Hochschild cohomology, modular tensor categories and mapping class groups

by

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Abstract

It is known that the mapping class group of a compact oriented surface of genus g and n boundary components acts projectively on certain Hom-spaces, called spaces of chiral conformal blocks, arising from a modular tensor category. If the tensor category is nonsemisimple, the Hom-functor is not exact and one can ask if there is a similar construction that gives a projective action of the mapping class group on the right derived functors. We will approach this question in two steps:

In the first step, which is carried out in Chapter 2, we consider the case where the modular tensor category is given by the category of finite-dimensional modules over a factorizable ribbon Hopf algebra, and take the torus as the surface. In this case, the mapping class group is the modular group $\mathrm{SL}(2, \mathbb{Z})$, and one of the Hom-spaces mentioned above is the center of the factorizable ribbon Hopf algebra. The center is the zeroth Hochschild cohomology group, and we show that it is indeed possible to extend this projective action of the modular group to an arbitrary Hochschild cohomology group of a factorizable ribbon Hopf algebra.

The second step, which is carried out in Chapter 3, is to consider the general case. We construct from an arbitrary modular tensor category and for every compact oriented surface with finitely many labeled boundary components a cochain complex of finite-dimensional vector spaces. Its zeroth cohomology group is the vector space appearing in the original construction. We show that the mapping class group of the surface acts projectively on the cohomology groups of this cochain complex in such a way that it reduces to the original action in degree zero. As we explain, this generalizes the results in Chapter 2.

Zusammenfassung

Es wurde gezeigt, dass die Abbildungsklassengruppe einer kompakten, orientierten Fläche vom Geschlecht g mit n Randkomponenten projektiv auf gewissen Hom-Räumen, genannt Räume von chiral konformen Blöcken, einer modularen Tensorkategorie wirkt. Ist die Tensorkategorie nicht halbeinfach, so ist der Hom-Funktor nicht exakt und es stellt sich die Frage, ob es eine ähnliche Konstruktion gibt, die eine projektive Wirkung der Abbildungsklassengruppe auf den rechtsabgeleiteten Funktoren liefert. Wir nähern uns dieser Frage in zwei Schritten:

Im ersten Schritt, der sich in Kapitel 2 findet, betrachten wir den Fall, dass die modulare Tensorkategorie durch die Kategorie der endlichdimensionalen Moduln über einer faktorisierbaren Band-Hopfalgebra gegeben ist, und wählen den Torus als Fläche. In diesem Fall ist die Abbildungsklassengruppe die modulare Gruppe $\mathrm{SL}(2, \mathbb{Z})$, und einer der oben erwähnten Hom-Räume ist das Zentrum der faktorisierbaren Band-Hopfalgebra. Das Zentrum ist die nullte Hochschild-Kohomologiegruppe, und wir zeigen, dass es in der Tat möglich ist, die projektive Wirkung der modularen Gruppe auf eine beliebige Hochschild-Kohomologiegruppe zu verallgemeinern.

Im zweiten Schritt, dem Kapitel 3 gewidmet ist, betrachten wir den allgemeinen Fall. Wir konstruieren, ausgehend von einer beliebigen modularen Tensorkategorie, für eine beliebige kompakte, orientierte Fläche einen Kokettenkomplex von endlichdimensionalen Vektorräumen. Die nullte Kohomologiegruppe dieses Kokettenkomplexes stimmt mit dem Vektorraum aus der ursprünglichen Konstruktion überein. Wir zeigen, dass die Abbildungsklassengruppe der Fläche auf den Kohomologiegruppen des Kokettenkomplexes in einer Weise wirkt, die sich in Grad Null auf die ursprüngliche Wirkung reduziert. Wie wir erklären, verallgemeinert das unsere Ergebnisse aus Kapitel 2.

Acknowledgements

Contents

Introduction

This thesis investigates actions of mapping class groups of surfaces on cochain complexes of vector spaces. By studying these actions, this thesis lies at the crossroads of different areas of mathematics: mapping class groups appearing in topology, Hochschild cohomology as a part of algebra, modular tensor categories—in particular the category of finite-dimensional modules over a factorizable ribbon Hopf algebra—and modular functors, appearing as a weaker notion of topological field theories.

The study of representations of mapping class groups of surfaces has a long history. They naturally act on the first homology of the surface. Furthermore, mapping class groups of surfaces act for example on Teichmüller spaces [FM, p. 342–343] and any three-dimensional topological field theory gives a linear representation of the mapping class group via the mapping cylinder construction. In the following, we will introduce the main players of this thesis and their interaction.

Hochschild cohomology

Hochschild cohomology is a cohomology theory for algebras over commutative rings with coefficients in a bimodule over the corresponding algebra. It was introduced by Gerhard Hochschild in 1945 in [Ho1] for algebras over a field and generalized to algebras over commutative rings by Henri Cartan and Samuel Eilenberg in 1956 in [CE, Chap. IX]. There is also a corresponding homology theory. Originally, Hochschild cohomology was defined using the bar complex, which is reviewed in Section 1.1, but with the interpretation as an Ext functor it can be calculated

using any projective resolution of the algebra consisting of bimodules. Explicitly, the n-th Hochschild cohomology group of the algebra A with coefficients in the A-bimodule M is given by $\operatorname{Ext}^n_{A \otimes A^{\mathrm{op}}}(A, M)$ [CE, Chap. IX, § 4].

Hochschild cohomology is used to understand the structure of the underlying algebra and to identify essential information about its representations. For example, an algebra is separable if and only if its first Hochschild cohomology group vanishes for any bimodule [Ho1, Thm. 4.1]. An historical overview and basic definitions of Hochschild cohomology, along with an overview of its connection to many current topics of interest, can be found in [Wi].

Every associative algebra is a bimodule over itself in a natural way, where the bimodule structure is given by multiplication. In this case, the zeroth Hochschild cohomology group is given by the center of the algebra, which, in the case of factorizable ribbon Hopf algebras, will appear later because it carries an action of the modular group, which is the mapping class group of the torus. More generally, the zeroth Hochschild cohomology group with coefficients in an arbitrary bimodule is given by the so-called space of invariants of that bimodule. As another example, the first Hochschild cohomology group with coefficients in any bimodule is given by the space of derivations modulo inner derivations and the second Hochschild cohomology group is given by square zero extensions [CE, Chap. IX, Prop. 4.1] and [Ho1, Prop. 6.2]. If we again consider the algebra as a bimodule over itself, the second Hochschild cohomology group can be related to formal deformations of the algebra. A formal deformation is a new product on the algebra with values in the formal power series whose coefficients come from the algebra [Wi, Def. 5.1.2]. Associated with any formal deformation of the algebra is an element in the second Hochschild cohomology group. In particular, if the second Hochschild cohomology group of the algebra vanishes, then the algebra has no nontrivial formal deformations [Wi, Sec. 5.1 and Prop. 5.2.8].

Hopf algebras and tensor categories

For two modules over an arbitrary algebra A, their tensor product over the ground field K is a vector space, but it does not carry an A-module structure in a canonical way. In order to endow this tensor product with an A-module structure, one needs an additional map, namely a so-called coproduct, that satisfies certain compatibility conditions and turns the algebra into a bialgebra. The category of finite-dimensional modules over such a bialgebra then becomes a monoidal category [EGNO, p. 92]. Furthermore, if one also wants to be able to define dual representations, one needs a so-called antipode, which is an analog of the inversion map in a group. Bialgebras that possess such an antipode are called Hopf algebras. Important examples for Hopf algebras are group algebras and universal enveloping algebras of Lie algebras. The category of finite-dimensional modules over a finite-dimensional Hopf algebra is a tensor category [EGNO, Cor. 5.3.7]. Hopf algebras were first introduced by Heinz Hopf in 1941 in algebraic topology [Ho] to study the topology of compact Lie groups. The term Hopf algebra was coined by Armand Borel in [Bo]. Around 1957, Gerhard Hochschild and George D. Mostow wrote a series of articles [HM1, HM2, HM3], in which they studied the algebra of representative functions $R(G)$ on a Lie group G. Throughout these articles, the authors work—without explicitly mentioning it—with the natural Hopf algebra structure on $R(G)$. However, in Hochschild's later book *The structure of Lie groups* [Ho2], the Hopf algebra structure is explicitly mentioned and plays an important role. Not long after Hopf algebras were first introduced, they appeared in several other settings, for example as quantum groups which are deformations of the universal enveloping algebra of a semisimple Lie algebra introduced by Vladimir Drinfel'd at the International Congress of Mathematicians in Berkeley (1986). Hopf algebras also appear in the theory of invariants for knots and links.

Moreover, if our Hopf algebra is quasitriangular, i.e. possesses a so-called R-matrix, this structure can be used to define a braiding on its representation category and vice versa [EGNO, p. 199]. The braiding gives solutions of the Yang-Baxter equation, which can be used to define representations of the braid group. The Hopf algebras that we will consider in this thesis will not only be quasitriangular, but are also required to possess a ribbon element, which leads to a ribbon structure in the

associated tensor category [EGNO, Prop. 8.11.2]. Furthermore, we will assume that our category satisfies a certain non-degeneracy condition on the braiding, namely being modular as discussed in Paragraph 1.2.1 and 3.2.1. Originally, the notion of a modular tensor category incorporated the condition that the category is semisimple, and modularity was defined by requiring that the Verlinde matrix is invertible [EGNO, Def. 8.13.4]. The notion of a nonsemisimple modular tensor category was introduced in [L0] by using instead the non-degeneracy condition on the braiding mentioned above. In [Sh1, Thm. 1.1], it was shown that this condition is also equivalent to the factorizability of the category, as defined in [EGNO, Def. 8.6.2]. Furthermore, Shimizu shows that a category is modular if and only if the only transparent objects are isomorphic to finite direct sums of the unit object, which we will use as a definition for a modular tensor category. Here, an object is called transparent if its double braiding with any other object is the identity. Combining these facts with [EGNO, Prop. 8.20.12], we see that in the semisimple case, all these conditions are equivalent to the original definition of modularity. We are mainly interested in the case that our modular tensor category is not semisimple, so that the objects in our category do not need to be the direct sum of simple objects, because otherwise the spaces that we construct will frequently vanish.

The category of finite-dimensional modules over a quasitriangular ribbon Hopf algebra is modular if and only if the quasitriangular ribbon Hopf algebra is factorizable ([EGNO, Exerc. 8.6.4.(i)]). In particular, this implies that the Hopf algebra under consideration is finite-dimensional. The category of finite-dimensional representations of a factorizable ribbon Hopf algebras will be our primary example for modular tensor categories. For any finite-dimensional Hopf algebra, its Drinfel'd double is a factorizable Hopf algebra. Modular tensor categories are related to topological field theories and modular functors as explained in the section after next.

Mapping class groups

Mapping class groups arise naturally in topology and were first studied in the first half of the twentieth century to investigate the intersection of closed curves on hyperbolic surfaces. They can be defined for any topological space, but we are interested in the case of a compact oriented surface. The mapping class group of a compact oriented surface is an important algebraic invariant of the surface. It is defined as the group of isotopy classes of orientation-preserving diffeomorphisms, as spelled out in greater detail in Definition 1.3.1. For a surface of genus g with n boundary components (each with a marked point), we denote its mapping class group by $\Gamma_{g,n}$. We will also need the notion of the pure mapping class group $P\Gamma_{g,n}$, which consists of isotopy classes of diffeomorphisms that restrict to the identity on the boundary, as explained in Paragraph 1.3.3. The mapping class group of a surface is related to various other groups, in particular braid groups and outer automorphism groups. Mapping class groups were first analyzed by Jakob Nielsen and Max Dehn. Jakob Nielsen proved that, up to conjugation, all automorphisms of the fundamental group of a surface can be represented by homeomorphisms, compare the Dehn-Nielsen-Baer theorem [FM, Thm. 8.1]. Max Dehn proved that the mapping class group of a closed surface is finitely generated by Dehn twists, originally called "Schraubungen" [D].

Mapping class groups are rather complicated objects, as is for example illustrated by the following fact: Since any finite group can be realized as the group of isometries of some compact Riemann surface, it is a subgroup of the mapping class group of the underlying topological surface [Gr, Thm. 4].

Mapping class groups are already interesting in low genus, although the mapping class groups of the disk and of the sphere are trivial [FM, Lem. 2.1 and p. 51]. For the annulus, the mapping class group is generated by the Dehn twist performed in a collar neighborhood of one of the boundary components, which generates an infinite cyclic group [FM, Prop. 2.4]. For the disk with n punctures, we obtain the braid group B_n on n strands [FM, p. 256], which we have already encountered in the context of quasitriangular Hopf algebras above and which also plays an important role in other areas of mathematics, for example in knot theory. By forgetting how

the strands twist and cross, we get a surjective group homomorphism from the braid group on n strands to the symmetric group on n elements, whose kernel is given by the pure braid group.

The mapping class group of the torus and the once-punctured torus is the modular group $\mathrm{SL}(2,\mathbb{Z})$ [FM, Thm. 2.5 and p. 57]. The modular group consists of special Möbius transformations and is used in the definition of modular forms, which are certain analytic functions on the upper half complex plane satisfying some functional equation involving the group action of the modular group. Modular forms provide an important link between complex analysis and number theory.

The modular group appears as our motivating example throughout this thesis: On the one hand, there is an action of $\mathrm{SL}(2,\mathbb{Z})$ on the center of a factorizable ribbon Hopf algebra, which can be defined in a purely algebraic way [SZ]. On the other hand, this action appears as part of a bigger picture in topological field theories and modular functors, as we will explain below. As already mentioned above, the center of an algebra is the zeroth Hochschild cohomology group, so there is a connection between Hochschild cohomology and the mapping class group of the torus, which will be further studied in Chapter 2 and extended to general surfaces with the help of modular functors in Chapter 3.

For the torus with one boundary component, one can compare the capping sequence appearing in Proposition 3.1.5, where we glue a punctured disk into the boundary component, with the short exact sequence $\mathbb{Z} \to B_3 \to \mathrm{SL}(2,\mathbb{Z})$ to see that the mapping class group of the torus with one boundary component is given by the braid group B_3 on 3 strands [FM, p. 92]. This fact is a key result that allows us to link topological and algebraic results.

Topological field theories and modular functors

A d-dimensional topological field theory is a symmetric monoidal functor from (a variant of) the category of d-dimensional cobordisms to the category of vector spaces as defined in [At, Se]. The category of vector spaces can be replaced by any symmetric monoidal category. Objects of the cobordism category are closed

oriented $(d-1)$-dimensional smooth manifolds, while morphisms are classes of cobordisms, which are oriented d-dimensional smooth manifolds, possibly with boundary. A topological field theory then assigns a finite-dimensional vector space to each object of this category and a linear map to each morphism. Included in the data of a monoidal functor is a coherent isomorphism between the image of the monoidal unit in the source category—the empty $(d-1)$-dimensional manifold in this case—and the monoidal unit in the target category—the ground field K in this case. A closed oriented manifold of dimension d can be viewed as a cobordism from the empty $(d-1)$-dimensional manifold to itself. Therefore, the topological field theory assigns to every closed oriented manifold of dimension d an element in $\mathrm{Hom}_K(K,K) \cong K$, i.e. an invariant.

In low dimensions, topological field theories are well understood and we can explicitly relate them to interesting algebraic structures: First, a 1-dimensional topological field theory assigns a finite-dimensional vector space to every closed oriented 0-manifold. As the functor is monoidal, its value on any closed oriented 0-manifold can be computed from its values on a positively and a negatively oriented point, which are finite dimensional vector spaces dual to each other. For 1-manifolds, it is enough to consider the closed unit interval and the unit circle. Depending on how we regard the unit interval as a cobordism, it gives 4 different maps: The identity on the vector space V associated to the positive oriented point, the identity on the dual vector space V^*, the evaluation map, or the coevalution map. The unit circle is a cobordism from the empty set to itself. So as in the general case, it gives an invariant, in this case given by the dimension of V, compare [Lu, Example 1.1.9].

Now, we will consider a 2-dimensional topological field theory: Its value on any closed oriented 1-manifold is uniquely determined by its value on the unit circle, which is a finite-dimensional vector space A. It can be shown that all 2-dimensional cobordisms can be generated by composition and disjoint union of the following six cobordisms: the cap, the pair of pants with two ingoing and one outgoing circle, the cylinder, the pair of pants with one ingoing and two outgoing circles, the cup, and two exchanging cylinders. They give a unit map on A, a multiplication on A, the identity on A, a comultiplication on A, a counit on A, and the twist map on $A \otimes A$. The relations between these maps imply that in dimension two,

topological field theories are equivalent to commutative K-Frobenius algebras [Ab, Sec. 5, Thm. 3].

We are interested in so called oriented 3-dimensional extended topological field theories $\mathcal{Z}$ with values in the category of 2-vector spaces [Lu, Def. 1.2.6]. The category of 2-vector spaces is a strict 2-category, as defined and explained in [Lu, Def. 1.2.3 and Ex. 1.2.4]. This allows us to consider surfaces with boundaries. In particular, such a topological field theory $\mathcal{Z}$ assigns to any oriented surface $\Sigma_{g,n}$, of genus g with n labeled boundary components, a finite-dimensional vector space $\mathcal{Z}(\Sigma_{g,n}^{X_1,\dots,X_n})$. These spaces are called spaces of chiral conformal blocks, or briefly block spaces. Furthermore, it assigns to any 3-dimensional cobordism between these surfaces a linear map between the vector spaces, which is compatible with concatenation. The block spaces depend functorially on the labels $X_1,\dots,X_n$ assigned to the boundary components, which are objects of a given modular tensor category $\mathcal{C}$. We therefore see that a topological field theory indeed assigns functors to surfaces. To 3-dimensional manifolds, possibly with corners, it assigns natural transformations between these functors. In particular, to cylinders twisted by elements of the mapping class group, the topological field theory assigns natural transformations that give rise to projective representations of the pure mapping class group $\mathrm{P}\Gamma_{g,n}$ of $\Sigma_{g,n}$ on the finite-dimensional vector space $\mathcal{Z}(\Sigma_{g,n}^{X_1,\dots,X_n})$. These data obey factorization constraints; they form a structure often called a modular functor.

For every semisimple modular tensor category $\mathcal{C}$, the Reshetikhin-Turaev construction provides an extended 3-dimensional topological field theory [RT, BK, T]. At least for certain aspects of this construction, it is not necessary that the category $\mathcal{C}$ be semisimple. We can still construct a modular functor, which in the nonsemisimple case does not directly lead to a construction of a topological field theory as in the semisimple case, as explained in greater detail in [L1, p. 468] and [L2, p. 312]. To construct the modular functor, we will use the framework created by V. Lyubashenko in his articles [L1] and [L2], which uses nets and ribbon graphs as described in Paragraph 3.2.3 and Paragraph 3.2.4. As already mentioned briefly, we assume that each of the n boundary components of the compact oriented surface $\Sigma_{g,n}$ is labeled by an object X_i of $\mathcal{C}$. Associated with such a labeled surface

is the vector space

$$\mathcal{Z}(\Sigma_{g,n}^{X_1,\dots,X_n}) := \operatorname{Hom}_{\mathcal{C}}(X_n \otimes \cdots \otimes X_1, L^{\otimes g}),$$

where L denotes a certain coend of $\mathcal{C}$, which is naturally a Hopf algebra in the braided tensor category $\mathcal{C}$ and will be defined in Paragraph 3.2.1. These spaces are again called spaces of chiral conformal blocks or briefly block spaces. They depend functorially on the labels and can therefore be viewed as the value of a left exact contravariant functor $\mathcal{Z}(\Sigma_{g,n})$ on the object $(X_1, \dots, X_n) \in \mathcal{C}^n$. The block spaces carry an action of the pure mapping class group $\mathrm{P}\Gamma_{g,n}$, where the whole mapping class group $\Gamma_{g,n}$ acts on

$$\bigoplus_{\tau \in \mathbb{S}_n} \mathcal{Z}(\Sigma_{g,n}^{X_{\tau(1)},\dots,X_{\tau(n)}}) = \bigoplus_{\tau \in \mathbb{S}_n} \operatorname{Hom}_{\mathcal{C}}(X_{\tau(n)} \otimes \cdots \otimes X_{\tau(1)}, L^{\otimes g})$$

[KL, L1, L2]. The construction is compatible with gluing of surfaces. We remark that these vector spaces differ from the ones constructed by V. Lyubashenko in the order of the X_i's which will be discussed further in Paragraph 3.2.3. As the category $\mathcal{C}$ is nonsemisimple, the Hom-functor is not exact and it is therefore natural to ask whether there is a similar construction that gives an action of the mapping class group on the right derived functors in such a way that the original action is recovered in degree zero.

We will approach this question in two steps: If the genus of the surface is 1, the vector space $\mathcal{Z}(\Sigma_{1,0}) = \operatorname{Hom}_{\mathcal{C}}(\mathbb{1}, L)$ is assigned to the torus and carries an action of the group $\Gamma_{1,0} \cong \mathrm{SL}(2, \mathbb{Z})$. If we consider the modular tensor category of finite-dimensional modules over a factorizable ribbon Hopf algebra, the vector space assigned to the torus is isomorphic to the center of the Hopf algebra as an $\mathrm{SL}(2, \mathbb{Z})$-module. As already mentioned above, the center of an algebra is the zeroth Hochschild cohomology group and the first goal of this thesis is to construct a projective action of $\mathrm{SL}(2, \mathbb{Z})$ on all Hochschild cohomology groups, which will be done explicitly in Chapter 2. In Chapter 3, we use a more abstract approach and work with an arbitrary modular tensor category to construct a cochain complex for any compact oriented surface such that its pure mapping class group acts projectively on the cohomology groups of this cochain complex, as stated in

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Corollary 3.3.6. In degree zero, the original action is recovered and it is therefore appropriate to call the cohomology groups derived block spaces. We will now discuss the main results of this thesis in greater detail:

Main results

The modular group $\mathrm{SL}(2,\mathbb{Z})$ acts projectively on the center of a factorizable ribbon Hopf algebra (see for example [CW1, CW2, Ke, KL, LM, T]). This action can be described explicitly by using the Drinfel'd map, the Radford map and the antipode of the Hopf algebra [SZ]. As the center is the zeroth Hochschild cohomology group of the Hopf algebra, it is natural to ask whether there is a corresponding action on the higher cohomology groups. In Chapter 2, we answer this question affirmatively by showing that the modular group acts projectively on all Hochschild cohomology groups. This action will be constructed quite explicitly: In Section 2.3, we will generalize the Radford map, the Drinfel'd map and the antipode to cochain maps of Hochschild cochain complexes. We will use these maps to generalize the action of the modular group on the center to an action on all Hochschild cohomology groups of our factorizable ribbon Hopf algebra in Section 2.4. In particular, we obtain explicit formulas for the action of the generators $\mathfrak{s} = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ and $\mathfrak{t} = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ of $\mathrm{SL}(2,\mathbb{Z})$, which are stated in Corollary 2.4.3 and Definition 2.4.4.

This result is a special case of a more general result on mapping class groups. The general case will be addressed in Chapter 3, where we work with an arbitrary modular tensor category $\mathcal{C}$. As we already said, we assume that each of the n boundary components of the compact oriented surface $\Sigma_{g,n}$ is labeled by an object X_i of $\mathcal{C}$. We will construct a cochain complex with the property that the pure mapping class group $\mathrm{P}\Gamma_{g,n}$ acts on all its cohomology groups

$$\mathcal{Z}^\bullet(\Sigma_{g,n}^{X_1,\ldots,X_n}) := \mathrm{Ext}_{\mathcal{C}}^\bullet(X_n \otimes \cdots \otimes X_1, L^{\otimes g}),$$

and the mapping class group $\Gamma_{g,n}$ acts on

$$\bigoplus_{\tau \in \mathbb{S}_n} \mathcal{Z}^\bullet(\Sigma_{g,n}^{X_{\tau(1)},\ldots,X_{\tau(n)}}) = \bigoplus_{\tau \in \mathbb{S}_n} \mathrm{Ext}_{\mathcal{C}}^\bullet(X_{\tau(n)} \otimes \cdots \otimes X_{\tau(1)}, L^{\otimes g}).$$

In degree zero, we recover the original action defined by V. Lyubashenko, as recorded in Corollary 3.3.6.

Our construction in Paragraph 3.3.2, is inspired by the principle of 'propagation of vacua' (cf. [TUY, Par. 2.2, p. 476]). This technique introduces an additional boundary component on the surface that, if labeled with the monoidal unit of the category, leads to block spaces $\mathcal{Z}(\Sigma_{g,n+1}^{X_1,\ldots,X_n,\mathbb{1}})$ that are canonically isomorphic to the block spaces without the additional boundary component. In our construction, the new boundary component serves as the position where we insert a projective resolution

$$\mathbb{1} \longleftarrow P_0 \longleftarrow P_1 \longleftarrow \cdots$$

of the monoidal unit. By the functoriality of the construction, an element of the group $\Gamma_{g,n+1}(n+1)$ fixing the additional $(n+1)$-st boundary component, gives rise to a cochain map between the cochain complexes $\mathrm{Hom}_{\mathcal{C}}(P_i \otimes X_n \otimes \cdots \otimes X_1, L^{\otimes g})$ and $\mathrm{Hom}_{\mathcal{C}}(P_i \otimes X_{\tau(n)} \otimes \cdots \otimes X_{\tau(1)}, L^{\otimes g})$ and so induces a homomorphism

$$\mathrm{Ext}^m(X_n \otimes \cdots \otimes X_1, L^{\otimes g}) \to \mathrm{Ext}^m(X_{\tau(n)} \otimes \cdots \otimes X_{\tau(1)}, L^{\otimes g})$$

(cf. Paragraph 3.3.2). Here, $\tau \in \mathbb{S}_{n+1}$ is the inverse of the permutation associated with the element in the mapping class group determined by its action on the boundary components. The main point of the argument will be that the additional boundary component can be closed again when passing to cohomology: Our main result is Theorem 3.3.5, where we show that we can associate with a mapping class in $\Gamma_{g,n}$ a projective class of morphisms not only between the Hom-spaces, but rather between the derived spaces $\mathrm{Ext}^m(X_n \otimes \cdots \otimes X_1, L^{\otimes g})$ and $\mathrm{Ext}^m(X_{\tau(n)} \otimes \cdots \otimes X_{\tau(1)}, L^{\otimes g})$ by choosing a preimage in $\Gamma_{g,n+1}(n+1)$ under the map that caps off the $(n+1)$-st boundary component with a full disk. To establish this result, we proceed in two steps, using standard techniques from the theory of mapping class groups, namely the capping sequence in the first step and the Birman sequence in the second step. An important consequence of Theorem 3.3.5 is Corollary 3.3.6, where we show that on $\mathrm{Ext}_{\mathcal{C}}^{\bullet}(X_n \otimes \cdots \otimes X_1, L^{\otimes g})$, the action of $P\Gamma_{g,n+1}(n+1)$ factors to an action of $P\Gamma_{g,n}$.

In Section 3.3.4, we show that the mapping class group actions obtained in Chapter 3 generalize the projective $SL(2, \mathbb{Z})$-representation on the Hochschild cohomology groups obtained in Chapter 2.

The ultimate goal of our construction in Chapter 3 would be to obtain a 'derived topological field theory' associated to any modular tensor category. This picture is not yet complete:

First, in order to have a true modular functor with values in a category of (co)chain complexes, the action of $\Gamma_{g,n+1}(n+1)$ on the cochain complex should factor to $\Gamma_{g,n}$ up to coherent homotopy. For the torus, this goal has been achieved in [SW]. Moreover, a homotopy coherent framework will be needed to formulate and prove gluing properties for the complexes.

Second, in order to extend the derived modular functor to a derived topological field theory, one would like to repeat the Reshetikhin-Turaev construction for any 3-manifold with boundaries in this derived setting. We note that there has been remarkable progress in using modified traces to give a nonsemisimple version of the Reshetikhin-Turaev construction [DGGPR, GPV], but this requires that at least one decoration X_i is a projective object, which shows similarities to our construction, as we label our additional boundary component by a projective resolution.

Publications

The chapters of this thesis are based on the following publications:

Chapter 2 S. Lentner, S.N. Mierach, C. Schweigert, Y. Sommerhäuser, *Hochschild cohomology and the modular group*, J. Algebra 507 (2018), 400–420

Chapter 3 S. Lentner, S.N. Mierach, C. Schweigert, Y. Sommerhäuser, *Hochschild cohomology, modular tensor categories, and mapping class groups, I. General Theory*, Preprint, `arXiv:2003.06527`, 2020

Chapter 1

Preliminaries

We will start with fixing some notation. Throughout this thesis, we will work over an arbitrary base field that is denoted by K. We will impose further restrictions on K in Section 1.2 and Chapter 3. In general, we compose mappings and morphisms in a category from right to left, i.e., we have $(g \circ f)(x) = g(f(x))$, so that f is applied first. In contrast, we concatenate paths in the fundamental group from left to right, i.e., in the concatenation $\kappa\gamma$, the path κ is traced out first, while the path γ is traced out afterwards. We will use the symbol $\mathbb{S}_n$ for the symmetric group in n letters.

1.1 Hochschild cohomology

In this section, we briefly review the Hochschild cohomology of an algebra A with coefficients in an A-bimodule M, as found for example in [CE] or [W]. We then show in Proposition 1.1.2 that two particular cochain maps are homotopic, a fact that will be important later for the verification of the relations that appear in our presentation of the modular group.

Throughout this section, all unadorned tensor products are taken over our base field K. The dual of a vector space V is denoted by $V^* := \operatorname{Hom}_K(V, K)$.

We begin by briefly recalling the approach to Hochschild cohomology via the standard resolution. Further details can be found for example in [CE, Chap. IX] or [W, Chap. 9]. We consider an associative algebra A over our base field K and an A-bimodule M. As in [CE, Chap. IX, § 3, p. 167], we assume that the left and the right action of A on M become equal when restricted to K, so that an A-bimodule is the same as a module over $A \otimes A^{\mathrm{op}}$. Here A^{op} denotes the opposite algebra, in which the product is modified by interchanging the factors.

The n-th Hochschild cohomology group $HH^n(A, M)$ of the algebra A with coefficients in the bimodule M is, by definition, the group $\mathrm{Ext}^n_{A \otimes A^{\mathrm{op}}}(A, M)$. To make this definition set-theoretically unambiguous, it is necessary to fix a resolution of A as an A-bimodule. We will use here the standard resolution from [CE, Chap. IX, § 6, p. 174f], which is called the bar resolution in [L, Par. 1.1.12, p. 12] and which we denote by S: For $n \geq 1$, we define $S_n := A \otimes A^{\otimes n} \otimes A$ and extend this definition to $n = 0$ and $n = -1$ by setting $S_0 := A \otimes A$ and $S_{-1} := A$. All these spaces are left A-modules by acting on the leftmost tensorand and right A-modules by acting on the rightmost tensorand. For $n \geq 0$, we define the boundary operator $d_n \colon S_n \to S_{n-1}$ via

$$d_n(a_0 \otimes a_1 \otimes \cdots \otimes a_n \otimes a_{n+1}) := \sum_{i=0}^{n} (-1)^i \, a_0 \otimes a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n \otimes a_{n+1}.$$

Then

$$A \xleftarrow{d_0} S_0 \xleftarrow{d_1} S_1 \xleftarrow{d_2} S_2 \xleftarrow{d_3} \cdots$$

is a free resolution of A with augmentation d_0. If we apply the contravariant functor $\mathrm{Hom}_{A \otimes A^{\mathrm{op}}}(-, M)$ to this resolution, we can, as explained in [CE, loc. cit.], make for $n \geq 1$ the identifications

$$\mathrm{Hom}_{A \otimes A^{\mathrm{op}}}(S_n, M) = \mathrm{Hom}_{A \otimes A^{\mathrm{op}}}(A \otimes A^{\otimes n} \otimes A, M) \cong \mathrm{Hom}_K(A^{\otimes n}, M)$$

and for $n = 0$ the identification $\mathrm{Hom}_{A \otimes A^{\mathrm{op}}}(A \otimes A, M) \cong M$. In this way, we obtain a cochain complex $C(A, M)$

$$M \xrightarrow{d^0} \mathrm{Hom}_K(A, M) \xrightarrow{d^1} \mathrm{Hom}_K(A \otimes A, M) \xrightarrow{d^2} \cdots$$

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with cochain groups $C^n(A, M) = \text{Hom}_K(A^{\otimes n}, M)$ for $n > 0$ and $C^0(A, M) = M$. Its coboundary operator $d^{n-1}\colon C^{n-1}(A, M) \to C^n(A, M)$, also called the differential, takes in view of this identification the form

$$d^{n-1}(f)(a_1 \otimes \cdots \otimes a_n) = a_1.f(a_2 \otimes \cdots \otimes a_n)$$
$$+ \sum_{i=1}^{n-1}(-1)^i f(a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n)$$
$$+ (-1)^n f(a_1 \otimes \cdots \otimes a_{n-1}).a_n.$$

The n-th cohomology group of this cochain complex is then an explicit realization of the Hochschild cohomology group $HH^n(A, M)$. We note that for finite-dimensional separable algebras, and therefore in particular for finite-dimensional semisimple algebras over fields of characteristic zero, the higher Hochschild cohomology groups $HH^n(A, M)$ for $n \geq 1$ vanish, as shown for example in [CE, Chap. IX, Thm. 7.10, p. 179].

The following special cases will be particularly important in the sequel:

Example 1.1.1 For the zeroth Hochschild cohomology group, we find

$$HH^0(A, M) = \ker(d^0\colon C^0(A, M) \to C^1(A, M))$$
$$= \{m \in M \mid a.m = m.a \text{ for all } a \in A\},$$

a subspace that is often called the space of invariants of M. For $M = A$, where the bimodule structure is given by multiplication, we get in particular that

$$HH^0(A, A) = Z(A),$$

the center of the algebra A.

For any bimodule M, the dual space $M^* = \text{Hom}_K(M, K)$ is again a bimodule with respect to the action $(a.\varphi.b)(m) = \varphi(b.m.a)$. According to the preceding computation, we then have

$$HH^0(A, M^*) = \{\varphi \in M^* \mid \varphi(m.a) = \varphi(a.m) \text{ for all } a \in A \text{ and all } m \in M\}.$$

By composition on the left, any bimodule homomorphism $g\colon M \to N$ induces a homomorphism

$$g_*\colon C^n(A, M) \to C^n(A, N), \ f \mapsto g \circ f$$

between the cochain groups, where in general we use a lower star for the map induced by composition on the left and an upper star for the map induced by composition on the right. Because these homomorphisms g_* commute with the coboundary operators, they can be combined to a cochain map. An element $c \in Z(A)$ in the center of A gives rise to two natural choices for g on every bimodule M, namely the left and right actions

$$l_c^M\colon M \to M, \ m \mapsto c.m \qquad \text{and} \qquad r_c^M\colon M \to M, \ m \mapsto m.c.$$

The induced maps on the Hochschild cochain complex are related as follows:

Proposition 1.1.2 The cochain maps $(l_c^M)_*$ and $(r_c^M)_*$ are homotopic.

Proof. We give a proof that works not only for the standard resolution, but more generally for any projective resolution

$$A \xleftarrow{\ \xi\ } P_0 \xleftarrow{\ d_1\ } P_1 \xleftarrow{\ d_2\ } P_2 \xleftarrow{\ d_3\ } \cdots$$

of A as an A-bimodule, which we briefly denote by P. As already pointed out above, the fact that c is central implies that the maps $l_c^{P_n}$ and $r_c^{P_n}$ are bimodule homomorphisms. Because ξ and the boundary operators d_n are bimodule homomorphisms, the maps $l_c^{P_n}$ commute with them, and therefore lift the left multiplication of c on A to the entire resolution:

$$
\begin{array}{ccccccc}
A & \xleftarrow{\ \xi\ } & P_0 & \xleftarrow{\ d_1\ } & P_1 & \xleftarrow{\ d_2\ } & \cdots \\
\downarrow{\scriptstyle l_c^A} & & \downarrow{\scriptstyle l_c^{P_0}} & & \downarrow{\scriptstyle l_c^{P_1}} & & \\
A & \xleftarrow{\ \xi\ } & P_0 & \xleftarrow{\ d_1\ } & P_1 & \xleftarrow{\ d_2\ } & \cdots
\end{array}
$$

Analogously, we can lift the right multiplication of c on A to the entire resolution:

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$$
\begin{array}{ccccccc}
A & \xleftarrow{\ \xi\ } & P_0 & \xleftarrow{\ d_1\ } & P_1 & \xleftarrow{\ d_2\ } & \cdots \\
\Big\downarrow{r_c^A} & & \Big\downarrow{r_c^{P_0}} & & \Big\downarrow{r_c^{P_1}} & & \\
A & \xleftarrow{\ \xi\ } & P_0 & \xleftarrow{\ d_1\ } & P_1 & \xleftarrow{\ d_2\ } & \cdots
\end{array}
$$

Because c is central, we have $r_c^A = l_c^A$. Therefore, the comparison theorem found in [ML1, Chap. III, Thm. 6.1, p. 87] or [W, Thm. 2.2.6, p. 35] yields that the chain maps $l_c^P = (l_c^{P_n})$ and $r_c^P = (r_c^{P_n})$ are chain homotopic.

The contravariant functor $\operatorname{Hom}_{A \otimes A^{\mathrm{op}}}(-, M)$ coming from our bimodule M turns this homotopy of chain maps into a homotopy of cochain maps, so that we get that the cochain maps $(l_c^P)^*$ and $(r_c^P)^*$ are cochain homotopic. But we have $(l_c^P)^* = (l_c^M)_*$: For $f \in \operatorname{Hom}_{A \otimes A^{\mathrm{op}}}(P_n, M)$ and $p \in P_n$, we have

$$
((l_c^M)_*(f))(p) = l_c^M(f(p)) = c.f(p) = f(c.p) = f(l_c^{P_n}(p)) = ((l_c^{P_n})^*(f))(p).
$$

A similar computation shows that $(r_c^P)^* = (r_c^M)_*$, so that our assertion follows. $\quad\square$

In the case where $P = S$ is the standard resolution, there is an explicit chain homotopy that can be used at the point where we invoked the comparison theorem in the proof above: If we define for $n \geq -1$ the map

$$
k_n \colon S_n = A^{\otimes(n+2)} \to S_{n+1} = A^{\otimes(n+3)}
$$

via

$$
k_n(a_0 \otimes \cdots \otimes a_{n+1}) := \sum_{j=0}^{n} (-1)^j (a_0 \otimes \cdots \otimes a_j \otimes c \otimes a_{j+1} \otimes \cdots \otimes a_{n+1}),
$$

so that $k_{-1}(a_0) = 0$, we have that $l_c^{S_0} - r_c^{S_0} = k_{-1} \circ d_0 + d_1 \circ k_0$ and also

$$
l_c^{S_n} - r_c^{S_n} = k_{n-1} \circ d_n + d_{n+1} \circ k_n
$$

for $n > 0$. This follows by induction from the recursion formulas

$$
k_n(x \otimes a_{n+1}) = k_{n-1}(x) \otimes a_{n+1} + (-1)^n x \otimes c \otimes a_{n+1}
$$

and
$$d_n(x \otimes a_{n+1}) = d_{n-1}(x) \otimes a_{n+1} + (-1)^n x.a_{n+1}$$

for $x \in A^{\otimes(n+1)}$, as we then have

$$(k_{n-1} \circ d_n)(x \otimes a_{n+1}) = k_{n-1}(d_{n-1}(x) \otimes a_{n+1}) + (-1)^n k_{n-1}(x.a_{n+1})$$
$$= k_{n-2}(d_{n-1}(x)) \otimes a_{n+1} + (-1)^{n-1}d_{n-1}(x) \otimes c \otimes a_{n+1} + (-1)^n k_{n-1}(x).a_{n+1}$$

as well as

$$(d_{n+1} \circ k_n)(x \otimes a_{n+1}) = d_{n+1}(k_{n-1}(x) \otimes a_{n+1}) + (-1)^n d_{n+1}(x \otimes c \otimes a_{n+1})$$
$$= d_n(k_{n-1}(x)) \otimes a_{n+1} + (-1)^{n+1}k_{n-1}(x).a_{n+1} + (-1)^n d_{n+1}(x \otimes c \otimes a_{n+1}).$$

By adding these two expressions and using the induction hypothesis, we obtain

$$(k_{n-1} \circ d_n + d_{n+1} \circ k_n)(x \otimes a_{n+1}) = c.x \otimes a_{n+1} - x.c \otimes a_{n+1}$$
$$+ (-1)^{n-1}d_{n-1}(x) \otimes c \otimes a_{n+1} + (-1)^n d_{n+1}(x \otimes c \otimes a_{n+1}).$$

Because

$$d_{n+1}(x \otimes c \otimes a_{n+1}) = d_n(x \otimes c) \otimes a_{n+1} + (-1)^{n+1} x \otimes ca_{n+1}$$
$$= d_{n-1}(x) \otimes c \otimes a_{n+1} + (-1)^n x.c \otimes a_{n+1} + (-1)^{n+1} x \otimes a_{n+1}c,$$

this gives the assertion.

As pointed out in the proof, applying the contravariant functor $\mathrm{Hom}_{A \otimes A^{\mathrm{op}}}(-, M)$ now gives a homotopy

$$k_n^* \colon \mathrm{Hom}_{A \otimes A^{\mathrm{op}}}(A^{\otimes(n+3)}, M) \to \mathrm{Hom}_{A \otimes A^{\mathrm{op}}}(A^{\otimes(n+2)}, M),$$

which, under the identifications described at the beginning of the section, yields the homotopy $h^{n+1} \colon C^{n+1}(A, M) \to C^n(A, M)$ explicitly given by

$$h^{n+1}(f)(a_1 \otimes \cdots \otimes a_n) = \sum_{j=0}^{n}(-1)^j f(a_1 \otimes \cdots \otimes a_j \otimes c \otimes a_{j+1} \otimes \cdots \otimes a_n).$$

We would like to emphasize that neither the preceding proposition nor the explicit form of the homotopy are new; rather, as explained in [F, Prop. 2.2, p. 655], this homotopy can be applied more generally in a situation where c is not necessarily central. As mentioned there, a related computation can be found in [CE, Chap. XIII, Prop. 6.1, p. 278]. Moreover, a similar homotopy for Hochschild homology is described in [L, Par. 1.1.5, p. 10; Exerc. 1.1.2, p. 15]. We note that generalizations of this proposition can be found in the literature, for example in [SS, Cor. 1.3, p. 709]. However, we will only need the facts derived above in the sequel.

1.2 Hopf algebras and tensor categories

In this section, we will introduce the type of categories we will work with, namely, modular tensor categories and give an important example.

1.2.1 Finiteness

As in [KL], we assume that K is a perfect field. This assures that together with the assumptions that follow, the Deligne product $\mathcal{C} \boxtimes \mathcal{D}$ is again a tensor category [De, Proposition 5.17] and that the functor $\mathcal{C} \boxtimes \mathcal{C} \to \mathcal{C}$ induced by the tensor product $\otimes \colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is exact [De, Proposition 5.13]. The perfectness of K is explicitly used in Lemma 3.3.1 but in the case that $\mathcal{C}$ is the category of finite-dimensional representations of a factorizable ribbon Hopf algebra, we give a proof that does not rely on this fact.

Furthermore, let $\mathcal{C}$ be an essentially small abelian K-linear category. Here, $\mathcal{C}$ is called essentially small if there is a set, not a class, of objects with the property that every object in $\mathcal{C}$ is isomorphic to an object in this set. We assume that $\mathcal{C}$ is finite in the sense of [EGNO, Def. 1.8.6, p. 9], i.e., that it has finite-dimensional spaces of morphisms, that every object has finite length, that it has enough projectives, and that there are only finitely many isomorphism classes of simple objects. That it has enough projectives means that for every object X,

there is an epimorphism $f\colon P \to X$ from a projective object P (cf. [Mi, Sec. II.14, p. 70]). A standard, but not completely trivial argument shows that, under the assumption of finite length, the epimorphism f can be chosen to be essential, where an epimorphism $f\colon P \to X$ is called essential if, given a morphism $g\colon Y \to P$, we can conclude that g is an epimorphism if $f \circ g$ is an epimorphism. An essential epimorphism $f\colon P \to X$ from a projective object P is usually called a projective cover of X; it follows from the assumption of finite length that this definition is equivalent to the one given in [EGNO, Def. 1.6.6, p. 6].

As explained in [DSS, Prop. 1.4, p. 3], an essentially small abelian K-linear category is finite if and only if it is equivalent to the category of finite-dimensional modules over a finite-dimensional K-algebra (cf. also [EGNO, p. 9f]). We note that finite categories are called bounded categories in [KL].

We will assume that $\mathcal{C}$ is a strict tensor category in the sense of [Ka, Def. XI.2.1, p. 282], so that we have in particular a tensor product functor $\otimes$ from $\mathcal{C} \times \mathcal{C}$ to $\mathcal{C}$ and a unit object $\mathbb{1}$. We note that tensor categories are called monoidal categories in [EGNO, Def. 2.2.8, p. 25]; the term 'tensor category' is used there for a category that satisfies additional restrictions (cf. [EGNO, Def. 4.1.1, p. 65]), all of which will be satisfied in our situation, as we start to discuss now: First, we require the tensor product to be K-bilinear. Second, we require that the unit object $\mathbb{1}$ is simple and that $\operatorname{End}(\mathbb{1})$ is one-dimensional over K.

In addition, we require that every object X has a left dual X^* (cf. [EGNO, Def. 2.10.1, p. 40]; [Ka, Def. XIV.2.1, p. 342]). The corresponding evaluation and coevaluation morphisms will be denoted by

$$\operatorname{ev}_X\colon X^* \otimes X \to \mathbb{1} \qquad \text{and} \qquad \operatorname{coev}_X\colon \mathbb{1} \to X \otimes X^*.$$

We also assume that $\mathcal{C}$ is braided with braiding

$$c_{X,Y}\colon X \otimes Y \to Y \otimes X$$

(cf. [EGNO, Def. 8.1.1, p. 195]; [Ka, Def. XIII.1.1, p. 315]) and that it carries a

ribbon structure

$$\theta_X \colon X \to X$$

(cf. [EGNO, Def. 8.10.1, p. 216]; [Ka, Def. XIV.3.2, p. 349]), which by definition satisfies the equations

$$\theta_{X \otimes Y} = (\theta_X \otimes \theta_Y) \circ c_{Y,X} \circ c_{X,Y}$$

and $\theta_{X^*} = \theta_X^*$. Under these assumptions, every object X does not only have a left dual X^*, but also a right dual ${}^*X$, which is defined by using the same object ${}^*X := X^*$, but introducing the evaluation morphism

$$\mathrm{ev}'_X := \mathrm{ev}_X \circ c_{X,X^*} \circ (\theta_X \otimes \mathrm{id}_{X^*}) \colon X \otimes {}^*X \to \mathbb{1}$$

and the coevaluation morphism

$$\mathrm{coev}'_X := (\mathrm{id}_{X^*} \otimes \theta_X) \circ c_{X,X^*} \circ \mathrm{coev}_X \colon \mathbb{1} \to {}^*X \otimes X$$

(cf. [Ka, Prop. XIV.3.5, p. 352]). The axioms of left and right duality imply that for every object X, the functor $X \otimes -$ has the left adjoint $X^* \otimes -$ and the right adjoint ${}^*X \otimes -$, so that

$$\mathrm{Hom}(X^* \otimes Y, Z) \cong \mathrm{Hom}(Y, X \otimes Z) \quad \text{and} \quad \mathrm{Hom}(X \otimes Y, Z) \cong \mathrm{Hom}(Y, {}^*X \otimes Z).$$

Similarly, the functor $- \otimes X$ has the right adjoint $- \otimes X^*$ and the left adjoint $- \otimes {}^*X$, so that

$$\mathrm{Hom}(Y \otimes X, Z) \cong \mathrm{Hom}(Y, Z \otimes X^*) \quad \text{and} \quad \mathrm{Hom}(Y \otimes {}^*X, Z) \cong \mathrm{Hom}(Y, Z \otimes X)$$

(cf. [EGNO, Prop. 2.10.8, p. 42]; [Ka, Prop. XIV.2.2, p. 343f and p. 346f]). This implies that the functors $X \otimes -$ and $- \otimes X$ are exact (cf. [HS, Chap. II, Thm. 7.7, p. 68]; [P, Par. 2.7, Satz 3, p. 72]). This in turn implies that $X \otimes P$ and $P \otimes X$ are projective if P is projective (cf. [HS, Chap. II, Prop. 10.2, p. 82]). We will also need the following related result on adjunctions:

Proposition 1.2.1 Suppose that $F \colon \mathcal{C} \to \mathcal{D}$ is a K-linear functor between finite K-linear categories. Then F has a right adjoint if and only if it is right exact.

Proof. If F has a right adjoint, it is right exact (cf. [HS, Chap. II, Thm. 7.7, p. 68]). As pointed out above, we can assume for the converse that $\mathcal{C}$ and $\mathcal{D}$ are the categories of finite-dimensional left modules over finite-dimensional algebras A and B, respectively. Then the Eilenberg-Watts theorem (cf. [Ro2, Thm. 5.45, p. 261]; [I, Thm. 2.4, p. 677]) states that F is naturally equivalent to the functor $M \otimes_A -$ for a B-A-bimodule M. But by the standard adjunction (cf. [Ro2, Thm. 2.76, p. 93]), this functor has the right adjoint $\mathrm{Hom}_B(M, -)$. $\qquad\square$

A different proof of this proposition that connects it to the adjoint functor theorem is given in [DSS, Cor. 1.9, p. 6].

An object X is called transparent if $c_{Y,X} \circ c_{X,Y} = \mathrm{id}_{X \otimes Y}$ for all objects $Y \in \mathcal{C}$ (cf. [B, p. 224]). Transparent objects are called central in [Mue, Rem. 2.10, p. 296]. The full subcategory consisting of all transparent objects is called the Müger center of $\mathcal{C}$.

The unit object is transparent (cf. [EGNO, Exerc. 8.1.6, p. 196]; [Ka, Prop. XIII.1.2, p. 316]), and the direct sum of two transparent objects is transparent. Therefore, an object that is isomorphic to a finite direct sum of copies of the unit object is contained in the Müger center. We say that $\mathcal{C}$ is modular if the Müger center does not contain any other objects. In this definition, we do not require that $\mathcal{C}$ be semisimple. In the semisimple case, this condition is equivalent to the invertibility of the modular S-matrix (cf. [EGNO, Prop. 8.20.12, p. 243]; [Mue, Cor. 2.16, p. 297]), which is in this case usually taken as the definition of modularity (cf. [EGNO, Def. 8.13.4, p. 224]).

1.2.2   Factorizable Hopf algebras

A fundamental example for categories with the properties that we have just listed are the categories of finite-dimensional representations of factorizable ribbon Hopf

algebras. This will be our main example throughout the thesis, especially in Chapter 2. Suppose that A is a Hopf algebra with coproduct Δ, counit ε, and antipode S. For the coproduct, we will use the sigma notation of R. Heyneman and M. Sweedler in the modified form $\Delta(a) = a_{(1)} \otimes a_{(2)}$. If we take for $\mathcal{C}$ the category of finite-dimensional left A-modules, it is not only a finite K-linear abelian category, but also a tensor category, where the tensor product of two A-modules X and Y becomes an A-module via

$$a.(x \otimes y) := a_{(1)}.x \otimes a_{(2)}.y$$

for $x \in X$, $y \in Y$, and $a \in A$. The unit object $\mathbb{1}$ is the base field K, endowed with the trivial A-module structure coming from the counit. The left dual X^* of X is the dual vector space $X^* := \operatorname{Hom}_K(X, K)$, endowed with structures described in [Ka, Examp. XIV.2.1, p. 347]. Although this category is not strict, our considerations below apply to this category, because we have the notion of the tensor product $X_1 \otimes X_2 \otimes \cdots \otimes X_n$ of n vector spaces or A-modules, which is not built as an iteration of the tensor product of two vector spaces and therefore does not depend on the insertion of parentheses. The tensor products that appear below have to be understood in this way.

We assume that A is quasitriangular with R-matrix $R = R^1 \otimes R^2 \in A \otimes A$, where the notation follows the spirit of the notation for the coproduct above and writes this tensor, which is in general not decomposable, as if it were decomposable (cf. [Ka, Def. VIII.2.1, p. 173]). From the R-matrix, we obtain a braiding on $\mathcal{C}$ via the formula

$$c_{X,Y}(x \otimes y) := R^2.y \otimes R^1.x$$

for $x \in X$ and $y \in Y$ (cf. [Ka, Prop. XIII.1.2, p. 318]). From the R-matrix, we also obtain the monodromy matrix $Q := R'R$, where $R' := R^2 \otimes R^1$. As in the case of the R-matrix, we write $Q = Q^1 \otimes Q^2$. The Hopf algebra A is called factorizable if the map

$$\Phi \colon A^* \to A, \quad \varphi \mapsto (\operatorname{id}_A \otimes \varphi)(Q)$$

is bijective (cf. [SZ, Par. 3.2, p. 26] and the references given there). Note that this condition implies that A is finite-dimensional. We will require that A be

factorizable, which is equivalent to the requirement that the category $\mathcal{C}$ be modular, as we will explain in Paragraph 3.2.1.

Furthermore, we assume that A contains a ribbon element, i.e., a nonzero central element v that satisfies

$$\Delta(v) = Q(v \otimes v) \qquad \text{and} \qquad S(v) = v$$

(cf. [SZ, Par. 4.3, p. 37]; note the difference to [Ka, Def. XIV.6.1, p. 361]). Our ribbon element gives rise to a ribbon structure of $\mathcal{C}$ by defining

$$\theta_X \colon X \to X, \ x \mapsto v.x$$

(cf. [Ka, Prop. XIV.6.2, p. 361]).

1.3   Mapping class groups

In this section, we will review some facts about surfaces and consider their fundamental group. We will introduce their mapping class groups and important mapping class group elements, i.e. Dehn twists and braidings.

1.3.1   The classification of surfaces

The principal result of the classification of surfaces asserts that every compact, connected, orientable, smooth 2-dimensional manifold is diffeomorphic to the connected sum Σ_g of a sphere with $g \geq 0$ tori, where it is understood that a diffeomorphism is arbitrarily often differentiable, and smoothness is understood in the same way. The number g of the attached tori, which are often called handles, is uniquely determined by the surface and is called its genus. If we include manifolds with boundary, then the classification theorem asserts that a compact, connected, orientable, smooth 2-dimensional manifold with boundary is diffeomorphic to Σ_g with $n \geq 0$ open disks removed, where the disks have to satisfy the requirement that their closures do not intersect. We will denote this surface by $\Sigma_{g,n}$, so

that $\Sigma_g = \Sigma_{g,0}$. If the boundary is not empty, it is the finite disjoint union of n connected components, called the boundary components. We assume that on each of the boundary components, a point has been distinguished, or marked, as we also say. The Euler characteristic of $\Sigma_{g,n}$ is $\chi(\Sigma_{g,n}) = 2 - 2g - n$.. By assumption, the surfaces above are orientable, but when we refer to $\Sigma_{g,n}$, we will always assume that an orientation has been chosen. In the pictures of $\Sigma_{g,n}$ that we will encounter below, $\Sigma_{g,n}$ will be drawn embedded into 3-dimensional space, in which case we will assume that the orientation is represented by the outward-pointing normal vector field.

In the standard proof of the classification theorem, the surface is realized not as a connected sum, but rather as a quotient space of a polygon whose edges are labeled in a certain normal form. In the normal form for the closed surface Σ_g, the edges are labeled consecutively as

$$\alpha_1, \beta_1, \alpha_1^{-1}, \beta_1^{-1}, \alpha_2, \beta_2, \alpha_2^{-1}, \beta_2^{-1}, \ldots, \alpha_g, \beta_g, \alpha_g^{-1}, \beta_g^{-1},$$

while for the surface $\Sigma_{g,n}$ with boundary, the edges are labeled consecutively as

$$\alpha_1, \beta_1, \alpha_1^{-1}, \beta_1^{-1}, \ldots, \alpha_g, \beta_g, \alpha_g^{-1}, \beta_g^{-1}, \xi_1, \rho_1, \xi_1^{-1}, \ldots, \xi_n, \rho_n, \xi_n^{-1}.$$

In the case $g = n = 3$, this polygon has the form
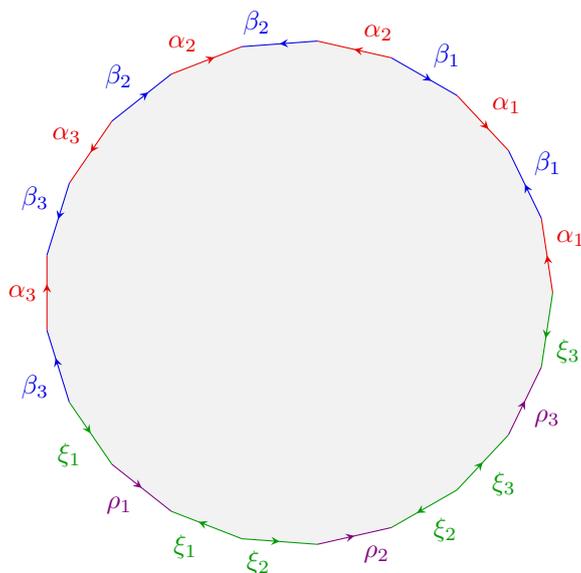
where the inverse signs have been depicted by reversing the orientation of the corresponding arrow.

In the case of a closed surface, the quotient space is then realized in such a way that all the vertices of the polygon become a single point x, while the edges labeled α_i and β_i are, respectively, identified with their counterparts labeled α_i^{-1} and β_i^{-1}, which are traced out in the opposite direction, as we use the general convention that $\gamma^{-1}(t) := \gamma(1 - t)$ for a path defined on the unit interval. In the case of the surface $\Sigma_{g,n}$ with boundary, in addition an edge labeled ξ_j is identified in the same way with its counterpart labeled ξ_j^{-1}. The edges labeled ρ_j then become closed curves in the quotient that represent the n boundary components (cf. [ST, Kap. 6, § 40, p. 142ff], [Mas, Chap. I, Sec. 10, p. 37ff]). In this case, not all vertices of the polygon map to the same point x, but only those that are not the start point or the end point of one of the edges $\rho_1, \ldots, \rho_n$. The start point of ρ_j is instead identified with the end point of ρ_j and in this way yields the marked point on the j-th boundary component.

If we carry out only the second identification in the case depicted above, i.e., the identification of ξ_j with ξ_j^{-1}, we obtain the following picture of the intermediate stage:
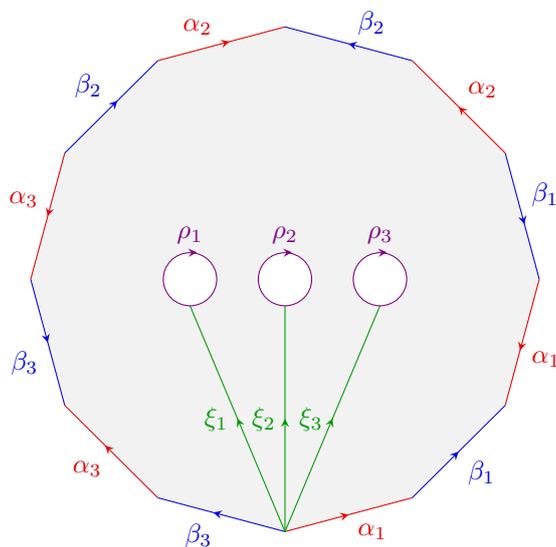
When we carry out all the identifications, the edges labeled α_i and β_i become in the quotient the closed curves that appear in the following picture:
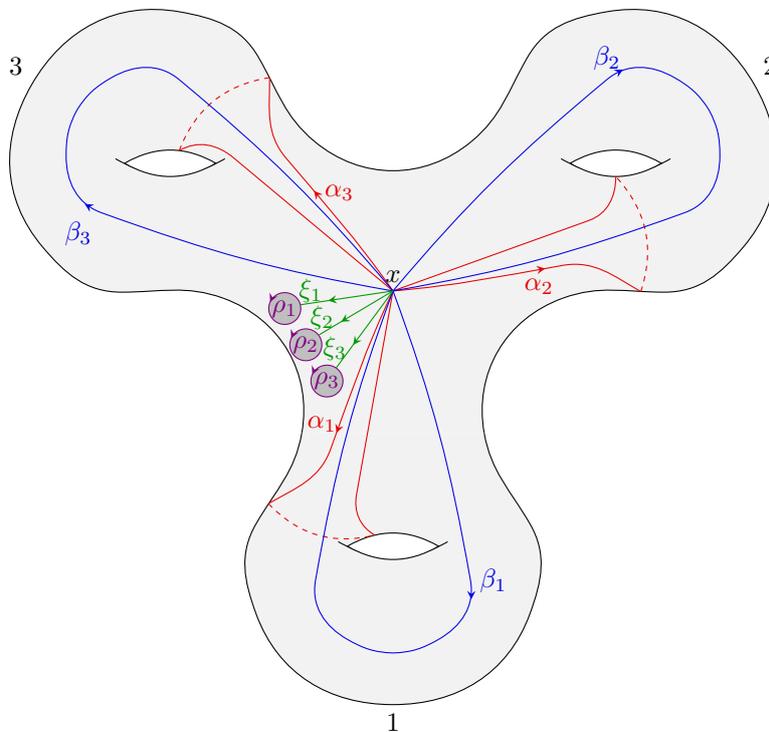
This realization of the surface is called the polygon model of the surface, as opposed to the connected sum model that we briefly described at the beginning of the paragraph. Note that, in order to obtain the outward-pointing orientation of the surface mentioned above, the surface of the polygon needs to be oriented by a normal vector field that is orthogonal to the page and is pointing away from the reader, which is not the standard orientation of the plane used elsewhere in the text, for example in Paragraph 3.1.1. Furthermore, in this model the boundary curves ρ_j do not carry the orientation that is induced by the surface on its boundary, but rather the opposite orientation.

1.3.2 The fundamental group

In the polygon model of a closed surface, we use the point x in the quotient that all vertices of the polygon map to as the base point of the fundamental group $\pi_1(\Sigma_g, x)$. The elements of the fundamental group are the homotopy classes $[\gamma]$ of closed paths γ defined on the unit interval, where homotopy is relative to this base point x. It turns out that the relative homotopy classes of the $2g$ elements $\alpha_1, \beta_1, \ldots, \alpha_g, \beta_g$ just defined generate the fundamental group. In the connected sum model, each pair α_i, β_i lies on one of the g tori that we attached to a sphere at the beginning of Paragraph 1.3.1. Two neighboring tori are connected by the path $\mu_i := \alpha_{i+1}^{-1} \beta_i \alpha_i \beta_i^{-1}$, where it is understood that $\mu_g := \alpha_1^{-1} \beta_g \alpha_g \beta_g^{-1}$ if $i = g$. To see this, we first observe that the path $\beta_i \alpha_i \beta_i^{-1}$ is homotopic to a kind of mirror image of α_i:
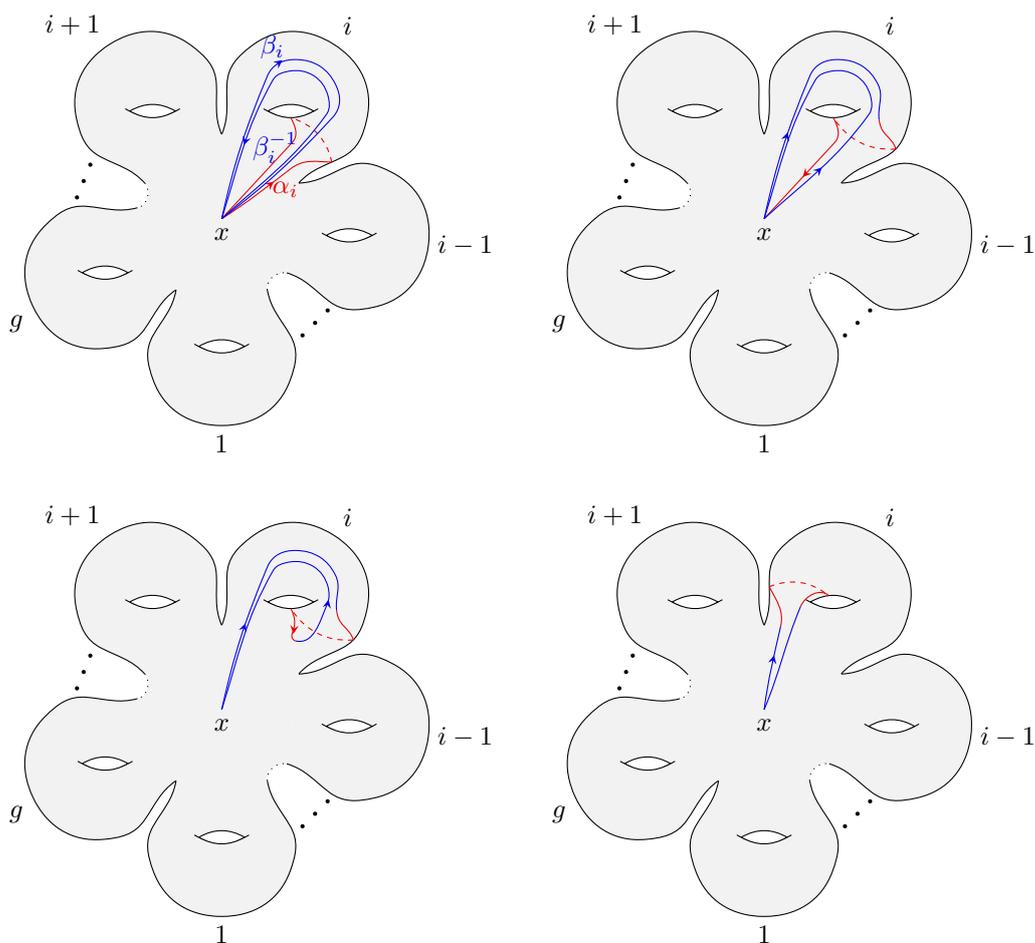
If we now also concatenate with the curve α_{i+1}^{-1}, we see that the curve μ_i is homotopic to a curve that connects the i-th and $(i+1)$-st handle:
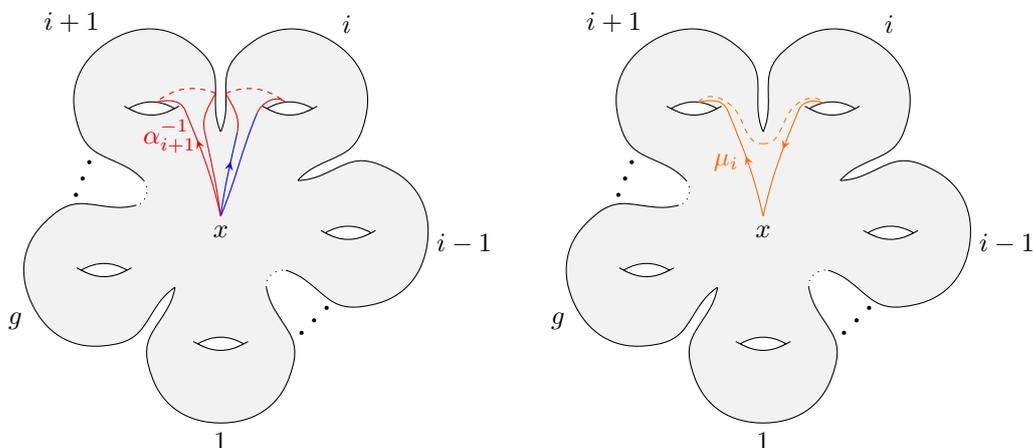
These pictures in fact illustrate that the concatenated path $\mu_g \mu_{g-1} \cdots \mu_1$ is homotopic to the constant path based at x, which means that

$$(\alpha_1^{-1}\beta_g\alpha_g\beta_g^{-1})(\alpha_g^{-1}\beta_{g-1}\alpha_{g-1}\beta_{g-1}^{-1})\cdots(\alpha_2^{-1}\beta_1\alpha_1\beta_1^{-1})$$

represents the unit element in the fundamental group. Alternatively, by conjugating with α_1 and inverting, we see that

$$\alpha_1\beta_1\alpha_1^{-1}\beta_1^{-1}\alpha_2\beta_2\alpha_2^{-1}\beta_2^{-1}\cdots\alpha_g\beta_g\alpha_g^{-1}\beta_g^{-1}$$

represents the unit element in the fundamental group. This fact is even easier to see in the polygon model, where this path exactly traces out the boundary of the polygon and can therefore be contracted to a point in the interior of the polygon. Using the Seifert-van Kampen theorem, one can show that this relation is a defining relation for the fundamental group (cf. [Fu, Prop. 17.6, p. 242]).

For the surfaces $\Sigma_{g,n}$ with boundary, additional generators are needed for the fundamental group, which we denote by $\delta_1, \ldots, \delta_n$. Starting at the base point, δ_j circles around the j-th boundary component. Together $\delta_1, \ldots, \delta_n$ form a so-called bouquet of circles. In the polygon model, δ_j arises by mapping the concatenation

of ξ_j, ρ_j, and ξ_j^{-1} to the quotient. We therefore see that in this case,

$$\alpha_1\beta_1\alpha_1^{-1}\beta_1^{-1}\alpha_2\beta_2\alpha_2^{-1}\beta_2^{-1}\cdots\alpha_g\beta_g\alpha_g^{-1}\beta_g^{-1}\delta_1\cdots\delta_n$$

is homotopic to the constant path based at x, and again this yields a defining relation for the fundamental group (cf. [AS, Chap. I, No. 43B, p. 100]). Because it is possible to solve in this relation for the last of the new generators, this implies that, in the presence of boundary components, the fundamental group of $\Sigma_{g,n}$ is a free group on $2g + n - 1$ generators. We note that, by a slight movement, it is always possible to find a representative in a relative homotopy class that does not intersect the boundary. In the sequel, we will assume in particular that this has been done for the generators δ_j.

1.3.3   Mapping class groups

An important object associated with the surface $\Sigma_{g,n}$ is its mapping class group $\Gamma_{g,n}$. There are several variants for the definition of this group in the literature; we will now explain which one is used in this thesis and how it compares to other variants.

By the boundary invariance theorem (cf. [Do, Chap. IV, Prop. 3.9, p. 61]), a diffeomorphism of $\Sigma_{g,n}$ will necessarily map a boundary component to a boundary component, but we require that the diffeomorphisms we consider also map the marked point on each boundary component to the corresponding marked point on the other boundary component. The group $\mathrm{Diffeo}^+(\Sigma_{g,n})$ of orientation-preserving diffeomorphisms of $\Sigma_{g,n}$ that permute the marked points becomes a topological group when endowed with the compact-open topology (cf. [A, Thm. 4, p. 598]). By the fundamental property of the compact-open topology (cf. [Q, Satz 14.17, p. 167]), the path-component $\mathrm{Diffeo}_0^+(\Sigma_{g,n})$ of the identity mapping consists precisely of those orientation-preserving diffeomorphisms that are isotopic to the identity. It is a normal subgroup. Continuity implies that the elements of $\mathrm{Diffeo}_0^+(\Sigma_{g,n})$ cannot permute the marked points, but rather need to preserve them individually. We define the mapping class group as the corresponding quotient group:

Definition 1.3.1 The mapping class group $\Gamma_{g,n}$ of $\Sigma_{g,n}$ is the quotient group

$$\Gamma_{g,n} := \mathrm{Diffeo}^+(\Sigma_{g,n}) / \mathrm{Diffeo}_0^+(\Sigma_{g,n})$$

In accordance with our notation for surfaces in Paragraph 1.3.1, we will also write Γ_g if $n = 0$. The mapping class of a diffeomorphism $\psi \in \mathrm{Diffeo}^+(\Sigma_{g,n})$ will be denoted by $[\psi]$.

In view of this definition, two diffeomorphisms represent the same element in the mapping class group if and only if they are contained in the same path-component of $\mathrm{Diffeo}^+(\Sigma_{g,n})$; i.e., if and only if they are isotopic. Because the path connecting them is contained in $\mathrm{Diffeo}^+(\Sigma_{g,n})$, the corresponding isotopy must permute the marked points at each time of the deformation process. By continuity, this is only possible if all diffeomorphisms occurring in this process, and in particular those at the beginning and the end, permute the marked points in the same way. By enumerating the marked points, we can therefore obtain a group homomorphism

$$p\colon \Gamma_{g,n} \to \mathbb{S}_n$$

to the symmetric group $\mathbb{S}_n$. We call the kernel of p the pure mapping class group and denote it by $\mathrm{P}\Gamma_{g,n}$. An element in the pure mapping class group can be represented by a diffeomorphism that restricts to the identity on the entire boundary, not only on the marked points. As explained in [FM, Sec. 1.4, p. 42], the mapping class group can alternatively be defined by using homeomorphisms instead of diffeomorphisms.

For a subset U of $\Sigma_{g,n}$, we also consider the subgroup of $\mathrm{Diffeo}^+(\Sigma_{g,n})$ consisting of those diffeomorphisms that restrict to the identity on U. If we divide it by its path component of the identity mapping, the arising quotient group $\Gamma_{g,n}(U)$ is called the relative mapping class group modulo U. It comes with a canonical group homomorphism

$$F_U\colon \Gamma_{g,n}(U) \to \Gamma_{g,n}$$

called the forgetful map, because it arises by forgetting the information about the restriction to U. This homomorphism is not necessarily injective, because

it might happen that a diffeomorphism is isotopic to the identity, although the diffeomorphisms that occur during this deformation process cannot be chosen so that they restrict to the identity on U. If $U = \{u\}$ consists of a single point u, we will also write $\Gamma_{g,n}(u)$ and F_u instead of $\Gamma_{g,n}(\{u\})$ and $F_{\{u\}}$.

The definition of mapping class groups used here is the one from [L1, Sec. 4, p. 485]. Often, other definitions are used, for example in [FM, Sec. 2.1, p. 44] or [Ko, Sec. 1, p. 101]. In these definitions, there are additional marked points in the interior, not on the boundary like in our definition. These points are frequently called punctures. In contrast to our definition, both the definition in [FM] and the definition in [Ko] require that the diffeomorphisms restrict to the identity on the boundary. In [FM], the punctures in the interior may be permuted by a diffeomorphism, while they are required to be preserved individually in [Ko].

1.3.4   Dehn twists

An annulus can be defined as the Cartesian product $A := S^1 \times [0, 2\pi]$ of the unit circle $S^1 := \{z \in \mathbb{C} \mid |z| = 1\}$ and the interval $[0, 2\pi] \subset \mathbb{R}$. A is an orientable 2-manifold with boundary, which is diffeomorphic to $\Sigma_{0,2}$ and therefore sometimes called a binion. We single out one of the two possible orientations by requiring that in the tangent space of A at the point $(1, \pi)$ the tangent vector v_1 to the curve $t \mapsto (1, \pi + t)$ and the tangent vector v_2 to the curve $t \mapsto (e^{it}, \pi)$ form a positively oriented basis v_1, v_2. Note that both curves start at the specified point for the parameter value $t = 0$. On A, we define the twist map

$$\mathfrak{d}\colon A \to A, \ (z, t) \mapsto (ze^{it}, t)$$

Now suppose that $\gamma\colon S^1 \to \Sigma_{g,n}$ is a simple closed curve that does not intersect the boundary. It follows from the existence of tubular neighborhoods (cf. [H, Chap. 4, Thm. 5.2, p. 110]) that there is an orientation-preserving embedding $\phi\colon A \to \Sigma_{g,n}$ with the property that $\phi(z, 0) = \gamma(z)$.

Definition 1.3.2 We define $\mathfrak{d}_\gamma \colon \Sigma_{g,n} \to \Sigma_{g,n}$ as the map

$$p \mapsto \begin{cases} (\phi \circ \mathfrak{d} \circ \phi^{-1})(p) & : p \in \phi(A) \\ p & : p \notin \phi(A) \end{cases}$$

Note that this map is a homeomorphism, not a diffeomorphism. But as we mentioned above, the mapping class group can also be defined using homeomorphisms, so that $\mathfrak{d}_\gamma$ determines an element $[\mathfrak{d}_\gamma]$ in the mapping class group, called the Dehn twist along γ. Since $\mathfrak{d}(z,0) = (z,0)$, we have $\mathfrak{d}_\gamma \circ \gamma = \gamma$.

It follows from the isotopy of tubular neighborhoods (cf. [H, Chap. 4, Thm. 5.3, p. 112]) that the mapping class of $\mathfrak{d}_\gamma$ does not depend on ϕ. Furthermore, any curve isotopic to γ yields the same Dehn twist in the mapping class group. Further details on Dehn twists and their geometric meaning can be found in [FM, Sec. 3.1, p. 64ff]. It is important to note that in some references the other orientation for the annulus A is used in the definition of a Dehn twist, for example in [L2, Par. 2.2, p. 316f] and [Ko, Sec. 2, p. 102]. Dehn twists defined using the other orientation are the inverses of the Dehn twists as they are defined here.

It is almost immediate from this definition that non-intersecting curves γ_1 and γ_2 give rise to commuting Dehn-twists $\mathfrak{d}_{\gamma_1}$ and $\mathfrak{d}_{\gamma_2}$. This and other basic properties of Dehn-twists are discussed in [FM, Sec. 3.2, p. 73ff].

We will introduce some special notation for a number of Dehn twists along certain curves. The first of these are the following: In the construction of $\Sigma_{g,n}$ from Σ_g, the removal of the j-th open disk from Σ_g leaves a connected component of the boundary that is diffeomorphic to the unit circle S^1. In the polygon model discussed in Paragraph 1.3.1, this curve is parametrized by ρ_j. By moving this curve slightly to the interior, for example by using a collar (cf. [H, Chap. 4, Thm. 6.1, p. 113]), we obtain a simple closed curve ∂_j that is freely homotopic to ρ_j, and therefore also to δ_j, but does not intersect the boundary. We will use the notation $\mathfrak{d}_j$ for the Dehn twist along ∂_j. As explained in [FM, Par. 4.2.5, p. 102], $\mathfrak{d}_j$ is contained in the center of the mapping class group.

1.3.5 Braidings

Besides Dehn twists, the second type of elements of the mapping class group that we will need are the braidings of the boundary components. To define them, we consider the surface B in $\mathbb{R}^2$ that consists of the closed unit disk from which two open disks with radius $1/4$ and centers $(1/2, 0)$ and $(-1/2, 0)$ on the x-axis, respectively, have been removed. B is an orientable 2-manifold with boundary, which is sometimes called a trinion, as it is diffeomorphic to $\Sigma_{0,3}$. We orient B by requiring that the canonical basis at the origin is positively oriented. On the boundary components, i.e., the unit circle and the two smaller circles, we choose the base points $(0, 1)$, $(1/2, 1/4)$, and $(-1/2, 1/4)$, respectively. As discussed in [T, Sec. V.2.5, p. 251], there is an orientation-preserving diffeomorphism $\mathfrak{b} \colon B \to B$, which we call the braiding map, that is the identity on the unit circle, interchanges the two smaller circles as well as their base points while preserving their orientations, and transforms the line segments connecting the base point of the unit circle with the base points of the smaller circles.

The braiding map is indicated in the following picture:
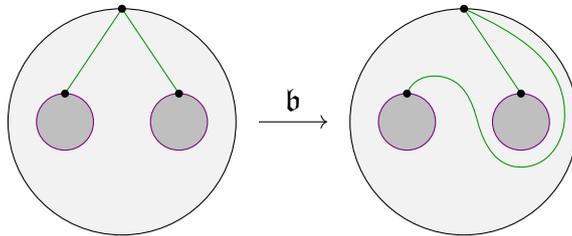
(cf. also [T, Fig. 2.9, p. 254]). As also stated in [T, loc. cit.], these conditions determine $\mathfrak{b}$ up to isotopy. It should be noted that the braiding map we define here is the inverse of the braiding map in [T, Sec. V.2.5, p. 251], but coincides with the map used in [FM, Par. 5.1.1, p. 119].

We will now use the braiding map to define the braiding of two boundary components of $\Sigma_{g,n}$. It should be noted that it is not possible to define such a braiding if the boundary components are distributed over the surface in a somewhat arbitrary fashion, as in our description of the surface in terms of the connected sum model,

as this definition depends not only on these two boundary components alone, but also on their position relative to the other boundary components. Rather, it is necessary that the boundary components are not only numbered, but in fact arranged in a certain order, like in the polygon model described in Paragraph 1.3.1. We use the standard arrangement of the boundary components in this model as follows: For two indices satisfying $1 \leq i < j \leq n$, we choose an orientation-preserving embedding $\varphi_{i,j} \colon B \to \Sigma_{g,n}$ of surfaces that maps the circle with center $(-1/2, 0)$ to the i-th boundary component of $\Sigma_{g,n}$ and the circle with center $(1/2, 0)$ to the j-th boundary component of $\Sigma_{g,n}$, mapping base points to base points. The unit circle is mapped to a smooth curve $\gamma_{i,j}$ on $\Sigma_{g,n}$ that is defined as follows: Starting at x, we follow a path slightly right from ξ_i until we reach the curve ∂_i, which we follow until we need to return along ξ_i, which we now do on its left side, which is the right side for the reversed orientation of ξ_i. Shortly before reaching x, however, we turn right and follow a path slightly right from ξ_j until we reach ∂_j, which we follow in the same way as before until we need to return along ξ_j, which we now do on its left side until we reach the base point x. The curve $\gamma_{i,j}$ is illustrated in the following picture:
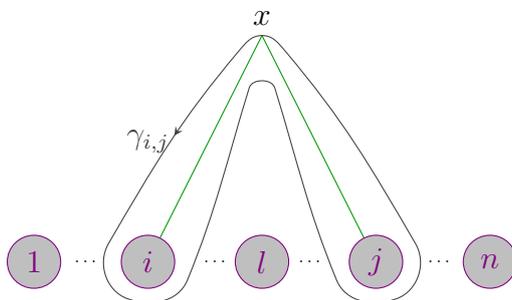
We now define the braiding of the i-th and the j-th boundary component in the same way as we defined Dehn twists in Paragraph 1.3.4:

Definition 1.3.3 We define $\mathfrak{b}_{i,j} \colon \Sigma_{g,n} \to \Sigma_{g,n}$ as the map

$$
p \mapsto \begin{cases} (\varphi_{i,j} \circ \mathfrak{b} \circ \varphi_{i,j}^{-1})(p) & : p \in \varphi_{i,j}(B) \\ p & : p \notin \varphi_{i,j}(B) \end{cases}
$$

Again, this map is a homeomorphism, not a diffeomorphism, but as for Dehn twists, it nonetheless determines an element $[\mathfrak{b}_{i,j}]$ in the mapping class group, called the braiding of the i-th and the j-th boundary component. It is obvious that under the projection $p \colon \Gamma_{g,n} \to \mathbb{S}_n$ to the symmetric group introduced in Paragraph 1.3.3, $[\mathfrak{b}_{i,j}]$ maps to the transposition of i and j.

We note that a mapping class very similar to $[\mathfrak{b}_{i,j}]$, using punctures instead of boundary components, is called a half-twist in [FM, Sec. 9.4, p. 255]. There are several relations between half-twists and Dehn twists; one is explained in the source just quoted, another one will be discussed in Paragraph 3.1.3.

Chapter 2

Hochschild cohomology and the modular group

It has been shown in previous work that the modular group acts projectively on the center of a factorizable ribbon Hopf algebra (see for example [CW1], [CW2], [Ke], [KL], [LM] and [T]). As seen in Example 1.1.1, the center is the zeroth Hochschild cohomology group. In this chapter, which is based on [LMSS1], we extend this projective action of the modular group to an arbitrary Hochschild cohomology group of a factorizable ribbon Hopf algebra.

This result is a special case of a more general result on mapping class groups which will be addressed in the next chapter.

This chapter is organized as follows: In the first section, we will consider Hochschild cohomology where the algebra A is a Hopf algebra and introduce a way to modify the bimodule structure of M while leaving the Hochschild cohomology groups essentially unchanged. In the second section, we turn to the case where A is a factorizable ribbon Hopf algebra and recall the action of the modular group on its center. In particular, we introduce the Radford and the Drinfel'd maps. Our treatment here follows largely the exposition in [SZ], to which the reader is referred for references to the original work. In the third section, we take advantage of our modification of the bimodule structure introduced in the first section to generalize

the Radford and the Drinfel'd maps to the Hochschild cochain complex. In the fourth and final section, we use these maps to generalize the action of the modular group on the center to an action on all Hochschild cohomology groups of our factorizable ribbon Hopf algebra.

Throughout this chapter, we will work over an arbitrary base field that is denoted by K, and all unadorned tensor products are taken over K. For a vector space V over K, we denote the associated projective space, i.e., the set of its one-dimensional subspaces, by $P(V)$, and the projectivity, or homography, induced by a bijective linear map f by $P(f)$. The set of all projectivities from $P(V)$ to itself forms the projective linear group $\mathrm{PGL}(V)$, which is isomorphic to the general linear group $\mathrm{GL}(V)$ modulo the scalar multiples of the identity transformation. By a projective representation or projective action, we mean a group homomorphism from a given group to the projective linear group. The dual of a vector space V is denoted by $V^* := \mathrm{Hom}_K(V, K)$.

2.1 Hochschild cohomology of Hopf algebras

We now consider Hochschild cohomology in the case where the algebra A is a Hopf algebra. We will denote the coproduct of A by Δ, its counit by ε, and its antipode by S. For the coproduct of $a \in A$, we will use Heyneman-Sweedler notation in the form $\Delta(a) = a_{(1)} \otimes a_{(2)}$.

Because A is a Hopf algebra, every A-bimodule M can be considered as a right A-module via the right adjoint action

$$\mathrm{ad} \colon M \otimes A \to M, \ m \otimes a \mapsto \mathrm{ad}(m \otimes a),$$

which is defined as $\mathrm{ad}(m \otimes a) := S(a_{(1)}).m.a_{(2)}$. We denote M by M_{ad} if it is considered as a right A-module in this way.

In general, a right A-module N becomes an A-bimodule with respect to the trivial left action, i.e., the action defined as $a.n := \varepsilon(a)n$. We denote N by ${}_\varepsilon N$ if it is

considered as a bimodule in this way. By combining the two operations, we can associate with an A-bimodule M the A-bimodule ${}_\varepsilon M_{\mathrm{ad}} := {}_\varepsilon(M_{\mathrm{ad}})$.

From [S, Def. 3.2, p. 876], we recall the notion of a two-sided Hopf module M over our Hopf algebra A. This is an A-bimodule that is simultaneously a left A-comodule whose coaction is denoted by

$$\delta_M \colon M \to A \otimes M, \ m \mapsto m^{(-1)} \otimes m^{(0)}.$$

This coaction is assumed to be compatible with the left and right actions via the respective Hopf module conditions; i.e., it is required that we have

$$\delta_M(a.m) = a_{(1)}m^{(-1)} \otimes a_{(2)}.m^{(0)} \quad \text{and} \quad \delta_M(m.a) = m^{(-1)}a_{(1)} \otimes m^{(0)}.a_{(2)}.$$

The simplest example of a two-sided Hopf module is A itself with respect to the left and right regular module structures, i.e., the module structures given by multiplication, and the left regular comodule structure, for which the coaction is just the coproduct. Other examples arise from the following construction, which provides the form of most two-sided Hopf modules that will be needed in the sequel:

Example 2.1.1 If V is a left A-comodule, then $M := A \otimes V \otimes A$ is a two-sided Hopf module with respect to the codiagonal coaction

$$\delta_M \colon M \to A \otimes M, \ a \otimes v \otimes b \mapsto a_{(1)}v^{(-1)}b_{(1)} \otimes a_{(2)} \otimes v^{(0)} \otimes b_{(2)}$$

and the left and right actions on the outer tensorands.

Two-sided Hopf modules can be used to relate the adjoint action introduced above with the original actions:

Lemma 2.1.2 Let M be a two-sided Hopf module and N an A-bimodule. Then the map

$$\omega_{M,N} \colon \operatorname{Hom}_{A \otimes A^{\mathrm{op}}}(M, {}_\varepsilon N_{\mathrm{ad}}) \to \operatorname{Hom}_{A \otimes A^{\mathrm{op}}}(M, N)$$

given by $\omega_{M,N}(f)(m) := m^{(-1)}.f(m^{(0)})$ is bijective.

Proof. Note that the dot that appears in the definition of $\omega_{M,N}$ denotes the original left action on N, not the trivial action. We therefore have

$$\omega_{M,N}(f)(a.m) = a_{(1)}m^{(-1)}.f(a_{(2)}.m^{(0)}) = a_{(1)}m^{(-1)}\varepsilon(a_{(2)}).f(m^{(0)}) = am^{(-1)}.f(m^{(0)})$$

and

$$\omega_{M,N}(f)(m.a) = m^{(-1)}a_{(1)}.f(m^{(0)}.a_{(2)}) = m^{(-1)}a_{(1)}S(a_{(2)}).f(m^{(0)}).a_{(3)}$$
$$= m^{(-1)}.f(m^{(0)}).a,$$

which shows that $\omega_{M,N}$ is well-defined. It is bijective because an inverse map is given by $\omega_{M,N}^{-1}(f)(m) := S(m^{(-1)}).f(m^{(0)})$; calculations similar to the ones above show that also this map is well-defined. $\square$

This lemma can be applied to the computation of the Hochschild cohomology of a Hopf algebra if the projective resolution of A that is used for this computation consists of two-sided Hopf modules, and if in addition its boundary operators are not only left and right A-linear, but also A-colinear. This happens in particular in the case of the standard resolution that we have used in Section 1.1 to define Hochschild cohomology: As already mentioned, A is a left comodule over itself via the left regular comodule structure, for which the coaction is just the comultiplication. As the tensor product of left comodules is again a left comodule with respect to the codiagonal comodule structure, we see that the n-th tensor power $A^{\otimes n}$ of A is a left A-comodule with coaction

$$\delta_{A^{\otimes n}} \colon A^{\otimes n} \to A \otimes A^{\otimes n}, \ a_1 \otimes \cdots \otimes a_n \mapsto a_{1(1)} \ldots a_{n(1)} \otimes (a_{1(2)} \otimes \cdots \otimes a_{n(2)}).$$

In view of Example 2.1.1, this implies that $S_n = A \otimes A^{\otimes n} \otimes A$ is a two-sided Hopf module whose coaction is

$$\delta_{S_n} \colon S_n \to A \otimes S_n, \ a_0 \otimes \cdots \otimes a_{n+1} \mapsto a_{0(1)} \ldots a_{n+1(1)} \otimes (a_{0(2)} \otimes \cdots \otimes a_{n+1(2)}).$$

Here, it is understood that $n \geq 1$, but in accordance with Section 1.1, we can extend this definition to the cases $n = 0$ and $n = -1$ by setting $\delta_{S_{-1}}(a) := a_{(1)} \otimes a_{(2)}$

and $\delta_{S_0}(a_0 \otimes a_1) := a_{0(1)} a_{1(1)} \otimes (a_{0(2)} \otimes a_{1(2)})$. Therefore, if M is an A-bimodule, the preceding lemma yields a family of mappings

$$\omega_{S_n, M} \colon \operatorname{Hom}_{A \otimes A^{\mathrm{op}}}(S_n, {}_\varepsilon M_{\mathrm{ad}}) \to \operatorname{Hom}_{A \otimes A^{\mathrm{op}}}(S_n, M)$$

which combine to an isomorphism of cochain complexes, as ω is clearly natural in its two arguments. Under the identification $\operatorname{Hom}_{A \otimes A^{\mathrm{op}}}(S_n, M) \cong \operatorname{Hom}_K(A^{\otimes n}, M)$ discussed at the beginning of Section 1.1, this isomorphism of cochain complexes takes the following form:

Proposition 2.1.3 The maps $\Omega^n \colon C^n(A, {}_\varepsilon M_{\mathrm{ad}}) \to C^n(A, M)$ defined via the formula

$$\Omega^n(f)(a_1 \otimes \cdots \otimes a_n) := a_{1(1)} \ldots a_{n(1)}.f(a_{1(2)} \otimes \cdots \otimes a_{n(2)})$$

give rise to an isomorphism $\Omega = (\Omega^n)$ between the Hochschild cochain complex of ${}_\varepsilon M_{\mathrm{ad}}$ and the Hochschild cochain complex of M. The inverse of Ω^n is given by

$$(\Omega^n)^{-1}(f)(a_1 \otimes \cdots \otimes a_n) = S(a_{1(1)} \ldots a_{n(1)}).f(a_{1(2)} \otimes \cdots \otimes a_{n(2)}).$$

If the antipode of A is bijective, the coopposite Hopf algebra A^{cop}, in which the product remains unaltered, but the coproduct is modified by interchanging the tensor factors, is a Hopf algebra, and its antipode is the inverse of the antipode of A. For an A-bimodule M, we denote the right adjoint action that arises from this Hopf algebra structure by cad; in terms of the original structure elements, this action is given by the formula

$$\mathrm{cad}(m \otimes a) = S^{-1}(a_{(2)}).m.a_{(1)}.$$

If we apply the preceding proposition to this situation, we obtain the following corollary:

Corollary 2.1.4 If the antipode of A is bijective, the maps Ω'^n from $C^n(A, {}_\varepsilon M_{\mathrm{cad}})$ to $C^n(A, M)$ defined via the formula

$$\Omega'^n(f)(a_1 \otimes \cdots \otimes a_n) := a_{1(2)} \ldots a_{n(2)}.f(a_{1(1)} \otimes \cdots \otimes a_{n(1)})$$

give rise to an isomorphism $\Omega' = (\Omega'^n)$ between the Hochschild cochain complex of $_\varepsilon M_{\text{cad}}$ and the Hochschild cochain complex of M. The inverse of Ω'^n is given by

$$(\Omega'^n)^{-1}(f)(a_1 \otimes \cdots \otimes a_n) = S^{-1}(a_{1(2)} \ldots a_{n(2)}).f(a_{1(1)} \otimes \cdots \otimes a_{n(1)}).$$

Proposition 2.1.3 generalizes a result found in [FS, Sec. 1, p. 2862f]. We note that further results related to this proposition can be found in the literature: In the case where the Hopf algebra is a group ring, a similar discussion is given in [EM, § 5, p. 60f], one of the foundational articles for group cohomology. A homology version of the proposition can be found in [FT, Prop. (2.4), p. 488], at least in the case where the bimodule is the underlying algebra. Similar statements for cohomology appear in [GK, Par. 5.5, p. 197] and [PW, Lem. 12, p. 591]. These last two references, however, rather state a combination of Proposition 2.1.3 with the following lemma, whose proof can be found in [CE, Chap. X, Thm. 2.1, p. 185]:

Lemma 2.1.5 For a right A-module N, we have $HH^n(A, {}_\varepsilon N) \cong \text{Ext}^n_A(K, N)$, where the base field K is given the trivial right A-module structure via the counit ε.

2.2   The action on the center

We now turn to the case of a factorizable ribbon Hopf algebra A with R-matrix R and ribbon element v. Our notational conventions in this section largely follow [SZ], to which we also refer for a more detailed account. A textbook that covers factorizable Hopf algebras and ribbon Hopf algebras is [R]. Besides the R-matrix and the ribbon element, the constructions in this section also depend on a nonzero right integral $\rho \in A^*$, whose normalization can be arbitrary for the most part, but for certain aspects needs to be fixed, as discussed at the end of the section.

Even though the R-matrix is in general not a pure tensor, we use the notation $R = R^1 \otimes R^2$. If τ denotes the flip map, we therefore have $\tau(R) = R^2 \otimes R^1$. This element in turn can be used to introduce the monodromy matrix $Q := \tau(R)R$, and

as for the R-matrix, we write $Q = Q^1 \otimes Q^2$. An important role will be played by the Drinfel'd and Radford maps, which are defined as follows:

Definition 2.2.1 We call the map

$$\bar{\Phi} \colon A^* \to A, \ \varphi \mapsto \varphi(Q^1)Q^2$$

the Drinfel'd map, and define the subalgebra

$$\bar{C}(A) := \{\varphi \in A^* \mid \varphi(bS^{-2}(a)) = \varphi(ab) \text{ for all } a, b \in A\},$$

whose elements we call generalized class functions. With the help of our nonzero right integral $\rho \in A^*$, we introduce the Radford map

$$\iota \colon A \to A^*, \ a \mapsto \rho_{(1)}(a)\rho_{(2)}.$$

By definition, A is factorizable if and only if $\bar{\Phi}$ is bijective, which implies in particular that A is finite-dimensional. The basic properties of the Drinfel'd map can be found in [SZ, Par. 3.2, p. 26], and the basic properties of the Radford map can also be found there, namely in [SZ, Par. 4.1, p. 35]. In particular, the Drinfel'd map restricts to an algebra isomorphism from $\bar{C}(A)$ to $Z(A)$, the center of A, while the Radford map restricts to a K-linear isomorphism from $Z(A)$ to $\bar{C}(A)$. A consequence of this last fact is that $\rho = \iota(1_A) \in \bar{C}(A)$, which is a special case of a general result found in [R, Thm. 10.5.4, p. 307] that arises when combined with [R, Prop. 12.4.2, p. 405].

Following [SZ, Par. 4.1, p. 35], we introduce the endomorphism $\mathfrak{S} := S \circ \bar{\Phi} \circ \iota$ of A, where as before S denotes the antipode of A. For $a \in A$, we have explicitly

$$\mathfrak{S}(a) = S(\bar{\Phi}(\rho_{(1)}(a)\rho_{(2)})) = S(\rho_{(1)}(a)\rho_{(2)}(Q^1)Q^2) = \rho(aQ^1)\,S(Q^2),$$

or $\mathfrak{S}(a) = \rho(aR^2R'^1)\,S(R^1R'^2)$ if we insert the definition of the monodromy matrix by using a second copy R' of the R-matrix.

As in [SZ, Par. 4.3, p. 37], we introduce a second such map, namely the multiplication

$$\mathfrak{T} \colon A \to A, \ a \mapsto va$$

with the ribbon element $v \in A$. Because v is central by definition, we could also have written av instead of va. The endomorphisms $\mathfrak{S}$ and $\mathfrak{T}$ will be used to encode the action of the two generators of the modular group described below.

We will need a third endomorphism of A, namely the antipode of the transmutation of A. The transmutation of a quasitriangular Hopf algebra was described by S. Majid in several articles, among them [M1], and is discussed in his monograph [M2]. The transmutation of A is equal to A as a set and even has the same vector space and algebra structure, but not the same coalgebra structure. We will use a slightly modified version in which A is replaced by $A^{\mathrm{op\,cop}}$. Upon this replacement, the formula for the antipode $\underline{S}$ of the transmutation given in [M2, Ex. 9.4.9, p. 504] becomes

$$\underline{S}(a) = S(S(R^1{}_{(1)})aR^1{}_{(2)})R^2.$$

If $u \coloneqq S(R^2)R^1$ is the Drinfel'd element of A, then the element $S(u) = R^1 S(R^2)$ is the Drinfel'd element of $A^{\mathrm{op\,cop}}$. Therefore, the alternative form of $\underline{S}$ given in [M2, Eq. (9.42), p. 507] yields in our case that

$$\underline{S}(a) = R^1 S(a) S(R^2) S(u^{-1}).$$

These three endomorphisms are related as follows:

Proposition 2.2.2 The maps $\mathfrak{S}$ and $\mathfrak{T}$ satisfy the relations

$$\mathfrak{S} \circ \mathfrak{T} \circ \mathfrak{S} = \rho(v)\, \mathfrak{T}^{-1} \circ \mathfrak{S} \circ \mathfrak{T}^{-1} \quad \text{and} \quad \mathfrak{S}^2 = (\rho \otimes \rho)(Q)\, \underline{S}^{-1}.$$

Proof. A proof of the first relation can be found in [SZ, Prop. 4.3, p. 37]. To prove the second relation, we use four copies R, R', R'' and R''' of the R-matrix. Because $\rho \in \bar{C}(A)$, the map $\mathfrak{S}$ is alternatively given by

$$\mathfrak{S}(a) = \rho(aR^2 R'^1)S(R'^2)S(R^1) = \rho(aR^2 S^{-1}(R'^1))R'^2 S(R^1) = \rho(S(R'^1)aR^2)R'^2 S(R^1),$$

where we have used the fact that $(S \otimes S)(R) = R$ proved in [M, Prop. 10.1.8, p. 180]. From [SZ, Prop. 4.1, p. 35], we know that $\mathfrak{S}$ is A-linear with respect to the right adjoint action. Therefore, we have

$$\underline{S}(\mathfrak{S}(a)) = S(\mathfrak{S}(S(R''^1{}_{(1)})aR''^1{}_{(2)}))R''^2$$
$$= \rho(S(R'^1)S(R'''^1{}_{(1)})aR'''^1{}_{(2)}R^2)S(R'^2 S(R^1))R''^2.$$

If we use one of the axioms for the R-matrix, namely [M, Eq. 10.1.6, p. 180], and the fact that the antipode is antimultiplicative, this equation can be rewritten in the form

$$\underline{S}(\mathfrak{S}(a)) = \rho(S(R'^1)S(R'''^1)aR''^1 R^2)S^2(R^1)S(R'^2)R'''^2 R''^2$$
$$= \rho(R'^1 S(R'''^1)aR''^1 R^2)S^2(R^1)R'^2 R'''^2 R''^2.$$

Another fact proved in [M, Prop. 10.1.8, p. 180] is that $(S \otimes \mathrm{id})(R) = R^{-1}$, so that this equation reduces to

$$\underline{S}(\mathfrak{S}(a)) = \rho(aR''^1 R^2)S^2(R^1)R''^2 = (\rho \otimes \rho)(Q)\mathfrak{S}^{-1}(a),$$

where the last step follows from [SZ, Prop. 4.2, p. 36]. Our claim is a minor rearrangement of this equation. $\qquad\square$

We would like to emphasize that this proposition is not new; rather, it is a variant of [LM, Thm. 4.4, p. 523]. We also note that it follows directly from another elementary property of R-matrices also proved in [M, Prop. 10.1.8, p. 180] that $\underline{S}$ agrees with the ordinary antipode S on the center of A, and because the square of the antipode is given by conjugation with the Drinfel'd element u, as shown in [M, Prop. 10.1.4, p. 179], we have $S = S^{-1}$ on the center. Therefore, the second relation in the previous proposition generalizes [SZ, Cor. 4.2, p. 37].

The fact that the square of $\underline{S}$ restricts to the identity on the center can also be seen from the fact that, in general, it is given by the right adjoint action of our ribbon element:

Lemma 2.2.3 For all $a \in A$, we have $\underline{S}^2(a) = \mathrm{ad}(a \otimes v)$.

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Proof. With the help of the alternative form of $\underline{S}$, we get

$$\underline{S}^2(a) = \underline{S}(R^1 S(a) S(R^2) S(u^{-1})) = R'^1 S(R^1 S(a) S(R^2) S(u^{-1})) S(R'^2) S(u^{-1})$$
$$= R'^1 S^2(u^{-1}) S^2(R^2) S^2(a) S(R^1) S(R'^2) S(u^{-1}) = R'^1 u^{-1} S^2(R^2 a) S(u^{-1} R'^2 R^1).$$

Using the definition of the monodromy matrix Q, the basic properties of ribbon elements found in [SZ, Par. 4.3, p. 37] and the above-mentioned fact that the square of the antipode is given by conjugation with the Drinfel'd element u, this becomes

$$\underline{S}^2(a) = Q^2 a u^{-1} S(u^{-1} Q^1) = S^2(Q^2) a u^{-1} S(u^{-1} S^2(Q^1)) = S^2(Q^2) a u^{-1} S(Q^1 u^{-1})$$
$$= S^2(Q^2) a u^{-1} S(u^{-1}) S(Q^1) = S^2(Q^2) a v^2 S(Q^1) = S(Q^1) a v^2 Q^2 = S(v_{(1)}) a v_{(2)}$$

as asserted. $\qquad\square$

We note that this equation is stated in [Ke, Eq. (2.60), p. 370], at least in the case of Drinfel'd doubles. A version in the framework of coends can be found in [L0, Cor. 3.10, p. 306].

The proposition above implies that the (homogeneous) modular group

$$\mathrm{SL}(2, \mathbb{Z}) = \{M \in \mathrm{GL}(2, \mathbb{Z}) \mid \det(M) = 1\}$$

acts projectively on the center of A. The modular group is generated by the two elements $\mathfrak{s} := \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ and $\mathfrak{t} := \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$, which satisfy the relations

$$\mathfrak{s}^4 = 1 \quad \text{and} \quad \mathfrak{s}\mathfrak{t}\mathfrak{s} = \mathfrak{t}^{-1}\mathfrak{s}\mathfrak{t}^{-1},$$

and these relations constitute a presentation of the group, as shown for example in [FR, Thm. 3.2.3.2, p. 97], [KT, Thm. A.2, p. 312] or [Ma, Sec. II.1, Thm. 8, p. 53].

If we denote the projective space associated to $Z(A)$ by $P(Z(A))$ and the automorphisms of this projective space arising from $\mathfrak{S}$ and $\mathfrak{T}$ by $P(\mathfrak{S})$ and $P(\mathfrak{T})$, the above proposition implies immediately the following fact:

Corollary 2.2.4 There is a unique homomorphism from $\mathrm{SL}(2,\mathbb{Z})$ to $\mathrm{PGL}(Z(A))$ that maps $\mathfrak{s}$ to $P(\mathfrak{S})$ and $\mathfrak{t}$ to $P(\mathfrak{T})$.

This result holds for any ribbon element $v \in A$ and any nonzero right integral $\rho \in A^*$. As shown in [R, Cor. 12.4.4, p. 407], we have $\rho(v) \neq 0$; this is obviously also a consequence of the proposition above. Because right integrals are only unique up to scalar multiples, we can choose a right integral that satisfies $\rho(v) = 1$; following [SZ, Def. 4.4, p. 39], we call such a right integral ribbon-normalized with respect to v. If we use a ribbon-normalized right integral, the proposition above shows that the action of the modular group on the center is linear, and not only projective, if and only if $(\rho \otimes \rho)(Q) = \pm 1$. By [SZ, Lem. 4.4, p. 39], this condition is equivalent to the condition $\rho(v^{-1}) = \pm 1$.

2.3 The Radford and the Drinfel'd maps for complexes

We remain in the situation described in Section 2.2 and consider a factorizable ribbon Hopf algebra A with R-matrix R, ribbon element v and nonzero right integral $\rho \in A^*$. Our first goal is to generalize the Radford map, the Drinfel'd map and the antipode to cochain maps of Hochschild cochain complexes. We begin with the Radford map, for which this is particularly easy.

By $A_{S^{-2}}$, we denote A considered as an A-bimodule with the left action given by multiplication, but the right action modified via the square of the inverse antipode, so that the right action is given by $b.a := bS^{-2}(a)$ for $a, b \in A$. As explained in Example 1.1.1, we then have

$$HH^0(A, (A_{S^{-2}})^*) = \{\varphi \in A^* \mid \varphi(bS^{-2}(a)) = \varphi(ab) \text{ for all } a, b \in A\} = \bar{C}(A),$$

the algebra of generalized class functions introduced in Definition 2.2.1. The bimodule $A_{S^{-2}}$ is related to the Radford map in the following way:

Proposition 2.3.1 The Radford map ι is a bimodule isomorphism from A to $(A_{S^{-2}})^*$.

Proof. By [M, Thm. 2.1.3, p. 18], A is a Frobenius algebra with Frobenius homomorphism ρ, so that ι is bijective. It is a bimodule homomorphism because

$$\iota(a_1aa_2)(b) = \rho(a_1aa_2b) = \rho(aa_2bS^{-2}(a_1)) = \iota(a)(a_2bS^{-2}(a_1)) = (a_1.\iota(a).a_2)(b)$$

for all $a, a_1, a_2, b \in A$, where the second equality holds because $\rho \in \bar{C}(A)$, a fact already pointed out in Section 2.2. $\square$

As bimodule isomorphisms induce isomorphisms between the corresponding Hochschild cochain complexes, this proposition enables us to generalize the Radford map to a cochain map as follows:

Definition 2.3.2 We define the Radford map for Hochschild cochain complexes as the cochain map from $C(A, A)$ to $C(A, (A_{S^{-2}})^*)$ with components

$$\iota^n \colon C^n(A, A) \to C^n(A, (A_{S^{-2}})^*), \;\; f \mapsto \iota \circ f.$$

In other words, we set $\iota^n := \iota_*$, the composition with ι on the left.

In order to compare this definition with the treatment of the Drinfel'd map and the antipode below, it will be important to relate this cochain map to another one defined between different cochain complexes. From Proposition 2.1.3, we get a cochain map $\Omega = (\Omega^n)$ from the cochain complex $C(A, {}_\varepsilon A_{\mathrm{ad}})$ to the cochain complex $C(A, A)$, but also a cochain map from the cochain complex $C(A, {}_\varepsilon((A_{S^{-2}})^*)_{\mathrm{ad}})$ to the cochain complex $C(A, (A_{S^{-2}})^*)$, which we denote by $\Omega'' = (\Omega''^n)$. The bimodule ${}_\varepsilon((A_{S^{-2}})^*)_{\mathrm{ad}}$ admits a slightly simpler description: For $\varphi \in (A_{S^{-2}})^*$, $a \in A$ and $b \in A_{S^{-2}}$, we have

$$\begin{aligned} \mathrm{ad}(\varphi \otimes a)(b) &= (S(a_{(1)}).\varphi.a_{(2)})(b) = \varphi(a_{(2)}.b.S(a_{(1)})) \\ &= \varphi(a_{(2)}bS^{-1}(a_{(1)})) = \varphi_{(1)}(a_{(2)})\varphi_{(2)}(b)\varphi_{(3)}(S^{-1}(a_{(1)})). \end{aligned}$$

This shows that the right adjoint action in ${}_\varepsilon((A_{S^{-2}})^*)_{\mathrm{ad}}$ coincides with the right coadjoint action of the coopposite Hopf algebra A^{cop}, which we denote by

$$\mathrm{coad} \colon A^* \otimes A \to A^*, \;\; \varphi \otimes a \mapsto \varphi_{(1)}(a_{(2)})\varphi_{(3)}(S^{-1}(a_{(1)})) \, \varphi_{(2)}.$$

In other words, we have $_\varepsilon((A_{S^{-2}})^*)_{\mathrm{ad}} = {}_\varepsilon(A^*)_{\mathrm{coad}}$. The Radford map now relates the two isomorphisms Ω and Ω'' as follows:

Lemma 2.3.3 The diagram

$$
\begin{array}{ccc}
C^n(A, {}_\varepsilon(A^*)_{\mathrm{coad}}) & \xrightarrow{\Omega''^n} & C^n(A, (A_{S^{-2}})^*) \\
\iota_* \uparrow & & \iota_* \uparrow \\
C^n(A, {}_\varepsilon A_{\mathrm{ad}}) & \xrightarrow{\Omega^n} & C^n(A, A)
\end{array}
$$

commutes.

Proof. We first note that it follows from Proposition 2.3.1 above that the Radford map ι is also a bimodule isomorphism from $_\varepsilon A_{\mathrm{ad}}$ to $_\varepsilon((A_{S^{-2}})^*)_{\mathrm{ad}} = {}_\varepsilon(A^*)_{\mathrm{coad}}$, so that the map on the left is well-defined. For $f \in C^n(A, {}_\varepsilon A_{\mathrm{ad}}) = \mathrm{Hom}_K(A^{\otimes n}, {}_\varepsilon A_{\mathrm{ad}})$ and $a_1, \ldots, a_n, b \in A$, we now have on the one hand

$$
\begin{aligned}
\iota(\Omega^n(f)(a_1 \otimes \cdots \otimes a_n))(b) &= \iota(a_{1(1)} \ldots a_{n(1)} f(a_{1(2)} \otimes \cdots \otimes a_{n(2)}))(b) \\
&= \rho(a_{1(1)} \ldots a_{n(1)} f(a_{1(2)} \otimes \cdots \otimes a_{n(2)})b).
\end{aligned}
$$

On the other hand, we have

$$
\begin{aligned}
((\Omega''^n \circ \iota_*)(f)(a_1 \otimes \cdots \otimes a_n))(b) &= (a_{1(1)} \ldots a_{n(1)}.\iota_*(f)(a_{1(2)} \otimes \cdots \otimes a_{n(2)}))(b) \\
&= \iota_*(f)(a_{1(2)} \otimes \cdots \otimes a_{n(2)})(b.a_{1(1)} \ldots a_{n(1)}) \\
&= \iota(f(a_{1(2)} \otimes \cdots \otimes a_{n(2)}))(bS^{-2}(a_{1(1)} \ldots a_{n(1)})) \\
&= \rho(f(a_{1(2)} \otimes \cdots \otimes a_{n(2)})bS^{-2}(a_{1(1)} \ldots a_{n(1)})).
\end{aligned}
$$

Since $\rho \in \bar{C}(A)$, these expressions are equal. $\qquad\square$

To generalize the Drinfel'd map to a cochain map between Hochschild cochain complexes, we first recall from [SZ, Par. 3.2, p. 26] that the Drinfel'd map $\bar{\Phi}$ is a bimodule isomorphism between $_\varepsilon(A^*)_{\mathrm{coad}}$ and $_\varepsilon A_{\mathrm{cad}}$, so that we obtain an isomorphism of cochain complexes

$$
\bar{\Phi}_* \colon C^n(A, {}_\varepsilon(A^*)_{\mathrm{coad}}) \to C^n(A, {}_\varepsilon A_{\mathrm{cad}}), \ f \mapsto \bar{\Phi} \circ f
$$

by composing with $\bar{\Phi}$ on the left. Now the isomorphism Ω' from Corollary 2.1.4 enables us to obtain a cochain map between the original cochain complexes:

Definition 2.3.4 We define the Drinfel'd map for Hochschild cochain complexes as the unique cochain map from $C(A, (A_{S^{-2}})^*)$ to $C(A, A)$ with components given by $\bar{\Phi}^n := \Omega'^n \circ \bar{\Phi}_* \circ (\Omega''^n)^{-1}$. In other words, it is the unique cochain map whose components make the diagram

$$
\begin{array}{ccc}
C^n(A, {}_\varepsilon(A^*)_{\mathrm{coad}}) & \xrightarrow{\Omega''^n} & C^n(A, (A_{S^{-2}})^*) \\
\downarrow{\scriptstyle \bar{\Phi}_*} & & \downarrow{\scriptstyle \bar{\Phi}^n} \\
C^n(A, {}_\varepsilon A_{\mathrm{cad}}) & \xrightarrow{\Omega'^n} & C^n(A, A)
\end{array}
$$

commutative.

With the help of the monodromy matrix Q, the map $\bar{\Phi}^n$ can be calculated explicitly: For $f \in C^n(A, (A_{S^{-2}})^*) = \mathrm{Hom}_K(A^{\otimes n}, (A_{S^{-2}})^*)$ and $a_1, \ldots, a_n \in A$, we have

$$
\begin{aligned}
\bar{\Phi}^n(f)(a_1 \otimes \cdots \otimes a_n) &= (\Omega'^n \circ \bar{\Phi}_* \circ (\Omega''^n)^{-1})(f)(a_1 \otimes \cdots \otimes a_n) \\
&= a_{1(2)} \ldots a_{n(2)} \bar{\Phi}((\Omega''^n)^{-1}(f)(a_{1(1)} \otimes \cdots \otimes a_{n(1)})) \\
&= a_{1(3)} \ldots a_{n(3)} \bar{\Phi}(S(a_{n(1)}) \ldots S(a_{1(1)}).f(a_{1(2)} \otimes \cdots \otimes a_{n(2)})) \\
&= a_{1(3)} \ldots a_{n(3)}(S(a_{n(1)}) \ldots S(a_{1(1)}).f(a_{1(2)} \otimes \cdots \otimes a_{n(2)}))(Q^1)Q^2 \\
&= a_{1(3)} \ldots a_{n(3)}f(a_{1(2)} \otimes \cdots \otimes a_{n(2)})(Q^1.(S(a_{n(1)}) \ldots S(a_{1(1)})))Q^2 \\
&= f(a_{1(2)} \otimes \cdots \otimes a_{n(2)})(Q^1 S^{-1}(a_{n(1)}) \ldots S^{-1}(a_{1(1)})) a_{1(3)} \ldots a_{n(3)}Q^2.
\end{aligned}
$$

In a similar way, we can generalize the antipode S to a cochain map between Hochschild cochain complexes: Since we have

$$
S(\mathrm{cad}(b \otimes a)) = S(S^{-1}(a_{(2)})ba_{(1)}) = S(a_{(1)})S(b)a_{(2)} = \mathrm{ad}(S(b) \otimes a),
$$

the antipode is a bimodule isomorphism from ${}_\varepsilon A_{\mathrm{cad}}$ to ${}_\varepsilon A_{\mathrm{ad}}$. Composition with S therefore yields an isomorphism

$$
S_* \colon C^n(A, {}_\varepsilon A_{\mathrm{cad}}) \to C^n(A, {}_\varepsilon A_{\mathrm{ad}}), \ f \mapsto S \circ f
$$

of cochain complexes. Now the isomorphisms Ω and Ω' from Section 2.1 enable us to obtain a cochain map between the original cochain complexes:

Definition 2.3.5 We define the antipode map for Hochschild cochain complexes as the cochain map from $C(A, A)$ to itself with components $S^n := \Omega^n \circ S_* \circ (\Omega'^n)^{-1}$. In other words, it is the unique cochain map whose components make the diagram

$$
\begin{array}{ccc}
C^n(A, {}_\varepsilon A_{\mathrm{cad}}) & \xrightarrow{\Omega'^n} & C^n(A, A) \\
\downarrow{\scriptstyle S_*} & & \downarrow{\scriptstyle S^n} \\
C^n(A, {}_\varepsilon A_{\mathrm{ad}}) & \xrightarrow{\Omega^n} & C^n(A, A)
\end{array}
$$

commutative.

As in the case of the Drinfel'd map, there is an explicit expression for the antipode map for cochain complexes: For $f \in C^n(A, A) = \mathrm{Hom}_K(A^{\otimes n}, A)$ and $a_1, \ldots, a_n \in A$, we have

$$
\begin{aligned}
S^n(f)(a_1 \otimes \cdots \otimes a_n) &= (\Omega^n \circ S_* \circ (\Omega'^n)^{-1})(f)(a_1 \otimes \cdots \otimes a_n) \\
&= a_{1(1)} \ldots a_{n(1)} S((\Omega'^n)^{-1}(f)(a_{1(2)} \otimes \cdots \otimes a_{n(2)})) \\
&= a_{1(1)} \ldots a_{n(1)} S(S^{-1}(a_{n(3)}) \ldots S^{-1}(a_{1(3)}) f(a_{1(2)} \otimes \cdots \otimes a_{n(2)})) \\
&= a_{1(1)} \ldots a_{n(1)} S(f(a_{1(2)} \otimes \cdots \otimes a_{n(2)})) a_{1(3)} \ldots a_{n(3)}.
\end{aligned}
$$

2.4 The action on the Hochschild cochain complex

We still remain in the situation described in Section 2.2 and Section 2.3. Our goal is to use the Radford map, the Drinfel'd map and the antipode map for Hochschild cochain complexes introduced in Section 2.3 in order to construct a projective action of the modular group $\mathrm{SL}(2, \mathbb{Z})$ on each Hochschild cohomology group $HH^n(A, A)$ in such a way that the action on the zeroth Hochschild cohomology group, which is, as we saw in Example 1.1.1, equal to the center $Z(A)$, coincides with the action described in Section 2.2. Up to homotopy, we will in

fact construct a projective action of the modular group on the entire Hochschild cochain complex.

To define a projective representation of the modular group, we have to specify the images of the generators $\mathfrak{s}$ and $\mathfrak{t}$ introduced in Section 2.2 and prove that they satisfy the relations stated there. For the first generator $\mathfrak{s}$, we use the same approach as in Section 2.2 and map it to the composition of the Radford map, the Drinfel'd map and the antipode:

Definition 2.4.1 We define the endomorphisms $\mathfrak{S}^n \colon C^n(A, A) \to C^n(A, A)$ as the following composition $\mathfrak{S}^n := S^n \circ \bar{\Phi}^n \circ \iota^n$.

Because the cochain complex versions of the Radford map, the Drinfel'd map and the antipode are cochain isomorphisms by construction, the maps $\mathfrak{S}^n$ are the components of a cochain automorphism of the Hochschild cochain complex. Its basic property is the following:

Lemma 2.4.2 The diagram

$$
\begin{array}{ccc}
C^n(A, {}_\varepsilon A_{\mathrm{ad}}) & \xrightarrow{\ \Omega^n\ } & C^n(A, A) \\
\downarrow{\scriptstyle \mathfrak{S}_*} & & \downarrow{\scriptstyle \mathfrak{S}^n} \\
C^n(A, {}_\varepsilon A_{\mathrm{ad}}) & \xrightarrow{\ \Omega^n\ } & C^n(A, A)
\end{array}
$$

commutes.

Proof. This is immediate from Lemma 2.3.3, Definition 2.3.4 and Definition 2.3.5: We have

$$
\begin{aligned}
\mathfrak{S}^n \circ \Omega^n &= S^n \circ \bar{\Phi}^n \circ \iota^n \circ \Omega^n = S^n \circ \bar{\Phi}^n \circ \Omega''^n \circ \iota_* \\
&= S^n \circ \Omega'^n \circ \bar{\Phi}_* \circ \iota_* = \Omega^n \circ S_* \circ \bar{\Phi}_* \circ \iota_* \\
&= \Omega^n \circ (S \circ \bar{\Phi} \circ \iota)_* = \Omega^n \circ \mathfrak{S}_*
\end{aligned}
$$

since successive composition with ι, $\bar{\Phi}$ and S is the same as composition with $\mathfrak{S}$. It may be noted that $\mathfrak{S}$, as the composition of the bimodule isomorphisms ι, $\bar{\Phi}$

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and S, is a bimodule automorphism of ${}_{\varepsilon}A_{\mathrm{ad}}$, so that the map $\mathfrak{S}_*$ on the left is indeed the component of a cochain map. $\qquad\square$

It is not difficult to compute $\mathfrak{S}^n$ explicitly in terms of the monodromy matrix Q:

Corollary 2.4.3 For $f \in C^n(A, A) = \mathrm{Hom}_K(A^{\otimes n}, A)$ and $a_1, \ldots, a_n \in A$, we have

$$\mathfrak{S}^n(f)(a_1 \otimes \cdots \otimes a_n) = \rho(S(a_{1(2)} \ldots a_{n(2)}) f(a_{1(3)} \otimes \cdots \otimes a_{n(3)}) Q^1) a_{1(1)} \ldots a_{n(1)} S(Q^2).$$

Proof. From the preceding lemma, we get

$$\begin{aligned}
\mathfrak{S}^n(f)(a_1 \otimes \cdots \otimes a_n) &= (\Omega^n \circ \mathfrak{S}_* \circ (\Omega^n)^{-1})(f)(a_1 \otimes \cdots \otimes a_n) \\
&= a_{1(1)} \ldots a_{n(1)} \mathfrak{S}((\Omega^n)^{-1}(f)(a_{1(2)} \otimes \cdots \otimes a_{n(2)})) \\
&= a_{1(1)} \ldots a_{n(1)} \mathfrak{S}(S(a_{1(2)} \ldots a_{n(2)}) f(a_{1(3)} \otimes \cdots \otimes a_{n(3)})),
\end{aligned}$$

so the assertion follows from the formula $\mathfrak{S}(a) = \rho(aQ^1)S(Q^2)$ obtained in Section 2.2. $\qquad\square$

For the second generator $\mathfrak{t}$ of the modular group, we also proceed as in Section 2.2 and let it act on the cochain groups by multiplication with the ribbon element v:

Definition 2.4.4 We define $\mathfrak{T}^n \colon C^n(A, A) \to C^n(A, A)$ as

$$\mathfrak{T}^n(f)(a_1 \otimes \cdots \otimes a_n) := v f(a_1 \otimes \cdots \otimes a_n)$$

for $f \in C^n(A, A) = \mathrm{Hom}_K(A^{\otimes n}, A)$. In other words, using the map $\mathfrak{T}$ introduced in Section 2.2, we set $\mathfrak{T}^n := \mathfrak{T}_*$, the composition with $\mathfrak{T}$ on the left.

Because v is central, the maps $\mathfrak{T}^n$ commute with the differentials d^n and therefore constitute the components of a cochain map. The centrality of v also implies that the diagram

$$\begin{array}{ccc}
C^n(A, {}_{\varepsilon}A_{\mathrm{ad}}) & \xrightarrow{\ \Omega^n\ } & C^n(A, A) \\
{\scriptstyle \mathfrak{T}_*}\downarrow & & \downarrow{\scriptstyle \mathfrak{T}^n} \\
C^n(A, {}_{\varepsilon}A_{\mathrm{ad}}) & \xrightarrow{\ \Omega^n\ } & C^n(A, A)
\end{array}$$

is commutative. Here, it is understood that $\mathfrak{T}_*$ is also given on $C^n(A, {}_\varepsilon A_{\mathrm{ad}})$ by multiplication with v, and not via the left or right action of v on ${}_\varepsilon A_{\mathrm{ad}}$.

The key result that relates these maps to the modular group is the following theorem:

Theorem 2.4.5

1. We have $\mathfrak{S}^n \circ \mathfrak{T}^n \circ \mathfrak{S}^n = \rho(v)\,(\mathfrak{T}^n)^{-1} \circ \mathfrak{S}^n \circ (\mathfrak{T}^n)^{-1}$.

2. The cochain maps with components $(\mathfrak{S}^n)^4$ and $((\rho \otimes \rho)(Q))^2\,\mathrm{id}_{C^n(A,A)}$ are homotopic.

Proof. As recalled in Proposition 2.2.2, we have $\mathfrak{S} \circ \mathfrak{T} \circ \mathfrak{S} = \rho(v)\,\mathfrak{T}^{-1} \circ \mathfrak{S} \circ \mathfrak{T}^{-1}$. Combining the commutativity of the preceding diagram with Lemma 2.4.2, we therefore get

$$\mathfrak{S}^n \circ \mathfrak{T}^n \circ \mathfrak{S}^n \circ \Omega^n = \Omega^n \circ \mathfrak{S}_* \circ \mathfrak{T}_* \circ \mathfrak{S}_* = \Omega^n \circ (\mathfrak{S} \circ \mathfrak{T} \circ \mathfrak{S})_*$$
$$= \rho(v)\,\Omega^n \circ (\mathfrak{T}^{-1} \circ \mathfrak{S} \circ \mathfrak{T}^{-1})_* = \rho(v)\,\Omega^n \circ \mathfrak{T}_*^{-1} \circ \mathfrak{S}_* \circ \mathfrak{T}_*^{-1}$$
$$= \rho(v)\,(\mathfrak{T}^n)^{-1} \circ \mathfrak{S}^n \circ (\mathfrak{T}^n)^{-1} \circ \Omega^n.$$

Because Ω^n is bijective, this proves our first assertion.

To prove the second assertion, it suffices to show that the cochain maps with components $(\Omega^n)^{-1} \circ (\mathfrak{S}^n)^4 \circ \Omega^n$ and $((\rho \otimes \rho)(Q))^2\,\mathrm{id}_{C^n(A, {}_\varepsilon A_{\mathrm{ad}})}$ are homotopic, because (Ω^n) is an isomorphism of cochain complexes. By Lemma 2.4.2, the following equation holds $(\Omega^n)^{-1} \circ (\mathfrak{S}^n)^4 \circ \Omega^n = (\mathfrak{S}^4)_*$, and we also have $\mathfrak{S}^2 = (\rho \otimes \rho)(Q)\,\underline{S}^{-1}$ by Proposition 2.2.2. Therefore our second assertion will hold if we can show that the cochain map $(\underline{S}^{-2})_*$ is homotopic to the identity on the cochain complex $C(A, {}_\varepsilon A_{\mathrm{ad}})$, or equivalently that the cochain map $(\underline{S}^2)_*$ is homotopic to the identity.

We know from Lemma 2.2.3 that $\underline{S}^2(a) = \mathrm{ad}(a \otimes v)$ for all $a \in A$. Because by definition the right action on the bimodule $M \coloneqq {}_\varepsilon A_{\mathrm{ad}}$ is given by the right adjoint action, this means in the notation of Proposition 1.1.2 that $(\underline{S}^2)_* = (r_v^M)_*$. Now

Proposition 1.1.2 states that $(r_v^M)_*$ is homotopic to $(l_v^M)_*$. But $(l_v^M)_*$ is the identity: For $f \in C^n(A, M) = \mathrm{Hom}_K(A^{\otimes n}, {}_\varepsilon A_{\mathrm{ad}})$ and $a_1, \ldots, a_n \in A$, we have

$$(l_v^M)_*(f)(a_1 \otimes \cdots \otimes a_n) = v.f(a_1 \otimes \cdots \otimes a_n) = \varepsilon(v)f(a_1 \otimes \cdots \otimes a_n)$$
$$= f(a_1 \otimes \cdots \otimes a_n)$$

because $\varepsilon(v) = 1$. $\qquad\square$

As a consequence, we can generalize the projective action of the modular group on the center $Z(A)$ obtained in Corollary 2.2.4, which is by Example 1.1.1 equal to $HH^0(A, A)$, to an arbitrary Hochschild cohomology group $HH^n(A, A)$. For this, we denote the automorphisms of $HH^n(A, A)$ induced by the cochain maps $(\mathfrak{S}^n)$ and $(\mathfrak{T}^n)$ by $\overline{\mathfrak{S}^n}$ and $\overline{\mathfrak{T}^n}$, respectively, and by $P(\overline{\mathfrak{S}^n})$ and $P(\overline{\mathfrak{T}^n})$ we denote the corresponding automorphisms of the projective space $P(HH^n(A, A))$. We then have the following generalization of Corollary 2.2.4:

Corollary 2.4.6 There is a unique homomorphism from the modular group $\mathrm{SL}(2, \mathbb{Z})$ to $\mathrm{PGL}(HH^n(A, A))$ that maps $\mathfrak{s}$ to $P(\overline{\mathfrak{S}^n})$ and $\mathfrak{t}$ to $P(\overline{\mathfrak{T}^n})$.

Exactly as in the analogous discussion at the end of Section 2.2, this representation of the modular group is linear, and not only projective, if the unique ribbon-normalized right integral ρ satisfies $\rho(v^{-1}) = \pm 1$, or equivalently $(\rho \otimes \rho)(Q) = \pm 1$.

We would like to emphasize that the proof of Theorem 2.4.5 and discussion preceding it show the following: The maps $\mathfrak{S}$ and $\mathfrak{T}$ are endomorphisms of the bimodule ${}_\varepsilon A_{\mathrm{ad}}$, although they do not in general give rise to a projective representation of the modular group on this bimodule. But they induce endomorphisms $\mathfrak{S}_*$ and $\mathfrak{T}_*$ of the cochain complex $C(A, {}_\varepsilon A_{\mathrm{ad}})$, and here the necessary relations are satisfied at least up to homotopy, so that we obtain a projective representation of the modular group in terms of homotopy classes of cochain maps. Via Ω, the cochain complex $C(A, {}_\varepsilon A_{\mathrm{ad}})$ is isomorphic to the standard Hochschild cochain complex $C(A, A)$, so that conjugation by Ω yields a projective representation of the same nature on this cochain complex. Passing to cohomology then leads to a projective representation of the modular group on each of the Hochschild cohomology groups $HH^n(A, A)$.

Chapter 3

Hochschild cohomology, modular tensor categories and mapping class groups

In this Chapter, which is based on [LMSS2], we show that the mapping class group of an arbitrary compact oriented surface acts projectively on the derived spaces of chiral conformal blocks associated to a topological field theory. Given a finite modular tensor category as introduced in Paragraph 1.2.1, we define, for every compact oriented surface with finitely many labeled boundary components, a cochain complex of finite-dimensional vector spaces. Its zeroth cohomology group is the vector space appearing in Lyubashenko's construction of spaces of chiral conformal blocks. We show that the mapping class group of the surface acts projectively on the cohomology groups of the complex in such a way that it reduces to Lyubashenko's action in degree zero. The construction reduces, in the case of the torus and if the category is given as the category of finite-dimensional representations of a factorizable ribbon Hopf algebra, to our previous result on the action of the modular group on the Hochschild cohomology.

The present work sets out the general theory needed to establish the result just described, but also prepares the ground for the computation of explicit examples of these mapping class group representations, which will be addressed in our forthcoming article [LMSS3]. A new aspect of our approach is that we consider an action of the mapping class group on the fundamental group by requiring that the base point of the fundamental group be kept fixed. In this way, it is possible to avoid the necessity to identify homomorphisms that are related by simultaneous conjugation that can be found in many of the other articles on topological or conformal field theory.

The material is organized as follows: In Section 3.1, we introduce the action of the mapping class group on the fundamental group just mentioned and introduce special elements in the mapping class group. We will also review the capping sequence and the Birman sequence and give some examples of mapping class groups. Section 3.2 explains how a topological field theory assigns representations of mapping class groups to certain tensor categories. For this, we use the framework created by V. Lyubashenko in his articles [L1] and [L2]; in particular, we use the approach to surfaces via nets and ribbon graphs described in his articles. In Section 3.3, we first state and prove our main result and then explain why this result generalizes our previous one from Chapter 2.

Throughout this chapter, the word 'projective' is used frequently. It has two very different meanings: When speaking about projective modules and projective resolutions, the term is used in the sense of ring theory and homological algebra. When speaking about projective space and projective representations or actions, the term is used in the sense of projective geometry as explained in greater detail in the introduction of Chapter 2.

We will assume throughout that our base field K is perfect, i.e., that all of its algebraic extensions are separable (cf. [J, Chap. IV, § 1, p. 146]). In particular, algebraically closed fields are perfect, as are fields of characteristic zero.

We use the symbol $\mathcal{V}$ for the category of finite-dimensional vector spaces. Additional notation will be explained in the text.

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3.1 Mapping class groups

We recall that we denote a surface of genus g with n boundary components by $\Sigma_{g,n}$ and its mapping class group by $\Gamma_{g,n}$ which was introduced in Paragraph 1.3.3.

3.1.1 The action on the fundamental group

During certain steps of our argument, it will be necessary to cut out small disks from our surface or at least to avoid certain points. For a Dehn twist, it makes a difference how we avoid a point. Suppose that a homotopy class is represented by a smooth simple closed curve γ that starts and ends at a point x on our surface. We choose a neighborhood U of x that is diffeomorphic to the open unit disk in $\mathbb{R}^2$ via an orientation-preserving diffeomorphism ϕ that sends x to the origin. As before, we assume that the unit disk has the orientation in which the canonical basis is a positively oriented basis for the tangent space at the origin. By perhaps passing to a smaller neighborhood, we can assume that γ passes through the unit disk as indicated on the left of the picture below. Because ϕ is orientation-preserving, it is meaningful to talk about pushing off γ to the left or to the right, as indicated in the middle and right picture below:
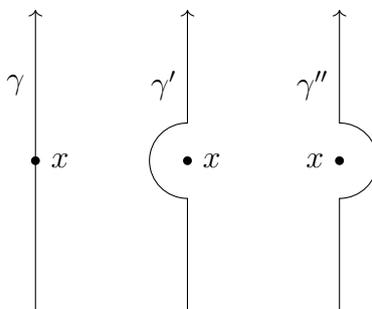
In this way, we obtain two curves γ' and γ'' whose free homotopy classes in $\Sigma_{g,n}\setminus\{x\}$ do not depend on our choices. Both curves avoid a small neighborhood of the point x and can be chosen so that they are still simple closed curves. Although we have in general $[\eth_{\gamma'}] \neq [\eth_{\gamma''}] \in \Gamma_{g,n}(x)$, we clearly have

$$F_x([\eth_{\gamma'}]) = F_x([\eth_{\gamma''}]) = [\eth_\gamma]$$

for their images under the forgetful map $F_x \colon \Gamma_{g,n}(x) \to \Gamma_{g,n}$.

As explained in [FM, Par. 8.2.7, p. 235], we can choose for x the base point of the fundamental group to get an action of $\Gamma_{g,n}(x)$ on $\pi_1(\Sigma_{g,n}, x)$. In particular, the Dehn twists $\mathfrak{d}_{\gamma'}$ and $\mathfrak{d}_{\gamma''}$ act on $\pi_1(\Sigma_{g,n}, x)$. We now give explicit formulas for this action in the case where the homotopy class acted upon can be represented by a curve κ that does not intersect γ, except at the base point x. It should be noted that, in the present situation, the curves κ and γ, and therefore also γ' and γ'', are oriented, although, according to our construction, Dehn twists like $\mathfrak{d}_{\gamma'}$ do not depend on the orientation of γ'. In each homotopy class, we can choose smooth representatives (cf. [FM, Par. 1.2.2, p. 26]), which by definition of smoothness have nowhere a vanishing tangent vector. If we in addition can choose these representatives so that their tangent vectors $\dot{\kappa}_x$ and $\dot{\gamma}_x$ at x are linearly independent, we say that κ and γ intersect transversely at x. If $\dot{\kappa}_x$ and $\dot{\gamma}_x$ form a positively oriented basis of the tangent space at x, we define the algebraic intersection number $i_A(\kappa, \gamma)$ as $+1$, and if they form a negatively oriented basis of the tangent space at x, we define the algebraic intersection number $i_A(\kappa, \gamma)$ as -1. This definition can be extended to curves with finitely many intersection points by defining $i_A(\kappa, \gamma)$ as the sum of the so-determined signs at all intersection points. This definition is illustrated in the picture

$$i_A(\kappa, \gamma) = +1 \qquad\qquad i_A(\kappa, \gamma) = -1$$

in which the standard orientation of the plane has been used, i.e., the orientation in which the canonical basis is positive and the normal vector points towards the reader.

The action of the two Dehn twists on the homotopy class $[\kappa]$ of κ is summarized in the following table:

	$i_A(\kappa, \gamma) = 1$	$i_A(\kappa, \gamma) = -1$
$\mathfrak{d}_{\gamma'}([\kappa])$	$[\kappa\gamma]$	$[\gamma^{-1}\kappa]$
$\mathfrak{d}_{\gamma''}([\kappa])$	$[\gamma\kappa]$	$[\kappa\gamma^{-1}]$

Here, expressions like $[\kappa\gamma] = [\kappa][\gamma]$ mean the product in the fundamental group: The concatenation of first κ, followed by γ. To understand the table, let us consider how the entry for $\mathfrak{d}_{\gamma'}([\kappa])$ in the case $i_A(\kappa, \gamma) = 1$ comes about: The curve starts at x with κ. In view of how γ' has been pushed off γ and how the curves intersect, κ is moving away from γ' and traces out κ almost completely before approaching the base point x upon returning. But before reaching x, it encounters γ' and at this point is forced to turn left in view of our definition of a Dehn twist. But because $i_A(\kappa, \gamma) = 1$, turning left is in this case the same as following γ in the direction of its orientation. The other entries arise from analogous considerations.

It is of course possible that we cannot choose representatives whose tangent vectors $\dot{\kappa}_x$ and $\dot{\gamma}_x$ at x are linearly independent. In this case, the two curves do not intersect transversely, and by pushing off γ either to the left or to the right, we can avoid intersection completely. Depending on the orientation of γ, the situation is described by one of the following two pictures:
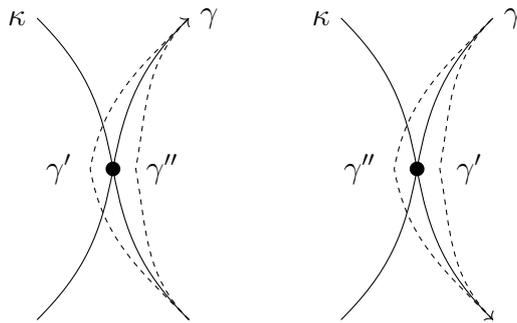
In the first case, the actions of the Dehn twists are given by

$$\mathfrak{d}_{\gamma'}([\kappa]) = [\gamma^{-1}\kappa\gamma] \qquad \text{and} \qquad \mathfrak{d}_{\gamma''}([\kappa]) = [\kappa],$$

while in the second case they are given by

$$\mathfrak{d}_{\gamma'}([\kappa]) = [\kappa] \qquad \text{and} \qquad \mathfrak{d}_{\gamma''}([\kappa]) = [\gamma\kappa\gamma^{-1}].$$

In both cases, these formulas are valid regardless of the orientation of κ.

3.1.2 Dehn twists for special curves

In Paragraph 1.3.4, we have already introduced the notation $\mathfrak{d}_j$ for the Dehn twist determined by the curve ∂_j. We now also assign names to the Dehn twists derived from the other curves stemming from the polygon model discussed in Paragraph 1.3.1 and define

$$\mathfrak{t}_i := \mathfrak{d}_{\alpha_i}, \qquad \mathfrak{r}_i := \mathfrak{d}_{\beta_i}, \qquad \mathfrak{n}_i := \mathfrak{d}_{\mu_i}.$$

Because the curves α_i and β_i intersect exactly once, the corresponding Dehn twists satisfy the braid relation $[\mathfrak{t}_i\mathfrak{r}_i\mathfrak{t}_i] = [\mathfrak{r}_i\mathfrak{t}_i\mathfrak{r}_i]$ (cf. [FM, Par. 3.5.1, Prop. 3.11, p. 77]). For the elements

$$\mathfrak{s}_i := \mathfrak{t}_i^{-1}\mathfrak{r}_i^{-1}\mathfrak{t}_i^{-1}$$

the braid relation implies that $[\mathfrak{s}_i\mathfrak{t}_i\mathfrak{s}_i^{-1}] = [\mathfrak{r}_i]$.

These special Dehn twists are important in view of the following result, which is known as the Dehn-Lickorish theorem:

Theorem 3.1.1 If $g \geq 1$, the Dehn twists $[\mathfrak{t}_i]$, $[\mathfrak{r}_i]$, and $[\mathfrak{n}_l]$, for $i = 1, \ldots, g$ and $l = 1, \ldots, g-1$, which are called the Lickorish generators, generate the mapping class groups $\Gamma_g = \Gamma_{g,0}$ and $\Gamma_{g,1}$.

A proof of this result can be found in [FM, Par. 4.4.4, p. 113ff]. We note that in the case $g = 0$, the groups Γ_g and $\Gamma_{g,1}$ are trivial anyway as a consequence of the Alexander lemma (cf. [FM, Par. 2.2.1, p. 47ff]). The case $g = 1$ will be discussed in more detail in Paragraph 3.1.7.

In the case where there is more than one boundary component, we need additional Dehn twists around the curves ζ_l, which we denote by $\mathfrak{z}_l$ for $l = 1, \ldots, n-1$. The curve ζ_l separates the l-th and $(l+1)$-st boundary component and connects to the g-th handle as shown in the picture for the case $g = 3$ and $n = 2$:
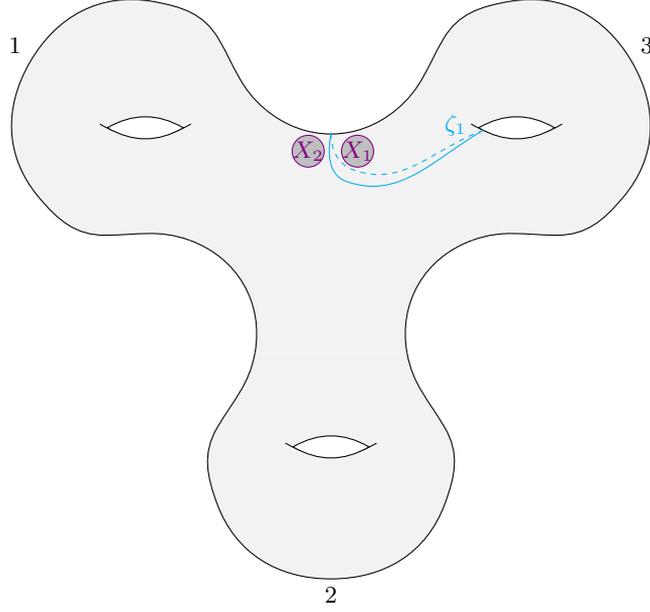
For this case, we have the following variant of the Dehn-Lickorish theorem (cf. [FM, Par. 4.4.4, p. 113f]):

Theorem 3.1.2 If $g \geq 1$ and $n \geq 2$, the mapping classes $[\mathfrak{t}_i]$, $[\mathfrak{r}_i]$, $[\mathfrak{n}_j]$, $[\mathfrak{d}_k]$, $[\mathfrak{z}_l]$, and $[\mathfrak{b}_{l,l+1}]$, for $i = 1, \ldots, g$, $j = 1, \ldots, g-1$, $k = 1, \ldots, n$, and $l = 1, \ldots, n-1$, generate the mapping class group $\Gamma_{g,n}$.

We can assume without loss of generality that the curve ∂_j does not only avoid the n boundary components, but also the base point x. Then the corresponding Dehn twist $\mathfrak{d}_j$ is contained in $\Gamma_{g,n}(x)$ and therefore can act on $\pi_1(\Sigma_{g,n}, x)$, as explained in Paragraph 3.1.1. Since any homotopy class in $\pi_1(\Sigma_{g,n}, x)$ can be represented by a curve that does not intersect ∂_j, this action is trivial. However, the curves α_i, β_i, and μ_i start and end at the base point x, and as explained in Paragraph 3.1.1, it makes a difference for the action on $\pi_1(\Sigma_{g,n}, x)$ whether the curves meet before or after the base point x. We will now make specific choices for these curves. For $\mathfrak{t}_i = \mathfrak{d}_{\alpha_i}$, we choose $\mathfrak{t}_i' := \mathfrak{d}_{\alpha_i'}$, for $\mathfrak{r}_i = \mathfrak{d}_{\beta_i}$, we choose $\mathfrak{r}_i'' := \mathfrak{d}_{\beta_i''}$, and for $\mathfrak{n}_i = \mathfrak{d}_{\mu_i}$, we choose $\mathfrak{n}_i'' := \mathfrak{d}_{\mu_i''}$. For $\mathfrak{s}_i$, we mix the choices and define $\mathfrak{s}_i' := \mathfrak{t}_i'^{-1} \mathfrak{r}_i''^{-1} \mathfrak{t}_i'^{-1}$. These maps act on the generators of the fundamental group as follows:

Proposition 3.1.3 Suppose that $i \leq g$.

1. For $j \leq n$, we have $\mathfrak{t}'_i([\delta_j]) = \mathfrak{r}''_i([\delta_j]) = [\delta_j]$. If $i \neq g$, we also have $\mathfrak{n}''_i([\delta_j]) = [\delta_j]$.

2. We have $\mathfrak{t}'_i([\beta_i]) = [\beta_i \alpha_i]$ and $\mathfrak{t}'_i([\alpha_i]) = [\alpha_i]$. For $j \neq i$, we have $\mathfrak{t}'_i([\beta_j]) = [\beta_j]$ and $\mathfrak{t}'_i([\alpha_j]) = [\alpha_j]$.

3. We have $\mathfrak{r}''_i([\alpha_i]) = [\alpha_i \beta_i^{-1}]$ and $\mathfrak{r}''_i([\beta_i]) = [\beta_i]$. For $j \neq i$, we have $\mathfrak{r}''_i([\alpha_j]) = [\alpha_j]$ and $\mathfrak{r}''_i([\beta_j]) = [\beta_j]$.

4. For $i = 1, \ldots, g-1$, we have $\mathfrak{n}''_i([\alpha_i]) = [\alpha_i]$ as well as

$$\mathfrak{n}''_i([\beta_i]) = [\mu_i \beta_i], \quad \mathfrak{n}''_i([\alpha_{i+1}]) = [\mu_i \alpha_{i+1} \mu_i^{-1}], \quad \mathfrak{n}''_i([\beta_{i+1}]) = [\beta_{i+1} \mu_i^{-1}].$$

   For $j \neq i$ and $j \neq i+1$, we have $\mathfrak{n}''_i([\alpha_j]) = [\alpha_j]$ and $\mathfrak{n}''_i([\beta_j]) = [\beta_j]$.

5. We have $\mathfrak{s}'_i([\alpha_i]) = [\alpha_i \beta_i \alpha_i^{-1}]$ and $\mathfrak{s}'_i([\beta_i]) = [\alpha_i^{-1}]$. For $j \neq i$, we have that $\mathfrak{s}'_i([\alpha_j]) = [\alpha_j]$ and $\mathfrak{s}'_i([\beta_j]) = [\beta_j]$.

Proof. (1) We first note that α_i and β_i are oriented so that α'_i and β''_i arise by pushing α_i and β_i off the base point x in the direction of the i-th attached torus, so that we could cut the i-th attached torus off again in such a way that the base point x on the one hand and α'_i and β''_i on the other hand would lie on different connected components. If $j \neq i$, this implies that the curves α'_i and β''_i do not intersect the curves α_j and β_j. This implies that $\mathfrak{t}'_i([\beta_j]) = [\beta_j]$ and $\mathfrak{t}'_i([\alpha_j]) = [\alpha_j]$ as well as $\mathfrak{r}''_i([\alpha_j]) = [\alpha_j]$ and $\mathfrak{r}''_i([\beta_j]) = [\beta_j]$. For the same reason, $\mathfrak{t}'_i$ and $\mathfrak{r}''_i$ preserve the homotopy class of δ_j.

(2) We have $i_A(\beta_i, \alpha_i) = -i_A(\alpha_i, \beta_i) = 1$, so that it follows from the table in Paragraph 3.1.1 that $\mathfrak{t}'_i([\beta_i]) = [\beta_i \alpha_i]$ and $\mathfrak{r}''_i([\alpha_i]) = [\alpha_i \beta_i^{-1}]$. Clearly, we have $\mathfrak{t}'_i([\alpha_i]) = [\alpha_i]$ and $\mathfrak{r}''_i([\beta_i]) = [\beta_i]$. Therefore, the second and the third assertions are now completely proved.

(3) For the fourth assertion, we see from the discussion in Paragraph 1.3.2 that μ_i is oriented in such a way that μ''_i arises by pushing μ_i off the base point x in the direction of the i-th and the $i+1$-st attached tori, so that we could cut these two tori

off again in such a way that the base point x and μ_i'' would lie on different connected components. For $j \neq i$ and $j \neq i+1$, this implies that μ_i'' does not intersect α_j or β_j, so that $\mathfrak{n}_i''([\alpha_j]) = [\alpha_j]$ and $\mathfrak{n}_i''([\beta_j]) = [\beta_j]$. This discussion also shows that μ_i'' does not intersect α_i, so that $\mathfrak{n}_i''([\alpha_i]) = [\alpha_i]$. On the other hand, μ_i intersects β_i and β_{i+1} exactly once, the intersection is transversal, and the intersection numbers are $i_A(\beta_i, \mu_i) = 1$ and $i_A(\beta_{i+1}, \mu_i) = -1$, respectively. It therefore follows from the table in Paragraph 3.1.1 that $\mathfrak{n}_i''([\beta_i]) = [\mu_i \beta_i]$ and $\mathfrak{n}_i''([\beta_{i+1}]) = [\beta_{i+1} \mu_i^{-1}]$. In contrast, the curves μ_i and α_{i+1} do not intersect transversally, but rather as in the second picture in Paragraph 3.1.1, so that $\mathfrak{n}_i''([\alpha_{i+1}]) = [\mu_i \alpha_{i+1} \mu_i^{-1}]$. Since $i \neq g$, the curve μ_i'' does not intersect δ_j, so that $\mathfrak{n}_i''([\delta_j]) = [\delta_j]$.

(4) It remains to show the fifth assertion. For $j \neq i$, the claim follows easily from the second and the third assertion. For the case $j = i$, we argue as follows: By inverting part of the third assertion, we have $\mathfrak{r}_i''^{-1}([\alpha_i \beta_i^{-1}]) = [\alpha_i]$, which implies that $\mathfrak{r}_i''^{-1}([\alpha_i])[\beta_i]^{-1} = \mathfrak{r}_i''^{-1}([\alpha_i])\mathfrak{r}_i''^{-1}([\beta_i]^{-1}) = [\alpha_i]$. This in turn yields

$$\mathfrak{s}_i'([\alpha_i]) = (\mathfrak{t}_i'^{-1}\mathfrak{r}_i''^{-1}\mathfrak{t}_i'^{-1})([\alpha_i]) = (\mathfrak{t}_i'^{-1}\mathfrak{r}_i''^{-1})([\alpha_i]) = \mathfrak{t}_i'^{-1}([\alpha_i \beta_i]) = \mathfrak{t}_i'^{-1}([\alpha_i])\mathfrak{t}_i'^{-1}([\beta_i]).$$

A second inversion yields $\mathfrak{t}_i'^{-1}([\beta_i]) = [\beta_i \alpha_i^{-1}]$, so that

$$\mathfrak{s}_i'([\alpha_i]) = [\alpha_i]\mathfrak{t}_i'^{-1}([\beta_i]) = [\alpha_i \beta_i \alpha_i^{-1}]$$

as asserted. The formula for $\mathfrak{s}_i'([\beta_i])$ follows from a similar computation. $\qquad \square$

3.1.3 Dehn twists related to two boundary components

According to our conventions in Paragraph 1.3.4, the curve $\delta_i \delta_j$, for $i < j$, cannot be used to define a Dehn twist for two reasons: On the one hand, it intersects the boundary, and on the other hand, it is not a simple closed curve. The first problem can be addressed by moving it slightly to the interior with the help of a collar, in the same way as in our treatment of ∂_j at the end of Paragraph 1.3.4. As a consequence of the orientation of ρ_i and ρ_j, a movement to the interior is a movement to the right of the given curve.

The second problem arises not only from the fact that the curve returns to the base point x at the time of concatenation, when δ_i ends and δ_j begins, but also from the fact that the paths ξ_i and ξ_j are both traced out twice, in opposite directions. To address this problem, we consider the curve $\gamma_{i,j}$ defined in Paragraph 1.3.5, which does not intersect the boundary, starts and ends in x, and represents the same relative homotopy class as $\delta_i\delta_j$ in the fundamental group $\pi_1(\Sigma_{g,n}, x)$. Passing to $\gamma'_{i,j}$ as indicated in Paragraph 3.1.1, we in addition avoid the base point x. In contrast to $\delta_i\delta_j$, the modification $\gamma'_{i,j}$ is a simple closed curve that can be used to define a Dehn twist. This curve is shown in the picture that appears in the proof below. We denote the Dehn twist along $\gamma'_{i,j}$ by $\mathfrak{d}_{i,j}$.

As explained in [FM, Par. 5.1.1, p. 118f] or [PS, § 7, p. 63], the Dehn twist $\mathfrak{d}_{i,j}$ is related to the braiding introduced in Paragraph 1.3.5: If we define the double braiding as $\mathfrak{q}_{i,j} := \mathfrak{b}_{j,i}\mathfrak{b}_{i,j}$, it is related to our Dehn twist via the formula

$$[\mathfrak{q}_{i,j}] = [\mathfrak{d}_{i,j}\mathfrak{d}_i^{-1}\mathfrak{d}_j^{-1}].$$

This formula holds because the Dehn twist $\mathfrak{d}_{i,j}$ not only interchanges the i-th and the j-th boundary components twice, but also twists these boundary components themselves, while the double braiding moves the boundary components in a parallel fashion, without introducing a twist. The action of $\mathfrak{q}_{i,j}$ on the generators of the fundamental group is given as follows:

Proposition 3.1.4 Suppose that $1 \leq i < j \leq n$.

1. We have

$$\mathfrak{q}_{i,j}([\delta_i]) = [(\delta_i\delta_j)^{-1}\delta_i(\delta_i\delta_j)] = [\delta_j^{-1}\delta_i\delta_j] \quad \text{and} \quad \mathfrak{q}_{i,j}([\delta_j]) = [(\delta_i\delta_j)^{-1}\delta_j(\delta_i\delta_j)].$$

2. If $l \neq i$ and $l \neq j$, we have $\mathfrak{q}_{i,j}([\delta_l]) = [\delta_l]$.

3. The double braiding $\mathfrak{q}_{i,j}$ acts trivially on the homotopy classes of $\alpha_1, \ldots, \alpha_g$ and $\beta_1, \ldots, \beta_g$ in $\pi_1(\Sigma_{g,n}, x)$.

Proof. We have already seen in Paragraph 3.1.2 that the Dehn twists $\mathfrak{d}_i$ and $\mathfrak{d}_j$ act trivially, so that the action of $\mathfrak{q}_{i,j}$ coincides with the action of $\mathfrak{d}_{i,j}$. If $l = i$ or $l = j$,

then $\gamma_{i,j}$ and δ_l intersect only in x, and the intersection is described by the first picture at the end of Paragraph 3.1.1. Therefore, the discussion of the first case there yields that $\mathfrak{d}_{i,j}([\delta_l]) = [\gamma_{i,j}^{-1}\delta_l\gamma_{i,j}]$. Because $\gamma_{i,j}$ is homotopic to $\delta_i\delta_j$ relative to the base point x, this proves the first assertion.

If $l < i$ or $l > j$, the generator δ_l also intersects $\gamma_{i,j}$ only in x, but now the intersection is described by the second picture at the end of Paragraph 3.1.1, so that $\gamma_{i,j}'$ does not intersect δ_l, which implies the second assertion in these cases. The case $i < l < j$ is more complicated and is illustrated by the picture
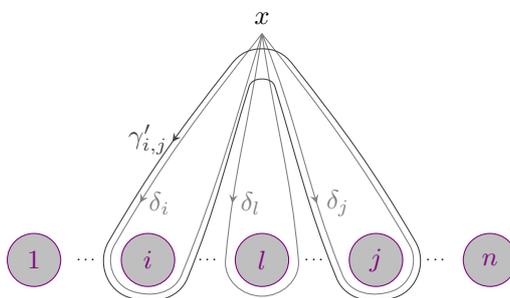
in which we have replaced the curves δ_i, δ_l, and δ_j by slight modifications within their relative homotopy class. It shows that the curve $\mathfrak{d}_{i,j}(\delta_l)$ comes about as follows: It starts at x with δ_l, but then soon meets $\gamma_{i,j}'$. At that point, it turns left and follows $\gamma_{i,j}'$ against its orientation. Upon returning, it follows δ_l again, but soon encounters $\gamma_{i,j}'$ a second time. Again, it turns left, but now follows $\gamma_{i,j}'$ in the direction of its orientation. Up to relative homotopy, these two encounters with $\gamma_{i,j}'$ cancel each other. While continuing along δ_l, the curve has two similar encounters with $\gamma_{i,j}'$, which again cancel each other. In summary, the curve is homotopic to δ_l, which finishes the proof of the second assertion.

The curves α_i and β_i intersect $\gamma_{i,j}$ again only in x, and the intersection is again described by the second picture at the end of Paragraph 3.1.1, so that $\gamma_{i,j}'$ does not intersect them at all. This yields the third assertion. $\qquad\square$

3.1.4 The capping homomorphism

In the process of defining $\Sigma_{g,n}$, we have removed n open disks from Σ_g. If we assume that our surface is realized via the polygon model, the boundary components come with a natural enumeration, and the j-th boundary component is parametrized by ρ_j. In the place of the j-th removed disk, we now glue a punctured disk back, i.e., a disk whose interior contains a marked point denoted by y. If a homeomorphism of $\Sigma_{g,n}$ restricts to the identity on the image of ρ_j, we can extend it to $\Sigma_{g,n-1}$ by requiring that the extension restricts to the identity on the newly inserted punctured disk. In this way, we obtain a group homomorphism

$$C_j \colon \Gamma_{g,n}(\rho_j) \to \Gamma_{g,n-1}(y)$$

which we call the j-th capping homomorphism. Here, we have not only written $\Gamma_{g,n-1}(y)$ for $\Gamma_{g,n-1}(\{y\})$, as indicated in Paragraph 1.3.3, but also briefly $\Gamma_{g,n}(\rho_j)$ for $\Gamma_{g,n}(\mathrm{Im}(\rho_j))$.

It is a consequence of the Alexander lemma (cf. [FM, Sec. 2.2, Lem. 2.1, p. 47f]) that $\mathfrak{d}_j$ is contained in the kernel of C_j. The following result, which can be found in [FM, Sec. 3.6, Prop. 3.19, p. 85] or [Ko, Sec. 3, p. 104f], states that $\mathfrak{d}_j$ in fact generates the kernel:

Proposition 3.1.5 The sequence

$$1 \longrightarrow \langle \mathfrak{d}_j \rangle \longrightarrow \Gamma_{g,n}(\rho_j) \xrightarrow{C_j} \Gamma_{g,n-1}(y) \longrightarrow 1$$

is short exact.

Almost always, $\mathfrak{d}_j \in \Gamma_{g,n}(\rho_j)$ is a nontrivial mapping class, in which case $\langle \mathfrak{d}_j \rangle$ is isomorphic to $\mathbb{Z}$. The only exception is the case $g = 0$ and $n = 1$: Here, it is easy to see that $\Gamma_{0,0}(y)$ is trivial (cf. [FM, p. 49]), and $\Gamma_{0,1}$ is trivial by the Alexander lemma.

Clearly, we can compose C_j with the forgetful map $F_y \colon \Gamma_{g,n-1}(y) \to \Gamma_{g,n-1}$ to obtain a homomorphism $D_j := F_y \circ C_j$ from $\Gamma_{g,n}(\rho_j)$ to $\Gamma_{g,n-1}$ that caps off the boundary component not with a punctured, but rather with a full disk. The effect of the additional mapping F_y will be studied in the next paragraph.

3.1.5 The Birman sequence

Because we can always modify a diffeomorphism by isotopy so that it fixes a point, the forgetful map

$$F_x \colon \Gamma_{g,n}(x) \to \Gamma_{g,n}$$

is surjective. If $\phi \in \mathrm{Diffeo}^+(\Sigma_{g,n})$ not only fixes x, but in addition represents a mapping class in the kernel of F_x, it can be connected to the identity in $\mathrm{Diffeo}^+(\Sigma_{g,n})$, i.e., there is a path

$$[0,1] \to \mathrm{Diffeo}^+(\Sigma_{g,n}), \ t \mapsto \phi_t$$

with $\phi_0 = \mathrm{id}$ and $\phi_1 = \phi$. Since ϕ_t is not required to fix the point x, we get a closed path

$$\gamma \colon [0,1] \to \Sigma_{g,n}, \ t \mapsto \phi_t(x)$$

based at x, which represents a homotopy class in the fundamental group $\pi_1(\Sigma_{g,n}, x)$.

Using the long exact sequence of a fibration (cf. [Ro1, Thm. 11.48, p. 358]), one can show that the homotopy class of γ determines the mapping class of ϕ. More precisely, there is a group antihomomorphism

$$P_x \colon \pi_1(\Sigma_{g,n}, x) \to \Gamma_{g,n}(x)$$

that maps the class of γ to the class of ϕ. This map is called the point-pushing map, or briefly the pushing map. The arising exact sequence

$$\pi_1(\Sigma_{g,n}, x) \xrightarrow{P_x} \Gamma_{g,n}(x) \xrightarrow{F_x} \Gamma_{g,n} \longrightarrow 1,$$

is called the Birman sequence. If the Euler characteristic of $\Sigma_{g,n}$ is strictly negative, the Birman sequence is in fact short exact, i.e., P_x is injective. The Birman sequence is discussed in greater detail in [FM, Sec. 4.2, p. 96ff].

As we have explained in Paragraph 3.1.1, the mapping class group $\Gamma_{g,n}(x)$ acts on the fundamental group $\pi_1(\Sigma_{g,n}, x)$, and so in particular $P_x([\gamma])$ acts on the fundamental group. If $P_x([\gamma]) = [\phi]$ and $[\beta] \in \pi_1(\Sigma_{g,n}, x)$, it can be shown that $\phi(\beta)$, i.e., the closed path $t \mapsto \phi(\beta(t))$, is homotopic to $\gamma^{-1}\beta\gamma$ relative to the basepoint x,

so that

$$P_x([\gamma])([\beta]) = [\gamma^{-1}\beta\gamma].$$

In other words, $P_x([\gamma])$ acts on the fundamental group by an inner automorphism. The Birman sequence therefore implies that we have a homomorphism from $\Gamma_{g,n}$ to $\mathrm{Out}(\pi_1(\Sigma_{g,n}, x))$ that makes the following diagram commutative:

$$
\begin{array}{ccc}
\Gamma_{g,n}(x) & \longrightarrow & \mathrm{Aut}(\pi_1(\Sigma_{g,n}, x)) \\
\downarrow{\scriptstyle F_x} & & \downarrow \\
\Gamma_{g,n} & \longrightarrow & \mathrm{Out}(\pi_1(\Sigma_{g,n}, x))
\end{array}
$$

Further aspects of this diagram are discussed in [FM, Par. 8.2.7, p. 235].

The action of the mapping class group $\Gamma_{g,n}(x)$ on the fundamental group $\pi_1(\Sigma_{g,n}, x)$ is compatible with the pushing map in another way: Clearly, $\Gamma_{g,n}(x)$ acts on the kernel of the group homomorphism F_x by conjugation. This action is compatible with the pushing map in the sense that

$$P_x([\psi(\gamma)]) = [\psi]P_x([\gamma])[\psi]^{-1}$$

for $[\psi] \in \Gamma_{g,n}(x)$ (cf. [FM, Par. 4.2.2, Fact 4.8, p. 99]). This relation is a direct consequence of our description above.

If a homotopy class in the fundamental group can be represented by a simple closed curve γ, there is a formula for its image under the pushing map in terms of Dehn twists: Using the notation from Paragraph 3.1.1, we have

$$P_x([\gamma]) = [\mathfrak{d}_{\gamma'}\mathfrak{d}_{\gamma''}^{-1}]$$

(cf. [FM, Par. 4.2.2, Fact 4.7, p. 99]).

3.1.6   Singular homology

If we think of a closed curve as being defined on the interval $[0, 1]$, it can be considered as a singular cycle. In this way, we obtain a group homomorphism

$$\pi_1(\Sigma_{g,n}, x) \to H_1(\Sigma_{g,n}, \mathbb{Z})$$

from the fundamental group to the first singular homology group that is called the (first) Hurewicz homomorphism. Hurewicz' theorem asserts in this situation that the Hurewicz homomorphism is surjective and induces an isomorphism from the abelianization of the fundamental group to the first singular homology group (cf. [Ro1, Thm. 4.29, p. 83]).

It follows from its universal property that the commutator factor group of a finitely presented group is presented by the same relations, but now understood as a presentation of an abelian group. Therefore, Hurewicz' theorem and the presentations of the fundamental group discussed in Paragraph 1.3.2 together imply that $H_1(\Sigma_g, \mathbb{Z})$ is a free abelian group of rank $2g$ with the homology classes of the curves $\alpha_1, \beta_1, \alpha_2, \beta_2, \ldots, \alpha_g, \beta_g$ as a basis. In the presence of boundary components, Hurewicz' theorem yields that $H_1(\Sigma_{g,n}, \mathbb{Z})$ is generated by the homology classes of the curves $\alpha_1, \beta_1, \alpha_2, \beta_2, \ldots, \alpha_g, \beta_g, \delta_1, \ldots, \delta_n$ subject to the relation that the homology classes of the curves $\delta_1, \ldots, \delta_n$ sum up to zero. By solving for one of these generators, we see that the first homology group is free abelian of rank $2g + n - 1$. We note that the homology class of δ_j is equal to the homology class of ∂_j, because these curves are freely homotopic.

The algebraic intersection number introduced in Paragraph 3.1.1 depends only on the homology classes of the curves involved. It therefore defines an alternating bilinear form on the abelian group $H_1(\Sigma_{g,n}, \mathbb{Z})$ which is symplectic if $n = 0$. For the generators, we have on the one hand

$$i_A(\beta_i, \alpha_j) = -i_A(\alpha_j, \beta_i) = \delta_{i,j} \quad \text{and} \quad i_A(\alpha_i, \alpha_j) = 0 = i_A(\beta_i, \beta_j)$$

(cf. [FM, Par. 6.1.2, p. 165]), while on the other hand we have $i_A(\delta_j, \gamma) = 0$ for

every closed curve γ, because its free homotopy class has a representative that does not intersect δ_j.

Because the outer automorphism group of any group evidently acts on the corresponding commutator factor group, Hurewicz' theorem implies that the commutative square obtained in Paragraph 3.1.5 can be enlarged to the diagram

$$
\begin{array}{ccc}
\Gamma_{g,n}(x) & \longrightarrow & \mathrm{Aut}(\pi_1(\Sigma_{g,n}, x)) \\
\downarrow{\scriptstyle F_x} & & \downarrow \\
\Gamma_{g,n} & \longrightarrow & \mathrm{Out}(\pi_1(\Sigma_{g,n}, x)) \\
\| & & \downarrow \\
\Gamma_{g,n} & \longrightarrow & \mathrm{Aut}(H_1(\Sigma_{g,n}, \mathbb{Z}))
\end{array}
$$

It is immediate from the construction that the action of $\Gamma_{g,n}$ on $H_1(\Sigma_{g,n}, \mathbb{Z})$ preserves the intersection form and therefore takes values in the symplectic group if $n = 0$. The discussion in Paragraph 3.1.2 and Paragraph 3.1.3 also implies how the special mapping classes introduced there act on the first homology group. The mapping classes $[\mathfrak{d}_j]$ and the mapping classes $[\mathfrak{q}_{i,j}]$ act as the identity on the entire first homology group. From Proposition 3.1.3, we see that the mapping classes $[\mathfrak{t}_i]$, $[\mathfrak{r}_i]$, and $[\mathfrak{s}_i]$ preserve the two subgroups generated by the homology classes of the curves $\alpha_1, \beta_1, \alpha_2, \beta_2, \ldots, \alpha_g, \beta_g$ on the one hand and the curves $\delta_1, \ldots, \delta_n$ on the other hand, and act as the identity on the second subgroup. As an abelian group, the first homology group is the direct sum of these two subgroups, and the first subgroup is free of rank $2g$ on the given generating set. We can therefore represent the action on the first subgroup by matrices in $\mathrm{Sp}(2g, \mathbb{Z})$. It follows from Proposition 3.1.3 that these matrix representations are block-diagonal with g blocks of 2×2-matrices. For $\mathfrak{t}_i$, $\mathfrak{r}_i$, and $\mathfrak{s}_i$, the blocks corresponding to the generators α_i and β_i are

$$
\mathfrak{t} := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \qquad \mathfrak{r} := \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \qquad \mathfrak{s} := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}
$$

respectively, while the other blocks are 2×2-identity matrices.

For $i = 1, \ldots, g-1$, the matrix representation of $\mathfrak{n}_i$ is also block-diagonal, but now contains a 4×4-block

$$\mathfrak{n} := \begin{pmatrix} 1 & 1 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

corresponding to the homology classes of the curves α_i, β_i, α_{i+1}, and β_{i+1}.

3.1.7 The mapping class group of the torus

The surface Σ_1 is a torus. Because the defining relation of the fundamental group discussed in Paragraph 1.3.2 in this case states that the relative homotopy class of $\alpha_1 \beta_1 \alpha_1^{-1} \beta_1^{-1}$ is trivial, the fundamental group is abelian here, and therefore Hurewicz' theorem mentioned in Paragraph 3.1.6 yields that it is isomorphic to the first singular homology group. According to the Dehn-Lickorish theorem 3.1.1, the Dehn twists $[\mathfrak{t}_1]$ and $[\mathfrak{r}_1]$, or alternatively $[\mathfrak{t}_1]$ and $[\mathfrak{s}_1] = [\mathfrak{t}_1^{-1} \mathfrak{r}_1^{-1} \mathfrak{t}_1^{-1}]$, generate the mapping class group Γ_1. As we have just seen, the actions of these generators on the first homology group are, respectively, represented by the matrices

$$\mathfrak{t} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \qquad \mathfrak{r} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \qquad \mathfrak{s} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

with respect to the basis consisting of the singular homology classes of α_1 and β_1. Because $\mathrm{Sp}(2g, \mathbb{Z}) = \mathrm{SL}(2, \mathbb{Z})$ if $g = 1$, the group homomorphism $\Gamma_{g,n} \to \mathrm{Sp}(2g, \mathbb{Z})$ described in Paragraph 3.1.6 becomes in this situation a group homomorphism

$$\Gamma_1 \to \mathrm{SL}(2, \mathbb{Z})$$

that maps $[\mathfrak{t}_1]$, $[\mathfrak{r}_1]$, and $[\mathfrak{s}_1]$ to $\mathfrak{t}$, $\mathfrak{r}$, and $\mathfrak{s}$, respectively. It is a classical fact that the modular group $\mathrm{SL}(2, \mathbb{Z})$ is generated by the matrices $\mathfrak{s}$ and $\mathfrak{t}$ and that

the relations $\mathfrak{s}^4 = 1$ and $\mathfrak{s}\mathfrak{t}\mathfrak{s} = \mathfrak{t}^{-1}\mathfrak{s}\mathfrak{t}^{-1}$ that they satisfy are defining (cf. [KT, Thm. A.2, p. 312]). These relations imply that $\mathfrak{s}^2$ is central, which can also easily be seen directly. Alternatively, the modular group is generated by the elements $\mathfrak{r}$ and $\mathfrak{t}$, and the defining relations for $\mathfrak{s}$ and $\mathfrak{t}$ just stated translate into the defining relations $\mathfrak{t}\mathfrak{r}\mathfrak{t} = \mathfrak{r}\mathfrak{t}\mathfrak{r}$ and $(\mathfrak{r}\mathfrak{t})^6 = 1$ for the generators $\mathfrak{r}$ and $\mathfrak{t}$ (cf. [SZ, Prop. 1.1, p. 7]). This implies immediately that our group homomorphism $\Gamma_1 \to \mathrm{SL}(2, \mathbb{Z})$ is surjective, a fact that is also a special case of a more general result for mapping class groups (cf. [FM, Par. 6.3.2, Thm. 6.4, p. 170]). In the case of the torus, the homomorphism is even bijective (cf. [FM, Par. 2.2.4, Thm. 2.5, p. 53]), so that $\Gamma_1 \cong \mathrm{SL}(2, \mathbb{Z})$.

To discuss the mapping class group $\Gamma_{1,1}$, we need to introduce the braid group B_3 on three strands, which we define as the group generated by two generators r and t subject to the one defining relation $rtr = trt$. We will refer to this relation as the braid relation. Geometrically, r can be interpreted as the interchange of the first two strands, while t can be interpreted as the interchange of the last two strands (cf. [FM, Sec. 9.2, p. 246f]). In view of the second presentation of the modular group given above, there is a surjective group homomorphism

$$B_3 \to \mathrm{SL}(2, \mathbb{Z})$$

that maps r to $\mathfrak{r}$ and t to $\mathfrak{t}$. If we define $s := (trt)^{-1}$, then on the one hand s is mapped to $\mathfrak{s}$ under our homomorphism, and on the other hand the braid relation is equivalent to the relation $sts = t^{-1}st^{-1}$ for the generators s and t of B_3. The braid relation implies relatively easily that the element $(rt)^3 = (rtr)(trt)$ is central in B_3. This yields, again in view of the second presentation of the modular group, not only that $(rt)^6$ is contained in the kernel of this homomorphism, but that it even generates the kernel. We therefore have the short exact sequence

$$1 \longrightarrow \langle (rt)^6 \rangle \longrightarrow B_3 \longrightarrow \mathrm{SL}(2, \mathbb{Z}) \longrightarrow 1.$$

We want to compare this homomorphism from B_3 to $\mathrm{SL}(2, \mathbb{Z})$ with the homomorphism $D_1 \colon \Gamma_{1,1}(\rho_1) = \Gamma_{1,1} \to \Gamma_1$ that caps off the one boundary component with a full disk. As discussed in Paragraph 3.1.4, D_1 is the composition

of $C_1 \colon \Gamma_{1,1} \to \Gamma_{1,0}(y)$, which caps off the boundary component with a punctured disk that contains a marked point y in its interior, and the forgetful map

$$F_y \colon \Gamma_{1,0}(y) \to \Gamma_{1,0} = \Gamma_1.$$

At this point, there is a small conceptual difficulty with the polygon model. In the polygon model, the capped-off boundary component is parametrized by ρ_1, and the point y in its interior is different from the base point x that is the common image of all the vertices of the polygon that are different from the start point and the end point of the edge labeled by ρ_1. To account for this difference, we choose a diffeomorphism $\phi \colon \Sigma_{1,0} \to \Sigma_{1,0}$ that is isotopic to the identity and satisfies $\phi(x) = y$. We then have the isomorphism

$$\Gamma_{1,0}(x) \to \Gamma_{1,0}(y), \ [\psi] \mapsto [\phi \circ \psi \circ \phi^{-1}]$$

that satisfies $F_y([\phi \circ \psi \circ \phi^{-1}]) = F_x([\psi])$. Moreover, the map

$$\pi_1(\Sigma_{1,0}, x) \to \pi_1(\Sigma_{1,0}, y), \ [\gamma] \mapsto [\phi \circ \gamma]$$

is an isomorphism of fundamental groups. Because the curves γ and $\phi \circ \gamma$ are freely homotopic, they map to the same singular homology class under the respective Hurewicz maps. If γ is a smooth simple closed curve, we have

$$[\eth_{\phi \circ \gamma}] = [\phi][\eth_\gamma][\phi^{-1}] = [\eth_\gamma]$$

in Γ_1. These considerations show that the diagram

$$
\begin{array}{ccc}
\Gamma_{1,0}(y) \longrightarrow \mathrm{Aut}(\pi_1(\Sigma_{1,0}, y)) \longrightarrow \mathrm{Aut}(H_1(\Sigma_{1,0}, \mathbb{Z})) \cong \mathrm{GL}(2, \mathbb{Z}) \\
\Big\downarrow{\scriptstyle F_y} \qquad\qquad\qquad\qquad\qquad\qquad \Big\| \\
\Gamma_{1,0} \longrightarrow \mathrm{Aut}(H_1(\Sigma_{1,0}, \mathbb{Z})) \cong \mathrm{GL}(2, \mathbb{Z})
\end{array}
$$

commutes. Although it does not prove this fact, this diagram suggests that the map $\Gamma_{1,0}(y) \to \mathrm{Aut}(H_1(\Sigma_{1,0}, \mathbb{Z}))$ in the top row might be injective, and this indeed turns out to be the case (cf. [FM, Par. 2.2.4, p. 54f]). Therefore the forgetful

map F_y must be bijective in our present case. For the kernel of the homomorphism $D_1 = F_y \circ C_1$, we therefore have $\mathrm{Ker}(D_1) = \mathrm{Ker}(C_1) = \langle \mathfrak{d}_1 \rangle$ by Proposition 3.1.5. For the Birman sequence

$$\pi_1(\Sigma_1, y) \xrightarrow{P_y} \Gamma_1(y) \xrightarrow{F_y} \Gamma_1 \longrightarrow 1$$

discussed in Paragraph 3.1.5, this implies that the pushing map P_y vanishes identically, and is in particular not injective. This does not contradict the claims made there, because the Euler characteristic $\chi(\Sigma_1) = 0$ is not strictly negative.

If we compose the monomorphism $\Gamma_{1,0}(y) \to \mathrm{Aut}(H_1(\Sigma_{1,0}, \mathbb{Z}))$ just discussed with the capping map $C_1 \colon \Gamma_{1,1} \to \Gamma_{1,0}(y)$, we obtain a homomorphism from $\Gamma_{1,1}$ to $\mathrm{Aut}(H_1(\Sigma_{1,0}, \mathbb{Z}))$. This map can also be constructed in another way. As discussed in Paragraph 1.3.2, the fundamental group $\pi_1(\Sigma_{1,1}, x)$ of a torus with one boundary component is a free (nonabelian) group whose generators are the relative homotopy classes $[\alpha_1]$ and $[\beta_1]$. But by Hurewicz' theorem, we then have that the first homology group $H_1(\Sigma_{1,1}, \mathbb{Z})$ is again free abelian with the homology classes of α_1 and β_1 as generators. This shows that the inclusion map $\Sigma_{1,1} \to \Sigma_1$ induces an isomorphism between the first homology groups, so that also $\mathrm{Aut}(H_1(\Sigma_{1,1}, \mathbb{Z})) \cong \mathrm{GL}(2, \mathbb{Z})$ with respect to these generators. In view of our considerations above, the Dehn twists $\mathfrak{t}_1$ and $\mathfrak{r}_1$ in $\Gamma_{1,1}$ are then represented by the same matrices as the corresponding Dehn twists in Γ_1.

As we said at the beginning of Paragraph 3.1.2, the Dehn twists $\mathfrak{r}_1$ and $\mathfrak{t}_1$ satisfy the braid relation, so that we obtain a group homomorphism

$$B_3 \to \Gamma_{1,1}$$

that maps r to $\mathfrak{r}_1$ and t to $\mathfrak{t}_1$. By the Dehn-Lickorish theorem 3.1.1, this homomorphism is surjective. Now the 2-chain relation (cf. [FM, Par. 4.4.1, p. 107f]) states that $(\mathfrak{r}_1 \mathfrak{t}_1)^6 = \mathfrak{d}_1$ (cf. [S, Par. 3.4.1, Fig. 141, p. 124] for an illustration).

We therefore have the commutative diagram

$$
\begin{array}{ccccccccc}
1 & \longrightarrow & \langle (rt)^6 \rangle & \longrightarrow & B_3 & \longrightarrow & \mathrm{SL}(2,\mathbb{Z}) & \longrightarrow & 1 \\
& & \downarrow & & \downarrow & & \downarrow {\scriptstyle \cong} & & \\
1 & \longrightarrow & \langle \mathfrak{d}_1 \rangle & \longrightarrow & \Gamma_{1,1} & \xrightarrow{\ D_1\ } & \Gamma_1 & \longrightarrow & 1
\end{array}
$$

in which the rightmost vertical map is the isomorphism described above. The five-lemma now implies that $\Gamma_{1,1} \cong B_3$ (cf. [FM, Par. 3.6.4, p. 87f]).

3.1.8 The mapping class group of the sphere

In the preceding paragraph, we have introduced the braid group on three strands. More generally, there is, for $n \geq 2$, a braid group B_n on n strands, which can be defined as the group generated by elements $\sigma_1, \ldots, \sigma_{n-1}$ subject to the relations

$$
\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}
$$

for $i = 1, \ldots, n-2$ and $\sigma_i \sigma_j = \sigma_j \sigma_i$ if $j > i+1$. Geometrically, σ_i can be interpreted as the interchange of the i-th and the $(i+1)$-st strand (cf. [FM, Sec. 9.2, p. 246f]). In the case $n = 3$ considered in the preceding paragraph, we have $r = \sigma_1$ and $t = \sigma_2$.

From this definition, we see that there is a group homomorphism from the braid group B_n to the symmetric group $\mathbb{S}_n$ that maps σ_i to the transposition of i and $i + 1$. Pulling the natural action of $\mathbb{S}_n$ back along this homomorphism, we get an action of B_n on the set $\{1, 2, \ldots, n\}$ that we denote by $i \mapsto \sigma.i$. If we view $\mathbb{Z}^n$ as the set of functions from $\{1, 2, \ldots, n\}$ to $\mathbb{Z}$, we can form the corresponding wreath product, which we denote by $\mathbb{Z}^n \rtimes B_n$. If $E_i = (e_i, 1_{B_n})$ denotes the i-th canonical basis vector, considered as an element of the wreath product, and if we identify $\sigma \in B_n$ with $(0, \sigma) \in \mathbb{Z}^n \rtimes B_n$, we have the commutation relation $\sigma E_i = E_{\sigma.i} \sigma$. The wreath product is therefore generated by the elements $\sigma_1, \ldots, \sigma_{n-1}$ together with $E_1, \ldots, E_n$. Besides the defining relations of the braid group stated above, these generators satisfy the relations $E_i E_j = E_j E_i$ as well as

$$\sigma_i E_j = \begin{cases} E_{i+1}\sigma_i & : j = i \\ E_i\sigma_i & : j = i+1 \\ E_j\sigma_i & : j \neq i \text{ and } j \neq i+1 \end{cases}$$

It is not too complicated to show that these relations are defining.

It turns out that the mapping class group $\Gamma_{0,n}$ is a quotient group of this wreath product. If we map σ_i to $\mathfrak{b}_{i,i+1}$ and E_i to $\mathfrak{d}_i$, we obtain a surjective group homomorphism from $\mathbb{Z}^n \rtimes B_n$ to $\Gamma_{0,n}$. However, the map is not bijective, because the generators $\mathfrak{b}_{i,i+1}$ and $\mathfrak{d}_i$ satisfy additional relations: Besides the relations

$$\mathfrak{b}_{i,i+1}\mathfrak{b}_{i+1,i+2}\mathfrak{b}_{i,i+1} = \mathfrak{b}_{i+1,i+2}\mathfrak{b}_{i,i+1}\mathfrak{b}_{i+1,i+2}$$

for $i = 1, \ldots, n-2$ and $\mathfrak{b}_{i,i+1}\mathfrak{b}_{j,j+1} = \mathfrak{b}_{j,j+1}\mathfrak{b}_{i,i+1}$ if $j > i+1$ for the braidings as well as $\mathfrak{d}_i\mathfrak{d}_j = \mathfrak{d}_j\mathfrak{d}_i$ and

$$\mathfrak{b}_{i,i+1}\mathfrak{d}_j = \begin{cases} \mathfrak{d}_{i+1}\mathfrak{b}_{i,i+1} & : j = i \\ \mathfrak{d}_i\mathfrak{b}_{i,i+1} & : j = i+1 \\ \mathfrak{d}_j\mathfrak{b}_{i,i+1} & : j \neq i \text{ and } j \neq i+1 \end{cases}$$

for the Dehn twists and their interaction with the braidings, we have the additional relations

$$\mathfrak{b}_{1,2}\mathfrak{b}_{2,3}\cdots \mathfrak{b}_{n-1,n}^2\cdots \mathfrak{b}_{2,3}\mathfrak{b}_{1,2} = \mathfrak{d}_1^{-2}$$

and $(\mathfrak{b}_{1,2}\mathfrak{b}_{2,3}\cdots \mathfrak{b}_{n-1,n})^n = \mathfrak{d}_1^{-1}\mathfrak{d}_2^{-1}\cdots \mathfrak{d}_n^{-1}$. These relations all together are again defining (cf. [L1, Par. 4.2, p. 487], see also [L2, Par. 3.1, p. 322f]).

Clearly, we can embed the wreath product $\mathbb{Z}^{n-1} \rtimes B_{n-1}$ into the wreath product $\mathbb{Z}^n \rtimes B_n$ by sending σ_i to σ_i and E_i to E_i. By composing this injection with the surjection just described, we get a group homomorphism from $\mathbb{Z}^{n-1} \rtimes B_{n-1}$ to $\Gamma_{0,n}$. It clearly takes values in the subgroup $\Gamma_{0,n}(\rho_n)$ that fixes the last boundary component pointwise. The fundamental fact about this map is the following:

Proposition 3.1.6 The group homomorphism $\mathbb{Z}^{n-1} \rtimes B_{n-1} \to \Gamma_{0,n}(\rho_n)$ that maps σ_i to $\mathfrak{b}_{i,i+1}$ and E_i to $\mathfrak{d}_i$ is an isomorphism.

This proposition follows from the fact that we can view a sphere with one boundary component as a closed disk, say of radius 1. The mapping class group $\Gamma_{0,n}(\rho_n)$ is therefore isomorphic to the mapping class group of a disk with $n-1$-holes in which the diffeomorphisms are required to restrict to the identity on the boundary unit circle. One of the standard topological descriptions of the braid group yields that this mapping class group is isomorphic to $\mathbb{Z}^{n-1} \rtimes B_{n-1}$ in the indicated way (cf. [PS, § 7, p. 64], see also [FM, Par. 9.1.3, p. 243f]).

3.2 Tensor categories

In this section, we will assigns representations of mapping class groups to modular tensor categories using the framework created by V. Lyubashenko in his articles [L1] and [L2]. In particular, we use the approach to surfaces via nets and ribbon graphs described in his articles. We will first need to introduce another important tool:

3.2.1 Coends

If $\mathcal{C}$ and $\mathcal{D}$ are categories and $F \colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{D}$ is a bifunctor, a coend of F is an object L of $\mathcal{D}$ together with a morphism $\iota_X \colon F(X,X) \to L$ for every object $X \in \mathcal{C}$ which is dinatural in the sense that, for every morphism $f \colon X \to Y$, the diagram

$$
\begin{array}{ccc}
F(Y,X) & \xrightarrow{F(\mathrm{id}_Y, f)} & F(Y,Y) \\
{\scriptstyle F(f,\mathrm{id}_X)}\big\downarrow & & \big\downarrow{\scriptstyle \iota_Y} \\
F(X,X) & \xrightarrow[\iota_X]{} & L
\end{array}
$$

commutes (cf. [ML2, Chap. IX, Sec. 6, p. 226f]). Moreover, this dinatural transformation is required to be universal, which means that for another dinatural transformation $\kappa_X \colon F(X,X) \to Z$, there is a unique morphism $g \colon L \to Z$ that makes for all $X \in \mathcal{C}$ the diagram

$$F(X, X) \xrightarrow{\iota_X} L$$

(with κ_X diagonal to Z and $g\colon L \to Z$)

commutative. We will also write $L = \int^X F(X, X)$.

From the universality, we see that coends behave well with respect to natural transformations:

Lemma 3.2.1 Suppose that $F'\colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{D}$ is a second bifunctor with coend $\iota'_X\colon F'(X, X) \to L'$, and that $\eta_{X,Y}\colon F(X, Y) \to F'(X, Y)$ is a natural transformation. Then there is a unique morphism $g\colon L \to L'$ such that the diagram

$$
\begin{array}{ccc}
F(X, X) & \xrightarrow{\iota_X} & L \\
\downarrow{\scriptstyle \eta_{X,X}} & & \downarrow{\scriptstyle g} \\
F'(X, X) & \xrightarrow{\iota'_X} & L'
\end{array}
$$

commutes for all objects $X \in \mathcal{C}$.

Proof. This lemma is the version of [ML2, Chap. IX, Sec. 7, Prop. 1, p. 228] for coends instead of ends. $\square$

When we speak of a coend without further specifications, we think of the case where $\mathcal{C} = \mathcal{D}$ is our base category satisfying the requirements stated in Paragraph 1.2.1, and the bifunctor F is given by $F(X, Y) = X^* \otimes Y$. It is shown in [KL, Par. 5.1.3, p. 266ff] that, under our assumptions on $\mathcal{C}$, a coend for this bifunctor F exists.

It is not too difficult to see that a functor that possesses a right adjoint preserves coends. It follows that

$$\mathrm{id}_{A^* \otimes B} \otimes \iota_X \colon A^* \otimes B \otimes X^* \otimes X \to A^* \otimes B \otimes L$$

is a coend for the functor $F(X, Y) = A^* \otimes B \otimes X^* \otimes Y$ for any objects $A, B \in \mathcal{C}$, and that

$$\iota_X \otimes \mathrm{id}_L \colon X^* \otimes X \otimes L \to L \otimes L$$

is a coend for the functor $F(X, Y) = X^* \otimes Y \otimes L$. Together with the Fubini theorem for coends (cf. [ML2, Chap. IX, Sec. 8, p. 230f]), this yields that the dinatural transformation

$$\iota_X \otimes \iota_Y \colon X^* \otimes X \otimes Y^* \otimes Y \to L \otimes L$$

is a coend for the functor $F(W, X, Y, Z) = W^* \otimes X \otimes Y^* \otimes Z$. Comparing this transformation with the dinatural transformation

$$(\mathrm{ev}_X \otimes \mathrm{ev}_Y) \circ (\mathrm{id}_{X^*} \otimes (c_{Y^*, X} \circ c_{X, Y^*}) \otimes \mathrm{id}_Y) \colon X^* \otimes X \otimes Y^* \otimes Y \to \mathbb{1},$$

the universal property yields a morphism $\omega_L \colon L \otimes L \to \mathbb{1}$ that, via the adjunctions stated in Paragraph 1.2.1, determines homomorphisms $\omega'_L \colon L \to L^*$ and $\omega''_L \colon L \to {}^*L$, which are duals of each other. It is shown in [Sh1, Thm. 1.1, p. 3] that the modularity of $\mathcal{C}$ is equivalent to the property that ω'_L, or alternatively ω''_L, is an isomorphism. This property is used in [KL, Def. 5.2.7, p. 276] as the definition of modularity.

The same method can be used to introduce other morphisms that will be needed in the sequel. The easiest of these morphisms arises from the dinatural transformation $\iota_X \circ (\mathrm{id}_{X^*} \otimes \theta_X) \colon X^* \otimes X \to L$. Applying the universal property of the coend yields a morphism $\mathfrak{T} \colon L \to L$ that makes all diagrams of the form

$$
\begin{array}{ccc}
X^* \otimes X & \xrightarrow{\iota_X} & L \\
& \searrow_{\iota_X \circ (\mathrm{id}_{X^*} \otimes \theta_X)} & \downarrow_{\mathfrak{T}} \\
& & L
\end{array}
$$

commutative. Similarly, the universal property of $L \otimes L$ explained above can be used to obtain a morphism $\mathfrak{N}' \colon L \otimes L \to L \otimes L$ that makes all diagrams of the form

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$$X^* \otimes X \otimes Y^* \otimes Y \xrightarrow{\iota_X \otimes \iota_Y} L \otimes L$$

$$\big(\iota_X \otimes \iota_Y\big) \circ \big(\mathrm{id}_{X^*} \otimes (c_{Y^*,X} \circ c_{X,Y^*}) \otimes \mathrm{id}_Y\big) \searrow \quad \downarrow \mathfrak{N}'$$

$$L \otimes L$$

commutative. Both of these morphisms are used when defining $\mathfrak{N} := \mathfrak{N}' \circ (\mathfrak{T} \otimes \mathfrak{T})$. An in a sense hybrid form of the dinatural transformations used in the definition of ω_L and of $\mathfrak{N}'$ is used in the definition of $\mathfrak{S}'$ from $L \otimes L$ to L: It arises by applying the universal property of $L \otimes L$ to a dinatural transformation in such a way that the diagram

$$X^* \otimes X \otimes Y^* \otimes Y \xrightarrow{\iota_X \otimes \iota_Y} L \otimes L$$

$$\big(\mathrm{ev}_X \otimes \iota_Y\big) \circ \big(\mathrm{id}_{X^*} \otimes (c_{Y^*,X} \circ c_{X,Y^*}) \otimes \mathrm{id}_Y\big) \searrow \quad \downarrow \mathfrak{S}'$$

$$L$$

becomes commutative.

Given an object $Z \in \mathcal{C}$, we define another morphism that is similar to $\mathfrak{N}'$: Again, because a functor that possesses a right adjoint preserves coends, the dinatural transformation $\mathrm{id}_Z \otimes \iota_X : Z \otimes X^* \otimes X \to Z \otimes L$ is a coend. Its universal property yields a morphism $\mathfrak{N}^l_{Z,L} : Z \otimes L \to Z \otimes L$ that makes all diagrams of the form

$$Z \otimes X^* \otimes X \xrightarrow{\mathrm{id}_Z \otimes \iota_X} Z \otimes L$$

$$\big(\mathrm{id}_Z \otimes \iota_X\big) \circ \big((c_{X^*,Z} \circ c_{Z,X^*}) \otimes \mathrm{id}_X\big) \searrow \quad \downarrow \mathfrak{N}^l_{Z,L}$$

$$Z \otimes L$$

commutative.

The coend L is a Hopf algebra inside the category $\mathcal{C}$. We will not need the arising product, coproduct, unit, counit, and antipode, which are constructed in a similar way as the morphisms ω_L, $\mathfrak{T}$, $\mathfrak{N}'$, and $\mathfrak{S}'$ by using the universal property of a coend and are described in [V, Par. 1.6, p. 478f] (cf. also [KL, Par. 5.2.2, p. 271ff]; note that the conventions are slightly different there). We will, however, need that, as a consequence of the Hopf algebra structure, there are two-sided integrals $\Lambda_L : \mathbb{1} \to L$

and $\lambda_L \colon L \to \mathbb{1}$. Here, the assumption on modularity is used for two points, namely on the one hand for the fact that the unit object $\mathbb{1}$ can indeed be used as the domain of Λ_L and the codomain of λ_L, and on the other hand for the fact that λ_L is two-sided (cf. [KL, Sec. 5.2, p. 270ff]). The integral Λ_L is used to define $\mathfrak{S} \in \mathrm{End}(L)$ as the composition

$$L \xrightarrow{\cong} L \otimes \mathbb{1} \xrightarrow{\mathrm{id}_L \otimes \Lambda_L} L \otimes L \xrightarrow{\mathfrak{S}'} L$$

(cf. [L1, Par. 1.3, p. 473]; the setting there is slightly more general as the modularity hypothesis is weakened).

3.2.2 Coends from Hopf algebras

In the case where $\mathcal{C}$ is the category of finite-dimensional left modules over a factorizable ribbon Hopf algebra A as reviewed in Paragraph 1.2.2, the coend can be described explicitly: The dual vector space $L := A^*$, viewed as a left A-module via the left coadjoint action

$$(a.\varphi)(a') := \varphi(S(a_{(1)})a'a_{(2)})$$

for $a, a' \in A$ and $\varphi \in A^*$, becomes a coend when endowed with the dinatural transformation

$$\iota_X(\xi \otimes x)(a) = \xi(a.x)$$

for a left A-module X and elements $\xi \in X^*$, $x \in X$, and $a \in A$ (cf. [V, Lem. 4.3, p. 498], see also [KL, Thm. 7.4.13, p. 331]). It is not difficult to find the explicit form of the morphisms introduced in Paragraph 3.2.1 in this model of the coend:

Proposition 3.2.2 Suppose that $\varphi, \psi \in A^*$ and that $a, a' \in A$. Then we have

1. $\mathfrak{T}(\varphi)(a) = \varphi(av)$

2. $\mathfrak{N}'(\varphi \otimes \psi)(a \otimes a') = \varphi(aQ^1)\psi(S(Q^2)a')$

3. $\mathfrak{S}'(\varphi \otimes \psi)(a) = \varphi(Q^1)\psi(S(Q^2)a)$

4. $\omega_L(\varphi \otimes \psi) = \varphi(Q^1)\psi(S(Q^2))$

where $v \in A$ and $Q \in A \otimes A$ denote the ribbon element and the monodromy matrix, respectively, which were introduced in Paragraph 1.2.2.

Proof. We prove these assertions by showing that the respective right-hand side indeed has the required universal property. For the first assertion, this holds since we have for an A-module X and $\xi \in X^*, x \in X$ that

$$((\iota_X \circ (\mathrm{id}_{X^*} \otimes \theta_X))(\xi \otimes x))(a) = \iota_X(\xi \otimes v.x)(a) = \xi(av.x) = \mathfrak{T}(\iota_X(\xi \otimes x))(a).$$

The proof of the second assertion is similar: If Y is another A-module and $\zeta \in Y^*$, $y \in X$ are elements, we have

$$([(\iota_X \otimes \iota_Y) \circ (\mathrm{id}_{X^*} \otimes (c_{Y^*,X} \circ c_{X,Y^*}) \otimes \mathrm{id}_Y)](\xi \otimes x \otimes \zeta \otimes y))(a \otimes a')$$
$$= ((\iota_X \otimes \iota_Y)(\xi \otimes Q^1.x \otimes Q^2.\zeta \otimes y))(a \otimes a') = \xi(aQ^1.x)(Q^2.\zeta)(a'.y)$$
$$= \xi(aQ^1.x)\zeta(S(Q^2)a'.y) = \mathfrak{N}'(\iota_X(\xi \otimes x) \otimes \iota_Y(\zeta \otimes y))(a \otimes a').$$

The proofs of the third and the fourth assertion are again very similar. $\square$

Because the antipode of a finite-dimensional Hopf algebra is bijective (cf. [M, Thm. 2.1.3, p. 18]), the fourth assertion shows that ω_L is nondegenerate if and only if the map Φ introduced in Paragraph 1.2.2 is bijective. This shows that the modularity of $\mathcal{C}$ is equivalent to the factorizability of A (cf. [KL, Par. 7.4.6, p. 332]).

We note that, because these maps are defined via the universal property of the coend, they must automatically be morphisms in $\mathcal{C}$, i.e., they must be A-linear. We also note that in the second assertion, we have chosen a special identification of $A^* \otimes A^*$ with $(A \otimes A)^*$, which is determined by the following equation $(\varphi \otimes \psi)(a \otimes a') = \varphi(a)\psi(a')$.

It is important to stress that the Hopf algebra structure on L is not the usual Hopf algebra structure on A^*. In fact, A^* is a Hopf algebra in the category of vector spaces, whereas L is a Hopf algebra in the category $\mathcal{C}$, which has a different braiding. The Hopf algebra structure on L is rather dual to the so-called transmuted

Hopf algebra structure of A (cf. [M1, Examp. 9.4.9, p. 504]; [V, Lem. 4.4, p. 499]). However, there is a very direct relation between the integrals of A^* and L: Because a factorizable Hopf algebra is unimodular (cf. [L1, Prop. 3.7.4, p. 482] and [R, Prop. 12.4.2, p. 405f]), there is a nonzero two-sided integral $\Lambda_A \in A$. This leads to the two-sided integral

$$\lambda_L \colon L \to \mathbb{1} = K, \ \varphi \mapsto \varphi(\Lambda_A).$$

On the other hand, a right integral $\rho \colon A \to K$ is by definition contained in $L = A^*$, and it can be shown that the unique morphism $\Lambda_L \colon \mathbb{1} = K \to L$ that maps 1_K to ρ is even a two-sided integral (cf. [BKLT, Prop. 6.6, p. 153]). As a consequence, the morphism $\mathfrak{S}$ is given by the formula

$$\mathfrak{S}(\varphi)(a) = \varphi(Q^1)\rho(S(Q^2)a).$$

3.2.3 The block spaces

In topological field theory, one associates to a modular category projective representations of mapping class groups of surfaces. There is a vast literature on this topic; our approach here is based on a construction given by V. Lyubashenko in his articles [L1] and [L2]. We will now briefly review some key aspects of his construction to the extent that we need them. Suppose that $\mathcal{C}$ is a category that satisfies the requirements stated in Paragraph 1.2.1. We assume that each of the n boundary components of the surface $\Sigma_{g,n}$ described in Paragraph 1.3.1 is labeled by an object X_i from $\mathcal{C}$. Associated with such a labeled surface is the vector space

$$\mathcal{Z}(\Sigma_{g,n}^{X_1,\dots,X_n}) := \mathrm{Hom}_{\mathcal{C}}(X_n \otimes \cdots \otimes X_1, L^{\otimes g}),$$

where L denotes the coend defined in Paragraph 3.2.1. This space, which we call the space of chiral conformal blocks or briefly the block space, obviously depends functorially on the labels and can therefore be viewed as the value of a left exact contravariant functor $\mathcal{Z}(\Sigma_{g,n})$ on the object $(X_1, \dots, X_n) \in \mathcal{C}^n$, by which we mean that the functor is left exact in each argument (cf. [P, Par. 4.6, p. 130f]).

The construction of a projective action of the mapping class group then starts from an oriented net (cf. [L2, Sec. 6, p. 355] and [L1, Par. 4.1, p. 486f]). We consider the following oriented net that encodes the structure of our surface:
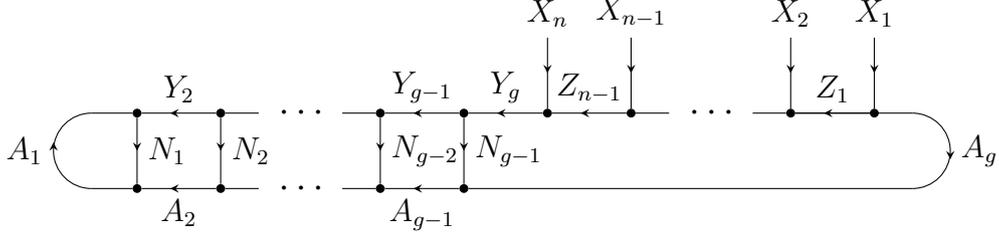
The block space arises from the oriented net by taking suitable coends over internal edges (cf. [L2, Par. 8.2, p. 374]). For our chosen net, the construction gives the vector space

$$T_1 :=$$

$$\int^{A_i, N_j, Z_l, Y_m} \mathrm{Hom}(A_1 \otimes Y_2, N_1) \otimes \mathrm{Hom}(Y_3, Y_2 \otimes N_2) \otimes \cdots \otimes \mathrm{Hom}(Y_g, Y_{g-1} \otimes N_{g-1})$$

$$\otimes \mathrm{Hom}(X_n \otimes Z_{n-1}, Y_g) \otimes \mathrm{Hom}(X_{n-1} \otimes Z_{n-2}, Z_{n-1}) \otimes \cdots \otimes \mathrm{Hom}(X_2 \otimes Z_1, Z_2)$$

$$\otimes \mathrm{Hom}(X_1, Z_1 \otimes A_g) \otimes \mathrm{Hom}(N_{g-1} \otimes A_g, A_{g-1}) \otimes \cdots \otimes \mathrm{Hom}(N_1 \otimes A_2, A_1)$$

where in the upper limit of the integral sign the abbreviation A_i has been used for $A_1, A_2, \ldots, A_g$. Similarly, the abbreviations N_j, Z_l, and Y_m have been used for the objects with indices $j = 1, \ldots, g-1$, $l = 1, \ldots, n-1$, and $m = 2, \ldots, g$. The coends are taken in the category of left exact functors: Obviously, the argument of the coend depends functorially on the objects, and the corresponding functor is left exact. A more detailed discussion of this aspect can be found in [L2, App. B, p. 398] and [FS1, Sec. 3, p. 72ff].

In view of [L2, Lem. B.1, p. 398] and the functor adjunctions discussed in Paragraph 1.2.1, the vector space T_1 is naturally isomorphic to

$$T_2 := \int^{A_i, N_j, Y_2, Y_g, Z_1} \mathrm{Hom}(Y_2, {}^*A_1 \otimes N_1) \otimes \mathrm{Hom}(Y_g, Y_2 \otimes N_2 \otimes \cdots \otimes N_{g-1})$$

$$\otimes \mathrm{Hom}(X_n \otimes \cdots \otimes X_2 \otimes Z_1, Y_g) \otimes \mathrm{Hom}(X_1 \otimes {}^*A_g, Z_1)$$

$$\otimes \mathrm{Hom}(N_{g-1}, A_{g-1} \otimes A_g^*) \otimes \cdots \otimes \mathrm{Hom}(N_1, A_1 \otimes A_2^*)$$

Taking also the coend over Y_g, we obtain a natural isomorphism with the vector space

$$T_3 := \int^{A_i, N_j, Z_1, Y_2} \mathrm{Hom}(Y_2, {}^*A_1 \otimes N_1)$$

$$\otimes \mathrm{Hom}(X_n \otimes \cdots \otimes X_2 \otimes Z_1, Y_2 \otimes N_2 \otimes \cdots \otimes N_{g-1}) \otimes \mathrm{Hom}(X_1 \otimes {}^*A_g, Z_1)$$

$$\otimes \mathrm{Hom}(N_{g-1}, A_{g-1} \otimes A_g^*) \otimes \cdots \otimes \mathrm{Hom}(N_1, A_1 \otimes A_2^*)$$

Treating $N_1, \ldots, N_{g-1}$ in the same way and moreover taking the coend over Z, we obtain the vector space

$$T_4 := \int^{A_i, Y_2} \mathrm{Hom}(Y_2, {}^*A_1 \otimes A_1 \otimes A_2^*)$$

$$\otimes \mathrm{Hom}(X_n \otimes \cdots \otimes X_1 \otimes {}^*A_g, Y_2 \otimes A_2 \otimes A_3^* \otimes \cdots \otimes A_{g-1} \otimes A_g^*)$$

Finally, if we also take the coend over Y_2 and apply the functor adjunctions to A_g, we arrive at the vector space

$$T_5 := \int^{A_i} \mathrm{Hom}(X_n \otimes \cdots \otimes X_1, {}^*A_1 \otimes A_1 \otimes A_2^* \otimes A_2 \otimes \cdots \otimes A_{g-1} \otimes A_g^* \otimes A_g)$$

It is now important to realize that we have introduced right duals in Paragraph 1.2.1 in such a way that left and right duals have the same underlying object. They differ only in their evaluation and coevaluation morphisms, which affect the functor adjunctions, but not the objects themselves. The vector space T_5 is therefore also equal to

$$T_5 = \int^{A_i} \mathrm{Hom}(X_n \otimes \cdots \otimes X_1, A_1^* \otimes A_1 \otimes \cdots \otimes A_g^* \otimes A_g)$$

Because the coends are taken in the category of left exact functors, the vector space T_5 is naturally isomorphic to our block space $\operatorname{Hom}(X_n \otimes \cdots \otimes X_1, L^{\otimes g})$ defined above (cf. [FS1, Prop. 3.4, p. 74]).

We have claimed that the specific oriented net given above encodes the structure of the surface introduced in Paragraph 1.3.1. We now explain this relation in the special case $g = 3$ and $n = 2$; the general case is not essentially different. The first step consists of the application of the fattening functor (cf. [L2, p. 341]), which turns the net into a so-called ribbon graph by replacing each trivalent vertex by a hexagon in which every second side is represented by a double line. These hexagons are arranged so that their double lines come to lie on the edges of the net, and if an edge connects two vertices, the corresponding hexagons are glued along their now common double line. If we apply this procedure to our net above, we obtain the ribbon graph
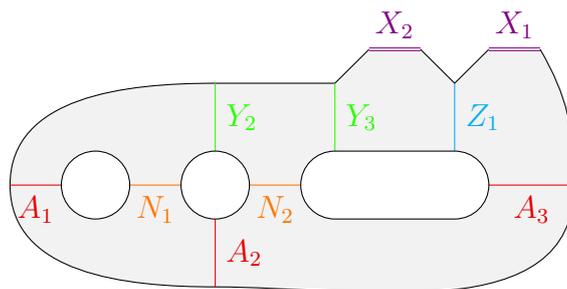
in which the colored lines indicate the glued double lines.

The second step consists in the application of the duplication functor (cf. [L2, Par. 2.2 and Par. 2.3, p. 315ff]). The duplication functor turns a ribbon graph into a surface by taking two copies of the graph and gluing them along the boundary components that do not arise from the double lines mentioned above. Applied to our ribbon graph, it yields a surface of genus 3 that has two boundary components, which arise from the two double lines still present in our ribbon graph and are labeled by X_1 and X_2. If we deform the arising surface slightly and bring the first and the third handle closer together by pulling the middle handle to the bottom, we obtain the standard surface
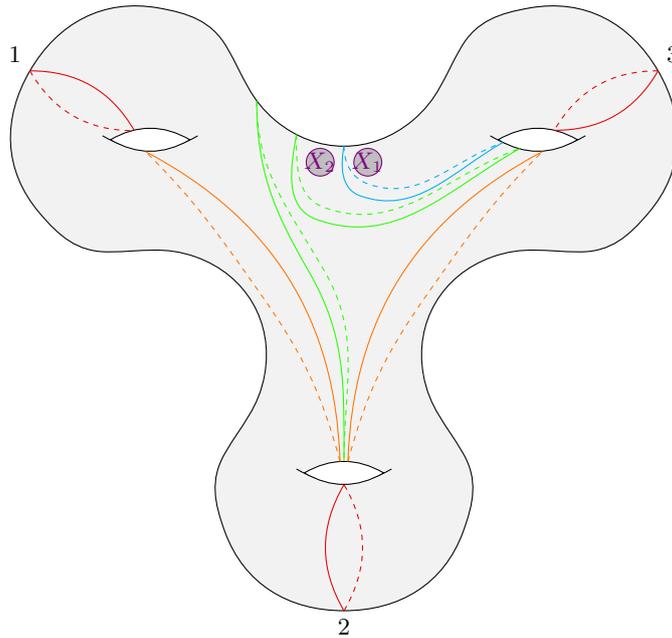
described in Paragraph 1.3.1, in which the curves arise from the duplication of the lines in the ribbon graph that have the same color. It should be noted that the red curves are freely homotopic to the curves denoted by α_1, α_2, and α_3 in Paragraph 1.3.1, and the orange curves are freely homotopic to the curves μ_1 and μ_2 from Paragraph 1.3.2, respectively.

It is possible to compare this discussion with the corresponding one in [L1, Par. 4.5, p. 494f]. To do that, it is necessary to deform the surface given in [L1, Fig. 6, p. 495] by pulling down the middle handles to the bottom in order to bring the first and the g-th handle together at the top, exactly as in our discussion of the relation between the net and the surface above. This leads to a surface in which the handles are labeled counterclockwise, but the boundary components are labeled clockwise, in contrast to the surface used in Paragraph 1.3.1, where the boundary components arose from the polygon and therefore were also labeled counterclockwise. As a consequence, the order of the labels $X_1, \ldots, X_n$ is reversed. The following table explains the relation between our surface and the surface in [L1, Fig. 2, p. 490 and Fig. 6, p. 495]:

Curves	α_i	β_i	μ_j	ζ_l	∂_k
Color	red	blue	orange	turquoise	purple
Label in the net	A_i		N_j	Z_l	X_k
Dehn twists	$\mathfrak{t}_i$	$\mathfrak{r}_i$	$\mathfrak{n}_j$	$\mathfrak{z}_l$	$\mathfrak{d}_k$
[L1]	e_i	b_i	a_{j+1}	$t_{n-l,g}$	R_{g-k+1}

Here, the index ranges are $i = 1, \ldots, g$, $j = 1, \ldots, g-1$, $k = 1, \ldots, n$, and $l = 1, \ldots, n-1$. The light green curves in our surface correspond to the curves labeled d_j in [L1, loc. cit.].

3.2.4 Mapping class group representations

As we just explained, the theory associates with a surface a certain left exact functor. But in addition, the theory associates with a mapping class $[\psi] \in \Gamma_{g,n}$ a projective class $[\mathcal{Z}(\psi)] = P(\mathcal{Z}(\psi))$ of natural equivalences between two of these functors, namely the functor

$$(X_1, \ldots, X_n) \mapsto \mathcal{Z}(\Sigma_{g,n}^{X_1, \ldots, X_n})$$

described at the beginning of Paragraph 3.2.3 on the one hand and the functor

$$(X_1, \ldots, X_n) \mapsto \mathcal{Z}(\Sigma_{g,n}^{X_{\tau(1)}, \ldots, X_{\tau(n)}})$$

on the other hand, where $\tau := p([\psi])^{-1} \in \mathbb{S}_n$ is the inverse permutation of the marked points introduced in Paragraph 1.3.3. A representative $\mathcal{Z}(\psi)$ of this projective class is a natural equivalence between these functors, i.e., a family of linear isomorphisms

$$\mathcal{Z}(\psi) \colon \mathcal{Z}(\Sigma_{g,n}^{X_1, \ldots, X_n}) \to \mathcal{Z}(\Sigma_{g,n}^{X_{\tau(1)}, \ldots, X_{\tau(n)}})$$

that are natural in the following sense:

If, for $i = 1, \ldots, n$, we are given morphisms $f_i \colon X_i \to Y_i$, then the diagram

$$
\begin{array}{ccc}
\mathcal{Z}(\Sigma_{g,n}^{Y_1,\ldots,Y_n}) & \xrightarrow{\ \mathcal{Z}(\psi)\ } & \mathcal{Z}(\Sigma_{g,n}^{Y_{\tau(1)},\ldots,Y_{\tau(n)}}) \\
\downarrow{\scriptstyle \circ(f_n \otimes \cdots \otimes f_1)} & & \downarrow{\scriptstyle \circ(f_{\tau(n)} \otimes \cdots \otimes f_{\tau(1)})} \\
\mathcal{Z}(\Sigma_{g,n}^{X_1,\ldots,X_n}) & \xrightarrow{\ \mathcal{Z}(\psi)\ } & \mathcal{Z}(\Sigma_{g,n}^{X_{\tau(1)},\ldots,X_{\tau(n)}})
\end{array}
$$

commutes. Clearly, a nonzero scalar multiple of such a natural equivalence is again a natural equivalence, and the set of all its nonzero scalar multiples constitutes its projective class $[\mathcal{Z}(\psi)] = P(\mathcal{Z}(\psi))$.

This assignment is also compatible with composition in the sense that, for a second mapping class $[\phi]$, we have $[\mathcal{Z}(\phi \circ \psi)] = [\mathcal{Z}(\phi) \circ \mathcal{Z}(\psi)]$. However, this does not imply that the mapping class group $\Gamma_{g,n}$ acts projectively on each block space $\mathcal{Z}(\Sigma_{g,n}^{X_1,\ldots,X_n})$, because the block spaces are not preserved by $\mathcal{Z}(\psi)$ if $[\psi]$ permutes two boundary components with different labels. In general, only the pure mapping class group $\mathrm{P}\Gamma_{g,n}$ acts projectively on each block space. However, the entire mapping class group $\Gamma_{g,n}$ acts projectively on the direct sum

$$
\bigoplus_{\tau \in \mathbb{S}_n} \mathcal{Z}(\Sigma_{g,n}^{X_{\tau(1)},\ldots,X_{\tau(n)}})
$$

of block spaces. In the case where all boundary components are labeled with the same object X, we also have an action of $\Gamma_{g,n}$ on the vector space $\mathrm{Hom}_{\mathcal{C}}(X^{\otimes n}, L^{\otimes g})$. We can in fact realize the direct sum above as a subspace of this vector space by setting $X := X_1 \oplus \cdots \oplus X_n$.

Via a sophisticated system of rules laid out in [L1] and [L2], the description of a surface in terms of a net makes it possible to describe the actions of the elements of the mapping class group on the block spaces explicitly. According to this formalism, the elements of the mapping class group introduced in Paragraph 1.3.4, Paragraph 1.3.5, Paragraph 3.1.2, and Paragraph 3.1.3 act on the block spaces as follows:

(1) For $j = 1, \ldots, n$, the Dehn twist $\mathfrak{d}_j$ acts by precomposition with the morphism $\mathrm{id}_{X_n \otimes \cdots \otimes X_{j+1}} \otimes \theta_{X_j} \otimes \mathrm{id}_{X_{j-1} \otimes \cdots \otimes X_1}$, i.e., via the map

$$\mathcal{Z}(\mathfrak{d}_j)\colon \mathrm{Hom}_{\mathcal{C}}(X_n \otimes \cdots \otimes X_1, L^{\otimes g}) \to \mathrm{Hom}_{\mathcal{C}}(X_n \otimes \cdots \otimes X_1, L^{\otimes g})$$
$$f \mapsto f \circ (\mathrm{id}_{X_n \otimes \cdots \otimes X_{j+1}} \otimes \theta_{X_j} \otimes \mathrm{id}_{X_{j-1} \otimes \cdots \otimes X_1}).$$

(2) For $j = 1, \ldots, n - 1$, the braiding $\mathfrak{b}_{j,j+1}$ acts by precomposition with the morphism $\mathrm{id}_{X_n \otimes \cdots \otimes X_{j+2}} \otimes c_{X_j, X_{j+1}} \otimes \mathrm{id}_{X_{j-1} \otimes \cdots \otimes X_1}$, i.e., it induces a map

$$\mathcal{Z}(\mathfrak{b}_{j,j+1})\colon \mathrm{Hom}_{\mathcal{C}}(X_n \otimes \cdots \otimes X_{j+1} \otimes X_j \otimes \cdots \otimes X_1, L^{\otimes g})$$
$$\to \mathrm{Hom}_{\mathcal{C}}(X_n \otimes \cdots \otimes X_j \otimes X_{j+1} \otimes \cdots \otimes X_1, L^{\otimes g})$$
$$f \mapsto f \circ (\mathrm{id}_{X_n \otimes \cdots \otimes X_{j+2}} \otimes c_{X_j, X_{j+1}} \otimes \mathrm{id}_{X_{j-1} \otimes \cdots \otimes X_1}).$$

(3) For $j = 1, \ldots, n - 1$, the element $\mathfrak{d}_{j,j+1}$ acts by precomposition with the morphism $\mathrm{id}_{X_n \otimes \cdots \otimes X_{j+2}} \otimes \theta_{X_{j+1} \otimes X_j} \otimes \mathrm{id}_{X_{j-1} \otimes \cdots \otimes X_1}$, a map that we denote by $\mathcal{Z}(\mathfrak{d}_{j,j+1})$.

(4) For $j = 1, \ldots, n - 1$, the element $\mathfrak{q}_{j,j+1}$ acts by precomposition with the morphism $\mathrm{id}_{X_n \otimes \cdots \otimes X_{j+2}} \otimes (c_{X_j, X_{j+1}} \circ c_{X_{j+1}, X_j}) \otimes \mathrm{id}_{X_{j-1} \otimes \cdots \otimes X_1}$, a map that we denote by $\mathcal{Z}(\mathfrak{q}_{j,j+1})$.

(5) For $i = 1, \ldots, g$, the Dehn twist $\mathfrak{t}_i$ acts by postcomposition with the morphism $\mathrm{id}_{L^{\otimes(i-1)}} \otimes \mathfrak{T} \otimes \mathrm{id}_{L^{\otimes(g-i)}}$, i.e., via the map

$$\mathcal{Z}(\mathfrak{t}_i)\colon \mathrm{Hom}_{\mathcal{C}}(X_n \otimes \cdots \otimes X_1, L^{\otimes g}) \to \mathrm{Hom}_{\mathcal{C}}(X_n \otimes \cdots \otimes X_1, L^{\otimes g})$$
$$f \mapsto (\mathrm{id}_{L^{\otimes(i-1)}} \otimes \mathfrak{T} \otimes \mathrm{id}_{L^{\otimes(g-i)}}) \circ f.$$

(6) For $i = 1, \ldots, g$, the element $\mathfrak{s}_i$ acts by postcomposition with the morphism $\mathrm{id}_{L^{\otimes(i-1)}} \otimes \mathfrak{S} \otimes \mathrm{id}_{L^{\otimes(g-i)}}$, a map that we denote by $\mathcal{Z}(\mathfrak{s}_i)$.

(7) For $i = 1, \ldots, g - 1$, the Dehn twist $\mathfrak{n}_i$ acts by postcomposition with the morphism $\mathrm{id}_{L^{\otimes(i-1)}} \otimes \mathfrak{N} \otimes \mathrm{id}_{L^{\otimes(g-i-1)}}$, a map that we denote by $\mathcal{Z}(\mathfrak{n}_i)$.

(8) For $j = 1, \ldots, n - 1$, the Dehn twist $\mathfrak{z}_j$ acts via the map

$$\mathcal{Z}(\mathfrak{z}_j)\colon \mathrm{Hom}_{\mathcal{C}}(X_n \otimes \cdots \otimes X_1, L^{\otimes g}) \to \mathrm{Hom}_{\mathcal{C}}(X_n \otimes \cdots \otimes X_1, L^{\otimes g})$$

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which is the unique linear map that makes the diagram

$$\mathrm{Hom}_{\mathcal{C}}(X_j\otimes\cdots\otimes X_1, {}^*(X_n\otimes\cdots\otimes X_{j+1})\otimes L^{\otimes g}) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(X_n\otimes\cdots\otimes X_1, L^{\otimes g})$$

$$\downarrow \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \downarrow {\scriptstyle \mathcal{Z}(\mathfrak{z}_j)}$$

$$\mathrm{Hom}_{\mathcal{C}}(X_j\otimes\cdots\otimes X_1, {}^*(X_n\otimes\cdots\otimes X_{j+1})\otimes L^{\otimes g}) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(X_n\otimes\cdots\otimes X_1, L^{\otimes g})$$

commutative. In this diagram, the horizontal arrows are the functor adjunctions from Paragraph 1.2.1, and the left vertical map is given by

$$h \mapsto (\theta_{*(X_n\otimes\cdots\otimes X_{j+1})\otimes L^{\otimes(g-1)}} \otimes \mathfrak{T}) \circ \mathfrak{N}^l_{*(X_n\otimes\cdots\otimes X_{j+1})\otimes L^{\otimes(g-1)},L} \circ h.$$

We now explain how some of the formulas come about. For any edge of a trivalent net, there is an automorphism called the twist (cf. [L2, Par. 4.3, p. 350]) that corresponds to a Dehn twist around the corresponding curve on the surface (cf. [L2, Prop. 2.2, p. 319]). Recall that, as already mentioned in Paragraph 1.3.4, Dehn twists as defined here are inverse Dehn twists as defined in [L1] and [L2]. By [L2, Par. 8.1, No. (x), p. 374], the twist acts by applying the ribbon twist to the corresponding variable. Because the isomorphisms between the vector spaces T_1 to T_5 introduced in Paragraph 3.2.3 and our block space are natural, this implies Claim (1). Similarly, the generators $\mathfrak{t}_i$, $\mathfrak{n}_j$, and $\mathfrak{z}_l$ act by applying a ribbon twist to the internal variables A_i, N_j, and Z_l (cf. [L1, p. 493]). We calculate the action of these generators explicitly in the case $g = 3$ and $n = 2$ to explain Claim (5), Claim (7), and Claim (8).

We first calculate the action of $\mathfrak{t}_1$: On the vector space T_1, it acts by postcomposition with the twist θ_{A_1} on the last tensor factor, which corresponds to postcomposition with $\theta_{A_1} \otimes \mathrm{id}_{A_2^*}$ on the last tensor factor on the vector spaces T_2 and T_3. Under the isomorphism to the vector space T_4, this corresponds to postcomposition with $\mathrm{id}_{*A_1} \otimes\theta_{A_1} \otimes \mathrm{id}_{A_2^*}$ on the first tensor factor. This in turn corresponds on the vector space T_5 to postcomposition with $\mathrm{id}_{*A_1} \otimes\theta_{A_1} \otimes \mathrm{id}_{A_2^*\otimes A_2\otimes A_3^*\otimes A_3}$. In view of the definition of $\mathfrak{T}$ in Paragraph 3.2.1, this becomes postcomposition with $\mathfrak{T} \otimes \mathrm{id}_{L\otimes L}$ on our block space $\mathrm{Hom}(X_2\otimes X_1, L\otimes L\otimes L)$. Very similar calculations yield the claim for $\mathfrak{t}_2$ and $\mathfrak{t}_3$.

Next, we calculate the action of $\mathfrak{n}_1$: On the vector space T_1, it acts by postcomposition with the twist θ_{N_1} on the first tensor factor. Under the isomorphism to the vector space T_2, this corresponds to postcomposition with $\mathrm{id}_{*A_1} \otimes \theta_{N_1}$ on the first tensor factor, which does not change under the isomorphism to the vector space T_3. On the vector space T_4, it becomes postcomposition with $\mathrm{id}_{*A_1} \otimes \theta_{A_1 \otimes A_2^*}$ on the first tensor factor. Under the isomorphism to the vector space T_5, this corresponds to postcomposition with $\mathrm{id}_{*A_1} \otimes \theta_{A_1 \otimes A_2^*} \otimes \mathrm{id}_{A_2 \otimes A_3^*}$. This is equal to postcomposition with $\mathrm{id}_{*A_1} \otimes ((\theta_{A_1} \otimes \theta_{A_2^*}) \circ c_{A_2^*, A_1} \circ c_{A_1, A_2^*}) \otimes \mathrm{id}_{A_2 \otimes A_3^* \otimes A_3}$, which corresponds on our block space $\mathrm{Hom}(X_2 \otimes X_1, L \otimes L \otimes L)$ to postcomposition with $\mathfrak{N} \otimes \mathrm{id}_L$. The calculations for $\mathfrak{n}_2$ are similar.

For the action of $\mathfrak{z}_1$, we proceed differently: On the vector space T_1, it acts by postcomposition with $\theta_{Z_1} \otimes \mathrm{id}_{A_3}$ on the tensor factor $\mathrm{Hom}(X_1, Z_1 \otimes A_3)$, which on the vector spaces T_2 and T_3 becomes postcomposition with θ_{Z_1} on the fourth and the third tensor factor, respectively. Under the isomorphism to the vector space T_4, this corresponds to precomposition with $\mathrm{id}_{X_2} \otimes \theta_{X_1 \otimes {}^*A_3}$. This vector space is isomorphic to

$$\int^{A_i} \mathrm{Hom}(X_1 \otimes {}^*A_3, {}^*X_2 \otimes A_1^* \otimes A_1 \otimes A_2^* \otimes A_2 \otimes A_3^*),$$

where the action becomes precomposition with $\theta_{X_1 \otimes {}^*A_3}$, which equals postcomposition with $\theta_{{}^*X_2 \otimes A_1^* \otimes A_1 \otimes A_2^* \otimes A_2 \otimes A_3^*}$. Under the isomorphism to the vector space

$$\int^{A_i} \mathrm{Hom}(X_1, {}^*X_2 \otimes A_1^* \otimes A_1 \otimes A_2^* \otimes A_2 \otimes A_3^* \otimes A_3),$$

it becomes postcomposition with $\theta_{{}^*X_2 \otimes A_1^* \otimes A_1 \otimes A_2^* \otimes A_2 \otimes A_3^*} \otimes \mathrm{id}_{A_3}$, which equals postcomposition with

$$((\theta_{{}^*X_2 \otimes A_1^* \otimes A_1 \otimes A_2^* \otimes A_2} \otimes \theta_{A_3^*}) \circ c_{A_3^*, {}^*X_2 \otimes A_1^* \otimes A_1 \otimes A_2^* \otimes A_2} \circ c_{{}^*X_2 \otimes A_1^* \otimes A_1 \otimes A_2^* \otimes A_2, A_3^*}) \otimes \mathrm{id}_{A_3}.$$

Under the isomorphism to the vector space $\mathrm{Hom}(X_1, {}^*X_2 \otimes L^{\otimes 3})$, this equals postcomposition with $(\theta_{{}^*X_2 \otimes L \otimes L} \otimes \mathfrak{T}) \circ \mathfrak{N}^l_{{}^*X_2 \otimes L \otimes L, L}$. But this means precisely that $\mathcal{Z}(\mathfrak{z}_1)$ acts as asserted on our block vector space $\mathrm{Hom}(X_2 \otimes X_1, L^{\otimes 3})$.

Using a fusing morphism (cf. [L2, Eq. (6.8), p. 359]), the braiding of two neighboring boundary components corresponds to a braiding morphism of a trivalent net (cf. [L2, Par. 4.3, p. 350f]), which in turn corresponds to the braiding of the corresponding labels of the boundary components on our block space (cf. [L2, Par. 8.1, No. (viii) and (ix), p. 373f]). In this way, we arrive at Claim (2) and Claim (4). Claim (3) follows from Claim (1), because we have seen in Paragraph 3.1.3 that $[\mathfrak{q}_{j,j+1}] = [\mathfrak{d}_{j,j+1}\mathfrak{d}_j^{-1}\mathfrak{d}_{j+1}^{-1}]$, and in a ribbon category we have

$$c_{X_j,X_{j+1}} \circ c_{X_{j+1},X_j} = (\theta_{X_{j+1}}^{-1} \otimes \theta_{X_j}^{-1}) \circ \theta_{X_{j+1}\otimes X_j} = \theta_{X_{j+1}\otimes X_j} \circ (\theta_{X_{j+1}}^{-1} \otimes \theta_{X_j}^{-1}).$$

The element $\mathfrak{s}_i$ corresponds to a switch (cf. [L2, Prop. 2.2, p. 319]) that acts by $\mathfrak{S}$ on the corresponding tensor factor of $L^{\otimes g}$ (cf. [L2, Prop. 8.8, p. 391]). This establishes Claim (6) and concludes our discussion about how the different elements of the mapping class group act.

It is important to note that, as we use a different net for the description of the surface, our formulas for the actions of the generators of the mapping class group on the block space are different from those in [L1, Par. 4.5, p. 494f]. Starting from the net in [L1, Fig. 5, p. 491] and using the internal variables called there E_i instead of those called there D_i to form the coends in the last step would give formulas similar to ours. To facilitate the comparison, we include a small table that relates our notation to the one used in [L1]:

Present notation	$\Gamma_{g,n}$	L	$\mathfrak{N}'$	$\mathfrak{N}_{X,L}^l$	$\mathfrak{S}$	$\mathfrak{T}$	$\mathfrak{b}_{l,l+1}$	$\mathfrak{q}_{l,l+1}$	$\mathfrak{s}_i$
[L1]	$M'_{g,n}$	$\mathbf{f}$	Ω	$\Omega_{X,\mathbf{f}}^l$	S	T	ω_l	ω_l^2	S_i

It must be emphasized that the morphisms in the first row are only analogous, but not strictly equal, to the ones in the second row.

In the case where $n = 1$, i.e., when there is only one boundary component, the projective action on the block space can be obtained by postcomposition from a projective action of $\Gamma_{g,1}$ on $L^{\otimes g}$: For a mapping class $[\psi] \in \Gamma_{g,1}$, a representative $\mathcal{Z}(\psi)$ of the associated projective class $[\mathcal{Z}(\psi)]$ is a natural equivalence from

the contravariant Hom-functor

$$X \mapsto \mathcal{Z}(\Sigma_{g,1}^X) = \operatorname{Hom}_{\mathcal{C}}(X, L^{\otimes g})$$

to itself, which according to the Yoneda lemma (cf. [ML2, Chap. III, Sec. 2, p. 61]; [P, Par. 1.15, p. 37]) is given by postcomposition with an automorphism of $L^{\otimes g}$. For the generators of the mapping class group whose projective action is described in the list above, this can be seen explicitly: The only generator that appears in this case and is not already given by postcomposition is $\mathfrak{d}_1$, which according to Claim (1) is given by precomposition with θ_X. However, by the naturality of the twist, this is equal to postcomposition with $\theta_{L^{\otimes g}}$.

3.2.5 Modular functors

The various mapping class group representations just described are not unrelated, but rather together form a modular functor (cf. [L2, Par. 8.1, p. 372ff]). In particular, they are compatible with gluing of surfaces (cf. [L2, Par. 8.1, No. (xi), p. 374]). We will need only one very special instance of this general property: Consider the last two boundary components of the surface $\Sigma_{g,n+2}$. They carry an orientation that is induced from the orientation of the surface, which is opposite to the orientation of the curves ρ_{n+1} and ρ_{n+2} in Paragraph 1.3.1. Up to isotopy, there is a unique orientation-reversing diffeomorphism between these two boundary components that maps the distinguished point on one boundary component to the distinguished point on the other. If we identify the two boundary components along this diffeomorphism, the arising quotient space is diffeomorphic to $\Sigma_{g+1,n}$, so that we have effectively attached a handle, which we consider as the first one. A diffeomorphism $\varphi \colon \Sigma_{g,n+2} \to \Sigma_{g,n+2}$ that restricts to the identity on the last two boundary components then induces a diffeomorphism $\psi \colon \Sigma_{g+1,n} \to \Sigma_{g+1,n}$. For $\tau := p([\varphi])^{-1} \in \mathbb{S}_{n+2}$, our hypothesis means that τ fixes $n+1$ and $n+2$.

Suppose we are given n objects $X_1, \ldots, X_n$ and another object X. From the adjunctions recalled in Paragraph 1.2.1, we get an isomorphism

$$\operatorname{Hom}_{\mathcal{C}}(X^* \otimes X^{**} \otimes X_n \otimes \cdots \otimes X_1, L^{\otimes g}) \to \operatorname{Hom}_{\mathcal{C}}(X_n \otimes \cdots \otimes X_1, X^* \otimes X \otimes L^{\otimes g}).$$

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If we postcompose with $\iota_X \otimes \mathrm{id}_{L^{\otimes g}}$, we obtain a morphism

$$\mathrm{Hom}_{\mathcal{C}}(X^* \otimes X^{**} \otimes X_n \otimes \cdots \otimes X_1, L^{\otimes g}) \to \mathrm{Hom}_{\mathcal{C}}(X_n \otimes \cdots \otimes X_1, L^{\otimes(g+1)})$$

that we call the handle gluing homomorphism. By construction, this homomorphism is natural with respect to the objects $X_1, \ldots, X_n$ of $\mathcal{C}$ in the sense that for given morphisms $f_i \colon X_i \to Y_i$, the diagram

$$
\begin{array}{ccc}
\mathcal{Z}(\Sigma_{g,n+2}^{Y_1,\ldots,Y_n,X^{**},X^*}) & \longrightarrow & \mathcal{Z}(\Sigma_{g+1,n}^{Y_1,\ldots,Y_n}) \\
{\scriptstyle \circ(\mathrm{id}_{X^* \otimes X^{**}} \otimes f_n \otimes \cdots \otimes f_1)} \downarrow & & \downarrow {\scriptstyle \circ(f_n \otimes \cdots \otimes f_1)} \\
\mathcal{Z}(\Sigma_{g,n+2}^{X_1,\ldots,X_n,X^{**},X^*}) & \longrightarrow & \mathcal{Z}(\Sigma_{g+1,n}^{X_1,\ldots,X_n})
\end{array}
$$

commutes.

The special case of the gluing property under consideration here then states that the diagram

$$
\begin{array}{ccc}
\mathcal{Z}(\Sigma_{g,n+2}^{X_1,\ldots,X_n,X^{**},X^*}) & \longrightarrow & \mathcal{Z}(\Sigma_{g+1,n}^{X_1,\ldots,X_n}) \\
\downarrow {\scriptstyle \mathcal{Z}(\varphi)} & & \downarrow {\scriptstyle \mathcal{Z}(\psi)} \\
\mathcal{Z}(\Sigma_{g,n+2}^{X_{\tau(1)},\ldots,X_{\tau(n)},X^{**},X^*}) & \longrightarrow & \mathcal{Z}(\Sigma_{g+1,n}^{X_{\tau(1)},\ldots,X_{\tau(n)}})
\end{array}
$$

commutes for suitably chosen representatives $\mathcal{Z}(\varphi)$ and $\mathcal{Z}(\psi)$ within their respective projective classes. As above, $\mathcal{Z}(\varphi)$ and $\mathcal{Z}(\psi)$ are viewed here as natural transformations of the corresponding functors, so that the nonzero scalar used in passing to a different representative does not depend on the spaces $X_1, \ldots, X_n$.

To illustrate this property, we consider the example $\varphi = \mathfrak{d}_{n+1}$, which also plays an important role later. Then the corresponding map on the glued surface is $\psi = \mathfrak{t}_1$. The actions of φ and ψ are described in the list in Paragraph 3.2.4. By its definition in Paragraph 3.2.1, we have $\mathfrak{T} \circ \iota_X = \iota_X \circ (\mathrm{id}_{X^*} \otimes \theta_X) = \iota_X \circ (\theta_X^* \otimes \mathrm{id}_X)$, where the second equality follows from the dinaturality of ι. Therefore, the diagram

$$\begin{CD} \mathrm{Hom}_\mathcal{C}(X_n \otimes \cdots \otimes X_1, X^* \otimes X \otimes L^{\otimes g}) @>(\iota_X \otimes \mathrm{id}_{L^{\otimes g}})\circ>> \mathrm{Hom}_\mathcal{C}(X_n \otimes \cdots \otimes X_1, L^{\otimes(g+1)}) \\ @V(\theta_X^* \otimes \mathrm{id}_X \otimes \mathrm{id}_{L^{\otimes g}})\circ VV @VV(\mathfrak{T} \otimes \mathrm{id}_{L^{\otimes g}})\circ V \\ \mathrm{Hom}_\mathcal{C}(X_n \otimes \cdots \otimes X_1, X^* \otimes X \otimes L^{\otimes g}) @>>(\iota_X \otimes \mathrm{id}_{L^{\otimes g}})\circ> \mathrm{Hom}_\mathcal{C}(X_n \otimes \cdots \otimes X_1, L^{\otimes(g+1)}) \end{CD}$$

commutes.

Combining this with the naturality of the adjunction, we see that the gluing property holds in this case.

We note that the choice $\varphi = \mathfrak{d}_{n+2}$ leads to the same map on the glued surface, namely $\psi = \mathfrak{t}_1$. A very similar reasoning shows that the gluing property also holds in this case.

3.2.6 The case of the torus

In order to understand how the results of this chapter generalize the results of Chapter 2, it will be important to consider the projective representation reviewed in Paragraph 3.2.4 in the case of the torus, where $g = 1$, and the case where $\mathcal{C}$ is the category of finite-dimensional representations of a Hopf algebra A with the properties described in Paragraph 1.2.2. Important ingredients of this projective representation are the endomorphisms $\mathfrak{S}$ and $\mathfrak{T}$ of $L = A^*$, whose explicit form we have determined in Paragraph 3.2.2. In Chapter 2, we have used the same symbols for endomorphisms of A, which we will now denote by $\hat{\mathfrak{S}}$ and $\hat{\mathfrak{T}}$ and which are given by

$$\hat{\mathfrak{S}}(a) = \rho(aQ^1)S(Q^2) \qquad \text{and} \qquad \hat{\mathfrak{T}}(a) = va$$

(cf. Section 2.2). In order to understand how these endomorphisms are related, we use a variant of the Radford map

$$\iota \colon A \to A^*, \ a \mapsto \iota(a)$$

defined by $\iota(a)(a') = \rho(aa')$, which was also introduced in the section just cited. Here ρ is the right integral already considered in Paragraph 3.2.2. As recalled in

Section 2.2, ρ is contained in the vector space

$$\bar{C}(A) := \{\varphi \in A^* \mid \varphi(aa') = \varphi(S^2(a')a) \text{ for all } a, a' \in A\},$$

which can be viewed as the set of invariants $(A^*)^A$ for the left coadjoint action discussed in Paragraph 3.2.2. The variant of the Radford map that we will need is the map $\bar{\iota} := \iota \circ S^2$, which then satisfies

$$\bar{\iota}(a)(a') = \iota(S^2(a))(a') = \rho(S^2(a)a') = \rho(a'a).$$

With the help of $\bar{\iota}$, we see that $\hat{\mathfrak{S}}$ and $\hat{\mathfrak{T}}$ are conjugate to $\mathfrak{S}$ and $\mathfrak{T}$:

Lemma 3.2.3 The diagrams

$$
\begin{array}{ccc}
A^* & \xrightarrow{\mathfrak{S}} & A^* \\
\bar{\iota} \uparrow & & \uparrow \bar{\iota} \\
A & \xrightarrow{\hat{\mathfrak{S}}} & A
\end{array}
\qquad \text{and} \qquad
\begin{array}{ccc}
A^* & \xrightarrow{\mathfrak{T}} & A^* \\
\bar{\iota} \uparrow & & \uparrow \bar{\iota} \\
A & \xrightarrow{\hat{\mathfrak{T}}} & A
\end{array}
$$

commute. Moreover, $\bar{\iota}$ is an isomorphism of A-modules, where A^* carries the left coadjoint action and A carries the left adjoint action of the coopposite Hopf algebra, given by $a.a' := a_{(2)}a'S^{-1}(a_{(1)})$.

Proof. To show the commutativity of the first diagram, we use that $\rho \in \bar{C}(A)$ to compute

$$
\begin{aligned}
\bar{\iota}(\hat{\mathfrak{S}}(a))(a') &= \rho(aQ^1)\bar{\iota}(S(Q^2))(a') = \rho(aQ^1)\rho(a'S(Q^2)) \\
&= \rho(S^2(Q^1)a)\rho(S^3(Q^2)a') = \rho(Q^1a)\rho(S(Q^2)a') \\
&= \bar{\iota}(a)(Q^1)\rho(S(Q^2)a') = \mathfrak{S}(\bar{\iota}(a))(a').
\end{aligned}
$$

The proof of the commutativity of the second diagram is even simpler: We have

$$\bar{\iota}(\hat{\mathfrak{T}}(a))(a') = \bar{\iota}(va)(a') = \rho(a'va) = \bar{\iota}(a)(a'v) = \mathfrak{T}(\bar{\iota}(a))(a')$$

by Proposition 3.2.2. Because ρ is a Frobenius homomorphism, ι and $\bar{\iota}$ are bijective.

To see that $\bar{\iota}$ is linear with respect to the specified A-actions, we compute

$$\bar{\iota}(a.a')(a'') = \rho(a''(a.a')) = \rho(a''a_{(2)}a'S^{-1}(a_{(1)})) = \rho(S(a_{(1)})a''a_{(2)}a')$$
$$= \bar{\iota}(a')(S(a_{(1)})a''a_{(2)}) = (a.(\bar{\iota}(a')))(a'')$$

and get the assertion. $\qquad\square$

By [SZ, Prop. 4.3, p. 37], we have $\hat{\mathfrak{S}} \circ \hat{\mathfrak{T}} \circ \hat{\mathfrak{S}} = \rho(v)\, \hat{\mathfrak{T}}^{-1} \circ \hat{\mathfrak{S}} \circ \hat{\mathfrak{T}}^{-1}$. The preceding lemma therefore implies that we also have

$$\mathfrak{S} \circ \mathfrak{T} \circ \mathfrak{S} = \rho(v)\, \mathfrak{T}^{-1} \circ \mathfrak{S} \circ \mathfrak{T}^{-1}.$$

It then follows from our discussion of the braid group B_3 in Paragraph 3.1.7 that the assignment

$$s \mapsto \mathfrak{S} \qquad \text{and} \qquad t \mapsto \mathfrak{T}$$

yields a projective representation of B_3 in $\mathrm{Aut}_A(L)$. If X is an A-module, post-composition with $\mathfrak{S}$ and $\mathfrak{T}$ therefore leads to a projective representation of B_3 on the vector space $\mathrm{Hom}_A(X, L)$. On the other hand, we have seen in Paragraph 3.1.7 that the assignment $r \mapsto \mathfrak{r}_1$ and $t \mapsto \mathfrak{t}_1$ yields an isomorphism between the braid group B_3 and the mapping class group $\Gamma_{1,1}$. By construction, this isomorphism maps s to $\mathfrak{s}_1$. Therefore, if we transport the projective action of B_3 to $\Gamma_{1,1}$ along this isomorphism, we obtain exactly the projective representation considered in Paragraph 3.2.4 in the case $g = 1$ and $n = 1$.

It is instructive to see why the 2-chain relation $(\mathfrak{r}_1\mathfrak{t}_1)^6 = \mathfrak{d}_1$ in the mapping class group $\Gamma_{1,1}$, which we discussed in Paragraph 3.1.7, is satisfied in our situation. We have already used in Paragraph 3.1.7 that $(\mathfrak{r}_1\mathfrak{t}_1)^6 = \mathfrak{s}_1^{-4}$. Now suppose that $f \in \mathrm{Hom}_A(X, L)$ is an A-linear map. For $x \in X$, we choose $a \in A$ so that $f(x) = \bar{\iota}(a)$. For example from Proposition 2.2.2 combined with Lemma 2.2.3, we know that

$$\hat{\mathfrak{S}}^4(a) = ((\rho \otimes \rho)(Q))^2 S(v_{(1)}^{-1})av_{(2)}^{-1}.$$

From the lemma above and the fact that $\rho \in \bar{C}(A)$, we then get

$$
\begin{aligned}
((\mathfrak{s}_1^4.f)(x))(a') = \mathfrak{S}^4(f(x))(a') &= \mathfrak{S}^4(\bar{\imath}(a))(a') = \bar{\imath}(\hat{\mathfrak{S}}^4(a))(a') = \rho(a'\hat{\mathfrak{S}}^4(a)) \\
&= ((\rho \otimes \rho)(Q))^2 \, \rho(a'S(v_{(1)}^{-1})av_{(2)}^{-1}) \\
&= ((\rho \otimes \rho)(Q))^2 \, \rho(S^2(v_{(2)}^{-1})a'S(v_{(1)}^{-1})a) \\
&= ((\rho \otimes \rho)(Q))^2 \, f(x)(S^2(v_{(2)}^{-1})a'S(v_{(1)}^{-1})) \\
&= ((\rho \otimes \rho)(Q))^2 \, (S(v^{-1}).f(x))(a') = ((\rho \otimes \rho)(Q))^2 \, (v^{-1}.f(x))(a') \\
&= ((\rho \otimes \rho)(Q))^2 \, f(v^{-1}.x)(a').
\end{aligned}
$$

On the other hand, we have

$$
((\mathfrak{d}_1^{-1}.f)(x))(a') = ((f \circ \theta_X^{-1})(x))(a') = f(v^{-1}.x)(a'),
$$

so that the actions of $\mathfrak{s}_1^4$ and $\mathfrak{d}_1^{-1}$ agree up to a scalar, as required.

In the case $g = 1$, but $n = 0$, the definition in Paragraph 3.2.3 is to be understood in such a way that the action is on the block space $\mathrm{Hom}_A(K, L)$, because the base field is the unit object of the category, as explained in Paragraph 1.2.2. For any left A-module Y, the map $\mathrm{Hom}_A(K, Y) \to Y^A$, $f \mapsto f(1_K)$ yields a bijection with the space of invariants

$$
Y^A = \{y \in Y \mid a.y = \varepsilon(a)y \text{ for all } a \in A\}.
$$

As we said above, in the case $Y = L$ we have $L^A = \bar{C}(A)$, while in the case $Y = A$, endowed with the action described in Lemma 3.2.3, we have $Y^A = Z(A)$, the center of A. In view of its A-linearity, $\bar{\imath}$ restricts to an isomorphism between $Z(A)$ and $\bar{C}(A)$.

In the case $X = K$, we have that θ_X is the identity map, because $\varepsilon(v) = 1$. This means that $\mathfrak{d}_1$ acts trivially on $\mathrm{Hom}_A(K, L)$. By the 2-chain relation that we have just checked explicitly, this implies that $\mathfrak{s}_1^4$ acts trivially. This means, in view of the discussion in Paragraph 3.1.7, that the projective action of the braid group B_3 factors through a projective action of the modular group $\mathrm{SL}(2, \mathbb{Z})$ on $\bar{C}(A)$, as it is required for the construction in Paragraph 3.2.4, because $\Gamma_1 \cong \mathrm{SL}(2, \mathbb{Z})$. Via $\bar{\imath}$, this

projective action on $\bar{C}(A)$ is isomorphic to an action of $\text{SL}(2, \mathbb{Z})$ on the center $Z(A)$. This is the action that was considered in Corollary 2.2.4.

Let us mention that using $\bar{\iota}$ is not the only way to relate the projective representations on $\bar{C}(A)$ and $Z(A)$. Other ways are discussed in [SZ, Par. 9.1, p. 87ff].

3.3   Derived functors

3.3.1   Projective resolutions

There are various approaches to the definition of the Ext-functors in abelian categories. We assume here that our category satisfies the assumptions listed in Paragraph 1.2.1; one of these assumptions was that it has enough projectives. In this case, we can use the approach via projective resolutions described in [Mi, Chap. VII, § 7, p. 182ff] and denoted there by $\overline{\text{Ext}}$, but here just denoted by Ext. In this approach, the group $\text{Ext}^m(X, Y)$ for two objects X and Y of $\mathcal{C}$ is defined by choosing a projective resolution

$$X \xleftarrow{\xi} P_0 \xleftarrow{d_1} P_1 \xleftarrow{d_2} P_2 \xleftarrow{d_3} \cdots$$

of X, which exists by the above hypothesis. The group $\text{Ext}^m(X, Y)$ is then defined as the m-cohomology group of the cochain complex

$$\text{Hom}_{\mathcal{C}}(P_0, Y) \longrightarrow \text{Hom}_{\mathcal{C}}(P_1, Y) \longrightarrow \text{Hom}_{\mathcal{C}}(P_2, Y) \longrightarrow \cdots$$

of abelian groups. Although the notation does not reflect this, the definition depends on the chosen resolution; different resolutions lead to Ext-groups that are canonically isomorphic, but not equal (cf. [ML1, Chap. XII, § 9, p. 390]). Note that $\text{Ext}^0(X, Y) \cong \text{Hom}_{\mathcal{C}}(X, Y)$, because the Hom-functor is left exact.

The following lemma about the ribbon structure θ will be important in the sequel:

Lemma 3.3.1 Suppose that

$$\mathbb{1} \xleftarrow{\xi} P_0 \xleftarrow{d_1} P_1 \xleftarrow{d_2} P_2 \xleftarrow{d_3} \cdots$$

is a projective resolution of the unit object. Then there exists, for every object $X \in \mathcal{C}$ and every $m \geq 0$, a morphism $h_m(X) \colon P_m \otimes X \to P_{m+1} \otimes X$ such that

$$\theta_{P_m \otimes X} - \mathrm{id}_{P_m} \otimes \theta_X = (d_{m+1} \otimes \mathrm{id}_X) \circ h_m(X) + h_{m-1}(X) \circ (d_m \otimes \mathrm{id}_X)$$

for all $m \geq 1$, and $\theta_{P_0 \otimes X} - \mathrm{id}_{P_0} \otimes \theta_X = (d_1 \otimes \mathrm{id}_X) \circ h_0(X)$. These morphisms are natural in X in the sense that, for a morphism $f \colon X \to Y$, the diagram

$$
\begin{array}{ccc}
P_m \otimes X & \xrightarrow{\mathrm{id}_{P_m} \otimes f} & P_m \otimes Y \\
{\scriptstyle h_m(X)}\big\downarrow & & \big\downarrow{\scriptstyle h_m(Y)} \\
P_{m+1} \otimes X & \xrightarrow[\mathrm{id}_{P_{m+1}} \otimes f]{} & P_{m+1} \otimes Y
\end{array}
$$

commutes.

Proof. We denote by $\mathrm{Rex}(\mathcal{C}, \mathcal{C})$ the category of K-linear right exact functors from $\mathcal{C}$ to $\mathcal{C}$, whose morphisms are natural transformations. By [Sh2, Cor. 2.6, p. 466], a result that relies on our assumption that the base field is perfect, the functor $X \mapsto P_m \otimes X$ is a projective object in the K-linear category $\mathrm{Rex}(\mathcal{C}, \mathcal{C})$, and therefore these functors form a projective resolution of the functor $X \mapsto \mathbb{1} \otimes X \cong X$ in the category $\mathrm{Rex}(\mathcal{C}, \mathcal{C})$. The result now follows from the comparison theorem (cf. [Ro2, Thm. 6.16, p. 340]) applied to the category $\mathrm{Rex}(\mathcal{C}, \mathcal{C})$. $\qquad\square$

If $\mathcal{C}$ is the category of finite-dimensional modules over a factorizable Hopf algebra discussed in Paragraph 1.2.2, the preceding proof can be given a more explicit form. By the Eilenberg-Watts theorem already mentioned in the proof of Proposition 1.2.1, right exact functors can in this case be represented by tensoring with bimodules. In particular, the functor $X \mapsto P_m \otimes X$ is represented as $X \mapsto (P_m \otimes A) \otimes_A X$, where the vector space $P_m \otimes A$ carries the A-bimodule structure

$$a.(p \otimes a').a'' = a_{(1)}.p \otimes a_{(2)}a'a''$$

for $p \in P_m$ and $a, a', a'' \in A$. We then have two liftings of θ_A, which appear in the diagram

$$A \xleftarrow{\xi \otimes \mathrm{id}_A} P_0 \otimes A \xleftarrow{d_1 \otimes \mathrm{id}_A} P_1 \otimes A \xleftarrow{d_2 \otimes \mathrm{id}_A} \cdots$$

$$\Big\downarrow \theta_A \qquad \theta_{P_0 \otimes A} \Big\Downarrow \mathrm{id}_{P_0} \otimes \theta_A \qquad \theta_{P_1 \otimes A} \Big\Downarrow \mathrm{id}_{P_1} \otimes \theta_A$$

$$A \xleftarrow{\xi \otimes \mathrm{id}_A} P_0 \otimes A \xleftarrow{d_1 \otimes \mathrm{id}_A} P_1 \otimes A \xleftarrow{d_2 \otimes \mathrm{id}_A} \cdots$$

and have the explicit form

$$\theta_{P_m \otimes A}(p \otimes a) = v_{(1)}.p \otimes v_{(2)}a \qquad \text{and} \qquad (\mathrm{id}_{P_m} \otimes \theta_A)(p \otimes a) = p \otimes va$$

in terms of the ribbon element v.

Now we can apply the more standard comparison theorem for bimodules, considered as $A \otimes A^{\mathrm{op}}$-modules, to see that these two liftings are chain-homotopic via a chain homotopy $h'_m \colon P_m \otimes A \to P_{m+1} \otimes A$. The mappings $h_m(X)$ are then induced from the mappings $h'_m \otimes_A \mathrm{id}_X$ via the isomorphism $(P_m \otimes A) \otimes_A X \cong P_m \otimes X$.

3.3.2 Derived block spaces

Our goal now is to extend the projective action of the pure mapping class group $\mathrm{P\Gamma}_{g,n}$ on the block spaces

$$\mathcal{Z}(\Sigma_{g,n}^{X_1,\ldots,X_n}) := \mathrm{Hom}_{\mathcal{C}}(X_n \otimes \cdots \otimes X_1, L^{\otimes g})$$

to the derived block spaces

$$\mathcal{Z}^m(\Sigma_{g,n}^{X_1,\ldots,X_n}) := \mathrm{Ext}^m(X_n \otimes \cdots \otimes X_1, L^{\otimes g}).$$

To do this, we consider the surface $\Sigma_{g,n+1}$ with one additional boundary component, and choose a projective resolution

$$\mathbb{1} \xleftarrow{\xi} P_0 \xleftarrow{d_1} P_1 \xleftarrow{d_2} P_2 \xleftarrow{d_3} \cdots$$

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of the unit object. As we mentioned in Paragraph 1.2.1, tensoring is exact and preserves projectives, so that

$$\mathbb{1} \otimes X_n \otimes \cdots \otimes X_1 \longleftarrow P_0 \otimes X_n \otimes \cdots \otimes X_1 \longleftarrow P_1 \otimes X_n \otimes \cdots \otimes X_1 \longleftarrow \cdots$$

is a projective resolution of $X_n \otimes \cdots \otimes X_1 \cong \mathbb{1} \otimes X_n \otimes \cdots \otimes X_1$. If we use the abbreviation $\Gamma_{g,n+1}(n+1)$ for the mapping class group $\Gamma_{g,n+1}(\mathrm{Im}(\rho_{n+1}))$, an element $[\psi]$ of this group fixes the $(n+1)$-st boundary component. Then the permutation $\tau := p([\psi])^{-1} \in \mathbb{S}_{n+1}$ introduced in Paragraph 1.3.3 fixes $n+1$, and therefore each representative $\mathcal{Z}(\psi)$ of the associated projective class $[\mathcal{Z}(\psi)]$ yields by naturality a cochain homomorphism between the cochain complexes

$$\mathcal{Z}(\Sigma_{g,n+1}^{X_1,\ldots,X_n,P_m}) = \mathrm{Hom}_{\mathcal{C}}(P_m \otimes X_n \otimes \cdots \otimes X_1, L^{\otimes g})$$

and

$$\mathcal{Z}(\Sigma_{g,n+1}^{X_{\tau(1)},\ldots,X_{\tau(n)},P_m}) = \mathrm{Hom}_{\mathcal{C}}(P_m \otimes X_{\tau(n)} \otimes \cdots \otimes X_{\tau(1)}, L^{\otimes g})$$

and so induces a homomorphism

$$\mathcal{Z}^m(\psi)\colon \mathrm{Ext}^m(X_n \otimes \cdots \otimes X_1, L^{\otimes g}) \to \mathrm{Ext}^m(X_{\tau(n)} \otimes \cdots \otimes X_{\tau(1)}, L^{\otimes g})$$

between the derived block spaces. Choosing a different representative of the projective class clearly rescales $\mathcal{Z}^m(\psi)$ by a nonzero scalar, so that the projective class $[\mathcal{Z}^m(\psi)]$ is well-defined.

However, we have associated this homomorphism between derived block spaces with a mapping class $[\psi] \in \Gamma_{g,n+1}(n+1)$, whereas the homomorphisms between the original block spaces were associated with a mapping class $[\psi] \in \Gamma_{g,n}$. As we will show now, we can also associate homomorphisms between the derived block spaces with a mapping class $[\psi] \in \Gamma_{g,n}$, namely by choosing a preimage in $\Gamma_{g,n+1}(n+1)$ under the homomorphism D_{n+1} defined at the end of Paragraph 3.1.4. For this, we obviously need to show that the arising homomorphisms are independent of the chosen preimage. We begin with a few auxiliary results:

Lemma 3.3.2 For the Dehn twist $\mathfrak{d}_{n+1} \in \Gamma_{g,n+1}(n+1)$, we have $[\mathcal{Z}^m(\mathfrak{d}_{n+1})] = [\mathrm{id}]$.

Proof. According to Paragraph 3.2.4, the Dehn twist $\mathfrak{d}_{n+1}$ acts by precomposition with the morphism $\theta_{X_{n+1}} \otimes \mathrm{id}_{X_n \otimes \cdots \otimes X_1}$. By [Ka, Lem. XIV.3.3, p. 350], we have $\theta_{\mathbb{1}} = \mathrm{id}_{\mathbb{1}}$. As we have discussed already at the end of Paragraph 3.3.1, the diagram

$$
\begin{array}{ccccccc}
\mathbb{1} & \xleftarrow{\;\xi\;} & P_0 & \xleftarrow{\;d_1\;} & P_1 & \xleftarrow{\;d_2\;} & \cdots \\
\downarrow{\scriptstyle\theta_{\mathbb{1}}} & & \downarrow{\scriptstyle\theta_{P_0}} & & \downarrow{\scriptstyle\theta_{P_1}} & & \\
\mathbb{1} & \xleftarrow{\;\xi\;} & P_0 & \xleftarrow{\;d_1\;} & P_1 & \xleftarrow{\;d_2\;} & \cdots
\end{array}
$$

commutes by the naturality of the twist. On the other hand, it is obvious that the diagram

$$
\begin{array}{ccccccc}
\mathbb{1} & \xleftarrow{\;\xi\;} & P_0 & \xleftarrow{\;d_1\;} & P_1 & \xleftarrow{\;d_2\;} & \cdots \\
\downarrow{\scriptstyle\mathrm{id}_{\mathbb{1}}} & & \downarrow{\scriptstyle\mathrm{id}_{P_0}} & & \downarrow{\scriptstyle\mathrm{id}_{P_1}} & & \\
\mathbb{1} & \xleftarrow{\;\xi\;} & P_0 & \xleftarrow{\;d_1\;} & P_1 & \xleftarrow{\;d_2\;} & \cdots
\end{array}
$$

commutes. Therefore, both the family (θ_{P_m}) and the family (id_{P_m}) lift the morphism $\theta_{\mathbb{1}} = \mathrm{id}_{\mathbb{1}}$ to the projective resolution. By the comparison theorem, the two lifts are chain-homotopic. This chain homotopy induces a cochain homotopy on the cochain complex $\mathrm{Hom}_{\mathcal{C}}(P_m \otimes X_n \otimes \cdots \otimes X_1, L^{\otimes g})$, which yields that the two maps induce the same map in cohomology, namely the identity. $\qquad\square$

We will need another lemma of a similar nature:

Lemma 3.3.3 For the two Dehn twists $\mathfrak{d}_n$ and $\mathfrak{d}_{n,n+1}$ in $\Gamma_{g,n+1}(n+1)$, we have $[\mathcal{Z}^m(\mathfrak{d}_n)] = [\mathcal{Z}^m(\mathfrak{d}_{n,n+1})]$.

Proof. From Lemma 3.3.1, we know that the chain maps $\theta_{P_m \otimes X_n}$ and $\mathrm{id}_{P_m} \otimes \theta_{X_n}$ are chain-homotopic. This implies that the chain maps

$$
\theta_{P_m \otimes X_n} \otimes \mathrm{id}_{X_{n-1} \otimes \cdots \otimes X_1} \qquad \text{and} \qquad \mathrm{id}_{P_m} \otimes \theta_{X_n} \otimes \mathrm{id}_{X_{n-1} \otimes \cdots \otimes X_1}
$$

are chain-homotopic. Because $\mathfrak{d}_{n,n+1}$ acts by precomposition with the first one and $\mathfrak{d}_n$ acts by precomposition with the second one, this implies the assertion. $\quad\square$

We now consider the curve α_i introduced in Paragraph 1.3.1, which begins and ends in the base point x for the surface $\Gamma_{g,n}$. If we push it off the base point to the left and to the right as described in Paragraph 3.1.1, we can cut out a small disk centered at x without intersecting these two curves, which we consider as the $(n+1)$-st boundary component. In this way, we obtain two curves α_i' and α_i'' on the surface $\Gamma_{g,n+1}$:
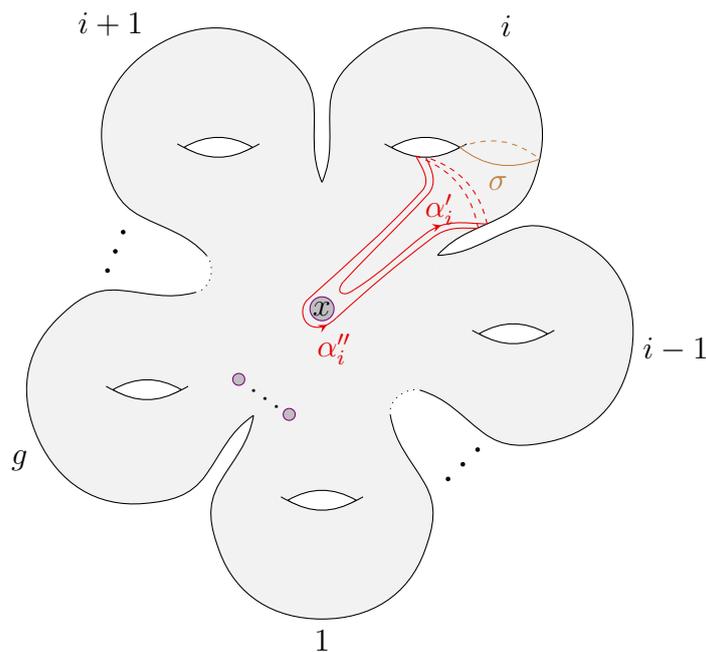
We now label the new boundary component with the elements of our projective resolution; i.e., for a given m, we set $X_{n+1} := P_m$. We want to show that the two Dehn twists $\mathfrak{t}_i' := \mathfrak{d}_{\alpha_i'}$ and $\mathfrak{t}_i'' := \mathfrak{d}_{\alpha_i''}$ induce the same map in cohomology. As we will see below, it is sufficient to consider the case $i = 1$:

Lemma 3.3.4 For the two Dehn twists $\mathfrak{t}_1' := \mathfrak{d}_{\alpha_1'}$ and $\mathfrak{t}_1'' := \mathfrak{d}_{\alpha_1''}$ in $\Gamma_{g,n+1}(n+1)$, we have $[\mathcal{Z}^m(\mathfrak{t}_1')] = [\mathcal{Z}^m(\mathfrak{t}_1'')]$.

Proof. (1) If we cut the surface along the curve that is denoted by σ in the picture above, we obtain a surface that is diffeomorphic to $\Sigma_{g-1,n+3}$:
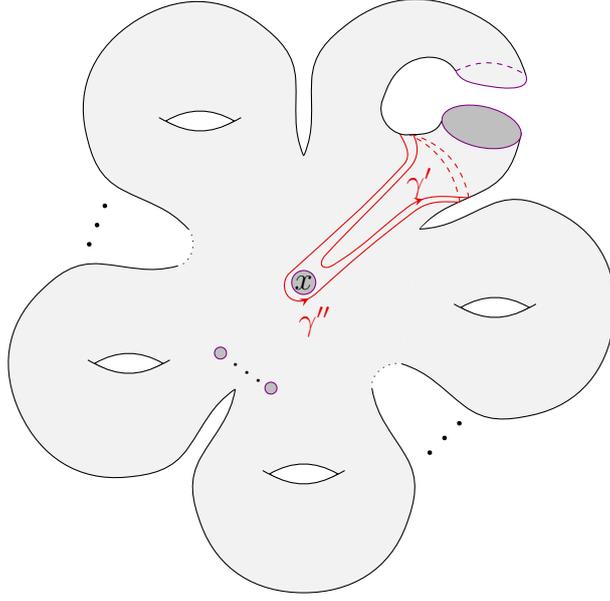
Here, we consider the lower boundary component arising from the cut in the picture above as the $(n+2)$-nd one and the upper boundary component as the $(n+3)$-rd one. The surface $\Sigma_{g,n+1}$ can be reconstructed from the surface $\Sigma_{g-1,n+3}$ by gluing in a handle, as described in Paragraph 3.2.5. Upon gluing, the Dehn twists along the curves denoted by γ' and γ'' in the second picture become the Dehn twists along the curves denoted by α_1' and α_1'' in the first picture, respectively. As we saw in Paragraph 3.2.5, the diagram

$$
\begin{array}{ccc}
\mathcal{Z}(\Sigma_{g-1,n+3}^{X_1,\ldots,X_n,P_m,X^{**},X^*}) & \longrightarrow & \mathcal{Z}(\Sigma_{g,n+1}^{X_1,\ldots,X_n,P_m}) \\
\mathcal{Z}(\mathfrak{d}_{\gamma'}) \downarrow & & \downarrow \mathcal{Z}(\mathfrak{t}_1') \\
\mathcal{Z}(\Sigma_{g-1,n+3}^{X_1,\ldots,X_n,P_m,X^{**},X^*}) & \longrightarrow & \mathcal{Z}(\Sigma_{g,n+1}^{X_1,\ldots,X_n,P_m})
\end{array}
$$

commutes for our choices of the representatives $\mathcal{Z}(\mathfrak{d}_{\gamma'})$ and $\mathcal{Z}(\mathfrak{t}_1')$ of the projective classes, because upon appropriate labeling $\mathfrak{d}_{\gamma'}$ is $\mathfrak{d}_{n+2}$ and $\mathfrak{t}_1'$ is $\mathfrak{t}_1$ in the notation used there. From the discussion in Paragraph 3.2.5, we know that a similar diagram commutes for $\mathfrak{d}_{\gamma''}$ and $\mathfrak{t}_1''$. Our goal is to show that $\mathcal{Z}(\mathfrak{t}_1')$ and $\mathcal{Z}(\mathfrak{t}_1'')$ induce the same map in cohomology.

(2) To see this, we apply Lemma 3.3.1 to the reversed category, in which tensor products are taken in the opposite order. In this way, we obtain for each object

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$X \in \mathcal{C}$ a chain homotopy $h_m(X)\colon X \otimes P_m \to X \otimes P_{m+1}$ between the chain maps $(\theta_{X \otimes P_m})$ and $(\theta_X \otimes \mathrm{id}_{P_m})$ that is natural in X. For two objects X and Y of $\mathcal{C}$, the Dehn twists $\mathfrak{d}_{\gamma'}$ and $\mathfrak{d}_{\gamma''}$ act on the vector space $\mathrm{Hom}_{\mathcal{C}}(Y^* \otimes X^{**} \otimes P_m \otimes Z, L^{\otimes(g-1)})$ by precomposition with $\mathrm{id}_{Y^*} \otimes \theta_{X^{**}} \otimes \mathrm{id}_{P_m \otimes Z}$ and $\mathrm{id}_{Y^*} \otimes \theta_{X^{**} \otimes P_m} \otimes \mathrm{id}_Z$, respectively, where, for brevity, we have used the notation $Z := X_n \otimes \cdots \otimes X_1$. By naturality, the adjunction isomorphism

$$\mathrm{Hom}_{\mathcal{C}}(Y^* \otimes X^{**} \otimes P_m \otimes Z, L^{\otimes(g-1)}) \to \mathrm{Hom}_{\mathcal{C}}(P_m \otimes Z, X^* \otimes Y \otimes L^{\otimes(g-1)})$$

is an isomorphism of cochain complexes, so that we obtain a cochain map $(f_{X,Y}'^m)$ of the cochain complex on the right-hand side whose defining property is that the diagram

$$
\begin{array}{ccc}
\mathrm{Hom}_{\mathcal{C}}(Y^* \otimes X^{**} \otimes P_m \otimes Z, L^{\otimes(g-1)}) & \longrightarrow & \mathrm{Hom}_{\mathcal{C}}(P_m \otimes Z, X^* \otimes Y \otimes L^{\otimes(g-1)}) \\
{\scriptstyle \mathcal{Z}(\mathfrak{d}_{\gamma'})} \downarrow {\scriptstyle = \circ(\mathrm{id}_{Y^*} \otimes \theta_{X^{**}} \otimes \mathrm{id}_{P_m \otimes Z})} & & \downarrow {\scriptstyle f_{X,Y}'^m} \\
\mathrm{Hom}_{\mathcal{C}}(Y^* \otimes X^{**} \otimes P_m \otimes Z, L^{\otimes(g-1)}) & \longrightarrow & \mathrm{Hom}_{\mathcal{C}}(P_m \otimes Z, X^* \otimes Y \otimes L^{\otimes(g-1)})
\end{array}
$$

commutes. For γ'', there is a second cochain map $(f_{X,Y}''^m)$ that makes a very similar diagram commutative. For γ', the naturality of the adjunction isomorphism implies that $f_{X,Y}'^m$ is given by postcomposition with $\theta_{X^*} \otimes \mathrm{id}_Y \otimes \mathrm{id}_{L^{\otimes(g-1)}}$. The chain homotopy $(\mathrm{id}_{Y^*} \otimes h_m(X^{**}) \otimes \mathrm{id}_Z)$ induces a cochain homotopy

$$h_m'(X,Y)\colon \mathrm{Hom}_{\mathcal{C}}(P_m \otimes Z, X^* \otimes Y \otimes L^{\otimes(g-1)}) \to \mathrm{Hom}_{\mathcal{C}}(P_{m-1} \otimes Z, X^* \otimes Y \otimes L^{\otimes(g-1)})$$

that is natural in X and Y and makes the diagram

$$
\begin{array}{ccc}
\mathrm{Hom}_{\mathcal{C}}(Y^* \otimes X^{**} \otimes P_m \otimes Z, L^{\otimes(g-1)}) & \longrightarrow & \mathrm{Hom}_{\mathcal{C}}(P_m \otimes Z, X^* \otimes Y \otimes L^{\otimes(g-1)}) \\
\downarrow {\scriptstyle \circ(\mathrm{id}_{Y^*} \otimes h_{m-1}(X^{**}) \otimes \mathrm{id}_Z)} & & \downarrow {\scriptstyle h_m'(X,Y)} \\
\mathrm{Hom}_{\mathcal{C}}(Y^* \otimes X^{**} \otimes P_{m-1} \otimes Z, L^{\otimes(g-1)}) & \longrightarrow & \mathrm{Hom}_{\mathcal{C}}(P_{m-1} \otimes Z, X^* \otimes Y \otimes L^{\otimes(g-1)})
\end{array}
$$

commutative.

(3) Now the functor $\mathrm{Hom}_{\mathcal{C}}(P_m \otimes Z, - \otimes L^{\otimes(g-1)})$ is an exact functor from the category $\mathcal{C}$ to the category $\mathcal{V}$ of finite-dimensional vector spaces. By Proposition 1.2.1,

it has a right adjoint and therefore preserves coends, as already mentioned in Paragraph 3.2.1. Therefore, the family of morphisms

$$\left(\iota_X \otimes \mathrm{id}_{L^{\otimes(g-1)}}\right) \circ \colon \, \mathrm{Hom}_{\mathcal{C}}(P_m \otimes Z, X^* \otimes X \otimes L^{\otimes(g-1)}) \to \mathrm{Hom}_{\mathcal{C}}(P_m \otimes Z, L^{\otimes g})$$

is a coend for the bifunctor

$$\mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{V}, \ (X, Y) \mapsto \mathrm{Hom}_{\mathcal{C}}(P_m \otimes Z, X^* \otimes Y \otimes L^{\otimes(g-1)}).$$

So we can apply Lemma 3.2.1 to this bifunctor and the corresponding bifunctor with $m - 1$ instead of m to obtain a K-linear map

$$h'_m \colon \, \mathrm{Hom}_{\mathcal{C}}(P_m \otimes Z, L^{\otimes g}) \to \mathrm{Hom}_{\mathcal{C}}(P_{m-1} \otimes Z, L^{\otimes g})$$

that makes the diagram

$$\begin{array}{ccc}
\mathrm{Hom}_{\mathcal{C}}(P_m \otimes Z, X^* \otimes X \otimes L^{\otimes(g-1)}) & \longrightarrow & \mathrm{Hom}_{\mathcal{C}}(P_m \otimes Z, L^{\otimes g}) \\
{\scriptstyle h'_m(X,X)} \downarrow & & \downarrow {\scriptstyle h'_m} \\
\mathrm{Hom}_{\mathcal{C}}(P_{m-1} \otimes Z, X^* \otimes X \otimes L^{\otimes(g-1)}) & \longrightarrow & \mathrm{Hom}_{\mathcal{C}}(P_{m-1} \otimes Z, L^{\otimes g})
\end{array}$$

commutative, i.e., satisfies

$$h'_m\big((\iota_X \otimes \mathrm{id}_{L^{\otimes(g-1)}}) \circ k\big) = (\iota_X \otimes \mathrm{id}_{L^{\otimes(g-1)}}) \circ h'_m(X, X)(k)$$

for all morphisms $k \colon P_m \otimes Z \to X^* \otimes X \otimes L^{\otimes(g-1)}$.

(4) By the definition of $h_m(X^{**})$, we have

$$\theta_{X^{**} \otimes P_m} - \theta_{X^{**}} \otimes \mathrm{id}_{P_m} = (\mathrm{id}_{X^{**}} \otimes d_{m+1}) \circ h_m(X^{**}) + h_{m-1}(X^{**}) \circ (\mathrm{id}_{X^{**}} \otimes d_m).$$

For a morphism $k' \colon Y^* \otimes X^{**} \otimes P_m \otimes Z \to L^{\otimes(g-1)}$, this implies that

$$\begin{aligned}
k' \circ \big[&(\mathrm{id}_{Y^*} \otimes \theta_{X^{**} \otimes P_m} \otimes \mathrm{id}_Z) - (\mathrm{id}_{Y^*} \otimes \theta_{X^{**}} \otimes \mathrm{id}_{P_m \otimes Z}) \big] \\
&= k' \circ \big[(\mathrm{id}_{Y^* \otimes X^{**}} \otimes d_{m+1} \otimes \mathrm{id}_Z) \circ (\mathrm{id}_{Y^*} \otimes h_m(X^{**}) \otimes \mathrm{id}_Z) \\
&\quad + (\mathrm{id}_{Y^*} \otimes h_{m-1}(X^{**}) \otimes \mathrm{id}_Z) \circ (\mathrm{id}_{Y^* \otimes X^{**}} \otimes d_m \otimes \mathrm{id}_Z) \big].
\end{aligned}$$

In view of the definition of $h'_m(X, Y)$ and the naturality of the adjunction isomorphism, this yields

$$f'''^m_{X,Y}(k) - f'^m_{X,Y}(k) = h'_{m+1}(X, Y)(k \circ (d_{m+1} \otimes \mathrm{id}_Z)) + h'_m(X, Y)(k) \circ (d_m \otimes \mathrm{id}_Z)$$

for all morphisms $k \colon P_m \otimes Z \to X^* \otimes Y \otimes L^{\otimes(g-1)}$. If we set $X = Y$ and compose with $\iota_X \otimes \mathrm{id}_{L^{\otimes(g-1)}}$, this equation becomes

$$\mathcal{Z}(\mathfrak{t}''_1) \circ (\iota_X \otimes \mathrm{id}_{L^{\otimes(g-1)}}) \circ k - \mathcal{Z}(\mathfrak{t}'_1) \circ (\iota_X \otimes \mathrm{id}_{L^{\otimes(g-1)}}) \circ k$$
$$= h'_{m+1}((\iota_X \otimes \mathrm{id}_{L^{\otimes(g-1)}}) \circ k \circ (d_{m+1} \otimes \mathrm{id}_Z)) + h'_m((\iota_X \otimes \mathrm{id}_{L^{\otimes(g-1)}}) \circ k) \circ (d_m \otimes \mathrm{id}_Z)$$

Both sides of this equation define a dinatural transformation from the bifunctor

$$\mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{V}, \ (X, Y) \mapsto \mathrm{Hom}_{\mathcal{C}}(P_m \otimes Z, X^* \otimes Y \otimes L^{\otimes(g-1)})$$

already considered above to its coend $\mathrm{Hom}_{\mathcal{C}}(P_m \otimes Z, L^{\otimes g})$. Our dinatural transformation factors over this coend, and the corresponding homomorphism, which is unique, can be read off directly from both the left and the right-hand side of the equation above. We get

$$\mathcal{Z}(\mathfrak{t}''_1) \circ k'' - \mathcal{Z}(\mathfrak{t}'_1) \circ k'' = h'_{m+1}(k'' \circ (d_{m+1} \otimes \mathrm{id}_Z)) + h'_m(k'') \circ (d_m \otimes \mathrm{id}_Z)$$

for all $k'' \in \mathrm{Hom}_{\mathcal{C}}(P_m \otimes Z, L^{\otimes g})$. Therefore, the family (h'_m) constitutes a cochain homotopy between the cochain maps induced by $\mathcal{Z}(\mathfrak{t}'_1)$ and $\mathcal{Z}(\mathfrak{t}''_1)$. When saying that, it should be noted that these maps are only determined up to a scalar; we have taken here for $\mathcal{Z}(\mathfrak{t}'_1)$ the representative that corresponds to the dinatural transformation $k \mapsto (\iota_X \otimes \mathrm{id}_{L^{\otimes(g-1)}}) \circ f'^m_{X,X}(k)$ via the universal property of the coend of our bifunctor, and have made a similar choice for the representative of $\mathcal{Z}(\mathfrak{t}''_1)$ using $f'''^m_{X,X}$. $\qquad\square$

By putting these auxiliary results together, we can now associate with a mapping class in $\Gamma_{g,n}$ a projective class of morphisms not only between the original block spaces, but rather between the derived block spaces. As already stated above, we do this by choosing a preimage under the epimorphism D_{n+1} defined at the end of Paragraph 3.1.4. The key fact that we need to prove is therefore the following:

Theorem 3.3.5 Suppose that the mapping classes $[\psi], [\psi'] \in \Gamma_{g,n+1}(n+1)$ satisfy $D_{n+1}([\psi]) = D_{n+1}([\psi'])$. Then we have $[\mathcal{Z}^m(\psi)] = [\mathcal{Z}^m(\psi')]$.

Proof. (1) Clearly, $[\psi]$ and $[\psi']$ differ by an element in the kernel of D_{n+1}, which is in particular an element in the pure mapping class group $\mathrm{P}\Gamma_{g,n+1}$. It therefore suffices to show that $[\mathcal{Z}^m(\psi)] = [\mathrm{id}]$ for each mapping class $[\psi]$ in the kernel of D_{n+1}. Recall that D_{n+1} is the composition

$$\Gamma_{g,n+1}(n+1) \xrightarrow{C_{n+1}} \Gamma_{g,n}(y) \xrightarrow{F_y} \Gamma_{g,n}$$

of the capping homomorphism, which arises from gluing a punctured disk in place of the missing disk, and the forgetful map F_y that, depending on the perspective, either fills this puncture or 'forgets' that the point was marked. According to Proposition 3.1.5, the kernel of C_{n+1} is generated by the Dehn twist $\mathfrak{d}_{n+1}$. But by Lemma 3.3.2 above, we have $[\mathcal{Z}^m(\mathfrak{d}_{n+1})] = [\mathrm{id}]$. This implies that the assignment $[\psi] \mapsto [\mathcal{Z}^m(\psi)]$ is well-defined for $[\psi] \in \Gamma_{g,n}(y)$, not only for $[\psi] \in \Gamma_{g,n+1}(n+1)$.

(2) To treat the second morphism F_y in the above composition, we use the Birman sequence from Paragraph 3.1.5. In order to do this, we proceed as in Paragraph 3.1.7 and first replace the additional puncture y by the base point x of the fundamental group that comes from the polygon model of the surface as the identification of all the vertices of the polygon, except for those that correspond to the marked points on the boundary components. The map that forgets the base point x instead of the puncture will be denoted by F_x instead of F_y. The Birman sequence then takes the form

$$\pi_1(\Sigma_{g,n}, x) \xrightarrow{P_x} \Gamma_{g,n}(x) \xrightarrow{F_x} \Gamma_{g,n} \longrightarrow 1,$$

where, as before, P_x denotes the pushing map. To complete the proof, we therefore have to show that $[\mathcal{Z}^m(P_x([\gamma]))] = [\mathrm{id}]$ for all homotopy classes $[\gamma] \in \pi_1(\Sigma_{g,n}, x)$. Because P_x is a group antihomomorphism, it suffices to prove this for all $[\gamma]$ in a generating set of the fundamental group.

(3) As discussed in Paragraph 1.3.2, the fundamental group $\pi_1(\Sigma_{g,n}, x)$ is generated by the homotopy classes of the simple closed curves $\alpha_1, \beta_1, \ldots, \alpha_g, \beta_g$, all

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of which are nonseparating, together with the curves $\delta_1, \ldots, \delta_n$, all of which are separating. From a formula stated at the end of Paragraph 3.1.5, we know that $P_x([\alpha_1]) = [\mathfrak{d}_{\alpha_1'} \mathfrak{d}_{\alpha_1''}^{-1}]$. Therefore Lemma 3.3.4 yields that $[\mathcal{Z}^m(P_x([\alpha_1]))] = [\mathrm{id}]$.

(4) For any of the other generators that correspond to nonseparating curves, say β_i, we use the change of coordinates principle (cf. [FM, Par. 1.3.1, p. 37]) to obtain a diffeomorphism $\varphi \colon \Sigma_{g,n} \to \Sigma_{g,n}$ satisfying $\varphi(\alpha_1) = \beta_i$. The discussion in [FM, loc. cit.] shows that we can assume that φ is orientation-preserving and satisfies $\varphi(x) = x$. From Paragraph 3.1.5, we know that $[P_x([\beta_i])] = [\varphi][P_x([\alpha_1])][\varphi^{-1}]$. Now the compatibility with composition described in Paragraph 3.2.4 implies that

$$[\mathcal{Z}^m(P_x([\beta_i]))] = [\mathcal{Z}^m(\varphi)][\mathcal{Z}^m(P_x([\alpha_1]))][\mathcal{Z}^m(\varphi^{-1})] = [\mathcal{Z}^m(\varphi)][\mathcal{Z}^m(\varphi^{-1})] = [\mathrm{id}].$$

(5) For the separating curves $\delta_1, \ldots, \delta_n$, the formula already mentioned above yields that $P_x([\delta_j]) = [\mathfrak{d}_{\delta_j'} \mathfrak{d}_{\delta_j''}^{-1}]$. We have $C_{n+1}([\mathfrak{d}_n]) = [\mathfrak{d}_{\delta_n'}]$ and $C_{n+1}([\mathfrak{d}_{n,n+1}]) = [\mathfrak{d}_{\delta_n''}]$. Therefore, Lemma 3.3.3 shows that $[\mathcal{Z}^m(P_x([\delta_n]))] = [\mathrm{id}]$. If $j \neq n$, we proceed as above and use the general change of coordinate principle (cf. [FM, loc. cit.]) to obtain a diffeomorphism $\varphi \colon \Sigma_{g,n} \to \Sigma_{g,n}$ with $\varphi(\delta_n) = \delta_j$ that is orientation-preserving and satisfies $\varphi(x) = x$. As above, we get $[P_x([\delta_j])] = [\varphi][P_x([\delta_n])][\varphi^{-1}]$, and a very similar computation as there then yields $[\mathcal{Z}^m(P_x([\delta_j]))] = [\mathrm{id}]$. $\qquad \square$

By construction, a mapping class that permutes the boundary components, does not in general carry a derived block space into itself. However, the pure mapping class group preserves these spaces:

Corollary 3.3.6 There is a projective action of $\mathrm{P\Gamma}_{g,n}$ on

$$\mathcal{Z}^m(\Sigma_{g,n}^{X_1,\ldots,X_n}) = \mathrm{Ext}^m(X_n \otimes \cdots \otimes X_1, L^{\otimes g})$$

that agrees with the original action on $\mathcal{Z}(\Sigma_{g,n}^{X_1,\ldots,X_n}) = \mathrm{Hom}_{\mathcal{C}}(X_n \otimes \cdots \otimes X_1, L^{\otimes g})$ if $m = 0$.

As in Paragraph 3.2.4, the entire mapping class group acts projectively on

$$\bigoplus_{\tau \in \mathbb{S}_n} \mathcal{Z}^m(\Sigma_{g,n}^{X_{\tau(1)},\ldots,X_{\tau(n)}}),$$

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the direct sum of derived block spaces. If all boundary components have the same label X, the mapping class group indeed acts projectively on a single derived block space:

Corollary 3.3.7 For $X \in \mathcal{C}$, there is a projective action of $\Gamma_{g,n}$ on $\mathrm{Ext}^m(X^{\otimes n}, L^{\otimes g})$ that agrees with the original action on $\mathrm{Hom}_{\mathcal{C}}(X^{\otimes n}, L^{\otimes g})$ if $m = 0$.

3.3.3 The case of the sphere

The construction of the mapping class group representations in the preceding paragraph was not based on generators and relations for the mapping class group, as such a presentation of the mapping class group is notoriously difficult. In the case $g = 0$, however, we gave such a presentation in Paragraph 3.1.8. It is instructive to see why the defining relations are satisfied in this case. In order to verify these relations, we will need a couple of lemmas. As stated in Paragraph 1.2.1, we are assuming that our category is strict.

Lemma 3.3.8 Suppose that $X_1, \ldots, X_n$ are objects of $\mathcal{C}$. Then we have

1. $c_{X_1 \otimes \cdots \otimes X_{n-1}, X_n} = (c_{X_1, X_n} \otimes \mathrm{id}_{X_2 \otimes \cdots \otimes X_{n-1}}) \circ (\mathrm{id}_{X_1} \otimes c_{X_2, X_n} \otimes \mathrm{id}_{X_3 \otimes \cdots \otimes X_{n-1}})$
$\circ \cdots \circ (\mathrm{id}_{X_1 \otimes \cdots \otimes X_{n-2}} \otimes c_{X_{n-1}, X_n})$

2. $c_{X_1, X_2 \otimes \cdots \otimes X_n} = (\mathrm{id}_{X_2 \otimes \cdots \otimes X_{n-1}} \otimes c_{X_1, X_n}) \circ \cdots \circ (\mathrm{id}_{X_2} \otimes c_{X_1, X_3} \otimes \mathrm{id}_{X_4 \otimes \cdots \otimes X_n})$
$\circ (c_{X_1, X_2} \otimes \mathrm{id}_{X_3 \otimes \cdots \otimes X_n})$

3. $c_{X_n, X_1 \otimes \cdots \otimes X_{n-1}} \circ \cdots \circ c_{X_2, X_3 \otimes \cdots \otimes X_n \otimes X_1} \circ c_{X_1, X_2 \otimes \cdots \otimes X_n}$
$= \theta_{X_1 \otimes \cdots \otimes X_n} \circ (\theta_{X_1}^{-1} \otimes \cdots \otimes \theta_{X_n}^{-1})$

4. $c_{X_2 \otimes \cdots \otimes X_n, X_1} \circ c_{X_3 \otimes \cdots \otimes X_n \otimes X_1, X_2} \circ \cdots \circ c_{X_1 \otimes \cdots \otimes X_{n-1}, X_n}$
$= \theta_{X_1 \otimes \cdots \otimes X_n} \circ (\theta_{X_1}^{-1} \otimes \cdots \otimes \theta_{X_n}^{-1})$

Proof. (1) The first assertion is clearly correct for $n = 2$ and then follows inductively from the equation

$$c_{X_1 \otimes \cdots \otimes X_{n-1}, X_n} = (c_{X_1 \otimes \cdots \otimes X_{n-2}, X_n} \otimes \mathrm{id}_{X_{n-1}}) \circ (\mathrm{id}_{X_1 \otimes \cdots \otimes X_{n-2}} \otimes c_{X_{n-1}, X_n}).$$

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(2) Associated with every braiding is another braiding in which $c_{X,Y}$ is replaced by $c_{Y,X}^{-1}$. If we apply the first assertion to this braiding instead, take inverses, and permute $X_1, \ldots, X_n$ cyclically, we obtain the second assertion.

(3) To prove the third assertion, we need the auxiliary statement that

$$c_{X_k, X_{k+1}\otimes\cdots\otimes X_n\otimes X_1\otimes\cdots\otimes X_{k-1}} \circ \cdots \circ c_{X_2, X_3\otimes\cdots\otimes X_n\otimes X_1} \circ c_{X_1, X_2\otimes\cdots\otimes X_n} =$$

$$c_{X_1\otimes\cdots\otimes X_k, X_{k+1}\otimes\cdots\otimes X_n} \circ (\theta_{X_1\otimes\cdots\otimes X_k} \otimes \mathrm{id}_{X_{k+1}\otimes\cdots\otimes X_n}) \circ (\theta_{X_1}^{-1} \otimes \cdots \otimes \theta_{X_k}^{-1} \otimes \mathrm{id}_{X_{k+1}\otimes\cdots\otimes X_n})$$

for $k = 1, \ldots, n-1$. For $k = 1$, this is obvious. For $k \leq n-2$, we have inductively

$$
\begin{aligned}
&c_{X_{k+1}, X_{k+2}\otimes\cdots\otimes X_n\otimes X_1\otimes\cdots\otimes X_k} \circ c_{X_1\otimes\cdots\otimes X_k, X_{k+1}\otimes\cdots\otimes X_n} \\
&\quad \circ (\theta_{X_1\otimes\cdots\otimes X_k} \otimes \mathrm{id}_{X_{k+1}\otimes\cdots\otimes X_n}) \circ (\theta_{X_1}^{-1} \otimes \cdots \otimes \theta_{X_k}^{-1} \otimes \mathrm{id}_{X_{k+1}\otimes\cdots\otimes X_n}) \\
&= (\mathrm{id}_{X_{k+2}\otimes\cdots\otimes X_n} \otimes c_{X_{k+1}, X_1\otimes\cdots\otimes X_k}) \circ (c_{X_{k+1}, X_{k+2}\otimes\cdots\otimes X_n} \otimes \mathrm{id}_{X_1\otimes\cdots\otimes X_k}) \\
&\quad \circ (\mathrm{id}_{X_{k+1}} \otimes c_{X_1\otimes\cdots\otimes X_k, X_{k+2}\otimes\cdots\otimes X_n}) \circ (c_{X_1\otimes\cdots\otimes X_k, X_{k+1}} \otimes \mathrm{id}_{X_{k+2}\otimes\cdots\otimes X_n}) \\
&\quad \circ (\theta_{X_1\otimes\cdots\otimes X_k} \otimes \theta_{X_{k+1}} \otimes \mathrm{id}_{X_{k+2}\otimes\cdots\otimes X_n}) \circ (\theta_{X_1}^{-1} \otimes \cdots \otimes \theta_{X_{k+1}}^{-1} \otimes \mathrm{id}_{X_{k+2}\otimes\cdots\otimes X_n}).
\end{aligned}
$$

By using the Yang-Baxter equation (cf. [Ka, Thm. XIII.1.3, p. 317]) on the first three terms, this expression becomes

$$
\begin{aligned}
&(c_{X_1\otimes\cdots\otimes X_k, X_{k+2}\otimes\cdots\otimes X_n} \otimes \mathrm{id}_{X_{k+1}}) \circ (\mathrm{id}_{X_1\otimes\cdots\otimes X_k} \otimes c_{X_{k+1}, X_{k+2}\otimes\cdots\otimes X_n}) \\
&\quad \circ (c_{X_{k+1}, X_1\otimes\cdots\otimes X_k} \otimes \mathrm{id}_{X_{k+2}\otimes\cdots\otimes X_n}) \circ (c_{X_1\otimes\cdots\otimes X_k, X_{k+1}} \otimes \mathrm{id}_{X_{k+2}\otimes\cdots\otimes X_n}) \\
&\quad \circ (\theta_{X_1\otimes\cdots\otimes X_k} \otimes \theta_{X_{k+1}} \otimes \mathrm{id}_{X_{k+2}\otimes\cdots\otimes X_n}) \circ (\theta_{X_1}^{-1} \otimes \cdots \otimes \theta_{X_{k+1}}^{-1} \otimes \mathrm{id}_{X_{k+2}\otimes\cdots\otimes X_n}),
\end{aligned}
$$

which by the defining property of a ribbon structure is equal to

$$
\begin{aligned}
&(c_{X_1\otimes\cdots\otimes X_k, X_{k+2}\otimes\cdots\otimes X_n} \otimes \mathrm{id}_{X_{k+1}}) \circ (\mathrm{id}_{X_1\otimes\cdots\otimes X_k} \otimes c_{X_{k+1}, X_{k+2}\otimes\cdots\otimes X_n}) \\
&\quad \circ (\theta_{X_1\otimes\cdots\otimes X_k\otimes X_{k+1}} \otimes \mathrm{id}_{X_{k+2}\otimes\cdots\otimes X_n}) \circ (\theta_{X_1}^{-1} \otimes \cdots \otimes \theta_{X_{k+1}}^{-1} \otimes \mathrm{id}_{X_{k+2}\otimes\cdots\otimes X_n}) \\
&= c_{X_1\otimes\cdots\otimes X_{k+1}, X_{k+2}\otimes\cdots\otimes X_n} \circ (\theta_{X_1\otimes\cdots\otimes X_{k+1}} \otimes \mathrm{id}_{X_{k+2}\otimes\cdots\otimes X_n}) \\
&\quad \circ (\theta_{X_1}^{-1} \otimes \cdots \otimes \theta_{X_{k+1}}^{-1} \otimes \mathrm{id}_{X_{k+2}\otimes\cdots\otimes X_n}),
\end{aligned}
$$

completing our inductive step.

(4) To derive the third assertion, we now use the case $k = n - 1$ of the auxiliary statement above and apply $c_{X_n, X_1 \otimes \cdots \otimes X_{n-1}}$ to obtain

$$c_{X_n, X_1 \otimes \cdots \otimes X_{n-1}} \circ c_{X_{n-1}, X_n \otimes X_1 \otimes \cdots \otimes X_{n-2}} \circ \cdots \circ c_{X_2, X_3 \otimes \cdots \otimes X_n \otimes X_1} \circ c_{X_1, X_2 \otimes \cdots \otimes X_n}$$

$$= c_{X_n, X_1 \otimes \cdots \otimes X_{n-1}} \circ c_{X_1 \otimes \cdots \otimes X_{n-1}, X_n} \circ (\theta_{X_1 \otimes \cdots \otimes X_{n-1}} \otimes \mathrm{id}_{X_n})$$
$$\circ (\theta_{X_1}^{-1} \otimes \cdots \otimes \theta_{X_{n-1}}^{-1} \otimes \mathrm{id}_{X_n})$$
$$= c_{X_n, X_1 \otimes \cdots \otimes X_{n-1}} \circ c_{X_1 \otimes \cdots \otimes X_{n-1}, X_n} \circ (\theta_{X_1 \otimes \cdots \otimes X_{n-1}} \otimes \theta_{X_n}) \circ (\theta_{X_1}^{-1} \otimes \cdots \otimes \theta_{X_n}^{-1})$$
$$= \theta_{X_1 \otimes \cdots \otimes X_n} \circ (\theta_{X_1}^{-1} \otimes \cdots \otimes \theta_{X_n}^{-1})$$

again by the defining property of a ribbon structure.

(5) A ribbon structure for the braiding $c_{Y,X}^{-1}$ considered in the proof of the second assertion is given by the morphisms $\theta_X^{-1} \colon X \to X$. If we apply the third assertion to this braiding and this ribbon structure instead and then take inverses, we obtain the fourth assertion. $\qquad\square$

Another lemma that we will need concerns precomposition with the twist:

Lemma 3.3.9 For all $f \in \mathrm{Hom}_{\mathcal{C}}(X \otimes Y, \mathbb{1})$, we have $f \circ (\theta_X \otimes \mathrm{id}_Y) = f \circ (\mathrm{id}_X \otimes \theta_Y)$. More generally, precomposition with $\theta_X \otimes \mathrm{id}_Y$ and precomposition with $\mathrm{id}_X \otimes \theta_Y$ induce the same homomorphism on $\mathrm{Ext}^m(X \otimes Y, \mathbb{1})$.

Proof. (1) As we recorded in Paragraph 1.2.1, the functor $- \otimes Y^*$ is the right adjoint of the functor $- \otimes Y$. The adjunction is given by

$$\eta \colon \mathrm{Hom}_{\mathcal{C}}(X \otimes Y, \mathbb{1}) \to \mathrm{Hom}_{\mathcal{C}}(X, Y^*), \;\; f \mapsto (f \otimes \mathrm{id}_{Y^*}) \circ (\mathrm{id}_X \otimes \mathrm{coev}_Y).$$

Therefore, the diagram

$$
\begin{array}{ccc}
\mathrm{Hom}_{\mathcal{C}}(X \otimes Y, \mathbb{1}) & \xrightarrow{\circ(\theta_X \otimes \mathrm{id}_Y)} & \mathrm{Hom}_{\mathcal{C}}(X \otimes Y, \mathbb{1}) \\
\eta \downarrow & & \downarrow \eta \\
\mathrm{Hom}_{\mathcal{C}}(X, Y^*) & \xrightarrow{\circ \theta_X} & \mathrm{Hom}_{\mathcal{C}}(X, Y^*)
\end{array}
$$

commutes. On the other hand, also the diagram

$$\text{Hom}_{\mathcal{C}}(X \otimes Y, \mathbb{1}) \xrightarrow{\circ(\text{id}_X \otimes \theta_Y)} \text{Hom}_{\mathcal{C}}(X \otimes Y, \mathbb{1})$$

$$\eta \downarrow \qquad\qquad\qquad \downarrow \eta$$

$$\text{Hom}_{\mathcal{C}}(X, Y^*) \xrightarrow[(\theta_{Y^*})\circ]{} \text{Hom}_{\mathcal{C}}(X, Y^*)$$

commutes, because

$$
\begin{aligned}
\eta(f \circ (\text{id}_X \otimes \theta_Y)) &= (f \otimes \text{id}_{Y^*}) \circ (\text{id}_X \otimes \theta_Y \otimes \text{id}_{Y^*}) \circ (\text{id}_X \otimes \text{coev}_Y) \\
&= (f \otimes \text{id}_{Y^*}) \circ (\text{id}_X \otimes \text{id}_Y \otimes \theta_Y^*) \circ (\text{id}_X \otimes \text{coev}_Y) \\
&= (f \otimes \theta_Y^*) \circ (\text{id}_X \otimes \text{coev}_Y) = \theta_Y^* \circ (f \otimes \text{id}_{Y^*}) \circ (\text{id}_X \otimes \text{coev}_Y) \\
&= \theta_Y^* \circ \eta(f) = \theta_{Y^*} \circ \eta(f).
\end{aligned}
$$

But by the naturality of the twist, we have $\theta_{Y^*} \circ g = g \circ \theta_X$ for $g \in \text{Hom}_{\mathcal{C}}(X, Y^*)$, so that the bottom rows in the two diagrams are equal. By comparing the two diagrams, we obtain the first assertion.

(2) From our projective resolution

$$\mathbb{1} \longleftarrow P_0 \longleftarrow P_1 \longleftarrow P_2 \longleftarrow \cdots$$

of the unit object, we get a projective resolution of X by tensoring with X, i.e., by defining $Q_m := P_m \otimes X$, and even a projective resolution of $X \otimes Y$ by tensoring again with Y. Then

$$
\begin{array}{ccccccc}
X \otimes Y & \longleftarrow & Q_0 \otimes Y & \longleftarrow & Q_1 \otimes Y & \longleftarrow & \cdots \\
\downarrow{\scriptstyle \text{id}_X \otimes \theta_Y} & & \downarrow{\scriptstyle \text{id}_{Q_0} \otimes \theta_Y} & & \downarrow{\scriptstyle \text{id}_{Q_1} \otimes \theta_Y} & & \\
X \otimes Y & \longleftarrow & Q_0 \otimes Y & \longleftarrow & Q_1 \otimes Y & \longleftarrow & \cdots
\end{array}
$$

is a lift of $\text{id}_X \otimes \theta_Y$ to this projective resolution. On the other hand, a lift of $\theta_X \otimes \text{id}_Y$ is given by

$$
\begin{array}{ccccccc}
X \otimes Y & \longleftarrow & Q_0 \otimes Y & \longleftarrow & Q_1 \otimes Y & \longleftarrow & \cdots \\
\downarrow{\scriptstyle \theta_X \otimes \text{id}_Y} & & \downarrow{\scriptstyle \theta_{Q_0} \otimes \text{id}_Y} & & \downarrow{\scriptstyle \theta_{Q_1} \otimes \text{id}_Y} & & \\
X \otimes Y & \longleftarrow & Q_0 \otimes Y & \longleftarrow & Q_1 \otimes Y & \longleftarrow & \cdots
\end{array}
$$

So the action of $\mathrm{id}_X \otimes \theta_Y$ on $\mathrm{Ext}^m(X \otimes Y, \mathbb{1})$ is induced by precomposition with $\mathrm{id}_{Q_m} \otimes \theta_Y$ on $\mathrm{Hom}_{\mathcal{C}}(Q_m \otimes Y, \mathbb{1})$, while the action of $\theta_X \otimes \mathrm{id}_Y$ is induced by precomposition with $\theta_{Q_m} \otimes \mathrm{id}_Y$ on this vector space. But by the preceding step, these two precomposition maps are equal. $\qquad\square$

The results of Paragraph 3.3.2 yield in the case $g = 0$ that the projective action of $\Gamma_{0,n+1}(n+1)$ on the direct sum

$$\bigoplus_{\tau \in \mathbb{S}_n} \mathrm{Ext}^m(X_{\tau(n)} \otimes \cdots \otimes X_{\tau(1)}, \mathbb{1})$$

descends to a projective action of $\Gamma_{0,n}$. In fact, the action is in this case not only projective, but rather an ordinary linear action. As we will explain now, we can see this explicitly from the presentations of these groups that we gave in Paragraph 3.1.8. According to Proposition 3.1.6, we have

$$\Gamma_{0,n+1}(n+1) \cong \mathbb{Z}^n \rtimes B_n.$$

It is not difficult to check that the defining relations of $\mathbb{Z}^n \rtimes B_n$ discussed there are strictly, not only projectively, satisfied on the direct sum

$$\bigoplus_{\tau \in \mathbb{S}_n} \mathrm{Hom}_{\mathcal{C}}(P_m \otimes X_{\tau(n)} \otimes \cdots \otimes X_{\tau(1)}, \mathbb{1}).$$

To show that the action descends to an action of $\Gamma_{0,n}$, we have to show that the additional relations

$$\mathfrak{b}_{1,2}\mathfrak{b}_{2,3} \cdots \mathfrak{b}_{n-1,n}^2 \cdots \mathfrak{b}_{2,3}\mathfrak{b}_{1,2} = \mathfrak{d}_1^{-2}$$

and $(\mathfrak{b}_{1,2}\mathfrak{b}_{2,3} \cdots \mathfrak{b}_{n-1,n})^n = \mathfrak{d}_1^{-1}\mathfrak{d}_2^{-1} \cdots \mathfrak{d}_n^{-1}$ hold on this vector space. It is sufficient to verify the relations on a single summand of the direct sum:

Proposition 3.3.10 For $f \in \mathrm{Hom}_{\mathcal{C}}(P_m \otimes X_n \otimes \cdots \otimes X_1, \mathbb{1})$, we have

$$\mathcal{Z}^m(\mathfrak{b}_{1,2}\mathfrak{b}_{2,3} \cdots \mathfrak{b}_{n-1,n}^2 \cdots \mathfrak{b}_{2,3}\mathfrak{b}_{1,2})(\bar{f}) = \mathcal{Z}^m(\mathfrak{d}_1^{-2})(\bar{f})$$

as well as

$$\mathcal{Z}^m((\mathfrak{b}_{1,2}\mathfrak{b}_{2,3} \cdots \mathfrak{b}_{n-1,n})^n)(\bar{f}) = \mathcal{Z}^m(\mathfrak{d}_1^{-1}\mathfrak{d}_2^{-1} \cdots \mathfrak{d}_n^{-1})(\bar{f}).$$

Proof. (1) It follows from the second assertion in Lemma 3.3.8 that

$$\mathcal{Z}(\mathfrak{b}_{n-1,n}\cdots\mathfrak{b}_{2,3}\mathfrak{b}_{1,2})(f)$$
$$= f \circ (\mathrm{id}_{P_m\otimes X_n\otimes\cdots\otimes X_3}\otimes c_{X_1,X_2}) \circ (\mathrm{id}_{P_m\otimes X_n\otimes\cdots\otimes X_4}\otimes c_{X_1,X_3}\otimes\mathrm{id}_{X_2})$$
$$\circ\cdots\circ(\mathrm{id}_{P_m}\otimes c_{X_1,X_n}\otimes\mathrm{id}_{X_{n-1}\otimes\cdots\otimes X_2})$$
$$= f \circ (\mathrm{id}_{P_m}\otimes c_{X_1,X_n\otimes\cdots\otimes X_2})$$

and then from the first assertion there that

$$\mathcal{Z}(\mathfrak{b}_{1,2}\mathfrak{b}_{2,3}\cdots\mathfrak{b}_{n-1,n}^2\cdots\mathfrak{b}_{2,3}\mathfrak{b}_{1,2})(f)$$
$$= f \circ (\mathrm{id}_{P_m}\otimes c_{X_1,X_n\otimes\cdots\otimes X_2}) \circ (\mathrm{id}_{P_m}\otimes c_{X_n,X_1}\otimes\mathrm{id}_{X_{n-1}\otimes\cdots\otimes X_2})$$
$$\circ\cdots\circ(\mathrm{id}_{P_m\otimes X_n\otimes\cdots\otimes X_4}\otimes c_{X_3,X_1}\otimes\mathrm{id}_{X_2})\circ(\mathrm{id}_{P_m\otimes X_n\otimes\cdots\otimes X_3}\otimes c_{X_2,X_1})$$
$$= f \circ (\mathrm{id}_{P_m}\otimes c_{X_1,X_n\otimes\cdots\otimes X_2}) \circ (\mathrm{id}_{P_m}\otimes c_{X_n\otimes\cdots\otimes X_2,X_1})$$
$$= f \circ (\mathrm{id}_{P_m}\otimes\theta_{X_n\otimes\cdots\otimes X_1}) \circ (\mathrm{id}_{P_m}\otimes\theta_{X_n\otimes\cdots\otimes X_2}^{-1}\otimes\theta_{X_1}^{-1}),$$

while $\mathcal{Z}(\mathfrak{d}_1^{-2})(f) = f \circ (\mathrm{id}_{P_m}\otimes\mathrm{id}_{X_n\otimes\cdots\otimes X_2}\otimes\theta_{X_1}^{-2})$. So, if we introduce the abbreviations $X := X_n \otimes \cdots \otimes X_2$ and $Y := X_1$, we have to show that the two cochain maps

$$f \mapsto f \circ (\mathrm{id}_{P_m}\otimes\theta_{X\otimes Y}) \circ (\mathrm{id}_{P_m}\otimes\theta_X^{-1}\otimes\theta_Y^{-1})$$

and $f \mapsto f \circ (\mathrm{id}_{P_m}\otimes\mathrm{id}_X\otimes\theta_Y^{-2})$ of the cochain complex $(\mathrm{Hom}_{\mathcal{C}}(P_m\otimes X\otimes Y,\mathbb{1}))$ induce the same homomorphism in cohomology.

By the first assertion in Lemma 3.3.9, the cochain map $f \mapsto f \circ (\mathrm{id}_{P_m}\otimes\theta_{X\otimes Y})$ is equal to the cochain map $f \mapsto f \circ (\theta_{P_m}\otimes\mathrm{id}_{X\otimes Y})$, and we saw in Lemma 3.3.2 that this cochain map is homotopic to the identity. This shows that our assertion will follow if we can show that the two cochain maps $f \mapsto f \circ (\mathrm{id}_{P_m}\otimes\theta_X^{-1}\otimes\mathrm{id}_Y)$ and $f \mapsto f \circ (\mathrm{id}_{P_m}\otimes\mathrm{id}_X\otimes\theta_Y^{-1})$ induce the same map in cohomology. But this follows directly from the second assertion in Lemma 3.3.9 by taking inverses.

(2) To prove the second relation, we note that the first assertion in Lemma 3.3.8 gives

$$
\begin{aligned}
\mathcal{Z}(\mathfrak{b}_{1,2}\mathfrak{b}_{2,3}\cdots\mathfrak{b}_{n-1,n})(f) \\
&= f \circ (\mathrm{id}_{P_m} \otimes c_{X_{n-1},X_n} \otimes \mathrm{id}_{X_{n-2}\otimes\cdots\otimes X_1}) \circ \cdots \\
&\quad \circ (\mathrm{id}_{P_m} \otimes \mathrm{id}_{X_{n-1}\otimes\cdots\otimes X_3} \otimes c_{X_2,X_n} \otimes \mathrm{id}_{X_1}) \circ (\mathrm{id}_{P_m} \otimes \mathrm{id}_{X_{n-1}\otimes\cdots\otimes X_2} \otimes c_{X_1,X_n}) \\
&= f \circ (\mathrm{id}_{P_m} \otimes c_{X_{n-1}\otimes\cdots\otimes X_1,X_n})
\end{aligned}
$$

and therefore the fourth assertion of this lemma yields

$$
\begin{aligned}
\mathcal{Z}((\mathfrak{b}_{1,2}\mathfrak{b}_{2,3}\cdots\mathfrak{b}_{n-1,n})^n)(f) \\
&= f \circ (\mathrm{id}_{P_m} \otimes c_{X_{n-1}\otimes\cdots\otimes X_1,X_n}) \circ (\mathrm{id}_{P_m} \otimes c_{X_{n-2}\otimes\cdots\otimes X_1\otimes X_n,X_{n-1}}) \\
&\quad \circ \cdots \circ (\mathrm{id}_{P_m} \otimes c_{X_n\otimes\cdots\otimes X_2,X_1}) \\
&= f \circ (\mathrm{id}_{P_m} \otimes \theta_{X_n\otimes\cdots\otimes X_1}) \circ (\mathrm{id}_{P_m} \otimes \theta_{X_n}^{-1} \otimes \cdots \otimes \theta_{X_1}^{-1}).
\end{aligned}
$$

As in the case of the first relation, the cochain map $f \mapsto f \circ (\mathrm{id}_{P_m} \otimes \theta_{X_n\otimes\cdots\otimes X_1})$ is equal to the cochain map $f \mapsto f \circ (\theta_{P_m} \otimes \mathrm{id}_{X_n\otimes\cdots\otimes X_1})$ by the first assertion in Lemma 3.3.9, which is homotopic to the identity by Lemma 3.3.2. Therefore, our cochain map is homotopic to the cochain map

$$
\mathcal{Z}(\mathfrak{d}_1^{-1}\mathfrak{d}_2^{-1}\cdots\mathfrak{d}_n^{-1})(f) = f \circ (\mathrm{id}_{P_m} \otimes \theta_{X_n}^{-1} \otimes \cdots \otimes \theta_{X_1}^{-1})
$$

as required. $\qquad\square$

3.3.4 Hochschild cohomology

In order to explain why Corollary 3.3.7 generalizes the main result of Chapter 2, we now specialize the situation to the case where $\mathcal{C}$ is the category of left modules over the factorizable ribbon Hopf algebra A described in Paragraph 1.2.2, and furthermore to the case where $g = 1$ and $n = 0$. From Paragraph 3.1.7, we know that $\Gamma_{1,0} \cong \mathrm{SL}(2,\mathbb{Z})$ and $\Gamma_{1,1} \cong B_3$. From Corollary 3.3.7, we then obtain a projective action of $\Gamma_{1,0} \cong \mathrm{SL}(2,\mathbb{Z})$ on $\mathrm{Ext}_A^m(K,L)$, which arises as a quotient of

the projective action of $\Gamma_{1,1} \cong B_3$ on $\operatorname{Hom}_A(P_m, L)$ for a projective resolution

$$K \longleftarrow P_0 \longleftarrow P_1 \longleftarrow P_2 \longleftarrow \cdots$$

of the base field K, considered as a trivial A-module, as explained in Paragraph 1.2.2. That the projective action of $\Gamma_{1,1}$ indeed descends to a projective action of $\Gamma_{1,0}$ on the cohomology groups can be seen quite explicitly in this case: According to Paragraph 3.2.4, the generators $\mathfrak{s}_1$ and $\mathfrak{t}_1$ of $\Gamma_{1,1}$ act by postcomposition with $\mathfrak{S}$ and $\mathfrak{T}$, respectively, and we have seen in Paragraph 3.2.6 that they satisfy the identity $\mathfrak{S} \circ \mathfrak{T} \circ \mathfrak{S} = \rho(v)\, \mathfrak{T}^{-1} \circ \mathfrak{S} \circ \mathfrak{T}^{-1}$, which means that they satisfy the defining identity of $\Gamma_{1,1} \cong B_3$ projectively. We have also seen there that the 2-chain relation is satisfied projectively, i.e., that the actions of $\mathfrak{s}_1^4$ and $\mathfrak{d}_1^{-1}$ agree up to a scalar. It therefore follows from Lemma 3.3.2 that $\mathfrak{s}_1^4$ acts on the cohomology groups as a scalar multiple of the identity, so that the defining relations of the modular group are satisfied projectively.

In order to relate this result to Hochschild cohomology, we choose our projective resolution of the base field in a special way. We first choose a projective resolution

$$A \longleftarrow Q_0 \longleftarrow Q_1 \longleftarrow Q_2 \longleftarrow \cdots$$

of the algebra A in the category of left $A \otimes A^{\mathrm{op}}$-modules, or equivalently the category of A-bimodules, where we require, as in [CE, Chap. IX, § 3, p. 167], that the left and the right action of A on a bimodule become equal when restricted to K. By [CE, Chap. X, Thm. 2.1, p. 185], we can then set $P_m := Q_m \otimes_A K$ to obtain a projective resolution of $A \otimes_A K \cong K$.

Now the left A-module L can be considered as an A-bimodule via the trivial right A-action, i.e., the action $\varphi.a := \varepsilon(a)\varphi$ for $\varphi \in L = A^*$. We denote L by L_ε if it is considered as an A-bimodule in this way. With the help of the cochain map

$$\operatorname{Hom}_{A \otimes A^{\mathrm{op}}}(Q_m, L_\varepsilon) \to \operatorname{Hom}_A(Q_m \otimes_A K, L), \ \ f \mapsto (q \otimes \lambda \mapsto \lambda f(q))$$

with inverse

$$\mathrm{Hom}_A(Q_m \otimes_A K, L) \to \mathrm{Hom}_{A \otimes A^{\mathrm{op}}}(Q_m, L_\varepsilon), \; g \mapsto (q \mapsto g(q \otimes 1_K))$$

we see that the cochain complexes $\mathrm{Hom}_A(Q_m \otimes_A K, L)$ and $\mathrm{Hom}_{A \otimes A^{\mathrm{op}}}(Q_m, L_\varepsilon)$ are isomorphic, so that also their cohomology groups $\mathrm{Ext}_A^m(K, L)$ and $\mathrm{Ext}_{A \otimes A^{\mathrm{op}}}^m(A, L_\varepsilon)$ are isomorphic. But the latter cohomology groups are, by definition, the Hochschild cohomology groups $HH^m(A, L_\varepsilon)$.

Via this isomorphism of cochain complexes, we can transfer the action of $\Gamma_{1,1}$ to the Hochschild cochain groups of the bimodule L_ε. The generators $\mathfrak{s}_1$ and $\mathfrak{t}_1$ of $\Gamma_{1,1}$ then clearly still act by postcomposition with the bimodule homomorphisms $\mathfrak{S}$ and $\mathfrak{T}$, while the Dehn twist $\mathfrak{d}_1$ acts by precomposition with the endomorphism $q \mapsto v.q$ of Q_m, or equivalently by postcomposition with the endomorphism $\varphi \mapsto v.\varphi$ of L_ε. From this correspondence, we see that our Lemma 3.3.2 can be considered as an analogue of Proposition 1.1.2.

While the two results are analogous, it is not yet clear that the projective representations of the modular group constructed here and in Corollary 2.4.6 are isomorphic; in fact, the action constructed above is on the vector space $HH^m(A, L_\varepsilon)$, while the action constructed in Chapter 2 is on the vector space $HH^m(A, A)$. However, it turns out that the two actions are indeed isomorphic:

Proposition 3.3.11 The projective actions of the modular group $\mathrm{SL}(2, \mathbb{Z})$ on the vector spaces $HH^m(A, L_\varepsilon)$ and $HH^m(A, A)$ are isomorphic.

Proof. (1) In general, for an A-bimodule N, we can modify the bimodule structure by defining the new left action as

$$a.n := n.S^{-1}(a)$$

and similarly the new right action as $n.a := S^{-1}(a).n$. We denote N by N' if we consider it endowed with this bimodule structure.

From the point of view of bimodules as $A \otimes A^{\mathrm{op}}$-modules, this operation is the pullback along the ring homomorphism

$$A \otimes A^{\mathrm{op}} \to A \otimes A^{\mathrm{op}}, \ a \otimes b \mapsto S^{-1}(b) \otimes S^{-1}(a).$$

(2) Now

$$A' \longleftarrow Q'_0 \longleftarrow Q'_1 \longleftarrow Q'_2 \longleftarrow \cdots$$

is a projective resolution of A', and the cochain complex $\mathrm{Hom}_{A \otimes A^{\mathrm{op}}}(Q'_m, N')$ is not only isomorphic to the cochain complex $\mathrm{Hom}_{A \otimes A^{\mathrm{op}}}(Q_m, N)$, but even set-theoretically equal. We therefore have that $\mathrm{Ext}^m_{A \otimes A^{\mathrm{op}}}(A', N') = \mathrm{Ext}^m_{A \otimes A^{\mathrm{op}}}(A, N)$.

(3) The antipode yields a bimodule isomorphism $S \colon A' \to A$, which by the comparison theorem lifts to a chain map

$$
\begin{array}{ccccccc}
A' & \longleftarrow & Q'_0 & \longleftarrow & Q'_1 & \longleftarrow & \cdots \\
\downarrow{\scriptstyle S} & & \downarrow{\scriptstyle S_0} & & \downarrow{\scriptstyle S_1} & & \\
A & \longleftarrow & Q_0 & \longleftarrow & Q_1 & \longleftarrow & \cdots
\end{array}
$$

By pulling back along the chain map (S_m), we obtain an isomorphism

$$\mathrm{Ext}^m_{A \otimes A^{\mathrm{op}}}(A, N') \to \mathrm{Ext}^m_{A \otimes A^{\mathrm{op}}}(A', N') = \mathrm{Ext}^m_{A \otimes A^{\mathrm{op}}}(A, N).$$

In other words, we obtain an isomorphism between $HH^m(A, N')$ and $HH^m(A, N)$.

(4) We know from Lemma 2.4.2 and the subsequent discussion that the projective action of $\mathrm{SL}(2, \mathbb{Z})$ on $HH^m(A, A)$ is isomorphic to the one on the cohomology groups $HH^m(A, {}_\varepsilon A_{\mathrm{ad}})$, where the generators $\mathfrak{s}$ and $\mathfrak{t}$ act by postcomposition with $\hat{\mathfrak{S}}$ and $\hat{\mathfrak{T}}$, respectively. Here, ${}_\varepsilon A_{\mathrm{ad}}$ is the A-bimodule with A as underlying vector space, trivial left action $a.a' := \varepsilon(a)a'$, and right action $\mathrm{ad}(a \otimes a') = S(a'_{(1)})aa'_{(2)}$ (cf. Section 2.1). We note that the preceding discussion shows that ${}_\varepsilon A_{\mathrm{ad}}$ itself is not an $\mathrm{SL}(2, \mathbb{Z})$-module, but only a $\Gamma_{1,1}$-module; however, the induced projective $\Gamma_{1,1}$-action on $HH^m(A, {}_\varepsilon A_{\mathrm{ad}})$ descends to a projective $\mathrm{SL}(2, \mathbb{Z})$-action.

If we set $N = {}_\varepsilon A_{\mathrm{ad}}$, we find that N' is the bimodule with underlying vector space A, left action given by $a.a' = a_{(2)} a' S^{-1}(a_{(1)})$, and trivial right action. From the way how we constructed the isomorphism between $HH^m(A, N')$ and $HH^m(A, N)$ in the preceding step, we see that it becomes equivariant if we let the generators $\mathfrak{s}$ and $\mathfrak{t}$ also act on $HH^m(A, N')$ by postcomposition with $\hat{\mathfrak{S}}$ and $\hat{\mathfrak{T}}$, respectively. But now Lemma 3.2.3 shows that $\bar\iota \colon N' \to L_\varepsilon$ is a bimodule isomorphism that is also equivariant under the projective $\Gamma_{1,1}$-action. Therefore postcomposition with $\bar\iota$ yields an isomorphism between $HH^m(A, N')$ and $HH^m(A, L_\varepsilon)$ that is SL$(2, \mathbb{Z})$-equivariant. Combining this with our isomorphism between $HH^m(A, {}_\varepsilon A_{\mathrm{ad}})$ and $HH^m(A, N')$ already obtained, our assertion follows. $\qquad\square$

In view of this proposition, we can indeed say that the mapping class group actions obtained here in Corollary 3.3.7 generalize the projective SL$(2, \mathbb{Z})$-representation on the Hochschild cohomology groups $HH^m(A, A)$ obtained in Corollary 2.4.6.

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Eidesstattliche Versicherung / Declaration on oath

Hiermit erkläre ich an Eides statt, dass ich die vorliegende Dissertationsschrift selbst verfasst und keine anderen als die angegebenen Quellen und Hilfsmittel benutzt habe.

I hereby declare upon oath that I have written the present dissertation independently and have not used further resources and aids than those stated.

St. John's, 23.03.2020
Ort, Datum | *City, date*

Unterschrift | *Signature*