Algebras of non-local screenings and diagonal Nichols algebras

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Chapter 1

Introduction

1.1 Physical background

Two dimensional conformal field theory is a very rich and wide subject where different branches of physics and mathematics come into play.
Physically it is a two dimensional quantum field theory which is invariant under conformal transformations. As any quantum field theory, it can be described by a compatible set of *correlation functions* $\langle \phi_1(x_1)\phi_2(x_2)\ldots\rangle$, i.e. complex numbers depending on fields ϕ_i which are inserted in points x_i of the Riemann sphere or some other surface (or respectively some other higher dimensional manifold). These functions, which in a conformal field theory have to be covariant under conformal transformations, can be seen as expectation values over all possible configurations of the fields and depend on the insertion points x_i. One way to compute correlation functions is via the so-called path integral formalism.
Conformal field theory has various applications in physics: in string theory where the world sheet theories of closed and open strings are conformal, in statistical mechanics where critical percolation may be described by logarithmic conformal models [CR13]. The path integral approach can also be applied to the stochastic dynamics in classical systems with many degrees of freedom. The study of an example, a classical system modelling molecular dynamics, was a side project of the PhD and is presented in chapter 3.

A conformal field theory in two dimensions factorises into a *chiral* and *anti-chiral* part which are described by holomorphic and anti-holomorphic functions, called chiral and anti-chiral *conformal blocks*. These are multi-valued functions. The symmetry algebra of the theory contains an infinite-dimensional algebra, called *Virasoro* algebra, generated by the conformal transformations. There are two copies of the Virasoro algebra Vir and $\overline{\text{Vir}}$ acting on the conformal field theory corresponding to the chiral and anti-chiral parts. The chiral and anti-chiral parts of a conformal field theory can be mathematically axiomatised by two copies $\mathcal{V}$ and $\overline{\mathcal{V}}$ of a vertex operator algebra.

A *vertex operator algebra* (VOA) is an algebraic structure with extra layer of analysis. Apart from the connection to physics, vertex operator algebras are furthermore interesting mathematical objects on their own. For example their representations provide, under some finiteness and semisimplicity conditions, examples of modular tensor categories [Hua08].
Other interesting vertex operator algebras are the so-called chiral *logarithmic* conformal field theories. These are vertex operator algebras with finite non-semisimple representation theory. Physically they correspond to chiral conformal field theories where the energy operator fails to be diagonalisable on the quantum state space and whose chiral conformal blocks may have logarithmic singularities. For a broader introduction to quantum field theory and vertex operator algebras, see chapter 2.

Chapter 4 contains the main body of this thesis.
We consider the example of the Heisenberg vertex algebra associated to a non-integral lattice and the corresponding non-local screening operators. Under certain *smallness* condition, these screening operators satisfy the relations of a Nichols algebra, with a diagonal braiding induced by the non-locality of the screening operators and non-integrality of the lattice.

One of the pursued goals is to take all finite-dimensional diagonal Nichols algebras, as classified by Heckenberger [Hec05], and find all realisations of the respective braidings by lattices, that are compatible with reflections.
A second goal is to study the associated algebra of screening operators when the *smallness* condition fails. For positive definite lattices, where *smallness* holds, we obtain the Nichols algebra, such as the positive part of the quantum group, and for negative definite lattices, where *smallness* fails, we obtain an extension thereof.

A motivation for this study is that for each Nichols algebra braiding, realising lattice and associated screening operators, we can conjecturally construct a logarithmic conformal field theory as kernel of the screening operators. Its representation theory should then be equivalent to the representation theory of the quantum group associated to the Nichols algebra. It is expected that the finiteness of the Nichols algebras coincides with the finiteness of the non-semisimple representation theory of the corresponding logarithmic conformal field theory. As resulting logarithmic conformal field theories we would then get p, p' models, super analogues and other new examples.

1.2 Mathematical tools

In this section we are going to give an overview of the main mathematical objects and tools that we use in chapter 4 in order to achieve the above-mentioned goals. These goals are presented in more detail in section 1.3.

1.2.1 Vertex algebras and representations

A *Vertex Operator Algebra (VOA)* [FBZ04] [Kac98] is a collection of the following data:
A $\mathbb{Z}$-graded vector space $\mathcal{V}$, a distinguished vector $|0\rangle \in \mathcal{V}$, a linear operator $\partial : \mathcal{V} \longrightarrow \mathcal{V}$, a conformal vector $\omega \in \mathcal{V}$ and a linear operator called *vertex operator*

$$\mathrm{Y} : \mathcal{V} \otimes_{\mathbb{C}} \mathcal{V} \to \mathcal{V}[[z, z^{-1}]] \tag{1.1}$$

taking values in formal power series with integer exponents and coefficients in the space $\mathcal{V}$. These data have to fulfil several compatibility axioms.
There is an action on $\mathcal{V}$ of the conformal symmetry algebra, the Virasoro algebra, defined via the conformal element ω.

A *module* $\mathcal{M}$ over a VOA $\mathcal{V}$ is a $\mathbb{C}$-graded vector space together with an operator

$$\mathrm{Y}_{\mathcal{M}} : \mathcal{V} \otimes_{\mathbb{C}} \mathcal{M} \to \mathcal{M}[[z, z^{-1}]] \tag{1.2}$$

fulfilling again compatibility axioms as Y. The map $\mathrm{Y}_{\mathcal{M}}$ takes values in formal power series with integer exponents and coefficients in $\mathcal{M}$.

An *intertwiner* between VOA modules $\mathcal{M}, \mathcal{N}, \mathcal{L}$ is a map

$$\mathrm{Y}_{\mathcal{M},\mathcal{N},\mathcal{L}} : \mathcal{M} \otimes_{\mathbb{C}} \mathcal{N} \to \mathcal{L}\{z\}[\log(z)] \tag{1.3}$$

where $\{z\}$ are power series with complex exponents, and $[\log(z)]$ denotes the space of polynomials in variable $\log(z)$. The intertwiner $\mathrm{Y}_{\mathcal{M},\mathcal{N},\mathcal{L}}$ must fulfil compatibility axioms (see [FHL93]).

In contrast with Y and $\mathrm{Y}_{\mathcal{M}}$, the intertwiners $\mathrm{Y}_{\mathcal{M},\mathcal{N},\mathcal{L}}$ are power series corresponding to multivalued functions. Intertwiners define, under some conditions on the vertex algebra, a tensor product on the modules and a braiding. The multivaluedness is the reason for getting non-trivial double braiding on the category of modules over $\mathcal{V}$. We now discuss it in more detail.

A result of [Hua08] and [HLZ14] tells us that under some finiteness conditions on the VOA $\mathcal{V}$, e.g. $\mathcal{V}$ being C_2-cofinite, the category $\mathrm{Rep}(\mathcal{V})$ of representations of $\mathcal{V}$ is a braided tensor category.
A braided tensor category is an abelian category with tensor product and braiding.
The tensor product $\mathcal{M} \otimes_{\mathcal{V}} \mathcal{N}$ is defined by having an intertwiner

$$\mathrm{Y}_{\mathcal{M}\otimes\mathcal{N}} : \mathcal{M} \otimes_{\mathbb{C}} \mathcal{N} \to (\mathcal{M} \otimes_{\mathcal{V}} \mathcal{N})\{z\}[\log(z)] \tag{1.4}$$

and being universal with respect to this property [HLZ10VI].
The braiding $c_{\mathcal{M},\mathcal{N}} : \mathcal{M} \otimes_{\mathcal{V}} \mathcal{N} \to \mathcal{N} \otimes_{\mathcal{V}} \mathcal{M}$ is roughly defined by

$$c_{\mathcal{M},\mathcal{N}} \circ \mathrm{Y}_{\mathcal{M}\otimes\mathcal{N}}(z) = \mathrm{Y}_{\mathcal{N}\otimes\mathcal{M}}(-z) \tag{1.5}$$

where we analytically continue z to $-z$ counter-clockwise.
The double braiding therefore measures, in some sense, the multivaluedness of the intertwiner $\mathrm{Y}_{\mathcal{M}\otimes\mathcal{N}}$: when it takes values in power series with integer exponents, e.g. when as module $\mathcal{M}$ we consider the VOA $\mathcal{V}$ itself, the double braiding is in fact trivial; when the exponents are fractional, once analytically continuing z to $-z$, the intertwiner catches the monodromies and the double-braiding is not trivial.

Moreover if $\mathrm{Rep}(\mathcal{V})$ is semisimple, then it is even a *modular tensor category*, i.e. it has a non-degenerate braiding

$$c_{\mathcal{M},\mathcal{N}} \circ c_{\mathcal{N},\mathcal{M}} = \mathrm{id} \ \ \forall \mathcal{N} \quad \Rightarrow \quad \mathcal{M} = \mathbb{I} \oplus \cdots \oplus \mathbb{I}.$$

This is conjectured in some form in the non-semisimple case as well.

One of the easiest examples of VOA is the n-dimensional *Heisenberg* vertex algebra $\mathcal{V}^n_{\mathsf{H}}$. It is defined as the space of polynomials $\mathcal{V}^n_{\mathsf{H}} := \mathbb{C}[\partial\phi_\lambda, \partial^2\phi_\lambda, \cdots]$ in the formal variable $\partial^m\phi_\lambda$, with $m \in \mathbb{N}$, $\lambda \in \mathbb{C}^n$, which is linear in the index variable $a\partial^m\phi_\lambda + b\partial^m\phi_\mu = \partial^m\phi_{a\lambda+b\mu}$. So it would be enough to consider $\partial^m\phi_{\alpha_i}$ for a basis α_i of $\mathbb{C}^n$.
The notation indicates that $\partial\phi_\lambda$ is a physical field in the corresponding chiral conformal field theory: the chiral algebra of n free bosons.
We define the following vertex operator on the generating element $\partial\phi_\lambda$:

$$\mathrm{Y}(\partial\phi_\lambda)\partial\phi_\mu = (\lambda, \mu) \cdot z^{-2} \cdot 1 + \sum_{k \geq 0} \frac{z^k}{k!} \partial\phi_\mu \, \partial^{1+k}\phi_\lambda \tag{1.6}$$

where $(\ ,\ )$ is the standard inner product on $\mathbb{C}^n$.
For every $a \in \mathbb{C}^n$ there is an irreducible module $\mathcal{V}_a := \mathbb{C}[\partial\phi_\lambda, \partial^2\phi_\lambda, \cdots]e^{\phi_a}$ with vertex map $\mathrm{Y}_{\mathcal{V}_a}(\partial\phi_\lambda)e^{\phi_a} = (\lambda, a) \cdot z^{-1} \cdot e^{\phi_a} + \cdots$.
The tensor product and the braiding follow from having some intertwiners

$$\mathrm{Y}_{\mathcal{V}_a \otimes \mathcal{V}_b}(e^{\phi_a})e^{\phi_b} = z^{ab} \cdot e^{\phi_{(a+b)}} + \cdots$$
$$\Rightarrow \mathcal{V}_a \otimes \mathcal{V}_b = \mathcal{V}_{a+b}, \qquad c_{\mathcal{V}_a, \mathcal{V}_b} : \ \mathcal{V}_a \otimes \mathcal{V}_b \xrightarrow{e^{i\pi(a,b)}} \mathcal{V}_b \otimes \mathcal{V}_a.$$

Notice that the double-braiding is trivial if and only if $(a, b) \in \mathbb{Z}$.

1.2.2 Screening operators

Screening operators are well known in vertex algebras and conformal field theory literature [DF84]. Normally one considers *local* screening operators, i.e. screening operators associated to the vacuum module of a vertex algebra $\mathcal{V}$. Such screening operators carry a Lie algebra structure.
In this thesis we are instead going to consider *non-local* screening operators associated to $\mathcal{V}$, i.e. screening operators associated to any module $\mathcal{M}$ of $\mathcal{V}$. Non-local screening operators appear in a set of conjectures [Wak86] [FGST06a] [AM08] [FT10] regarding logarithmic conformal field theories arising from their kernels in vertex algebras associated to Lie algebra root lattices, and conjecturally having the same representation theory of quantum

groups.

Let $\mathcal{V}$ be a VOA and $\mathcal{M},\mathcal{N}$ modules. Recall that the tensor product $\mathcal{M}\otimes_{\mathcal{V}}\mathcal{N}$ is defined by having an intertwiner

$$\mathrm{Y}_{\mathcal{M}\otimes\mathcal{N}}:\ \mathcal{M}\otimes_{\mathbb{C}}\mathcal{N}\to(\mathcal{M}\otimes_{\mathcal{V}}\mathcal{N})\{z\}[\log(z)].$$

Fix $m\in\mathcal{M}$. For all modules $\mathcal{N}$ of $\mathcal{V}$, we get a map

$$\mathrm{Y}_{\mathcal{M}\otimes\mathcal{N}}(m,z):\ \mathcal{N}\to(\mathcal{M}\otimes_{\mathcal{V}}\mathcal{N})\{z\}[\log(z)].$$

Integrating around the lift of the circle around $z=0$ in the multivalued covering, we get a map associated to $m\in\mathcal{M}$

$$\mathfrak{Z}_m:\mathcal{N}\to\overline{\mathcal{M}\otimes_{\mathcal{V}}\mathcal{N}} \tag{1.7}$$

which takes values in the algebraic closure of the tensor product.
We call the map $\mathfrak{Z}_m$ a *(non-local) screening operator.*

In what follows, we focus on the screening operators $\mathfrak{Z}_{v_i}$ associated to elements e^{v_i}, $v_i\in\mathbb{C}^n$ in the n-dimensional Heisenberg VOA modules $\mathcal{V}_{v_i}$ defined above. In particular we want the v_i to form a basis. We consider the non-integral lattice $\Lambda\subset\mathbb{C}^n$ spanned by them.

One of the aim of this thesis is to analyse the algebra generated by the screening operators $\mathfrak{Z}_{v_i}$ under composition, associated to a fixed lattice Λ. We saw that non-locality implies the multivaluedness of the intertwiners and thus a non-trivial double braiding. Therefore while local screening operators generate Lie algebras, non-local screening operators generate algebras largely determined by the braiding, such as Nichols algebras and extensions thereof.

1.2.3 Nichols algebras

We now briefly describe the notion of a Nichols algebra, postponing the rigorous definition until section 4.1.
Nichols algebras were first introduced in [Nic78].

Let (V,c) be a braided vector space. The *Nichols algebra* $\mathcal{B}(V)$ is the tensor algebra $\mathrm{T}(V)$ modulo the kernel of the *quantum symmetrizer*

$$\mathrm{III}_{q,n}:=\sum_{\tau\in\mathbb{S}_n}\rho_n(s(\tau)),$$

where $s(\tau)$ is the preimage of a permutation $\tau \in \mathbb{S}_n$ of shortest length in the braid group $\mathbb{B}_n$, which has an action ρ_n on $V^{\otimes n}$ using the initially given braiding c on V.
Note that $s : \mathbb{S}_n \to \mathbb{B}_n$ is usually just a map of sets. In the case of a symmetric braiding, the composition $\rho_n \circ s$ factorises to a group homomorphism.

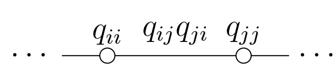
In particular in what follows we treat the case when (V, c) is a finite-dimensional vector space with *diagonal* braiding, i.e. there is a basis $\{x_1, \dots, x_n\}$ of V such that

$$c : x_i \otimes x_j \longmapsto q_{ij} \cdot x_j \otimes x_i \qquad q_{ij} \in \mathbb{C}^\times.$$

We call the matrix $q = (q_{ij})_{i,j}$ the *braiding* matrix. The Nichols algebra $\mathcal{B}(V)$ then depends just of the braiding q_{ij}, therefore we write $\mathcal{B} = \mathcal{B}(q)$.

Examples of Nichols algebras are $\mathcal{B}(V) = \mathbb{C}[x]/x^\ell$ when $V = \langle x \rangle_\mathbb{C}$ is a one dimensional vector space, and the braiding $q_{11} = q$ is a ℓ-th root of unity or $\mathcal{B}(V) = \mathrm{S}(V)$ the symmetric algebra (resp. $\mathcal{B}(V) = \Lambda(V)$ the exterior algebra) when the braiding is $q_{ij} = 1\ \forall i, j$ (resp. $q_{ij} = -1\ \forall i, j$).
Another example which is central in our work is of Nichols algebras associated to Lie algebras: consider a finite-dimensional complex semisimple Lie algebra $\mathfrak{g}$, with simple roots $\alpha_1, \dots, \alpha_n$, root lattice Λ, and inner product $(\, , \,)$. Let $q \in \mathbb{C}^\times$ be a primitive ℓ-th root of unity and the diagonal braiding $q_{ij} = q^{(\alpha_i, \alpha_j)}$. Then $\mathcal{B}(q) = u_q(\mathfrak{g})^+$ is the positive part of the small quantum group $u_q(\mathfrak{g})$.

Although the definition of Nichols algebras could look technical, they can be actually thought as a quite natural generalisation of Lie algebras. It was indeed proven that finite-dimensional Nichols algebras are endowed with generalised root systems, Cartan matrices and Weyl reflections. This result goes back to [Hec06b] for diagonal Nichols algebras and to [AHS10] otherwise. Moreover, [Hec06a] provides a classification of finite-dimensional Nichols algebras with diagonal braiding q_{ij} via generalised root systems and q-diagrams of the form:



The idea is, in the diagonal case, to define a root system by labelling the basis of the space V by what we call simple roots $\{x_{\alpha_1}, \dots, x_{\alpha_n}\}$.

A generalised Cartan matrix a_{ij} and a Weyl groupoid generated by reflections $\mathcal{R}^k$ can then be defined by analogy to Lie theory.

1.2.4 Algebra of screenings

Now that we have defined Nichols algebras, we can study under which condition the algebra of screening operators is in fact a finite-dimensional Nichols algebra. In particular we use the classification list of [Hec05].

Once again, the setting is: a non-integral lattice Λ generated by elements $v_1, \dots, v_n \in \Lambda$ with inner product $m_{ij} := (v_i, v_j)$; associated to them the elements e^{v_i} in modules over the Heisenberg vertex algebra $\mathcal{V}_{\mathsf{H}}^n$ with braiding

$$e^{v_i} \otimes e^{v_j} \mapsto q_{ij}\, e^{v_j} \otimes e^{v_i}, \qquad q_{ij} := e^{i\pi m_{ij}}.$$

A result by [Len17] tells us that if a certain *smallness* condition on m_{ij} (see theorem 4.2.1) is satisfied, corresponding to the poles of the intertwiners being not too severe, the screening operators $\mathfrak{Z}_{v_1}, \dots, \mathfrak{Z}_{v_n}$ form the diagonal finite-dimensional Nichols algebra $\mathcal{B}(q)$ with braiding q_{ij}. If this smallness condition fails, the screening algebra is an extension of $\mathcal{B}(q)$, which we would like to understand.

An immediate example of that is the one mentioned above:
if Λ is a root lattice of a semisimple finite-dimensional complex Lie algebra $\mathfrak{g}$ and the poles of the intertwiners are not too severe, then the screening algebra is the positive part $\mathcal{B}(q) = u_q(\mathfrak{g})^+$ of the small quantum group $u_q(\mathfrak{g})$.

1.3 Main goals of the thesis

In this section we are going to outline the main achievements of this work.

Our first goal is to find all lattices realising Nichols algebra braidings.
Let Λ be a lattice of rank n with basis $\{v_1, \dots, v_n\}$ and inner product $m_{ij} := (v_i, v_j)$. We say that $(\Lambda,\ m_{ij})$ *realise* a given braiding q_{ij} with generalised Cartan matrix a_{ij} if

$$e^{i\pi m_{ij}} = q_{ij} \qquad \text{and} \qquad 2m_{ij} = a_{ij} m_{ii} \ \text{ or } \ (1 - a_{ij}) m_{ii} = 2 \tag{1.8}$$

Moreover all the reflected matrices $\mathcal{R}^k(m_{ij})$ must fulfil again the second of (1.8). This condition expresses that the reflections lift in a suitable sense to the inner product m_{ij}.
This definition goes back to [Sem11], but there (1.8) is required to hold only for one specific Weyl chamber, while we want it to hold in all Weyl chambers.

As an example, we show (details in 4.3.5) which lattices realise the braiding of the Nichols algebra $\mathcal{B}(q) = u_q(\mathfrak{sl}_3)^+$.

The q-diagram of $\mathcal{B}(q)$ is given by $\overset{q^2}{\circ}\overset{q^{-2}}{\text{———}}\overset{q^2}{\circ}$ where $q = e^{i\pi r}, q^2 \neq -1$. This braiding is realised by rescaling a Lie algebra root lattice of type A_2, $\mathfrak{sl}_3$ with positive roots $\{\alpha_1, \alpha_2, \alpha_{12}\}$ by a parameter $r \in \mathbb{Q}$.

As inner product we get a family of realising solutions: $m_{ij} = \begin{bmatrix} 2r & -r \\ -r & 2r \end{bmatrix}$.

It is interesting to note that if we allow the value $q^2 = -1$, we have an additional family of realising solutions parametrised by $r = \frac{p'}{2}$, where p' is an odd integer:

$$m_{ij}^{\mathrm{I}} = \begin{bmatrix} 2r & -r \\ -r & 1 \end{bmatrix} \qquad m_{ij}^{\mathrm{II}} = \begin{bmatrix} 1 & -1+r \\ -1+r & 1 \end{bmatrix} = \mathcal{R}^2(m_{ij})$$

obtained by rescaling a Lie superalgebra root lattice of type $A(1,0)$, $\mathfrak{g} = \mathfrak{sl}(2|1)$ by the parameter $r = \frac{p'}{2}$.

We find for every finite-dimensional diagonal Nichols algebra as classified by [Hec05], [Hec06a], families of solutions (Λ, m_{ij}) realising the associated braiding. These solutions are mostly coming from rescaling Lie algebra (section 4.4) or Lie superalgebra (section 4.5) lattices by rational parameters. In the tables of section 4.9 we list all solutions for finite-dimensional diagonal Nichols algebras of rank 2 and 3.

Our second goal is to investigate when the screening operators form a finite-dimensional Nichols algebra and when they form a larger algebra, extension of a Nichols algebra. We start by refining the result of [Len17], namely weakening the smallness condition:

Theorem 1.3.1. *Let Λ be a non-integral lattice with $v_i \in \Lambda$, $m_{ij} := (v_i, v_j)$. Consider the braiding $q_{ij} := e^{i\pi m_{ij}}$ and the screening operators $\mathfrak{Z}_{v_i}$.*

- *The* truncation *relation* $(\mathfrak{Z}_{v_i})^n = 0$ *holds if:*

$$m_{ii} \notin -\mathbb{N}\frac{2}{k} \qquad k = 1, \ldots, n = \mathrm{ord}(q_{ii}).$$

- *The* Serre *relation* $\left[\mathfrak{Z}_{v_i}, \left[\ldots \left[\mathfrak{Z}_{v_i}, \mathfrak{Z}_{v_j}\right]\,\right]\ldots\right] = 0$ *holds if:*

$$\begin{aligned} m_{ii} &\notin -\mathbb{N}\frac{2}{k} && k = 1, \ldots, n-1 = (1 - a_{ij}) \\ m_{ij} + k\frac{m_{ii}}{2} &\notin -\mathbb{N} && k = 0, \ldots, n-2 = (1 - a_{ij}) - 1 \end{aligned}$$

The truncation and Serre relations are typical relations of Nichols algebras. The theorem, proven by analytic continuation, tells us that these relations always hold except for a set of values of m_{ij} depending on the rational parameters. For those values where they do not hold, it is interesting to understand what is the extension of the Nichols algebra that the screening operators form.

We find that when the braiding $q_{ij} := e^{i\pi(\alpha_i,\alpha_j)r}$ is realised by a rescaled Lie algebra root lattice $m_{ij} = (\alpha_i, \alpha_j)r$ then for $r \geq 0$ all Nichols algebra relations hold and the algebra of screening operators is therefore the small quantum group $u_q(\mathfrak{g})^+$; whereas for $r < 0$ all Nichols algebra relations, except the truncation relations, hold and the algebra of screening operators is a larger algebra, conjecturally the positive part of a quantum group with infinite centre $U_q(\mathfrak{g})^+$ (see 4.4.4).
We find an analogous result for braiding realised by rescaling a Lie superalgebra root lattice (see 4.5.4).

There are then a finite number of finite-dimensional Nichols algebras whose braidings q_{ij} are realised by lattices neither coming from Lie algebras, nor from Lie superalgebras root lattices. For those, the smallness condition always holds and the associated screening operators always form the Nichols algebras $\mathcal{B}(q)$.

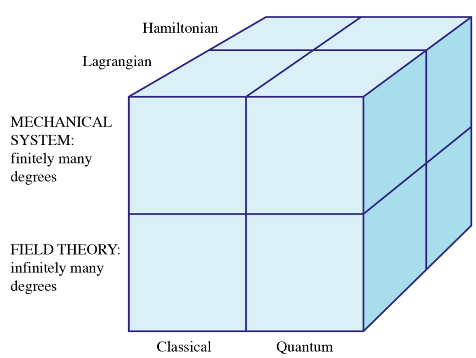
1.4 Structure of the thesis

The thesis is divided into three parts: in chapter 2 we give a more detailed introduction to quantum field theory and vertex algebras, in chapter 3 we give an overview of a side project of the work involving the study of a classical system with the tool of path integrals. Finally in chapter 4 we present the results of the main project of the work whose goals where introduced in the previous section.

Chapter 2

From classical to quantum

In this chapter we want to lead the reader through a more detailed introduction to quantum field theory and vertex algebras.
We start in section 2.1 by recalling the Hamiltonian and Lagrangian formalism in classical mechanics and classical field theory for the toy examples of the free particle and the free scalar field.
We then proceed quantising the two systems in sections 2.2 and 2.3 obtaining the quantum free particle and the quantum free scalar field.
The equivalence between the Hamiltonian and Lagrangian quantisation of the free particle is shown by comparing the Schroedinger solution and the path integral amplitude in 2.2.3.
In order to show some evidence of the equivalence between the two quantisations of the free scalar field we introduce the notion of vertex operator algebras and two point functions in sections 2.4 and 2.4.3.
This chapter is based on the following sources: [Ben18], [CM08], [Sch14].



2.1 Classical setup

2.1.1 Mechanical system

Lagrangian formulation

Let Γ be the phase space of a n-dimensional physical system. Let L be the *Lagrangian* of the system, i.e. a regular function $L : \Gamma \longrightarrow \mathbb{R}$ depending on time-dependent coordinates in Γ.
For a particle moving in a manifold $\mathcal{M}$ with metric g, we can consider the phase space to be the tangent bundle of the manifold, $\Gamma = T\mathcal{M}$.
In what follows we simply consider $\mathcal{M} = \mathbb{R}^n$, then $\Gamma = T\mathcal{M} = \mathbb{R}^n \times \mathbb{R}^n$ with coordinates $(\dot{q}, q) \in \Gamma$. These coordinates are time-dependent in the sense that $\dot{q}, q : \mathcal{I} \to \mathbb{R}^n$ with $\mathcal{I}$ one-dimensional oriented time manifold.

Definition 2.1.1. Consider the set of C^1 time-dependent functions with fixed boundary values

$$\mathcal{A} := \{w \in C^1([t_0, t_1], \mathbb{R}^n), w(t_0) = x_0, w(t_1) = x_1\}. \tag{2.1}$$

The *action functional* is defined on $w \in \mathcal{A}$ as

$$S[w] = \int_{t_0}^{t_1} L(\dot{w}(t), w(t))dt. \tag{2.2}$$

The classical problem in the calculus of variation is to minimise S, namely:

Problem 2.1.2. *Find $X \in \mathcal{A}$ such that $S[X] = min_{w \in \mathcal{A}} S[w]$*

Theorem 2.1.3. *If X satisfies problem 2.1.2, then X solves the Euler-Lagrange equation, which is the equation of motion of the system:*

$$-\frac{d}{dt} L_{\dot{q}}(\dot{X}(t), X(t)) + L_q(\dot{X}(t), X(t)) = 0 \qquad t_0 < t < t_1. \tag{2.3}$$

Hamiltonian formulation

Definition 2.1.4. Let us consider again the phase space $\Gamma = \mathbb{R}^n \times \mathbb{R}^n$ of a physical system. Let H be the *Hamiltonian* of the system, i.e. a regular function $H : \Gamma \longrightarrow \mathbb{R}$ depending on $(p, q) \in \Gamma$, the time-dependent coordinate q and conjugate momentum p.

The following system of equations is called *Hamiltonian system* of Hamiltonian H:

$$\begin{cases} \dot{q} &= \frac{\partial H}{\partial p} \\ \\ \dot{p} &= -\frac{\partial H}{\partial q} \end{cases} \tag{2.4}$$

and describes the motion of the system.

Definition 2.1.5. In an Hamiltonian system it is natural to define an operation called *Poisson bracket* between regular functional, called *observables*, $F, G : \Gamma \longrightarrow \mathbb{R}$ as

$$\{F, G\} = \frac{\partial F}{\partial q}\frac{\partial G}{\partial p} - \frac{\partial F}{\partial p}\frac{\partial G}{\partial q} \tag{2.5}$$

satisfying some elementary properties.
In particular it holds:

$$\{F, H\} = \dot{F}. \tag{2.6}$$

Equivalence of the two formulations

The two formulations are equivalent under some standard hypothesis: we will now derive the Hamiltonian, starting from the Lagrangian.

Definition 2.1.6. We define, for $t_0 < t < t_1$, the *generalised momentum*

$$p(\dot{q}, q, t) := \frac{\partial L}{\partial \dot{q}}. \tag{2.7}$$

If L is convex in $\dot{q}$, then it exists $\forall q, p \in \mathbb{R}^n$ a unique, C^1 solution $\dot{q} =: \dot{Q}(p, q)$ in $\mathbb{R}^n$ which inverts (2.7). The Hamiltonian H associated to the Lagrangian L is obtained from the *Legendre* transform:

$$H(p, q) := p \cdot \dot{Q}(p, q) - L(\dot{Q}(p, q), q). \tag{2.8}$$

Theorem 2.1.7. *Under the same convexity condition, if X is solution of the Euler-Lagrange equation (2.3) and p is defined as in (2.7), then X, p is the solution of the Hamilton's equations:*

$$\begin{cases} \dot{X} &= \frac{\partial H}{\partial p} \\ \\ \dot{p} &= -\frac{\partial H}{\partial X} \end{cases}$$

Therefore the Lagrangian and Hamiltonian formulations are equivalent.

Example: particle subjected to a potential

The system describing the motion on a line of a particle with mass m subjected to a potential $V(q)$ has Lagrangian:

$$L(\dot{q}, q) = \frac{m\dot{q}^2}{2} - V(q)$$

and thus action functional $S[q] = \int_{t_0}^{t_1} (\frac{m\dot{q}^2}{2} - V(q))dt$.

Proposition 2.1.8. *The Euler-Lagrange equation is*

$$m\ddot{q} = -\frac{\partial V}{\partial q} \tag{2.9}$$

i.e. the Newton equation with external force $F(q) = -\frac{\partial V}{\partial q}$.

The generalised momentum is $p = \frac{\partial L}{\partial \dot{q}} = m\dot{q}$ and thus $\dot{Q} = \frac{p}{m}$. Substituting $\dot{Q}$ in (2.8) we obtain the Hamiltonian

$$H(p, q) = \frac{p^2}{2m} + V(q).$$

Lemma 2.1.9. *The two formulations coincide.*

Proof. We want to show that the equation of motion is in the Hamiltonian system again 2.9. The Hamilton's equations (2.4) are

$$\begin{cases} \dot{q} &= \frac{p}{m} \\ \dot{p} &= -\frac{\partial V}{\partial q} \end{cases}$$

which together give the equation (2.9). □

Definition 2.1.10. We call the system *free* particle if $V(q) = 0$, namely in absence of external potential.

2.1.2 Field theory

In section 2.1.1 we introduced two different formalisms to study a mechanical system which can be analogously applied to the study of classical field theory.

Definition 2.1.11. Let $\mathcal{A} = \{\phi(x) : \Sigma \to \mathcal{M}\}$ be the set of smooth functions from a manifold Σ with metric h to a target manifold $\mathcal{M}$ with metric g. The functions $\phi(x)$ are called *fields*.

Remark 2.1.12. Definition (2.1.11) is a higher dimensional generalisation of what we defined in (2.1.1): one can indeed obtained the system of a particle moving in $\mathbb{R}^n$, by considering as source space the one-dimensional oriented time manifold $\mathcal{I}$ and the set of smooth functions $\mathcal{A} = \{\phi(x) : \mathcal{I} \to \mathcal{M} = \mathbb{R}^n\}$.

Definition 2.1.13. If $\mathcal{M} = \mathbb{R}$ (or $\mathbb{C}$) the fields are called *scalar* fields.

Remark 2.1.14. Other examples of fields $\phi(x)$ are *vector* fields when e.g. $\mathcal{M} = T\Sigma$ the tangent bundle or *spinor* fields when $\mathcal{M} = Spin\Sigma$ the spinor bundle, and $\phi(x)$ is a section of $\mathcal{M}$. An example of spinor field is given by the *Dirac* field.

Definition 2.1.15. In field theory one uses the notions of Lagrangian and Hamiltonian *densities* $\mathcal{L}$ and $\mathcal{H}$, functions of the fields and their derivatives, for which hold the definitions and results of 2.1.1. From $\mathcal{L}$ and $\mathcal{H}$ one can obtain the Lagrangian and Hamiltonian L and H by integrating with respect to the spatial coordinate x:

$$L = \int dx\, \mathcal{L} \qquad H = \int dx\, \mathcal{H}. \tag{2.10}$$

Example: the scalar field

We now assume $\phi(x,t) : \Sigma \to \mathbb{R}$ to be a free massless scalar field and the source manifold Σ to be $\mathbb{R} \times S^1$ with Lorentz metric.
The Lagrangian density of a free massless scalar field is

$$\mathcal{L} := \frac{1}{2}\partial_\mu \phi \partial^\mu \phi.$$

and therefore the action functional:

$$S := \int dt\, L = \int dt dx \mathcal{L} = \int dt dx \left(\frac{1}{2}\partial_\mu \phi \partial^\mu \phi \right).$$

The generalised momentum field is defined via the functional derivative

$$\Pi(x,t) := \frac{\delta \mathcal{L}}{\delta \frac{\partial \phi}{\partial t}} = \frac{\partial \phi}{\partial t}.$$

The Hamiltonian density is thus given by

$$\begin{aligned}\mathcal{H} :&= \Pi \cdot \frac{\partial \phi}{\partial t} - \mathcal{L} \\ &= \left(\frac{\partial \phi}{\partial t}\right)^2 - \frac{1}{2}\left(\frac{\partial \phi}{\partial t}\right)^2 + \frac{1}{2}\left(\frac{\partial \phi}{\partial x}\right)^2 \\ &= \frac{1}{2}\Pi^2 + \frac{1}{2}\left(\frac{\partial \phi}{\partial x}\right)^2.\end{aligned}$$

Therefore the Hamiltonian of the system is:

$$H(\phi, \Pi) = \frac{1}{2}\int dx\ (\Pi^2 + \left(\frac{\partial \phi}{\partial x}\right)^2) \tag{2.11}$$

2.2 Quantisation of the free particle

We now show the Lagrangian and Hamiltonian quantisation of a mechanical system, focusing in particular on the example of the free particle.
The quantisation is in both approaches not completely rigorous: in the Hamiltonian approach one has to provide a Hilbert space and operators with commutator rules according to the Poisson bracket defined in 2.1.5, and there is no general recipe to do it; in the Lagrangian approach one has to define a measure $\mathcal{D}x(t)$ on the space of all functions: this is typically mathematically not well-defined and therefore the path integral which uses that measure is not well-defined, yet nevertheless useful.

2.2.1 Lagrangian quantisation

Definition 2.2.1. The Lagrangian *quantisation* consists of defining on the space of all the functions $\mathcal{A} = \{x(t)\}$ the complex valued density $e^{\frac{i}{\hbar}S[x(t)]}$ where $S[x(t)]$ is the classical action functional.

Once introduced this density, the system is no longer deterministic. We are then interested in computing the expectation values of the functional, the observables.

Definition 2.2.2. The expectation value of an observable F is a *path integral*, namely the integral on all the possible configurations of the system

$$\langle F[x(t)] \rangle := \frac{1}{\mathcal{Z}} \int_{\{x(t)\}} \mathcal{D}x(t) \cdot e^{\frac{i}{\hbar} S[x(t)]} \cdot F[x(t)]$$

where the normalization term $\mathcal{Z}$, also called path partition function, is defined as

$$\mathcal{Z} := \langle 1 \rangle = \int_{\{x(t)\}} \mathcal{D}x(t) \cdot e^{\frac{i}{\hbar} S[x(t)]}$$

Remark 2.2.3. We recall that this integral is not mathematically rigorous.

Remark 2.2.4. In the limit $\hbar \to 0$, the path integral is dominated by the path x_{min} which minimises the action S, i.e. the path solution of the classical system

$$\lim_{\hbar \to 0} \langle F[x(t)] \rangle = F[x_{min}(t)].$$

Example 2.2.5. The probability of a free particle to be in x_1 at the time t_1 if it is in x_0 at the time t_0 is given by

$$\langle x_1, t_1 \mid x_0, t_0 \rangle = \frac{1}{\mathcal{Z}} \int\limits_{\substack{x(t_0)=x_0 \\ x(t_1)=x_1}} \mathcal{D}x(t) e^{\frac{i}{\hbar} \int_{t_0}^{t_1} dt \frac{\dot{x}}{2} m}.$$

2.2.2 Hamiltonian quantisation

Definition 2.2.6. The Hamiltonian quantisation consists of replacing

- the classical deterministic state (p, q) by a quantum state $\psi(x, t)$ in a Hilbert space $\mathbf{H}$.
- the observables F by self-adjoint operators $\widehat{F}$ acting on $\mathbf{H}$ such that $[\widehat{F}, \widehat{G}] = i\hbar \widehat{\{F, G\}}$ where $\hbar$ is the Planck constant.
  In particular we replace the variables of the classical system q, p by operators $\widehat{q}, \widehat{p}$ with commutators $[\widehat{q}, \widehat{p}] = i\hbar$, since $\{q, p\} = 1$.

Example 2.2.7. We now quantise the free particle system.
The classical deterministic state (p, q), describing the position and momentum of the particle, is substituted by a quantum state $\psi(x, t)$, describing the probability to find the particle in a certain position x at time t.

Let $\mathbf{H}$ be the space of functions in a variable x, define $\widehat{q} := x\cdot$, $\widehat{p} := -i\hbar\partial_x$. The functional of the system are then polynomial in the variable x and ∂_x. The quantised Hamiltonian is:

$$\widehat{H}(\widehat{p},\widehat{q}) = \frac{\widehat{p}^2}{2m} = -\frac{\hbar^2}{2m}\partial_x^2 \tag{2.12}$$

and the related Poisson equation: $[\widehat{H},\widehat{F}] = i\hbar\dot{\widehat{F}}$.
The solution of the system will be a wave function $\psi(x,t)$ satisfying the Poisson equation

$$\left[\widehat{H},\psi(x,t)\right] = i\hbar\partial_t\psi(x,t)$$

which yields the *Schroedinger* equation

$$-\frac{\hbar^2}{2m}\partial_x^2\psi(x,t) = i\hbar\partial_t\psi(x,t). \tag{2.13}$$

2.2.3 Equivalence of the quantised systems

In the previous sections we quantised the free particle system following two different approaches. In the following we will see that they are equivalent: the wave function $\psi(x,t)$ with fixed initial data (x_0,t_0) and final data (x_1,t_1), solution of the Schroedinger equation, coincides with the *propagator*, i.e. the amplitude of the operator representing the state with those initial and final data computed through the path integral approach. We recall that, as the path integral has to be understood in a heuristic sense, so is the equivalence we are going to show.

Lemma 2.2.8. *We have, heuristically, the following equivalence:*

$$\langle x_1,t_1 \mid x_0,t_0\rangle = \sqrt{\frac{m}{i\hbar(t_1-t_0)2\pi}}\,e^{-\frac{m}{i\hbar}\frac{(x_1-x_0)^2}{2(t_1-t_0)}} = \psi(x_1,t_1) \tag{2.14}$$

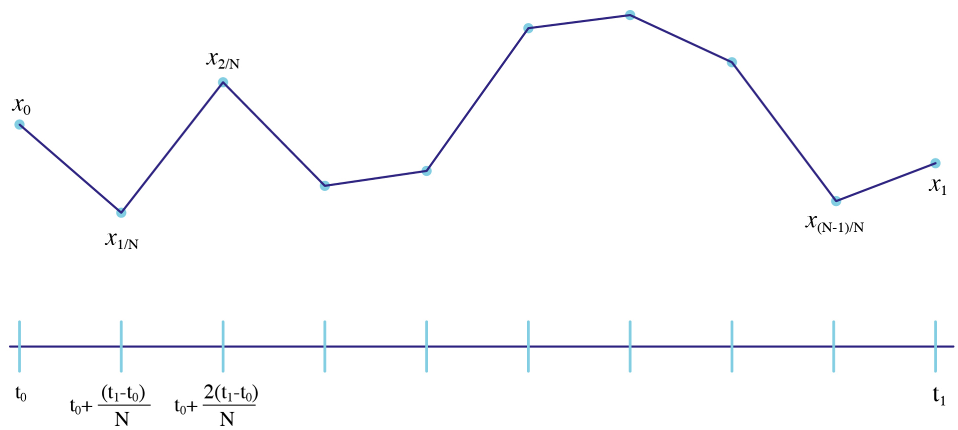
Proof. Lagrangian side:
We want to compute the left side of equation (2.14), i.e. the probability of a free particle to be in x_1 at the time t_1 if it is in x_0 at the time t_0. As we saw in 2.2.5, this is given by the following path integral:

$$\langle x_1,t_1 \mid x_0,t_0\rangle = \frac{1}{\mathcal{Z}}\int\limits_{\substack{x(t_0)=x_0\\ x(t_1)=x_1}} \mathcal{D}x(t)\,e^{\frac{i}{\hbar}\int_{t_0}^{t_1} dt\,\frac{\dot{x}}{2}m}$$

that we discretise in the standard way (see e.g. [PS95]) thinking of the path as the limit for $N \to \infty$ of N little paths.

$$= \lim_{N\to\infty} \frac{1}{\mathcal{Z}} \int_{\mathbb{R}} dx_{1/N} \int_{\mathbb{R}} dx_{2/N} \dots$$

$$\dots \int_{\mathbb{R}} dx_{(N-1)/N} \quad e^{\frac{im}{\hbar}\frac{t_1-t_0}{N}\left[\left(\frac{x_{1/N}-x_0}{\frac{t_1-t_0}{N}}\right)^2\frac{1}{2}+\left(\frac{x_{2/N}-x_{1/N}}{\frac{t_1-t_0}{N}}\right)^2\frac{1}{2}+\dots\right]}$$



Now, calling $t := t_1 - t_0$ and defining

$$z_1 := x_{1/N} - x_0, \quad z_2 := x_{2/N} - x_{1/N} \quad \dots \quad z_n := x_1 - x_{(N-1)/N}$$

we can write the limit as:

$$= \lim_{N\to\infty} \frac{1}{\mathcal{Z}} \int_{\sum_k z_k = x_1 - x_0} dz_1 \dots dz_n \prod_k e^{\frac{im}{\hbar}\frac{z_k^2}{2(t/N)^2}\frac{t}{N}}$$

We, thus have a function of the form

$$f(x,t) = \lim_{N\to\infty} \frac{1}{\mathcal{Z}} g(x) * \dots * g(x)$$

convolution product of N functions $g(x) = e^{\frac{im}{\hbar}\frac{x^2}{2t/N}}$, $x = x_1 - x_0$.
The normalization term is defined as:

$$\mathcal{Z} = \int \mathcal{D}x(t) e^{\frac{i}{\hbar}\int_{t_0}^{t_1} dt \frac{\dot{x}}{2} m}.$$

Proceeding as before, we can compute it and obtain something of the form

$$\mathcal{Z} = \left(\int_{\mathbb{R}} e^{-\frac{m}{i\hbar}\frac{y^2}{2t/N}} dy \right)^N = \left(\sqrt{\frac{2\pi i t\hbar}{Nm}} \right)^N .$$

We now take the Fourier transform $\mathcal{F}$ of $f(x,t)$ and use the convolution theorem

$$\mathcal{F}(f(x,t)) = \lim_{N\to\infty} \frac{1}{\mathcal{Z}} \mathcal{F}(g(x)) * \ldots * \mathcal{F}(g(x)) = \lim_{N\to\infty} \frac{1}{\mathcal{Z}} \left(\mathcal{F}(g(x))\right)^N .$$

Hence,

$$\hat{f}(\xi,t) = \lim_{N\to\infty} \left(\sqrt{\frac{Nm}{2\pi t\hbar i}} \int_{\mathbb{R}} e^{\frac{im}{\hbar}\frac{x^2}{2t/N}} e^{-2\pi i x\xi} dx \right)^N$$

and transforming the exponent in the square of a binomial we get:

$$= \lim_{N\to\infty} \left(\sqrt{\frac{Nm}{2\pi t\hbar i}} e^{\frac{\pi^2\xi^2 2t\hbar}{mNi}} \int_{\mathbb{R}} e^{-y^2} dy \sqrt{\frac{2ti\hbar}{Nm}} \right)^N = e^{\frac{\pi^2\xi^2 2t\hbar}{mi}}$$

where the last equivalence follows from $\int_{\mathbb{R}} e^{-y^2} dy = \sqrt{\pi}$. Now we take the inverse of the Fourier transform and use the Fourier transform theorem:

$$f(x,t) = \int_{\mathbb{R}} e^{\frac{\pi^2\xi^2 2t\hbar}{mi}} e^{-2\pi i x\xi} d\xi$$

and again treating it as the square of a binomial we obtain as expected:

$$= e^{\frac{mix^2}{2t\hbar}} \int_{\mathbb{R}} e^{-y^2} dy \sqrt{\frac{m}{2it\hbar\pi^2}} = \sqrt{\frac{m}{2it\hbar\pi}} e^{\frac{mix^2}{2t\hbar}} = \sqrt{\frac{m}{2i(t_1-t_0)\hbar\pi}} e^{\frac{mi(x_1-x_0)^2}{2(t_1-t_0)\hbar}} .$$

Hamiltonian side:
We want to compute the right side of equation (2.14). We consider the quantised Hamiltonian described in (2.12) and compute by hand a solution for fixed initial and final data of the Schroedinger equation (2.13):

$$-\frac{\hbar^2}{2m}\partial_x^2 \psi(x,t) = i\hbar\partial_t\psi(x,t).$$

The solution will have a Gaussian form since it describes a wave package moving in time

$$\psi(x,t) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \tag{2.15}$$

where $\sigma = \sigma(t)$ and $\mu = \mu(t)$ are function of the time.
Computing the derivatives and substituting in (2.13) we obtain

$$\sigma = \sqrt{\frac{i\hbar}{m}(t - t_0)}, \qquad \mu = x_0$$

and therefore inserting $\psi(x,t)$ in the final state (x_1, t_1) we obtain the expected result. □

2.3 Quantisation of the free scalar field

We now show the Lagrangian and Hamiltonian quantisation of a field theory, focusing in particular on the example of the free scalar massless field.

2.3.1 Lagrangian quantisation

The path integral quantisation of a field theory generalises the mechanical one. The idea is to replace the path of a particle $x(t)$ by a field configuration $\phi(x)$.

Definition 2.3.1. The Lagrangian quantisation consists of defining on the space of all field configurations $\mathcal{A} = \{\phi(x)\}$ the density $e^{\frac{iS[\phi(x)]}{\hbar}}$.

Definition 2.3.2. The expectation value of an observable F is the path integral over all possible configurations of the field:

$$\langle F[\phi(x)]\rangle := \frac{1}{\mathcal{Z}} \int_{\{\phi(x)\}} \mathcal{D}\phi(x) e^{\frac{iS[\phi(x)]}{\hbar}} F[\phi(x)]$$

where the normalization term $\mathcal{Z}$ is defined as

$$\mathcal{Z} := \langle 1 \rangle = \int_{\{\phi(x)\}} \mathcal{D}\phi(x) e^{\frac{iS[\phi(x)]}{\hbar}}.$$

In particular the n-point functions depending on a sequence (x_i) of points in the source manifold Σ can be computed as follows:

$$\langle\phi(x_1)\cdots\phi(x_n)\rangle := \frac{1}{\mathcal{Z}}\int_{\{\phi(x)\}}\mathcal{D}\phi(x)e^{\frac{iS[\phi(x)]}{\hbar}}\phi(x_1)\cdots\phi(x_n).$$

Example 2.3.3. The two-point function of a free massless scalar field ϕ results

$$\langle\phi(x_1)\phi(x_2)\rangle \sim ln(x_1 - x_2)^2 + \text{const} \tag{2.16}$$

which taking the derivatives yields

$$\langle\partial\phi(x_1)\partial\phi(x_2)\rangle \sim \frac{1}{(x_1 - x_2)^2}. \tag{2.17}$$

We do not present the details which can be found e.g. in [FMS96].
The same result is obtained by computing the two point function using vertex operators, which is the ultimate output of the Hamiltonian quantisation, as we will see in the next sections.

2.3.2 Hamiltonian quantisation

The Hamiltonian canonical quantisation of a field theory consists of a first quantisation which closely generalises the mechanical one, and a second quantisation which leads to an algebraic study of the system.

Definition 2.3.4. The Hamiltonian field quantisation consists of replacing the classical observables F by self-adjoint operators $\widehat{F}$ acting on the Hilbert space of quantum states.

In particular we replace the quantum variables, the field $\phi(x,t)$ and the canonical momentum $\Pi(x,t)$, by two self-adjoint operators $\widehat{\phi}(x,t)$ and $\widehat{\Pi}(x,t)$ with equal-time commutators given by

$$\left[\widehat{\phi}(x,t), \widehat{\Pi}(y,t)\right] = i\hbar\delta(x-y).$$

Example 2.3.5. The first quantisation ot the free massless scalar field yields to the Hamiltonian operator

$$\widehat{H} = \int dx(\frac{1}{2}\widehat{\Pi}^2 + \frac{1}{2}\left(\frac{\partial\widehat{\phi}}{\partial x}\right)^2) \tag{2.18}$$

which corresponds to the classical Hamiltonian (2.11) after substituting ϕ and Π by $\widehat{\phi}$ and $\widehat{\Pi}$.

Remark 2.3.6. We quantise the free massless scalar field on a space with Lorentz metric. The example where the space has Euclidean metric is very similar.

Second quantisation - free massless scalar field

Definition 2.3.7. The *second* quantisation of the free massless scalar field theory consists of

- taking the Fourier transform of $\widehat{\phi}(x)$ and $\widehat{\Pi}(x)$:

  $$\widehat{\phi}_k = \int \widehat{\phi}(x)e^{-ikx}dx \qquad \widehat{\Pi}_k = \int \widehat{\Pi}(x)e^{-ikx}dx$$

  where $\widehat{\phi}_{-k} = \widehat{\phi}_k^\dagger$ and $\widehat{\Pi}_{-k} = \widehat{\Pi}_k^\dagger$ and $[\widehat{\phi}_k, \widehat{\Pi}_k^\dagger] = i\hbar$. The Hamiltonian results

  $$\widehat{H} = \frac{1}{2}\sum_{k\in\mathbb{Z}}\left[\widehat{\Pi}_k\widehat{\Pi}_k^\dagger + k^2\widehat{\phi}_k\widehat{\phi}_k^\dagger\right]$$

- defining the *creation* and *annihilation* operators:

  $$\widehat{a}_k = \frac{1}{\sqrt{2\hbar k}}\left(k\widehat{\phi}_k + i\widehat{\Pi}_k\right)$$
  $$\widehat{a}_k^\dagger = \frac{1}{\sqrt{2\hbar k}}\left(k\widehat{\phi}_k^\dagger - i\widehat{\Pi}_k^\dagger\right)$$

  with commutators

  $$\left[\widehat{a}_\alpha, \widehat{a}_\beta^\dagger\right] = \delta_{\alpha\beta}.$$

  The quantised Hamiltonian 2.18 in terms of the mode operators is

  $$\widehat{H} = \sum_{k\in\mathbb{Z}}\hbar k\widehat{a}_k^\dagger\widehat{a}_k \tag{2.19}$$

The creation and annihilation operators, or *mode* operators, $\widehat{a}_k, \widehat{a}_k^\dagger$ generate the algebra of observables of the quantised field theory which we are going to study in more detail in the next section.

The Heisenberg algebra

We now focus on the Heisenberg algebra H, the algebra of the mode operators found through the second quantisation, and on its representation $\mathcal{V}_{\mathsf{H}}$.
We rescale the mode operators

$$a_n = -i\sqrt{n}\widehat{a}_{-n} \qquad a_{-n} = i\sqrt{n}\widehat{a}^{\dagger}_{-n}$$

Definition 2.3.8. Let $\mathsf{H} := \langle a_n \rangle_{n\in\mathbb{Z}}$ be the Lie algebra spanned by the creation and annihilation operators with commutation rule $[a_\alpha, a_\beta] = \alpha\delta_{\alpha,-\beta}$. We call H *Heisenberg* algebra.

Proposition 2.3.9. *The* Fock space $\mathcal{V}_{\mathsf{H}}$*, defined as the space of polynomials* $\mathbb{C}[a_{-1}, a_{-2}, \ldots]$*, is a representation of the Heisenberg algebra with the following action of the mode operators:*

- a_k *act by derivation* $\frac{\partial}{\partial a_{-k}}$ *for* $k \geq 0$ *(annihilating)*
- a_k *act by multiplication* $a_k\cdot$ *for* $k \leq 0$ *(creating)*
- a_0 *acts by* 0*.*

In the next chapter we will give the definition of *vertex operator algebra* and see that the representation $\mathcal{V}_{\mathsf{H}}$ can be enriched with a vertex algebra structure. We will then call $\mathcal{V}_{\mathsf{H}}$ the *Heisenberg vertex algebra*.
This is then, from the mathematical side, the algebra describing the chiral quantum (conformal) field theory of the free massless scalar field.

In section 2.4.3 we use vertex algebras to compute again (2.17).

2.4 Vertex operator algebras

We now introduce vertex operator algebras and modules, focusing in particular on the example of the Heisenberg vertex algebra.

2.4.1 First definitions

Definition 2.4.1. A *vertex algebra* is a collection of data:

- a $\mathbb{Z}$-graded vector space $\mathcal{V}$ called *space of states*

- a distinguished vector $|0\rangle \in \mathcal{V}$ called *vacuum* vector
- a linear operator $\partial : \mathcal{V} \longrightarrow \mathcal{V}$ called *translation* operator
- a linear operator $Y(\cdot, z)$ called *vertex operator* defined as

$$\begin{aligned} Y(\cdot, z) : \quad \mathcal{V} &\longrightarrow \mathrm{End}\mathcal{V}[[z, z^{-1}]] \\ A &\longmapsto Y(A, z) = \sum_{n\in\mathbb{Z}} A_n z^{-n-1}. \end{aligned}$$

taking values in formal power series in z and coefficients in $\mathrm{End}\mathcal{V}$.
These data are subject to the following axioms:

(i) the *vacuum* axiom:

$$Y(|0\rangle, z) = \mathrm{id}_{\mathcal{V}} \quad \text{and} \quad Y(A, z)|0\rangle \in A + z\mathcal{V}[[z]]$$

where the latter implies $Y(A, z)|0\rangle|_{z=0} = A$.

(ii) the *translation* axiom

$$[\partial, Y(A, z)] = \frac{\partial}{\partial z} Y(A, z) \quad \text{and} \quad \partial|0\rangle = 0$$

(iii) the *locality* axiom:

$$\exists N = N_{a,b} \in \mathbb{Z}_{\geq 0} \qquad (z - w)^N [Y(A, z), Y(B, w)] = 0$$

which tells us that all the fields are mutually *local* with each other, in the sense just meant.

Commutative associative unital algebras with a derivation are examples of vertex algebras.

Remark 2.4.2. The operator $Y(A, z)$ is a formal power series with coefficients $A_n \in \mathrm{End}\mathcal{V}$ and is also called *field*, referring to the role it plays in physics. Therefore the map $A \longmapsto Y(A, z)$, together with the vacuum axiom, defines what in physics is called the *state-field* correspondence.

Remark 2.4.3. Another way to define the vertex operator is

$$Y : \quad \mathcal{V} \otimes_{\mathbb{C}} \mathcal{V} \longrightarrow \mathcal{V}[[z, z^{-1}]] \tag{2.20}$$

which highlight that Y is a multiplication depending on the point of insertion.

Vertex algebras describe the chiral and anti-chiral parts of conformal field theories. Therefore a crucial notion is the one of conformal symmetry algebra acting on vertex algebras. The symmetry algebra is called *Virasoro* algebra and is given by the central extension of the Witt algebra of infinite conformal transformations.

Definition 2.4.4. The Virasoro algebra is a Lie algebra with generators L_n, $n \in \mathbb{Z}$ with commutators

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0}$$

and c called *central charge* commuting element, $[c, L_n] = 0 \ \forall n$.

Definition 2.4.5. A vertex algebra $\mathcal{V}$ is said to be a vertex *operator* algebra (VOA) of central charge $c \in \mathbb{C}$ if there is a non-zero element $\omega \in \mathcal{V}_2$, called *conformal vector*, such that the coefficients $L_n^\vee$ of the associated field

$$Y(\omega, z) = \sum_{n\in\mathbb{Z}} L_n^\vee z^{-n-2} =: T(z)$$

satisfy the defining relations of the Virasoro algebra with central charge c. Moreover we demand $L_{-1}^\vee = \partial$, and $L_0^\vee|_{\mathcal{V}_n} = n \cdot \mathrm{Id}_{\mathcal{V}_n}$ where we denoted by $\mathcal{V}_n$ the n-th grading layer.

Remark 2.4.6. In physics literature $T(z)$ is called *energy momentum tensor.*

An important example of vertex operator algebra is the one of the Heisenberg vertex algebra mentioned above.

We change the notation writing $\frac{1}{(n-1)!}\partial^n\phi$ instead of a_{-n}

Example 2.4.7. The 1-dimensional *Heisenberg* VOA is defined as collection of the data:

- the Fock space $\mathcal{V}_{\mathsf{H}} = \mathbb{C}[\partial\phi, \partial^2\phi \ldots]$
- the vacuum vector $|0\rangle = 1 \in \mathbb{C}$
- the translation operator

$$\begin{aligned} \partial : \mathcal{V}_{\mathsf{H}} &\longrightarrow \mathcal{V}_{\mathsf{H}} \\ \partial^k\phi &\longmapsto \partial^{k+1}\phi \end{aligned}$$

- one parameter family of conformal vectors $\omega_Q = \frac{1}{2}\partial\phi\partial\phi + Q\partial^2\phi$, $Q \in \mathbb{C}$
- the vertex operator $Y(\partial\phi, z) = \sum\limits_{n\in\mathbb{N}} a_n\ z^{-n-1}$, where a_n are the mode operators.

Moreover the space is graded $\deg(\partial^{k_1}\phi \dots \partial^{k_n}\phi) = \sum\limits_{i=1}^{n} k_i$.

This example is the mathematical formalisation of the free scalar field in physics, with background charge Q. We adopted this notation to suggest that the variable $\partial\phi : \Sigma \longrightarrow \mathbb{C}$ is the field of the associated chiral conformal field theory.

Example 2.4.8. The n-dimensional Heisenberg VOA is given by the space $\mathcal{V}^n_{\mathsf{H}} := \mathbb{C}[\partial\phi_\lambda, \partial^2\phi_\lambda, \cdots]$, with $\lambda \in \mathbb{C}^n$. Explicitly $\mathcal{V}^n_{\mathsf{H}}$ is the polynomial ring in the variables $\partial\phi_{e_1}, \dots, \partial\phi_{e_n}, \partial^2\phi_{e_1}, \dots, \partial^2\phi_{e_n}, \dots$ with $e_1, \dots, e_n$ basis of $\mathbb{C}^n$.

2.4.2 Modules over a VOA

Definition 2.4.9. A vertex algebra module over a VOA $\mathcal{V}$ is a $\mathbb{C}$-graded vector space $\mathcal{M}$ together with an operator

$$Y_{\mathcal{M}} : \quad \mathcal{V} \longrightarrow End\mathcal{M}[[z, z^{-1}]]$$

satisfying similar axioms as in definition 2.4.1.

Remark 2.4.10. We can write $Y_{\mathcal{M}}$ as we did for the vertex operator Y in (2.20):

$$Y_{\mathcal{M}} : \quad \mathcal{V} \otimes_{\mathbb{C}} \mathcal{M} \longrightarrow \mathcal{M}[[z, z^{-1}]] \tag{2.21}$$

which makes clear the module structure of $\mathcal{M}$.

Remark 2.4.11. The Virasoro algebra acts on $\mathcal{M}$ automatically via

$$Y_{\mathcal{M}}(\omega, z) = \sum_{n\in\mathbb{Z}} L^{\mathcal{M}}_n z^{-n-2}$$

Example 2.4.12. The easiest example of module over a vertex operator algebra is given by the vertex operator algebra over itself.

Example 2.4.13. The 1-dimensional Heisenberg vertex algebra $\mathcal{V}_{\mathsf{H}}$ has an irreducible representation $\mathcal{V}_a = \mathbb{C}[\partial\phi, \partial^2\phi, \dots]e^{\phi_a}$ for every complex number $a \in \mathbb{C}$, where e^{ϕ_a} is a formal variable.
Analogously the n-dimensional Heisenberg algebra $\mathcal{V}^n_{\mathsf{H}}$ has an irreducible representation $\mathcal{V}_a := \mathbb{C}[\partial\phi_\lambda, \partial^2\phi_\lambda, \cdots]e^{\phi_a}$ for every $a \in \mathbb{C}^n$.

2.4.3 Two point function

We now want to compute the two point function of a free field in the vertex algebras notation, i.e. $\langle 0| Y(\partial\phi, z)Y(\partial\phi, w)|0\rangle$, and compare it to (2.17).

The vertex operator of the Heisenberg vertex algebra (2.4.7) is defined as $Y(\partial\phi, z) = \sum\limits_{n\in\mathbb{N}} a_n\, z^{-n-1}$ and therefore applied:

$$
\begin{aligned}
Y(\partial\phi, z)|0\rangle &= \sum_{k\geq 0} \frac{z^k}{k!}\partial^{1+k}\phi \\
Y(\partial\phi, z)\partial\phi &= z^{-2}\cdot 1 + \sum_{k\geq 0} \frac{z^k}{k!}\partial^{1+k}\phi\ \partial\phi \\
Y(\partial\phi, z)\partial^{1+k}\phi &= (-1)^k \binom{-2}{k} z^{-2-k}\cdot 1 + \sum_{j\geq 0} \frac{z^j}{j!}\partial^{1+j}\phi\ \partial^{1+k}\phi.
\end{aligned}
$$

and thus we have:

$$
\begin{aligned}
\langle 0| Y(\partial\phi, z)Y(\partial\phi, w)|0\rangle &= \langle 0| \partial\phi(z)\partial\phi(w)|0\rangle = \sum_{k\geq 0} w^k \langle 0| \partial\phi(z) |\frac{\partial^{1+k}\phi}{k!}\rangle \\
&= \sum_{k\geq 0} w^k \langle 0| \left((-1)^k \binom{-2}{k} z^{-2-k}\cdot 1 + \sum_{j\geq 0} \frac{z^j}{j!}\partial^{1+j}\phi\ \frac{\partial^{1+k}\phi}{k!} \right) \\
&= \sum_{k\geq 0} w^k (-1)^k \binom{-2}{k} z^{-2-k}\cdot 1 \langle 0|0\rangle + \sum_{k\geq 0} w^k \langle 0| \sum_{j\geq 0} \frac{z^j}{j!}\partial^{1+j}\phi\ \frac{\partial^{1+k}\phi}{k!}
\end{aligned}
$$

projecting on the vacuum state, the second term vanishes while the first term gives exactly $\frac{1}{(z-w)^2}$ as in 2.17.

Chapter 3

Path integral approach to classical dynamics of molecules

In this chapter we are going to summarise in big lines a side project of the thesis. This project is based on the collaboration with the physical and theoretical chemistry group of Prof. Dr. Bettina G. Keller of the Freie Universitaet Berlin.

The aim is to use the path integral formalism to describe the stochastic dynamics of a classical molecular system with many degrees of freedom.
We start by describing the system of a macro-molecule with many degrees of freedom using the Langevin equation and its over-damped version. This can be then translated into the stochastic Fokker-Planck equation by considering the probability density functions of the previous system. In this way we consider most degrees of freedom as stochastic noise [Zwa01]. The classical stochastic dynamics of the system turns out to be equivalent to a path integral and a Schrödinger equation.

The results of this chapter are not new, but a collection and blend of classical molecular systems knowledge and path integrals manipulation. As outlook of further collaboration, we plan to match this study with numerical simulations of macro-molecules and develop a perturbative model of a peptide bond.

3.1 Differential equations for molecular processes

3.1.1 The Langevin equation

The dynamics of a molecular system can be modelled by Langevin equation. Consider a system with N particles with mass matrix $\mathbf{M}$ and coordinates $\mathbf{x} = \mathbf{x}(t) \in \Omega \subset \mathbb{R}^{3N}$, where Ω denotes the configuration space. The corresponding Langevin equation is given by

$$\mathbf{M}\ddot{\mathbf{x}} = -\nabla V_L(\mathbf{x}) - \gamma\dot{\mathbf{x}} + \sqrt{2D\gamma^2}\,\mathbf{R}(t) \tag{3.1}$$

where D is the diffusion constant, γ is the friction coefficient, and $V(\mathbf{x})$ the potential which models the interaction of the particles. $\mathbf{R}(t)$ denotes a delta-correlated stationary Gaussian random force with zero-mean, i.e. satisfying

$$\langle \mathbf{R}(t) \rangle = 0, \qquad \langle \mathbf{R}(t)\mathbf{R}(t') \rangle = \delta\left(t - t'\right). \tag{3.2}$$

3.1.2 The overdamped Langevin equation

Over-damped Langevin dynamics are a simplified version of Langevin dynamics. They correspond to the limit where in equation (3.1) no average acceleration takes place, i.e. $\mathbf{M}\ddot{\mathbf{x}} \ll \gamma\dot{\mathbf{x}}$, either small $\mathbf{M}$ (mass-less limit) or large γ (high-friction limit). The over-damped Langevin equation is then given as

$$\dot{\mathbf{x}} = -\frac{\nabla V_L(\mathbf{x})}{\gamma} + \sqrt{2D}\,\mathbf{R}(t). \tag{3.3}$$

These dynamics are also called Brownian dynamics with drift $V_L(\mathbf{x})$. Discretising equation (3.3) in time using the Euler-Maruyama method [KP92] yields the iterative equation

$$x_{k+1} = x_k - \frac{\nabla V_L(x_k)}{\gamma}\Delta t + \sqrt{2D}\,\sqrt{\Delta t}\,\eta_k \tag{3.4}$$

with iteration time step Δt, $t_{\text{tot}} = n \cdot \Delta t$ and random number η_k drawn from a standard Gaussian distribution $\eta_k \sim \mathcal{N}(0, 1)$ with zero mean and unit variance.

3.2 The Fokker-Planck and the Schroedinger equations

3.2.1 The Fokker-Planck equation and operator

The Fokker-Planck (FP) equation is a partial differential equation which describes the time evolution of a probability density function $p(\mathbf{x}, t)$ of the stochastic process modelled by the over-damped Langevin equation (3.3):

$$\partial_t p(\mathbf{x}, t) = \sum_{i=1}^{N} \partial_{x_i} \left(\frac{\partial_{x_i} V_L(\mathbf{x})}{\gamma} p(\mathbf{x}, t) \right) + D \sum_{i=1}^{N} \partial_{x_i x_i} p(\mathbf{x}, t). \qquad (3.5)$$

As interpretation, imagine an ensemble of infinitely many systems. Each system can be in a different state $\mathbf{x} \in \Omega$. The probability density function $p(\mathbf{x}, t)$ represents the distribution of the systems over Ω at time t. Each individual particle moves according to eq. (3.3), and thus $p(\mathbf{x}, t)$ evolves in time.

For the sake of simplicity, we consider from now on a one-dimensional Fokker-Planck-equation

$$\partial_t p(x, t) = \left(\frac{\partial_{xx} V_L}{\gamma} + \frac{\partial_x V_L}{\gamma} \partial_x + D \partial_{xx} \right) p(x, t) = \hat{L}\, p(x, t) \qquad (3.6)$$

where we have defined the Fokker-Planck operator

$$\hat{L} = \frac{\partial_{xx} V_L}{\gamma} + \frac{\partial_x V_L}{\gamma} \partial_x + D \partial_{xx}. \qquad (3.7)$$

The following functions formally solve the Fokker-Planck equation

$$\varphi_k(x, t) = e^{\lambda_k t} \cdot \varphi_k(x), \qquad (3.8)$$

with (λ_k, φ_k) eigenvalues and eigenfunctions of the Fokker-Planck operator, $\lambda_k \leq 0$. The solutions $p(x, t)$ are appropriate linear combinations of $\varphi_k(x, t)$. They are non-negative and normalised, in order to be interpreted as probability density.

It can be shown, that the smallest eigenvalue is $\lambda_0 = 0$, and the corresponding eigenfunction thus gives a stationary solution

$$\varphi_0(x,t) = e^{0t} \cdot \varphi_0 = \frac{1}{\mathcal{Z}}\, e^{-\frac{V_L(x)}{D\gamma}} \tag{3.9}$$

where $\mathcal{Z}$ is a normalisation constant to ensure $\int_\Omega dx\, \varphi_0(x) = 1$, so the $\varphi_0(x,t)$ can be interpreted as a probability density.

3.2.2 Equivalence to the Schroedinger equation

We now show that the eigenvalue problem for the Fokker-Planck operator $\hat{L}$ is equivalent to the eigenvalue problem of a corresponding Schroedinger operator $\hat{H}$ of the general form

$$\hat{H} = a\, \partial_{xx} + V_H(x), \tag{3.10}$$

where different values of a are simple rescaling.
This means that the time-independent Fokker-Planck equation is equivalent to the time-independent Schroedinger equation [Ris89]:

$$\hat{L}\, \varphi_k(x) = \lambda_k\, \varphi_k(x) \quad \Leftrightarrow \quad \hat{H}\, \psi_k(x) = \lambda_k\, \psi_k(x). \tag{3.11}$$

Theorem 3.2.1. *We have the following similarity transformation on an appropriate space of functions*

$$\hat{H} = e^{\frac{V_L(x)}{2D\gamma}}\, \hat{L} e^{-\frac{V_L(x)}{2D\gamma}} \tag{3.12}$$

between the Fokker-Planck operator $\hat{L}$ and a Schroedinger operator $\hat{H}$ for $a = D$, where we define the effective potential

$$V_H(x) := \frac{\partial_{xx} V_L(x)}{2\gamma} - \frac{(\partial_x V_L(x))^2}{4D\gamma^2}. \tag{3.13}$$

In particular, the eigenvalues are the same and the Fokker-Planck eigenfunctions φ and Hamiltonian eigenfunctions ψ can be transformed into each other:

$$\psi(x) := e^{\frac{V_L(x)}{2D\gamma}}\, \varphi(x). \tag{3.14}$$

Analytical examples

We now present two examples: in the first one we start with the Fokker-Planck side and get the Schroedinger one by using theorem 3.2.1, while in the second we proceed reversely.

Example 3.2.2 (Free particle)**.** For a system describing the dynamics of a free particle the potential is zero $V_L(x) = 0\ \forall x$. By equation (3.13), follows $V_H(x) = 0$. Consequently, the Fokker-Planck operator $\hat{L}(x)$ and the Schroedinger operator $\hat{H}(x)$ are identical

$$\hat{L}(x) = \hat{H}(x) = D\,\partial_{xx}. \tag{3.15}$$

The eigenfunctions are $\psi(x) = \varphi(x) = e^{ikx}$ where $k = \pm\sqrt{\lambda/D}$, and λ eigenvalue.

Example 3.2.3 (Harmonic oscillator)**.** We next consider the operator $\hat{H}$ of to the harmonic oscillator with potential $V_H(x) = \frac{kx^2}{2}$ with $k > 0$.
We have the following eigenfunctions for every $n = 0, 1, \ldots$ with eigenvalue $\lambda_n = -2\sqrt{ak}\left(n + \frac{1}{2}\right)$:

$$\psi_n(x) = \frac{1}{\sqrt{2^n\,n!}} \cdot \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \cdot e^{-\frac{m\omega x^2}{2\hbar}} \cdot H_n\left(\sqrt{\frac{m\omega}{\hbar}}x\right)$$

where we defined the Hermite polynomials

$$H_n(z) = (-1)^n\ e^{z^2}\frac{d^n}{dz^n}\left(e^{-z^2}\right).$$

Having now solved the Schroedinger equation, we wish to find a corresponding Fokker-Planck equation and its solutions, by applying theorem 3.2.1.

The Fokker-Planck operator is given by

$$\hat{L} = e^{-\frac{V_L(x)}{2D\gamma}}\hat{H}e^{\frac{V_L(x)}{2D\gamma}}$$

where the potential $V_L(x)$ can be obtained as

$$V_L(x) = -2D\gamma\ln(\psi_0) - 2D\gamma\ln(\mathcal{Z})$$

with ψ_0 the (correctly normalised) lowest eigenfunction of the Schroedinger equation. In this case:

$$V_L(x) = -2D\gamma\ln(\psi_0) \tag{3.16}$$

$$= -2D\gamma \cdot (-\frac{m\omega x^2}{2\hbar}) + 2D\gamma \cdot \ln\left(\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\right). \tag{3.17}$$

The eigenfunctions $\psi_n(x)$ are mapped consistently to

$$\varphi_n(x) = e^{-\frac{V_L(x)}{2D\gamma}} \cdot \frac{1}{\sqrt{2^n\, n!}} \cdot \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \cdot e^{-\frac{m\omega x^2}{2\hbar}} \cdot H_n\left(\sqrt{\frac{m\omega}{\hbar}}x\right).$$

We remark that after substitution with $a = -\frac{\hbar^2}{2m}$ these are the usual formulae of quantum mechanics.

3.3 Path integral formalism

We recall from definition (2.2.2) that the expectation value of $F[x(t)]$ in the path integral approach is given by

$$\langle F[x(t)]\rangle := \frac{1}{\mathcal{Z}} \int \mathcal{D}x(t)\; e^{-\beta S[x(t)]}\; F[x(t)], \qquad S[x(t)] = \int_{-\infty}^{\infty} dt\; L(x(t)).$$

Remark 3.3.1. In this chapter we allow the path integral to be *Euclidean* with real $\beta \in \mathbb{R}^+$ or *Lorentzian* with imaginary $\beta \in i\mathbb{R}$. The two versions can be formally transformed into each other by substituting an imaginary time (Wick rotation). The Euclidean path integral can be made rigorous while the Lorentzian, treated in section 2.2.1, is always ill-defined.

In particular, the propagator is

$$\langle x_1, t_1 \mid x_0, t_0\rangle := \frac{1}{\mathcal{Z}} \int_{\substack{x(t_0)=x_0\\ x(t_1)=x_1}} \mathcal{D}x(t)\; e^{-\beta S[x(t)]}.$$

If we fix some x_0, t_0, or integrate them over some probability function or probability amplitude function $f(x_0, t_0)$, then

$$f(x_1, t_1) = \int_{-\infty}^{\infty} dx_0\; \langle x_1, t_1 \mid x_0, t_0\rangle \cdot f(x_0, t_0).$$

Example 3.3.2 (Free particle)**.** The free particle $V_H(x) = 0$ has the associated Schroedinger equation

$$D\partial_{xx}\psi(x) = \lambda\psi(x) \tag{3.18}$$

and, as we saw, it has eigenfunctions e^{ikx}, $k^2 = \lambda/D$ for any eigenvalue λ, where normalisation puts restrictions on λ.

The propagator is a Gaussian distribution, which means if the particle is localised at t_0 in x_0, then at a later time t_1 it will be a Gauss probability density or amplitude with standard deviation proportional to $t_1 - t_0$

$$\langle x_1, t_1 \mid x_0, t_0 \rangle = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x_1-x_0)^2}{2\sigma^2}}, \qquad \sigma^2 = \beta^{-1} m^{-1} (t_1 - t_0). \tag{3.19}$$

This can be proven by checking that $\langle x_1, t_1 \mid x_0, t_0 \rangle$ satisfies the Schroedinger equation or by explicitly computing the path integral, as we saw in section 2.2.3 for the Lorentzian path integral with $\beta = i/\hbar$.

This example can be generalised in the following

Theorem 3.3.3. *Given the Lagrangian $L(x(t)) = \frac{1}{2} m \dot{x}(t)^2 - V(x(t))$, then the propagator solves the Schroedinger equation*

$$\beta^{-1} \frac{\partial}{\partial t_1} \langle x_1, t_1 | x_0, t_0 \rangle = \left(\frac{1}{2m\beta^2} \frac{\partial^2}{\partial x_1^2} + V(x_1) \right) \langle x_1, t_1 \mid x_0, t_0 \rangle.$$

Differently spoken, if $\psi(x_1, t_1) = \int_{-\infty}^{+\infty} dt_0 \langle x_1, t_1 \mid x_0, t_0 \rangle \psi(x_0, t_0)$ for some given initial probability density function or amplitude $\psi(x_0, t_0)$ at time t_0, then $\psi(x_1, t_1)$ solves the time-dependent Schroedinger equation

$$\beta^{-1} \frac{\partial}{\partial t_1} \psi(x_1, t_1) = \left(\frac{1}{2m\beta^2} \frac{\partial^2}{\partial x_1^2} + V(x_1) \right) \psi(x_1, t_1).$$

3.4 Ongoing work

This project is still ongoing, with two main aims to be reached. First we want to study in examples the correspondence between the path integral and the Schroedinger formulations and compare it to the numerical simulations. Secondly, we want to develop a toy model of a peptide bond. Every peptide bond between two amino-acids in a protein has a single degree of freedom, the angle θ_i, subjected to some standard potential with two preferred states. The other degrees of freedom (hydrogen atoms, side chains, etc.), should be integrated out, or treated stochastically with output a Schrödinger equation in the variables θ_i. Long homogeneous chains of peptide bonds, i.e. a continuous set of angles θ_i, should be treated as a quantum field theory in the variable θ_i. The side chains, large scale geometry of the protein, etc. give additional terms in the potential. Our goal is to treat this system perturbatively, using technology from quantum field theory.

Chapter 4

Algebra of screenings and Nichols algebras

This chapter presents the main project of the PhD and it is based on the preprint [FL19].
We first give a deeper introduction to Nichols algebras following [CL16]. We then state the *smallness* condition on m_{ij} and prove in theorem 4.2.4 a refined version of it. In section 4.3 we present the classification problem. We then classify all *realising* lattices and study the associated algebra of screening operators for braidings of Cartan type (section 4.4), Super Lie type (section 4.5) and in general for all other finite-dimensional diagonal Nichols algebras in rank 2 (section 4.6), rank 3 (section 4.7) and higher rank (section 4.8). Final tables show all realising Λ, m_{ij} for rank 2 and 3.

4.1 Preliminaries on Nichols algebras

4.1.1 Definition and properties

Let $V = \langle x_1, \ldots, x_{rank} \rangle_{\mathbb{C}}$ be a complex vector space and let $(q_{ij})_{i,j=1,\ldots rank}$ be an arbitrary matrix with $q_{ij} \in \mathbb{C}^{\times}$. This defines a *braiding of diagonal type* on V via:

$$c: \; c(x_i \otimes x_j) = q_{ij} x_j \otimes x_i.$$

Hence we get an action ρ_n of the *braid group* $\mathbb{B}_n$ on $V^{\otimes n}$ via:

$$c_{i,i+1} := \mathrm{id} \otimes \cdots \otimes c \otimes \cdots \otimes \mathrm{id}.$$

Definition 4.1.1. Let (V, c) be a braided vector space. We consider the canonical projections $\mathbb{B}_n \twoheadrightarrow \mathbb{S}_n$ sending the braiding $c_{i,i+1}$ to the transposition $(i, i+1)$. There exists the *Matsumoto section* of sets $s : \mathbb{S}_n \to \mathbb{B}_n$ given by $(i, i+1) \mapsto c_{i,i+1}$ which has the property $s(xy) = s(x)s(y)$ whenever $\text{length}(xy) = \text{length}(x) + \text{length}(y)$. Then we define the *quantum symmetrizer* by

$$Ш_{q,n} := \sum_{\tau \in \mathbb{S}_n} \rho_n(s(\tau)) \tag{4.1}$$

where ρ_n is the representation of $\mathbb{B}_n$ on $V^{\otimes n}$ induced by the braiding c. Then the *Nichols algebra* or *quantum shuffle algebra* generated by (V, c) is defined by

$$\mathcal{B}(V) := \bigoplus_n V^{\otimes n} / \ker(Ш_{q,n}).$$

Remark 4.1.2. This characterisation enables one in principle to compute $\mathcal{B}(V)$ in each degree, but it is very difficult to find generators and relations for $\mathcal{B}(V)$ since in general the kernel of the map $Ш_{q,n}$ is hard to calculate in explicit terms.
In fact $\mathcal{B}(V)$ is a Hopf algebra in a braided sense and as such it enjoys several equivalent universal properties.

4.1.2 Examples

Example 4.1.3 (Rank 1)**.** [Nic78] Let $V = x\mathbb{C}$ be a 1-dimensional vector space with braiding given by $q_{11} = q \in \mathbb{C}^\times$, then

$$\mathbb{C} \ni Ш_{q,n} = \sum_{\tau \in \mathbb{S}_n} q_{11}^{|\tau|} = \prod_{k=1}^{n} \frac{1 - q^k}{1 - q} =: [n]_q!$$

Because this polynomial has zeros all $q \neq 1$ of order $\leq n$ the Nichols algebra is

$$\mathcal{B}(V) = \begin{cases} \mathbb{C}[x]/(x^\ell), & q_{11} \text{ primitive } \ell\text{-th root of unity} \\ \mathbb{C}[x], & \text{else} \end{cases}$$

Example 4.1.4 (Quantum group)**.** [Lus93, AHS10] Let $\mathfrak{g}$ be a finite-dimensional complex semisimple Lie algebra of rank n with simple roots $\alpha_1, \ldots, \alpha_n$ and inner product $(\ , \ )$. Let q be a primitive ℓ-th root of unity. Consider the n-dimensional vector space V with diagonal braiding $q_{ij} := \ q^{(\alpha_i, \alpha_j)}$ Then the

Nichols algebra $\mathcal{B}(V)$ is isomorphic to the positive part $u_q(\mathfrak{g})^+$ of the small quantum group $u_q(\mathfrak{g})$, which is a deformation of the universal enveloping of a Lie algebra $U(\mathfrak{g})$.

4.1.3 Generalised root systems and Weyl groupoids

Every finite-dimensional Nichols algebra comes with a generalised *root system*, a *Weyl groupoid* and a *PBW-type basis* [Kha00], [Hec06b], [HS08], [AHS10].
The Weyl groupoid plays a similar role as the Weyl group does for ordinary root systems in Lie algebras, but in the general case not all Weyl chambers look the same: different braiding matrices, different Cartan matrices and even different Dynkin diagrams are attached to different Weyl chambers (i.e. groupoid objects). This behaviour already appears for Lie superalgebras.
The finite Weyl groupoids are classified in [CH09], [CH10]; apart from the finite Weyl groups there are additional series $D_{n,m}$ and 74 sporadic examples.

Remark 4.1.5. We remark that the generalised root systems do not provide a complete classification as they do in the theory of complex semisimple Lie algebras: there are non-isomorphic Nichols algebras whose corresponding Weyl groupoids are equivalent and there are Weyl groupoids to which no finite-dimensional diagonal Nichols algebra corresponds.

We now introduce the notions of Cartan matrix and Weyl reflections for Nichols algebras. The details can be found e.g. in [Hec06b]. Where not otherwise stated one treats root systems in analogy to Lie algebra theory. In particular the notions of simple and positive roots, the correspondence of Weyl chambers to choices of bases of simple roots and the notation $\alpha_{ij} = \alpha_i + \alpha_j$ is analogous to Lie algebra theory.

Definition 4.1.6. To every braiding matrix q_{ij} we define the associated Cartan matrix (a_{ij}) for all $i \neq j$ by

$$a_{ii} = 2 \quad \text{and} \quad a_{ij} := -\min\left\{ m \in \mathbb{N} \mid q_{ii}^{-m} = q_{ij}q_{ji} \quad \text{or} \quad q_{ii}^{(1+m)} = 1 \right\}. \tag{4.2}$$

Definition 4.1.7. We call a root α_i q-Cartan, respectively q-truncation, if it satisfies:

$$q_{ii}^{a_{ij}} = q_{ij}q_{ji}, \qquad \text{respectively} \qquad q_{ii}^{1-a_{ij}} = 1. \tag{4.3}$$

We observe that a root can be both q-Cartan *and* q-truncation. In particular we will call a root *only* q-Cartan, respectively *only* q-truncation, if it is exclusively so.

Definition 4.1.8. The Weyl groupoid is generated by reflections, defined for every k as:

$$\begin{aligned} \mathcal{R}^k : \mathbb{Z}^n &\longrightarrow \mathbb{Z}^n \\ \alpha_i &\longmapsto \alpha_i - a_{ki}\alpha_k \end{aligned}$$

Remark 4.1.9. The braiding matrix entry q_{ij} extends uniquely to a bicharacter $\chi_q : \mathbb{Z}^n \times \mathbb{Z}^n \longrightarrow \mathbb{C}^\times$ with:

$$q_{ij} = \chi_q(\alpha_i, \alpha_j).$$

Then the reflection $\mathcal{R}^k$ transforms q_{ij} into the bicharacter

$$\mathcal{R}^k(q_{ij}) = \mathcal{R}^k(\chi_q(\alpha_i, \alpha_j)) = \chi_q(\mathcal{R}^k(\alpha_i), \mathcal{R}^k(\alpha_j)).$$

As we said, this is a new braiding matrix, possibly different from the original one and with possibly different associated Cartan matrix. However, the Nichols algebras have the same dimension and are closely related [HS11, BLS15].
Analogously the scalar product between roots $m_{ij} = (\alpha_i, \alpha_j)$ extends uniquely to a bicharacter $\chi_m(\alpha_i, \alpha_j)$ and the reflection as the reflected bicharacter $\mathcal{R}^k(m_{ij}) = \chi_m(\mathcal{R}^k(\alpha_i), \mathcal{R}^k(\alpha_j))$.

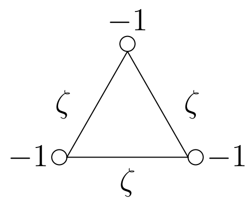
Remark 4.1.10. With $\mathcal{R}^k$ we mean the reflection around the k-th simple root in the respective Weyl chamber, which can be again expressed in coordinates with respect to the simple roots $\alpha_1, \ldots, \alpha_n$ in some fixed initial Weyl chamber.

Example 4.1.11 (D(2,1;α))**.** We consider, as an example, the finite-dimensional diagonal Nichols algebra of rank 3 with the following braiding in an initial Weyl chamber

$$q_{ii} = -1, \qquad q_{ij}q_{ji} = \zeta,$$

with $i \neq j$ and $\zeta \in \mathcal{R}_3$ a primitive third root of unity. There is a more general version of this example, including three roots of unity q, r, s of order greater than three, which corresponds to a different choice of the parameter α.

Following Heckenberger, we write the braiding as a diagram, where nodes correspond to the simple roots α_i and are decorated by the braiding q_{ii} and each edge is decorated by the double braiding $q_{ij}q_{ji}$:



As it turns out, the overall root system has seven positive roots. If $\{\alpha_1, \alpha_2, \alpha_3\}$ are the simple roots in the Weyl chamber shown above, then the positive roots in this basis are:

$$\{\alpha_1,\ \alpha_2,\ \alpha_3,\ \alpha_{12},\ \alpha_{23},\ \alpha_{13},\ \alpha_{123}\}$$

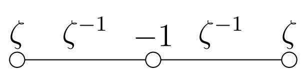
and the Cartan matrix attached to this Weyl chamber, which we label by the upper index I is:

$$a^I_{ij} = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$$

We now reflect around α_2. Then the new simple roots are $\{\alpha_{12}, -\alpha_2, \alpha_{23}\}$ and the new braiding matrix is:

$$\begin{aligned} q_{12,12} &= q_{23,23} = \zeta & \qquad q_{22} &= -1 \\ q_{12,2}q_{2,12} &= q_{23,2}q_{2,23} = \zeta^{-1} & \qquad q_{12,23}q_{23,12} &= 1 \end{aligned}$$

which is in diagram notation:



In this new basis the positive roots are:

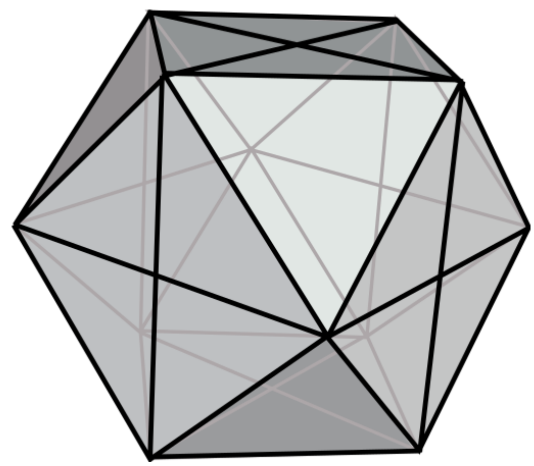
$$\{\alpha_{12},\ -\alpha_2,\ \alpha_{23},\ \alpha_1,\ \alpha_3,\ \alpha_{123},\ \alpha_{13}\}$$

and the Cartan matrix attached to this second Weyl chamber II is hence

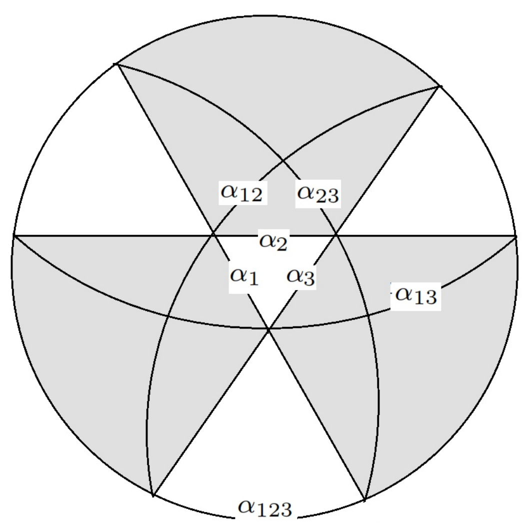
$$a^{II}_{ij} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

Even though this Cartan matrix is of standard type A_3, the root system has one additional root.

The following figures show the hyperplane arrangement of the root system in $\mathbb{R}^3$:



and its projection on the plane:



Each of the seven lines corresponds to the hyperplane orthogonal to a root. Each triangle is a Weyl chamber with the three adjacent hyperplanes corresponding to the three simple roots. Equilateral triangles (white) correspond to the Cartan matrix I and right triangles (grey) to the Cartan matrix II.

4.2 Screening operators

As introduced in section 1.2.4, we have the following theorem by [Len17]:

Theorem 4.2.1. *Given a non-integral lattice Λ and elements $v_1, \ldots, v_n \in \Lambda$, we consider the elements e^{v_i} in the modules $\mathcal{V}_{v_i}$ of the associated Heisenberg VOA $\mathcal{V}_{\mathsf{H}}^n$. The braiding between two elements is*

$$e^{v_i} \otimes e^{v_j} \mapsto q_{ij}\, e^{v_j} \otimes e^{v_i},$$

$$\text{where } q_{ij} := e^{i\pi m_{ij}}, \quad m_{ij} := (v_i, v_j).$$

Consider the diagonal Nichols algebra $\mathcal{B}(q)$ for braiding matrix $q = (q_{ij})_{i,j}$ generated by elements x_{v_i}, then any relation in the Nichols algebra, in degree $(d_1, \ldots, d_n) \in \mathbb{N}^n$, holds for the screening operators $\mathfrak{Z}_{v_i}$, under the additional assumption of smallness*:*

$$\forall J \subseteq I, \;\; i, j \in J \qquad \sum_{i<j} d_i d_j m_{ij} + \sum_i \binom{d_i}{2} m_{ii} > 1 - \sum_i d_i$$

where $I = \{1, \ldots, n\}$ is the index set.

Example 4.2.2. In the case $\Lambda = \frac{1}{\sqrt{p}}\Lambda_{\mathfrak{g}}$, with $\Lambda_{\mathfrak{g}}$ the root-lattice of a complex finite-dimensional simple Lie algebra $\mathfrak{g}$, and $\ell = 2p$ even integer, we obtain as $\mathcal{B}(q)$ the positive part of the small quantum group $u_q(\mathfrak{g})^+$ where q is a primitive ℓ-th root of unity and the braiding is

$$q_{ij} = e^{i\pi(\frac{1}{\sqrt{p}}\alpha_i, \frac{1}{\sqrt{p}}\alpha_j)} = e^{\frac{2i\pi}{\ell}(\alpha_i, \alpha_j)} = q^{(\alpha_i, \alpha_j)},$$

where $\alpha_i \in \Lambda_{\mathfrak{g}}$.

In particular, by theorem 6.1 of [Len17]:

Lemma 4.2.3. *If Λ is positive definite and $m_{ii} = (v_i, v_i) \leq 1$ for v_i in a fixed basis, then the smallness condition holds.*

Theorem 4.2.1 is a general result. We will now present the refined version, mentioned in 1.3, which will appear in our examples. Roughly, it shows that for the definition of smallness the assumption not-too-negative can be replaced by not-a-negative-integer, by analytic continuation. We prove this only in two special cases.

We remark that the composition of operators each mapping to an algebraic closure is convergent under the assumption of smallness and the following statement is about the analytic continuation of this quantity in the variable m_i and m_{ij}.

Theorem 4.2.4 (Continued Smallness)**.** *As in the previous theorem, we consider the action of linear combinations of monomials $\mathfrak{Z}_{v_1}\cdots\mathfrak{Z}_{v_n}$ of n screening operators on the module V_λ, we will denote $m_i := (v_i, \lambda)$ and $m_{ij} := (v_i, v_j)$ for $1 \leq i, j \leq n$.*

a) If all m_i are equal $\forall i \in I$ and all m_{ij} are equal $\forall i, j \in I$, then a relation in the Nichols algebra $\mathcal{B}(q)$ holds for the screening operators $\mathfrak{Z}_{v_i}$, under the weaker assumptions which we call continued smallness

$$m_{ij} \notin -\mathbb{N}\frac{2}{k} \qquad k = 1, \ldots, n.$$

b) If there is a distinguished element $1 \in I$ such that all m_i are equal $\forall i \in I, i \neq 1$ and all m_{ij} are equal $\forall i, j \in I, i \neq 1$, then a relation in the Nichols algebra $\mathcal{B}(q)$ holds for the screening operators $\mathfrak{Z}_{v_i}$, under the weaker assumption of continued smallness

$$m_{ij} \notin -\mathbb{N}\frac{2}{k} \qquad k = 1, \ldots, n-1$$
$$m_{1j} + k\frac{m_{ij}}{2} \notin -\mathbb{N} \qquad k = 0, \ldots, n-2.$$

Proof. Retaking the steps in the proof of theorem 4.2.1 in [Len17] we consider the following function (which play roughly the role of structure constants for multiplying screenings) given by integrating around the multivalued covering of the circle

$$F((m_i, m_{ij})_{ij}) = \int \cdots \int_{[e^0, e^{2\pi}]^n} dz_1 \ldots dz_n \prod_i z_i^{m_i} \prod_{i<j} (z_i - z_j)^{m_{ij}}$$

We express this function as quantum symmetrizer of another function:

$$F(m_i, m_{ij}) = \text{Ш}\ \tilde{F}(m_i, m_{ij})$$

$$\tilde{F}((m_i; m_{ij})_{ij}) := \frac{1}{(2\pi i)^n} \sum_{k=0}^{n} (-1)^k \left(\prod_{i=k+1}^{n} e^{2\pi i\, m_i} \right) \sum_{\eta \in \mathbb{S}_{k,\overline{n-k}}} \left(\prod_{i<j,\ \eta(i)>\eta(j)} e^{\pi i\, m_{ij}} \right)$$
$$\cdot \operatorname{Sel}((m_{\eta^{-1}(i)}; 0; m_{\eta^{-1}(i)\eta^{-1}(j)})_{ij})$$

where Sel indicates the Selberg integral and $\mathbb{S}_{k,\overline{n-k}}$ denotes a variant of the $(k, n-k)$ shuffles as in [Len17].

$$\operatorname{Sel}(m_i, \bar{m}_i, m_{ij}) = \operatorname{Sel}((m_i; \bar{m}_i; m_{ij})_{i<j})$$
$$:= \int \cdots \int_{1>z_1>\ldots>z_n>0} dz_1 \cdots dz_n \prod_i z_i^{m_i} \prod_i (1-z_i)^{\bar{m}_i} \prod_{i<j} (z_i - z_j)^{m_{ij}}.$$

By this result, the Nichols algebra relations are thereby proven to hold if $\tilde{F}$ is analytic at the parameters m_i, m_{ij} under consideration.

a) In our special situation with equal $m_{ij} =: m_{vv}$ and $m_i =: m_{v\lambda}$ we find from the factorisation in [Len17] resp. from the Selberg integral formula:

$$\tilde{F}(m_{v\lambda}; m_{vv}) := \prod_{s=0}^{n-1} \left((e^{\pi i m_{vv}})^s e^{2\pi i m_{v\lambda}} - 1 \right) \cdot \operatorname{Sel}(m_{v\lambda}; 0; m_{vv})$$

$$\operatorname{Sel}(a-1, b-1, 2c) = \prod_{k=0}^{n-1} \frac{\Gamma(a+kc)\Gamma(b+kc)\Gamma(1+(k+1)c)}{\Gamma(a+b+(n+k-1)c)\Gamma(1+c)}$$

Our goal is to prove that under the assumptions on $m_{vv}, m_{v\lambda}$ the function $\tilde{F}$ is analytic.

The Gamma function does not have zeros, and it has poles for negative integer values of z. Thus the only possible poles are for:

- Poles:

$$a + kc \in -\mathbb{N}_0, \qquad k = 0, \ldots, n-1.$$

These simple poles cancel with the factors $((e^{\pi i m_{vv}})^s e^{2\pi i m_{v\lambda}} - 1)$. So at these values $\tilde{F}$ is analytic and thus F vanishes according to the quantum symmetrizer formula and Nichols algebra relations hold. We remark however, that these exceptionally non-zero values of $\tilde{F}$ give rise to reflection operators [Len17].

- Poles:

$$\begin{aligned} 1 + kc &\in -\mathbb{N}_0, && k = 0, \dots, n-1 \\ \iff kc &\in -\mathbb{N}, && k = 0, \dots, n-1 \\ \iff kc &\in -\mathbb{N}, && k = 1, \dots, n-1 \end{aligned}$$

- Poles:

$$\begin{aligned} 1 + (k+1)c &\in -\mathbb{N}_0, && k = 0, \dots, n-1 \\ \iff (k+1)c &\in -\mathbb{N}, && k = 0, \dots, n-1 \\ \iff \quad kc &\in -\mathbb{N}, && k = 1, \dots, n \end{aligned}$$

where in the last step we substituted $k+1$ with k.

We thus found that to avoid poles we need to ask the condition

$$k \frac{m_{ij}}{2} \notin -\mathbb{N}, \qquad k = 1, \dots, n.$$

b) To prove the second point we proceed in the same way, this time isolating the distinguish element with index equals to 1.

$$\begin{aligned} &\mathrm{Sel}(m_i, m_{ij}, m_1, m_{1j}) \\ &= \int_0^1 \dots \int_0^1 \prod_{i=2}^n z_1^{m_1} z_i^{m_i} \prod_{j=2}^n (z_1 - z_j)^{m_{1j}} \prod_{2 \le i < j \le n} (z_i - z_j)^{m_{ij}} dz_1 \cdot dz_2 \dots dz_n \\ &= \int_0^1 dz_1 \; z_1^{m_1 + (n-1) + \sum m_{1j} + \sum m_{ij} + \sum m_i} \\ &\qquad \cdot \int_0^1 \dots \int_0^1 \prod_{i=2}^n \tilde{z}_i^{m_i} \prod_{j=2}^n (1 - \tilde{z}_j)^{m_{1j}} \prod_{2 \le i < j \le n} (\tilde{z}_i - \tilde{z}_j)^{m_{ij}} d\tilde{z}_2 \dots d\tilde{z}_n. \end{aligned}$$

Calling m the power of z_1, we have:

$$\begin{aligned}
&\mathrm{Sel}(m_i, m_{ij}, m_1, m_{1j}) \\
&= \frac{(e^{2\pi i m}-1)/2\pi i}{1+m} \int_0^1 \cdots \int_0^1 \prod_{i=2}^{n} \tilde{z}_i^{m_i} \prod_{j=2}^{n} (1-\tilde{z}_j)^{m_{1j}} \\
&\qquad\qquad\qquad\qquad \cdot \prod_{2\le i<j\le n} (\tilde{z}_i - \tilde{z}_j)^{m_{ij}} d\tilde{z}_2 \ldots d\tilde{z}_n \\
&= \frac{(e^{2\pi i m}-1)/2\pi i}{1+m} \prod_{k=0}^{n-2} \frac{\Gamma(a+kc)\Gamma(b+kc)\Gamma(1+(k+1)c)}{\Gamma(a+b+(n+k-1)c)\Gamma(1+c)}
\end{aligned}$$

A similar calculation as in the previous case disregard the pole at $1+m=0$ and $a+kc \in -\mathbb{N}_0$.
Then we have poles just for:

- Poles:

$$\begin{aligned}
& b+kc \in -\mathbb{N}_0, && k=0,\ldots,n-2 \\
\Longleftrightarrow\ & m_{1j} + k\frac{m_{ij}}{2} \notin -\mathbb{N} && k=0,\ldots,n-2.
\end{aligned}$$

- Poles:

$$\begin{aligned}
& 1+(k+1)c \in -\mathbb{N}_0, && k=0,\ldots,n-2 \\
\Longleftrightarrow\ & kc \in -\mathbb{N}, && k=1,\ldots,n-1 \\
\Longleftrightarrow\ & k\frac{m_{ij}}{2} \in -\mathbb{N} && k=1,\ldots,n-1
\end{aligned}$$

which are the asserted conditions.

□

4.2.1 Central charge

The output of our study are elements $v_1, \dots v_n \in \mathbb{C}^n$, the respective screening operators of the Heisenberg algebra $\mathcal{V}_{\mathsf{H}}^n$, and their algebra relations. We also wish to fix an action of the Virasoro algebra at a certain central charge on $\mathcal{V}_{\mathsf{H}}^n$. As discussed in [FL17] it is usually desirable to choose the Virasoro structure in such a way, that it is compatible with the screening operators associated to the $v_1, \dots, v_n$. In this way the screening operators are Virasoro algebra homomorphisms and thus preserve the conformal invariance of the system they are acting on. The compatibility condition gives a unique solution of Virasoro structure and a characteristic central charge for the situation at hand, as follows:

Proposition 4.2.5. *For the Heisenberg algebra, there is a family of Virasoro structures parametrised by the choice of an element $Q \in \mathbb{C}^n$, called background charge [FT10] such that the central charge is*

$$c = rank - 12(Q, Q).$$

The compatibility condition reads

$$\frac{1}{2}(v_i, v_i) - (v_i, Q) = 1 \qquad i = 1, \dots, n$$

i.e. says that the modules $\mathcal{V}_{v_i}$ have conformal weight equals to 1.

In particular for rank = 2, if we make the formula above explicit, we get the following, as in [Sem11]:

$$c = 2 - 3\frac{|v_1(m_{22} - 2) - v_2(m_{11} - 2)|^2}{m_{11}m_{22} - m_{12}^2} \tag{4.4}$$

Remark 4.2.6. In [Sem11], they notice that the central charge (4.4) is invariant under the reflections $\mathcal{R}^k$ on the roots. This result was the first evidence suggesting a tighter connection between Nichols algebras and conformal field theories.

4.3 Formulation of the classification problem

We now give the precise definition of lattices realising a given braiding, concept that we introduced in section 1.3.

Definition 4.3.1. Let Λ be a lattice of rank n, basis $\{v_1, \ldots, v_n\}$, bilinear form $(\ ,\ )$ and let $m_{ij} := (v_i, v_j)$. Given a braiding matrix q_{ij} and associated Cartan matrix a_{ij}, we say that the lattice Λ and the matrix m_{ij} *realise* q_{ij} iff

- we have: $e^{i\pi m_{ij}} = q_{ij}$
- m_{ij} satisfies:

$$\text{A: } 2m_{ij} = a_{ij}m_{ii} \qquad \text{or} \qquad \text{B: } (1 - a_{ij})m_{ii} = 2 \tag{4.5}$$

- all the reflected matrices $\mathcal{R}^k(m_{ij})$ fulfil again (4.5).

We will say with respect to a realisation m_{ij} that a root v_i is *m-Cartan* if m_{ii} satisfies (4.5)A, and *m-truncation* if it satisfies (4.5)B.

[Sem11] asks this condition only for one specific Weyl chamber.

Remark 4.3.2. We observe that condition (4.5) is the logarithmic version of (4.3).

Remark 4.3.3. Clearly m-Cartan implies q-Cartan and m-truncation implies q-truncation. The converse is not always true. If a root is both q-Cartan and q-truncation, then there are two possible solutions in terms of the m_{ij} matrix.
An example is the $\mathfrak{sl}(2|1)$ superalgebra, presented below in example (4.3.5).

Proposition 4.3.4. *Clearly, if v_k is m-Cartan, then $\mathcal{R}^k(m_{ij}) = m_{ij}$.*

Our goals are as follows:

- Given a braiding q_{ij} from Heckenberger lists in [Hec05], [Hec06a], construct all realising Λ, m_{ij}. In sections 4.4, 4.5 and 4.6.1 we construct the m_{ij} while in section 4.6.2 we prove that the constructed m_{ij} exhaust all cases of Heckenberger list in rank 2. In section 4.7 we do the same for rank 3, and in section 4.8 for higher rank.
- We compute the central charges for each solution in rank 2.

- We analyse which Nichols algebras relations hold and which don't, for the associated screening operators. This may depend on a free parameter in the family of solutions.

Example 4.3.5. We now show an example of this procedure. We consider row 3 of table 1 in [Hec05], described by the braiding matrices:

$$q^{\mathrm{I}}_{ij} = \begin{bmatrix} q^2 & q^{-1} \\ q^{-1} & -1 \end{bmatrix} \qquad q^{\mathrm{II}}_{ij} = \begin{bmatrix} -1 & q \\ q & -1 \end{bmatrix}$$

and corresponding diagrams:

I II

with $q \in \mathbb{C}^\times,\ q^2 \neq \pm 1$, simple roots $\{\alpha_1, \alpha_2\}$ and $\{\alpha_{12}, \alpha_2\}$ respectively, and a unique associated Cartan matrix

$$a^{\mathrm{I}}_{ij} = a^{\mathrm{II}}_{ij} = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}.$$

This is the Lie superalgebra $\mathfrak{sl}(2|1)$ root system. The set of positive roots is $\{\alpha_1, \alpha_2, \alpha_{12}\}$ where α_1 is only q-Cartan and α_2, α_{12} are only q-truncation (for $q^2 = -1$ this is not true: all roots are both q-Cartan and q-truncation, which gives more solution, see remark 4.3.7).

Proposition 4.3.6. *The following m_{ij} matrices are realising solutions of the given braiding and its reflections:*

$$m^{I}_{ij} = \begin{bmatrix} 2r & -r \\ -r & 1 \end{bmatrix}, \qquad m^{II}_{ij} = \begin{bmatrix} 1 & -1+r \\ -1+r & 1 \end{bmatrix}$$

for all $r = \frac{p'}{p} \in \mathbb{Q}$ with $(p', p) = 1$ such that $e^{i\pi r} = q$.

Proof. We check that condition (4.5)B is satisfied for α_2, α_{12}:

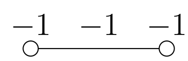
$$m_{22} = \frac{2}{1 - a_{21}} = 1$$
$$m_{12,12} = \frac{2}{1 - a_{12,2}} = 1$$

while condition (4.5)A is satisfied for the root α_1:

$$m_{11} = \frac{2m_{12}}{a_{1,2}} = 2r.$$

The reflection on α_1 preserves q_{ij} as well as m_{ij}, because α_1 is m-Cartan. We check that reflections on α_2 and α_{12}, which interchange q^I_{ij} and q^{II}_{ij}, also interchange our choices of m^I_{ij} and m^{II}_{ij}. □

Remark 4.3.7. As we will see in section 4.6.2, for $q^2 \neq \pm 1$ this family gives all solutions.
For $q^2 = -1$, we have more choices for the m_{ij}-matrices because the roots become both q-truncation and q-Cartan. Thus we may have solutions fulfilling either (4.5A) or (4.5B). The new (unique) diagram in this case is:



to whom correspond several solutions of m_{ij}-matrices; for simple roots α_1, α_2: let $p' \in \mathbb{Z}$ with $(p', 2) = 1$,

- if we assume α_1 and α_2 m-truncation, the unique family of solutions is given by

$$m_{ij} = \begin{bmatrix} 1 & -\frac{p''}{2} \\ -\frac{p''}{2} & 1 \end{bmatrix} \qquad m_{ij} = \begin{bmatrix} 1 & -\frac{p'}{2} \\ -\frac{p'}{2} & p' \end{bmatrix} \qquad m_{ij} = \begin{bmatrix} p' & -\frac{p'}{2} \\ -\frac{p'}{2} & 1 \end{bmatrix}.$$

  These m_{ij} are reflections one of the other by Proposition 4.3.6 with $p'' = 2 - p'$. Other combinations bring to the same solution in different Weyl chambers.

- if we assume α_1 and α_2 m-Cartan, the unique family of solutions is given by

$$m_{ij} = \begin{bmatrix} p' & -\frac{p'}{2} \\ -\frac{p'}{2} & p' \end{bmatrix}$$

  which can be interpreted as coming from $\mathfrak{sl}_3$ for $p = 2$.

4.4 Cartan type

4.4.1 q diagram

Let $\mathfrak{g}$ be a simple Lie algebra with simple roots $\alpha_1, \ldots, \alpha_n$ and denote the Killing form by $(\alpha_i, \alpha_j)_\mathfrak{g}$ in the standard scaling taking values in $\{2, 4, 6\}$ for $i = j$ and $\{-3, -2, -1, 0\}$ for $i \neq j$.
Let $q \in \mathbb{C}^\times$ be a primitive ℓ-th root of unity with $\ell \in \mathbb{Z}$ and let $\mathrm{ord}(q^2) > d$ with d half length of the long roots. Define a braiding matrix by

$$q_{ij} = q^{(\alpha_i, \alpha_j)_\mathfrak{g}}.$$

Definition 4.4.1. The finite-dimensional Nichols algebra $\mathcal{B}(q)$ is called of *Cartan type*.

We have that:

- q_{ij} is invariant under reflections $\mathcal{R}^k$,
- the Weyl groupoid is the Weyl group associated to $\mathfrak{g}$,
- the set of positive roots is the set of roots associated to $\mathfrak{g}$,
- the Cartan matrix a_{ij} is exactly the Cartan matrix for $\mathfrak{g}$.

4.4.2 Construction of m_{ij}

Definition 4.4.2. Given $r \in \mathbb{Q}$, such that $\frac{r}{2} = \frac{k}{\ell}$, with $k \in \mathbb{Z}$, $(k, \ell) = 1$ we define

$$m_{ij} := (\alpha_i, \alpha_j)r.$$

Differently spoken, the lattice Λ of definition 4.3.1 is, in this case, exactly the root lattice of $\mathfrak{g}$ rescaled by r.

Remark 4.4.3. Usually in literature $r = \frac{p'}{p}$, e.g. $r = \frac{1}{p}$ and $\ell = 2p$, $q = e^{\frac{\pi i}{p}}$.

Lemma 4.4.4. *The matrix m_{ij} realises the braiding q_{ij} for all reflections, and every simple root is m-Cartan.*

Proof. Condition (4.5) asks

$$2m_{ij} = a_{ij}m_{ii} \quad \text{or} \quad (1 - a_{ij})m_{ii} = 2 \tag{4.6}$$

$$2m_{ji} = a_{ji}m_{jj} \quad \text{or} \quad (1 - a_{ji})m_{jj} = 2. \tag{4.7}$$

But from the last point of enumeration 4.4.1 we have $a_{ij} = \frac{2(\alpha_i,\alpha_j)}{(\alpha_i,\alpha_i)}$. Hence

$$m_{ii} = (\alpha_i, \alpha_i)r = 2(\alpha_i, \alpha_j)r\frac{(\alpha_i, \alpha_i)}{2(\alpha_i, \alpha_j)} = 2\frac{m_{ij}}{a_{ij}}$$

which is (4.5)A, saying that the roots are m-Cartan.
Since any reflection leaves the m_{ij} invariant (not just the q_{ij}) because is a m-Cartan reflection, condition (4.5) holds also after reflections. □

Lemma 4.4.5. *If $\ell_i > 1 - a_{ij}$ for $i = 1, \ldots, n$, with $\ell_i := \frac{\ell}{gdc(\ell,2d_i)}$ as in [Lus90], then none of the simple roots are m-truncation.*

Proof. Assume the root α_i is m-truncation, i.e. $(1-a_{ij})m_{ii} = 2$, this implies: $q_{ii}^{(1-a_{ij})} = e^{i\pi m_{ii}(1-a_{ij})} = e^{i\pi\cdot 2} = 1$. But $ord(q_{ii}) = ord(q^{2d_i}) = \ell_i > 1 - a_{ij}$ and we find a contradiction. □

Lemma 4.4.6. *If all simple roots are just m-Cartan, then the unique solution for the matrix m_{ij} is the one of definition 4.4.2. In particular this is the case if $\ell_i > 1 - a_{ij}$ for $i = 1, \ldots, n$.*

Proof. If all roots are m-Cartan then if we fix m_{ii} for some root α_i, the mixed term m_{ij} is fixed by condition 4.5(A) and so is m_{jj} by the same condition with reversed indices. Moreover the reflections around m-Cartan roots leave the system invariant, so the m_{ij} are fixed $\forall i, j$. But then, up to a rescaling there is a unique solution for m_{ij} and this is the one defined in 4.4.2. □

Example 4.4.7. As a counterexample of the condition of lemma 4.4.5, we consider $\mathfrak{sl}_3$. In this case, $a_{ij} = -1\ \forall i, j$ and for $2\ell_i = \ell = 2p = 4$, i.e. $q_{ii} = -1$, the roots can be considered as m-truncation as well. We thus obtain an additional solution of m_{ij}, which will be understood from reinterpreting $\mathfrak{sl}_3$, $\ell = 2p = 4$, as the Lie superalgebra $\mathfrak{sl}(2|1)$, $\ell = 2p = 4$, treated in the remark of example 4.3.5.

4.4.3 Central charge

Recall $\{v_1, \ldots, v_n\}$ as basis of Λ with $m_{ij} = (v_i, v_j)$.

Proposition 4.4.8. *The central charge of the system is*

$$c = \mathrm{rank}\mathfrak{g} - 12(\frac{1}{r} \mid \rho^\vee \mid^2 - 2(\rho, \rho^\vee) + r \mid \rho \mid^2) \tag{4.8}$$

where ρ is the sum of all positive roots.

Proof. The central charge is:

$$c = rank - 12(Q, Q)$$

where $Q = \sum_j a_j v_j$ is the unique combination such that for every i

$$\frac{1}{2}(v_i, v_i)_\Lambda - (v_i, Q) = 1$$
$$\frac{1}{2}(v_i, v_i)_\Lambda - \sum_j a_j (v_i, v_j)_\Lambda = 1$$

Rewriting $v_i = -\sqrt{r}\alpha_i$, with α_i root of $\mathfrak{g}$, this set of equations bring us to

$$Q = \sqrt{\frac{1}{r}}\rho^\vee - \sqrt{r}\rho$$

that on turn gives the central charge as in the statement. □

Remark 4.4.9. The central charge matches with the one of the affine Lie algebra $\hat{\mathfrak{g}}_k$ at level $k + h^\vee = \frac{1}{r}$ as in [Ara07].

Remark 4.4.10. For rank 2 the central charge is

$$c = 1 - 3\frac{(2p' - 2p)^2}{2pp'} = 13 - 6\frac{p}{p'} - 6\frac{p'}{p}$$

which is the central charge of the p, p' model.

4.4.4 Algebra relations

We now want to determine when the algebra of screenings satisfies Nichols algebra relations. We will again denote d the half length of the long roots.

With the definition of smallness and the results in [Len17], see theorem 4.2.3, we get for a rescaled root lattice $m_{ij} = (\alpha_i, \alpha_j)r$:

Corollary 4.4.11. *If $\frac{1}{2d} \geq r > 0$, then all Nichols algebra relations hold.*

Proof. Since we are rescaling by $\sqrt{r}$ a positive definite lattice Λ, the only condition for the new lattice to be positive definite is $r > 0$. We ask moreover $m_{ii} = 2dr \leq 1$ for all i. This implies $r \leq \frac{1}{2d}$. □

Now we want to analyse the algebra relations in the screening algebra for arbitrary values of r. To do so we study relation by relation using theorem 4.2.4.

Definition 4.4.12. A generator x_i is said to satisfy the *truncation* relation if

$$x_i^{\ell_i} = 0, \qquad \ell_i = ord(q_{ii}).$$

A pair of generators x_i, x_j are said to satisfy the *Serre* relation if

$$(ad_c x_i)^{1-a_{ij}} x_j = 0, \qquad a_{ij} = -\min \left\{ m \in \mathbb{Z} \mid q_{ii}^{-m} = q_{ij}q_{ji} \quad \text{or} \quad q_{ii}^{(1+m)} = 1. \right\}$$

We denoted the braided commutator by

$$(ad_c x_i)x_j := [x_i, x_j]_c = [x_i, x_j]_q = x_i x_j - q_{ij} x_j x_i.$$

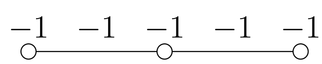
Theorem 5.25 of [Ang08] states a set of defining relations for each finite-dimensional Nichols algebra of Cartan type:

Theorem 4.4.13. *For finite-dimensional Nichols algebra of Cartan type $u_q(\mathfrak{g})^+$, i.e. with diagonal braiding $q_{ij} = q^{(\alpha_i,\alpha_j)}$ associated to the root system of a Lie algebra $\mathfrak{g}$, the defining relations are as follows*

1. *For each root α the truncation relation and for each pair of simple roots α_i, α_j with $q_{ii}^{1-a_{ij}} \neq 1$ the Serre relation.*

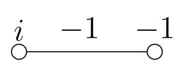
2. *For the following subdiagrams the following additional relations:*

    - *For type A_3 with $q = -1$*

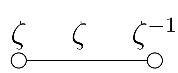
      

      $$[(ad_c x_2)x_1, (ad_c x_2)x_3]_c = 0$$

    - *For type B_2 or C_2 with $q = i$*

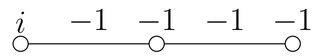
      

      or with $q = \zeta \in \mathcal{R}_3$

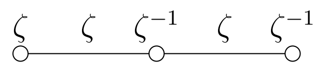
      

$$[(ad_c x_1)^2 x_2, (ad_c x_1) x_2]_c = 0$$

- *For type* B_3 *with* $q = i$

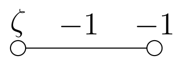


or with $q = \zeta \in \mathcal{R}_3$

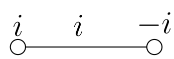


$$[(ad_c x_1)^2 (ad_c x_2) x_3, (ad_c x_1) x_2]_c = 0$$

- *For type* G_2 *with* $q = \zeta \in \mathcal{R}_6$



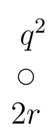
or with $q = i$



$$\begin{aligned} [(ad_c x_1)^3 x_2, (ad_c x_1)^2 x_2]_c &= 0 \\ [x_1, [x_1^2 x_2 x_1 x_2]_c]_c &= 0 \\ [[x_1^2 x_2 x_1 x_2]_c, [x_1, x_2]_c]_c &= 0 \\ [[x_1^2 x_2]_c, [x_1^2 x_2 x_1 x_2]_c]_c &= 0. \end{aligned}$$

We now apply our refined smallness criteria of theorem 4.2.4 to these explicit set of relations to determine the algebra of screening operators in comparison to the Nichols algebra.

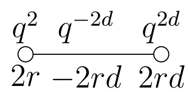
Example 4.4.14. Let us consider a rank 1 Cartan q-diagram and corresponding m-solution:



The truncation relation $(\mathfrak{Z}_1)^n = 0$, $n = ord(q^2)$ holds, according to 4.2.4, iff $r > 0$.

For $r < 0$ it is further calculated in [Len17] that $(\mathfrak{Z}_{\sqrt{r}\alpha_1})^n = \mathfrak{Z}_{n\sqrt{r}\alpha_1}$ which is a *local* screening. The algebra of screenings is therefore an extension of the Nichols algebra by a long screening.

Example 4.4.15. Let us consider a rank 2 Cartan q-diagram and corresponding m-solution:



– By the previous example the simple truncation relations hold for $r \geq 0$.

  We conjecture that in this case the non-simple truncation ($[\mathfrak{Z}_1, \mathfrak{Z}_2]_c^n$ etc.) also hold for $r > 0$. But this would either require a reflection theory for algebra of screenings or a generalisation of theorem 4.2.4.

– The long Serre relation $[\mathfrak{Z}_2, [\mathfrak{Z}_2, \mathfrak{Z}_1]_c]_c = 0$ holds if $2dr \notin -\mathbb{N}$. Does the long Serre relation may fail if $q_{22} = -1$ and $r < 0$, which is when the long root α_2 is both q-Cartan and q-truncation and when the truncation relation fails. But for these cases the Serre relation was in theorem 4.4.13 not required as an independent relation.

– The short Serre relation $[\mathfrak{Z}_1, \ldots [\mathfrak{Z}_1, \mathfrak{Z}_2]_c \ldots]_c = 0$ which involves $d + 1$ times the first screening, holds if

$$2r, 3r, \ldots, (d+1)r \notin -\mathbb{N}$$
$$dr, (d-1)r, (d-2)r, \ldots, 2r \notin \mathbb{N}.$$

  In particular:

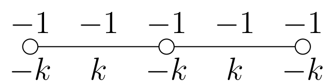
  * for $d = 1$ see the long Serre relations.
  * for $d = 2$ holds iff $3r \notin -\mathbb{N}$.
  * for $d = 3$ holds iff $2r, 4r \notin -\mathbb{N}$ and $2r \notin \mathbb{N}$. But $2r \in \mathbb{Z}$ is not admissible because $q \neq -1$.

So again the short Serre relation may fail if the short root α_1 is both q-Cartan and q-truncation and the truncation relation fails.

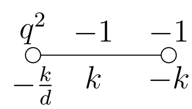
– Extra relations as listed in point (2) of theorem 4.4.13 apply exactly in the exceptional cases for the Serre relations above.

Summarizing we have the following possible exceptions:

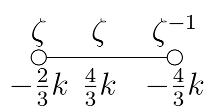
- for $q^2 = -1$, $k \in \mathbb{N}$, k odd, ($r < 0$, $d = 1$):



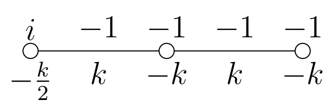
- for $q^2 \in \mathcal{R}_{2d}$, $k \in \mathbb{N}$, k odd, $\forall d$ ($r < 0$):



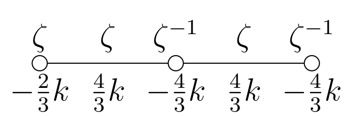
- for $q^2 = \zeta \in \mathcal{R}_3$, $k \in \mathbb{N}$, k odd, ($r < 0$, $d = 2$):



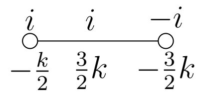
- for $q^2 = i$, $k \in \mathbb{N}$, k odd, ($r < 0$, $d = 2$):



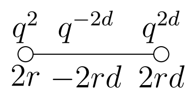
- for $q^2 = \zeta \in \mathcal{R}_3$, $k \in \mathbb{N}$, k odd, ($r < 0$, $d = 2$):



- for $k \in \mathbb{N}$, k odd, ($r < 0$, $d = 3$):



Proposition 4.4.16. *We consider again a rank 2 Cartan q-diagram*



1. *If* $q^{2d} = -1$, $r < 0$ *the long Serre relation holds.*
2. *If ord*(q^2)$= d + 1$, $r < 0$ *the short Serre relation holds.*

Proof. 1. The long Serre relation reads

$$[\mathfrak{z}_2, [\mathfrak{z}_2, \mathfrak{z}_1]_{-1}]_{+1} = (\mathfrak{z}_2)^2\mathfrak{z}_1 + \mathfrak{z}_2\mathfrak{z}_1\mathfrak{z}_2 - \mathfrak{z}_2\mathfrak{z}_1\mathfrak{z}_2 - \mathfrak{z}_1(\mathfrak{z}_2)^2 = [(\mathfrak{z}_2)^2, \mathfrak{z}_1]_{+1}.$$

Since $r < 0$ this is not automatically zero because $(\mathfrak{z}_2)^2 \neq 0$. Despite this it was studied in [Len17] that $(\mathfrak{z}_2)^2 \sim \mathfrak{z}_{22}$. Then standard OPE calculations give: $[(\mathfrak{z}_2)^2, \mathfrak{z}_1]_c = [\mathfrak{z}_{22}, \mathfrak{z}_1]_c = 0$.

2. This point is a generalisation of the previous. We have:

$$\begin{aligned}
[\mathfrak{z}_1, \ldots, [\mathfrak{z}_1, \mathfrak{z}_2]_c \ldots]_c &= (\mathfrak{z}_1)^{d+1}\mathfrak{z}_2 - (q^{-d} + q^{-d+2} + \ldots + q^d)(\mathfrak{z}_1)^d\mathfrak{z}_2\mathfrak{z}_1 + \ldots \\
&= \sum_{i+j=d+1} (-1)^i (\mathfrak{z}_1)^i \mathfrak{z}_2 (\mathfrak{z}_1)^j \frac{[i+j]_q!}{[i]_q![j]_q!} = [(\mathfrak{z}_1)^{d+1}, \mathfrak{z}_2]_c \\
&= [\mathfrak{z}_{(d+1)\alpha_1}, \mathfrak{z}_2]_c = 0
\end{aligned}$$

where $[i]_q! := 1(1+q)(1+q+q^2)\cdots(1+\ldots+q^{i-1})$ and where for the penultimate equality we used again results from [Len17] and for the last one theorem 4.2.1.

□

Remark 4.4.17. Alternatively this follows conceptually from the fact that this holds for generic q. One can argue similarly for the other relations or proceed as in the previous proposition.

In conclusion:

Corollary 4.4.18. *The screening operators algebra is as follows:*

- *For $r \geq 0$ all Nichols algebra relations hold (conjecturally also the non-simple truncation relations).*
- *For $r < 0$ all Nichols algebra relations hold except the truncation relations.*

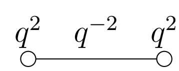
  Conjecturally, the non-zero result of the truncation relation are, as above, themselves local screenings and in the centre of the algebra of screenings. Hence in these cases we get the positive part of the infinite-dimensional Kac-Procesi-DeConcini quantum group, also called non-restricted specialisation *[CP94].*

Remark 4.4.19. We remark that for $r < 0$ products of screenings can be not well defined.

4.4.5 Examples in rank 2

Heckenberger row 2

This case of the list is described by the braiding diagram:



with $q \in \mathbb{C}$ $q^2 \neq 1$ and simple roots $\{\alpha_1, \alpha_2\}$. The realising lattice is a rescaled A_2 root lattice i.e. $\mathfrak{sl}_3$.

The set of positive roots is given by $\{\alpha_1, \alpha_2, \alpha_{12}\}$ with unique associate Cartan matrix:

$$a_{ij} = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}.$$

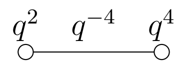
Proposition 4.4.20. *Defining r as in 4.4.2 we find that the following m_{ij}-matrix is a realising solution:*

$$m_{ij} = \begin{bmatrix} 2r & -r \\ -r & 2r \end{bmatrix}.$$

Remark 4.4.21. For $q^2 = -1$, these are all solutions and the roots are both q-Cartan *and* q-truncation.
This case is shown in detail in remark 4.3.7 of example 4.3.5.

Heckenberger row 4

This case of the list is described by the braiding diagram:



with $q \in \mathbb{C}$ $q^2 \neq \pm 1$ and simple roots $\{\alpha_1, \alpha_2\}$. The realising lattice is a rescaled B_2 root lattice.

The set of positive roots is given by $\{\alpha_1, \alpha_2, \alpha_{12}, \alpha_{112}\}$ with unique associate Cartan matrix:

$$a_{ij} = \begin{bmatrix} 2 & -2 \\ -1 & 2 \end{bmatrix}.$$

Proposition 4.4.22. *If $q^2 \neq \pm 1$, then for every possible r defined as in 4.4.2 the following m_{ij}-matrix*

$$m_{ij} = \begin{bmatrix} 2r & -2r \\ -2r & 4r \end{bmatrix}$$

is a realising solution for the braiding.

Remark 4.4.23.

1. When $q^2 \in \mathcal{R}_4$, the root α_2 is q-Cartan *and* q-truncation. There is an additional family of solutions when it is m-truncation:

$$m^{\mathrm{I}}_{ij} = \begin{bmatrix} 2r & -2r \\ -2r & 1 \end{bmatrix} \qquad m^{\mathrm{II}}_{ij} = \begin{bmatrix} -2r+1 & 2r-1 \\ 2r-1 & 1 \end{bmatrix} \qquad \text{for } r = \frac{p'}{4},\ \ p' \text{ odd},$$

   with simple roots I: $\{\alpha_1, \alpha_2\}$ and II: $\{\alpha_{12}, -\alpha_2\}$.

   This lattice can be interpreted as lattice realising the Lie superalgebra $B(1,1)$ described in case Heckenberger row 5, which for this choice of q^2 has the same q-diagram.

2. When $q^2 \in \mathcal{R}_3$, the root α_1 is q-Cartan *and* q-truncation. There is an additional family of solutions when it is m-truncation:

$$m^{\mathrm{I}}_{ij} = \begin{bmatrix} \frac{2}{3} & -2r \\ -2r & 4r \end{bmatrix} \qquad m^{\mathrm{II}}_{ij} = \begin{bmatrix} \frac{2}{3} & -\frac{4}{3}+2r \\ -\frac{4}{3}+2r & \frac{8}{3}-4r \end{bmatrix}$$

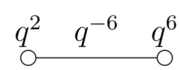
for $r = \frac{2+3p'}{6}$, $p' \in \mathbb{Z}$, with simple roots I: $\{\alpha_1, \alpha_2\}$ and II: $\{-\alpha_1, \alpha_{112}\}$.

This lattice can be interpreted as lattice realising the case Heckenberger row 6 (a colour Lie algebra), which for this choice of q^2 has the same q-diagram.

Remark 4.4.24. Note that $q^2 = -1$ is excluded. Indeed for that value, the system degenerates and the short truncation roots form a lattice of type A_1^n as described in [FL17]. Physically it corresponds to n pair of symplectic fermions.

Heckenberger row 11

This case of the list is described by the braiding diagram:



with $q^2 \neq \pm 1, q^2 \notin \mathcal{R}_3$ and simple roots $\{\alpha_1, \alpha_2\}$.
The realising lattice is of type G_2.

The set of positive roots is given by $\{\alpha_1, \alpha_2, \alpha_{12}, \alpha_{112}, \alpha_{1112}, \alpha_{11122}\}$ with unique associate Cartan matrix:

$$a_{ij} = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}.$$

Proposition 4.4.25. *If $q^2 \neq \pm 1, q^2 \notin \mathcal{R}_3$, then for every possible r defined as in 4.4.2 the following m_{ij}-matrix*

$$m_{ij} = \begin{bmatrix} 2r & -3r \\ -3r & 6r \end{bmatrix}$$

is a realising solution for the braiding.

Remark 4.4.26. When $q^2 \in \mathcal{R}_4$, the root α_1 is q-Cartan *and* q-truncation. When it is m-truncation we get:

$$m_{ij}^{\mathrm{I}} = \begin{bmatrix} \frac{1}{2} & -3r \\ -3r & 6r \end{bmatrix} \qquad m_{ij}^{\mathrm{II}} = \begin{bmatrix} \frac{1}{2} & -\frac{3}{2} + 3r \\ -\frac{3}{2} + 3r & \frac{9}{2} - 12r \end{bmatrix}$$

with simple roots I: $\{\alpha_1, \alpha_2\}$ and II: $\{-\alpha_1, \alpha_{1112}\}$.

The root α_{1112} is never m-truncation and it is m-Cartan iff $r = \frac{1}{4}$. But for this value of r, α_1 is also m-Cartan and thus this is *not* a new solution.

Remark 4.4.27. When $q^2 \in \mathcal{R}_6$, the root α_2 is m-Cartan *and* m-truncation. When it is m-truncation we get:

$$m^{\mathrm{I}}_{ij} = \begin{bmatrix} 2r & -3r \\ \\ -3r & 1 \end{bmatrix} \qquad m^{\mathrm{II}}_{ij} = \begin{bmatrix} 1-4r & -1+3r \\ \\ -\frac{3}{2}+3r & 1 \end{bmatrix}$$

with simple roots I: $\{\alpha_1, \alpha_2\}$ and II: $\{\alpha_{12}, -\alpha_2\}$.

The root α_{12} is never m-truncation and it is m-Cartan iff $r = \frac{1}{6}$. But for this value of r, α_2 is also m-Cartan and thus this is *not* a new solution.

4.5 Super Lie type

4.5.1 q diagram

Let $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ be a simple Lie superalgebra of *classical, basic* type [FSS96], i.e. of type $A(m,n), B(m,n), C(n+1), D(m,n), F(4), G(3), D(2,1;\alpha)$. For these Lie superalgebras a (non degenerate or zero) Killing form $(\ ,\ )_{\mathfrak{g}}$ is defined.

We now choose a Weyl chamber $\alpha_1, \dots, \alpha_{f-1},\ \alpha_f,\ \alpha_{f+1}, \dots, \alpha_n$ with just one simple fermionic root α_f. We call it the *standard chamber* according to [Kac77]. Given α positive root in the standard chamber, we define $f(\alpha)$ the multiplicity of α_f in α.

We can then split $\mathfrak{g}$ as the direct sum of vector spaces

$$\mathfrak{g} = \mathfrak{g}' \oplus \mathfrak{g}'' \oplus \mathfrak{m},$$

where $\mathfrak{g}'$ and $\mathfrak{g}''$ are two bosonic connected component generated by the simple roots $\alpha_1, \dots, \alpha_{f-1}$ and $\alpha_{f+1}, \dots, \alpha_n$ respectively, while $\mathfrak{m}$ is the $\mathfrak{g}' \oplus \mathfrak{g}''$-module spanned by all other roots.

We have that $\mathfrak{m}$ contains $\mathfrak{g}_1$ and thus in particular contains the $\mathfrak{g}' \oplus \mathfrak{g}''$-submodule generated by the fermion α_f, i.e. the vector space of fermions γ, with $f(\gamma) = 1$. Moreover $\mathfrak{m}$ may contain bosonic roots δ, with $f(\delta)$ positive even.

Definition 4.5.1. We can write the inner product $(\ ,\ )_{\mathfrak{g}}$ of two arbitrary simple roots as

$$(\alpha_i, \alpha_j)_{\mathfrak{g}} = (\alpha_i, \alpha_j)_{\mathfrak{g}'} + (\alpha_i, \alpha_j)_{\mathfrak{g}''} = \begin{cases} (\alpha_i, \alpha_j)_{\mathfrak{g}'} & \text{if } i \leq f,\ j < f \\ 0 & \text{if } i \leq f \leq j \\ (\alpha_i, \alpha_j)_{\mathfrak{g}''} & \text{if } i \geq f,\ j > f. \end{cases}$$

In particular we assume $(\alpha_f, \alpha_f)_{\mathfrak{g}} = (\alpha_f, \alpha_f)_{\mathfrak{g}'} = (\alpha_f, \alpha_f)_{\mathfrak{g}''} = 0$.

Definition 4.5.2. Let q', q'' be primitive roots of unity of the same order. Then to the above data in the standard chamber we associate the braiding matrix q_{ij} with

$$q_{ij} = \begin{cases} (q')^{(\alpha_i, \alpha_j)_{\mathfrak{g}'}} & \text{if } i \leq f,\ j < f \\ (q'')^{(\alpha_i, \alpha_j)_{\mathfrak{g}''}} & \text{if } i \geq f,\ j > f \\ 1 & \text{if } i > f > j \\ -1 & \text{if } i = f = j. \end{cases}$$

Under certain conditions on the q_{ij}, this braiding gives a finite-dimensional Nichols algebra $\mathcal{B}(q_{ij})$, which we call of *Super Lie type*.

The reflections will act on the braiding as follow:

- Reflections $\mathcal{R}^k$ around bosonic roots α_k leave q_{ij} invariant.
- Reflections $\mathcal{R}^k$ around fermionic roots α_k interchange fermionic and bosonic roots and may produce a braiding containing $-q$.

Remark 4.5.3. In the classification of Nichols algebras in [Hec05] and [Hec06a] we find that the fermion (as in the Lie superalgebra sense of the term) in the standard chamber α_f has $q_{ff} = -1$, i.e. it is q-truncation. This is not true in general for every fermion as we can see in the following example.

Example 4.5.4. The case Heckenberger row 5 of table 1 in [Hec05] is described by two diagrams:

I II

corresponding to the simple roots:

$$\mathrm{I} : \{\alpha_1, \alpha_2\} \qquad \mathrm{II} : \{-\alpha_1, \alpha_{12}\}.$$

This is the Lie superalgebra **B(1,1)** and α_{12} is a fermion with $q_{12,12} \neq -1$. We will describe this example in detail later on in this section.

4.5.2 Construction of m_{ij}

Definition 4.5.5. Given $p', p'' \in \mathbb{Z}$ such that $(p', p) = (p'', p) = 1$, we define $r' := \frac{p'}{p}$, $r'' := \frac{p''}{p}$ and in the standard chamber:

$$m_{ij}^S = \begin{cases} (\alpha_i, \alpha_j)_{\mathfrak{g}'} r' & \text{if } i \leq f,\ j < f \\ (\alpha_i, \alpha_j)_{\mathfrak{g}''} r'' & \text{if } i \geq f,\ j > f \\ 0 & \text{if } i > f > j \\ 1 & \text{if } i = f = j. \end{cases}$$

We notice that if we restrict to $\mathfrak{g}'$ or $\mathfrak{g}''$, we get exactly the same result as in the Cartan type section for p', p respectively p'', p.

Lemma 4.5.6. *If we call $q' = e^{i\pi r'}$ and $q'' = e^{i\pi r''}$, then $q_{ij} = e^{i\pi m_{ij}}$ is the braiding defined in definition 4.5.2.*

Proof. We have $m_{ij} = 0$ if α_i and α_j are disconnected, so that $1 = e^{i\pi\cdot 0}$ and $m_{ff} = 1$ for the fermionic root which gives $-1 = e^{i\pi\cdot 1}$ as demanded. □

Lemma 4.5.7. *In an arbitrary chamber $C_{\gamma_1,\ldots,\gamma_{rank}}$ we have*

$${m_{ij}}^C = (\gamma_i, \gamma_j)_{\mathfrak{g}'} r' + (\gamma_i, \gamma_j)_{\mathfrak{g}''} r'' + f(\gamma_i) f(\gamma_j).$$

Proof. We write $\gamma_i = \sum_k x_{ik}\alpha_k$ and $\gamma_j = \sum_l x_{jl}\alpha_l$ and we extend for linearity:

$$\begin{aligned} {m_{ij}}^C = & \sum_{k,l} x_{ik} x_{jl} {m_{kl}}^S \\ = & \sum_{k,l \in \mathfrak{g}' \cup \{f\}} x_{ik} x_{jl} (\alpha_k, \alpha_l)_{\mathfrak{g}'} r' + \sum_{k,l \in \mathfrak{g}'' \cup \{f\}} x_{ik} x_{jl} (\alpha_k, \alpha_l)_{\mathfrak{g}''} r'' + x_{if} x_{jf} = \\ = & \ (\gamma_i, \gamma_j)_{\mathfrak{g}'} r' + (\gamma_i, \gamma_j)_{\mathfrak{g}''} r'' + f(\gamma_i) f(\gamma_j) \end{aligned}$$

where the last equality follows from the definition of $f(\gamma)$ as the multiplicity of α_f in γ and the fact that on each component $\mathfrak{g}'$ and $\mathfrak{g}''$ the roots are spanned as in a Lie algebra. □

Corollary 4.5.8. *A root γ in an arbitrary chamber is*

- *m-truncation if* $\quad (\gamma,\gamma)_{\mathfrak{g}'}r' + (\gamma,\gamma)_{\mathfrak{g}''}r'' + f(\gamma)f(\gamma) = 1$
- *m-Cartan if, for every simple root β_i in the standard chamber,*

$$\begin{aligned}&(\gamma,\beta_i)_{\mathfrak{g}'}r' + 2(\gamma,\beta_i)_{\mathfrak{g}''}r'' + 2f(\gamma)f(\beta_i)\\ &= a_{\gamma,\beta_i}((\gamma,\gamma)_{\mathfrak{g}'}r' + (\gamma,\gamma)_{\mathfrak{g}''}r'' + f(\gamma)f(\gamma)).\end{aligned}$$

Example 4.5.9. We consider as an example the Lie superalgebra **A(1,1)** of rank 3. The simple roots in the standard chamber are $\{\alpha_1,\ \alpha_2 = \alpha_f,\ \alpha_3\}$ with inner product:

$$(\alpha_i,\alpha_j) = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 0 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

Hence:

$$m^S_{ij} = \begin{bmatrix} 2r' & -r' & 0 \\ -r' & 1 & -r'' \\ 0 & -r'' & 2r'' \end{bmatrix}, \qquad q_{ij} = \begin{bmatrix} (q')^2 & (q')^{-1} & 1 \\ (q')^{-1} & -1 & (q'')^{-1} \\ 1 & (q'')^{-1} & (q'')^2 \end{bmatrix}.$$

Remark 4.5.10. According to [Kac77] we can write the simple roots as

$$\alpha_1 = \epsilon_1 - \epsilon_2, \quad \alpha_2 = \alpha_f = \epsilon_2 - \delta_1, \quad \alpha_3 = \delta_1 - \delta_2,$$

with vectors ϵ_i generating $\mathfrak{g}'$ and δ_i generating $\mathfrak{g}''$.

What remains to do is to see under which conditions the defined m_{ij} are realising solutions of the given braidings.

Lemma 4.5.11. *If $\gamma \in \mathfrak{g}'$ i.e. $\gamma = \sum_{i=1}^{f-1} a_i\alpha_i$ or $\gamma \in \mathfrak{g}''$ i.e. $\gamma = \sum_{i=f+1}^{n} a_i\alpha_i$, then γ is m-Cartan.*

Proof. Suppose $\gamma \in \mathfrak{g}'$; then

$$\begin{aligned}(\gamma,\gamma) &= (\gamma,\gamma)_{\mathfrak{g}'} \qquad (\gamma,\gamma)_{\mathfrak{g}''} = 0\\ (\gamma,\alpha_i) &= (\gamma,\alpha_i)_{\mathfrak{g}'} \qquad (\gamma,\alpha_i)_{\mathfrak{g}''} = 0\end{aligned}$$

for every arbitrary simple root α_i. Moreover $f(\gamma) = 0$. So we have that (4.5)A:

$$2(\gamma, \alpha_i)_{\mathfrak{g}'} r' + 2(\gamma, \alpha_i)_{\mathfrak{g}''} r'' + 2f(\gamma)f(\alpha_i) = a_{\gamma,i}((\gamma, \gamma)_{\mathfrak{g}'} r' + (\gamma, \gamma)_{\mathfrak{g}''} r'' + f(\gamma)f(\gamma))$$

becomes:

$$2(\gamma, \alpha_i)r' = a_{\gamma,i}((\gamma, \gamma)r').$$

Since $\gamma \in \mathfrak{g}'$, we are restricting to one bosonic sector and thus the latter is true because of definition of $a_{\gamma,i}$ in the Lie algebra setting.
By linearity in the simple roots α_i it is possible to extend this result to every arbitrary root $\alpha = \sum b_i \alpha_i$. □

Lemma 4.5.12. *If $\gamma \neq \alpha_f$ is isotropic, i.e. $(\gamma, \gamma) = (\gamma, \gamma)_{\mathfrak{g}'} = (\gamma, \gamma)_{\mathfrak{g}''} = 0$, and $f(\gamma) = \pm 1$ then γ is m-truncation.*

Proof. Condition (4.5)B for a root to be m-truncation reads:

$$(\gamma, \gamma)_{\mathfrak{g}'} + (\gamma, \gamma)_{\mathfrak{g}''} + f(\gamma)f(\gamma) = 1$$

which is clearly true under these hypothesis. □

We summarise these results in the following:

Corollary 4.5.13. *The matrix m_{ij} defined in 4.5.5 realises the braiding q_{ij} for every root α, with the following possible exceptions:*

1. *α is a boson in $\mathfrak{g}' \cup \mathfrak{g}''$, i.e. $f(\alpha)$ is a strictly positive even integer.*
2. *α is an isotropic fermion with $f \neq \pm 1$.*
3. *α is a non-isotropic fermion.*
4. *α is a fermion strong orthogonal to another fermion γ, i.e. in their real span $\langle \alpha, \gamma \rangle_{\mathbb{R}}$ there aren't roots.*

Proof.
- If a boson α belongs to only one bosonic side $\mathfrak{g}'$ or $\mathfrak{g}''$, then lemma 4.5.11 tells us it must be m-Cartan. Otherwise, α is like in (1) and must be spanned by the standard fermion as well, thus $f(\alpha) > 0$ even. In this case lemma 4.5.11 fails since no Lie algebra Killing form is a priori holding. We then have to check explicitly for which r' and r'' one of condition (4.5) holds using Corollary 4.5.8.

- Let now α be a fermion which is never strong orthogonal to other fermions. If it is isotropic and $f(\alpha) = \pm 1$, thanks to lemma 4.5.12, it satisfies the M-condition truncation. If $f \neq \pm 1$ or it is non-isotropic, we are back to the points (2) and (3) of the lemma and we have to check explicitly for which r' and r'' one of condition (4.5) holds using Corollary 4.5.8.

- If α and γ are two strong orthogonal fermions, then $a_{\alpha\beta} = 0$. In this case we have to check for which r' and r''

$$m_{\alpha,\beta} = (\alpha,\beta)_{\mathfrak{g}'} r' + (\alpha,\beta)_{\mathfrak{g}''} r'' + f(\alpha) f(\beta) = 0$$

□

Remark 4.5.14. In the examples we didn't find any boson with $f > 2$ and any fermion with $f > 1$. Thus, point (1) concerns then just bosons with $f = 2$ and point (2) never happens.

In conclusion we will now have to look, in every example, if one or more of the situations described by lemma 4.5.13 is happening.

Now as last result we state a classification Lemma:

Lemma 4.5.15. *If all the bosonic roots are m-Cartan, then the unique possible realising solution for the given braiding is the matrix m_{ij} of definition 4.5.5. In particular this is the case if $\ell_i > 1 - a_{ij}$ for $\forall i \neq f$.*

Proof. Condition (4.5) gives a unique solution for the m_{ij} in the standard chamber: the fermionic root is m-truncation and thus fixed to $m_{ff} = 1$, while, since all other roots are m-Cartan, restricting our study to the two bosonic sectors separately we end up in the same situation of lemma 4.4.6. Moreover the compatibility with the reflections fixes the m_{ij} in all chambers. □

Example 4.5.16. We apply lemma 4.5.13 to example 4.5.9: after reflecting the standard chamber set of roots around the fermion α_2, we find for new simple roots: $\{\alpha_{12}, -\alpha_2, \alpha_{23}\}$ the matrix:

$$m^C_{ij} = \begin{bmatrix} 1 & -1 + r' & -1 + r' + r'' \\ -1 + r' & 1 & -1 + r'' \\ -1 + r' + r'' & -1 + r'' & 1 \end{bmatrix}.$$

Exception (4) of lemma 4.5.13 appears. We then have to ask $m_{23} = 0$, i.e. $r' + r'' = 1$. In that case m_{ij} is a realising solution.
This construction realises the Nichols algebra $\mathcal{B}(q)$ described by case row 8 of table 2 in [Hec05] when $q \neq \pm 1$.

4.5.3 Central charge

We will compute the central charge of systems associated to Lie superalgebras $\mathfrak{g}$, with non degenerate Killing form (,).

Proposition 4.5.17. *The central charge of the system is $c = rank - 12(Q,Q)$ with*

$$Q = \frac{\rho^\vee_{\mathfrak{g}'}}{\sqrt{r'}} - \rho_{\mathfrak{g}'}\sqrt{r'} + \frac{\rho^\vee_{\mathfrak{g}''}}{\sqrt{r''}} - \rho_{\mathfrak{g}''}\sqrt{r''} - \rho^\vee_{rest}$$

where we denoted by $\rho_{\mathfrak{g}'}$ the sum of positive roots in $\mathfrak{g}'$, $\rho_{\mathfrak{g}''}$ the sum of positive roots in $\mathfrak{g}''$ and ρ_{rest} the sum of the remaining positive roots of $\mathfrak{g}$.

Proof. The central charge is $c = rank - 12(Q,Q)$ if Q is such that $\forall \alpha_i$ simple root of $\mathfrak{g}$

$$\frac{1}{2}(-\sqrt{r_i}\alpha_i, -\sqrt{r_i}\alpha_i) - (-\sqrt{r_i}\alpha_i, Q) = 1 \qquad \text{where} \quad r_i = \begin{cases} \frac{p'}{p} \text{ if } i < f \\ 1 \text{ if } i = f \\ \frac{p''}{p} \text{ if } i > f. \end{cases}$$

Let $\lambda^\vee_j = \frac{\lambda_j}{d_j}$ be such that $(\alpha_i, \lambda^\vee_j) = \delta_{ij}$. Since $\rho_{\mathfrak{g}} = \sum_{i=1}^{n} \lambda_i$, we have that $\rho_{\mathfrak{g}'} = \sum_{i<f} \lambda_i$, $\rho_{\mathfrak{g}''} = \sum_{i>f} \lambda_i$ and then $\rho_{\text{rest}} = \lambda_f$. We can thus rewrite Q as:

$$Q = \frac{\rho^\vee_{\mathfrak{g}'}}{\sqrt{r'}} - \rho_{\mathfrak{g}'}\sqrt{r'} + \frac{\rho^\vee_{\mathfrak{g}''}}{\sqrt{r''}} - \rho_{\mathfrak{g}''}\sqrt{r''} - \rho^\vee_{\text{rest}} = \sum_i \left(\frac{1}{\sqrt{r_i}} - \sqrt{r_i} d_i \right) \lambda^\vee_i.$$

Hence the previous equation becomes:

$$\begin{aligned} &\frac{1}{2}(-\alpha_i\sqrt{r_i}, -\alpha_i\sqrt{r_i}) - (-\alpha_i\sqrt{r_i}, Q) \\ &= \frac{1}{2} 2 d_i r_i + \sum_j \sqrt{r_i}(\frac{1}{\sqrt{r_j}} - \sqrt{r_j} d_j)(\alpha_i, \lambda^\vee_j) = 1 \end{aligned}$$

□

4.5.4 Algebra relations

We now want to determine when the algebra of screenings satisfies Nichols algebras relations for braiding q_{ij}.
We will denote again d', d'' the half length of the long bosonic root in $\mathfrak{g}', \mathfrak{g}''$.

Lemma 4.5.18. *For m_{ij} as above, lemma 4.2.3 holds under the condition*

$$\frac{1}{2d'} \geq r' > 0, \qquad \frac{1}{2d''} \geq r'' > 0, \qquad \det(m_{ij}) > 0.$$

Proof. Lemma 4.2.3 holds for all monomials under the assumptions $|\alpha_i| \leq 1$, which means $2d'r' \leq 1$, $2d''r'' \leq 1$, and m_{ij} positive definite. By Sylvester's criterion, this is equivalent to $\det(m_{ij}) > 0$ and to the principal minor being positive definite. The principal minor is a rescaling of the root lattices $\mathfrak{g}', \mathfrak{g}''$, so it is positive definite for $r', r'' > 0$.

□

Example 4.5.19. For type $A(n,m)$ these conditions read

$$\frac{1}{2} \geq r' > 0, \qquad \frac{1}{2} \geq r'' > 0, \qquad \frac{n}{n+1}r' + \frac{m}{m+1}r'' < 1.$$

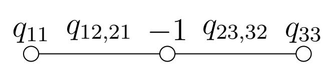
In [Ang15], theorem 3.1, we find a set of defining relations for each finite-dimensional Nichols algebra of super Lie type. We report them in the following theorem.

Theorem 4.5.20. *For finite-dimensional Nichols algebra of super Lie type with diagonal braiding $q_{ij} = q^{(\alpha_i,\alpha_j)_{\mathfrak{g}',\mathfrak{g}''}}$ for bosonic roots and $q_{ii} = -1$ for the fermionic root in the standard chamber, associated to the root system of a Lie superalgebra $\mathfrak{g}$, the defining relations are as follows*

1. *For each root α the truncation relation and for each pair of simple roots α_i, α_j with $q_{ii}^{1-a_{ij}} \neq 1$ the Serre relation.*

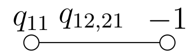
2. *For the following subdiagrams the following additional relations:*

   - *For type $A(2,0)$, $A(1,1)$, $D(2,1;\alpha)$:*



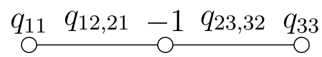
$$[(ad_c x_2)x_1, (ad_c x_2)x_3]_c = 0$$

- *For type* $B(1,1)$*:*



$$[(ad_c x_1)^2 x_2, (ad_c x_1)x_2]_c = 0$$

- *For type* $B(2,1)$

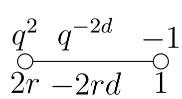


$$[(ad_c x_1)^2 (ad_c x_2)x_3, (ad_c x_1)x_2]_c = 0$$

Example 4.5.21. Let us consider a rank 1 q-diagram and corresponding m-solution, for a bosonic and fermionic root respectively:

The bosonic truncation relation $(\mathfrak{z}_b)^n = 0$, $n = ord(q^2)$ holds, according to 4.2.4, iff $r > 0$.
The fermionic truncation relation $(\mathfrak{z}_f)^2 = 0$ always holds according to 4.2.4.

Example 4.5.22. Let us consider a rank 2 super Lie q-diagram and corresponding m-solution:



In the examples we will found such a diagram just if $d = 1, 2$.

– By the previous example the simple truncation relations hold for $r \geq 0$.

We conjecture that in this case the non-simple truncation ($[\mathfrak{z}_1, \mathfrak{z}_2]_c^n$ etc.) also hold for $r > 0$. But this would either require a reflection theory for algebra of screenings or a generalisation of theorem 4.2.4.

– The bosonic Serre relation $[\mathfrak{z}_1, \dots [\mathfrak{z}_1, \mathfrak{z}_2]_c \dots]_c = 0$ (already studied in the Cartan section), involves $d+1$ times the first screening and holds

  * for $d = 1$ iff $2r \notin -\mathbb{N}$.
  * for $d = 2$ iff $3r \notin -\mathbb{N}$.

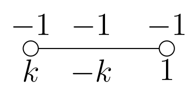
  So it may fail if the bosonic root α_1 is both q-Cartan and q-truncation and the truncation relation fails.

– The fermionic Serre relation $[\mathfrak{z}_2, [\mathfrak{z}_2, \mathfrak{z}_1]_c]_c = 0$ holds

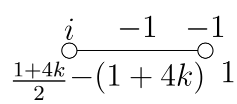
  * for $d = 1$ iff $-r + \frac{1}{2} \notin -\mathbb{N}$.
  * for $d = 2$ iff $2r, -2r + \frac{1}{2} \notin -\mathbb{N}$.
    But $2r \in \mathbb{Z}$ is not admissible because $q^{-2d} \neq 1$.

Summarising we have the following possible exceptions:

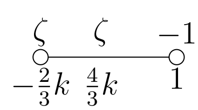
- for $k \in \mathbb{Z}$, k odd, $d = 1$:



- for $q^2 = i$, $k \in \mathbb{N}$, $d = 2$:



- for $q^2 = \zeta \in \mathcal{R}_3$, $k \in \mathbb{N}$, k odd, $d = 2$:



In conclusion:

Corollary 4.5.23. *Apart from the possible exceptions above, the screening operators algebra is as follows:*

- *For $r', r'' \geq 0$ all Nichols algebra relations hold (conjecturally also the non-simple truncation relations).*
- *For $r', r'' < 0$ all Nichols algebra relations hold except the bosonic truncation relations.*

  Conjecturally the algebra of screenings is again the positive part of an infinite-dimensional Kac-Procesi-DeConcini quantum super group.
- *For $r' > 0, r'' < 0$ or $r' < 0, r'' > 0$ the truncation relations on one side of the Dynkin diagram of the standard chamber fail, and we conjecturally get the positive part of an corresponding version of an infinite-dimensional Kac-Procesi-DeConcini quantum super group.*

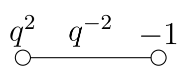
Regarding the Kac-Procesi-DeConcini version of an arbitrary Nichols algebra, see the concept of a pre-Nichols algebra in [Ang14].

4.5.5 Examples in rank 2

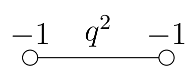
We now present the cases of table 1 in [Hec05] rising from Lie superalgebras of rank 2. We will check in every case whether the exceptions of corollary 4.5.13 appear.
In rank 2, there is obviously always just one bosonic sector $\mathfrak{g}'$.
In the respective remarks we will express the simple roots in the standard chamber using as in [Kac77] the standard basis ϵ_i and δ_i.

Heckenberger row 3

The case row 3 of table 1 in [Hec05], studied in example 4.3.5, is realised by the Lie superalgebra lattice $\mathbf{A(1,0)}$. This case is described by the diagrams:



I



II

with $q^2 \neq \pm 1$ and simple roots I : $\{\alpha_1, \alpha_2\}$, II : $\{\alpha_{12}, -\alpha_2\}$. The set of positive roots is given by $\{\alpha_1, \alpha_2, \alpha_{12}\}$ with unique associate Cartan matrix and inner products

$$a_{ij} = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}, \qquad (\alpha_i, \alpha_j) = \begin{bmatrix} 2 & -1 \\ -1 & 0 \end{bmatrix}.$$

Therefore the m_{ij} matrix in the standard basis and after reflecting around α_2 are given by:

$$m^{\mathrm{I}}_{ij} = \begin{bmatrix} 2r & -r \\ -r & 1 \end{bmatrix}, \qquad m^{\mathrm{II}}_{ij} = \begin{bmatrix} 1 & -1+r \\ -1+r & 1 \end{bmatrix}.$$

None of the exceptions of lemma 4.5.13 appears; therefore m_{ij} is a realising solution $\forall r$. This result matches with example 4.3.5.

Remark 4.5.24. As observed in example 4.3.5, if we allow the value $q^2 = -1$ we obtain row 2 of table 1 in [Hec05].

Remark 4.5.25. The simple roots in the standard chamber of $A(1,0)$ can be expressed by

$$\alpha_1 = \epsilon_1 - \epsilon_2, \qquad \alpha_2 = \alpha_f = \epsilon_2 - \delta_1.$$

Heckenberger row 5

Row 5 of table 1 in [Hec05] is realised by the Lie superalgebra lattice **B(1,1)**. This case is described by the diagrams:

I II

with $q^2 \neq \pm 1, q^2 \notin \mathcal{R}_4$ and simple roots I : $\{\alpha_1, \alpha_2\}$, II : $\{\alpha_{12}, -\alpha_2\}$.

The set of positive roots is given by $\{\alpha_1, \alpha_2, \alpha_{12}, \alpha_{112}\}$ with unique associate Cartan matrix:

$$a_{ij} = \begin{bmatrix} 2 & -2 \\ -1 & 2 \end{bmatrix}$$

and inner product:

$$(\alpha_i, \alpha_j) = \begin{bmatrix} 2 & -2 \\ -2 & 0 \end{bmatrix}.$$

Therefore the m_{ij} matrix in the standard basis and after reflecting around α_1 are given by:

$$m^{\mathrm{I}}_{ij} = \begin{bmatrix} 2r & -2r \\ -2r & 1 \end{bmatrix} \qquad m^{\mathrm{II}}_{ij} = \begin{bmatrix} -2r+1 & 2r-1 \\ 2r-1 & 1 \end{bmatrix}.$$

None of the exceptions of lemma 4.5.13 appears; therefore m_{ij} is a realising solution $\forall r$.

Remark 4.5.26. When $q^2 \in \mathcal{R}_3$, the root α_1 is q-Cartan *and* q-truncation. When it is m-truncation we get:

$$m_{ij}^{\mathrm{I}} = \begin{bmatrix} \frac{2}{3} & -2r \\ & \\ -2r & 1 \end{bmatrix} \qquad m_{ij}^{\mathrm{II}} = \begin{bmatrix} \frac{5}{3} - 4r & 2r - 1 \\ & \\ 2r - 1 & 1 \end{bmatrix} \qquad m_{ij}^{\mathrm{III}} = \begin{bmatrix} \frac{2}{3} & 2r - \frac{4}{3} \\ & \\ 2r - \frac{4}{3} & \frac{11}{3} - 8r \end{bmatrix}$$

where III: $\{-\alpha_1, \alpha_{112}\}$ comes after reflecting around α_1. The root α_{112} is never m-Cartan and it is m-truncation iff $r = \frac{1}{3}$. But for this value of r, α_1 is also m-Cartan and thus this is *not* a new solution.

Remark 4.5.27. If we allow $q = i$ the system is the one described in row 4 in section 4.4. Also in this case it corresponds to the Lie superalgebra $B(1,1)$.

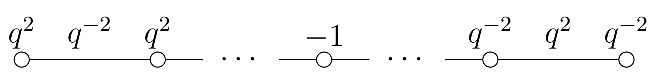
Remark 4.5.28. The roots can be expressed by

$$\alpha_1 = \epsilon_1, \qquad \alpha_2 = \alpha_f = \delta_1 - \epsilon_1.$$

4.5.6 Arbitrary rank

We generalise our study to arbitrary rank cases. In every case we will see under which assumptions the constructed m_{ij} matrices are realising solutions.

A(m,n)



The simple roots in the standard chamber are:

$$\alpha_1, \dots, \alpha_f = \alpha_{m+1}, \dots, \alpha_{m+n+1}$$

with inner product matrix

$$(\alpha_i, \alpha_j) = \begin{bmatrix} 2 & -1 & & & & \\ -1 & \ddots & \ddots & & & \\ & \ddots & \ddots & \ddots & & \\ & & \ddots & 0 & \ddots & \\ & & & \ddots & \ddots & -1 \\ & & & & -1 & 2 \end{bmatrix}$$

We list all positive roots. We denote by Δ_0 the set of bosons and by Δ_1 the set of fermions according to the literature [Kac77].

$$\Delta_0 = \{\alpha_l + \ldots + \alpha_k,\ \text{with } l, k < f \text{ or } l, k > f\}$$
$$\Delta_1 = \{\alpha_l + \ldots + \alpha_k,\ \text{with } l \leq f \leq k,\ l \neq k\}$$

We now apply the lemmas of the previous section to determine possible conditions on r' and r'' such that the m_{ij} matrix defined as in 4.5.5 is a realising solution.

- All the bosons are either in $\mathfrak{g}'$ or $\mathfrak{g}''$. Then, thanks to lemma 4.5.11, we know they are always m-Cartan.
- All the fermions are isotropic and have $f = \pm 1$. Thanks to lemma 4.5.12 we know that if they are not strong orthogonal to any other root they are m-truncation.
- We now focus on the case of strong orthogonal fermions. Let us consider two fermions:

$$\gamma_1 = \alpha_{l_1} + \ldots + \alpha_{k_1} \qquad \text{with } l_1 \leq f \leq k_1,$$
$$\gamma_2 = \alpha_{l_2} + \ldots + \alpha_{k_2} \qquad \text{with } l_2 \leq f \leq k_2.$$

  They are strong orthogonal if $l_1 \neq l_2,\ k_1 \neq k_2$. In this case we have to check that $m_{12} = (\gamma_1, \gamma_2)_{\mathfrak{g}'} r' + (\gamma_1, \gamma_2)_{\mathfrak{g}''} r'' + f(\gamma_1) f(\gamma_2) = 0$.
  We thus compute the inner products in the two bosonic sides. We assume $l_1 < l_2$ and $k_1 < k_2$, because every other combination works analogously and gives the same result.
  Wlog we can assume $l_2 = l_1 + 1$ and $k_2 = k_1 + 1$ and thus

$$\begin{aligned}(\gamma_1, \gamma_2) =& (\alpha_{l_1}, \gamma_2) + (\alpha_{l_1+1}, \gamma_2) + \ldots + (\alpha_f, \gamma_2) + \ldots + (\gamma_{k_1}, \gamma_2) \\ =& (\alpha_{l_1}, \alpha_{l_1+1})_{\mathfrak{g}'} \\ &+ (\alpha_{l_1+1}, \alpha_{l_1+1})_{\mathfrak{g}'} + (\alpha_{l_1+1}, \alpha_{l_1+2})_{\mathfrak{g}'} \\ &+ \ldots \\ &+ (\alpha_f, \alpha_{f-1})_{\mathfrak{g}'} + (\alpha_f, \alpha_f) + (\alpha_f, \alpha_{f+1})_{\mathfrak{g}''} \\ &+ \ldots \\ &+ (\alpha_{k_1}, \alpha_{k_1} - 1)_{\mathfrak{g}''} + (\alpha_{k_1}, \alpha_{k_1})_{\mathfrak{g}''} + (\alpha_{k_1}, \alpha_{k_1+1})_{\mathfrak{g}''}\end{aligned}$$

The only term that contributes is $(\alpha_f, \alpha_{f-1})_{\mathfrak{g}'} + (\alpha_f, \alpha_f) + (\alpha_f, \alpha_{f+1})_{\mathfrak{g}''}$ since the previous terms sum up to zero in $\mathfrak{g}'$, and the following terms sum up to zero in $\mathfrak{g}''$. Hence we have $(\gamma_1, \gamma_2) = -1_{\mathfrak{g}'} - 1_{\mathfrak{g}''}$. Asking m_{12} to be zero, means to ask

$$-1 \cdot r' - 1 \cdot r'' + 1 = 0 \qquad \Rightarrow \qquad r' + r'' = 1$$

To conclude, the only condition needed for the m_{ij} matrix to be a realising solution is $r' + r'' = 1$.

Remark 4.5.29. This condition matches with the formulation of $A(m, n)$ in terms of Nichols algebra diagram ([Hec06a], Table C, row 2), where $q_{\mathfrak{g}'} = q$ and $q_{\mathfrak{g}''} = q^{-1}$. Indeed, if $r' + r'' = 1$ then

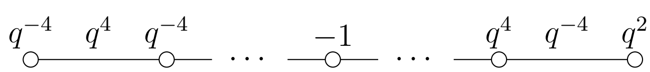
$$q_{\mathfrak{g}'} q_{\mathfrak{g}''} = e^{i\pi(\alpha_i,\alpha_i)r'} e^{i\pi(\alpha_j,\alpha_j)r''} = e^{i\pi 2r'} e^{i\pi 2r''} = e^{i\pi 2(r'+r'')} = 1,$$

calling α_i a root in $\mathfrak{g}'$ and α_j a root in $\mathfrak{g}''$.

Remark 4.5.30. We can write the simple roots in the standard chamber using as in [Kac77] the standard basis $\epsilon_1, \ldots, \epsilon_{m+1}, \delta_1, \ldots, \delta_{n+1}$:

$$\{\alpha_1 = \epsilon_1 - \epsilon_2, \;\; \alpha_2 = \epsilon_2 - \epsilon_3, \ldots, \;\; \alpha_{m+1} = \epsilon_{m+1} - \delta_1,$$
$$\alpha_{m+2} = \delta_1 - \delta_2, \ldots, \;\; \alpha_{m+n+1} = \delta_n - \delta_{n+1}\}$$

B(m,n)



The simple roots in the standard chamber are:

$$\alpha_1, \ldots, \alpha_f = \alpha_n, \ldots, \alpha_{m+n}$$

with inner product matrix

$$(\alpha_i, \alpha_j) = \begin{bmatrix} 4 & -2 & & & & \\ -2 & \ddots & \ddots & & & \\ & \ddots & \ddots & \ddots & & \\ & & \ddots & 0 & \ddots & \\ & & & \ddots & \ddots & -2 \\ & & & & -2 & 2 \end{bmatrix}$$

All the positive roots are:

$$\begin{aligned}\Delta_0 = \{&\alpha_l + \ldots + \alpha_k, && \text{with } l, k < f\\ &\alpha_l + \ldots + \alpha_k, && \text{with } l, k > f,\ k \neq m+n\\ &\alpha_l + \ldots + \alpha_{m+n}, && \text{with } l > f\\ &\alpha_l + \ldots + 2\alpha_k + \ldots + 2\alpha_{m+n}, && \text{with } l < f,\ k \leq f\\ &\alpha_l + \ldots + 2\alpha_k + \ldots + 2\alpha_{m+n}, && \text{with } l, k > f\}\end{aligned}$$

$$\begin{aligned}\Delta_1 = \{&\alpha_l + \ldots + \alpha_{m+n}, && \text{with } l \leq f\\ &\alpha_l + \ldots + 2\alpha_k + \ldots + 2\alpha_{m+n}, && \text{with } l < f < k\\ &\alpha_l + \ldots + \alpha_k, && \text{with } l < f < k,\ k \neq m+n\}\end{aligned}$$

We now apply the lemmas of the previous section to determine possible conditions on r' and r'' such that the m_{ij} matrix defined as in 4.5.5 is a realising solution.

- All the bosons which are not of the type $\gamma_{lk} := \alpha_l + \ldots + 2\alpha_k + \ldots + 2\alpha_{m+n}$, with $l < f$, $k \leq f$, are either in $\mathfrak{g}'$ or $\mathfrak{g}''$. Then, thanks to lemma 4.5.11, we know they are always m-Cartan.

- For γ_{lk}, we need to explicitly ask condition (4.5).
  The inner product is $(\gamma_{lk}, \gamma_{lk}) = -2_{\mathfrak{g}'} - 4_{\mathfrak{g}''}$.

  - γ_{lk} is m-truncation if $2r' + 4r'' = 3$.
  - γ_{lk} is m-Cartan if $r' + r'' = 1$.

- All the fermions which are not of the type $\gamma_l := \alpha_l + \ldots + \alpha_{m+n}$, are isotropic and have $f = \pm 1$. Thanks to lemma 4.5.12 we then know that if they are not strong orthogonal to any other root they are m-truncation.

- For γ_l, we need to explicitly ask condition (4.5).
  The inner product is $(\gamma_l, \gamma_l) = -1_{\mathfrak{g}''}$.

  - γ_l is m-truncation holds if $r'' = 0$.
  - γ_l is m-Cartan holds if $r' + r'' = 1$.

- We now focus on the case of strong orthogonal fermions. Let us consider the fermions:

$$\{\gamma_1 := \alpha_{l_1} + \ldots + \alpha_{m+n}$$
$$\gamma_2 := \alpha_{l_2} + \ldots + 2\alpha_{k_2} + \ldots + 2\alpha_{m+n}$$
$$\gamma_3 := \alpha_{l_3} + \ldots + \alpha_{k_3}\}$$

  The fermions γ_1 and γ_2 are strong orthogonal iff $l_1 \neq l_2$;
  The fermions γ_2 and γ_3 are strong orthogonal iff $l_2 \neq l_3$ or $k_2 \neq k_3 + 1$;
  The fermions γ_1 and γ_3 are strong orthogonal iff $l_1 \neq l_3$;
  Two fermions of type γ_2 are strong orthogonal iff have different l_2 and k_2;
  Two fermions of type γ_3 are strong orthogonal iff have different l_3 and k_3; Asking the condition $m_{ij} = 0$ for those cases, we find again the condition $r' + r'' = 1$.

In conclusion, the only condition needed for the m_{ij} matrix to be a realising solution is $r' + r'' = 1$. If this condition is satisfied the bosons with $f = 2$ as well as the non isotropic fermions are m-Cartan. If moreover $r' = r'' = \frac{1}{2}$ then the bosons with $f = 2$ are also m-truncation.

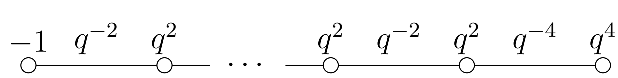
Remark 4.5.31. As in the case of the Lie superalgebras of type $A(m,n)$, the condition $r' + r'' = 1$ matches with the formulation of $B(m,n)$ in terms of Nichols algebra diagram ([Hec06a], Table C, row 4), where $q_{\mathfrak{g}'} = q$ and $q_{\mathfrak{g}''} = q^{-1}$.

Remark 4.5.32. We can write the simple roots in the standard chamber using as in [Kac77] the standard basis $\epsilon_1, \ldots, \epsilon_m, \delta_1, \ldots, \delta_n$:

$$\{\alpha_1 = \delta_1 - \delta_2, \ \ \alpha_2 = \delta_2 - \delta_3, \ldots, \ \ \alpha_n = \delta_n - \epsilon_1,$$
$$\alpha_{n+1} = \epsilon_1 - \epsilon_2, \ldots, \ \ \alpha_{m+n} = \epsilon_m.\}$$

The bosons with $f = 2$ will be of the form $\delta_i + \delta_j$, while the non isotropic fermions will be δ_i.

C(n)



The simple roots in the standard chamber are:

$$\alpha_f = \alpha_1, \ldots, \alpha_n$$

with inner product matrix

$$(\alpha_i, \alpha_j) = \begin{bmatrix} 0 & -1 & & & & \\ -1 & 2 & \ddots & & & \\ & \ddots & \ddots & \ddots & & \\ & & \ddots & \ddots & -1 & \\ & & & -1 & 2 & -2 \\ & & & & -2 & 4 \end{bmatrix}.$$

All the positive roots are:

$$\begin{aligned} \Delta_0 = \{ & \alpha_l + \ldots + \alpha_k, && \text{with } l \neq 1 \; k \neq n \\ & \alpha_l + \ldots + 2\alpha_k + \ldots + 2\alpha_{n-1} + \alpha_n, && \text{with } l \neq 1 \; k \neq n \\ & \alpha_l + \ldots + \alpha_n, && \text{with } l \neq 1 \\ & 2\alpha_l + \ldots + 2\alpha_{n-1} + \alpha_n, && \text{with } l \neq 1 \} \end{aligned}$$

$$\begin{aligned} \Delta_1 = \{ & \alpha_1 + \ldots + \alpha_n && \\ & \alpha_1 + \ldots + \alpha_k, && \text{with } k \neq 1 \\ & \alpha_1 + \ldots + 2\alpha_k + \ldots + 2\alpha_{n-1} + \alpha_n, && \text{with } k \neq n \} \end{aligned}$$

We now apply the lemmas of the previous section to determine possible conditions on r' such that the m_{ij} matrix defined as in 4.5.5 is a realising solution.

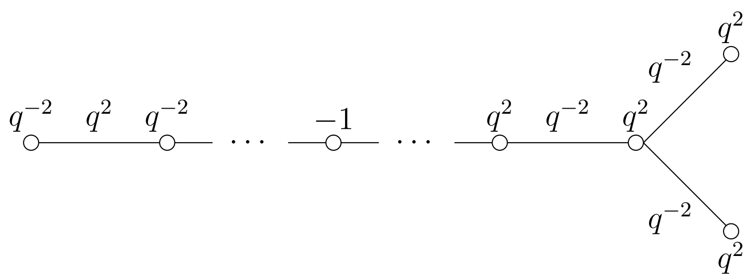
- Since there is just one bosonic side it is obvious that all the bosons are m-Cartan.
- All the fermions are isotropic, non strong orthogonal to each other, and have $f = \pm 1$. Thanks to lemma 4.5.12 we then know that they are m-truncation.

To conclude, the m_{ij} matrix is always a realising solution.

Remark 4.5.33. We can write the simple roots in the standard chamber using as in [Kac77] the standard basis $\epsilon_1, \delta_1 \ldots, \delta_{n-1}$:

$$\{\alpha_1 = \epsilon_1 - \delta_1, \;\; \alpha_2 = \delta_1 - \delta_2, \ldots, \;\; \alpha_{n-1} = \delta_{n-2} - \delta_{n-1}, \;\; \alpha_n = 2\delta_{n-1}\}$$

D(m,n)



The simple roots in the standard chamber are:

$$\alpha_1, \ldots, \alpha_n = \alpha_f, \ldots, \alpha_{n+m}$$

with inner product matrix

$$(\alpha_i, \alpha_j) = \begin{bmatrix} 2 & -1 & & & & \\ -1 & \ddots & \ddots & & & \\ & \ddots & 0 & \ddots & & \\ & & \ddots & 2 & -1 & -1 \\ & & & -1 & 2 & 0 \\ & & & -1 & 0 & 2 \end{bmatrix}$$

All the positive roots are:

$$\begin{aligned}\Delta_0 = \{ & \alpha_l + \ldots + \alpha_k, && \text{with } l,k < f\\ & \alpha_l + \ldots + \alpha_k, && \text{with } l,k > f\\ & \alpha_l + \ldots + \alpha_{m+n-2} + \alpha_{m+n}, && \text{with } l > f\\ & \alpha_l + \ldots + 2\alpha_k + \ldots + 2\alpha_{m+n-2} + \alpha_{m+n-1} + \alpha_{m+n}, && \text{with } l < f,\ k \leq f\\ & \alpha_l + \ldots + 2\alpha_k + \ldots + 2\alpha_{m+n-2} + \alpha_{m+n-1} + \alpha_{m+n}, && \text{with } l,k > f\\ & 2\alpha_l + \ldots + 2\alpha_k + \ldots + 2\alpha_{m+n-2} + \alpha_{m+n-1} + \alpha_{m+n}, && \text{with } l < f,\ k \leq f\}\end{aligned}$$

$$\begin{aligned}\Delta_1 = \{ & \alpha_l + \ldots + \alpha_k, && \text{with } l \leq f \leq k\\ & \alpha_l + \ldots + \alpha_{n+m-2} + \alpha_{n+m}, && \text{with } l \leq f\\ & \alpha_l + \ldots + 2\alpha_k + \ldots + 2\alpha_{m+n-2} + \alpha_{n+m-1} + \alpha_{n+m}, && \text{with } l < f < k\}\end{aligned}$$

We now apply the lemmas of the previous section to determine possible conditions on r' and r'' such that the m_{ij} matrix defined as in 4.5.5 is a realising solution.

- All bosons except the IV or VI type in the list, are either in $\mathfrak{g}'$ or $\mathfrak{g}''$. Then, thanks to lemma 4.5.11, we know they are always m-Cartan.
- The bosons of type IV have inner product $-2_{\mathfrak{g}'} - 4_{\mathfrak{g}''}$.
  - it is m-truncation if $2r' + 4r'' = 3$.
  - it is m-Cartan if $r' + r'' = 1$.

  The bosons of type VI have inner product $-4_{\mathfrak{g}''}$.
  - it is m-truncation if $4r'' = 3$.
  - it is m-Cartan if $r' + r'' = 1$.
- All fermions are isotropic and have $f = \pm 1$. Thanks to lemma 4.5.12 we then know that if they are not strong orthogonal to any other root they are m-truncation.
- There are many possibility for two fermions to be strong orthogonal. Asking the condition $m_{ij} = 0$ for those cases, we find again the condition $r' + r'' = 1$.

In conclusion, the only condition needed for the m_{ij} matrix to be a realising solution is $r' + r'' = 1$. If this condition is satisfied the bosons with $f = 2$ are m-Cartan. If moreover $r' = r'' = \frac{1}{2}$ then the boson of type IV are also m-truncation. Instead if $r' = \frac{1}{4}$, $r'' = \frac{3}{4}$ then the boson of type VI are also m-truncation.

Remark 4.5.34. As in the previous cases the condition $r' + r'' = 1$ matches with the formulation of $D(m, n)$ in terms of Nichols algebra diagram ([Hec06a], Table C, row 10), where $q_{\mathfrak{g}'} = q$ and $q_{\mathfrak{g}''} = q^{-1}$.

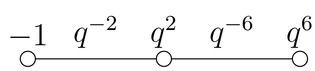
Remark 4.5.35. We can write the simple roots in the standard chamber using as in [Kac77] the standard basis $\epsilon_1, \dots, \epsilon_m, \delta_1, \dots, \delta_n$:

$$\{\alpha_1 = \delta_1 - \delta_2, \dots, \quad \alpha_n = \delta_n - \epsilon_1, \quad \alpha_{n+1} = \epsilon_1 - \epsilon_2, \; \dots \\ \dots \; \alpha_{m+n-1} = \epsilon_{m-1} - \epsilon_m, \quad \alpha_{m+n} = \epsilon_{m-1} + \epsilon_m\}.$$

The bosons of type IV will be of the form $\delta_i + \delta_j$, while the one of type VI will be of the form $2\delta_i$.

Sporadic cases

G(3)



The simple roots in the standard chamber are $\{\alpha_1 = \alpha_f, \alpha_2, \alpha_3\}$ with inner product

$$(\alpha_i, \alpha_j) = \begin{bmatrix} 0 & -1 & 0 \\ -1 & 2 & -3 \\ 0 & -3 & 6 \end{bmatrix}.$$

There is only one bosonic part $\mathfrak{g}'$ and the positive roots are:

$$\{\alpha_1, \; \alpha_2, \; \alpha_3, \; \alpha_{12}, \; \alpha_{23}, \; \alpha_{223}, \; \alpha_{123}, \; \alpha_{1223}, \\ \alpha_{12223}, \; \alpha_{2223}, \; \alpha_{22233}, \; \alpha_{1222233}, \; \alpha_{122233}.\}$$

The m_{ij} matrix is given by

$$m^{\mathrm{I}}_{ij} = \begin{bmatrix} 1 & -r & 0 \\ -r & 2r & -3r \\ 0 & -3r & 6r \end{bmatrix}.$$

- Since there is just one bosonic side it is obvious that all the bosons satisfy are m-Cartan.
- All the fermions, except for α_{1223}, are isotropic and have $f = \pm 1$. Thanks to lemma 4.5.12 we then know that they are m-truncation.
- The fermion α_{1223} is m-Cartan without further assumptions.
- There are no couples of strong orthogonal fermions.

To conclude the m_{ij} matrix is a realising solution $\forall r$.
This construction realise the Nichols algebra $\mathcal{B}(q)$ described row 7 of table 2 in [Hec05] when $q \neq \pm 1,\ q \notin \mathcal{R}_3$.

For this lower rank case we can also show explicitly all the reflections of the m_{ij} matrix: reflecting m^{I}_{ij} around α_1 we find the following

$$m^{\mathrm{II}}_{ij} = \begin{bmatrix} 1 & -1+r & 0 \\ -1+r & 1 & -3r \\ 0 & -3r & 6r \end{bmatrix}.$$

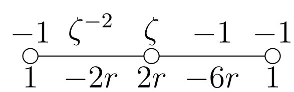
Reflecting it around α_{12} we find the following

$$m^{\mathrm{III}}_{ij} = \begin{bmatrix} 2r & -r & -2r \\ -r & 1 & -1+3r \\ -2r & -1+3r & 1 \end{bmatrix}.$$

Reflecting it around α_{123} we find the following

$$m^{\mathrm{IV}}_{ij} = \begin{bmatrix} 6r & -3r & 0 \\ -3r & 1 & -1+2r \\ 0 & -1+2r & 1-2r. \end{bmatrix}$$

Remark 4.5.36. If $q^2 \in \mathcal{R}_6$, α_3 is both q-Cartan and q-truncation. When it is m-truncation we find

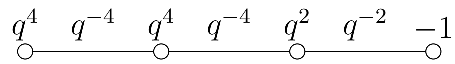


with $\zeta \in \mathcal{R}_6$. This is a solution iff $r = \frac{1}{6}$. But for this value of r, α_3 is also m-Cartan and thus this is not a new solution.

Remark 4.5.37. The roots can be expressed by

$$\alpha_1 = \alpha_f = \delta + \epsilon_1, \qquad \alpha_2 = \epsilon_2 \qquad \alpha_3 = \epsilon_3 - \epsilon_2$$

F(4)



The simple roots in the standard chamber are $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4 = \alpha_f\}$ with inner product

$$(\alpha_i, \alpha_j) = \begin{bmatrix} 4 & -2 & & \\ -2 & 4 & -2 & \\ & -2 & 2 & -1 \\ & & -1 & 0 \end{bmatrix}.$$

There is only one bosonic part $\mathfrak{g}'$ and the rest of the positive roots are:

$$\{\alpha_{12},\ \alpha_{23},\ \alpha_{34},\ \alpha_{233},\ \alpha_{123},\ \alpha_{234},\ \alpha_{1233},\ \alpha_{2334},$$
$$\alpha_{1234},\ \alpha_{12233},\ \alpha_{12334},\ \alpha_{1223334},\ \alpha_{122334},\ \alpha_{12233344}.\}$$

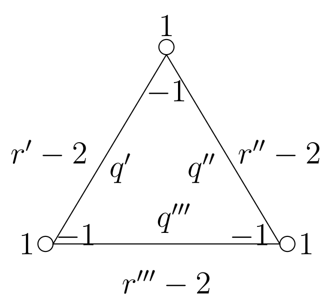
- All bosons except $\alpha_{12233344}$ are completely in the bosonic sector and thus are m-Cartan.
- The boson $\alpha_{12233344}$ is m-Cartan without further assumptions.
- All fermions are isotropic and have $f = \pm 1$. Thanks to lemma 4.5.12 we then know they are m-truncation.
- We have two couples of strong orthogonal fermions:

  $$\{\alpha_{34}, \alpha_{122334}\} \qquad \{\alpha_{234}, \alpha_{12334}\}$$

  which give the condition $r = \frac{1}{3}$.

To conclude, the condition for the m_{ij} matrix to be a realising solution is $r = \frac{1}{3}$.

D(2,1; α)



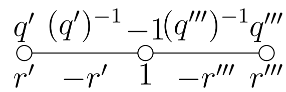
The simple roots in the standard chamber are $\{\alpha_1, \alpha_2 = \alpha_f, \alpha_3\}$ with inner product:

$$(\alpha_i, \alpha_j) = \begin{bmatrix} 2 & -2 & 0 \\ -2 & 0 & -2 \\ 0 & -2 & 2 \end{bmatrix}$$

The positive roots are:

$$\{\alpha_1,\ \alpha_2,\ \alpha_3,\ \alpha_{12},\ \alpha_{23},\ \alpha_{123},\ \alpha_{1223}.\}$$

Reflecting the diagram around one of the root (the system is completely symmetric in the three roots), we obtain:



Exception (4) of lemma 4.5.13 appears. Imposing that the first and the third roots are not connected we find the condition $r' + r'' + r''' = 2$. In this case these m_{ij} matrices are realising solution.
This corresponds to the condition $q' \cdot q'' \cdot q''' = 1$ of case 9 (as well as 10 and 11), rank 3, in table 2 of [Hec05].

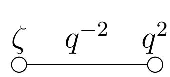
4.6 Rank 2

4.6.1 Other cases in rank 2: construction

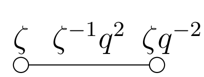
In this section we are going to present the examples of $rank = 2$ Nichols algebra which don't belong to the Cartan and super Lie study of the previous two sections.

Heckenberger row 6

This case of table 1 in [Hec05] is described by two diagrams:



I



II

where $\zeta \in \mathcal{R}_3$ and $q^2 \neq 1, \zeta, \zeta^2$ and with respectively simple roots:

$$\mathrm{I} : \{\alpha_1, \alpha_2\} \qquad \mathrm{II} : \{-\alpha_1, \alpha_{112}\}.$$

There is just one associate Cartan matrix:

$$a_{ij} = \begin{bmatrix} 2 & -2 \\ -1 & 2 \end{bmatrix}.$$

The set of positive roots is $\{\alpha_1, \alpha_2, \alpha_{12}, \alpha_{112}\}$ where α_2 and α_{112} are only q-Cartan while the others are only q-truncation.

Proposition 4.6.1. *The following m_{ij} matrices are realising solutions of the given braiding and its reflections:*

$$m^{I}_{ij} = \begin{bmatrix} \frac{2}{3} & -r \\ -r & 2r \end{bmatrix} \qquad m^{II}_{ij} = \begin{bmatrix} \frac{2}{3} & -\frac{4}{3}+r \\ -\frac{4}{3}+r & \frac{8}{3}-2r \end{bmatrix}.$$

Proof. First we check that condition (4.5)B is satisfied for α_1:

$$m_{11} = \frac{2}{1 - a_{12}} = \frac{2}{3}$$

and condition (4.5)A is satisfied for α_{22} and α_{112}:

$$m_{22,22} = \frac{2m_{12}}{a_{21}} = 2r$$
$$m_{112,112} = \frac{2m_{112,-1}}{a_{112,1}} = \frac{8}{3} - 2r.$$

We then check that the reflection around α_1 send one m_{ij}-matrix to the other as follows:

$$m^{\mathrm{II}}_{ij} = \begin{bmatrix} \frac{2}{3} & -\frac{4}{3}+r \\ -\frac{4}{3}+r & \frac{8}{3}-2r \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix}^T \begin{bmatrix} \frac{2}{3} & -r \\ -r & 2r \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix} = \mathcal{R}^1(m^{\mathrm{I}}_{ij})$$

□

Remark 4.6.2. When $q^2 \in \mathcal{R}_2$, the root α_2 is q-Cartan *and* q-truncation. When it is m-truncation we get:

$$m^{\text{I}}_{ij} = \begin{bmatrix} \frac{2}{3} & -r \\ \\ -r & 1 \end{bmatrix} \qquad m^{\text{II}}_{ij} = \begin{bmatrix} \frac{2}{3} & -\frac{4}{3}+r \\ \\ -\frac{4}{3}+r & \frac{11}{3}-4r \end{bmatrix} \qquad m^{\text{III}}_{ij} = \begin{bmatrix} \frac{5}{3}-2r & r-1 \\ \\ r-1 & 1 \end{bmatrix}.$$

with III: $\{\alpha_{12}, -\alpha_2\}$.
The root α_{112} is never m-truncation and it is m-Cartan iff $r = \frac{1}{2}$. But for this value of r, α_2 is also m-Cartan and thus this is *not* a new solution.

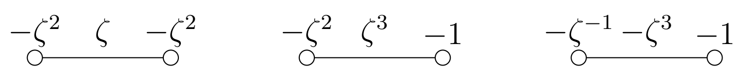
As we can see in [Hel10] truncation and Serre relations are the only defining relations. We have the following:

Proposition 4.6.3. *The truncation relations hold for every* $r \geq 0$*, while the Serre relations hold for* $2r \notin -\mathbb{N}$ *and* $r \neq \frac{1+3k}{3}, \frac{2+3k}{3}$.

Remark 4.6.4. We could call this case of *colour* type. It indeed behaves as a super Lie case except for the fact that $m_{ff} = \frac{2}{3}$, and not 1. In particular lemma 4.5.15 trivially extends to this case as a classification lemma, with the appropriate changes.

Heckenberger row 9

This case of table 1 in [Hec05] is described by three diagrams:



I II III

where $\zeta \in \mathcal{R}_{12}$ and with respectively simple roots:

$$\text{I} : \{\alpha_1, \alpha_2\} \qquad \text{II} : \{-\alpha_1, \alpha_{112}\} \qquad \text{III} : \{\alpha_{12}, -\alpha_{122}\}.$$

The associate Cartan matrices are:

$$a^{\text{I}}_{ij} = \begin{bmatrix} 2 & -2 \\ \\ -2 & 2 \end{bmatrix} \qquad a^{\text{II}}_{ij} = \begin{bmatrix} 2 & -2 \\ \\ -1 & 2 \end{bmatrix} \qquad a^{\text{III}}_{ij} = \begin{bmatrix} 2 & -3 \\ \\ -1 & 2 \end{bmatrix}.$$

The set of positive roots is $\{\alpha_1, \alpha_2, \alpha_{12}, \alpha_{112}, \alpha_{122}\}$ where α_{12} is only q-Cartan while the others are only q-truncation.

Proposition 4.6.5. *The following m_{ij} matrices are realising solutions of the given braiding and its reflections:*

$$m^I_{ij} = \begin{bmatrix} \frac{2}{3} & -\frac{7}{12} \\ -\frac{7}{12} & \frac{2}{3} \end{bmatrix} \qquad m^{II}_{ij} = \begin{bmatrix} \frac{2}{3} & -\frac{3}{4} \\ -\frac{3}{4} & 1 \end{bmatrix} \qquad m^{III}_{ij} = \begin{bmatrix} \frac{1}{6} & -\frac{1}{4} \\ -\frac{1}{4} & 1 \end{bmatrix}$$

Proof. First we check that condition (4.5)B is satisfied for all the roots:

$$\begin{aligned} m_{11} &= \frac{2}{1-a_{12}} = \frac{2}{3} \\ m_{22} &= \frac{2}{1-a_{21}} = \frac{2}{3} \\ m_{112,112} &= \frac{2}{1-a_{112,1}} = 1 \\ m_{122,122} &= \frac{2}{1-a_{122,12}} = 1 \end{aligned}$$

and condition (4.5)A is satisfied for the root α_{12}:

$$m_{12,12} = \frac{2m_{-122,12}}{a_{12,112}} = \frac{1}{6}.$$

We then check that the reflections send one m_{ij}-matrix to the other as follows:

$$m^{\text{II}}_{ij} = \begin{bmatrix} \frac{2}{3} & -\frac{3}{4} \\ -\frac{3}{4} & 1 \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix}^T \begin{bmatrix} \frac{2}{3} & -\frac{7}{12} \\ -\frac{7}{12} & \frac{2}{3} \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix} = \mathcal{R}^1(m^{\text{I}}_{ij})$$

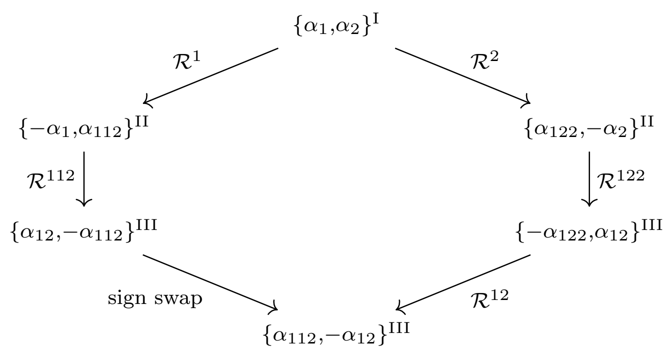
$$m^{\text{III}}_{ij} = \begin{bmatrix} \frac{1}{6} & -\frac{1}{4} \\ -\frac{1}{4} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}^T \begin{bmatrix} \frac{2}{3} & -\frac{3}{4} \\ -\frac{3}{4} & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} = \mathcal{R}^{122} \circ \mathcal{R}^2(m^{\text{I}}_{ij})$$

□

Corollary 4.6.6. *By formula (4.4) for rank 2, we have that the central charge of the system is $c = -126$.*

Proposition 4.6.7. *Truncation and Serre relations always hold, by lemma 4.2.3.*

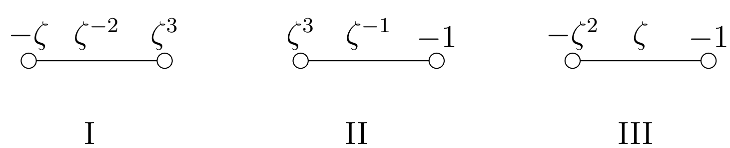
We conclude this case with a picture illustrating how the set of simple roots behave under reflections. We write I, II, III, to indicate to which diagram do the simple roots in each case belong.



(4.9)

Heckenberger row 10

This case of table 1 in [Hec05] is described by three diagrams:



where $\zeta \in \mathcal{R}_9$ and with respectively simple roots:

$$\mathrm{I} : \{\alpha_1, \alpha_2\} \qquad \mathrm{II} : \{-\alpha_2, \alpha_{122}\} \qquad \mathrm{III} : \{\alpha_{12}, -\alpha_{122}\}.$$

The associate Cartan matrices are:

$$a^{\mathrm{I}}_{ij} = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} \qquad a^{\mathrm{II}}_{ij} = \begin{bmatrix} 2 & -2 \\ -1 & 2 \end{bmatrix} \qquad a^{\mathrm{III}}_{ij} = \begin{bmatrix} 2 & -4 \\ -1 & 2 \end{bmatrix}.$$

The set of positive roots is $\{\alpha_1, \alpha_2, \alpha_{12}, \alpha_{112}, \alpha_{122}, \alpha_{11122}\}$ where α_1 and α_{12} are only q-Cartan while the others are only q-truncation.

Proposition 4.6.8. *The following m_{ij} matrices are realising solutions of the given braiding and its reflections:*

$$m^{I}_{ij} = \begin{bmatrix} \frac{5}{9} & -\frac{5}{9} \\ -\frac{5}{9} & \frac{2}{3} \end{bmatrix} \qquad m^{II}_{ij} = \begin{bmatrix} \frac{2}{3} & -\frac{7}{9} \\ -\frac{7}{9} & 1 \end{bmatrix} \qquad m^{III}_{ij} = \begin{bmatrix} \frac{1}{9} & -\frac{2}{9} \\ -\frac{2}{9} & 1 \end{bmatrix}$$

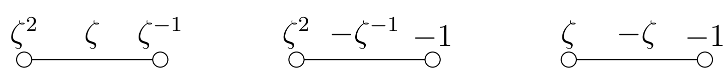
Proof. We check that the roots $\{\alpha_2, \alpha_{112}, \alpha_{122}, \alpha_{11122}\}$ satisfy condition (4.5)B, while the root α_1 and α_{12} satisfy condition (4.5)A.
We check that the reflections send one m_{ij}-matrix to the other. □

Corollary 4.6.9. *By formula (4.4) for rank 2, we have that the central charge of the system is $-\frac{1088}{5}$.*

Proposition 4.6.10. *Truncation and Serre relations always hold, by lemma 4.2.3.*

Heckenberger row 12

This case of table 1 in [Hec05] is described by three diagrams:



I II III

where $\zeta \in \mathcal{R}_8$ and with respectively simple roots:

$$\text{I} : \{\alpha_1, \alpha_2\} \qquad \text{II} : \{-\alpha_1, \alpha_{1112}\} \qquad \text{III} : \{\alpha_{112}, -\alpha_{1112}\}.$$

There is just one associate Cartan matrix:

$$a_{ij} = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}.$$

The set of positive roots is $\{\alpha_1, \alpha_2, \alpha_{12}, \alpha_{112}, \alpha_{1112}, \alpha_{11122}\}$ where α_2 and α_{112} are only q-Cartan while the others are only q-truncation.

Proposition 4.6.11. *The following m_{ij} matrices are realising solutions of the given braiding and its reflections:*

$$m^{I}_{ij} = \begin{bmatrix} \frac{1}{2} & -\frac{7}{8} \\ -\frac{7}{8} & \frac{7}{4} \end{bmatrix} \qquad m^{II}_{ij} = \begin{bmatrix} \frac{1}{2} & -\frac{5}{8} \\ -\frac{5}{8} & 1 \end{bmatrix} \qquad m^{III}_{ij} = \begin{bmatrix} \frac{1}{4} & -\frac{3}{8} \\ -\frac{3}{8} & 1 \end{bmatrix}$$

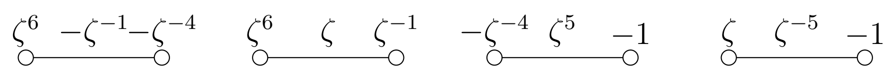
Proof. We check that the roots $\{\alpha_1, \alpha_{12}, \alpha_{1112}, \alpha_{11122}\}$ satisfy condition (4.5)B, while the root α_2 and α_{112} satisfy condition (4.5)A.
We check that the reflections send one m_{ij}-matrix to the other. □

Corollary 4.6.12. *By formula (4.4) for rank 2, we have that the central charge of the system is* $-\frac{874}{7}$.

Proposition 4.6.13. *Truncation and Serre relations always hold, by lemma 4.2.3.*

Heckenberger row 13

This case of table 1 in [Hec05] is described by four diagrams:



I II III IV

where $\zeta \in \mathcal{R}_{24}$ and with respectively simple roots:

$\mathrm{I} : \{\alpha_1, \alpha_2\}$ $\mathrm{II} : \{-\alpha_1, \alpha_{1112}\}$ $\mathrm{III} : \{-\alpha_2, \alpha_{122}\}$ $\mathrm{IV} : \{\alpha_{12}, -\alpha_{122}\}$..

The associate Cartan matrices are:

$$a^{\mathrm{I}}_{ij} = \begin{bmatrix} 2 & -3 \\ -2 & 2 \end{bmatrix} \quad a^{\mathrm{II}}_{ij} = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix} \quad a^{\mathrm{III}}_{ij} = \begin{bmatrix} 2 & -2 \\ -1 & 2 \end{bmatrix} \quad a^{\mathrm{IV}}_{ij} = \begin{bmatrix} 2 & -5 \\ -1 & 2 \end{bmatrix}.$$

The set of positive roots is $\{\alpha_1, \alpha_2, \alpha_{12}, \alpha_{112}, \alpha_{122}, \alpha_{1112}, \alpha_{11122}, \alpha_{1111222}\}$ where α_{12} and α_{1112} are the only q-Cartan roots while the others are only q-truncation.

Proposition 4.6.14. *The following m_{ij} matrices are realising solutions of the given braiding and its reflections:*

$$m^{I}_{ij} = \begin{bmatrix} \frac{1}{2} & -\frac{13}{24} \\ -\frac{13}{24} & \frac{2}{3} \end{bmatrix} \qquad m^{II}_{ij} = \begin{bmatrix} \frac{1}{2} & -\frac{23}{24} \\ -\frac{23}{24} & \frac{23}{12} \end{bmatrix}$$

$$m^{III}_{ij} = \begin{bmatrix} 1 & -\frac{19}{24} \\ -\frac{19}{24} & \frac{2}{3} \end{bmatrix} \qquad m^{IV}_{ij} = \begin{bmatrix} 1 & -\frac{5}{24} \\ -\frac{5}{24} & \frac{1}{12} \end{bmatrix}$$

Proof. We check that the roots α_{12} and α_{1112} satisfy condition (4.5)A, while the rest condition (4.5)B.
We check that the reflections send one m_{ij}-matrix to the other. □

Corollary 4.6.15. *By formula (4.4) for rank 2, we have that the central charge of the system is* $-\frac{7826}{23}$.

Proposition 4.6.16. *Truncation and Serre relations always hold, by lemma 4.2.3.*

Heckenberger row 14

This case of table 1 in [Hec05] is described by two diagrams:

I II

where $\zeta \in \mathcal{R}_5$ and with respectively simple roots:

$$\mathrm{I} : \{\alpha_1, \alpha_2\} \qquad \mathrm{II} : \{\alpha_{12}, -\alpha_2\}.$$

The associate Cartan matrices are:

$$a^{\mathrm{I}}_{ij} = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix} \qquad a^{\mathrm{II}}_{ij} = \begin{bmatrix} 2 & -4 \\ -1 & 2 \end{bmatrix}.$$

The set of positive roots is $\{\alpha_1, \alpha_2, \alpha_{12}, \alpha_{112}, \alpha_{1112}, \alpha_{1111222}, \alpha_{11122}, \alpha_{11111222}\}$ where $\alpha_1, \alpha_{12}, \alpha_{112}$ and α_{11122} are only q-Cartan while the others are only q-truncation.

Proposition 4.6.17. *The following* m_{ij} *matrices are realising solutions of the given braiding and its reflections:*

$$m^{I}_{ij} = \begin{bmatrix} \frac{2}{5} & -\frac{3}{5} \\ -\frac{3}{5} & 1 \end{bmatrix} \qquad m^{II}_{ij} = \begin{bmatrix} \frac{1}{5} & -\frac{2}{5} \\ -\frac{2}{5} & 1 \end{bmatrix}.$$

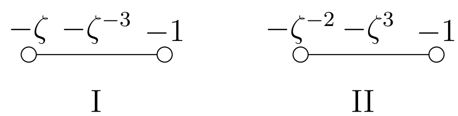
Proof. We check that the roots $\alpha_1, \alpha_{12}, \alpha_{112}$ and α_{11122} satisfy condition (4.5)A, while the others satisfy condition (4.5)B.
We check that the reflections send one m_{ij}-matrix to the other. □

Corollary 4.6.18. *By formula (4.4) for rank 2, we have that the central charge of the system is* -364*.*

Proposition 4.6.19. *Truncation and Serre relations always hold, by lemma 4.2.3.*

Heckenberger row 17

This case of table 1 in [Hec05] is described by two diagrams:



where $\zeta \in \mathcal{R}_7$ and with respectively simple roots:

$$\mathrm{I} : \{\alpha_1, \alpha_2\} \qquad \mathrm{II} : \{\alpha_{12}, -\alpha_2\}.$$

The associate Cartan matrices are:

$$a^{\mathrm{I}}_{ij} = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix} \qquad a^{\mathrm{II}}_{ij} = \begin{bmatrix} 2 & -5 \\ -1 & 2 \end{bmatrix}.$$

The set of positive roots is

$$\{\alpha_1, \alpha_2, \alpha_{12}, \alpha_{112}, \alpha_{1112}, \alpha_{11122}, \alpha_{1111222}, \alpha_{111112222}, \alpha_{111111122222},$$
$$\alpha_{11111222}, \alpha_{1111111122222}, \alpha_{11111112222}\}$$

where $\{\alpha_1, \alpha_{12}, \alpha_{112}, \alpha_{11122}, \alpha_{1111222}, \alpha_{11111222}\}$ are only q-Cartan while the others are only q-truncation.

Proposition 4.6.20. *The following* m_{ij} *matrices are realising solutions of the given braiding and its reflections:*

$$m^{I}_{ij} = \begin{bmatrix} \frac{6}{14} & -\frac{9}{14} \\ -\frac{9}{14} & 1 \end{bmatrix} \qquad m^{II}_{ij} = \begin{bmatrix} \frac{2}{14} & -\frac{5}{14} \\ -\frac{5}{14} & 1 \end{bmatrix}.$$

Proof. We check that the roots $\{\alpha_1, \alpha_{12}, \alpha_{112}, \alpha_{11122}, \alpha_{1111222}, \alpha_{11111222}\}$ satisfy condition (4.5)A, while the others satisfy condition (4.5)B.
We check that the reflections send one m_{ij}-matrix to the other. □

Corollary 4.6.21. *By formula (4.4) for rank 2, we have that the central charge of the system is* -962*.*

Proposition 4.6.22. *Truncation and Serre relations always hold, by lemma 4.2.3.*

4.6.2 Classification: rank 2

In this section we are going to prove the following

Theorem 4.6.23. *For all finite-dimensional diagonal Nichols algebras of rank* $= 2$*, all* m_{ij} *matrices which are realising solutions of the given braiding are the ones constructed in sections 4.4, 4.5 or 4.6.1.*

In order to prove it, we are going to go through table 1 in [Hec05], see which roots are q-truncation, q-Cartan and compute for every diagram the corresponding m_{ij}. We will see that for every case, the m_{ij} match with one of the constructed in the previous sections, and that there are no other possible solutions.

To prove this result we will need the following tools:

Proposition 4.6.24. *We consider a diagram*

where we assume that both $\{\alpha_i, \alpha_j\}$ *are q-truncation, and apply a reflection* $\mathcal{R}^i$ *around the root* α_i

$$\mathcal{R}^i : \qquad \begin{aligned} \alpha_i &\longmapsto -\alpha_i \\ \alpha_j &\longmapsto \alpha \end{aligned}$$

arriving to a new diagram with simple roots $\{-\alpha_i,\ \alpha := \alpha_j - a_{ij}\alpha_i\}$.
We have:

1. *if* β *is m-truncation then*

$$m_{ij} = \frac{a_{ij}}{1-a_{ij}} - \frac{1}{a_{ij}(1-a_{\beta,-i})} + \frac{1}{a_{ij}(1-a_{ji})} \tag{4.10}$$

2. *if* β *is m-Cartan then*

$$m_{ij} = \frac{a_{ij}}{1-a_{ij}} + \frac{\left(\frac{1}{1-a_{ji}} - \frac{a_{ij}}{(1-a_{ij})a_{\beta i}}\right)}{(-\frac{1}{a_{\beta i}} + a_{ij})} \tag{4.11}$$

Proof. Since $\{\alpha_i, \alpha_j\}$ are only q-truncation, thus m-truncation, we have the relations

$$m_{ii} = \frac{2}{1-a_{ij}} \qquad\qquad m_{jj} = \frac{2}{1-a_{ji}}.$$

1. If β is m-truncation then $m_{\beta\beta} = \frac{2}{1-a_{\beta,-i}}$. But for definition of β we have:

   $$m_{\beta\beta} = m_{jj} - 2a_{ij}m_{ij} + a_{ij}^2 m_{ii}.$$

   Gathering all the information together we get:

   $$\frac{2}{1-a_{\beta,-i}} = \frac{2}{1-a_{ji}} - 2a_{ij}m_{ij} + a_{ij}^2\frac{2}{1-a_{ij}}$$

   and from this the final result.

2. This case is completely analogous, with the only difference that β is m-Cartan and thus $m_{\beta\beta} = \frac{2m_{\beta,-i}}{a_{\beta i}}$ we will then have:

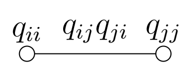
   $$m_{\beta\beta} = \frac{2m_{\beta,-i}}{a_{\beta i}} = -2\frac{m_{ij}}{a_{\beta i}} + \frac{2a_{ij}(\frac{2}{1-a_{ij}})}{a_{\beta i}}$$
   $$m_{\beta\beta} = \frac{2}{1-a_{ji}} - 2a_{ij}m_{ij} + a_{ij}^2\frac{2}{1-a_{ij}}.$$

   The two equations together give the thesis.

□

Analogously:

Proposition 4.6.25. *We consider a diagram*



where we assume that $\{\alpha_i, \alpha_j\}$ are the first q-Cartan and the latter q-truncation. We apply a reflection around the q-truncation root α_j,

$$\mathcal{R}^j : \qquad \alpha_j \longmapsto -\alpha_j$$
$$\alpha_i \longmapsto \beta$$

arriving to a new diagram associated to the roots: $\{\beta := \alpha_i - a_{ji}\alpha_j,\ -\alpha_j\}$. We have:

1. *if* β *is m-truncation then*

$$m_{ij} = \frac{a_{ij}}{1 - a_{ij}a_{ji}} \left(\frac{1}{1 - a_{\beta,-j}} - \frac{a_{ji}^2}{1 - a_{ji}} \right) \tag{4.12}$$

2. *if* β *is m-Cartan then*

$$m_{ij} = \frac{a_{ij}a_{ji}}{1 - a_{ji}} \cdot \frac{a_{ji}a_{\beta,-j} - 2}{a_{ji}a_{ij}a_{\beta,-j} - a_{\beta,-j} - a_{ij}} \tag{4.13}$$

Heckenberger row 2

We have $d = 1$ and then $\ell_1 = \ell_2 = \frac{\ell}{gdc(\ell,2)}$. Therefore $\ell \neq 2$ and since $a_{ij} = -1$ we have the following:
If $\ell > 4$ or $\ell = 3$ then by classification lemma 4.4.6 we get a unique solution, presented in section 4.4 Heckenberger row 2.
If $\ell = 4$ then $q_{ii} = q^2 = -1$ and the roots are both q-Cartan and q-truncation:

- If both are m-Cartan, we find a unique solution, by lemma 4.4.6 presented in section 4.4 Heckenberger row 2, in the limit case $q^2 = -1$.
- If one of the two is m-truncation, we find a unique solution, presented in section 4.5, Heckenberger row 3, in the limit case $q^2 = -1$. This result is a consequence of lemma 4.5.15.
- If both are only m-truncation we recognise the matrix $\begin{bmatrix} 1 & -\frac{p'}{2} \\ -\frac{p'}{2} & 1 \end{bmatrix}$ which is the other Weyl chamber in example 4.3.5.

Heckenberger row 3

We have $d = 1$ and then $\ell_1 = \ell_2 = \frac{\ell}{gdc(\ell,2)}$. Therefore $\ell \neq 2$ and since $a_{12} = -1$ we have the following:
If $\ell > 4$ or $\ell = 3$ then by classification lemma 4.5.15 we get a unique solution, presented in section 4.5 case Heckenberger row 3.
If $\ell = 4$, α_1 is both q-Cartan and q-truncation.

- If it is m-Cartan, we find again the unique solution presented in section 4.5 Heckenberger row 3, in the limit case $q^2 = -1$. This result is a consequence of lemma 4.5.15.

- If it is m-truncation we recognise again the matrix $\begin{bmatrix} 1 & -\frac{p'}{2} \\ -\frac{p'}{2} & 1 \end{bmatrix}$ which is the other Weyl chamber in example 4.3.5.

Heckenberger row 4

We have $d = d_2 = 2$ and then $\ell_1 = \frac{\ell}{gdc(\ell,2)}$, $\ell_2 = \frac{\ell}{gdc(\ell,4)}$. Moreover $\ell \neq 2, 4$, because $q^2 \neq \pm 1$, and since $a_{12} = -2, a_{21} = -1$ we have the following:
If $\ell > 8$ or $\ell = 5, 7$ then by classification lemma 4.4.6 we get a unique solution, presented in section 4.4 Heckenberger row 4.
If $\ell = 8$ then the long root α_2 is both q-Cartan and q-truncation, while α_1 is only q-Cartan.

- If α_2 is m-Cartan, we find again the unique solution presented in section 4.4, Heckenberger row 4, by lemma 4.4.6.
- If α_2 is m-truncation, we find the unique solution presented in section 4.5, Heckenberger row 5, in the limit case $q^2 = i$, by lemma 4.5.15.

If $\ell = 3, 6$ then the short root α_1 is both q-Cartan and q-truncation, while α_2 is only q-Cartan.

- If α_1 is m-Cartan, we find a unique solution, presented in section 4.4 Heckenberger row 4, again thanks to lemma 4.4.6.
- If α_1 is m-truncation, we find a family of solutions, presented in section 4.6.1, Heckenberger row 6, up to rescaling. The uniqueness follows from lemma 4.5.15, as observed in remark 4.6.4.

Heckenberger row 5

We have $d = 1$ and then $\ell_1 = \frac{\ell}{gdc(\ell,2)}$. Moreover $\ell \neq 2, 4$, because $q^2 \neq \pm 1$, and since $a_{12} = -2$ we have the following:
If $\ell > 6$ or $\ell = 5$ then by classification lemma 4.5.15 we get a unique solution, presented in section 4.5 Heckenberger row 5.
If $\ell = 3, 6$ then the bosonic root α_1 is both q-Cartan and q-truncation.

- If α_1 is m-Cartan, we find again the unique solution presented in section 4.5 Heckenberger row 5, by lemma 4.5.15.

- If α_1 is m-truncation, we recognise the matrix $\begin{bmatrix} \frac{2}{3} & -2r \\ -2r & 1 \end{bmatrix}$ of remark 4.5.26 which is a solution only for $r = \frac{1}{3}$.

Heckenberger row 6

We have $d = 1$ and then $\ell_2 = \frac{\ell}{gdc(\ell,2)}$. Moreover $\ell \neq 2, 3, 6$, because $q^2 \neq 1, \zeta, \zeta^2$, with $\zeta \in \mathcal{R}_3$. Since $a_{12} = -1$ we have the following:
If $\ell > 6$ or $\ell = 5$ then by classification lemma 4.5.15 we get a unique solution, presented in section 4.6.1 Heckenberger row 6 (see remark 4.6.4).
If $\ell = 4$ then the root α_2 is both q-Cartan and q-truncation.

- If α_2 is m-Cartan, we find again the unique solution presented in section 4.6.1 Heckenberger row 6, by lemma 4.5.15.
- If α_2 is m-truncation, we recognise the matrix $\begin{bmatrix} \frac{2}{3} & -r \\ -r & 1 \end{bmatrix}$ of remark 4.6.2 which is a solution only for $r = \frac{1}{2}$.

Heckenberger row 7

We apply formula (4.10) to the reflection $\mathcal{R}^1$ and $\mathcal{R}^2$, since the simple roots α_1 and α_2 as well as the ones after reflections are only q-truncation and thus m-truncation. From the first reflection we obtain $m_{12} = -\frac{2}{3}$, while from the latter $m_{12} = -\frac{1}{2}$. Since these results don't match, it means that there is no possible formulation of the Nichols Algebra braiding in terms of the m_{ij} matrix.

Remark 4.6.26. We have q-truncation roots α_i, α_j, with $q_{ii} = \zeta$, $q_{jj} = \zeta^{-1}$, both third roots of unity and it is not possible to realise both of them with $m_{ii} = m_{jj} = \frac{2}{3}$. This is another way to see that this case is not realisable.

Heckenberger row 8

We apply formula (4.10) to the reflections $\mathcal{R}^1$ and $\mathcal{R}^2$, since the simple roots α_1 and α_2 as well as the ones after reflections are only q-truncation and thus m-truncation. From the first reflection we obtain $m_{12} = -\frac{3}{4}$, while from the latter $m_{12} = -\frac{7}{12}$. Since these results don't match, it means that there is no possible formulation of the Nichols Algebra braiding in terms of the m_{ij} matrix.

Heckenberger row 9

We apply formula (4.10) to the reflection $\mathcal{R}^1$ or $\mathcal{R}^2$, since the simple roots α_1 and α_2 as well as the ones after reflections are only q-truncation and thus m-truncation. The resulting m_{12} shows that this is the m_{ij} appearing in section 4.6.1. This is thus the only possible solution.

Heckenberger row 10

We apply formula (4.12) to the reflection $\mathcal{R}^2$, since the simple root α_1 is only q-Cartan and thus m-Cartan, while α_2 as well as the ones after reflections are only q-truncation and thus m-truncation. The resulting m_{12} shows that this is the m_{ij} appearing in section 4.6.1. This is thus the only possible solution.

Heckenberger row 11

We have $d = d_2 = 3$ and then $\ell_1 = \frac{\ell}{gdc(\ell,2)}$, $\ell_2 = \frac{\ell}{gdc(\ell,6)}$. Moreover $\ell \neq 2, 3, 4, 6$ because $q^2 \neq \pm 1$, $q^2 \notin \mathcal{R}_3$. Since $a_{12} = -3$ and $a_{21} = -1$ we have the following:
If $\ell > 12$ or $\ell = 5, 7, 9, 10, 11$ then by classification lemma 4.4.6 we get a unique solution, presented in section 4.4 Heckenberger row 11.
If $\ell = 12$ then the root α_2 is both q-Cartan and q-truncation, while the root α_1 is only q-Cartan.

- If α_2 is m-Cartan, we find again the unique solution presented in section 4.4 Heckenberger row 11, by lemma 4.4.6.
- If α_2 is m-truncation, we recognise the matrix $\begin{bmatrix} 2r & -3r \\ -3r & 1 \end{bmatrix}$ of remark 4.4.26 which is a solution only for $r = \frac{1}{6}$.

If $\ell = 8$ then the root α_1 is both q-Cartan and q-truncation, while the root α_2 is only q-Cartan.

- If α_1 is m-Cartan, we find again the unique solution presented in section 4.4 Heckenberger row 11, by lemma 4.4.6.

- If α_1 is m-truncation, we recognise the matrix $\begin{bmatrix} \frac{1}{2} & -3r \\ -3r & 6r \end{bmatrix}$ of remark 4.4.27 which is a solution only for $r = \frac{1}{4}$.

Heckenberger row 12

We apply formula (4.12) to the reflections $\mathcal{R}^1$, since the simple roots α_1 as well as the ones after reflections are only q-truncation and thus m-truncation, while α_2 is only q-Cartan, and thus m-Cartan. The result is $m_{12} = -\frac{7}{8}$, which matches with the one of section 4.6.1.

Heckenberger row 13

We apply formula (4.10) to the reflection $\mathcal{R}^1$ or $\mathcal{R}^2$, since the simple roots α_1 and α_2 as well as the ones after reflections are only q-truncation and thus m-truncation. The resulting m_{12} shows that this is the m_{ij} appearing in section 4.6.1. This is thus the only possible solution.

Heckenberger row 14

We apply formula (4.13) to the reflections $\mathcal{R}^2$, since the simple roots α_1 as well as the ones after reflections are only q-Cartan and thus m-Cartan, while α_2 is only q-truncation, and thus m-truncation. The result is $m_{12} = -\frac{3}{5}$, which matches with the one of section 4.6.1.

Heckenberger row 15

We apply formula (4.10) to the reflections $\mathcal{R}^1$ and (4.11) to $\mathcal{R}^2$ since the simple roots α_1 and α_2 as well as the ones after $\mathcal{R}^1$ are only q-truncation and thus m-truncation, while the ones after $\mathcal{R}^2$ are only q-Cartan, and thus m-Cartan. From the first reflection we obtain $m_{12} = -\frac{4}{5}$, while from the latter $m_{12} = -\frac{11}{20}$. Since these results don't match, it means that there is no possible formulation of the Nichols Algebra braiding in terms of the m_{ij} matrix.

Heckenberger row 16

The root α_1 is q-Cartan so we can't start with the system of simple roots α_1, α_2 if we want to compare the results of the reflections around them. We then start with the simple roots α_{122} and $-\alpha_2$ which are only q-truncation and thus m-truncation. After reflection $\mathcal{R}^{122}$ we obtain a only q-Cartan, and thus m-Cartan, simple root. While after reflection $\mathcal{R}^2$ we obtain a only q-truncation, and thus m-truncation, simple root. We then apply (4.11) to $\mathcal{R}^{122}$ and (4.10) to $\mathcal{R}^2$ obtaining to different results. Hence there is no possible formulation of the Nichols Algebra braiding in terms of the m_{ij} matrix.

Heckenberger row 17

We apply formula (4.12) to the reflections $\mathcal{R}^2$, since the simple roots α_2 as well as the ones after reflections are only q-truncation and thus m-truncation, while α_1 is only q-Cartan, and thus m-Cartan. The result is $m_{12} = -\frac{5}{14}$, which matches with the one of section 4.6.1.

4.7 Rank 3

We now rise the rank by one and construct all m_{ij}-matrices which realise finite-dimensional diagonal Nichols algebras of rank 3, listed in table 2 of [Hec05].

For Cartan type we will refer to the study of section 4.4. For super Lie type we will explicitly compute the realising solutions.
For the other cases, we will see that the m_{ij} matrices are completely fixed by the lower rank: this will imply uniqueness of the solution and make it not just a construction result but also a classification one.

In particular for these latter cases we will proceed as follows:

- Given a q-diagram in rank 3, we will consider it as two rank 2 q-diagrams joined in the middle node. We will then associate to both sides the m_{ij}-matrices realising them, found in the rank 2 study. For these m_{ij}-matrices to be compatible, some restriction on the parameter of which they depend will possibly appear.

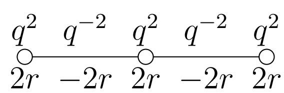
- We will then reflect the q-diagram on its q-truncation roots and proceed again as in the first point for the new diagram.
  We reflect until we arrive not just to an already found q-diagram, but also when the m_{ij} realisation is repeated (the m_{ij} matrix can be different also if associated to the same q-diagram).

- We will then have to make sure that all the conditions found on the parameters are compatible and acceptable, in order for the rank 3 m_{ij}-matrices to be realising solutions.

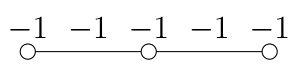
The q-diagrams and the associated realising solutions are listed in table 4.2 of the Appendix.

Heckenberger row 1

This case belongs to the Cartan section. In particular it corresponds to the Lie algebras A_3 and it is described by the following q-diagram with corresponding m_{ij} solution:

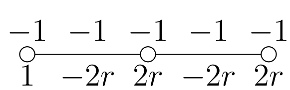


Remark 4.7.1. When $q^2 \in \mathcal{R}_2$ the roots are both q-Cartan and q-truncation and the q-diagram reads



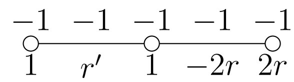
We have the following extra solutions:

– When α_1 is m-truncation and α_2, α_3 are m-Cartan we find



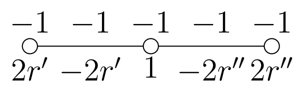
which is one chamber of the Lie superalgebra $A(2,0)$ described in Heckenberger row 4.

– When α_1, α_2 are m-truncation and α_3 is m-Cartan we find



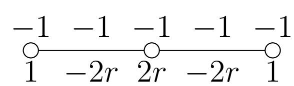
which is a m-solution just for $r = \frac{1}{2}$ and $r' = -1$. But for these values of r, r' the roots α_1, α_2 are also m-Cartan and thus this is not a new solution.

– When α_2 is m-truncation and α_1, α_3 are m-Cartan we find



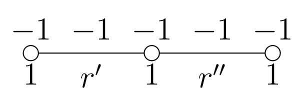
This is a solution either for $r' = \frac{1}{2}$ for which we end up again in the previous point, or for $r' = 1 - r''$, which gives us one chamber of the Lie superalgebra $A(1,1)$ described in Heckenberger row 8.

– When α_1, α_3 are m-truncation and α_2 is m-Cartan we find



which is another chamber of the Lie superalgebra $A(1,1)$ described in Heckenberger row 8.

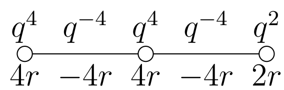
– When the roots are all m-truncation we find



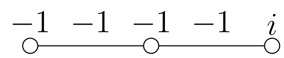
This is a solution either for $r' = -r'' - 2$ which is again a chamber of the Lie superalgebra $A(1,1)$, or for $r' = r'' = -1$ for which the roots are also m-Cartan and thus does not give a new solution.

Heckenberger row 2

This case belongs to the Cartan section. In particular it corresponds to the Lie algebras B_3 and it is described by the following q-diagram with corresponding m_{ij} solution:

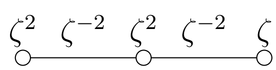


Remark 4.7.2. When $q^2 \in \mathcal{R}_4$ the roots α_1, α_2 are both q-Cartan and q-truncation and the q-diagram reads



For all the possible combinations of m-truncation and m-Cartan roots, no new solution is found. In some cases we find the Lie superalgebra $B(2,1)$ described in Heckenberger row 5.

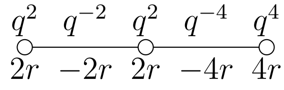
Remark 4.7.3. When $q^2 \in \mathcal{R}_3$ the root α_3 is both q-Cartan and q-truncation and the q-diagram reads



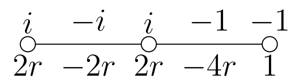
with $\zeta \in \mathcal{R}_3$. The case when it is m-truncation is a solution only for $r = \frac{1}{3}$ for which the root is also m-Cartan and thus does not give a new solution.

Heckenberger row 3

This case belongs to the Cartan section. In particular it corresponds to the Lie algebras C_3 and it is described by the following q-diagram with corresponding m_{ij} solution:



Remark 4.7.4. If $q^2 \in \mathcal{R}_4$, α_3 is both q-Cartan and q-truncation and the q-diagram reads



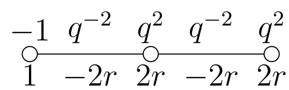
The case when it is m-truncation is a solution iff $r = \frac{1}{4}$ for which it is actually also m-Cartan. So this is not a new solution.

Heckenberger row 4

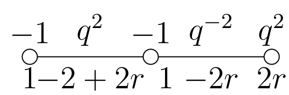
Row 4 of table 2 in [Hec05] corresponds to the Lie superalgebra $A(2,0)$. The simple roots in the standard chamber are $\{\alpha_1 = \alpha_f, \alpha_2, \alpha_3\}$. We then have just a bosonic part $\mathfrak{g}'$. The inner products is given by:

$$(\alpha_i, \alpha_j) = \begin{bmatrix} 0 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

and therefore:



Reflecting around α_1 we find the following



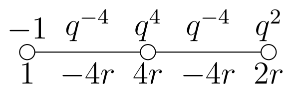
Reflecting around the second root we find a symmetric result.
The roots satisfy condition (4.5) $\forall r$ and therefore this m_{ij} is a realising solution.

Heckenberger row 5

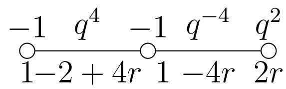
Row 5 of table 2 in [Hec05] corresponds to the Lie superalgebra $B(2,1)$. The simple roots in the standard chamber are $\{\alpha_1 = \alpha_f, \alpha_2, \alpha_3\}$. We then have just a bosonic part $\mathfrak{g}'$. The inner products is given by:

$$(\alpha_i, \alpha_j) = \begin{bmatrix} 0 & -2 & 0 \\ -2 & 4 & -2 \\ 0 & -2 & 2 \end{bmatrix}$$

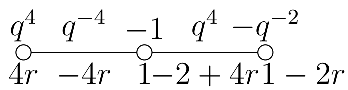
and therefore:



Reflecting around α_1 we find the following



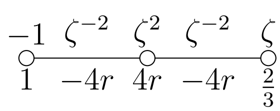
and after another reflection around the second root we find the following



The roots satisfy condition (4.5) $\forall r$ and therefore this m_{ij} is a realising solution.

Remark 4.7.5. If $q^2 \in \mathcal{R}_4$ then the root α_2 is both q-Cartan and q-truncation. This case has been already studied in details in Heckenberger row 2 remark 4.7.2.

Remark 4.7.6. If $q^2 \in \mathcal{R}_3$ then the root α_3 is both q-Cartan and q-truncation. When it is m-truncation we get:



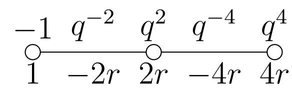
This is a solution iff $r = \frac{1}{3}$. But for this value of r, α_3 is also m-Cartan and thus this is not a new solution.

Heckenberger row 6

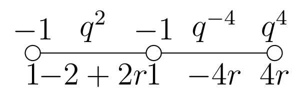
Row 6 of table 2 in [Hec05] corresponds to the Lie superalgebra $C(3)$.
The simple roots in the standard chamber are $\{\alpha_1 = \alpha_f, \alpha_2, \alpha_3\}$. We then have just a bosonic part $\mathfrak{g}'$. The inner products is given by:

$$(\alpha_i, \alpha_j) = -\begin{bmatrix} 0 & -1 & 0 \\ -1 & 2 & -2 \\ 0 & -2 & 4 \end{bmatrix}$$

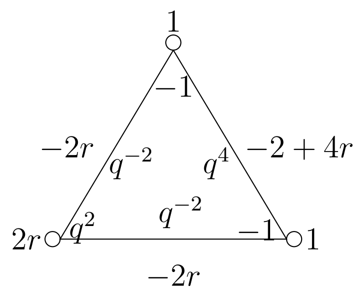
and therefore:



Reflecting around α_1 we find the following

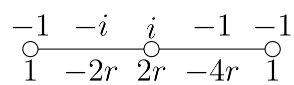


Reflecting around α_{12} we find the following



The roots satisfy condition (4.5) $\forall r$ and therefore this m_{ij} is a realising solution.

Remark 4.7.7. If $q^2 \in \mathcal{R}_4$, α_3 is both q-Cartan and q-truncation. When it is m-truncation we find



This is a solution iff $r = \frac{1}{4}$. But for this value of r, α_3 is also m-Cartan and thus this is not a new solution.

Remark 4.7.8. The simple roots in the standard chamber can be expressed according to [Kac77] by

$$\alpha_1 = \alpha_f = \epsilon_1 - \delta_1, \qquad \alpha_2 = \delta_1 - \delta_2 \qquad \alpha_3 = 2\delta_2.$$

Heckenberger row 7

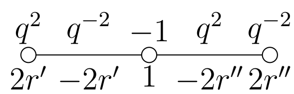
Row 7 of table 2 in [Hec05] corresponds to the Lie superalgebra $G(3)$ and it has been already explicitly treated as sporadic case of super Lie type in section 4.5.6.

Heckenberger row 8

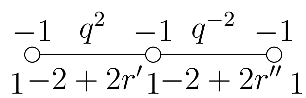
Row 8 of table 2 in [Hec05] corresponds to the Lie superalgebra $A(1,1)$. The simple roots in the standard chamber are $\{\alpha_1,\ \alpha_2 = \alpha_f,\ \alpha_3\}$. We then have two bosonic parts $\mathfrak{g}'$ and $\mathfrak{g}''$. The inner products is given by:

$$(\alpha_i, \alpha_j) = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 0 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

and therefore:



Reflecting around α_2 we find the following



Other reflections give different m_{ij} matrices as shown in table 4.2. However, exception (4) of lemma 4.5.13, already appears. Indeed to the latter diagram is associated the following:

$$m_{ij}^C = \begin{bmatrix} 1 & -1+r' & -1+r'+r'' \\ -1+r' & 1 & -1+r'' \\ -1+r'+r'' & -1+r'' & 1 \end{bmatrix}.$$

We then have to ask $m_{13}^C = 0$, i.e. $r' + r'' = 1$. In this case these m_{ij} matrices are realising solution.

Remark 4.7.9. The simple roots in the standard chamber can be expressed according to [Kac77] by

$$\alpha_1 = \epsilon_1 - \epsilon_2, \quad \alpha_2 = \alpha_f = \epsilon_2 - \delta_1, \quad \alpha_3 = \delta_1 - \delta_2,$$

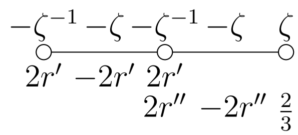
with vectors ϵ_i generating $\mathfrak{g}'$ and δ_i generating $\mathfrak{g}''$.

Heckenberger row 9-10-11

Rows 9,10,11 of table 2 in [Hec05] correspond to the Lie superalgebra $D(2,1;\alpha)$ and it has been already explicitly treated as sporadic case of super Lie type in section 4.5.6.

Heckenberger row 12

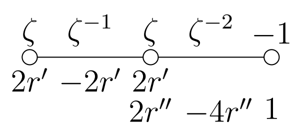
The first diagram is a composition of the diagrams of rank 2: #2 with $q = -\zeta^{-1}$ and #6 with $q = -\zeta^{-1}$, with $\zeta \in \mathcal{R}_3$.



For them to be joint in the middle circle we find $r' = r'' =: r$.
The only q-truncation root is the third. Reflecting on it we find the same diagram and as matching condition $2r = \frac{8}{3} - 2r$, i.e. $r = \frac{2}{3}$. But $q = e^{i\pi r} \in \mathcal{R}_6$.
So this case is not realisable.

Heckenberger row 13

This case has two sub cases: $\zeta \in \mathcal{R}_3$ and $\zeta \in \mathcal{R}_6$ and diagram:



1. Suppose $\zeta \in \mathcal{R}_3$. The first diagram is a composition of the diagrams of rank 2: #2 with $q = \zeta$ and #5 with $q = \zeta$. For them to be joint in the middle circle we find $r' = r'' =: r$.
   The only q-truncation root is the third. Reflecting on it we find a diagram composition of #4 with $q = -\zeta^{-1}$ and #5 with $q = \zeta$. As matching condition we find $r = -2r + 1$, i.e. $r = \frac{1}{3}$ which is an acceptable condition.
   This case is thus realisable by the unique solution with parameter $r = \frac{1}{3}$.

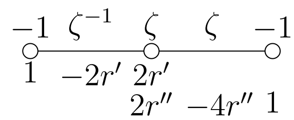
2. Suppose $\zeta \in \mathcal{R}_6$. We proceed analogously, but after reflecting around the third root we find a diagram which is composition of #6 with $q = \zeta$ and #5 with $q = \zeta$. The condition now is $r = \frac{1}{6}$ which is an acceptable condition.
   This case is thus realisable by the unique solution with parameter $r = \frac{1}{6}$.

Heckenberger row 14

This case is not realisable, since one of the diagrams contains diagram #7 of rank 2 which is on turn not realisable.

Heckenberger row 15

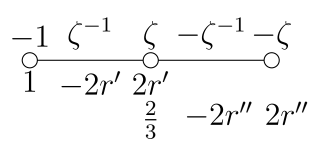
The first diagram is a composition of the diagrams of rank 2: #3 with $q = \zeta$ and #5 with $q = \zeta$, where $\zeta \in \mathcal{R}_3$.



For them to be joint in the middle circle we find $r' = r'' =: r$.
After the reflections around $\mathcal{R}^{12} \circ \mathcal{R}^1$ we find the condition $r = \frac{1}{3}$ which is acceptable and gives a unique realisable solution.

Heckenberger row 16

The first diagram is a composition of the diagrams of rank 2: #3 with $q = \zeta$ and #6 with $q = -\zeta$, where $\zeta \in \mathcal{R}_3$.



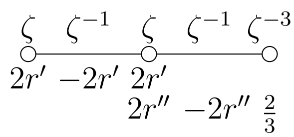
For them to be joint in the middle circle we find $r' = \frac{1}{3}$.
After reflecting on the second root we find the condition $r'' = \frac{5}{6}$.
This case is thus realisable by the unique solution with parameters $r' = \frac{1}{3}$ and $r'' = \frac{5}{6}$.

Heckenberger row 17

This case is not realisable, since one of the diagrams contains diagram #7 of rank 2 which is on turn not realisable.

Heckenberger row 18

The first diagram is a composition of the diagrams of rank 2: #2 with $q = \zeta$ and #6 with $q = \zeta$, with $\zeta \in \mathcal{R}_9$.



For them to be joint in the middle circle we find $r' = r'' =: r$.
The only q-truncation root is the third. Reflecting on it we find the same diagram and as matching condition $r = -\frac{8}{3} + 2r$, i.e. $r = \frac{8}{9}$.
This case is thus realisable by the unique solution with parameter $r = \frac{8}{9}$.

4.8 Rank ≥ 4

The construction of all m_{ij}-matrices, which realise finite-dimensional diagonal Nichols algebras of rank ≥ 4 can be obtained directly from rank 3. Namely, for a given q-diagram of [Hec06a] one has to combine in a coherent way the m_{ij} for some overlapping subdiagrams. It is indeed enough to know rank 3 because the effect of a reflection $\mathcal{R}^k$ on a pair of roots α_i, α_j and q_{ij}, m_{ij} only depends on the rank 3 subdiagram α_i, α_j, α_k.

4.9 Tables: realising lattices of Nichols algebras in rank 2 and 3

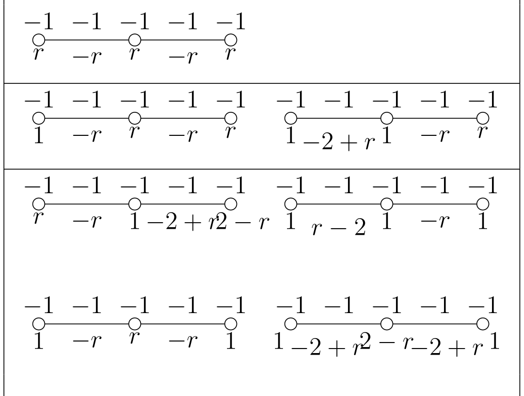
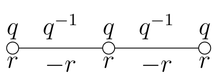
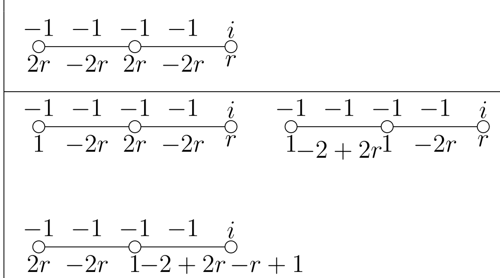
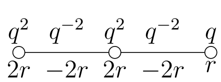
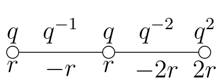
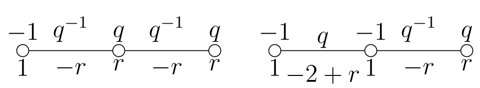
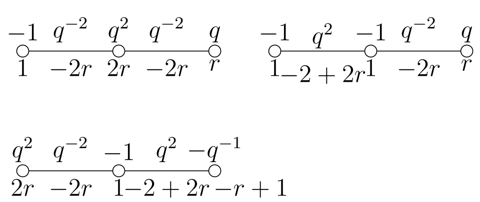
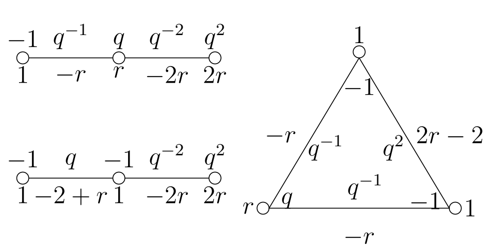
We now list from [Hec05] all finite-dimensional diagonal Nichols algebras in rank 2 and 3 in terms of their q-diagrams, and below each of them we display the corresponding realising lattice in terms of m_{ij}-diagrams, such that $q_{ij} = e^{i\pi m_{ij}}$ and the reflection compatibility 4.5 holds.
The numbers of the rows are Heckenberger's numbering, but sometimes we subdivide the cases, e.g. $2'$, $2''$. Note that we display the Nichols algebras associated to quantum groups as Heckenberger, in contrast to the notation used for quantum groups and used in section 4.4, 4.5, which means that there is an additional 2 factor in the q-exponent missing.

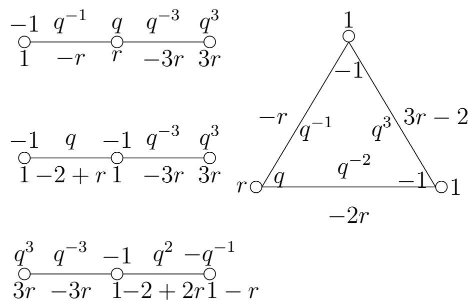
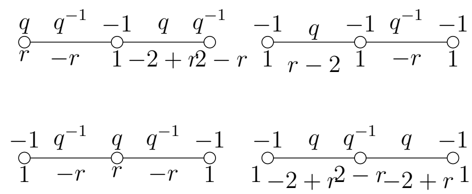
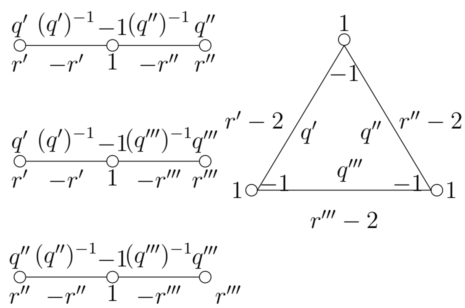
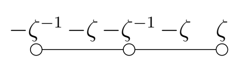
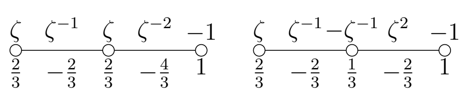
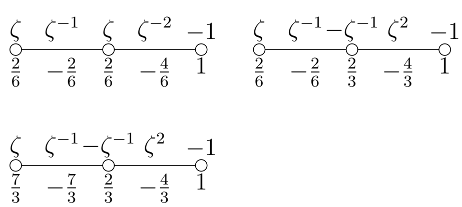
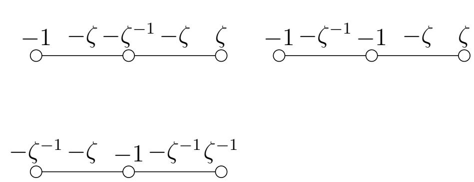
Table 4.1: Realisation of finite-dimensional diagonal Nichols algebras of rank 2.

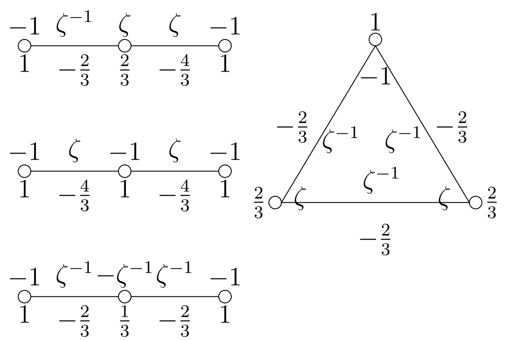
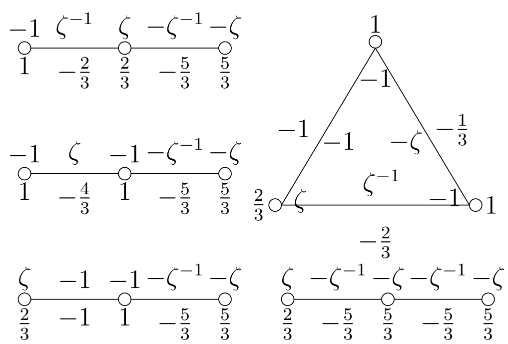
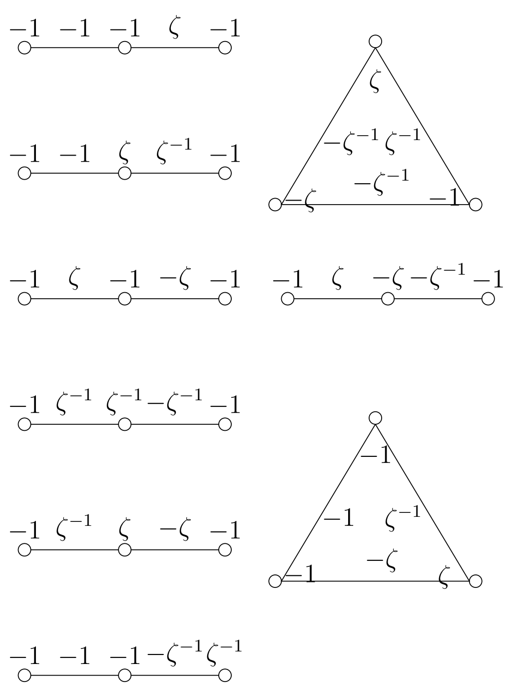
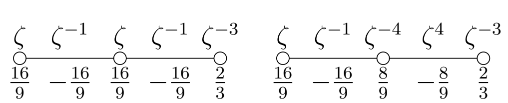
row	**Braiding**	**Conditions**	
$2'$	$\underset{r}{\overset{-1}{\circ}}\overset{-1}{\underset{-r}{\text{———}}}\underset{r}{\overset{-1}{\circ}}$ $\underset{r}{\overset{-1}{\circ}}\overset{-1}{\underset{-r}{\text{———}}}\underset{1}{\overset{-1}{\circ}}$ $\underset{1}{\overset{-1}{\circ}}\overset{-1}{\underset{-2+r}{\text{———}}}\underset{1}{\overset{-1}{\circ}}$		One solution according to A_2 (see $2''$). One solution according to $A(1,0)$ (see 3).
$2''$	$\underset{r}{\overset{q}{\circ}}\overset{q^{-1}}{\underset{-r}{\text{———}}}\underset{r}{\overset{q}{\circ}}$	$q \neq \pm 1$	Cartan, A_2
3	$\underset{r}{\overset{q}{\circ}}\overset{q^{-1}}{\underset{-r}{\text{———}}}\underset{1}{\overset{-1}{\circ}}$ $\underset{1}{\overset{-1}{\circ}}\overset{q}{\underset{-2+r}{\text{———}}}\underset{1}{\overset{-1}{\circ}}$	$q \neq \pm 1$	Super Lie, $A(1,0)$
$4'$	$\underset{r}{\overset{i}{\circ}}\overset{-1}{\underset{-2r}{\text{———}}}\underset{2r}{\overset{-1}{\circ}}$ $\underset{r}{\overset{i}{\circ}}\overset{-1}{\underset{-2r}{\text{———}}}\underset{1}{\overset{-1}{\circ}}$ $\underset{-r+1}{\overset{i}{\circ}}\overset{-1}{\underset{-2+2r}{\text{———}}}\underset{1}{\overset{-1}{\circ}}$	$i \in \mathcal{R}_4$	One solution according to B_2 (see $4'''$). One solution according to $B(1,1)$ (see 5).
$4''$	$\underset{r}{\overset{\zeta}{\circ}}\overset{\zeta}{\underset{-2r}{\text{———}}}\underset{2r}{\overset{\zeta^{-1}}{\circ}}$ $\underset{\frac{2}{3}}{\overset{\zeta}{\circ}}\overset{\zeta}{\underset{-2r}{\text{———}}}\underset{2r}{\overset{\zeta^{-1}}{\circ}}$ $\underset{\frac{2}{3}}{\overset{\zeta}{\circ}}\overset{\zeta}{\underset{-\frac{8}{3}+2r}{\text{———}}}\underset{\frac{8}{3}-2r}{\overset{\zeta^{-1}}{\circ}}$	$\zeta \in \mathcal{R}_3$	One solution according to B_2 (see $4'''$). One solution according to 6.
$4'''$	$\underset{r}{\overset{q}{\circ}}\overset{q^{-2}}{\underset{-2r}{\text{———}}}\underset{2r}{\overset{q^2}{\circ}}$	$q \neq \pm 1,\ q \notin \mathcal{R}_3, \mathcal{R}_4$	Cartan, B_2
5	$\underset{r}{\overset{q}{\circ}}\overset{q^{-2}}{\underset{-2r}{\text{———}}}\underset{1}{\overset{-1}{\circ}}$ $\underset{1-r}{\overset{-q^{-1}}{\circ}}\overset{q^2}{\underset{-2+2r}{\text{———}}}\underset{1}{\overset{-1}{\circ}}$	$q \neq \pm 1,\ q \notin \mathcal{R}_4$	Super Lie, $B(1,1)$
6	$\underset{\frac{2}{3}}{\overset{\zeta}{\circ}}\overset{q^{-1}}{\underset{-r}{\text{———}}}\underset{r}{\overset{q}{\circ}}$ $\underset{\frac{2}{3}}{\overset{\zeta}{\circ}}\overset{\zeta^{-1}q}{\underset{-\frac{8}{3}+r}{\text{———}}}\underset{\frac{8}{3}-r}{\overset{\zeta q^{-1}}{\circ}}$	$\zeta \notin \mathcal{R}_3,\ q \neq 1, \zeta, \zeta^2$	
7	$\overset{\zeta}{\circ}\overset{-\zeta}{\text{———}}\overset{-1}{\circ}$ $\overset{\zeta^{-1}}{\circ}\overset{-\zeta^{-1}}{\text{———}}\overset{-1}{\circ}$	$\zeta \in \mathcal{R}_3$	No solution
8	$\overset{-\zeta^{-2}}{\circ}\overset{-\zeta^{3}}{\text{———}}\overset{-\zeta^{2}}{\circ}$ $\overset{-\zeta^{-2}}{\circ}\overset{\zeta^{-1}}{\text{———}}\overset{-1}{\circ}$ $\overset{-\zeta^{2}}{\circ}\overset{-\zeta}{\text{———}}\overset{-1}{\circ}$ $\overset{-\zeta^{3}}{\circ}\overset{\zeta}{\text{———}}\overset{-1}{\circ}$ $\overset{-\zeta^{3}}{\circ}\overset{-\zeta^{-1}}{\text{———}}\overset{-1}{\circ}$	$\zeta \in \mathcal{R}_{12}$	No solution

9	$\underset{\frac{2}{3}}{\overset{-\zeta^2}{\circ}}\underset{-\frac{7}{6}}{\overset{\zeta}{\text{———}}}\underset{\frac{2}{3}}{\overset{-\zeta^2}{\circ}}$ $\underset{\frac{2}{3}}{\overset{-\zeta^2}{\circ}}\underset{-\frac{3}{2}}{\overset{\zeta^3}{\text{———}}}\underset{1}{\overset{-1}{\circ}}$ $\underset{\frac{1}{6}}{\overset{-\zeta^{-1}}{\circ}}\underset{-\frac{1}{2}}{\overset{-\zeta^3}{\text{———}}}\underset{1}{\overset{-1}{\circ}}$	$\zeta \in \mathcal{R}_{12}$	
10	$\underset{\frac{5}{9}}{\overset{-\zeta}{\circ}}\underset{-\frac{10}{9}}{\overset{\zeta^{-2}}{\text{———}}}\underset{\frac{2}{3}}{\overset{\zeta^3}{\circ}}$ $\underset{\frac{2}{3}}{\overset{\zeta^3}{\circ}}\underset{-\frac{14}{9}}{\overset{\zeta^{-1}}{\text{———}}}\underset{1}{\overset{-1}{\circ}}$ $\underset{\frac{1}{9}}{\overset{-\zeta^2}{\circ}}\underset{-\frac{4}{9}}{\overset{\zeta}{\text{———}}}\underset{1}{\overset{-1}{\circ}}$	$\zeta \in \mathcal{R}_9$	
11	$\underset{r}{\overset{q}{\circ}}\underset{-3r}{\overset{q^{-3}}{\text{———}}}\underset{3r}{\overset{q^3}{\circ}}$	$q \notin \mathcal{R}_3,\ q \neq \pm 1$	Cartan, G_2
12	$\underset{\frac{1}{2}}{\overset{\zeta^2}{\circ}}\underset{-\frac{7}{4}}{\overset{\zeta}{\text{———}}}\underset{\frac{7}{4}}{\overset{\zeta^{-1}}{\circ}}$ $\underset{\frac{1}{2}}{\overset{\zeta^2}{\circ}}\underset{-\frac{5}{4}}{\overset{-\zeta^{-1}}{\text{———}}}\underset{1}{\overset{-1}{\circ}}$ $\underset{\frac{1}{4}}{\overset{\zeta}{\circ}}\underset{-\frac{3}{4}}{\overset{-\zeta}{\text{———}}}\underset{1}{\overset{-1}{\circ}}$	$\zeta \in \mathcal{R}_8$	
13	$\underset{\frac{1}{2}}{\overset{\zeta^6}{\circ}}\underset{-\frac{13}{12}}{\overset{-\zeta^{-1}}{\text{———}}}\underset{\frac{2}{3}}{\overset{-\zeta^{-4}}{\circ}}$ $\underset{\frac{1}{2}}{\overset{\zeta^6}{\circ}}\underset{-\frac{23}{12}}{\overset{\zeta}{\text{———}}}\underset{\frac{23}{12}}{\overset{\zeta^{-1}}{\circ}}$ $\underset{\frac{2}{3}}{\overset{-\zeta^{-4}}{\circ}}\underset{-\frac{19}{12}}{\overset{\zeta^5}{\text{———}}}\underset{1}{\overset{-1}{\circ}}$ $\underset{\frac{1}{12}}{\overset{\zeta}{\circ}}\underset{-\frac{5}{12}}{\overset{\zeta^{-5}}{\text{———}}}\underset{1}{\overset{-1}{\circ}}$	$\zeta \in \mathcal{R}_{24}$	
14	$\underset{\frac{2}{5}}{\overset{\zeta}{\circ}}\underset{-\frac{6}{5}}{\overset{\zeta^2}{\text{———}}}\underset{1}{\overset{-1}{\circ}}$ $\underset{\frac{1}{5}}{\overset{-\zeta^{-2}}{\circ}}\underset{-\frac{4}{5}}{\overset{\zeta^{-2}}{\text{———}}}\underset{1}{\overset{-1}{\circ}}$	$\zeta \in \mathcal{R}_5$	
15	$\overset{\zeta}{\circ}\overset{\zeta^{-3}}{\text{———}}\overset{-1}{\circ}$ $\overset{-\zeta}{\circ}\overset{-\zeta^{-3}}{\text{———}}\overset{-1}{\circ}$ $\overset{-\zeta^{-2}}{\circ}\overset{\zeta^3}{\text{———}}\overset{-1}{\circ}$ $\overset{-\zeta^{-2}}{\circ}\overset{-\zeta^{-3}}{\text{———}}\overset{-1}{\circ}$	$\zeta \in \mathcal{R}_{20}$	No solution
16	$\overset{-\zeta}{\circ}\overset{-\zeta^{-3}}{\text{———}}\overset{\zeta^5}{\circ}$ $\overset{\zeta^3}{\circ}\overset{-\zeta^4}{\text{———}}\overset{-\zeta^{-4}}{\circ}$ $\overset{\zeta^5}{\circ}\overset{-\zeta^{-2}}{\text{———}}\overset{-1}{\circ}$ $\overset{\zeta^3}{\circ}\overset{-\zeta^2}{\text{———}}\overset{-1}{\circ}$	$\zeta \in \mathcal{R}_{15}$	No solution
17	$\underset{\frac{6}{14}}{\overset{-\zeta}{\circ}}\underset{-\frac{9}{7}}{\overset{-\zeta^{-3}}{\text{———}}}\underset{1}{\overset{-1}{\circ}}$ $\underset{\frac{2}{14}}{\overset{-\zeta^{-2}}{\circ}}\underset{-\frac{5}{7}}{\overset{-\zeta^3}{\text{———}}}\underset{1}{\overset{-1}{\circ}}$	$\zeta \in \mathcal{R}_7$	

Table 4.2: Realisation of finite-dimensional diagonal Nichols algebras of rank 3.

<table>
<tr><th>row</th><th>Braiding</th><th>Conditions</th><th></th></tr>
<tr><td>1′</td><td>
</td><td></td><td>One solution according to A_3 (see 1″).
One solution according to $A(2,0)$ (see 4).
One solution according to $A(1,1)$ (see 8).</td></tr>
<tr><td>1″</td><td>
</td><td>$q \neq \pm 1$</td><td>Cartan, A_3</td></tr>
<tr><td>2′</td><td>
</td><td>$i \in \mathcal{R}_4$</td><td>One solution according to B_3 (see 2″).
One solution according to $B(2,1)$ (see 5).</td></tr>
<tr><td>2″</td><td>
</td><td>$q \neq \pm 1$,
$q \notin \mathcal{R}_4$</td><td>Cartan, B_3</td></tr>
<tr><td>3</td><td>
</td><td>$q \neq \pm 1$</td><td>Cartan, C_3</td></tr>
<tr><td>4</td><td>
</td><td>$q \neq \pm 1$</td><td>Super Lie, A(2,0)</td></tr>
<tr><td>5</td><td>
</td><td>$q \neq \pm 1$,
$q \notin \mathcal{R}_4$</td><td>Super Lie, B(2,1)</td></tr>
<tr><td>6</td><td>
</td><td>$q \neq \pm 1$</td><td>Super Lie, C(3)</td></tr>
</table>

7		$q \neq \pm 1$, $q \notin \mathcal{R}_3$	Super Lie, $G(3)$
8		$q \neq \pm 1$	Super Lie, $A(1,1)$
$9, 10, 11$		$q', q'', q''' \neq 1$, $q' \cdot q'' \cdot q''' = 1$	Super Lie, D(2, 1;α), $r' + r'' + r''' = 2$
12		$\zeta \in \mathcal{R}_3$	No solution.
$13'$		$\zeta \in \mathcal{R}_3$	$r = \frac{1}{3}$
$13''$		$\zeta \in \mathcal{R}_6$	$r = \frac{1}{6}$
14		$\zeta \in \mathcal{R}_3$	No solution.

15		$\zeta \in \mathcal{R}_3$	$r = \frac{1}{3}$
16		$\zeta \in \mathcal{R}_3$	$r = \frac{5}{6}$
17		$\zeta \in \mathcal{R}_3$	No solution.
18		$\zeta \in \mathcal{R}_9$	$r = \frac{8}{9}$

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Summary

The first part of the thesis focuses on the introduction to its two main topics: path integrals and vertex algebras.
Starting with the toy examples of the classical free particle and the classical free scalar field, we show how to quantise them following two formulations: the Lagrangian which makes use of path integrals and the Hamiltonian which in the mechanical example yields the Schroedinger equation of a free particle, whereas in the field theory example yields the Heisenberg algebra. In order to describe the latter we introduce vertex algebras.
In a ongoing work with a theoretical chemistry group at the Freie Universitaet Berlin, we apply the path integral formulation to the study of stochastic dynamics in a classical system with many degrees of freedom. We consider the Fokker-Planck equation, a partial differential equation which stochastically describes the dynamics of a molecular system and discuss its equivalence to the Schroedinger equation. This in turn is equivalent to the path integral formalism which can therefore be applied to find solutions of the original system. The outlook is to further develop this study in specific examples.

In the second and main part of the thesis we proceed with a construction and classification study. For every finite-dimensional Nichols algebra with diagonal braiding q_{ij}, we find all lattices Λ with Gram matrix $m_{ij} = (v_i, v_j)$ which realise the braiding, i.e. such that $q_{ij} = e^{\pi i m_{ij}}$ and the Weyl reflections on q_{ij} lift to reflections on m_{ij} in a suitable sense.
For every Nichols algebra braiding q_{ij} and realising lattice Λ we then consider the Heisenberg vertex algebra and the corresponding non-local screening operators associated to Λ. Under certain *smallness* condition, these screening operators satisfy the relations of the Nichols algebra.
We then study for every finite-dimensional diagonal Nichols algebra the algebra of screening operators by analysing the smallness condition which does not always hold. When it fails the algebra of screenings is an extension of the Nichols algebra depending on the free parameters in the realisation.

Zusammenfassung

Der erste Teil meiner Doktorarbeit befasst sich mit der Einführung in ihre Hauptthemen, Pfadintegrale und Vertexalgebren.
Beginnend mit den Toy-Beispielen vom klassischen mechanischen Teilchen und dem klassischen freien skalaren Feld, zeigen wir, wie man die klassischen Beispiele quantisieren kann. Wir betrachten dazu zwei verschiedene Formulierungen, die Lagrangesche Formulierung, die Pfadintegrale benutzt, und die Hamiltonsche Formulierung, die in dem mechanischen Beispiel zur Schrödingergleichung von einem freien Teilchen und in dem Beispiel der Feldtheorie zur Heisenbergalgebra führt. Um die Heisenbergalgebra zu beschreiben, führen wir Vertexalgebren ein.
In einer laufenden Zusammenarbeit mit der Arbeitsgruppe für theoretische Chemie der Freien Universität Berlin, wenden wir die Pfadintegralformulierung auf das Studium der stochastischen Dynamik in einem klassischen System mit vielen Freiheitsgraden an. Wir betrachten die Fokker-Planck-Gleichung, eine Partielle Differentialgleichung, die die Dynamiken eines Molekularsystems stochastisch beschreibt, und wir zeigen, dass sie zur Schrödingergleichung äquivalent ist. Die Schrödingergleichung ist wiederum zur Pfadintegralformulierung äquivalent, die deswegen verwendet werden kann, um Lösungen des ursprünglichen Systems zu finden. Wir entwickeln dieses Studium in Beispielen weiter.

Im zweiten und Hauptteil der Arbeit fahren wir mit einem Studium von Konstruktion und Klassifikation fort. Für jede endlichdimensionale Nicholsalgebra mit diagonaler Verzopfung q_{ij}, finden wir alle Gitter Λ mit Gramschen Matrizen $m_{ij} = (v_i, v_j)$, die die Verzopfung so realisieren, dass die $q_{ij} = e^{\pi i m_{ij}}$ und die Weylreflexionen von q_{ij} sich zu den Reflexionen von m_{ij} geeignet hochheben.
Für jede Verzopfung q_{ij} einer Nicholsalgebra und realisierende Gitter Λ betrachten wir dann die Heisenbergsche Vertexalgebra und die zu dem Gitter zugehörigen nicht lokalen Screeningoperatoren. Unter den sogenannten *smallness* Bedingung, erfüllen diese Screeningoperatoren die Beziehungen der Nicholsalgebra.
Für jede endlichdimensionale diagonale Nicholsalgebra, studieren wir dann die Algebra der Screeningoperatoren indem wir die smallness Bedingung analysieren, die nicht immer erfüllt ist. Wenn sie nicht erfüllt ist, ist die Algebra der Screeningoperatoren eine Erweiterung der Nicholsalgebra, die von den freien Parametern in der Realisierung abhängt.

Publication list

I. Flandoli, S. Lentner, *Algebras of non-local screenings and diagonal Nichols algebras*, (2019) arXiv:1911.11040

I. Flandoli, S. Lentner, *Logarithmic conformal field theories of type* B_n , $\ell = 4$ *and symplectic fermions*, (2017) Journal of Mathematical Physics