Higher Categorical and Operadic Concepts for Orbifold Constructions

A Study at the Interface of Topology and Representation Theory

Dissertation with the aim of achieving a doctoral degree
at the Department of Mathematics
Faculty of Mathematics, Computer Science and Natural Sciences
University of Hamburg

submitted by Lukas Jannik Woike

2020 Hamburg

Submitted on: March 25, 2020

Day of thesis defense: June 12, 2020

The following evaluators recommend the admission of the dissertation:

Prof. Dr. Christoph Schweigert,
Prof. Dr. Claudia Scheimbauer,
Prof. Dr. Alexis Virelizier

Eidesstattliche Versicherung

Hiermit erkläre ich an Eides statt, dass ich die vorliegende Dissertationsschrift selbst verfasst und keine anderen als die angegebenen Quellen und Hilfsmittel benutzt habe.

Hamburg, den 25. März 2020

Lukas Jannik Woike

Contents

1 Introduction and summary

Algebraic topology is informed by the idea that topological spaces or manifolds may be studied by extracting from them algebraic objects. These algebraic objects cover a wide spectrum ranging from numbers like Betti numbers over groups and vector spaces to higher categories that contain multiple (possibly infinitely many) layers of algebraic information.

The reverse principle is also well-established: Manifolds and spaces (often low-dimensional ones) can be used to describe and investigate algebraic objects. For example, the covering theory of graphs yields a topological proof of the Nielsen-Schreier Theorem [BL36], the little disks operads [BV68, May72, BV73] provide a topological origin for monoidal categories, braided monoidal categories and Gerstenhaber algebras [SW03, Fre17], and factorization homology for the circle allows us to describe Hochschild homology [AF15], see [Sch14, BZBJ18a, BZBJ18b] for the relation of factorization homology to topological field theories and quantum algebra.

In this thesis, a particular instance of the principle of describing algebraic structures via topology will be present throughout, namely the intimate relation between low-dimensional topology and representation theory. This relation is afforded by three-dimensional extended topological field theory [At88, RT91, Tur10a] – a notion that will be described informally below and then in detail in the main part of the thesis.

The correspondence between algebra and topology is not limited to *objects*, but extends to *operations*: If an algebraic object is described by a topological structure, then it is only reasonable to also describe the natural *operations* that an algebraic object admits in topological terms. This thesis is concerned with the search for a topological version of an algebraic operation on representation categories, namely the concept of an *orbifold category* [Kir04, Müg05, GNN09] that will be described in more detail towards the end of the introduction and then later in Section 6.1.2. The *topological orbifold construction* that we develop in this thesis is formulated on the level of extended topological field theories. It enjoys the properties that topological counterparts of algebraic constructions often have:

- They are more conceptual, compact and easier to handle.
- They lead to generalizations and unifications.
- One often gains insight through the interplay of the topological and the algebraic construction.

Especially the last point will be illustrated in detail.

As a key ingredient to establish the relation between the topological and the algebraic orbifold construction, we will encounter another example of how to encode complicated algebraic data topologically, the *little bundles operad* which describes braided crossed categories, a type of category studied in representation theory and of great relevance to the algebraic orbifold construction.

Extended (equivariant) topological field theories. In order to describe the orbifold construction in more precise terms, let us first give an informal description of topological field theories (referred to in the literature also as topological *quantum* field theories). The notion of a topological field theory can be defined in all dimensions, but our informal description will concentrate on the three-dimensional case for concreteness: A (non-extended) three-dimensional topological field theory Z assigns to every compact oriented surface Σ a vector space $Z(\Sigma)$ and to each compact oriented bordism M (Figure 1.1) whose ingoing and outgoing boundaries are compact oriented surfaces Σ_0 and Σ_1, respectively, a linear map $Z(M) : Z(\Sigma_0) \longrightarrow Z(\Sigma_1)$. These assignments respect the gluing of bordisms and their disjoint union. This allows to compute Z on a given manifold by cutting the manifold into simpler pieces. A three-dimensional topological field theory with values in vector spaces may be formally described [At88] as a symmetric monoidal functor $Z : \mathsf{Cob}(3,2) \longrightarrow \mathsf{Vect}$ from the symmetric monoidal three-dimensional bordism category $\mathsf{Cob}(3,2)$ to the symmetric monoidal category Vect of vector spaces (over a field that in this thesis will always be given by the complex numbers).

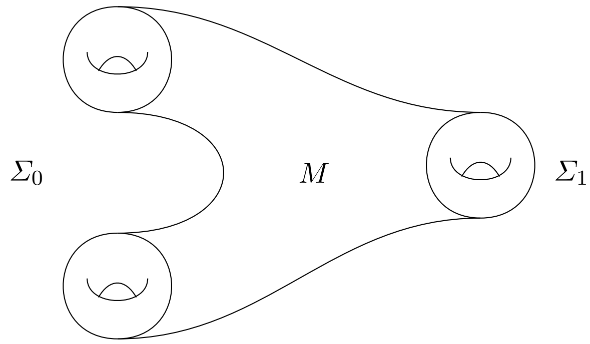


Figure 1.1: A three-dimensional bordism M with ingoing boundary Σ_0 (consisting of the disjoint union of two tori) and outgoing boundary Σ_1 (consisting of a single torus).

The notion of an *extended* three-dimensional topological field theory enhances this picture and assigns also special types of linear categories, so-called *2-vector spaces*, to compact oriented one-dimensional manifolds, i.e. disjoint unions of circles. The formal definition describes extended three-dimensional topological field theories as symmetric monoidal functors $\mathsf{Cob}(3,2,1) \longrightarrow \mathsf{2Vect}$ from the bordism bicategory to the bicategory of 2-vector spaces. No assignments to even lower-dimensional manifolds, i.e. points, will be made; in technical terms, the extended topological field theories of this thesis will be *once-extended*, but not *fully extended*.

The notion of a topological field theory is inspired by quantum field theories and provides a mathematical axiomatization of a certain class of quantum field theories. These important relations to physics are covered e.g. in [FQ93, FHLT10, Kap10], see additionally [RW18] for the relations to quantum computing. In this thesis, however, the emphasis lies on the deep connections of extended three-dimensional topological field theories to representation theory that we will explain in more detail now: Starting from a semisimple modular category (a non-degenerate kind of ribbon category that we may obtain, for instance, as the representation category of certain Hopf algebras [EGNO15, Kas95]), the Reshetikhin-Turaev construction [RT91, Tur10a] allows us to build an extended three-dimensional topological field theory (possibly containing anomalies). Conversely, if we evaluate an extended three-dimensional topological field theory on the circle, we obtain a semisimple modular tensor category (this structure comes from the evaluation on manifolds of higher dimension). This map from three-dimensional extended topological field theories to semisimple modular tensor categories is an equivalence [BDSPV15].

The notion of a topological field theory admits many variants. For example, one may consider bordisms that are decorated with additional data. The following type of topological field theory will be crucial in this thesis as it will form the input datum for the topological orbifold construction: *Homotopy quantum field theories*, as introduced in [Tur99] and further developed in the monograph [Tur10b], are topological field theories defined on bordisms equipped with maps to a fixed topological space, called the *target space*. In the most investigated special case, this target space is chosen to be aspherical, i.e. to be the classifying space of a (finite) group G. Homotopy quantum field theories with such a choice of target space are called *G-equivariant topological* *field theories* in this thesis. Of course, decorating bordisms with a map to BG amounts to decorating them with a G-bundle.

The topological orbifold construction. The *topological orbifold construction* or *orbifoldization* for equivariant topological field theories that we will develop in this thesis is now a construction which assigns to a given equivariant topological field theory a non-equivariant topological field theory, its *orbifold theory*. Such an orbifold construction should be understood as a sum over twisted sectors combined with a computation of the invariants of the theory in the appropriate sense, see [DVVV89] for this perspective on orbifoldization including the relation to sigma models with orbifold target, and e.g. [FKS92, Ban98, Ban02, CGPW16, EG18] for the study of orbifold theories, in particular permutation orbifolds.

Our construction is set up as follows: For a given finite group G, the construction takes as input an extended G-equivariant topological field theory, i.e. a symmetric monoidal functor $Z : G\text{-}\mathsf{Cob}(n,n-1,n-2) \longrightarrow \mathsf{2Vect}$ from the symmetric monoidal bicategory $G\text{-}\mathsf{Cob}(n,n-1,n-2)$ of n-dimensional bordisms equipped with a map to BG to the symmetric monoidal bicategory $\mathsf{2Vect}$ of 2-vector spaces. The output of our construction is the orbifold theory $Z/G : \mathsf{Cob}(n,n-1,n-2) \longrightarrow \mathsf{2Vect}$, a non-equivariant topological field theory. For the construction of Z/G, we perform the following two steps:

(1) First we produce from the equivariant theory Z a symmetric monoidal functor $\widehat{Z} : \mathsf{Cob}(n,n-1,n-2) \longrightarrow \mathsf{2VecBunGrpd}$ from the cobordism category to the symmetric monoidal bicategory $\mathsf{2VecBunGrpd}$ built in Section 3.3.1 from 2-vector bundles over essentially finite groupoids and (higher) spans of groupoids, see also [Hau18] for related concepts. Hence, this step changes the target category of the theory (also referred to as *coefficients*) from $\mathsf{2Vect}$ to the more complicated $\mathsf{2VecBunGrpd}$ which, in exchange, now contains information about the equivariance. The domain category changes from $G\text{-}\mathsf{Cob}(n,n-1,n-2)$ to $\mathsf{Cob}(n,n-1,n-2)$ and hence becomes simpler. This step will be referred to as *change to equivariant coefficients* and will be explained in Section 4.1.

(2) To produce topological field theories valued in $\mathsf{2Vect}$, we need the symmetric monoidal parallel section functor

$$\mathsf{Par} : \mathsf{2VecBunGrpd} \longrightarrow \mathsf{2Vect}$$

whose construction will be the main result of Chapter 3. It takes (homotopy) invariants of 2-vector bundles and sends (higher) spans of groupoids to certain pull-push maps combined with (higher) intertwiners. To some extent, it makes the idea of the 'Sum functor' in [FHLT10] precise. By restriction to the endomorphisms of the respective monoidal units one obtains the functor developed in [Tro16].

Now we can define the orbifold theory as the concatenation of symmetric monoidal functors

$$\frac{Z}{G} : \mathsf{Cob}(n,n-1,n-2) \xrightarrow{\widehat{Z}} \mathsf{2VecBunGrpd} \xrightarrow{\mathsf{Par}} \mathsf{2Vect} \ .$$

The construction is functorial in Z, so the orbifoldization takes the form of a functor

$$-/G : \mathsf{HSym}(G\text{-}\mathsf{Cob}(n,n-1,n-2),\mathsf{2Vect}) \longrightarrow \mathsf{Sym}(\mathsf{Cob}(n,n-1,n-2),\mathsf{2Vect}) \qquad (1.1)$$
$$Z \longmapsto \frac{Z}{G}$$

from the 2-groupoid $\mathsf{HSym}(G\text{-}\mathsf{Cob}(n,n-1,n-2),\mathsf{2Vect})$ of extended G-equivariant topological field theories to the 2-groupoid $\mathsf{Sym}(\mathsf{Cob}(n,n-1,n-2),\mathsf{2Vect})$ of extended topological field theories. An explicit description of the orbifold construction is given in Proposition 4.3. In

Section 4.3, finally, we generalize the orbifold construction to a pushforward operation

$$\lambda_* : \mathsf{HSym}(G\text{-}\mathsf{Cob}(n, n-1, n-2), \mathsf{2Vect}) \longrightarrow \mathsf{HSym}(H\text{-}\mathsf{Cob}(n, n-1, n-2), \mathsf{2Vect}) \quad (1.2)$$

for equivariant topological field theories along any morphism $\lambda : G \longrightarrow H$ of finite groups. Both the orbifold construction (1.1) and the push construction (1.2) lift the orbifold construction for non-extended equivariant topological field theories from [SW19] to a bicategorical setting. This makes the construction considerably more involved, but also algebraically richer.

As outlined in the beginning, the main motivation for developing an orbifold construction for *extended* topological field theories comes from the 3-2-1-dimensional case. The goal is to relate in this specific dimension the orbifold construction on field theory level, as developed in this thesis, with an algebraic orbifoldization procedure for monoidal representation categories [Kir04, Müg05, GNN09]. This relation will be worked out in Chapter 6 and will also be summarized below.

The precise formulation of this relation, however, requires a rather long preparation, namely the thorough investigation of the structure that is present on the category obtained by evaluation of an extended G-equivariant topological field theory on the circle. This investigation is of interest independently of the orbifold construction.

The little bundles operad and braided crossed categories. The key technical ingredient for the investigation of the category obtained by the evaluation of an equivariant topological field theory on the circle is a colored topological operad, the *little bundles operad*. We introduce this operad in Chapter 5. The little bundles operad is motivated and constructed in the following way: Consider for $r \geq 0$ an r-ary operation $f \in E_2(r)$ of the little disks operad E_2 [BV68, BV73], i.e. an affine embedding of r disks into another disk, and the groupoid $\mathsf{PBun}_G(\mathsf{C}(f))$ of G-bundles over the closed complement $\mathsf{C}(f)$ of the image of the embedding f. Then the (pure) braid group on r strands acts on the space $\mathsf{PBun}_G(\mathsf{C}(f))$. The homotopy quotient is known as a *Hurwitz space*, see [Cle72, Hur91] and e.g. [EVW16] for an overview. We consider a model $W_2(r)$ for this homotopy quotient which, by restriction to the boundary circles, comes with a Serre fibration $W_2(r) \longrightarrow \mathrm{Map}(\mathbb{S}^1, BG)^{r+1}$ to the $r+1$-fold product of the free loop space of the classifying space of G. This allows us to prove that the fibers of this Serre fibration, considered for varying $r \geq 0$, combine into a topological $\mathrm{Map}(\mathbb{S}^1, BG)$-colored operad E_2^G that we call the *operad of little G-bundles*.

The operad E_2^G of little bundles is aspherical, and we exhibit a presentation as a groupoid-valued operad in terms of generators and relations (Section 5.3) using so-called *parenthesized G-braids*. This allows us to prove in Theorem 5.32 that the categorical little bundles algebras (i.e. little bundles algebras in categories) are precisely braided G-crossed categories – a G-equivariant and G-graded version of a braided monoidal category which is *not* a braided category itself in the usual sense. Roughly, a braided G-crossed category $\mathcal{C}$ is G-graded with the component of $g \in G$ being denoted by $\mathcal{C}_g$, and it carries a homotopy coherent G-action, where the action by $g \in G$ carries $X \in \mathcal{C}_h$ to $g.X \in \mathcal{C}_{ghg^{-1}}$. The monoidal product sends $X \in \mathcal{C}_g$ and $Y \in \mathcal{C}_h$ to $X \otimes Y \in \mathcal{C}_{gh}$. The crossed braiding consists of coherent isomorphisms

$$X \otimes Y \cong g.Y \otimes X \quad \text{for} \quad X \in \mathcal{C}_g \ , \quad Y \in \mathcal{C}_h \ .$$

Again, this will not define an actual braiding except on the neutral sector $\mathcal{C}_e$. This notion is due to Turaev [Tur00, Tur10b]; we, however, use a version of this notion [Gal17, Definition 5.4] allowing for more general coherence data and omitting the requirement of rigidity (existence of duals).

Theorem 5.32. *There is a Quillen equivalence*

$$\{\textit{braided } G\textit{-crossed categories}\} \underset{\Phi^*}{\overset{\Phi_!}{\rightleftarrows}} \{\textit{categorical little } G\textit{-bundles algebras}\} \ .$$

The model structures on algebras over an operad are the transferred ones, see page 98.

Braided crossed categories have been studied in [Müg04, Kir04, GNN09, ENOM10, Tur10b]. One ingredient of our proof is a coherence result for G-equivariant categories [Gal17]. In the existing literature, the bookkeeping of the coherence data of a G-crossed braided monoidal category is done manually. Our operadic approach encodes this data in a compact way.

For the application to equivariant topological field theories that we have in mind, it will be necessary to prove a statement similar to Theorem 5.32 for an enhancement of the little bundles operad, the *framed little bundles operad* (Section 5.4.2), that also allows for a rotation of little decorated disks, thereby generalizing the usual enhancement of the little disks operad by the framed little disks operad to the equivariant setting.

Theorem 5.39. *There is a Quillen equivalence*

$$\{\textit{balanced braided } G\textit{-crossed categories}\} \underset{\Phi^{\mathsf{f}*}}{\overset{\Phi^{\mathsf{f}}_!}{\rightleftarrows}} \{\textit{categorical framed little } G\textit{-bundles algebras}\} \ .$$

Balanced braided G-crossed categories are defined in Section 5.4.2.

The above results on the categorical algebras of the (framed) little bundles operad allow us to prove the following result on the structure present on the category $\mathcal{C}^Z$ (more precisely: 2-vector space) obtained as the value of a 3-2-1-dimensional G-equivariant topological field theory Z on the circle:

Theorem 5.49. *For any finite group G and any extended G-equivariant topological field theory $Z : G\text{-}\mathsf{Cob}(3,2,1) \longrightarrow \mathsf{2Vect}$, the 2-vector space $\mathcal{C}^Z$ that is obtained from Z by evaluation on the circle with varying G-bundle decoration is a finitely semisimple G-ribbon category.*

The notion of a G-ribbon category is given in Definition 5.48; a stricter version of this definition appears in [Tur10b, VI.2.3].

Topological versus algebraic orbifoldization. In Proposition 6.2 we explicitly compute how the G-ribbon structure of $\mathcal{C}^Z$ behaves under the topological orbifold construction, and in Theorem 6.4 we prove our main result on the relation of this topological orbifold structure to the one obtained via the purely algebraic orbifoldization procedure in terms of orbifold categories [Kir04, Müg05, GNN09]. Roughly, the algebraic orbifoldization sends a braided G-crossed category $\mathcal{C}$ with components $(\mathcal{C}_g)_{g\in G}$ to the *orbifold category* $\mathcal{C}/G$ whose objects are objects $X \in \mathcal{C}$ together with coherent isomorphisms $X \cong g.X$ for all $g \in G$ (this is the category of homotopy fixed points under the G-action). If $\mathcal{C}$ is a finitely semisimple G-ribbon category, then $\mathcal{C}/G$ comes with the structure of a finitely semisimple ribbon category. A strong motivation for the concept of an orbifold category, that will, however, not play a role in this thesis, comes from the study of vertex operator algebras. More precisely, for a vertex operator algebra V with a G-action, the category of modules over the vertex subalgebra V^G of invariants can be described as an orbifold category, see [Kir04] for a more detailed explanation. Algebraic orbifoldization procedures are also used in an operator algebraic approach to conformal field theory [Müg05].

Our result on the relation of the 3-2-1-dimensional topological orbifold construction and the algebraic orbifold construction may now be stated as follows:

Theorem 6.4. *The square*

$$\begin{tikzcd}
\text{\textit{3-2-1-dimensional G-equivariant topological field theories}} \arrow[r, "\substack{\text{evaluation}\\ \text{on the circle}\\ \text{(Theorem 5.49)}}"] \arrow[d, "\substack{\text{orbifoldization } -/G \text{, see (1.1)}\\ \text{(topological orbifoldization)}}"'] & \text{\textit{finitely semisimple G-ribbon categories}} \arrow[d, "\substack{\text{orbifold category}\\ \text{(algebraic orbifoldization)}\\ \text{[Kir04, Müg05, GNN09]}}"] \\
\text{\textit{3-2-1-dimensional topological field theories}} \arrow[r, "\substack{\text{evaluation}\\ \text{on the circle}\\ \text{[BDSPV15]}}"'] & \text{\textit{finitely semisimple ribbon categories}}
\end{tikzcd}$$

commutes up to natural isomorphism.

We make the following statements about the modularity of the categories appearing on the right hand side:

Theorem 6.20. *Let G be a finite group. For any extended G-equivariant topological field theory Z, the category $\mathcal{C}^Z$ obtained by evaluation on the circle is*

(a) *G-modular if its monoidal unit is simple,*

(b) *and in the general case still G-multimodular.*

We refer to Definition 6.17 for the notion of G-(multi)modularity. This result extends one of the main results of [BDSPV15] to the equivariant case. The proof of (a) makes explicit use of the interplay between the topological and algebraic orbifoldization.

As a consequence, we obtain a functor from 3-2-1-dimensional G-equivariant topological field theories to G-multimodular categories and hence a first step towards the classification of 3-2-1-dimensional G-equivariant topological field theories (Remark 6.22).

As a further application, our construction provides a uniform topological formulation for the following two instances of orbifoldization:

- In combination with the cover functor [BS11], our orbifold construction yields permutation orbifolds [FKS92, Ban98, Ban02, EG18], see Example 6.10.
- The orbifoldization of extended cohomological homotopy quantum field theories leads to the twisted Drinfeld doubles of a finite group from [DPR90], as discussed in [MW20a], see also Example 4.5.

Our construction ensures the existence of these orbifold theories as extended topological field theories and makes them explicitly computable. For example, we provide a formula for the number of simple objects of the orbifold theory (Theorem 6.8), which as a byproduct yields restrictions for manifold invariants coming from homotopy quantum field theories (Corollary 6.5).

The concepts developed in this thesis have found important applications in work of other authors: In [MS19] the topological orbifold construction is used in the construction of topological field theories corresponding to anomalies in quantum field theory, and in [You19] the parallel section functor is one of the key technical ingredients for the construction of orientation twisted homotopy quantum field theories.

Background. While the central notions of this thesis, like the notion of an extended (equivariant) topological field theory, will be defined in detail in later chapters, familiarity with some more standard notions and techniques will be assumed. For the basic notions of category theory, we refer to [ML98, Chapter I-V] and for an introduction to (braided) monoidal categories, their associated graphical calculus and connections to topological field theory to [TV17, Chapter 1-3]. A large part of this thesis will be concerned with constructions in symmetric monoidal bicategories for which we refer to [SP09, Chapter 2]. Finally, an introduction to operads can be found e.g. in [Fre17, Chapter 1-2].

This thesis is based on the following three publications:

[1] Extended Homotopy Quantum Field Theories and their Orbifoldization. With Christoph Schweigert. *J. Pure Appl. Algebra* 224(1), 2020. arXiv:1802.08512 [math.QA]

[2] A Parallel Section Functor for 2-Vector Bundles. With Christoph Schweigert. *Theory Appl. Categ.* 33(23):644–690, 2018. arXiv:1711.08639 [math.CT]

[3] The Little Bundles Operad. With Lukas Müller. 2019. Accepted for publication in *Algebr. Geom. Topol.* arXiv:1901.04850 [math.AT]

Other publications of the author:

[4] Orbifold Construction for Topological Field Theories. With Christoph Schweigert. *J. Pure Appl. Algebra* 223:1167–1192, 2019. arXiv:1705.05171 [math.QA]

[5] Parallel Transport of Higher Flat Gerbes as an Extended Homotopy Quantum Field Theory. With Lukas Müller. *J. Homotopy Relat. Str.* 15(1):113-142, 2020. arXiv:1802.10455 [math.QA]

[6] Equivariant Higher Hochschild Homology and Topological Field Theories. With Lukas Müller. *Homology Homotopy Appl.* 22(1):27–54, 2020. arXiv:1809.06695 [math.AT]

[7] Operads for algebraic quantum field theory. With Marco Benini and Alexander Schenkel. 2017. Accepted for publication in *Comm. Contemp. Math.* arXiv:1709.08657 [math-ph]

[8] Involutive categories, colored $*$-operads and quantum field theory. With Marco Benini and Alexander Schenkel. *Theory Appl. Categ.* 34(2):13-57, 2019. arXiv:1802.09555 [math.CT]

[9] Homotopy theory of algebraic quantum field theories. With Marco Benini and Alexander Schenkel. *Lett. Math. Phys.* 109:1487–1532, 2019. arXiv:1805.08795 [math-ph]

Further preprint:

[10] The Hochschild Complex of a Finite Tensor Category. With Christoph Schweigert. 2019. arXiv:1910.00559 [math.QA]

We will also give a very brief summary of the papers [4]-[10] that are *not* being reported on in this thesis and explain the relations of these papers to the topics of this thesis:

In [4] we give an orbifold construction for *non-extended* topological field theories which, on a technical level, is a lot less demanding because all construction are 1-categorical. While establishing in this thesis the orbifold construction in the *extended case*, which lives in a bicategorical framework, we will at several occasions explain how the results of [4] may be recovered.

Papers [5] and [6] form an example-driven study of various higher categorical aspects of equivariant topological field theories. In [5] we construct an extended G-equivariant topological field theory from a group cocycle on G. We prove that the orbifoldization of this theory is precisely twisted Dijkgraaf-Witten theory, see also Example 4.5. This opens a perspective on twisted Drinfeld doubles through topological orbifoldization. In [6] we define an $(\infty, 1)$-categorical version of equivariant topological field theories and provide an example via equivariant higher Hochschild homology. In [6, Section 3.4.2], we explore how orbifoldization procedures could be understood in the $(\infty, 1)$-categorical framework.

The publications [7]-[9] are not concerned with topological field theories, but as in this thesis, operadic techniques play an important role: We develop an operad whose algebras are precisely algebraic quantum field theories in the sense of Haag-Kastler, thereby paving the way towards a notion of a homotopy coherent algebraic quantum field theory. For ordinary algebraic quantum field theories, observables assigned to causally independent spacetime regions commute, which is referred to as *Einstein causality*. For homotopy coherent algebraic quantum field theories, this axiom is relaxed in a consistent way. For both topological field theories and algebraic quantum field theories, orbifoldization is one of the main tools for the construction of new field theories from existing equivariant field theories. We present general concepts of orbifoldization for algebraic quantum field theories in [7, Section 4.5]. In [9, Section 5] we use a derived version of orbifoldization to construct one of the first non-trivial examples of a homotopy coherent algebraic quantum field theory. It can be understood as a fiberwise groupoid cohomology of a category fibered in groupoids with coefficients in an ordinary algebraic quantum field theory.

Finally, paper [10] aims at generalizing the relations between low-dimensional topology and representation categories from the semisimple case (that we consider throughout this thesis) to the non-semisimple case. The technical and conceptual challenges that this leads to are approached by using techniques from homological algebra and homotopy theory. As the main result of [10], we show that the Hochschild complex of a not necessarily semisimple modular category carries a homotopy coherent projective action of the mapping class group of the torus. When this modular category is given by the category of modules over a ribbon factorizable Hopf algebra, the induced action on Hochschild homology is dual to the one considered in [LMSS18]. It should be noted, however, that the construction in [10] yields really a canonical homotopy coherent action at chain level and therefore goes beyond [LMSS18] in many respects. For the treatment of non-semisimple braided crossed categories in [10], we make extensive use of the little bundles operad developed in this thesis. To this end, it is advantageous that this operad is constructed in a general way that does not make reference in any way to semisimplicity.

Conventions. All vector spaces or higher analogues thereof encountered in this thesis will be over the field of complex numbers. Therefore, we suppress the field in the notation and write Vect instead of $\mathsf{Vect}_{\mathbb{C}}$. Still, all constructions would also work over an algebraically closed field of characteristic zero.

We will refer to weak 2-functors between bicategories just as functors unless we want to stress the categorical level.

As mentioned in the introduction, the word *extended* in connection with topological field theories will always mean *once-extended*.

Acknowledgments

I would like to thank my advisor Christoph Schweigert for his excellent guidance, continuous support and valuable advice throughout my PhD studies. I am very grateful for the opportunity to work on a topological version of the orbifold construction, a project combining my interests in topology, algebra and mathematical physics, and appreciated the freedom I was given when working towards such a construction. I cannot imagine any better style of supervision!

To Claudia Scheimbauer and Alexis Virelizier I am grateful for acting as referees for this thesis.

Among the many people I have enjoyed working with during my PhD studies, I would like to single out my coauthors: I would like to thank Lukas Müller for a productive collaboration, for countless hours of staring at a problem together until everything falls into place and for helpful comments on probably every mathematical project I was ever involved in (including this thesis). I would like to thank Marco Benini and Alexander Schenkel with whom I am working on homotopical algebraic quantum field theory. This project has significantly extended my mathematical toolkit and my understanding of quantum field theory.

Over the last few years, I enjoyed making contact with the very welcoming communities of algebra, topology and mathematical physics, heard inspiring talks that motivated me to learn new techniques, profited from illuminating comments on problems I was facing in my projects and was invited to conferences and research stays. On these occasions, I have met a great number of amazing researchers. In particular, I would like to thank Adrien Brochier for helpful conversations on various topics in quantum algebra, Jürgen Fuchs for careful explanations on topological field theory and modular functors, César Galindo for discussions on applications of orbifoldization, Najib Idrissi for inviting me to Paris and for helpful comments on the little bundles operad, Ehud Meir for answering a lot of questions on representation theory, Claudia Scheimbauer for a lot of always encouraging and pleasant exchanges on topological field theory and homotopy theory and for inviting me to Munich, Alexis Virelizier for inviting me to Lille and for suggesting a generalization of the orbifold construction to a push operation along a group morphism, and Nathalie Wahl for the interest in my work, numerous valuable comments, long insightful discussions and for inviting me to Copenhagen twice.

I gratefully acknowledge the financial support of the DFG Research Training Group 1670 'Mathematics inspired by string theory and QFT' at the University of Hamburg. In this environment, I found not only ideal working conditions, but also enjoyed the numerous activities and the company of competent and friendly colleagues. In particular, I would like to thank Severin Bunk, Simon Lentner and Lóránt Szegedy, my fellow PhD students Vincent Koppen, Manasa Manjunatha, Svea Nora Mierach, Yang Yang, my office mate Ilaria Flandoli, and also Gerda Mierswa Silva for helping me navigate through administrative issues.

Finally, for support in countless different ways, I thank my family Nicole, Thies, Marga, Peter, Daniel, Sarah, Clara and especially Rosalie.

2 Extended equivariant topological field theories

In this short chapter, we develop a bicategorical version of the notion of an equivariant topological field theory and thereby define the mathematical objects that this thesis will be mainly concerned with.

2.1 Extended homotopy quantum field theories

In order to define extended equivariant topological field theories, we first define an extended version of the homotopy quantum field theories in [Tur10b] for arbitrary target spaces (Definition 2.3). By specializing to aspherical targets we obtain extended equivariant topological field theories. In the 3-2-1-dimensional case, equivariant topological field theories have also been defined in [MNS12] using the language of principal fiber bundles. The present generalization to arbitrary dimension and target space seems to be new.

The definition of an extended homotopy quantum field theory requires a suitable symmetric monoidal bordism bicategory $T\text{-}\mathsf{Cob}(n, n-1, n-2)$ for an arbitrary target space T. It will generalize the symmetric monoidal bordism bicategory $\mathsf{Cob}(n, n-1, n-2)$ used as the domain of extended topological field theories, see e.g. [SP09], in the sense that all manifolds involved are additionally equipped with continuous maps to T.

For the definition of $T\text{-}\mathsf{Cob}(n, n-1, n-2)$, we need not only manifolds and manifolds with boundary, but also manifolds with corners whose definition we briefly recall, see also [SP09, Section 3.1.1]: An *n-dimensional manifold with corners of codimension 2* is a second countable Hausdorff space M together with a maximal atlas of charts of the form

$$M \supseteq U \xrightarrow{\varphi} V \subset \mathbb{R}^{n-2} \times (\mathbb{R}_{\geq 0})^2 \ .$$

Given $x \in M$ we define the *index* of x to be the number of coordinates of $(\mathrm{pr}_{(\mathbb{R}_{\geq 0})^2} \circ \varphi)(x)$ equal to 0 for some chart φ (and hence for all charts). The *corners* are points of index 2. A *connected face* of M is the closure of a maximal connected subset of points of index 1. A *face* is the disjoint union of connected faces. A *manifold with faces* is a manifold with corners such that every point of index 2 belongs to exactly two different connected faces.

Finally, an *n-dimensional $\langle 2 \rangle$-manifold* is an n-dimensional manifold M with faces together with a decomposition $\partial M = \partial_0 M \cup \partial_1 M$ of its topological boundary into faces such that $\partial_0 M \cap \partial_1 M$ is the set of corners of M. We call $\partial_0 M$ the *0-boundary* of M and $\partial_1 M$ the *1-boundary* of M.

Definition 2.1 (Bordism bicategory for arbitrary target space)**.** Let $n \geq 2$. For a non-empty topological space T, referred to as the *target space*, the *bicategory $T\text{-}\mathsf{Cob}(n, n-1, n-2)$ of bordisms with maps to T* is defined in the following way:

(0) Objects, also called 0-cells, are pairs (S, ξ), where S is an $(n-2)$-dimensional oriented closed manifold and $\xi : S \longrightarrow T$ is a map (by a map between topological spaces we always mean a continuous map).

(1) A 1-morphism or 1-cell $(\Sigma, \varphi) : (S_0, \xi_0) \longrightarrow (S_1, \xi_1)$ is an oriented compact collared bordism $(\Sigma, \chi_-, \chi_+) : S_0 \longrightarrow S_1$, i.e. a compact oriented $(n-1)$-dimensional manifold Σ with boundary together with orientation preserving diffeomorphisms $\chi_- : S_0 \times [0, 1) \longrightarrow \Sigma_-$

and $\chi_+ : S_1 \times (-1, 0] \longrightarrow \Sigma_+$, where $\Sigma_- \cup \Sigma_+$ is a collar of $\partial\Sigma$, and a continuous map $\varphi : \Sigma \longrightarrow T$ such that the diagram

$$\begin{tikzcd} & \Sigma \arrow[dd, "\varphi"] & \\ S_0 \times \{0\} \arrow[ur, "\chi_-"] \arrow[dr, "\xi_0"'] & & S_1 \times \{0\} \arrow[ul, "\chi_+"'] \arrow[dl, "\xi_1"] \\ & T & \end{tikzcd}$$

commutes. Here and in the sequel, the restrictions to subspaces are often suppressed in the notation. We do not assume any compatibility on the collars. Composition of 1-morphisms is by gluing of bordisms along collars and maps, respectively. Note that the collars are necessary to define the composition. Identities are given by cylinders decorated with the homotopy which is constant along the cylinder axis.

(2) A 2-morphism or 2-cell $(\Sigma, \varphi) \Longrightarrow (\Sigma', \varphi')$ between 1-morphisms $(S_0, \xi_0) \longrightarrow (S_1, \xi_1)$ is an equivalence class of pairs (M, ψ), where $M : \Sigma \longrightarrow \Sigma'$ is an n-dimensional collared compact oriented bordism with corners and $\psi : M \longrightarrow T$ is a map. Here an n-dimensional collared compact oriented bordism with corners is a $\langle 2\rangle$-manifold M together with
- a decomposition of its 0-boundary $\partial_0 M = \partial_0 M_- \cup \partial_0 M_+$ and corresponding orientation preserving diffeomorphisms $\delta_- : \Sigma \times [0, 1) \longrightarrow M_-$ and $\delta_+ : \Sigma' \times (-1, 0] \longrightarrow M_+$ onto collars of this decomposition,
- a decomposition of its 1-boundary $\partial_1 M = \partial_1 M_- \cup \partial_1 M_+$ and corresponding orientation preserving diffeomorphisms $\alpha_- : S_0 \times [0, 1) \times [0, 1] \longrightarrow M_-$ and $\alpha_+ : S_1 \times (-1, 0] \times [0, 1] \longrightarrow M_+$ onto collars of this decomposition such that there is an $\varepsilon > 0$ and commutative triangles

$$\begin{tikzcd} S_0 \times [0,1) \times [0,\varepsilon) \arrow[r, "\alpha_-"] \arrow[dr, "\chi_- \times \mathrm{id}"'] & M & S_1 \times (-1,0] \times [0,\varepsilon) \arrow[l, "\alpha_+"'] \arrow[dl, "\chi_+ \times \mathrm{id}"] \\ & \Sigma \times [0,\varepsilon) \arrow[u, "\delta_-"] & \end{tikzcd} \tag{2.1}$$

and

$$\begin{tikzcd} & \Sigma' \times (-\varepsilon, 0] \arrow[d, "\delta_+"] & \\ S_0 \times [0,1) \times (1-\varepsilon, 1] \arrow[ur, "\chi'_- \times \mathrm{id} - 1"] \arrow[r, "\alpha_-"] & M & S_1 \times (-1,0] \times (1-\varepsilon, 1] \arrow[ul, "\chi'_+ \times \mathrm{id} - 1"'] \arrow[l, "\alpha_+"'] \end{tikzcd} . \tag{2.2}$$

Furthermore, we require the diagram

$$\begin{tikzcd} & M \arrow[dd, "\psi"] & \\ S_0 \times [0,1] \sqcup \Sigma \arrow[ur, "\alpha_- \sqcup \delta_-"] \arrow[dr, "\xi_0 \circ \mathrm{pr}_{S_0} \sqcup \varphi"'] & & S_1 \times [0,1] \sqcup \Sigma' \arrow[ul, "\alpha_+ \sqcup \delta_+"'] \arrow[dl, "\xi_1 \circ \mathrm{pr}_{S_1} \sqcup \varphi'"] \\ & T & \end{tikzcd}$$

to commute. The name for the map $\alpha_- \sqcup \delta_-$ involves a slight abuse of notation: On the summand $S_0 \times [0, 1]$ is given by the composition $S_0 \times [0, 1] \longrightarrow S_0 \times [0, 1) \times [0, 1] \xrightarrow{\alpha_-}$

M induced by the injection $\{0\} \longrightarrow [0,1)$ and α_-. Note again that we do not assume any compatibility on the collars.

Two such pairs (M,ψ) and $(\widetilde{M},\widetilde{\psi})$ are defined to be equivalent if there is an orientation-preserving diffeomorphism $\Phi : M \longrightarrow M$ making the diagram

$$\begin{tikzcd}
& M \arrow[dd, "\Phi"] & \\
\Sigma \times [0,1) \arrow[ur, "\delta_-"] \arrow[dr, "\widetilde{\delta}_-"'] & & \Sigma' \times (-1,0] \arrow[ul, "\delta_+"'] \arrow[dl, "\widetilde{\delta}_+"] \\
& \widetilde{M} &
\end{tikzcd}$$

and a similar diagram for the collars of the 1-boundary commute such that additionally $\psi = \widetilde{\psi} \circ \Phi$.

To define the vertical composition of 2-morphisms, we fix once and for all a diffeomorphism $[0,2] \longrightarrow [0,1]$ which is the identity on a neighborhood of 0, and near 2 given by $x \longmapsto x-1$. Now the vertical composition is given by gluing using the collars of 0-boundaries. Furthermore, we can use the diffeomorphism fixed above to rescale the ingoing and outgoing 1-collars. As for 1-morphisms, there is no problem in gluing maps to T because the maps to T are only continuous (T is not even assumed to have a smooth structure).

Horizontal composition of 2-morphisms is defined by gluing manifolds and maps along 1-boundaries. The new 0-collars can be constructed from the old ones by restricting them to $[0,\varepsilon)$ (such that condition (2.1) and (2.2) ensure that we can glue them along the boundary) and then rescaling the interval keeping a neighborhood of 0 fixed.

Disjoint union endows the bicategory T-$\mathsf{Cob}(n,n-1,n-2)$ with the structure of a symmetric monoidal bicategory with duals. The empty manifold with its unique map to T is the monoidal unit.

Remark 2.2. (a) Following standard conventions, we will denote the composition of 1-morphisms and 2-morphisms from right to left by using the concatenation symbol $\circ$. Whenever we draw pictures of bordisms, however, composition has to be read from left to right.

(b) To maintain readability, we will often suppress the collars in the notation.

(c) Consider a 1-morphism $(\Sigma,\varphi) : (S_0,\xi_0) \longrightarrow (S_1,\xi_1)$, a compact collared bordism $\Sigma' : S_0 \longrightarrow S_1$ and a diffeomorphism $\Phi : \Sigma \longrightarrow \Sigma'$ preserving orientation and the collars. This data gives rise to an invertible 2-morphism $(M,\psi) : (\Sigma,\varphi) \longrightarrow (\Sigma', \Phi_*\varphi := \varphi \circ \Phi^{-1})$ as follows: As the underlying compact collared bordism with corners M, we take the result of gluing $\Sigma \times [0,1]$ and $\Sigma' \times [0,1]$ via Φ. Moreover, $\psi : M \longrightarrow T$ is the map that φ and $\Phi_*\varphi$ give rise to; for details on this mapping cylinder construction see [MS18, Appendix A.2].

Having defined our bordism bicategory with target T we are now ready to lift the definition of a homotopy quantum field theory from [Tur10b] to a bicategorical setting.

Definition 2.3 (Extended homotopy quantum field theory)**.** An *n-dimensional extended homotopy quantum field theory with target space T taking values in a symmetric monoidal bicategory $\mathcal{S}$* is a symmetric monoidal functor $Z : T\text{-}\mathsf{Cob}(n,n-1,n-2) \longrightarrow \mathcal{S}$ satisfying the *homotopy invariance property*: For two 2-morphisms $(M,\psi),(M,\psi') : (\Sigma_a,\varphi_a) \Longrightarrow (\Sigma_b,\varphi_b)$ between the 1-morphisms $(\Sigma_a,\varphi_a),(\Sigma_b,\varphi_b) : (S_0,\xi_0) \longrightarrow (S_1,\xi_1)$ with $\psi \simeq \psi'$ relative ∂M, we have the equality

$$\begin{tikzcd}[column sep=huge] Z(S_0,\xi_0) \arrow[r, bend left=80, "Z(\Sigma_a,\varphi_a)"] \arrow[r, bend right=80, "Z(\Sigma_b,\varphi_b)"'] \arrow[r, phantom, "Z(M,\psi)\Downarrow"] & Z(S_1,\xi_1) \end{tikzcd} \quad = \quad \begin{tikzcd}[column sep=huge] Z(S_0,\xi_0) \arrow[r, bend left=80, "Z(\Sigma_a,\varphi_a)"] \arrow[r, bend right=80, "Z(\Sigma_b,\varphi_b)"'] \arrow[r, phantom, "Z(M,\psi')\Downarrow"] & Z(S_1,\xi_1) \end{tikzcd}$$

of 2-morphisms. We denote by $\mathsf{HSym}(T\text{-}\mathsf{Cob}(n,n-1,n-2),\mathcal{S})$ the bicategory of n-dimensional extended homotopy quantum field theories (also called $(n,n-1,n-2)$-dimensional homotopy quantum field theories), i.e. the bicategory of homotopy invariant symmetric monoidal functors $T\text{-}\mathsf{Cob}(n,n-1,n-2) \longrightarrow \mathcal{S}$.

Remark 2.4. (a) This definition contains the appropriate bicategorical version of the homotopy invariance property in [Tur10b]. It is made in such a way that we recover the usual homotopy invariance property if we pass from extended homotopy quantum field theories to non-extended ones by restriction to the endomorphisms of the monoidal unit in both domain and codomain.

(b) If T is a just the space with one point, we recover the notion of an extended topological field theory. Recall that by *extended* we mean *once-extended* in this thesis.

(c) As for non-extended homotopy quantum field theories, the homotopy invariance can be built in by decorating the top-dimensional bordisms with relative homotopy classes of maps rather than actual maps. For technical reasons, however, we work with the above definition which requires homotopy invariance as an additional property just as in [Tur10b].

(d) The symmetric monoidal bicategory $\mathcal{S}$, which is the codomain of Z, will be referred to as the *coefficients* or *coefficient category of* Z.

(e) The bicategory $\mathsf{HSym}(T\text{-}\mathsf{Cob}(n,n-1,n-2),\mathcal{S})$ is in fact a 2-groupoid.

(f) Let Z be an n-dimensional extended homotopy quantum field theory, $\Sigma : S_0 \longrightarrow S_1$ a 1-morphism in $\mathsf{Cob}(n,n-1,n-2)$ and φ and φ' two maps $\Sigma \longrightarrow T$. Then for any homotopy $\varphi \overset{h}{\simeq} \varphi'$ relative $\partial\Sigma$, we obtain an invertible 2-isomorphism $Z(h) : Z(\Sigma,\varphi) \Longrightarrow Z(\Sigma,\varphi')$ by evaluation of Z on $\Sigma \times [0,1]$ equipped with h. Note that $Z(h)$ only depends on the equivalence class of the homotopy h.

Example 2.5 (The symmetric monoidal bicategory $\mathsf{2Vect}$)**.** Let us review the main example of a symmetric monoidal bicategory that will be relevant as the coefficients of an extended homotopy quantum field theory in the sequel, namely the symmetric monoidal bicategory $\mathsf{2Vect}$ of *2-vector spaces (of Kapranov-Voevodsky type[1])*, see [KV94, Mor11]:

(0) Objects are 2-vector spaces, i.e. $\mathbb{C}$-linear additive semisimple categories with biproducts, finite-dimensional morphism spaces and finitely many simple objects up to isomorphism.

[1] There are other types of 2-vector spaces, but throughout this text we will always mean 2-vector spaces of Kapranov-Voevodsky type when talking about 2-vector spaces. In particular, we always work over the complex field.

(1) 1-Morphisms are $\mathbb{C}$-linear functors, which are also called *2-linear maps*.
(2) 2-Morphisms are natural transformations of $\mathbb{C}$-linear functors.

The monoidal product is the Deligne product, the monoidal unit is the category $\mathsf{FinVect}$ of finite-dimensional complex vector spaces. For any 2-vector space $\mathcal{V}$, we can choose a *basis*, i.e. a family of representatives for the finitely many isomorphism classes of simple objects. Having chosen a basis $\mathscr{B}$ of a 2-vector space $\mathcal{V}$ we can write any object X in $\mathcal{V}$ as a biproduct

$$X \cong \bigoplus_{B \in \mathscr{B}} V_B \otimes B \ ,$$

where the V_B are finite-dimensional complex vector spaces and where $V_B \otimes B$ is the tensoring of the vector spaces V_B with the object B; essentially, this is the $\dim V_B$-fold biproduct of B with itself. The $\mathbb{C}$-linearity of a functor between 2-vector spaces is equivalent to the preservation of biproducts. Consequently, any 2-linear map $\mathcal{V} \longrightarrow \mathcal{W}$ is determined by its values on this basis, which allows us the describe 2-linear maps in terms of matrices with vector space valued entries. Moreover, note that 2-linear maps $\mathcal{V} \longrightarrow \mathcal{W}$ are precisely the exact functors. Indeed, exactness implies preservations of biproducts. The converse holds since all short exact sequences in $\mathcal{V}$ split by semisimplicity.

Up to $\mathbb{C}$-linear equivalence, a 2-vector space is determined by the cardinality of its basis, which we will also refer to as *dimension*. For instance, any n-dimensional 2-vector space is equivalent to the category $\mathbb{C}[\mathbb{Z}_n]$-Mod of finite-dimensional complex modules over the group algebra of the cyclic group $\mathbb{Z}_n$.

For later use, we recall that the symmetric monoidal category obtained by restriction of $\mathsf{2Vect}$ to the endomorphisms of the monoidal unit is the category $\mathsf{FinVect}$ of finite-dimensional complex vector spaces.

For two topological spaces X and Y, we denote by Y^X the space of maps $X \longrightarrow Y$ equipped with the compact-open topology. Depending on what is convenient, we can see X and Y and Y^X also as Kan complexes. For any space or Kan complex Z, we denote by $\Pi(Z) = \Pi_1(Z)$ and $\Pi_2(Z)$ the fundamental groupoid and the fundamental 2-groupoid, respectively, and also set $\Pi_j(X,Y) := \Pi_j(Y^X)$ for $j = 1, 2$.

From the definition of an extended homotopy quantum field theory, we obtain the following statement (see also Remark 2.4 (f)):

Proposition 2.6. *For any extended homotopy quantum field theory $Z : T\text{-}\mathsf{Cob}(n, n-1, n-2) \longrightarrow \mathcal{S}$ and any closed oriented $(n-2)$-dimensional manifold S, we naturally obtain a representation*

$$\widehat{Z}(S) := Z(S, -) : \Pi_2(S,T) = \Pi_2\left(T^S\right) \longrightarrow \mathcal{S} \ ,$$

i.e. a 2-functor $\Pi_2\left(T^S\right) \longrightarrow \mathcal{S}$ sending $\xi : S \longrightarrow T$ to $Z(S,\xi)$. The definition on homotopies is by evaluation of Z on the cylinder $S \times [0,1]$ over S; the definition on equivalences classes of homotopies of homotopies is by evaluation of Z on the cylinder $S \times [0,1]^2$ over the cylinder over S.

The fact that this is well-defined on equivalences classes of homotopies of homotopies makes uses of homotopy invariance, compare also with [SW19, Proposition 2.8].

2.2 Aspherical targets: Extended equivariant topological field theories

Specifying for the target space an aspherical space leads to equivariant topological field theories, see also [Tur10b] for the non-extended case. Here a space or simplicial set T is called *aspherical* if $\pi_k(T) = 0$ for $k \geq 2$ and all choices of base points. In this case, T is equivalent to the disjoint union of classifying spaces of groups. Without loss of generality, we will just consider the connected case, i.e. $T = BG$ for a group G. In context of the orbifold construction, we will later on require G to be finite, but the following definition can be made for an arbitrary group G:

Definition 2.7 (Extended equivariant topological field theory)**.** For a group G, set

$$G\text{-}\mathsf{Cob}(n, n-1, n-2) := BG\text{-}\mathsf{Cob}(n, n-1, n-2)$$

for the classifying space BG of G. An *n-dimensional extended G-equivariant topological field theories with values in a symmetric monoidal bicategory $\mathcal{S}$* is a homotopy quantum field theory $Z : G\text{-}\mathsf{Cob}(n, n-1, n-2) \longrightarrow \mathcal{S}$ with target space BG and values in $\mathcal{S}$.

Remark 2.8. (a) A G-equivariant topological field theory assigns data to manifolds decorated with maps to BG. Homotopy classes of such maps correspond to isomorphism classes of principal G-bundles, and in Lemma 2.9 below it is explained that this identification extends to groupoids of bundles.

(b) A class of examples of extended G-equivariant topological field theories is constructed in [MNS12] in a language slightly different from the one used here. Another class is constructed in [MW20a] from cohomological data. Two very important construction procedures for non-extended three-dimensional G-equivariant topological field theories have been given by Turaev and Virelizier in [TV12, TV14], see also Remark 6.22 at the end of this thesis.

In the sequel, it will be crucial to know the following basic fact about mapping spaces with aspherical target space, i.e. classifying space of a group (or more generally a groupoid):

Lemma 2.9. *Let Γ be a groupoid. For any space X, the mapping space $B\Gamma^X$ is equivalent to the nerve of the functor groupoid $[\Pi(X), \Gamma]$. In particular, for every (discrete) group G and every manifold M (with boundary or corners) the space BG^M is equivalent to the nerve $B\mathsf{PBun}_G(M)$ of the groupoid $\mathsf{PBun}_G(M)$ of G-bundles over M.*

Proof. We can see X as a Kan complex. Since the fundamental groupoid functor $\Pi : \mathsf{Kan} \longrightarrow \mathsf{Grpd}$ from the category Kan of Kan complexes to the category Grpd of groupoids is left adjoint to the nerve functor $B : \mathsf{Grpd} \longrightarrow \mathsf{Kan}$, we find

$$\begin{aligned}
\mathrm{Hom}_{\mathsf{Kan}}(Y, B\Gamma^X) &\cong \mathrm{Hom}_{\mathsf{Kan}}(Y \times X, B\Gamma) \\
&\cong \mathrm{Hom}_{\mathsf{Grpd}}(\Pi(Y \times X), \Gamma) \\
&\cong \mathrm{Hom}_{\mathsf{Grpd}}(\Pi(Y) \times \Pi(X), \Gamma) \\
&\cong \mathrm{Hom}_{\mathsf{Grpd}}(\Pi(Y), [\Pi(X), \Gamma]) \\
&\cong \mathrm{Hom}_{\mathsf{Kan}}(Y, B[\Pi(X), \Gamma]) \ .
\end{aligned}$$

The Yoneda Lemma implies that $B\Gamma^X$ is isomorphic to the nerve $B[\Pi(X), \Gamma]$ of the groupoid $[\Pi(X), \Gamma]$ of functors from $\Pi(X)$ to Γ. The additional statement involving the groupoid of bundles now follows from the holonomy description of bundles, i.e. the fact that for any manifold

M (with boundary or corners) the groupoid $\mathsf{PBun}_G(M)$ is equivalent to $[\Pi(M), \star//G]$. □

Remark 2.10. This result says that for an extended G-equivariant topological field theory $Z : G\text{-}\mathsf{Cob}(n, n-1, n-2) \longrightarrow \mathcal{S}$ the representation $Z(S, -) : \Pi_2\,(S, BG) \longrightarrow \mathcal{S}$ from Proposition 2.6 can and will be treated as a representation of the groupoid $\Pi(S, BG)$ (or rather as a 2-vector bundle over $\Pi(S, BG)$ in the sense of Definition 3.4 below). This will turn out to be a tremendous simplification (which is one of the reasons why the restriction to aspherical targets is so common in the literature).

Example 2.11 (The cover functor)**.** For a finite group G, there is a canonical symmetric monoidal functor

$$\mathrm{Cov} : G\text{-}\mathsf{Cob}(n, n-1, n-2) \longrightarrow \mathsf{Cob}(n, n-1, n-2),$$

the so-called *cover functor*, which is studied in [BS11] and defined as follows: For a closed oriented $(n-2)$-dimensional manifold S with a map $\xi : S \longrightarrow BG$, we take the pullback bundle $\xi^*EG \longrightarrow S$. This G-bundle is a covering map and by [Lee12, Proposition 4.40 and 15.35] the total space ξ^*EG inherits the structure of a closed oriented manifold of dimension $n-2$. The assignment $\mathrm{Cov}(S, \xi) := \xi^*EG$ extends to a symmetric monoidal functor. If we are given an extended topological field theory $Z : \mathsf{Cob}(n, n-1, n-2) \longrightarrow \mathcal{S}$, its pullback $\mathrm{Cov}^* Z$ along the cover functor is a G-equivariant topological field theory. This provides an important class of examples of G-equivariant field theories. In Example 6.10 we will use the cover functor to formalize the idea of the permutation orbifolds appearing in [FKS92, Ban98, Ban02].

3 A parallel section functor for 2-vector bundles

One of most important and by far the technically most challenging ingredient of the orbifold construction is the bicategorical parallel section functor. The necessity of this tool was explained in the introduction and will become even clearer in the next chapter once we introduce the *change to equivariant coefficients* (Section 4.1).

Despite of its relevance to the orbifold construction, we will make an effort to motivate the parallel section functor independently: A representation of a group G on, say, a complex vector space V can be seen as a functor $\star//G \longrightarrow \mathsf{Vect}$ from the groupoid $\star//G$ with one object $\star$ and automorphism group G to the category Vect of complex vector spaces sending $\star$ to V. It is an obvious generalization to replace $\star//G$ by a groupoid $\varGamma$ and call any functor $\varrho : \varGamma \longrightarrow \mathsf{Vect}$ a representation of $\varGamma$. The limit of the functor ϱ yields the invariants of the representation.

A functor $\varrho : \varGamma \longrightarrow \mathsf{Vect}$, i.e. a representation of $\varGamma$, is a purely algebraic object. However, it can also be seen a (flat) vector bundle over the groupoid $\varGamma$. This profitable point of view is for instance emphasized in [Wil05]. It allows us to think of the algebraic notion of invariants of a representation in a geometric way, namely in terms of parallel sections. We will take this convenient geometric point of view below.

If we denote by $\mathsf{VecBun}(\varGamma) = [\varGamma, \mathsf{FinVect}]$ the category of finite-dimensional vector bundles over a groupoid $\varGamma$, then taking parallel sections yields a functor

$$\mathsf{Par}_\varGamma : \mathsf{VecBun}(\varGamma) \longrightarrow \mathsf{FinVect} , \tag{3.1}$$

namely the limit functor on the functor category $\mathsf{VecBun}(\varGamma)$.

There is a higher analogue of a vector bundle over a groupoid, namely a 2-vector bundle over a groupoid, i.e. a 2-functor from a given groupoid (seen as a bicategory) to the bicategory $\mathsf{2Vect}$ of 2-vector spaces, see [BBFW12] and [Kir04] for related notions. To a 2-vector bundle $\varrho : \varGamma \longrightarrow \mathsf{2Vect}$ over a groupoid $\varGamma$, we associate the category of parallel sections (Definition 3.6) and prove that this category is naturally a 2-vector space if $\varGamma$ is essentially finite (Proposition 3.9). Hence, as a categorification of (3.1), we obtain a 2-functor

$$\mathsf{Par}_\varGamma : \mathsf{2VecBun}(\varGamma) \longrightarrow \mathsf{2Vect} \tag{3.2}$$

from the bicategory of 2-vector bundles over a fixed groupoid $\varGamma$ to the category of 2-vector spaces.

The parallel section functors (3.1) and (3.2) are *not* our main concern. Instead, we are interested in a variant of parallel section functors meeting the requirements determined by our motivation, namely the orbifoldization of equivariant topological field theories: The orbifold construction for (non-extended) equivariant topological field theories was formulated in [SW19] by means of a parallel section functor

$$\mathsf{Par} : \mathsf{VecBunGrpd} \longrightarrow \mathsf{FinVect} \tag{3.3}$$

for vector bundles over varying groupoids. Here, $\mathsf{VecBunGrpd}$ is the symmetric monoidal bicategory from [SW19, Section 3.2] whose objects are vector bundles over essentially finite groupoids and whose morphisms come from spans of groupoids and intertwiners. The key point about this functor is that it provides pull-push maps between the vector spaces of parallel sections of vector bundles over different groupoids which are related by a span of groupoids. Having in mind that our parallel section functor is tailored to the application in equivariant topological field theory also explains the importance of spans of groupoids: Equivariant topological field theories assign quantities to bordisms equipped with principal fiber bundles, and the application of the bundle stack to bordisms, which can be seen as cospans in manifolds, yields exactly spans of groupoids. Hence, the biased reader may think of all groupoids below as groupoids of principal fiber bundles over some manifold.

In order to give an orbifoldization procedure for extended field theories, which is one of the main objectives of this thesis, we need a higher analogue of the parallel section functor (3.3) or, in other words, the extension of the 2-functor (3.2) to a symmetric monoidal 2-functor defined on a symmetric monoidal bicategory $\mathsf{2VecBunGrpd}$ of 2-vector bundles over varying groupoids. The construction of this symmetric monoidal 2-functor

$$\mathsf{Par} : \mathsf{2VecBunGrpd} \longrightarrow \mathsf{2Vect} \ , \tag{3.4}$$

is the main result of this chapter (Theorem 3.25). The relation between the different parallel section functors is summarized in the diagram

$$\begin{tikzcd}
\begin{array}{c}\mathsf{Par}_\Gamma : \mathsf{VecBun}(\Gamma) \longrightarrow \mathsf{FinVect} \\ \text{for any groupoid } \Gamma\end{array} \arrow[r, dashed, "\text{passing to spans of groupoids}"] \arrow[d, dashed, "\text{categorification}"'] & \mathsf{Par} : \mathsf{VecBunGrpd} \longrightarrow \mathsf{FinVect} \arrow[d, dashed, "\text{categorification}"] \\
\begin{array}{c}\mathsf{Par}_\Gamma : \mathsf{2VecBun}(\Gamma) \longrightarrow \mathsf{2Vect} \\ \text{for any groupoid } \Gamma\end{array} \arrow[r, dashed, "\text{passing to spans of groupoids}"] & \mathsf{Par} : \mathsf{2VecBunGrpd} \longrightarrow \mathsf{2Vect},
\end{tikzcd}$$

in which the object of main interest is sitting in the right lower corner. Note that the upper half of the diagram was already discussed in [SW19].

Concretely, we proceed as follows: In Section 3.1 we first recall ordinary vector bundles over groupoids and their parallel sections including various pull-push operations. Afterwards, we discuss the higher analogues of these notions, i.e. 2-vector bundles over groupoids and their parallel sections and hence the left lower corner of the above diagram.

Section 3.2 is devoted to the introduction of pullback and pushforward maps on two different categorical levels needed for the construction of the parallel section functor (3.4). The discussion of pullback and pushforward 2-morphisms in Section 3.2.2 leads to a higher version of the equivariant Beck-Chevalley condition (Proposition 3.17), which is of independent interest.

In Section 3.3 we construct the parallel section functor (3.4), i.e. the right lower corner of the above diagram. To this end, we first have to introduce the domain symmetric monoidal bicategory $\mathsf{2VecBunGrpd}$ in Section 3.3.1. The objects are 2-vector bundles over essentially finite groupoids, 1-morphisms arise from spans of essentially finite groupoids and intertwiners and 2-morphisms from spans of spans of essentially finite groupoids and higher intertwiners. Section 3.3.2 contains the formulation and proof of the main result (Theorem 3.25). Finally, we show how to recover the categorical parallel section functor (3.3) from the bicategorical one (3.4) in Proposition 3.26.

3.1 2-Vector bundles and their parallel sections

After recalling the notion of a vector bundle over a groupoid, we fix the definition of 2-vector bundle used in this text and define the category of parallel sections of a 2-vector bundle.

3.1.1 A brief reminder on vector bundles over groupoids

Let us review some of the notions and constructions used or introduced in [SW19] while implementing also some mild generalizations: A vector bundle over a groupoid Ω with values in a 2-vector space $\mathcal{V}$ (see Example 2.5) is a functor $\xi : \Omega \longrightarrow \mathcal{V}$.

By $\mathsf{VecBun}(\Gamma, \mathcal{V})$ we denote the category of $\mathcal{V}$-valued vector bundles over Γ. In case Γ is essentially finite, this category naturally carries the structure of a 2-vector space [Mor11, Lemma 4.1.1].

If $\xi : \Omega \longrightarrow \mathcal{V}$ is a vector bundle and $\Phi : \Gamma \longrightarrow \Omega$ a functor between groupoids, then we can form the pullback $\Phi^*\xi := \xi \circ \Phi$ of ξ to Γ. In fact, Φ gives rise to a pullback functor

$$\Phi^* : \mathsf{VecBun}(\Omega, \mathcal{V}) \longrightarrow \mathsf{VecBun}(\Gamma, \mathcal{V}) \ . \tag{3.5}$$

More concisely,

$$\mathsf{VecBun}(-, \mathcal{V}) : \mathsf{FinGrpd}^{\mathrm{opp}} \longrightarrow \mathsf{2Vect}$$

naturally extends to a 2-functor defined on the bicategory of essentially finite groupoids, functors and natural transformations. It sends a groupoid Γ to the 2-vector space $\mathsf{VecBun}(\Gamma, \mathcal{V})$, a functor $\Phi : \Gamma \longrightarrow \Omega$ to the pullback functor Φ^* and a natural transformation $\eta : \Phi \Longrightarrow \Phi'$ of functors $\Phi, \Phi' : \Gamma \longrightarrow \Omega$ to the obvious natural transformation $-(\eta) : \Phi^* \Longrightarrow \Phi'^*$ whose component

$$\xi(\eta) : \Phi^*\xi \longrightarrow \Phi'^*\xi$$

for ξ in $\mathsf{VecBun}(\Gamma, \mathcal{V})$ consists of the maps $\xi(\eta_x) : \xi(\Phi(x)) \longrightarrow \xi(\Phi'(x))$ for all $x \in \Gamma$.

The space $\mathsf{Par}\,\xi$ of parallel sections of a $\mathcal{V}$-valued vector bundle ξ over Ω is defined as the limit of the functor ξ,

$$\mathsf{Par}\,\xi := \lim \xi \ ,$$

see [SW19, Section 3.1]. This construction yields a functor

$$\mathsf{Par}_\Omega : \mathsf{VecBun}(\Omega, \mathcal{V}) \longrightarrow \mathcal{V}$$

for each essentially finite groupoid Ω. These functors constitute a 1-morphism

$$\mathsf{Par} : \mathsf{VecBun}(-, \mathcal{V}) \longrightarrow \mathcal{V}$$

in the bicategory of 2-functors $\mathsf{FinGrpd}^{\mathrm{opp}} \longrightarrow \mathsf{2Vect}$, where we use $\mathcal{V}$ to denote the constant 2-functor with value $\mathcal{V}$.

By the following standard fact limits can be pulled back:

Lemma 3.1. *Let $\mathcal{C}$ be a complete category and $X : \mathcal{J} \longrightarrow \mathcal{C}$ a functor from a small category $\mathcal{J}$ to $\mathcal{C}$. Then any functor $\Phi : \mathcal{I} \longrightarrow \mathcal{J}$ of small categories induces a morphism*

$$\lim X \longrightarrow \lim \Phi^* X \ .$$

If Φ is an equivalence, then this morphism is an isomorphism. The dual statement for colimits is true if $\mathcal{C}$ is cocomplete.

Stronger statements can be made using final and initial functors, but this is not needed here.

For a functor $\varPhi : \varGamma \longrightarrow \varOmega$ between groupoids, we obtain a natural map

$$\varPhi^* : \mathsf{Par}\,\xi \longrightarrow \mathsf{Par}\,\varPhi^*\xi, \quad s \longmapsto \varPhi^* s = s \circ \varPhi \ ,$$

the *pullback map*. By abuse of notation it is denoted by the same symbol as the pullback functor (3.5), but should not be confused with the latter.

In case that $\varGamma$ and $\varOmega$ are essentially finite, we introduced in [SW19, Section 3.4] also a *pushforward map*

$$\varPhi_* : \mathsf{Par}\,\varPhi^*\xi \longrightarrow \mathsf{Par}\,\xi$$

by integration over the homotopy fiber $\varPhi^{-1}[y]$ over $y \in \varOmega$. Recall that for a given $y \in \varOmega$, an object $(x, g) \in \varPhi^{-1}[y]$ in the homotopy fiber of $\varPhi$ over y is an object $x \in \varGamma$ together with a morphism $g : \varPhi(x) \longrightarrow y$ in $\varGamma$. A morphism $(x, g) \longrightarrow (x', g')$ in $\varPhi^{-1}[y]$ is a morphism $h : x \longrightarrow x'$ such that $g'\varPhi(h) = g$.

Since $\mathsf{Par}\,\varPhi^*\xi$ is the limit of $\varPhi^*\xi$, it comes equipped with maps $\pi_x : \mathsf{Par}\,\varPhi^*\xi \longrightarrow \xi(\varPhi(x))$ for each $x \in \varGamma$, which we can use to form the concatenation

$$\nu_{x,g} : \mathsf{Par}\,\varPhi^*\xi \xrightarrow{\pi_x} \xi(\varPhi(x)) \xrightarrow{\xi(g)} \xi(y) \ .$$

An easy computation shows that the morphism $\nu_{x,g}$ only depends on the isomorphism class of (x, g) in $\varPhi^{-1}[y]$. This allows us to define

$$\int_{\varPhi^{-1}[y]} \nu_{x,g}\, \mathrm{d}(x, g) := \sum_{[x,g]\in\pi_0(\varPhi^{-1}(y))} \frac{\nu_{x,g}}{|\mathrm{Aut}(x, g)|} : \mathsf{Par}\,\varPhi^*\xi \longrightarrow \xi(y) \ . \tag{3.6}$$

The morphisms $\nu_{x,g}$ can be added and multiplied by scalars since $\mathrm{Hom}_{\mathcal{V}}(\varPhi^*\xi, \xi(y))$ is a complex vector space. Formula (3.6) provides us with an instance of an *integral with respect to groupoid cardinality*, i.e. a sum over the values of an invariant function on an essentially finite groupoid, here $\varPhi^{-1}[y] \ni (x, g) \longmapsto \nu_{x,g}$, taking values in a complex vector space, here $\mathrm{Hom}_{\mathcal{V}}(\varPhi^*\xi, \xi(y))$, weighted by the cardinalities of the automorphism groups in our groupoid. For more background on groupoid cardinality we refer to [BHW10]. The integral with respect to groupoid cardinality was also an essential concept for the construction of the parallel section functor in [SW19] and is also fully recalled there.

An easy computation shows that for any morphism $a : y \longrightarrow y'$

$$\xi(a) \int_{\varPhi^{-1}[y]} \nu_{x,g}\, \mathrm{d}(x, g) = \int_{\varPhi^{-1}[y']} \nu_{x',g'}\, \mathrm{d}(x', g') \ .$$

This implies that the maps (3.6) induce a natural map

$$\varPhi_* : \mathsf{Par}\,\varPhi^*\xi \longrightarrow \mathsf{Par}\,\xi \ ,$$

the so-called *pushforward map*.

The most important properties of pullback and the pushforward map include the composition laws and the equivariant Beck-Chevalley property (the name is justified by the fact that it reduces to the ordinary Beck-Chevalley property that is discussed for example in [Mor11, Appendix A.2] in context of Dijkgraaf-Witten theory). The proofs are slight generalizations of those in [SW19]. We will discuss in Section 3.1 generalizations to the bicategorical setting.

Proposition 3.2. *Let $\mathcal{V}$ be a 2-vector space and let $\varPhi : \varGamma \longrightarrow \varOmega$ and $\varPsi : \varOmega \longrightarrow \varLambda$ be functors between essentially finite groupoids.*

(a) *For any $\mathcal{V}$-valued vector bundle ξ over Λ, the composition law $(\Psi \circ \Phi)^* = \Phi^* \circ \Psi^*$ for the pullback maps holds.*
(b) *For any $\mathcal{V}$-valued vector bundle ξ over Γ, the composition law $(\Psi \circ \Phi)_* = \Psi_* \circ \Phi_*$ for the pushforward maps holds.*

For a cospan $\Lambda \xrightarrow{\Psi} \Omega \xleftarrow{\Phi} \Gamma$ of groupoids, one can form the homotopy pullback

$$\begin{tikzcd} \Gamma \times_\Omega \Lambda \arrow[r, "\pi_\Gamma"] \arrow[d, "\pi_\Lambda"'] & \Gamma \arrow[d, "\Phi"] \arrow[dl, Rightarrow, "\eta"'] \\ \Lambda \arrow[r, "\Psi"'] & \Omega \end{tikzcd} .$$

The homotopy pullback $\Gamma \times_\Omega \Lambda$ completes the cospan $\Lambda \xrightarrow{\Psi} \Omega \xleftarrow{\Phi} \Gamma$ to a square commuting up to natural isomorphism, and it is universal with this property in the appropriate weak sense (we do not introduce an extra notation to highlight a *homotopy* pullback as opposed to an ordinary one – in fact, for groupoids, we will only consider homotopy pullbacks and never ordinary ones). Instead of spelling out this universal property, we will use the following explicit model for the homotopy pullback $\Gamma \times_\Omega \Lambda$: It is the the groupoid of triples (x, y, η) of $x \in \Gamma$, $y \in \Omega$ and a morphism $\eta : \Phi(x) \longrightarrow \Psi(y)$. A morphism $(x, y, \eta) \longrightarrow (x', y', \eta')$ in $\Gamma \times_\Omega \Lambda$ is a pair (α, β) of a morphism $\alpha : x \longrightarrow x'$ and a morphism $\beta : y \longrightarrow y'$ such that $\Psi(\beta)\eta = \eta'\Phi(\alpha)$. The functors π_Γ and π_Λ are the obvious projection functors. The third component of the triples that $\Gamma \times_\Omega \Lambda$ consists of is responsible for filling the square with the natural isomorphism η that is indicated in the diagram.

Note that if $\Lambda = \star$, then Ψ just selects an object $z \in \Omega$. In that case, the homotopy pullback is precisely the homotopy fiber $\Phi^{-1}[z]$ of Φ over z as defined on page 28.

Proposition 3.3 (Equivariant Beck-Chevalley condition)**.** *For the homotopy pullback*

$$\begin{tikzcd} \Gamma \times_\Omega \Lambda \arrow[r, "\pi_\Gamma"] \arrow[d, "\pi_\Lambda"'] & \Gamma \arrow[d, "\Phi"] \arrow[dl, Rightarrow, "\eta"'] \\ \Lambda \arrow[r, "\Psi"'] & \Omega \end{tikzcd}$$

of a cospan $\Lambda \xrightarrow{\Psi} \Omega \xleftarrow{\Phi} \Gamma$ of essentially finite groupoids and any $\mathcal{V}$-valued vector bundle ξ over Ω, the pentagon relating different pull-push combinations

$$\begin{tikzcd} \mathsf{Par}\, \Phi^*\xi \arrow[rr, "\Phi_*"] \arrow[d, "\pi_\Gamma^*"'] & & \mathsf{Par}\, \xi \arrow[dd, "\Psi^*"] \\ \mathsf{Par}\, \pi_\Gamma^*\Phi^*\xi \arrow[dr, "\xi(\eta)_*"'] & & \\ & \mathsf{Par}\, \pi_\Lambda^*\Psi^*\xi \arrow[r, "\pi_{\Lambda *}"] & \mathsf{Par}\, \Psi^*\xi \end{tikzcd}$$

commutes.

3.1.2 2-Vector bundles

The goal of this chapter is a bicategorical generalization of the parallel section functor. Hence, we need a higher analogue of a groupoid representation (or a vector bundle over a groupoid). This will be the notion of a 2-vector bundle below. For related notions see [BBFW12] and [Kir04].

Definition 3.4 (2-Vector bundle)**.** A *2-vector bundle ϱ over a groupoid Γ with values in a symmetric monoidal bicategory $\mathcal{S}$* is a representation of Γ on $\mathcal{S}$, i.e. a 2-functor $\varrho : \Gamma \longrightarrow \mathcal{S}$, where Γ is seen as a bicategory without non-trivial 2-morphisms. (There are no monoidality requirements on ϱ.) By $\mathsf{2VecBun}(\Gamma, \mathcal{S})$ we denote the symmetric monoidal bicategory of $\mathcal{S}$-valued 2-vector bundles over Γ.

Remark 3.5. (a) We use the term '2-vector bundle' although we do not assume any (higher) linear structure on the target $\mathcal{S}$.

(b) Let us partly unpack the definition of a 2-vector bundle $\varrho : \Gamma \longrightarrow \mathcal{S}$:
- To $x \in \Gamma$ the 2-vector bundle ϱ assigns an object $\varrho(x)$ in $\mathcal{S}$, also called the *fiber of ϱ over x*.
- To a morphism $g : x \longrightarrow y$ in Γ the 2-vector bundle assigns a 1-morphism $\varrho(g) : \varrho(x) \longrightarrow \varrho(y)$, which in geometric terms can be thought of as a parallel transport operator.
- The data comprises natural isomorphisms

$$\begin{aligned} \eta_x : \varrho(\mathrm{id}_x) &\cong \mathrm{id}_{\varrho(x)} \ , \\ \alpha_{g,h} : \varrho(g) \circ \varrho(h) &\cong \varrho(gh) \end{aligned}$$

for composable morphisms in Γ. These natural isomorphisms are subject to obvious coherence conditions.

(c) Let us describe the bicategory $\mathsf{2VecBun}(\Gamma, \mathcal{S})$ in more detail:

(0) Objects are 2-vector bundles over Γ.

(1) 1-morphisms are 2-vector bundles morphisms or, equivalently, intertwiners. An intertwiner $\phi : \varrho \longrightarrow \xi$ of 2-vector bundles over Γ consists of 1-morphisms $\phi_x : \varrho(x) \longrightarrow \xi(x)$ for each $x \in \Gamma$ and natural morphisms

$$\xi(g) \circ \phi_x \xrightarrow{\theta_g} \phi_y \circ \varrho(g) \quad \text{for all} \quad g : x \longrightarrow y$$

subject to obvious coherence conditions. These coherence conditions entail in particular that all θ_g are 2-isomorphisms. For this it is crucial that Γ is a groupoid.

(2) A 2-morphism $\eta : \phi \longrightarrow \psi$ between 1-morphisms (ϕ, θ) and (ψ, κ) between the 2-vector bundles ϱ and ξ consists of 2-morphisms $\eta_x : \phi_x \longrightarrow \psi_x$ such that for all $g : x \longrightarrow y$ the square

$$\begin{tikzcd} \xi(g) \circ \phi_x \arrow[r, "\theta_g"] \arrow[d, "\eta_x"'] & \phi_y \circ \varrho(g) \arrow[d, "\eta_y"] \\ \xi(g) \circ \psi_x \arrow[r, "\kappa_g"] & \psi_y \circ \varrho(g) \end{tikzcd}$$

commutes. Here $\eta_x : \xi(g) \circ \phi_x \longrightarrow \xi(g) \circ \psi_x$ is the 2-morphism induced by η_x and the identity on $\varrho(x)$, but we suppress the identity morphism in the notation for readability.

The monoidal product in $\mathsf{2VecBun}(\Gamma, \mathcal{S})$ is the monoidal product in $\mathcal{S}$ applied object-wise. The monoidal unit $\mathbb{I}_\Gamma$ in $\mathsf{2VecBun}(\Gamma, \mathcal{S})$ assigns to each $x \in \Gamma$ is the monoidal unit $\mathbb{I}$ in $\mathcal{S}$ and to all morphisms in Γ the identity 1-morphism.

Since the symmetric monoidal bicategory $\mathsf{2Vect}$ (Example 2.5) will be the most important one for us in the sequel, we agree on the notation $\mathsf{2VecBun}(\Gamma) := \mathsf{2VecBun}(\Gamma, \mathsf{2Vect})$, i.e. 2-vector bundles with unspecified target category always have to be understood as $\mathsf{2Vect}$-valued 2-vector bundles.

3.1.3 Parallel sections of 2-vector bundles

Parallel sections of a given vector bundle ϱ (or, equivalently, invariants of the representation ϱ) can be obtained by taking the morphisms from the trivial vector bundle to ϱ. This principle can be directly generalized to 2-vector bundles.

Definition 3.6 (Parallel sections of a 2-vector bundle)**.** Let $\mathcal{S}$ be a symmetric monoidal bicategory. The *category of parallel sections* of an $\mathcal{S}$-valued 2-vector bundle ϱ over a groupoid Γ is the category

$$\mathsf{Par}\,\varrho := \mathrm{Hom}_{\mathsf{2VecBun}(\Gamma,\mathcal{S})}(\mathbb{I}_\Gamma, \varrho) \ .$$

Remark 3.7. (a) A parallel section $s \in \mathsf{Par}\,\varrho$ gives us a 1-morphism $s(x) : \mathbb{I} \longrightarrow \varrho(x)$ in $\mathcal{S}$ for each $x \in \Gamma$ and coherent isomorphisms $s(y) \cong \varrho(g) \circ s(x)$ for all $g : x \longrightarrow y$ in Γ. Instead of $\varrho(g) \circ s(x)$ we will often write $\varrho(g)s(x)$ or even $g.s(x)$ if the vector bundle is clear from the context. For $\mathcal{S} = \mathsf{2Vect}$ the monoidal unit $\mathbb{I}$ is given by the category $\mathsf{FinVect}$ of finite-dimensional complex vector spaces. Note that 1-morphisms $\mathsf{FinVect} \longrightarrow \varrho(x)$ can be identified with the value on $\mathbb{C}$ and hence with an object in the fiber $\varrho(x)$.

(b) The parallel sections of a 2-vector bundle are thus 'parallel up to isomorphism', where the isomorphism is part of the data. Hence, being parallel is no longer a property, but structure. In other contexts, the parallel sections considered here would be called homotopy fixed points, see e.g. [HSV17].

For a $\mathsf{2Vect}$-valued 2-vector bundle, we would like to find conditions under which the category of parallel sections is naturally a 2-vector space again. It is easy to see that the category of parallel sections inherits all the needed structure and properties from the 2-vector bundle except for finite semisimplicity. In order to look at this last missing point more closely, we use techniques and results from [Kir01].

First of all, we note that it suffices to study 2-vector bundles over connected groupoids, i.e. we can concentrate on 2-vector bundles $\varrho : \star//G \longrightarrow \mathsf{2Vect}$ for a finite group G. In this case we obtain a 2-vector space $\mathcal{V} := \varrho(\star)$, and any $g \in G$ yields a 2-linear equivalence $\varrho(g) : \mathcal{V} \longrightarrow \mathcal{V}$. These equivalences fulfill the properties of a representation only up to isomorphism as discussed in Remark 3.5 (b), but still we will refer to this as a representation of G on $\mathcal{V}$. We denote the action of $g \in G$ on some object $X \in \mathcal{V}$ by $g.X$ and the evaluation of the coherence isomorphisms on X by

$$\beta^X_{g,h} : g.h.X \longrightarrow (gh).X \ .$$

According to Definition 3.6, $\mathsf{Par}\,\varrho$ is the category of pairs $(X, \phi = (\phi_g)_{g \in G})$, where X is in $\mathcal{V}$ and ϕ is a family of coherent isomorphisms $\phi_g : g.X := \varrho(g)(X) \longrightarrow X$.

Let $(X_s)_{s \in \mathscr{S}}$ be a basis of $\mathcal{V}$. Since any $g \in G$ acts as an equivalence, it maps simple objects to simple objects. Hence, when forgetting about the coherence data, $g \in G$ just acts as a permutation of the basis. Consequently, we obtain an action of G on $\mathscr{S}$. We denote the corresponding action groupoid by $\mathscr{S}//G$ and the set of orbits by $\mathscr{S}/G$.

For a given orbit $\mathcal{O} \in \mathscr{S}/G$ and $s \in \mathcal{O}$, there is an isomorphism $\xi_g^s : g.X_s \longrightarrow X_{g.s}$. It is unique up to multiplication by an element in $\mathbb{C}^\times = \mathbb{C} \setminus \{0\}$ since $X_{g.s}$ is simple, and we fix such an isomorphism. Now for $g, h \in G$ we find $\xi_g^{h.s} \circ g.\xi_h^s \circ \left(\beta_{gh}^{X_s}\right)^{-1} = \alpha_{gh}^s \xi_{gh}^s$ for some $\alpha_{gh}^s \in \mathbb{C}^\times$ since $X_{(gh).s} = X_{g.h.s}$ is simple. The scalars α_{gh}^s form a cocycle $\alpha_\mathcal{O} \in Z^2(G; \mathrm{Map}(\mathcal{O}, \mathbb{C}^\times))$ with coefficients in the Abelian group of functions $\mathcal{O} \longrightarrow \mathbb{C}^\times$, and the class $[\alpha_\mathcal{O}] \in H^2(G; \mathrm{Map}(\mathcal{O}, \mathbb{C}^\times))$ does not depend on the chosen isomorphisms ξ_g^s. This cocycle (or its class) is used to define the *twisted group algebra* $\mathfrak{A}_{\alpha_\mathcal{O}}(G, \mathcal{O})$, a semisimple finite-dimensional complex algebra, see [Kir01] and references therein. Using the category of modules over these twisted group algebras we can state a version of the following result of [Kir01]:

Proposition 3.8 ([Kir01, Theorem 3.5]). *Let $\varrho : \star//G \longrightarrow \mathsf{2Vect}$ be a 2-vector bundle, i.e. a representation of the group G on a 2-vector space $\mathcal{V} := \varrho(\star)$. Then there is an equivalence*

$$\mathsf{Par}\, \varrho \simeq \bigoplus_{\mathcal{O} \in \mathscr{S}/G} \mathfrak{A}_{\alpha_\mathcal{O}}(G, \mathcal{O})\text{-}\mathsf{Mod}$$

of Abelian categories. Hence if G is finite, $\mathsf{Par}\, \varrho$ is semisimple with finitely many simple objects and hence a 2-vector space.

Since

$$\mathrm{Map}(\mathscr{S}, \mathbb{C}^\times) \cong \mathrm{Map}\left(\coprod_{\mathcal{O} \in \mathscr{S}/G} \mathcal{O}, \mathbb{C}^\times\right) \cong \prod_{\mathcal{O} \in \mathscr{S}/G} \mathrm{Map}(\mathcal{O}, \mathbb{C}^\times) \ ,$$

we can combine the cocycles $\alpha_\mathcal{O}$ with coefficients in $\mathrm{Map}(\mathcal{O}, \mathbb{C}^\times)$ into a cocycle α with coefficients in $\mathrm{Map}(\mathscr{S}, \mathbb{C}^\times)$. In the sequel, we will rather use the cocycle α and write $\mathfrak{A}_\alpha(G, \mathcal{O})\text{-}\mathsf{Mod}$ instead of $\mathfrak{A}_{\alpha_\mathcal{O}}(G, \mathcal{O})\text{-}\mathsf{Mod}$ (the dependence on the orbit is still present in the notation, so there is no risk of confusion).

Summarizing and extending to non-connected groupoids, we conclude that for an essentially finite groupoid Γ taking parallel sections of a $\mathsf{2Vect}$-valued 2-vector bundle ϱ over Γ produces a 2-vector space $\mathsf{Par}\, \varrho$, which is entirely determined by the action groupoid $\mathscr{S}//\Gamma$ and a gerbe on Γ, i.e. a class $H^2(\Gamma; \mathrm{Map}(\mathscr{S}, \mathbb{C}^\times))$ with coefficients in the Abelian group of functions $\mathscr{S} \longrightarrow \mathbb{C}^\times$. Obviously, taking parallel sections is also 2-functorial. Hence, together with Proposition 3.8, we find:

Proposition 3.9. *Taking parallel sections of 2-vector bundles over an essentially finite groupoid Γ naturally extends to a 2-functor*

$$\mathsf{Par}_\Gamma : \mathsf{2VecBun}(\Gamma) \longrightarrow \mathsf{2Vect} \ .$$

The image of a 1-morphism λ or a 2-morphism η will be denoted by λ_* or η_*, respectively.

We should emphasize that this is 'just' the parallel section functor for 2-vector bundles over *one fixed groupoid.* It is *not* the parallel section functor we intend to construct in this chapter.

3.2 Pullback and pushforward

The construction of the parallel section functor in Section 3.3 relies on certain pullback and pushforward maps that we will introduce in this section on two different categorical levels.

3.2.1 Pullback and pushforward 1-morphisms

Just like ordinary bundles, 2-vector bundles have an obvious notion of pullback: For any functor $\Phi : \Gamma \longrightarrow \Omega$ between groupoids, we obtain a pullback functor

$$\Phi^* : \mathsf{2VecBun}(\Omega, \mathcal{S}) \longrightarrow \mathsf{2VecBun}(\Gamma, \mathcal{S})$$

by precomposition.

Additionally, for any 2-vector bundle ϱ over Ω, we get a pullback 1-morphism in $\mathcal{S}$

$$\Phi^* : \mathsf{Par}\, \varrho \longrightarrow \mathsf{Par}\, \Phi^* \varrho, \quad s \longmapsto \Phi^* s$$

denoted by the same symbol and given by

$$(\Phi^* s)(x) := s(\Phi(x)) \quad \text{for all} \quad x \in \Gamma$$

together with the isomorphisms

$$(\Phi^* s)(y) = s(\Phi(y)) \cong \Phi(g).s(\Phi(x)) = (\Phi^* \varrho)(g)(\Phi^* s)(x) \quad \text{for all} \quad g : x \longrightarrow y \quad \text{in } \Gamma.$$

It is now easy to prove the following statements:

Proposition 3.10 (Pullback 1-morphism)**.** *Let $\Phi : \Gamma \longrightarrow \Omega$ be a functor between groupoids.*

(a) *Contravariance: If $\Psi : \Omega \longrightarrow \Lambda$ is another functor, then we have $(\Psi \circ \Phi)^* = \Phi^* \circ \Psi^*$ for both the functors as well as the pullback 1-morphisms induced by them.*
(b) *If $\eta : \Phi \longrightarrow \Phi'$ is a natural transformation of functors $\Gamma \longrightarrow \Omega$, then for any 2-vector bundle ϱ over Ω, we obtain a 1-isomorphism $\varrho(\eta) : \Phi^* \varrho \longrightarrow \Phi'^* \varrho$.*
(c) *Naturality: The pull maps are natural in the sense that for any 1-morphism $\lambda : \varrho \longrightarrow \xi$ of 2-vector bundles over Ω the square*

$$\begin{tikzcd}
\mathsf{Par}\, \varrho \arrow[r, "\Phi^*"] \arrow[d, "\lambda_*"'] & \mathsf{Par}\, \Phi^* \varrho \arrow[d, "(\Phi^* \lambda)_*"] \\
\mathsf{Par}\, \xi \arrow[r, "\Phi^*"] & \mathsf{Par}\, \Phi^* \xi
\end{tikzcd}$$

commutes strictly. The vertical arrows are the images of 2-vector bundle morphisms under the functor from Proposition 3.9.

We define the pushforward 1-morphisms via the limit of diagrams with shape of a homotopy fiber and values in spaces of 1-morphisms:

Definition 3.11 (Pushforward 1-morphism)**.** Let $\mathcal{S}$ be a symmetric monoidal bicategory with complete categories of 1-morphisms between any two objects. Let $\Phi : \Gamma \longrightarrow \Omega$ be a functor between groupoids. For an $\mathcal{S}$-valued 2-vector bundle ϱ over Ω and $s \in \mathsf{Par}\, \Phi^* \varrho$ we define the parallel section $\Phi_* s \in \mathsf{Par}\, \varrho$ by

$$(\Phi_* s)(y) := \lim_{(x,g) \in \Phi^{-1}[y]} g.s(x) \quad \text{for all} \quad y \in \Omega\ .$$

The limit is taken in $\mathrm{Hom}_{\mathcal{S}}(\mathbb{I}, \varrho(y))$ and has the shape of the homotopy fiber $\Phi^{-1}[y]$ of Φ over $y \in \Omega$. We call the resulting 2-linear map

$$\Phi_* : \mathsf{Par}\, \Phi^* \varrho \longrightarrow \mathsf{Par}\, \varrho, \quad s \longmapsto \Phi_* s$$

pushforward 1-morphism.

Remark 3.12. (a) An easy computation shows that Φ_* actually takes values in parallel sections.
(b) In the special case $\mathcal{S} = \mathsf{2Vect}$, the functor

$$\Phi^{-1}[y] \longrightarrow \mathrm{Hom}_{\mathsf{2Vect}}(\mathsf{FinVect}, \varrho(y)) \simeq \varrho(y) \ , \quad (x,g) \longmapsto g.s(x)$$

is a $\varrho(y)$-valued vector bundle over the homotopy fiber $\Phi^{-1}[y]$ of Φ over y. Its limit, by definition, coincides with $(\Phi_* s)(y)$ and is the space of parallel sections of this vector bundle as recalled in Section 3.1.1.

As an immediate consequence of the pasting law for homotopy pullbacks, we obtain:

Lemma 3.13. *For composable functors $\Phi : \Gamma \longrightarrow \Omega$ and $\Psi : \Omega \longrightarrow \Lambda$ between groupoids, there is a canonical equivalence*

$$(\Psi \circ \Phi)^{-1}[z] \simeq \Psi^{-1}[z] \times_\Omega \Gamma$$

for every $z \in \Lambda$.

Proposition 3.14. *Let $\Phi : \Gamma \longrightarrow \Omega$ and $\Psi : \Omega \longrightarrow \Lambda$ be composable functors between groupoids.*

(a) *Covariance: The pushforward 1-morphisms obey the composition law $(\Psi \circ \Phi)_* \cong \Psi_* \circ \Phi_*$ by a canonical isomorphism.*
(b) *Naturality: The pushforward 1-morphisms are natural in the sense that for any 1-morphism $\lambda : \varrho \longrightarrow \xi$ of 2-vector bundles over Ω the square*

$$\begin{tikzcd}
\mathsf{Par}\, \Phi^* \varrho \arrow[r, "\Phi_*"] \arrow[d, "(\Phi^*\lambda)_*"'] & \mathsf{Par}\, \varrho \arrow[d, "\lambda_*"] \arrow[dl, Rightarrow, "\cong"'] \\
\mathsf{Par}\, \Phi^* \xi \arrow[r, "\Phi_*"'] & \mathsf{Par}\, \xi
\end{tikzcd}$$

commutes up to a canonical natural isomorphism arising from the coherence isomorphism that λ comes equipped with.
(c) *The naturality isomorphisms and the composition of pushforward 1-morphisms are compatible in the sense that for a 1-morphism $\lambda : \varrho \longrightarrow \xi$ of 2-vector bundles over Λ we have the equality of 2-isomorphisms*

$$
\begin{tikzcd}
\mathsf{Par}\, \Phi^*\Psi^*\varrho \arrow[r, "\Phi_*"] \arrow[d, "(\Phi^*\Psi^*\lambda)_*"'] \arrow[rr, bend left=60, "(\Psi\circ\Phi)_*"] \arrow[rr, phantom, bend left=35, "\cong"] & \mathsf{Par}\, \Psi^*\varrho \arrow[r, "\Psi_*"] \arrow[d, "(\Psi^*\lambda)_*"'] \arrow[dl, phantom, "\cong"] & \mathsf{Par}\, \varrho \arrow[d, "\lambda_*"] \arrow[dl, phantom, "\cong"] & = & \mathsf{Par}\, \Phi^*\Psi^*\varrho \arrow[r, "(\Psi\circ\Phi)_*"] \arrow[d, "(\Phi^*\Psi^*\lambda)_*"'] & \mathsf{Par}\, \varrho \arrow[d, "\lambda_*"] \arrow[dl, phantom, "\cong"] \\
\mathsf{Par}\, \Phi^*\Psi^*\xi \arrow[r, "\Phi_*"'] \arrow[rr, bend right=60, "(\Psi\circ\Phi)_*"'] \arrow[rr, phantom, bend right=35, "\cong"] & \mathsf{Par}\, \Psi^*\xi \arrow[r, "\Psi_*"'] & \mathsf{Par}\, \xi & & \mathsf{Par}\, \Phi^*\Psi^*\xi \arrow[r, "(\Psi\circ\Phi)_*"'] & \mathsf{Par}\, \xi \ .
\end{tikzcd}
$$

Proof. For a parallel section s of a 2-vector bundle ϱ over Λ and $z \in \Lambda$, we find by definition

$$
\begin{aligned}
(\Psi_*\Phi_* s)(z) &\cong \lim_{(y,h)\in\Psi^{-1}[z]} \lim_{(x,g)\in\Phi^{-1}[y]} (h\Psi(g)).s(x) \ , \qquad (3.7)\\
((\Psi\circ\Phi)_* s)(z) &= \lim_{(x,k)\in(\Psi\circ\Phi)^{-1}[z]} k.s(x) \ .
\end{aligned}
$$

The double limit (3.7) can be seen as a limit over the homotopy pullback $\Psi^{-1}[z] \times_\Omega \Gamma$, which by Lemma 3.13 is canonically equivalent to $(\Psi\circ\Phi)^{-1}[z] \simeq \Psi^{-1}[z] \times_\Omega \Gamma$. This equivalence yields the needed isomorphism $(\Psi\circ\Phi)_* \cong \Psi_* \circ \Phi_*$ by Lemma 3.1. The remaining assertions can be directly verified. □

3.2.2 Pullback and pushforward 2-morphisms and the equivariant Beck-Chevalley condition

So far, we have established pullback and pushforward 1-morphisms. In the next step, we will provide pull and push 2-morphisms between different pull-push combinations.

We consider a weakly commuting square

$$
\begin{tikzcd}
\Pi \arrow[r, "P"] \arrow[d, "Q"'] & \Gamma \arrow[d, "\Phi"] \arrow[dl, Rightarrow, "\eta"'] \\
\Lambda \arrow[r, "\Psi"'] & \Omega
\end{tikzcd}
$$

of essentially finite groupoids. By the definition of the homotopy fiber $Q^{-1}[y]$ we obtain a weakly commutative square

$$
\begin{tikzcd}
Q^{-1}[y] \arrow[r] \arrow[d] & \Pi \arrow[r, "P"] \arrow[d, "Q"'] & \Gamma \arrow[d, "\Phi"] \arrow[dl, Rightarrow, "\eta"'] \\
\star \arrow[r, "y"] & \Lambda \arrow[r, "\Psi"] & \Omega
\end{tikzcd}
$$

(the natural isomorphism being part of the square for the homotopy fiber is suppressed in the notation). The universal property of the homotopy fiber $\Phi^{-1}[\Psi(y)]$ gives us a functor F :

$Q^{-1}[y] \longrightarrow \Phi^{-1}[\Psi(y)]$, which is explicitly given by

$$F(z,g) := (P(z), \Psi(g)\eta_z) \quad \text{for all} \quad (z,g) \in Q^{-1}[y] \; . \tag{3.8}$$

Now for any 2-vector bundle ϱ over Ω taking values in 2-vector spaces, $s \in \mathsf{Par}\, \Phi^* \varrho$ and $y \in \Lambda$, we define the vector bundle with values in the 2-vector space $\varrho(\Psi(y))$

$$\xi : \Phi^{-1}[\Psi(y)] \longrightarrow \varrho(\Psi(y)) \; , \quad (x,h) \longmapsto \varrho(h)s(x) = h.s(x) \; .$$

We should emphasize that ξ is an ordinary vector bundle, so we obtain a pullback map

$$F^* : \mathsf{Par}\, \xi \longrightarrow \mathsf{Par}\, F^* \xi \; , \tag{3.9}$$

and a pushforward map

$$F_* : \mathsf{Par}\, F^* \xi \longrightarrow \mathsf{Par}\, \xi \tag{3.10}$$

by the constructions recalled in Section 3.1.1. Of course, the auxiliary objects ξ and F depend on all the functors involved and on s and y, which is suppressed in the notation. Next observe

$$\mathsf{Par}\, \xi = \lim_{(x,h) \in \Phi^{-1}[\Psi(y)]} \varrho(h)s(x) = (\Psi^* \Phi_* s)(y)$$

and

$$\mathsf{Par}\, F^* \xi = \lim_{(z,g) \in Q^{-1}[y]} \varrho(\Psi(g)\eta_z)s(P(z)) = (Q_* \varrho(\eta) P^* s)(y) \; .$$

Restoring the previously suppressed dependence on s and y we obtain maps

$$(\eta^*)^y_s := F^* : (\Psi^* \Phi_* s)(y) \longrightarrow (Q_* \varrho(\eta)_* P^* s)(y) \tag{3.11}$$

and

$$(\eta_*)^y_s := F_* : (Q_* \varrho(\eta)_* P^* s)(y) \longrightarrow (\Psi^* \Phi_* s)(y) \; . \tag{3.12}$$

If we let s and y run over all parallel sections of $\Phi^* \varrho$ and all objects of Λ, respectively, they combine into the following natural transformations:

Proposition 3.15. *Consider a weakly commuting square*

$$\begin{tikzcd} \Pi \arrow[r, "P"] \arrow[d, "Q"'] & \Gamma \arrow[d, "\Phi"] \arrow[dl, Rightarrow, "\eta"'] \\ \Lambda \arrow[r, "\Psi"'] & \Omega \end{tikzcd}$$

of essentially finite groupoids and a 2-vector bundle ϱ over Ω. Then we have two natural *transformations*

$$\begin{tikzcd}[column sep=huge] \mathsf{Par}\,\Phi^*\varrho \arrow[r, bend left=60, "\Psi^*\Phi_*"] \arrow[r, bend right=60, "Q_*\varrho(\eta)_*P^*"'] \arrow[r, phantom, "\eta^* \Downarrow \quad \Uparrow \eta_*"] & \mathsf{Par}\,\Psi^*\varrho \end{tikzcd}$$

of 2-linear functors by the construction given above. We call η^* the pull map and η_* the push map.

We will need the special case in which the square in Proposition 3.15 is a homotopy pullback square. For the investigation of this special case, we need the following easy Lemma:

Lemma 3.16. *Let $\Phi : \Gamma \longrightarrow \Omega$ be an equivalence of groupoids, then $\Phi^{-1}[y]$ is equivalent to the groupoid consisting of one object (x, g), where $x \in \Gamma$ and $g : \Phi(x) \longrightarrow y$, and trivial automorphism group.*

Now if the square in Proposition 3.15 is a homotopy pullback square, then the functor F from (3.8) is an equivalence by the pasting law for homotopy pullbacks. Using Lemma 3.16 we can deduce that in this case F^* from (3.9) is inverse to F_* from (3.10). Interpreting this in terms of η^* and η_*, see (3.11) and (3.12), we obtain a generalization of the equivariant Beck-Chevalley (Proposition 3.3) to 2-vector bundles:

Proposition 3.17 (Equivariant Beck-Chevalley condition for 2-vector bundles)**.** *For the homotopy pullback*

$$\begin{tikzcd} \Gamma \times_\Omega \Lambda \arrow[r, "\pi_\Gamma"] \arrow[d, "\pi_\Lambda"'] & \Gamma \arrow[d, "\Phi"] \arrow[dl, Rightarrow, "\eta"'] \\ \Lambda \arrow[r, "\Psi"'] & \Omega \end{tikzcd}$$

of a cospan $\Lambda \xrightarrow{\Psi} \Omega \xleftarrow{\Phi} \Gamma$ of essentially finite groupoids and any 2-vector bundle ϱ over Ω, the pentagon relating different pull-push combinations

$$
\begin{tikzcd}
\mathsf{Par}\,\Phi^*\varrho \arrow[rr, "\Phi_*"] \arrow[d, "\pi_\Gamma^*"'] & & \mathsf{Par}\,\varrho \arrow[dd, "\Psi^*"] \arrow[ddl, phantom, "\Swarrow \eta^*"] \\
\mathsf{Par}\,\pi_\Gamma^*\Phi^*\varrho \arrow[dr, "\varrho(\eta)_*"'] & & \\
& \mathsf{Par}\,\pi_\Lambda^*\Psi^*\varrho \arrow[r, "\pi_{\Lambda *}"] & \mathsf{Par}\,\Psi^*\varrho
\end{tikzcd}
$$

commutes up to the natural isomorphism

$$\eta^* : \Psi^*\Phi_* \Longrightarrow \pi_{\Lambda *}\varrho(\eta)_*\pi_\Gamma^*$$

with inverse

$$\eta_* : \pi_{\Lambda *}\varrho(\eta)_*\pi_\Gamma^* \Longrightarrow \Psi^*\Phi_* \ .$$

For later purposes, we work out a special case of Proposition 3.15. To this end, let us look at the pushforward maps from Section 3.1.1: Let V be an object in a 2-vector space $\mathcal{V}$ and Γ an essentially finite groupoid. Then there is a constant vector bundle ξ_V assigning V to all objects in Γ and the identity on V to all morphisms. Obviously, $\mathsf{Par}\,\xi_V$, i.e. the limit of the functor ξ_V, is given by the product $\prod_{\pi_0(\Gamma)} V$, but this is also a coproduct, hence

$$\mathsf{Par}\,\xi_V = \coprod_{\pi_0(\Gamma)} V \ .$$

Now note that $\xi_V = t^*V$, where $t : \Gamma \longrightarrow \star$ is the functor to the terminal groupoid $\star$ and V is the object seen as vector bundle over $\star$. The pushforward along t is now a map

$$\int_\Gamma := t_* : \coprod_{\pi_0(\Gamma)} V \longrightarrow V$$

that we call integral with respect to groupoid cardinality, see also page 28. Recalling the definition of the pushforward (Section 3.1.1), we see that on the summand belonging to $[x] \in \pi_0(\Gamma)$ it is given by the endomorphism

$$\frac{1}{|\mathrm{Aut}(x)|} \cdot \mathrm{id}_V : V \longrightarrow V \ .$$

Corollary 3.18. *Consider a weakly commuting square*

$$
\begin{tikzcd}
\Pi \arrow[r, "P"] \arrow[d, "Q"'] & \Gamma \arrow[d, "\mathrm{id}_\Gamma"] \arrow[dl, phantom, "\Swarrow \eta"] \\
\Lambda \arrow[r, "\Psi"'] & \Gamma
\end{tikzcd}
$$

of essentially finite groupoids and a given 2-vector bundle ϱ over Γ. Then the natural transfor-

mation

$$\eta_* : Q_* \varrho(\eta)_* P^* \Longrightarrow \Psi^*$$

admits the following explicit description: For $s \in \mathsf{Par}\,\varrho$ and $y \in \Lambda$ we obtain the commuting diagram

$$\begin{tikzcd}
(Q_* \varrho(\eta)_* P^* s)(y) \arrow[r, "\cong"] \arrow[dr, "\eta_*"'] & \coprod_{\pi_0(Q^{-1}[y])} s(\Psi(y)) \arrow[d, "\int_{Q^{-1}[y]}"] \\
& (\Psi^* s)(y) = s(\Psi(y)) \ ,
\end{tikzcd}$$

where $\cong$ denotes a natural isomorphism to $\coprod_{\pi_0(Q^{-1}[y])} s(\Psi(y))$. This expresses η_* as an integral with respect to groupoid cardinality.

Proof. We use that s is parallel to see

$$(Q_* \varrho(\eta) P^* s)(y) = \lim_{(z,g) \in Q^{-1}[y]} \varrho(\Psi(g)\eta) s(P(z)) \cong \lim_{(z,g) \in Q^{-1}[y]} s(\Psi(y)) \ .$$

The limit of this last constant diagram is given by the finite product $\prod_{\pi_0(Q^{-1}[y])} s(\Psi(y))$, which coincides with the finite coproduct $\coprod_{\pi_0(Q^{-1}[y])} s(\Psi(y))$ since $s(\Psi(y))$ is an object in a 2-vector space. By definition and Lemma 3.16 the component $(Q_* \varrho(\eta) P^* s)(y) \longrightarrow s(\Psi(y))$ of η_* is the pushforward map along the functor $Q^{-1}[y] \longrightarrow \Psi(y)$, where $\Psi(y)$ is the discrete groupoid with one object $\Psi(y)$. This implies the claim. $\square$

3.3 The parallel section functor on the symmetric monoidal bicategory 2VecBunGrpd

In this section, we formulate and prove the main result of this chapter: We introduce the symmetric monoidal bicategory 2VecBunGrpd, which is built from 2-vector bundles and spans of groupoids (Section 3.3.1) and construct the parallel section functor

$$\mathsf{Par} : \mathsf{2VecBunGrpd} \longrightarrow \mathsf{2Vect}$$

with values in 2-vector spaces.

3.3.1 The symmetric monoidal bicategory 2VecBunGrpd

For the definition of the parallel section functor, we need to introduce the symmetric monoidal bicategory 2VecBunGrpd. Objects are 2-vector bundles over essentially finite groupoids. The 1-morphisms and 2-morphisms come from spans of groupoids and spans of spans of groupoids decorated with (higher) intertwiners.

Definition 3.19 (The symmetric monoidal bicategory 2VecBunGrpd)**.** We define the symmetric monoidal bicategory 2VecBunGrpd as follows:

1. Objects are 2-vector bundles over essentially finite groupoids, i.e. pairs (Γ, ϱ) where ϱ is a 2-vector bundle over an essentially finite groupoid Γ.
2. A 1-morphism $(\Gamma_0, \varrho_0) \longrightarrow (\Gamma_1, \varrho_1)$ is a span

$$\Gamma_0 \xleftarrow{r_0} \Lambda \xrightarrow{r_1} \Gamma_1$$

of essentially finite groupoids together with an intertwiner

$$\lambda : r_0^*\varrho_0 \longrightarrow r_1^*\varrho_1 \ ,$$

i.e. a 1-morphism of the 2-vector bundles $r_0^*\varrho_0$ and $r_1^*\varrho_1$ over Λ. We denote such a 1-morphism by $(\Gamma_0, \varrho_0) \xleftarrow{r_0} (\Lambda, \lambda) \xrightarrow{r_1} (\Gamma_1, \varrho_1)$.

3. A 2-morphism from $(\Gamma_0, \varrho_0) \xleftarrow{r_0} (\Lambda, \lambda) \xrightarrow{r_1} (\Gamma_1, \varrho_1)$ to $(\Gamma_0, \varrho_0) \xleftarrow{r_0'} (\Lambda, \lambda) \xrightarrow{r_1'} (\Gamma_1, \varrho_1)$ is an equivalence class (as explained in Remark 3.20 (b) below) of
   - a span of spans, i.e. a diagram

$$\begin{tikzcd}
& & \Lambda \arrow[dll, "r_0"'] \arrow[drr, "r_1"] & & \\
\Gamma_0 & & \Omega \arrow[u, "t"'] \arrow[d, "t'"] \arrow[ll, Rightarrow, "\alpha_0"'] \arrow[rr, Rightarrow, "\alpha_1"] & & \Gamma_1 \\
& & \Lambda' \arrow[ull, "r_0'"] \arrow[urr, "r_1'"'] & &
\end{tikzcd}$$

     in essentially finite groupoids commutative up to the indicated natural isomorphisms (in this way of presentation, the direction of the natural isomorphism is obtained by reading from top to bottom, e.g. α_0 is a natural isomorphism $r_0t \Rightarrow r_0't'$).
   - together with a natural transformation

$$\begin{tikzcd}
(r_0t)^*\varrho_0 = t^*r_0^*\varrho_0 \arrow[r, "t^*\lambda"] \arrow[d, "\varrho_0(\alpha_0)"'] & t^*r_1^*\varrho_1 = (r_1t)^*\varrho_1 \arrow[d, "\varrho_1(\alpha_1)"] \arrow[dl, Rightarrow, "\omega"'] \\
(r_0't')^*\varrho_0 = t'^*r_0'^*\varrho_0 \arrow[r, "t'^*\lambda'"] & t'^*r_1'^*\varrho_1 = (r_1't')^*\varrho_1
\end{tikzcd}$$

     We will denote this 2-morphism by

$$\begin{tikzcd}
& & (\Lambda, \lambda) \arrow[dll, "r_0"'] \arrow[drr, "r_1"] & & \\
(\Gamma_0, \varrho_0) & & (\Omega, \omega) \arrow[u, "t"'] \arrow[d, "t'"] \arrow[ll, Rightarrow, "\alpha_0"'] \arrow[rr, Rightarrow, "\alpha_1"] & & (\Gamma_1, \varrho_1) \\
& & (\Lambda', \lambda') \arrow[ull, "r_0'"] \arrow[urr, "r_1'"'] & &
\end{tikzcd}$$

4. Composition of 1-morphisms: For 1-morphisms

$$(\Gamma_0, \varrho_0) \xleftarrow{r_0} (\Lambda, \lambda) \xrightarrow{r_1} (\Gamma_1, \varrho_1)$$

   and

$$(\Gamma_1, \varrho_1) \xleftarrow{r_1'} (\Lambda', \lambda') \xrightarrow{r_2'} (\Gamma_2, \varrho_2)$$

   the composition $(\Lambda', \lambda') \circ (\Lambda, \lambda)$ is by homotopy pullback. More precisely, the span part of the composition is the outer span of

$$\begin{tikzcd}
 & & \varLambda \times_{\varOmega} \varLambda' \arrow[dl, "\pi"'] \arrow[dr, "\pi'"] & & \\
 & \varLambda \arrow[rr, Rightarrow, "\eta"] \arrow[dl, "r_0"'] \arrow[dr, "r_1"] & & \varLambda' \arrow[dl, "r'_1"'] \arrow[dr, "r'_2"] & \\
\varGamma & & \varGamma_1 & & \varGamma_2 \ ,
\end{tikzcd}$$

where η is the natural transformation the homotopy pullback comes equipped with, together with the intertwiner

$$\lambda \times_{\varGamma_1} \lambda' : \pi^* r_0^* \varrho_0 \xrightarrow{\pi^* \lambda} \pi^* r_1^* \varrho_1 \xrightarrow{\varrho_1(\eta)} {\pi'}^* {r'_1}^* \varrho_1 \xrightarrow{{\pi'}^* \lambda'} {\pi'}^* {r'_2}^* \varrho_2 \ .$$

5. Vertical composition of 2-morphisms: The vertical composition of the 2-morphisms

$$\begin{tikzcd}
 & (\varLambda, \lambda) \arrow[dl, "r_0"'] \arrow[dr, "r_1"] & \\
(\varGamma_0, \varrho_0) & (\varOmega, \omega) \arrow[u, "t"'] \arrow[d, "t'"] \arrow[l, Rightarrow, "\alpha_0"'] \arrow[r, Rightarrow, "\alpha_1"] & (\varGamma_1, \varrho_1) \\
 & (\varLambda', \lambda') \arrow[ul, "r'_0"] \arrow[ur, "r'_1"'] &
\end{tikzcd}$$

and

$$\begin{tikzcd}
 & (\varLambda', \lambda') \arrow[dl, "r'_0"'] \arrow[dr, "r'_1"] & \\
(\varGamma_0, \varrho_0) & (\widetilde{\varOmega}, \widetilde{\omega}) \arrow[u, "u'"'] \arrow[d, "u''"] \arrow[l, Rightarrow, "\beta_0"'] \arrow[r, Rightarrow, "\beta_1"] & (\varGamma_1, \varrho_1) \\
 & (\varLambda'', \lambda'') \arrow[ul, "r''_0"] \arrow[ur, "r''_1"'] &
\end{tikzcd}$$

is the 2-morphism

$$\begin{tikzcd}
 & (\varLambda, \lambda) \arrow[dl, "r_0"'] \arrow[dr, "r_1"] & \\
(\varGamma_0, \varrho_0) & (\varOmega \times_{\varLambda'} \widetilde{\varOmega}, \omega \times_{\lambda'} \widetilde{\omega}) \arrow[u, "v"'] \arrow[d, "v''"] \arrow[l, Rightarrow, "\gamma_0"'] \arrow[r, Rightarrow, "\gamma_1"] & (\varGamma_1, \varrho_1) \\
 & (\varLambda'', \lambda'') \arrow[ul, "r''_0"] \arrow[ur, "r''_1"'] &
\end{tikzcd} \ , \qquad (3.13)$$

whose components are defined as follows:

- $\Omega \times_{\Lambda'} \widetilde{\Omega}$ denotes the homotopy pullback

$$\begin{tikzcd}
\Omega \times_{\Lambda'} \widetilde{\Omega} \arrow[r, "q"] \arrow[d, "\widetilde{q}"'] \arrow[dr, phantom, "\swarrow \eta"] & \Omega \arrow[d, "t'"] \\
\widetilde{\Omega} \arrow[r, "u'"'] & \Lambda'
\end{tikzcd}$$

  (this introduces q, $\widetilde{q}$ and η),

- the functors v and v'' in (3.13) are defined by $v := tq$ and $v'' := u''\widetilde{q}$,

- the natural transformations γ_0 and γ_1 in (3.13) are obtained by

$$\gamma_0 : r_0 v = r_0 tq \overset{\alpha_0}{\Longrightarrow} r_0' t' q \overset{\eta}{\Longrightarrow} r_0' u' \widetilde{q} \overset{\beta_0}{\Longrightarrow} r_0'' u'' \widetilde{q} = r_0'' v''$$

  and

$$\gamma_1 : r_1 v = r_1 tq \overset{\alpha_1}{\Longrightarrow} r_1' t' q \overset{\eta}{\Longrightarrow} r_1' u' \widetilde{q} \overset{\beta_1}{\Longrightarrow} r_1'' u'' \widetilde{q} = r_1'' v'' \ ,$$

  where identity transformations are suppressed in the notation,

- and the natural morphism $\omega \times_{\lambda'} \widetilde{\omega}$ is obtained as the composition

$$\begin{tikzcd}
(r_0 v)^* \varrho_0 = q^* t^* r_0^* \varrho_0 \arrow[r, "q^* t^* \lambda"] \arrow[d, "q^* \varrho_0(\alpha_0)"'] \arrow[dr, phantom, "\swarrow q^*\omega"] & q^* t^* r_1^* \varrho_1 = (r_1 v)^* \varrho_1 \arrow[d, "q^* \varrho_1(\alpha_1)"] \\
q^* t'^* r_0'^* \varrho_0 \arrow[r, "q^* t'^* \lambda'"] \arrow[d, "\varrho_0(\eta)"'] \arrow[dr, phantom, "\swarrow \theta'"] & q^* t'^* r_1'^* \varrho_1 \arrow[d, "\varrho_1(\eta)"] \\
\widetilde{q}^* u'^* r_0'^* \varrho_0 \arrow[r, "\widetilde{q}^* u'^* \lambda'"] \arrow[d, "\widetilde{q}^* \varrho_0(\beta_0)"'] \arrow[dr, phantom, "\swarrow \widetilde{q}^*\omega'"] & \widetilde{q}^* u'^* r_1'^* \varrho_1 \arrow[d, "\widetilde{q}^* \varrho_1(\beta_1)"] \\
\widetilde{q}^* u''^* r_0''^* \varrho_0 \arrow[r, "\widetilde{q}^* u''^* \lambda''"] & \widetilde{q}^* u''^* r_1''^* \varrho_1 \ ,
\end{tikzcd}$$

  where the middle square is decorated with the isomorphism θ' belonging to λ' (Remark 3.5 (c)), i.e. the evaluation of the 2-morphism in the middle square on $(z, \widetilde{z}, g) \in \Omega \times_{\Lambda'} \widetilde{\Omega}$ is given by

$$\begin{tikzcd}
\varrho_0(r_0' t'(z)) \arrow[r, "\lambda'_{t'(z)}"] \arrow[d, "\varrho_0(r_0'(g))"'] \arrow[dr, phantom, "\swarrow \theta_g"] & \varrho_1(r_1' t'(z)) \arrow[d, "\varrho_1(r_1'(g))"] \\
\varrho_0(r_0' u'(\widetilde{z})) \arrow[r, "\lambda'_{u'(\widetilde{z})}"'] & \varrho_1(r_1' u'(\widetilde{z})) \ .
\end{tikzcd}$$

6. Horizontal composition of 2-morphisms: The horizontal composition of the 2-morphisms

$$\begin{tikzcd}
 & (\Lambda,\lambda) \arrow[dl, "r_0"'] \arrow[dr, "r_1"] & \\
(\Gamma_0,\varrho_0) & (\Omega,\omega) \arrow[u, "t"'] \arrow[d, "t'"] \arrow[l, Rightarrow, "\alpha_0"'] \arrow[r, Rightarrow, "\alpha_1"] & (\Gamma_1,\varrho_1) \\
 & (\Lambda',\lambda') \arrow[ul, "r_0'"] \arrow[ur, "r_1'"'] &
\end{tikzcd}$$

and

$$\begin{tikzcd}
 & (\Pi,\pi) \arrow[dl, "v_1"'] \arrow[dr, "v_2"] & \\
(\Gamma_1,\varrho_1) & (\widetilde{\Omega},\widetilde{\omega}) \arrow[u, "u"'] \arrow[d, "u'"] \arrow[l, Rightarrow, "\beta_0"'] \arrow[r, Rightarrow, "\beta_1"] & (\Gamma_2,\varrho_2) \\
 & (\Pi',\pi') \arrow[ul, "v_1'"] \arrow[ur, "v_2'"'] &
\end{tikzcd}$$

is

$$\begin{tikzcd}
 & (\Lambda\times_{\Gamma_1}\Pi,\lambda\times_{\Gamma_1}\pi) \arrow[dl, "r_0p"'] \arrow[dr, "v_2q"] & \\
(\Gamma_0,\varrho_0) & (\Omega\times_{\Gamma_1}\widetilde{\Omega},\omega\times_{\Gamma_1}\widetilde{\omega}) \arrow[u, "c"'] \arrow[d, "d"] \arrow[l, Rightarrow, "\delta_0"'] \arrow[r, Rightarrow, "\delta_1"] & (\Gamma_2,\varrho_2) \\
 & (\Lambda'\times_{\Gamma_1}\Pi',\lambda'\times_{\Gamma_1}\pi') \arrow[ul, "r_0'p'"] \arrow[ur, "v_2'q'"'] &
\end{tikzcd}\ ,$$

where

- $p:\Lambda\times_{\Gamma_1}\Pi\longrightarrow\Lambda$, $q:\Lambda\times_{\Gamma_1}\Pi\longrightarrow\Pi$, $p':\Lambda'\times_{\Gamma_1}\Pi'\longrightarrow\Lambda'$, $q':\Lambda'\times_{\Gamma_1}\Pi'\longrightarrow\Pi'$ are the projection functors defined on the respective homotopy pullbacks,

- $\Omega\times_{\Gamma_1}\widetilde{\Omega}$ denotes the homotopy pullback

$$\begin{tikzcd}
\Omega\times_{\Gamma_1}\widetilde{\Omega} \arrow[r, "\ell"] \arrow[d, "\widetilde{\ell}"'] & \Omega \arrow[d, "r_1t"] \arrow[dl, Rightarrow, "\varepsilon"'] \\
\widetilde{\Omega} \arrow[r, "v_1u"'] & \Gamma_1
\end{tikzcd}$$

  (this introduces ℓ, $\widetilde{\ell}$ and ε); note that the groupoid $\Omega\times_{\Gamma_1}\widetilde{\Omega}$ together with the natural isomorphism ε' given as the composition

$$\begin{tikzcd}
\varOmega \times_{\varGamma_1} \widetilde{\varOmega} \arrow[r, "\ell"] \arrow[d, "\widetilde{\ell}"'] & \varOmega \arrow[dl, Rightarrow, "\varepsilon"'] \arrow[d, "r_1 t"'] \arrow[d, bend left=80, "r_1' t'"] \arrow[d, phantom, shift left=5ex, "\overset{\alpha_1^{-1}}{\Longleftarrow}"] \\
\widetilde{\varOmega} \arrow[r, "v_1 u"] \arrow[r, bend right=60, "v_1' u'"'] \arrow[r, phantom, shift right=5ex, "\Downarrow \beta_0"] & \varGamma_1
\end{tikzcd}$$

is also a homotopy pullback of the cospan defined by the primed functors $r_1' t'$ and $v_1' u'$; we denote this homotopy pullback by $(\varOmega \times_{\varGamma_1} \widetilde{\varOmega})'$,

- the functor $c : \varOmega \times_{\varGamma_1} \widetilde{\varOmega} \longrightarrow \varLambda \times_{\varGamma_1} \varPi$ is defined using the universal property of the homotopy pullback and makes the diagram

$$\begin{tikzcd}
\varOmega \times_{\varGamma_1} \widetilde{\varOmega} \arrow[rr, "\ell"] \arrow[dr, "c"] \arrow[dd, "\widetilde{\ell}"'] & & \varOmega \arrow[d, "t"] \\
& \varLambda \times_{\varGamma_1} \varPi \arrow[r, "p"] \arrow[d, "q"'] & \varLambda \arrow[dl, Rightarrow, "\eta"'] \arrow[d, "r_1"] \\
\widetilde{\varOmega} \arrow[r, "u"] & \varPi \arrow[r, "v_1"'] & \varGamma_1
\end{tikzcd}$$

commute up to ε (the squares left blank are also labeled by natural isomorphisms arising from the universal property of the homotopy pullback, but we suppress them in the notation); note here that c is induced by the product functor $t \times u$,

- the functor $d : \varOmega \times_{\varGamma_1} \widetilde{\varOmega} \longrightarrow \varLambda' \times_{\varGamma_1} \varPi'$ is defined analogously using the universal property of the homotopy pullback and makes the diagram

$$\begin{tikzcd}
\varOmega \times_{\varGamma_1} \widetilde{\varOmega} \arrow[rr, "\ell"] \arrow[dr, "d"] \arrow[dd, "\widetilde{\ell}"'] & & \varOmega \arrow[d, "t'"] \arrow[dd, bend left=80, "r_1 t"] \\
& \varLambda' \times_{\varGamma_1} \varPi' \arrow[r, "p'"] \arrow[d, "q'"'] & \varLambda' \arrow[dl, Rightarrow, "\eta'"'] \arrow[d, "r_1'"] \arrow[r, phantom, "\overset{\alpha_1}{\Longleftarrow}"] & {} \\
\widetilde{\varOmega} \arrow[r, "u'"'] \arrow[rr, bend right=60, "v_1 u"'] & \varPi' \arrow[r, "v_1'"'] \arrow[d, phantom, "\Uparrow \beta_0"] & \varGamma_1 \\
& {} &
\end{tikzcd}$$

commute up to ε (again, the squares left blank are also labeled by natural isomorphisms arising from the universal property of the homotopy pullback, but we suppress them in the notation); note here that d arises from the product functor $t' \times u'$ and that the composition of the natural isomorphisms in the inner square is ε',

- $\lambda \times_{\Gamma_1} \pi$ is the intertwiner

$$\lambda \times_{\Gamma_1} \pi : (r_0 p)^* \varrho_0 \cong p^* r_0^* \varrho_0 \xrightarrow{p^*\lambda} p^* r_1^* \varrho_1 \xrightarrow{\varrho_1(\eta)} q^* v_1^* \varrho_1 \xrightarrow{q^*\pi} q^* v_2^* \varrho_2$$

  and $\lambda' \times_{\Gamma_1} \pi'$ is defined analogously (these definitions are recalled here for convenience, although they already follow from the definition of 1-morphisms, see Definition 3.19, 4),

- $\delta_0 : r_0 pc \Rightarrow r_0' p' d$ is the natural transformation

$$r_0 pc = r_0 t\ell \overset{\alpha_0}{\Longrightarrow} r_0' t' \ell = r_0' p' d,$$

  and, analogously, $\delta_1 : v_2 qc \Rightarrow v_2' q' d$ is the natural transformation

$$v_2 qc = v_2 u\widetilde{\ell} \overset{\beta_1}{\Longrightarrow} v_2' u' \widetilde{\ell} = v_2' q' d \ ,$$

- the natural transformation $\omega \times_{\Gamma_1} \widetilde{\omega}$ is defined by the commutativity of the diagram

$$\begin{tikzcd}
(r_0 pc)^* \varrho_0 \arrow[r, "(pc)^*\lambda"] \arrow[d, "\varrho_0(\delta_0) = \varrho_0(\ell^*\alpha_0)"'] & (r_1 pc)^* \varrho_1 \arrow[r, "\varrho_1(\varepsilon)"] \arrow[d, "\varrho_1(\ell^*\alpha_1)"'] \arrow[dl, Rightarrow, "\ell^*\omega"'] & (v_1 qc)^* \varrho_1 \arrow[r, "(qc)^*\pi"] \arrow[d, "\varrho_1(\widetilde{\ell}^*\beta_0)"'] & (v_2 qc)^* \varrho_2 \arrow[d, "\varrho_2(\widetilde{\ell}^*\beta_1) = \varrho_2(\delta_1)"] \arrow[dl, Rightarrow, "\widetilde{\ell}^*\widetilde{\omega}"'] \\
(r_0' p' d)^* \varrho_0 \arrow[r, "(p'd)^*\lambda'"] & (r_1' p' d)^* \varrho_1 \arrow[r, "\varrho_1(d^*\eta')"] & (v_1' q' d)^* \varrho_1 \arrow[r, "(q'd)^*\pi'"] & (v_2' q' d)^* \varrho_2 \ .
\end{tikzcd}$$

7. The symmetric monoidal structure is inherited from the symmetric monoidal structure of 2VecBun(Γ) for one fixed groupoid (Definition 3.4) and the symmetric monoidal structure on the bicategory of spans of essentially finite groupoids: For objects $X = (\Gamma, \varrho)$ and $Y = (\Omega, \xi)$ the monoidal product $X \boxtimes Y$ is the bundle over $\Gamma \times \Omega$ assigning to $(x, y) \in \Gamma \times \Omega$ to 2-vector space $\varrho(x) \boxtimes \xi(y)$. The monoidal product of 1- and 2-morphisms is defined analogously by using the Cartesian product of groupoids and the Deligne product. The monoidal unit object is the trivial representation of the groupoid $\star$ with one object and trivial automorphism group on the 2-vector space FinVect of finite-dimensional complex vector spaces; we denote it again by $\star$.

Remark 3.20. (a) The composition of 1-morphisms in 2VecBunGrpd requires a model for the homotopy pullback to be chosen. For definiteness, we choose the one given on page 29. Choosing a different model yields a composition naturally 2-isomorphic to the initial one.

(b) For readability, we did not spell out the equivalence relation needed to define 2-morphisms, but only worked with representatives. We justify this by the fact that in [Mor15] such issues were addressed for pure span bicategories (without 2-vector bundles) and in [SW19] it was explicitly explained in the categorical case how these equivalence relations have to be generalized to take vector bundles into account. This generalization can be done in the bicategorical case as well following the exact same strategy one categorical level higher.

Let us now explain the relation between the symmetric monoidal bicategory 2VecBunGrpd and the symmetric monoidal category VecBunGrpd from [SW19, Definition 3.7]:

Proposition 3.21. *The category* $\mathrm{End}_{\mathsf{2VecBunGrpd}}(\star)$ *of endomorphisms of the monoidal unit is canonically equivalent, as a symmetric monoidal category, to* VecBunGrpd.

Proof. The claim holds more or less by construction, so we only give the main arguments: Denote by $\tau : \star \longrightarrow \mathsf{FinVect}$ the trivial representation of the terminal groupoid $\star$ and by $t : \Gamma \longrightarrow \star$ the unique functor. The category $\mathrm{End}_{\mathsf{2VecBunGrpd}}(\star)$ is symmetric monoidal. Its objects are spans $\star \xleftarrow{t} \Gamma \xrightarrow{t} \star$ together with a 1-morphism $\lambda : t^*\tau \longrightarrow t^*\tau$. But this means that we specify for each $x \in \Gamma$ a 2-linear map $\lambda : \mathsf{FinVect} \longrightarrow \mathsf{FinVect}$, i.e. a vector space $\varrho_\lambda(x)$ (by evaluation of λ on the ground field). For a morphism $x \longrightarrow y$ in Γ we obtain a natural transformation $\lambda_x \longrightarrow \lambda_y$, which is equivalent to a linear map $\varrho_\lambda(x) \longrightarrow \varrho_\lambda(y)$. This shows that the objects of $\mathrm{End}_{\mathsf{2VecBunGrpd}}(\star)$ can be identified with vector bundles over essentially finite groupoids.

A 1-morphism $(\Gamma_0, \lambda_0) \longrightarrow (\Gamma_1, \lambda_1)$ in $\mathrm{End}_{\mathsf{2VecBunGrpd}}(\star)$ is a span of spans

$$\begin{tikzcd} & \Gamma_0 \arrow[dl, "t"'] \arrow[dr, "t"] & \\ \star & \Omega \arrow[u, "r_0"'] \arrow[d, "r_1"] & \star \\ & \Gamma_1 \arrow[ul, "t"] \arrow[ur, "t"'] & \end{tikzcd}$$

together with a natural morphism ω between the intertwiners $t^*\lambda_0, t^*\lambda_1 : r_0^* t^*\tau \longrightarrow r_1^* t^* \tau$, which is just an intertwiner from $r_0^*\varrho_{\lambda_0}$ to $r_1^*\varrho_{\lambda_1}$. All these identifications extend naturally to the composition and the symmetric monoidal structure. □

In a symmetric monoidal bicategory, there is a notion of a dualizable object, see [Lur09, Definition 2.3.5]: An object in a symmetric monoidal bicategory is called *dualizable* if it is dualizable in the homotopy category. So informally speaking, a dualizable object has evaluation and coevaluation 1-morphisms which obey the triangle identities up to 2-isomorphism. The 2-isomorphisms are not required to be coherent.

Exactly the same arguments which proved the existence of duals in $\mathsf{VecBunGrpd}$ in [SW19, Proposition 3.8] prove the following result:

Proposition 3.22. *Every object in* $\mathsf{2VecBunGrpd}$ *is dualizable.*

3.3.2 The parallel section functor

This subsection is devoted to the construction of the parallel section functor

$$\mathsf{Par} : \mathsf{2VecBunGrpd} \longrightarrow \mathsf{2Vect}$$

It will send an object (Γ, ϱ), i.e. a 2-vector bundle ϱ over a groupoid Γ to its space of parallel sections (Definition 3.6). We have already shown that this is indeed a 2-vector space (Proposition 3.9). It remains to define Par on 1-morphisms and 2-morphisms in $\mathsf{2VecBunGrpd}$. This is accomplished in the following two definitions using the pullback and pushforward constructions from Section 3.2.

Definition 3.23 (Parallel section functor on 1-morphisms)**.** Let $(\Gamma_0, \varrho_0) \xleftarrow{r_0} (\Lambda, \lambda) \xrightarrow{r_1} (\Gamma_1, \varrho_1)$ be a 1-morphism in $\mathsf{2VecBunGrpd}$. Denote by $\mathsf{Par}(\Lambda, \lambda)$ the 2-linear map

$$\mathsf{Par}(\Lambda, \lambda) : \mathsf{Par}\, \varrho_0 \xrightarrow{r_0^*} \mathsf{Par}\, r_0^*\varrho_0 \xrightarrow{\lambda_*} \mathsf{Par}\, r_1^*\varrho_1 \xrightarrow{r_{1*}} \mathsf{Par}\, \varrho_1 \ .$$

Here we use the 2-linear pull map from Proposition 3.10, the operation of intertwiners on parallel sections from Proposition 3.9 and the 2-linear push map from Proposition 3.11.

Definition 3.24 (Parallel section functor on 2-morphisms)**.** Let

$$
\begin{tikzcd}
& (\Lambda,\lambda) \arrow[dl, "r_0"'] \arrow[dr, "r_1"] & \\
(\Gamma_0,\varrho_0) & (\Omega,\omega) \arrow[u, "t"'] \arrow[d, "t'"] \arrow[l, Rightarrow, "\alpha_0"'] \arrow[r, Rightarrow, "\alpha_1"] & (\Gamma_1,\varrho_1) \\
& (\Lambda',\lambda') \arrow[ul, "r_0'"] \arrow[ur, "r_1'"'] &
\end{tikzcd}
$$

be a 2-morphism in 2VecBunGrpd. Then we define the 2-morphism

$$
\mathsf{Par}(\Omega,\omega) : \mathsf{Par}(\Lambda,\lambda) \longrightarrow \mathsf{Par}(\Lambda',\lambda')
$$

to be

$$
\begin{tikzcd}
& \mathsf{Par}\, r_0^*\varrho_0 \arrow[r, "\lambda_*"] \arrow[d, "t^*"'] & \mathsf{Par}\, r_1^*\varrho_1 \arrow[d, "t^*"] \arrow[ddr, "r_{1*}"] & \\
& \mathsf{Par}\, t^*r_0^*\varrho_0 \arrow[r, "t^*\lambda_*"] \arrow[dd, "\varrho_0(\alpha_0)_*"'] & \mathsf{Par}\, t^*r_1^*\varrho_1 \arrow[dd, "\varrho_1(\alpha_1)_*"] \arrow[ddl, Rightarrow, "\omega_*"'] & \\
\mathsf{Par}\,\varrho_0 \arrow[uur, "r_0^*"] \arrow[ddr, "{r_0'}^*"'] \arrow[r, phantom, "\Downarrow \alpha_{0*}"] & & & \mathsf{Par}\,\varrho_1 \arrow[l, phantom, "\Downarrow \alpha_1^*"'] \; , \\
& \mathsf{Par}\, {t'}^*{r_0'}^*\varrho_0 \arrow[r, "{t'}^*\lambda'_*"] \arrow[d, "t'_*"'] & \mathsf{Par}\, {t'}^*{r_1'}^*\varrho_1 \arrow[d, "t'_*"] \arrow[dl, Rightarrow, "\cong"'] & \\
& \mathsf{Par}\, {r_0'}^*\varrho_0 \arrow[r, "\lambda'_*"'] & \mathsf{Par}\, {r_1'}^*\varrho_1 \arrow[uur, "r'_{1*}"'] &
\end{tikzcd}
$$

where

- the commutativity of the top square corresponds to the naturality of the pullback maps (Proposition 3.10 (c)) and the commutativity of the lowest square corresponds to the naturality of the pushforward maps up to natural isomorphism (Proposition 3.14 (b)),

- ω_* is the application of the functor from Proposition 3.9 to ω,

- $\alpha_{0*} : t'_*\varrho_0(\alpha_0)(r_0t)^* \longrightarrow {r_0'}^*$ comes from the application of Proposition 3.15 to the square

$$
\begin{tikzcd}
\Omega \arrow[r, "r_0 t"] \arrow[d, "t'"'] & \Gamma_0 \arrow[d, "\mathrm{id}_{\Gamma_0}"] \arrow[dl, Rightarrow, "\alpha_0"'] \\
\Lambda' \arrow[r, "r_0'"'] & \Gamma_0 \; ,
\end{tikzcd}
$$

- and $\alpha_1^* : r_{1*} \longrightarrow (r_1't')_*\varrho_1(\alpha_1)t^*$ comes similarly from the application of Proposition 3.15 to the square

$$\begin{tikzcd} \Omega \arrow[r, "t"] \arrow[d, "r_1' t'"'] & \Lambda \arrow[d, "r_1"] \arrow[dl, Rightarrow, "\alpha_1"'] \\ \Gamma_1 \arrow[r, "\mathrm{id}_{\Gamma_1}"'] & \Gamma_1 \end{tikzcd} .$$

We are now ready to formulate the main result of this chapter:

Theorem 3.25 (Parallel section functor)**.** *The assignments of Definition 3.23 for 1-morphisms and Definition 3.24 for 2-morphisms extend to a symmetric monoidal 2-functor*

$$\mathsf{Par} : \mathsf{2VecBunGrpd} \longrightarrow \mathsf{2Vect}$$

that we call parallel section functor.

Proof. (i) First we prove that Par is functorial on 1-morphisms. Obviously, it respects identities up to natural isomorphism. For the proof of the compatibility with composition, we start with two composable 1-morphisms

$$(\Gamma_0, \varrho_0) \xleftarrow{r_0} (\Lambda, \lambda) \xrightarrow{r_1} (\Gamma_1, \varrho_1)$$

and

$$(\Gamma_1, \varrho_1) \xleftarrow{r_1'} (\Lambda', \lambda') \xrightarrow{r_2'} (\Gamma_2, \varrho_2)$$

in 2VecBunGrpd. According to the definition of the composition of 1-morphisms (Definition 3.19, 4), we need to form the homotopy pullback

$$\begin{tikzcd} & & \Lambda \times_\Omega \Lambda' \arrow[dl, "\pi"'] \arrow[dr, "\pi'"] & & \\ & \Lambda \arrow[rr, Rightarrow, "\eta"] \arrow[dl, "r_0"'] \arrow[dr, "r_1"] & & \Lambda' \arrow[dl, "r_1'"'] \arrow[dr, "r_2'"] & \\ \Gamma_0 & & \Gamma_1 & & \Gamma_2 \end{tikzcd} ,$$

By the definition of the parallel section functor we find the following isomorphisms

$$\begin{aligned} \mathsf{Par}((\Lambda', \lambda') \circ (\Lambda, \lambda)) &= (r_2' \pi')_* (\pi'^* \lambda')_* \varrho_1(\eta)_* (\pi^* \lambda)_* (r_0 \pi)^* \\ &\cong r_{2*}' \pi_*' (\pi'^* \lambda')_* \varrho_1(\eta)_* (\pi^* \lambda)_* \pi^* r_0^* \quad \begin{pmatrix} \text{Proposition 3.10 (a) and} \\ \text{Proposition 3.14 (a)} \end{pmatrix} \\ &= r_{2*}' \pi_*' (\pi'^* \lambda')_* \varrho_1(\eta)_* \pi^* \lambda_* r_0^* \quad \text{(Proposition 3.10 (c))} \\ &\cong r_{2*}' \lambda_*' \pi_*' \varrho_1(\eta)_* \pi^* \lambda_* r_0^* \quad \text{(Proposition 3.14 (b))} \\ &\cong r_{2*}' \lambda_*' {r_1'}^* r_{1*} \lambda_* r_0^* \quad \text{(Proposition 3.17)} \\ &= \mathsf{Par}(\Lambda', \lambda') \circ \mathsf{Par}(\Lambda, \lambda) \ . \end{aligned}$$

Their composition defines an isomorphism that we take as part of data of the 2-functor Par.

(ii) Now we prove that the vertical composition of 2-morphisms is preserved. Again, it is obvious that identities are respected. For the proof of the composition law, we take 2-

morphisms

$$\begin{tikzcd}
& (\Lambda,\lambda) \arrow[dl, "r_0"'] \arrow[dr, "r_1"] & \\
(\Gamma_0,\varrho_0) & (\Omega,\omega) \arrow[u, "t"'] \arrow[d, "t'"] \arrow[l, Rightarrow, "\alpha_0"'] \arrow[r, Rightarrow, "\alpha_1"] & (\Gamma_1,\varrho_1) \\
& (\Lambda',\lambda') \arrow[ul, "r_0'"] \arrow[ur, "r_1'"'] &
\end{tikzcd}$$

and

$$\begin{tikzcd}
& (\Lambda',\lambda') \arrow[dl, "r_0'"'] \arrow[dr, "r_1'"] & \\
(\Gamma_0,\varrho_0) & (\widetilde{\Omega},\widetilde{\omega}) \arrow[u, "u'"'] \arrow[d, "u''"] \arrow[l, Rightarrow, "\beta_0"'] \arrow[r, Rightarrow, "\beta_1"] & (\Gamma_1,\varrho_1) \\
& (\Lambda'',\lambda'') \arrow[ul, "r_0''"] \arrow[ur, "r_1''"'] &
\end{tikzcd}$$

as well as the composition

$$\begin{tikzcd}
& (\Lambda,\lambda) \arrow[dl, "r_0"'] \arrow[dr, "r_1"] & \\
(\Gamma_0,\varrho_0) & (\Omega \times_{\Lambda'} \widetilde{\Omega}, \omega \times_{\lambda'} \widetilde{\omega}) \arrow[u, "v"'] \arrow[d, "v''"] \arrow[l, Rightarrow, "\gamma_0"'] \arrow[r, Rightarrow, "\gamma_1"] & (\Gamma_1,\varrho_1) \\
& (\Lambda'',\lambda'') \arrow[ul, "r_0''"] \arrow[ur, "r_1''"'] &
\end{tikzcd},$$

as given in Definition 3.19, 5 with the same notation that we used there. According to Definition 3.24 the natural transformation $\mathsf{Par}(\Omega \times_{\Lambda'} \widetilde{\Omega}, \omega \times_{\lambda'} \widetilde{\omega})$ for the composition is given by

$$\begin{tikzcd}
& \mathsf{Par}\, r_0^*\varrho_0 \arrow[r, "\lambda_*"] \arrow[d, "v^*"'] & \mathsf{Par}\, r_1^*\varrho_1 \arrow[d, "v^*"] \arrow[ddr, "r_{1*}"] & \\
& \mathsf{Par}\, v^*r_0^*\varrho_0 \arrow[r, "v^*\lambda_*"] \arrow[dd, "\varrho_0(\gamma_0)_*"'] & \mathsf{Par}\, v^*r_1^*\varrho_1 \arrow[dd, "\varrho_1(\gamma_1)_*"] \arrow[ddl, Rightarrow, "(\omega \times_{\lambda'} \widetilde{\omega})_*"'] & \\
\mathsf{Par}\, \varrho_0 \arrow[uur, "r_0^*"] \arrow[ddr, "r_0^{\prime\prime*}"'] \arrow[r, phantom, "\Downarrow\gamma_{0*}"] & & & \mathsf{Par}\, \varrho_1 \arrow[l, phantom, "\Downarrow\gamma_1^*"'] \\
& \mathsf{Par}\, v^{\prime\prime*}r_0^{\prime\prime*}\varrho_0 \arrow[r, "v^{\prime\prime*}\lambda''_*"] \arrow[d, "v''_*"'] & \mathsf{Par}\, v^{\prime\prime*}r_1^{\prime\prime*}\varrho_1 \arrow[d, "v''_*"] \arrow[dl, Rightarrow, "\cong"'] & \\
& \mathsf{Par}\, r_0^{\prime\prime*}\varrho_0 \arrow[r, "\lambda''_*"] & \mathsf{Par}\, r_1^{\prime\prime*}\varrho_1 \arrow[uur, "r''_{1*}"'] &
\end{tikzcd},$$

In a first step, let us look at the inner ladder of this diagram. The ladder is equal to

$$\begin{tikzcd}
\mathsf{Par}\, r_0^* \varrho_0 \arrow[r, "\lambda_*"] \arrow[d, "t^*"'] & \mathsf{Par}\, r_1^* \varrho_1 \arrow[d, "t^*"] \\
\mathsf{Par}\, t^* r_0^* \varrho_0 \arrow[r, "t^* \lambda_*"] \arrow[d, "q^*"'] & \mathsf{Par}\, t^* r_1^* \varrho_1 \arrow[d, "q^*"] \\
\mathsf{Par}\, v^* r_0^* \varrho_0 \arrow[r, "v^* \lambda_*"] \arrow[d, "\varrho_0(\alpha_0)_*"'] \arrow[dr, phantom, "\Swarrow (q^*\omega)_*"] & \mathsf{Par}\, v^* r_1^* \varrho_1 \arrow[d, "\varrho_1(\alpha_1)_*"] \\
\mathsf{Par}\, q^* t'^* {r'_0}^* \varrho_0 \arrow[r, "q^* t'^* \lambda'_*"] \arrow[d, "\varrho_0(\eta)_*"'] \arrow[dr, phantom, "\Swarrow \theta'_*"] & \mathsf{Par}\, q^* t'^* {r'_1}^* \varrho_1 \arrow[d, "\varrho_1(\eta)_*"] \\
\mathsf{Par}\, \widetilde{q}^* u'^* {r'_0}^* \varrho_0 \arrow[r, "\widetilde{q}^* u'^* \lambda'_*"] \arrow[d, "\varrho_0(\beta_0)_*"'] \arrow[dr, phantom, "\Swarrow (\widetilde{q}^*\widetilde{\omega})_*"] & \mathsf{Par}\, \widetilde{q}^* u'^* {r'_1}^* \varrho_1 \arrow[d, "\varrho_1(\beta_1)_*"] \\
\mathsf{Par}\, \widetilde{q}^* u''^* {r''_0}^* \varrho_0 \arrow[r, "\widetilde{q}^* u''^* \lambda''_*"] \arrow[d, "\widetilde{q}_*"'] \arrow[dr, phantom, "\Swarrow \cong"] & \mathsf{Par}\, \widetilde{q}^* u''^* {r''_1}^* \varrho_1 \arrow[d, "\widetilde{q}_*"] \\
\mathsf{Par}\, u''^* {r''_0}^* \varrho_0 \arrow[r, "u''^* \lambda''_*"] \arrow[d, "u''_*"'] \arrow[dr, phantom, "\Swarrow \cong"] & \mathsf{Par}\, u''^* {r''_1}^* \varrho_1 \arrow[d, "u''_*"] \\
\mathsf{Par}\, {r''_0}^* \varrho_0 \arrow[r, "\lambda''_*"] & \mathsf{Par}\, {r''_1}^* \varrho_1\ .
\end{tikzcd}$$

Here we have used the composition behavior and naturality of the pull and push 1-morphisms (Proposition 3.10 and Proposition 3.14), but we suppress the isomorphism $v''_* \cong u''_* \widetilde{q}_*$ for readability. Additionally, we have unpacked the definition of $\omega \times_{\lambda'} \widetilde{\omega}$ (Definition 3.19, 5). Recall that the isomorphism θ' is the datum that λ' comes equipped with.

We investigate this ladder and obtain the equality

$$
\begin{tikzcd}[column sep=small]
\mathsf{Par}\, r_0^*\varrho_0 \arrow[rr, "\lambda_*"] \arrow[d, "t^*"'] & & \mathsf{Par}\, r_1^*\varrho_1 \arrow[d, "t^*"] \\
\mathsf{Par}\, t^*r_0^*\varrho_0 \arrow[rr, "t^*\lambda_*"] \arrow[d, "q^*"'] & & \mathsf{Par}\, t^*r_1^*\varrho_1 \arrow[d, "q^*"] \\
\mathsf{Par}\, v^*r_0^*\varrho_0 \arrow[rr, "v^*\lambda_*"] \arrow[d, "\varrho_0(\alpha_0)_*"'] \arrow[dr, phantom, "\Swarrow (q^*\omega)_*"] & & \mathsf{Par}\, v^*r_1^*\varrho_1 \arrow[d, "\varrho_1(\alpha_1)_*"] \\
\mathsf{Par}\, q^*t'^*{r'_0}^*\varrho_0 \arrow[rr, "q^*t'^*\lambda'_*"] \arrow[d, "\varrho_0(\eta)_*"'] \arrow[dr, phantom, "\Swarrow \theta'_*"] & & \mathsf{Par}\, q^*t'^*{r'_1}^*\varrho_1 \arrow[d, "\varrho_1(\eta)_*"] \\
\mathsf{Par}\, \widetilde{q}^*u'^*{r'_0}^*\varrho_0 \arrow[rr, "\widetilde{q}^*u'^*\lambda'_*"] \arrow[d, "\varrho_0(\beta_0)_*"'] \arrow[dr, phantom, "\Swarrow (\widetilde{q}^*\widetilde{\omega})_*"] & & \mathsf{Par}\, \widetilde{q}^*u'^*{r'_1}^*\varrho_1 \arrow[d, "\varrho_1(\beta_1)_*"] \\
\mathsf{Par}\, \widetilde{q}^*u''^*{r''_0}^*\varrho_0 \arrow[rr, "\widetilde{q}^*u''^*\lambda''_*"] \arrow[d, "\widetilde{q}_*"'] \arrow[dr, phantom, "\Swarrow \cong"] & & \mathsf{Par}\, \widetilde{q}^*u''^*{r''_1}^*\varrho_1 \arrow[d, "\widetilde{q}_*"] \\
\mathsf{Par}\, u''^*{r''_0}^*\varrho_0 \arrow[rr, "u''^*\lambda''_*"] \arrow[d, "u''_*"'] \arrow[dr, phantom, "\Swarrow \cong"] & & \mathsf{Par}\, u''^*{r''_1}^*\varrho_1 \arrow[d, "u''_*"] \\
\mathsf{Par}\, {r''_0}^*\varrho_0 \arrow[rr, "\lambda''_*"] & & \mathsf{Par}\, {r''_1}^*\varrho_1
\end{tikzcd}
\quad = \quad
\begin{tikzcd}[column sep=small]
\mathsf{Par}\, r_0^*\varrho_0 \arrow[rr, "\lambda_*"] \arrow[d, "t^*"'] & & \mathsf{Par}\, r_1^*\varrho_1 \arrow[d, "t^*"] \\
\mathsf{Par}\, t^*r_0^*\varrho_0 \arrow[rr, "t^*\lambda_*"] \arrow[d, "\varrho_0(\alpha_0)_*"'] \arrow[dr, phantom, "\Swarrow \omega_*"] & & \mathsf{Par}\, t^*r_1^*\varrho_1 \arrow[d, "\varrho_1(\alpha_1)_*"] \\
\mathsf{Par}\, t'^*{r'_0}^*\varrho_0 \arrow[rr, "t'^*\lambda'_*"] \arrow[d, "q^*"'] & & \mathsf{Par}\, t'^*{r'_1}^*\varrho_1 \arrow[d, "q^*"] \\
\mathsf{Par}\, q^*t'^*{r'_0}^*\varrho_0 \arrow[rr, "q^*t'^*\lambda'_*"] \arrow[d, "\varrho_0(\eta)"'] \arrow[dr, phantom, "\Swarrow \theta'_*"] & & \mathsf{Par}\, q^*t'^*{r'_1}^*\varrho_1 \arrow[d, "\varrho_1(\eta)"] \\
\mathsf{Par}\, \widetilde{q}^*u'^*{r'_0}^*\varrho_0 \arrow[rr, "\widetilde{q}^*u'^*\lambda'_*"] \arrow[d, "\widetilde{q}_*"'] \arrow[dr, phantom, "\Swarrow \cong"] & & \mathsf{Par}\, \widetilde{q}^*u'^*{r'_1}^*\varrho_1 \arrow[d, "\widetilde{q}_*"] \\
\mathsf{Par}\, u'^*{r'_0}^*\varrho_0 \arrow[rr, "u'^*\lambda'_*"] \arrow[d, "\varrho_0(\beta_0)_*"'] \arrow[dr, phantom, "\Swarrow \widetilde{\omega}_*"] & & \mathsf{Par}\, u'^*{r'_1}^*\varrho_1 \arrow[d, "\varrho_1(\beta_1)_*"] \\
\mathsf{Par}\, u''^*{r''_0}^*\varrho_0 \arrow[rr, "u''^*\lambda''_*"] \arrow[d, "u''_*"'] \arrow[dr, phantom, "\Swarrow \cong"] & & \mathsf{Par}\, u''^*{r''_1}^*\varrho_1 \arrow[d, "u''_*"] \\
\mathsf{Par}\, {r''_0}^*\varrho_0 \arrow[rr, "\lambda''_*"] & & \mathsf{Par}\, {r''_1}^*\varrho_1
\end{tikzcd}
\ ,
$$

where the changes only involve the second and the third as well as the fifth and the sixth square. By re-inserting the ladder we obtain

$$
\begin{tikzcd}[column sep=small]
 & & \mathsf{Par}\, r_0^*\varrho_0 \arrow[r, "\lambda_*"] \arrow[d, "t^*"'] & \mathsf{Par}\, r_1^*\varrho_1 \arrow[d, "t^*"] \arrow[rrdddd, "r_{1*}"] & & \\
 & & \mathsf{Par}\, t^*r_0^*\varrho_0 \arrow[r, "t^*\lambda_*"] \arrow[d, "\varrho_0(\alpha_0)_*"'] \arrow[dr, phantom, "\Swarrow \omega_*"] & \mathsf{Par}\, t^*r_1^*\varrho_1 \arrow[d, "\varrho_1(\alpha_1)_*"] & \arrow[d, phantom, "\Swarrow \alpha_1^*"] & \\
 & \arrow[d, phantom, "\Swarrow \alpha_{0*}"] & \mathsf{Par}\, t'^*{r'_0}^*\varrho_0 \arrow[r, "t'^*\lambda'_*"] \arrow[d, "q^*"'] \arrow[ldd, "t'_*"'] & \mathsf{Par}\, t'^*{r'_1}^*\varrho_1 \arrow[d, "q^*"] \arrow[rdd, "t'_*"] & & \\
 & & \mathsf{Par}\, q^*t'^*{r'_0}^*\varrho_0 \arrow[r, "q^*t'^*\lambda'_*"] \arrow[dd, "\varrho_0(\eta)_*"'] \arrow[ddr, phantom, "\Swarrow \theta'_*"] & \mathsf{Par}\, q^*t'^*{r'_1}^*\varrho_1 \arrow[dd, "\varrho_1(\eta)_*"] & & \\
\mathsf{Par}\, \varrho_0 \arrow[r, "{r'_0}^*"] \arrow[rruuuu, "r_0^*"] \arrow[rrdddd, "{r''_0}^*"'] & \mathsf{Par}\, {r'_0}^*\varrho_0 \arrow[r, phantom, "\overset{\cong}{\Leftarrow}"] \arrow[rdd, "u'^*"] & {} & {} \arrow[r, phantom, "\overset{\cong}{\Leftarrow}"] & \mathsf{Par}\, {r'_1}^*\varrho_1 \arrow[r, "r'_{1*}"] \arrow[ldd, "u'^*"'] & \mathsf{Par}\, \varrho_1 \\
 & & \mathsf{Par}\, \widetilde{q}^*u'^*{r'_0}^*\varrho_0 \arrow[r, "\widetilde{q}^*u'^*\lambda'_*"] \arrow[d, "\widetilde{q}_*"'] \arrow[dr, phantom, "\Swarrow \cong"] & \mathsf{Par}\, \widetilde{q}^*u'^*{r'_1}^*\varrho_1 \arrow[d, "\widetilde{q}_*"] & & \\
 & \arrow[u, phantom, "\Swarrow \beta_{0*}"] & \mathsf{Par}\, u'^*{r'_0}^*\varrho_0 \arrow[r, "u'^*\lambda'_*"] \arrow[d, "\varrho_0(\beta_0)_*"'] \arrow[dr, phantom, "\Swarrow \widetilde{\omega}_*"] & \mathsf{Par}\, u'^*{r'_1}^*\varrho_1 \arrow[d, "\varrho_1(\beta_1)_*"] & \arrow[u, phantom, "\Swarrow \beta_1^*"] & \\
 & & \mathsf{Par}\, u''^*{r''_0}^*\varrho_0 \arrow[r, "u''^*\lambda''_*"] \arrow[d, "u''_*"'] \arrow[dr, phantom, "\Swarrow \cong"] & \mathsf{Par}\, u''^*{r''_1}^*\varrho_1 \arrow[d, "u''_*"] & & \\
 & & \mathsf{Par}\, {r''_0}^*\varrho_0 \arrow[r, "\lambda''_*"] & \mathsf{Par}\, {r''_1}^*\varrho_1 \arrow[rruuuu, "{r''_1}_*"'] & &
\end{tikzcd}
\ .
\tag{3.14}
$$

Here we have also decomposed the triangles containing γ_{0*} and γ_1^* into three smaller triangles each, which we will justify in step (iii). If we accept this for a moment, we can observe that the natural isomorphisms in the inner hexagon yield a natural isomorphism

$$u'^* t'_*(t'^* \lambda'_*) \longrightarrow (u'^* \lambda'_*) u'^* t'_*$$

between 2-linear maps $\mathsf{Par}\, t'^* {r'_0}^* \varrho_0 \longrightarrow \mathsf{Par}\, u'^* {r'_1}^* \varrho_1$. When evaluated on $s \in \mathsf{Par}\, t'^* {r'_0}^* \varrho_0$ and $\widetilde{z} \in \widetilde{\Omega}$, it consists of the isomorphism from

$$((u'^* t'_*(t'^* \lambda'_*))s)(\widetilde{z}) = \lim_{(z,g) \in t'^{-1}[u'(\widetilde{z})]} \varrho_1(r'_1(g)) \lambda'_{t'(z)} s(z)$$

to

$$((u'^* \lambda'_*) u'^* t'_*)s)(\widetilde{z}) = \lim_{(z,g) \in t'^{-1}[u'(\widetilde{z})]} \lambda'_{u'(\widetilde{z})} \varrho_0(r'_0(g)) s(z)$$

described as follows: We have to pull back the diagram underlying the first limit along the equivalence $t'^{-1}[u'(\widetilde{z})] \simeq \widetilde{q}^{-1}[\widetilde{z}]$ coming from the pasting law for homotopy pullbacks to a diagram over $q^{-1}[\widetilde{z}]$, which amounts just to a change of variables. Then we apply the (pulled back version of) θ'. Finally, we pull back the diagram to $t'^{-1}[u'(\widetilde{z})]$ using (the inverse of) $t'^{-1}[u'(\widetilde{z})] \simeq \widetilde{q}^{-1}[\widetilde{z}]$. But this isomorphism is equal to the isomorphism

$$\lim_{(z,g) \in t'^{-1}[u'(\widetilde{z})]} \varrho_1(r'_1(g)) \lambda'_{t'(z)} s(z) \longrightarrow \lim_{(z,g) \in t'^{-1}[u'(\widetilde{z})]} \lambda'_{u'(\widetilde{z})} \varrho_0(r'_0(g)) s(z)$$

just coming from θ'.

This allows us to simplify the inner hexagon in (3.14) and gives us

$$\begin{tikzcd}
 & \mathsf{Par}\, r_0^* \varrho_0 \arrow[r, "\lambda_*"] \arrow[d, "t^*"'] & \mathsf{Par}\, r_1^* \varrho_1 \arrow[d, "t^*"] \arrow[dddr, "r_{1*}"] & \\
 & \mathsf{Par}\, t^* r_0^* \varrho_0 \arrow[r, "t^* \lambda_*"] \arrow[d, "\varrho_0(\alpha_0)_*"'] \arrow[dr, phantom, "\Swarrow \omega_*"] & \mathsf{Par}\, t^* r_1^* \varrho_1 \arrow[d, "\varrho_1(\alpha_1)_*"] & \\
 & \mathsf{Par}\, t'^* {r'_0}^* \varrho_0 \arrow[r, "t'^* \lambda'_*"] \arrow[d, "t'_*"'] \arrow[l, phantom, "\Swarrow \alpha_{0*}"] & \mathsf{Par}\, t'^* {r'_1}^* \varrho_1 \arrow[d, "t'_*"] \arrow[r, phantom, "\Swarrow \alpha_1^*"] & \\
\mathsf{Par}\, \varrho_0 \arrow[r, "{r'_0}^*"] \arrow[uuur, "r_0^*"] \arrow[dddr, "{r''_0}^*"'] & \mathsf{Par}\, {r'_0}^* \varrho_0 \arrow[r, "\lambda'_*"] \arrow[d, "u'^*"'] \arrow[dr, phantom, "\Swarrow \cong"] & \mathsf{Par}\, {r'_1}^* \varrho_1 \arrow[r, "{r'_1}_*"] \arrow[d, "u'^*"] & \mathsf{Par}\, \varrho_1 \, . \\
 & \mathsf{Par}\, u'^* {r'_0}^* \varrho_0 \arrow[r, "u'^* \lambda'_*"] \arrow[d, "\varrho_0(\beta_0)_*"'] \arrow[l, phantom, "\Swarrow \beta_{0*}"] \arrow[dr, phantom, "\Swarrow \widetilde{\omega}_*"] & \mathsf{Par}\, u'^* {r'_1}^* \varrho_1 \arrow[d, "\varrho_1(\beta_1)_*"] \arrow[r, phantom, "\Swarrow \beta_1^*"] & \\
 & \mathsf{Par}\, u''^* {r''_0}^* \varrho_0 \arrow[r, "u''^* \lambda''_*"] \arrow[d, "u''_*"'] \arrow[dr, phantom, "\Swarrow \cong"] & \mathsf{Par}\, u''^* {r''_1}^* \varrho_1 \arrow[d, "u''_*"] & \\
 & \mathsf{Par}\, {r''_0}^* \varrho_0 \arrow[r, "u''^* \lambda''_*"] & \mathsf{Par}\, {r''_1}^* \varrho_1 \arrow[uuur, "{r''_1}^*"'] &
\end{tikzcd}$$

Here we have replaced the inner hexagon by two squares. One of them commutes strictly (Proposition 3.10 (c)), the other up to the natural isomorphism from Proposition 3.14 (b) (recall that this was induced was θ'). This proves the preservation of the vertical composition.

(iii) We still have to justify the decomposition of γ_{0*} and γ_1^* that we have used to obtain (3.14). We only do this for γ_{0*} because it is the more difficult case (involving pushforward maps instead of only pullback maps). First note that the small inner triangles (the ones being part of the inner hexagon in (3.14)) come from a homotopy pullback, so the corresponding natural transformations are actually isomorphisms by Proposition 3.17. To prove that γ_{0*} is equal to the composition of the transformation living on the three triangles on the left side of (3.14), we choose $s \in \mathsf{Par}\,\varrho_0$ and $y'' \in \Lambda$. Now both transformations in question correspond to maps

$$(u''_* \varrho_0(\beta_0)_* \widetilde{q}_* \varrho_0(\eta)_* q^* \varrho(\alpha_0)_* t^* r_0^* s)(y'') \longrightarrow s(r''_0(y'')) \ . \tag{3.15}$$

Using the definition of γ_0 in Definition 3.19, 5 we can identify

$$(u''_* \varrho_0(\beta_0)_* \widetilde{q}_* \varrho_0(\eta)_* q^* \varrho(\alpha_0)_* t^* r_0^* s)(y'')$$

with $(v''_* \varrho_0(\gamma_0)_* v^* r_0^* s)(y'')$. Hence, we will see the maps (3.15) as maps

$$(v''_* \varrho_0(\gamma_0)_* v^* r_0^* s)(y'') \longrightarrow s(r''_0(y'')) \ .$$

Now the composition of the three left triangles in (3.14) amounts to the composition

$$\begin{aligned}(v''_* \varrho_0(\gamma_0)_* v^* r_0^* s)(y'') &\xrightarrow{\eta_*} (u''_* \varrho_0(\beta_0)_* u'^* t'_* \varrho(\alpha_0)_* t^* r_0^* s)(y'') \\ &\xrightarrow{\alpha_{0*}} (u''_* \varrho_0(\beta_0)_* u'^* r_0'^* s)(y'') \\ &\xrightarrow{\beta_{0*}} \ s(r''_0(y'')) \ ,\end{aligned} \tag{3.16}$$

and we have to show that it is given by γ_{0*}. To see this, observe that the object $(v''_* \varrho_0(\gamma_0)_* v^* r_0^* s)(y'')$ is a limit over the groupoid $v''^{-1}[y'']$, whereas

$$(u''_* \varrho_0(\beta_0)_* u'^* t'_* \varrho(\alpha_0)_* t^* r_0^* s)(y'')$$

is a limit over $u''^{-1}[y''] \times_{\Lambda'} \Omega$. The first map η_* is the pushforward along the equivalence

$$v''^{-1}[y''] = (u'' \circ \widetilde{q})^{-1}[y''] \simeq u''^{-1}[y''] \times_{\widetilde{\Omega}} (\Omega \times_{\Lambda'} \widetilde{\Omega}) \simeq u''^{-1}[y''] \times_{\Lambda'} \Omega.$$

Here, by pushforward we always mean pushforward of limits, i.e. pushforward of sections of ordinary vector bundles over groupoids in the sense of Section 3.1.1. Next, $(u''_* \varrho_0(\beta_0)_* u'^* r_0'^* s)(y'')$ is a limit over $u''^{-1}[y'']$ and α_{0*} is the pushforward along the projection $u''^{-1}[y''] \times_{\Lambda'} \Omega \longrightarrow u''^{-1}[y'']$. Finally, $s(r''_0(y''))$ is a limit over the terminal groupoid $\star$ and β_{0*} is the pushforward along the functor $u''^{-1}[y''] \longrightarrow \star$. By Proposition 3.2, (b) we conclude that the composition (3.16) is the pushforward along the composition

$$v''^{-1}[y''] \simeq u''^{-1}[y''] \times_{\Lambda'} \Omega \longrightarrow u''^{-1}[y''] \longrightarrow \star$$

of functors, i.e. it integrates over the homotopy fiber $v''^{-1}[y'']$ with respect to groupoid cardinality. Hence, by Corollary 3.18 it is equal to γ_{0*}. This gives us the missing step in the derivation of (3.14).

(iv) Next we prove that the horizontal composition of 2-morphisms is respected up to the isomorphisms specified for the composition of 1-morphisms. To this end, we take 2-morphisms

$$\begin{tikzcd}
 & (\varLambda,\lambda) \arrow[dl, "r_0"'] \arrow[dr, "r_1"] & \\
(\varGamma_0,\varrho_0) & (\varOmega,\omega) \arrow[l, Rightarrow, "\alpha_0"'] \arrow[r, Rightarrow, "\alpha_1"] \arrow[u, "t"] \arrow[d, "t'"'] & (\varGamma_1,\varrho_1) \\
 & (\varLambda',\lambda') \arrow[ul, "r'_0"] \arrow[ur, "r'_1"'] &
\end{tikzcd}$$

and

$$\begin{tikzcd}
 & (\varPi,\pi) \arrow[dl, "v_1"'] \arrow[dr, "v_2"] & \\
(\varGamma_1,\varrho_1) & (\widetilde{\varOmega},\widetilde{\omega}) \arrow[l, Rightarrow, "\beta_0"'] \arrow[r, Rightarrow, "\beta_1"] \arrow[u, "u"] \arrow[d, "u'"'] & (\varGamma_2,\varrho_2) \\
 & (\varPi',\pi') \arrow[ul, "v'_1"] \arrow[ur, "v'_2"'] &
\end{tikzcd}$$

and their horizontal composition

$$\begin{tikzcd}
 & (\varLambda \times_{\varGamma_1} \varPi, \lambda \times_{\varGamma_1} \pi) \arrow[dl, "r_0 p"'] \arrow[dr, "v_2 q"] & \\
(\varGamma_0,\varrho_0) & (\varOmega \times_{\varGamma_1} \widetilde{\varOmega}, \omega \times_{\varGamma_1} \widetilde{\omega}) \arrow[l, Rightarrow, "\delta_0"'] \arrow[r, Rightarrow, "\delta_1"] \arrow[u, "c"] \arrow[d, "d"'] & (\varGamma_2,\varrho_2) \\
 & (\varLambda' \times_{\varGamma_1} \varPi', \lambda' \times_{\varGamma_1} \pi') \arrow[ul, "r'_0 p'"] \arrow[ur, "v'_2 q'"'] &
\end{tikzcd}$$

as given in Definition 3.19, 6. We have to show the equality of natural transformations

$$\mathsf{Par}(\varOmega \times_{\varGamma_1} \widetilde{\varOmega}, \omega \times_{\varGamma_1} \widetilde{\omega}) \quad = \quad
\begin{tikzcd}[column sep=huge]
\mathsf{Par}\,\varrho_0 \arrow[r, bend left=50, "\mathsf{Par}(\varLambda,\lambda)", ""{name=A1, below}] \arrow[r, bend right=50, "\mathsf{Par}(\varLambda',\lambda')"', ""{name=B1, above}] \arrow[rr, bend left=80, "\mathsf{Par}(\varLambda \times_{\varGamma_1} \varPi, \lambda \times_{\varGamma_1} \pi)"] \arrow[rr, bend right=80, "\mathsf{Par}(\varLambda' \times_{\varGamma_1} \varPi', \lambda' \times_{\varGamma_1} \pi')"'] \arrow[r, phantom, "\Downarrow \mathsf{Par}(\varOmega,\omega)"] \arrow[rr, phantom, bend left=65, "\Downarrow \cong"] \arrow[rr, phantom, bend right=65, "\Uparrow \cong"'] & \mathsf{Par}\,\varrho_1 \arrow[r, bend left=50, "\mathsf{Par}(\varPi,\pi)"] \arrow[r, bend right=50, "\mathsf{Par}(\varPi',\pi')"'] \arrow[r, phantom, "\Downarrow \mathsf{Par}(\widetilde{\varOmega},\widetilde{\omega})"] & \mathsf{Par}\,\varrho_2
\end{tikzcd} . \tag{3.17}$$

We abbreviate the left hand side by L and the right hand side by R. Using Definition 3.19, 6 and the labels therein we find for the left hand side

$$
L \; = \;
\begin{tikzcd}
 & \mathsf{Par}(r_0p)^*\varrho_0 \arrow[r, "p^*\lambda_*"] \arrow[dd, "c^*"] & \mathsf{Par}(r_1p)^*\varrho_1 \arrow[r, "\varrho_1(\eta)"] \arrow[dd, "c^*"] & \mathsf{Par}(r_1q)^*\varrho_1 \arrow[r, "q^*\pi_*"] \arrow[dd, "c^*"] & \mathsf{Par}(v_2q)^*\varrho_2 \arrow[dd, "c^*"] \arrow[ddddr, "(v_2q)_*"] & \\
 & & & & & \\
 & \mathsf{Par}(r_0pc)^*\varrho_0 \arrow[r, "(pc)^*\lambda_*"] \arrow[dddd, "\varrho_0(\ell^*\alpha_0)_*"] & \mathsf{Par}(r_1pc)^*\varrho_1 \arrow[r, "\varrho_1(\varepsilon)"] \arrow[dddd, "\varrho_1(\ell^*\alpha_1)_*"] \arrow[ddddl, phantom, "\Swarrow \ell^*\omega_*"] & \mathsf{Par}(v_1pc)^*\varrho_1 \arrow[r, "(qc)^*\pi"] \arrow[dddd, "\varrho_1(\widetilde{\ell}^*\beta_0)_*"] & \mathsf{Par}(v_2qc)^*\varrho_2 \arrow[dddd, "\varrho_2(\widetilde{\ell}^*\beta_1)_*"] \arrow[ddddl, phantom, "\Swarrow \widetilde{\ell}^*\widetilde{\omega}_*"] & \\
 & & & & & \\
\mathsf{Par}\,\varrho_0 \arrow[uuuur, "(r_0p)^*"] \arrow[ddddr, "(r'_0p')^*"'] \arrow[r, phantom, "\delta_{0*}\Swarrow"] & {} & & & {} \arrow[r, phantom, "\delta_1^*\Swarrow"] & \mathsf{Par}\,\varrho_2 \\
 & & & & & \\
 & \mathsf{Par}(r'_0pd)^*\varrho_0 \arrow[r, "(p'd)^*\lambda'_*"] \arrow[dd, "d_*"] & \mathsf{Par}(r'_1p'd)^*\varrho_1 \arrow[r, "\varrho_1(d^*\eta')"] \arrow[dd, "d_*"] \arrow[ddl, phantom, "\cong"] & \mathsf{Par}(v'_1q'd)^*\varrho_1 \arrow[r, "(q'd)^*\pi'_*"] \arrow[dd, "d_*"] & \mathsf{Par}(v'_2q'd)^*\varrho_1 \arrow[dd, "d_*"] \arrow[ddl, phantom, "\cong"] & \\
 & & & & & \\
 & \mathsf{Par}(r'_0p')^*\varrho_0 \arrow[r, "p'^*\lambda'_*"] & \mathsf{Par}(r'_1p')^*\varrho_1 \arrow[r, "\varrho_1(\eta')"] & \mathsf{Par}(v'_1q')^*\varrho_1 \arrow[r, "q'^*\pi'_*"] & \mathsf{Par}(v'_2q')^*\varrho_1 \arrow[uuuur, "(v'_2q')_*"'] &
\end{tikzcd}
\; ,
$$

while the right hand side of (3.17) is given by

$$
R \; = \;
\begin{tikzcd}
 & \mathsf{Par}(r_0p)^*\varrho_0 \arrow[r, "p^*\lambda_*"] & \mathsf{Par}(r_1p)^*\varrho_2 \arrow[rr, "\varrho_1(\eta)"] & & \mathsf{Par}(v_1q)^*\varrho_1 \arrow[r, "q^*\pi_*"] \arrow[d, "q_*"] & \mathsf{Par}(v_2q)^*\varrho_1 \arrow[d, "q_*"] \arrow[dl, phantom, "\cong"] \arrow[dddr, "(v_2q)_*"] & \\
 & \mathsf{Par}\,r_0^*\varrho_0 \arrow[r, "\lambda_*"] \arrow[u, "p^*"] \arrow[d, "t^*"] & \mathsf{Par}\,r_1^*\varrho_1 \arrow[u, "p^*"] \arrow[d, "t^*"] \arrow[ddr, "r_{1*}"] & \arrow[dd, phantom, "\eta_*\Downarrow"] & \mathsf{Par}\,v_1^*\varrho_1 \arrow[r, "\pi_*"] \arrow[d, "t^*"] & \mathsf{Par}\,v_2^*\varrho_2 \arrow[d, "t^*"] \arrow[ddr, "v_{2*}"] & \\
 & \mathsf{Par}(r_0t)^*\varrho_0 \arrow[r, "t^*\lambda_*"] \arrow[dd, "\varrho_0(\alpha_0)_*"] & \mathsf{Par}(r_1t)^*\varrho_1 \arrow[dd, "\varrho_1(\alpha_1)_*"] \arrow[ddl, phantom, "\Swarrow\omega_*"] & & \mathsf{Par}(v_1u)^*\varrho_1 \arrow[r, "u^*\pi_*"] \arrow[dd, "\varrho_1(\beta_0)_*"] & \mathsf{Par}(v_2u)^*\varrho_1 \arrow[dd, "\varrho_2(\beta_1)_*"] \arrow[ddl, phantom, "\Swarrow\widetilde{\omega}_*"] & \\
\mathsf{Par}\,\varrho_0 \arrow[uuur, "(r_0p)^*"] \arrow[uur, "r_0^*"'] \arrow[ddr, "r'_0{}^*"] \arrow[dddr, "(r'_0p')^*"'] \arrow[r, phantom, "\alpha_{0*}\Swarrow"] & {} & {} \arrow[r, phantom, "\alpha_1^*\Downarrow"] & \mathsf{Par}\,\varrho_1 \arrow[uur, "v_1^*"] \arrow[ddr, "v'_1{}^*"'] \arrow[r, phantom, "\Downarrow\beta_{0*}"] & {} & {} \arrow[r, phantom, "\beta_1^*\Swarrow"] & \mathsf{Par}\,\varrho_2 \\
 & \mathsf{Par}(r'_0t')^*\varrho_0 \arrow[r, "t'^*\lambda'_*"] \arrow[d, "t'_*"] & \mathsf{Par}(r'_1t')^*\varrho_1 \arrow[d, "t'_*"] \arrow[dl, phantom, "\cong"] & \arrow[dd, phantom, "\eta^*\Downarrow"] & \mathsf{Par}(v'_1u')^*\varrho_1 \arrow[r, "u'^*\pi'_*"] \arrow[d, "t'_*"] & \mathsf{Par}(v'_2u')^*\varrho_2 \arrow[d, "t'_*"] \arrow[dl, phantom, "\cong"] & \\
 & \mathsf{Par}\,r'_0{}^*\varrho_0 \arrow[r, "\lambda'_*"] \arrow[d, "p'^*"] & \mathsf{Par}\,r'_1{}^*\varrho_1 \arrow[d, "p'^*"] \arrow[uur, "r'_{1*}"] & & \mathsf{Par}\,v'_1{}^*\varrho_1 \arrow[r, "\pi'_*"] & \mathsf{Par}\,v'_2{}^*\varrho_2 \arrow[uur, "v'_{2*}"'] \arrow[dl, phantom, "\cong"] & \\
 & \mathsf{Par}(r'_0p')^*\varrho_0 \arrow[r, "p'^*\lambda'_*"] & \mathsf{Par}(r'_1p')^*\varrho_2 \arrow[rr, "\varrho_1(\eta')"] & & \mathsf{Par}(v'_1q')^*\varrho_1 \arrow[r, "q'^*\pi'_*"] \arrow[u, "q'_*"] & \mathsf{Par}(v'_2q')^*\varrho_1 \arrow[u, "q'_*"] \arrow[uuur, "(v'_2q')_*"'] &
\end{tikzcd}
\; ,
$$

We now describe the 2-morphisms

$$L,\ R:\quad (v_2q)_*(q^*\pi_*)\varrho_1(\eta)(p^*\lambda_*)(r_0p)^* \longrightarrow (v'_2q')_*(q'^*\pi'_*)\varrho_1(\eta')(p'^*\lambda'_*)(r'_0p')^*$$

explicitly by chasing through the diagrams. We will look at the component for a parallel section $s \in \mathsf{Par}\,\varrho_0$. The image of s under the transformations will be evaluated at $x_2 \in \Gamma_2$. In the following step-by-step description of both transformations some of the obvious isomorphisms will not be mentioned explicitly in order to not obscure the main ideas:

Description of L: Using the canonical equivalences

$$(v_2q)^{-1}[x_2] \simeq v_2^{-1}[x_2] \times_\Pi (\Lambda \times_{\Gamma_1} \Pi) \simeq v_2^{-1}[x_2] \times_{\Gamma_1} \Lambda$$

that follow from the pasting law or more specifically Lemma 3.13, we obtain

$$((v_2q)_*(q^*\pi_*)\varrho_1(\eta)(p^*\lambda_*)(r_0p)^*s)(x_2) = \lim_{\substack{v_2^{-1}[x_2]\times_{\Gamma_1}\Lambda: \\ \bar{y}\in\Pi,\, v_2(\bar{y}) \overset{\xi}{\cong} x_2 \\ y\in\Lambda,\, r_1(y)\overset{\nu}{\cong} v_1(\bar{y})}} \varrho_2(\xi)\pi_{\bar{y}}\varrho_1(\nu)\lambda_y s(r_0(y)) \; . \tag{3.18}$$

The groupoid $v_2^{-1}[x_2] \times_{\Gamma_1} \Lambda$ is the index groupoid for the diagram that we need to compute the limit of. We have written the index groupoid below the limit symbol. After a double point we also listed all the dummy variables. We will use this notation in the sequel. The first transformation we have to apply is δ_1^*. Since

$$(v_2' q' d)^{-1}[x_2] \simeq {v_2'}^{-1}[x_2] \times_{\Pi'} (\Omega \times_{\Gamma_1} \widetilde{\Omega}) \ ,$$

its target is

$$\begin{aligned} &((v_2' q' d)_* \varrho_2(\widetilde{\ell^*\beta_1}) c^*(q^*\pi_*) \varrho_1(\eta)(p^*\lambda_*)(r_0 p)^* s)(x_2) \\ = &\lim_{\substack{{v_2'}^{-1}[x_2]\times_{\Pi'}(\Omega\times_{\Gamma_1}\widetilde{\Omega}): \\ \bar{y}'\in\Pi', v_2'(\bar{y}')\overset{\xi'}{\cong} x_2 \\ z\in\Omega, \widetilde{z}\in\widetilde{\Omega}, r_1 t(z)\overset{\mu}{\cong} v_1 u(\widetilde{z}) \\ \bar{y}'\overset{\kappa}{\cong} u'(\widetilde{z})}} \varrho_2(\xi' v_2'(\kappa)\beta_{1,\widetilde{z}}) \pi_{u(\widetilde{z})} \varrho_1(\mu) \lambda_{t(z)} s(r_0 t(z)) \ . \end{aligned} \tag{3.19}$$

The needed map from (3.18) to (3.19) is the pullback along the functor

$${v_2'}^{-1}[x_2] \times_{\Pi'} (\Omega \times_{\Gamma_1} \widetilde{\Omega}) \longrightarrow v_2^{-1}[x_2] \times_{\Gamma_1} \Lambda$$

which, on the level of dummy variables as established in (3.18) and (3.19), sends

$$\left(\bar{y}', v_2'(\bar{y}') \overset{\xi'}{\cong} x_2, z, \widetilde{z}, r_1 t(z) \overset{\mu}{\cong} v_1 u(\widetilde{z}), \bar{y}' \overset{\kappa}{\cong} u'(\widetilde{z})\right) \in {v_2'}^{-1}[x_2] \times_{\Pi'} (\Omega \times_{\Gamma_1} \widetilde{\Omega})$$

to

$$\left(u(\widetilde{z}), v_2 u(\widetilde{z}) \overset{\beta_1}{\cong} v_2' u'(\widetilde{z}) \overset{\kappa}{\cong} v_2'(\bar{y}') \overset{\xi'}{\cong} x_2, t(z), r_1 t(z) \overset{\mu}{\cong} v_1 u(\widetilde{z})\right) \in v_2^{-1}[x_2] \times_{\Gamma_1} \Lambda \ .$$

The next transformation does not change the index groupoids, but is the vertex-wise transformation

$$\varrho_2(\xi' v_2'(\kappa)) \varrho_2(\beta_{1,\widetilde{z}}) \pi_{u(\widetilde{z})} \varrho_1(\mu) \lambda_{t(z)} s(r_0 t(z)) \xrightarrow{\widetilde{\omega}_{\widetilde{z}}} \varrho_2(\xi' v_2'(\kappa)) \pi'_{u'(\widetilde{z})} \varrho_1(\beta_{0,\widetilde{z}}\mu) \lambda_{t(z)} s(r_0 t(z))$$

applied to the diagram that we take the limit of on the right hand side of (3.19). In the next step, we have to replace the groupoid $\Omega \times_{\Gamma_1} \widetilde{\Omega}$ as a part of the index groupoid in (3.19) by the canonically equivalent groupoid $(\Omega \times_{\Gamma_1} \widetilde{\Omega})'$, see Definition 3.19, 6 for the notation. More concretely, we replace $z \in \Omega$ and $\widetilde{z} \in \widetilde{\Omega}$ together with $r_1 t(z) \overset{\mu}{\cong} v_1 u(\widetilde{z})$ by the same pair $(z, z') \in \Omega \times \widetilde{\Omega}$, but now with $r_1' t'(z) \overset{\mu'}{\cong} r_1' t'(z)$, where $\mu' \alpha_{1,z} = \beta_{0,\widetilde{z}}\mu$. This leaves us with vertices

$$\varrho_2(\xi' v_2'(\kappa)) \pi'_{u'(\widetilde{z})} \varrho_1(\mu' \alpha_{1,z}) \lambda_{t(z)} s(r_0 t(z))$$

such that we can apply the vertex-wise transformation

$$\begin{aligned} \varrho_2(\xi' v_2'(\kappa)) \pi'_{u'(\widetilde{z})} \varrho_1(\mu' \alpha_{1,z}) \lambda_{t(z)} s(r_0 t(z)) &\xrightarrow{\omega_z} \varrho_2(\xi' v_2'(\kappa)) \pi'_{u'(\widetilde{z})} \varrho_1(\mu') \lambda_{t'(z)} \varrho_0(\alpha_{\alpha_0,z}) s(r_0 t(z)) \\ &\cong \varrho_2(\xi' v_2'(\kappa)) \pi'_{u'(\widetilde{z})} \varrho_1(\mu') \lambda_{t'(z)} s(r_0' t'(z)) \ , \end{aligned}$$

where the last isomorphism comes from parallelity of s. We need to perform one last step, namely the application of δ_{0*}: To this end, we have to pass from the index groupoid is ${v_2'}^{-1}[x_2] \times_{\Pi'} (\Omega \times_{\Gamma_1} \widetilde{\Omega})'$ to the index groupoid for the final result

$$((v_2' q')_* ({q'}^* \pi'_*) \varrho_1(\eta') ({p'}^* \lambda'_*) (r_0' p')^* s)(x_2)$$

is ${v_2'}^{-1}[x_2] \times_{\Gamma_1} \Lambda'$. The needed map comes from pushing along the functor

$$ {v_2'}^{-1}[x_2] \times_{\Pi'} (\Omega \times_{\Gamma_1} \widetilde{\Omega})' \longrightarrow {v_2'}^{-1}[x_2] \times_{\Gamma_1} \Lambda' \ , $$

which is induced from projection to Ω and $t' : \Omega \longrightarrow \Lambda'$. In summary, the natural transformation $L = \mathsf{Par}(\Omega \times_{\Gamma_1} \widetilde{\Omega}, \omega \times_{\Gamma_1} \widetilde{\omega})$ consists of the maps

$$ ((v_2 q)_*(q^*\pi_*)\varrho_1(\eta)(p^*\lambda_*)(r_0 p)^* s)(x_2) \longrightarrow ((v_2' q')_*({q'}^*\pi_*')\varrho_1(\eta')({p'}^*\lambda_*')(r_0' p')^* s)(x_2) $$

obtained by performing two operations:

- Apply ω and $\widetilde{\omega}$ vertex-wise to the diagrams involved.
- Compute on the level of the index groupoids the pull-push map along the span

$$ v_2^{-1}[x_2] \times_{\Gamma_1} \Lambda \longleftarrow {v_2'}^{-1}[x_2] \times_{\Pi'} (\Omega \times_{\Gamma_1} \widetilde{\Omega}) \simeq {v_2'}^{-1}[x_2] \times_{\Pi'} (\Omega \times_{\Gamma_1} \widetilde{\Omega})' \longrightarrow {v_2'}^{-1}[x_2] \times_{\Gamma_1} \Lambda' \ . \tag{3.20} $$

These two operations obviously commute.

Description of R: The maps

$$ ((v_2 q)_*(q^*\pi_*)\varrho_1(\eta)(p^*\lambda_*)(r_0 p)^* s)(x_2) \longrightarrow ((v_2' q')_*({q'}^*\pi_*')\varrho_1(\eta')({p'}^*\lambda_*')(r_0' p')^* s)(x_2) $$

that R consists of can be described similarly as for L. Since no new ideas enter, we just give the result. Again, we have to perform two commuting operations:

- Apply ω and $\widetilde{\omega}$ vertex-wise to the diagrams involved.
- Compute on the level of the index groupoids the pull-push map along the two composable spans

$$ \begin{array}{ccccc} {v_2}^{-1}[x_2] \times_{\Gamma_1} \Lambda & \longleftarrow & {v_2'}^{-1}[x_2] \times_{\Pi'} (\widetilde{\Omega} \times_{\Gamma_1} \Lambda) & \longrightarrow & {v_2'}^{-1}[x_2] \times_{\Gamma_1} \Lambda \\ {v_2'}^{-1}[x_2] \times_{\Gamma_1} \Lambda & \longleftarrow & {v_2'}^{-1}[x_2] \times_{\Gamma_1} \Omega & \longrightarrow & {v_2'}^{-1}[x_2] \times_{\Gamma_1} \Lambda' \ . \end{array} \tag{3.21} $$

The composition of the spans in (3.21) is (equivalent to) the span in (3.20). Indeed, we find the canonical equivalences

$$ \begin{aligned} & \left({v_2'}^{-1}[x_2] \times_{\Pi'} (\widetilde{\Omega} \times_{\Gamma_1} \Lambda)\right) \times_{{v_2'}^{-1}[x_2] \times_{\Gamma_1} \Lambda} \left({v_2'}^{-1}[x_2] \times_{\Gamma_1} \Omega\right) \\ \simeq & \left(\left({v_2'}^{-1}[x_2] \times_{\Gamma_1} \Lambda\right) \times_{\Pi'} \widetilde{\Omega}\right) \times_{{v_2'}^{-1}[x_2] \times_{\Gamma_1} \Lambda} \left({v_2'}^{-1}[x_2] \times_{\Gamma_1} \Omega\right) \\ \simeq & \widetilde{\Omega} \times_{\Pi'} \left({v_2'}^{-1}[x_2] \times_{\Gamma_1} \Omega\right) \\ \simeq & {v_2'}^{-1}[x_2] \times_{\Pi'} (\Omega \times_{\Gamma_1} \widetilde{\Omega}) \end{aligned} $$

Applying Proposition 3.3 (equivariant Beck-Chevalley condition) now finishes the proof of (3.17).

(v) We endow Par with a monoidal structure. For this we use for any 2-vector bundle ϱ over Γ and ξ over Ω the obvious 2-linear maps

$$ \Phi : \mathsf{Par}\, \varrho \boxtimes \mathsf{Par}\, \xi \longrightarrow \mathsf{Par}(\varrho \boxtimes \xi) $$

defined using the universal property of the Deligne product. Note that we suppress the groupoids in the notation, i.e. we use the shorthand $\mathsf{Par}\, \varrho = \mathsf{Par}(\Gamma, \varrho)$ etc. It remains to show that these 2-linear maps are equivalences. For the proof, we can assume without loss of generality that Γ and Ω are finite groups G and H, in which case ϱ and ξ send the one object to a 2-vector space $\mathcal{V}$ and $\mathcal{W}$, respectively. Now we use Proposition 3.8 and the notation used therein to write the relevant 2-vector spaces of parallel sections as

$$\mathsf{Par}\,\varrho \simeq \bigoplus_{\mathcal{O}\in\mathscr{S}/G} \mathfrak{A}_\alpha(G,\mathcal{O})\text{-}\mathsf{Mod}\ ,$$
$$\mathsf{Par}\,\xi \simeq \bigoplus_{\mathcal{P}\in\mathscr{T}/H} \mathfrak{A}_\beta(H,\mathcal{P})\text{-}\mathsf{Mod}\ ,$$

where $\mathscr{S}$ and $\mathscr{T}$ is the set of isomorphism classes of simple objects in $\mathcal{V}$ and $\mathcal{W}$ with representing systems $(X_s)_{s\in\mathscr{S}}$ and $(Y_t)_{t\in\mathscr{T}}$, respectively, and $\alpha \in H^2(G;\mathrm{Map}(\mathscr{S},\mathbb{C}^\times))$ and $\beta \in H^2(H;\mathrm{Map}(\mathscr{T},\mathbb{C}^\times))$ are the corresponding cohomology classes, see Section 3.1.3. The set of isomorphism classes of $\mathcal{V}\boxtimes\mathcal{W}$ is given by $\mathscr{S}\times\mathscr{T}$ with a representing system $(X_s\boxtimes Y_t)_{(s,t)\in\mathscr{S}\times\mathscr{T}}$. Denote the corresponding cohomology class by $\gamma \in H^2(G\times H;\mathrm{Map}(\mathscr{S}\times\mathscr{T},\mathbb{C}^\times))$. Using $(\mathscr{S}\times\mathscr{T})//(G\times H) \simeq \mathscr{S}//G\times\mathscr{T}//H$ we obtain

$$\mathsf{Par}(\varrho\boxtimes\xi) \simeq \bigoplus_{(\mathcal{O},\mathcal{P})\in\mathscr{S}/G\times\mathscr{T}/H} \mathfrak{A}_\gamma(G\times H;(\mathcal{O},\mathcal{P}))\text{-}\mathsf{Mod}.$$

This yields the weakly commutative diagram

$$\begin{tikzcd}
\mathsf{Par}\,\varrho\boxtimes\mathsf{Par}\,\xi \arrow[r, "\Phi"] \arrow[d, "\simeq"'] & \mathsf{Par}(\varrho\boxtimes\xi) \arrow[d, "\simeq"] \\
\left(\bigoplus_{\mathcal{O}\in\mathscr{S}} \mathfrak{A}_\alpha(G,\mathcal{O})\text{-}\mathsf{Mod}\right)\boxtimes\left(\bigoplus_{\mathcal{P}\in\mathscr{T}} \mathfrak{A}_\beta(H,\mathcal{P})\text{-}\mathsf{Mod}\right) \arrow[r, "\Psi"] & \bigoplus_{(\mathcal{O},\mathcal{P})\in\mathscr{S}/G\times\mathscr{T}/H} \mathfrak{A}_\gamma(G\times H;(\mathcal{O},\mathcal{P}))\text{-}\mathsf{Mod}\ ,
\end{tikzcd}$$

in which the vertical equivalences are the ones just discussed. The functor Ψ admits the following description: The projections $\mathscr{S}\times\mathscr{T}\longrightarrow\mathscr{S}$ and $\mathscr{S}\times\mathscr{T}\longrightarrow\mathscr{T}$ yield maps

$$\mathrm{Map}(\mathscr{S},\mathbb{C}^\times)\longrightarrow\mathrm{Map}(\mathscr{S}\times\mathscr{T},\mathbb{C}^\times),$$
$$\mathrm{Map}(\mathscr{T},\mathbb{C}^\times)\longrightarrow\mathrm{Map}(\mathscr{S}\times\mathscr{T},\mathbb{C}^\times).$$

Together with the projections $G\times H\longrightarrow G$ and $G\times H\longrightarrow H$, they induce a map

$$\begin{aligned}&H^2(G;\mathrm{Map}(\mathscr{S},\mathbb{C}^\times))\oplus H^2(H;\mathrm{Map}(\mathscr{T},\mathbb{C}^\times))\\ \longrightarrow &H^2(G\times H;\mathrm{Map}(\mathscr{S}\times\mathscr{T},\mathbb{C}^\times))\oplus H^2(G\times H;\mathrm{Map}(\mathscr{S}\times\mathscr{T},\mathbb{C}^\times))\ .\end{aligned}$$

Using the group operation in $H^2(G\times H;\mathrm{Map}(\mathscr{S}\times\mathscr{T},\mathbb{C}^\times))$ we obtain a map

$$H^2(G;\mathrm{Map}(\mathscr{S},\mathbb{C}^\times))\oplus H^2(H;\mathrm{Map}(\mathscr{T},\mathbb{C}^\times))\longrightarrow H^2(G\times H;\mathrm{Map}(\mathscr{S}\times\mathscr{T},\mathbb{C}^\times))$$

sending (α,β) to γ. With this observation in mind, Ψ is the composition of equivalences

$$\begin{aligned}&\left(\bigoplus_{\mathcal{O}\in\mathscr{S}} \mathfrak{A}_\alpha(G,\mathcal{O})\text{-}\mathsf{Mod}\right)\boxtimes\left(\bigoplus_{\mathcal{P}\in\mathscr{T}} \mathfrak{A}_\beta(H,\mathcal{P})\text{-}\mathsf{Mod}\right)\\
\simeq &\bigoplus_{(\mathcal{O},\mathcal{P})\in\mathscr{S}/G\times\mathscr{T}/H} \mathfrak{A}_\alpha(G,\mathcal{O})\text{-}\mathsf{Mod}\boxtimes\mathfrak{A}_\beta(H,\mathcal{P})\text{-}\mathsf{Mod}\\
\simeq &\bigoplus_{(\mathcal{O},\mathcal{P})\in\mathscr{S}/G\times\mathscr{T}/H} \left(\mathfrak{A}_\alpha(G,\mathcal{O})\otimes\mathfrak{A}_\beta(H,\mathcal{P})\right)\text{-}\mathsf{Mod}\\
\simeq &\bigoplus_{(\mathcal{O},\mathcal{P})\in\mathscr{S}/G\times\mathscr{T}/H} \mathfrak{A}_\gamma(G\times H;(\mathcal{O},\mathcal{P}))\text{-}\mathsf{Mod}\ .\end{aligned}$$

Since Ψ is an equivalence, so is Φ.

To fully specify the monoidal structure, we need to exhibit for 1-morphisms

$$(\Gamma_0, \varrho_0) \xleftarrow{r_0} (\Lambda, \lambda) \xrightarrow{r_1} (\Gamma_1, \varrho_1)$$

and

$$(\Gamma_0', \varrho_0') \xleftarrow{r_0'} (\Lambda', \lambda') \xrightarrow{r_1'} (\Gamma_1', \varrho_1')$$

in 2VecBunGrpd natural 2-isomorphisms

$$\begin{tikzcd}
\mathsf{Par}(\Gamma_0, \varrho_0) \boxtimes \mathsf{Par}(\Gamma_0', \varrho_0') \arrow[r, "\mathsf{Par}(\Lambda,\lambda)\boxtimes\mathsf{Par}(\Lambda',\lambda')"] \arrow[d, "\Phi"'] & \mathsf{Par}(\Gamma_1, \varrho_1) \boxtimes \mathsf{Par}(\Gamma_1', \varrho_1') \arrow[d, "\Phi"] \arrow[dl, Rightarrow, "\cong"'] \\
\mathsf{Par}((\Gamma_0, \varrho_0) \boxtimes (\Gamma_0', \varrho_0')) \arrow[r, "\mathsf{Par}((\Lambda,\lambda)\boxtimes(\Lambda,\lambda))"'] & \mathsf{Par}((\Gamma_1', \varrho_1') \boxtimes (\Gamma_1', \varrho_1'))
\end{tikzcd} .$$

When evaluated on $s \in \mathsf{Par}(\Gamma_0, \varrho_0)$, $s' \in \mathsf{Par}(\Gamma_0, \varrho_0)$ and $(y, y') \in \Gamma_1 \times \Gamma_1'$, they are given by

$$\begin{aligned}
&((\Phi \circ (\mathsf{Par}(\Lambda,\lambda) \boxtimes \mathsf{Par}(\Lambda',\lambda')))\,(s \boxtimes s'))\,(y, y') \\
=&(\mathsf{Par}(\Lambda,\lambda)s)(y) \boxtimes (\mathsf{Par}(\Lambda',\lambda')s')(y') \\
=& \lim_{(x,g)\in r_1^{-1}[y]} g.\lambda_x s(r_0(x)) \boxtimes \lim_{(x',g')\in r_1'^{-1}[y']} g'.\lambda_{x'} s'(r_0'(x')) \\
\cong& \lim_{((x,g),(x',g'))\in r_1^{-1}[y]\times r_1'^{-1}[y']} g.\lambda_x s(r_0(x)) \boxtimes g'.\lambda_{x'} s'(r_0'(x')) \\
\cong& \lim_{((x,g),(x',g'))\in r_1^{-1}[y]\times r_1'^{-1}[y']} (g.\lambda_x \boxtimes g'.\lambda_{x'})(s(r_0(x)) \boxtimes s'(r_0'(x'))) \\
\cong& \lim_{((x,x'),(g,g'))\in (r_1\times r_1')^{-1}[y,y']} (g.\lambda_x \boxtimes g'.\lambda_{x'})(s(r_0(x)) \boxtimes s'(r_0'(x'))) \\
=&(((\mathsf{Par}((\Lambda,\lambda)\boxtimes(\Lambda,\lambda)) \circ \Phi))(s \boxtimes s'))\,(y, y') \ .
\end{aligned}$$

Here we used that the Deligne product preserves limits (because $\boxtimes$-tensoring is exact). This concludes the definition of the monoidal structure. This monoidal structure is also symmetric: For a monoidal functor between symmetric monoidal bicategories the symmetry is *structure* and is given by natural isomorphisms

$$\begin{tikzcd}
\mathsf{Par}\,\varrho \boxtimes \mathsf{Par}\,\xi \arrow[r, "\Phi"] \arrow[d, "c_{\mathsf{Par}\,\varrho,\mathsf{Par}\,\xi}"'] & \mathsf{Par}(\varrho \boxtimes \xi) \arrow[d, "\mathsf{Par}\,c_{\varrho,\xi}"] \arrow[dl, Rightarrow, "\cong"'] \\
\mathsf{Par}\,\xi \boxtimes \mathsf{Par}\,\varrho \arrow[r, "\Phi"'] & \mathsf{Par}(\xi \boxtimes \xi)
\end{tikzcd}$$

for all 2-vector bundles ϱ over Γ and ξ over Ω, where the horizontal maps are the monoidal structure and the vertical maps are given by the braiding and the image thereof under the parallel section functor. In fact, it can be seen that this square even commutes strictly, so we obtain the needed symmetric structure from the identity maps.

□

The parallel section functors for 2-vector bundles generalizes the parallel section functor from [SW19]:

Proposition 3.26. *The restriction of the parallel section functor* Par : 2VecBunGrpd $\longrightarrow$ 2Vect *to the endomorphisms of the respective monoidal units of* 2VecBunGrpd *and* 2Vect *yields the* *parallel section functor* $\mathsf{VecBunGrpd} \longrightarrow \mathsf{FinVect}$ *from [SW19, Theorem 3.17] (see also [Tro16]), i.e. the square*

$$\begin{tikzcd} \mathrm{End}_{\mathsf{2VecBunGrpd}}(\star) \arrow[r, "\mathsf{Par}"] \arrow[d, "\Phi"'] & \mathrm{End}_{\mathsf{2Vect}}(\mathsf{FinVect}) \arrow[d, "\Psi"] \\ \mathsf{VecBunGrpd} \arrow[r, "\mathsf{Par}"'] & \mathsf{FinVect} \end{tikzcd}$$

featuring the equivalences from Proposition 3.21 and Example 2.5 commutes up to natural isomorphism.

Proof. The proof proceeds very much like the proof of Proposition 3.21, and we also use the notation established therein. An object in $\mathrm{End}_{\mathsf{2VecBunGrpd}}(\star)$ is a span $\star \xleftarrow{t} \Gamma \xrightarrow{t} \star$ together with an intertwiner $\lambda : t^*\tau \longrightarrow t^*\tau$, where τ is the trivial representation of the terminal groupoid $\star$ on $\mathsf{FinVect}$. The restriction

$$\mathsf{Par} : \mathrm{End}_{\mathsf{2VecBunGrpd}}(\star) \longrightarrow \mathrm{End}_{\mathsf{2Vect}}(\mathsf{FinVect})$$

of the parallel section functor sends this object to a 2-linear map $\mathsf{FinVect} \longrightarrow \mathsf{FinVect}$. Under Ψ this 2-linear map is identified with the vector space

$$(t_*\lambda_*t^*s)(\star) \ ,$$

where $s \in \mathsf{Par}(\star, \tau)$ is the parallel section sending $\star$ to $\mathbb{C}$. But by definition

$$(t_*\lambda_*t^*s)(\star) = \lim_{x\in\Gamma} \varrho_\lambda(x) \ ,$$

where the representation ϱ_λ of Γ is the image of Γ and λ under Φ (Proposition 3.21), and the limit of ϱ_λ is just the space of parallel sections of ϱ_λ.

Consider now a morphism

$$\begin{tikzcd} & (\Gamma_0, \lambda_0) \arrow[dl, "t"'] \arrow[dr, "t"] & \\ (\star, \tau) & (\Omega, \omega) \arrow[u, "r_1"'] \arrow[d, "r_0"] & (\star, \tau) \\ & (\Gamma_1, \lambda_1) \arrow[ul, "t"] \arrow[ur, "t"'] & \end{tikzcd}$$

in $\mathrm{End}_{\mathsf{2VecBunGrpd}}(\star)$. By what we have already seen in this proof, the parallel section functor assigns to this morphism the transformation

$$\begin{tikzcd} \mathsf{FinVect} \arrow[rr, bend left=60, "\mathbb{C} \longmapsto \lim\limits_{x_0 \in \Gamma_0} \varrho_{\lambda_0}(x_0)"] \arrow[rr, bend right=60, "\mathbb{C} \longmapsto \lim\limits_{x_1 \in \Gamma_1} \varrho_{\lambda_1}(x_1)"'] \arrow[rr, phantom, "\Downarrow"] & & \mathsf{FinVect} \end{tikzcd} ,$$

whose image under Ψ is the linear map

$$\mathsf{Par}\, \varrho_{\lambda_0} = \lim_{x_0 \in \Gamma_0} \varrho_{\lambda_0}(x_0) \longrightarrow \mathsf{Par}\, \varrho_{\lambda_1} = \lim_{x_1 \in \Gamma_1} \varrho_{\lambda_1}(x_1)$$

which by Definition 3.24 is the composition

$$\mathsf{Par}\, \varrho_{\lambda_0} \xrightarrow{r_0^*} \mathsf{Par}\, r_0^* \varrho_{\lambda_0} \xrightarrow{\omega_*} \mathsf{Par}\, r_1^* \varrho_{\lambda_1} \xrightarrow{r_{1*}} \mathsf{Par}\, \varrho_{\lambda_1} .$$

The names chosen for these three maps suggestively already coincide with the corresponding maps used for the parallel section functor in [SW19]. For the first map (the pullback map) this is indeed obvious. For the second map this follows from the fact that ω can be seen as an intertwiner $r_0^* \varrho_{\lambda_0} \longrightarrow r_1^* \varrho_{\lambda_1}$ as observed in the proof of Proposition 3.21. Finally, the fact that r_{1*} is really given by integral over homotopy fibers of r_1 with respect to groupoid cardinality (as the pushforward maps in [SW19]) follows from the application of Corollary 3.18 to the square

$$\begin{tikzcd} \Omega \arrow[r] \arrow[d, "r_1"'] & \star \arrow[d] \\ \Gamma_1 \arrow[r] & \star \end{tikzcd} .$$

□

4 The topological orbifold construction

As outlined in the introduction, the topological orbifold construction is a two-step procedure. One of the ingredients is the parallel section functor from Theorem 3.25 in the preceding chapter. In this chapter, we provide the second key ingredient, namely the change to equivariant coefficients which is a construction on the level of field theories. Afterwards, we combine both constructions into the topological orbifold construction, thereby achieving one of the main goals of this thesis.

4.1 Change to equivariant coefficients

Given a G-equivariant topological field theory $Z : G\text{-}\mathsf{Cob}(n, n-1, n-2) \longrightarrow \mathsf{2Vect}$ we will produce an ordinary topological field theory $\widehat{Z} : \mathsf{Cob}(n, n-1, n-2) \longrightarrow \mathsf{2VecBunGrpd}$ whose coefficients are the symmetric monoidal bicategory $\mathsf{2VecBunGrpd}$ from Definition 3.19. We will refer to these coefficients as *equivariant coefficients.*

As the parallel section functor, this construction has a precursor for non-extended equivariant field theories that is given in [SW19, Section 3.3].

Theorem 4.1. *For any finite group G, the assignment $Z \longmapsto \widehat{Z}$ from Proposition 2.6 naturally extends to a functor*

$$\widehat{-} : \mathsf{HSym}(G\text{-}\mathsf{Cob}(n, n-1, n-2), \mathsf{2Vect}) \longrightarrow \mathsf{Sym}(\mathsf{Cob}(n, n-1, n-2)\ , \mathsf{2VecBunGrpd}) \tag{4.1}$$

assigning to an extended G-equivariant topological field theory an extended topological field theory with values in $\mathsf{2VecBunGrpd}$*. We call* (4.1) *the change to equivariant coefficients.*

Proof. In the first step, we specify all the data needed to define $\widehat{Z}$ for an extended G-equivariant topological field theory $Z : G\text{-}\mathsf{Cob}(n, n-1, n-2) \longrightarrow \mathsf{2Vect}$:

(0) To an object S in $\mathsf{Cob}(n, n-1, n-2)$ we assign the 2-vector bundle $\widehat{Z}(S) : \Pi(S, BG) \longrightarrow \mathsf{2Vect}$ from Proposition 2.6 taking into account Remark 2.10.

(1) To a 1-morphism $\Sigma : S_0 \longrightarrow S_1$ in $\mathsf{Cob}(n, n-1, n-2)$ we assign the span

$$\Pi(S_0, BG) \xleftarrow{r_0} \Pi(\Sigma, BG) \xrightarrow{r_1} \Pi(S_1, BG)$$

(r_0 and r_1 are the obvious restriction functors) and the intertwiner

$$Z(\Sigma, -) : r_0^* \widehat{Z}(S_0) \longrightarrow r_1^* \widehat{Z}(S_1)$$

consisting of the map $Z(\Sigma, \varphi) : Z(S_0, \varphi|_{S_0}) \longrightarrow Z(S_1, \varphi|_{S_1})$ for each map $\varphi : \Sigma \longrightarrow BG$ and natural isomorphisms

$$\begin{tikzcd}
Z(S_0, \varphi|_{S_0}) \arrow[r, "Z(\Sigma, \varphi)"] \arrow[d, "Z(S_0 \times I, h|_{S_0})"'] & Z(S_1, \varphi|_{S_1}) \arrow[d, "Z(S_1 \times I, h|_{S_1})"] \arrow[dl, Rightarrow, "\cong"'] \\
Z(S_0, \varphi'|_{S_0}) \arrow[r, "Z(\Sigma, \varphi')"'] & Z(S_1, \varphi'|_{S_1})
\end{tikzcd} \tag{4.2}$$

for every equivalence class $\varphi \overset{h}{\simeq} \varphi'$ of homotopies between maps $\varphi, \varphi' : \Sigma \longrightarrow BG$. These isomorphisms (4.2) are obtained as follows: We will define below an invertible 2-morphism

$$((S_1 \times I) \circ \Sigma \circ (S_0 \times I), h|_{S_1} \cup \varphi \cup \mathrm{id}_{\varphi_0|_{S_0}}) \overset{\widehat{h}}{\Longrightarrow} ((S_1 \times I) \circ \Sigma \circ (S_0 \times I), \mathrm{id}_{\varphi'|_{S_1}} \cup \varphi' \cup h|_{S_0}) \quad (4.3)$$

in G-$\mathsf{Cob}(n, n-1, n-2)$ and use it together with the functoriality of Z to obtain the isomorphisms (4.2) as

$$\begin{aligned}
Z(S_1 \times I, h|_{S_1}) \circ Z(\Sigma, \varphi) &\cong Z(S_1 \times I, h|_{S_1}) \circ Z(\Sigma, \varphi) \circ Z(S_0 \times I, \mathrm{id}_{\varphi_0|_{S_0}}) \\
&\cong Z((S_1 \times I) \circ \Sigma \circ (S_0 \times I), h|_{S_1} \cup \varphi \cup \mathrm{id}_{\varphi_0|_{S_0}}) \\
&\overset{Z(\widehat{h})}{\cong} Z((S_1 \times I) \circ \Sigma \circ (S_0 \times I), \mathrm{id}_{\varphi'|_{S_1}} \cup \varphi' \cup h|_{S_0}) \\
&\cong Z(S_1 \times I, \mathrm{id}_{\varphi'|_{S_1}}) \circ Z(\Sigma, \varphi') \circ Z(S_0 \times I, h|_{S_0}) \\
&\cong Z(S_1 \times I, \mathrm{id}_{\varphi'|_{S_1}}) \circ Z(\Sigma, \varphi') \ .
\end{aligned}$$

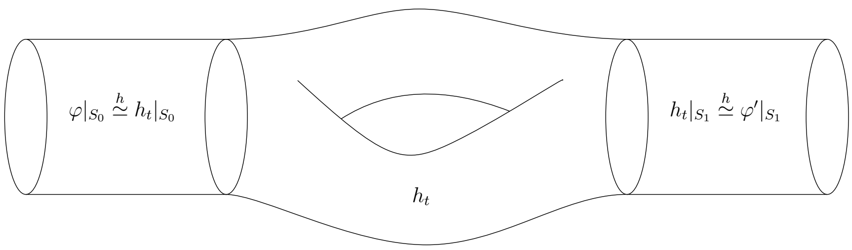
The needed 2-isomorphism (4.3) will be obtained as a homotopy

$$h|_{S_1} \cup \varphi \cup \mathrm{id}_{\varphi_0|_{S_0}} \overset{\widehat{h}}{\simeq} \mathrm{id}_{\varphi'|_{S_1}} \cup \varphi' \cup h|_{S_0} : (S_1 \times I) \circ \Sigma \circ (S_0 \times I) \longrightarrow BG$$

relative boundary, see also Remark 2.4 (f) to see how such a homotopy gives rise to an invertible 2-morphism. For the definition of this homotopy, we note that h gives rise to homotopies

$$\begin{aligned}
\varphi|_{S_0} &\overset{h}{\simeq} h_t|_{S_0} \ , \\
h_t|_{S_1} &\overset{h}{\simeq} \varphi'|_{S_1}
\end{aligned}$$

for all $t \in I$, which by abuse of notation we just denote by h again. Now the map $\widehat{h}_t : (S_0 \times I) \circ \Sigma \circ (S_1 \times I) \longrightarrow BG$ is defined by gluing together h_t and these two auxiliary homotopies as indicated in the picture



,

in which we see Σ with the cylinders over S_0 and S_1 glued to it. A direct computation shows that the isomorphisms (4.2) are coherent.

(2) To a 2-morphism $M : \Sigma_a \Longrightarrow \Sigma_b$ between 1-morphisms $\Sigma_a, \Sigma_b : S_0 \longrightarrow S_1$ the functor $\widehat{Z}$ assigns the strict span of spans

$$
\begin{tikzcd}
 & \Pi(\Sigma, BG) \arrow[dl, "r_0"'] \arrow[dr, "r_1"] & \\
\Pi(S_0, BG) & \Pi(M, BG) \arrow[u, "t"'] \arrow[d, "t'"] & \Pi(S_1, BG) \\
 & \Pi(\Sigma', BG) \arrow[ul, "r'_0"] \arrow[ur, "r'_1"'] &
\end{tikzcd}
$$

coming from restriction of maps together with the map

$$Z(M, -) : t^* Z(\Sigma, -) \longrightarrow t'^* Z(\Sigma', -)$$

of intertwiners coming from evaluation of Z on maps $M \longrightarrow BG$. For this to be really a map of intertwiners we need to verify the condition given in Remark 3.5 (c). Combining this with Remark 3.5 (b) we see that we need to prove that for every equivalence class $\psi \overset{h}{\simeq} \psi' : M \longrightarrow BG$ of homotopies the 2-cell

$$
\begin{tikzcd}
Z(S_0, \psi'|_{S_0}) \arrow[rr, "Z(\Sigma_a, \psi'|_{\Sigma_a})"] \arrow[dd, bend right=60, "\mathrm{id}_{Z(S_0, \psi'|_{S_0})}"'] \arrow[drr, phantom, "\cong"] & & Z(S_1, \psi'|_{S_1}) \arrow[dd, bend left=60, "\mathrm{id}_{Z(S_1, \psi'|_{S_1})}"] \\
Z(S_0, \psi|_{S_0}) \arrow[u, "Z(S_0 \times I, h|_{S_0})"] \arrow[d, "Z(S_0 \times I, h|_{S_0})"'] \arrow[rr, bend left, "Z(\Sigma_a, \psi|_{\Sigma_a})"] \arrow[rr, bend right, "Z(\Sigma_b, \psi|_{\Sigma_b})"'] \arrow[rr, phantom, "\Downarrow Z(M, \psi)"] & & Z(S_1, \psi|_{S_1}) \arrow[u, "Z(S_1 \times I, h|_{S_1})"'] \arrow[d, "Z(S_1 \times I, h|_{S_1})"] \\
Z(S_0, \psi'|_{S_0}) \arrow[rr, "Z(\Sigma_b, \psi'|_{\Sigma_b})"'] \arrow[urr, phantom, "\cong"] & & Z(S_1, \psi'|_{S_1})
\end{tikzcd}
\tag{4.4}
$$

in $\mathsf{2Vect}$, in which the 2-cells occupying the two squares in the middle block come from the definition of $\widehat{-}$ on 1-morphisms, is equal to $Z(M, \psi') : Z(\Sigma_a, \psi'|_{\Sigma_a}) \longrightarrow Z(\Sigma_b, \psi'|_{\Sigma_b})$. Indeed, this follows from homotopy invariance because (4.4) can be described by evaluation of Z on a map on M homotopic to ψ' relative ∂M.

In the next step, one needs to prove that $\widehat{Z}$ is a symmetric monoidal functor. The proof is relies on the gluing property of the stack $\Pi(-, BG)$ and the fact that Z is symmetric monoidal. In more detail, for two 1-morphisms $\Sigma : S_0 \longrightarrow S_1$ and $\Sigma' : S_1 \longrightarrow S_2$ in $\mathsf{Cob}(n, n-1, n-2)$ consider the diagram

$$\begin{tikzcd}
 & & \Pi(\Sigma' \circ \Sigma, BG) \arrow[dddll, "s_0"'] \arrow[d, "R"] \arrow[dddrr, "s_2"] & & \\
 & & \Pi(\Sigma, BG) \times_{\Pi(S_1,BG)} \Pi(\Sigma', BG) \arrow[dl, "p"'] \arrow[dr, "p'"] & & \\
 & \Pi(\Sigma, BG) \arrow[rr, Rightarrow, "\eta"] \arrow[dl, "r_0"'] \arrow[dr, "r_1"] & & \Pi(\Sigma', BG) \arrow[dl, "r_1'"'] \arrow[dr, "r_2'"] & \\
\Pi(S_0, BG) & & \Pi(S_1, BG) & & \Pi(S_2, BG)
\end{tikzcd} ,$$

where r_0, r_1, r_1', r_2, s_0 and s_2 are the restriction functors, the inner square is the homotopy pullback and R also comes from restriction. The gluing property of $\Pi(-, BG)$ says that R is an equivalence, which exhibits $\Pi(\Sigma' \circ \Sigma)$ as another model for the homotopy pullback (for this model the pullback square commutes strictly). Now by Remark 3.20 (a) the composition $\widehat{Z}(\Sigma') \circ \widehat{Z}(\Sigma)$ is canonically 2-isomorphic to the 1-morphism in $\mathsf{2VecBunGrpd}$ with span part

$$\Pi(S_0, BG) \xleftarrow{s_0} \Pi(\Sigma \circ \Sigma', BG) \xrightarrow{s_2} \Pi(S_2, BG)$$

and intertwiner $s_0^* \widehat{Z}(S_0) \longrightarrow s_2^* \widehat{Z}(S_2)$ whose evaluation on $\varphi : \Sigma' \circ \Sigma \longrightarrow BG$ is given by

$$Z(\Sigma', \varphi|_{\Sigma'}) \circ Z(\Sigma, \varphi|_{\Sigma}) \cong Z(\Sigma' \circ \Sigma, \varphi) ,$$

where this last isomorphism is part of the data of Z. This gives us the needed isomorphism $\widehat{Z}(\Sigma') \circ \widehat{Z}(\Sigma) \cong \widehat{Z}(\Sigma' \circ \Sigma)$.

The proof of the strict preservation of vertical composition of 2-morphisms and the preservation of the horizontal composition of 2-morphisms up to the 2-isomorphisms for the composition of 1-morphisms just constructed proceeds in an analogous way using the gluing property of $\Pi(-, BG)$, see also [SW19, Theorem 3.9] for the non-extended case which uses similar arguments.

The symmetric monoidal structure comes from the canonical equivalences $\Pi(X \sqcup Y, BG) \simeq \Pi(X, BG) \times \Pi(Y, BG)$, where X and Y are spaces, and the monoidal structure of Z.

Finally, we observe that $\widehat{Z}$ is functorial in Z. □

4.2 Definition and explicit description of the orbifold construction

The orbifold construction for equivariant topological field theories is obtained by first changing to equivariant coefficients using Theorem 4.1 and then applying the parallel section functor

$$\mathsf{Par} : \mathsf{2VecBunGrpd} \longrightarrow \mathsf{2Vect}$$

from Theorem 3.25 which extends taking parallel sections of 2-vector bundles to a symmetric monoidal functor by means of pull-push constructions. We are now ready to state the following central definition:

Definition 4.2 (Orbifold construction for extended G-equivariant topological field theories)**.** Let G be a finite group. Then the *orbifold construction for extended G-equivariant topological* *field theories* is the functor

$$-/G : \mathsf{HSym}(G\text{-}\mathsf{Cob}(n,n-1,n-2),\mathsf{2Vect}) \longrightarrow \mathsf{Sym}(\mathsf{Cob}(n,n-1,n-2),\mathsf{2Vect})$$

from the 2-groupoid of extended G-equivariant topological field theories to the 2-groupoid of extended topological field theories defined as the concatenation

$$\begin{tikzcd}
\mathsf{HSym}(G\text{-}\mathsf{Cob}(n,n-1,n-2),\mathsf{2Vect}) \arrow[r, "\widehat{-}"] \arrow[dr, "-/G"'] & \mathsf{Sym}(\mathsf{Cob}(n,n-1,n-2),\mathsf{2VecBunGrpd}) \arrow[d, "\mathsf{Par}_* = \mathsf{Par}\circ -"] \\
 & \mathsf{Sym}(\mathsf{Cob}(n,n-1,n-2),\mathsf{2Vect})\ .
\end{tikzcd}$$

For an extended G-equivariant topological field theory Z, we call the extended topological field theory Z/G the *orbifold theory of* Z.

From the prescriptions for the change of coefficients and the definition of the parallel section functor, it is straightforward to deduce the following explicit description of the orbifold construction which we are going to need in Chapter 6:

Proposition 4.3. *For any finite group G and an extended G-equivariant topological field theory $Z : G\text{-}\mathsf{Cob}(n,n-1,n-2) \longrightarrow \mathsf{2Vect}$, the orbifold theory $Z/G : \mathsf{Cob}(n,n-1,n-2) \longrightarrow \mathsf{2Vect}$ admits the following description:*

(a) *To an object S in $\mathsf{Cob}(n,n-1,n-2)$ the orbifold theory assigns the 2-vector space*

$$\frac{Z}{G}(S) = \mathsf{Par}\,\widehat{Z}(S)$$

of parallel sections of the 2-vector bundle $\widehat{Z}(S) = Z(S,-)$ over the groupoid $\Pi(S,BG)$, see Proposition 2.6.

(b) *To a 1-morphism $\Sigma : S_0 \longrightarrow S_1$ in $\mathsf{Cob}(n,n-1,n-2)$ the orbifold theory assigns the 2-linear map (i.e. a linear functor)*

$$\frac{Z}{G}(\Sigma) \ :\ \frac{Z}{G}(S_0) = \mathsf{Par}\,\widehat{Z}(S_0) \longrightarrow \frac{Z}{G}(S_1) = \mathsf{Par}\,\widehat{Z}(S_1)$$

given by

$$\left(\frac{Z}{G}(\Sigma)s\right)(\xi_1) = \lim_{(\varphi,h)\in r_1^{-1}[\xi_1]} Z(S_1\times[0,1],h)Z(\Sigma,\varphi)s(\varphi|_{S_0}) \quad \text{for all} \quad s \in \mathsf{Par}\,\widehat{Z}(S_0),$$
$$\xi_1 : S_1 \longrightarrow BG,$$

where $r_1 : \Pi(\Sigma,BG) \longrightarrow \Pi(S_1,BG)$ is the restriction functor.

(c) *For a 2-morphism $M : \Sigma_a \Longrightarrow \Sigma_b$ between 1-morphisms $\Sigma_a, \Sigma_b : S_0 \longrightarrow S_1$ in $\mathsf{Cob}(n,n-1,n-2)$, the value of the 2-morphism*

$$\frac{Z}{G}(M) \ :\ \frac{Z}{G}(\Sigma_a) \longrightarrow \frac{Z}{G}(\Sigma_b)$$

on $s \in \mathsf{Par}\,\widehat{Z}(S_0)$ and $\xi_1 : S_1 \longrightarrow BG$ is given by the commutativity of the square

$$\begin{tikzcd}
\left(\dfrac{Z}{G}(\Sigma_a)s\right)(\xi_1) \arrow[r, "pull"] \arrow[d, "\frac{Z}{G}(M)"'] & \lim\limits_{\substack{(\varphi_b,h_b,\psi,g) \\ \in (r_1^b)^{-1}[\xi_1]\times_{\Pi(\Sigma_b,BG)}\Pi(M,BG)}} Z(S_1\times[0,1],h_b * g|_{S_1})Z(\Sigma_a,\psi|_{\Sigma_a})s(\psi|_{S_0}) \arrow[d, "Z(M,-)"] \\
\left(\dfrac{Z}{G}(\Sigma_b)s\right)(\xi_1) & \lim\limits_{\substack{(\varphi_b,h_b,\psi,g) \\ \in (r_1^b)^{-1}[\xi_1]\times_{\Pi(\Sigma_b,BG)}\Pi(M,BG)}} Z(S_1\times[0,1],h_b)Z(\Sigma_b,\psi|_{\Sigma_b})s(\psi|_{S_0}) \ , \arrow[l, "push"]
\end{tikzcd}$$

where

- *the pull map uses the pullback of limits along the functor* $(r_1^b)^{-1}[\xi_1] \times_{\Pi(\Sigma_b,BG)} \Pi(M,BG) \longrightarrow (r_1^a)^{-1}[\xi_1]$ *defined using the universal property of the homotopy pullbacks involved,*
- *the map labeled with* $Z(M,-)$ *uses the vertex-wise transformation coming from the transformation* $Z(\Sigma_a,\psi|_{\Sigma_a}) \xrightarrow{Z(M,\psi)} Z(\Sigma_b,\psi|_{\Sigma_b})$, *the isomorphism*

$$Z(S_1\times[0,1],g|_{S_1})Z(\Sigma_b,\psi|_{\Sigma_b}) \cong Z(\Sigma_b,\psi|_{\Sigma_b})Z(S_0\times[0,1],g|_{S_0})$$

  and the fact that s *is parallel,*
- *and the push map uses the pushforward of limits in 2-vector spaces, see Section 3.1.1, along the functor* $(r_1^b)^{-1}[\xi_1] \times_{\Pi(\Sigma_b,BG)} \Pi(M,BG) \longrightarrow (r_1^a)^{-1}[\xi_1]$ *defined using the universal property of the homotopy pullbacks involved.*

The orbifold construction for extended equivariant topological field theories generalizes previous work in [SW19]. Indeed, it can be compared with the orbifoldization in the non-extended case if we take into account that an extended field theory can be restricted to the endomorphisms of the empty set to obtain a non-extended field theory. Recalling how we generalized the change of coefficients in Theorem 4.1 and the parallel section functor, see in particular Proposition 3.26, we obtain the following statement:

Proposition 4.4. *For any finite group* G, *the square*

$$\begin{tikzcd}
\mathsf{HSym}(G\text{-}\mathsf{Cob}(n,n-1,n-2),\mathsf{2Vect}) \arrow[r, "-/G"] \arrow[d, "restriction"'] & \mathsf{Sym}(\mathsf{Cob}(n,n-1,n-2),\mathsf{2Vect}) \arrow[d, "restriction"] \\
\mathsf{HSym}(G\text{-}\mathsf{Cob}(n,n-1),\mathsf{Vect}) \arrow[r, "-/G"'] & \mathsf{Sym}(\mathsf{Cob}(n,n-1),\mathsf{Vect})
\end{tikzcd}$$

commutes up to natural isomorphism of functors. The upper horizontal functor is the bicategorical orbifold construction in the extended case from Definition 4.2, the lower horizontal functor is the categorical orbifold construction in the non-extended case from [SW19, Definition 4.3].

Example 4.5. Twisted Dijkgraaf-Witten theories form a class of extended topological field theories that naturally arise as an orbifold theory: In [MW20a] we construct from a cocycle $\theta \in Z^n(G;\mathrm{U}(1))$ an extended G-equivariant topological field theory $Z_\theta : G\text{-}\mathsf{Cob}(n,n-1,n-2) \longrightarrow \mathsf{2Vect}$, thereby lifting the construction in [Tur10b, I.2.1] to a bicategorical setting. We prove that its orbifold theory Z_θ/G is the twisted Dijkgraaf-Witten theory from [FQ93, Mor15].

4.3 Generalization of the orbifold construction to a pushforward along a group morphism

In [SW19] we generalized the non-extended orbifold construction to a push operation along an arbitrary morphism $\lambda : G \longrightarrow H$ of groups in the sense that the orbifold construction corresponds to the pushforward along the morphism $G \longrightarrow \{e\}$ to the trivial group. This is also possible for the extended orbifold construction as we will sketch now: First denote by

$$\lambda_* : \Pi(-, BG) \longrightarrow \Pi(-, BH)$$

the stack morphism induced by λ. For an extended G-equivariant topological field theory $Z : G\text{-}\mathsf{Cob}(n, n-1, n-2) \longrightarrow \mathsf{2Vect}$, we would like to define a symmetric monoidal functor $\widehat{Z}^\lambda : H\text{-}\mathsf{Cob}(n, n-1, n-2) \longrightarrow \mathsf{2VecBunGrpd}$. To an object (S, ξ) in $H\text{-}\mathsf{Cob}(n, n-1, n-2)$, i.e. an $(n-2)$-dimensional closed oriented manifold together with a map $\xi : S \longrightarrow BH$ it assigns the pullback $q^* Z(S, -)$ of the 2-vector bundle $Z(S, -) : \Pi(S, BG) \longrightarrow \mathsf{2Vect}$ along the functor $q : \lambda_*^{-1}[\xi] \longrightarrow \Pi(S, BG)$ featuring in the defining square of the homotopy fiber

$$\begin{tikzcd}
\lambda_*^{-1}[\xi] \arrow[r, "q"] \arrow[d] & \Pi(S, BG) \arrow[d, "\lambda_*"] \\
\star \arrow[r, "\xi"'] & \Pi(S, BH)
\end{tikzcd}$$

of $\lambda_* : \Pi(S, BG) \longrightarrow \Pi(S, BH)$ over ξ. On 1-morphisms and 2-morphisms one straightforwardly generalizes the assignments made for the change to equivariant coefficients to obtain $\widehat{Z}^\lambda$. The construction is functorial in Z, so we obtain the following result:

Proposition 4.6. *For any morphism $\lambda : G \longrightarrow H$ of finite groups, the assignment $Z \longmapsto \widehat{Z}^\lambda$ extends to a functor*

$$\widehat{-}^\lambda : \mathsf{HSym}(G\text{-}\mathsf{Cob}(n, n-1, n-2), \mathsf{2Vect}) \longrightarrow \mathsf{HSym}(H\text{-}\mathsf{Cob}(n, n-1, n-2), \mathsf{2VecBunGrpd}).$$

Definition 4.7. For a morphism $\lambda : G \longrightarrow H$ of finite groups, we define the *pushforward of G-equivariant topological field theories*

$$\lambda_* : \mathsf{HSym}(G\text{-}\mathsf{Cob}(n, n-1, n-2), \mathsf{2Vect}) \longrightarrow \mathsf{HSym}(H\text{-}\mathsf{Cob}(n, n-1, n-2), \mathsf{2Vect}) \tag{4.5}$$

along λ as the concatenation

$$\begin{tikzcd}
\mathsf{HSym}(G\text{-}\mathsf{Cob}(n, n-1, n-2), \mathsf{2Vect}) \arrow[r, "\widehat{-}^\lambda"] \arrow[dr, "\lambda_*"'] & \mathsf{HSym}(H\text{-}\mathsf{Cob}(n, n-1, n-2), \mathsf{2VecBunGrpd}) \arrow[d, "\mathsf{Par}_*"] \\
 & \mathsf{HSym}(H\text{-}\mathsf{Cob}(n, n-1, n-2), \mathsf{2Vect}) .
\end{tikzcd}$$

As announced above, the orbifold construction can be identified with the pushforward along the group morphism $G \longrightarrow \{e\}$ to the trivial group.

The results in [SW19, Section 4] generalize to the present context of extended field theories although the details are involved and will not be pursued further in this thesis: For composable morphisms $\lambda : G \longrightarrow H$ and $\mu : H \longrightarrow J$ of finite groups, we obtain the composition law $(\mu \circ \lambda)_* \simeq \mu_* \circ \lambda_*$, where $\simeq$ denotes a canonical coherent equivalence of functors between 2-groupoids. Then by sending a finite group G to the 2-groupoid of extended G-equivariant topological field theories and a morphism of finite groups to the corresponding push functor (4.5) we obtain a 3-functor

$$\mathsf{FinGrp} \longrightarrow 2\text{-}\mathsf{Grpd}$$

from the category of finite groups (seen as a tricategory with only identity 2-cells and 3-cells) to the tricategory of 2-groupoids.

In [MW20a] we use the pushforward for the construction of examples of extended homotopy quantum field theories.

5 The little bundles operad and evaluation on the circle

If we are given an extended equivariant topological field theory $Z : G\text{-}\mathsf{Cob}(3,2,1) \longrightarrow \mathsf{2Vect}$ in dimension three, we obtain a category for each G-bundle over the circle. In order to relate the topological orbifold construction in the 3-2-1-dimensional case with the algebraic orbifoldization, it is necessary to investigate the structure that is present on this family of categories obtained by evaluation on the circle. The description of this structure will be accomplished in this chapter, more precisely in Theorem 5.49.

The main technical tool will be a colored topological operad, the *little bundles operad*, for which a presentation in terms of generators and relations will be given. This will allow us to conclude that categorical little bundles algebras are equivalent to braided G-crossed categories. The latter is the key step towards Theorem 5.49.

While the results of this chapter are essential to profit from the interplay of topological and algebraic orbifoldization in Chapter 6, they are not only relevant in the context of orbifoldization. For example, Theorem 5.32 and the results of Section 5.4 should be of independent interest.

5.1 Maps on complements of little disks

The first section of this chapter will consist of rather technical preparations for the definition of the little bundles operad.

For $n \geq 1$ let E_n be the little n-disks operad, see e.g. [Fre17, Chapter 4]. Recall that the operations $E_n(r)$ in arity $r \geq 0$ for this topological operad are given as follows: Denote by $\mathbb{D}^n$ the closed n-dimensional disk $\mathbb{D}^n := \{x \in \mathbb{R}^n \,|\, |x|^2 \leq 1\}$. Then $E_n(r)$ is given by the space of all maps $f : \coprod_{k=1}^r \mathbb{D}^n \longrightarrow \mathbb{D}^n$ such that

- the restriction $f_k : \mathbb{D}^n \longrightarrow \mathbb{D}^n$ of f to the k-th disk is an affine embedding, i.e. given by a rescaling of the radius and a translation,
- for $1 \leq j < k \leq r$ the interiors of the images of f_j and f_k do not intersect, i.e. $\overset{\circ}{\operatorname{im}} f_j \cap \overset{\circ}{\operatorname{im}} f_k = \emptyset$.

We write $\mathsf{C}(f) := \mathbb{D}^n \setminus \overset{\circ}{\operatorname{im}} f$ for the complement of the interior of the image of f in the closed n-disk. This allows us to the define the subspace

$$W_n(r) := \{(f,x) \in E_n(r) \times \mathbb{D}^n \,|\, x \in \mathsf{C}(f)\} \subset E_n(r) \times \mathbb{D}^n \ .$$

The boundary $\partial\mathsf{C}(f)$ of $\mathsf{C}(f)$ consists of r spheres $\mathbb{S}^{n-1}$ (the *ingoing boundary* or *ingoing spheres*) possibly wedged together and sitting inside a bigger sphere $\mathbb{S}^{n-1}$ (the *outgoing boundary* or *outgoing sphere*).

The goal of this section is to define an operad E_n^T whose colors are maps from $\mathbb{S}^{n-1}$ to some fixed topological space T and whose operations from an r-tuple $\underline{\varphi} = (\varphi_1, \ldots, \varphi_r)$ of maps $\mathbb{S}^{n-1} \longrightarrow T$ (generally, we will use an underline to indicate tuples) to a single map $\psi : \mathbb{S}^{n-1} \longrightarrow T$ will be given by all operations $f \in E_n(r)$ together with maps $\mathsf{C}(f) \longrightarrow T$ whose restriction to $\partial\mathsf{C}(f)$ is $(\underline{\varphi}, \psi)$.

We will introduce the operad E_n^T formally in Section 5.1.2. The definition will not be as a homotopy quotient of a braid group action (as mentioned in the introduction), but will make use of the auxiliary constructions in Section 5.1.1. The description using a homotopy quotient is then discussed in Section 5.2.1.

5.1.1 The auxiliary spaces $W_n^T(r)$

For a topological space T, we define $W_n^T(r)$ as the set of pairs (f, ξ), where $f \in E_n(r)$ and $\xi : \mathsf{C}(f) \longrightarrow T$ is a continuous map. Projection onto the first factor yields a map

$$p : W_n^T(r) \longrightarrow E_n(r)$$

of sets. We equip $W_n^T(r)$ with the final topology with respect to all maps of sets $g : Y \longrightarrow W_n^T(r)$ from arbitrary topological spaces Y to the set W_n^T such that

(I) $p \circ g$ is a continuous map,

(II) and the natural map $W_n(r) \times_{E_n(r)} Y \longrightarrow T$ is continuous.

Proposition 5.1. *The map $p : W_n^T(r) \longrightarrow E_n(r)$ is a Serre fibration.*

The proof of this statement and some similar statements in the sequel involve some elementary, but tedious point-set topology. In order to not interrupt the general line of thought, the lengthy proofs are deferred to Section 5.1.3. The above statement will appear there as Proposition 5.8.

Next we define the subspace

$$\partial W_n(r) := \{(f, x) \in E_n(r) \times \mathbb{D}^n \,|\, x \in \partial\mathsf{C}(f)\} \subset W_n(r)$$

and note that there is a natural map

$$E_n(r) \times \coprod^{r+1} \mathbb{S}^{n-1} \longrightarrow \partial W_n(r) \tag{5.1}$$

of spaces over $E_n(r)$ identifying the first r copies of $\mathbb{S}^{n-1}$ with the ingoing boundary spheres and the last copy with the outgoing boundary spheres. This map is generally not a homeomorphism since some of the boundary spheres might be wedged together. By combining the restriction of the evaluation map

$$\partial W_n(r) \times_{E_n(r)} W_n^T(r) \longrightarrow T$$

(which will be shown to be continuous in Lemma 5.6 in the technical Section 5.1.3) with (5.1) we obtain a map

$$\left(\coprod^{r+1} \mathbb{S}^{n-1}\right) \times W_n^T(r) \longrightarrow T$$

which, by adjunction, gives us a map

$$q : W_n^T(r) \longrightarrow \prod^{r+1} \mathrm{Map}(\mathbb{S}^{n-1}, T) \ . \tag{5.2}$$

Proposition 5.2. *The map $q : W_n^T(r) \longrightarrow \prod^{r+1} \mathrm{Map}(\mathbb{S}^{n-1}, T)$ is a Serre fibration.*

This is another technical result whose proof will be given in Section 5.1.3 below (Proposition 5.9).

5.1.2 The operad E_n^T

From the map q we can construct a topological operad colored over the set of maps $\mathbb{S}^{n-1} \longrightarrow T$: Let $\underline{\varphi} = (\varphi_1, \ldots, \varphi_r)$ be an r-tuple of maps $\mathbb{S}^{n-1} \longrightarrow T$. Then for another map $\psi : \mathbb{S}^{n-1} \longrightarrow T$ we consider the fiber $E_n^T\binom{\psi}{\underline{\varphi}}$ of q over $(\underline{\varphi}, \psi)$, i.e. the pullback

$$\begin{tikzcd} E_n^T\binom{\psi}{\underline{\varphi}} \arrow[r] \arrow[d] & W_n^T(r) \arrow[d, "q"] \\ \star \arrow[r, "(\underline{\varphi}{,}\psi)"] & \prod^{r+1} \mathrm{Map}(\mathbb{S}^{n-1}, T) \ , \end{tikzcd}$$

which is also a homotopy pullback since q is a Serre fibration by Proposition 5.2. Explicitly, $E_n^T\binom{\psi}{\underline{\varphi}}$ consists of elements $f \in E_n(r)$ together with a map $\xi : \mathsf{C}(f) \longrightarrow T$ whose restriction to $\partial\mathsf{C}(f)$ is given by $(\underline{\varphi}, \psi)$. We will denote the point in $E_n^T\binom{\psi}{\underline{\varphi}}$ formed by f and ξ by $\langle f, \xi \rangle$.

The operad structure on E_n^T, which will be defined now, makes use of the operad structure of E_n for which we refer to [Fre17, Chapter 4]. The operadic identity $\star \longrightarrow E_n^T\binom{\varphi}{\varphi}$ is the operadic identity in $E_n(1)$, namely the identity embedding $I : \mathbb{D}^n \longrightarrow \mathbb{D}^n$, together with $\varphi : \mathsf{C}(I) = \mathbb{S}^{n-1} \longrightarrow T$. Moreover, the action of the symmetric group Σ_r on r letters on $E_n(r)$, for all $r \geq 0$, turns E_n^T into a $\mathrm{Map}(\mathbb{S}^{n-1}, T)$-colored symmetric sequence.

The operadic composition consists of maps

$$\circ : E_n^T\begin{pmatrix} \psi \\ \underline{\varphi} \end{pmatrix} \times \prod_{j=1}^r E_n^T\begin{pmatrix} \varphi_j \\ \underline{\lambda}_j \end{pmatrix} \longrightarrow E_n^T\begin{pmatrix} \psi \\ \otimes_{j=1}^r \underline{\lambda}_j \end{pmatrix} ,$$

where $\otimes$ denotes the juxtaposition of tuples. It sends

$$\left(\langle f, \xi \rangle, \prod_{j=1}^r \langle g_j, \mu_j \rangle \right) \in E_n^T\begin{pmatrix} \psi \\ \underline{\varphi} \end{pmatrix} \times \prod_{j=1}^r E_n^T\begin{pmatrix} \varphi_j \\ \underline{\lambda}_j \end{pmatrix}$$

to

$$\left\langle f \circ \underline{g}, \xi \cup_{\coprod^r \mathbb{S}^{n-1}} \underline{\mu} \right\rangle \in E_n^T\begin{pmatrix} \psi \\ \otimes_{j=1}^r \underline{\lambda}_j \end{pmatrix} ,$$

where

- the composition of f with the r-tuple $\underline{g}$ of embeddings is formed via the composition in E_n,
- we use that for $1 \leq j \leq r$ the restriction of $\mu_j : \mathsf{C}(g_j) \longrightarrow T$ to the last copy of $\mathbb{S}^{n-1}$ (the outer sphere) is precisely φ_j in order to glue ξ and $\underline{\mu}$ along r copies of $\mathbb{S}^{n-1}$.

Proposition 5.3. *Let T be any space. With the above definitions E_n^T is a topological operad colored over* $\mathrm{Map}(\mathbb{S}^{n-1}, T)$*.*

Proof. The only non-trivial point is the continuity of the composition maps. It suffices to prove that the partial compositions

$$\circ_j : E_n^T\begin{pmatrix} \psi \\ \varphi_1, \ldots, \varphi_j, \ldots, \varphi_r \end{pmatrix} \times E_n^T\begin{pmatrix} \varphi_j \\ \underline{\lambda} \end{pmatrix} \longrightarrow E_n^T\begin{pmatrix} \psi \\ \varphi, \ldots, \underline{\lambda}, \ldots, \varphi_r \end{pmatrix} \tag{5.3}$$

are continuous.

To this end, set $r' := |\underline{\lambda}|$ and consider the restriction $W_n^T(r) \longrightarrow \mathrm{Map}(\mathbb{S}^{n-1}, T)$ to the outer boundary sphere and the restriction $W_n^T(r') \longrightarrow \mathrm{Map}(\mathbb{S}^{n-1}, T)$ to the j-th ingoing boundary sphere and observe that $E_n^T\binom{\psi}{\varphi_1,\dots,\varphi_j,\dots,\varphi_r} \times E_n^T\binom{\varphi_j}{\underline{\lambda}}$ is a subspace of the pullback

$$W_n^T(r) \times_{\mathrm{Map}(\mathbb{S}^{n-1},T)} W_n^T(r')$$

such that (5.3) is the restriction of the map

$$\widehat{\circ}_j : W_n^T(r) \times_{\mathrm{Map}(\mathbb{S}^{n-1},T)} W_n^T(r') \longrightarrow W_n^T(r+r'-1), \quad \big(\langle f, \xi\rangle, \langle f', \xi'\rangle\big) \longmapsto \langle f \circ f', \xi \cup^j_{\mathbb{S}^{n-1}} \xi'\rangle \ .$$

Here $f \circ_j f'$ is the (partial) operadic composition in E_n and $\xi \cup^j_{\mathbb{S}^{n-1}} \xi'$ is the map obtained from gluing ξ and ξ' along the j-th sphere $\mathbb{S}^{n-1}$ in the domain of definition of ξ. The statement follows now from Lemma 5.10 in Section 5.1.3 below asserting that $\widehat{\circ}_j$ is continuous. □

Remark 5.4. An operad is called *Σ-cofibrant* if the underlying symmetric sequence is cofibrant in the projective model structure. This property is important for the homotopy theory of algebras over this operad and needed later in Section 5.3.2. The model for E_n used in this thesis is Σ-cofibrant [Fre17, page 140]. By the same arguments, E_n^T is Σ-cofibrant.

5.1.3 Some technicalities on the auxiliary spaces $W_n^T(r)$

This subsection contains a detailed investigation of the auxiliary spaces $W_n^T(r)$. This will lead to the technical statements whose proofs we had omitted in the preceding two subsections.

In order to investigate the spaces $W_n^T(r)$, we will need the following construction for a pair $\langle f, \xi\rangle \in W_n^T(r)$, i.e. for $f \in E_n(r)$ and a map $\xi : \mathsf{C}(f) \longrightarrow T$: First note that $\mathsf{C}(f)$ arises from $\mathbb{D}^n$ by cutting out r open disks specified by their radii and centers. We now reduce each of these radii by half. Additionally, we double the radius of the outer disk. The resulting manifold with boundary is a 'fattening' of $\mathsf{C}(f)$ and denoted by $\widehat{\mathsf{C}}(f)$, see Figure 5.1. One can use the value of ξ on the boundary of $\mathsf{C}(f)$ to extend it to a map $\widehat{\xi} : \widehat{\mathsf{C}}(f) \longrightarrow T$. This extension will be referred to as *radial extension*.

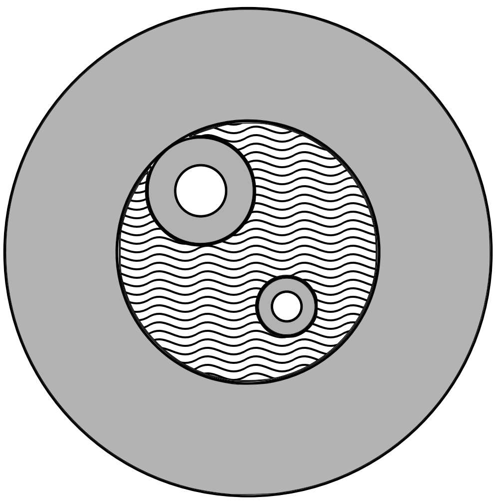


Figure 5.1: The fattening $\widehat{\mathsf{C}}(f)$ for an element f in $E_2(2)$. The complement $\mathsf{C}(f)$ of f is the area filled with wavy lines. The space $\widehat{\mathsf{C}}(f)$ is the union of wavy area and the gray areas.

We define the subspace

$$\widehat{W}_n(r) := \{(f,x) \in E_n(r) \times \mathbb{D}_2^n \mid x \in \widehat{\mathsf{C}}(f)\} \subset E_n(r) \times \mathbb{D}_2^n \ ,$$

where we denote by $\mathbb{D}_2^n$ the n-dimensional disk of radius 2.

Lemma 5.5 (Continuity of radial extension)**.** *For a topological space Y, let $g : Y \longrightarrow W_n^T(r)$ be a map of sets satisfying condition (I) and (II) on page 72. Then the radial extension*

$$\widehat{W}_n(r) \times_{E_n(r)} Y \longrightarrow T, \quad (f,x,y) \longmapsto \widehat{g(y)}(x) \tag{5.4}$$

is continuous.

Proof. Let $U \subset T$ be open and let (f_0, x_0, y_0) be in its preimage under (5.4). We need to exhibit a neighborhood of (f_0, x_0, y_0) in $\widehat{W}_n(r) \times_{E_n(r)} Y$ whose image under (5.4) is contained in U.

If $x_0 \in \overset{\circ}{\mathsf{C}}(f_0)$, then (f_0, x_0, y_0) is contained in the subspace $W_n(r) \times_{E_n(r)} Y \subset \widehat{W}_n(r) \times_{E_n(r)} Y$. By assumption the map $W_n(r) \times_{E_n(r)} Y \longrightarrow T$ is continuous, hence we find open neighborhoods V of f_0 in $E_n(r)$, V'' of y_0 in Y and, additionally, for some $\varepsilon > 0$ an open ball $\mathbb{B}_\varepsilon(x_0)$ of radius $\varepsilon > 0$ around x_0 such that the image of the open neighborhood $(V' \times \mathbb{B}_\varepsilon(x_0) \times V'') \cap W_n(r) \times_{E_n(r)} Y$ of (f_0, x_0, y_0) under $W_n(r) \times_{E_n(r)} Y \longrightarrow T$ is contained in U. But since

$$(V' \times \mathbb{B}_\varepsilon(x_0) \times V'') \cap W_n(r) \times_{E_n(r)} Y = (V' \times \mathbb{B}_\varepsilon(x_0) \times V'') \cap \widehat{W}_n(r) \times_{E_n(r)} Y$$

the set $(V' \times \mathbb{B}_\varepsilon(x_0) \times V'') \cap W_n(r) \times_{E_n(r)} Y$ is also an open neighborhood of (f_0, x_0, y_0) in $\widehat{W}_n(r) \times_{E_n(r)} Y$ being mapped to U under (5.4).

If $x_0 \in \partial \mathsf{C}(f)$, then, as in the case $x_0 \in \overset{\circ}{\mathsf{C}}(f_0)$, we find suitable neighborhoods V', V'' and $\mathbb{B}_\varepsilon(x_0)$ such that $(V' \times \mathbb{B}_\varepsilon(x_0) \times V'') \cap W_n(r) \times_{E_n(r)} Y$ is mapped to U (again by continuity of $W_n(r) \times_{E_n(r)} Y \longrightarrow T$). By construction of the radial extension, $(V' \times \mathbb{B}_\varepsilon(x_0) \times V'') \cap \widehat{W}_n(r) \times_{E_n(r)} Y$ is also mapped to U, which gives us the desired neighborhood is this case.

Now assume that x_0 is not in $\mathsf{C}(f_0)$. Then the corresponding point on the boundary of $\mathsf{C}(f)$ obtained by following a straight line in radial direction is also in the preimage of U, again by construction of the radial extension. For this point, there exists the desired open neighborhood as argued above. We can translate this neighborhood to x_0, rescale it to get the desired neighborhood for x_0. □

Lemma 5.6. *The evaluation map* $\mathrm{ev} : W_n(r) \times_{E_n(r)} W_n^T(r) \longrightarrow T$ *is continuous.*

Proof. Let $U \subset T$ be open and $(f_0, x_0, \xi_0) \in \mathrm{ev}^{-1}(U)$. We need to show that there exists an open neighborhood of (f_0, x_0, ξ_0) in $W_n(r) \times_{E_n(r)} W_n^T(r)$ such that its image under the evaluation is contained in U.

In a first step, consider the radial extension $\widehat{\xi}_0 : \widehat{\mathsf{C}}(f_0) \longrightarrow T$ of ξ_0 to the fattening $\widehat{\mathsf{C}}(f_0)$ of $\mathsf{C}(f_0)$ as discussed above on page 74. By continuity of $\widehat{\xi}_0$ there is an $\varepsilon > 0$ such that

(a) $\mathbb{B}_\varepsilon(x_0) \subset \widehat{\mathsf{C}}(f_0)$,

(b) $\overline{\mathbb{B}_\varepsilon(x_0)} \cap \partial\widehat{\mathsf{C}}(f_0) = \emptyset$

(c) and $\widehat{\xi}_0\left(\overline{\mathbb{B}_\varepsilon(x_0)}\right) \subset U$.

Now define the subset $U_W \subset W_n^T(r)$ of those $(f,\xi) \in W_n^T(r)$ for which

(a′) $\mathbb{B}_\varepsilon(x_0) \subset \widehat{\mathsf{C}}(f)$, (b′) $\overline{\mathbb{B}_\varepsilon(x_0)} \cap \partial\widehat{\mathsf{C}}(f) = \emptyset$

(c′) and $\widehat{\xi}(\overline{\mathbb{B}_\varepsilon(x_0)}) \subset U$.

Next recall that $W_n(r) \times_{E_n(r)} W_n^T(r)$ is a subspace of $E_n(r) \times \mathbb{D}_2^n \times W_n^T(r)$, where $\mathbb{D}_2^n$ is the closed n-disk of radius 2. The intersection of $E_n(r) \times \mathbb{B}_\varepsilon(x_0) \times U_W \subset E_n(r) \times \mathbb{D}_2^n \times W_n^T(r)$ with $W_n(r) \times_{E_n(r)} W_n^T(r)$ contains (f_0, x_0, ξ_0) and is mapped to U under the evaluation. Hence, it remains to show that $E_n(r) \times \mathbb{B}_\varepsilon(x_0) \times U_W$ is open in $W_n(r) \times \mathbb{D}_2^n \times W_n^T(r)$. For this, it suffices to prove that U_W is open in $W_n^T(r)$.

By definition of the final topology, a subset $V \subset W_n^T(r)$ is open if and only if for all maps $g : Y \longrightarrow W_n^T(r)$ of sets from a topological space Y satisfying the conditions (I) and (II) on page 72 the preimage $g^{-1}(V)$ of V is open. For the proof that U_W meets this requirement, let $g : Y \longrightarrow W_n^T(r)$ be a map satisfying conditions (I) and (II). First we remark that it follows from conditions (a′) and (b′) above that the image $p(U_W)$ of U_W under the projection $p : W_n^T(r) \longrightarrow E_n(r)$ is open in $E_n(r)$. Hence, by (I) the set $Y' := (p \circ g)^{-1}(p(U_W)) \subset Y$ is also open. This implies that a subset of Y' is open if and only if it is open seen as a subset of Y. By (II) the map $W_n(r) \times_{E_n(r)} Y \longrightarrow T$ is continuous. From this we conclude that its restriction $W_n(r) \times_{E_n(r)} Y' \longrightarrow T$ is continuous as well. It naturally gives rise to a map

$$\widehat{W}_n(r) \times_{E_n(r)} Y' \longrightarrow T, \quad (f, x, y) \longmapsto \widehat{g(y)}(x)$$

which is continuous by Lemma 5.5.

There is a natural embedding

$$\overline{\mathbb{B}_\varepsilon(x_0)} \times Y' \longrightarrow \widehat{W}_n(r) \times_{E_n(r)} Y', \quad (x, y) \longmapsto (p \circ g(y), x, y)$$

which is continuous by the universal property of the subspace and product topology. This implies that the composition

$$\begin{aligned} \overline{\mathbb{B}_\varepsilon(x_0)} \times Y' &\longrightarrow T \\ (x, y) &\longmapsto \widehat{g(y)}(x) \end{aligned}$$

is continuous and hence, by adjunction, gives rise to a continuous map

$$Y' \longrightarrow \mathrm{Map}(\overline{\mathbb{B}_\varepsilon(x_0)}, T) \ . \tag{5.5}$$

By definition the subspace $M(\overline{\mathbb{B}_\varepsilon(x_0)}, U) \subset \mathrm{Map}(\overline{\mathbb{B}_\varepsilon(x_0)}, T)$ of all maps $\overline{\mathbb{B}_\varepsilon(x_0)} \longrightarrow T$ sending $\overline{\mathbb{B}_\varepsilon(x_0)}$ to U is open in the compact-open topology. This implies that the preimage of $M(\overline{\mathbb{B}_\varepsilon(x_0)}, U)$ under (5.5) is open. But this preimage is just $g^{-1}(U_W)$ showing that $g^{-1}(U_W)$ is open and finishing the proof. □

Lemma 5.7. *For $f \in E_n(r)$, denote by $p^{-1}(f)$ the fiber of $p : W_n^T(r) \longrightarrow E_n(r)$ over f endowed with the subspace topology induced from $W_n^T(r)$. Then the evaluation $\mathsf{C}(f) \times p^{-1}(f) \longrightarrow T$ is continuous, and the topology on $p^{-1}(f)$ agrees with the compact-open topology, i.e. $p^{-1}(f) = \mathrm{Map}(\mathsf{C}(f), T)$.*

Proof. The map $\mathsf{C}(f) \times p^{-1}(f) \longrightarrow T$ is the restriction of the map $\mathrm{ev} : W_n(r) \times_{E_n(r)} W_n^T(r) \longrightarrow T$ from Lemma 5.6 to the fiber of $W_n(r) \times_{E_n(r)} W_n^T(r) \longrightarrow E_n(r)$ over f and hence continuous.

Moreover, $p^{-1}(f) = \mathrm{Map}(\mathsf{C}(f), T)$ as sets, so it remains to show that the identity map is continuous in both directions.

The identity is continuous as a map $\mathrm{Map}(\mathsf{C}(f), T) \longrightarrow p^{-1}(f)$: For this it suffices to show that the composition $\mathrm{Map}(\mathsf{C}(f), T) \longrightarrow W_n^T(r)$ with the inclusion $p^{-1}(f) \longrightarrow W_n^T(r)$ is continuous. The composition of $\mathrm{Map}(\mathsf{C}(f), T) \longrightarrow W_n^T(r) \longrightarrow E_n(r)$ factors through $\{f\} \longrightarrow E_n(r)$ and is therefore continuous, so condition (I) is fulfilled. For condition (II) to be fulfilled, we need the evaluation map $\mathsf{C}(f) \times \mathrm{Map}(\mathsf{C}(f), T) \longrightarrow T$ to be continuous. But this is the case because $\mathsf{C}(f)$ is locally compact.

The identity is continuous as a map $p^{-1}(f) \longrightarrow \mathrm{Map}(\mathsf{C}(f), T)$: By adjunction (and since $\mathsf{C}(f)$ is locally compact), continuity of $p^{-1}(f) \longrightarrow \mathrm{Map}(\mathsf{C}(f), T)$ is equivalent to continuity of $\mathsf{C}(f) \times p^{-1}(f) \longrightarrow T$, which we have already established. $\square$

Proposition 5.8. *The map $p : W_n^T(r) \longrightarrow E_n(r)$ is a Serre fibration.*

Proof. Let us fix $f_0 \in E_n(r)$ and an $\varepsilon > 0$ smaller than the radii of all disks in the image of f_0. Let $X := \bigcup_{i=1}^r \mathbb{B}_\varepsilon(c_i)$ be the union of ε-balls around the centers $c_1, \ldots, c_r$ of f_0. We define an open neighborhood U_ε of f_0 in $E_n(r)$ consisting of those $f \in E_n(r)$ satisfying the following requirements (illustrated in Figure 5.2):

- $X \cap \mathsf{C}(f) = \emptyset$.
- The center of any disk belonging to f is contained in X (by the first requirement each center of f is contained in $\mathbb{B}_\varepsilon(c_i)$ for a unique i).

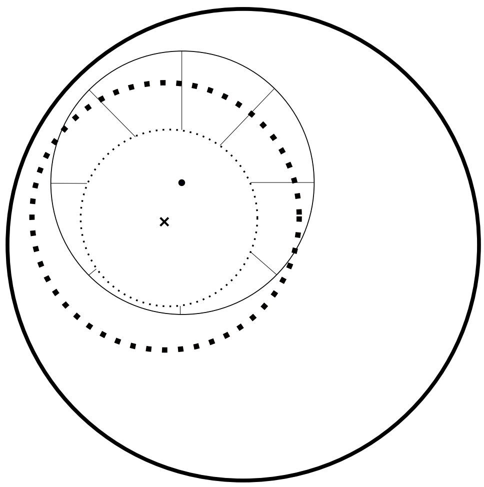


Figure 5.2: For the proof of Proposition 5.8. The large dashed circle corresponds to f_0, the solid circle to f and the small dashed circle to a circle of radius ε around the center of f_0 represented by a cross. The solid lines indicate the direction of the radial extension.

We set $\mathsf{C}(\varepsilon, f_0) := \mathbb{D}^n \setminus X$. Since being a Serre fibration is a local property [Bre93, Theorem VII.6.11], it suffices to prove that for $m \geq 0$ the lifting problem

$$\begin{tikzcd}
\mathbb{D}^m \times 0 \arrow[r, "K"] \arrow[d] & p^{-1}(U_\varepsilon) \arrow[d, "p"] \\
\mathbb{D}^m \times I \arrow[r, "L"] \arrow[ur, dashed, "\widetilde{L}"] & U_\varepsilon
\end{tikzcd}$$

can be solved. For every $x \in \mathbb{D}^m$ we write $K(x) = (K'(x), K''(x))$, where $K'(x) \in E_n(r)$ and $K''(x) : \mathsf{C}(K'(x)) \longrightarrow T$. We can continuously extend $K''(x)$ radially in the direction of the centers of $K'(x)$ to a map $\widetilde{K}''(x) : \mathsf{C}(\varepsilon, f_0) \longrightarrow T$ (constantly along the radial lines in Figure 5.2). Since $\mathsf{C}(L(x,t)) \subset \mathsf{C}(\varepsilon, f_0)$, we can set $\widetilde{L}(x,t) := (L(x,t), \widetilde{K}''(x)|_{\mathsf{C}(L(x,t))})$ for $x \in \mathbb{D}^m$ and $t \in I$. This is obviously a p-lift of L as a map of sets. It remains to show that $\widetilde{L}$ is continuous: Indeed, condition (I) is satisfied by definition. For (II) we investigate the map

$$W_n(r) \times_{E_n(r)} (\mathbb{D}^m \times I) \longrightarrow T \tag{5.6}$$

and realize that its domain is a subspace

$$W_n(r) \times_{E_n(r)} (\mathbb{D}^m \times I) \subset U_\varepsilon \times \mathsf{C}(\varepsilon, f_0) \times \mathbb{D}^m \times I$$

and that (5.6) is the restriction of

$$U_\varepsilon \times \mathsf{C}(\varepsilon, f_0) \times \mathbb{D}^m \times I \xrightarrow{\text{pr}} U_\varepsilon \times \mathsf{C}(\varepsilon, f_0) \times \mathbb{D}^m \xrightarrow{F} T \ ,$$

where

$$\begin{aligned} F : U_\varepsilon \times \mathsf{C}(\varepsilon, f_0) \times \mathbb{D}^m &\longrightarrow T \\ (f, x, x') &\longmapsto (\widetilde{K}''(x'))(x) \ . \end{aligned}$$

Hence, it suffices to prove that F is continuous. This follows from a slight modification of the proof of Lemma 5.5 showing that the radial extension used here is continuous as well. $\square$

We also need to prove the following statement about the map q from (5.2):

Proposition 5.9. *The map $q : W_n^T(r) \longrightarrow \prod^{r+1} \mathrm{Map}(\mathbb{S}^{n-1}, T)$ obtained by restriction to the boundary is a Serre fibration.*

Proof. We need to prove that for $m \geq 0$ the lifting problem

$$\begin{tikzcd} \mathbb{D}^m \times 0 \arrow[rr, "K"] \arrow[d] & & W_n^T(r) \arrow[d, "q"] \\ \mathbb{D}^m \times I \arrow[rr, "L"] \arrow[urr, dashed, "\widetilde{L}"] & & \prod^{r+1} \mathrm{Map}(\mathbb{S}^{n-1}, T) \end{tikzcd}$$

can be solved. For this we write $K(x) = (K'(x), K''(x))$ for $x \in \mathbb{D}^m$, where $K'(x) \in E_n(r)$ and $K''(x) \in \mathrm{Map}(\mathsf{C}(K'(x)), T)$. Next note that L gives us for each $x \in \mathbb{D}^m$ paths $h_1^x, \ldots, h_{r+1}^x$ in $\mathrm{Map}(\mathbb{S}^{n-1}, T)$, and for $1 \leq j \leq r+1$ the path h_j^x is a homotopy of maps $\mathbb{S}^{n-1} \longrightarrow T$ starting at the j-th component $q_j(K''(x))$ of the restriction of $K''(x) : \mathsf{C}(K'(x)) \longrightarrow T$ to the boundary of $\mathsf{C}(K'(x))$.

The desired lift $\widetilde{L} : \mathbb{D}^m \times I \longrightarrow W_n^T(r)$ can now be described as follows: For $(x,t) \in \mathbb{D}^m \times I$ the $E_n(r)$-part of $\widetilde{L}(x,t)$ is obtained from $K'(x)$ by enhancing the radius of the outer disk by t and reducing the radii of the inner disks by multiplying them by $1 - t/2$. Afterwards, we rescale by the factor $1/(1+t)$ to really obtain a point in $E_n(r)$. The needed map from the complement of this point in $E_n(r)$ to T is obtained by gluing together $K''(x)$ and the restriction of the homotopies $h_1^x, \ldots, h_{r+1}^x$ to $[0,t]$. $\square$

The spaces W_n^T allow for a gluing map that we need in order to prove that the composition of the little bundles operad is continuous. First observe that for $r \geq 1$ and $1 \leq j \leq r$, there are maps $W_n^T(r) \longrightarrow \mathrm{Map}(\mathbb{S}^{n-1}, T)$ and $W_n^T(r') \longrightarrow \mathrm{Map}(\mathbb{S}^{n-1}, T)$ by restriction to the j-th boundary sphere and the outer boundary sphere, respectively. The gluing map will be defined on the pullback $W_n^T(r) \times_{\mathrm{Map}(\mathbb{S}^{n-1},T)} W_n^T(r')$.

Lemma 5.10. *Let r be a positive integer. For $1 \leq j \leq r$ the gluing map*

$$\widehat{\circ}_j : W_n^T(r) \times_{\mathrm{Map}(\mathbb{S}^{n-1},T)} W_n^T(r') \longrightarrow W_n^T(r+r'-1),\ \big(\langle f,\xi\rangle, \langle f',\xi'\rangle\big) \longmapsto \langle f \circ_j f', \xi \cup^j_{\mathbb{S}^{n-1}} \xi'\rangle$$

is continuous. Here $f \circ_j f'$ is the operadic composition in E_n, and $\xi \cup^j_{\mathbb{S}^{n-1}} \xi'$ is the map obtained from gluing ξ and ξ' along the j-th sphere $\mathbb{S}^{n-1}$ in the domain of definition of ξ.

Proof. By definition of the topology of the spaces W_n^T continuity of the gluing map amounts to proving that the composition with $W_n^T(r+r'-1) \longrightarrow E_n(r+r'-1)$ is continuous (which is obvious) and that the evaluation

$$W_n(r+r'-1) \times_{E_n(r+r'-1)} \big(W_n^T(r) \times_{\mathrm{Map}(\mathbb{S}^{n-1},T)} W_n^T(r')\big) \longrightarrow T \tag{5.7}$$

is continuous.

We will prove the latter by factorizing (5.7) into continuous maps. As a first step, we will describe the left hand side of (5.7) as the pushout

$$\begin{array}{ccc}
\mathbb{S}^{n-1} \times W_n^T(r) \times_{\mathrm{Map}(\mathbb{S}^{n-1},T)} W_n^T(r') & \longrightarrow & W_n(r') \times_{E_n(r')} W_n^T(r) \times_{\mathrm{Map}(\mathbb{S}^{n-1},T)} W_n^T(r') \\
\Big\downarrow & & \Big\downarrow \\
W_n(r) \times_{E_n(r)} W_n^T(r) \times_{\mathrm{Map}(\mathbb{S}^{n-1},T)} W_n^T(r') & \longrightarrow & \begin{array}{c}\Big(W_n(r) \times_{E_n(r)} \big(W_n^T(r) \times_{\mathrm{Map}(\mathbb{S}^{n-1},T)} W_n^T(r')\big)\Big) \\ \cup \Big(W_n(r') \times_{E_n(r')} \big(W_n^T(r) \times_{\mathrm{Map}(\mathbb{S}^{n-1},T)} W_n^T(r')\big)\Big)\end{array}
\end{array},$$

where the left vertical map is given by

$$\begin{aligned}\mathbb{S}^{n-1} \times W_n^T(r) \times_{\mathrm{Map}(\mathbb{S}^{n-1},T)} W_n^T(r') &\longrightarrow W_n(r) \times_{E_n(r)} W_n^T(r) \times_{\mathrm{Map}(\mathbb{S}^{n-1},T)} W_n^T(r') \\ (x, \langle f_1,\xi_1\rangle, \langle f_2,\xi_2\rangle) &\longmapsto ((f_1,x), \langle f_1,\xi_1\rangle, \langle f_2,\xi_2\rangle)\end{aligned}$$

(here x is seen as point on the j-th boundary sphere of $\mathsf{C}(f_1)$). Analogously, the upper horizontal map is given

$$\begin{aligned}\mathbb{S}^{n-1} \times W_n^T(r) \times_{\mathrm{Map}(\mathbb{S}^{n-1},T)} W_n^T(r') &\longrightarrow W_n(r') \times_{E_n(r')} W_n^T(r) \times_{\mathrm{Map}(\mathbb{S}^{n-1},T)} W_n^T(r') \\ (x, \langle f_1,\xi_1\rangle, \langle f_2,\xi_2\rangle) &\longmapsto ((f_2,x), \langle f_1,\xi_1\rangle, \langle f_2,\xi_2\rangle)\ ,\end{aligned}$$

where x is seen as point on the outer boundary sphere of $\mathsf{C}(f_2)$. Indeed, there is a homeomorphism

$$\begin{aligned}&W_n(r+r'-1) \times_{E_n(r+r'-1)} \big(W_n^T(r) \times_{\mathrm{Map}(\mathbb{S}^{n-1},T)} W_n^T(r')\big) \\ \xrightarrow{\cong}\ &\big(W_n(r) \times_{E_n(r)} W_n^T(r) \times_{\mathrm{Map}(\mathbb{S}^{n-1},T)} W_n^T(r')\big) \cup \big(W_n(r') \times_{E_n(r')} W_n^T(r) \times_{\mathrm{Map}(\mathbb{S}^{n-1},T)} W_n^T(r')\big)\end{aligned} \tag{5.8}$$

which by means of the natural embeddings of $\mathsf{C}(f_1)$ and $\mathsf{C}(f_2)$ into $\mathsf{C}(f_1 \circ f_2)$ is given by

$$((f_1 \circ f_2, x), \langle f_1,\xi_1\rangle, \langle f_2,\xi_2\rangle) \longmapsto \begin{cases} ((f_1,x), \langle f_1,\xi_1\rangle, \langle f_2,\xi_2\rangle)\ , & \text{if } x \in \mathrm{im}(\mathsf{C}(f_1) \longrightarrow \mathsf{C}(f_1 \circ f_2)) \\ ((f_2,x), \langle f_1,\xi_1\rangle, \langle f_2,\xi_2\rangle)\ , & \text{if } x \in \mathrm{im}(\mathsf{C}(f_2) \longrightarrow \mathsf{C}(f_1 \circ f_2))\ . \end{cases}$$

By slight abuse of notation we use here the same symbol for a point in $\mathsf{C}(f_i)$ for $i = 1, 2$ and its image in $\mathsf{C}(f_1 \circ f_2)$. Thanks to $\mathsf{C}(f_1 \circ f_2) \cong \mathsf{C}(f_1) \cup_{\mathbb{S}^{n-1}} \mathsf{C}(f_2)$, this is continuous. The formula for the continuous inverse can also be written down directly, so (5.8) is a homeomorphism.

The projections

$$\begin{aligned}W_n(r) \times_{E_n(r)} W_n^T(r) \times_{\mathrm{Map}(\mathbb{S}^{n-1},T)} W_n^T(r') &\longrightarrow W_n(r) \times_{E_n(r)} W_n^T(r)\ , \\ W_n(r') \times_{E_n(r')} W_n^T(r) \times_{\mathrm{Map}(\mathbb{S}^{n-1},T)} W_n^T(r') &\longrightarrow W_n(r') \times_{E_n(r')} W_n^T(r')\end{aligned}$$

induce a map

$$\begin{aligned} &\big(W_n(r)\times_{E_n(r)} W_n^T(r)\times_{\mathrm{Map}(\mathbb{S}^{n-1},T)} W_n^T(r')\big)\cup\big(W_n(r')\times_{E_n(r')} W_n^T(r)\times_{\mathrm{Map}(\mathbb{S}^{n-1},T)} W_n^T(r')\big)\\ \longrightarrow\ &\big(W_n(r)\times_{E_n(r)} W_n^T(r)\big)\cup\big(W_n(r')\times_{E_n(r')} W_n^T(r')\big)\ , \end{aligned} \tag{5.9}$$

where the pushout on the right hand side is also over $\mathbb{S}^{n-1}\times W_n^T(r)\times_{\mathrm{Map}(\mathbb{S}^{n-1},T)} W_n^T(r')$.

Now (5.7) is the composition of continuous maps

$$\begin{array}{c} W_n(r+r'-1)\times_{E_n(r+r'-1)}\big(W_n^T(r)\times_{\mathrm{Map}(\mathbb{S}^{n-1},T)} W_n^T(r')\big)\\ \Big\downarrow{\scriptstyle (5.8)}\\ \big(W_n(r)\times_{E_n(r)} W_n^T(r)\times_{\mathrm{Map}(\mathbb{S}^{n-1},T)} W_n^T(r')\big)\cup\big(W_n(r')\times_{E_n(r')} W_n^T(r)\times_{\mathrm{Map}(\mathbb{S}^{n-1},T)} W_n^T(r')\big)\\ \Big\downarrow{\scriptstyle (5.9)}\\ \big(W_n(r)\times_{E_n(r)} W_n^T(r)\big)\cup\big(W_n(r')\times_{E_n(r')} W_n^T(r')\big)\\ \Big\downarrow{\scriptstyle \text{evaluation map}}\\ T\ . \end{array}$$

In the last step, we use the evaluation map from Lemma 5.6 on $W_n(r)\times_{E_n(r)} W_n^T(r)$ and $W_n(r')\times_{E_n(r')} W_n^T(r')$. Both maps agree on $\mathbb{S}^{n-1}\times W_n^T(r)\times_{\mathrm{Map}(\mathbb{S}^{n-1},T)} W_n^T(r')$ and therefore actually descend to a map on the pushout. □

5.2 The operad E_2^G of little G-bundles

Let us specialize the operad from Proposition 5.3 to aspherical spaces T (see page 22 for the definition) to obtain what we will refer to as *little bundles operad*.

If T is an aspherical space (which we will assume to be connected without loss of generality), then, up to equivalence, T is the classifying space of its fundamental group G. We set $E_n^G := E_n^{BG}$ for any (discrete) group G.

For a manifold X (possibly with boundary), the mapping space $\mathrm{Map}(X,BG)$ is the nerve $B\mathsf{PBun}_G(X)$ of the groupoid of principal G-bundles over X, i.e.

$$\mathrm{Map}(X,BG)\simeq B\mathsf{PBun}_G(X)\ , \tag{5.10}$$

which implies

$$\varPi\,\mathrm{Map}(X,BG)\simeq \mathsf{PBun}_G(X)\ , \tag{5.11}$$

where $\varPi$ denotes the fundamental groupoid functor. A proof of these well-known facts follows e.g. from Lemma 2.9.

In particular, $\mathrm{Map}(X,BG)$ is aspherical again with

$$\pi_0(\mathrm{Map}(X,BG))\cong \pi_0(\mathsf{PBun}_G(X))\ , \tag{5.12}$$

$$\pi_1(\mathrm{Map}(X,BG),\varphi)\cong \mathrm{Aut}(\varphi^*EG)\ , \tag{5.13}$$

where φ^*EG is the pullback of the universal G-bundle $EG\longrightarrow BG$ along a map $\varphi : X\longrightarrow BG$ and where we denote by $\mathrm{Aut}(P)$ the group of automorphisms of a G-bundle P (the group of *gauge transformations*).

Recall that if X is connected, we find

$$\mathsf{PBun}_G(X)\simeq \mathrm{Hom}(\pi_1(X),G)//G \tag{5.14}$$

by the holonomy classification of G-bundles, see e.g. [Tau11, Theorem 13.2]. More precisely, after choice of a base point in X (suppressed here in the notation), the bundle groupoid $\mathsf{PBun}_G(X)$ is equivalent to the action groupoid associated to the action of G by conjugation on the set of group morphisms $\pi_1(X) \longrightarrow G$.

Note that for $n > 2$ the operad E_n^G is not really interesting since all G-bundles over $\mathbb{S}^{n-1}$ for $n > 2$ are trivializable. The case relevant to us is $n = 2$:

Definition 5.11. For any group G, we call the topological operad E_2^G the *little bundles operad.*

In the remaining subsections of this section, we will show that E_2^G is aspherical (Proposition 5.19), where we call an operad in spaces or simplicial sets aspherical if all its components are aspherical. Moreover, we will explicitly describe the components of the little bundles operad as action groupoids (Proposition 5.20).

5.2.1 The space W_2^G as a Hurwitz space

In a first step, we investigate the spaces $W_2^G(r) := W_2^{BG}(r)$ for a group G.

For this recall from [Fre17, Chapter 5] that $E_2(r)$ is the classifying space of the pure braid group P_r on r strands, i.e.

$$E_2(r) \simeq BP_r \ . \tag{5.15}$$

Alternatively (and for our applications more conveniently), we can describe the fundamental groupoid $\Pi E_2(r)$ as the action groupoid

$$\Pi E_2(r) \simeq \Sigma_r // B_r \ , \tag{5.16}$$

where the braid group B_r acts on Σ_r by $c.\sigma := \pi(c)\sigma$ for $c \in B_r$ and $\sigma \in \Sigma_r$, i.e. via the projection $\pi : B_r \longrightarrow \Sigma_r$ fitting into the short exact sequence

$$0 \longrightarrow P_r \longrightarrow B_r \xrightarrow{\pi} \Sigma_r \longrightarrow 0 \ .$$

If we consider the long exact sequence of homotopy groups for the Serre fibration from Proposition 5.1 whose fibers we computed in Lemma 5.7 and take (5.12), (5.13) and (5.15) into account, we arrive at:

Lemma 5.12. *The space $W_2^G(r)$ is aspherical, and for $f \in E_2(r)$ and $\varphi \in \mathrm{Map}(\mathsf{C}(f), BG)$, there is an exact sequence*

$$0 \longrightarrow \mathrm{Aut}(\varphi^* EG) \longrightarrow \pi_1(W_2^G(r), \langle f, \varphi \rangle) \longrightarrow P_r \longrightarrow \pi_0(\mathsf{PBun}_G(\mathsf{C}(f))) \longrightarrow \pi_0(W_2^G(r)) \longrightarrow 0 \ .$$

We will denote the homotopy fiber of a map $q : X \longrightarrow Y$ over $y \in Y$ by $q^{-1}[y]$. If X and Y are aspherical, we can make the following elementary observation:

Lemma 5.13. *Let $q : X \longrightarrow Y$ be a map between aspherical spaces, then for $y \in Y$ the natural map*

$$\Pi(q^{-1}[y]) \longrightarrow \Pi(q)^{-1}[y]$$

from the fundamental groupoid of the homotopy fiber $q^{-1}[y]$ to the homotopy fiber $\Pi(q)^{-1}[y]$ of $\Pi(q) : \Pi(X) \longrightarrow \Pi(Y)$ over $y \in Y$ is an equivalence.

Proof. Since Π sends Serre fibrations to isofibrations, it suffices to prove the statement for a Serre fibration and the actual fibers instead of homotopy fibers.

The spaces involved are aspherical, hence we only need to prove that the map

$$\pi_0(q^{-1}(y)) \longrightarrow \pi_0(\Pi(q)^{-1}(y)) \tag{5.17}$$

is bijective and that the map

$$\pi_1(q^{-1}(y), x) \longrightarrow \pi_1(\Pi(q)^{-1}(y), x) \tag{5.18}$$

is a group isomorphism for all $x \in X$ such that $q(x) = y$.

Surjectivity of (5.17) follows from the definitions. To see injectivity of (5.17), let x and x' be points such that $q(x) = y = q(x')$ and $x \cong x'$ in $\Pi(q)^{-1}(y)$ via a morphism $g : x \longrightarrow x'$. Then g can be represented as a path in X from x to x' such that $q(g)$ is homotopic relative boundary to the constant path at y. This homotopy has a q-lift starting at g. The endpoint is a path from x to x' in $q^{-1}(y)$ proving that x and x' lie in the same component of $q^{-1}(f)$.

The fact that (5.18) is an isomorphism follows by comparing the exact sequences that both $q^{-1}(y)$ and $\Pi(q)^{-1}(y)$ give rise to. □

The following result will be the key for understanding the auxiliary spaces $W_2^G(r)$. It provides a link to a certain flavor of Hurwitz spaces, see also Remark 5.18 below.

Proposition 5.14. *There is an equivalence*

$$W_2^G(r) \simeq \operatorname*{hocolim}_{f \in \Pi E_2(r)} \operatorname{Map}(\mathsf{C}(f), BG) . \tag{5.19}$$

Here by an equivalence we mean that there exists a zigzag of equivalences, i.e. the objects are isomorphic in the homotopy category.

Proof. Since $p : W_2^G(r) \longrightarrow E_2(r)$ is a Serre fibration (Lemma 5.1), $\Pi(p) : \Pi W_2^G(r) \longrightarrow \Pi E_2(r)$ is an isofibration. In fact, it is also a category fibered in groupoids in the sense of [DM69], see also [Hol08, Definition 3.1]. Corresponding to this category fibered in groupoids, we have by [Hol08, Section 3.3] a (pseudo-)functor $X : (\Pi E_2(r))^{\mathrm{opp}} \longrightarrow \mathsf{Grpd}$ (we can also see this as a $\Pi E_2(r)$-shaped diagram since any groupoid is equivalent to its opposite) such that for $f \in E_2(r)$ the groupoid $X(f)$ is equivalent to the fiber of $\Pi W_2^G(r) \longrightarrow \Pi E_2(r)$ over f. This fiber is equivalent to $\Pi p^{-1}(f)$ by Lemma 5.13 and, finally, to $\Pi \operatorname{Map}(\mathsf{C}(f), BG)$ by Lemma 5.7. If we denote by $\int$ the Grothendieck construction (see e.g. [MM92, Section I.5]), we conclude from [Hol08, Theorem 3.12] that there is a canonical fiberwise equivalence

$$\int X \longrightarrow \Pi W_2^G(r) \tag{5.20}$$

of groupoids over $\Pi E_2(r)$. It is then straightforward to verify that this is also an equivalence of groupoids.

By Thomason's Theorem [Th79, Theorem 1.2] we obtain a canonical equivalence

$$\operatorname*{hocolim}_{f \in \Pi E_2(r)} BX(f) \xrightarrow{\simeq} B \int X .$$

Combining this with the equivalence (5.20) yields the assertion if we additionally take into account that $\operatorname{Map}(\mathsf{C}(f), BG)$ and $W_2^G(r)$ are aspherical by (5.10) and Lemma 5.12, respectively. □

Since $\mathsf{C}(f)$ is equivalent to a wedge $\bigvee_{j=1}^r \mathbb{S}^1$ of r circles, we conclude from (5.10) and (5.14)

$$\operatorname{Map}(\mathsf{C}(f), BG) \simeq B\left(\operatorname{Hom}(\mathbb{Z}^{*r}, G)//G\right) \simeq B(G^{\times r}//G) , \tag{5.21}$$

where $\mathbb{Z}^{*r}$ is the r-fold free product of $\mathbb{Z}$ with itself.

Lemma 5.15. *Under the identifications* (5.16) *and* (5.21), *the diagram from $\Pi E_2(r)$ to spaces underlying the homotopy colimit* (5.19) *is point-wise the nerve of the diagram*

$$\Sigma_r//B_r \longrightarrow \mathsf{Grpd}$$

sending $\sigma \in \Sigma_r$ to $G^{\times r}//G$. The generator $c_{j,j+1} \in B_r$ braiding strand j and $j+1$ acts as the automorphism

$$G^{\times r}//G \longrightarrow G^{\times r}//G, \quad (g_1, \ldots, g_j, g_{j+1}, \ldots, g_r) \longmapsto (g_1, \ldots, g_j g_{j+1} g_j^{-1}, g_j, \ldots, g_r) \ . \tag{5.22}$$

Proof. We only have to observe that the transformation of holonomies under the braid group action is given by the formula (5.22) (sometimes called *Hurwitz formula*). For a detailed proof of this fact (given without loss of generality for two embedded disks), we refer to e.g. [MNS12, Lemma 3.25]. □

By Proposition 5.14 $W_2^G(r)$ is the homotopy colimit of the nerve of the diagram presented in Lemma 5.15. For later purposes, we need to describe $W_2^G(r)$ explicitly as a groupoid. To this end, we will need an explicit formula for some specific homotopy colimits.

First recall that for any diagram X from a groupoid Ω to spaces, the homotopy colimit is given by the realization of the simplicial space with level n given by

$$\coprod_{\vec{y}:[n]\longrightarrow\Omega} X(y_0) \ ,$$

where the coproduct runs over all strings $\vec{y} : [n] \longrightarrow \Omega$ of length $n \geq 0$, see e.g. [Rie14, Corollary 5.1.3] for the definition of the face and degeneracy maps.

Lemma 5.16. *Let Γ be a diagram from a groupoid Ω to groupoids. The homotopy colimit of $B\Gamma$ is the nerve of a groupoid admitting the following description:*

- *The objects are given by pairs (y_0, x_0), where $y_0 \in \Omega$ and $x_0 \in \Gamma_0(y_0)$.*
- *For every pair (g_0, f_0), where $g_0 : y_0 \longrightarrow y_1$ is a morphism in Ω and $f_0 : x_0 \longrightarrow x_1$ a morphism in $\Gamma(y_0)$, we get a morphism $(y_0, x_0) \longrightarrow (y_1, g_0.x_1)$, where $g_0.x_1 = \Gamma(g_0)(x_1)$.*
- *The composition of morphisms is given by*

$$\left(y_1 \xrightarrow{g_1} y_2, g_0.x_1 \xrightarrow{f_1} x_2\right) \circ \left(y_0 \xrightarrow{g_0} y_1, x_0 \xrightarrow{f_0} x_1\right) := \left(y_0 \xrightarrow{g_1 g_0} y_1, x_0 \xrightarrow{(g_0^{-1}.f_1)f_0} g_0^{-1}x_2\right)$$

Proof. As just explained, the desired homotopy colimit is the realization of the simplicial space with

$$\coprod_{\vec{y}:[n]\longrightarrow\Omega} B\Gamma(y_0)$$

in level n. Since the realization can be computed as the diagonal, we find

$$\left(\operatorname*{hocolim}_{\Omega} B\Gamma\right)_n = \coprod_{\vec{y}:[n]\longrightarrow\Omega} B_n\Gamma(y_0) \ .$$

Carefully writing out the low degree face and degeneracy maps yields the claim. □

If we combine Lemma 5.15 with the homotopy colimit formula provided in Lemma 5.16, we obtain:

Lemma 5.17. *The groupoid $\Pi W_2^G(r)$ is equivalent to the groupoid with objects $\Sigma_r \times G^{\times r}$ and pairs $(c,h) \in B_r \times G$ as morphisms $(\sigma, g_1, \dots, g_r) \longrightarrow (\pi(c)\sigma, c.(hg_1h^{-1}, \dots, hg_rh^{-1}))$, where the action of the braid group on tuples of group elements is given by* (5.22).

Remark 5.18. The homotopy quotient of the space of G-bundles over a punctured plane by the braid group action (or its description in terms of holonomies) first appeared in [Cle72, Hur91] and is called a *Hurwitz space*, see [EVW16] for an overview.

5.2.2 Groupoid description of E_2^G

Our investigation of W_2^G is the key to the computation of the homotopy groups of the little bundles operad E_2^G. Using the long exact sequence for the Serre fibration $q : W_2^G(r) \longrightarrow \prod^{r+1} \operatorname{Map}(\mathbb{S}^1, BG)$ from Proposition 5.2 combined with Lemma 5.12 we obtain:

Proposition 5.19. *For any group G, the operad E_2^G is aspherical.*

Therefore, without loss of homotopical information, it suffices to compute the groupoid-valued operad ΠE_2^G. To this end, recall from (5.11) that the groupoid $\Pi \operatorname{Map}(\mathbb{S}^1, BG)$ is canonically equivalent to the groupoid of G-bundles over $\mathbb{S}^1$, hence for any fixed choice of base point, we obtain an equivalence

$$\Pi \operatorname{Map}(\mathbb{S}^1, BG) \xrightarrow{\simeq} G//G \ .$$

In the sequel, we choose a weak inverse

$$\widehat{-} : G//G \xrightarrow{\simeq} \Pi \operatorname{Map}(\mathbb{S}^1, BG) \ . \tag{5.23}$$

We make our choices such that the unit element e of the group G is mapped to the constant loop at the base point, and such that all loops in the image map the point $(0,1) \in \mathbb{S}^1 \subset \mathbb{R}^2$ to the base point of BG. The object function $\widehat{-} : G \longrightarrow \operatorname{Map}(\mathbb{S}^1, BG)$ can be used to pull back E_2^G to a G-colored operad whose components are given as follows:

Proposition 5.20. *For $\underline{g} \in G^r$ and $h \in G$, the groupoid $\Pi E_2^G\binom{\widehat{h}}{\underline{\widehat{g}}}$ is equivalent to the action groupoid of the B_r-action specified in Lemma 5.17 on the subset*

$$\Sigma_r \times_h G^r := \left\{ (\sigma, \underline{b}) \in \Sigma_r \times G^r \;\middle|\; \prod_{j=1}^r b_{\sigma(j)} g_j b_{\sigma(j)}^{-1} = h \right\} \subset \Sigma_r \times G^r \ .$$

Proof. By Lemma 5.13, $\Pi E_2^G\binom{h}{\underline{g}}$ is equivalent to the homotopy fiber of

$$\Pi W_2^G(r) \longrightarrow \Pi \prod^{r+1} \operatorname{Map}(\mathbb{S}^1, BG) \simeq (G//G)^{r+1} \tag{5.24}$$

over $(\underline{g}, h)$. Using the presentation of $\Pi W_2^G(r)$ given in Lemma 5.17, the functor (5.24) sends $(\sigma, a_1, \dots, a_r)$ to $(a_1, \dots, a_r, a_{\sigma(1)} \cdots a_{\sigma(r)})$. Therefore, the homotopy fiber of (5.24) over $(\underline{g}, h)$

consists of all

$$\begin{aligned}&(\sigma, \underline{a} = (a_1, \ldots, a_r)) \in \Sigma_r \times G^r, \quad \underline{b} = (b_1, \ldots, b_{r+1}) \in G^{r+1}\\ &\text{such that} \quad b_j a_j b_j^{-1} = g_j, \quad 1 \le j \le r, \quad b_{r+1} a_{\sigma(1)} \cdots a_{\sigma(r)} b_{r+1}^{-1} = h \ .\end{aligned}$$

From Lemma 5.17 it follows that, up to equivalence, we can concentrate on the full subgroupoid of the homotopy fiber spanned by those objects satisfying $b_{r+1} = 1$; and a morphism $(\sigma, \underline{a}, \underline{b}) \longrightarrow (\sigma', \underline{a}', \underline{b}')$ in that full subgroupoid, i.e. with $b_{r+1} = b'_{r+1} = 1$, is just an element of B_r. Of course, for an object $(\sigma, \underline{a}, \underline{b})$, the tuple $\underline{a}$ is redundant because $a_j = b_j^{-1} g_j b_j$. Also, we may work with the tuple $\underline{b}^{-1} = (b_1^{-1}, \ldots, b_r^{-1})$ instead of $\underline{b}$. □

Remark 5.21. The above statement just gives the components of ΠE_2^G, but does not give a description as an operad. This problem will be addressed in Section 5.3.

In the sequel, we will need a lifting result for the functor $\Pi E_2^G\binom{\psi}{\underline{\varphi}} \longrightarrow \Pi E_2(r)$. We denote by $[1]$ the category with objects 0 and 1 and a morphism $0 \longrightarrow 1$ as the only non-identity morphism.

Proposition 5.22. *For $\underline{\varphi} \in \prod^r \mathrm{Map}(\mathbb{S}^1, BG)$ and $\psi \in \mathrm{Map}(\mathbb{S}^1, BG)$, the forgetful functor*

$$\Pi E_2^G\begin{pmatrix}\psi\\ \underline{\varphi}\end{pmatrix} \longrightarrow \Pi E_2(r) \tag{5.25}$$

admits lifts of the form

$$\begin{tikzcd} 0 \arrow[r, "x_0"] \arrow[d] & \Pi E_2^G\binom{\psi}{\underline{\varphi}} \arrow[d] \\ {[1]} \arrow[r, "g"] \arrow[ur, dashed] & \Pi E_2(r) \end{tikzcd}$$

as long as the end and starting point of g are points in $E_2(r)$ whose little disks have non-intersecting boundaries.

However in general, (5.25) is not a fibration as the following counterexample shows: Consider for $g, h \in G$ the diagram

$$\begin{tikzcd} 0 \arrow[r, "\widehat{h}"] \arrow[d] & \Pi E_2^G\binom{\widehat{hgh^{-1}}}{\widehat{g}} \arrow[d] \\ {[1]} \arrow[r, "L"] & \Pi E_2(1) \ , \end{tikzcd}$$

where L is the homotopy sketched in Figure 5.6b on page 97 (to be read from bottom (deformation parameter $t = 0$) to top ($t = 1$)) and $\widehat{h}$ is seen as a point in $\Pi E_2^G\binom{\widehat{hgh^{-1}}}{\widehat{g}}$ by placing the homotopy corresponding to h on the complement of L_0. Clearly, for this square, there is no lift to $\Pi E_2^G\binom{\widehat{hgh^{-1}}}{\widehat{g}}$ whenever $h \neq e$.

Proof of Proposition 5.22. We start by showing that for $\underline{\varphi} \in \prod^r \mathrm{Map}(\mathbb{S}^{n-1}, BG)$ and $\psi \in \mathrm{Map}(\mathbb{S}^{n-1}, T)$ the composition $E_n^T\binom{\psi}{\underline{\varphi}} \longrightarrow W_n^T(r) \longrightarrow E_n(r)$ admits lifts for paths $I \longrightarrow E_n(r)$

whose little disks have non-intersecting boundaries. Indeed, the needed lifts

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & E_2^G\binom{\psi}{\underline{\varphi}} \arrow[d] \\ I \arrow[r] \arrow[ur, dashed] & E_2(r) \end{tikzcd}$$

could be constructed thanks to Lemma 5.1 if we allow the lift to take values in $W_2^G(r)$ without making sure that we hit the correct fiber. The parameter $t \in I$ will then describe a path in $W_2^G(r)$ whose restriction to the boundary circles will describe homotopies of $\underline{\varphi}$ and ψ. In order to remain in the fiber $E_n^T\binom{\psi}{\underline{\varphi}}$, these restrictions would have to be constant. We can easily achieve that by fixing small non-intersecting collars around the boundary circles (this is possible since we assumed that the boundaries of the disks do not intersect) on which we place the inverses of the homotopies of $\underline{\varphi}$ and ψ mentioned above (this strategy appeared also in the proof of Proposition 5.9).

More generally, a path in ΠE_2^G, for which the start and end point do not contain disks with intersecting boundaries, admits a representative in E_2^G whose little disks do not touch (by rescaling in the interior). Then the argument just given applies. □

Lemma 5.23. *The forgetful functor $(\Sigma_r \times_h G^r)//B_r \longrightarrow \Sigma_r//B_r$ admits a unique solution to the lifting problem*

$$\begin{tikzcd} 0 \arrow[r, "{(\sigma,\underline{b})}"] \arrow[d] & (\Sigma_r \times_h G^r)//B_r \arrow[d] \\ {[1]} \arrow[r, "c:\sigma \longrightarrow \pi(c)\sigma"'] \arrow[ur, dashed, "\exists!"] & \Sigma_r//B_r \ . \end{tikzcd}$$

Proof. The unique lift is $c : (\sigma, \underline{b}) \longrightarrow c.(\sigma, \underline{b})$. □

Remark 5.24 (Uniqueness of the lifts in Proposition 5.22)**.** For two lifts $\widetilde{g} : x_0 \longrightarrow x_1$ and $\widetilde{g}' : x_0 \longrightarrow x_1'$ of $g : [1] \longrightarrow \Pi E_2(r)$ under (5.25), there is a unique morphism $h = \widetilde{g}'\widetilde{g}^{-1} : x_1 \longrightarrow x_1'$ such that $h\widetilde{g} = \widetilde{g}'$ and $h = \mathrm{id}_{x_1}$ whenever $x_1 = x_1'$, i.e. the lift is completely determined by its start and end point. For the proof of this uniqueness statement, we consider the commutative diagram

$$\begin{tikzcd} 0 \arrow[r, "x_0"] \arrow[d] & \Pi E_2^G\binom{\psi}{\underline{\varphi}} \arrow[r, "\simeq"] \arrow[d] & (\Sigma_r \times_h G^r)//B_r \arrow[d] \\ {[1]} \arrow[r, "g"] & \Pi E_2(r) \arrow[r, "\simeq"] & \Sigma_r//B_r \ , \end{tikzcd}$$

where $r = |\underline{\varphi}|$. Moreover, we have used a holonomy description of $(\underline{\varphi}, \psi)$ to apply Proposition 5.20. We deduce from Lemma 5.23 that the images of two lifts $\widetilde{g} : x_0 \longrightarrow x_1$ and $\widetilde{g}' : x_0 \longrightarrow x_1'$ under

$$\Pi E_2^G\begin{pmatrix}\psi \\ \underline{\varphi}\end{pmatrix} \longrightarrow (\Sigma_r \times_h G^r)//B_r \tag{5.26}$$

agree. This implies that (5.26) maps h to the identity and hence $h = \mathrm{id}_{x_1}$ whenever $x_1 = x_1'$.

Furthermore, the set of allowed endpoints is the preimage of the endpoint of the lift from Lemma 5.23 under the equivalence $\Pi E_2^G\binom{\psi}{\underline{\varphi}} \longrightarrow (\Sigma_r \times_h G^r)//B_r$ intersected with the preimage of the endpoint of g under the projection $\Pi E_2^G\binom{\psi}{\underline{\varphi}} \longrightarrow \Pi E_2(r)$.

5.3 Categorical algebras over the little bundle operad

Since any aspherical operad can be seen as an operad in groupoids, it is natural to consider its categorical algebras (algebras in the symmetric monoidal category of categories). For the little disks operad, this leads to braided monoidal categories, see [Fre17, Chapter 5 and 6] for a detailed discussion that will also be briefly summarized below. For the little bundles operad, as we prove in this section, this leads to braided G-crossed categories. This type of category, which is of great importance in equivariant representation theory [GNN09, ENOM10] and topological field theory [TV12, TV14], is based on work by Turaev [Tur00, Tur10b]. Various flavors of the notion exist [MNS12, Gal17] differing, for example, by the type of coherence data considered. We will give the precise version used in this thesis below.

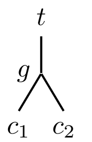
5.3.1 Groupoid-valued operads in terms of generators and relations

In this subsection we recall the definition of an operad in terms of generators and relations; we refer to [Fre17, Yau16] for details.

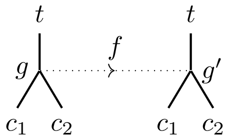
For a fixed non-empty set $\mathfrak{C}$ of colors we denote by $U : \mathsf{Op}_{\mathfrak{C}}(\mathcal{M}) \longrightarrow \mathsf{Sym}_{\mathfrak{C}}(\mathcal{M})$ the forgetful functor from the category of $\mathfrak{C}$-colored operads valued in a bicomplete closed symmetric monoidal category $\mathcal{M}$ to the category of symmetric sequences in $\mathcal{M}$. This functor admits a left adjoint $F : \mathsf{Sym}_{\mathfrak{C}}(\mathcal{M}) \longrightarrow \mathsf{Op}_{\mathfrak{C}}(\mathcal{M})$, the free operad functor.

The free operad functor and the cocompleteness of the category of operads can be used to define an operad via generators and relations: Fix a collection of generators $G \in \mathsf{Sym}_{\mathfrak{C}}(\mathcal{M})$ and relations $R \in \mathsf{Sym}_{\mathfrak{C}}(\mathcal{M})$ together with two morphisms $r_1, r_2 : R \longrightarrow UF(G)$. Via the adjunction $F \dashv U$, this defines two morphisms $F(R) \rightrightarrows F(G)$. The operad generated by G and R is the coequalizer of the parallel pair $F(R) \rightrightarrows F(G)$.

In the case that $\mathcal{M}$ is the category of groupoids, $\mathcal{M} = \mathsf{Grpd}$, we draw an object g of the groupoid $G\binom{t}{(c_1,\dots,c_n)}$ as a planar graph with one vertex labeled by g, n ingoing legs labeled by $c_1, \dots, c_n$ and one outgoing edge labeled by t. For example, we depict an object $g \in G\binom{t}{(c_1,c_2)}$ as

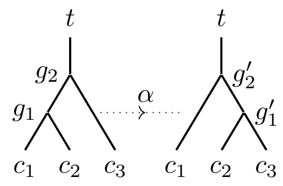


We will draw morphisms as dotted lines between trees. For example, we depict a morphism $f : g \longrightarrow g' \in G\binom{t}{(c_1,c_2)}$ as

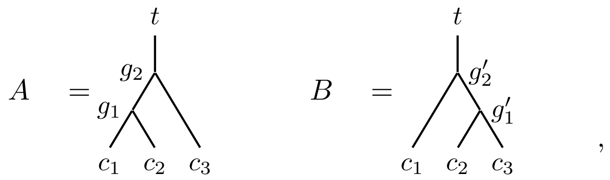


Furthermore, when we draw a list of generators, we only draw 'elementary' generators and add elements corresponding to the action of the permutation group and the composition of morphisms. Put more formally, we only specify the groupoid as a directed graph, take the free groupoid generated by this graph and add elements corresponding to the action of the permutation group freely. Note that this automatically adds inverses for every morphism.

To simplify the notation later on, we draw diagrams like



with generators g_1, g_2, g_1', g_2' to describe the following: We formally add objects A and B in $G\binom{t}{(c_1,c_2,c_3)}$, a morphism $\alpha : A \longrightarrow B$ between them and afterwards impose the relation



where the respective right hand sides of the equations describe the operadic composition of generators of G in the operad FG via trees.

Recall that the universal property of the coequalizer and the universal property of the free operad allow us to describe algebras over an operad defined in terms of generators and relations very concretely:

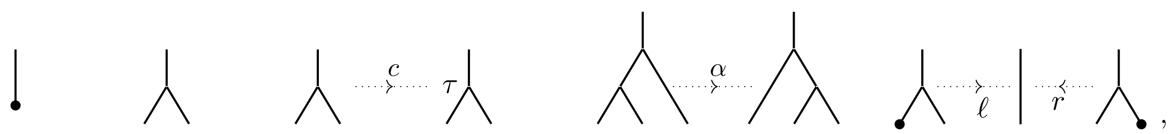
Proposition 5.25. *An algebra A in the category* Cat *of small categories over a $\mathfrak{C}$-colored operad in* Grpd *described by generators G and relations R consists of*

- *a category A_c for every $c \in \mathfrak{C}$,*
- *a functor $A_g : A_{c_1} \times \cdots \times A_{c_n} \longrightarrow A_t$ for every generating object $g \in G\binom{t}{\underline{c}}$,*
- *a natural isomorphism $A_f : A_g \Longrightarrow A_g'$ for every generating morphism $f : g \longrightarrow g'$,*

such that all relations described by R are satisfied.

5.3.2 E_2^G in terms of generators and relations

As a preparation for the description of the little bundle operad in terms of generators and relations, we briefly recall the corresponding known description for the little disks operad [Fre17, Chapter 5 and 6]: One introduces the groupoid-valued operad PBr of *parenthesized braids* with the generators



where τ denotes the application of the non-trivial permutation of two elements. As relations, we impose the pentagon identity for α, the hexagon identities for c and the triangle identity on ℓ and r. Proposition 5.25 implies that algebras over PBr are by construction braided monoidal categories.

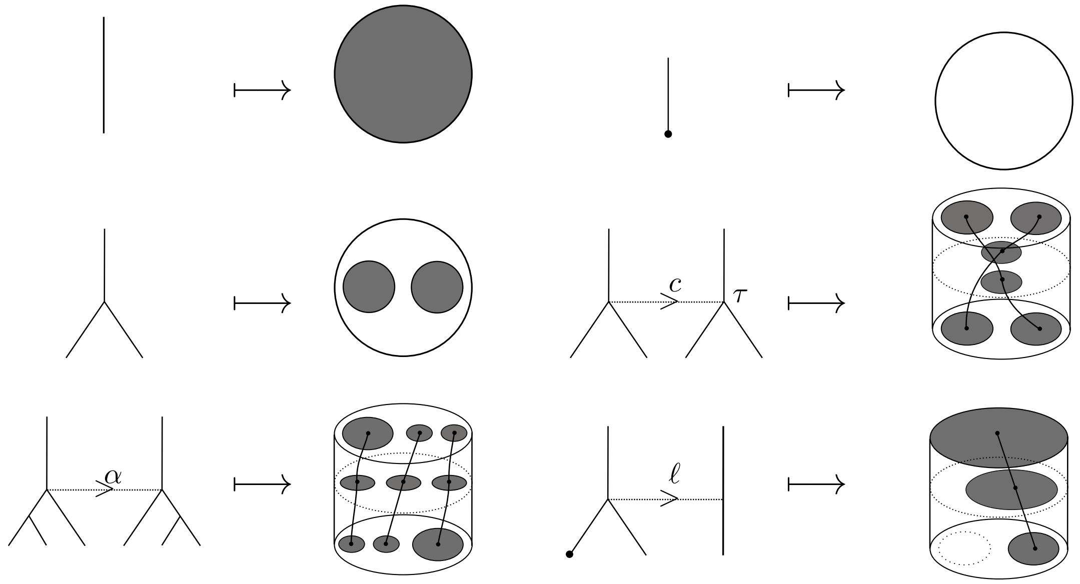


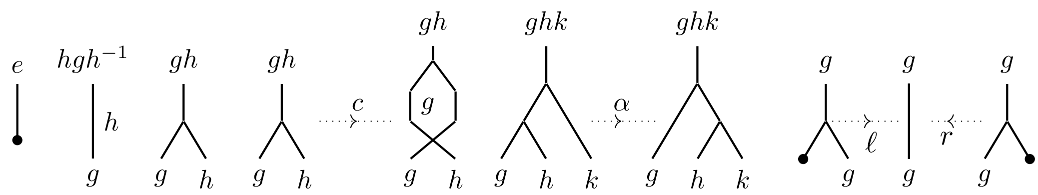
Figure 5.3: Definition of the morphism $\mathsf{PBr} \longrightarrow \varPi E_2$ on generators. We only draw the center of the circles for paths in E_2. The definition for r is analogous to the definition for ℓ.

Figure 5.3 indicates the definition of a morphism of operads from the free operad on the depicted generators to $\varPi E_2$ which by [Fre17, Theorem 6.2.4] descends to a morphism

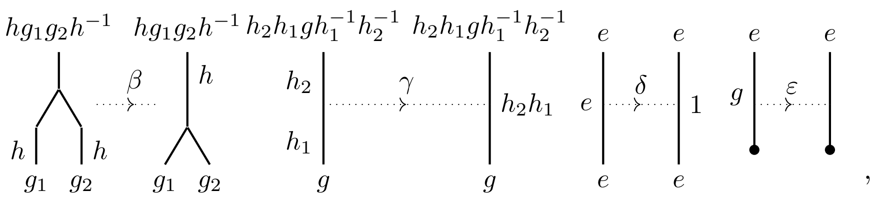
$$\mathsf{PBr} \longrightarrow \varPi E_2 \ .$$

By [Fre17, Proposition 6.2.2] this morphism is an equivalence, thereby giving us a presentation of $\varPi E_2$ in terms of generators and relations.

After this short review of the non-equivariant case, we generalize this description in terms of generators and relations to the little bundle operad E_2^G by introducing the G-colored operad PBr^G of *parenthesized G-braids*. Its generating operations and isomorphisms are given by



(5.27)

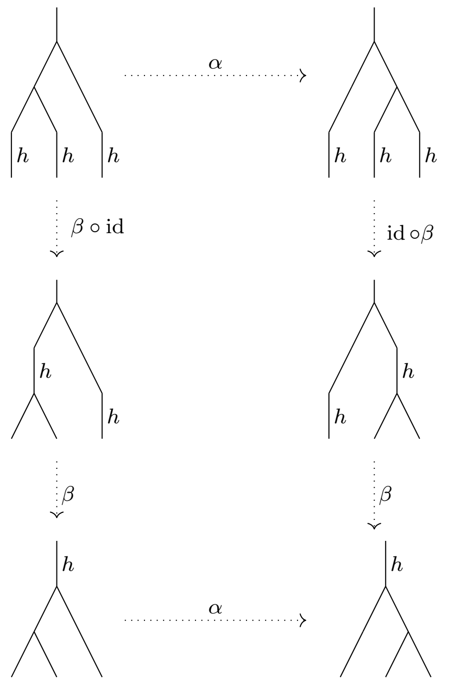


, (5.28)

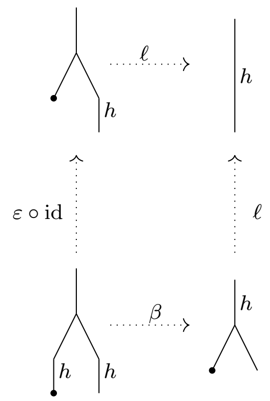
where 1 denotes the operadic unit.

As relations, we introduce the pentagon equation for α, the triangle equations for ℓ and r and, *additionally*, the following relations (the labels only indicate the corresponding morphisms and

$\circ$ denotes the operadic composition):

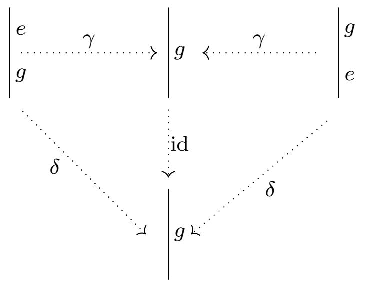


$(G1)$

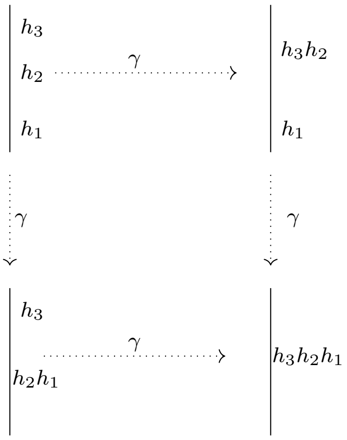


$(G2)$

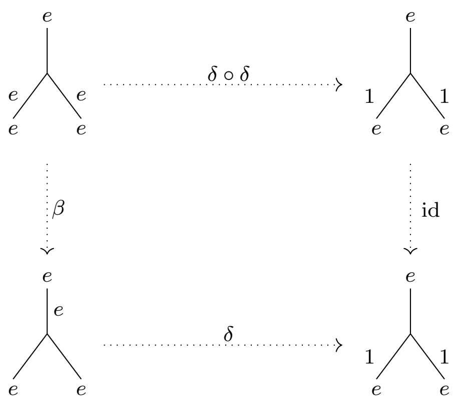
+ a similar relation involving r instead of ℓ



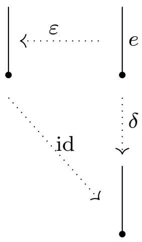
(G3)



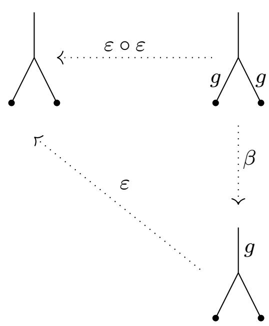
(G4)



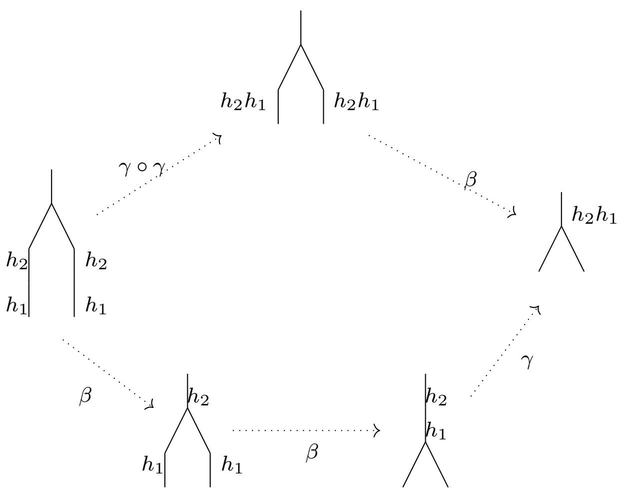
(G5)



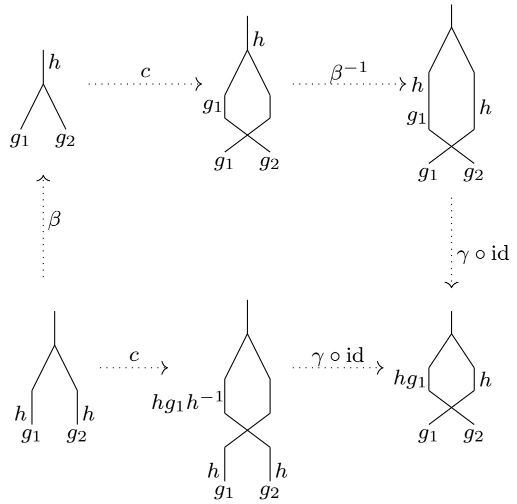
(G6)



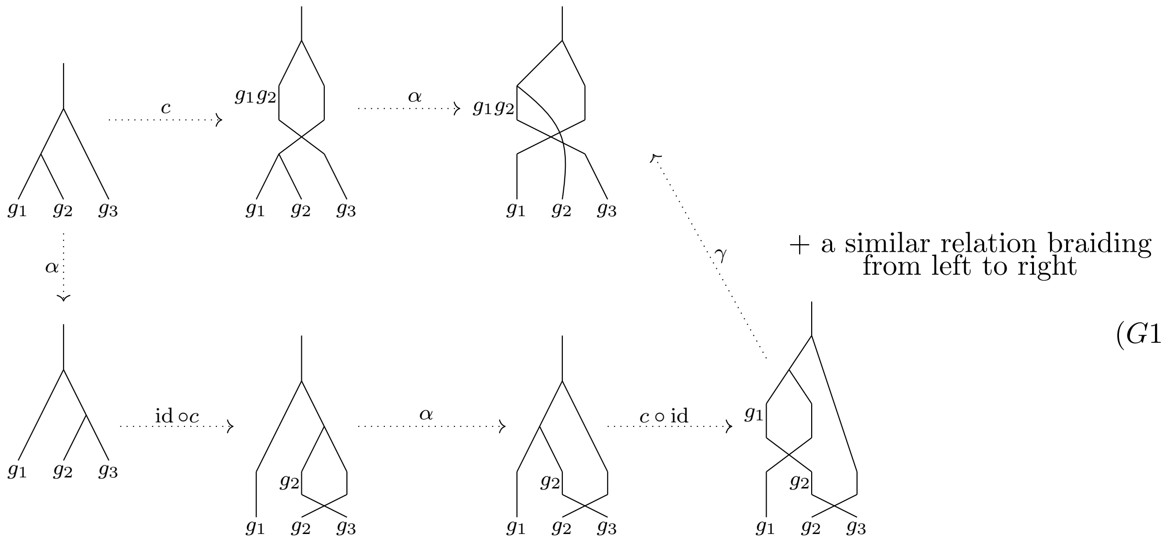
(G7)



(G8)



(*G*9)



(*G*10)

In the next step, we describe categorical algebras over PBr^G. To this end, we introduce the auxiliary operad P^G of *G-parentheses* which differs from PBr^G by the omission of the isomorphism c in (5.27) and all the relations it is involved in. From the construction of P^G and Proposition 5.25 we will be able to read off that categorical P^G-algebras are a type of equivariant monoidal categories, called *G-crossed monoidal categories*. This notion is based on [Tur10b, Section VI.1] and was further developed by other authors [MNS12, Gal17]; we will use the version given in [Gal17, Definition 5.1]. In particular, our notion of a G-crossed monoidal category does *not* include rigidity (the existence of dual objects).

Let us recall the basic definitions: For a group G, consider a family $(\mathcal{C}_g)_{g\in G}$ of categories $\mathcal{C}_g$ indexed by G. The category $\mathcal{C}_g$ is often referred to as the *twisted sector* for $g \in G$; the sector of the neutral element $e \in G$ is also called the *neutral sector*. Suppose now that $(\mathcal{C}_g)_{g\in G}$ is endowed with a homotopy coherent G-action shifting the sectors by conjugation, i.e. $g \in G$ acts as an equivalence $\mathcal{C}_h \longrightarrow \mathcal{C}_{ghg^{-1}}$ for every $h \in G$ such that the composition of these equivalences respects the group multiplication up to coherent isomorphism. Now a *G-equivariant monoidal product* on $(\mathcal{C}_g)_{g\in G}$ consists of functors $\otimes_{g,h} : \mathcal{C}_g \times \mathcal{C}_h \longrightarrow \mathcal{C}_{gh}$, which are associative and unital (with the unit as an object in $\mathcal{C}_e$) up to coherent isomorphism such that the G-action and the monoidal product intertwine up to coherent isomorphism [Gal17, Section 3.1]. If $(\mathcal{C}_g)_{g\in G}$ is equipped with a homotopy coherent G-action shifting the sectors by conjugation and an equivariant monoidal product, we call $(\mathcal{C}_g)_{g\in G}$ a *G-crossed monoidal category*; it is also called *G-equivariant monoidal category* in [MNS12]. The definition of P^G just translates the description of G-crossed monoidal categories in [Gal17, Definition 5.1] into the language of operads:

Lemma 5.26. *A categorical algebra over P^G is the same as a G-crossed monoidal category.*

Proof. We use the description for algebras over operads in terms of generators and relations given in Proposition 5.25 and provide a concrete dictionary to the conditions in [Gal17] (some of the conditions in [Gal17] are only explicitly spelled out in their strict form, which is justified by a strictification result proven before listing the conditions; our description gives all relations in their weakest form): The functors $\otimes_{g,h}$ correspond to the third generator in (5.27) and the unit is given by the first generator in (5.27). The morphisms corresponding to the associativity and unitality of $\otimes_{g,h}$ correspond to the generators α, ℓ and r, respectively.

The homotopy coherent G-action is obtained from the second generator in (5.27) and γ from (5.28) (in [Gal17] the action is denoted by g_* and the natural isomorphisms γ correspond to ϕ^{-1}). The coherence conditions for the action [Gal17, (4) on page 123 and Equation (3.1)] correspond to the relations ($G3$) and ($G4$).

The compatibility between the action and the monoidal product is encoded in the generators β and ε (the natural isomorphism corresponding to β is called ψ^g in [Gal17]). Relations ($G1$) and ($G2$) correspond to the condition that the action is via monoidal functors (which is [Gal17, Equation (3.2) and (2) on page 123]). The further compatibility between the action and the monoidal product is implemented via the relations ($G5$)-($G8$) (here ($G5$) is [Gal17, (3) on page 123], ($G6$) and ($G7$) are [Gal17, (1) on page 123] and ($G8$) is [Gal17, Equation (3.3)]). □

In the sequel, we will need the following fact about P^G:

Lemma 5.27. *The operad P^G is discrete and $\pi_0\mathsf{P}^G\binom{h}{\underline{g}}$ is given by the set $\Sigma_r \times_h G^r$ from Proposition 5.20, i.e. by the set of all pairs $(\sigma, \underline{b}) \in \Sigma_r \times G^r$ with $r := |\underline{g}|$ such that*

$$\prod_{j=1}^{r} b_{\sigma(j)} g_j b_{\sigma(j)} = h \ . \tag{5.29}$$

Proof. The Set-valued operad $\pi_0\mathsf{P}^G$ has the same generators as P^G, but all the isomorphisms introduced in (5.27) and (5.28) have to be replaced by actual equalities. Recalling the generators and relations for the associative operad, we see that $\pi_0\mathsf{P}^G$ is a colored version of the associative operad where all ingoing legs can be labeled by a group element. Group elements can be pushed through the multiplication (relation β) and be composed (relation γ) according to the group law. A label by the neutral element is treated as 'no label' (relation δ) and a label on a leg over the unit can be deleted (relation ε). These relations allow to bring each of the operations in $\pi_0\mathsf{P}^G\binom{h}{\underline{g}}$ into a unique standard form where we can describe them as a pair $(\sigma, \underline{b})$ of a permutation and an r-tuple of group elements by arguments analogous to those in [BSW17, Section 4.3]. The prescription of ingoing and outgoing colors leads to the requirement (5.29) for $(\sigma, \underline{b})$.

We still have to prove that all fundamental groups of $\mathsf{P}^G\binom{h}{\underline{g}}$ are trivial: For this we have to make sure that the given coherence diagrams for $\alpha, \beta, \gamma, \delta$ and ε ensure that, for each object in $\mathsf{P}^G\binom{h}{\underline{g}}$, there is only one morphism starting and ending at that object. The needed arguments amount precisely to the Coherence Theorem for G-crossed monoidal categories which is the main result of [Gal17]. □

A *G-braiding* on a G-crossed monoidal category $(\mathcal{C}_g)_{g\in G}$ is a family of coherent isomorphisms

$$X \otimes Y \cong g.Y \otimes X$$

for $X \in \mathcal{C}_g$ and $Y \in \mathcal{C}_h$. A G-crossed monoidal category equipped with a G-braiding is called a *braided G-crossed category*. The necessary coherence conditions are given in [Gal17, Definition 5.4], but the following results tells us that they are equivalently described by the relations in PBr^G:

Proposition 5.28. *A categorical algebra over* PBr^G *is the same as a braided* G*-crossed category.*

Proof. We use again Proposition 5.25. The G-braiding is operadically captured by the generator c in (5.27). The relation ($G9$) corresponds to [Gal17, Equation (5.1)], the first relation in ($G10$) to [Gal17, Equation (5.3)] and the equation not spelled out in detail in ($G10$) to [Gal17, Equation (5.2)]. □

For PBr^G, we have a description analogous to the one given for P^G in Lemma 5.27:

Proposition 5.29. *Color-wise there is an equivalence*

$$\mathsf{PBr}^G\binom{h}{\underline{g}} \simeq (\Sigma_r \times_h G^r)//B_r, \quad r = |\underline{g}|$$

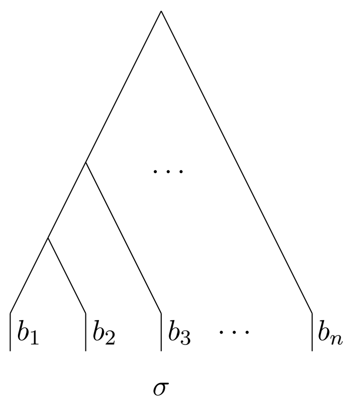
of groupoids.

Proof. We can describe PBr^G by adding to P^G the isomorphism c and the hexagon axiom that it has to satisfy.

For each $(\sigma, \underline{b}) \in \pi_0\mathsf{P}^G\binom{h}{\underline{g}}$, see Lemma 5.27, and $1 \leq j \leq r-1$ the isomorphism c induces an isomorphism

$$(\sigma, \underline{b}) \xrightarrow{c_{j,j+1}} (\tau_{j,j+1}\sigma, (b_1, \ldots, b_j b_{j+1} b_j^{-1}, b_j, \ldots, b_r)) =: c_{j,j+1}.(\sigma, \underline{b}) \ , \tag{5.30}$$

where $\tau_{j,j+1}$ is the transposition of j and $j+1$. Formally, this is achieved by choosing a standard representative for the classes, say



,

applying operations in P^G to bring it into a form such that the braiding can be applied to the legs j and $j+1$ and restoring the standard form by operations in P^G. The P^G-operations are always uniquely determined by starting point and endpoint thanks to discreteness of P^G (Lemma 5.27). Now $\mathsf{PBr}^G\binom{h}{\underline{g}}$ is equivalent to the groupoid whose objects are given by the set $\pi_0\mathsf{P}^G\binom{h}{\underline{g}} \cong \Sigma_r \times_h G^r$ and whose morphisms are words in the $c_{j,j+1}$ modulo the (induced) hexagon relations which – as in the non-equivariant case – amount precisely to the braid group relations. This proves that $\mathsf{PBr}^G\binom{h}{\underline{g}}$ is equivalent to the action groupoid of the B_r-action on $\Sigma_r \times_h G^r$ given by (5.30). □

Remark 5.30. We can read off from Proposition 5.29 that $\pi_0\mathsf{PBr}^G$-algebras with values in vector spaces are G-crossed algebras as considered in [Kau02, Tur10b].

Next we construct an operad morphism $\Phi : \mathsf{PBr}^G \longrightarrow \Pi E_2^G$ generalizing the corresponding construction for E_2. As the underlying map of colors we use the object function $\widehat{-} : G \longrightarrow$

$\mathrm{Map}(\mathbb{S}^1, BG)$ of the equivalence

$$\widehat{-} : G//G \xrightarrow{\simeq} \Pi\,\mathrm{Map}(\mathbb{S}^1, BG)$$

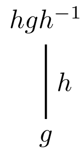
from (5.23).

Next we specify the images of the generators given in (5.27) and (5.28) (we will prove as part of Theorem 5.31 below that this assignment is compatible with the relations ($G1$)-($G10$)):

1. The generator

is mapped by Φ to the embedding of an empty collection of disks (as in the non-equivariant case, see Figure 5.3) together with the constant map to BG.

2. The generator

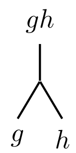


is mapped by Φ to the embedding $\mathbb{D}^2 \longrightarrow \mathbb{D}^2, x \longmapsto x/2$ and an arbitrary choice

$$\mathbb{D}^2 \setminus \frac{\mathbb{D}^2}{2} \cong \mathbb{S}^1 \times \left[\frac{1}{2}, 1\right] \longrightarrow BG$$

of a representative in the homotopy class $\widehat{h}$ corresponding to the morphism $h : g \longrightarrow hgh^{-1}$ in $G//G$.

3. The generator



is mapped by Φ to the embedding (see also Figure 5.3)

$$\begin{aligned} f : \mathbb{D}^2 \sqcup \mathbb{D}^2 &\longrightarrow \mathbb{D}^2 \\ x_1 &\longmapsto \frac{3}{8} \cdot x_1 - \frac{1}{2}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ x_2 &\longmapsto \frac{3}{8} \cdot x_2 + \frac{1}{2}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad . \end{aligned}$$

To equip $\mathsf{C}(f)$ with a continuous map φ to BG, we consider the decomposition of $\mathsf{C}(f)$ sketched in Figure 5.4. The value of φ on the boundary is given by $\widehat{g}, \widehat{h}$ and $\widehat{gh}$. On the wavy triangle we choose φ to be constant. Note that the gray area is homeomorphic to the standard 2-simplex. The 2-simplices of BG are described by pairs of group elements, and we equip the gray simplex with the BG-valued map corresponding to (g, h).

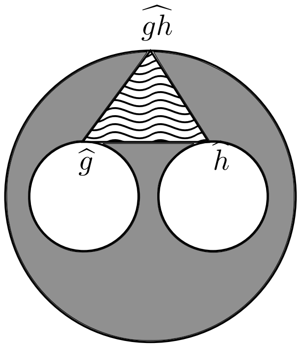
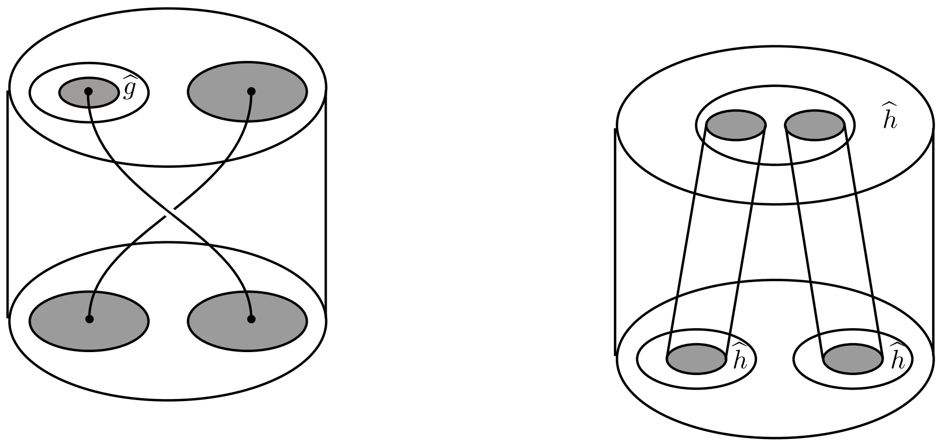


Figure 5.4: A sketch for the definition of the map φ.



(a) c (b) β

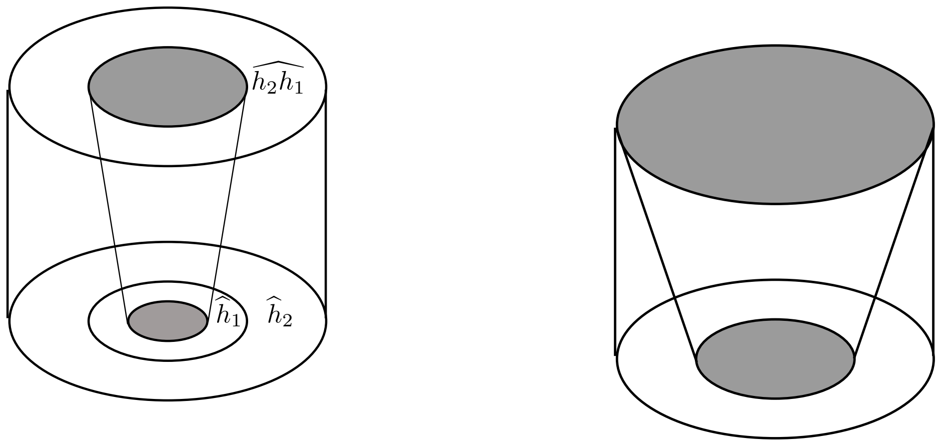
Figure 5.5: Definition of the morphism $\mathsf{PBr}^G \longrightarrow \Pi E_2^G$ on c and β. In (a) the left circle at the bottom is labeled with $\widehat{g}$; the circle on the right with $\widehat{h}$. Again, we have only drawn the center of every disk when depicting paths.

4. The path in $E_2(2)$ underlying the Φ-image of the braiding c in $\Pi E_2^G\binom{\widehat{gh}}{(\widehat{g},\widehat{h})}$ is sketched in Figure 5.5a. By Proposition 5.22 there exists a lift to the fundamental groupoid of the little bundles operad which by Remark 5.24 is unique once we specify starting point and endpoint. The starting point is the point defined in 3. The endpoint is determined by the images under Φ of the generators that the target of c in (5.27) is built from and their operadic composition. This describes the image of c in ΠE_2^G.

5. The image of the morphisms ℓ and r cannot be constructed using Proposition 5.22 since the disks touch at the end point. However, we can use Proposition 5.22 to get a path from the start point of ℓ and r to the disk embedding $x \longmapsto x/2$ equipped with a map to BG which is constant in the radial direction. Now we can rescale the disk as in Figure 5.6b and leave the map constant. The composition of these two paths defines the image of ℓ and r.

6. To define the image of α, we first define the underlying path in $E_2(3)$ to agree with the corresponding path for PBr, see Figure 5.3. The corresponding morphism of $\Pi E_2^G\binom{\widehat{g_1g_2g_3}}{(\widehat{g_1},\widehat{g_2},\widehat{g_3})}$ is again the unique lift which exists by Proposition 5.22 and Remark 5.24.

7. The image of β is again constructed by lifting a path in E_2 sketched in Figure 5.5b using



(a) γ (b) δ

Figure 5.6: Definition of the morphism $\mathsf{PBr}^G \longrightarrow \Pi E_2^G$ on γ and δ.

Proposition 5.22.

8. The path in $E_2(1)$ underlying the image of γ is drawn in Figure 5.6a. We equip it with a representative for the unique homotopy class of maps to BG constructed from the unique homotopy relative boundary of maps to BG between the composition of the homotopies corresponding to $\widehat{h_1}$ and $\widehat{h_2}$ and the homotopy corresponding to $\widehat{h_1 h_2}$.

9. The image of δ is defined by a simple rescaling sketched in Figure 5.6b equipped with the constant map to BG.

10. To define the image of ε, note that if we consider a disk such that any radial path from the origin to the boundary is labeled by g, the path in BG corresponding to the diameter of the disk is the composition of $\widehat{g}$ with $\widehat{g}^{-1}$ and hence homotopic to the constant map. We use such a homotopy to define the image of ε under Φ.

Theorem 5.31. *This assignment yields an equivalence*

$$\Phi : \mathsf{PBr}^G \longrightarrow \Pi E_2^G$$

of operads in groupoids.

Proof. (i) To show that Φ is a map of operads, we need to show that the assignments above are compatible with the relations listed on page 90 ff. Verifying a relation amounts to proving that two morphisms in components of ΠE_2^G (namely those prescribed by the above assignments) are equal. This can be achieved by observing that they have the same source and target object and that they lift the same morphism in components of ΠE_2. The latter follows by construction and the fact that in the non-equivariant case $\mathsf{PBr} \longrightarrow \Pi E_2$ is a map of operads. Now we invoke Remark 5.24 to get the desired equality of morphisms (note that the uniqueness statement of Remark 5.24 can even be used in those cases where it does not grant existence of the lifts). Hence, we have shown that Φ descends to a morphism $\Phi : \mathsf{PBr}^G \longrightarrow \Pi E_2^G$ of operads.

(ii) In the next step, we prove that Φ is an equivalence. First observe that Φ induces an equivalence of the categories enriched in groupoids built from PBr^G and ΠE_2^G by discarding all non-unary operations. In fact, both these categories have discrete morphism spaces and the functor induced by Φ is actually the equivalence $\widehat{-} : G//G \xrightarrow{\simeq} \Pi \operatorname{Map}(\mathbb{S}^1, BG)$ fixed in (5.23). Therefore, to conclude the proof that $\varPhi$ is a equivalence, it suffices to prove that its components

$$\varPhi : \mathsf{PBr}^G \binom{h}{\underline{g}} \longrightarrow E_2^G \binom{\widehat{h}}{\widehat{\underline{g}}}$$

are equivalences of groupoids. This follows from the 2-out-of-3 property because these components fit into the weakly commutative triangle

$$\begin{tikzcd}
 & \varSigma_r \times_h G^r//B_r & \\
\mathsf{PBr}^G\binom{h}{\underline{g}} \arrow[ur, "\simeq"] \arrow[rr, "\varPhi"'] & & E_2^G\binom{\widehat{h}}{\widehat{\underline{g}}} \arrow[ul, "\simeq"']
\end{tikzcd} ,$$

where $r = |\underline{g}|$. Here the equivalence $\mathsf{PBr}^G\binom{h}{\underline{g}} \simeq \varSigma_r \times_h G^r//B_r$ comes from Proposition 5.29 and the equivalence $E_2^G\binom{\widehat{h}}{\widehat{\underline{g}}} \simeq \varSigma_r \times_h G^r//B_r$ from Proposition 5.20.

□

An operad valued in a model category is called *admissible* if its category of algebras inherits a model structure in which equivalences and fibrations are created by the forgetful functor to colored objects. This model structure is also referred to as the *transferred model structure*. From [BM07, Theorem 2.1] one can deduce that operads valued in Cat with its canonical model structure are admissible. Hence, via operadic left Kan extension $\varPhi$ induces a Quillen adjunction

$$\mathsf{Alg}(\mathsf{PBr}^G) \underset{\varPhi^*}{\overset{\varPhi_!}{\rightleftarrows}} \mathsf{Alg}(\varPi E_2^G) \tag{5.31}$$

between the categories of algebras over PBr^G and $\varPi E_2^G$, respectively. As a consequence of Theorem 4.11, we arrive at our main result:

Theorem 5.32. *The operad map $\varPhi : \mathsf{PBr}^G \longrightarrow \varPi E_2^G$ induces a Quillen equivalence*

$$\{\textit{braided } G\textit{-crossed categories}\} \underset{\varPhi^*}{\overset{\varPhi_!}{\rightleftarrows}} \{\textit{categorical little } G\textit{-bundles algebras}\} .$$

Proof. Taking into account Proposition 5.28, we need to show that (5.31) is a Quillen equivalence. By [BM07, Theorem 4.1] this follows from $\varPhi$ being an equivalence (Theorem 5.31) if PBr^G and $\varPi E_2^G$ are $\varSigma$-cofibrant.

To see the latter, observe $\varPi$ sends $\varSigma$-cofibrant topological operads to $\varSigma$-cofibrant categorical operads. Now $\varPi E_2^G$ is $\varSigma$-cofibrant thanks to Remark 5.4. It remains to prove that PBr^G is $\varSigma$-cofibrant, which easily follows from the fact that the permutation action is free. □

Remark 5.33. For later purposes, we record the following fact about the operad of parenthesized G-braids: Let $\varLambda : \mathcal{P} \longrightarrow \mathcal{O}$ be a trivial fibration of groupoid-valued colored operads (this means it is component-wise an isofibration *and* equivalence of categories) that is the identity on colors, then the lifting problem

$$\begin{tikzcd}
 & \mathcal{P} \arrow[d, "\varLambda"] \\
\mathsf{PBr}^G \arrow[ur, dashed, "\widetilde{\varPsi}"] \arrow[r, "\varPsi"'] & \mathcal{O}
\end{tikzcd} \tag{5.32}$$

can be solved for any operad map $\varPsi : \mathsf{PBr}^G \longrightarrow \mathcal{O}$. The set X of colors for $\mathcal{O}$ need not be G, and the map $f : G \longrightarrow X$ of colors underlying $\varPsi$ can be arbitrary.

Of course, this is a cofibrancy statement in spirit of [BM03], where a model structure on operads is defined. We will, however, not use this language here to formulate the statement because that would require us to develop a model category of operads for varying colors that also takes into account the subtleties coming from the requirements on the maps of colors in (5.32) which is the identity for $\varLambda$, but not for $\varPsi$.

Let us prove now that the lifting problem (5.32) can be solved: Since $\varLambda$ is the identity on colors, it is clear how to define $\widetilde{\varPsi}$ on colors. Next observe that the object layer of PBr^G forms a free G-colored operad in the category of sets (no relations are imposed on object level). Since $\varLambda$ is component-wise surjective on objects, we can define $\widetilde{\varPsi}(o)$ on every object operation o such that operadic composition is respected and such that $\varLambda\widetilde{\varPsi}(o) = \varPsi(o)$. It remains to define $\widetilde{\varPsi}$ on every morphism $\alpha : o \longrightarrow o'$ of objects $o, o' \in \mathsf{PBr}^G\binom{h}{\underline{g}}$, where $\underline{g}$ is a tuple of elements in G and $h \in G$. We define $\widetilde{\varPsi}(\alpha)$ to be the preimage of $\varPsi(\alpha)$ under the bijection

$$\mathrm{Hom}_{\mathcal{P}\binom{f(h)}{f(\underline{g})}}\left(\widetilde{\varPsi}(o), \widetilde{\varPsi}(o')\right) \xrightarrow{\cong} \mathrm{Hom}_{\mathcal{O}\binom{f(h)}{f(\underline{g})}}\left(\varPsi(o), \varPsi(o')\right)$$

induced by $\varLambda$. From this construction, it follows that $\widetilde{\varPsi}$ is an operad map with underlying map of colors f and $\varLambda\widetilde{\varPsi} = \varPsi$. Also note that the lift $\widetilde{\varPsi}$ is essentially unique.

5.4 Application to topological field theories and the framed little bundles operad

The little bundles operad describes the genus zero part of surfaces decorated with G-bundles, hence it is intimately related to equivariant topological field theories. We will make this precise and then give an enhancement using the framed little bundles operad.

5.4.1 Applications of the little bundles operad to topological field theories

We will formulate our first statement in terms of the homotopical analogues of equivariant topological field theories which were introduced in [MW20b] using a Segal space model for the $(\infty, 1)$-category G-$\mathsf{Cob}(n)$ of G-cobordisms based on [GTMW09, CS19].

Definition 5.34. For any group G, an *n-dimensional homotopical equivariant topological field theory with values in a symmetric monoidal $(\infty, 1)$-category $\mathcal{S}$* is a symmetric monoidal ∞-functor

$$Z : G\text{-}\mathsf{Cob}(n) \longrightarrow \mathcal{S}$$

For us, the case $n = 2$ will be relevant. Informally, the objects in G-$\mathsf{Cob}(2)$ are finite disjoint union of oriented circles equipped with maps to BG. For two such collections of decorated circles, the morphism space between them is the space of two-dimensional compact oriented bordisms between these two collections of circles equipped with a map to BG extending the ones prescribed on the boundary components. We refer to [MW20b] for the technical details and an example of such a field theory constructed using an equivariant version of higher derived Hochschild chains.

We will now make the relation between the little bundles operad and topological field theory precise by proving that the value of a two-dimensional homotopical equivariant topological field theory on the circle is a homotopy little bundles algebra, thereby generalizing [BZFN10, Proposition 6.3] or the earlier version of this result phrased in terms of Gerstenhaber algebras [Ge94].

Theorem 5.35. *For any homotopical two-dimensional G-equivariant topological field theory Z, the values of Z on the circle with varying G-bundle decoration combine into a homotopy algebra over the little bundles operad E_2^G.*

A homotopy E_2^G-algebra is here to be understood as an algebra over the Boardman-Vogt resolution of E_2^G [BM07].

Proof. Any operation in $E_2^G\binom{\psi}{\underline{\varphi}}$ can be seen as bordism $\left(\mathbb{S}^1\right)^{\sqcup r} \longrightarrow \mathbb{S}^1$ (namely the complement of the little disk embedding), where $r := |\underline{\varphi}|$, decorated with a map to BG whose restriction to the ingoing and outgoing boundary is $\underline{\varphi}$ and ψ, respectively. Strictly speaking, we can see only those operations as bordisms whose E_2-part consists of little disks with non-intersecting boundary. This leads us to an equivalent suboperad of E_2^G (which is not strictly unital any more, but only up to homotopy), but we will suppress this in the notation. In summary, we find maps

$$E_2^G\begin{pmatrix}\psi\\ \underline{\varphi}\end{pmatrix} \longrightarrow G\text{-}\mathsf{Cob}(2)(((\mathbb{S}^1)^{\sqcup r},\underline{\varphi}),(\mathbb{S}^1,\psi)) \ , \tag{5.33}$$

where G-$\mathsf{Cob}(n)(-,-)$ denotes the morphism spaces of G-$\mathsf{Cob}(2)$. The operadic composition on the left hand side is mapped to the composition of G-bordisms on the right hand side.

Now by definition a homotopical two-dimensional G-equivariant topological field theory gives us a map

$$G\text{-}\mathsf{Cob}(2)(((\mathbb{S}^1)^{\sqcup r},\underline{\varphi}),(\mathbb{S}^1,\psi)) \longrightarrow [Z(\mathbb{S}^1,\varphi_1)\otimes\dots\otimes Z(\mathbb{S}^1,\varphi_r), Z(\mathbb{S}^1,\psi)] \ , \tag{5.34}$$

where $[-,-]$ denotes the mapping space of $\mathcal{S}$. This map, by definition, respects the composition up to coherent homotopy. Concatenating (5.33) and (5.34) we obtain maps

$$E_2^G\begin{pmatrix}\psi\\ \underline{\varphi}\end{pmatrix} \longrightarrow [Z(\mathbb{S}^1,\varphi_1)\otimes\dots\otimes Z(\mathbb{S}^1,\varphi_r), Z(\mathbb{S}^1,\psi)] \ .$$

This would endow the $\mathrm{Map}(\mathbb{S}^1,BG)$-colored object $\left(Z(\mathbb{S}^1,\varphi)\right)_{\varphi\in\mathrm{Map}(\mathbb{S}^1,BG)}$ with an E_2^G-algebra structure if composition were respected strictly. Instead, it is only respected up to coherent homotopy – with the coherence data coming from Z. But such a structure of an algebra over an operad respecting the operadic structure only up to coherent homotopy is precisely an (ordinary) algebra over the Boardman-Vogt resolution of that operad [BM07]. This is also made precise in great detail in [Yau18, Chapter 7]. □

We can use the little bundles operad also to obtain results about ordinary (non-homotopical) 3-2-1-dimensional topological field theories with non-aspherical target space. Statements about the non-aspherical case are scarce in the literature, and the following Proposition is supposed to indicate that we can make at least a statement about the value of such theories on the circle.

Proposition 5.36. *Let T be a space such that $\pi_k(T)=0$ for $k\geq 3$ and $Z : T\text{-}\mathsf{Cob}(3,2,1)\longrightarrow$ $\mathsf{2Vect}$ a 3-2-1-dimensional topological field theory with target T valued in the symmetric monoidal bicategory of complex 2-vector spaces. Then the operad E_2^T takes values in 2-groupoids, and the values of Z on the circle with varying G-bundle decoration combine into a homotopy E_2^T-algebra in 2-vector spaces.*

Proof. Similar arguments as those for Proposition 5.19 show that E_2^T takes values in 2-groupoids.

Next we observe that we can restrict Z to a two-dimensional non-extended $(\infty,1)$-topological field theory with target T and values in $\mathsf{2Vect}$ by discarding the definition of Z on *non-invertible* three-dimensional bordisms. More precisely, we just remember the definition of Z on decorated compact oriented one-dimensional manifolds, decorated compact oriented two-dimensional bordisms and structure-preserving diffeomorphisms between those. It is understood here that the diffeomorphisms are seen as invertible three-dimensional bordisms via the mapping cylinder construction Remark 2.2 (c).

Having made this observation, we can proceed as in the proof of Theorem 5.35. □

We will, however, not spell out the data of a (homotopy) E_2^T-algebra; a presentation of E_2^T in terms of generators and relations is beyond the scope of this thesis. A first approach to E_2^T-algebras might be through the examples that we can produce from a cohomology class in $H^3(T;\mathrm{U}(1))$ using Proposition 5.36 and [MW20a, Theorem 3.19].

In the case $T = BG$, we deduce from Proposition 5.36 that the value of an extended G-equivariant topological field theory $Z : G\text{-}\mathsf{Cob}(3,2,1) \longrightarrow \mathsf{2Vect}$ on the circle is a homotopy little bundles algebra. The cofibrancy property proven for PBr^G in Remark 5.33 implies that the equivalence $\mathsf{PBr}^G \longrightarrow \mathit{\Pi} E_2^G$ from Theorem 5.31 lifts to the Boardman-Vogt resolution of $\mathit{\Pi} E_2^G$. From this, we conclude:

Corollary 5.37. *For any group G and an extended G-equivariant topological field theory $Z : G\text{-}\mathsf{Cob}(3,2,1) \longrightarrow \mathsf{2Vect}$, the values of Z on the circle with varying G-bundle decoration combine into a braided G-crossed category in* $\mathsf{2Vect}$.

5.4.2 The framed little bundles operad

There is an extension of the little disks operad E_2, namely the *framed little disks operad* $\mathsf{f}E_2$ whose operations are not only given by affine embeddings of disks into a bigger disk, but also rotations in a fixed direction; i.e.

$$\mathsf{f}E_2(r) = E_2(r) \times (\mathbb{S}^1)^{\times r}$$

as topological spaces. The r copies of the circle give the rotation parameter for the r disks. We refer to [SW03] for an elegant construction of the operad structure via the semidirect product construction. If $f = (f_0, R) \in \mathsf{f}E_2(r)$ is a framed little disks operation, then we can see $\mathsf{C}(f) = \mathsf{C}(f_0)$ as an oriented bordism $(\mathbb{S}^1)^{\sqcup r} \longrightarrow \mathbb{S}^1$, where we use the rotation part R to identify $(\mathbb{S}^1)^{\sqcup r}$ with the ingoing boundary circles of $\mathsf{C}(f)$. It is crucial here that the circles are oriented such that R can be understood as an orientation-preserving diffeomorphism. From this we read off that the statements on equivariant topological field theories made in the previous subsection should actually be enhanced to a framed version because we are, in this thesis, always interested in oriented[1] topological field theories.

First we observe that in the constructions and definitions of Section 5.1, we may replace the operad E_2 of little disks by the operad $\mathsf{f}E_2$ of framed little disks. This way we obtain the obtain the *operad $\mathsf{f}E_2^G$ of framed little bundles*. For an r-tuple $\underline{\varphi} = (\varphi_1, \dots, \varphi_r)$ of maps $\mathbb{S}^1 \longrightarrow BG$ and another map $\psi : \mathbb{S}^1 \longrightarrow BG$ the space of operations $\mathsf{f}E_2^G\binom{\psi}{\underline{\varphi}}$ is given by pairs (f, ξ), where $f = (f_0, R) \in \mathsf{f}E_2(r)$ and $\xi : \mathsf{C}(f) \longrightarrow BG$ is a map on the complement of the image of f such that the restriction

$$(\mathbb{S}^1)^{\sqcup r} \sqcup \mathbb{S}^1 \xrightarrow{R \sqcup \mathrm{id}_{\mathbb{S}^1}} \mathsf{C}(f) \xrightarrow{\xi} BG$$

is given by $(\underline{\varphi}, \psi)$. The construction can also be made for higher-dimensional disks and arbitrary

[1] The nomenclature is very unfortunate, but established: Operations in the *framed* little disks operad give *oriented* bordisms and not *framed* ones.

target spaces, but for concreteness we do not discuss this here. Note that there is a natural map

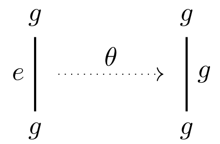
$$E_2^G \longrightarrow \mathsf{f}E_2^G \tag{5.35}$$

of $\mathrm{Map}(\mathbb{S}^1, BG)$-colored operads whose underlying map on colors is the identity.

Since we find

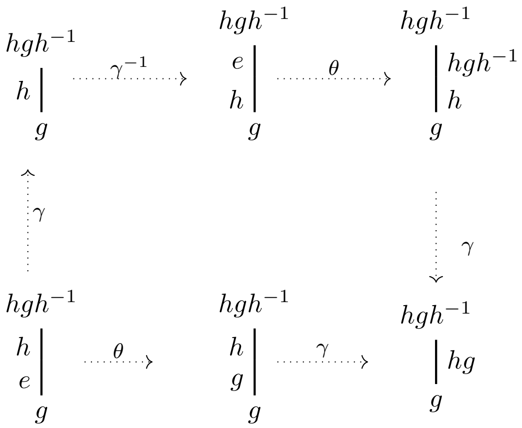
$$\mathsf{f}E_2^G\begin{pmatrix}\psi\\ \underline{\varphi}\end{pmatrix} = E_2^G\begin{pmatrix}\psi\\ \underline{\varphi}\end{pmatrix} \times (\mathbb{S}^1)^{\times r} \tag{5.36}$$

as spaces, the operad $\mathsf{f}E_2^G$ is also aspherical. Therefore, it is natural to ask for a groupoid model for $\mathsf{f}E_2^G$. We obtain this by replacing the operad PBr^G of parenthesized G-braids from Section 5.3.2 by the *operad* fPBr^G *of framed parenthesized* G*-braids*: To this end, we add to PBr^G an additional generator θ

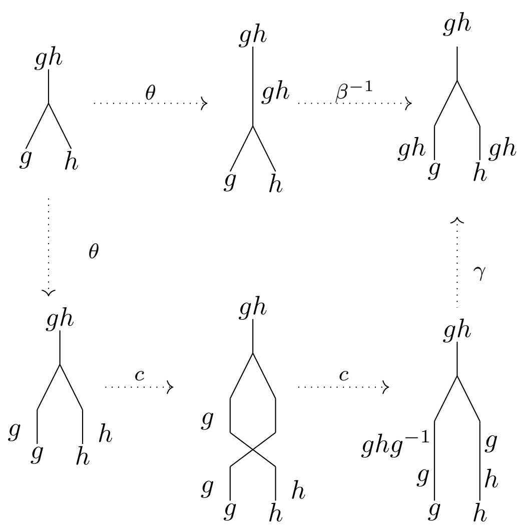


(5.37)

for every $g \in G$ that we will refer to as *balancing* (for reasons that will become clear below). It will be subject to the following relations (B1)-(B3):

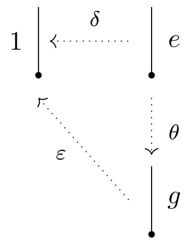


(B1)



(B2)

(In (B2) we suppress a few δ isomorphisms for readability.)



(B3)

As for PBr^G, there is a non-equivariant version fPBr for fPBr^G, the *framed parenthesized braid operad*. It is known that fPBr provides a model for the framed little disks operad [SW03]; i.e. there is an equivalence

$$\mathsf{fPBr} \xrightarrow{\simeq} \varPi \mathsf{f}E_2 \ . \tag{5.38}$$

It sends the balancing (by which we mean the non-equivariant version) to a disk rotation. Since categorical algebras over fPBr are by definition *balanced braided categories*, (5.38) tells us that categorical framed little disks algebras are equivalent to balanced braided categories (recall that balanced braided categories are closely related to ribbon categories, but are not necessarily rigid; in [SW03] they are called ribbon braided categories).

We will refer to a categorical algebra over fPBr^G as a *balanced braided G-crossed category*. A stricter version of these axioms appears in [Tur10b, Section VI.1] as part of the notion of a ribbon G-category; however, as in the non-framed case, we do not require rigidity.

We will now extend the functor $\varPhi : \mathsf{PBr}^G \longrightarrow \varPi E_2^G$ from Theorem 5.31 to a functor $\varPhi^{\mathsf{f}}$: $\mathsf{fPBr}^G \longrightarrow \varPi \mathsf{f}E_2^G$. In order to define $\varPhi^{\mathsf{f}}$ on the generators that already appear in PBr^G, we just use $\varPhi$ and the natural map $\varPi E_2^G \longrightarrow \varPi \mathsf{f}E_2^G$ induced by (5.35).

Hence, it remains to define $\varPhi^{\mathsf{f}}$ on the balancing θ from (5.37). For this, recall that the functor $\varPhi : \mathsf{PBr}^G \longrightarrow \varPi E_2^G$ sends the source object of θ to the embedding $\mathbb{D}^2 \longrightarrow \mathbb{D}^2, x \longmapsto x/2$ and a map $\mathbb{S}^1 \times [1/2,1] \longrightarrow BG$ that we denote by I_g. The restriction I_g to the boundary circle yields a map $\varphi_g : \mathbb{S}^1 \longrightarrow BG$ that classifies the G-bundle over $\mathbb{S}^1$ with holonomy g. The map I_g is equivalent to the constant homotopy. Consider now the homotopy

$$D : \mathbb{S}^1 \times \left[\frac{1}{2},1\right] \times I \longmapsto \mathbb{S}^1 \times \left[\frac{1}{2},1\right] , \quad (z,t,s) \longmapsto \left(z\mathrm{e}^{-2\pi\,\mathrm{i}(2-2t)s}, t \right) \tag{5.39}$$

from the identity of the cylinder $\mathbb{S}^1 \times [1/2,1]$ to the Dehn twist of that cylinder. The composition $I_g D$ describes a path in $\mathsf{f}W_2^G(1)$ (which is the version of W_2^G with $\mathsf{f}E_2$ instead of E_2), but not in $E_2^G\binom{\varphi_g}{\varphi_g}$ because D rotates the circle $\mathbb{S}^1 \times \{1/2\} \subset \mathbb{S}^1 \times [1/2,1]$, and during the rotation the bundle decoration for this circle changes. However, if we simultaneously rotate the disk embedding (which is allowed in $\mathsf{f}E_2$, but not in E_2) and thereby precisely undo the effect of the rotation on the decoration, we *do* get a path in $\mathsf{f}E_2^G\binom{\varphi_g}{\varphi_g}$. We define this path to be $\varPhi^{\mathsf{f}}(\theta)$.

Let us verify that the target object of $\varPhi^{\mathsf{f}}(\theta)$ is indeed the image of the right hand side of (5.37) under $\varPhi$ (and hence under $\varPhi^{\mathsf{f}}$): This amounts to proving that for the radial line

$$r : [1/2,1] \longrightarrow \mathbb{S}^1 \times [1/2,1] , \quad t \longmapsto (z_0, t)$$

at the base point z_0 of $\mathbb{S}^1$ (recall the conventions from Section 5.2.2), the path $I_g D|_{s=1} r$, when seen as a loop in BG, is given by φ_g up to homotopy. Indeed, for $t \in [1/2,1]$

$$I_g D|_{s=1} r(t) = I_g \left(z_0 \, \mathrm{e}^{-2\pi\,\mathrm{i}(2-2t)}, t \right) = I_g \left(z \, \mathrm{e}^{4\pi\,\mathrm{i}\,t}, t \right) \ .$$

By definition of I_g this loop is homotopic to $[1/2,1] \ni t \longmapsto \varphi_g \left(z_0 \, \mathrm{e}^{4\pi\,\mathrm{i}\,t} \right)$, which is the desired result.

Theorem 5.38. *The assignment $\theta \longmapsto \Phi^{\mathsf{f}}(\theta)$ extends $\Phi : \mathsf{PBr}^G \longrightarrow \Pi E_2^G$ from Theorem 5.31 to an equivalence $\Phi^{\mathsf{f}} : \mathsf{fPBr}^G \longrightarrow \Pi \mathsf{f} E_2^G$ of operads in groupoids.*

Proof. All the steps in the proof are a slight extension of those given in the proof of Theorem 5.31:

(i) Let us make the following two observations:

- The assignments of Φ^{f} extend the assignments for the non-equivariant situation (5.38), as follows directly from the definitions.
- The map

$$\Pi \mathsf{f} E_2^G \begin{pmatrix} \psi \\ \underline{\varphi} \end{pmatrix} \longrightarrow \Pi \mathsf{f} E_2(r) \ , \quad r = |\underline{\varphi}| \ . \tag{5.40}$$

has the following lifting properties: From (5.36) it follows that (5.40) is the product map of

$$\Pi E_2^G \begin{pmatrix} \psi \\ \underline{\varphi} \end{pmatrix} \longrightarrow \Pi E_2(r) \tag{5.41}$$

and the identity on $\Pi(\mathbb{S}^1)^{\times r}$. This entails that (5.40) inherits the lifting properties from (5.41); more precisely, the existence statement from Proposition 5.22 and the uniqueness statement from Remark 5.24.

Now the same arguments as in the proof of Theorem 5.31 show that Φ^{f} is a functor.

(ii) For the proof that Φ^{f} is an equivalence, first observe that the functor induced by Φ^{f} on the categories of unary operations is $G//G \times \star//\mathbb{Z} \longrightarrow \Pi \operatorname{Map}(\mathbb{S}^1, BG) \times \Pi\mathbb{S}^1$, i.e. the product of the equivalence (5.23) with the equivalence $\star//\mathbb{Z} \longrightarrow \Pi\mathbb{S}^1$ coming from our choice of a base point for the circle. Finally, the components $\Phi^{\mathsf{f}} : \mathsf{fPBr}^G\binom{h}{\underline{g}} \longrightarrow \mathsf{f}E_2^G\binom{\widehat{h}}{\underline{\widehat{g}}}$ are equivalences as well because they fit into the commutative diagram

$$\begin{tikzcd}
 & \mathsf{PBr}^G\binom{h}{\underline{g}} \times (\star//\mathbb{Z})^{\times r} \arrow[dl, "\simeq"'] \arrow[dr, "\simeq"] & \\
\mathsf{fPBr}^G\binom{h}{\underline{g}} \arrow[ur] \arrow[rr, "\Phi^{\mathsf{f}}"] & & \mathsf{f}E_2^G\binom{\widehat{h}}{\underline{\widehat{g}}} \arrow[ul]
\end{tikzcd} ,$$

where $r = |\underline{g}|$. The right diagonal equivalence comes from (5.36) and the left diagonal equivalence arises from the fact that the relations (B1)-(B3) for the balancing allow us to write every morphism in fPBr^G uniquely as a collection of balancings on the ingoing legs followed by a morphism in PBr^G (informally, this just uses that the balancing can be pushed past all the other morphisms). Now it follows from the 2-out-of-3 property that the components of Φ^{f} are equivalences.

□

As a direct analogue of Theorem 5.32, we deduce from Theorem 5.38:

Theorem 5.39. *The operad map $\Phi^{\mathsf{f}} : \mathsf{fPBr}^G \longrightarrow \Pi \mathsf{f} E_2^G$ induces a Quillen equivalence*

$$\{\textit{balanced braided } G\textit{-crossed categories}\} \underset{\Phi^{\mathsf{f}*}}{\overset{\Phi^{\mathsf{f}}_!}{\rightleftarrows}} \{\textit{categorical framed little } G\textit{-bundles algebras}\} \ .$$

If we now use that we can see a framed little disks operation as an oriented two-dimensional bordism (as outlined at the beginning of this subsection), we obtain the following enhancements of the statements in the previous subsection:

Theorem 5.40. *For any homotopical two-dimensional G-equivariant topological field theory Z, the values of Z on the circle with varying G-bundle decoration combine into a homotopy algebra over the framed little bundles operad E_2^G.*

Corollary 5.41. *For any group G and an extended G-equivariant topological field theory $Z : G\text{-}\mathsf{Cob}(3,2,1) \longrightarrow \mathsf{2Vect}$, the values of Z on the circle with varying G-bundle decoration combine into a balanced braided G-crossed category in* $\mathsf{2Vect}$.

Recall that this uses that equivariant topological field theories in this thesis are always oriented.

The implications of Corollary 5.41 will be spelled out (in the language of topological field theory, rather than the one of operads) in the next section.

5.5 The evaluation of 3-2-1-dimensional equivariant topological field theories on the circle

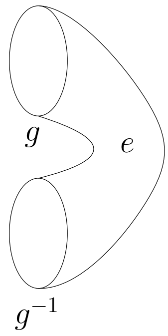
In this section, we proceed with the investigation of the structure that is present on the category that we obtain by evaluation of an extended equivariant topological field theory on the circle. Most of the work has already been done in the previous sections through the discussion of the (framed) little bundles operad. We will now turn to some properties that are not yet part of the operadic description. The goal will be to further refine the statement of Corollary 5.41.

5.5.1 A more explicit translation from little bundles operations to decorated bordisms

The translation from little bundles operations to decorated bordisms was generally described in Section 5.4. In the sequel, it will be necessary to make this translation slightly more explicit in some relevant cases.

To this end, let $Z : G\text{-}\mathsf{Cob}(3,2,1) \longrightarrow \mathsf{2Vect}$ be an extended G-equivariant topological field theory for a finite group G. Then by the construction from Proposition 2.6 and Remark 2.10 we obtain a 2-vector bundle over the groupoid of G-bundles over $\mathbb{S}^1$ by evaluation of Z on the circle $\mathbb{S}^1$ equipped with G-bundles over the circle. This structure corresponds precisely to the unary little bundles operations as we will explain now.

Recall first that the groupoid $\mathsf{PBun}_G(\mathbb{S}^1)$ of G-bundles over $\mathbb{S}^1$ is non-canonically equivalent to the action groupoid $G//G$ (see Section 5.2.2). More precisely, the equivalence chooses a base point and orientation of $\mathbb{S}^1$ and assigns to a given bundle the holonomy of the based loop surrounding $\mathbb{S}^1$ once in the positive direction. So whenever a bundle is characterized by a group element, we actually mean the holonomy with respect to the loop determined by the base point and the orientation. To illustrate this point, consider the bent cylinder (as a bordism $\mathbb{S}^1 \coprod \mathbb{S}^1 \longrightarrow \emptyset$)



with the identity homotopy on it. On all circle-shaped slices of the cylinder, we have the same bundle. But since the upper and the lower copy of the circle carry different orientations, we obtain holonomy values which are inverse to each other.

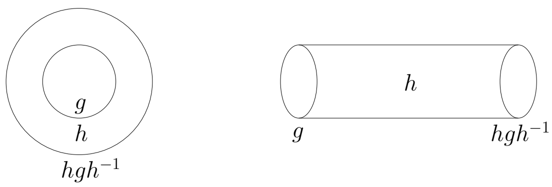
By means of a fixed equivalence $\mathsf{PBun}_G(\mathbb{S}^1) \simeq G//G$, we obtain from Z a 2-vector bundle

$$G//G \longrightarrow \mathsf{2Vect}$$

sending an object $g \in G$ in $G//G$ to a 2-vector space $\mathcal{C}_g^Z = Z(\mathbb{S}^1, g)$ and a morphism $h : g \longrightarrow hgh^{-1}$ in $G//G$ to a 2-linear equivalence

$$\phi_h : \mathcal{C}_g^Z = Z(\mathbb{S}^1, g) \longrightarrow \mathcal{C}_{hgh^{-1}}^Z = Z(\mathbb{S}^1, hgh^{-1}) \ . \tag{5.42}$$

We use the notation $h.X := \phi_h X$. By construction, this 2-linear equivalence arises by evaluation of Z on the cylinder with ingoing holonomy g and outgoing holonomy hgh^{-1}. The two bundles characterized by these holonomies are isomorphic by a gauge transformation h. Technically, we have to understand h as a homotopy of the classifying maps for the bundles characterized by the holonomies g and hgh^{-1}. This homotopy is put on the cylinder such that we can evaluate Z on it. Depending on what is convenient, we will switch between the pictorial representations



for the corresponding 1-morphism in G-$\mathsf{Cob}(3,2,1)$. By construction, the equivalence (5.42) comes from the second generator from the left in (5.27) of the operad of parenthesized G-braids or, equivalently, the little bundles operad.

In yet another language,

- the category

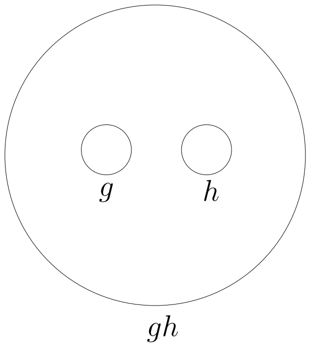
$$\mathcal{C}^Z := \bigoplus_{g \in G} \mathcal{C}_g^Z$$

- together with the equivalences $\phi_h : \mathcal{C}^Z \longrightarrow \mathcal{C}^Z$ obtained from the equivalences (5.42)

- and the coherence data of our 2-vector bundle consisting of natural isomorphisms $\alpha_{g,h}$: $\phi_g \circ \phi_h \cong \phi_{gh}$ and $\phi_e \cong \mathrm{id}_{\mathcal{C}^Z}$

form a *G-equivariant category* in the terminology of [Kir04]. As mentioned on page 92, we call $\mathcal{C}_g$ the *twisted sector* for the group element $g \in G$. The sector $\mathcal{C}_e^Z$ of the neutral element $e \in G$ is called the *neutral sector*.

By evaluation of Z on the pair of pants, the 2-vector space $\mathcal{C}^Z$ comes with a monoidal product that is compatible with the G-action; this is already contained in Corollary 5.37. In more detail, the pair of pants with bundles $g, h \in G$ on the ingoing circles has the bundle gh on the outgoing circle; pictorially:



(5.43)

Or, in other words, the evaluation of the stack $\Pi(-, BG)$ of G-bundles on the pair of pants yields the span

$$G//G \times G//G \xleftarrow{B} (G \times G)//G \xrightarrow{M} G//G \ , \tag{5.44}$$

where B is the obvious functor and M the multiplication. Hence, evaluation of Z on the pair of pants decorated with ingoing bundles g and h yields a 2-linear functor $\otimes_{g,h} : \mathcal{C}_g^Z \boxtimes \mathcal{C}_h^Z \longrightarrow \mathcal{C}_{gh}^Z$. These functors assemble to give the monoidal product $\otimes : \mathcal{C}^Z \boxtimes \mathcal{C}^Z \longrightarrow \mathcal{C}^Z$. It carries $X \in \mathcal{C}_g^Z$ and $Y \in \mathcal{C}_h^Z$ to $X \otimes Y \in \mathcal{C}_{gh}^Z$. Evaluation on the disk seen as bordism $\emptyset \longrightarrow \mathbb{S}^1$ decorated with the trivial G-bundle yields a 2-linear functor $\eta : \mathsf{FinVect} \longrightarrow \mathcal{C}_e^Z$, which is determined by the object $I := \eta(\mathbb{C})$ in $\mathcal{C}_e^Z$. This object is the monoidal unit. The relation between the G-action and the monoidal product on $\mathcal{C}^Z$ can be made precise by saying that $\mathcal{C}^Z$ is G-crossed monoidal category or, equivalently (see Lemma 5.26), an algebra over the G-colored operad P^G of G-parentheses. This just rephrases a part of the information contained in Corollary 5.37 in a different language.

In fact, Corollary 5.37 also makes a statement about the equivariant braiding for the monoidal product. Also for this piece of structure it will be helpful to translate from the description as a little bundles operation to the language of topological field theories. To this end, observe that rotating the inner circles in (5.43) counterclockwise around each other while keeping the outgoing circle fixed yields a diffeomorphism of the pair of pants relative boundary. In the sense of Remark 2.2 (c) this diffeomorphism gives rise to an invertible 2-morphism $G\text{-}\mathsf{Cob}(3,2,1)$, also described in detail in [MNS12, Lemma 3.25], on which we can evaluate Z. This invertible 2-morphism corresponds to the generator c in the operad PBr^G (or, equivalently, the little bundles operad) as follows from the proof of Theorem 5.31.

As a result, the evaluation of Z on this invertible 2-morphism yields natural isomorphisms

$$c_{X,Y} : X \otimes Y \longrightarrow g.Y \otimes X \quad \text{for all} \quad X \in \mathcal{C}_g^Z, \quad Y \in \mathcal{C}_h^Z$$

that, again by Corollary 5.37, form a G-braiding, i.e. make $\mathcal{C}^Z$ into a braided G-crossed monoidal category.

Of course, it is implicit here that $\mathcal{C}^Z$ lives in 2-vector spaces, i.e. it is a PBr^G-algebra in $\mathsf{2Vect}$. We will here and in the sequel indicate the target category $\mathsf{2Vect}$ by adding the additional qualifier *finitely semisimple*. In that language, we obtain in summary the following statement:

Corollary 5.42. *The category $\mathcal{C}^Z$ is a finitely semisimple braided G-crossed monoidal category.*

Let us emphasizes that this just rephrases Corollary 5.41.

5.5.2 The language of bimodules

The following observations allow us to compute a 3-2-1-dimensional G-equivariant topological field theory Z on surfaces decorated with G-bundles just by means of the monoidal structure on $\mathcal{C}^Z$: The monoidal product in $\mathcal{C}^Z$ is built from the 2-linear maps

$$\otimes_{g,h} \ : \ \mathcal{C}_g^Z \boxtimes \mathcal{C}_h^Z \longrightarrow \mathcal{C}_{gh}^Z$$

obtained by evaluation on the pair of pants decorated with bundles as explained above. Evaluation Z on the same manifold read backwards yields 2-linear maps

$$\Delta_{g,h} \ : \ \mathcal{C}_{gh}^Z \longrightarrow \mathcal{C}_g^Z \boxtimes \mathcal{C}_h^Z \ .$$

The direct generalization of the adjunction relation in [BDSPV15] gives us the adjunction

$$\otimes_{g,h} \dashv \Delta_{g,h} \tag{5.45}$$

in $\mathsf{2Vect}$. The same arguments apply to the monoidal unit

$$\eta : \mathsf{FinVect} \longrightarrow \mathcal{C}_e^Z$$

and the evaluation of Z on the manifold read backwards, namely

$$\varepsilon : \mathcal{C}_e^Z \longrightarrow \mathsf{FinVect} \ ,$$

i.e. we obtain the adjunction

$$\eta \dashv \varepsilon \tag{5.46}$$

in $\mathsf{2Vect}$

In order to use these adjunctions, we recall from [BDSPV15, Section 2.2] some needed facts on the symmetric monoidal bicategory Bimod of 2-vector spaces, bimodules (here a bimodule from $\mathcal{V}$ to $\mathcal{W}$ between $\mathbb{C}$-linear categories $\mathcal{V}$ and $\mathcal{W}$ is a functor $P : \mathcal{V}^{\mathrm{opp}} \boxtimes \mathcal{W} \longrightarrow \mathsf{FinVect}$) and natural transformations. The composition of bimodules $P : \mathcal{U}^{\mathrm{opp}} \boxtimes \mathcal{V} \longrightarrow \mathsf{FinVect}$ and $Q : \mathcal{V}^{\mathrm{opp}} \boxtimes \mathcal{W} \longrightarrow \mathsf{FinVect}$ is the bimodule $Q \circ P : \mathcal{U}^{\mathrm{opp}} \boxtimes \mathcal{W} \longrightarrow \mathsf{FinVect}$ given by the coend

$$(Q \circ P)(U, W) := \int^{V \in \mathcal{V}} Q(V, W) \otimes P(U, V) \quad \text{for all} \quad U \in \mathcal{U}, \quad W \in \mathcal{W} \ ,$$

see [ML98] for an introduction to coends. Any 2-linear map $F : \mathcal{V} \longrightarrow \mathcal{W}$ gives rise to a bimodule $F_* : \mathcal{W}^{\mathrm{opp}} \boxtimes \mathcal{V} \longrightarrow \mathsf{FinVect}$ by

$$F_*(W, V) := \mathrm{Hom}_{\mathcal{W}}(W, FV) \quad \text{for all} \quad V \in \mathcal{V}, \quad W \in \mathcal{W} \ .$$

This assignment extends to a 2-functor

$$-_* : \mathsf{2Vect} \longrightarrow \mathsf{Bimod}.$$

The functoriality of $-_*$ entails that for 2-linear maps $F : \mathcal{U} \longrightarrow \mathcal{V}$ and $G : \mathcal{V} \longrightarrow \mathcal{W}$

$$\mathrm{Hom}_{\mathcal{W}}(W, GFU) \cong \int^{V \in \mathcal{V}} \mathrm{Hom}_{\mathcal{W}}(W, GV) \otimes \mathrm{Hom}_{\mathcal{V}}(V, FU) \text{ for all } U \in \mathcal{U},\ W \in \mathcal{W} \quad (5.47)$$

by a canonical isomorphism of vector spaces. Note that $F : \mathcal{V} \longrightarrow \mathcal{W}$ also gives rise to a bimodule $F^* : \mathcal{V}^{\mathrm{opp}} \boxtimes \mathcal{W} \longrightarrow \mathsf{Vect}$ by

$$F^*(V, W) := \mathrm{Hom}_{\mathcal{W}}(FV, W) \quad \text{for all} \quad V \in \mathcal{V}, \quad W \in \mathcal{W},$$

which is related to F_* by the adjunction

$$F_* \dashv F^* \quad (5.48)$$

in Bimod.

Now from (5.45) we first deduce

$$(\otimes_{g,h})_* \dashv (\Delta_{g,h})_*,$$

but by (5.48) also

$$(\otimes_{g,h})_* \dashv \otimes_{g,h}^*.$$

Uniqueness of adjoints yields a canonical isomorphism

$$(\Delta_{g,h})_* \cong \otimes_{g,h}^*.$$

If we apply this also to (5.46), we have proven the following:

Proposition 5.43. *Let G be a finite group and $g, h \in G$. For a 3-2-1-dimensional G-equivariant topological field theory Z, we obtain the following adjunction relations for the functors obtained from surfaces with boundary:*

(a) $(\Delta_{g,h})_* \cong \otimes_{g,h}^*$,
(b) $\varepsilon_* \cong \eta^*$

If we use the notation $\eta_*(X) := \eta_*(X, \mathbb{C})$ and the dual convention for ε, we arrive at:

Corollary 5.44. *For $g, h \in G$, we have*

(a) $\otimes_{g,h_*}(W, X \boxtimes Y) = \mathrm{Hom}_{\mathcal{C}_{gh}^Z}(W, X \otimes Y)$ *for all* $X \in \mathcal{C}_g^Z, Y \in \mathcal{C}_h^Z$ *and* $W \in \mathcal{C}_{gh}^Z$,
(b) $\Delta_{g,h_*}(Y \boxtimes W, X) = \mathrm{Hom}_{\mathcal{C}_{gh}^Z}(Y \otimes W, X)$ *for all* $X \in \mathcal{C}_{gh}^Z$, $Y \in \mathcal{C}_g^Z$ *and* $W \in \mathcal{C}_h^Z$,
(c) $\eta_*(X) = \mathrm{Hom}_{\mathcal{C}_e^Z}(X, I)$ *for all* $X \in \mathcal{C}_e^Z$,
(d) $\varepsilon_*(X) = \mathrm{Hom}_{\mathcal{C}_e^Z}(I, X)$ *for all* $X \in \mathcal{C}_e^Z$.

Corollary 5.44 allows us to compute the evaluation of an extended G-equivariant topological field theory on any surface decorated with bundles in terms of the monoidal structure.

Example 5.45. As an illustration, let us compute the evaluation

$$Z(B_g) : \mathcal{C}_g^Z \boxtimes \mathcal{C}_{g^{-1}}^Z \longrightarrow \mathsf{FinVect}$$

of a 3-2-1-dimensional extended G-equivariant topological field theory Z on the bent cylinder B_g decorated with bundles as on page 106. By cutting B_g into a pair of pants and a cup we find via functoriality of Z, (5.47) and Corollary 5.44

$$Z(B_g)(X,Y) \cong \int^{W \in \mathcal{C}_e^Z} \mathrm{Hom}_{\mathcal{C}_e^Z}(I,W) \otimes \mathrm{Hom}_{\mathcal{C}_e^Z}(W, X \otimes Y) \quad \text{for all} \quad X \in \mathcal{C}_g^Z, \quad Y \in \mathcal{C}_{g^{-1}}^Z.$$

By the Yoneda Lemma, see e.g. [Rie14, Example 1.4.6], this implies

$$Z(B_g)(X,Y) \cong \mathrm{Hom}_{\mathcal{C}_e^Z}(I, X \otimes Y) \ ,$$

i.e. $Z(B_g)(X,Y)$ is given by the invariants in the monoidal product $X \otimes Y$.

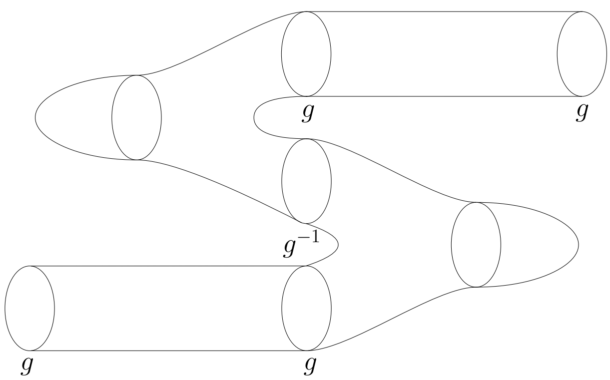
5.5.3 Duality

In the next step, we prove that $\mathcal{C}^Z$ is also rigid:

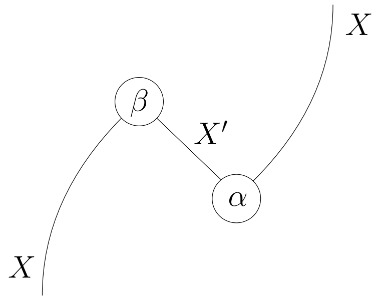
Proposition 5.46. *For any extended G-equivariant topological field theory $Z : G\text{-}\mathsf{Cob}(3,2,1) \longrightarrow \mathsf{2Vect}$, the monoidal category $\mathcal{C}^Z$ has duals.*

The duals considered here are *left* duals in the terminology of [EGNO15]. An argument analogous to the one given in the proof below shows the existence of right duals. We will see later that left and right duals actually agree in this case.

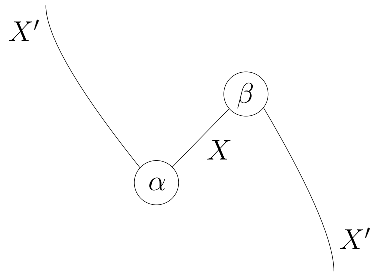
Proof. The proof uses the appropriate equivariant versions of the arguments given in [BDSPV15] for the non-equivariant case: For a group element $g \in G$, we denote the 1-morphism



in $G\text{-}\mathsf{Cob}(3,2,1)$ by N_g. This is the 1-morphism appearing in the proof of [BDSPV15, Proposition 4.2] appropriately decorated with bundles. It is diffeomorphic to the cylinder with g on the ingoing and outgoing circle and the identity homotopy on it. This gives us a natural isomorphism $\mathrm{id}_{\mathcal{C}_g^Z} \cong Z(N_g)$ of 2-linear maps $\mathcal{C}_g^Z \longrightarrow \mathcal{C}_g^Z$. By slicing up N_g as indicated in the above picture and using the functoriality and monoidality of Z we find yet another 2-linear map $\mathcal{C}_g^Z \longrightarrow \mathcal{C}_g^Z$, which is also naturally isomorphic to the identity functor. By looking at the resulting isomorphism for the corresponding bimodules $(\mathcal{C}_g^Z)^{\mathrm{opp}} \boxtimes \mathcal{C}_g^Z \longrightarrow \mathsf{Vect}$ one deduces as in [BDSPV15, Propositions 4.2 and 4.8] that for any $X \in \mathcal{C}_g^Z$ there is an object $X' \in \mathcal{C}_{g^{-1}}^Z$ together with morphisms $\alpha : I \longrightarrow X' \otimes X$ and $\beta : X \otimes X' \longrightarrow I$ such that



is the identity of X (these string diagrams have to be read from bottom to top). Again, as in the proof of [BDSPV15, Propositions 4.8], this implies that the endomorphism



of X' is an idempotent, and by finite (co)completeness of $\mathcal{C}^Z_{g^{-1}}$ it splits into morphisms $\gamma : X' \longrightarrow X^*$ and $\delta : X^* \longrightarrow X'$ in $\mathcal{C}^Z_{g^{-1}}$ such that $\gamma \circ \delta = \mathrm{id}_{X^*}$. A direct computation in the graphical calculus shows that $X^* \in \mathcal{C}^Z_{g^{-1}}$ is dual to X with evaluation $\mathrm{eva}_X := \beta \circ (\mathrm{id}_X \otimes \delta)$ and coevaluation $\mathrm{coeva}_X := (\gamma \otimes \mathrm{id}_X) \circ \alpha$. □

Corollary 5.47. *For any extended G-equivariant topological field theory $Z : G\text{-}\mathsf{Cob}(3,2,1) \longrightarrow \mathsf{2Vect}$, the duality in the category $\mathcal{C}^Z$ has the following properties:*

(a) *The dual X^* of an object $X \in \mathcal{C}^Z_g$ lives in the sector $\mathcal{C}^Z_{g^{-1}}$.*
(b) *For $g, h \in G$ and $X \in \mathcal{C}^Z_g$ the object $h.X^*$ is dual to $h.X$, i.e. $(h.X)^* \cong h.X^*$.*

Proof. Assertion (a) is clear from the proof of Proposition 5.46 and also a necessity since $X \otimes X^*$ needs to be in the neutral sector. Assertion (b) follows directly from the fact that G acts by monoidal functors (which holds true because $\mathcal{C}^Z$ is G-crossed monoidal). □

5.5.4 Balancing and ribbon structure

In Section 5.4.2 we have introduced the operad of framed little bundles and proved in Corollary 5.41 that the evaluation $\mathcal{C}^Z$ of an extended G-equivariant topological field theory Z on the circle is even a balanced braided G-crossed category. By definition the balancing is a natural isomorphism

$$\theta_X : X \longrightarrow g.X \quad \text{for all} \quad X \in \mathcal{C}^Z_g \ , \quad g \in G \tag{5.49}$$

compatible with the G-action, the braiding and the unit as prescribed by the conditions (B1)-(B3) listed on page 102.

As for the braiding, we may describe this isomorphism as the evaluation of Z on an invertible 2-morphism: Compare the identity of $\mathcal{C}^Z_g$ to the equivalence $\phi_g : \mathcal{C}^Z_g \longrightarrow \mathcal{C}^Z_g$. Both are obtained by evaluation of Z on a cylinder with ingoing and outgoing circle labeled by g. But the 1-morphism which yields the identity carries the constant homotopy while the 1-morphism giving us $\phi_g : \mathcal{C}_g^Z \longrightarrow \mathcal{C}_g^Z$ carries g seen as a homotopy. More precisely, if g is represented by the loop $\gamma : \mathbb{S}^1 \longrightarrow BG$, then $\phi_g : \mathcal{C}_g^Z \longrightarrow \mathcal{C}_g^Z$ is the evaluation of Z on the cylinder together with the map $\widetilde{\gamma} : \mathbb{S}^1 \times [0,1] \longrightarrow BG$ with $\widetilde{\gamma}(z,t) = \gamma(z\,\mathrm{e}^{2\pi\,\mathrm{i}\,t})$ for all $(z,t) \in \mathbb{S}^1 \times I$. Consider now the Dehn twist of the cylinder, i.e. the diffeomorphism

$$D : \mathbb{S}^1 \times I \longrightarrow \mathbb{S}^1 \times I, \quad (z,t) \longmapsto \left(z\,\mathrm{e}^{2\pi\,\mathrm{i}\,t}, t\right) \tag{5.50}$$

keeping the boundary circles fixed, and observe that the pullback of the constant homotopy from γ to γ along D is $\widetilde{\gamma}$. Now by the mapping cylinder construction (Remark 2.2 (c)) we obtain a natural isomorphism from the identity of $\mathcal{C}_g^Z$ to $g : \mathcal{C}_g^Z \longrightarrow \mathcal{C}_g^Z$ and thereby the natural isomorphism (5.49). The fact that this actually corresponds to the balancing in the operad fPBr^G of framed parenthesized G-braids or, equivalently, the framed little bundles operad follows from the concrete definitions of the functor Φ^{f} from Theorem 5.38 which also uses the Dehn twist on a cylinder. More precisely, this is seen as follows: Up to a rescaling $[0,1] \cong [1/2,1]$ the end of the homotopy (5.39) is the Dehn twist (5.50). From this we can deduce that the homotopy (5.39) and the Dehn twist (5.50) give rise to the same invertible 2-morphism in $G\text{-}\mathsf{Cob}(3,2,1)$.

In summary, we have deduced from Corollary 5.41 that $\mathcal{C}^Z$ is a finitely semisimple balanced braided G-crossed category. The purpose of this subsection is to prove that an even stronger statement can be made: In Proposition 5.46 we have shown that $\mathcal{C}^Z$ is actually rigid, and we may ask for the following compatibility of balancing and duality: Since $X^* \in \mathcal{C}_{g^{-1}}^Z$ for $X \in \mathcal{C}_g^Z$ by Proposition 5.46, the twist evaluated on $g.X^*$ together with the coherence isomorphisms yields an isomorphism

$$\theta_{g.X^*} : g.X^* \longrightarrow g^{-1}.g.X^* \cong X^*. \tag{5.51}$$

Here we also used the coherence isomorphisms, but by abuse of notation refrain from giving a new name to the composite. On the other hand, there is the dual map

$$\theta_X^* : g.X^* \cong (g.X)^* \longrightarrow X^* \tag{5.52}$$

of $\theta_X : X \longrightarrow g.X$; we implicitly used here the isomorphism $g.X^* \cong (g.X)^*$ from Proposition 5.46). This leads to the following definition:

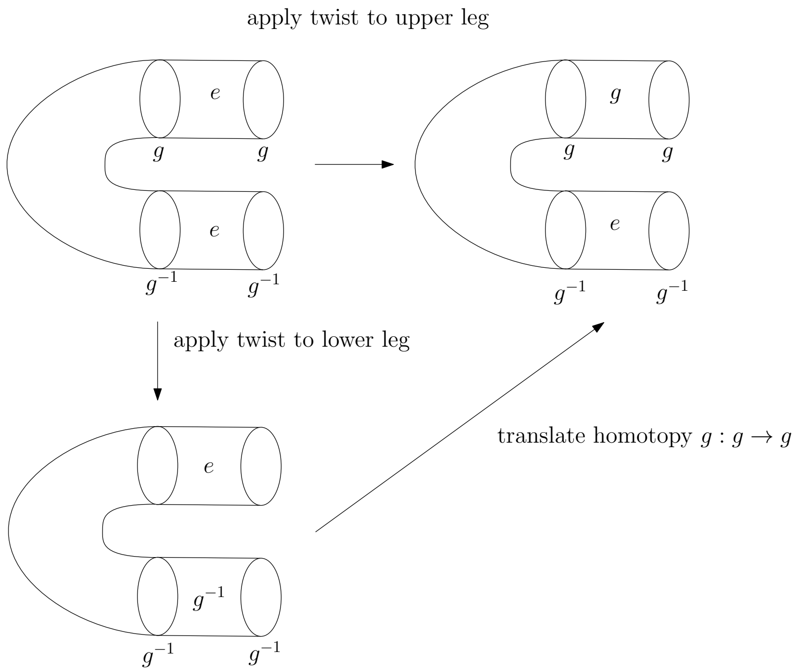
Definition 5.48. For a group G, a *G-ribbon category* is a balanced braided G-crossed category which is rigid and in which the morphisms (5.51) and (5.52) are equal for all choices of objects in all sectors. In this case, the balancing is also called *(ribbon) G-twist*.

A stricter version of this definition appears in [Tur10b, VI.2.3], see also [MNS12, Definition 4.8]. A priori, Definition 5.48 can be made for a left and a right duality. However, as in the non-equivariant case, the ribbon structure will result in a pivotal structure. As a consequence, left and right duals will coincide. Therefore, we will not make this distinction in the sequel.

Now the following result concludes the investigation of $\mathcal{C}^Z$ is this chapter:

Theorem 5.49. *For any extended G-equivariant topological field theory $Z : G\text{-}\mathsf{Cob}(3,2,1) \longrightarrow \mathsf{2Vect}$, the finitely semisimple balanced braided G-crossed category $\mathcal{C}^Z$ is a finitely semisimple G-ribbon category.*

Proof. For $g \in G$ and $X \in \mathcal{C}_g^Z$, it remains to prove that (5.51) and (5.52) are equal. To this end, we evaluate the commutative triangle



on the level of bimodules, see page 108. By Corollary 5.44 we translate it to the commutative triangle

$$\begin{tikzcd} \mathrm{Hom}_{\mathcal{C}_e^Z}(X \otimes X^*, I) \arrow[r, "f \longmapsto f \circ (\theta_X^{-1} \otimes \mathrm{id}_{X^*})"] \arrow[d, "f \longmapsto f \circ (\mathrm{id}_X \otimes \theta_{X^*}^{-1})"'] & \mathrm{Hom}_{\mathcal{C}_e^Z}(g.X \otimes X^*, I) \\ \mathrm{Hom}_{\mathcal{C}_e^Z}(X \otimes g^{-1}.X^*, I) \ . \arrow[ur, "g"'] & \end{tikzcd} \tag{5.53}$$

Here by abuse of notation we denote by g the map induced by the functor $g : \mathcal{C}_g^Z \longrightarrow \mathcal{C}_g^Z$ on morphism spaces and coherence isomorphisms, i.e. the map

$$\mathrm{Hom}_{\mathcal{C}_e^Z}(X \otimes g^{-1}.X^*, I) \xrightarrow{g} \mathrm{Hom}_{\mathcal{C}_e^Z}(g.(X \otimes g^{-1}.X^*), g.I) \cong \mathrm{Hom}_{\mathcal{C}_e^Z}(g.X \otimes X^*, I) \ .$$

Since the evaluation $\mathrm{eva}_X : X \otimes X^* \longrightarrow I$ is an element of $\mathrm{Hom}_{\mathcal{C}_e^Z}(X \otimes X^*, I)$, we obtain from (5.53)

$$g.\left(\mathrm{eva}_X \circ (\mathrm{id}_X \otimes \theta_{X^*}^{-1})\right) = \mathrm{eva}_X \circ (\theta_X^{-1} \otimes \mathrm{id}_{X^*}) \ .$$

Using that $g : \mathcal{C}_g^Z \longrightarrow \mathcal{C}_g^Z$ is a monoidal functor and the compatibility of balancing and G-action, this implies

$$\mathrm{eva}_{g.X} \circ (\theta_X \otimes \mathrm{id}_{g.X^*}) = \mathrm{eva}_X \circ (\mathrm{id}_X \otimes \theta_{g.X^*}) \ .$$

Now a straightforward computation in the graphical calculus using the snake identities for the duality morphisms shows that (5.51) is indeed equal to (5.52). □

6 The orbifold construction in dimension 3-2-1

In Theorem 5.49 we have established that the evaluation of an extended three-dimensional G-equivariant topological field theory on the circle yields a finitely semisimple G-ribbon category. For this type of category, an *algebraic* orbifold construction [Kir04, GNN09] is available that produces a finitely semisimple ribbon category. We will compare this algebraic construction to the topological orbifoldization in this chapter. Moreover, we present applications of this comparison result.

6.1 Topological orbifoldization on the circle versus algebraic orbifoldization

Given an extended G-equivariant topological field theory $Z : G\text{-}\mathsf{Cob}(3,2,1) \longrightarrow \mathsf{2Vect}$ we can evaluate the orbifold theory $Z/G : \mathsf{Cob}(3,2,1) \longrightarrow \mathsf{2Vect}$ from Definition 4.2 on the circle and obtain a 2-vector space $Z/G(\mathbb{S}^1)$. This 2-vector space is endowed with the structure of a finitely semisimple ribbon category in two ways which are a priori different. We will describe and compare these structures.

6.1.1 Topological orbifoldization on the circle

By [BDSPV15] the topological field theory Z/G can be used to endow $Z/G(\mathbb{S}^1)$ with the structure of a finitely semisimple ribbon category. Using the explicit description of the orbifold theory Z/G in Proposition 4.3 we will now characterize $Z/G(\mathbb{S}^1)$ in terms of the category $\mathcal{C}^Z = Z(\mathbb{S}^1, -)$ that we obtain by evaluation of Z on the circle with varying G-bundle decoration, see Section 5.5. This will allow us in Theorem 6.4 to relate the topological orbifold construction of this thesis to the concept of an orbifold category appearing e.g. in [Kir04] or [GNN09].

The following observation can be verified by a direct computation:

Lemma 6.1. *For the multiplication functor $M : (G \times G)//G \longrightarrow G//G$ the homotopy fiber $M^{-1}[g]$ over any $g \in G$ is equivalent to the discrete groupoid with object set $\{(a,b) \in G\times G \mid ab = g\}$.*

Recall from Proposition 4.3 that the orbifold theory Z/G assigns to the circle the 2-vector space of parallel sections of $\mathcal{C}^Z$. The data of a parallel section of $\mathcal{C}^Z$ is an object $s(g) \in Z(\mathbb{S}^1, g)$ for each $g \in G$ together with coherent isomorphisms $h.s(g) \cong s(hgh^{-1})$ for each $h \in G$. These isomorphisms describe the parallelity up to isomorphism.

Proposition 6.2. *Let G be a finite group and $Z : G\text{-}\mathsf{Cob}(3,2,1) \longrightarrow \mathsf{2Vect}$ an extended G-equivariant topological field theory. The value $Z/G(\mathbb{S}^1)$ of the orbifold theory Z/G on the circle naturally carries the structure of a finitely semisimple ribbon category (by being the value of an extended three-dimensional topological field theory on the circle [BDSPV15]). This structure arises in the following way from the structure of $\mathcal{C}^Z$:*

(a) *For $s, s' \in Z/G(\mathbb{S}^1)$, up to natural isomorphism, the monoidal product is given by*

$$(s \otimes s')(g) = \coprod_{ab=g} s(a) \otimes s'(b) \quad \text{for all} \quad g \in G \ .$$

The unit of this monoidal product is the unit of $\mathcal{C}^Z$ seen as a parallel section in the obvious way. If $\mathcal{C}^Z$ has a simple unit, then so has $Z/G(\mathbb{S}^1)$.

(b) *For $s, s' \in Z/G(\mathbb{S}^1)$ the braiding isomorphism $s \otimes s' \cong s' \otimes s$ is given by the isomorphisms*

$$(s \otimes s')(g) = \coprod_{ab=g} s(a) \otimes s'(b) \longrightarrow \coprod_{uv=g} s'(u) \otimes s(v) = (s' \otimes s)(g) \quad \text{for all} \quad g \in G$$

which map the summand (a, b) to the summand (aba^{-1}, a) by

$$s(a) \otimes s'(b) \xrightarrow{c_{s(a),s'(b)}} a.s'(b) \otimes s(a) \xrightarrow{\text{parallelity}} s'(aba^{-1}) \otimes s(a).$$

(c) *For $s \in Z/G(\mathbb{S}^1)$ the twist is given by*

$$s(g) \xrightarrow{\theta_{s(g)}} g.s(g) \xrightarrow{\text{parallelity}} s(ggg^{-1}) = s(g) \quad \text{for all} \quad g \in G \ .$$

Proof. (a) The monoidal product is obtained from the pair of pants. Hence, using the span (5.44) and the concrete description of the orbifold construction in Proposition 4.3 (b) we find

$$(s \otimes s')(g) = \lim_{(a,b,h) \in M^{-1}[g]} h.(s(a) \otimes s(b)) \ .$$

Since G acts by monoidal functors and s and s' are parallel, this reduces to

$$(s \otimes s')(g) \cong \lim_{(a,b,h) \in M^{-1}[g]} s(hah^{-1}) \otimes s(hbh^{-1}) \ .$$

Now Lemma 6.1 yields the assertion if we take into account that finite coproducts and finite products in a 2-vector space coincide. The monoidal unit can also be obtained by Proposition 4.3, (b). Alternatively, we can just use that the given object is a unit for the monoidal product and hence the unique one up to isomorphism.

We need to prove the additional statement on the simplicity of units: The unit of $Z/G(\mathbb{S}^1)$ is I with the canonical isomorphisms $\phi_g : g.I \cong I$ coming from the fact that G acts by monoidal functors. Hence, an endomorphism of the unit of $Z/G(\mathbb{S}^1)$ is a morphism $\psi : I \longrightarrow I$ such that $\phi_g \circ (g.\psi) = \psi \circ \phi_g$ for all $g \in G$. If I is simple, then $\psi = \lambda \, \mathrm{id}_I$ for some $\lambda \in \mathbb{C}$ and the requirement $\phi_g \circ (g.\psi) = \psi \circ \phi_g$ is automatically satisfied since g acts as a $\mathbb{C}$-linear functor. This proves that an endomorphism of the unit of $\mathcal{C}^Z$ is the same as an endomorphism of the unit of $Z/G(\mathbb{S}^1)$. Therefore, the unit of $Z/G(\mathbb{S}^1)$ is simple as well.

(b) The evaluation of the stack $\Pi(-, BG)$ on the 2-cell in $\mathsf{Cob}(3, 2, 1)$ that we used to produce the braiding yields the span of spans

$$\begin{tikzcd}
 & (G\times G)//G \arrow[dl, "B"'] \arrow[dr, "M"] & \\
G//G\times G//G & (G\times G)//G \arrow[l, Rightarrow, "\alpha"'] \arrow[u, "="'] \arrow[d, "R"] & G//G \ , \\
 & (G\times G)//G \arrow[ul, "B"] \arrow[ur, "M"'] &
\end{tikzcd}$$

where $R : (G \times G)//G \longrightarrow (G \times G)//G$ is the functor $(g, h) \longmapsto (ghg^{-1}, g)$ and α is the obvious natural transformation. By Proposition 4.3 (c) the braiding isomorphism $(s \otimes s')(g) \cong (s' \otimes s)(g)$ is given as follows: We start with

$$(s \otimes s')(g) = \lim_{(a,b,h)\in M^{-1}[g]} s(hah^{-1}) \otimes s'(hbh^{-1}) \ ,$$

apply vertex-wise the equivariant braiding, i.e. the isomorphisms

$$s(hah^{-1}) \otimes s'(hbh^{-1}) \cong (hah^{-1}).s'(hbh^{-1}) \otimes s(hah^{-1}) \ ,$$

use parallelity

$$(hah^{-1}).s'(hbh^{-1}) \otimes s(hah^{-1}) \cong s'(haba^{-1}h^{-1}) \otimes s(hah^{-1})$$

and push the resulting limit

$$\lim_{(a,b,h)\in M^{-1}[g]} (hah^{-1}).s'(hbh^{-1}) \otimes s(hah^{-1}) \cong \lim_{(a,b,h)\in M^{-1}[g]} s'(haba^{-1}h^{-1}) \otimes s(hah^{-1})$$

along the equivalence $M^{-1}[g] \cong M^{-1}[g]$ induced by R. Using the identifications made in (a) based on Lemma 6.1 the assertion follows.

(c) The proof of this assertion follows also from Proposition 4.3.

□

6.1.2 Algebraic orbifoldization

In order to compare Proposition 6.2 to the concept of an orbifold category, let us recall the latter from [Kir04, Theorem 3.9], see also [Müg05, GNN09].

Proposition 6.3 (Algebraic orbifoldization of an equivariant ribbon category from [Kir04])**.** *Let G be a finite group and $\mathcal{C}$ a finitely semisimple G-ribbon category, then the orbifold category $\mathcal{C}/G$, which is the category of homotopy fixed points (the category of objects X in $\mathcal{C}$ together with a family of coherent isomorphisms $(\chi_g : g.X \longrightarrow X)_{g\in G}$ in the sense of [Kir04, Definition 3.1]), inherits the following structure from $\mathcal{C}$:*

(a) *By*

$$(X, (\chi_g)_{g\in G}) \otimes (Y, (\lambda_g)_{g\in G}) := (X \otimes Y, (\chi_g \otimes \lambda_g)_{g\in G})$$

for all $(X, (\chi_g)_{g\in G}), (Y, (\lambda_g)_{g\in G}) \in \mathcal{C}/G$ it is made into a monoidal category with the monoidal unit in $\mathcal{C}$ (seen as a homotopy fixed point) as the monoidal unit. The monoidal category $\mathcal{C}/G$ has duals.

(b) *The monoidal category $\mathcal{C}/G$ is braided and the underlying isomorphism $X \otimes Y \longrightarrow Y \otimes X$*

for objects

$$\left(X = \bigoplus_{g\in G} X_g, (\chi_g)_{g\in G}\right), \quad \left(Y = \bigoplus_{g\in G} Y_g, (\lambda_g)_{g\in G}\right) \in \mathcal{C}/G$$

is given by

$$X_g \otimes Y_h \xrightarrow{c_{X_g,Y_h}} g.Y_h \otimes X_g \xrightarrow{\lambda_g \otimes \mathrm{id}_{X_g}} Y_{ghg^{-1}} \otimes X_g \quad \text{for all} \quad g,h \in G.$$

(c) *The braided monoidal category $\mathcal{C}/G$ comes with a twist which on the object*

$$\left(X = \bigoplus_{g\in G} X_g, (\chi_g)_{g\in G}\right)$$

arises from the equivariant twist by

$$X_g \xrightarrow{\theta_{X_g}} g.X_g \xrightarrow{\chi_g} X_g.$$

6.1.3 The comparison result

We will now state our comparison result, derive some immediate consequences and present applications.

Theorem 6.4. *For any extended G-equivariant topological field theory $Z : G\text{-}\mathsf{Cob}(3,2,1) \longrightarrow \mathsf{2Vect}$, the evaluation of the orbifold theory $Z/G : \mathsf{Cob}(3,2,1) \longrightarrow \mathsf{2Vect}$ on $\mathbb{S}^1$ yields an equivalence*

$$\frac{Z}{G}(\mathbb{S}^1) \simeq \frac{\mathcal{C}^Z}{G} \tag{6.1}$$

as 2-vector spaces. Both categories carry the structure of a finitely semisimple ribbon category:

- *$Z/G(\mathbb{S}^1)$ by being the value of an extended topological field theory on the circle in the sense of Proposition 6.2.*
- *$\mathcal{C}^Z/G$ by Proposition 6.3.*

Both structures agree, i.e. (6.1) *is an equivalence of finitely semisimple ribbon categories.*

Proof. The equivalence $Z/G(\mathbb{S}^1) \simeq \mathcal{C}^Z/G$ of 2-vector spaces holds by definition of the orbifold construction and the definition of the orbifold category in [Kir04]. By Proposition 6.3 the category $\mathcal{C}^Z/G$ naturally inherits from $\mathcal{C}^Z$ the structure of an finitely semisimple ribbon category, and by Proposition 6.2 the category $Z/G(\mathbb{S}^1)$ has the same type of structure. Comparing the description of these structures as given in Proposition 6.3 and Proposition 6.2 shows that they agree. □

Diagrammatically, the above Theorem means that the square

$$\begin{tikzcd}
\text{3-2-1-dimensional } G\text{-equivariant topological field theories} \arrow[r, "\text{evaluation on the circle}"] \arrow[d, "\text{orbifoldization } -/G"'] & \text{finitely semisimple } G\text{-ribbon categories} \arrow[d, "\text{orbifold category}"] \\
\text{3-2-1-dimensional topological field theories} \arrow[r, "\text{evaluation on the circle}"'] & \text{finitely semisimple ribbon categories}
\end{tikzcd}$$

commutes up to natural isomorphism.

Corollary 6.5. *For any extended G-equivariant topological field theory $Z : G\text{-}\mathsf{Cob}(3,2,1) \longrightarrow \mathsf{2Vect}$, the orbifold theory $Z/G : \mathsf{Cob}(3,2,1) \longrightarrow \mathsf{2Vect}$ is determined up to equivalence by the orbifold category $\mathcal{C}^Z/G$.*

Proof. This follows from Theorem 6.4 if we take into account that by [BDSPV15] any 3-2-1-dimensional topological field theory is determined up to equivalence by the finitely semisimple ribbon category it yields on the circle. □

As an application, we can give a generalization of [SW19, Example 4.7] concerned with the orbifoldization of equivariant Dijkgraaf-Witten theories:

Proposition 6.6. *Let $Z_\lambda : J\text{-}\mathsf{Cob}(3,2,1) \longrightarrow \mathsf{2Vect}$ be the extended J-equivariant Dijkgraaf-Witten theory constructed in [MNS12] from a short exact sequence $0 \longrightarrow G \longrightarrow H \xrightarrow{\lambda} J \longrightarrow 0$ of finite groups. The orbifold theory Z_λ/J is equivalent to the extended Dijkgraaf-Witten theory Z_H for the group H, i.e.*

$$\frac{Z_\lambda}{J} \simeq Z_H \ .$$

Proof. In [MNS12, Proposition 35] the orbifold category $\mathcal{C}^{Z_\lambda}/J$ of $\mathcal{C}^{Z_\lambda}$ is computed to be category $D(H)\text{-}\mathsf{Mod}$ of finite-dimensional modules over the Drinfeld double $D(H)$ of the group H. By Theorem 6.4 this is the category that Z_λ/J assigns to the circle. Since this category is also the value of Z_H on the circle, we can use Corollary 6.5 to deduce the desired assertion. □

One should appreciate that this statement, although more general, admits a significantly simpler and more conceptual proof than the corresponding statement in [SW19, Example 4.7] because it can be completely played back to the categories obtained on the circle.

In another application, we will use topological field theory as a counting device: For this let us first recall the following well-known fact which in a different language appears for instance in [Tur10a, Corollary IV.12.1.2]:

Lemma 6.7. *Let $Z : \mathsf{Cob}(n, n-1, n-2) \longrightarrow \mathsf{2Vect}$ be an extended topological field theory, then*

$$Z(\mathbb{T}^n) = \#\ \textit{simple objects in}\ Z(\mathbb{T}^{n-2}) \ .$$

Proof. Set $\mathcal{C} := Z(\mathbb{T}^{n-2})$, then $\mathcal{C}$ is dualizable in (the homotopy category of) $\mathsf{2Vect}$ and the vector space assigned to $\mathbb{T}^{n-1} = \mathbb{T}^{n-2} \times \mathbb{S}^1$ is the concatenation of the coevaluation and evaluation of $\mathcal{C}$, which is given by $\bigoplus_{j=1}^n \mathrm{Hom}_\mathcal{C}(X_j, X_j)$ where the sum runs over the simple objects. The dimension of this vector space is the number of simple objects of $\mathcal{C}$. By [Tur10a, Theorem III.2.1.3] this number is also the invariant that Z assigns to the top-dimensional manifold $\mathbb{T}^n$. □

In order to combine this fact with the orbifold construction, we recall that the groupoid of G-bundles over $\mathbb{T}^n$ for $n \geq 1$ is equivalent to the action groupoid $\mathrm{Com}(G^n)//G$ of the action of G on n-tuples of mutually commuting group elements by conjugation. Hence, a G-bundle over $\mathbb{T}^n$ can be described by n group elements $g_1, \ldots, g_n \in G$ such that $g_i g_j = g_j g_i$ for all $1 \leq i, j \leq n$.

Theorem 6.8. *Let G be a finite group and $Z : G\text{-}\mathsf{Cob}(n, n-1, n-2) \longrightarrow \mathsf{2Vect}$ an extended G-equivariant topological field theory. Then*

$$\#\text{ simple objects in } \frac{Z}{G}(\mathbb{T}^{n-2}) = \frac{1}{|G|} \sum_{(g_1,\dots,g_n)\in \mathrm{Com}(G^n)} Z(\mathbb{T}^n, g_1, \dots, g_n) \ . \tag{6.2}$$

For $n = 3$, we also find the formula

$$\#\text{ simple objects in } \frac{\mathcal{C}^Z}{G} = \frac{1}{|G|} \sum_{(g_1,g_2,g_3)\in \mathrm{Com}(G^3)} Z(\mathbb{T}^3, g_1, g_2, g_3) \tag{6.3}$$

using the orbifold category $\mathcal{C}^Z/G$ of the G-ribbon category $\mathcal{C}^Z$ that Z gives rise to.

Proof. Once we prove (6.2), formula (6.3) will follow from Theorem 6.4. Hence, we only have to prove (6.2): By Lemma 6.7 we find

$$\#\text{ simple objects in } \frac{Z}{G}(\mathbb{T}^{n-2}) = \frac{Z}{G}(\mathbb{T}^n) \ .$$

The number $Z/G(\mathbb{T}^n)$ can be computed using the non-extended orbifold construction. Knowing the groupoid of G-bundles over $\mathbb{T}^n$ we can use [SW19, Corollary 4.4 (c)] to express $Z/G(\mathbb{T}^n)$ as the integral

$$\begin{aligned}\frac{Z}{G}(\mathbb{T}^n) &= \int_{(g_1,\dots,g_n)\in \mathrm{Com}(G^n)//G} Z(\mathbb{T}^n, g_1, \dots, g_n)\, \mathrm{d}(g_1, \dots, g_n) \\ &= \sum_{[g_1,\dots,g_n]\in \pi_0(\mathrm{Com}(G^n)//G)} \frac{Z(\mathbb{T}^n, g_1, \dots, g_n)}{|\mathrm{Aut}(g_1, \dots, g_n)|}\end{aligned}$$

with respect to groupoid cardinality. By the orbit stabilizer Theorem we obtain

$$|\mathrm{Aut}(g_1, \dots, g_n)| = \frac{|G|}{|\mathcal{O}(g_1, \dots, g_n)|} \ ,$$

where $\mathcal{O}(g_1, \dots, g_n)$ is the orbit of $(g_1, \dots, g_n)$ in $\mathrm{Com}(G^n)//G$. This implies

$$\frac{Z}{G}(\mathbb{T}^n) = \frac{1}{|G|} \sum_{(g_1,\dots,g_n)\in \mathrm{Com}(G^n)} Z(\mathbb{T}^n, g_1, \dots, g_n)$$

and hence the result. □

Even in the non-extended case, we can read off from the above proof that

$$\frac{1}{|G|} \sum_{(g_1,\dots,g_n)\in \mathrm{Com}(G^n)} Z(\mathbb{T}^n, g_1, \dots, g_n) = \frac{Z}{G}(\mathbb{T}^n) = \dim \frac{Z}{G}(\mathbb{T}^{n-1})$$

is a non-negative integer. This provides constraints for manifold invariants which arise from a (not necessarily extended) equivariant topological field theory:

Corollary 6.9. *Consider an invariant of closed oriented n-dimensional manifolds decorated with G-bundles for a finite group G which yields on the torus $\mathbb{T}^n$ decorated with the bundle specified by commuting group elements $(g_1, \dots, g_n) \in G^n$ the number $z_{g_1,\dots,g_n} \in \mathbb{C}$. If the invariant arises from an G-equivariant topological field theory, then $\sum_{(g_1,\dots,g_n)\in \mathrm{Com}(G^n)} z_{g_1,\dots,g_n}$ is a non-negative integer multiple of $|G|$.*

Example 6.10 (Permutation orbifolds)**.** Let $\mathcal{C}$ be a modular category and $Z : \mathsf{Cob}(3,2,1) \longrightarrow \mathsf{2Vect}$ the extended topological field theory giving us $\mathcal{C}$ upon evaluation on the circle (Z is unique up to equivalence). Consider now a finite group, which for illustration purposes we take to be the permutation group S_n on n letters (this is not really a restriction because any finite group embeds into a permutation group). The pullback $\mathrm{Cov}^* Z$ of Z along the cover functor $\mathrm{Cov} : \mathrm{S}_n\text{-}\mathsf{Cob}(3,2,1) \longrightarrow \mathsf{Cob}(3,2,1)$ from Example 2.11 is a S_n-equivariant topological field theory. Using Theorem 6.4 we see that the evaluation of the orbifold theory $(\mathrm{Cov}^* Z)/\mathrm{S}_n$ on the circle is what is commonly referred to as the *permutation orbifold* of $\mathcal{C}$ and which is denoted by $\mathcal{C} \wr \mathrm{S}_n$ in [Ban98, Ban02]. Since a permutation orbifold is a special case of an orbifold theory, we can use Theorem 6.8 to compute the number of simple objects of $\mathcal{C} \wr \mathrm{S}_n$.

To this end, note that for any finite group G and mutually commuting groups elements $g_1, g_2, g_3 \in G$ we can define the quotient P_{g_1,g_2,g_3} of $\mathbb{R}^3 \times G$ by

$$\begin{aligned}
(x_1+1, x_2, x_3, h) &\sim (x_1, x_2, x_3, hg_1) \ , \\
(x_1, x_2+1, x_3, h) &\sim (x_1, x_2, x_3, hg_2) \ , \\
(x_1, x_2, x_3+1, h) &\sim (x_1, x_2, x_3, hg_3)
\end{aligned}$$

for all $x_1, x_2, x_3 \in \mathbb{R}$ and $h \in G$. The projection $\mathbb{R}^3 \times G \longrightarrow \mathbb{R}^3$ induces a map $P_{g_1,g_2,g_3} \longrightarrow \mathbb{T}^3$, which is a G-bundle with holonomy values g_1, g_2 and g_3 along the generators of the fundamental group of $\mathbb{T}^3$. The subgroup $\langle g_1, g_2, g_3\rangle \subset G$ generated by g_1, g_2 and g_3 acts from the right on G. It is easy to see that

$$P_{g_1,g_2,g_3} \cong \coprod_{|G/\langle g_1,g_2,g_3\rangle|} \mathbb{T}^3$$

as manifolds.

Going back to $G = \mathrm{S}_n$ we find by Theorem 6.8

$$\#\text{ simple objects in } \mathcal{C} \wr \mathrm{S}_n = \frac{1}{n!} \sum_{\substack{\text{mutually commuting} \\ \text{permutations} \\ \sigma_1,\sigma_3,\sigma_3 \\ \text{on } n \text{ letters}}} (\#\text{ simple objects in } \mathcal{C})^{|\,\mathrm{S}_n/\langle \sigma_1,\sigma_2,\sigma_3\rangle|} \ .$$

Hence, Theorem 6.8 specializes to the formula given in [Ban02, Equation (3)]. In fact, our orbifold construction allows for a uniform treatment of the entire theory of permutation orbifolds.

In [MW20a] we also explain how Theorem 6.8 yields the formulae for the number of simple twisted representations of finite groups and the number of simple representations of twisted Drinfeld doubles of finite groups found in [Wil05].

6.2 Equivariant Verlinde algebra and modularity

The evaluation of a 3-2-1-dimensional topological field theory on the circle yields not only a ribbon category, but a *modular category* by [BDSPV15] (possibly with non-simple unit, see however [BDSPV15, Lemma 5.3]).

In this section we give the equivariant version of this result. To make contact to an equivariant modularity, we use the equivariant Verlinde algebra from [Kir04] whose definition can be understood by evaluation of the modular functor corresponding to the equivariant theory on the 2-torus $\mathbb{T}^2$, see [Kir04, Section 8], which is inspired by [Tur10b, Section 8.6]. We begin by working out these ideas in the language of coends and based on a strong topological motivation.

6.2.1 The equivariant Verlinde algebra

Let $Z : G\text{-}\mathsf{Cob}(n, n-1, n-2) \longrightarrow \mathsf{2Vect}$ be an extended G-equivariant topological field theory. Any $(n-1)$-dimensional closed oriented manifold Σ together with a map $\varphi : \Sigma \longrightarrow BG$ gives rise to a 2-linear map $Z(\Sigma, \varphi) : \mathsf{FinVect} \longrightarrow \mathsf{FinVect}$ and hence to a vector space, which by abuse of notation we will also denote by $Z(\Sigma, \varphi)$. The dependence on φ is functorial, so we get a functor

$$Z(\Sigma, -) : \Pi(\Sigma, BG) \longrightarrow \mathsf{FinVect}, \quad \varphi \longmapsto Z(\Sigma, \varphi),$$

i.e. a representation of (or in more geometric terms: a vector bundle over) the groupoid of G-bundles over Σ. Clearly, this is the representation we obtain be seeing Z as a non-extended theory and applying [SW19, Proposition 2.8].

These vector bundles enjoy the following gluing properties which follow directly from the functoriality of Z and (5.47):

Lemma 6.11. *Let G be a finite group, $Z : G\text{-}\mathsf{Cob}(n, n-1, n-2) \longrightarrow \mathsf{2Vect}$ an extended G-equivariant topological field theory and Σ a closed oriented $(n-1)$-dimensional manifold obtained by gluing the oriented $(n-1)$-dimensional manifolds Σ' and Σ'' along the $(n-2)$-dimensional closed oriented manifold S. Then for two maps $\varphi' : \Sigma' \longrightarrow BG$ and $\varphi'' : \Sigma'' \longrightarrow BG$ with $\varphi'|_S = \varphi''|_S =: \xi$ we have*

$$Z(\Sigma, \varphi' \cup_S \varphi'') \cong \int^{X \in Z(S,\xi)} Z(\Sigma'', \varphi'')X \otimes \operatorname{Hom}_{Z(S,\xi)}(X, Z(\Sigma', \varphi')\mathbb{C})$$

by a canonical isomorphism of vector spaces.

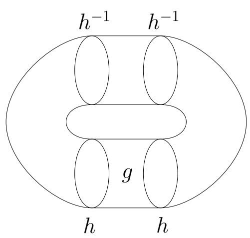
A particularly important special case arises if Σ is the 2-torus $\mathbb{T}^2$. By the holonomy classification of flat bundles the groupoid of G-bundles over the torus is equivalent to the full subgroupoid of $\operatorname{Com}(G^n)//G \subset (G \times G)//G$ consisting of pairs of commuting group elements, see also the explanations before Theorem 6.8.

Proposition 6.12. *Let $Z : G\text{-}\mathsf{Cob}(3, 2, 1) \longrightarrow \mathsf{2Vect}$ be an extended G-equivariant topological field theory, then for all $g, h \in G$ with $gh = hg$*

$$Z(\mathbb{T}^2, (g, h)) \cong \int^{X \in \mathcal{C}_h^Z} \operatorname{Hom}_{\mathcal{C}_h^Z}(g.X, X)$$

by a canonical isomorphism of vector spaces.

Proof. We can cut the torus with bundle decoration (g, h), i.e. with a G-bundle with holonomies g and h, respectively, along the generators of the fundamental group, as indicated in the following picture:



We want to apply Lemma 6.11 with

- (Σ'', φ'') given by the bent cylinder B_h as described in Example 5.45 (that is the right third of the above picture),

- (Σ', φ') given by the same bent cylinder read backwards with two cylinders glued to it such that the lower leg is equipped with g (that is the left and the middle third of the above picture glued together).

Hence, (S, ξ) is given by two copies of the circle with h and h^{-1} on it. By Example 5.45 we find for $X \in \mathcal{C}_h^Z$ and $Y \in \mathcal{C}_{h^{-1}}^Z$

$$Z(\Sigma'', \varphi'')(X \boxtimes Y) \cong \mathrm{Hom}_{\mathcal{C}_e^Z}(I, X \otimes Y)$$

and similarly (i.e. by means of Corollary 5.44)

$$\mathrm{Hom}_{Z(S,\xi)}(X \boxtimes Y, Z(\Sigma', \varphi')\mathbb{C}) \cong \mathrm{Hom}_{\mathcal{C}_e^Z}(g.X \otimes Y, I) \ .$$

Now by applying Lemma 6.11 we obtain

$$Z(\mathbb{T}^2, (g,h)) \cong \int^{X \boxtimes Y \in \mathcal{C}_h^Z \boxtimes \mathcal{C}_{h^{-1}}^Z} \mathrm{Hom}_{\mathcal{C}_e^Z}(I, X \otimes Y) \otimes \mathrm{Hom}_{\mathcal{C}_e^Z}(g.X \otimes Y, I) \ .$$

By [FSS20, Lemma 3.11] and $\mathrm{Hom}_{\mathcal{C}_e^Z}(I, X \otimes Y) \cong \mathrm{Hom}_{\mathcal{C}_h^Z}(Y^*, X)$ we find

$$\begin{aligned} Z(\mathbb{T}^2, (g,h)) &\cong \int^{X \in \mathcal{C}_h^Z} \int^{Y \in \mathcal{C}_{h^{-1}}^Z} \mathrm{Hom}_{\mathcal{C}_h^Z}(Y^*, X) \otimes \mathrm{Hom}_{\mathcal{C}_e^Z}(g.X \otimes Y, I) \\ &\cong \int^{X \in \mathcal{C}_h^Z} \int^{Y \in \mathcal{C}_h^Z} \mathrm{Hom}_{\mathcal{C}_h^Z}(Y, X) \otimes \mathrm{Hom}_{\mathcal{C}_h^Z}(g.X, Y) \ , \end{aligned}$$

where in the last step we used the substitution $Y \longmapsto Y^*$ and

$$\mathrm{Hom}_{\mathcal{C}_e^Z}(g.X \otimes Y, I) \cong \mathrm{Hom}_{\mathcal{C}_h^Z}(g.X, Y^*) \ .$$

By the Yoneda Lemma (compare to Example 5.45) we arrive at

$$Z(\mathbb{T}^2, (g,h)) \cong \int^{X \in \mathcal{C}_h^Z} \mathrm{Hom}_{\mathcal{C}_h^Z}(g.X, X) \ .$$

□

Remark 6.13. A map from the surface Σ_g of genus g to BG can equivalently be described by a morphism $\varphi : \pi_1(\Sigma_g) \longrightarrow G$ from the fundamental group of Σ_g to G. We denote by $a_1, \ldots, a_g, b_1, \ldots, b_g$ usual generators of $\pi_1(\Sigma_g)$ subject to the relation $\prod_{j=1}^g [a_j, b_j] = e$. With similar methods, duality and the fact $\mathrm{Hom}_{\mathcal{C}_e^Z}(I, -)$ is exact and hence preserves finite colimits we find

$$Z(\Sigma_g, \varphi) \cong \mathrm{Hom}_{\mathcal{C}_e^Z}(I, L_\varphi) \ ,$$

where L_φ is the coend

$$L_\varphi := \bigotimes_{j=1}^g L_\varphi^j, \quad L_\varphi^j := \int^{X_j \in \mathcal{C}_{\varphi(a_j)}^Z} X_j \otimes \varphi(b_j).X_j^* \ .$$

These formulae can be found in [Tur10b, VII.3.3], where they are used as a definition to build a G-modular functor from an appropriate type of G-category. Above we have followed the converse logic and started with a given extended G-equivariant topological field theory, extracted this category and the corresponding modular functor and derived these formulae.

If we denote by P the pair of pants, then evaluation of Z on the bordism $\mathbb{S}^1 \times P : \mathbb{T}^2 \coprod \mathbb{T}^2 \longrightarrow \mathbb{T}^2$ appropriately decorated with G-bundles yields linear maps

$$Z(\mathbb{T}^2, (g,h)) \otimes Z(\mathbb{T}^2, (g,h')) \longrightarrow Z(\mathbb{T}^2, (g,hh')) \quad \text{for all} \quad g,h,h' \in G ,$$

which extend by zero to an associative multiplication on the total space

$$\bigoplus_{\substack{g,h\in G\\ gh=hg}} Z(\mathbb{T}^2, (g,h)) \cong \bigoplus_{\substack{g,h\in G\\ gh=hg}} \int^{X\in\mathcal{C}_h^Z} \mathrm{Hom}_{\mathcal{C}_h^Z}(g.X, X) . \tag{6.4}$$

The vector space (6.4) together with this multiplication is called the *equivariant Verlinde algebra of Z*. It is the key to the proof of the following statement:

Proposition 6.14. *Let G be a finite group and $Z : G\text{-}\mathsf{Cob}(3,2,1) \longrightarrow \mathsf{2Vect}$ an extended G-equivariant topological field theory such that the monoidal unit of $\mathcal{C}^Z$ is simple. Then all twisted sectors $\mathcal{C}_g^Z$ for $g \in G$ are non-trivial, i.e. different from the zero 2-vector space.*

Proof. It is well-known that the mapping class group of the torus has an element $\phi_s : \mathbb{T}^2 \longrightarrow \mathbb{T}^2$ such that the bundle (g,h) is pulled back along ϕ_s to the bundle (h^{-1}, g). Hence, the evaluation of Z on the invertible 2-morphism in $G\text{-}\mathsf{Cob}(3,2,1)$ built from ϕ_s (Remark 2.2 (c)) yields an isomorphism $Z(\mathbb{T}^2, (g,h)) \cong Z(\mathbb{T}^2, (h^{-1}, g))$ for $g,h \in G$; in particular

$$Z(\mathbb{T}^2, (g,e)) \cong Z(\mathbb{T}^2, (e,g)) \quad \text{for all} \quad g \in G . \tag{6.5}$$

Suppose now $\mathcal{C}_g^Z = 0$ for some $g \neq e$. Then $Z(\mathbb{T}^2, (e,g)) = 0$ by Proposition 6.12 and hence $Z(\mathbb{T}^2, (g,e)) = 0$ by (6.5). On the other hand, if we complete the unit $I \in \mathcal{C}_e^Z$ to a basis $(I, (B_j)_{j\in J})$ of simple objects for $\mathcal{C}_e^Z$, we find by Proposition 6.12

$$Z(\mathbb{T}^2, (g,1)) \cong \mathrm{Hom}_{\mathcal{C}_e^Z}(g.I, I) \oplus \bigoplus_{j\in J} \mathrm{Hom}_{\mathcal{C}_e^Z}(g.B_j, B_j) .$$

We are using here the standard fact that coends over finitely semisimple categories can be expressed by a sum over the simple objects, see [KL01, Corollary 5.1.9]. The element g acts as a monoidal functor, so $\mathrm{Hom}_{\mathcal{C}_e^Z}(g.I, I) \cong \mathrm{Hom}_{\mathcal{C}_e^Z}(I, I) \cong \mathbb{C}$ leading to $Z(\mathbb{T}^2, (g,e)) \neq 0$ and hence to a contradiction. □

Example 6.15. The statement of Proposition 6.14 is false if we do not assume the simplicity of the monoidal unit: Let $Z : \mathsf{Cob}(3,2,1) \longrightarrow \mathsf{2Vect}$ be a non-equivariant extended topological field theory such that the unit of $\mathcal{C}^Z := Z(\mathbb{S}^1)$ is simple. Then by [BDSPV15] the category $\mathcal{C}^Z$ is modular. If we push Z along the group morphism $\iota : \{e\} \longrightarrow G$ for some finite group G using the pushforward construction of Section 4.3, we obtain a G-equivariant topological field theory $\iota_* Z$. Evaluation of $\iota_* Z$ on the circle yields the category $\mathcal{C}^{\iota_* Z}$ with trivial twisted sectors and neutral sector $\mathcal{C}_e^{\iota_* Z} = \bigoplus_{g\in G} \mathcal{C}^Z$. The action by $h \in G$ sends the copy for g to the copy for hg. If we denote by I^g the unit I of $\mathcal{C}^Z$ in the copy for $g \in G$, then the unit of $\mathcal{C}^{\iota_* Z}$ is given by $J = \bigoplus_{g\in G} I^g$, so it is not simple for $|G| \geq 2$. As a semisimple braided monoidal category, $\mathcal{C}^{\iota_* Z}$ decomposes into semisimple braided monoidal categories with simple unit, see [BDSPV15, Lemma 5.3], but this decomposition is not preserved by the G-action.

The twisted sectors of $\mathcal{C}^{\iota_* Z}$ are allowed to be trivial because the argument given in the proof of Proposition 6.14 fails. More precisely, in contrast to the proof of Proposition 6.14, we find $Z(\mathbb{T}^2)(g,e) = 0$ for $g \neq e$ because $\mathcal{C}^{\iota_* Z}$ has no simple objects invariant (up to isomorphism) under g.

We have seen in Proposition 6.14 and Example 6.15 that it is important to know whether the unit of the equivariant monoidal category coming from an equivariant topological field theory is simple. The situation is under control for those theories arising from our pushforward construction:

Proposition 6.16. *Let $\lambda : G \longrightarrow H$ be a morphism of finite groups and $Z : G\text{-}\mathsf{Cob}(3,2,1) \longrightarrow \mathsf{2Vect}$ an extended G-equivariant topological field theory such that the monoidal unit $I \in \mathcal{C}^Z$ is simple. The monoidal unit in the category $\mathcal{C}^{\lambda_* Z}$ associated to the pushforward*

$$\lambda_* Z : H\text{-}\mathsf{Cob}(3,2,1) \longrightarrow \mathsf{2Vect}$$

of Z along λ in the sense of Definition 4.7 has the endomorphism space $\mathbb{C}^{|H/\operatorname{im}\lambda|}$. In particular, the unit of $\mathcal{C}^{\lambda_* Z}$ is simple if and only if λ is surjective.

Proof. The group morphism λ induces a functor $\lambda_* : G//G \longrightarrow H//H$ for the groupoids of G-bundles and H-bundles over the circle, respectively. An easy computation shows that the homotopy fiber over $e \in H$ is given by $(\ker\lambda \times H)//G$, where G acts on $\ker\lambda \times H$ by

$$a.(g,h) = \left(aga^{-1}, h\lambda\left(a^{-1}\right)\right) \quad \text{for all} \quad a \in G, \quad g \in \ker\lambda, \qquad h \in H.$$

By the definition of the pushforward, $\mathcal{C}_e^{\lambda_* Z}$ is the 2-vector space of parallel sections of the 2-vector bundle obtained by pullback of $\mathcal{C}^Z : G//G \longrightarrow \mathsf{2Vect}$ along the projection $(\ker\lambda \times H)//G \longrightarrow G//G$. The evaluation of $\lambda_* Z$ on the disk decorated with the trivial H-bundle yields a map $\mathsf{FinVect} \longrightarrow \mathcal{C}_e^{\lambda_* Z}$ whose image on $\mathbb{C}$ is the monoidal unit J of $\mathcal{C}^{\lambda_* Z}$. Again, by the definition of the pushforward, this map $\mathsf{FinVect} \longrightarrow \mathcal{C}_e^{\lambda_* Z}$ and its image on $\mathbb{C}$ are computed as follows: The morphism λ induces the functor $\star//G \longrightarrow \star//H$ for the G-bundles and H-bundles over the disk, respectively. Its homotopy fiber over $\star$ is given by $(\{e\} \times H)//G$. By restriction to the boundary, this groupoid embeds into the homotopy fiber $(\ker\lambda \times H)//G$ that we computed for the circle. Denote by $\iota : (\{e\} \times H)//G \longrightarrow (\ker\lambda \times H)//G$ the embedding. Now the monoidal unit $J \in \mathcal{C}_e^{\lambda_* Z}$ is the parallel section given on $(g,h) \in \ker\lambda \times H$ by

$$J(g,h) = \lim_{\iota^{-1}[g,h]} I \ .$$

This parallel section is supported on $\{e\} \times H$, where it has constant value I. Since H acts on $\mathcal{C}^{\lambda_* Z}$ by linear functors, we see that the endomorphism space of J of is given by $\mathbb{C}^{|\pi_0((\{e\}\times H)//G)|} = \mathbb{C}^{|H/\operatorname{im}\lambda|}$. $\square$

The right hand side of (6.4) makes sense for any G-ribbon category (regardless of whether it comes from an equivariant topological field theory) and inspires the following definition:

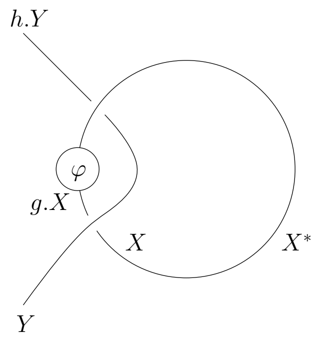
Definition 6.17 (Equivariant modularity, after [Kir04])**.** Let G be a finite group and $\mathcal{C}$ a finitely semisimple G-equivariant ribbon category. We define as in [Kir04, Section 8]

$$\widetilde{\mathcal{V}}(\mathcal{C})_{g,h} := \int^{X \in \mathcal{C}_h} \operatorname{Hom}_{\mathcal{C}_h}(g.X, X)$$

and the *equivariant Verlinde algebra*

$$\widetilde{\mathcal{V}}(\mathcal{C}) := \bigoplus_{\substack{g,h \in G \\ gh=hg}} \widetilde{\mathcal{V}}(\mathcal{C})_{g,h} \ .$$

For $g,h \in G$ with $gh = hg$, $X \in \mathcal{C}_h$, $Y \in \mathcal{C}_g$ and a morphism $\varphi : g.X \longrightarrow X$ we define the morphism $\widetilde{s}(\varphi) : Y \longrightarrow h.Y$ using the graphical calculus as



This assignment induces a linear map

$$\widetilde{s} : \widetilde{\mathcal{V}}(\mathcal{C})_{g,h} = \int^{X \in \mathcal{C}_h} \mathrm{Hom}_{\mathcal{C}_h}(g.X, X) \longrightarrow \int^{Y \in \mathcal{C}_g} \mathrm{Hom}_{\mathcal{C}_g}(Y, h.Y) \cong \widetilde{\mathcal{V}}(\mathcal{C})_{h^{-1},g} \ .$$

We denote the induced map $\widetilde{\mathcal{V}}(\mathcal{C}) \longrightarrow \widetilde{\mathcal{V}}(\mathcal{C})$ also by $\widetilde{s}$. We call the finitely semisimple G-equivariant ribbon category $\mathcal{C}$ a *G-multimodular category* if the map $\widetilde{s} : \widetilde{\mathcal{V}}(\mathcal{C}) \longrightarrow \widetilde{\mathcal{V}}(\mathcal{C})$ is invertible. A *G-modular category* is a G-multimodular category with simple monoidal unit.

Remark 6.18. (a) The name equivariant Verlinde *algebra* is also justified in the purely algebraic case because $\widetilde{\mathcal{V}}(\mathcal{C})$ comes with a multiplication, see [Kir04, Section 8], which is in accordance with the multiplication provided by Proposition 6.12 in the case where our category comes from a topological field theory.

(b) A $\{e\}$-multimodular category is just a modular category without the requirement that the unit is simple. However by [BDSPV15, Lemma 5.3], such a category decomposes into a sum of modular categories. For $G \neq \{e\}$ such a decomposition need not be possible, see Example 6.15, so the simplicity of the unit is an important requirement for equivariant categories.

(c) In [TV14] a G-modular category is defined to be a finitely semisimple G-equivariant ribbon category with simple unit such that the twisted sectors are non-trivial and the neutral sector is modular. This notion of G-modularity turns out to be equivalent to the one defined above as follows from a result by Müger in [Tur10b, Appendix 5, Theorem 4.1 (ii)], see also [Müg04], and the characterization of G-modularity as defined above in terms of the orbifold theory given in [Kir04] and recalled as Theorem 6.19 below.

6.2.2 Equivariant modularity of $\mathcal{C}^Z$

Now we can prove the main result of this section, namely the equivariant modularity of the category $\mathcal{C}^Z$ that a 3-2-1-dimensional G-equivariant topological field theory Z yields on the circle. We will have two versions of the result depending on whether the unit in $\mathcal{C}^Z$ is simple. The proofs will be totally independent.

If the unit of $\mathcal{C}^Z$ is simple, then we will prove that $\mathcal{C}^Z$ is G-modular. The method of proof demonstrates that the topological orbifold construction provides a link between the purely algebraic understanding of equivariant modular categories in [Kir04] to the topological results of [BDSPV15]. To this end, we use that the notion of equivariant modularity is completely governed by the following strong algebraic result from [Kir04] that we slightly rephrase:

Theorem 6.19 ([Kir04, Theorem 10.5]). *Let G be a finite group. For any finitely semisimple G-equivariant ribbon category $\mathcal{C}$, the orbifold category $\mathcal{C}/G$ naturally inherits by Proposition 6.3* *the structure of a finitely semisimple ribbon category and*

$$\mathcal{C} \text{ is } G\text{-modular} \iff \mathcal{C}/G \text{ is modular.}$$

Theorem 6.20. *Let G be a finite group. For any extended G-equivariant topological field theory Z the category $\mathcal{C}^Z$ obtained by evaluation on the circle is*

(a) *G-modular if its monoidal unit is simple,*
(b) *and in the general case still G-multimodular.*

Proof. If the unit of $\mathcal{C}^Z$ is simple, the monoidal unit of $Z/G(\mathbb{S}^1)$ is simple as well by Proposition 6.2. Now Theorem 6.4 yields an equivalence

$$\frac{Z}{G}(\mathbb{S}^1) \simeq \frac{\mathcal{C}^Z}{G}$$

of finitely semisimple ribbon categories. But by [BDSPV15] the category $Z/G(\mathbb{S}^1)$ is even modular, hence so is $\mathcal{C}^Z/G$. Now Theorem 6.19 implies that $\mathcal{C}^Z$ is G-modular. This proves (a).

For the proof of (b), by Theorem 5.49 we have to show that the operator $\widetilde{s} : \widetilde{\mathcal{V}}(\mathcal{C}^Z) \longrightarrow \widetilde{\mathcal{V}}(\mathcal{C}^Z)$ is invertible.

In the non-equivariant case, this follows from the fact that $\widetilde{s}$ is obtained by evaluation of Z on an invertible 2-automorphism of the torus $\mathbb{T}^2$. More precisely, the non-equivariant version of $\widetilde{s}$ is discussed in [Tur10a, II.1.4 & 3.9] and sometimes also referred to as the 'S-matrix'. In [Tur10a, IV.5.4] it is explained that this map is the evaluation of the topological field theory on an element of the mapping class group $\mathrm{SL}(2,\mathbb{Z})$ of the torus seen as invertible three-dimensional bordism via the mapping cylinder construction. We denote this mapping class group element by ϕ_s. Now $\widetilde{s}$ is invertible since ϕ_s is.

This is adapted to the equivariant case as follows: By the mapping cylinder construction in the form of Remark 2.2 (c) the element ϕ_s gives rise to an invertible 2-morphism in $G\text{-}\mathsf{Cob}(3,2,1)$ from $\mathbb{T}^2$ with bundle decoration (g,h) for $g,h \in G$ with $gh = hg$ to $\mathbb{T}^2$ with bundle decoration (h^{-1},g), compare to the proof of Proposition 6.14. By the same arguments as in the non-equivariant case, the evaluation of Z on this 2-morphism is the map $\widetilde{s} : \widetilde{\mathcal{V}}(\mathcal{C}^Z)_{g,h} \longrightarrow \widetilde{\mathcal{V}}(\mathcal{C}^Z)_{h^{-1},g}$, which shows us that this map is invertible. But then $\widetilde{s} : \widetilde{\mathcal{V}}(\mathcal{C}^Z) \longrightarrow \widetilde{\mathcal{V}}(\mathcal{C}^Z)$ is also invertible. $\square$

Remark 6.21. We can give another proof of Theorem 6.20 (a): By Remark 6.18 (c) it suffices to show the following two things:

- The neutral sector of $\mathcal{C}^Z$ is modular: This follows from the fact that we can pull Z back along the symmetric monoidal functor $\mathsf{Cob}(3,2,1) \longrightarrow G\text{-}\mathsf{Cob}(3,2,1)$ equipping all manifolds with the trivial G-bundle. This yields an ordinary extended topological field theory whose value on the circle is $\mathcal{C}^Z_e$, which is a modular category by [BDSPV15].

- The twisted sectors of $\mathcal{C}^Z$ are non-trivial: This was proven directly in Proposition 6.14 based on modular invariance.

Note that (b) generalizes (a) if we take the statement in Proposition 6.2 on the simplicity of the units into account.

Remark 6.22. There are two main constructions for three-dimensional G-equivariant topological field theories due to Turaev and Virelizier:

- The *state sum construction* [TV12] takes as input a spherical G-fusion category $\mathcal{S}$ and yields a G-equivariant Turaev-Viro type theory $\mathrm{TV}^G_{\mathcal{S}}$.

- The *surgery construction* [TV14] takes as input an (anomaly-free) G-modular category $\mathcal{C}$ and yields a G-equivariant Reshetikhin-Turaev type theory $\mathrm{RT}^G_{\mathcal{C}}$.

In [TV12, TV14] both constructions are not given using the language of *extended* equivariant topological field theories, but in [TV19, Remark 8.5] it is explained how these constructions can be lifted to this framework. Then the surgery construction will give an extended equivariant topological field theory $\mathrm{RT}^G_{\mathcal{C}}$ in the sense of this thesis such that the value of $\mathrm{RT}^G_{\mathcal{C}}$ on the circle is $\mathcal{C}$; and the state sum construction will also give an extended equivariant topological field theory $\mathrm{TV}^G_{\mathcal{S}}$ such that the evaluation of $\mathrm{TV}^G_{\mathcal{S}}$ on the circle will be given by the G-center $Z_G(\mathcal{S})$ of $\mathcal{S}$ according to [TV19, Theorem 8.2]

$$\mathrm{TV}^G_{\mathcal{S}} \simeq \mathrm{RT}^G_{Z_G(\mathcal{S})} \tag{6.6}$$

which is a generalization of the non-equivariant case.

If G is finite, we can compute the orbifold theories of $\mathrm{RT}^G_{\mathcal{C}}$ and $\mathrm{TV}^G_{\mathcal{S}}$ for a G-modular category $\mathcal{C}$ and a G-fusion category $\mathcal{S}$: By Theorem 6.4 the orbifold theory $\mathrm{RT}^G_{\mathcal{C}}/G : \mathsf{Cob}(3,2,1) \longrightarrow \mathsf{2Vect}$ of $\mathrm{RT}^G_{\mathcal{C}}$ is the Reshetikhin-Turaev theory for the orbifold category $\mathcal{C}/G$, i.e.

$$\frac{\mathrm{RT}^G_{\mathcal{C}}}{G} \simeq \mathrm{RT}_{\mathcal{C}/G} \ . \tag{6.7}$$

For $\mathrm{TV}^G_{\mathcal{S}}$, we find

$$\frac{\mathrm{TV}^G_{\mathcal{S}}}{G}(\mathbb{S}^1) \overset{(6.6)}{\simeq} \frac{\mathrm{RT}^G_{Z_G(\mathcal{S})}}{G}(\mathbb{S}^1) \overset{(6.7)}{\simeq} \frac{Z_G(\mathcal{S})}{G} \simeq Z(\mathcal{S})$$

as modular categories, where in the last step we used [GNN09, Theorem 3.5]. Hence, the orbifold theory $\mathrm{TV}^G_{\mathcal{S}}/G$ is the non-equivariant Turaev-Viro theory for $\mathcal{S}$ seen as spherical fusion category (recall that a G-fusion category is fusion if and only if G is finite, see [TV12, Section 4.2]). Hence, on the level of spherical fusion categories, orbifoldization amounts to forgetting the G-grading.

Furthermore, we remark that a generalization of RT^G_{-} taking G-multimodular categories as input should provide a weak inverse to the functor from G-equivariant 3-2-1-dimensional topological field theories to G-multimodular categories by evaluation on the circle, see Theorem 6.20 (when restricting to the anomaly-free case). Hence, G-equivariant 3-2-1-dimensional topological field theories should be classified by (anomaly-free) G-multimodular categories.

Bibliography

[At88] M. F. Atiyah. Topological quantum field theory. *Publ. Math. IHÉS*, 68:175–186, 1988.

[AF15] D. Ayala, J. Francis. Factorization homology of topological manifolds. *J. Topol.* 8(4):1045–1084, 2015.

[BL36] R. Baer, F. Levi. Freie Produkte und ihre Untergruppen. *Comp. Math.* 3:391–398, 1936.

[BHW10] J. C. Baez, A. E. Hoffnung, C. D. Walker. Higher dimensional algebra. VII: Groupoidification. *Theory Appl. Categ.* 24(18):489–553, 2010.

[BBFW12] J. C. Baez, A. Baratin, L. Freidel, D. K. Wise. Infinite-Dimensional Representations of 2-Groups. *Mem. Amer. Math. Soc.* 219(1032), 2012.

[Ban98] P. Bantay. Characters and modular properties of permutation orbifolds. *Phys. Lett. B* 419:175–178, 1998.

[Ban02] P. Bantay. Permutation orbifolds. *Nucl. Phys. B* 633:365–378, 2002.

[BS11] T. Barmeier, C. Schweigert. A Geometric Construction for Permutation Equivariant Categories from Modular Functors. *Transform. Groups* 16:287–337, 2011.

[BDSPV15] B. Bartlett, C. L. Douglas, C. J. Schommer-Pries, J. Vicary. *Modular Categories as Representations of the 3-dimensional Bordism Category*. 2015. arXiv:1509.06811 [math.AT]

[BSW17] M. Benini, A. Schenkel, L. Woike. Operads for algebraic quantum field theory. Accepted for publication in *Commun. Contemp. Math.* arXiv:1709.08657 [math-ph]

[BZBJ18a] D. Ben-Zvi, A. Brochier, D. Jordan. Integrating quantum groups over surfaces. *J. Top.* 11(4):874–917, 2018.

[BZBJ18b] D. Ben-Zvi, A. Brochier, D. Jordan. Quantum character varieties and braided module categories. *Sel. Math. New Ser.* 24:4711–4748, 2018.

[BZFN10] D. Ben-Zvi, J. Francis, D. Nadler. Integral Transforms and Drinfeld Centers in Derived Algebraic Geometry. *J. Amer. Math. Soc.* 23(4):909–966, 2010.

[BM03] C. Berger and I. Moerdijk. Axiomatic homotopy theory for operads. *Comment. Math. Helv.* 78(4):805–831, 2003.

[BM07] C. Berger and I. Moerdijk. Resolution of coloured operads and rectification of homotopy algebras. in: A. Davydov, M. Batanin, M. Johnson, S. Lack and A. Neeman (eds.), Categories in algebra, geometry and mathematical physics, *Contemp. Math.* 431:31–58, American Mathematical Society, Providence, 2007.

[BV68] J. M. Boardman, R. M. Vogt. Homotopy-everything H-spaces. *Bull. Amer. Math. Soc.* 74:1117–1122, 1968.

[BV73] J. M. Boardman, R. M. Vogt. *Homotopy invariant algebraic structures on topological spaces.* Lecture Notes in Mathematics, 347. Springer, 1973.

[Bre93] G. E. Bredon. *Topology and Geometry*. Springer Graduate Texts in Mathematics 139, 1993.

[ENOM10] P. Etingof, D. Nikshych, V. Ostrik, E. Meir. Fusion categories and homotopy theory. *Quantum Topology* 1:209-273, 2010.

[CS19] D. Calaque, C. Scheimbauer. A note on the (∞, n)-category of cobordisms. *Algebr. Geom. Topol.* 19:533–655, 2019.

[CRS17] N. Carqueville, I. Runkel, G. Schaumann. Orbifolds of n-dimensional defect TQFTs. 2017. arXiv:1705.06085 [math.QA]

[Cle72] A. Clebsch. Zur Theorie der Riemann'schen Flächen. *Math. Ann.* 6:216–230, 1872.

[CGPW16] S. X. Cui, C. Galindo, J. Y. Plavnik, Z. Wang. On Gauging Symmetry of Modular Categories. *Commun. Math. Phys.* 348(3):1043–1064, 2016.

[DM69] P. Deligne, D. Mumford. The Irreducibility of the Space of Curves of Given Genus. *Publ. Math. IHÉS* 36:75–110, 1969.

[DPR90] R. Dijkgraaf, V. Pasquier, P. Roche. Quasi Hopf algebras, group cohomology and orbifold models. *Nuclear Phys. B Proc. Suppl.* 18B, 60–72, 1990.

[DVVV89] R. Dijkgraaf, C. Vafa, E. Verlinde, H. Verlinde. The Operator Algebra of Orbifold Models. *Commun. Math. Phys.* 123:485–526, 1989.

[EVW16] J. S. Ellenberg, A. Venkatesh, C. Westerland. Homological stability for Hurwitz spaces and the Cohen-Lenstra conjecture over function fields. *Ann. Math.* 183(3):729–786, 2016.

[EGNO15] P. Etingof, S. Gelaki, D. Nikshych, V. Ostrik. Tensor categories. Mathematical Surveys and Monographs 205, American Mathematical Society, 2015.

[EG18] D. E. Evans, T. Gannon. Reconstruction and Local Extensions for Twisted Group Doubles, and Permutation Orbifolds. 2018. arXiv:1804.11145 [math.QA]

[FHLT10] D. S. Freed, M. J. Hopkins, J. Lurie, C. Teleman. Topological quantum field theories from compact Lie groups. In P. R. Kotiuga (ed.): *A celebration of the mathematical legacy of Raoul Bott*, AMS, 2010.

[FQ93] D. S. Freed, F. Quinn. Chern-Simons theory with finite gauge group. *Comm. Math. Phys.* 156(3):435–472, 1993.

[Fre17] B. Fresse. *Homotopy of operads and Grothendieck-Teichmüller groups. Part 1: The Algebraic Theory and its Topological Background.* Mathematical Surveys and Monographs 217, American Mathematical Society, Providence, 2017.

[FFRS09] J. Fröhlich, J. Fuchs, I. Runkel, C. Schweigert. Defect lines, dualities, and generalised orbifolds. *Proceedings of the XVI International Congress on Mathematical Physics*, 2009.

[FKS92] J. Fuchs, A. Klemm, M. G. Schmidt. Orbifolds by cyclic permutations in Gepner type superstrings and in the corresponding Calabi-Yau manifolds. *Ann. Physics* 214(2):221–257, 1992.

[FSS20] J. Fuchs, G. Schaumann, C. Schweigert. Eilenberg-Watts calculus for finite categories and a bimodule Radford S^4 theorem. *Trans. Am. Math. Soc.* 373:1–40, 2020.

[GTMW09] S. Galatius, U. Tillmann, I. Madsen, M. Weiss. The homotopy type of the cobordism category. *Acta Math.* 202(2):195–239, 2009.

[Gal17] C. Galindo. Coherence for monoidal G-categories and braided G-crossed categories. *J. Algebra* 487:118–137, 2017.

[GNN09] S. Gelaki, D. Naidu, D. Nikshych. Centers of Graded Fusion Categories. *Algebra & Number Theory* 3(8):959–990, 2009.

[Ge94] E. Getzler. Batalin-Vilkovisky algebras and two-dimensional topological field theories. *Comm. Math. Phys.* 159:265–285, 1994.

[Hau18] R. Haugseng. Iterated spans and classical topological field theories. *Math. Z.* 289(3):1427–1488, 2018.

[HSV17] J. Hesse, C. Schweigert, A. Valentino. Frobenius Algebras and Homotopy Fixed Points of Group Actions on Bicategories. *Theory Appl. Categ.* 32(18):652–681, 2017.

[Hol08] S. Hollander. A homotopy theory for stacks. *Israel J. Math.* 163:93–124, 2008.

[Hur91] A. Hurwitz. Über Riemann'sche Flächen mit gegebenen Verzweigungspunkten. *Math. Ann.* 39:1–61, 1891.

[KV94] M. Kapranov, V. Voevodsky. Braided monoidal 2-categories and Manin-Schechtman higher braid groups. *J. Pure Appl. Algebra* 92:241–267, 1994.

[Kap10] A. Kapustin. Topological Field Theory, Higher Categories, and Their Applications. Proceedings of the International Congress of Mathematicians Hyderabad, India, 2010.

[Kas95] C. Kassel. *Quantum Groups*. Springer-Verlag New York, 1995.

[Kau02] R. M. Kaufmann. Orbifold Frobenius Algebras, Cobordisms and Monodromies. *Commun. Contemp. Math.*, 310:135–161, 2002.

[KL01] T. Kerler, V. V. Lyubashenko. *Non-Semisimple Topological Quantum Field Theories for 3-Manifolds with Corners*. Lecture Notes in Mathematics 1765, Springer-Verlag Berlin Heidelberg, 2001.

[Kir01] A. A. Kirillov. *Modular categories and orbifold models II*. 2001. arXiv:math/0110221 [math.QA]

[Kir04] A. A. Kirillov. On G-equivariant modular categories. 2004. arXiv:math/0401119v1 [math.QA]

[Lee12] J. M. Lee. *Introduction to Smooth Manifolds*. Springer, 2012.

[LMSS18] S. Lentner, S. N. Mierach, C. Schweigert, Y. Sommerhäuser. Hochschild Cohomology and the Modular Group. *J. Algebra* 507:400–420, 2018.

[Lur09] J. Lurie. On the Classification of Topological Field Theories. *Current Devel. Math.* 2008:129–280, 2008.

[MM92] S. Mac Lane, I. Moerdijk. *Sheaves in Geometry and Logic*. Springer, 1992.

[ML98] S. Mac Lane. *Categories for the Working Mathematician*. Springer, 1998.

[MNS12] J. Maier, T. Nikolaus, C. Schweigert. Equivariant Modular Categories via Dijkgraaf-Witten theory. *Adv. Theor. Math. Phys.* 16:289-358, 2012.

[May72] P. May. *The geometry of iterated loop spaces*. Lecture Notes in Mathematics 271, Springer, 1972.

[Mor11] J. C. Morton. Two-vector spaces and groupoids. *Appl. Categ. Structures* 19:659–707, 2011.

[Mor15] J. C. Morton. Cohomological Twisting of 2-Linearization and Extended TQFT. *J. Homotopy Relat. Struct.* 10:127–187, 2015.

[Müg04] M. Müger. Galois extensions of braided tensor categories and braided crossed G-categories. *J. Alg.* 277:256–281, 2004.

[Müg05] M. Müger. Conformal Orbifold Theories and Braided Crossed G-Categories. *Commun. Math. Phys.* 260:727–762, 2005.

[MS18] L. Müller, R. J. Szabo. Extended Quantum Field Theory, Index Theory, and the Parity Anomaly. *Commun. Math. Phys.* 362:1049–1109, 2018.

[MS19] L. Müller, R. J. Szabo. 't Hooft Anomalies of Discrete Gauge Theories and Non-abelian Group Cohomology. *Commun. Math. Phys.* (online first), 2019.

[MW20a] L. Müller, L. Woike. Parallel Transport of Higher Flat Gerbes as an Extended Homotopy Quantum Field Theory. *J. Homotopy Relat. Str.* 15(1):113-142, 2020.

[MW20b] L. Müller, L. Woike. Equivariant Higher Hochschild Homology and Topological Field Theories. *Homology Homotopy Appl.* 22(1):27–54, 2020.

[RT91] N. Reshetikhin, V. Turaev. Invariants of 3-manifolds via link polynomials and quantum groups. *Invent. Math.* 103:547–598, 1991.

[Rie14] E. Riehl. *Categorical Homotopy Theory*. Cambridge University Press, 2014.

[RW18] E. C. Rowell, Z. Wang. Mathematics of Topological Quantum Computing. *Bull. Am. Math. Soc.* 55(2):183–238, 2018.

[SW03] P. Salvatore, N. Wahl. Framed discs operads and Batalin-Vilkovisky algebras. *Quart. J. Math. 54*, 213-231, 2003.

[Sch14] C. Scheimbauer. *Factorization Homology as a Fully Extended Topological Field Theory*. PhD Thesis, ETH Zürich, 2014.

[SP09] C. J. Schommer-Pries. *The classification of two-dimensional extended topological field theories*. Ph.D. thesis, University of California, Berkeley, 2009. arXiv:1112.1000 [math.AT]

[SW19] C. Schweigert, L. Woike. Orbifold Construction for Topological Field Theories. *J. Pure Appl. Algebra* 223:1167–1192, 2019.

[Tau11] C. H. Taubes. *Differential Geometry – Bundles, Connections, Metrics and Curvature*. Graduate Text in Mathematics 23. Oxford University Press, 2011.

[Th79] R. W. Thomason. Homotopy colimits in the category of small categories. *Math. Proc. Cambridge Philos. Soc.* 85(1):91–109, 1979.

[Tro16] F. Trova. Nakayama categories and groupoid quantization. 2016. arXiv:1602.01019 [math.CT]

[Tur99] V. Turaev. Homotopy field theory in dimension 2 and group-algebras. 1999. arXiv:math/9910010 [math.QA]

[Tur00] V. Turaev. Homotopy field theory in dimension 3 and crossed group-categories. 2000. arXiv:math/0005291 [math.GT]

[Tur10a] V. Turaev. *Quantum Invariants of Knots and 3-Manifolds*. De Gruyter, Berlin 2010.

[Tur10b] V. Turaev. *Homotopy Quantum Field Theory*. With appendices by M. Müger and A. Virelizier. European Mathematical Society, Zürich, 2010.

[TV12] V. Turaev, A. Virelizier. On 3-dimensional homotopy quantum field theory, I. *Int. J. Math.* 23(9):1–28, 2012.

[TV14] V. Turaev, A. Virelizier. On 3-dimensional homotopy quantum field theory II: The surgery approach. *Int. J. Math.* 25(4):1–66, 2014.

[TV17] V. Turaev, A. Virelizier. *Monoidal Categories and Topological Field Theory*. Birkhäuser, 2017.

[TV19] V. Turaev, A. Virelizier. On 3-dimensional homotopy quantum field theory III: comparison of two approaches. arXiv:1911.10257 [math.GT]

[Wil05] S. Willerton. The twisted Drinfeld double of a finite group via gerbes and finite groupoids. *Algebr. Geom. Topol.* 8:1419–1457, 2008.

[Yau16] D. Yau. *Colored operads.* Graduate Studies in Mathematics 170, American Mathematical Society, Providence, 2016.

[Yau18] D. Yau. *Homotopical algebraic quantum field theory.* arXiv:1802.08101v1 [math-ph]

[You19] M. B. Young. Orientation twisted homotopy field theories and twisted unoriented Dijkgraaf-Witten theory. *Comm. Math. Phys.* 374:1645–1691, 2020.

Summary

The notion of a topological field theory lies at the interface of topology, algebra and mathematical physics. In the three-dimensional case, it is intimately related to representation theory.

In this thesis, we develop an *orbifold construction* for extended topological field theories with particular focus on applications in the three-dimensional case. In order to describe this construction, the notion of an *equivariant* topological field theory, a particular flavor of topological field theory, is needed: For a finite group G, an extended equivariant topological field theory is an extended topological field theory defined on a bordism bicategory in which all manifolds are equipped with a G-bundle. The topological orbifold construction is a functorial construction that assigns to a given extended *G-equivariant* topological field theory an extended *non-equivariant* topological field theory.

We develop a two-step procedure for the construction of the orbifold theory of a given extended equivariant topological field theory: First, we produce from the input theory a non-equivariant topological field theory with values in a certain symmetric monoidal bicategory built from 2-vector bundles over groupoids and their higher spans (we refer to this step as *change to equivariant coefficients*). Afterwards, we develop and apply a parallel section functor that assigns 2-vector spaces of parallel sections to 2-vector bundles and certain pull-push maps to higher spans of 2-vector bundles. Since the orbifold construction is given at the level of topological field theories, we will refer to it as the *topological orbifold construction*.

While this construction itself can be formulated in any dimension, a large part of the thesis is concerned with the three-dimensional (more precisely 3-2-1-dimensional) case. In this specific dimension, we can profit from the deep connection between topological field theories and representation theory. We prove that, when restricted to the circle, the topological orbifold construction corresponds to the purely algebraic concept of an orbifold category, thereby opening a topological perspective on this widely used and well-investigated construction in representation theory. In fact, one of the strengths of the topological orbifold construction lies precisely in this relation to the concept of an orbifold category. As an illustration of the interplay between topological and algebraic orbifoldization, we prove that the evaluation of a 3-2-1-dimensional G-equivariant topological field theory on the circle is a G-(multi)modular category.

Already the formulation of the relation between topological and algebraic orbifoldization requires proving a number of results on the structure that is present on the category obtained by evaluation of a 3-2-1-dimensional G-equivariant topological field theory on the circle. We accomplish this by introducing the so-called *little bundles operad*, a topological operad built from Hurwitz spaces generalizing the little disks operad. By exhibiting a presentation in terms of generators and relations for this aspherical operad, we prove that its categorical algebras are precisely braided crossed categories.

The applications of the topological orbifold construction go beyond the ones presented in this thesis. In combination with subsequent work of the author with L. Müller on extended topological field theories from cohomological data, it allows us to understand twisted Drinfeld doubles in a topological way. Moreover, it has been extensively used in work of Müller-Szabo for the description of anomalies in quantum field theories, and by Young for the construction of orientation twisted homotopy quantum field theories.

Zusammenfassung

Der Begriff einer topologischen Feldtheorie liegt an der Schnittstelle von Topologie, Algebra und mathematischer Physik. Im dreidimensionalen Fall besitzt er enge Beziehungen zur Darstellungstheorie.

In dieser Arbeit entwickeln wir eine *Orbifoldkonstruktion* für erweiterte topologische Feldtheorien mit besonderem Augenmerk auf dem dreidimensionalen Fall. Um diese Konstruktion zu beschreiben, benötigen wir den Begriff einer *äquivarianten* topologischen Feldtheorie, eine spezielle Ausprägung von topologischer Feldtheorie: Für eine endliche Gruppe G ist eine erweiterte äquivariante topologische Feldtheorie eine erweiterte topologische Feldtheorie definiert auf einer Bikategorie von Bordismen, deren Mannigfaltigkeiten mit G-Bündeln ausgestattet sind. Die topologische Orbifoldkonstruktion ist eine funktorielle Konstruktion, die einer gegebenen erweiterten *G-äquivarianten* topologischen Feldtheorie eine erweiterte *nicht-äquivariante* topologische Feldtheorie zuweist.

Wir entwickeln ein aus zwei Schritten bestehendes Verfahren für die Konstruktion der Orbifoldtheorie einer gegebenen erweiterten äquivarianten topologischen Feldtheorie: Zuerst produzieren wir aus der gegebenen Theorie eine nicht-äquivariante topologische Feldtheorie mit Werten in einer bestimmten symmetrisch monoidalen Bikategorie bestehend aus 2-Vektorbündeln über Gruppoiden und ihren höheren Korrespondenzen (wir bezeichnen diesen Schritt als *Wechsel zu äquivarianten Koeffizienten*). Anschließend entwickeln und verwenden wir einen Funktor der parallelen Schnitte, der 2-Vektorbündeln die 2-Vektorräume ihrer parallelen Schnitte und höheren Korrespondenzen gewisse Pull-Push-Abbildungen zuweist. Da die Orbifoldkonstruktion auf der Ebene von topologischen Feldtheorien gegeben wird, bezeichnen wir sie als *topologische Orbifoldkonstruktion*.

Während die Konstruktion selbst in jeder Dimension gegeben werden kann, befasst sich ein großer Teil dieser Arbeit mit dem dreidimensionalen (genauer 3-2-1-dimensionalen) Fall. In dieser bestimmten Dimension können wir von der tiefen Verbindung zwischen topologischen Feldtheorien und Darstellungstheorie profitieren. Wir beweisen, dass die topologische Orbifoldkonstruktion, wenn auf den Kreis eingeschränkt, dem rein algebraischen Konzept einer Orbifoldkategorie entspricht, womit sich eine topologische Perspektive auf diese viel benutzte und gut untersuchte darstellungstheoretische Konstruktion eröffnet. Tatsächlich liegt eine der Stärken der topologischen Orbifoldkonstruktion genau in dieser Beziehung zum Konzept einer Orbifoldkategorie. Als Illustration des Zusammenspiels zwischen topologischer und algebraischer Orbifoldisierung beweisen wir, dass die Auswertung einer 3-2-1-dimensionalen äquivarianten topologischen Feldtheorie auf dem Kreis eine G-(multi)modulare Kategorie ist.

Bereits die Formulierung der Beziehung zwischen topologischer und algebraischer Orbifoldisierung verlangt den Beweis einer Vielzahl von Ergebnissen über die anzutreffende Struktur auf der Kategorie, die durch Auswertung einer 3-2-1-dimensionalen G-äquivarianten topologischen Feldtheorie auf dem Kreis erhalten wird. Uns gelingt dies durch die Einführung der sogenannten *Operade der kleinen Bündel*, einer topologischen Operade, die die Operade der kleinen Scheiben verallgemeinert. Durch Angabe einer Darstellung dieser asphärischen Operade durch Erzeuger und Relationen beweisen wir, dass die kategoriellen Algebren dieser Operade gerade verzopfte gekreuzte Kategorien sind.

Die Anwendungen der topologischen Orbifoldkonstruktion gehen über die in dieser Arbeit präsentierten hinaus. In Kombination mit nachfolgender Arbeit des Autors mit L. Müller zu erweiterten topologischen Feldtheorien aus kohomologischen Daten erlaubt sie uns, getwistete Drinfeld-Doppel auf topologische Art zu verstehen. Weiterhin geht sie zentral ein in Arbeiten von Müller-Szabo für die Beschreibung von Anomalien in Quantenfeldtheorien und von Young für die Konstruktion von orientierungsgetwisteten Homotopiequantenfeldtheorien.