to define local projectors via these local representations. One of our main insights, established in Subsection 2.2.2, is that, in the presence of defects, the suitable generalization of the Haar integral to semisimple bicomodule algebras is given by the symmetric separability idempotent, see Definition 15. The symmetric separability idempotent of a semisimple algebra is unique, which we recall in Proposition 17. Furthermore, we show in Proposition 19 that for a semisimple (bi-)comodule algebra, the symmetric separability idempotent satisfies a compatibility with the (bi-)comodule structure which generalizes a basic property of the Haar integral of a semisimple Hopf algebra. In the absence of defects, the symmetric separability idempotent reduces to the Haar integral, as we show in Example 18.

Using such separability idempotents, in Subsection 2.2.3 we finally construct projectors for each vertex, as usual called *vertex operators*, and for each plaquette, as usual called *plaquette operators*. Our main result in Chapter 2, Theorem 25, is that all vertex operators and plaquette operators commute – giving rise to an exactly solvable Hamiltonian defined as a sum of commuting projectors, which project to the ground states of the model.

Concerning the ground states, our construction can be seen as a concrete representation-theoretic realization of the category-theoretic construction in [FSS2]. While in [FSS2] more general categories than representation categories of Hopf algebras and bicomodule algebras are considered, for us the additional structure of fibre functors on the categories is necessary in order to define a larger vector space which contains the pre-block space and block space as subspaces. Moreover, while for the construction in [FSS2] no semisimplicity is required, in this chapter semisimplicity is essential for the construction of commuting local projectors, since we define them in terms of the symmetric separability idempotents. (In Chapter 3 we achieve partial results towards constructing projectors for non-semisimple Hopf algebras.) Lastly, since semisimple Hopf algebras have an involutive antipode, they have a canonical trivial pivotal structure. Hence, we can define our model on any surface with orientation. The approach in [FSS2] is to assume no pivotal structure on the tensor categories, but instead to assume more geometric structure, namely a 2-framing, on the surfaces.

In the standard Kitaev model based on a semisimple Hopf algebra, the Haar integral is the main algebraic ingredient defining the commuting projectors. The importance of such an idempotent leads to the problem which we study in the second part of this thesis, Chapter 3, and which can be phrased in purely representation-theoretic terms. While in Chapter 2 we considered the symmetric separability idempotent of a bicomodule algebra as a generalization of the Haar integral of a semisimple Hopf algebra, in this chapter we give a generalization of the Haar integral to a class of not necessarily semisimple Hopf algebras.

In Chapter 3 we study the decomposition of the regular module of a finite-dimensional Hopf algebra into isotypic components. Recall that $\Bbbk$ is an algebraically closed field of characteristic zero and H a finite-dimensional Hopf algebra over $\Bbbk$. If H is semisimple, the Artin-Wedderburn theorem implies that as a left H-module H decomposes into the direct sum of submodules H_i isomorphic to the $\dim(S_i)$-fold direct sum $S_i^{\oplus \dim S_i}$ of the simple H-module S_i. Here, i runs over the set I of isomorphism classes of simple H-modules. The decomposition $H = \bigoplus_{i\in I} H_i$ is called the *isotypic decomposition* of H, seen as a left H-module, into its *isotypic components* H_i. It can be also described by the central orthogonal idempotents $(e_i)_{i\in I}$ in H such that $e_i \in H_i$ and $\sum_{i\in I} e_i = 1$. Then $H_i = He_i$ and the projection from $H = \bigoplus_{i\in I} H_i$ onto the direct summand H_j is given by right multiplication by e_j for all $j \in I$.

So far this only uses the algebra structure of H. The following idea is well-known and lies at the heart of the theory of representations of a finite group. For a semisimple Hopf algebra H over $\Bbbk$ with antipode S, the central orthogonal idempotents e_i can be described explicitly in terms of the Haar integral and the irreducible characters of H by the following *character-projector* *formula* [S, Cor. 4.6]

$$e_i = \dim(S_i)\chi_i(S(\ell_{(1)}))\ell_{(2)}. \tag{1.1}$$

Here, Sweedler notation is understood, and $\ell \in H$ is the *Haar integral* for H, the unique (two-sided) integral of H, normalised to $\varepsilon(\ell) = 1$, which exists due to the Maschke theorem for semisimple Hopf algebras [Sw, Theorem 5.1.8]. The functional $\chi_i : H \longrightarrow \Bbbk$ here is the character of the simple H-module S_i.

In this thesis, we study generalizations of the character-projector formula (1.1) for finite-dimensional Hopf algebras that are not necessarily semisimple. Hence, we do not have a Haar integral at our disposal. It is a central insight of this thesis, instead to use the character of the regular representation of the Hopf algebra H^* dual to H. While for semisimple algebras there is a unique isotypic decomposition, in the non-semisimple case such decompositions are in general not unique anymore. Our aim in this chapter is to nevertheless construct one *explicit* decomposition using the Hopf-algebraic structure.

We obtain the strongest results for Hopf algebras with the Chevalley property, see Definition 45. This is a large class of finite-dimensional Hopf algebras, including semisimple Hopf algebras and basic Hopf algebras, i.e. Hopf algebras for which all simple modules are one-dimensional, the Hopf algebras dual to pointed Hopf algebras.

Our main results are as follows: for a finite-dimensional Hopf algebra with the Chevalley property, we give in Theorem 52 an explicit idempotent for each one-dimensional simple module. In Theorem 56, we exhibit a necessary and sufficient condition involving the so-called Hecke algebra of the trivial representation (see Definition 55) ensuring that these idempotents form a complete set in the sense that they sum up to the identity.

In Conjecture 48, we propose an explicit generalization of the character-projector formula (1.1) for finite-dimensional Hopf algebras with the Chevalley property. (The Chevalley property is essential, as witnessed by the counterexample given in Example 50.) The two main theorems 52 and 56 imply our Conjecture 48 for basic Hopf algebras that satisfy the condition on the Hecke algebra, as summarized in Corollary 57. Furthermore, in Proposition 60 we prove that Conjecture 48 holds for Hopf algebras which have the Chevalley property and the dual Chevalley property. Lastly, we provide further evidence for Conjecture 48 by studying in Subsection 3.3.2 an example of a Hopf algebra with the Chevalley property that is not covered by our general results in Section 3.2. We do this by performing some of the more computationally complex calculations using the computer algebra software Magma.

Publications

The chapters of this thesis are based on the following pre-prints:

Chapter 2: *Defects in Kitaev models and bicomodule algebras.*
arXiv:2001.10578 [math.QA]

Chapter 3: *On isotypic decompositions for non-semisimple Hopf algebras.*
With Ehud Meir and Christoph Schweigert.
arXiv:1910.13161 [math.QA]

2 Defects and boundaries in Kitaev models

This chapter is organised as follows. In Section 2.1 we introduce the Hopf-algebraic and representation-theoretic data labelling defects in our construction of the Kitaev model. In Definition 1 we define line defects to be labeled by bicomodule algebras, for which we give a category-theoretic motivation in Subsubsection 2.1.1.1 using Tannaka-Krein duality. In Definition 5 we introduce an algebra, which we call vertex algebra, whose representations we define in Definition 7 to be the labels for a point defect, at which line defects intersect. We show in Theorem 8 that its representation category is equivalent to the category of generalized Wilson lines at the intersection of surface defects in a corresponding three-dimensional field theory.

In Section 2.2 we give our construction of the Kitaev model based on the Hopf-algebraic data introduced in the first section. In Definition 9 we define the vector spaces that are assigned to surfaces, which function as the state spaces of the Kitaev model. In Subsection 2.2.1 and, in particular Theorem 13, we prove that the state spaces admit natural bimodule structures over the vertex algebras. We use these bimodule structures in Subsection 2.2.3 to define in Definitions 22 and 24 local projectors on the state spaces. We prove in Proposition 19 a compatibility of the symmetric separability idempotent of a semisimple comodule algebra with the comodule structure. These results finally culminate in our main result, Theorem 25, which shows that the local projectors we defined commute pairwise and, hence, give rise to a Hamiltonian, defined in Definition 26.

2.1 Hopf-algebraic and representation-theoretic labels for surfaces with cell decomposition

Following the discussion in the introduction, we will explain in the first section the input data for our construction.

Let Σ be a compact oriented surface together with a cell decomposition $(\Sigma^0, \Sigma^1, \Sigma^2)$ with non-empty sets of 0-cells (or *vertices*), 1-cells (or *edges*) and 2-cells (or *plaquettes*), respectively. This can be thought of as an embedding of a graph (Σ^0, Σ^1) into Σ such that its complement in Σ is the disjoint union of a set Σ^2 of disks. Furthermore, let the edges be oriented, i.e. there are source and target maps $s, t : \Sigma^1 \longrightarrow \Sigma^0$. If the surface Σ has a boundary, then we require that the 1-skeleton of the cell decomposition be contained in the boundary.

For the construction of a Kitaev model one needs as a further input Hopf-algebraic and representation-theoretic data labelling the various strata of the cell decomposition. In the ordinary Kitaev model without defects as in [BMCA], all edges of the cell decomposition are labeled by a single semisimple Hopf algebra H, and wherever point-like excitations are considered [BK2], a vertex is labeled by a representation of the Drinfeld double $D(H)$ of the Hopf algebra H. In this thesis we consider more general labels for the edges, thereby implementing arbitrary line defects (also known as *domain walls* in condensed matter theory) and boundaries in the Kitaev model. Accordingly we also consider more general labels for vertices, implementing point defects (also known as *point-like excitations*) inside defect lines or boundaries. For the remainder of this section we will specify the three types of Hopf-algebraic and representation-theoretic data that label the plaquettes, edges and vertices of a cell decomposition.

2.1.1 Bicomodule algebras over Hopf algebras for line defects

We fix once and for all an algebraically closed field $\Bbbk$ of characteristic zero. For the necessary background on Hopf algebras and conventions regarding the notation, see [Mon, Ka, BMCA].

Definition 1.

- Let H_1 and H_2 be Hopf algebras over $\Bbbk$. An H_1-H_2-*bicomodule algebra* K is a $\Bbbk$-algebra K together with an H_1-H_2-bicomodule structure, i.e. with co-associative co-action written in Sweedler notation for comodules as

$$\begin{aligned} K &\longrightarrow H_1 \otimes K \otimes H_2, \\ k &\longmapsto k_{(-1)} \otimes k_{(0)} \otimes k_{(1)}, \end{aligned}$$

  which is required to be a morphism of algebras. If $H_1 = \Bbbk$ or $H_2 = \Bbbk$, then K is just a right H_2-comodule or a left H_1-comodule algebra, respectively.

  A *semisimple bicomodule algebra* is a bicomodule algebra whose underlying algebra is semisimple.

- Let Σ be an oriented surface with a cell decomposition with oriented edges. A *label H_p for a plaquette $p \in \Sigma^2$* is a semisimple Hopf algebra H_p over $\Bbbk$.

  For any edge $e \in \Sigma^1$ let $p_1 \in \Sigma^2$ and $p_2 \in \Sigma^2$ be the labelled plaquettes on the left and on the right of e, respectively, with respect to the orientation of e relative to the orientation of Σ. Then a *label K_e for the edge e* is a finite-dimensional semisimple H_{p_1}-H_{p_2}-bicomodule algebra K_e over $\Bbbk$.

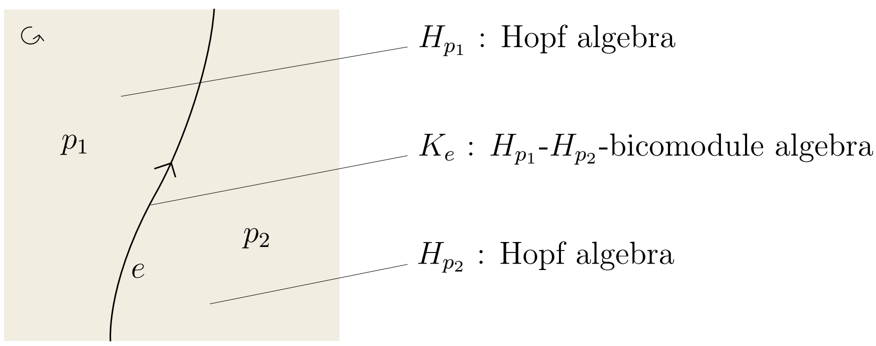


Figure 2.1: An edge e and the adjacent plaquettes p_1 and p_2 with their algebraic data. The two arrows denote the orientations of the edge and, respectively, of the surface Σ into which the edge is embedded.

  If the edge e lies in the boundary of Σ and hence only has a plaquette p on one side (left or right), then K_e is just a left or right H_p-comodule algebra, respectively.

Examples 2.

1. Let H be a Hopf algebra. The *regular* H-bicomodule algebra is the algebra underlying the Hopf algebra H together with left and right co-action given by the co-multiplication of H. Note that the regular H-bicomodule algebra is semisimple if and only if the Hopf algebra H is semisimple, since both are defined by the semisimplicity of the underlying algebra.

2. Let G be a finite group and $\Bbbk G$ its group algebra, which has a basis $(b_g)_{g\in G}$ parametrized by G and multiplication induced by the group multiplication. $\Bbbk G$ is a semisimple Hopf algebra with comultiplication given by the diagonal map $b_g \mapsto b_g \otimes b_g$ for all $g \in G$. Further, let $U \subseteq G$ be a subgroup and $\zeta \in Z^2(U, \Bbbk^\times)$ a group 2-cocycle. Then the cocycle-twisted group algebra $\Bbbk U_\zeta$ with multiplication $b_u \cdot b_v := \zeta(u,v)b_{uv}$ for all $u, v \in U$ is a $\Bbbk G$-comodule algebra with co-action given by the diagonal map $b_u \mapsto b_u \otimes b_u$.

2.1.1.1 A category-theoretic motivation for bicomodule algebras via Tannaka-Krein duality

Let us explain the emergence of bicomodule algebras from the point of view of Tannaka-Krein duality, as outlined in the Introduction. We thereby relate the algebraic input data for our construction, as defined in Definition 1, to the category-theoretic data for the state-sum construction of a modular functor in [FSS2]. For the relevant category-theoretic notions and background, see e.g. [EGNO].

First of all, for a finite-dimensional Hopf algebra H over $\Bbbk$, it is well known that the category H–mod of finite-dimensional left H-modules is a finite $\Bbbk$-linear tensor category. This tensor category comes equipped with a forgetful functor $H\text{–mod} \longrightarrow \mathrm{vect}(\Bbbk)$ into the tensor category of finite-dimensional vector spaces. The forgetful functor is monoidal, exact and faithful.

In fact, it is known [EGNO] that the datum of a finite-dimensional Hopf algebra H over $\Bbbk$ is equivalent to the datum of a finite $\Bbbk$-linear tensor category $\mathcal{A}$ together with a monoidal fiber functor $\omega : \mathcal{A} \longrightarrow \mathrm{vect}(\Bbbk)$, i.e. an exact and faithful $\Bbbk$-linear tensor functor to the category of finite-dimensional vector spaces. More precisely, the Hopf algebra H can be reconstructed as the algebra of natural endo-transformations of the fiber functor ω and the tensor structure on the fiber functor ω induces the additional coalgebra structure on the algebra H, such that $\mathcal{A} \cong H\text{–mod}$ as tensor categories.

We extend this idea to bimodule categories as follows. For a finite-dimensional H_1-H_2-bicomodule algebra K for Hopf algebras H_1 and H_2, the category K–mod has the structure of an (H_1–mod)-(H_2–mod)-bimodule category in a natural way. Indeed, if X_1 is an H_1-module, X_2 is an H_2-module and M is a K-module, then $X_1 \triangleright M \triangleleft X_2 := X_1 \otimes_\Bbbk M \otimes_\Bbbk X_2$ becomes a K-module by pulling back the natural $(H_1 \otimes K \otimes H_2)$-action on it along the co-action map $K \longrightarrow H_1 \otimes K \otimes H_2$ that belongs to K.

On the other hand, let $(\mathcal{A}_1, \omega_1 : \mathcal{A}_1 \longrightarrow \mathrm{vect}(\Bbbk))$ and $(\mathcal{A}_2, \omega_2 : \mathcal{A}_2 \longrightarrow \mathrm{vect}(\Bbbk))$ be finite $\Bbbk$-linear tensor categories together with monoidal fiber functors. Consider $\mathrm{vect}(\Bbbk)$ as an $\mathcal{A}_1$-$\mathcal{A}_2$-bimodule category via the monoidal functors ω_1 and ω_2. Let $\mathcal{M}$ be a finite $\Bbbk$-linear $\mathcal{A}_1$-$\mathcal{A}_2$-bimodule category. Then we define a *bimodule fiber functor* $\omega : \mathcal{M} \longrightarrow \mathrm{vect}(\Bbbk)$ for $\mathcal{M}$ to be an exact and faithful $\mathcal{A}_1$-$\mathcal{A}_2$-bimodule functor from $\mathcal{M}$ to the category of finite-dimensional vector spaces. Let H_1 and H_2 be the corresponding finite-dimensional Hopf algebras over $\Bbbk$ corresponding to $(\mathcal{A}_1, \omega_1)$ and $(\mathcal{A}_2, \omega_2)$. Then, by the same argument as for tensor categories *mutatis mutandis*, the bimodule structure on the fiber functor ω induces the structure of an H_1-H_2-bicomodule algebra K on the algebra of natural endo-transformations of ω, such that ω induces an equivalence of bimodule categories $\mathcal{M} \cong K$–mod.

Hence, we conclude that bicomodule algebras emerge naturally as the algebraic input data for Kitaev models, if one follows the following idea in order to obtain concrete Hopf-algebraic data: Take the category-theoretic data underlying the corresponding topological field theories or modular functors, which are tensor categories and bimodule categories [FSS2, KK], and equip them with fiber functors of the appropriate type.

2.1.2 Algebraic structure at half-edges and sites

It remains to determine the possible labels for the vertices of the cell decomposition. This is the content of Subsection 2.1.3. Before that, in this Subsection 2.1.2, we first introduce suitable notation and terminology in order to extract and conveniently speak about the combinatorial information contained in the cell decomposition.

Fix a vertex $v \in \Sigma^0$. Then let $\Sigma^{0.5}_v$ be the set of *half-edges incident to* v. This is the set of incidences of an edge with the given vertex $v \in \Sigma^0$. (A loop at v yields two half-edges incident to v.) Note that we have a map $\Sigma^{0.5}_v \longrightarrow \Sigma^1$, assigning to any half-edge its underlying edge, which is in general not injective due to the possible existence of loops. We will denote by Σ^1_v its image in Σ^1, that is the set of edges starting or ending at the given vertex v.

We will say that $e \in \Sigma^{0.5}_v$ is *directed away from* $v \in \Sigma^0$ if $v = s(e)$ and, that $e \in \Sigma^{0.5}_v$ is *directed towards* $v \in \Sigma^0$ if $v = t(e)$. Then for any half-edge $e \in \Sigma^{0.5}_v$ incident to the vertex $v \in \Sigma^0$, let the sign $\varepsilon(e) \in \{+1, -1\}$ be positive if the half-edge $e \in \Sigma^{0.5}_v$ is directed away from the vertex v:

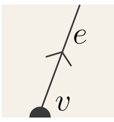


Figure 2.2: A half-edge $e \in \Sigma^{0.5}_v$ incident to v with sign $\varepsilon(e) := +1$

and negative if $e \in \Sigma^{0.5}_v$ is directed towards v:

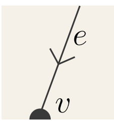


Figure 2.3: A half-edge $e \in \Sigma^{0.5}_v$ incident to v with sign $\varepsilon(e) := -1$

Let $p \in \Sigma^2$ be the plaquette on the left of the half-edge $e \in \Sigma^{0.5}_v$, as seen from the vertex $v \in \Sigma^0$, and let $p' \in \Sigma^2$ be the plaquette on the right, as in Figure 2.4.

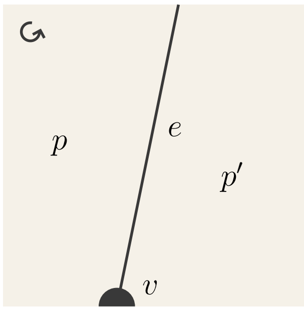


Figure 2.4: A half-edge e at v with neighboring plaquettes p and p'

What we have not represented in the figure is that the half-edge e comes with an orientation, expressed by the sign $\varepsilon := \varepsilon(e)$. By our assignment of labels, if the half-edge e is directed away from the vertex v, i.e. $\varepsilon = +1$, then it is labeled with an H_p-$H_{p'}$-bicomodule algebra K_e, with co-action written in Sweedler notation for comodules:

$$\left.\begin{array}{rcl} K_e & \longrightarrow & H_p \otimes K_e \otimes H_{p'} \\ k & \longmapsto & k_{(-1)} \otimes k_{(0)} \otimes k_{(1)} \end{array}\right\} \text{ if } \varepsilon(e) = +1.$$

If, on the other hand, the half-edge e points towards v, that is $\varepsilon = -1$, then K_e is an $H_{p'}$-H_p-bicomodule algebra:

$$\left.\begin{array}{rcl} K_e & \longrightarrow & H_{p'} \otimes K_e \otimes H_p \\ k & \longmapsto & k_{(-1)} \otimes k_{(0)} \otimes k_{(1)} \end{array}\right\} \text{ if } \varepsilon(e) = -1.$$

We shall introduce notation which allows us to treat both cases $\varepsilon = +1$ and $\varepsilon = -1$ at once. Let

$$K_e^{+1} := K_e$$
$$K_e^{-1} := K_e^{\mathrm{op}},$$

where K_e^{op} is the algebra with opposite multiplication. Moreover, let

$$H_p^{+1} := H_p,$$
$$H_p^{-1} := H_p^{\mathrm{opcop}},$$

where H_p^{opcop} is the Hopf algebra with opposite multiplication and opposite comultiplication. If K_e is a left (or right, respectively) H_p-comodule algebra, then K_e^{-1} is canonically a left (or right, respectively) H_p^{-1}-comodule algebra.

Hence, in both above cases we can write that K_e^ε is an H_p^ε-$H_{p'}^\varepsilon$-bicomodule algebra, with co-action in Sweedler notation:

$$\begin{aligned} K_e^\varepsilon &\longrightarrow H_p^\varepsilon \otimes K_e^\varepsilon \otimes H_{p'}^\varepsilon, \\ k &\longmapsto k_{(-\varepsilon)} \otimes k_{(0)} \otimes k_{(\varepsilon)}. \end{aligned}$$

Denote by Σ_v^{sit} the set of *sites incident to* v. These are incidences of a plaquette $p \in \Sigma^2$ with the given vertex $v \in \Sigma^0$. (Note that a single plaquette $p \in \Sigma^2$ can have two separate incidences with the vertex v. This happens when an edge in its boundary is a loop.) Dually, for a plaquette $p \in \Sigma^2$ denote by Σ_p^{sit} the set of *sites incident to* p. These are incidences of a vertex

$v \in \Sigma^0$ with the given plaquette p. It is justified to use the name *site* for both notions: To any site $p \in \Sigma_v^{\mathrm{sit}}$ at a vertex $v \in \Sigma^0$ corresponds a unique site $\widetilde{v} \in \Sigma_p^{\mathrm{sit}}$ with underlying vertex v at the plaquette that underlies the site $p \in \Sigma_v^{\mathrm{sit}}$.

Now let $p \in \Sigma_v^{\mathrm{sit}}$ be such a site at the vertex $v \in \Sigma^0$. There is a half-edge $e'_p \in \Sigma_v^{0.5}$ bounding p on the left as seen from the vertex v and there is a half-edge $e_p \in \Sigma_v^{0.5}$ bounding p on the right. For an example consider Figure 2.5.

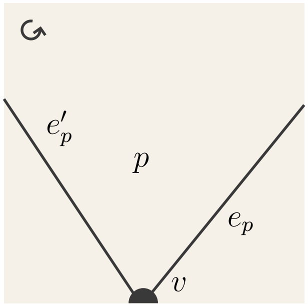


Figure 2.5: A site $p \in \Sigma_v^{\mathrm{sit}}$ with neighboring half-edges e'_p and e_p.

Then, in consideration of the respective signs $\varepsilon := \varepsilon(e_p)$ and $\varepsilon' := \varepsilon(e'_p)$ of the half-edges e_p and e'_p, we have by our assignment of labels that $K_{e'_p}^{\varepsilon'}$ is a right $H_p^{\varepsilon'}$-comodule algebra and that $K_{e_p}^{\varepsilon}$ is a left H_p^{ε}-comodule algebra. In other words, we have a left $((H_p^{\varepsilon'})^{\mathrm{cop}} \otimes H_p^{\varepsilon})$-comodule structure on the algebra

$$K_{\{e_p, e'_p\}} := \bigotimes_{e \in \{e_p, e'_p\} \subseteq \Sigma_v^{0.5}} K_e^{\varepsilon(e)} = \begin{cases} K_{e'_p}^{\varepsilon'} \otimes K_{e_p}^{\varepsilon}, & e_p \neq e'_p \in \Sigma_v^{0.5} \\ K_{e_p}^{\varepsilon}, & e_p = e'_p \in \Sigma_v^{0.5} \end{cases}. \tag{2.1}$$

Next we introduce, for a fixed site $p \in \Sigma_v^{\mathrm{sit}}$, a canonical left $((H_p^{\varepsilon'})^{\mathrm{cop}} \otimes H_p^{\varepsilon})$-module algebra, which we think of as associated to the site p:

Definition 3. Let $v \in \Sigma^0$ be a vertex and $p \in \Sigma_v^{\mathrm{sit}}$ a site at v with neighboring half-edges $e_p, e'_p \in \Sigma_v^{0.5}$ with signs $\varepsilon, \varepsilon' \in \{+1, -1\}$ as before.

The ε'-ε-*balancing algebra* H_p^*, or more explicitly $(H_p)^*_{(\varepsilon', \varepsilon)}$, is the left $((H_p^{\varepsilon'})^{\mathrm{cop}} \otimes H_p^{\varepsilon})$-module algebra, whose underlying $\Bbbk$-algebra is the dual algebra of the Hopf algebra H_p, with the following action.

$$\begin{aligned} ((H_p^{\varepsilon'})^{\mathrm{cop}} \otimes H_p^{\varepsilon}) \otimes H_p^* &\longrightarrow H_p^*, \\ a' \otimes a \otimes f &\longmapsto f(a'^{\langle -\varepsilon' \rangle} \cdot ? \cdot a^{\langle \varepsilon \rangle}), \end{aligned}$$

where

$$a^{\langle \varepsilon \rangle} := \begin{Bmatrix} a, & \varepsilon = +1 \\ S(a), & \varepsilon = -1 \end{Bmatrix} \quad \text{for all } a \in H_p$$

and where $S : H_p \longrightarrow H_p$ denotes the antipode.

Together, the $((H_p^{\varepsilon'})^{\mathrm{cop}} \otimes H_p^{\varepsilon})$-comodule algebra $K_{\{e_p, e'_p\}}$, associated to the half-edges $e_p \in \Sigma_v^{0.5}$ and $e'_p \in \Sigma_v^{0.5}$, and the $((H_p^{\varepsilon'})^{\mathrm{cop}} \otimes H_p^{\varepsilon})$-module algebra H_p^*, associated to the site $p \in \Sigma_v^{\mathrm{sit}}$ situated between the edges e_p and e'_p, can be *coupled* into a single $\Bbbk$-algebra, denoted by

$$H_p^* \ogreaterthan K_{\{e_p, e'_p\}} \tag{2.2}$$

which has underlying vector space $H_p^* \otimes K_{\{e_p,e'_p\}}$ and which is an instance of the following general construction. For related constructions see [Mon].

Definition 4. Let H be a Hopf algebra over $\Bbbk$, let A be a left H-module algebra and let K be a left H-comodule algebra. Then the *crossed product algebra* $A \otimes K$ is the $\Bbbk$-algebra with underlying vector space $A \otimes K$ and multiplication

$$(a \otimes k) \cdot (a' \otimes k') := a(k_{(-1)}.a') \otimes k_{(0)}k' \quad \text{for } (a \otimes k), (a' \otimes k') \in A \otimes K.$$

In particular, the algebra $H_p^* \otimes K_{\{e_p,e'_p\}}$ contains H_p^* and $K_{\{e_p,e'_p\}}$ as subalgebras and the commutation relation between these is

$$k \cdot f = f(k_{(\varepsilon')}^{\langle -\varepsilon' \rangle} \cdot ? \cdot k_{(-\varepsilon)}^{\langle \varepsilon \rangle}) \cdot k_{(0)} \quad \forall f \in H_p^*, k \in K_{\{e_p,e'_p\}}, \tag{2.3}$$

the so-called *straightening formula*. This generalizes the straightening formula of the Drinfeld double of a Hopf algebra, see Example 6.

2.1.3 Vertex algebras and their representations as labels for vertices

In this subsection we introduce, for each vertex $v \in \Sigma^0$, an algebra over $\Bbbk$, which is constructed from the algebraic labelling in the neighbourhood of the vertex v. The representations of this algebra will serve as possible labels for the vertex v. In a corresponding three-dimensional topological field theory these are the possible labels for generalized Wilson lines.

Let us collect the algebras $K_e^{\varepsilon(e)}$ of all half-edges $e \in \Sigma_v^{0.5}$ incident to the vertex $v \in \Sigma^0$ into a tensor product

$$K_{\Sigma_v^{0.5}} := \bigotimes_{e \in \Sigma_v^{0.5}} K_e^{\varepsilon(e)}.$$

With the notation of the previous subsection, for each site $p \in \Sigma_v^{\mathrm{sit}}$ with neighboring half-edges e_p and e'_p as in Figure 2.5, the algebra $K_{\{e'_p,e_p\}}$ is a left comodule over

$$\big(H_p^{\varepsilon(e'_p)}\big)^{\mathrm{cop}} \otimes H_p^{\varepsilon(e_p)}.$$

This trivially extends to an $((H_p^{\varepsilon(e'_p)})^{\mathrm{cop}} \otimes H_p^{\varepsilon(e_p)})$-comodule structure on the tensor product $K_{\Sigma_v^{0.5}}$ of $K_{\{e,e'\}}$ with the algebras attached to the remaining half-edges in $\Sigma_v^{0.5}$. The co-actions on $K_{\Sigma_v^{0.5}}$ for different sites commute with each other, because they come from the bicomodule structures of the tensor factors $(K_e)_{e \in \Sigma_v^{0.5}}$, making $K_{\Sigma_v^{0.5}}$ a left comodule algebra over the tensor product of Hopf algebras

$$\bigotimes_{p \in \Sigma_v^{\mathrm{sit}}} \big(H_p^{\varepsilon(e'_p)}\big)^{\mathrm{cop}} \otimes H_p^{\varepsilon(e_p)}. \tag{2.4}$$

For each site $p \in \Sigma_v^{\mathrm{sit}}$ we want to *couple* the balancing algebra H_p^* to $K_{\Sigma_v^{0.5}}$, similarly as in (2.2). For this we collect the balancing algebras of the sites around the vertex v into a tensor product

$$H^*_{\Sigma_v^{\mathrm{sit}}} := \bigotimes_{p \in \Sigma_v^{\mathrm{sit}}} H_p^*.$$

This is a left module algebra over the tensor product of Hopf algebras as in (2.4). Now we have all the ingredients to introduce:

Definition 5. Let $v \in \Sigma^0$. The $\Bbbk$-*algebra* C_v *associated to the vertex* v, or *vertex algebra*, is defined as follows. For any site $p \in \Sigma_v^{\mathrm{sit}}$ denote by e'_p and $e_p \in \Sigma_v^{0.5}$ the half-edges bounding p on the left and on the right, respectively, from the perspective of the vertex v, as illustrated in Figure 2.5. Then let

$$C_v := H^*_{\Sigma_v^{\mathrm{sit}}} \oslash K_{\Sigma_v^{0.5}} = \left(\bigotimes_{p\in\Sigma_v^{\mathrm{sit}}} H_p^*\right) \oslash \left(\bigotimes_{e\in\Sigma_v^{0.5}} K_e^{\varepsilon(e)}\right)$$

be the crossed product algebra, as introduced in Definition 4, for the left module algebra $H^*_{\Sigma_v^{\mathrm{sit}}}$ and the left comodule algebra $K_{\Sigma_v^{0.5}}$ over the tensor product (2.4) of Hopf algebras.

In particular, the algebra contains $H^*_{\Sigma_v^{\mathrm{sit}}} = \otimes_{p\in\Sigma_v^{\mathrm{sit}}} H_p^*$ and $K_{\Sigma_v^{0.5}} = \otimes_{e\in\Sigma_v^{0.5}} K_e^{\varepsilon(e)}$ as subalgebras and, for each site $p' \in \Sigma_v^{\mathrm{sit}}$, we have the commutation relation (2.3); so in other words,

$$H^*_{p'} \oslash K_{\{e_{p'},e'_{p'}\}} \subseteq \left(\bigotimes_{p\in\Sigma_v^{\mathrm{sit}}} H_p^*\right) \oslash \left(\bigotimes_{e\in\Sigma_v^{0.5}} K_e^{\varepsilon(e)}\right) = C_v \tag{2.5}$$

is a subalgebra of C_v.

Example 6. Let us consider the situation where the vertex $v \in \Sigma^0$ has precisely one half-edge e, which is directed away from the vertex and which is labeled by the regular H-bicomodule algebra H, the *transparent* label.

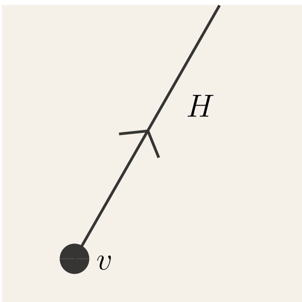


Figure 2.6: A vertex v with a single half-edge transparently labeled by H;
the associated algebra C_v is the Drinfeld double $D(H)$

Then for the algebra C_v at the vertex v we have $H^*_{\Sigma_v^{\mathrm{sit}}} \oslash K_{\Sigma_v^{0.5}} = H^* \oslash H$ and the commutation relation (2.3) gives

$$h \cdot f = f(S(h_{(3)}) \cdot ? \cdot h_{(1)}) \cdot h_{(2)}. \tag{2.6}$$

This is precisely the so-called *straightening formula* of the Drinfeld double $D(H)$ of a semisimple Hopf algebra H [Ka]. In the Kitaev model without defects as in [BMCA, BK2], representations of the Drinfeld double $D(H)$ label point-like excitations.

Up to this point we have explained how, for a given vertex $v \in \Sigma^0$, the algebraic labelling of the edges and plaquettes and the combinatorial structure of the cell decomposition around that vertex gives rise to the $\Bbbk$-algebra $C_v = H^*_{\Sigma_v^{\mathrm{sit}}} \oslash K_{\Sigma_v^{0.5}}$.

Definition 7. We declare the *category of possible labels* for a vertex $v \in \Sigma^0$ for the Kitaev construction to be the $\Bbbk$-linear category C_v–mod of finite-dimensional left modules over the $\Bbbk$-algebra C_v.

Indeed, in [FSS2], the category-theoretic data assigned to a vertex $v \in \Sigma^0$ is as follows. In the language of [FSS2], a vertex v corresponds to a boundary circle $\mathbb{L}_v$ with marked points on which defect lines end. A 2-cell $p \in \Sigma^2$ is labelled by a finite tensor category; in our context this is the representation category H_p–mod of a finite-dimensional Hopf algebra H_p. An edge $e \in \Sigma^1$ is labelled by a finite bimodule category; in our context this is the representation category K_e–mod of a bicomodule algebra K_e. Then according to [FSS2, Definitions 3.4 and 3.9] the category of possible labels of a vertex $v \in \Sigma^0$ is given by the category $\mathrm{T}(\mathbb{L}_v)$ of so-called balancings on the Deligne tensor product $\boxtimes_{e \in \Sigma_v^{0.5}}(K_e^{\varepsilon(e)}\text{–mod})$ of the bimodule categories labelling the half-edges around the vertex v.

Theorem 8. *Let $v \in \Sigma^0$. There is a canonical equivalence of $\Bbbk$-linear categories*

$$\mathrm{T}(\mathbb{L}_v) \cong C_v\text{–mod}$$

between the category assigned by the modular functor T*, constructed in [FSS2], to the circle* $\mathbb{L}_v$ *with marked points corresponding to the half-edges incident to v and the representation category of the algebra C_v.*

Proof. The proof requires the introduction of significant additional notation and is therefore relegated to the Appendix 2.A, see Theorem 33. □

This theorem paves the way for comparing our construction with the modular functor constructed in [FSS2].

Furthermore, in the case that the edges incident to the vertex v are labeled transparently by a single Hopf algebra H seen as the regular H-bicomodule algebra, then the category C_v–mod is equivalent to the Drinfeld center $Z(H\text{–mod})$ [FSS2, Remarks 3.5 (iii) and 5.23], which is equivalent to the category of representations of the Drinfeld double $D(H)$. These are also the possible labels for point-like excitations in the Kitaev model without defects, cf. [BK2].

2.2 Construction of a Kitaev model with defects

Having specified in the preceding subsections the algebraic input data for the Kitaev model and, in particular, having identified the possible labels for vertices, we are now in a position to construct, for any oriented surface Σ with labeled cell decomposition, the vector space and local projectors of the model.

We recall that we have for each plaquette $p \in \Sigma^2$ a semisimple Hopf algebra H_p, for each edge $e \in \Sigma^1$ a semisimple algebra K_e with a compatible bicomodule structure over the Hopf algebras of the incident plaquettes, and for each vertex $v \in \Sigma^0$ a left module Z_v over the algebra $C_v = H^*_{\Sigma_v^{\mathrm{sit}}} \oslash K_{\Sigma_v^{0.5}}$, introduced in Definition 5. We abbreviate

$$K_{\Sigma^1} := \bigotimes_{e \in \Sigma^1} K_e,$$
$$Z_{\Sigma^0} := \bigotimes_{v \in \Sigma^0} Z_v,$$

for the tensor products as vector spaces over $\Bbbk$. More precisely, K_{Σ^1} enters our construction of the local projectors and the Hamiltonian of the model not only as a vector space, but together with its structure as the regular $(\bigotimes_{e \in \Sigma^1} K_e)$-bimodule and its various co-actions with respect to the Hopf algebras labeling the plaquettes. Similarly, we will regard Z_{Σ^0} together with its C_v-module structure for every vertex $v \in \Sigma^0$.

The first thing we construct is the vector space, on which subsequently the commuting local projectors and the Hamiltonian will be defined.

Definition 9. The *state space* assigned to an oriented surface Σ with labeled cell decomposition as above is the vector space

$$\mathcal{H} := \mathrm{Hom}_{\Bbbk}(K_{\Sigma^1}, Z_{\Sigma^0}) = (\bigotimes_{e\in\Sigma^1} K_e^*) \otimes (\bigotimes_{v\in\Sigma^0} Z_v). \tag{2.7}$$

We refer to a tensor factor associated to an edge e or to a vertex v as a *local degree of freedom* associated to e or v, respectively.

Remarks 10.

1. In the standard Kitaev construction without defects, the vector space is a tensor product of copies of a single Hopf algebra H for every edge, which we interpret in our context as the regular bicomodule algebra over H (the transparent labeling), and for every vertex the dual vector space of a module over $D(H)$ [BMCA, BK2]. In our construction, we instead consider a module over the algebra C_v for every vertex $v \in \Sigma^0$ and the vector space duals of the bicomodule algebras for the edges. This dual version will make it easier to compare our ground-state spaces with the block spaces of [FSS2].

2. In order to define the state space $\mathcal{H}$ we are implicitly using that we do not only have the categories $(K_e\text{–mod})_{e\in\Sigma^1}$ and $(H_p\text{–mod})_{p\in\Sigma^2}$ as algebraic input data, but we also have the algebras $(K_e)_{e\in\Sigma^1}$ and $(H_p)_{p\in\Sigma^2}$, of which they are the representation categories. In other words, we need fibre functors on the categories $(K_e\text{–mod})_{e\in\Sigma^1}$ and $(H_p\text{–mod})_{p\in\Sigma^2}$ to the category of vector spaces in order to define $\mathcal{H}$ as a space of $\Bbbk$-linear homomorphisms.

3. Note that we are only defining a vector space over $\Bbbk$, and not a Hilbert space, i.e. we do not consider a scalar product here. Accordingly, when we speak of *projectors* on this vector space we always mean idempotent endomorphisms. By a *Hamiltonian* we mean a diagonalizable endomorphism.

2.2.1 Local representations of the vertex algebras on the state space

Next, we exhibit on the vector space $\mathcal{H}$ a natural C_v-bimodule structure for each vertex $v \in \Sigma^0$, that is *local* in the sense that it acts non-trivially only on the local degrees of freedom in a neighborhood of the vertex $v \in \Sigma^0$. This is analogous to the existence of local actions of the Drinfeld double $D(H)$ on the state space in the ordinary Kitaev model without defects for a semisimple Hopf algebra H [BMCA, BK2]. In our construction, however, the algebras C_v are in general not Hopf algebras and we only obtain *bi*module structures on $\mathcal{H}$. (A C_v-bimodule structure is equivalent to a left $(C_v \otimes C_v^{\mathrm{op}})$-action, where C_v^{op} has the opposite multiplication of C_v. Whenever C_v is a Hopf algebra, such as $D(H)$, any C_v-bimodule structure can be pulled back to a left C_v-action via the algebra map $(\mathrm{id} \otimes S) \circ \Delta : C_v \to C_v \otimes C_v^{\mathrm{op}}$, using the co-multiplication Δ and the antipode S of the Hopf algebra.)

Let $v \in \Sigma^0$ be any vertex. Recall from Subsection 2.1.3 that the algebra

$$C_v = H^*_{\Sigma_v^{\mathrm{sit}}} \oslash K_{\Sigma_v^{0.5}}$$

is a crossed product of $H^*_{\Sigma_v^{\mathrm{sit}}}$ and $K_{\Sigma_v^{0.5}}$ and contains these as subalgebras, and that

$$H^*_{\Sigma_v^{\mathrm{sit}}} = \bigotimes_{p\in\Sigma_v^{\mathrm{sit}}} H^*_p$$

is the tensor product of the algebras H^*_p for each site $p \in \Sigma_v^{\mathrm{sit}}$. A C_v-bimodule structure on $\mathcal{H}$ is therefore fully determined by a $K_{\Sigma_v^{0.5}}$-bimodule structure and H^*_p-bimodule structures for each site $p \in \Sigma_v^{\mathrm{sit}}$, provided that for each $p \in \Sigma_v^{\mathrm{sit}}$ the left and right actions of $K_{\Sigma_v^{0.5}}$ and H^*_p each satisfy the straightening formula (2.3) of the crossed product algebra $H^*_p \otimes K_{\Sigma_v^{0.5}}$, which we prove in Theorem 13.

We start by exhibiting a $K_{\Sigma_v^{0.5}}$-bimodule structure on the vector space $\mathcal{H}$. This is the analogon of the action of the Hopf algebra H for every vertex in the ordinary Kitaev model for a semisimple Hopf algebra H.

Definition 11. Let $v \in \Sigma^0$. The $K_{\Sigma_v^{0.5}}$*-bimodule structure on* $\mathcal{H}$

$$\widetilde{A}_v : K_{\Sigma_v^{0.5}} \otimes K^{\mathrm{op}}_{\Sigma_v^{0.5}} \otimes \mathcal{H} \longrightarrow \mathcal{H},$$

is defined on the vector space of linear maps $\mathcal{H} = \mathrm{Hom}_{\Bbbk}(K_{\Sigma^1}, Z_{\Sigma^0})$ in the standard way by pre-composing with the left action on K_{Σ^1} and post-composing with the left action on Z_{Σ^0}, which are defined as follows:

- Firstly, the vector space K_{Σ^1} becomes a left $K_{\Sigma_v^{0.5}}$-module as follows. Restrict the regular K_{Σ^1}-bimodule structure of K_{Σ^1}, seen as a left $(K_{\Sigma^1} \otimes K^{\mathrm{op}}_{\Sigma^1})$-action, to the subalgebra $K_{\Sigma_v^{0.5}} \subseteq K_{\Sigma^1} \otimes K^{\mathrm{op}}_{\Sigma^1}$.
- Secondly, the vector space Z_{Σ^0} becomes a left $K_{\Sigma_v^{0.5}}$-module as follows. Restrict the given C_v-module structure on Z_v to the subalgebra $K_{\Sigma_v^{0.5}} \subseteq \bigotimes_{v\in\Sigma^0}(H^*_{\Sigma_v^{\mathrm{sit}}} \otimes K_{\Sigma_v^{0.5}}) = C_v$ and extend the action trivially to the vector space $Z_{\Sigma^0} = Z_v \otimes \bigotimes_{w\in\Sigma^0\setminus\{v\}} Z_w$.

Next we will exhibit, for any site $p \in \Sigma_v^{\mathrm{sit}}$ incident to a vertex $v \in \Sigma^0$, an H^*_p-bimodule structure on $\mathcal{H}$.

Recall that Σ_p^{sit} denotes the set of incidences of a vertex with a given plaquette p (which we also call *sites*) and denote by $\Sigma_p^{1.5}$ the set of incidences of an edge with the given plaquette p (which we call *plaquette edges*). We consider their union $\Sigma_p^{\mathrm{sit}} \cup \Sigma_p^1$ together with a cyclic order on it, given by the clockwise direction along the boundary of p with respect to the orientation of Σ, as illustrated in Figure 2.7

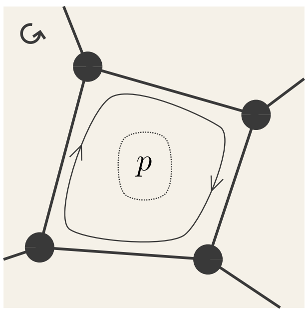


Figure 2.7: Cyclic order on the set $\Sigma_p^{\mathrm{sit}} \cup \Sigma_p^1$ of sites and plaquette edges of a plaquette p

Furthermore, for any plaquette edge $e \in \Sigma_p^1$ at the plaquette p, let the sign $\varepsilon_p(e) \in \{+1, -1\}$ be positive if the plaquette edge $e \in \Sigma_p^1$ is clockwise directed around the plaquette p:

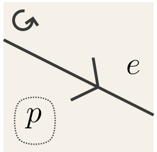


Figure 2.8: A plaquette edge e with sign $\varepsilon_p(e) := +1$

and negative if $e \in \Sigma^1_p$ is directed counter-clockwise around p:

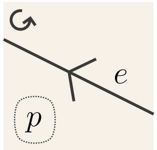


Figure 2.9: A plaquette edge e with sign $\varepsilon_p(e) := -1$

Recall that, attached to each plaquette $p \in \Sigma^2$, there is a Hopf algebra H_p. Now, depending on choice of a site $v \in \Sigma^{\mathrm{sit}}_p$ at p, we define an H^*_p-bimodule structure on the vector space $\mathcal{H}$. This is the analogon of the action of the dual Hopf algebra H^* for every site in the ordinary Kitaev model for a semisimple Hopf algebra H.

Definition 12. Let $p \in \Sigma^2$. We define, for each site $v \in \Sigma^{\mathrm{sit}}_p$, the H^*_p-*bimodule structure on* $\mathcal{H}$, or left action of the enveloping algebra $H^*_p \otimes (H^*_p)^{\mathrm{op}}$,

$$\widetilde{B}_{(p,v)} : H^*_p \otimes (H^*_p)^{\mathrm{op}} \otimes \mathcal{H} \longrightarrow \mathcal{H},$$

by the following left and right H^*_p-actions on $\mathcal{H}$.

- We start by declaring that H^*_p acts from the left on $\mathcal{H} = (\bigotimes_{e\in\Sigma^1} K^*_e) \otimes (\bigotimes_{w\in\Sigma^0} Z_w)$ by the action of $H^*_p \subseteq H^*_{\Sigma^{\mathrm{sit}}_v} \circleddash K_{\Sigma^{0.5}_v}$ on the $(H^*_{\Sigma^{\mathrm{sit}}_v} \circleddash K_{\Sigma^{0.5}_v})$-module Z_v and by acting as the identity on the remaining tensor factors of $\mathcal{H}$.
- For the right action of H^*_p on $\mathcal{H}$, we use the total order on the set $(\Sigma^{\mathrm{sit}}_p \cup \Sigma^{1.5}_p) \setminus \{v\}$ starting right after $v \in \Sigma^{\mathrm{sit}}_p$ in $\Sigma^{\mathrm{sit}}_p \cup \Sigma^{1.5}_p$ with respect to the cyclic order declared above, given by the clockwise direction around the plaquette p. We first exhibit individual right H^*_p-actions on the tensor factors of $(\bigotimes_{e\in\Sigma^1_p} K^*_e) \otimes (\bigotimes_{w\in\Sigma^0_p\setminus\{v\}} Z_w)$:
  - For any $e \in \Sigma^{1.5}_p$, the vector space K^*_e becomes a right H^*_p-module as follows. K_e is a right $H_p^{\varepsilon_p(e)}$-comodule and, hence, a left $(H^*_p)^{\varepsilon_p(e)}$-module. Thus the vector space dual K^*_e becomes a right $(H^*_p)^{\varepsilon_p(e)}$-module, and finally, by pulling back along the algebra isomorphism $?^{\langle\varepsilon_p(e)\rangle} : H^*_p \to H_p^{*\varepsilon_p(e)}$, a right H^*_p-module.

    Recall that $?^{\langle+1\rangle} \overset{\text{def}}{=} \mathrm{id}_{H^*_p}$ and $?^{\langle-1\rangle} \overset{\text{def}}{=} S$, the antipode of H^*_p. Explicitly, this right H^*_p-action is given by

    $$\begin{aligned} K^*_e \otimes H^*_p &\longrightarrow K^*_e, \\ \varphi \otimes f &\longmapsto \left(k \mapsto \varphi\left(k_{(0)} f\left(k_{(\varepsilon_p(e))}^{\langle\varepsilon_p(e)\rangle}\right)\right)\right). \end{aligned}$$

  - For any $w \in \Sigma^{\mathrm{sit}}_p \setminus \{v\}$, the vector space Z_w becomes a right H^*_p-module as follows. The $(H^*_{\Sigma^2_w} \circleddash K_{\Sigma^1_w})$-module Z_w comes with a left H^*_p-action since $H^*_p \subseteq H^*_{\Sigma^2_w} \circleddash K_{\Sigma^1_w}$ is a subalgebra. We let H^*_p act on Z_w from the right by pulling back this left action along the antipode $?^{\langle-1\rangle} = S : H^*_p \to H^*_p$.

Then we declare H_p^* to act from the right on the tensor product $(\bigotimes_{e\in\Sigma_p^1} K_e^*)\otimes(\bigotimes_{w\in\Sigma_p^0\setminus\{v\}} Z_w)$ by applying the co-multiplication on H_p^* suitably many times and then acting individually on the tensor factors in the sequence given by the image of the clockwise linear order that we have prescribed on the set $(\Sigma_p^{\mathrm{sit}} \cup \Sigma_p^{1.5}) \setminus \{v\}$ under the map $(\Sigma_p^{\mathrm{sit}} \cup \Sigma_p^{1.5}) \setminus \{v\} \to (\Sigma_p^0 \cup \Sigma_p^1) \setminus \{v\}$ that assigns to a site its underlying vertex and to a plaquette edge its underlying edge. Finally, this gives a right H_p^*-action on $\mathcal{H} = (\bigotimes_{e\in\Sigma^1} K_e^*) \otimes (\bigotimes_{w\in\Sigma^0} Z_w)$ by acting with the identity on all remaining tensor factors.

So far we have defined, in Definitions 11 and 12, on the vector space $\mathcal{H}$ an $K_{\Sigma_v^{0.5}}$-bimodule structure $\widetilde{A}_v$ for each vertex $v \in \Sigma^0$ and an H_p^*-bimodule structure $\widetilde{B}_{(p,v)}$ for each site $p \in \Sigma_v^{\mathrm{sit}}$. These are analogous to the actions of the Hopf algebra H and the dual Hopf algebra H^* defined for each site in the ordinary Kitaev model without defects. Just as the latter are shown to interact with each other non-trivially, giving a representation of the Drinfeld double $D(H)$ at each site [BMCA], we will now proceed to study how the bimodule structures $\widetilde{A}_v$ and $\widetilde{B}_{(p,v')}$ of $K_{\Sigma_v^{0.5}}$ and H_p^* for various v and (p, v') interact with each other.

In order to simplify the proof we will make a certain regularity assumption on the cell decomposition of the surface Σ: We call a cell decomposition *regular* if it has no looping edges, i.e. there is no edge which has the same source vertex as target vertex and if the Poincaré-dual cell decomposition also has no looping edges, i.e. in the original cell decomposition there is no plaquette that has two incidences with one and the same edge (on its two sides).

Theorem 13. *Let $\mathcal{H}$ be the vector space defined in Definition 9 for an oriented surface Σ with a labelled cell decomposition. Recall from Definitions 11 and 12 the $K_{\Sigma_v^{0.5}}$-bimodule structure $\widetilde{A}_v$ on $\mathcal{H}$ for every vertex $v \in \Sigma^0$, and the H_p^*-bimodule structure $\widetilde{B}_{(p,v)}$ on $\mathcal{H}$ for every plaquette $p \in \Sigma^2$ together with incident site $v' \in \Sigma_p^{sit}$. Then*

- *For any pair of vertices $v_1 \neq v_2 \in \Sigma^0$, the actions $\widetilde{A}_{v_1}$ and $\widetilde{A}_{v_2}$ commute with each other.*
- *For any pair of sites $(p_1 \in \Sigma^2, v_1 \in \Sigma_{p_1}^{sit})$ and $(p_2 \in \Sigma^2, v_2 \in \Sigma_{p_2}^{sit})$ such that $p_1 \neq p_2$, the actions $\widetilde{B}_{(p_1,v_1)}$ and $\widetilde{B}_{(p_2,v_2)}$ commute with each other.*
- *Assume that the cell decomposition of Σ is regular. For any site $(p \in \Sigma^2, v \in \Sigma_p^{sit})$, the actions $\widetilde{A}_v$ and $\widetilde{B}_{(p,v)}$ compose to give on $\mathcal{H}$ a bimodule structure over the crossed product algebra $H_{(p,v)}^* \circledcirc K_{\Sigma_v^{0.5}}$,*

$$\begin{aligned}\widetilde{B}_{(p,v)}\widetilde{A}_v : H_p^* \otimes K_{\Sigma_v^{0.5}} \otimes (H_p^* \otimes K_{\Sigma_v^{0.5}})^{\mathrm{op}} \otimes \mathcal{H} &\longrightarrow \mathcal{H},\\ f \otimes k \otimes f' \otimes k' \otimes x &\longmapsto \widetilde{B}_{(p,v)}^{f\otimes f'}\widetilde{A}_v^{k\otimes k'}(x).\end{aligned}$$

Proof.

- The left $K_{\Sigma_{v_1}^{0.5}}$- and $K_{\Sigma_{v_2}^{0.5}}$-actions act as the identity on all tensor factors of $\mathcal{H}$ except on Z_{v_1} and Z_{v_2}, respectively. It is thus clear that they commute for $v_1 \neq v_2$.

  The right $K_{\Sigma_{v_1}^{0.5}}$- and $K_{\Sigma_{v_2}^{0.5}}$-actions only have a common tensor factor on which they do not act by the identity for every edge $e \in \Sigma^1$ that joins the vertices v_1 and v_2. Such an edge is directed away from one of the vertices and directed towards the other. Hence, the action for one of the vertices comes from left multiplication of K_e and the other one from right multiplication, so they commute.

- The left $H^*_{p_1}$- and $H^*_{p_2}$-actions act as the identity on all tensor factors of $\mathcal{H}$ except on Z_{v_1} and Z_{v_2}, respectively. It is thus clear that they commute for $v_1 \neq v_2$. In the remaining case $v_1 = v_2 =: v$, $H^*_{p_1}$ and $H^*_{p_2}$ are commuting subalgebras in C_v. Since their actions on Z_v are by Definition 12 the restrictions of the C_v-action that Z_v comes with, they must therefore commute.

  The right $H^*_{p_1}$- and $H^*_{p_2}$-actions only have a common tensor factor on which they do not act by the identity for every vertex $v \in \Sigma^0$ and for every edge $e \in \Sigma^1$ that lies in the boundaries of both plaquettes p_1 and p_2. For any such vertex v, the two actions come from the $(H^*_{\Sigma^{\mathrm{sit}}_v} \otimes K_{\Sigma^{0.5}_v})$-action on Z_v restricted to the two subalgebras $H^*_{p_1}$ and $H^*_{p_2}$, respectively. These subalgebras commute inside $H^*_{\Sigma^{\mathrm{sit}}_v} \otimes K_{\Sigma^{0.5}_v}$, therefore showing the claim.

- The left $K_{\Sigma^{0.5}_v}$- and H^*_p-actions on $\mathcal{H}$ are simply the restrictions of the left C_v-action on Z_v to $K_{\Sigma^{0.5}_v}$ and H^*_p, respectively, and the identity on all other tensor factors of $\mathcal{H}$. Hence, by construction they satisfy the commutation relations of the crossed product algebra $H^*_p \otimes K_{\Sigma^{0.5}_v} \subseteq C_v$, see also (2.5).

  The right $K_{\Sigma^{0.5}_v}$- and H^*_p-actions on $\mathcal{H}$ are non-trivial only on the tensor factors $\bigotimes_{e\in\Sigma^1_v} K^*_e$ and $(\bigotimes_{e\in\Sigma^1_p} K^*_e) \otimes (\bigotimes_{w\in\Sigma^0_p\setminus\{v\}} Z_w)$, respectively. We can therefore restrict our attention to the vector space $(\bigotimes_{e\in\Sigma^1_v\cup\Sigma^1_p} K^*_e) \otimes (\bigotimes_{w\in\Sigma^0_p\setminus\{v\}} Z_w)$, on which $K_{\Sigma^{0.5}_v}$ and H^*_p act from the right.

  For convenience, for the remainder of the proof we now switch to the dual vector space $(\bigotimes_{e\in\Sigma^1_v\cup\Sigma^1_p} K_e) \otimes (\bigotimes_{w\in\Sigma^0_p\setminus\{v\}} Z^*_w)$, with the corresponding left actions of $K_{\Sigma^{0.5}_v}$ and H^*_p. With the notation of Subsection 2.1.2, let $e_p, e'_p \in \Sigma^{0.5}_v$ be the half-edges at v on the two sides of the site $p \in \Sigma^{\mathrm{sit}}_v$, with signs $\varepsilon := \varepsilon(e_p)$ and $\varepsilon' := \varepsilon(e'_p)$. The $K_{\Sigma^{0.5}_v}$- and H^*_p-actions only overlap on the tensor factors $(K_e)_{e\in\Sigma^1_v\cap\Sigma^1_p}$ corresponding to the edges underlying the half-edges $e_p, e'_p \in \Sigma^{0.5}_v$. Due to our regularity assumption on the cell decomposition, the half-edges e_p and e'_p have distinct underlying edges. Then the action of $K_{\Sigma^{0.5}_v} = (K^{\varepsilon}_{e_p} \otimes K^{\varepsilon'}_{e'_p}) \otimes \bigotimes_{e\in\Sigma^{0.5}_v\setminus\{e_p,e'_p\}} K^{\varepsilon(e)}_e$ on $\bigotimes_{e\in\Sigma^1_v} K_e$, which is a tensor product of algebras, decomposes into a tensor product of the action of $K^{\varepsilon}_{e_p} \otimes K^{\varepsilon'}_{e'_p}$ on $K_{e_p} \otimes K_{e'_p}$ and the action of $\bigotimes_{e\in\Sigma^{0.5}_v\setminus\{e_p,e'_p\}} K^{\varepsilon(e)}_e$ on $\bigotimes_{e\in\Sigma^1_v\setminus\{e_p,e'_p\}} K_e$. On the latter vector space, H^*_p does not act non-trivially by our regularity assumption on the cell decomposition. Hence, it remains to consider the interactions of the left actions of $K^{\varepsilon}_{e_p} \otimes K^{\varepsilon'}_{e'_p}$ and H^*_p on the vector space $K_{e_p} \otimes K_{e'_p} \otimes (\bigotimes_{e\in\Sigma^1_p\setminus\{e_p,e'_p\}} K_e) \otimes (\bigotimes_{w\in\Sigma^0_p\setminus\{v\}} Z^*_w)$. We abbreviate by $V := (\bigotimes_{e\in\Sigma^1_p\setminus\{e_p,e'_p\}} K_e) \otimes (\bigotimes_{w\in\Sigma^0_p\setminus\{v\}} Z^*_w)$ the tensor factor on which only H^*_p acts non-trivially. Furthermore, without loss of generality, we write the left H^*_p-action on V in terms of the Sweedler notation for the corresponding right H_p-coaction, $V \to V \otimes H_p, v \mapsto v_{(0)} \otimes v_{(1)}$:

  $$H^*_p \otimes V \longrightarrow V, v \longmapsto f.v =: f(v_{(1)})v_{(0)}.$$

  Finally, it is left to analyze the interaction between the H^*_p-action

  $$\begin{aligned} H^*_p \otimes K_{e_p} \otimes K_{e'_p} \otimes V &\longrightarrow K_{e_p} \otimes K_{e'_p} \otimes V, \\ f \otimes x \otimes x' \otimes v &\longmapsto f_{(3)}.x \otimes f_{(1)}.x' \otimes f_{(2)}.v \\ &\qquad = f\left(x'^{\langle\varepsilon'\rangle}_{(\varepsilon')} v_{(1)} x^{\langle-\varepsilon\rangle}_{(-\varepsilon)} \right) x_{(0)} \otimes x'_{(0)} \otimes v_{(0)}, \end{aligned}$$

and the $(K^{\varepsilon}_{e_p} \otimes K^{\varepsilon'}_{e'_p})$-action

$$\begin{aligned}(K^{\varepsilon}_{e_p} \otimes K^{\varepsilon'}_{e'_p}) \otimes K_{e_p} \otimes K_{e'_p} \otimes V &\longrightarrow K_{e_p} \otimes K_{e'_p} \otimes V,\\ a \otimes a' \otimes x \otimes x' \otimes v &\longmapsto a.x \otimes a'.x' \otimes v\\ &\qquad (a \cdot_{\varepsilon} x) \otimes (a' \cdot_{\varepsilon'} x') \otimes v,\end{aligned}$$

where $\cdot_{\varepsilon}$ and $\cdot_{\varepsilon'}$ denote the multiplication in $K^{\varepsilon}_{e_p}$ and $K^{\varepsilon'}_{e'_p}$, respectively, that is

$$a \cdot_{\varepsilon} x := \begin{cases} ax, & \varepsilon = +1,\\ xa, & \varepsilon = -1.\end{cases}$$

It remains to show that that these actions satisfy the straightening formula

$$f(a'^{\langle -\varepsilon' \rangle}_{(\varepsilon')} \cdot ? \cdot a^{\langle \varepsilon \rangle}_{(-\varepsilon)}).(a_{(0)} \otimes a'_{(0)}).(x \otimes x' \otimes v) = (a \otimes a').f.(x \otimes x' \otimes v),$$

for all $f \in H^*_p$, $a \otimes a' \in K^{\varepsilon}_{e_p} \otimes K^{\varepsilon'}_{e'_p}$ and $x \otimes x' \otimes v \in K_{e_p} \otimes K_{e'_p} \otimes V$. Indeed, the following calculation, which is analogous to the calculation in the proof of [BMCA, Theorem 1] but more general and at the same time shorter, verifies this.

$$\begin{aligned} &f\left(a'^{\langle -\varepsilon' \rangle}_{(\varepsilon')} \cdot ? \cdot a^{\langle \varepsilon \rangle}_{(-\varepsilon)}\right).(a_{(0)} \otimes a'_{(0)}).(x \otimes x' \otimes v)\\ &\quad = f\left(a'^{\langle -\varepsilon' \rangle}_{(\varepsilon')} \cdot ? \cdot a^{\langle \varepsilon \rangle}_{(-\varepsilon)}\right).((a_{(0)} \cdot_{\varepsilon} x) \otimes (a'_{(0)} \cdot_{\varepsilon'} x') \otimes v)\\ &\quad = f\left(a'^{\langle -\varepsilon' \rangle}_{(2\varepsilon')} \cdot (a'_{(0)} \cdot_{\varepsilon'} x')^{\langle \varepsilon' \rangle}_{(\varepsilon')} \cdot v_{(1)} \cdot (a_{(0)} \cdot_{\varepsilon} x)^{\langle -\varepsilon \rangle}_{(-\varepsilon)} \cdot a^{\langle \varepsilon \rangle}_{(-2\varepsilon)}\right)\\ &\qquad ((a_{(0)} \cdot_{\varepsilon} x)_{(0)} \otimes (a'_{(0)} \cdot_{\varepsilon'} x')_{(0)} \otimes v_{(0)})\\ &\quad = f\left(a'^{\langle -\varepsilon' \rangle}_{(2\varepsilon')} \cdot a'^{\langle \varepsilon' \rangle}_{(\varepsilon')} \cdot x'^{\langle \varepsilon' \rangle}_{(\varepsilon')} \cdot v_{(1)} \cdot x^{\langle -\varepsilon \rangle}_{(-\varepsilon)} \cdot a^{\langle -\varepsilon \rangle}_{(-\varepsilon)} \cdot a^{\langle \varepsilon \rangle}_{(-2\varepsilon)}\right)\\ &\qquad ((a_{(0)} \cdot_{\varepsilon} x_{(0)}) \otimes (a'_{(0)} \cdot_{\varepsilon'} x'_{(0)}) \otimes v_{(0)})\\ &\quad = f\left(x'^{\langle \varepsilon' \rangle}_{(\varepsilon')} \cdot v_{(1)} \cdot x^{\langle -\varepsilon \rangle}_{(-\varepsilon)}\right)((a \cdot_{\varepsilon} x_{(0)}) \otimes (a' \cdot_{\varepsilon'} x'_{(0)}) \otimes v_{(0)})\\ &\quad = (a \otimes a').\left(f\left(x'^{\langle \varepsilon' \rangle}_{(\varepsilon')} \cdot v_{(1)} \cdot x^{\langle -\varepsilon \rangle}_{(-\varepsilon)}\right)(x_{(0)} \otimes x'_{(0)} \otimes v_{(0)})\right)\\ &\quad = (a \otimes a').\big(f.(x \otimes x' \otimes v)\big).\end{aligned}$$

This proves that H^*_p and $K^{\varepsilon}_{e_p} \otimes K^{\varepsilon'}_{e'_p}$ together give a representation of the crossed product algebra $H^*_p \circledcirc (K^{\varepsilon}_{e_p} \otimes K^{\varepsilon'}_{e'_p})$, as claimed.

□

Remark 14. Taking all sites $p \in \Sigma^{\mathrm{sit}}_v$ around a given vertex $v \in \Sigma^0$ together, we thus get, due to Theorem 13, on $\mathcal{H}$ a bimodule structure over the vertex algebra C_v. It is remarkable that this makes the crossed product algebra structure on C_v show up naturally – analogous to the appearance of the algebra structure of the Drinfeld double in the commutation relation of the vertex and plaqette actions in the standard Kitaev model without defects.

2.2.2 Towards local projectors: Symmetric separability idempotents for bicomodule algebras

Before we proceed to use the bimodule structures on the state space $\mathcal{H}$ defined in Subsection 2.2.1 to define commuting local projectors on the vector space $\mathcal{H}$, we need to invoke another algebraic ingredient.

The standard Kitaev construction for a semisimple Hopf algebra H makes use of the Haar integrals of H and of H^*, in order to define commuting local projectors on the state space via the actions of H and H^*. The *Haar integral* of a semisimple Hopf algebra H over $\mathbb{k}$ is the unique element $\ell \in H$ satisfying $x\ell = \varepsilon(x)\ell = \ell x$ for all $x \in H$ and $\varepsilon(\ell) = 1$. This means that ℓ is the central idempotent which projects to the H-invariants: for any H-module M, we have $\ell.M = M^H := \{m \in M \mid h.m = \varepsilon(h)m \quad \forall h \in H\}$. Furthermore, $\ell \in H$ is cocommutative, i.e. $\ell_{(1)} \otimes \ell_{(2)} = \ell_{(2)} \otimes \ell_{(1)}$ in Sweedler notation. The idempotence, centrality and cocommutativity of the Haar integral are crucial in showing that the Haar integral gives rise to commuting local projectors in the standard Kitaev construction [BMCA].

In our setting, instead of a semisimple Hopf algebra acting on the state space, we have, for each vertex $v \in \Sigma^0$, a bimodule structure on the state space over a semisimple (bi-)comodule algebra $K_{\Sigma^{0.5}_v}$. Hence, we need a notion replacing the Haar integral, that works in this setting. Our main insight is that the suitable generalization of the Haar integral to our setting is the unique symmetric separability idempotent, which exists for any semisimple algebra over an algebraically closed field $\mathbb{k}$ with characteristic zero.

Definition 15. Let A be an algebra over a field $\mathbb{k}$. A *symmetric separability idempotent* for A is an element $p \in A \otimes A$, which we write as $p = p^1 \otimes p^2 \in A \otimes A$ omitting the summation symbol, satisfying

$$(x \cdot p^1) \otimes p^2 = p^1 \otimes (p^2 \cdot x) \quad \forall x \in A, \tag{2.8}$$

$$p^1 \cdot p^2 = 1, \tag{2.9}$$

$$p^1 \otimes p^2 = p^2 \otimes p^1, \qquad \text{(symmetry)} \tag{2.10}$$

where on both sides of equation (2.8) and in equation (2.9) we are using the multiplication in A.

The properties (2.8) and (2.9) immediately imply that $p^1 \otimes p^2$ is an idempotent when seen as an element of the enveloping algebra $A \otimes A^{\mathrm{op}}$.

Remarks 16.

1. The structure of a separability idempotent, i.e. an element $p^1 \otimes p^2 \in A \otimes A$ satisfying (2.8) and (2.9), is equivalent to an A-bimodule map $s : A \longrightarrow A \otimes A$ that is a section of the multiplication $m : A \otimes A \longrightarrow A$, by defining $s(x) := p^1 \otimes p^2 x$ for all $x \in A$. An algebra endowed with such a structure is called *separable* and, in general, such a separability structure might not exist or be unique. A symmetric separability structure, however, is always unique – see the end of the proof of Proposition 17.

2. Representation-theoretically, a separability idempotent $p^1 \otimes p^2 \in A \otimes A^{\mathrm{op}}$ plays the role of projecting to the subspace of invariants for any A-bimodule M. Indeed, due to property (2.8), one has

$$p^1.M.p^2 = M^A := \{m \in M \mid a.m = m.a \quad \forall a \in A\} \subseteq M.$$

This is in analogy to the Haar integral $\ell \in H$ of a semisimple Hopf algebra H which projects to the invariants $\ell.M = M^H := \{m \in M \mid h.m = \varepsilon(h)m \quad \forall h \in H\}$ of any left H-module M.

Just as every finite-dimensional semisimple Hopf algebra over a field $\Bbbk$ has a unique Haar integral, for every finite-dimensional semisimple $\Bbbk$-algebra there exists a unique symmetric separability idempotent:

Proposition 17 ([A])**.** *Let A be a finite-dimensional semisimple algebra over a field $\Bbbk$ which is algebraically closed and of characteristic zero. Then there exists a unique symmetric separability idempotent $p^1 \otimes p^2 \in A \otimes A^{\mathrm{op}}$ for A.*

Proof. For a more detailed proof, see [A, Thm. 3.1, Cor. 3.1.1]. Here we recall the main idea that the unique symmetric separability idempotent can be described in terms of the trace form on A, because we will use this description in Proposition 19.

Due to semisimplicity, the following symmetric bilinear pairing on A is non-degenerate:

$$\begin{aligned} T : A \otimes A &\longrightarrow \Bbbk, \\ a \otimes b &\longmapsto t(a \cdot b) := \mathrm{tr}_A(L_{a \cdot b}), \end{aligned}$$

defined in terms of the trace form $t : A \to \Bbbk, a \mapsto \mathrm{tr}_A(L_a)$, where $L_?$ denotes the left multiplication of A. In fact, this non-degenerate bilinear pairing turns A into a symmetric special Frobenius algebra. Consider the isomorphism $\#_T : A \xrightarrow{\sim} A^*, a \mapsto t(a \cdot -)$, induced by this non-degenerate bilinear pairing. This is an isomorphism of A-bimodules. It induces an isomorphism $A \otimes A \xrightarrow{\sim} A^* \otimes A \cong \mathrm{End}_{\Bbbk}(A)$. Consider the pre-image $p \in A \otimes A$ of the identity id_A under this isomorphism. As usual, we write an element $p \in A \otimes A$ as $p = p^1 \otimes p^2$, omitting the summation symbol. In fact, if we choose a basis $(p_i^1)_i$ for A and let $(p_i^2)_i$ be its dual basis of A with respect to the non-degenerate pairing T, then $p^1 \otimes p^2$ is the sum $\sum_i p_i^1 \otimes p_i^2$. With this definition of $p^1 \otimes p^2 \in A \otimes A$ it is straightforward to verify the defining properties (2.8), (2.9) and (2.10) of a symmetric separability idempotent.

To prove that the symmetric separability idempotent is unique, let $p^1 \otimes p^2, q^1 \otimes q^2 \in A \otimes A^{\mathrm{op}}$ be any two symmetric separability idempotents for A. Then they are equal by the following computation:

$$\begin{aligned} p^1 \otimes p^2 &\overset{(2.9)}{=} q^1 q^2 p^1 \otimes p^2 \overset{(2.8)}{=} q^1 p^1 \otimes p^2 q^2 \overset{(2.10)}{=} q^2 p^1 \otimes p^2 q^1 \\ &\overset{(2.8)}{=} q^2 \otimes p^2 p^1 q^1 \overset{(2.10)}{=} q^2 \otimes p^1 p^2 q^1 \overset{(2.9)}{=} q^2 \otimes q^1 \overset{(2.10)}{=} q^1 \otimes q^2, \end{aligned}$$

using the defining properties (2.8), (2.9) and (2.10). □

Example 18. Let H be a finite-dimensional semisimple Hopf algebra over $\Bbbk$ with Haar integral $\ell \in H$. Then the symmetric separability idempotent for H is $\ell_{(1)} \otimes S(\ell_{(2)}) \in H \otimes H^{\mathrm{op}}$.

Indeed, the invariance property of the Haar integral, $x\ell = \varepsilon(x)\ell$ for all $x \in H$, implies the corresponding invariance property (2.8) of $\ell_{(1)} \otimes S(\ell_{(2)})$. The normalization $\varepsilon(\ell) = 1$ of the Haar integral implies the corresponding normalization property 2.9 for the separability idempotent. Finally, using that the Haar integral is two-sided, which implies $S(\ell) = \ell$, it can be shown that $\ell_{(1)} \otimes S(\ell_{(2)})$ is symmetric.

Hence we see that, in the sense of this example, the symmetric separability idempotent of a semisimple algebra generalizes the Haar integral of a semisimple Hopf algebra.

In our construction of a Kitaev model, however, we are not only dealing with semisimple algebras, but semisimple algebras together with a compatible bicomodule structure. On the other hand, the Haar integral $\ell \in H$ has the property of being cocommutative, $\ell_{(1)} \otimes \ell_{(2)} = \ell_{(2)} \otimes \ell_{(1)}$, which is crucial in showing that it gives rise to commuting projectors in [BMCA] and we have not exhibited an analogous property of the symmetric separability idempotent. In the following proposition we prove such a property, which holds for the symmetric separability idempotent of a semisimple (bi-)comodule algebra and which generalizes the cocommutativity of the Haar integral, see Example 20.

Proposition 19. *Let H be a semisimple Hopf algebra over $\Bbbk$ and let K be a semisimple right H-comodule algebra with symmetric separability idempotent $p^1 \otimes p^2 \in K \otimes K^{\mathrm{op}}$. Consider the right H-coaction on the tensor product $K \otimes K^{\mathrm{op}}$:*

$$\begin{aligned} K \otimes K^{\mathrm{op}} &\longrightarrow K \otimes K^{\mathrm{op}} \otimes H, \\ k \otimes k' &\longmapsto k_{(0)} \otimes k'_{(0)} \otimes k_{(1)} k'_{(1)}. \end{aligned}$$

Then $p^1 \otimes p^2 \in K \otimes K^{\mathrm{op}}$ is an H-coinvariant element of $K \otimes K^{\mathrm{op}}$, i.e. $p^1_{(0)} \otimes p^2_{(0)} \otimes p^1_{(1)} p^2_{(1)} = p^1 \otimes p^2 \otimes 1_H \in K \otimes K^{\mathrm{op}} \otimes H$, and this is equivalent to

$$p^1_{(0)} \otimes p^1_{(1)} \otimes p^2 = p^1 \otimes S(p^2_{(1)}) \otimes p^2_{(0)} \in K \otimes H \otimes K^{\mathrm{op}}. \tag{2.11}$$

Analogously, if K is a left H-comodule algebra, then

$$p^1_{(0)} \otimes p^1_{(-1)} \otimes p^2 = p^1 \otimes S(p^2_{(-1)}) \otimes p^2_{(0)} \in K \otimes H \otimes K^{\mathrm{op}}. \tag{2.12}$$

Proof. Without loss of generality we only show the case where K is a right H-comodule algebra. Recall from the proof of Proposition 17 that the symmetric separability idempotent $p^1 \otimes p^2 \in K \otimes K^{\mathrm{op}}$ for K can be characterized in terms of the multiplication and the trace form $t : K \longrightarrow \Bbbk$ on K, namely by $t(p^1 \cdot x)p^2 = x \ \forall x \in K$, as explained in the proof of Proposition 17. Another way of phrasing this is that the map $K^* \longrightarrow K$ defined by $f \longmapsto f(p^1)p^2$ is the inverse of the isomorphism $K \longrightarrow K^*, k \longmapsto t(? \cdot k)$ induced by the non-degenerate pairing $t \circ \mu$, where $\mu : K \otimes K \longrightarrow K$ is the multiplication on K.

The crucial step for the present proof is the observation that the multiplication and the trace form on K are morphisms of H-comodules if K is an H-comodule algebra. For the multiplication this means that $x_{(0)} y_{(0)} \otimes x_{(1)} y_{(1)} = (xy)_{(0)} \otimes (xy)_{(1)} \ \forall x, y \in K$, which holds by definition of a comodule algebra, see Definition 1. As for the H-colinearity of the trace form, note that $t = \mathrm{ev}_K \circ (\mu \otimes \mathrm{id}_{K^*}) \circ (\mathrm{id}_K \otimes \mathrm{coev}_K)$, where $\mu : K \otimes K \to K$ denotes the multiplication, and $\mathrm{coev}_K : \Bbbk \longrightarrow K \otimes K^*$ and $\mathrm{ev}_K : K \otimes K^* \longrightarrow \Bbbk$ are the standard coevaluation and evaluation morphisms for vector spaces. Due the involutivity of the antipode S of H, both ev_K and coev_K are morphisms of right H-comodules for the H-comodule structure on the dual K^* given by $K^* \longrightarrow K^* \otimes H, \varphi \longmapsto \varphi_{(0)} \otimes \varphi_{(1)}$, where $\varphi_{(0)}(x)\varphi_{(1)} := \varphi(x_{(0)})S(x_{(1)})$ for all $x \in K$. (We are here implicitly using the canonical trivial pivotal structure on the tensor category of right H-comodules, which exists due to the involutivity of the antipode of H.) Since therefore the trace form t is composed only of morphisms of right H-comodules, it is itself a morphism of right H-comodules, i.e.

$$t(k_{(0)})k_{(1)} = t(k)1_H \quad \forall k \in K. \tag{2.13}$$

As a consequence, the isomorphism $K \longrightarrow K^*, k \longmapsto t(? \cdot k)$ induced by the pairing $t \circ \mu$ is an isomorphism of H-comodules. Indeed, for all $x \in K$ one has $t(xk_{(0)})k_{(1)} = t(x_{(0)}k_{(0)})S(x_{(2)})x_{(1)}k_{(1)} \overset{(2.13)}{=} t(x_{(0)}k)S(x_{(1)}) \overset{\mathrm{def}}{=} (t(? \cdot k))_{(0)}(x)(t(? \cdot k))_{(1)}$.

This immediately implies that the inverse map, $K^* \longrightarrow K, \varphi \longmapsto \varphi(p^1)p^2$, must also be a morphism of H-comodules, which spelled out means that $\varphi(p^1_{(0)})p^2 \otimes S(p^1_{(1)}) \overset{\text{def}}{=} \varphi_{(0)}(p^1)p^2 \otimes \varphi_{(1)} = \varphi(p^1)p^2_{(0)} \otimes p^2_{(1)}$ for all $\varphi \in K^*$. This implies the equation (2.11) of the claim. To show that this is equivalent to $p^1 \otimes p^2 \in K \otimes K^{\mathrm{op}}$ being H-coinvariant, we compute

$$p^1_{(0)} \otimes p^2_{(0)} \otimes p^1_{(1)}p^2_{(1)} \overset{(2.11)}{=} p^1 \otimes p^2_{(0)} \otimes S(p^2_{(1)})p^2_{(2)} = p^1 \otimes p^2 \otimes 1_H.$$

□

Example 20. Let H be a semisimple Hopf algebra and consider it as the regular H-bicomodule algebra, as in Example 2.(1). Recall that for H the symmetric separability idempotent is $p^1 \otimes p^2 = \ell_{(1)} \otimes S(\ell_{(2)}) \in H \otimes H$. Let us spell out Proposition 19 for the left and right H-comodule structures on the regular bicomodule algebra H. Equation (2.11) boils down to the equation $(\ell_{(1)})_{(1)} \otimes (\ell_{(1)})_{(2)} \otimes S(\ell_{(3)}) = \ell_{(1)} \otimes S(S(\ell_{(2)})_{(2)}) \otimes S(\ell_{(2)})_{(1)}$. But due to $S^2 = \mathrm{id}_H$ both sides of the equation are equal to $\ell_{(1)} \otimes \ell_{(2)} \otimes S(\ell_{(3)})$. On the other hand, equation (2.12) boils down to the equation $(\ell_{(1)})_{(2)} \otimes (\ell_{(1)})_{(1)} \otimes S(\ell_{(3)}) = \ell_{(1)} \otimes S(S(\ell_{(2)})_{(1)}) \otimes S(\ell_{(2)})_{(2)}$, which in turn due to $S^2 = \mathrm{id}_H$ simplifies to $\ell_{(2)} \otimes \ell_{(1)} \otimes S(\ell_{(3)}) = \ell_{(1)} \otimes \ell_{(3)} \otimes S(\ell_{(2)})$. This is equivalent to the cocommutativity property $\ell_{(1)} \otimes \ell_{(2)} = \ell_{(2)} \otimes \ell_{(1)}$.

Hence we have shown that the coinvariance property of the symmetric separability idempotent for a bicomodule algebra, proven in Proposition 19, is the appropriate analogue of the cocommutativity of the Haar integral. In the proof of Lemma 21 we will use it in a crucial way, on the way towards proving in Theorem 25 that symmetric separability idempotents allow for defining commuting projectors.

Lemma 21. *Let H be a semisimple Hopf algebra over $\Bbbk$ and let K be a semisimple left H-comodule algebra and A a semisimple left H-module algebra. Let $p^1 \otimes p^2 \in K \otimes K^{\mathrm{op}}$ and $\pi^1 \otimes \pi^2 \in A \otimes A^{\mathrm{op}}$ be the symmetric separability idempotents for K and A, respectively.*

Then $(1_A \otimes p^1) \otimes (1_A \otimes p^2)$ and $(\pi^1 \otimes 1_K) \otimes (\pi^2 \otimes 1_K)$ commute in the algebra $(A \ominus K) \otimes (A \ominus K)^{\mathrm{op}}$, where $A \ominus K$ is the crossed product algebra defined in Definition 4.

Proof. Due to the co-invariance of the symmetric separability idempotent of a semisimple comodule algebra over $\Bbbk$, proven in Proposition 19, we have

$$p^1_{(-1)} \otimes p^1_{(0)} \otimes p^2 \overset{(2.12)}{=} S(p^2_{(-1)}) \otimes p^1 \otimes p^2_{(0)}$$

and

$$(h.\pi^1) \otimes \pi^2 = \pi^1 \otimes (S(h).\pi^2)$$

for all $h \in H$, where the latter can be derived from equation (2.11) by regarding A as a right H^*-comodule algebra, which is equivalent to a left H-module algebra [Mon]. By definition of the multiplication in $(A \ominus K) \otimes (A \ominus K)^{\mathrm{op}}$ we have:

$$(1_A \otimes p^1) \otimes (1_A \otimes p^2) \cdot (\pi^1 \otimes 1_K) \otimes (\pi^2 \otimes 1_K) = (p^1_{(-1)}.\pi^1 \otimes p^1_{(0)}) \otimes (\pi^2 \otimes p^2)$$

and

$$(\pi^1 \otimes 1_K) \otimes (\pi^2 \otimes 1_K) \cdot (1_A \otimes p^1) \otimes (1_A \otimes p^2) = (\pi^1 \otimes p^1) \otimes (p^2_{(-1)}.\pi^2 \otimes p^2_{(0)})$$

But the right-hand sides of these equations are equal by the following computation:

$$(p^1_{(-1)}.\pi^1 \otimes p^1_{(0)}) \otimes (\pi^2 \otimes p^2) = (S(p^2_{(-1)}).\pi^1 \otimes p^1) \otimes (\pi^2 \otimes p^2_{(0)})$$

$$= (\pi^1 \otimes p^1) \otimes (S^2(p^2_{(-1)}).\pi^2 \otimes p^2_{(0)})$$
$$= (\pi^1 \otimes p^1) \otimes (p^2_{(-1)}.\pi^2 \otimes p^2_{(0)}).$$

$\square$

2.2.3 Local commuting projector Hamiltonian from vertex and plaquette operators

In this subsection we define on the vector space $\mathcal{H}$ assigned to a surface Σ with a labelled cell decomposition a set of commuting local projectors and finally, in the spirit of Kitaev models, a Hamiltonian on $\mathcal{H}$ as the sum of commuting projectors.

Recall that in Subsection 2.2.1 we have defined on $\mathcal{H}$ a $K_{\Sigma_v^{0.5}}$-bimodule structure $\widetilde{A}_v$ for each vertex $v \in \Sigma^0$ and a H_p^*-bimodule structure $\widetilde{B}_{(p,v)}$ for each site (p,v), $p \in \Sigma^2$, $v \in \Sigma_p^{\mathrm{sit}}$.

A $K_{\Sigma_v^{0.5}}$-bimodule structure is equivalent to a left $(K_{\Sigma_v^{0.5}} \otimes K^{\mathrm{op}}_{\Sigma_v^{0.5}})$-action on $\mathcal{H}$, so that specifying an element of the so-called enveloping algebra $(K_{\Sigma_v^{0.5}} \otimes K^{\mathrm{op}}_{\Sigma_v^{0.5}})$ determines an endomorphism of $\mathcal{H}$. By assumption, all bicomodule algebras K_e labelling the cell decomposition of Σ are semisimple and, hence, the tensor product $K_{\Sigma_v^{0.5}}$ is semisimple and possesses a unique symmetric separability idempotent $p_v^1 \otimes p_v^2 \in (K_{\Sigma_v^{0.5}} \otimes K^{\mathrm{op}}_{\Sigma_v^{0.5}})$ according to Proposition 17.

Definition 22. Let $v \in \Sigma^0$. The *vertex operator* for the vertex v is the idempotent endomorphism of the state space $\mathcal{H}$

$$A_v := \widetilde{A}_v(p_v^1 \otimes p_v^2) : \mathcal{H} \longrightarrow \mathcal{H}$$

given by acting with the unique symmetric separability idempotent

$$p_v^1 \otimes p_v^2 \in K_{\Sigma_v^{0.5}} \otimes K^{\mathrm{op}}_{\Sigma_v^{0.5}}$$

via the $K_{\Sigma_v^{0.5}}$-bimodule structure $\widetilde{A}_v$, defined in Definition 11.

This operator is *local* in the sense that it acts as the identity on all tensor factors in $\mathcal{H} = (\otimes_{e \in \Sigma^1} K_e^*) \otimes (\otimes_{w \in \Sigma^0} Z_w)$ except for those associated to the vertex $v \in \Sigma^0$ and to the edges $e \in \Sigma_v^1$ incident to v. Since the symmetric separability idempotent of a semisimple bicomodule algebra generalizes the Haar integral of a semisimple Hopf algebra, as explained in Subsection 2.2.2, we see that the vertex operator defined here provides a suitable analogon to the vertex operators in the ordinary Kitaev model for a semisimple Hopf algebra.

Next we want to define a projector on $\mathcal{H}$ for each plaquette $p \in \Sigma^2$ in analogy to the plaquette operators of the ordinary Kitaev model for a semisimple Hopf algebra H, which are defined by acting with the Haar integral of the dual Hopf algebra H^*. In our construction, we have defined in Definition 12 an H_p^*-bimodule structure $\widetilde{B}_{(p,v)}$ on $\mathcal{H}$ for every plaquette $p \in \Sigma^2$ with incident site $v \in \Sigma_p^{\mathrm{sit}}$ and we can again use this to define a projector $\widetilde{B}_{(p,v)}(\lambda_{p(1)} \otimes S(\lambda_{p(2)}))$ on $\mathcal{H}$ by acting with the symmetric separability idempotent of the semisimple algebra H_p^*, which is $\lambda_{p(1)} \otimes S(\lambda_{p(2)}) \in H_p^* \otimes (H_p^*)^{\mathrm{op}}$, see Example 18. However note that, as opposed to the vertex operator here it is actually not necessary to invoke the concept of the symmetric separability idempotent, since H_p^* is a Hopf algebra just as in the ordinary Kitaev model, and its symmetric separability idempotent is given by the Haar integral.

When considering the projector $\widetilde{B}_{(p,v)}(\lambda_{p(1)} \otimes S(\lambda_{p(2)}))$ on $\mathcal{H}$, it seems that a priori it depends not only on the plaquette p but also on the site $v \in \Sigma_p^{\mathrm{sit}}$ that we had to choose in Definition 12 in order to define the bimodule structure $\widetilde{B}_{(p,v)}$. Just like the plaquette operators in the ordinary Kitaev model, we will show that due to the properties of the Haar integral the projector only depends on the plaquette p:

Lemma 23. *Let $p \in \Sigma^2$. If $\lambda_p \in H_p^*$ is the Haar integral of H_p^*, then the endomorphism*

$$\widetilde{B}_{(p,v)}(\lambda_{p(1)} \otimes S(\lambda_{p(2)})) : \mathcal{H} \longrightarrow \mathcal{H}$$

does not depend on the choice of the site $v \in \Sigma_p^{sit}$.

Proof. The endomorphism $\widetilde{B}_{(p,v)}(\lambda_{p(1)} \otimes S(\lambda_{p(2)}))$ is equal to the endomorphism of $\mathcal{H}$ obtained by acting with the Haar integral λ via the left H_p^*-action $B'_{(p,v)}$ on $\mathcal{H}$ that is the pullback of the left $(H_p^* \otimes (H_p^*)^{\mathrm{op}})$-action $\widetilde{B}_{(p,v)}$ along the algebra map $(\mathrm{id}_{H_p^*} \otimes S) \circ \Delta : H_p^* \longrightarrow H_p^* \otimes (H_p^*)^{\mathrm{op}}$. Next we observe that the action $B'_{(p,v)}$ is independent of v for any cocommutative element λ of the Hopf algebra H_p^*. Indeed, looking carefully at Definition 12, we extract from it that $B'_{(p,v)}(\lambda)$ acts with the multiple coproduct of λ on the degrees of freedom of $\mathcal{H}$ in the boundary of the plaquette p in a cyclic order starting at the vertex v. Therefore, for a different vertex $v' \in \Sigma_p^{\mathrm{sit}}$, the endomorphism $B'_{(p,v')}(\lambda)$ will only differ by a cyclic shift in the multiple coproduct of λ. But since λ is cocommutative, any multiple coproduct of it is invariant under such cyclic shifts of its tensor factors. □

Thus we have shown that the following is well-defined.

Definition 24. Let $p \in \Sigma^2$. The *plaquette operator* for the plaquette p is the idempotent endomorphism of the state space $\mathcal{H}$

$$B_p := \widetilde{B}_{(p,v)}(\lambda_{p(1)} \otimes S(\lambda_{p(2)})) : \mathcal{H} \longrightarrow \mathcal{H}$$

given by acting via the $H_p^* \otimes (H_p^*)^{\mathrm{op}}$-action $\widetilde{B}_{(p,v)}$ introduced in Definition 12 with the unique symmetric separability idempotent $\lambda_{p(1)} \otimes S(\lambda_{p(2)}) \in H_p^* \otimes (H_p^*)^{\mathrm{op}}$ for H_p^*. Here $\lambda_p \in H_p^*$ is the Haar integral for H_p^*.

This operator is *local* in the sense that it acts as the identity on all tensor factors in $\mathcal{H} = (\otimes_{e \in \Sigma^1} K_e^*) \otimes (\otimes_{v \in \Sigma^0} Z_v)$ except for those associated to the edges $e \in \Sigma_p^1$ and the vertices $v \in \Sigma_p^0$ incident to the plaquette p.

We have thus defined a family of projectors $(A_v)_{v \in \Sigma^0}$ and $(B_p)_{p \in \Sigma^2}$ on the vector space $\mathcal{H}$. We now finally reach our main result that they all commute with each other.

Theorem 25. *Let Σ be an oriented compact surface with a regular cell decomposition labeled by semisimple Hopf algebras, semisimple bicomodule algebras and representations of the vertex algebras, and let $\mathcal{H}$ be the associated vector space defined in Definition 9 with vertex and plaquette operators $\{(A_v)_{v \in \Sigma^0}, (B_p)_{p \in \Sigma^2}\}$ defined in Definitions 22 and 24.*

Then any pair of vertex or plaquette operators commutes.

Proof. Due to Theorem 13, the only non-trivial commutation relations between a $K_{\Sigma_v^{0.5}}$-action and an H_p^*-action on $\mathcal{H}$ may occur when v and p are incident to each other. In that case, the $K_{\Sigma_v^{0.5}}$-bimodule structure $\widetilde{A}_v$ and the H_p^*-bimodule structure $\widetilde{B}_{(p,v)}$ together form a bimodule structure over the crossed product algebra $H_p^* \circledcirc K_{\Sigma_v^{0.5}}$. However, due to Lemma 21 the symmetric separability idempotents for $K_{\Sigma_v^{0.5}}$ and H_p^* commute in $(H_p^* \circledcirc K_{\Sigma_v^{0.5}}) \otimes (H_p^* \circledcirc K_{\Sigma_v^{0.5}})^{\mathrm{op}}$ and, hence, the vertex operator A_v and the plaquette operator B_p commute with each other. □

This is completely analogous to the standard Kitaev model without defects: We have a family of commuting projectors on the state space. Since any family of commuting projectors is simultaneously diagonalizable, this allows for the definition of an exactly solvable Hamiltonian as the sum of commuting projectors. We thus conclude our construction of the Kitaev model with defects as follows:

Definition 26. The *Hamiltonian* on the state space $\mathcal{H}$ assigned to an oriented surface Σ with labeled cell decomposition as above is the diagonalizable endomorphism

$$h := \sum_{v\in\Sigma^0} (\mathrm{id}_{\mathcal{H}} - A_v) + \sum_{p\in\Sigma^2} (\mathrm{id}_{\mathcal{H}} - B_p) : \mathcal{H} \longrightarrow \mathcal{H}.$$

The associated *ground-state space* is its kernel,

$$\mathcal{H}_0 := \ker h,$$

i.e. the simultaneous 1-eigenspace for all the projectors $\{(A_v)_{v\in\Sigma^0}, (B_p)_{p\in\Sigma^2}\}$.

Such a Hamiltonian is also called *frustration-free*, as its lowest eigenvalue is not lower than any eigenvalue of its summands.

Remark 27. The ground-state space $\mathcal{H}_0$ is isomorphic to the vector space that is category-theoretically realized by the modular functor constructed in [FSS2] for the defect surface Σ labeled by the corresponding representation categories of the Hopf algebras and bicomodule algebras. We leave the detailed proof of this statement for future investigations.

As a consequence, the ground-state space $\mathcal{H}_0$ is invariant under fusion of defects and independent of the transparently labeled part of the cell decomposition. Moreover, due to the results of [FSS2], there will be a mapping class group action on $\mathcal{H}_0$ that can be explicitly computed. This allows to define quantum gates on the ground-state space in terms of the mapping class group action, as has been proposed before, and to address questions of universality of such gates. We have thus constructed an explicit Hamiltonian model which offers the possibility for quantum computation, realizing a general framework for theories of the type discussed e.g. in [BJQ].

A detailed investigation of the above and related questions remain for future work.

2.A Appendix: a category-theoretic motivation for the vertex algebras

The construction in this chapter takes as its input a compact oriented surface Σ, whose 2-cells are labelled by Hopf algebras and whose 1-cells are labelled by bicomodule algebras. Furthermore, we have introduced in Definition 5, for every vertex $v \in \Sigma^0$, an algebra C_v, which we call vertex algebra. The category of possible labels for a vertex $v \in \Sigma^0$ of the cell decomposition is the category of modules over the relevant vertex algebra C_v, see Definition 7.

On the other hand, in three-dimensional topological field theories and modular functors defined on surfaces with defects such as in [FSS2, KK], the strata are labelled by category-theoretic data: 2-cells by finite tensor categories and 1-cells by finite bimodule categories, which in our setting arise as the representation categories of the Hopf algebras and bicomodule algebras that we use as labels for our construction.

Furthermore, in [FSS2], a category is assigned to any boundary circle of a surface with defects, which is equivalent to a Drinfeld center in the absence of defects. Such a boundary circle can be intersected by defect lines labelled by bimodule categories, leading to marked points on the circle. In our construction this situation corresponds to a vertex $v \in \Sigma^0$ at which a number of edges labelled by bicomodule algebras meet. We can regard such a vertex as a boundary circle $\mathbb{L}_v$, cut into the surface Σ, at which defect lines end which are labelled by the representation categories of the corresponding bicomodule algebras.

The main result of this section, Theorem 33, is that the category assigned to such a decorated circle with marked points $\mathbb{L}_v$ according to the prescription of [FSS2], defined in Definition 30, is canonically isomorphic to the category of labels that we have defined in Definition 7 for such a vertex $v \in \Sigma^0$ in a labeled cell decomposition.

First we must explain the category that is assigned to a boundary circle of a defect surface in the construction of [FSS2]. For the category-theoretic background, see also [EGNO]. We adapt the notions and notation to our setting, since it slightly differs from the one in [FSS2]. Here, the tensor categories we consider are pivotal and the underlying defect surface is oriented, whereas in the reference no pivotal structures are used and instead the surfaces are framed.

For a tensor category $\mathcal{A}$ and a sign $\varepsilon \in \{+1, -1\}$, write

$$\mathcal{A}^{\varepsilon} := \begin{cases} \mathcal{A}, & \text{if } \varepsilon = +1, \\ \overline{\mathcal{A}}, & \text{if } \varepsilon = -1, \end{cases}$$

where $\overline{\mathcal{A}} := \mathcal{A}^{\mathrm{op,mop}}$ is the tensor category whose underlying linear category is the opposite category of $\mathcal{A}$ and whose tensor product is also opposite to the one of $\mathcal{A}$, i.e. $\overline{a} \otimes \overline{b} := \overline{b \otimes a}$ for $a, b \in \mathcal{A}$, where for any object $a \in \mathcal{A}$ we denote its corresponding object in the opposite category $\overline{\mathcal{A}}$ by $\overline{a}$, and likewise for morphisms. If $\mathcal{A} = H$–mod for a Hopf algebra H, then $\overline{\mathcal{A}} \cong \overline{H}$–mod canonically as tensor categories, where $\overline{H} := H^{\mathrm{op,cop}}$ is the Hopf algebra that has the opposite multiplication as well as the opposite co-multiplication with respect to H. For $X \in H$–mod, the corresponding object $\overline{X}$ in $\overline{H}$–mod is given by the vector space dual $\mathrm{Hom}_{\Bbbk}(X, \Bbbk)$ of X with the natural induced $\overline{H}$-action. For $\varepsilon \in \{+1, -1\}$, we also write $H^{\varepsilon} := \overline{H}$ if $\varepsilon = -1$, and $H^{\varepsilon} := H$ if $\varepsilon = +1$.

The right duality functor induces a monoidal equivalence, $\mathcal{A} \longrightarrow \overline{\mathcal{A}}, x \longmapsto \overline{x^{\vee}}$. For $\mathcal{A} = H$–mod for a Hopf algebra H, this equivalence takes an H-module X and turns it into an $\overline{H}$-module by pulling back the H-action along the antipode $S : \overline{H} \longrightarrow H$. Note that instead of the right dual functor one can also take any other odd-fold right or left dual. For our purposes this choice does not matter, since the tensor categories which we will consider are pivotal, where all these odd-fold duals are canonically identified. Indeed, for a semisimple Hopf algebra H, the antipode is involutive, so that all odd powers of the antipode are the same. (This is in contrast to [FSS2] where no pivotal structures on the tensor categories are used, but instead 2-framings on the underlying surfaces are used to determine which multiple of the duality functor to use in a given moment in the construction.)

If $\mathcal{A}_1$ and $\mathcal{A}_2$ are two tensor categories and $\mathcal{M}$ is an $\mathcal{A}_1$-$\mathcal{A}_2$-bimodule category, then the opposite linear category $\overline{\mathcal{M}} := \mathcal{M}^{\mathrm{op}}$ canonically becomes an $\overline{\mathcal{A}_2}$-$\overline{\mathcal{A}_1}$-bimodule category by defining $\overline{a_2} \triangleright \overline{m} \triangleleft \overline{a_1} := \overline{a_1 \triangleright m \triangleleft a_2}$ for $a_1 \in \mathcal{A}_1$, $m \in \mathcal{M}$, $a_2 \in \mathcal{A}_2$ and likewise for morphisms.

For $\varepsilon \in \{+1, -1\}$, we write

$$\mathcal{M}^\varepsilon := \begin{cases} \mathcal{M} \text{ as an } \mathcal{A}_1\text{-}\mathcal{A}_2\text{-bimodule category}, & \text{if } \varepsilon = +1, \\ \overline{\mathcal{M}} \text{ as an } \overline{\mathcal{A}_2}\text{-}\overline{\mathcal{A}_1}\text{-bimodule category}, & \text{if } \varepsilon = -1. \end{cases}$$

If $\mathcal{M} = K$–mod for an H_1-H_2-bicomodule algebra K, then $\overline{\mathcal{M}} \cong \overline{K}$–mod canonically as ($H_1$–mod)-($H_2$–mod)-bimodule categories, where $\overline{K} := K^{\mathrm{op}}$ is the opposite algebra with respect to K considered as an $\overline{H_2}$-$\overline{H_1}$-bicomodule algebra. For $M \in K$–mod, the corresponding object $\overline{M}$ in $\overline{K}$–mod is given by the vector space dual $\mathrm{Hom}_{\Bbbk}(M, \Bbbk)$ of M with the natural induced $\overline{K}$-action. For $\varepsilon \in \{+1, -1\}$, we also write $K^\varepsilon := \overline{K}$ if $\varepsilon = -1$, and $K^\varepsilon := K$ if $\varepsilon = +1$.

A boundary circle of an oriented surface with defect lines labeled by bimodule categories gives rise to the following data. Consider an oriented circle with n marked points $(e_i)_{i \in \mathbb{Z}_n}$ that are each labelled with a sign $\varepsilon_i \in \{+1, -1\}$, so that we call these points *oriented.* Label each segment between two marked points e_i and e_{i+1} by a finite pivotal tensor category $\mathcal{A}_{i,i+1}$ and label each marked point e_i with a finite bimodule category $\mathcal{M}_i$, which is an $\mathcal{A}_{i-1,i}$-$\mathcal{A}_{i,i+1}$-bimodule category if $\varepsilon_i = +1$, and an $\mathcal{A}_{i+1,i}$-$\mathcal{A}_{i,i-1}$-bimodule category if $\varepsilon_i = -1$. In other words, then $\mathcal{M}_i^{\varepsilon_i}$ is an $\mathcal{A}_{i-1,i}^{\varepsilon_i}$-$\mathcal{A}_{i,i+1}^{\varepsilon_i}$-bimodule category, using the notation we have introduced above for opposite tensor categories and opposite bimodule categories. The set $(\mathcal{M}_i^{\varepsilon_i})_{i \in \mathbb{Z}_n}$ is called a *string of cyclically composable bimodule categories*, according to [FSS2].

To this decorated circle with marked points, by the prescription of [FSS2], one associates a linear category, which we will explain now, see Definition 30. First we consider the Deligne product $\mathcal{M}_1^{\varepsilon_1} \boxtimes \cdots \boxtimes \mathcal{M}_n^{\varepsilon_n}$ of the categories $(\mathcal{M}_i^{\varepsilon_i})_{i \in \mathbb{Z}_n}$. Following the above notation, corresponding to each segment between two marked points e_i and e_{i+1} in the circle there is the structure of an $\mathcal{A}_{i,i+1}^{\varepsilon_{i+1}}$-$\mathcal{A}_{i,i+1}^{\varepsilon_i}$-bimodule category on this Deligne product. These n bimodule category structures on the Deligne product commute with each other (up to canonical coherent isomorphisms), since they act either on different Deligne factors or on two different sides of one of the bimodule categories.

For each of these bimodule category structures on the Deligne product we can consider so-called *balancings*; e.g. for a $\boxtimes$-factorized object $(\overline{m_1}^{\varepsilon_1} \boxtimes \cdots \boxtimes \overline{m_n}^{\varepsilon_n})$ these are natural isomorphisms $(\overline{m_1}^{\varepsilon_1} \boxtimes \cdots \boxtimes \overline{m_i}^{\varepsilon_i} \boxtimes (\overline{a^{\varepsilon_{i+1}}}^{\varepsilon_{i+1}} \triangleright \overline{m_{i+1}}^{\varepsilon_{i+1}}) \boxtimes \cdots \boxtimes \overline{m_n}^{\varepsilon_n} \longrightarrow \overline{m_1}^{\varepsilon_1} \boxtimes \cdots \boxtimes (\overline{m_i}^{\varepsilon_i} \triangleleft \overline{a^{\varepsilon_i}}^{\varepsilon_i}) \boxtimes \overline{m_{i+1}}^{\varepsilon_{i+1}} \boxtimes \cdots \boxtimes \overline{m_n}^{\varepsilon_n})_{a \in \mathcal{A}_{i,i+1}}$ Here, for any category $\mathcal{X}$ and $\varepsilon \in \{+1, -1\}$, we use the notation

$$\overline{x}^\varepsilon := \begin{cases} x \in \mathcal{X}, & \text{if } \varepsilon = +1, \\ \overline{x} \in \overline{\mathcal{X}}, & \text{if } \varepsilon = -1. \end{cases}$$

for the object in $\mathcal{X}^\varepsilon$ that corresponds to the object $x \in \mathcal{X}$, and for a pivotal tensor category $\mathcal{A}$ we use the notation

$$a^\varepsilon := \begin{cases} a, & \text{if } \varepsilon = +1, \\ a^\vee, & \text{if } \varepsilon = -1. \end{cases}$$

(While this notation would make sense for any tensor category that is not necessarily pivotal, it would be unnatural as it would arguably favor the right dual functor over all other odd-fold duals. Therefore we assume that $\mathcal{A}$ is pivotal, which is the case of our interest anyway.)

Let us recall the general definition of such balancings for bimodule categories.

Definition 28. Let $\mathcal{A}$ be a pivotal tensor category, let $\varepsilon, \varepsilon' \in \{+1, -1\}$ and let $\mathcal{M}$ be an $\mathcal{A}^\varepsilon$-$\mathcal{A}^{\varepsilon'}$-bimodule category.

Then the *category* $\mathcal{Z}_{\varepsilon,\varepsilon'}(\mathcal{M})$ *of balancings in* $\mathcal{M}$ has as objects pairs (m, β), where m is an object of $\mathcal{M}$ and the *balancing* $(\beta_a : \overline{a^{\varepsilon}}^{\varepsilon} \triangleright m \xrightarrow{\sim} m \triangleleft \overline{a^{\varepsilon'}}^{\varepsilon'})_{a\in\mathcal{A}}$ is a natural isomorphism satisfying

$$\begin{tikzcd}
\overline{(a\otimes b)}^{\varepsilon\varepsilon} \triangleright m \cong \overline{a^{\varepsilon}}^{\varepsilon} \triangleright \overline{b^{\varepsilon}}^{\varepsilon} \triangleright m \arrow[dd, "\beta_{a\otimes b}"'] \arrow[dr, "\mathrm{id}_{\overline{a^{\varepsilon}}^{\varepsilon}} \triangleright \beta_b"] & \\
& \overline{a^{\varepsilon}}^{\varepsilon} \triangleright m \triangleleft \overline{b^{\varepsilon'}}^{\varepsilon'} \arrow[dl, "\beta_a \triangleleft \mathrm{id}_{\overline{b^{\varepsilon'}}^{\varepsilon'}}"] \\
m \triangleleft \overline{(a\otimes b)}^{\varepsilon'\varepsilon'} \cong m \triangleleft \overline{a^{\varepsilon'}}^{\varepsilon'} \triangleleft \overline{b^{\varepsilon'}}^{\varepsilon'} &
\end{tikzcd}
\qquad
\begin{tikzcd}
\overline{\mathbb{I}^{\varepsilon}}^{\varepsilon} \triangleright m \arrow[dd, "\beta_{\mathbb{I}}"'] \arrow[dr, "\cong"] & \\
& m \arrow[dl, "\cong"] \\
m \triangleleft \overline{\mathbb{I}^{\varepsilon'}}^{\varepsilon'} &
\end{tikzcd}$$

or, in formulas,

$$\beta_{a\otimes b} = (\beta_a \triangleleft \mathrm{id}_{\overline{b^{\varepsilon'}}^{\varepsilon'}}) \circ (\mathrm{id}_{\overline{a^{\varepsilon}}^{\varepsilon}} \triangleright \beta_b) \qquad \forall a, b \in \mathcal{A}, \tag{2.14}$$

$$\beta_{\mathbb{I}} = \mathrm{id}_m, \tag{2.15}$$

where we have omitted the bimodule constraint isomorphisms.

The morphisms in the category of balancings are defined to be the morphisms in $\mathcal{M}$ that are compatible with the balancings.

Remark 29. While this definition does not require any pivotal structure on the tensor category – one can consider every dual to be the right dual, for example – we will consider it only for a pivotal tensor category, since otherwise it would not coincide with the definition of the category of κ-balancings from [FSS2] for an integer $\kappa \in \mathbb{Z}$. In the construction in [FSS2] this integer comes from a framing of the underlying surface and determines which of the various multiples of the double-dual functor, which are trivialised by a pivotal structure, we would need to insert in the above definition.

The category that one finally assigns to the decorated circle with marked points, according to the prescription of [FSS2] is as follows:

Definition 30 (c.f. Definition 3.4 in [FSS2])**.** Let $\mathbb{L}$ be an oriented circle with marked oriented points $\{e_i\}_{i\in\mathbb{Z}_n}$ labelled by bimodule categories – giving rise to a string $(\mathcal{M}_i^{\varepsilon_i})_{i\in\mathbb{Z}_n}$ of cyclically composable bimodule categories. The *category* $\mathrm{T}(\mathbb{L})$ *assigned to the circle* $\mathbb{L}$ is the category of balancings on the Deligne product $(\boxtimes_{i\in\mathbb{Z}_n}\mathcal{M}_i^{\varepsilon_i})$ with respect to the $\mathcal{A}_{i,i+1}^{\varepsilon_{i+1}}$-$\mathcal{A}_{i,i+1}^{\varepsilon_i}$-bimodule category structures for all $i \in \mathbb{Z}_n$. In formulas,

$$\mathrm{T}(\mathbb{L}) := \mathcal{Z}_{\varepsilon_1,\varepsilon_n}(\cdots \mathcal{Z}_{\varepsilon_2,\varepsilon_1}(\boxtimes_{i\in\mathbb{Z}_n}\mathcal{M}_i^{\varepsilon_i})). \tag{2.16}$$

Remarks 31.

- This category is well-defined because the bimodule category structures on the Deligne product, with respect to which the balancings are considered, all commute with each other (up to canonical coherent natural isomorphisms). In [FSS2] it is explained that the category of balancings is monadic and that the monads for the balancings for the different bimodule category structures on the Deligne product satisfy a distributivity law, which also shows that (2.16) does not depend on the order in which we consider the balancings.

- The category assigned to a decorated circle with marked points reduces to the well-known Drinfeld center $\mathcal{Z}(\mathcal{A})$, as shown in [FSS2], if all bimodule categories $\mathcal{M}_i$ are given by a single tensor category $\mathcal{A}$.

In Theorem 33 we want to give a realization of such a category assigned to a decorated circle with marked points, in terms of representations of a $\Bbbk$-algebra, namely the vertex algebra C_v, if the bimodule categories $(\mathcal{M}_i)_i$ are the representation categories of bicomodule algebras $(K_e)_{e\in\Sigma_v^{0.5}}$.

To this end, we first show generally that the category of balancings, as in Definition 28, can be realized in such a representation-theoretic way. For this, let H be a finite-dimensional Hopf algebra over $\Bbbk$, let $\varepsilon, \varepsilon' \in \{+1, -1\}$ and let K be an H^ε-$H^{\varepsilon'}$-bicomodule algebra. Recall from Subsubsection 2.1.1.1 that the category K–mod is an H^ε-$H^{\varepsilon'}$-bimodule category, so that we can consider the category of balancings $\mathcal{Z}_{\varepsilon,\varepsilon'}(K\text{–mod})$ as defined in Definition 28. On the other hand, recall from Definition 3 the so-called balancing algebra $H^*_{\varepsilon,\varepsilon'}$, which is an $((H^{\varepsilon'})^{\mathrm{cop}} \otimes H^\varepsilon)$-module algebra, and recall from Definition 4 the crossed product algebra $H^*_{\varepsilon,\varepsilon'} \oslash K$, for which we consider K as an $((H^{\varepsilon'})^{\mathrm{cop}} \otimes H^\varepsilon)$-comodule algebra. This $\Bbbk$-algebra $H^*_{\varepsilon,\varepsilon'} \oslash K$ with underlying vector space $H^* \otimes K$ is characterized by having H^* and K as subalgebras, and by the following instance of the straightening formula for the multiplication of an element $f \in H^*$ with an element $k \in K$:

$$k \cdot f = f(k_{(1)}^{\langle -\varepsilon' \rangle} \cdot ? \cdot k_{(-1)}^{\langle \varepsilon \rangle}) \cdot k_{(0)} \tag{2.17}$$

The following proposition proves that the category of balancings on K–mod is isomorphic to the representation category of the $\Bbbk$-algebra $H^*_{\varepsilon,\varepsilon'} \oslash K$. This justifies the name “balancing algebra” for $H^*_{\varepsilon,\varepsilon'}$ and will be used in Theorem 33 to establish a connection between the vertex algebras defined in this thesis and the categories assigned to circles in [FSS2].

Proposition 32. *Let H be a semisimple finite-dimensional Hopf algebra over $\Bbbk$, let $\varepsilon, \varepsilon' \in \{+1, -1\}$ and let K be an H^ε-$H^{\varepsilon'}$-bicomodule algebra. Then there is a canonical equivalence of $\Bbbk$-linear categories*

$$\mathcal{Z}_{\varepsilon,\varepsilon'}(K\text{–mod}) \cong (H^*_{\varepsilon,\varepsilon'} \oslash K)\text{–mod}\,.$$

Proof. Let $(M, \beta = (\beta_X : \overline{X^{\varepsilon}}^{\varepsilon} \triangleright M \xrightarrow{\sim} M \triangleleft \overline{X^{\varepsilon'}}^{\varepsilon'})_{X\in H\text{–mod}})$ be an object in $\mathcal{Z}_{\varepsilon,\varepsilon'}(K\text{–mod})$. Recall that the vector spaces underlying the modules $\overline{X^{\varepsilon}}^{\varepsilon} \in H^\varepsilon$–mod and $\overline{X^{\varepsilon'}}^{\varepsilon'} \in H^{\varepsilon'}$–mod are the same as $X \in H$–mod. In this proof, to simplify notation, we will often write β_X as a map $X \otimes M \longrightarrow M \otimes X$, keeping implicit the module structures on the respective vector spaces.

We define, using β, a left H^*-module structure on M as follows. We denote by $H_{\mathrm{reg}} \in H$–mod the left regular H-module with underlying vector space H, whose H-action is defined by left multiplication.

$$\begin{aligned} \rho : H^* \otimes M &\longrightarrow M, \\ f \otimes m &\longmapsto (\mathrm{id}_M \otimes f)\beta_{H_{\mathrm{reg}}}(1_H \otimes m) \end{aligned} \tag{2.18}$$

We show that this indeed satisfies the axioms of a left H^*-module: On the one hand we have, for $f, g \in H^*$ and $m \in M$,

$$\begin{aligned} \rho(f \otimes \rho(g \otimes m)) &\overset{\mathrm{def}}{=} (\mathrm{id}_M \otimes f)\beta_{H_{\mathrm{reg}}}(1_H \otimes (\mathrm{id}_M \otimes g)\beta_{H_{\mathrm{reg}}}(1_H \otimes m)) \\ &= (\mathrm{id}_M \otimes f \otimes g)(\beta_{H_{\mathrm{reg}}} \otimes \mathrm{id}_H)(\mathrm{id}_H \otimes \beta_{H_{\mathrm{reg}}})(1_H \otimes 1_H \otimes m). \end{aligned}$$

On the other hand, we have

$$\begin{aligned}
\rho((f\cdot g)\otimes m) &= (\mathrm{id}_M \otimes (f\cdot g))\beta_{H_{\mathrm{reg}}}(1_H\otimes m)\\
&= (\mathrm{id}_M \otimes f\otimes g)(\mathrm{id}_M\otimes\Delta)\beta_{H_{\mathrm{reg}}}(1_H\otimes m)\\
&\overset{\beta\text{ natural}}{=} (\mathrm{id}_M \otimes f\otimes g)\beta_{H_{\mathrm{reg}}\otimes H_{\mathrm{reg}}}(\Delta(1_H)\otimes m)\\
&= (\mathrm{id}_M \otimes f\otimes g)\beta_{H_{\mathrm{reg}}\otimes H_{\mathrm{reg}}}(1_H\otimes 1_H\otimes m)\\
&\overset{(2.14)}{=} (\mathrm{id}_M \otimes f\otimes g)(\beta_{H_{\mathrm{reg}}}\otimes \mathrm{id}_H)(\mathrm{id}_H\otimes\beta_{H_{\mathrm{reg}}})(1_H\otimes 1_H\otimes m),
\end{aligned}$$

where we use in the third line that the coproduct of H is an H-module morphism $\Delta : H_{\mathrm{reg}} \longrightarrow H_{\mathrm{reg}}\otimes H_{\mathrm{reg}}$. This shows one of the two axioms of an H^*-module. For the other axiom, let again $m\in M$. Then, indeed, we have

$$\begin{aligned}
\rho(1_{H^*}\otimes m) &= \rho(\varepsilon\otimes m)\\
&\overset{\text{def}}{=} (\mathrm{id}_M\otimes\varepsilon)\beta_{H_{\mathrm{reg}}}(1_H\otimes m)\\
&\overset{\beta\text{ natural}}{=} \beta_{\Bbbk}(\varepsilon(1_H)\otimes m)\\
&= m,
\end{aligned}$$

where we use in the third line that the co-unit of H is an H-module morphism $\varepsilon : H_{\mathrm{reg}} \longrightarrow \Bbbk$. Hence, we have shown that ρ endows M with the structure of an H^*-module.

To prove that (M,ρ) is an object of $(H^*_{\varepsilon,\varepsilon'} \circledcirc K)$–mod we have to show that the just defined H^*-action ρ and the given K-action on M, which we simply denote by $K\otimes M \to M, k\otimes m \mapsto k.m$, satisfy the straightening formula (2.17). That is, we have to show that, for all $f\in H^*$, $k\in K$ and $m\in M$,

$$k.((\mathrm{id}_M\otimes f)\beta_{H_{\mathrm{reg}}}(1_H\otimes m)) = (\mathrm{id}_M\otimes f(k_{(1)}^{\langle-\varepsilon'\rangle}\cdot ?\cdot k_{(-1)}^{\langle\varepsilon\rangle}))\beta_{H_{\mathrm{reg}}}(1_H\otimes k_{(0)}.m) \tag{2.19}$$

We start with the right-hand side:

$$\begin{aligned}
(\mathrm{id}_M\otimes f(k_{(1)}^{\langle-\varepsilon'\rangle}\cdot ?\cdot k_{(-1)}^{\langle\varepsilon\rangle}))\beta_{H_{\mathrm{reg}}}(1_H\otimes k_{(0)}.m) &\overset{\beta\text{ natural}}{=} (\mathrm{id}_M\otimes f(k_{(1)}^{\langle-\varepsilon'\rangle}\cdot ?))\beta_{H_{\mathrm{reg}}}(k_{(-1)}^{\langle\varepsilon\rangle}\otimes k_{(0)}.m)\\
&\overset{\beta_{H_{\mathrm{reg}}}\ K\text{-linear}}{=} ((k_{(0)}.?)\otimes f(k_{(2)}^{\langle-\varepsilon'\rangle}k_{(1)}^{\langle\varepsilon'\rangle}\cdot ?))\beta_{H_{\mathrm{reg}}}(1_H\otimes m)\\
&= k.((\mathrm{id}_M\otimes f)\beta_{H_{\mathrm{reg}}}(1_H\otimes m)).
\end{aligned}$$

Here we use in the first line that right multiplication by any element $h\in H$ is an H-module morphism $(?\cdot h) : H_{\mathrm{reg}} \longrightarrow H_{\mathrm{reg}}$ for the left regular H-module H_{reg}, and in the last line we use the defining property of the antipode of H. This concludes the proof that $(M,\rho)\in (H^*_{\varepsilon,\varepsilon'}\circledcirc K)$–mod.

Conversely, assume that $M \in (H^*_{\varepsilon,\varepsilon'}\circledcirc K)$–mod and let us define on M a balancing $\beta_X : \overline{X^{\varepsilon}}^{\varepsilon} \triangleright M \longrightarrow M \triangleleft \overline{X^{\varepsilon'}}^{\varepsilon'}$ for all $X\in H$–mod. Denoting by $(e^i\in H^*)_i$ and $(e_i\in H)_i$ a pair of dual bases, we define

$$\begin{aligned}
\beta_X : X\otimes M &\longrightarrow M\otimes X,\\
x\otimes m &\longmapsto \sum_i e^i.m\otimes e_i.x,
\end{aligned}$$

where $e_i.x$ refers to X as an H-module, not $\overline{X^{\varepsilon'}}^{\varepsilon'}$ as an $H^{\varepsilon'}$-module, even though we will show that β_X is a K-module morphism $\overline{X^{\varepsilon}}^{\varepsilon} \triangleright M \longrightarrow M \triangleleft \overline{X^{\varepsilon'}}^{\varepsilon'}$. Indeed, for $k \in K, x \in X, m \in M$, we calculate

$$
\begin{aligned}
k.(\beta_X(x \otimes m) &\overset{\text{def}}{=} \sum_i (k_{(0)}.e^i.m) \otimes (k_{(1)}^{\langle\varepsilon'\rangle}.e_i.x) \\
&\overset{(2.17)}{=} \sum_i (e^i(k_{(1)}^{\langle-\varepsilon'\rangle} \cdot ? \cdot k_{(-1)}^{\langle\varepsilon\rangle}).k_{(0)}.m) \otimes (k_{(2)}^{\langle\varepsilon'\rangle}.e_i.x) \\
&= \sum_i (e^i.k_{(0)}.m) \otimes (k_{(2)}^{\langle\varepsilon'\rangle}.k_{(1)}^{\langle-\varepsilon'\rangle}.e_i.k_{(-1)}^{\langle\varepsilon\rangle}.x) \\
&= \sum_i (e^i.k_{(0)}.m) \otimes (e_i.k_{(-1)}^{\langle\varepsilon\rangle}.x) \\
&\overset{\text{def}}{=} \beta_X(k.(x \otimes m))
\end{aligned}
$$

Furthermore, it can be seen directly that $(\beta_X)_{X \in H\text{–mod}}$ is a natural family. Indeed, for any H-module morphism $f : X \longrightarrow Y$ and $x \in X, m \in M$, we have $\beta_Y(f(x) \otimes m) \overset{\text{def}}{=} \sum_i e^i.m \otimes e_i.(f(x)) = \sum_i e^i.m \otimes f(e_i.x) \overset{\text{def}}{=} (\mathrm{id}_M \otimes f)\beta_X(x \otimes m)$.

It remains to show that $(\beta_X)_{X \in H\text{–mod}}$ satisfies axioms (2.14) and (2.15), i.e. $\beta_{X \otimes Y} = (\beta_X \otimes \mathrm{id}_Y)(\mathrm{id}_X \otimes \beta_Y)$ for all $X, Y \in H$–mod and $\beta_{\Bbbk} = \mathrm{id}_M$.

For the first identity, let $x \in X$, $y \in Y$ and $m \in M$. Then on the one hand we have $\beta_{X \otimes Y}(x \otimes y \otimes m) \overset{\text{def}}{=} \sum_i e^i.m \otimes e_i.(x \otimes y) = \sum_i e^i.m \otimes (e_{i(1)}.x) \otimes (e_{i(2)}.y)$. On the other hand, $(\beta_X \otimes \mathrm{id}_Y)(\mathrm{id}_X \otimes \beta_Y)(x \otimes y \otimes m) \overset{\text{def}}{=} \sum_{i,j} e^j.e^i.m \otimes e_j.x \otimes e_i.y = \sum_i e^i.m \otimes (e_{i(1)}.x) \otimes (e_{i(2)}.y)$, where the last identity uses that the multiplication of H^* is defined as the dual of the co-multiplication of H.

In order to show (2.15), we use that the unit of H^* is the co-unit $\varepsilon : H \to \Bbbk$ of H. For $\lambda \in \Bbbk$ and $m \in M$ we thus have $\beta_{\Bbbk}(m \otimes \lambda) \overset{\text{def}}{=} \sum_i e^i.m \otimes \varepsilon(e_i)\lambda = 1_{H^*}.m = m$.

So far in this proof, we have shown that on $M \in K$–mod one can construct out of a balancing on M an H^*-action such that M becomes an $(H^*_{\varepsilon,\varepsilon'} \otimes K)$-module, and that conversely out of an $(H^*_{\varepsilon,\varepsilon'} \otimes K)$-module structure one can construct a balancing on $M \in K$–mod. To conclude the proof of the proposition we have to show that these two assignments are inverse to each other.

First, assume that $(M, \beta) \in \mathcal{Z}_{\varepsilon,\varepsilon'}(K\text{–mod})$. Consider the balancing β' on M that is constructed from the H^*-action on M which in turn is constructed from β, as shown above. For $X \in H$–mod, $x \in X$ and $m \in M$ we have

$$
\begin{aligned}
\beta'_X(x \otimes m) &\overset{\text{def}}{=} \sum_i (\mathrm{id}_M \otimes e^i)\beta_{H_{\mathrm{reg}}}(1_H \otimes m) \otimes e_i.x \\
&= (\beta_{H_{\mathrm{reg}}}(1_H \otimes m))_{(M)} \otimes (\beta_{H_{\mathrm{reg}}}(1_H \otimes m))_{(X)}.x \\
&\overset{\beta \text{ natural}}{=} \beta_X(x \otimes m),
\end{aligned}
$$

where use the notation $(\beta_{H_{\mathrm{reg}}}(1_H \otimes m))_{(M)} \otimes (\beta_{H_{\mathrm{reg}}}(1_H \otimes m))_{(X)} := \beta_{H_{\mathrm{reg}}}(1_H \otimes m) \in M \otimes X$, and in the third line we use that $(?.x) : H_{\mathrm{reg}} \longrightarrow X$ is an H-module morphism for any $x \in X$.

Finally, assume that $M \in (H^*_{\varepsilon,\varepsilon'} \otimes K)$–mod with H^*-action $\rho : H^* \otimes M \longrightarrow M$. Consider the H^*-action ρ' on M that is constructed from the balancing on M which in turn is constructed from ρ, as shown above. For $f \in H^*$ and $m \in M$ we then have

$$\rho'(f \otimes m) \stackrel{\text{def}}{=} \sum_i (\mathrm{id}_M \otimes f)(\rho(e^i \otimes m) \otimes e_i.1_H) = \sum_i \rho(e^i \otimes m) f(e_i) = \rho(f \otimes m),$$

which concludes the proof of the proposition. □

Now, finally, we can prove the main result of this appendix. Most of the work for this has already been done in the proof of Proposition 32. Let $v \in \Sigma^0$ be a vertex of a labeled cell decomposition of Σ so that $(K_e)_{e \in \Sigma_v^{0.5}}$ are bicomodule algebras labelling the incident edges at v. Let $\mathbb{L}_v$ be the corresponding circle with marked points which are labeled by cyclically composable bimodule categories $(K_e\text{–mod})_{e \in \Sigma_v^{0.5}}$.

Theorem 33. *Let $v \in \Sigma^0$ be a vertex in a labelled (as defined in Definition 1) cell decomposition of a compact oriented surface Σ. There is a canonical equivalence of $\Bbbk$-linear categories*

$$\mathrm{T}(\mathbb{L}_v) \cong C_v\text{–mod}$$

between the category assigned by the modular functor T*, constructed in [FSS2], to the circle $\mathbb{L}_v$ with marked points corresponding to the half-edges incident to a vertex $v \in \Sigma^0$ and the representation category of the algebra C_v.*

Proof. Consider the bicomodule algebra $(\bigotimes_{e \in \Sigma_v^{0.5}} K_e^{\varepsilon(e)})$, which realizes the Deligne product $\boxtimes_{e \in \Sigma_v^{0.5}} (K_e\text{–mod})^{\varepsilon(e)} = (\bigotimes_{e \in \Sigma_v^{0.5}} K_e^{\varepsilon(e)})\text{–mod}$ as a representation category. For each incident site $p \in \Sigma_v^{\mathrm{sit}}$, which corresponds to a segment between two marked points of the corresponding decorated circle $\mathbb{L}_v$ and is labeled by a Hopf algebra H_p, it has an $H_p^{\varepsilon(e_p)}$-$H_p^{\varepsilon(e'_p)}$-bicomodule structure, where e_p and e'_p are half-edges incident to v in the boundary of the plaquette p, cf. Figure 2.5. Denote the sites in Σ_v^{sit} in clockwise order around v by $(p_{1,2}, \dots, p_{n,1})$ and abbreviate $\varepsilon(e_{p_{i,i+1}}) =: \varepsilon_{i+1}$ and $\varepsilon(e'_{p_{i,i+1}}) =: \varepsilon_i$. We then repeatedly apply Proposition 32 for each of these $H_{p_{i,i+1}}^{\varepsilon_{i+1}}$-$H_{p_{i,i+1}}^{\varepsilon_i}$-bicomodule structures. This is well-defined and does not depend on the order, since for different $p \in \Sigma_v^{\mathrm{sit}}$ the bicomodule structures commute with each other. We hence obtain an equivalence of categories

$$\mathcal{Z}_{\varepsilon_1,\varepsilon_n}(\cdots \mathcal{Z}_{\varepsilon_2,\varepsilon_1}(\boxtimes_{e \in \Sigma_v^{0.5}} (K_e\text{–mod})^{\varepsilon(e)})) \cong \Big(\big((H_{p_{n,1}})^*_{\varepsilon_1,\varepsilon_n} \otimes \cdots \otimes (H_{p_{1,2}})^*_{\varepsilon_2,\varepsilon_1}\big) \circledcirc \big(\bigotimes_{e \in \Sigma_v^{0.5}} K_e^{\varepsilon(e)}\big)\Big)\text{–mod}$$
$$\stackrel{\text{def}}{=} C_v\text{–mod},$$

which concludes the proof. □

Remark 34. Since the category of balancings reduces to the Drinfeld center $\mathcal{Z}(\mathcal{A})$ if all bimodule categories $\mathcal{M}_i$ are given by a single tensor category $\mathcal{A}$, as shown in [FSS2], we see that also in our construction in case of only transparently labeled edges incident to the vertex v, the category of labels is the representation category of the Drinfeld double, just as in the Kitaev construction without defects, see e.g. [BK2].

3 Isotypic decompositions for non-semisimple Hopf algebras

This chapter is organised as follows. In Section 3.1 we first review the definition of isotypic decompositions for finite-dimensional algebras and then obtain preliminary results about them. In particular, in Proposition 40 we give a characterisation of the semisimplicity of a Hopf algebra in terms of the centrality of the idempotent associated to the trivial isotypic component. Section 3.2 contains our main results for general finite-dimensional Hopf algebras with the Chevalley property. Finally, in Section 3.3 we first illustrate our results with an example of a basic Hopf algebra (Subsection 3.3.1) and then provide further evidence for Conjecture 48 by studying in Subsection 3.3.2 an example of a Hopf algebra with the Chevalley property that is not covered by our general results in Section 3.2.

3.1 Isotypic decompositions for finite-dimensional algebras

Let H be a (not necessarily semisimple) finite-dimensional algebra over $\Bbbk$. Let, as before, I denote the (finite) set of isomorphism classes of simple H-modules. Then, as a projective left H-module, H possesses a direct sum decomposition into projective H-submodules H_i,

$$H = \bigoplus_{i \in I} H_i, \tag{3.1}$$

where $H_i \cong P_i^{\oplus n_i}$ is a direct sum of projective indecomposable submodules of the same isomorphism type P_i, the projective cover of the simple H-module given by $i \in I$.

Definition 35. We call H_i an *i-isotypic component* of H, for $i \in I$, and a direct sum decomposition into isotypic components an *isotypic decomposition* of H.

Specifying an isotypic decomposition is equivalent to specifying the corresponding orthogonal idempotents $(p_i)_{i \in I}$ such that $p_i \in H_i$ and $\sum_{i \in I} p_i = 1$.

Remark 36. Isotypic decompositions can clearly be defined for any projective left module over H. However, in general there does not exist a description in terms of orthogonal idempotents in H, since for this we use that left H-module endomorphisms of H are in bijection with right multiplications with elements of H: $\mathrm{End}_H(H) \cong H^{\mathrm{op}}$.

By the Krull-Schmidt theorem, the multiplicities n_i of the indecomposable modules inside each H_i are unique for any isotypic decomposition. In fact, they are given by the dimensions of the simple H-modules. Indeed, let S_i be a simple H-module in the isomorphism class $i \in I$ and let $H_i \cong P_i^{\oplus n_i}$, where P_i is the projective cover of S_i and $n_i \in \mathbb{N}$. Then we have an isomorphism of vector spaces

$$S_i \cong \mathrm{Hom}_H(H, S_i) \cong \mathrm{Hom}_H\left(\oplus_{j \in I} P_j^{\oplus n_j}, S_i\right) \cong \mathrm{Hom}_H(P_i, S_i)^{\oplus n_i}.$$

Since $\mathrm{Hom}_H(P_i, S_i)$ is one-dimensional, this implies that $n_i = \dim(S_i)$.

Another point of view on isotypic decompositions is the following. Let $J \subseteq H$ be the Jacobson radical of the finite-dimensional algebra H, i.e. the maximal nilpotent ideal of H (as a general reference see [Br]). Then the algebra H/J is the maximal semisimple quotient algebra of H with natural surjection of algebras $\pi : H \longrightarrow H/J$.

Lemma 37. *An isotypic decomposition of H is equivalent to an algebra map $s : Z(H/J) \longrightarrow H$ such that $\pi \circ s = \mathrm{id}_{Z(H/J)}$.*

Proof. Indeed, given an isotypic decomposition $H = \bigoplus_{i\in I} Hp_i$, mapping $Z(H/J) \ni e_i \mapsto p_i \in H$, where e_i are the central orthogonal idempotents of the semisimple algebra H/J, gives us such an algebra map $s : Z(H/J) \longrightarrow H$ because $Z(H/J) = \mathrm{span}_{\Bbbk}\{e_i\}$.

Conversely, given an algebra map $s : Z(H/J) \longrightarrow H$ such that $\pi \circ s = \mathrm{id}_{Z(H/J)}$, the images of the central orthogonal idempotents $e_i \in H/J$ give us the orthogonal idempotents $p_i := s(e_i)$ of an isotypic decomposition of H. For this we need to show that Hp_i is a projective cover of $S_i^{\oplus \dim(S_i)}$. Indeed, Hp_i is a projective cover of $Hp_i/J(Hp_i)$ and we have an H-module isomorphism $Hp_i/J(Hp_i) \cong (H/J)e_i \cong S_i^{\oplus \dim S_i}$. The first isomorphism follows from $J(Hp_i) = Hp_i \cap J(H) = \ker(\pi|_{Hp_i})$, together with the fact that $(H/J)e_i$ is the image of the restricted quotient map $\pi|_{Hp_i} : Hp_i \longrightarrow H/J$. □

3.1.1 (Non-)uniqueness of isotypic decompositions

In general, an isotypic decomposition is not unique. We can characterize the uniqueness of such a decomposition as follows.

Lemma 38. *Let H be a finite-dimensional algebra over $\Bbbk$. A direct sum decomposition $H = \bigoplus_{i\in J} H_i$ into left H-submodules, where the isomorphism types of the summands H_i are prescribed, is unique if and only if any left H-module automorphism of H commutes with the projections of the direct sum $\bigoplus_{i\in J} H_i$.*

Proof. Let $H = \bigoplus_{i\in J} H_i$ be a unique decomposition into components of prescribed isomorphism type and let $\varphi : H \longrightarrow H$ be an H-module automorphism. Then $H = \bigoplus_i \varphi(H_i)$ together with the projections $(\varphi \circ p_i \circ \varphi^{-1} : H \to \varphi(H_i))_i$ is also such a decomposition and therefore $H_i = \varphi(H_i) \in H$ and $p_i = \varphi \circ p_i \circ \varphi^{-1}$ for all $i \in J$.

It remains to prove the implication in the other direction. For this assume that any H-automorphism $\varphi : H \longrightarrow H$ commutes with the projections $p_i : H \longrightarrow H_i$ for all $i \in J$. Now let $H = \bigoplus_i H_i'$ be another decomposition into components of the prescribed isomorphism type with projections $p_i' : H \longrightarrow H_i'$. There are isomorphisms $\varphi_i : H_i \longrightarrow H_i'$. Together they give an isomorphism $\varphi = \bigoplus_i \varphi_i : H = \bigoplus_i H_i \longrightarrow \bigoplus_i H_i' = H$, which by construction satisfies $\varphi \circ p_i = p_i' \circ \varphi$ for all $i \in J$. But by assumption an H-automorphism φ commutes with the projections $p_i : H \longrightarrow H_i$ for all $i \in J$, i.e. we have $\varphi \circ p_i = p_i \circ \varphi$ for all $i \in J$. Together this implies $p_i' \circ \varphi = p_i \circ \varphi$ for all $i \in J$ and by invertibility of φ this proves the claim that $p_i = p_i'$. □

Remark 39. Furthermore, we can describe the set of decompositions of H into isotypic components as follows. Choose one such decomposition $(p_i : H \longrightarrow H_i)_i$. Mapping an H-linear automorphism $\varphi \in \mathrm{Aut}_H(H)$ to the decomposition $(\varphi \circ p_i \circ \varphi^{-1} : H \longrightarrow \varphi(H_i))_i$ induces a bijection

$$\mathrm{Aut}_H(H) \,/\, \prod_i \mathrm{Aut}_H(H_i) \xrightarrow{\sim} \Big\{(p_i' : H \to H_i')_i \text{ isotypic decomposition}\Big\}.$$

Denoting by $\mathrm{Cent}_{H^\times}\{p_i | i \in I\}$ the centralizer of the set $(p_i)_{i\in I}$ in $H^\times$, we can thus also describe the set of isotypic decompositions as the homogeneous set

$$H^\times / \,\mathrm{Cent}_{H^\times}\{p_i | i \in I\}.$$

If H is a semisimple Hopf algebra, then the idempotents e_i in equation (1.1) giving us the isotypic decomposition are central (as they are for any semisimple algebra by the Artin-Wedderburn theorem), implying by Lemma 38 the uniqueness of the isotypic decomposition in the semisimple case.

Conversely, we obtain the following characterization of semisimplicity for a Hopf algebra H:

Proposition 40. *A finite-dimensional Hopf algebra H over $\Bbbk$ is semisimple if and only if there exists a decomposition $H = \bigoplus_{i\in I} He_i$ into isotypic components such that $e_{\mathbb{I}} \in H$, the idempotent corresponding to the trivial H-module, is central.*

Proof. The only-if part of the statement is implied by the Artin-Wedderburn theorem.

For the rest of the proof assume that $e_{\mathbb{I}}$ is central. This implies that $\mathrm{Ext}^1_H(\mathbb{I}, S) = 0$ for any non-trivial simple H-module S as we show next. Let $0 \to S \to M \to \mathbb{I} \to 0$ be a short exact sequence in H–mod, where M is an arbitrary H-module. Since $e_{\mathbb{I}} \in H$ is central, acting with this element defines an H-module morphism on any H-module, in particular $e_{\mathbb{I}} : S \to S$. If there were an $x \in S$ such that $e_{\mathbb{I}}.x \neq 0$, then this would define a non-zero H-module map $He_{\mathbb{I}} \longrightarrow S, he_{\mathbb{I}} \longmapsto he_{\mathbb{I}}.x$. But for a simple H-module S non-isomorphic to the trivial one $\mathbb{I}$, this does not exist, since $He_{\mathbb{I}}$ is the projective cover of $\mathbb{I}$. Hence, we must have $e_{\mathbb{I}}.S = 0$. This implies that we obtain a well-defined morphism $e_{\mathbb{I}} : M/S \to M$, which provides a splitting of the short exact sequence, since $M/S \cong \mathbb{I}$ by exactness of the sequence and, hence, $e_{\mathbb{I}}$ acts on M/S as the identity. We have thus shown that $\mathrm{Ext}^1_H(\mathbb{I}, S) = 0$ for any non-trivial simple H-module S, using that $e_{\mathbb{I}}$ is central.

Due to Theorem 4.4.1 in [EGNO], we also have $\mathrm{Ext}^1_H(\mathbb{I}, \mathbb{I}) = 0$.

Now let N be an arbitrary H-module. Since a short exact sequence of H-modules induces a long exact sequence of corresponding Ext groups, we can use a composition series for N to inductively show that there exists a simple H-module S (the smallest module in the composition series) such that $\mathrm{Ext}^1_H(\mathbb{I}, S)$ surjects to $\mathrm{Ext}^1_H(\mathbb{I}, N)$. Since we have shown that $\mathrm{Ext}^1_H(\mathbb{I}, S) = 0$, this implies that $\mathrm{Ext}^1_H(\mathbb{I}, N) = 0$ for all N, and hence the trivial H-module $\mathbb{I}$ is projective. This implies that every H-module is projective, since the tensor product of a projective module with any other module is projective. We conclude that H is semisimple. □

Example 41. Let $H = H_4 = \Bbbk\langle g, x\rangle/(g^2 = 1, x^2 = 0, gx = -xg)$ be Sweedler's four-dimensional Hopf algebra, which reappears in more detail in Example 44. Consider the decomposition $H = P_0 \oplus P_1 := H\frac{1+g}{2} \oplus H\frac{1-g}{2}$. Then there exists an automorphism $\varphi : H \to H$ of H as a left H-module such that $\varphi(P_0) \neq P_0$. Indeed, let φ be given by right multiplication by the invertible element $1 + \frac{1+g}{2}x$. This does not commute with the element $\frac{1+g}{2}$, as can be easily computed. Hence $\frac{1+g}{2}$ is not central and H_4 is not semisimple.

We can therefore see that

$$H = H\frac{1+g}{2} \oplus H\frac{1-g}{2}$$

and

$$H = H\frac{1+g}{2}(1+x) \oplus H\frac{1-g}{2}$$

are two different isotypic decompositions for H_4.

3.1.2 Isotypic decompositions for self-injective algebras

When we are dealing with a self-injective algebra H, i.e. the regular H-module is injective, then finding an isotypic decomposition simplifies to finding isotypic components individually, as we will show in this subsection. In particular, this applies to Hopf algebras since they are Frobenius algebras and hence self-injective.

For a simple H-module S_i, $i \in I$, write $\phi(i)$ for the (isomorphism class of) the socle $\mathrm{soc}(P_i)$ of the projective cover P_i of S_i. Since P_i is also injective, because H is a self-injective algebra, it holds that P_i is the injective envelope of $\phi(i)$. Since P_i is indecomposable, this means that $\phi(i)$ is simple. So this gives us a bijection $\phi : I \longrightarrow I$, which is called the *Nakayama permutation* of H(cf. [F]). In particular, if $i \neq j$, we have $\mathrm{soc}(P_i) \neq \mathrm{soc}(P_j)$.

Proposition 42. *Assume that for any $i \in I$, $H_i \subseteq H$ is an i-isotypic component of H, i.e. $H_i \cong P_i^{\oplus \dim(S_i)}$. Then the map*

$$\Psi : \bigoplus_{i \in I} H_i \longrightarrow H$$

is an isomorphism of left H-modules.

Proof. Due to dimension considerations it is enough to prove that Ψ is injective. Assume the opposite. Then choose a minimal subset $J \subseteq I$ such that $\Psi| : \bigoplus_{i\in J} H_i \longrightarrow H$ is not injective. Then for some $i \in J$ we have that $\Psi| : \bigoplus_{j\in J\setminus\{i\}} H_j \longrightarrow H$ is injective and $H_i \cap \Psi(\bigoplus_{j\in J\setminus\{i\}} H_j) \neq 0$. If two modules intersect, then there is a simple module contained in the intersection. But the biggest semisimple submodule of H_i is the socle $\mathrm{soc}(H_i) \cong \mathrm{soc}(P_i)^{\oplus \dim(S_i)} = \phi(i)^{\oplus \dim(S_i)}$. The biggest semisimple submodule of the other module is

$$\bigoplus_{j\in J\setminus\{i\}} \mathrm{soc}(H_j) \cong \bigoplus_{j\in J\setminus\{i\}} \mathrm{soc}(P_j)^{\oplus \dim(S_j)} = \bigoplus_{j\in J\setminus\{i\}} \phi(j)^{\oplus \dim(S_j)}.$$

Since these two modules do not have any common submodule (up to isomorphism), we arrive at a contradiction. □

So the conclusion of this proposition is: Once we pick for every $i \in I$ a submodule H_i which is of the right isomorphism type (that is, it is an i-isotypic component), this will give us a direct sum decomposition of H into isotypic components.

3.2 Isotypic decompositions for Hopf algebras with the Chevalley property

As we have seen, isotypic decompositions of non-semisimple Hopf algebras are in general not unique. Our goal is to nevertheless construct one explicit isotypic decomposition for a Hopf algebra by generalizing to the non-semisimple case the idempotents given by the character-projector formula (1.1), which makes use of the additional Hopf-algebraic structure such as the Haar integral.

Note that the idempotent $e_{\mathbb{I}}$ for the isotypic component corresponding to the trivial module is given by the Haar integral: $e_{\mathbb{I}} = (\varepsilon(S(\ell_{(1)})\ell_{(2)})(\Delta(\ell)) = \ell$. In a non-semisimple Hopf algebra any (say, left) integral $\ell \in H$ satisfies $\varepsilon(\ell) = 0$ and, hence, $\ell^2 = \varepsilon(\ell)\ell = 0$. So ℓ is not an idempotent anymore. Therefore the character-projector formula (1.1) does not generalize to anything desirable in terms of idempotents for an isotypic decomposition in the non-semisimple case.

Instead we want to take into account that for a semisimple Hopf algebra the Haar integral coincides with the (appropriately normalized) character of the regular representation of the dual algebra:

Proposition 43 (Prop. 1 b) in [LR]). *Let H be a semisimple finite-dimensional Hopf algebra over $\Bbbk$. Then the Haar integral of H is equal to $p := \frac{1}{\dim(H)}\chi_{H^*}$, where $\chi_{H^*} \in H^{**} \cong H$ is the character of the regular H^*-module.*

This proposition motivates us to consider, for our purposes, the character of the regular H^*-module as the appropriate generalization of the Haar integral to the non-semisimple case:

$$p := \frac{1}{\dim(H)}\chi_{H^*} \in H^{**} \cong H. \tag{3.2}$$

Indeed,

- We still have that p is an idempotent:

$$\begin{aligned} p^2 &= \frac{1}{\dim(H)^2}\chi_{H^*}\chi_{H^*} = \frac{1}{\dim(H)^2}\chi_{H^*\otimes H^*} = \frac{1}{\dim(H)^2}\chi_{H^*\otimes H^*_{\mathrm{triv}}} \\ &= \frac{1}{\dim(H)^2}\chi_{H^*}\chi_{H^*_{\mathrm{triv}}} = \frac{1}{\dim(H)^2}\chi_{H^*}\dim(H) = \frac{1}{\dim(H)}\chi_{H^*} = p, \end{aligned}$$

  where we have used the isomorphism of H^*-modules $H^* \otimes H^* \to H^* \otimes H^*_{\mathrm{triv}}, f \otimes g \mapsto f_{(1)} \otimes S(f_{(2)}).g$.

- Another basic property of p is that $\varepsilon(p) = 1$, since $\varepsilon(\chi_{H^*}) = \chi_{H^*}(\varepsilon) = \dim(H)$.

- Moreover, the cyclicity of the trace implies that $p \in H$ is cocommutative.

- A difference from the semisimple case is that p is in general not central when H is not semisimple. But, due to Proposition 40, this is exactly what we expect.

Example 44. Let us consider as a non-semisimple example where the element p is not equal to any integral, the four-dimensional Sweedler Hopf algebra (cf. Example 41)

$$H_4 = \Bbbk\langle g, x\rangle/(g^2 = 1, x^2 = 0, gx = -xg)$$

The co-multiplication is given by $\Delta(g) = g \otimes g$ and $\Delta(x) = x \otimes 1 + g \otimes x$.

As can be straightforwardly verified, the space of left integrals of H_4 is $I_\ell = \Bbbk(1+g)x$ and the space of right integrals is $\Bbbk x(1+g)$. However, $p \in H_4$ is equal to $\frac{1}{2}(1+g)$.

For later reference, let us list the simple modules of Sweedler's Hopf algebra H_4. The Jacobson radical J of H_4 is $\Bbbk\{x, gx\} \subseteq H_4$ and the maximal semisimple quotient H_4/J of H_4 can then be identified with the group algebra $\Bbbk\{1, g\}$ of the cyclic group of order 2. Hence, there are two non-isomorphic simple H_4-modules, both one-dimensional: the trivial one, denoted $\Bbbk_+$, sending $g \mapsto 1$ and $x \mapsto 0$ and the other one, denoted $\Bbbk_-$, mapping $g \mapsto -1$ and $x \mapsto 0$.

Let us see what the character-projector formula (1.1) would give us in this example if we replace the Haar integral ℓ in the formula by the idempotent p. For the trivial module, we of course get $p = \frac{1}{2}(1 + g)$ itself and for the non-trivial simple H_4-module we obtain $\frac{1}{2}(1 - g)$. These are indeed orthogonal and both idempotent.

The Hopf algebra H_4 in the previous example has only one-dimensional simple modules, so in particular satisfies the so-called Chevalley property. For the remainder of this chapter, we will restrict our attention to the better behaved subclass of Hopf algebras with the Chevalley property.

Definition 45. Let H be a Hopf algebra. H has the *Chevalley property* if the tensor product of any two semisimple H-modules is again a semisimple H-module.

Remark 46. A Hopf algebra H, all of whose simple modules are one-dimensional, clearly possesses the Chevalley property. We call such an algebra *basic*.

Another characterization of the Chevalley property is the following (cf. [AEG, Proposition 4.2]):

Lemma 47. *H has the Chevalley property if and only if its Jacobson radical J is a Hopf ideal in H, that is J is an ideal such that $\Delta(J) \subseteq J \otimes H + H \otimes J$, $\varepsilon(J) = 0$ and $S(J) \subseteq J$, or in other words, H/J has the structure of a Hopf algebra such that the quotient map $\pi : H \longrightarrow H/J$ is a Hopf algebra morphism.*

The class of Hopf algebras with the Chevalley property in particular includes the semisimple Hopf algebras, but is much larger than that. In fact, many known examples of finite-dimensional Hopf algebras over $\Bbbk$ have either the Chevalley property or a dual Hopf algebra with the Chevalley property. The latter case includes in particular the pointed Hopf algebras that have been much studied by N. Andruskiewitsch, H.-J. Schneider et al. For example the pointed Hopf algebras with abelian group of group-like elements were classified in [AS] (under a mild restriction on the order of the group). Hopf algebras with the Chevalley property were also studied in [AEG, AGM, AV].

In view of Proposition 43 and the observations following it, we conjecture the following character-projector formula for non-semisimple Hopf algebras:

Conjecture 48. *Let H be a finite-dimensional Hopf algebra over $\Bbbk$ with the Chevalley property (cf. Def. 45). Then we conjecture that the elements*

$$p_i := \dim(S_i)\chi_i(S(p_{(1)}))p_{(2)} \in H, \tag{3.3}$$

for $i \in I$, where I is the set of isomorphism classes of simple H-modules, define a set of orthogonal idempotents of H such that $H = \bigoplus_{i \in I} Hp_i$ is an isotypic decomposition.

Remark 49. When H has the Chevalley property, the quotient map $\pi : H \longrightarrow H/J$ sends $\{p_i\}_{i \in I}$ to the canonical basis of central orthogonal idempotents of the center of the semisimple algebra H/J. The restriction of the quotient map to the subalgebra $\Bbbk\langle p_i\rangle \subseteq H$ generated by $\{p_i\}_{i\in I}$ then gives us a surjection

$$\pi : \Bbbk\langle p_i\rangle \longrightarrow Z(H/J).$$

onto a semisimple algebra. It is known that any such surjection splits as an algebra map, and if $s : Z(H/J) \to \Bbbk\langle p_i\rangle$ is a section, then $s\pi(p_i)$ will give us a basis of orthogonal idempotents which are moreover non-commutative polynomials in the elements p_i. However, our conjecture asks if the character-projector formula generalises directly, without any adjustments.

In order to justify our assumption of the Chevalley property for this conjecture, we will give a counter-example:

Example 50. Let us give a counter-example to the above conjecture for a Hopf algebra that does not possess the Chevalley property.

Let $\mu \in \Bbbk$. Then we define an 8-dimensional Hopf algebra $H(\mu)$ over $\Bbbk$, a deformation of the so-called *double cover* of Sweedler's 4-dimensional Hopf algebra, as follows. As an algebra it is generated by elements g and x with the relations

$$\begin{aligned} g^4 &= 1 \\ gxg^{-1} &= -x \\ x^2 &= \frac{\mu}{2}(1-g^2), \end{aligned}$$

and the co-multiplication is given by $\Delta(g) = g\otimes g$ and $\Delta(x) = x\otimes 1 + g\otimes x$. From this one can compute that $p = \frac{1}{\dim(H_\mu)}\chi_{H(\mu)^*} = \frac{1}{4}(1+g+g^2+g^3)$. Let us compute the remaining p_i from the character-projector formula (3.3):

To this end we have to determine the simple $H(\mu)$-modules and compute their characters. Since $\frac{1+g^2}{2}$ and $\frac{1-g^2}{2}$ are central orthogonal idempotents, $H(\mu)$ decomposes as a direct sum of algebras

$$H(\mu) = H(\mu)\frac{1+g^2}{2} \oplus H(\mu)\frac{1-g^2}{2} =: H(\mu)^+ \oplus H(\mu)^-.$$

Denoting $\gamma := g(\frac{1+g^2}{2})$ and $\xi := x(\frac{1+g^2}{2})$, as an algebra $H(\mu)^+$ has the unit $1_+ := \frac{1+g^2}{2}$ and is generated by γ and ξ satisfying the relations

$$\begin{aligned} \gamma^2 &= 1_+, \\ \gamma\xi\gamma^{-1} &= -\xi, \\ \xi^2 &= 0, \end{aligned}$$

and hence is isomorphic to the four-dimensional Sweedler algebra H_4.

On the other hand, denoting $G := g(\frac{1-g^2}{2})$ and $X := x(\frac{1-g^2}{2})$, as an algebra $H(\mu)^-$ has the unit $1_- := \frac{1-g^2}{2}$ and is generated by G and X satisfying the relations

$$\begin{aligned} G^2 &= -1_-, \\ GXG^{-1} &= -X, \\ X^2 &= \mu 1_- \end{aligned}$$

Now we have to distinguish the two cases $\mu = 0$ and $\mu \neq 0$.

Let us first assume that $\mu = 0$. Then setting $\widetilde{G} := iG$, the elements $\widetilde{G}$ and X satisfy the relations of the four-dimensional Sweedler algebra H_4, so that we have that, as an algebra $H(0)$, is isomorphic to the direct sum of two Sweedler algebras: $H(0) \cong H_4 \oplus H_4$. We have seen in Example 44 that Sweedler's Hopf algebra has two one-dimensional simple modules $\Bbbk_+$ and $\Bbbk_-$, so that $H(0)$ therefore has four one-dimensional simple modules: $\Bbbk_+^+$ and $\Bbbk_-^+$ for the $H(0)^+$-part and $\Bbbk_+^-$ and $\Bbbk_-^-$ for the $H(0)^-$-part. The corresponding orthogonal idempotents p_i according to the character-projector formula (3.3) are $p = p_{\Bbbk_+^+} = \frac{1}{4}(1 + g + g^2 + g^3)$ and $p_{\Bbbk_-^+} = \frac{1}{4}(1-g+g^2-g^3)$ for the two simple modules of $H(0)^+ \cong H_4$, and $p_{\Bbbk_+^-} = \frac{1}{4}(1+ig-g^2-ig^3)$ and $p_{\Bbbk_-^-} = \frac{1}{4}(1 - ig - g^2 + ig^3)$ for the two simple modules of $H(0)^- \cong H_4$.

Now consider the case $\mu \neq 0$. Then it turns out that the algebra $H(\mu)^-$ is isomorphic to the matrix algebra $\mathrm{Mat}_2(\Bbbk)$, by identifying G with $\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$ and X with $\begin{pmatrix} 0 & \mu \\ 1 & 0 \end{pmatrix}$. The matrix algebra has only one simple module, the standard two-dimensional one, which we shall denote by V, even as a module over $H(\mu)$. Its character χ_V maps $G \mapsto 0$, $G^2 \mapsto -2$ and $G^3 \mapsto 0$. In total then, $H(\mu)$ has three simple modules $\Bbbk_+^+$, $\Bbbk_-^+$ and V and the corresponding p_i of the character-projector formula (3.3) are $p = \frac{1}{4}(1 + g + g^2 + g^3)$ and $p_{\Bbbk_-^+} = \frac{1}{4}(1 - g + g^2 - g^3)$ for the two simple modules of $H(\mu)^+ \cong H_4$, and $p_V = 2 \cdot \frac{1}{4}(2 - 2g^2) = 1 - g^2$ for the unique simple module of $H(\mu)^- \cong \mathrm{Mat}_2(\Bbbk)$. Here we see now that ${p_V}^2 = (1 - g^2)^2 = 1 - 2g^2 + g^4 = 2(1 - g^2)$ is not an idempotent. The reason for this failure to be idempotent is, one could argue, the factor 2 in $p_V = 2 \cdot \frac{1}{4}(2 - 2g^2)$, which comes from the factor $\dim(V)$ in the character-projector formula $p_V := \dim(V)\chi_V(S(p_{(1)}))p_{(2)}$.

Indeed, if we set $\mu = 0$, then V is not a simple $H(\mu)$-module anymore, but rather fits in a short exact sequence $0 \to \Bbbk_-^- \to V \to \Bbbk_+^- \to 0$, so that the character splits into $\chi_V = \chi_{\Bbbk_-^-} + \chi_{\Bbbk_+^-}$. Therefore, for $\mu = 0$, p_V becomes $\dim(V)(S(\chi_{\Bbbk_-^-} + \chi_{\Bbbk_+^-}) \otimes \mathrm{id})(\Delta(p)) = \dim(V)(p_{\Bbbk_-^-} + p_{\Bbbk_+^-})$. The sum of orthogonal idempotents $p_{\Bbbk_-^-}$ and $p_{\Bbbk_+^-}$ is of course again an idempotent, so here we do not get an idempotent precisely because of the factor $\dim(V) = 2$.

3.2.1 Idempotence of the conjectured idempotents

Making use the Chevalley property we will now arrive at further results concerning the idempotent $p \in H$ and isotypic decompositions of H, towards proving our Conjecture 48.

We reiterate that if H has the Chevalley property, then H/J is a semisimple Hopf algebra and the quotient map $\pi : H \longrightarrow H/J$ is a surjective morphism of Hopf algebras. We want to determine the image $\pi(p)$ of the idempotent $p \in H$ under this surjection.

Lemma 51. *The image of the element $p \in H$ under the quotient map $\pi : H \to H/J$ is $p_{H/J} := \frac{1}{\dim(H/J)}\chi_{(H/J)^*} \in H/J$, and hence by Proposition 43 the Haar integral of the semisimple Hopf algebra H/J.*

Proof. Considering p as an element of H^{**}, $\pi(p) \in (H/J)^{**}$ is the restriction of $p : H^* \to \Bbbk$ to the subalgebra $(H/J)^*$. According to the Nichols-Zoeller theorem H^* is free as an $(H/J)^*$-module, i.e.: $H^* \cong ((H/J)^*)^{\oplus N}$, for $N \in \mathbb{N}$ such that $\dim(H) = N \dim(H/J)$. This implies for the characters: $\chi_{H^*}|_{(H/J)^*} = N\chi_{(H/J)^*}$ and thus we have $\pi(p) = \frac{1}{\dim(H)}\chi_{H^*}|_{(H/J)^*} = \frac{N}{\dim(H)}\chi_{(H/J)^*} = \frac{1}{\dim(H/J)}\chi_{(H/J)^*} = p_{H/J}$. □

From this follows the main result of this subsection:

Theorem 52. *Let H be a Hopf algebra with the Chevalley property and let $\chi : H \to \Bbbk$ be the character of a non-zero one-dimensional (hence, simple) H-module. Then the element $p_\chi = (S(\chi) \otimes \mathrm{id}_H)(\Delta(p)) \in H$ is an idempotent such that $Hp_\chi \subseteq H$ is a χ-isotypic component of H.*

Proof. Since $\chi : H \to \Bbbk$ is the character of a one-dimensional H-module, it is an algebra morphism. This implies that the element $p_\chi = (S(\chi) \otimes \mathrm{id}_H)(\Delta(p_H)) \in H$ is an idempotent as it is the image of the idempotent p under an algebra morphism. Furthermore $\chi : H \to k$ factors through H/J as χ corresponds to a simple H-module and, hence, $\pi(p_\chi) = \pi((S(\chi) \otimes \mathrm{id}_H)(\Delta(p))) = (S(\chi) \otimes \mathrm{id}_H)(\Delta(\pi(p))) = (S(\chi) \otimes \mathrm{id}_H)(\Delta(p_{H/J}))$, using Lemma 51 in the last step.

Since $p_{H/J}$ is the Haar integral of the semisimple Hopf algebra H/J, we thus know that $\pi(p_\chi) \in H/J$ is the central idempotent projecting to the isotypic component of χ in H/J.

To conclude the proof we show that, if $\widetilde{e_S} \in H$ is an idempotent preimage of the central idempotent $e_S \in H/J$ corresponding to a simple H-module (and, hence, also H/J-module) S, then the submodule $H\widetilde{e_S} \subseteq H$ is a projective cover of $S^{\oplus \dim S}$. Indeed, $H\widetilde{e_S}$ is a projective cover of $H\widetilde{e_S}/J(H\widetilde{e_S})$. Moreover, note that we have the isomorphism of H-modules $H\widetilde{e_S}/J(H\widetilde{e_S}) \cong (H/J)e_S \cong S^{\oplus \dim(S)}$. The first isomorphism follows from $J(H\widetilde{e_S}) = H\widetilde{e_S} \cap J(H) = \ker(\pi|_{H\widetilde{e_S}})$, where $\pi : H \longrightarrow H/J$ denotes the quotient map, together with the fact that $(H/J)e_S$ is the image of the restricted quotient map $\pi|_{H\widetilde{e_S}} : H\widetilde{e_S} \longrightarrow H/J$. □

In this subsection we have shown that, for a one-dimensional simple H-module $i \in I$, the conjectured idempotent p_i is indeed an idempotent projecting to an i-isotypic component of H. However, we do not yet know whether these idempotents for one-dimensional simple modules are orthogonal to each other.

3.2.2 Orthogonality of the conjectured idempotents

In particular, if we assume that H has only one-dimensional simple modules, i.e. H is *basic*, then we know so far, combining Propositions 42 and 52, that $H = \bigoplus_{i\in I} Hp_i$ is an isotypic decomposition for H. However, we do not yet know whether the natural projections of $\bigoplus_{i\in I} Hp_i$ onto the direct summands Hp_i are the same as the projections given by right multiplication with the idempotents p_i. This is the case if and only if the p_i are orthogonal to each other, i.e. $p_ip_j = \delta_{i,j}p_i$ for all $i, j \in I$. In this subsection we prove a result which implies in particular for a basic Hopf algebra H, using our results from Subsection 3.2.1, that under a certain additional assumption they are.

Due to the following lemma, showing that $\sum_{i\in I} p_i = 1$ is sufficient to show that the idempotents p_i are pairwise orthogonal to each other.

Lemma 53. *Let H be an algebra with decomposition $H = \bigoplus_i H_i$ into left H-submodules H_i and let $p_i \in H_i$ be elements such that $\sum_i p_i = 1$. Then $p_ip_j = \delta_{i,j}p_i$ for all i, j and $H_i = Hp_i$.*

Proof. We have for any i that $p_i = p_i1 = \sum_j p_ip_j$. Then $p_i \in H_i$ by assumption and $p_ip_j \in H_j$ because H_j is an H-submodule together imply that $p_ip_j = \delta_{i,j}p_i$ using the direct sum property of $\bigoplus_i H_i$.

It is left to show that $H_i = Hp_i$. $Hp_i \subseteq H_i$ follows immediately from the facts that $p_i \in H_i$ and that H_i is an H-submodule. In order to show that also $H_i \subseteq Hp_i$, assume that $h_i \in H_i$. We have $h_i = h_i1 = \sum_j h_ip_j$. Since $h_ip_j \in H_j$, this implies that $h_i = h_ip_i$, concluding the proof that $H_i \subseteq Hp_i$. □

In this subsection we therefore want to show that $\sum_{i\in I} p_i = 1$ (Thm. 56).

First we need a lemma. Note that the regular character $\chi_H : H \longrightarrow \Bbbk$ lifts to H/J via the quotient map $\pi : H \longrightarrow H/J$ (as all characters of H do, since the Jacobson radical J is a nil ideal). In fact, we can furthermore show that on H/J it is proportional to the character $\chi_{H/J}$ of the regular H/J-module, i.e. we have:

Lemma 54. *Let H be a Hopf algebra with the Chevalley property. Then: $\chi_H = \frac{\dim(H)}{\dim(H/J)}\chi_{H/J}\circ\pi$.*

Proof. On the hand, we have for any left H-module M a canonical H-module isomorphism $H \otimes M \cong H \otimes M_{\mathrm{triv}}$. Applying this to $M = \pi^*(H/J)$, by which we denote H/J with the action of H via the quotient map $\pi : H \longrightarrow H/J$, we obtain the equality of characters $\chi_H \cdot \chi_{\pi^*(H/J)} = \chi_H \dim(H/J) \in H^*$.

On the other hand, for any H/J-module N we have a canonical isomorphism of H/J-modules $N \otimes H/J \cong N_{\mathrm{triv}} \otimes H/J$, which implies for the characters: $\chi_N \cdot \chi_{H/J} = \dim(N)\chi_{H/J}$.

Next, observe that $\pi : H \longrightarrow H/J$ induces an isomorphism $\pi^* : G_0(H/J) \longrightarrow G_0(H)$ of the Grothendieck rings of H–mod and (H/J)–mod. Since the character of a module only depends on its class in the Grothendieck ring, this implies that there exists an H/J-module V such that $\chi_H = \pi^*\chi_V$. Moreover, $\dim(V) = \dim(H)$, since modules in the same class in the Grothendieck ring have the same dimension.

In summary, we obtain

$$\begin{aligned}\chi_H \cdot \chi_{\pi^*(H/J)} &= \pi^*(\chi_V \cdot \chi_{H/J}) \\ &= \pi^*(\dim(V)\chi_{H/J}) \\ &= \dim(H)\pi^*(\chi_{H/J}).\end{aligned}$$

Together with the first paragraph of the proof this shows the claim. □

As always denote by I the set of isomorphism classes of simple H-modules and for $i \in I$ write $p_i := \frac{\dim(i)}{\dim(H)}(S(\chi_i) \otimes \mathrm{id}_H)(\Delta(\chi_{H^*}))$, as in Conjecture 48, where $\chi_i \in H^*$ is the character of the simple H-module S_i and where $\chi_{H^*} \in H^{**} \cong H$ is the regular character of H^*.

The following Theorem 56 proves that $\sum_{i\in I} p_i = 1$ holds for a Hopf algebra H with the Chevalley property, under an additional assumption on H. In order to formulate this assumption, we have to introduce the so-called *Hecke algebra* associated to H^*. Since H has the Chevalley property, H/J is its maximal semisimple quotient-Hopf-algebra. Dually this means that $H_0^* := (H/J)^* \subseteq H^*$ is the maximal semisimple sub-Hopf-algebra of H^*. (In other words, $H_0^* = (H/J)^*$ is in particular the coradical [Mon] of H^*.) Hence we can consider the unique Haar integral $\Lambda_0 \in H_0^*$ of this semisimple Hopf algebra H_0^*. Now the space $\Lambda_0 H^* \Lambda_0 \subseteq H^*$ is an (in general, not unital) subalgebra of H^* with unit Λ_0. It can also be characterised as the endomorphism algebra $\mathrm{End}_{H^*}(H^*\Lambda_0) \cong \Lambda_0 H^* \Lambda_0$ of the H^*-module $H^*\Lambda_0$ induced from the trivial H_0^*-module along the inclusion $H_0^* \subseteq H^*$. Hence:

Definition 55. We call the algebra $\Lambda_0 H^* \Lambda_0$ with unit Λ_0 the *Hecke algebra* $\mathcal{H}(H^*, H_0^*)$ associated to the trivial representation of $H_0^* \subseteq H^*$.

Now we can state our result.

Theorem 56. *Let H be a Hopf algebra with the Chevalley property. Let $\Lambda_0 \in H^*$ be the Haar integral of the maximal semisimple sub-Hopf-algebra $H_0^* = (H/J)^*$.*

Then

$$\sum_{i \in I} p_i = 1_H$$

if and only if the Hecke algebra $\Lambda_0 H^* \Lambda_0$ has up to isomorphism only one simple module.

Proof. Since for the regular character $\chi_H \in H^*$ of H we have by Lemma 54

$$\chi_H = \frac{\dim(H)}{\dim(H/J)} \pi^*(\chi_{H/J}) = \frac{\dim(H)}{\dim(H/J)} \sum_{i \in I} \dim(i) \chi_i = \frac{\dim(H)}{\dim(H/J)} \sum_{i \in I} \dim(i) S(\chi_i),$$

the equation $\sum_{i \in I} p_i = 1_H$ is equivalent to

$$\frac{\dim(H/J)}{\dim(H)^2} (\chi_H \otimes \mathrm{id}_H)(\Delta(\chi_{H^*})) = 1_H.$$

Using that $\Lambda_0 = \frac{1}{\dim(H/J)} \chi_{H/J}$ by semisimplicity of the Hopf algebra H/J, and $\frac{1}{\dim(H/J)} \chi_{H/J} = \frac{1}{\dim(H)} \chi_H$ by Lemma 54, we rewrite this equation to

$$(\Lambda_0 \otimes \mathrm{id}_H)(\Delta(\chi_{H^*})) = \frac{\dim(H)}{\dim(H/J)} 1_H,$$

which can be rewritten as

$$\chi_{H^*}(\Lambda_0 \cdot -) = \frac{\dim(H)}{\dim(H/J)} \varepsilon_{H^*}. \tag{3.4}$$

Since the subalgebra H_0^* is semisimple, we can decompose H^* as an H_0^*-bimodule as

$$H^* = \bigoplus_{i,j \in I'} e_i H^* e_j =: \bigoplus_{i,j \in I'} H^*_{i,j},$$

where $(e_i)_{i \in I'}$ are the central orthogonal idempotents of the semisimple algebra H_0^* (in particular, $e_{\mathbb{I}} = \Lambda_0$, where $e_{\mathbb{I}}$ is the idempotent corresponding to the trivial H_0^*-module). Therefore, with respect to this decomposition of H^* we have:

$$H^*_{i,j} \cdot H^*_{k,l} \subseteq \begin{cases} H^*_{i,l} & : \quad j = k, \\ 0 & : \quad j \neq k. \end{cases}$$

In particular, if $i \neq j$, then $H^*_{i,j}$ contains only nilpotent elements. From this it follows that both sides of equation (3.4) vanish on

$$\bigoplus_{\substack{i,j \in I' \\ (i,j) \neq (\mathbb{I}, \mathbb{I})}} H^*_{i,j}$$

Indeed, both χ_{H^*} (being a character) and ε_{H^*} (being an algebra map) vanish on nilpotent elements of H^*. Furthermore, for $i \neq \mathbb{I}$, $\chi_{H^*}(\Lambda_0 \cdot -)$ vanishes on $H^*_{i,j}$ by orthogonality of $(e_i)_{i \in I'}$ and so does ε_{H^*} for the same reason, since $\varepsilon_{H^*}(\Lambda_0) = 1$.

Therefore, equation (3.4) is equivalent to

$$\chi_{H^*}|_{\Lambda_0 H^* \Lambda_0} = \frac{\dim(H)}{\dim(H/J)} \varepsilon_{H^*}|_{\Lambda_0 H^* \Lambda_0}. \tag{3.5}$$

For this, note that left multiplication by $\Lambda_0 H^*\Lambda_0$ on H^* is non-zero only on the direct summand $\Lambda_0 H^* \subseteq \bigoplus_{i\in I'} e_i H^* = H^*$. This defines an action of the algebra $\Lambda_0 H^*\Lambda_0$ (with unit Λ_0) on $\Lambda_0 H^*$. Thus equation (3.5) is equivalent to the statement that the character of $\Lambda_0 H^*$ as a left $\Lambda_0 H^*\Lambda_0$-module is equal to $\frac{\dim(H)}{\dim(H/J)}\varepsilon_{H^*}|_{\Lambda_0 H^*\Lambda_0}$. We can show this to be equivalent to the statement that up to isomorphism, the algebra $\Lambda_0 H^*\Lambda_0$ has only one simple module: the trivial one defined on $\Bbbk$ via $\varepsilon_{H^*}|_{\Lambda_0 H^*\Lambda_0} : \Lambda_0 H^*\Lambda_0 \longrightarrow \Bbbk$.

Indeed, if this is the case, then the character of the $\Lambda_0 H^*\Lambda_0$-module $\Lambda_0 H^*$ must be equal to $n \cdot \varepsilon_{H^*}|_{\Lambda_0 H^*\Lambda_0}$, where $n \in \mathbb{N}$ is the length of the Jordan-Hölder series of the module $\Lambda_0 H^*$. Evaluating on Λ_0, which is the unit for the algebra $\Lambda_0 H^*\Lambda_0$, gives $n = \dim(\Lambda_0 H^*)$. Therefore, we obtain $\chi_{H^*}|_{\Lambda_0 H^*\Lambda_0} = \dim(\Lambda_0 H^*)\varepsilon_{H^*}|_{\Lambda_0 H^*\Lambda_0}$. It remains to verify that $\dim(\Lambda_0 H^*) = \frac{\dim(H)}{\dim(H/J)}$. Indeed, by Nichols-Zoeller $H^* \cong (H_0^*)^N$ as a left H_0^*-module, for $N = \frac{\dim(H)}{\dim(H/J)}$. Under this isomorphism we have $\Lambda_0 H^* \cong (\Lambda_0 H_0^*)^N = (\Lambda_0 \Bbbk)^N$, since Λ_0 is the Haar integral of H_0^*. Hence, $\dim(\Lambda_0 H^*) = N = \frac{\dim(H)}{\dim(H/J)}$.

Conversely, if there is another simple $\Lambda_0 H^*\Lambda_0$-module, not isomorphic to the trivial one given by $\varepsilon_{H^*}|_{\Lambda_0 H^*\Lambda_0}$, then it is also a quotient of the regular $\Lambda_0 H^*\Lambda_0$-module and, hence, of $\Lambda_0 H^*$. But then the character of $\Lambda_0 H^*$ cannot be equal to $\frac{\dim(H)}{\dim(H/J)}\varepsilon_{H^*}|_{\Lambda_0 H^*\Lambda_0}$. □

Finally, we conclude from Theorems 52 and 56 the validity of Conjecture 48 for a certain subclass of the Hopf algebras with the Chevalley property:

Corollary 57. *Let H be a finite-dimensional basic Hopf algebra over $\Bbbk$ and denote by $H_0^* := (H/J)^*$ the maximal semisimple sub-Hopf-algebra of its dual H^*. Assume that the associated Hecke algebra $\mathcal{H}(H^*, H_0^*)$ (cf. Definition 55) has, up to isomorphism, a unique simple $\mathcal{H}(H^*, H_0^*)$-module. Then Conjecture 48 holds for H, i.e. $(p_i = \dim(S_i)\chi_i(S(p_{(1)}))p_{(2)})_{i\in I}$ are orthogonal idempotents such that $H = \bigoplus_{i\in I} Hp_i$ is an isotypic decomposition for H.*

Proof. Theorem 52 and Proposition 42 imply that the $(p_i)_{i\in I}$ are idempotents and that $H = \bigoplus_{i\in I} Hp_i$ is an isotypic decomposition, since H has only one-dimensional simple H-modules. Furthermore, Theorem 56 and Lemma 53 together imply that the $(p_i)_{i\in I}$ are orthogonal. □

3.2.3 Hopf algebras with the Chevalley property and the dual Chevalley property

Let H be a Hopf algebra over $\Bbbk$ with both the Chevalley property and the *dual Chevalley property* (i.e. also the dual Hopf algebra H^* has the Chevalley property). Then a lot more can be said about the structure of H and, in particular, our conjecture that the elements p_i give an isotypic decomposition of H can be verified.

Lemma 58. *Let H be a Hopf algebra over $\Bbbk$ with both the Chevalley property and the dual Chevalley property. Then there exists a Hopf algebra section $\iota : H/J \longrightarrow H$ of the quotient map $\pi : H \longrightarrow H/J$, identifying the maximal semisimple quotient Hopf algebra H/J with the maximal semisimple sub-Hopf-algebra $(H^*/J_{H^*})^* \subseteq H$.*

Remark 59. By Radford's projection theorem this implies that H is isomorphic to the Radford biproduct $R\#(H/J)$, where $R := H^{\mathrm{co}(H/J)}$, the subspace of right (H/J)-coinvariants of H.

Proof. Since H^* has the Chevalley property, its maximal semisimple Hopf algebra quotient is $H^* \twoheadrightarrow H^*/J_{H^*}$, where J_{H^*} is the Jacobson radical of H^*. This means that

$$(H^*/J_{H^*})^* \subseteq H^{**} \cong H$$

is the maximal semisimple sub-Hopf-algebra of H. Consider the composition $(H^*/J_{H^*})^* \hookrightarrow H \twoheadrightarrow H/J$ of inclusion and quotient map. We will show that it is an isomorphism of Hopf algebras $(H^*/J_{H^*})^* \cong H/J$.

Firstly, it is injective because $(H^*/J_{H^*})^* \cap J = 0$, because $(H^*/J_{H^*})^* \cap J \subseteq (H^*/J_{H^*})^*$ is a nilpotent ideal of $(H^*/J_{H^*})^*$, but $(H^*/J_{H^*})^*$ is semisimple and therefore has no non-zero nilpotent ideal.

Secondly, we see that $(H^*/J_{H^*})^* \hookrightarrow H \twoheadrightarrow H/J$ is also surjective, because its dual is

$$H_0^* \overset{\mathrm{def}}{=} (H/J)^* \hookrightarrow H^* \twoheadrightarrow H^*/J_{H^*},$$

which is just the inclusion of the maximal semisimple sub-Hopf-algebra followed by the surjection to the maximal semisimple quotient Hopf algebra for the dual Hopf algebra H^*, and this is injective by the above argument. □

Proposition 60. *Let H be a Hopf algebra over $\Bbbk$ with both the Chevalley property and the dual Chevalley property. Then the family $p_i \in H$, as described in Conjecture 48, gives a set of orthogonal idempotents of an isotypic decomposition of H.*

Proof. We will prove this by proving that $p_i = \iota(e_i)$ for the Hopf algebra inclusion $\iota : H/J \longrightarrow H$, which we have shown to exist in Lemma 58, where $e_i \in H/J$ are the central orthogonal idempotents of the isotypic decomposition of the semisimple Hopf algebra H/J. We have $p_i = \frac{\dim(S_i)}{\dim(H)}\chi_i(S(\chi_{H^*(1)}))\chi_{H^*(2)}$ and $e_i = \frac{\dim(S_i)}{\dim(H/J)}\widetilde{\chi_i}(S(\chi_{(H/J)^*(1)}))\chi_{(H/J)^*(2)}$, where $\chi_i \in H^*$ is the character of the i-th simple H-module and $\widetilde{\chi_i} \in (H/J)^*$ is the character of the corresponding H/J-module, i.e. $\chi_i = \widetilde{\chi_i} \circ \pi$.

What we thus have to show is that

$$\frac{1}{\dim(H)}\chi_i(S(\chi_{H^*(1)}))\chi_{H^*(2)} = \frac{1}{\dim(H/J)}\widetilde{\chi_i}(S(\chi_{(H/J)^*(1)}))\iota(\chi_{(H/J)^*(2)}).$$

Using $\pi \circ \iota = \mathrm{id}_{H/J}$ and that ι is a morphism of Hopf algebras, we obtain

$$\begin{aligned}
\frac{1}{\dim(H/J)}\widetilde{\chi_i}(S(\chi_{(H/J)^*(1)}))\iota(\chi_{(H/J)^*(2)}) &= \frac{1}{\dim(H/J)}\widetilde{\chi_i}(\pi(\iota(S(\chi_{(H/J)^*(1)}))))\iota(\chi_{(H/J)^*(2)}) \\
&= \frac{1}{\dim(H/J)}\chi_i(\iota(S(\chi_{(H/J)^*(1)})))\iota(\chi_{(H/J)^*(2)}) \\
&= \frac{1}{\dim(H/J)}\chi_i(S(\iota(\chi_{(H/J)^*})_{(1)}))\iota(\chi_{(H/J)^*})_{(2)}.
\end{aligned}$$

Thus it is left to show that

$$\frac{1}{\dim(H/J)}\iota(\chi_{(H/J)^*}) = \frac{1}{\dim(H)}\chi_{H^*}$$

Denote by $\Pi : H^* \longrightarrow H^*/J_{H^*}$ the quotient map, which is a morphism of Hopf algebras due to the Chevalley property of H^*. Then by Lemma 54 applied to H^*, we have

$$\frac{1}{\dim(H)}\chi_{H^*} = \frac{1}{\dim(H^*/J_{H^*})}\Pi^*(\chi_{H^*/J_{H^*}}).$$

But in Lemma 58 we have identified the Hopf algebras $(H^*/J_{H^*})^*$ and H/J and via this identification, by definition, the injection $\iota : H/J \longrightarrow H$ corresponds to

$$\Pi^* : (H^*/J_{H^*})^* \longrightarrow H^{**} \cong H.$$

Hence, $\frac{1}{\dim(H^*/J_{H^*})}\Pi^*(\chi_{H^*/J_{H^*}}) = \frac{1}{\dim(H/J)}\iota(\chi_{(H/J)^*})$, concluding the proof. $\square$

3.3 Examples

In this last section of the chapter we discuss two examples of Hopf algebras with the Chevalley property that we can show to satisfy our conjecture. The first example is a basic Hopf algebra for which we can verify the assumptions of Theorem 56, which implies that the conjecture holds. The second example is not basic and it does not follow directly from our general results that it satisfies our conjecture, but we carry out explicit computations to show that it does.

3.3.1 The dual of a deformation of the double cover of Sweedler's Hopf algebra: a basic Hopf algebra

Recall the 8-dimensional Hopf algebra $H(\mu)$ from Example 50. Without loss of generality let us set $\mu = 2$, since for all $\mu \in \Bbbk^\times$, $H(\mu)$ is in the same isomorphism class of Hopf algebras. Here we are interested in its dual Hopf algebra, which does satisfy the Chevalley property, and which by slight abuse of notation we will denote by H, so that $H^* = H(2)$. Recall that H^* is generated as an algebra by g and x subject to the relations

$$\begin{aligned} g^4 &= 1 \\ gxg^{-1} &= -x \\ x^2 &= (1-g^2), \end{aligned}$$

with the co-multiplication given by $\Delta(g) = g \otimes g$ and $\Delta(x) = x \otimes 1 + g \otimes x$.

H^* has a $\mathbb{Z}_2$-grading as an algebra, $H^* = (H^*)_0 \oplus (H^*)_1$, where $(H^*)_0 = \mathrm{span}_\Bbbk\{1, g, g^2, g^3\} = \Bbbk G$ is the group algebra of the group G of group-like elements and $(H^*)_1 = (H^*)_0 \cdot x$. Furthermore, we have $\Delta(H^*)_0 \subseteq (H^*)_0 \otimes (H^*)_0$ and $\Delta(H^*)_1 \subseteq (H^*)_1 \otimes (H^*)_0 \oplus (H^*)_0 \otimes (H^*)_1$.

The simple H^*-comodules are given by the simple comodules of the coradical of H^*, which is $(H^*)_0 = \Bbbk G \subseteq H^*$, and therefore we have four one-dimensional simple H^*-comodules corresponding to the four group elements of G.

For the dual Hopf algebra H this means that there are four simple H-modules and they are one-dimensional and given by the four group elements $1, g, g^2, g^3 \in G$ interpreted as elements of H^*.

Let us consider the corresponding four elements $p, p_g, p_{g^2}, p_{g^3} \in H$ given by the character-projector formula (3.3). Since all four simple modules are one-dimensional, our results from Subsection 3.2.1 imply that these four elements are idempotents projecting to appropriate isotypic components. The remaining question of their orthogonality is answered affirmatively by Theorem 56. Indeed, we can verify that the Hecke algebra $\Lambda_0 H^* \Lambda_0$, where here $\Lambda_0 = \frac{1}{4}(1 + g + g^2 + g^3) \in H_0^*$, satisfies the condition of Theorem 56, since

$$\Lambda_0 H^* \Lambda_0 = \mathrm{span}_\Bbbk\{\Lambda_0, \Lambda_0 x \Lambda_0\} \cong \Bbbk[x]/(x^2).$$

Finally we can also compute the orthogonal idempotents $p, p_g, p_{g^2}, p_{g^3} \in H$ explicitly: With respect to the basis $\{1, g, g^2, g^3, x, gx, g^2x, g^3x\}$ for H^*, with dual basis

$$\{\delta_1, \delta_g, \delta_{g^2}, \delta_{g^3}, \delta_x, \delta_{gx}, \delta_{g^2x}, \delta_{g^3x}\}$$

for H, it is easy to see by computation that we have:

$$\begin{aligned} p &= \delta_1 \\ p_g &= \delta_g \\ p_{g^2} &= \delta_{g^2} \\ p_{g^3} &= \delta_{g^3} \end{aligned}$$

Therefore, indeed, the idempotents are orthogonal to each other.

3.3.2 A Hopf algebra with the Chevalley property which is not basic

Finally we consider an example of a Hopf algebra with the Chevalley property that is not basic, i.e. it does not have the property that all its simple modules are one-dimensional. We show by computation that it still satisfies Conjecture 48. This gives evidence that our conjecture holds for all Hopf algebras with the Chevalley property, even though our general results only cover basic Hopf algebras (with an additional assumption on the associated Hecke algebra).

We describe the Hopf algebra H that we want to consider, first by its dual Hopf algebra H^*. As an algebra, H^* is generated by the elements a, b, c and $\{e_g \mid g \in S_3\}$ subject to the following relations:

$$\begin{aligned} e_g e_h &= \delta_{g,h} e_g \quad \forall g, h \in S_3 \\ \sum_{g \in S_3} e_g &= 1 \\ a e_g &= e_{(12)g} a \quad \forall g \in S_3 \\ b e_g &= e_{(23)g} b \quad \forall g \in S_3 \\ c e_g &= e_{(31)g} c \quad \forall g \in S_3 \\ ab + bc + ca &= 0 \\ ac + cb + ba &= 0 \\ a^2 &= \lambda_{ab}(e_{13} + e_{132}) + \lambda_{ac}(e_{23} + e_{123}) \\ b^2 &= \lambda_{bc}(e_{12} + e_{132}) + \lambda_{ba}(e_{13} + e_{123}) \\ c^2 &= \lambda_{ca}(e_{23} + e_{132}) + \lambda_{cb}(e_{12} + e_{123}), \end{aligned}$$

where $\lambda_{ab}, \lambda_{bc}, \lambda_{ca} \in \Bbbk$ are the deformation parameters. The co-multiplication on H^* is given by

$$\begin{aligned} \Delta(e_g) &= \sum_{h \in S_3} e_{gh^{-1}} \otimes e_h \quad \forall g \in S_3 \\ \Delta(a) &= a \otimes 1 + (e_1 - e_{12}) \otimes a + (e_{132} - e_{13}) \otimes b + (e_{123} - e_{23}) \otimes c \\ \Delta(b) &= b \otimes 1 + (e_1 - e_{23}) \otimes b + (e_{132} - e_{12}) \otimes c + (e_{123} - e_{13}) \otimes a \\ \Delta(c) &= c \otimes 1 + (e_1 - e_{13}) \otimes c + (e_{132} - e_{23}) \otimes a + (e_{123} - e_{12}) \otimes b. \end{aligned}$$

We see that H^* contains the dual group algebra $\Bbbk^{S_3} = \operatorname{span}_\Bbbk\{e_g \mid g \in S_3\}$ of the symmetric group S_3 as a sub-Hopf-algebra. In fact, H^* is a cocycle deformation of the Radford biproduct $B\#\Bbbk^{S_3}$, where B is the Nichols algebra generated by a, b and c and the dual group algebra $\Bbbk^{S_3}$ is the maximal semisimple sub-Hopf-algebra. For more details about this Hopf algebra see [M, AV].

For the dual Hopf algebra H with the Chevalley property this means that it possesses the group algebra $\Bbbk S_3$ as its maximal semisimple quotient Hopf algebra and, hence, the simple H-modules are given by the irreducible representations of S_3: the trivial representation, the sign representation and the two-dimensional simple H-module V, which is defined by:

$$(12) \longmapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$
$$(123) \longmapsto \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}.$$

For later reference we note that the character $\chi_V \in H^*$ of this H-module is $\chi_V = 2e_1 - e_{123} - e_{132}$.

Now we want to compute the three corresponding elements p, p_{sgn} and $p_V \in H$ from the conjectured character-projector formula (3.3).

Let us start with $p = \frac{1}{\dim(H)}\chi_{H^*}$, that is we have to compute the trace of the regular representation of H^*. A convenient basis for H^* as a vector space is given by

$$\{1, a, b, c, ab, bc, ac, cb, aba, abc, bac, abac\} \times \{e_g \mid g \in S_3\}. \tag{3.6}$$

Hence we can think of H^* as being $\mathbb{N}_{\geq 0}$-graded as a coalgebra (the grading comes from the grading of the Nichols algebra B), where the degree is determined by the length of the word in the letters a, b and c.

The basis elements $e_g, g \in S_3$, of degree 0 are idempotents with 12-dimensional image, that is $\chi_{H^*}(e_g) = 12$. This determines the trace of all elements of degree 0.

The basis elements of degree 1 are nilpotent: For example,

$$ae_g ae_g = a^2 e_{(12)g} e_g = a^2 \delta_{(12)g,g} e_g = 0$$

for all $g \in S_3$ since $(12)g \neq g$. This implies that $\chi_{H^*}(ae_g) = 0$ and likewise $\chi_{H^*}(be_g) = 0$ and $\chi_{H^*}(ce_g) = 0$ for all $g \in S_3$.

A similar argument shows that the trace is zero on the degree 2 and degree 3 parts of H^*. On the degree 4 component this argument does not work anymore, since $(31)(12)(23)(12) = 1$ and hence

$$abace_g abace_g = (abac)^2 e_g, \tag{3.7}$$

which we cannot immediately see to be zero by orthogonality of the $(e_g)_{g\in S_3}$ as before. In order to obtain an explicit expression for $\chi_{H^*}(abace_g)$, we start by computing $(abac)^2$.

Lemma 61.

$$(abac)^2 = abac((\lambda_{ab}^2 + \lambda_{ca}^2 - \lambda_{bc}^2)e_{23} + 2\lambda_{ab}\lambda_{ac}e_{132}) - \lambda_{ab}^2\lambda_{ac}^2 e_{23} - \lambda_{ab}^2\lambda_{ac}^2 e_{132}$$

Proof. We straightforwardly calculate using the relations of the algebra H^*:

$$abacabac = -aba(ab + bc)bac$$

$$
\begin{aligned}
&= -abaabbac - ababcbac \\
&= -ab(\lambda_{ab}(e_{13}+e_{132}) + \lambda_{ac}(e_{23}+e_{123}))(\lambda_{bc}(e_{12}+e_{132}) + \lambda_{ba}(e_{13}+e_{123}))ac - ababcbac.
\end{aligned} \tag{3.8}
$$

We further calculate

$$
\begin{aligned}
ababcbac &= -ababc(ac+cb)c \\
&= -ababcacc - ababccbc \\
&= -ababca(\lambda_{ca}(e_{23}+e_{132}) + \lambda_{cb}(e_{12}+e_{123})) - abab(\lambda_{ca}(e_{23}+e_{132}) + \lambda_{cb}(e_{12}+e_{123}))bc \\
&= -ababca(\lambda_{ca}(e_{23}+e_{132}) + \lambda_{cb}(e_{12}+e_{123})) \\
&\quad - ababbc(\lambda_{ca}(e_{(31)(23)(23)} + e_{(31)(23)(132)}) + \lambda_{cb}(e_{(31)(23)(12)} + e_{(31)(23)(123)})) \\
&= -ababca(\lambda_{ca}(e_{23}+e_{132}) + \lambda_{cb}(e_{12}+e_{123})) \\
&\quad - ababbc(\lambda_{ca}(e_{31}+e_{123}) + \lambda_{cb}(e_{23}+e_{1})) \\
&= -ababca(\lambda_{ca}(e_{23}+e_{132}) + \lambda_{cb}(e_{12}+e_{123})) \\
&\quad - aba(\lambda_{bc}(e_{12}+e_{132}) + \lambda_{ba}(e_{13}+e_{123}))c(\lambda_{ca}(e_{31}+e_{123}) + \lambda_{cb}(e_{23}+e_{1})) \\
&= -ababca(\lambda_{ca}(e_{23}+e_{132}) + \lambda_{cb}(e_{12}+e_{123})) \\
&\quad - abac(\lambda_{bc}(e_{(31)(12)} + e_{(31)(132)}) + \lambda_{ba}(e_{(31)(13)} + e_{(31)(123)}))(\lambda_{ca}(e_{31}+e_{123}) + \lambda_{cb}(e_{23}+e_{1})) \\
&= -ababca(\lambda_{ca}(e_{23}+e_{132}) + \lambda_{cb}(e_{12}+e_{123})) \\
&\quad - abac(\lambda_{bc}(e_{123}+e_{23}) + \lambda_{ba}(e_{1}+e_{12}))(\lambda_{ca}(e_{31}+e_{123}) + \lambda_{cb}(e_{23}+e_{1}))
\end{aligned} \tag{3.9}
$$

and

$$
\begin{aligned}
ababca &= -aba(ab+ca)a \\
&= -abaaba - abacaa \\
&= -ab(\lambda_{ab}(e_{13}+e_{132}) + \lambda_{ac}(e_{23}+e_{123}))ba - abac(\lambda_{ab}(e_{13}+e_{132}) + \lambda_{ac}(e_{23}+e_{123})) \\
&= -abba(\lambda_{ab}(e_{(12)(23)(13)} + e_{(12)(23)(132)}) + \lambda_{ac}(e_{(12)(23)(23)} + e_{(12)(23)(123)})) \\
&\quad - abac(\lambda_{ab}(e_{13}+e_{132}) + \lambda_{ac}(e_{23}+e_{123})) \\
&= -abba(\lambda_{ab}(e_{23}+e_{1}) + \lambda_{ac}(e_{12}+e_{132})) \\
&\quad - abac(\lambda_{ab}(e_{13}+e_{132}) + \lambda_{ac}(e_{23}+e_{123})) \\
&= -a(\lambda_{bc}(e_{12}+e_{132}) + \lambda_{ba}(e_{13}+e_{123}))a(\lambda_{ab}(e_{23}+e_{1}) + \lambda_{ac}(e_{12}+e_{132})) \\
&\quad - abac(\lambda_{ab}(e_{13}+e_{132}) + \lambda_{ac}(e_{23}+e_{123})) \\
&= -aa(\lambda_{bc}(e_{(12)(12)} + e_{(12)(132)}) + \lambda_{ba}(e_{(12)(13)} + e_{(12)(123)}))(\lambda_{ab}(e_{23}+e_{1}) + \lambda_{ac}(e_{12}+e_{132})) \\
&\quad - abac(\lambda_{ab}(e_{13}+e_{132}) + \lambda_{ac}(e_{23}+e_{123})) \\
&= -aa(\lambda_{bc}(e_{1}+e_{13}) + \lambda_{ba}(e_{132}+e_{23}))(\lambda_{ab}(e_{23}+e_{1}) + \lambda_{ac}(e_{12}+e_{132})) \\
&\quad - abac(\lambda_{ab}(e_{13}+e_{132}) + \lambda_{ac}(e_{23}+e_{123})) \\
&= -(\lambda_{ab}(e_{13}+e_{132}) + \lambda_{ac}(e_{23}+e_{123}))(\lambda_{bc}(e_{1}+e_{13}) + \lambda_{ba}(e_{132}+e_{23})) \\
&\qquad \cdot (\lambda_{ab}(e_{23}+e_{1}) + \lambda_{ac}(e_{12}+e_{132})) \\
&\quad - abac(\lambda_{ab}(e_{13}+e_{132}) + \lambda_{ac}(e_{23}+e_{123})) \\
&= -(\lambda_{ab}\lambda_{bc}e_{13} + \lambda_{ab}\lambda_{ba}e_{132} + \lambda_{ac}\lambda_{ba}e_{23})(\lambda_{ab}(e_{23}+e_{1}) + \lambda_{ac}(e_{12}+e_{132})) \\
&\quad - abac(\lambda_{ab}(e_{13}+e_{132}) + \lambda_{ac}(e_{23}+e_{123})) \\
&= -\lambda_{ab}\lambda_{ba}\lambda_{ac}e_{132} - \lambda_{ac}\lambda_{ba}\lambda_{ab}e_{23} - abac(\lambda_{ab}(e_{13}+e_{132}) + \lambda_{ac}(e_{23}+e_{123})).
\end{aligned} \tag{3.10}
$$

Putting the above computations together, we finally obtain:

$$
\begin{aligned}
abacabac &= -ab(\lambda_{ab}(e_{13}+e_{132})+\lambda_{ac}(e_{23}+e_{123}))(\lambda_{bc}(e_{12}+e_{132})+\lambda_{ba}(e_{13}+e_{123}))ac - ababcbac \\
&\overset{(3.9)}{=} -ab(\lambda_{ab}(e_{13}+e_{132})+\lambda_{ac}(e_{23}+e_{123}))(\lambda_{bc}(e_{12}+e_{132})+\lambda_{ba}(e_{13}+e_{123}))ac \\
&\quad + ababca(\lambda_{ca}(e_{23}+e_{132})+\lambda_{cb}(e_{12}+e_{123})) \\
&\quad + abac(\lambda_{bc}(e_{123}+e_{23})+\lambda_{ba}(e_{1}+e_{12}))(\lambda_{ca}(e_{31}+e_{123})+\lambda_{cb}(e_{23}+e_{1})) \\
&\overset{(3.10)}{=} -ab(\lambda_{ab}(e_{13}+e_{132})+\lambda_{ac}(e_{23}+e_{123}))(\lambda_{bc}(e_{12}+e_{132})+\lambda_{ba}(e_{13}+e_{123}))ac \\
&\quad + (-\lambda_{ab}\lambda_{ba}\lambda_{ac}e_{132} - \lambda_{ac}\lambda_{ba}\lambda_{ab}e_{23} - abac(\lambda_{ab}(e_{13}+e_{132})+\lambda_{ac}(e_{23}+e_{123}))) \\
&\qquad \cdot (\lambda_{ca}(e_{23}+e_{132})+\lambda_{cb}(e_{12}+e_{123})) \\
&\quad + abac(\lambda_{bc}(e_{123}+e_{23})+\lambda_{ba}(e_{1}+e_{12}))(\lambda_{ca}(e_{31}+e_{123})+\lambda_{cb}(e_{23}+e_{1})) \\
&= -ab(\lambda_{ab}(e_{13}+e_{132})+\lambda_{ac}(e_{23}+e_{123}))(\lambda_{bc}(e_{12}+e_{132})+\lambda_{ba}(e_{13}+e_{123}))ac \\
&\quad - \lambda_{ab}\lambda_{ba}\lambda_{ac}\lambda_{ca}e_{132} - \lambda_{ac}\lambda_{ba}\lambda_{ab}\lambda_{ca}e_{23} - abac(\lambda_{ab}\lambda_{ca}e_{132}+\lambda_{ac}\lambda_{ca}e_{23}+\lambda_{ac}\lambda_{cb}e_{123}) \\
&\quad + abac(\lambda_{bc}(e_{123}+e_{23})+\lambda_{ba}(e_{1}+e_{12}))(\lambda_{ca}(e_{31}+e_{123})+\lambda_{cb}(e_{23}+e_{1})) \\
&= -abac(\lambda_{ab}(e_{(31)(12)(13)}+e_{(31)(12)(132)})+\lambda_{ac}(e_{(31)(12)(23)}+e_{(31)(12)(123)})) \\
&\qquad \cdot (\lambda_{bc}(e_{(31)(12)(12)}+e_{(31)(12)(132)})+\lambda_{ba}(e_{(31)(12)(13)}+e_{(31)(12)(123)})) \\
&\quad - \lambda_{ab}\lambda_{ba}\lambda_{ac}\lambda_{ca}e_{132} - \lambda_{ac}\lambda_{ba}\lambda_{ab}\lambda_{ca}e_{23} - abac(\lambda_{ab}\lambda_{ca}e_{132}+\lambda_{ac}\lambda_{ca}e_{23}+\lambda_{ac}\lambda_{cb}e_{123}) \\
&\quad + abac(\lambda_{bc}(e_{123}+e_{23})+\lambda_{ba}(e_{1}+e_{12}))(\lambda_{ca}(e_{31}+e_{123})+\lambda_{cb}(e_{23}+e_{1})) \\
&= -abac(\lambda_{ab}(e_{23}+e_{1})+\lambda_{ac}(e_{12}+e_{132}))(\lambda_{bc}(e_{31}+e_{1})+\lambda_{ba}(e_{23}+e_{132})) \\
&\quad - \lambda_{ab}\lambda_{ba}\lambda_{ac}\lambda_{ca}e_{132} - \lambda_{ac}\lambda_{ba}\lambda_{ab}\lambda_{ca}e_{23} - abac(\lambda_{ab}\lambda_{ca}e_{132}+\lambda_{ac}\lambda_{ca}e_{23}+\lambda_{ac}\lambda_{cb}e_{123}) \\
&\quad + abac(\lambda_{bc}(e_{123}+e_{23})+\lambda_{ba}(e_{1}+e_{12}))(\lambda_{ca}(e_{31}+e_{123})+\lambda_{cb}(e_{23}+e_{1})) \\
&= -abac(\lambda_{ab}\lambda_{ba}e_{23}+\lambda_{ab}\lambda_{bc}e_{1}+\lambda_{ac}\lambda_{ba}e_{132}) \\
&\quad - \lambda_{ab}\lambda_{ba}\lambda_{ac}\lambda_{ca}e_{132} - \lambda_{ac}\lambda_{ba}\lambda_{ab}\lambda_{ca}e_{23} - abac(\lambda_{ab}\lambda_{ca}e_{132}+\lambda_{ac}\lambda_{ca}e_{23}+\lambda_{ac}\lambda_{cb}e_{123}) \\
&\quad + abac(\lambda_{bc}\lambda_{ca}e_{123}+\lambda_{bc}\lambda_{cb}e_{23}+\lambda_{ba}\lambda_{cb}e_{1}) \\
&= abac(-\lambda_{ab}\lambda_{ba}e_{23} - \lambda_{ab}\lambda_{bc}e_{1} - \lambda_{ac}\lambda_{ba}e_{132} - \lambda_{ab}\lambda_{ca}e_{132} - \lambda_{ac}\lambda_{ca}e_{23} - \lambda_{ac}\lambda_{cb}e_{123} \\
&\qquad\quad + \lambda_{bc}\lambda_{ca}e_{123} + \lambda_{bc}\lambda_{cb}e_{23} + \lambda_{ba}\lambda_{cb}e_{1}) \\
&\quad - \lambda_{ab}\lambda_{ba}\lambda_{ac}\lambda_{ca}e_{132} - \lambda_{ac}\lambda_{ba}\lambda_{ab}\lambda_{ca}e_{23} \\
&= abac((-\lambda_{ab}\lambda_{ba} - \lambda_{ac}\lambda_{ca} + \lambda_{bc}\lambda_{cb})e_{23} + (-\lambda_{ab}\lambda_{bc}+\lambda_{ba}\lambda_{cb})e_{1} + (-\lambda_{ac}\lambda_{ba} - \lambda_{ab}\lambda_{ca})e_{132} \\
&\qquad\quad + (-\lambda_{ac}\lambda_{cb}+\lambda_{bc}\lambda_{ca})e_{123}) \\
&\quad - \lambda_{ab}\lambda_{ba}\lambda_{ac}\lambda_{ca}e_{132} - \lambda_{ac}\lambda_{ba}\lambda_{ab}\lambda_{ca}e_{23} \\
&= abac((\lambda_{ab}^2+\lambda_{ca}^2-\lambda_{bc}^2)e_{23} + (-\lambda_{ab}\lambda_{bc}+\lambda_{ab}\lambda_{bc})e_{1} + (2\lambda_{ac}\lambda_{ab})e_{132} \\
&\qquad\quad + (-\lambda_{ca}\lambda_{bc}+\lambda_{bc}\lambda_{ca})e_{123}) \\
&\quad - \lambda_{ab}^2\lambda_{ac}^2 e_{132} - \lambda_{ac}\lambda_{ab}\lambda_{ab}\lambda_{ac}e_{23} \\
&= abac((\lambda_{ab}^2+\lambda_{ca}^2-\lambda_{bc}^2)e_{23} + 2\lambda_{ab}\lambda_{ac}e_{132}) - \lambda_{ab}^2\lambda_{ac}^2 e_{23} - \lambda_{ab}^2\lambda_{ac}^2 e_{132}. \qquad (3.11)
\end{aligned}
$$

□

Combining (3.7) and Lemma 61 we immediately obtain

$$(abac\ e_{12})^2 = 0 \text{ and } (abac\ e_{31})^2 = 0.$$

This implies that

$$\chi_{H^*}(abac\ e_{12}) = 0 \text{ and } \chi_{H^*}(abac\ e_{31}) = 0. \qquad (3.12)$$

Furthermore, using the relations of the algebra H^* – in particular, we will use several times (we will indicate it when we do) that

$$\begin{aligned} baca &= -b(cb+ba)a \\ &= -bcba - b^2a^2 \\ &= bc(ac+cb) - b^2a^2 \\ &= bc(ac+cb) - b^2a^2 \\ &= bcac + bc^2b - b^2a^2 \\ &= -(ab+ca)ac + bc^2b - b^2a^2 \\ &= -abac - ca^2c + bc^2b - b^2a^2, \end{aligned} \tag{3.13}$$

and using the cyclicity of the trace, we can compute:

$$\begin{aligned} \chi_{H^*}(abac\ e_{23}) &= \chi_{H^*}(baca\ e_{(12)(23)}) \\ &= \chi_{H^*}(acab\ e_{(23)(12)(23)}) \\ &= \chi_{H^*}(acab\ e_{31}) \\ &= \chi_{H^*}(-a(ab+bc)b\ e_{31}) \\ &= \chi_{H^*}(-a^2b^2\ e_{31} - abcb\ e_{31}) \\ &= \chi_{H^*}(-a^2b^2\ e_{31} + ab(ac+ba)\ e_{31}) \\ &= \chi_{H^*}(-a^2b^2\ e_{31} + ab^2a\ e_{31} + abac\ e_{31}) \\ &\overset{(3.12)}{=} \chi_{H^*}(-a^2b^2\ e_{31} + a(\lambda_{bc}(e_{12}+e_{132}) + \lambda_{ba}(e_{13}+e_{123}))a\ e_{31}) \\ &= \chi_{H^*}(-a^2b^2\ e_{31} \\ &\qquad + a^2(\lambda_{bc}(e_{(12)(12)} + e_{(12)(132)}) + \lambda_{ba}(e_{(12)(13)} + e_{(12)(123)}))e_{31}) \\ &= \chi_{H^*}(-a^2b^2\ e_{31} + a^2(\lambda_{bc}(e_1+e_{31}) + \lambda_{ba}(e_{132}+e_{23}))e_{31}) \\ &= \chi_{H^*}(-a^2b^2\ e_{31} + a^2\lambda_{bc}e_{31}) \\ &= \chi_{H^*}(-(\lambda_{ab}(e_{13}+e_{132}) + \lambda_{ac}(e_{23}+e_{123}))(\lambda_{bc}(e_{12}+e_{132}) + \lambda_{ba}(e_{13}+e_{123}))e_{31} \\ &\qquad + (\lambda_{ab}(e_{13}+e_{132}) + \lambda_{ac}(e_{23}+e_{123}))\lambda_{bc}e_{31}) \\ &= \chi_{H^*}(-\lambda_{ab}\lambda_{ba}e_{31} + \lambda_{ab}\lambda_{bc}e_{31}) \\ &= \chi_{H^*}(\lambda_{ab}(\lambda_{ab}+\lambda_{bc})e_{31}) \\ &= \chi_{H^*}(\lambda_{ab}\lambda_{ac}e_{31}) \\ &= 12\lambda_{ab}\lambda_{ac}. \end{aligned} \tag{3.14}$$

$$\begin{aligned} \chi_{H^*}(abac\ e_1) &= \chi_{H^*}(baca\ e_{12}) \\ &\overset{(3.13)}{=} \chi_{H^*}(-abac\ e_{12} - ca^2c\ e_{12} + bc^2b\ e_{12} - b^2a^2\ e_{12}) \\ &\overset{(3.12)}{=} \chi_{H^*}(-ca^2c\ e_{12} + bc^2b\ e_{12} - b^2a^2\ e_{12}) \\ &= \chi_{H^*}(-c(\lambda_{ab}(e_{13}+e_{132}) + \lambda_{ac}(e_{23}+e_{123}))c\ e_{12} \\ &\qquad + b(\lambda_{ca}(e_{23}+e_{132}) + \lambda_{cb}(e_{12}+e_{123}))b\ e_{12} \\ &\qquad - (\lambda_{bc}(e_{12}+e_{132}) + \lambda_{ba}(e_{13}+e_{123})) \\ &\qquad\quad \cdot (\lambda_{ab}(e_{13}+e_{132}) + \lambda_{ac}(e_{23}+e_{123}))\ e_{12}) \\ &= \chi_{H^*}(-c(\lambda_{ab}(e_{13}+e_{132}) + \lambda_{ac}(e_{23}+e_{123}))c\ e_{12} \end{aligned}$$

$$
\begin{aligned}
&\quad + b(\lambda_{ca}(e_{23}+e_{132})+\lambda_{cb}(e_{12}+e_{123}))b\ e_{12})\\
&= \chi_{H^*}(-c^2(\lambda_{ab}(e_{(13)(13)}+e_{(13)(132)})+\lambda_{ac}(e_{(13)(23)}+e_{(13)(123)}))e_{12}\\
&\quad + b(\lambda_{ca}(e_{23}+e_{132})+\lambda_{cb}(e_{12}+e_{123}))b\ e_{12})\\
&= \chi_{H^*}(-c^2(\lambda_{ab}(e_1+e_{23})+\lambda_{ac}(e_{132}+e_{12}))e_{12}\\
&\quad + b(\lambda_{ca}(e_{23}+e_{132})+\lambda_{cb}(e_{12}+e_{123}))b\ e_{12})\\
&= \chi_{H^*}(-c^2\lambda_{ac}e_{12}\\
&\quad + b(\lambda_{ca}(e_{23}+e_{132})+\lambda_{cb}(e_{12}+e_{123}))b\ e_{12})\\
&= \chi_{H^*}(-(\lambda_{ca}(e_{23}+e_{132})+\lambda_{cb}(e_{12}+e_{123}))\lambda_{ac}e_{12}\\
&\quad + b^2(\lambda_{ca}(e_{(23)(23)}+e_{(23)(132)})+\lambda_{cb}(e_{(23)(12)}+e_{(23)(123)}))e_{12}\\
&= \chi_{H^*}(-\lambda_{cb}\lambda_{ac}e_{12}+b^2(\lambda_{ca}(e_1+e_{12})+\lambda_{cb}(e_{132}+e_{31}))e_{12}\\
&= \chi_{H^*}(-\lambda_{cb}\lambda_{ac}e_{12}+b^2\lambda_{ca}e_{12})\\
&= \chi_{H^*}(-\lambda_{cb}\lambda_{ac}e_{12}+(\lambda_{bc}(e_{12}+e_{132})+\lambda_{ba}(e_{13}+e_{123}))\lambda_{ca}e_{12})\\
&= \chi_{H^*}(-\lambda_{cb}\lambda_{ac}e_{12}+\lambda_{bc}\lambda_{ca}e_{12})\\
&= \chi_{H^*}(-\lambda_{cb}\lambda_{ac}e_{12}+\lambda_{bc}\lambda_{ca}e_{12})\\
&= 0.
\end{aligned}
$$

$$
\begin{aligned}
\chi_{H^*}(abac\ e_{123}) &= \chi_{H^*}(baca\ e_{(12)(123)})\\
&= \chi_{H^*}(baca\ e_{23})\\
&\overset{(3.13)}{=} \chi_{H^*}(-abac\ e_{23}-ca^2c\ e_{23}+bc^2b\ e_{23}-b^2a^2e_{23})\\
&\overset{(3.14)}{=} \chi_{H^*}(-ca^2c\ e_{23}+bc^2b\ e_{23}-b^2a^2e_{23})-12\lambda_{ab}\lambda_{ac}\\
&= \chi_{H^*}(-c(\lambda_{ab}(e_{13}+e_{132})+\lambda_{ac}(e_{23}+e_{123}))c\ e_{23}\\
&\quad\quad + b(\lambda_{ca}(e_{23}+e_{132})+\lambda_{cb}(e_{12}+e_{123}))b\ e_{23}-b^2a^2e_{23})\\
&\quad - 12\lambda_{ab}\lambda_{ac}\\
&= \chi_{H^*}(-c^2(\lambda_{ab}(e_{(13)(13)}+e_{(13)(132)})+\lambda_{ac}(e_{(13)(23)}+e_{(13)(123)}))e_{23}\\
&\quad\quad + b^2(\lambda_{ca}(e_{(23)(23)}+e_{(23)(132)})+\lambda_{cb}(e_{(23)(12)}+e_{(23)(123)}))e_{23}-b^2a^2e_{23})\\
&\quad - 12\lambda_{ab}\lambda_{ac}\\
&= \chi_{H^*}(-c^2(\lambda_{ab}(e_1+e_{23})+\lambda_{ac}(e_{132}+e_{12}))e_{23}\\
&\quad\quad + b^2(\lambda_{ca}(e_1+e_{12})+\lambda_{cb}(e_{132}+e_{31}))e_{23}-b^2a^2e_{23})\\
&\quad - 12\lambda_{ab}\lambda_{ac}\\
&= \chi_{H^*}(-(\lambda_{ca}(e_{23}+e_{132})+\lambda_{cb}(e_{12}+e_{123}))(\lambda_{ab}(e_1+e_{23})+\lambda_{ac}(e_{132}+e_{12}))e_{23}\\
&\quad\quad + (\lambda_{bc}(e_{12}+e_{132})+\lambda_{ba}(e_{13}+e_{123}))(\lambda_{ca}(e_1+e_{12})+\lambda_{cb}(e_{132}+e_{31}))e_{23}\\
&\quad\quad - b^2a^2e_{23})\\
&\quad - 12\lambda_{ab}\lambda_{ac}\\
&= \chi_{H^*}(-\lambda_{ca}\lambda_{ab}e_{23}-b^2a^2e_{23})-12\lambda_{ab}\lambda_{ac}\\
&= \chi_{H^*}(-\lambda_{ca}\lambda_{ab}e_{23}\\
&\quad\quad - (\lambda_{bc}(e_{12}+e_{132})+\lambda_{ba}(e_{13}+e_{123}))(\lambda_{ab}(e_{13}+e_{132})+\lambda_{ac}(e_{23}+e_{123}))e_{23})\\
&\quad - 12\lambda_{ab}\lambda_{ac}\\
&= \chi_{H^*}(-\lambda_{ca}\lambda_{ab}e_{23})-12\lambda_{ab}\lambda_{ac}
\end{aligned}
$$

$$
\begin{aligned}
&= \chi_{H^*}(\lambda_{ac}\lambda_{ab}e_{23}) - 12\lambda_{ab}\lambda_{ac}\\
&= 12\lambda_{ac}\lambda_{ab} - 12\lambda_{ab}\lambda_{ac}\\
&= 0.
\end{aligned}
$$

$$
\begin{aligned}
\chi_{H^*}(abac\ e_{132}) &= \chi_{H^*}(baca\ e_{(12)(132)})\\
&= \chi_{H^*}(baca\ e_{31})\\
&\overset{(3.13)}{=} \chi_{H^*}(-abac\ e_{31} - ca^2c\ e_{31} + bc^2b\ e_{31} - b^2a^2e_{31})\\
&\overset{(3.12)}{=} \chi_{H^*}(-ca^2c\ e_{31} + bc^2b\ e_{31} - b^2a^2e_{31})\\
&\overset{c^2e_{31}=0}{=} \chi_{H^*}(bc^2b\ e_{31} - b^2a^2e_{31})\\
&= \chi_{H^*}(b(\lambda_{ca}(e_{23}+e_{132}) + \lambda_{cb}(e_{12}+e_{123}))b\ e_{31} - b^2a^2e_{31})\\
&= \chi_{H^*}(b^2(\lambda_{ca}(e_{(23)(23)} + e_{(23)(132)}) + \lambda_{cb}(e_{(23)(12)} + e_{(23)(123)}))e_{31} - b^2a^2e_{31})\\
&= \chi_{H^*}(b^2(\lambda_{ca}(e_1+e_{12}) + \lambda_{cb}(e_{132}+e_{31}))e_{31} - b^2a^2e_{31})\\
&= \chi_{H^*}(b^2\lambda_{cb}e_{31} - b^2a^2e_{31})\\
&= \chi_{H^*}((\lambda_{bc}(e_{12}+e_{132}) + \lambda_{ba}(e_{13}+e_{123}))\lambda_{cb}e_{31}\\
&\qquad - (\lambda_{bc}(e_{12}+e_{132}) + \lambda_{ba}(e_{13}+e_{123}))a^2e_{31})\\
&= \chi_{H^*}(\lambda_{ba}\lambda_{cb}e_{31} - \lambda_{ba}a^2e_{31})\\
&= \chi_{H^*}(\lambda_{ba}\lambda_{cb}e_{31} - \lambda_{ba}(\lambda_{ab}(e_{13}+e_{132}) + \lambda_{ac}(e_{23}+e_{123}))e_{31})\\
&= \chi_{H^*}(\lambda_{ba}\lambda_{cb}e_{31} - \lambda_{ba}\lambda_{ab}e_{31})\\
&= \chi_{H^*}(\lambda_{ba}(\lambda_{cb} - \lambda_{ba})e_{31})\\
&= \chi_{H^*}(\lambda_{ba}\lambda_{ca}e_{31})\\
&= \chi_{H^*}(\lambda_{ab}\lambda_{ac}e_{31})\\
&= 12\lambda_{ab}\lambda_{ac}.
\end{aligned}
$$

Summarizing our above calculations we finally obtain:

Proposition 62.

$$
\begin{aligned}
p = \frac{1}{\dim H}\chi_{H^*} &= \frac{1}{72}\Bigg(\sum_{g\in S_3} 12g + 12\lambda_{ab}\lambda_{ac}(\delta_{abac\ e_{23}} + \delta_{abac\ e_{132}})\Bigg)\\
&= \frac{1}{6}\Bigg(\sum_{g\in S_3} g + \lambda_{ab}\lambda_{ac}(\delta_{abac\ e_{23}} + \delta_{abac\ e_{132}})\Bigg).
\end{aligned}
$$

Since the character of the sign representation is $\chi_{\mathrm{sgn}} = e_1 - e_{12} - e_{23} - e_{31} + e_{123} + e_{132}$, we have therefore:

$$
p_{\mathrm{sgn}} = p(\chi_{\mathrm{sgn}}\cdot -) = \frac{1}{6}\Big(1 - (12) - (23) - (31) + (123) + (132) + \lambda_{ab}\lambda_{ac}(-\delta_{abac\ e_{23}} + \delta_{abac\ e_{132}})\Big).
$$

Furthermore, we have:

$$
p_V = p(\chi_V\cdot -) = \frac{1}{6}\Bigg(\sum_{g\in S_3} g + \lambda_{ab}\lambda_{ac}(\delta_{abac\ e_{23}} + \delta_{abac\ e_{132}})\Bigg)((2e_1 - e_{123} - e_{132})\cdot -)
$$

$$= \frac{1}{6}(2 - (123) - (132) - \lambda_{ab}\lambda_{ac}\delta_{abac\ e_{132}}).$$

Evidently, the images of these elements p_i under the surjection $H \longrightarrow \Bbbk S_3$ to the maximal semisimple quotient algebra $\Bbbk S_3$ are the orthogonal idempotents of $\Bbbk S_3$ for the unique isotypic decomposition of $\Bbbk S_3$. Hence, by Lemma 37, it only remains to check that the p_i are orthogonal idempotents in H in order to prove that they provide an isotypic decomposition of H.

3.3.2.1 Idempotence and orthogonality of p, p_{sgn} and p_V

Concerning the question whether these elements p, p_{sgn} and $p_V \in H$ satisfy our Conjecture 48, we can apply our general results from Subsections 3.2.1 and 3.2.2. We obtain from Theorem 52 that p and p_{sgn} are idempotents projecting to isotypic components of H of trivial type and sign representation type, respectively. Moreover, we obtain from Theorem 56 that $p+p_{\mathrm{sgn}}+p_V = 1$, if we can verify that H satisfies the assumptions of that proposition.

Indeed, we can ensure that the subalgebra $\Lambda_0 H^* \Lambda_0 \subseteq H^*$ has only one simple representation up to isomorphism. The Haar integral of the semisimple sub-Hopf-algebra $H_0^* = \Bbbk^{S_3} \subseteq H^*$ is given by the idempotent $e_1 \in \Bbbk^{S_3}$. With this we can compute that $\Lambda_0 H^* \Lambda_0 = e_1 H^* e_1 = \Bbbk\{e_1, abace_1\}$, since 1 and $abac \in B(V)$ span the subspace of S_3-degree 1 in $B(V)$. e_1 is the unit of the algebra $\Lambda_0 H^* \Lambda_0$ and, furthermore, we have $(abace_1)^2 = 0$ by Lemma 61. Essentially, the reason for this is that the deformed relations for the squares a^2, b^2 and c^2 take values in the kernel of right (or left) multiplication by $\Lambda_0 = e_1$. The algebra $\Lambda_0 H^* \Lambda_0$ is therefore isomorphic to the two-dimensional algebra $\Bbbk[x]/(x^2)$, which indeed has a unique simple representation.

Finally, with the help of a computation with the computer algebra system Magma, as can be seen in the Appendix, we can extend these results to the statement that all three p, p_{sgn} and p_V are idempotents and pairwise orthogonal.

3.3.2.2 Calculations with Magma

We describe here a Magma code for calculating explicitly the products of the different conjectured idempotents p_i for the Hopf algebra discussed in Subsection 3.3.2. We begin with the remark that, when considering the grading we have for H^*,

$$H^* = \bigoplus_{k=0}^{4} H_k^*,$$

the only direct summand on which the product $p_i p_j$ (where $i \neq j$) might not vanish is H_4^*. This follows from the fact that all p_i's vanish on H_k for $k \neq 0, 4$, and on the coalgebra grading. The calculation with Magma will be done in the following way: for different values of $\lambda_a, \lambda_b, \lambda_c$ we will define the algebra $A = H^*$ in Magma. Then we will present it in a matrix form, and calculate the trace of the regular representation, as well as the translations of this trace by multiples of irreducible characters. For the calculations of the product we will calculate $(p_i \otimes 1)\Delta(abace_g)$ where $g \in G$ and $p_i \in \{p, p_V\}$ by hand, and apply the relevant functionals $p_j \in \{p, p_{\mathrm{sgn}}, p_V\}$ to them. Finally, since all the relevant values are polynomials of degree at most 3 in $\lambda_{ab}\lambda_{ac}$ it will be enough to show that they vanish on four different values of $\lambda_{ab}\lambda_{ac}$.

The code is enclosed here. We ran it on `http://magma.maths.usyd.edu.au/calc/`, the online version of Magma.

```
/* Values of lambdas */
lam_a:=0;
lam_b:=23;
lam_c:=11;

K:=RationalField();
A<e1,e2,e3,e4,e5,e6,a,b,c>:= FPAlgebra<K, e1,e2,e3,e4,e5,e6,a,b,c|
e1*e1-e1, e2*e1, e3*e1, e4*e1, e5*e1, e6*e1,
e1*e2, e2*e2-e2, e3*e2, e4*e2, e5*e2, e6*e2,
e1*e3, e2*e3, e3*e3-e3, e4*e3, e5*e3, e6*e3,
e1*e4, e2*e4, e3*e4, e4*e4-e4, e5*e4, e6*e4,
e1*e5, e2*e5, e3*e5, e4*e5, e5*e5-e5, e6*e5,
e1*e6, e2*e6, e3*e6, e4*e6, e5*e6, e6*e6-e6,e1+e2+e3+e4+e5+e6-1,
a*e1-e2*a,a*e2-e1*a, a*e3-e5*a,e5*a-a*e3,a*e4-e6*a,a*e6-e4*a,
b*e1-e3*b,b*e3-e1*b, b*e4-e5*b,e5*b-b*e4,b*e2-e6*b,b*e6-e2*b,
c*e1-e4*c,c*e4-e1*c, c*e2-e5*c,e5*c-c*e2,c*e3-e6*c,c*e6-e3*c,
a*b+b*c+c*a, a*c+c*b+b*a,
a^2 - (lam_a-lam_b)*(e4+e6) - (lam_a-lam_c)*(e3+e5),
b^2 - (lam_b-lam_c)*(e2+e6) - (lam_b-lam_a)*(e4+e5),
c^2 - (lam_c-lam_a)*(e3+e6) - (lam_c-lam_b)*(e2+e5)>;
/* Defining A=H^* by generators and relations */
D:=Dimension(A);
S,f:= Algebra(A); /* S is now the algebra A considered as a subalgebra of the
72 x 72 matrix algebra. f:A\to S is the natural isomorphism */
Y:=AssociativeArray();

B,h:=ChangeBasis(S,[f(e1),f(e2),f(e3),f(e4),f(e5),f(e6),
f(a*e1),f(a*e2),f(a*e3),f(a*e4),f(a*e5),f(a*e6),
f(b*e1),f(b*e2),f(b*e3),f(b*e4),f(b*e5),f(b*e6),
f(c*e1),f(c*e2),f(c*e3),f(c*e4),f(c*e5),f(c*e6),
f(a*b*e1),f(a*b*e2),f(a*b*e3),f(a*b*e4),f(a*b*e5),f(a*b*e6),
f(b*c*e1),f(b*c*e2),f(b*c*e3),f(b*c*e4),f(b*c*e5),f(b*c*e6),
f(a*c*e1),f(a*c*e2),f(a*c*e3),f(a*c*e4),f(a*c*e5),f(a*c*e6),
f(c*b*e1),f(c*b*e2),f(c*b*e3),f(c*b*e4),f(c*b*e5),f(c*b*e6),
f(a*b*a*e1),f(a*b*a*e2),f(a*b*a*e3),f(a*b*a*e4),f(a*b*a*e5),f(a*b*a*e6),
f(a*b*c*e1),f(a*b*c*e2),f(a*b*c*e3),f(a*b*c*e4),f(a*b*c*e5),f(a*b*c*e6),
f(b*a*c*e1),f(b*a*c*e2),f(b*a*c*e3),f(b*a*c*e4),f(b*a*c*e5),f(b*a*c*e6),
f(a*b*a*c*e1),f(a*b*a*c*e2),f(a*b*a*c*e3),
f(a*b*a*c*e4),f(a*b*a*c*e5),f(a*b*a*c*e6)]);

/* we now fix for S the basis described above.
This is given by the algebra B. The map h:S\to B is then the isomorphism */

for i:=1 to D do
Y[i]:=0;
for j:=1 to D do
```

```
Y[i]:= Y[i] + BasisProduct(B,i,j)[j]/72;
end for;
end for;

/* We calculate p as the trace of the regular representation
divided by the dimension. Notice that we think of p as an element in H=A^*. */

"print p";
for i:=1 to D do
Y[i];
end for;
"end p";
"";

chi:= e1-e2-e3-e4+e5+e6;
chiV:= 2*e1 - e5-e6;

/* the characters of the two non-trivial representations of A^*.
Both are elements of A */

Z:=AssociativeArray();
for i:=1 to D do
Z[i]:=0;
for j:=1 to D do
Z[i]:= Z[i] + (h(f(chi))*BasisProduct(B,i,j))[j]/72;
end for;
end for;

/* The array Z contains now the translation of p by the sign representation.
In other words, it is p_{sign}, considered as an element of A^*. */

W:=AssociativeArray();
for i:=1 to D do
W[i]:=0;
for j:=1 to D do
W[i]:= W[i] + 2*(h(f(chiV))*BasisProduct(B,i,j))[j]/72;
end for;
end for;

/* Similarly, we calculate p_V for the
2-dimensional irreducible representation of A. */

E2:=AssociativeArray();
for i:=1 to D do
E2[i]:=Y[i]+Z[i] + 2*W[i];
end for;
```

```
"print epsilon";
for i:=1 to D do
E2[i];
end for;
/* We calculate and print the sum p + p_{sign} + p_V.
If it is the counit, then we are on the right path. */

"print p_sign";
for i:=1 to D do
Z[i];
end for;
"end p_sign";
"";

"print p_V";
for i:=1 to D do
W[i];
end for;
"end p_V";
"";

/* Next, we calculated manually the
elements v_i:=(p \otimes 1)\Delta(a*b*a*c*ei). */
v1:= h(f(1/6*(lam_a-lam_b)*(lam_a-lam_c)*(e3 + e5) +
1/6*((lam_c-lam_a)*b*b*e6 - (lam_a-lam_b)*c*c*e4-
(lam_a-lam_b)*a*a*e5 - (lam_a-lam_c)*a*a*e3)+
1/6*(a*b*a*c*e1 + a*c*a*b*e2 + c*b*c*a*e3 +
b*a*b*c*e4 + c*a*c*b*e6 + b*c*b*a*e5)));

v2:= h(f(1/6*(lam_a-lam_b)*(lam_a-lam_c)*(e6 + e4) +
1/6*((lam_c-lam_a)*b*b*e3 - (lam_a-lam_b)*c*c*e5-
(lam_a-lam_b)*a*a*e4 - (lam_a-lam_c)*a*a*e6)+
1/6*(a*b*a*c*e2 + a*c*a*b*e1 + c*b*c*a*e6 +
b*a*b*c*e5 + c*a*c*b*e3 + b*c*b*a*e4)));

v3:= h(f(1/6*(lam_a-lam_b)*(lam_a-lam_c)*(e1 + e2) +
1/6*((lam_c-lam_a)*b*b*e4 - (lam_a-lam_b)*c*c*e6-
(lam_a-lam_b)*a*a*e2 - (lam_a-lam_c)*a*a*e1)+
1/6*(a*b*a*c*e3 + a*c*a*b*e5 + c*b*c*a*e1 +
b*a*b*c*e6 + c*a*c*b*e4 + b*c*b*a*e2)));

v4:= h(f(1/6*(lam_a-lam_b)*(lam_a-lam_c)*(e5 + e3) +
1/6*((lam_c-lam_a)*b*b*e2 - (lam_a-lam_b)*c*c*e1-
(lam_a-lam_b)*a*a*e3 - (lam_a-lam_c)*a*a*e5)+
1/6*(a*b*a*c*e4 + a*c*a*b*e6 + c*b*c*a*e5 +
```

```
b*a*b*c*e1 + c*a*c*b*e2 + b*c*b*a*e3)));

v5:= h(f(1/6*(lam_a-lam_b)*(lam_a-lam_c)*(e4 + e6) +
1/6*((lam_c-lam_a)*b*b*e1 - (lam_a-lam_b)*c*c*e2-
(lam_a-lam_b)*a*a*e6 - (lam_a-lam_c)*a*a*e4)+
1/6*(a*b*a*c*e5 + a*c*a*b*e3 + c*b*c*a*e4 +
b*a*b*c*e2 + c*a*c*b*e1 + b*c*b*a*e6)));

v6:=h(f(1/6*(lam_a-lam_b)*(lam_a-lam_c)*(e2 + e1) +
1/6*((lam_c-lam_a)*b*b*e5 - (lam_a-lam_b)*c*c*e3-
(lam_a-lam_b)*a*a*e1 - (lam_a-lam_c)*a*a*e2)+
1/6*(a*b*a*c*e6 + a*c*a*b*e4 + c*b*c*a*e2 +
b*a*b*c*e3 + c*a*c*b*e5 + b*c*b*a*e1)));

E:= AssociativeArray();
for i:=1 to 6 do
E[i]:=0;
end for;

for i:= 1 to D do
E[1]:= E[1] + Y[i]*v1[i];
E[2]:= E[2] + Y[i]*v2[i];
E[3]:= E[3] + Y[i]*v3[i];
E[4]:= E[4] + Y[i]*v4[i];
E[5]:= E[5] + Y[i]*v5[i];
E[6]:= E[6] + Y[i]*v6[i];
end for;

/* This calculates p (p \otimes 1)\Delta(a*b*a*c*ei) = p p (a*b*a*c*ei).
Since a*b*a*c*ei are the only elements on which p^2 might be non-zero,
it is enough to consider them.
After that we do a similar calculation for p*p_V and p*p_{sign}.*/
"results of p*p - p";
for i:=1 to 6 do
E[i]-Y[D-6+i];
end for;

for i:=1 to 6 do
E[i]:=0;
end for;

for i:= 1 to D do
E[1]:= E[1] + Z[i]*v1[i];
E[2]:= E[2] + Z[i]*v2[i];
E[3]:= E[3] + Z[i]*v3[i];
E[4]:= E[4] + Z[i]*v4[i];
```

```
E[5]:= E[5] + Z[i]*v5[i];
E[6]:= E[6] + Z[i]*v6[i];
end for;

"Results of p*p_{sign}";
for i:=1 to 6 do
E[i];
end for;

for i:=1 to 6 do
E[i]:=0;
end for;

for i:= 1 to D do
E[1]:= E[1] + W[i]*v1[i];
E[2]:= E[2] + W[i]*v2[i];
E[3]:= E[3] + W[i]*v3[i];
E[4]:= E[4] + W[i]*v4[i];
E[5]:= E[5] + W[i]*v5[i];
E[6]:= E[6] + W[i]*v6[i];
end for;
"results of p*p_V";
for i:=1 to 6 do
E[i];
end for;

/* Similarly to the elements vi from the previous part, we define
wi= (p_V \otimes 1)\Delta(a*b*a*c*ei) and similarly calculate the products.*/
w1:= 2*h(f(1/6*((lam_a-lam_c)*b*b*e6 + (lam_a-lam_b)*a*a*e5) -
1/6*((lam_a-lam_b)*(lam_a-lam_c)*e5)
+ 1/3*a*b*a*c*e1 - 1/6*(c*a*c*b*e6 + b*c*b*a*e5)));

w2:= 2*h(f(1/6*((lam_a-lam_c)*b*b*e3 + (lam_a-lam_b)*a*a*e4) -
1/6*((lam_a-lam_b)*(lam_a-lam_c)*e4)
+ 1/3*a*b*a*c*e2 - 1/6*(c*a*c*b*e3 + b*c*b*a*e4)));

w3:= 2*h(f(1/6*((lam_a-lam_c)*b*b*e4 + (lam_a-lam_b)*a*a*e2) -
1/6*((lam_a-lam_b)*(lam_a-lam_c)*e2)
+ 1/3*a*b*a*c*e3 - 1/6*(c*a*c*b*e4 + b*c*b*a*e2)));

w4:= 2*h(f(1/6*((lam_a-lam_c)*b*b*e2 + (lam_a-lam_b)*a*a*e3) -
1/6*((lam_a-lam_b)*(lam_a-lam_c)*e3)
+ 1/3*a*b*a*c*e4 - 1/6*(c*a*c*b*e2 + b*c*b*a*e3)));

w5:= 2*h(f(1/6*((lam_a-lam_c)*b*b*e1 + (lam_a-lam_b)*a*a*e6) -
1/6*((lam_a-lam_b)*(lam_a-lam_c)*e6)
```

```
+ 1/3*a*b*a*c*e5 - 1/6*(c*a*c*b*e1 + b*c*b*a*e6)));

w6:= 2*h(f(1/6*((lam_a-lam_c)*b*b*e5 + (lam_a-lam_b)*a*a*e1) -
1/6*((lam_a-lam_b)*(lam_a-lam_c)*e1)
+ 1/3*a*b*a*c*e6 - 1/6*(c*a*c*b*e5 + b*c*b*a*e1)));

/*
"print ws";
for i:= 1 to D do
w1[i], w2[i], w3[i], w4[i], w5[i], w6[i];
end for;
*/
for i:=1 to 6 do
E[i]:=0;
end for;

for i:= 1 to D do
E[1]:= E[1] + Y[i]*w1[i];
E[2]:= E[2] + Y[i]*w2[i];
E[3]:= E[3] + Y[i]*w3[i];
E[4]:= E[4] + Y[i]*w4[i];
E[5]:= E[5] + Y[i]*w5[i];
E[6]:= E[6] + Y[i]*w6[i];
end for;

" ";
"results of p_V*p";
for i:=1 to 6 do
E[i];
end for;

for i:=1 to 6 do
E[i]:=0;
end for;

for i:= 1 to D do
E[1]:= E[1] + Z[i]*w1[i];
E[2]:= E[2] + Z[i]*w2[i];
E[3]:= E[3] + Z[i]*w3[i];
E[4]:= E[4] + Z[i]*w4[i];
E[5]:= E[5] + Z[i]*w5[i];
E[6]:= E[6] + Z[i]*w6[i];
end for;

"Results of p_V*p_{sign}";
for i:=1 to 6 do
```

```
E[i];
end for;

for i:=1 to 6 do
E[i]:=0;
end for;

for i:= 1 to D do
E[1]:= E[1] + W[i]*w1[i];
E[2]:= E[2] + W[i]*w2[i];
E[3]:= E[3] + W[i]*w3[i];
E[4]:= E[4] + W[i]*w4[i];
E[5]:= E[5] + W[i]*w5[i];
E[6]:= E[6] + W[i]*w6[i];
end for;

"results of p_V*p_V - p_V";
for i:=1 to 6 do
E[i]-W[D-6+i];
end for;
```

Bibliography

[A] M. Aguiar, *A note on strongly separable algebras.* Colloquium on Homology and Representation Theory (Vaquerías, 1998). Bol. Acad. Nac. Cienc. (Córdoba) 65 (2000), 51–60. Retrieved April 15, 2020, from: `http://pi.math.cornell.edu/~maguiar/strongly.pdf`

[AEG] N. Andruskiewitsch, P. Etingof and S. Gelaki, *Triangular Hopf algebras with the Chevalley property.* Michigan Math. J. 49 (2001), no. 2, 277–298. arXiv:math/0008232 [math.QA]

[AGM] N. Andruskiewitsch, C. Galindo and M. Müller, *Examples of finite-dimensional Hopf algebras with the dual Chevalley property.* Publ. Mat. 61 (2017), no. 2, 445–474. arXiv:1509.01548 [math.QA]

[AS] N. Andruskiewitsch and H.-J. Schneider, *On the classification of finite-dimensional pointed Hopf algebras.* Annals of Mathematics, Volume 171 (2010) 375-417.

[AV] N. Andruskiewitsch and C. Vay, *On a family of Hopf algebras of dimension 72.* Bull. Belg. Math. Soc. Simon Stevin 19 (2012), 415-443, arXiv:1105.0394 [math.QA]

[At] M. F. Atiyah, *Topological quantum field theory.* Publ. Math. IHS 68 (1988) 175186.

[BK1] B. Balsam and A. Kirillov Jr, *Turaev-Viro invariants as an extended TQFT.* arXiv:1103.2963 [math.QA]

[BK2] B. Balsam and A. Kirillov Jr, *Kitaev's Lattice Model and Turaev-Viro TQFTs.* arXiv:1206.2308 [math.QA]

[BJQ] M. Barkeshli, C.-M. Jian and X.-L. Qi, *Genons, twist defects, and projective non-Abelian braiding statistics.* Phys. Rev. B 87, 045130 (2013). arXiv:1208.4834 [cond-mat.str-el]

[BW1] J. W. Barrett and B. W. Westbury, *Spherical categories.* Adv. Math. 143 (1999), no. 2, 357–375.

[BW2] J. W. Barrett and B. W. Westbury, *Invariants of piecewise-linear 3-manifolds.* Trans. Amer. Math. Soc. 348 (1996), no. 10, 3997–4022.

[BG] B. Bartlett and G. Goosen, *Extended TQFTs via Generators and Relations I: The Extended Toric Code.* arXiv:2001.08191 [math.QA]

[BSW] S. Beigi, P. W. Shor and D. Whalen, *The Quantum Double Model with Boundary: Condensations and Symmetries*. Communications in Mathematical Physics, Volume 306, Number 3, 663-694 (2011). arXiv:1006.5479 [quant-ph]

[BMD] H. Bombin and M. A. Martin-Delgado, *A Family of Non-Abelian Kitaev Models on a Lattice: Topological Confinement and Condensation*. Phys. Rev. B 78, 115421 (2008). arXiv:0712.0190 [cond-mat.str-el]

[BK] S. Bravyi and A. Kitaev, *Quantum codes on a lattice with boundary*. arXiv:quant-ph/9811052

[Br] M. Brešar, *Introduction to Noncommutative Algebra*. Universitext. Springer, Cham, 2014.

[BMCA] O. Buerschaper, J. M. Mombelli, M. Christandl and M. Aguado, *A hierarchy of topological tensor network states*. J. Math. Phys. 54, 012201 (2013). arXiv:1007.5283 [cond-mat.str-el]

[BR] Buffenoir and Roche, *Two dimensional lattice gauge theory based on a quantum group*. Commun.Math.Phys. 170 (1995) 669-698. arXiv:hep-th/9405126

[CMS] N. Carqueville, C. Meusburger and G. Schaumann, *3-dimensional defect TQFTs and their tricategories*. arXiv:1603.01171 [math.QA]

[CRS] N. Carqueville, I. Runkel and G. Schaumann, *Orbifolds of n-dimensional defect TQFTs*. Geom. Topol. 23 (2019) 781-864. arXiv:1705.06085 [math.QA]

[CCW] I. Cong, M. Cheng and Z. Wang, *Hamiltonian and Algebraic Theories of Gapped Boundaries in Topological Phases of Matter*. Comm. Math. Phys. 355 (2), 645-689 (2017). arXiv:1707.04564 [cond-mat.str-el]

[CDHPRRS] S. X. Cui, D. Ding, X. Han, G. Penington, D. Ranard, B. C. Rayhaun and Z. Shangnan, *Kitaev's quantum double model as an error correcting code*. arXiv:1908.02829 [quant-ph].

[D] P. Deligne, *Catégories tannakiennes*. The Grothendieck Festschrift, Vol. II, 111–195, Progr. Math., 87, Birkhäuser Boston, Boston, MA, 1990.

[EGNO] P. Etingof, S. Gelaki, D. Nikshych and V. Ostrik, *Tensor categories*. Mathematical Surveys and Monographs, 205. American Mathematical Society, Providence, RI, 2015. Retrieved April 15, 2020, from: `http://www-math.mit.edu/~etingof/egnobookfinal.pdf`

[F] R. Farnsteiner, *Self-injective algebras: The Nakayama permutation*. Lecture notes retrieved April 15, 2020, from: `https://www.math.uni-bielefeld.de/~sek/select/rf1.pdf`

[FQ] D.S. Freed and F. Quinn, *Chern-Simons theory with finite gauge group*. Comm. Math. Phys., 156(3):435–472, 1993. arXiv:hep-th/9111004

[FLW] M. H. Freedman, M. Larsen and Z. Wang, *A modular functor which is universal for quantum computation.* Comm. Math. Phys. 227 (2002), no. 3, 605–622. arXiv:quant-ph/0001108

[FSS1] J. Fuchs, G. Schaumann and C. Schweigert, *A trace for bimodule categories.* Appl. Categ. Structures 25 (2017), no. 2, 227–268. arXiv:1412.6968 [math.CT]

[FSS2] J. Fuchs, G. Schaumann and C. Schweigert, *A modular functor from state sums for finite tensor categories and their bimodules.* arXiv:1911.06214 [math.QA]

[FS] J. Fuchs and C. Schweigert, *A note on permutation twist defects in topological bilayer phases.* Lett. Math. Phys. 104 (2014), no. 11, 1385–1405. arXiv:1310.1329 [hep-th]

[FSV1] J. Fuchs, C. Schweigert and A. Valentino, *Bicategories for boundary conditions and for surface defects in 3-d TFT.* Comm. Math. Phys. 321 (2013), no. 2, 543–575. arXiv:1203.4568 [hep-th]

[FSV2] J. Fuchs, C. Schweigert and A. Valentino, *A geometric approach to boundaries and surface defects in Dijkgraaf-Witten theories.* Comm. Math. Phys. 332 (2014), 981–1015. arXiv:1307.3632 [hep-th]

[G] D. Gottesman, *An Introduction to Quantum Error Correction.* Quantum Computation: A Grand Mathematical Challenge for the Twenty-First Century and the Millennium, ed. S. J. Lomonaco, Jr., pp. 221-235 (American Mathematical Society, Providence, Rhode Island, 2002). arXiv:quant-ph/0004072

[HIZ] A. Hamma, R. Ionicioiu and P. Zanardi, *Ground state entanglement and geometric entropy in the Kitaev's model.* Phys.Lett. A 337, 22 (2005). arXiv:quant-ph/0406202v3

[Ka] C. Kassel, *Quantum Groups.* Graduate Texts in Mathematics, 155. Springer-Verlag, New York, 1995.

[Kir] A. Kirillov Jr, *String-net model of Turaev-Viro invariants.* arXiv:1106.6033 [math.AT]

[Kit1] A. Kitaev, *Fault-tolerant quantum computation by anyons.* Annals Phys. 303 (2003) 2-30. arXiv:quant-ph/9707021

[Kit2] A. Kitaev, *Anyons in an exactly solved model and beyond.* Annals Phys. 321 (2006) 2-111. arXiv:cond-mat/0506438

[KK] A. Kitaev and L. Kong, *Models for gapped boundaries and domain walls.* Comm. Math. Phys. 313 (2012), no. 2, 351–373. arXiv:1104.5047 [cond-mat.str-el]

[K] V. Koppen, *Defects in Kitaev models and bicomodule algebras.* arXiv:2001.10578 [math.QA]

[KMS] V. Koppen, E. Meir and C. Schweigert, *On isotypic decompositions for non-semisimple Hopf algebras.* arXiv:1910.13161 [math.QA]

[LR] R. G. Larson, D. E. Radford, *Semisimple cosemisimple Hopf algebras.* Amer. J. Math. 110 (1988), no. 1, 187–195.

[LLW] K. Laubscher, D. Loss and J. R. Wootton, *Universal quantum computation in the surface code using non-Abelian islands.* Phys. Rev. A 100, 012338 (2019). arXiv:1811.06738v2 [quant-ph]

[MNS] J. Maier, T. Nikolaus and C. Schweigert, *Equivariant Modular Categories via Dijkgraaf-Witten Theory.* arXiv:1103.2963 [math.QA]

[M] E. Meir, *Hopf cocycle deformations and invariant theory.* Math. Z. (2019). https://doi.org/10.1007/s00209-019-02326-5, arXiv:1804.00289 [math.QA]

[Me] C. Meusburger, *Kitaev lattice models as a Hopf algebra gauge theory.* Comm. Math. Phys. 353 (2017), no. 1, 413–468. arXiv:1607.01144 [math.QA]

[Mo] C. Mochon, *Anyon computers with smaller groups.* Phys. Rev. A 69, 032306 (2004). arXiv:quant-ph/0306063

[Mon] S. Montgomery, *Hopf Algebras and Their Actions on Rings.* CBMS Regional Conference Series in Mathematics, 82. Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the Amer. Math. Soc., Providence, RI, 1993.

[S] H.-J. Schneider, *Lectures on Hopf algebras.* Notes by Sonia Natale. Trabajos de Matemática 31/95, FaMAF, 1995. Retrieved April 15, 2020, from: `http://documents.famaf.unc.edu.ar/publicaciones/documents/serie_b/BMat31.pdf`

[Sw] M. E. Sweedler, *Hopf algebras.* Mathematics Lecture Note Series W. A. Benjamin, Inc., New York 1969.

[TSG] D. C. Tsui, H. L. Stormer, and A. C. Gossard, *Two-Dimensional Magnetotransport in the Extreme Quantum Limit.* Phys. Rev. Lett. 48, 1559 (1982).

[TVi] V. Turaev and A. Virelizier, *On two approaches to 3-dimensional TQFTs.* arXiv:1006.3501 [math.GT]

[TV] V. Turaev and O. Viro, *State sum invariants of 3-manifolds and quantum 6j-symbols.* Topology 31 (1992) 865-902.

[W] X.G. Wen, *Topological Orders in Rigid States.* Int. J. Mod. Phys. B 4 (1990) 239.

Eidesstattliche Versicherung

Hiermit erkläre ich an Eides statt, dass ich die vorliegende Dissertationsschrift selbst verfasst und keine anderen als die angegebenen Quellen und Hilfsmittel benutzt habe.

Hamburg, den 19. April 2020

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