Incomplete Information in Dynamic Stochastic General Equilibrium Models

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Abstract

In economics, researchers have not fully agreed upon the source of rigidities which are relevant for aggregate dynamics. Typically, rigidities are assumed to have different sources. For example, rigidities in prices are justified by menu costs and costs of collecting information, rigidities in consumption by habit formation and rigidities in capital by investment costs. More recently, the focus has shifted back to incomplete information as the source and the micro-foundation of frictions. The literature discusses various types of incomplete information. Early papers, i.e. Phelps (1970) and Lucas (1973), relax the full information assumption by assuming partial information. This means that all agents share the same but an incomplete information set.

At the beginning of the 2000's more complex structures of incomplete information were analysed. Morris and Shin (2002) assume heterogeneous information which makes the agents Bayesian learners that extract information from private and public signals.[1] Mankiw and Reis (2002) assume sticky-information in which agents are randomly selected to update their information set and, hence, part of the agents take their decision based on outdated information. Further, Sims (2003) introduces the idea of limited information processing capacity of agents. Under this constraint agents rationally allocate their attention to most informative private or public signals about the state of the world, while the constraint does not allow the agents to observe all variables perfectly.

In this dissertation, I focus on incomplete information in the form of partial and heterogeneous information. Models with heterogeneous information give rise to higher order expectations which lead to the so called infinite regress problem. In order to keep the problem tractable, one needs to impose certain assumptions about the information structure. One way to reduce the dimension of the state vector is to assume that the fundamentals are revealed after a finite number of periods. This method was suggested by Townsend (1983) and applied since the early days of this field of research. Nimark (2011) in comparison chooses to reduce

[1] I use the term of heterogeneous information for what is also found as noisy information and dispersed information in the literature.

the number of state variables by cropping higher orders from the state vector. Thereby, he argues that the cropped orders of expectations have no impact on the policy rules of the agents and hence for the dynamics of the model. In his approach the state of the world is never revealed, which implies a consistent treatment of the information that is available to the agents in the economy. Kasa (2000) and Rondina and Walker (2011) choose an alternative way to handle the infinite regress problem. They transform their model in the frequency domain. This way they find an exact solution to the problem. This approach generates more precise estimates, but it is very difficult to implement. Considering these arguments, I follow Nimark (2011) with regard to solving the infinite regress problem throughout the dissertation.

In Chapter 1, I revisit the results of Baxter et al. (2011) and Graham and Wright (2010) on the effects of incomplete information in the real business cycle (RBC) model. In these papers, I find a discussion of partial as well as heterogeneous information in the RBC model. The discussion encompasses convergence criteria for the model with partial information and pseudo-shocks in the case with heterogeneous information. I review the conclusions made and argue that one should treat capital separately from exogenous state variables, as a predetermined endogenous state variable. Then, I show that the filtering problem in the model with partial information can be simplified to one about the exogenous state variables only. To the heterogeneous information model a fundamental and a non-fundamental solution exists. Thereby, the dynamics of the fundamental solution simplifies to the one of the partial information model. The non-fundamental solution instead gives rise to pseudo-shocks to capital as reported in the literature. Furthermore, in all cases the signals have to reveal individual wages, idiosyncratic composite productivity and the return to capital to guarantee market clearing on the goods market. Moreover, I clarify some derivations made in their papers such that they are theoretically sound, but I do not develop an alternative framework which overcomes the practical issues that appear. Based on the theoretical insights of the first chapter, I generalise the framework and provide practical solutions in the second chapter.

In Chapter 2, I derive a solution algorithm that solves heterogeneous information dynamic stochastic equilibrium (HI-DSGE) models of a general form. The algorithm is general enough to handle both predetermined and contemporaneous endogenous as well as exogenous state variables. In addition, I derive general conditions on the structure of signals to ensure all markets to clear, and conditions for heterogeneous information to make a difference compared to partial information models. I illustrate the power of the methodology with an heterogeneous information new Keynesian model with capital, which has not been studied before. I analyse its dynamic implications compared to the full information version of the model and discuss three conjectures that Lorenzoni stated in his conlusion. The paper comes with an extensive Matlab toolkit[2], including documentation which allows any researcher to study heterogeneous information models without in-depth knowledge of the solution methodology.

In Chapter 3, I analyse the asset pricing implications of a heterogeneous information New Keynesian model. First, I address the equity premium and the risk free rate puzzle. Second, I relate the model to the finding in the full information literature that nominal rigidities reduce risk premia when the model is driven by supply shocks, and that risk premia increase with nominal rigidities in the presence of demand shocks. In a standard New Keynesian model, demand shocks appear only in the form of monetary policy shocks, while heterogeneous information models are shown to provide a broader theory of demand shocks.

I find that heterogeneous information can at least mitigate if not resolve the equity premium and the risk free rate puzzles. With regard to the second issue, I find opposing effects of supply and demand shocks to the ones in the full information literature.

[2]The Matlab toolkit can be downloaded under http://sites.google.com/site/hidsge/.

Zusammenfassung

Wirtschaftswissenschaftler sind sich weiterhin nicht einig über die grundlegenden Rigiditäten, die für aggregierte Konjunkturzyklen bedeutend sind. Typischerweise wird angenommen, dass Rigiditäten unterschiedliche Gründe haben. Zum Beispiel werden Rigiditäten bei Preisen mit Preisanpassungs- und Informationssammelkosten verbunden, Konsum passt sich aufgrund gebildeter Gewohnheiten nur langsam an neue wirtschaftliche Umstände an und der Kapitalstock wegen Investitionskosten. In den vergangenen Jahren wurde die Aufmerksamkeit hingegen wieder auf unvollständige Informationen als Quelle und Mikrofundierung von Friktionen gerichtet. Jedoch wurden bereits in den 70er Jahren verschiedene Formen von unvollständigen Informationen diskutiert. Phelps (1970) und Lucas (1973) haben die Annahme vollständiger Informationen aufgeweicht und Modelle mit partiellen Informationen entwickelt, in denen alle Agenten die gleichen, aber unvollständige Informationen über den Zustand der Volkswirtschaft teilen.

Anfang der 2000er wurden dann komplexere Strukturen unvollständiger Informationen analysiert. Morris and Shin (2002) modellieren die Annahme, dass Agenten heterogene Informationen erhalten. Das bedeutet, dass sie einerseits private und andererseits öffentliche Signale über den Zustand der Volkswirtschaft beobachten. Mankiw and Reis (2002) nehmen hingegen an, dass nur ein Teil der Agenten zu einem gegebenen Zeitpunkt ihre Informationen aktualisiert und damit der andere Teil seine Entscheidungen auf Grundlage von veralteten Informationen trifft. Darüber hinaus formuliert Sims (2003) die Idee, dass Agenten nur über begrenzte Informationsverarbeitungskapazitäten verfügen. Unter dieser Einschränkung wählen die Agenten selbst welchen privaten oder öffentlichen Signalen sie ihre Aufmerksamkeit widmen wollen. Dabei sorgt die Einschränkung dafür, dass sie nicht alle Bestandteile der Volkswirtschaft vollständig beobachten können.

In dieser Dissertation konzentriere ich mich auf Modelle mit unvollständige Informationen im Sinne von partiellen und heterogenen Informationen. Modelle mit heterogenen Informationen erzeugen Erwartungen höherer Ordnung, die zu dem so genannten Problem endloser Rekursion führen. Um dem Problem Herr zu werden, müssen bestimmte An-

nahmen über die Informationsstruktur getroffen werden. Eine Möglichkeit die Dimension der Zustandsvariablen in Grenzen zu halten ist die Annahme, dass der wahre Zustand der Volkswirtschaft nach einem bestimmten Zeitraum bekannt wird. Diese Methode wurde von Townsend (1983) vorgeschlagen und findet in der Literatur seither Anwendung. Nimark (2011) löst das Problem auf andere Weise. Er verringert die Anzahl der Zustandsvariablen dadurch, dass er diese einfach nach einer bestimmten Anzahl abschneidet. Er argumentiert, dass die Erwartungen höherer Ordnung ab einer bestimmten Ordnung keinen Einfluss mehr auf die Entscheidungsfindung der Agenten haben und damit auch nicht auf die volkswirtschaftliche Dynamik. Bei dieser Methode bleibt der wahre Zustand der Volkswirtschaft immer unbekannt und die Informationen werden über die Zeit konsistent behandelt. Kasa (2000) und Rondina and Walker (2011) wählen eine dritte Methode dem Problem unendlicher Rekursion zu begegnen. Sie transformieren das Problem der Erwartungsbildung in den Frequenzbereich. In Folge dessen finden sie eine exakte Lösung des Problems. Diese ist jedoch sehr schwierig zu implementieren. Nach Abwägung der Vor- und Nachteile der verschiedenen Methoden, verwende ich die Methode von Nimark (2011) im Rahmen meiner Dissertation.

In Kapitel 1 greife ich die Ergebnisse von Baxter et al. (2011) und Graham and Wright (2010) zu unvollständigen Informationen in Modellen realer Konjunkturzyklen (RBC Modelle) auf. In diesen Papieren werden ein Modell mit partiellen und eins mit heterogenen Informationen diskutiert. Dabei beinhaltet die Diskussion Konvergenzkriterien für Modelle mit partiellen Informationen und Pseudo-Schocks im Fall mit heterogenen Informationen. Ich überprüfe die gemachten Schlussfolgerungen und argumentiere, dass man Kapital als eine voraus bestimmte endogene Zustandsvariable getrennt von exogenen Zustandsvariablen behandeln sollte. Ich zeige unter diesem Umstand, dass das Filterproblem des Modells mit partiellen Informationen zu einem Problem von ausschließlich exogenen Prozessen vereinfacht werden kann. Bezüglich des Modells mit heterogenen Informationen bestehen eine fundamentale und eine nicht-fundamentale Lösung. Dabei führt die fundamentale Lösung zu der des Modells mit partiellen Informationen. Die nicht-fundamentale Lösung erzeugt hingegen Pseudo-Schocks in Bezug auf Kapital, wie es in der Literatur dokumentiert ist. In allen Fällen müssen die individuellen Löhne und die idiosynkratische Produktivität sowie die Kapitalrendite den Agenten bekannt sein, um die Markträumung auf dem Gütermarkt zu garantieren. Darüber hinaus stelle ich einige Herleitungen klar, so dass diese theoretisch konsistent sind. Die Anpassung der Herleitungen beeinflusst die Lösungsmethodik, die zu praktischen Probleme führen. Auf Grundlage der theoretischen Erkenntnisse dieses ersten Kapitels entwickele ich im zweiten Kapitel eine Methodik, die theoretisch konsistent und praktikabel ist.

In Kapital 2 entwickele ich einen Lösungsalgorithmus für stochastische allgemeine Gleichgewichtsmodelle mit heterogenen Informationen (HI-DSGE Modelle) in allgemeiner Form. Der Algorithmus kann voraus bestimmte und kontemporäre endogene sowie exogene Zustandsvariablen gemeinsam verarbeiten. Zusätzlich leite ich Bedingungen für die Signale her, die die Agenten beobachten, die sicherstellen, dass alle Märkte geräumt werden. Weiterhin zeige ich Bedingungen auf, unter denen Modelle mit heterogenen Informationen eine unterschiedliche Dynamik aufweisen als Modelle mit partiellen Informationen. Ich veranschauliche die Möglichkeitenn des Algorithmus anhand eines neukeynesianischen Modells mit heterogenen Informationen und Kapital, das bisher in der Literatur nicht betrachtet wurde. Ich analysiere die Bedeutung der heterogenen Informationen für die Dynamik des Modells im Vergleich zu dem gleichen Modell mit vollständigen Informationen und ich diskutiere drei Mutmaßungen die Lorenzoni (2010) in der Schlussfolgerung seines Papiers äußert. Mit dem Papier werde ich ein Matlab-Toolkit inklusive umfassender Dokumentation veröffentlichen mithilfe dessen jeder Wirtschaftswissenschaftler diese Klasse von Modellen studieren kann ohne ein tiefgreifendes Wissen der Lösungsmethodik entwickeln zu müssen.

In Kapitel 3 untersuche ich die Implikationen eines neukeynesianischen Modells mit heterogenen Informationen für die Preisbildung bei Vermögenswerten. Einerseits beziehe ich mich auf die Puzzles über Aktienprämien und den risikolosen Zinssatz. Andererseits untersuche ich die Implikationen des Modells für die Bedeutung von Angebots- und Nachfrageschocks für Aktienprämien. In der Literatur mit vollständigen Informationen wurde festgestellt, dass nominale Rigiditäten Aktienprämien erhöhen, wenn das Modell von Nachfrageschocks getrieben ist und diese verringern, wenn die treibende Kraft Angebotsschocks sind. Diese Beobachtung ist inbesondere interessant, da Modelle mit heterogenen Informationen eine allgemeinere Theorie von Nachfrageschocks bieten jene als unter vollständigen Informationen.

Im Ergebnis finde ich, dass heterogene Informationen das Aktienprämiums-Puzzle lindern, wenn nicht sogar lösen können. In Hinblick auf die zweite Frage finde ich gegensätzliche Ergebnisse bezüglich der Auswirkungen von Angebots- und Nachfrageschocks auf Aktienprämien im Vergleich zu denen aus der Literatur mit vollständigen Informationen.

Table of contents

List of figures

List of tables

Chapter 1

Incomplete Information in the Real Business Cycle Model

1.1 Introduction

In this paper, I analyze the effects of incomplete information in the Real Business Cycle (RBC) model, where I distinguish between partial and heterogeneous information. I refer to partial information when agents receive public signals about the state of the economy, and to heterogeneous information when agents receive private and public information about the economy. This implies that the model with partial information can be written as a representative agent model, while models with heterogeneous information generally cannot, as individual agents behave differently.

There is a large strand of literature on partial and heterogeneous information New Keynesian models. A selection of partial information New Keynesian models include Svensson and Woodford (2003), Svensson and Woodford (2004), Pearlman et al. (1986) and Pearlman (1992), on the one hand. On the other hand, Woodford (2003), Lorenzoni (2009) cover New Keynesian models with heterogeneous information. However, all these papers do not discuss capital as part of the model. There are only very few papers that cover models with capital. Baxter et al. (2011) refer to a partial information RBC model with capital and Blanchard et al. (2013) estimate a medium-scaled partial information DSGE model to identify noise and structural shocks from the data. Graham and Wright (2010) set up an RBC model with heterogeneous information. More recently, Hassan and Mertens (2014) suggest a methodology to incorporate heterogeneous information settings in DSGE models, but they do not allow for dynamic learning. Angeletos and La'O (2009) develop an RBC model without capital. Moreover, Rondina and Walker (2017) design a model with confounding dynamics in which the agents use a Wiener-Kolmogorov prediction formula to form their expectations.

In this paper, I specifically revisit the papers by Baxter et al. (2011) and Graham and Wright (2010). They are both of particular interest as they provide a compact and comprehensive approach to dealing with incomplete information in the RBC model. I clarify the conclusions made in their papers and show that they are imprecise. I alter them such that they are theoretically sound. However, I refrain from developing an alternative framework which overcomes the practical issues that appear. Based on the theoretical insights in the paper at hand, I extend the framework and provide practical solutions in my companion paper Schaefer (2019b).

Baxter et al. (2011) present a parsimonious solution algorithm which can be applied to solve partial information models that exhibit dynamic and measurement endogeneity. Dynamic and measurement endogeneity are found in DSGE models and refer to a model in which state variables, and the signals about them, depend on forward looking variables, respectively. Further, the authors discuss general stability criteria of the algorithm. Then, they use the algorithm to solve and discuss the RBC model with partial information.

Graham and Wright (2010) extend the approach of Baxter et al. (2011) to allow for heterogeneous information. They find that if agents observe prices only, namely individual wages and the return to capital, the model accommodates a strong negative response in expected capital to a positive productivity shock. This is surprising in two ways. First, it is surprising that the prices in the economy do not reveal the state of the economy.[1] Second, they point out that, as true capital is not observable, the expectation of capital is the reference to compare the data to. Therefore, it is surprising that the expectation of capital responds negatively to a positive productivity shock. This means that there is a counter-factual response between the model and the data. The reason for this negative response is that the agents cannot disentangle the underlying productivity shock from what Graham and Wright (2010) call a pseudo-shock. The pseudo-shock emerges from the situation that the agents believe that they previously overestimated the aggregate capital stock, so they expect capital to be low today and hence invest more compared to the full information case.

In revisiting their results and the conclusions made, I find three important differences. First, I find that it is important to distinguish between exogenous and predetermined endogenous state variables as they enter the signal extraction problem in different ways. With

[1] Radner (1979) in Graham and Wright (2010).

partial information predetermined state variables are known by the agents and with heterogeneous information two stable equilibria appear. Second, I show that the signals need to reveal idiosyncratic productivity, individual wages and the return to capital to ensure market clearing on the goods market. This fact implies a different guess for the individual forward looking variables. Third, I formulate the higher order expectations in such a way that they are consistent with the definition of private information.

These differences affect the results of the papers significantly. I show in this paper that the filtering problem of the RBC model with partial information presented in Baxter et al. (2011), when correctly specified, simplifies to a filtering problem of the productivity processes only. The reason is that individual capital is an individual predetermined state variable which is part of the agent's information set. With partial information all agents act alike and thus all agents choose the same capital. This implies that they do not only know their own individual capital but they also know aggregate capital. Consequently, they do not need to form expectation about capital and hence it drops out of the filtering problem completely. In addition, it is important to point out that agents face only a filtering problem when composite productivity includes at least two aggregate components of different persistence. Otherwise the model simplifies to the full information case.

Individual capital drops from the filtering problem of the RBC model with heterogeneous information presented by Graham and Wright (2010) for the same reason, as it is an individual predetermined state variable. However, agents have different productivity and receive different signals about the economy and hence the capital choices of the agents are different from one another. Nevertheless, there is a case, when all agents choose only the return to capital to infer the aggregate capital stock and the individual wages to infer the exogenous state variables, the solution to the heterogeneous information model simplifies to a fundamental solution, i.e. the one of partial information.[2] Instead, if the agents use the return to capital and the individual wages jointly to estimate aggregate capital and the exogenous state variables, then the heterogeneous information model accommodates a non-fundamental solution. The non-fundamental solution exhibits the same pseudo-shocks as they are reported in the literature, despite the separate treatment of individual capital as a predetermined state variable. Therefore, the counter-factual response does not appear for the reason mentioned

[2] Graham and Wright (2010) consider this equilibrium but deem it unstable based on their treatment of individual capital.

by Graham and Wright (2010).[3]

The remainder of the paper is organized as follows. In Section 1.2, I present a real business cycle model with incomplete markets which is suitable to analyse the effects of partial and heterogeneous information. The full information solution is briefly presented in Section 1.3. The partial information model is analysed in Section 1.4. In Section 1.5, I discuss the effects of heterogeneous information. Section 1.6 concludes.

1.2 The model

The model is an RBC model with incomplete markets following Graham and Wright (2010), including a household and a firm sector. However, I assume that the firm's production function includes composite productivity with two aggregate components of different persistence, instead of one, and one idiosyncratic component. The economy is assumed to consist of an infinite number of geographically separated regions, indicated by j, on which there is each one household and one firm. In the remainder of the paper, I will refer to the regions as islands, as it is common in the literature. Firms hire labour only from the households on their island while capital is traded across the whole economy. Moreover, the agents on island j share the same information set in period t, Ω_{jt}. The agents know the structure and parameters of the economy as well as the distribution of the innovations of the stochastic processes, but they do not directly observe the realization of the state variables. Instead they receive signals about them, which are a subset of the model's variables, Υ_{jt} and Υ_t. They can be individual variables (private signals), $\underline{\Upsilon}_{jt} \subseteq \Upsilon_{jt}$, and aggregate variables (public signals), $\underline{\Upsilon}_t \subseteq \Upsilon_t$. Further, it is important to understand that the choice variables of the agents are part of their information set, but do not add additional information to the filtering problem, because the choice is taken conditional on the filtering problem about the state variables. The exception from this fact are individual predetermined state variables.[4] They are also chosen conditional on the current period expectation of the state of the world, but today's choice of tomorrow's variable will be known at the beginning of the next period. This is an important aspect of the information set, therefore I treat individual predetermined state

[3] Collary 5 in Baxter et al. (2011) states that a model with dynamic endogeneity might exhibit non-invertible information sets and explosive eigenvalues. This does not hold true for the partial information version of the model as capital is not part of the filtering problem and neither for heterogeneous information version in Graham and Wright (2010), who refer to Baxter et al. (2011), as individual capital is not part of the filtering problem.

[4] For details, see Appendix A.2.3.

variables separately, $X^c_{j,t+1}$. Summarizing, the information set includes the whole history of signals and individual predetermined state variables $\Omega_{jt} = \{\underline{\Upsilon}_{j\tau}, \underline{\Upsilon}_{\tau}, X^c_{j,\tau+1} | \tau = 0,1,2,...,t\}$. For simplicity, I define $E[X_t|\Omega_{jt}] = E_{jt}[X_t] = X_{t|jt}$.

In addition, I define a variable to be revealed by a signal in t to the agents on island j if the variable becomes a member of the information set, Ω_{jt}, after observing the signal. The signal always reveals itself but can also reveal other variables if they are linear functions of the signals.

1.2.1 The household's problem

The household on island j has three choices to maximize its expected discounted utility subject to its budget constraint and the individual capital law of motion: consumption, C_{jt}, labour supply in terms of hours worked, H_{jt}, and tomorrow's capital stock, $K_{j,t+1}$. The subjective discount factor is denoted as β. The utility function is logarithmic in consumption and power in hours worked, where θ is a scaling factor between the utility of consumption and leisure. The household has one unit of time endowment per period which it can spend either on leisure or labour, $L_{jt} = 1 - H_{jt}$. γ determines the Frisch elasticity of labour supply. The maximisation problem reads:

$$\max_{\{C_{jt},H_{jt},K_{j,t+1}\}} E_{jt}\left[\sum_{t=0}^{\infty}\beta^t\left(\ln C_{jt} + \theta\frac{(1-H_{jt})^{1+\gamma}}{1+\gamma}\right)\right] \tag{1.1}$$

$$\text{s.t.} \quad C_{jt} + I_{jt} = R^k_t K_{jt} + W_{jt}H_{jt} \tag{1.2}$$

$$\text{and} \quad K_{j,t+1} = \big(1-\delta\big)K_{jt} + I_{jt}, \quad \text{where } k_{j0} = k_0 \text{ is known.} \tag{1.3}$$

The budget constraint requires that the households capital income and its labour income is equal to its consumption and investment, I_{jt}. Capital income is defined as the return to capital, R^k_t, times last period's choice of capital and wages, W_{jt}, times hours worked. Further, tomorrow's capital stock is equal to today's non-depreciated capital plus investment. The depreciation rate is denoted as δ.

The maximization problem yields a standard Euler equation for consumption and an equation for the optimal labour choice:

$$\frac{1}{C_{jt}} = \beta E_{jt}\left[\frac{R_{t+1}}{C_{j,t+1}}\right], \tag{1.4}$$

$$\frac{E_{jt}\left[W_{jt}\right]}{C_{jt}} = \theta\left(1-H_{jt}\right)^{\gamma}. \tag{1.5}$$

Define $R_t = \left(1-\delta\right) + R_t^k$ as the net return to capital. Note that the choice variables are part of the information set and hence can be pulled outside of the expectation operator. As I have not yet discussed the signals that the agents receive, I assume wages and the return to capital not to be part of the information set at this point.

1.2.2 The firm's problem

The firm on island j chooses labour demand, N_{jt}, and capital demand, J_{jt}, that maximize their profits, π_{jt}, subject to a Cobb-Douglas type of production technology, where α is the output elasticity of capital. Individual output is denoted by Y_{jt}. The firms maximisation problem reads:

$$\max_{\left\{N_{jt}J_{jt}\right\}} E_{jt}\left[\pi_{jt}\right] = E_{jt}\left[Y_{jt} - W_{jt}N_{jt} - R_t^k J_{jt}\right] \tag{1.6}$$

$$s.t. \quad Y_{jt} = \left(J_{jt}\right)^{\alpha}\left(e^{z_{jt}}N_{jt}\right)^{1-\alpha}. \tag{1.7}$$

In contrast to the model in Graham and Wright (2010), I assume composite productivity to consist of two aggregate components, while they assume one, and one idiosyncratic component, $z_{jt} = z_t + \varepsilon_{jt}$, where aggregate productivity itself has a persistent and an transitory component $z_t = a_t + \varepsilon_t$. The persistent component follows an AR(1) process with auto-correlation parameter ρ and innovation v_t.[5] The three types of innovations are i.i.d. normally distributed, $N(0,\sigma_{\varepsilon j}^2)$, $N(0,\sigma_{\varepsilon}^2)$ and $N(0,\sigma_{v}^2)$, respectively. Moreover, it is assumed that integrating over the idiosyncratic innovations of all islands yields zero, $\int \varepsilon_{jt}\, dj = 0$.

[5]The results are all unaffected by the assumption that ε_t is transitory. The results hold as long as the persistence of a_t and ε_t are different. If they were the same, they would enter the choice of consumption with equal weights and make it irrelevant to the agents to distinguish between them. Additionally, we choose ε_{jt} to be transitory. With ε_{jt} being persistent one needs to add an exogenous noisy private signal about ε_{jt} for the model to converge to a consistent solution.

The first order conditions of the firm's problem show that the return to capital is equal to the marginal product of capital and the wage rate is equal to the marginal product of labour:

$$E_{jt}\left[R_t^k\right] = \alpha \frac{E_{jt}\left[Y_{jt}\right]}{J_{jt}} \quad \text{and} \tag{1.8}$$

$$E_{jt}\left[W_{jt}\right] = \left(1-\alpha\right)\frac{E_{jt}\left[Y_{jt}\right]}{N_{jt}}. \tag{1.9}$$

Again, note that the return to capital, individual wages and individual production are not part of the information set of the agents, as I have not defined it yet.

1.2.3 Aggregate variables and market clearing

The equilibrium dynamics of the individual variables,

$$\Upsilon_{jt} = \left\{K_{jt}, z_{jt}, Y_{jt}, C_{jt}, I_{jt}, H_{jt}, J_{jt}, N_{jt}, W_{jt}\right\},$$

are described by the equations (1.2), (1.3), (1.4), (1.5), (1.7), (1.8), (1.9), as well as the definition of idiosyncratic composite productivity, z_{jt}, and the market clearing condition for labour. Aggregate variables are defined by the integral over the same individual variables, i.e. for any variable in the set Υ_{jt}, $\Upsilon_t = \int \Upsilon_{jt}\, dj$. The equilibrium dynamics of the aggregate variables,

$$\Upsilon_t = \left\{K_t, z_t, Y_t, C_t, I_t, H_t, J_t, N_t, W_t, R_t^k, R_t\right\},$$

are defined by the aggregated individual equilibrium dynamics, as well as the definition of the net return to capital and the market clearing condition for capital.

There are three markets to clear, namely the labour, the capital and the goods market. I follow Graham and Wright (2010) on the labour and capital market clearing conditions. Labour market clearing requires $H_{jt} = N_{jt}$ for all islands j, as it is assumed that firms demand labour only from the household of the same island. The capital market is cleared through the return to capital, R_t^k, which holds for all islands j. In consequence and in line with the household's Euler equation, the non-arbitrage assumption requires households to invest their capital in more productive firms such that the return to capital is the same across the economy. Hence, generally the capital demand and the capital stock on an island are not the same, $K_{jt} \neq J_{jt}$, but on the aggregate level supply equals demand, $K_t = J_t$. However, Graham and

Wright do not discuss market clearing on the goods market. I show that market clearing requires individual wages, idiosyncratic composite productivity and the return to capital to be part of the information set.

Proposition 1. *Goods market clearing requires that the return to capital, individual wages and idiosyncratic productivity are part of the information set, Ω_{jt}.*

Proof. See Appendix A.2.1. □

To understand this result, I first want to clarify the role of prices in a market. Prices are the result of the agent's interaction. Thus, all agents that participate in the same market know the prices on the market, i.e. the prices are part of the information set of all agents participating in the market. In addition, agents need to take their choice based on idiosyncratic composite productivity and not their expectation of its components for their decision to be consistent with its realisation. However, this does not mean that the return to capital, individual wages and idiosyncratic composite productivity have to be signals.

Additionally, with heterogeneous information individual wages reveal idiosyncratic composite productivity and thus, the price for capital and the price for labour are sufficient for market clearing. With partial information any of the two variables suffice.

Proposition 2. *The prices on the capital and labour market, being part of the information set, Ω_{jt}, are sufficient to guarantee market clearing on the goods market.*

Proof. See Appendix A.2.2. □

1.2.4 State space representation

For the remaining part of the paper, I log-linearise the equilibrium dynamics and write them in terms of log-deviations from steady state.[6]

The linearised equations can be cast in the following state space system, which is inspired by Graham and Wright (2010) and Baxter et al. (2011). The difference with regard to the state space representation of the equilibrium dynamics is that I explicitly account for endogenous predetermined state variables. This distinction between endogenous predetermined state variables and exogenous state variables is crucial for the correct solution of the model.

[6]To keep the notation as simple and clear as possible we use lower case letters for log-deviations and do not indicate them with a hat. In the appendix instead I am precise on the notation and indicate log-deviations from steady state with a hat $\hat{\Upsilon}_{jt}$ and logs of a variable with lower case letters.

First, I define the vector $X_{jt}^{c} = \begin{bmatrix} k_{jt} \end{bmatrix}'$ of individual predetermined state variables. Further, I define exogenous idiosyncratic state variables as $X_{jt} = \begin{bmatrix} \varepsilon_{jt} \end{bmatrix}'$. Analogue, I summarize aggregate predetermined and exogenous state variables in $X_{t}^{c} = \begin{bmatrix} k_{t} \end{bmatrix}'$ and $X_{t} = \begin{bmatrix} a_{t} \ \varepsilon_{t} \end{bmatrix}'$. Second, I define parameter matrices as expressions without a time subscript: M, N and G.[7] The matrices are constraint by zeros. The reason is that individual predetermined state variables can be written as functions of individual predetermined state variables, private and public signals as well as individual forward looking variables. Moreover, the exogenous state variables depend only on themselves. Private signals generally do not aggregate to public signals. They typically integrate to aggregate endogenous predetermined and exogenous state variables. In due consideration of these insights, I can cast all state variables in the following form:

$$\begin{bmatrix} X_{j,t+1}^{c} \\ X_{j,t+1} \\ X_{t+1}^{c} \\ X_{t+1} \end{bmatrix} = \begin{bmatrix} M_{cj}^{cj} & 0 & 0 & 0 \\ 0 & M_{xj}^{xj} & 0 & 0 \\ 0 & 0 & M_{c}^{c} & M_{c}^{x} \\ 0 & 0 & 0 & M_{x}^{x} \end{bmatrix} \begin{bmatrix} X_{jt}^{c} \\ X_{jt} \\ X_{t}^{c} \\ X_{t} \end{bmatrix} + \begin{bmatrix} M_{cj}^{\Upsilon j} & M_{cj}^{\Upsilon} \\ 0 & 0 \\ 0 & M_{c}^{\Upsilon} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \underline{\Upsilon}_{jt} \\ \underline{\Upsilon}_{t} \end{bmatrix} + \begin{bmatrix} M_{cj}^{fj} & 0 \\ 0 & 0 \\ 0 & M_{c}^{f} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} F_{jt} \\ F_{t} \end{bmatrix} + \begin{bmatrix} 0 \\ N_{xj} \\ 0 \\ N_{x} \end{bmatrix} \begin{bmatrix} s_{j,t+1} \\ s_{t+1} \end{bmatrix}, \tag{1.10}$$

where s_{jt} and s_t are i.i.d. Gaussian error terms, $\begin{bmatrix} s_{jt} \\ s_{t} \end{bmatrix} \sim N(0, I)$.

Contemporaneous variables, such as e.g. investment, labour, wages and the return to capital can be written as a function of the state variables and forward looking variables:

$$\begin{bmatrix} \Upsilon_{jt} \\ \Upsilon_{t} \end{bmatrix} = \begin{bmatrix} G_{cj} & G_{c} \end{bmatrix} \begin{bmatrix} X_{jt}^{c} \\ X_{t}^{c} \end{bmatrix} + \begin{bmatrix} G_{xj} & G_{x} \end{bmatrix} \begin{bmatrix} X_{jt} \\ X_{t} \end{bmatrix} + \begin{bmatrix} G_{fj} & G_{f} \end{bmatrix} \begin{bmatrix} F_{jt} \\ F_{t} \end{bmatrix}. \tag{1.11}$$

The state space representation is completed with the Euler equations of individual forward looking variables. The forward looking variables in the model at hand is consumption

[7] See Appendix A.1.3 for the linearised equilibrium dynamics and Appendix A.1.4 for details on the parameter matrices.

$F_{jt} = \left[c_{jt}\right]'$. All variants of R define parameter matrices:[8]

$$R_c^0 \begin{bmatrix} X_{jt}^c \\ E_{jt}X_t^c \end{bmatrix} + R_\Upsilon^0 \begin{bmatrix} \Upsilon_{jt} \\ \Upsilon_t \end{bmatrix} + R_f^0 \begin{bmatrix} F_{jt} \\ E_{jt}F_t \end{bmatrix} = E_{jt} \left\{ R_c^1 \begin{bmatrix} X_{j,t+1}^c \\ X_{t+1}^c \end{bmatrix} + R_\Upsilon^1 \begin{bmatrix} \Upsilon_{j,t+1} \\ \Upsilon_{t+1} \end{bmatrix} + R_f^1 \begin{bmatrix} F_{j,t+1} \\ F_{t+1} \end{bmatrix} \right\}. \tag{1.12}$$

It is worth noting that individual forward looking variables are chosen conditional on individual predetermined state variables, signals and the individual expectation of next periods variables, too.

For later reference, I use the definition of the contemporaneous variables (1.11) and the state law of motion (1.10) to rewrite (1.12) as:[9]

$$Q_c^0 \begin{bmatrix} X_{jt}^c \\ E_{jt}X_t^c \end{bmatrix} + Q_x^0 \begin{bmatrix} X_{jt} \\ E_{jt}X_t \end{bmatrix} + Q_\Upsilon^0 \begin{bmatrix} \Upsilon_{jt} \\ \Upsilon_t \end{bmatrix} + Q_f^0 \begin{bmatrix} F_{jt} \\ E_{jt}F_t \end{bmatrix} = Q_f^1 E_{jt} \begin{bmatrix} F_{j,t+1} \\ F_{t+1} \end{bmatrix}. \tag{1.13}$$

1.3 Full information

Let me start with the full information setting as the benchmark case. I will solve it analogue to the incomplete information cases.[10] The solution algorithm for incomplete information models is a numerical procedure to solve the model by the methods of undetermined coefficients which includes three steps. First, I guess the policy function for individual forward looking variables. Second, I use the guess to express contemporaneous variables in terms of the state variables and I use the guess to complete the state law of motion. Third, I confirm my guess for the policy function of forward looking variables, using the state law of motion and the Euler equation. With incomplete information the guess needs to additionally satisfy the filtering problem of the agents. Under full information the filtering problem is obsolete.

[8]For details see Appendix A.1.4.

[9]Details are to be found in Appendix A.1.4.

[10]Obviously, the full information version of the model can be solved easily using standard techniques presented in Blanchard and Kahn (1980), Sims (2002) and Klein (2000) for which there exist solvers being implemented in various computer programs.

Under full information all agents behave alike. Therefore, I drop the subscript j. I define $X_t^c = \left[k_t\right]'$, $X_t = \left[a_t \; \varepsilon_t\right]'$ and $F_t = \left[c_t\right]'$. The information set under full information is $\Omega_t = \{X_{\tau+1}^c, X_\tau | \tau = 0,1,2,...,t\}$ and the expectation operator becomes $E[X_t|\Omega_t] = E_t[X_t] = X_t$.

I guess that the forward looking variables are a function of predetermined endogenous and exogenous state variables, i.e:

$$F_t = \eta^* \left[X_t^{c\prime} \quad X_t^{\prime}\right]^{\prime}, \quad \text{where} \quad \eta^* = \left[\eta_c^* \quad \eta_x^*\right]. \tag{1.14}$$

Then, I can write the state space system as follows. Contemporaneous variables, (1.11), become a function of the state variables only:

$$\Upsilon_t = G^*\left(\eta^*\right) \left[X_t^{c\prime} \quad X_t^{\prime}\right]^{\prime},$$

where $G^*\left(\eta^*\right)$ is a matrix which is defined by the matrices of (1.11) and the parameters of the guess for the policy function of forward looking variables, η^*, defined in (1.14).

Further, I find the state law of motion to read:

$$\begin{bmatrix} X_{t+1}^c \\ X_{t+1} \end{bmatrix} = M^*\left(\eta^*\right) \begin{bmatrix} X_t^c \\ X_t \end{bmatrix} + \begin{bmatrix} N_c^* \\ N_x^* \end{bmatrix} s_{t+1},$$

where $M^*\left(\eta^*\right)$ is a matrix that is a function of $G^*\left(\eta^*\right)$ and the matrices of (1.10).

Finally, I substitute the guess for the policy function of the forward looking variables and the implied state space representation of the model in the Euler equation of the forward looking variables, (1.13). Then, I equate the parameters of identical variables, to find:

$$\eta^* = C^0 + C^1\left(\eta^*\right) M^*(\eta^*). \tag{1.15}$$

In order to find the solution, I guess initial values for η^* and solve for them iteratively, i.e. I determine the fixed point of equation (1.15).[11]

[11] The matrices of the full information solution are described in detail in Appendix A.3.1.

1.4 Partial Information

Similar to the solution of the full information version of the model, I first need to formulate a guess about the forward looking variables. Then, I express the contemporaneous variables in terms of the state variables and I use the guess to complete the state law of motion. I confirm my guess using the Euler equation. As mentioned before, with partial information the fixed point solution encompasses the policy function for forward looking variables and a filtering problem.

With partial information agents receive noisy signals about the state of the economy. However, all agents receive the same information. This implies that the agents behave all alike, as under full information. Therefore, I drop the subscript j here, too, and define $X_t^c = \begin{bmatrix} k_t \end{bmatrix}'$, $X_t = \begin{bmatrix} a_t \ \varepsilon_t \end{bmatrix}'$ and $F_t = \begin{bmatrix} c_t \end{bmatrix}'$. Partial and full information share the fact that endogenous predetermined state variables are part of the agent's information set as long as Proposition 1 is not violated, but under partial information agents observe only a subset of contemporaneous variables as signals, $\Omega_t = \{X_{\tau+1}^c, \underline{\Upsilon}_\tau | \tau = 0, 1, 2, ..., t\}$.

Proposition 3. *Capital is known to the agent in a model with partial information if market clearing on the goods market is assured.*

Proof. See Appendix A.2.3. □

Following up on Proposition 3, I guess that forward looking variables are a function of predetermined state variables and exogenous state variables as well as the expectation thereof:

$$F_t = \eta^p \begin{bmatrix} X_t^{c\prime} & X_t' & X_{t|t}' \end{bmatrix}', \quad \text{where} \quad \eta^p = \begin{bmatrix} \eta_c^p & \eta_x^p & \eta_e^p \end{bmatrix}. \tag{1.16}$$

The guess differs to the one in Baxter et al. (2011). Their guess, $F_t = \eta_p \begin{bmatrix} X_{t|t}^c \ X_{t|t} \end{bmatrix}$, is the certainty equivalent guess, while the one in this paper instead depends on the information structure. The guess depends on the information structure as the parameters η^p depend on the Kalman gain, as I will show.

The guess for forward looking variables implies an extended state space system, which also includes the expectation of exogenous state variables $X_{t|t}$. Consequently, contemporaneous variables can be expressed as:

$$Y_t = G^p\left(\eta^p\right)\left[X_t^{c\prime} \quad X_t^{\prime} \quad X_{t|t}^{\prime}\right]^{\prime}, \quad \text{where} \quad G^p\left(\eta^p\right) = \left[G_c\left(\eta_c^p\right) \quad G_x\left(\eta_x^p\right) \quad G_e\left(\eta_e^p\right)\right] \tag{1.17}$$

and the state law of motion can be written as:

$$\begin{bmatrix} X_{t+1}^c \\ X_{t+1} \\ X_{t+1|t+1} \end{bmatrix} = M^p\left(\mathcal{K},\eta^p\right)\begin{bmatrix} X_t^c \\ X_t \\ X_{t|t} \end{bmatrix} + N^p\left(\mathcal{K},\eta^p\right)s_{t+1}, \tag{1.18}$$

where

$$M^p\left(\mathcal{K},\eta^p\right) = \begin{bmatrix} M_c^c\left(\eta_c^p\right) & M_c^x\left(\eta_x^p\right) & M_c^e\left(\eta_e^p\right) \\ 0 & M_x^x & 0 \\ 0 & M_e^x\left(\mathcal{K},\eta_x^p\right) & M_e^e\left(\mathcal{K},\eta_x^p\right) \end{bmatrix} \quad \text{and} \quad N^p\left(\mathcal{K},\eta^p\right) = \begin{bmatrix} N_c \\ N_x \\ N_e\left(\mathcal{K},\eta_x^p\right) \end{bmatrix}.$$

$M_e^x\left(\mathcal{K},\eta_x^p\right)$, $M_e^e\left(\mathcal{K},\eta_x^p\right)$ and $N_e\left(\mathcal{K},\eta_x^p\right)$ depend on the solution of the fixed point problem between policy functions mapping the forward looking variables to the state variables and the Kalman gain, $\mathcal{K}$, derived as part of the filtering problem that is discussed in Section 1.4.1.

I confirm the guess of the forward looking variables analogue to the full information case:

$$\eta^p = C^0 + C^1\left(\eta^p\right)\left(T_k M^p\left(\mathcal{K},\eta^p\right) + T_{\neq k} M^p\left(\mathcal{K},\eta^p\right)T_e\right), \tag{1.19}$$

where T_k selects capital which follows the state law of motion outside of the expectation operator and $T_{\neq k}$ the the exogenous state variables that stay within the expectation operator. Moreover, T_e adds up the exogenous state variables and the expectations thereof.[12]

There is an important reason why the policy function of forward looking variables, (1.16), is not only a function of expectational variables as claimed by Baxter et al. (2011). I showed that market clearing requires that the choice of a variable is taken conditional on the observed signals and not the expectation of its components. In addition, I showed that capital is part of the information set. Therefore, I can pull it out of the expectation operator in $t+1$ and, hence, the period t variables in the state law of motion remain outside of the expectation operator, too. The expectational parts of the policy function for forward looking variables

[12] The concrete elements of the matrices T_k, $T_{\neq k}$ and T_e are presented in Appendix A.3.2.

stem from next periods aggregate exogenous state variables. The aggregate state variables follow their state law of motion and the current period realisations are not known. In this case, the agents need to form expectations about the distinct state variables today in order to anticipate its evolution and thus, to optimally choose the forward looking variable.

In effect, the matrix mapping the forward looking variables to the state variables has a different functional form than under full information. The matrix for predetermined state variables remains the same $\eta_c^p = \eta_c^*$, but the forward looking variables are mapped differently to exogenous state variables.[13]

Concluding, the solution of the partial information model is the fixed point of η^p derived from the Euler equation and $\mathscr{K}$ derived from the filtering problem.

1.4.1 Signal extraction problem

In this subsection, I discuss the filtering problem of the agents. I assume, as it is standard in the literature, that the agents form their expectation about the state variables by means of the Kalman filter and all agents know that everybody forms their expectation this way. Further, I assume that the agents have already observed a sufficiently long history of signals such that the Kalman gain has converged to the steady state Kalman gain. The Kalman updating equation is defined as follows:

$$X_{t+1|t+1} - X_{t+1|t} = \mathscr{K}\left(\underline{\Upsilon}_{t+1} - \underline{\Upsilon}_{t+t|t}\right). \tag{1.20}$$

The forecast error of the state variables is equal to the Kalman gain times the forecast error of the signals, while the forecast error of the predetermined state variable is zero. This underscores the importance to distinguish between predetermined state and exogenous state variables. Furthermore, I show that the filtering problem is also indirectly independent of predetermined state variables. The forecast error of the signals $\underline{\Upsilon}_{t+1} - \underline{\Upsilon}_{t+1|t}$ is defined as:

$$\underline{\Upsilon}_{t+1} - \underline{\Upsilon}_{t+1|t} = \underline{G}(\eta^p)\left(\begin{bmatrix} X^c_{t+1} \\ X_{t+1} \\ X_{t+1|t+1} \end{bmatrix} - \begin{bmatrix} X^c_{t+1|t} \\ X_{t+1|t} \\ X_{t+1|t} \end{bmatrix}\right), \tag{1.21}$$

[13]The matrices of the partial information solution are described in detail in Appendix A.3.2. As it is pointed out by Nimark (2011) in his model with heterogeneous information the parameters mapping the forward looking variables into the hierarchy of expectations add up to the full information solution. The same reason holds true also in this case at hand, i.e. $\eta_x^* = \eta_x^p + \eta_e^p$.

where predetermined state variables cancel out as well. Plugging (1.20) into (1.21) I find:

$$\underline{\Upsilon}_{t+1} - \underline{\Upsilon}_{t+1|t} = J^{-1}\underline{G}_x\left(\eta_x^p\right)\left(X_{t+1} - X_{t+1|t}\right), \tag{1.22}$$

where $J = \left(I - \underline{G}_e\left(\eta_e^p\right)\mathscr{K}\right)$. Then, I can plug (1.22) back into the Kalman updating equation (1.20) to find:

$$X_{t+1|t+1} = \left(I - \mathscr{K}J^{-1}\underline{G}_x\left(\eta_x^p\right)\right)X_{t+1|t} + \mathscr{K}J^{-1}\underline{G}_x\left(\eta_x^p\right)X_{t+1}.$$

If I use the state law of motion for X_{t+1} from (1.18), I can derive the state law of motion of the expectation of the state variables $X_{t+1|t+1}$:

$$\begin{aligned} X_{t+1|t+1} = &\left[0 \quad \mathscr{K}J^{-1}\underline{G}_x\left(\eta_x^p\right)M_x^x \quad \left(I - KJ^{-1}\underline{G}_x\left(\eta_x^p\right)\right)M_x^x\right]\left[X_t^{c\prime} \quad X_t' \quad X_{t|t}'\right]' \\ &+ \mathscr{K}J^{-1}\underline{G}_x\left(\eta_x^p\right)N_x s_{t+1}, \end{aligned} \tag{1.23}$$

which verifies my guess and identifies $M_e^x\left(\mathscr{K},\eta^p\right)$, $M_e^e\left(\mathscr{K},\eta^p\right)$ and $N_e\left(\mathscr{K},\eta^p\right)$ with one difference: The matrices depend on η^p, via J, and not on η_x^p only. I show shortly that one can eliminate J from the matrices.[14]

It remains to solve for the Kalman gain $\mathscr{K}$ as well as the corresponding mean square error (MSE) and the variance-covariance matrix of the one period ahead forecast error.[15]

First, I define the variance-covariance matrix of the one period ahead forecast error by using the state law of motion of X_{t+1} from (1.18):

$$P = M_x^x \hat{P} M_x^{x\prime} + N_x N_x'. \tag{1.24}$$

Second, the MSE, $\hat{P}$, is defined as:

$$\hat{P} = \left(I - \mathscr{K}J^{-1}\underline{G}_x\left(\eta_x^p\right)\right)P.$$

[14] The details on the matrices can be found in Appendix A.3.2.

[15] Here I follow common wisdom, for example see Hamilton (1994).

Finally, the Kalman gain can be computed using equations (1.22) and (1.24):[16]

$$\begin{aligned}\mathscr{K} &= \left\{P\underline{G}_x(\eta_x^p)'(J^{-1})'\right\}\left\{J^{-1}\left(\underline{G}_x(\eta_x^p)P\underline{G}_x(\eta_x^p)'\right)(J^{-1})'\right\}^{-1} \\ &= \left\{P\underline{G}_x(\eta_x^p)'\right\}\left\{\underline{G}_x(\eta_x^p)P\underline{G}_x(\eta_x^p)'\right\}^{-1}J.\end{aligned}$$

Baxter et al. (2011) show that one can formulate a parallel filtering problem, in which the Kalman gain, the MSE and the variance-covariance matrix of the one period forecast error can be written independently of the effects of predetermined state variables and the expectations of exogenous state variables. To find their formulation I just multiply out J in the Kalman gain equation. Then, I find the Kalman gain and the State MSE of the parallel problem as:

$$\tilde{\mathscr{K}} = \left\{P\underline{G}_x(\eta_x^p)'\right\}\left\{\underline{G}_x(\eta_x^p)P\underline{G}_x(\eta_x^p)'\right\}^{-1} = \mathscr{K}J^{-1} \quad \text{and} \quad \hat{P} = \left(I - \tilde{\mathscr{K}}\underline{G}_x(\eta_x^p)\right)P. \tag{1.25}$$

This also affects the state law of motion for the matrices that depend on the Kalman gain. $M_e^x(\mathscr{K},\eta_x^p)$, $M_e^e(\mathscr{K},\eta_x^p)$ and $N_e(\mathscr{K},\eta_x^p)$ eventually confirm my guess:

$$M_e^x = \tilde{\mathscr{K}}\underline{G}_x(\eta_x^p)M_x^x, \quad M_e^e = (I - \tilde{\mathscr{K}}\underline{G}_x(\eta_x^p))M_x^x, \quad \text{and} \quad N_e = \tilde{\mathscr{K}}\underline{G}_x(\eta_x^p)N_x.$$

1.4.2 Analytical and numerical results

While so far I have discussed the solution of the partial information version of the model to the general problem, I discuss in this section the solution of the RBC model as an explicit example. Given that the filtering problem is relatively simple, I can derive the results of the filtering problem analytically. However, the fixed point problem, including the solution of the policy function for forward looking variables, cannot be determined all by hand. Therefore, I derive the filtering problem analytically, for illustrative purposes, and describe the remaining part of the fixed point problem to be solved numerically.

[16] Recall that for symmetric matrices, $A = (J^{-1})\underline{G}_x(\eta_x^p)P\underline{G}_x(\eta_x^p)'(J^{-1})'$, it holds that $A = A'$. Further, for a non-singular matrix J^{-1} it holds that $(J')^{-1} = (J^{-1})'$.

The state variables of the RBC model can be cast in the form, (1.10). The exogenous state variables are $X_t = \left[a_t \; \varepsilon_t\right]'$, which follow the state law of motion:

$$\begin{bmatrix} a_{t+1} \\ \varepsilon_{t+1} \end{bmatrix} = M_x^x \begin{bmatrix} a_t \\ \varepsilon_t \end{bmatrix} + N_x s_{t+1}, \quad \text{where} \quad M_x^x = \begin{bmatrix} \rho & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad N_x = \begin{bmatrix} \sigma_\nu & 0 \\ 0 & \sigma_\varepsilon \end{bmatrix}.$$

The predetermined state variable is capital, $X_t^c = k_t$, which follows the state law of motion:

$$k_{t+1} = M_c^c k_t + M_c^\Upsilon \begin{bmatrix} z_t \\ r_t^k \end{bmatrix} + M_c^f c_t,$$

where

$$M_c^c = \left(1 - \delta + \left(1 - \alpha\right) \frac{Y}{K}\right),$$
$$M_c^\Upsilon = \left[\left(1 - \alpha\right)\left(1 + \xi\right) \tfrac{Y}{K} \quad -\alpha \xi \tfrac{Y}{K}\right] \quad \text{and}$$
$$M_c^f = -\left(\frac{C}{K} + \left(1 - \alpha\right) \xi \frac{Y}{K}\right),$$

where $\xi = \frac{1-H}{\gamma H}$ and where the forward looking variable is consumption: $F_t = c_t$.

I established that any of the two variables, wages and the return to capital, satisfies market clearing on the goods market. For illustration, I choose the return to capital r_t^k as the signal, which has the form (1.11):

$$r_t^k = \underline{G}_c k_t + \underline{G}_x \begin{bmatrix} a_t \\ \varepsilon_t \end{bmatrix} + \underline{G}_f c_t.$$

Making use of the guess for the policy function of forward looking variables, (1.16), I can write the return to capital as:

$$\begin{aligned} r_t^k &= \left(\underline{G}_c + \underline{G}_f \eta_c^p\right) k_t + \left(\underline{G}_x + \underline{G}_f \eta_x^p\right) \begin{bmatrix} a_t \\ \varepsilon_t \end{bmatrix} + \underline{G}_f \eta_e^p \begin{bmatrix} a_{t|t} \\ \varepsilon_{t|t} \end{bmatrix} \\ &= \underline{G}_c(\eta_c^p) X_t^c + \underline{G}_x(\eta_x^p) X_t + \underline{G}_e(\eta_e^p) X_{t|t}. \end{aligned}$$

Further, I define $\underline{G}_x(\eta_x^p) = \left[g_a^p \; g_\varepsilon^p\right]$. From the discussion in Section 1.4.1, I know that it is sufficient to solve the Kalman gain in the form of (1.25):

$$\tilde{\mathscr{K}} = \left\{P\underline{G}_x(\eta_x^p)'\right\}\left\{\underline{G}_x(\eta_x^p)P\underline{G}_x(\eta_x^p)'\right\}^{-1} = \begin{bmatrix} \frac{g_a^p P_{11}}{(g_a^p)^2 P_{11} + (g_\varepsilon^p)^2 P_{22}} \\ \frac{g_\varepsilon^p P_{22}}{(g_a^p)^2 P_{11} + (g_\varepsilon^p)^2 P_{22}} \end{bmatrix},$$

where I already anticipated (1.26) which shows that the variance-covariance matrix of the forecast error is a diagonal matrix:

$$\begin{aligned}\begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} &= \begin{bmatrix} \rho & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \hat{P}_{11} & \hat{P}_{12} \\ \hat{P}_{21} & \hat{P}_{22} \end{bmatrix}\begin{bmatrix} \rho & 0 \\ 0 & 0 \end{bmatrix}' + \begin{bmatrix} \sigma_v & 0 \\ 0 & \sigma_\varepsilon \end{bmatrix}\begin{bmatrix} \sigma_v & 0 \\ 0 & \sigma_\varepsilon \end{bmatrix}' \\ &= \begin{bmatrix} \rho^2 \hat{P}_{11} + \sigma_v^2 & 0 \\ 0 & \sigma_\varepsilon^2 \end{bmatrix}. \end{aligned} \tag{1.26}$$

Finally, I need to determine the MSE, (1.25), $\hat{P} = \left(I - \tilde{\mathscr{K}}\underline{G}_x(\eta_x^p)\right)P$:

$$\begin{bmatrix} \hat{P}_{11} & \hat{P}_{12} \\ \hat{P}_{21} & \hat{P}_{22} \end{bmatrix} = \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} \frac{g_a^p(\rho^2\hat{P}_{11}+\sigma_v^2)}{(g_a^p)^2(\rho^2\hat{P}_{11}+\sigma_v^2)+(g_\varepsilon^p)^2\sigma_\varepsilon^2} \\ \frac{g_\varepsilon^p \sigma_\varepsilon^2}{(g_a^p)^2(\rho^2\hat{P}_{11}+\sigma_v^2)+(g_\varepsilon^p)^2\sigma_\varepsilon^2} \end{bmatrix}\begin{bmatrix} g_a^p & g_\varepsilon^p \end{bmatrix}\right)\begin{bmatrix} \rho^2\hat{P}_{11}+\sigma_v^2 & 0 \\ 0 & \sigma_\varepsilon^2 \end{bmatrix}.$$

As only $\hat{P}_{11}$ is unknown, I determine the MSE simply by solving the quadratic equation associated with the first element in $\hat{P}$.

$$0 = \rho^2(g_a^p)^2\left(\hat{P}_{11}\right)^2 + \hat{P}_{11}\left((g_\varepsilon^p)^2\sigma_v^2 + (1-\rho^2)(g_\varepsilon^p)^2\sigma_\varepsilon^2\right) - (g_\varepsilon^p)^2\sigma_v^2\sigma_\varepsilon^2$$

With the solution to the Kalman gain, I find the matrices $M_e^x\left(\mathscr{K},\eta_x^p\right)$, $M_e^e\left(\mathscr{K},\eta_x^p\right)$ and $N_e\left(\mathscr{K},\eta_x^p\right)$ in (1.18) to be:

$$M_e^x\left(\mathscr{K},\eta_x^p\right) = \tilde{\mathscr{K}}\underline{G}_x(\eta_x^p)M_x^x = \begin{bmatrix} \frac{\rho(g_a^p)^2(\rho^2\hat{P}_{11}+\sigma_v^2)}{(g_a^p)^2(\rho^2\hat{P}_{11}+\sigma_v^2)+(g_\varepsilon^p)^2\sigma_\varepsilon^2} & 0 \\ \frac{\rho(g_\varepsilon^p)^2(\rho^2\hat{P}_{11}+\sigma_v^2)}{(g_a^p)^2(\rho^2\hat{P}_{11}+\sigma_v^2)+(g_\varepsilon^p)^2\sigma_\varepsilon^2} & 0 \end{bmatrix},$$

$$M_e^e\left(\mathscr{K},\eta_x^p\right) = (I - \tilde{\mathscr{K}}\underline{G}_x(\eta_x^p))M_x^x = \begin{bmatrix} \frac{\rho(g_\varepsilon^p)^2(\rho^2\hat{P}_{11}+\sigma_v^2)}{(g_a^p)^2(\rho^2\hat{P}_{11}+\sigma_v^2)+(g_\varepsilon^p)^2\sigma_\varepsilon^2} & 0 \\ \frac{-\rho(g_\varepsilon^p)^2(\rho^2\hat{P}_{11}+\sigma_v^2)}{(g_a^p)^2(\rho^2\hat{P}_{11}+\sigma_v^2)+(g_\varepsilon^p)^2\sigma_\varepsilon^2} & 0 \end{bmatrix} \text{ and}$$

$$N_e\left(\mathscr{K},\eta_x^p\right) = \tilde{\mathscr{K}}\underline{G}_x(\eta_x^p)N_x = \begin{bmatrix} \frac{(g_\varepsilon^p)^2(\rho^2\hat{P}_{11}+\sigma_v^2)\sigma_v^2}{(g_a^p)^2(\rho^2\hat{P}_{11}+\sigma_v^2)+(g_\varepsilon^p)^2\sigma_\varepsilon^2} & \frac{g_a^p g_\varepsilon^p(\rho^2\hat{P}_{11}+\sigma_v^2)\sigma_\varepsilon^2}{(g_a^p)^2(\rho^2\hat{P}_{11}+\sigma_v^2)+(g_\varepsilon^p)^2\sigma_\varepsilon^2} \\ \frac{g_\varepsilon^p g_a^p(\rho^2\hat{P}_{11}+\sigma_v^2)\sigma_v^2}{(g_a^p)^2(\rho^2\hat{P}_{11}+\sigma_v^2)+(g_\varepsilon^p)^2\sigma_\varepsilon^2} & \frac{(g_\varepsilon^p)^2(\rho^2\hat{P}_{11}+\sigma_v^2)\sigma_\varepsilon^2}{(g_a^p)^2(\rho^2\hat{P}_{11}+\sigma_v^2)+(g_\varepsilon^p)^2\sigma_\varepsilon^2} \end{bmatrix}.$$

Table 1.1 Calibration - structural parameters

Parameters	γ	α	δ	β	ρ_a	σ_v	σ_ε	$\sigma_{\varepsilon j}$	$\bar{H}$
Values	5	1/3	0.025	0.99	0.90	0.01	0.03	0.10	0.2

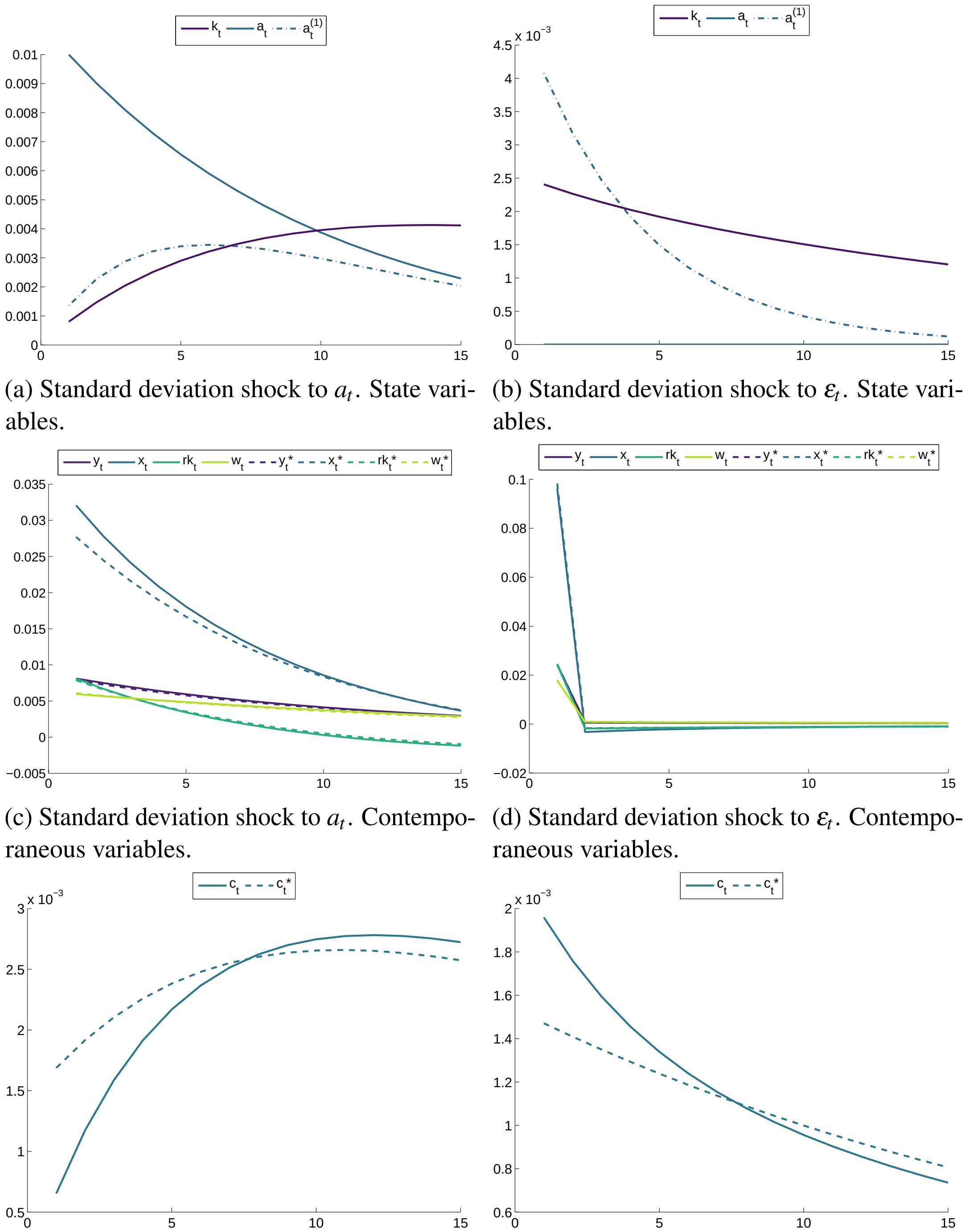
So far I solved the filtering problem conditional on η_x^p, which is part of the terms $\left[g_a^p\ g_\varepsilon^p\right] = \underline{G}_x\left(\eta_x^p\right)$. The matrix η_x^p, mapping the forward looking variables on the exogenous state variables, is not identified yet. However, it can be identified using the Euler equation, (1.19). I cannot identify η_x^p analytically. Therefore, I compute the solution numerically. To keep my results comparable to the literature, I borrow the parameter calibration from Graham and Wright (2010).[17] The benchmark calibration is presented in Table 1.1.

Graph (a) of Figure 1.1 shows the impulse response function of the state variables of the partial equilibrium model to a one standard deviation shock to the persistent component of productivity. The graph shows that the agents learn slowly about the components of productivity. The second graph (b) shows the response of the exogenous state variables to an one standard deviation innovation of the transitory component. Also here, the agents attribute weight to both processes. Capital is known under partial information. Therefore only the realization and not the expectation of capital is displayed.

Graph (c) shows the response of production, investment, the return to capital and wages to an innovation to the persistent productivity process under partial information (solid lines) as well as under full information solution (dashed lines). The impulse response of consumption is shown in graph (e). The actual difference between the dynamics of the partial to the full information solution is very small. The largest difference occurs with respect to consumption and investment. The reason is that as agents believe that the innovation in productivity is transitory. They invest relatively more and consume less relative to the full information model. The opposite phenomenon occurs when there is an innovation to the transitory component of productivity, shown in the graphs (d) and (f).

It is important to note that without the two aggregate components of composite productivity, the model does not induce learning and the impulse responses were equal to the full information version of the model.

[17]In the model at hand I do not allow for a balanced growth path and I assume idiosyncratic productivity to be transitory. These differences do not affect the quantitative results significantly, but they avoid unnecessary complications.



(a) Standard deviation shock to a_t. State variables.

(b) Standard deviation shock to ε_t. State variables.

(c) Standard deviation shock to a_t. Contemporaneous variables.

(d) Standard deviation shock to ε_t. Contemporaneous variables.

(e) Standard deviation shock to a_t. Forward looking variables.

(f) Standard deviation shock to ε_t. Forward looking variables.

Fig. 1.1 Impulse responses. Partial information model.

The graphs (a) and (b) show the impulse response functions of the state variables to innovations of the aggregate components of productivity. The solid lines represents the impulse responses of the realized state variables and the dashed lines the ones of the expectation of the state variables. The graphs (c) and (d) show the impulse responses of selected contemporaneous to the same innovations. The solid lines represent the responses under partial information and the dashed lines responses under full information. The graphs in the third row (e) and (f) display the impulse response functions of forward looking variables also under partial (solid line) and full information (dashed line).

1.5 Heterogeneous Information

The procedure to solve for the model with heterogeneous information follows the one for partial information with the only difference that the filtering problem becomes more complex. With heterogeneous information agents do not behave alike. The relevant state variables and the information set include the whole model presented in Section 1.2.

Before I proceed, I establish the fact that individual capital, as an individual predetermined state variable, is part of the information set while aggregate capital is not necessarily part of it.

Proposition 4. *Individual capital is known to the agents in a model with heterogeneous information.*

Proof. See Appendix A.2.3. □

The case is not so clear with aggregate capital as there is a fundamental and a non-fundamental solution to the model with heterogeneous information. Capital is only known to the agents in the fundamental solution.

Proposition 5. *There is a fundamental solution to the heterogeneous information model in which aggregate capital is known to the agents and a non-fundamental solution in which aggregate capital is unknown to the agents.*

Proof. See Appendix A.2.3. □

The intuition for this result is the following. If there is a public signal about at least two distinct aggregate state variables, there is an equilibrium in which the agents agree on using the signal to identify one of the state variables and form expectation about the other, using the remaining signals. Alternatively, they can use all signals to form expectation about the variables jointly. In the case at hand, the return to capital is a public signal about aggregate capital and aggregate productivity. The fundamental solution arises if the agents use the return to capital to identify capital and use the individual wage to form their expectation about aggregate productivity. The non-fundamental solution arises if the agents use the return to capital and individual wages jointly to form expectations about both aggregate capital and aggregate productivity.

Moreover, before I look at the details of the solution, I introduce the concept of higher order expectations.

1.5.1 Higher order expectations

With heterogeneous information the law of iterated expectations typically does not hold. This gives rise to higher order expectations. For any vector of aggregate state variables Γ_t, I define the first order expectation as the integral over the individual expectation of all agents in the economy about that variable, as it is common in the literature. Formally this means:

$$\Gamma_t^{(1)} = \int E\left[\Gamma_t | \Omega_{jt}\right] dj. \tag{1.27}$$

Any higher order expectation o is defined as:

$$\Gamma_t^{(o)} = \int E\left[\Gamma_t^{(o-1)} | \Omega_{jt}\right] dj. \tag{1.28}$$

For compactness, I summarize the complete hierarchy of higher order expectations in the vector $\Gamma_t^{(0:\infty)}$,

$$\Gamma_t^{(0:\infty)} = \begin{bmatrix} \Gamma_t' & \Gamma_t^{(1)'} & \Gamma_t^{(2)'} & \dots & \Gamma_t^{(\infty)'} \end{bmatrix}'. \tag{1.29}$$

The source of the higher order expectation is the Euler equation for forward looking variables (1.13). Individual forward looking variables might depend on future aggregate state variables either directly or indirectly via aggregate forward looking variables or aggregate contemporaneous variables. These variables might follow an aggregate state law of motion which by itself depends on aggregate forward looking variables. If one is aggregating over the individual forward looking variable in order to forward substitute the aggregate forward looking variable one finds that individual forward looking variables depend on higher order expectations.

As in Nimark (2011), I will approximate the infinite hierarchy of expectations by a finite dimension. I define the highest order of expectation to be considered as $\overline{o}$.

1.5.2 State space representation

Also with heterogeneous information, I need to guess the functional form of the individual forward looking variables. But before I proceed, I define $\Gamma_t = [X_t^{c\prime}\, X_t']'$ and $\Gamma_{jt} = \left[X_{jt}^{c\prime}\, X_{jt}'\right]'$ to keep the notation clean. I guess that individual forward looking variables depend on individual state variables, on the hierarchy of expectations of aggregate state variables, the individual expectation thereof and the individual expectation of idiosyncratic exogenous state variables:

$$F_{jt} = \eta_{fj} \left[\Gamma_t^{(0:\overline{o})\prime} \quad \Gamma_{t|jt}^{(0:\overline{o})\prime} \quad \Gamma_{jt}^{\prime} \quad X_{jt|jt}^{\prime} \right]^{\prime}, \text{ where } \eta_{fj} = \left[\eta_{fj}^{e\Gamma} \quad \eta_{fj}^{ej\Gamma} \quad \eta_{fj}^{\Gamma j} \quad \eta_{fj}^{ejxj} \right]. \quad (1.30)$$

Note that there is never a hierarchy of expectation of individual state variables, which stands in contrast to the formulation of Graham and Wright (2010). The reason is that individual variables are private information, hence the agent's best expectation from island j about idiosyncratic variables on island i is equal to the expectation about the aggregate variable, i.e. $E[X_{it}|\Omega_{jt}] = E[X_t|\Omega_{jt}]$ and the aggregate expectation is $\int E[X_t|\Omega_{jt}]\, dj = X_{t|t}^{(1)}$. Therefore, my guess for forward looking variables is different from the one of Graham and Wright (2010). They guess that $F_{jt} = \eta_{fj}\left[\Gamma_{t|jt}^{(0:\overline{o})\prime}\ \Gamma_{jt|jt}^{(0:\overline{o})\prime}\right]^{\prime}$, which neglects the effects of Proposition 4 that individual capital is part of the information set and they formulate a hierarchy of expectations of individual state variables. There is another complication which Graham and Wright (2010) do not consider. Individual and aggregate capital are among others a function of signals which appear outside the expectation operator.

I find the functional form of the aggregate forward looking variables by integrating over the individual ones of all islands:

$$F_t = \eta_f \left[\Gamma_t^{(0:\overline{o})\prime} \quad \Gamma_{t|jt}^{(0:\overline{o})\prime} \quad \Gamma_{jt}^{\prime} \quad X_{jt|jt}^{\prime} \right]^{\prime}, \text{ where } \quad \eta_f = \eta_{fj} T_e. \quad (1.31)$$

The matrix T_e maps the individual state variables and the individual expectation of the hierarchy of expectation to the aggregate hierarchy of expectation. The idiosyncratic exogenous state variables and the expectation thereof integrate to zero by assumption. The guess of the forward looking variables implies an extended state law of motion, which includes the full hierarchy of expectation, the individual expectations thereof, individual state variables and the individual expectation of idiosyncratic exogenous state variables:

$$\begin{bmatrix} \Gamma_{t+1}^{(0:\bar{o})} \\ \Gamma_{t+1|j,t+1}^{(0:\bar{o})} \\ \Gamma_{j,t+1} \\ X_{j,t+1|j,t+1} \end{bmatrix} = \begin{bmatrix} M_{e\Gamma}^{e\Gamma} & 0 & 0 & 0 \\ M_{ej\Gamma}^{e\Gamma} & M_{ej\Gamma}^{ej\Gamma} & M_{ej\Gamma}^{\Gamma j} & M_{ej\Gamma}^{ejxj} \\ M_{\Gamma j}^{e\Gamma} & M_{\Gamma j}^{ej\Gamma} & M_{\Gamma j}^{\Gamma j} & M_{\Gamma j}^{ejxj} \\ M_{ejxj}^{e\Gamma} & M_{ejxj}^{ej\Gamma} & M_{ejxj}^{\Gamma j} & M_{ejxj}^{ejxj} \end{bmatrix} \begin{bmatrix} \Gamma_{t}^{(0:\bar{o})} \\ \Gamma_{t|jt}^{(0:\bar{o})} \\ \Gamma_{jt} \\ X_{jt|jt} \end{bmatrix} + \begin{bmatrix} N_{e\Gamma} \\ N_{ej\Gamma} \\ N_{\Gamma j} \\ N_{ejxj} \end{bmatrix} \begin{bmatrix} s_{j,t+1} \\ s_{t+1} \end{bmatrix} \text{ and} \tag{1.32}$$

$$\begin{bmatrix} \Upsilon_{jt} \\ \Upsilon_{t} \end{bmatrix} = \begin{bmatrix} G_{e\Gamma} & G_{ej\Gamma} & G_{\Gamma j} & G_{ejxj} \end{bmatrix} \begin{bmatrix} \Gamma_{t}^{(0:\bar{o})\prime} & \Gamma_{t|jt}^{(0:\bar{o})\prime} & \Gamma_{jt}^{\prime} & X_{jt|jt}^{\prime} \end{bmatrix}^{\prime}.$$

The state space system depends, as with partial information, on the solution of the forward looking variables and on the Kalman gain, which is derived as part of the filtering problem derived in Section 1.5.3.[18]

I confirm my guess of the individual forward looking variables analogue to the partial information case:

$$\eta_{fj} = C^{0}\left(\eta_{f}\right) + C^{1}\left(\eta\right)\left(T_{k_j}M + T_{\neq k_j}MT_{ej}\right), \tag{1.33}$$

where T_{ej} is a matrix which adds up the parameters of the non-expectational state variables to the expectational ones. The solution of the heterogeneous information model depends on the fixed point of the policy function for forward looking variables identified by the Euler equation (1.33) and the Kalman gain derived as part of the filtering problem.

1.5.3 Signal extraction problem

In this subsection, I discuss the filtering problem of agents receiving heterogeneous information. As in the case with partial information, I assume that the agents have already observed a sufficiently long history of signals such that the Kalman gain has converged to the steady state Kalman gain. The Kalman updating equation is defined as follows:

$$\begin{bmatrix} \Gamma_{t+1|j,t+1}^{(0:\bar{o})} \\ X_{j,t+1|j,t+1} \end{bmatrix} - \begin{bmatrix} \Gamma_{t+1|jt}^{(0:\bar{o})} \\ X_{j,t+1|jt} \end{bmatrix} = \begin{bmatrix} \mathscr{K}_{ej\Gamma} \\ \mathscr{K}_{ejxj} \end{bmatrix} \left(\begin{bmatrix} \Upsilon_{t+1} \\ \Upsilon_{j,t+1} \end{bmatrix} - \begin{bmatrix} \Upsilon_{t+1|jt} \\ \Upsilon_{j,t+1|jt} \end{bmatrix} \right). \tag{1.34}$$

[18]The matrices of the heterogeneous information solution are described in detail in Appendix A.3.3.

In addition, I compute the forecast error of the signals as:

$$\begin{bmatrix} \underline{\Upsilon}_{t+1} \\ \underline{\Upsilon}_{j,t+1} \end{bmatrix} - \begin{bmatrix} \underline{\Upsilon}_{t+1|jt} \\ \underline{\Upsilon}_{j,t+1|jt} \end{bmatrix} = J^{-1}\underline{G}_1 \left(\begin{bmatrix} \Gamma^{(0:\overline{o})}_{t+1} \\ X_{j,t+1} \end{bmatrix} - \begin{bmatrix} \Gamma^{(0:\overline{o})}_{t+1|jt} \\ X_{j,t+1|jt} \end{bmatrix} \right), \tag{1.35}$$

where $J = \left(I - \begin{bmatrix} \underline{G}_{ej\Gamma} & \underline{G}_{ejxj} \end{bmatrix} \mathscr{K}\right)$, $\underline{G}_1 = \begin{bmatrix} \underline{G}_{e\Gamma} & \underline{G}_{xj} \end{bmatrix}$ and $\mathscr{K} = \begin{bmatrix} \mathscr{K}'_{ej\Gamma} & \mathscr{K}'_{ejxj} \end{bmatrix}'$.

Plug (1.35) back into the Kalman updating equation (1.34) to find:

$$\begin{bmatrix} \Gamma^{(0:\overline{o})}_{t+1|j,t+1} \\ X_{j,t+1|j,t+1} \end{bmatrix} = \left(I - \mathscr{K}J^{-1}\underline{G}_1\right) \begin{bmatrix} \Gamma^{(0:\overline{o})}_{t+1|jt} \\ X_{j,t+1|jt} \end{bmatrix} + \mathscr{K}J^{-1}\underline{G}_1 \begin{bmatrix} \Gamma^{(0:\overline{o})}_{t+1} \\ X_{j,t+1} \end{bmatrix}. \tag{1.36}$$

Now, I make use of the guess for the state law of motion, (1.32), from which I select the transition and the impact matrix of the hierarchy of expectation and the idiosyncratic exogenous state variables:

$$A = \begin{bmatrix} A_{e\Gamma} & A_{xj} \end{bmatrix} = \begin{bmatrix} M^{e\Gamma}_{e\Gamma} & 0 \\ 0 & M^{xj}_{xj} \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} N_{e\Gamma} \\ N_{xj} \end{bmatrix}.$$

Using the state law of motion of the hierarchy of expectations and of idiosyncratic exogenous state variables, I find the state law of motion of the individual expectation of the hierarchy of expectation and the individual expectation of idiosyncratic exogenous productivity as defined in (1.32):

$$\begin{aligned} \begin{bmatrix} \Gamma^{(0:\overline{o})}_{t+1|j,t+1} \\ X_{j,t+1|j,t+1} \end{bmatrix} &= \begin{bmatrix} \left(I - \mathscr{K}_{ej\Gamma}J^{-1}\underline{G}_1\right)A_{e\Gamma} & \left(I - \mathscr{K}_{ej\Gamma}J^{-1}\underline{G}_1\right)A_{xj} \\ \left(I - \mathscr{K}_{ejxj}J^{-1}\underline{G}_1\right)A_{e\Gamma} & \left(I - \mathscr{K}_{ejxj}J^{-1}\underline{G}_1\right)A_{xj} \end{bmatrix} \begin{bmatrix} \Gamma^{(0:\overline{o})}_{t|jt} \\ X_{jt|jt} \end{bmatrix} \\ &+ \begin{bmatrix} \mathscr{K}_{ej\Gamma}J^{-1}\underline{G}_1A_{e\Gamma} & \mathscr{K}_{ej\Gamma}J^{-1}\underline{G}_1A_{xj} \\ \mathscr{K}_{ejxj}J^{-1}\underline{G}_1A_{e\Gamma} & \mathscr{K}_{ejxj}J^{-1}\underline{G}_1A_{xj} \end{bmatrix} \begin{bmatrix} \Gamma^{(0:\overline{o})}_{t} \\ X_{jt} \end{bmatrix} + \begin{bmatrix} \mathscr{K}_{ej\Gamma}J^{-1}\underline{G}_1B \\ \mathscr{K}_{ejxj}J^{-1}\underline{G}_1B \end{bmatrix} \begin{bmatrix} s_{jt} \\ s_t \end{bmatrix}, \end{aligned} \tag{1.37}$$

To conclude the guess for the state law of motion I only need to find the expression for $M^{e\Gamma}_{e\Gamma}$ and $N_{e\Gamma}$, which are the transition matrix and the impact matrix of the hierarchy of expectations. To find the state law of motion of the aggregate hierarchy of expectations, I aggregate over (1.37),

$$\Gamma_{t+1}^{(1:\bar{o})} = M_{ej\Gamma}^{ej\Gamma}\Gamma_t^{(1:\bar{o})} + M_{ej\Gamma}^{e\Gamma}\Gamma_t^{(0:\bar{o})} + N_{ej\Gamma}T_s \begin{bmatrix} s_{j,t+1} \\ s_{t+1} \end{bmatrix}, \tag{1.38}$$

where T_s is a matrix that sets the parameters of idiosyncratic innovations equal to zero. Then, I verify my guess for the hierarchy of expectations by amending (1.38) with the state law of motion of the non-expectational state variables:

$$M_{e\Gamma}^{e\Gamma} = \begin{bmatrix} M_{\Gamma}^{e\Gamma} \\ M_{ej\Gamma}^{e\Gamma} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & M_{ej\Gamma}^{ej\Gamma} \end{bmatrix}, \quad \text{and} \quad N_{e\Gamma} = \begin{bmatrix} N_{\Gamma} \\ N_{ej\Gamma}T_s \end{bmatrix}.$$

The variance-covariance matrix of the one period ahead forecast error:

$$P = A\hat{P}A' + BB', \tag{1.39}$$

the MSE :

$$\hat{P} = \left(I - \mathscr{K} J^{-1}\underline{G}_1\right)P. \tag{1.40}$$

and the Kalman gain, can be derived analogue to the partial information case:

$$\mathscr{K} = \{P\underline{G}_1'\}\{\underline{G}_1 P\underline{G}_1'\}^{-1}J. \tag{1.41}$$

As with partial information, I multiply out J from the Kalman gain. This makes the filtering problem independent of individual endogenous predetermined state variables, the individual expectation of the hierarchy of expectations and the individual expectation of idiosyncratic exogenous state variables:

$$\tilde{\mathscr{K}} = \{P\underline{G}_1'\}\{\underline{G}_1 P\underline{G}_1'\}^{-1} = KJ^{-1} \quad \text{and} \quad \hat{P} = \left(I - \tilde{\mathscr{K}}\underline{G}_1\right)P. \tag{1.42}$$

1.5.4 Numerical results

In this subsection, I assess the model dynamics with heterogeneous information. I cannot solve the filtering problem analytically as in the previous section on partial information as the state space is a lot larger with the existence of higher order expectations. Therefore, I solve the fixed point problem of the policy function for the forward looking variables and the filtering extraction problem numerically altogether.

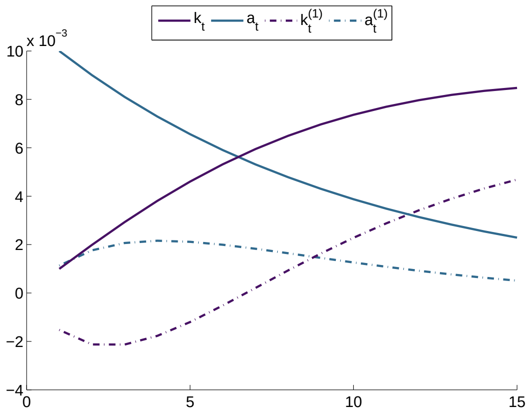
Further, I showed in Proposition 5 that there are two possible solutions to the model with heterogeneous information. As the model dynamics of the fundamental solution are equivalent to the partial information model, I only present the dynamics of the non-fundamental solution and refer to Figure 1.1 for the dynamics of the fundamental solution.

I choose the signals to be market consistent, as defined by Graham and Wright (2010), which means that the only signals about the state of the world are the return to capital and individual wages. The impulse response functions of the model are displayed in Figure 1.2. The impulse responses of the state variables (solid line) and the first order expectation thereof (dashed line) are shown in the first two graphs. Graph (a) shows the responses to a standard deviation shock to the persistent component of productivity and Graph (b) to a shock to the transitory component.

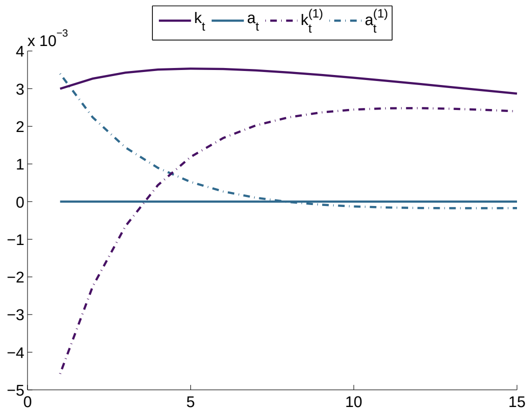
The most important aspect of the first two graphs is the expectation about capital. An increase in the return to capital can stem from an increase in productivity or because the agents previously overestimated the capital stock. Graham and Wright (2010) describe the latter as a pseudo-shock. In this case, the agents revise downwards their expectation about the aggregate capital stock and hence agents invest more and realized capital is higher than under full information. In the RBC model, the agents expectation that the pseudo-shock moved the return to capital weighs stronger than their expectation that productivity increased.

With regard to productivity, agents cannot clearly disentangle the source of the shocks. Graph (a) shows that agents expect the transitory component to be the source of the shock rather than the persistent component, even if the origin of the innovation was actually the persistent process. The reason is that the agents attribute a larger weight to the more volatile process. Consequently, the expectation to a shock to the transitory component, shown in graph (b), are closer to the underlying shock. Altogether, the impulse responses look different to the one in Graham and Wright (2010) as I include two aggregate components in composite productivity. If I set $\sigma_\varepsilon = 0$ and $\sigma_\nu = 1$ then the impulse responses are very similar to theirs. The remaining difference stems from the changes in the conclusions that I discussed with regard to the solution methodology.

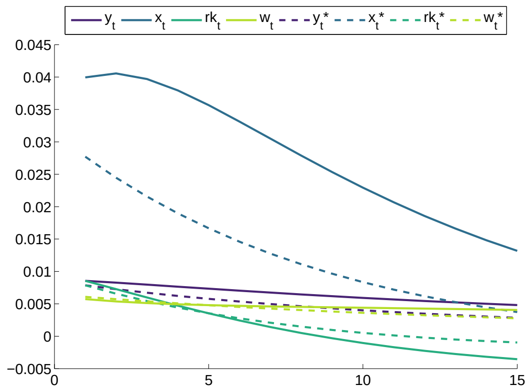
In the second row, the two graphs display the impulse response function of selected contemporaneous variables under heterogeneous information (solid line) and under full information (dashed line). Graph (c) shows the responses to a standard deviation innovation to the persistent component of productivity and graph (d) to the temporary one. The last two graphs (e) and (f) show the impulse responses of consumption, also under heterogeneous



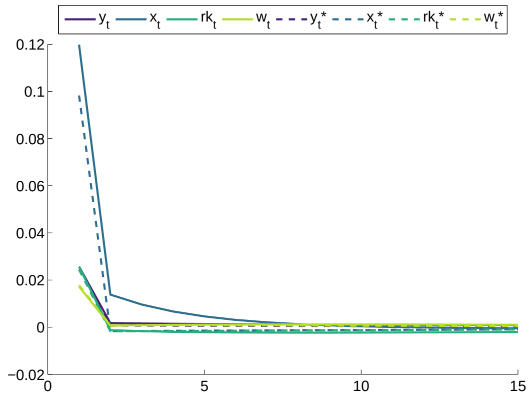
(a) Standard deviation shock to a_t. State variables.



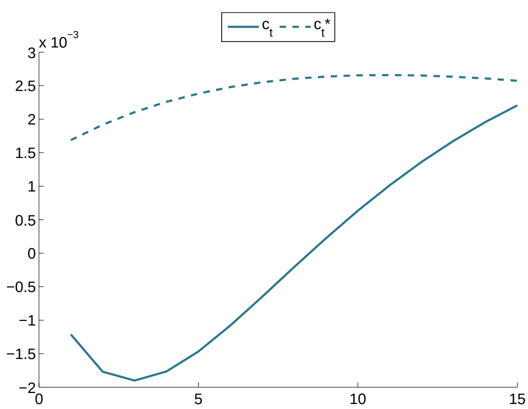
(b) Standard deviation shock to ε_t. State variables.



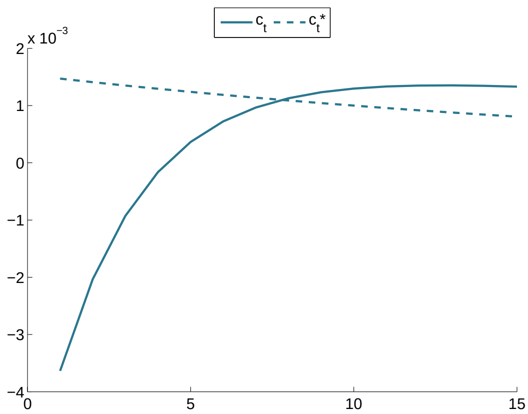
(c) Standard deviation shock to a_t. Contemporaneous variables.



(d) Standard deviation shock to ε_t. Contemporaneous variables.



(e) Standard deviation shock to a_t. Forward looking variables.



(f) Standard deviation shock to ε_t. Forward looking variables.

Fig. 1.2 Impulse responses of the heterogeneous information model. Non-fundamental solution.

The graphs (a) and (b) show the impulse response functions of the state variables to innovations of the aggregate components of productivity. The solid lines represents the impulse responses of the realized state variables and the dashed line of the expectation of the state variables. The graphs (c) and (d) show the impulse responses of selected contemporaneous to the same innovations. The solid lines represent the responses under heterogeneous information and the dashed lines the responses under full information. The graphs in the third row (e) and (f) display the impulse response functions of forward looking variables also under heterogeneous and full information.

and full information. If one compares the impulse responses to the partial information solution, which is equal to the fundamental solution, one sees that the pseudo-shock has a significant impact on the dynamics of the model dynamics. Investment increases significantly to persistent and to transitory shocks. With partial information instead, investment increased only in response to a persistent shock and decreased to a transitory shock compared to the full information solution.

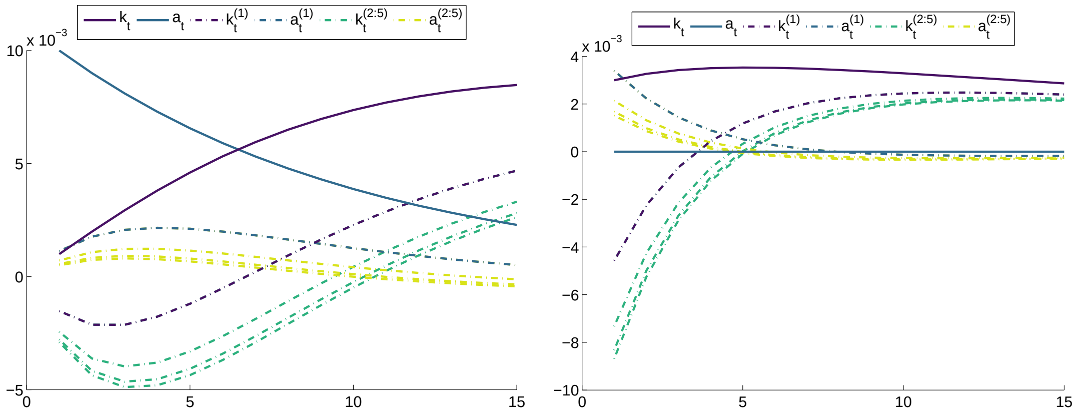
1.5.5 Discussion

The impulse responses of the state variables from the non-fundamental solution of the heterogeneous information model are shown in Figure 1.3. The solid lines show the realizations of capital and the aggregate persistent component of productivity. The first order expectation are plotted as dashed lines of the same colour. The first order of expectation is negative to the positive productivity shock and the first order expectation of productivity is positive but significantly lower than the realization of productivity. The higher order expectation of capital are plotted in green and follow in increasing order. In this model higher order expectations of capital respond stronger to the shock than lower ones. However, this leads to a stable solution as the expectations converge to one another. The higher order expectations of productivity are ordered from lower orders of expectations to higher orders and they are plotted in yellow.

The puzzle about the non-fundamental solution stated by Graham and Wright (2010) stems from the observation that the first order expectation of capital responds negative to a positive productivity shock. They argue that one should compare the first order expectation to the empirical data to be consistent with the model assumption that capital is not directly observable. Based on empirical analysis, however, one would expect a positive response of capital to a positive productivity shock.

Altogether, the dynamics of the non-fundamental solution that I present in this paper are in line with the ones in the literature. I make only different findings with regard to the potential source for the phenomenon. It cannot come from the explosive root in individual capital, as claimed by Baxter et al. (2011) and Graham and Wright (2010),[19] as it is not part of the filtering problem and the non-fundamental solution arises nevertheless.

[19] Collary 5 in Baxter et al. (2011) states that a model with dynamic endogeneity might exhibit explosive eigenvalues and non-invertible information sets.



(a) Unit shock to a_t. State variables. (b) Unit shock to ε_t. State variables.

Fig. 1.3 Impulse responses of the heterogeneous information model. Non-fundamental solution. Higher order expectations.

Graph (a) shows the impulse response functions of capital and productivity and the higher order expectation thereof to a positive shock to the persistent aggregate component of productivity. The realizations of the variables are plotted in solid lines, the first order condition in dashed lines of the same colour and the remaining order of expectations as dashed lines of another colour. Graph (b) shows the impulse responses of the same variables to a positive shock of the transitory aggregate component of productivity.

1.6 Conclusion

In this paper, I have revisited the results of the literature on the effects of incomplete information in the real business cycle (RBC) model. I introduce the notion of predetermined state variables in the context of incomplete information and show that the filtering problem of the model with partial information can be simplified to exogenous productivity process only. Further, the guess for the policy function of forward looking variables is a function of state variables and expectational variables and not only of expectational variables.

With regard to the heterogeneous information model my contribution extends even further. Also here, I show that individual capital, as the predetermined state variable, is never part of the signal extraction problem. This changes the guess for the forward looking variables, too. Moreover, I correct the formulation of higher order expectation with regard to individual state variables. Most importantly, I show that there are two stable solutions to the heterogeneous information model. One in which the state law of motion is a fundamental representation of the state variables and one in which it is not. The fundamental solution coincides with the one of the partial information model, while the second corresponds to the case in the literature. It is also worth mentioning that the non-fundamental solution appears also with the changes introduced in this paper. Hence, it cannot stem from an interaction between an explosive part in individual capital and the signal extraction problem, as individual capital is not part of it.

Finally, I show that the return to capital, individual wages and idiosyncratic composite productivity need to be revealed by the signals to guarantee market clearing on the goods market.

This means that the RBC model of Section 1.2 is not particularly interesting to include either partial or heterogeneous information, besides the appearance of the non-fundamental solution. However, the literature has followed other avenues to make use of incomplete information in DSGE models. Blanchard et al. (2013) use a medium scaled DSGE model with partial information to back out the impact of noise shocks on business cycle dynamics. Further, it has been shown by Woodford (2003), Lorenzoni (2009) and Angeletos and La'O (2009) that heterogeneous information play a significant role for the dynamics of a model if it exhibits strategic complementarity. Following this strand of literature, I analyse a heterogeneous information new Keynesian model with capital, which exhibits strategic complementaries in price setting in my companion paper. In Schaefer (2019b), I additionally develop a solution methodology that can handle contemporaneous exogenous and endogenous as well as predetermined endogenous state variables and that shows more robust properties than the one by Graham and Wright (2010).

Chapter 2

Solving Heterogeneous Information Dynamic Stochastic General Equilibrium Models Easily

2.1 Introduction

Heterogeneous information dynamic stochastic general equilibrium (HI-DSGE) models relax the assumption of perfect information compared to classic DSGE models, while the assumption of rationally behaving agents remains unchanged. This class of models is of particular interest for economists as heterogeneous information can serve as a micro-foundation for persistence in macroeconomic variables. The persistence is generated as agents need to learn about the underlying state of the economy from signals. Heterogeneous information models are characterized by the fact that agents receive private and public signals about the state of the economy. In this sense they are different from partial information models that are characterised by public signals only. Moreover, heterogeneous information models usually generate more persistence in macroeconomic variables compared to partial information models.

On the one hand, the literature on partial information includes, among others, the work of Blanchard et al. (2013) who estimate a new Keynesian model with capital under partial information and use it to back out noise and structural shocks from the data. Levine et al. (2012) also contribute to the estimation of partial information models. Collard and Dellas (2010) evaluate how partial information and the corresponding signal extraction problem can generate persistence in macroeconomic variables at the example of a new Keynesian model. Baxter et al. (2011) derive conditions for stability of models with partial information. Schaefer (2019a) revisits their analysis and shows that they are imprecise with respect to the treatment of capital in their model which affects the dynamics of the model. Further, Svensson and Woodford (2003) discuss optimal monetary policy in a new Keynesian model with partial information.

On the other hand, there is a growing strand of literature on heterogeneous information models. However, they are mostly relatively small-scaled. Moreover, each of the papers presents an individual approach on how to solve the model. Melosi (2014) estimates a new Keynesian model with heterogeneous information by Woodford (2003) with Bayesian methods. Nimark (2014) estimates a heterogeneous information model with man-bites-dog signals and shows that these signals can generate large non-fundamental changes in macroeconomic variables. Further, Nimark (2011) discusses an asset pricing model and proposes a methodology to handle higher order expectations, which I use to solve for higher order expectations that arise in HI-DSGE models. Lorenzoni (2010) derives optimal monetary policy in a new Keynesian model with heterogeneous information and in which agents can insure against their idiosyncratic risk. Moreover, Lorenzoni (2009) shows that heterogeneous information can generate a substantial amount of demand shocks driving the business cycle. Angeletos and La'O (2009) analyse a real business cycle model without capital and Hellwig and Venkateswaran (2009) discuss the price adjustment process in their model.

In a related field of research, Mackowiak and Wiederholt (2011) solve a DSGE model in which the agents are subject to rational inattention. Rational inattention models are more complex than HI-DSGE models, as the solution to the models includes a constraint optimization problem in which agents need to decide about optimal signals. This is theoretically appealing but it also makes the models more difficult to solve. However, one can understand rational inattention as a general explanation for heterogeneous information to be included in a model, as it is pointed out in the literature.

Only very few papers include capital as part of the model. Hassan and Mertens (2014) develop a solution methodology to solve an RBC model with capital in which the agents receive private information about future productivity and observe the stock price as a noisy public signal about the economy. The difference to the methodology that I present at the paper at hand is that my methodology can handle dynamic learning problems in which higher order expectations arise. Graham and Wright (2010) also study an RBC model with heterogeneous information. Further, they apply the insights of Baxter et al. (2011) to construct a solution methodology that allows to solve incomplete information models. Schaefer (2019a) revisits their analysis and shows that the model accommodates two possible equilibria from which one resembles to the partial information model.

In this paper, I build on the insights gained in Schaefer (2019a) and develop a solution methodology that allows for contemporaneous and predetermined endogenous state variables as well as exogenous state variables. Furthermore, I exploit the structure of the class of models to its fullest to generate stability of the solution, generalizing the idea of Lorenzoni (2009). The novelty of my approach is that it holds for the whole class of models and does not need to be derived for an explicit model. This allows researchers without in-depth knowledge of the solution methodology to study HI-DSGE models. I illustrate the power of the methodology by means of a new Keynesian model with capital that has not been studied before.

The partial information models and almost all the models with heterogeneous information that I present in the literature above[1] can be cast in the state space form that I present in this paper and hence can be solved without deriving the methodologies discussed in the individual papers.

The remainder of the paper is organized as follows. In Section 2.2, I outline the structure of HI-DSGE models that I cover. Section 2.3 defines the state space representation in which researcher can cast their model. Then, I derive the solution methodology in Section 2.4. In Section 2.5, I develop a new Keynesian model with capital that is used to illustrate the methodology. Furthermore, I examine the dynamic implications of the model in Section 2.6. Section 2.7 concludes.

2.2 The general setting

HI-DSGE models are based on the assumption that the agents in the economy differ from one another in two aspects. First, the stochastic processes driving the model consists of aggregate and idiosyncratic components and second, agents observe private as well as public

[1] For the model in Nimark (2014), the code would have to be modified as it allows for stochastic changes in the precision of signals. Changes might also apply for the model in Hassan and Mertens (2014).

signals about the state of the world. Private signals are by definition only known by the individual agent receiving them, while public signals are observable by all agents in the economy. Typically, it is assumed that a group of agents share the same information set. One can imagine these agents to live in the same region. The literature refers to such a region typically as an island. Moreover, due to heterogeneous information, the model cannot be reduced to a representative agent model and it mostly leads to a hierarchy of expectations as the law of iterated expectations fails. The conditions under which the law of iterated expectation actually fails are derived in Section 2.3.1.

I formalize this type of economy as follows. I assume there to be an infinite number of islands j. The agents on each island receive public, $\underline{\Upsilon}_t$, and private signals, $\underline{\Upsilon}_{jt}$, about aggregate state variables, Γ_t, and idiosyncratic exogenous state variables, X_{jt}. The agents on island j share the same information set Ω_{jt} in time period t. The information set includes the complete history of current and past signals, $\Omega_{jt} = \{\underline{\Upsilon}_{j\tau}, \underline{\Upsilon}_{\tau} | \tau = 0,1,2,...,t\}$ and the knowledge about the structure of the economy. While agents generally do not observe innovations of the stochastic processes in the economy, they know the distribution of separate innovations to occur.[2] Further, all agents apply and know that all agents apply the Kalman filter in order to form their expectation about the state variables.

I define the individual expectation of an arbitrary vector of variables X_t as follows: $E\{X_t|\Omega_{jt}\} = E_{jt}\{X_t\} = X_{t|jt}$. Moreover, I write aggregate expectations as $\int E\{X_t|\Omega_{jt}\}\, dj = E\{X_t|\Omega_t\} = \bar{E}_t\{X_t\} = X_{t|t}$.

2.3 State space representation

In this section, I describe the general state space representation in which researcher can cast their log-linearised HI-DSGE model and which I use to derive the solution.[3]

[2]The underlying idea of this assumption is that agents have received already a sufficiently long history of signals such that the Kalman filter converged to the steady state Kalman filter.

[3]Generally, it is possible to allow for a second order approximation of the model and still be able to solve the model. The complication of the second order approximation arises from the fact that for this purpose one needs to tackle a quadratic filtering problem, too. A good starting point to do this is the idea from Monfort et al. (2015). However, extending the methodology to include second order approximation is beyond the scope of this paper.

I distinguish between predetermined endogenous state variables, X^c_{jt}, X^c_t, contemporaneous endogenous state variables, X^n_{jt}, X^n_t and exogenous state variables, X_{jt}, X_t. Moreover, I collect aggregate and individual contemporaneous variables in the vectors Υ_t and Υ_{jt}. Aggregate and individual forward looking variables are collected in F_t and F_{jt}.

Proposition 1. *For all markets to clear it is necessary that agents take their choices based on variables that are part of their information set only.*

Proof. The budget constraint of the agents consists of prices and quantities. Agents choose their individual quantities and they are either price setters or price taker. In either case, they know the price in the markets in which they are active, which makes these prices part of their information set. Further, the quantities and prices (if they are price setter) may be set conditional on composite exogenous processes. If agents do not take their choices conditional on the composite exogenous process, but instead conditional on the individual expectation of the components, market clearing is not guaranteed. The reason is that the realisation of the process is not in line with the agents expectations, i.e. this creates a wedge in the budget constraints. Market clearing is then only achieved in the case in which the expectation coincides with the realisation. For further details, see Appendix B.2.1. □

Based on Proposition 1, I formulate the state space representation of the model. First, I define the matrices of the endogenous state variables. Realize that individual endogenous state variables come along in the following form:

$$\begin{bmatrix} A^{cj1}_{cj} & A^{nj1}_{cj} \\ 0 & A^{nj1}_{nj} \end{bmatrix} \begin{bmatrix} X^c_{j,t+1} \\ X^n_{jt} \end{bmatrix} = \begin{bmatrix} A^{cj0}_{cj} & 0 \\ A^{cj0}_{nj} & A^{nj0}_{nj} \end{bmatrix} \begin{bmatrix} X^c_{jt} \\ X^n_{j,t-1} \end{bmatrix} + \begin{bmatrix} A^{\Upsilon}_{cj} & A^{\Upsilon j}_{cj} \\ A^{\Upsilon}_{nj} & A^{\Upsilon j}_{nj} \end{bmatrix} \begin{bmatrix} \underline{\Upsilon}_t \\ \underline{\Upsilon}_{jt} \end{bmatrix} + \begin{bmatrix} A^{fj}_{cj} \\ A^{fj}_{nj} \end{bmatrix} F_{jt}. \tag{2.1}$$

Individual endogenous state variables are a function of last period's individual endogenous state variables, private and public signals and the individual forward looking variables. Private and public signals are a subset of contemporaneous variables, $\{\underline{\Upsilon}_{jt}, \underline{\Upsilon}_t\} \subseteq \{\Upsilon_{jt}, \Upsilon_t\}$. I define public signals to consist of at least one aggregate state variable and an aggregate exogenous noise term. Private signals consist of at least one aggregate or idiosyncratic exogenous state variable with an idiosyncratic exogenous noise term. Signals can also include aggregate forward looking variables. It is only important that they do not include individual endogenous state variables or individual forward looking variables. Intuitively, this is easy to understand. As the individual choices are made conditional on the state of the economy, the choice itself does not carry informational content to forecast the state of the economy. Even more importantly, it is necessary that endogenous state variables depend only on the signals and not on the unobservable components to ensure all markets to clear. Throughout the paper $s_t \sim N(0,I)$ and $s_{jt} \sim N(0,I)$ refer to aggregate and idiosyncratic innovations respectively.

Aggregate endogenous state variables can be of two types. First, integrated individual endogenous state variables define aggregate state variables, $\int X^c_{jt}\, dj = X^c_t$ and $\int X^n_{jt}\, dj = X^n_t$. Further, aggregate exogenous policy rules, such as the Taylor rule qualify by their structure as aggregate endogenous state variables:

$$\begin{bmatrix} A_c^{c1} & A_c^{n1} \\ 0 & A_n^{n1} \end{bmatrix} \begin{bmatrix} X^c_{t+1} \\ X^n_t \end{bmatrix} = \begin{bmatrix} A_c^{c0} & 0 & A_c^{x0} \\ 0 & A_n^{n0} & A_n^{x0} \end{bmatrix} \begin{bmatrix} X^c_t \\ X^n_{t-1} \\ X_{t-1} \end{bmatrix} + \begin{bmatrix} A_c^{\Upsilon} \\ A_n^{\Upsilon} \end{bmatrix} \underline{\Upsilon}_t + \begin{bmatrix} A_c^{f} \\ A_n^{f} \end{bmatrix} F_t + \begin{bmatrix} A_c^{s} \\ A_n^{s} \end{bmatrix} s_t . \tag{2.2}$$

The aggregate components of the private signals can be attributed to state variables, aggregate forward looking variables and shocks, without loss of generality. I add these to the state law of motion on the right hand side, as there are also private signals about exogenous state variables.

Exogenous idiosyncratic and exogenous aggregate state variables are simply defined as:

$$\begin{bmatrix} X_t \\ X_{jt} \end{bmatrix} = \begin{bmatrix} A_x^{x} & 0 \\ 0 & A_{xj}^{xj} \end{bmatrix} \begin{bmatrix} X_t \\ X_{jt} \end{bmatrix} + \begin{bmatrix} A_x^{s} & 0 \\ 0 & A_{xj}^{sj} \end{bmatrix} \begin{bmatrix} s_t \\ s_{jt} \end{bmatrix} . \tag{2.3}$$

For notational convenience, I stack aggregate and idiosyncratic state variables in the vectors $\Gamma_t = \left[X^{c\prime}_{t+1}\; X^{n\prime}_t\; X^{\prime}_t\right]^{\prime}$ and $\Gamma_{jt} = \left[X^{c\prime}_{j,t+1}\; X^{n\prime}_{jt}\; X^{\prime}_{jt}\right]^{\prime}$, respectively. This way, I can write the complete state law of motion as:

$$\begin{aligned} \begin{bmatrix} A_\Gamma^{\Gamma 1} & 0 \\ 0 & A_{\Gamma j}^{\Gamma j1} \end{bmatrix} \begin{bmatrix} \Gamma_t \\ \Gamma_{jt} \end{bmatrix} &= \begin{bmatrix} A_\Gamma^{\Gamma 0} & 0 \\ 0 & A_{\Gamma j}^{\Gamma j0} \end{bmatrix} \begin{bmatrix} \Gamma_{t-1} \\ \Gamma_{j,t-1} \end{bmatrix} + \begin{bmatrix} A_{\Gamma j}^{f} & 0 \\ 0 & A_{\Gamma j}^{fj} \end{bmatrix} \begin{bmatrix} F_t \\ F_{jt} \end{bmatrix} \\ &+ \begin{bmatrix} A_\Gamma^{\Upsilon} & 0 \\ A_{\Gamma j}^{\Upsilon} & A_{\Gamma j}^{\Upsilon j} \end{bmatrix} \begin{bmatrix} \underline{\Upsilon}_t \\ \underline{\Upsilon}_{jt} \end{bmatrix} + \begin{bmatrix} A_\Gamma^{s} & 0 \\ 0 & A_{\Gamma j}^{sj} \end{bmatrix} \begin{bmatrix} s_t \\ s_{jt} \end{bmatrix} . \end{aligned} \tag{2.4}$$

Second, I cast contemporaneous variables in the form:

$$\begin{bmatrix} B_{\Upsilon}^{\Upsilon} & 0 \\ B_{\Upsilon j}^{\Upsilon} & B_{\Upsilon j}^{\Upsilon j} \end{bmatrix} \begin{bmatrix} \Upsilon_t \\ \Upsilon_{jt} \end{bmatrix} = \begin{bmatrix} B_{\Upsilon}^{\Gamma 1} & 0 \\ B_{\Upsilon j}^{\Gamma 1} & B_{\Upsilon j}^{\Gamma j1} \end{bmatrix} \begin{bmatrix} \Gamma_t \\ \Gamma_{jt} \end{bmatrix} + \begin{bmatrix} B_{\Upsilon}^{\Gamma 0} & 0 \\ B_{\Upsilon j}^{\Gamma 0} & B_{\Upsilon j}^{\Gamma j0} \end{bmatrix} \begin{bmatrix} \Gamma_{t-1} \\ \Gamma_{j,t-1} \end{bmatrix} + \begin{bmatrix} B_{\Upsilon j}^{f} & 0 \\ B_{\Upsilon j}^{f} & B_{\Upsilon j}^{fj} \end{bmatrix} \begin{bmatrix} F_t \\ F_{jt} \end{bmatrix} + \begin{bmatrix} B_{\Upsilon}^{s} & 0 \\ B_{\Upsilon j}^{s} & B_{\Upsilon j}^{sj} \end{bmatrix} \begin{bmatrix} s_t \\ s_{jt} \end{bmatrix}. \tag{2.5}$$

Contemporaneous variables depend on contemporaneous and lagged state variables, forward looking variables and shocks or noise terms.

Third, one can cast the individual dynamic Euler equations in the form:

$$\begin{aligned} &\begin{bmatrix} 0 & C_{fj0}^{fj} \end{bmatrix} \begin{bmatrix} F_t \\ F_{jt} \end{bmatrix} + \begin{bmatrix} 0 & C_{fj0}^{\Gamma j1} \end{bmatrix} \begin{bmatrix} \Gamma_t \\ \Gamma_{jt} \end{bmatrix} + \begin{bmatrix} 0 & C_{fj0}^{\Gamma j0} \end{bmatrix} \begin{bmatrix} \Gamma_{t-1} \\ \Gamma_{j,t-1} \end{bmatrix} + C_{fj0}^{\Upsilon} \begin{bmatrix} \underline{\Upsilon}_t \\ \underline{\Upsilon}_{jt} \end{bmatrix} + \\ &E_{jt} \left\{ C_{fj1}^{F} \begin{bmatrix} F_{t+1} \\ F_{j,t+1} \end{bmatrix} + C_{fj1}^{\Upsilon} \begin{bmatrix} \underline{\Upsilon}_{t+1} \\ \underline{\Upsilon}_{j,t+1} \end{bmatrix} + C_{fj1}^{\Gamma} \begin{bmatrix} \Gamma_{t+1} \\ \Gamma_{j,t+1} \end{bmatrix} + \begin{bmatrix} C_{fj0}^{\Gamma 1} & 0 \end{bmatrix} \begin{bmatrix} \Gamma_t \\ \Gamma_{jt} \end{bmatrix} + \begin{bmatrix} C_{fj0}^{\Gamma 0} & 0 \end{bmatrix} \begin{bmatrix} \Gamma_{t-1} \\ \Gamma_{j,t-1} \end{bmatrix} \right\} = 0. \end{aligned} \tag{2.6}$$

The individual dynamic Euler equation, (2.6), is used to identify the policy function of the individual forward looking variables to which the same constraints apply as to individual endogenous state variables. The agents choose their individual forward looking variables conditional on individual endogenous state variables, signals and their individual expectation of tomorrow's, and in some cases the expectation of today's, state of the world.

After integrating (2.6), I find the aggregate dynamic Euler equations to be:

$$\begin{aligned} &\begin{bmatrix} C_{f0}^{f} & 0 \end{bmatrix} \begin{bmatrix} F_t \\ F_{jt} \end{bmatrix} + \begin{bmatrix} C_{f0}^{\Gamma 1} & 0 \end{bmatrix} \begin{bmatrix} \Gamma_t \\ \Gamma_{jt} \end{bmatrix} + \begin{bmatrix} C_{f0}^{\Gamma 0} & 0 \end{bmatrix} \begin{bmatrix} \Gamma_{t-1} \\ \Gamma_{j,t-1} \end{bmatrix} + \begin{bmatrix} C_{f0}^{\Upsilon} & 0 \end{bmatrix} \begin{bmatrix} \underline{\Upsilon}_t \\ \underline{\Upsilon}_{jt} \end{bmatrix} + \begin{bmatrix} C_{f0}^{s} & 0 \end{bmatrix} \begin{bmatrix} s_t \\ s_{jt} \end{bmatrix} + \\ &\bar{E}_t \left\{ \begin{bmatrix} C_{f1}^{f} & 0 \end{bmatrix} \begin{bmatrix} F_{t+1} \\ F_{j,t+1} \end{bmatrix} + \begin{bmatrix} C_{f1}^{\Gamma} & 0 \end{bmatrix} \begin{bmatrix} \Gamma_{t+1} \\ \Gamma_{j,t+1} \end{bmatrix} + \begin{bmatrix} C_{f1}^{\Upsilon} & 0 \end{bmatrix} \begin{bmatrix} \underline{\Upsilon}_{t+1} \\ \underline{\Upsilon}_{j,t+1} \end{bmatrix} + \begin{bmatrix} C_{f0}^{\Gamma 1} & 0 \end{bmatrix} \begin{bmatrix} \Gamma_t \\ \Gamma_{jt} \end{bmatrix} + \begin{bmatrix} C_{f0}^{\Gamma 0} & 0 \end{bmatrix} \begin{bmatrix} \Gamma_{t-1} \\ \Gamma_{j,t-1} \end{bmatrix} \right\} = 0. \end{aligned} \tag{2.7}$$

Any HI-DSGE model as outlined in Section 2.2 can be cast in this state space representation.

2.3.1 Discussion of information structures

In this subsection, I discuss the implications of the modelling choice for higher order expectations. Higher order expectations appear with heterogeneous information as the law of iterated expectation does not hold, i.e. $\int E[\Gamma_t|\Omega_{jt}]\,dj \neq E[\int E[\Gamma_t|\Omega_{jt}]\,dj|\Omega_{jt}]$. They enter into the solution of the model via the forward looking variables. Specifically, they appear if the model leads to the structure $F_{jt} = E[f(F_t,\cdot)|jt]$, which means that the individual forward looking variable is a function of the expectation of the aggregate forward looking variable. If one solves the equation forward, one finds a sum of an increasing number of terms of higher order expectations. The parameter loading in front of the aggregate forward looking variable determines the weight given to each order of expectations which is decaying exponentially.

The signal to noise ratios by itself also affect how higher order expectations enter the policy functions of the agents. If the signal to noise ratio is low, agents rather neglect the signal and with the signal to noise ratio high they put more weight on it. In the first case learning is slower and the latter it is faster. In addition, when agents observe a public signal about more than one aggregate state variable, they could coordinate on an equilibrium in which one of the aggregate state variables is common knowledge and agents form expectations only about the others. This is shown in Schaefer (2019a) for a specific real business cycle model. There are two equilibria in which the dynamics of one resembles to the one of the partial information version of the model, despite agents receiving both public and private signals.

2.4 Solution methodology

The solution methodology is a generalization of the method of undetermined coefficients that is commonly used for full information models and it includes two steps. First, I solve for parameter matrices of my guess of the policy function of individual forward looking variables that are invariant to the information available to the agents. Second, I solve the parameter matrices of the policy function of individual forward looking variables that depend on the signal extraction problem that agents face to form their expectation about the state of the economy, which includes an extended state space with a hierarchy of expectations.

2.4.1 Solve for matrices that are invariant to the information set

In this subsection, I solve for parameter matrices that are invariant to the information that agents receive. I specify which matrices I refer to after my conjecture about the policy function of individual forward looking variables.

I guess that the policy function of individual forward looking variables consist of four groups of variables: lagged state variables, signals, innovations and the hierarchy of expectations of the state of the economy. I denote the hierarchy of expectation by the vector $\left[Z_t' \, Z_{jt}'\right]'$ which will be specified in Section 2.4.2.

$$F_{jt} = \begin{bmatrix} \xi_{fj}^{\Gamma} & \xi_{fj}^{\Gamma j} \end{bmatrix} \begin{bmatrix} \Gamma_{t-1} \\ \Gamma_{j,t-1} \end{bmatrix} + \xi_{fj}^{\Upsilon} \begin{bmatrix} \underline{\Upsilon}_t \\ \underline{\Upsilon}_{jt} \end{bmatrix} + \begin{bmatrix} 0 & \xi_{fj}^{Z} \end{bmatrix} \begin{bmatrix} Z_t \\ Z_{jt} \end{bmatrix} + \xi_{fj}^{S} \begin{bmatrix} s_t \\ s_{jt} \end{bmatrix} \tag{2.8}$$

When aggregated, I find:

$$F_t = \begin{bmatrix} \xi_f^{\Gamma} & 0 \end{bmatrix} \begin{bmatrix} \Gamma_{t-1} \\ \Gamma_{j,t-1} \end{bmatrix} + \begin{bmatrix} \xi_f^{\Upsilon} & 0 \end{bmatrix} \begin{bmatrix} \underline{\Upsilon}_t \\ \underline{\Upsilon}_{jt} \end{bmatrix} + \begin{bmatrix} \xi_f^{Z} & 0 \end{bmatrix} \begin{bmatrix} Z_t \\ Z_{jt} \end{bmatrix} + \begin{bmatrix} \xi_f^{s} & 0 \end{bmatrix} \begin{bmatrix} s_t \\ s_{jt} \end{bmatrix}. \tag{2.9}$$

Stacking individual and aggregate forward looking variables in one vector yields:

$$\begin{bmatrix} F_t \\ F_{jt} \end{bmatrix} = \xi^{\Gamma} \begin{bmatrix} \Gamma_{t-1} \\ \Gamma_{j,t-1} \end{bmatrix} + \xi^{\Upsilon} \begin{bmatrix} \underline{\Upsilon}_t \\ \underline{\Upsilon}_{jt} \end{bmatrix} + \xi^{Z} \begin{bmatrix} Z_t \\ Z_{jt} \end{bmatrix} + \xi^{S} \begin{bmatrix} s_t \\ s_{jt} \end{bmatrix}. \tag{2.10}$$

I claim that ξ^{Γ}, ξ^{Υ} and ξ^{S} are invariant to the information available to the agents.

Next, I combine the state law of motion (2.4) and contemporaneous variables (2.5) and write them in terms of state variables, forward looking variables and shocks.[4]

$$\begin{bmatrix} \Gamma_t \\ \Gamma_{jt} \end{bmatrix} = A^{\Gamma} \begin{bmatrix} \Gamma_{t-1} \\ \Gamma_{j,t-1} \end{bmatrix} + A^{F} \begin{bmatrix} F_t \\ F_{jt} \end{bmatrix} + A^{S} \begin{bmatrix} s_t \\ s_{jt} \end{bmatrix} \tag{2.11}$$

$$\begin{bmatrix} \Upsilon_t \\ \Upsilon_{jt} \end{bmatrix} = B^{\Gamma} \begin{bmatrix} \Gamma_{t-1} \\ \Gamma_{j,t-1} \end{bmatrix} + B^{F} \begin{bmatrix} F_t \\ F_{jt} \end{bmatrix} + B^{S} \begin{bmatrix} s_t \\ s_{jt} \end{bmatrix} \tag{2.12}$$

To verify my guess for individual forward looking variables, I plug the state law of motion, (2.11), and the guess for forward looking variables, (2.10), into the individual Euler

[4] Details on the derivation can be found in Appendix B.3.1.

equation, (2.6), in period $t+1$. This yields the following expression:

$$
\begin{aligned}
& C^{fj}_{fj0} F_{jt} + \left(C^{\Gamma 1}_{fj0} + \left(C^{\Gamma}_{fj1} A^{\Gamma} + \left(C^{F}_{fj1} + C^{\Gamma}_{fj1} A^{F}\right) \xi^{\Gamma}\right) \mathbb{I}_{\Omega}\right) \begin{bmatrix} \Gamma_t \\ \Gamma_{jt} \end{bmatrix} + C^{\Gamma 0}_{fj0} \begin{bmatrix} \Gamma_{t-1} \\ \Gamma_{j,t-1} \end{bmatrix} + C^{\Upsilon}_{fj0} \begin{bmatrix} \underline{\Upsilon}_t \\ \underline{\Upsilon}_{jt} \end{bmatrix} \\
& + E_{jt} \left\{ \left(C^{\Upsilon}_{fj1} + \left(C^{F}_{fj1} + C^{\Gamma}_{fj1} A^{F}\right) \xi^{\Upsilon}\right) \begin{bmatrix} \underline{\Upsilon}_{t+1} \\ \underline{\Upsilon}_{j,t+1} \end{bmatrix} \right. \\
& \left. + \left(C^{\Gamma}_{fj1} A^{\Gamma} + \left(C^{F}_{fj1} + C^{\Gamma}_{fj1} A^{F}\right) \xi^{\Gamma}\right) \mathbb{I}_{\neq\Omega} \begin{bmatrix} \Gamma_t \\ \Gamma_{jt} \end{bmatrix} + \left(C^{F}_{fj1} + C^{\Gamma}_{fj1} A^{F}\right) \xi^{Z} \begin{bmatrix} Z_{t+1} \\ Z_{j,t+1} \end{bmatrix} \right\} = 0,
\end{aligned}
$$

where $\mathbb{I}_{\Omega}$ is a diagonal matrix that selects state variables that are part of the information set Ω_{jt}, while $\mathbb{I}_{\neq\Omega}$ is a diagonal matrix that selects all the state variables that are not part of it. Moreover, $C^{\Gamma 0}_{fj0} = \left[0 \; C^{\Gamma j0}_{fj0}\right]$ and $C^{\Gamma 1}_{fj0} = \left[0 \; C^{\Gamma j1}_{fj0}\right]$.

Next, I substitute the state law of motion, (2.11), in t, to find:

$$
\begin{aligned}
& \left(C^{fj}_{fj0} + \left(C^{\Gamma 1}_{fj0} + \left(C^{\Gamma}_{fj1} A^{\Gamma} + \left(C^{F}_{fj1} + C^{\Gamma}_{fj1} A^{F}\right) \xi^{\Gamma}\right) \mathbb{I}_{\Omega}\right) A_F\right) F_{jt} \\
& + \left(C^{\Gamma 0}_{fj0} + \left(C^{\Gamma 1}_{fj0} + \left(C^{\Gamma}_{fj1} A^{\Gamma} + \left(C^{F}_{fj1} + C^{\Gamma}_{fj1} A^{F}\right) \xi^{\Gamma}\right) \mathbb{I}_{\Omega}\right) A_{\Gamma}\right) \begin{bmatrix} \Gamma_{t-1} \\ \Gamma_{j,t-1} \end{bmatrix} + C^{\Upsilon}_{fj0} \begin{bmatrix} \underline{\Upsilon}_t \\ \underline{\Upsilon}_{jt} \end{bmatrix} \\
& + \left(C^{\Gamma 1}_{fj0} + \left(C^{\Gamma}_{fj1} A^{\Gamma} + \left(C^{F}_{fj1} + C^{\Gamma}_{fj1} A^{F}\right) \xi^{\Gamma}\right) \mathbb{I}_{\Omega}\right) A_S \begin{bmatrix} s_t \\ s_{jt} \end{bmatrix} + E_{jt} \left\{ \left(C^{\Upsilon}_{fj1} + \left(C^{F}_{fj1} + C^{\Gamma}_{fj1} A^{F}\right) \xi^{\Upsilon}\right) \begin{bmatrix} \underline{\Upsilon}_{t+1} \\ \underline{\Upsilon}_{j,t+1} \end{bmatrix} \right. \\
& \left. + \left(C^{\Gamma}_{fj1} A^{\Gamma} + \left(C^{F}_{fj1} + C^{\Gamma}_{fj1} A^{F}\right) \xi^{\Gamma}\right) \mathbb{I}_{\neq\Omega} \begin{bmatrix} \Gamma_t \\ \Gamma_{jt} \end{bmatrix} + \left(C^{F}_{fj1} + C^{\Gamma}_{fj1} A^{F}\right) \xi^{Z} \begin{bmatrix} Z_{t+1} \\ Z_{j,t+1} \end{bmatrix} \right\} = 0.
\end{aligned}
\tag{2.13}
$$

Then, I plug my guess for forward looking variables, (2.8), in (2.13) and match coefficients of identical variables. This directly identifies $\left[\xi^{\Gamma}_{fj} \; \xi^{\Gamma j}_{fj}\right]$, ξ^{Υ}_{fj} and ξ^{S}_{fj} and confirms my guess that they are invariant to the information agents receive.

$$0 = \left(C^{fj}_{fj0} + \left(C^{\Gamma 1}_{fj0} + \left(C^{\Gamma}_{fj1}A^{\Gamma} + (C^{F}_{fj1} + C^{\Gamma}_{fj1}A^{F})\,\boldsymbol{\xi}^{\Gamma}\right)\mathbb{I}_{\Omega}\right)A_F\right)\begin{bmatrix}\boldsymbol{\xi}^{\Gamma}_{fj} & \boldsymbol{\xi}^{\Gamma j}_{fj}\end{bmatrix} \tag{2.14}$$

$$+ \left(C^{\Gamma 0}_{fj0} + \left(C^{\Gamma 1}_{fj0} + \left(C^{\Gamma}_{fj1}A^{\Gamma} + (C^{F}_{fj1} + C^{\Gamma}_{fj1}A^{F})\,\boldsymbol{\xi}^{\Gamma}\right)\mathbb{I}_{\Omega}\right)A_{\Gamma}\right)$$

$$\boldsymbol{\xi}^{\Upsilon}_{fj} = -\left(C^{fj}_{fj0} + \left(C^{\Gamma 1}_{fj0} + \left(C^{\Gamma}_{fj1}A^{\Gamma} + (C^{F}_{fj1} + C^{\Gamma}_{fj1}A^{F})\,\boldsymbol{\xi}^{\Gamma}\right)\mathbb{I}_{\Omega}\right)A_F\right)^{-1} C^{\Upsilon}_{fj0} \tag{2.15}$$

$$\boldsymbol{\xi}^{S}_{fj} = -\left(C^{fj}_{fj0} + \left(C^{\Gamma 1}_{fj0} + \left(C^{\Gamma}_{fj1}A^{\Gamma} + (C^{F}_{fj1} + C^{\Gamma}_{fj1}A^{F})\,\boldsymbol{\xi}^{\Gamma}\right)\mathbb{I}_{\Omega}\right)A_F\right)^{-1} \tag{2.16}$$

$$\times \left(C^{\Gamma 1}_{fj0} + \left(C^{\Gamma}_{fj1}A^{\Gamma} + (C^{F}_{fj1} + C^{\Gamma}_{fj1}A^{F})\,\boldsymbol{\xi}^{\Gamma}\right)\mathbb{I}_{\Omega}\right)A_S$$

The expressions show that ξ^{Υ}_{fj} and ξ^{S}_{fj} depend only on ξ^{Γ} and can be determined by simple matrix inversion. The equation of the lagged state variables however, ξ^{Γ}, describe a quadratic equation system that one needs to solve numerically.

Proposition 2. *The solution to the quadratic equation system that identifies ξ^{nj}_{fj} and ξ^{cj}_{fj} coincides with the parameters of the full information solution.*

Proof. Individual endogenous state variables are part of the agent's information set. Through the state law of motion of individual endogenous state variables all the weight is passed through to ξ^{nj}_{fj} and ξ^{cj}_{fj}, the same way as under full information. For details, see Appendix B.2.2. □

This means that I can use the parameters for ξ^{nj}_{fj} and ξ^{cj}_{fj} directly from the solution of the full information solution, which can be easily computed with standard algorithms to solve rational expectation models, such as Sims (2002). ξ^{Γ}_{fj} and ξ^{xj}_{fj} instead do not coincide with the full information solution, but they can be solved for with numerical root finding routines as they are implemented in the Matlab function `fsolve`.

The guess for individual forward looking variables is completely verified as soon as ξ^{Z}_{fj} is identified. In order to do so, I use the results of the invariant matrices of the guess in combination with (2.11) and (2.12) to derive the state law of motion and the contemporaneous variables in terms of lagged state, expectational state variables and shocks only.

$$\begin{bmatrix}\Gamma_t\\ \Gamma_{jt}\end{bmatrix} = M^{\Gamma}\begin{bmatrix}\Gamma_{t-1}\\ \Gamma_{j,t-1}\end{bmatrix} + M^{S}\begin{bmatrix}s_t\\ s_{jt}\end{bmatrix} + M^{F}\xi^{Z}\begin{bmatrix}Z_t\\ Z_{jt}\end{bmatrix} \tag{2.17}$$

$$\begin{bmatrix}\Upsilon_t\\ \Upsilon_{jt}\end{bmatrix} = G^{\Gamma}\begin{bmatrix}\Gamma_{t-1}\\ \Gamma_{j,t-1}\end{bmatrix} + G^{S}\begin{bmatrix}s_t\\ s_{jt}\end{bmatrix} + G^{F}\xi^{Z}\begin{bmatrix}Z_t\\ Z_{jt}\end{bmatrix} \tag{2.18}$$

In addition, to complete verifying my guess for individual forward looking variables, I need to make a guess about the state law of motion of an extended state space including the hierarchy of expectations.

$$\begin{bmatrix}Z_t\\ Z_{jt}\end{bmatrix} = M\begin{bmatrix}Z_{t-1}\\ Z_{j,t-1}\end{bmatrix} + N\begin{bmatrix}s_t\\ s_{jt}\end{bmatrix} \quad \text{and} \quad \begin{bmatrix}\Upsilon_t\\ \Upsilon_{jt}\end{bmatrix} = G\begin{bmatrix}Z_{t-1}\\ Z_{j,t-1}\end{bmatrix} + H\begin{bmatrix}s_t\\ s_{jt}\end{bmatrix} \tag{2.19}$$

Making use of the rewritten state space (2.17), (2.18) and the guess of the extended state law of motion, I complete verifying my guess for the individual forward looking variables in (2.13). (2.13) includes signals which are a subset of contemporaneous variables. Analogue to the notation of signals carrying an underscore to indicate that they are a subset of contemporaneous variables, $\underline{G}^{\Gamma}$ and $\underline{G}^{F}$ select the rows of G^{Γ} and G^{F} that correspond to the signals. Equating identical variables, ξ^{Z}_{fj} must satisfy the equation:

$$\begin{aligned}\xi^{Z}_{fj} = &-\left(C^{fj}_{fj0} + (C^{\Gamma 1}_{fj0} + (C^{\Gamma}_{fj1}A^{\Gamma} + (C^{F}_{fj1} + C^{\Gamma}_{fj1}A^{F})\,\xi^{\Gamma})\,\mathbb{I}_{\Omega})\,A_F\right)^{-1}\\ &\times\left((C^{\Gamma}_{fj1}A^{\Gamma} + (C^{F}_{fj1} + C^{\Gamma}_{fj1}A^{F})\,\xi^{\Gamma})\,\mathbb{I}_{\neq\Omega} + \left(C^{\Upsilon}_{fj1} + (C^{F}_{fj1} + C^{\Gamma}_{fj1}A^{F})\,\xi^{\Upsilon}\right)\underline{G}^{\Gamma}\right.\\ &\left.+\left((C^{F}_{fj1} + C^{\Gamma}_{fj1}A^{F}) + \left(C^{\Upsilon}_{fj1} + (C^{F}_{fj1} + C^{\Gamma}_{fj1}A^{F})\,\xi^{\Upsilon}\right)\underline{G}^{F}\right)\xi^{Z}M\right).\end{aligned} \tag{2.20}$$

The solution to ξ^{Z}_{fj} depends on the informational content as it depends on the transition matrix M of the extended state law of motion, (2.19).

2.4.2 Solve the signal extraction problem

I find the solution to the signal extraction problem in a similar manner to Schaefer (2019a). The main difference with regard to the signal extraction problem is that the methodology in the paper at hand can handle contemporaneous and predetermined endogenous state variables as well as exogenous state variables, while the former cannot handle both types of endogenous variables at the same time.

Before I proceed, recall the fact that both types of individual endogenous state variables are part of the agents information set in t. It is an important point to understand which variables the agents need to form their expectation about and which variably carry informational content for the signal extraction problem. For dealing with the hierarchy of expectations, I follow the approach of Nimark (2011). He assumes that the state of the economy is never revealed. To cope with the infinite regress problem of heterogeneous information models he reduces the dimensionality problem by assuming that the effect of the $\overline{o}+1$ hierarchy of expectation does not have any significant effect on the policy function of the agents and hence the state space system can be truncated at $\overline{o}$. At this point, I define the variable Z_{jt} to be equal to $Z_{jt} = \left[\Gamma^{(0:\overline{o})\prime}_{t|jt}\, \Gamma'_{jt}\, X'_{jt|jt}\right]'$, where $\Gamma^{(0:\overline{o})}_{t} = \left[\Gamma'_t\, \Gamma^{(1)\prime}_t \,\dots\, \Gamma^{(\overline{o})\prime}_t\right]'$, $\Gamma^{(1)}_t = \int E\left[\Gamma_t|\Omega_{jt}\right]\, dj$ and $\Gamma^{(\overline{o})}_t = \int E\left[\Gamma^{(\overline{o}-1)}_t|\Omega_{jt}\right]\, dj$.

As it is shown in the literature, the signal extraction problem includes two steps. First, one guesses the state law of motion of the extended state law of motion, which includes non-expectational and expectational state variables. Second, one uses the guess for the state law of motion to compute the Kalman gain and the associated mean square error. The difference to the literature is that the signal extraction problem typically does not includes predetermined and contemporaneous endogenous state variables simultaneously.

The extended state law of motion for the class of models covered in this paper includes the hierarchy of expectation, the individual expectation thereof as well as individual state variables and the individual expectation of the exogenous idiosyncratic state variables. The extended state law of motion takes the form:

$$\begin{bmatrix} \Gamma^{(0:\infty)}_t \\ \Gamma^{(0:\infty)}_{t|jt} \\ \Gamma_{jt} \\ X_{jt|jt} \end{bmatrix} = \begin{bmatrix} M^{e\Gamma}_{e\Gamma} & 0 & 0 & 0 \\ M^{e\Gamma}_{ej\Gamma} & M^{ej\Gamma}_{ej\Gamma} & M^{\Gamma j}_{ej\Gamma} & M^{ejxj}_{ej\Gamma} \\ M^{e\Gamma}_{\Gamma j} & M^{ej\Gamma}_{\Gamma j} & M^{\Gamma j}_{\Gamma j} & M^{ejxj}_{\Gamma j} \\ M^{e\Gamma}_{ejxj} & M^{ej\Gamma}_{ejxj} & M^{\Gamma j}_{ejxj} & M^{ejxj}_{ejxj} \end{bmatrix} \begin{bmatrix} \Gamma^{(0:\infty)}_{t-1} \\ \Gamma^{(0:\infty)}_{t-1|j,t-1} \\ \Gamma_{j,t-1} \\ X_{j,t-1|j,t-1} \end{bmatrix} + \begin{bmatrix} N_{e\Gamma} \\ N_{ej\Gamma} \\ N_{\Gamma j} \\ N_{ejxj} \end{bmatrix} \begin{bmatrix} s_t \\ s_{jt} \end{bmatrix}. \tag{2.21}$$

Combining the guess for the extended state law of motion, (2.21), with the expressions for the non-expectational state variables, (2.17), and (2.18), I find:

$$\begin{bmatrix}\Gamma_t\\ \Gamma_{jt}\end{bmatrix}=\begin{bmatrix}M^{e\Gamma}_{\Gamma} & 0 & 0 & 0\\ M^{e\Gamma}_{\Gamma j} & M^{ej\Gamma}_{\Gamma j} & M^{\Gamma j}_{\Gamma j} & M^{ejxj}_{\Gamma j}\end{bmatrix}\begin{bmatrix}\Gamma^{(0:\infty)}_{t-1}\\ \Gamma^{(0:\infty)}_{t-1|j,t-1}\\ \Gamma_{j,t-1}\\ X_{j,t-1|j,t-1}\end{bmatrix}+\begin{bmatrix}N_{\Gamma}\\ N_{\Gamma j}\end{bmatrix}\begin{bmatrix}s_t\\ s_{jt}\end{bmatrix},\quad\text{and}\tag{2.22}$$

$$\begin{bmatrix}\Upsilon_t\\ \Upsilon_{jt}\end{bmatrix}=\begin{bmatrix}G_{e\Gamma} & G_{ej\Gamma} & G_{\Gamma j} & G_{ejxj}\end{bmatrix}\begin{bmatrix}\Gamma^{(0:\infty)}_{t-1}\\ \Gamma^{(0:\infty)}_{t-1|j,t-1}\\ \Gamma_{j,t-1}\\ X_{j,t-1|j,t-1}\end{bmatrix}+H\begin{bmatrix}s_t\\ s_{jt}\end{bmatrix},\tag{2.23}$$

which confirms the part of the guess (2.21) that refers to the non-expectational state variables.[5]

The agents use the Kalman filter to form their expectation about the state of the economy. The Kalman updating equation reads:

$$\begin{bmatrix}\Gamma^{(0:\bar{o})}_{t|jt}\\ X_{jt|jt}\end{bmatrix}=\begin{bmatrix}\Gamma^{(0:\bar{o})}_{t|j,t-1}\\ X_{jt|j,t-1}\end{bmatrix}+\begin{bmatrix}\mathscr{K}_{ej\Gamma}\\ \mathscr{K}_{ejxj}\end{bmatrix}\left(\begin{bmatrix}\underline{\Upsilon}_t\\ \underline{\Upsilon}_{jt}\end{bmatrix}-\begin{bmatrix}\underline{\Upsilon}_{t|j,t-1}\\ \underline{\Upsilon}_{jt|j,t-1}\end{bmatrix}\right).\tag{2.24}$$

Next, I use (2.23) and I recall that individual endogenous state variables are part of the information set to compute the forecast error of the signals as:[6]

$$\begin{bmatrix}\underline{\Upsilon}_t\\ \underline{\Upsilon}_{jt}\end{bmatrix}-\begin{bmatrix}\underline{\Upsilon}_{t|j,t-1}\\ \underline{\Upsilon}_{jt|j,t-1}\end{bmatrix}=\begin{bmatrix}\underline{G}_{e\Gamma} & \underline{G}_{xj}\end{bmatrix}\left(\begin{bmatrix}\Gamma^{(0:\bar{o})}_{t-1}\\ X_{j,t-1}\end{bmatrix}-\begin{bmatrix}\Gamma^{(0:\bar{o})}_{t-1|j,t-1}\\ X_{j,t-1|j,t-1}\end{bmatrix}\right)+\underline{H}\begin{bmatrix}s_t\\ s_{jt}\end{bmatrix}.\tag{2.25}$$

[5] For details, see Appendix B.3.2.

[6] The formulation here is significantly simpler compared to the formulation in Schaefer (2019a). The difference is that in Schaefer (2019a) the predetermined state variables enter the state vector in t, $\Gamma_t = \left[X^c_t\ X_t\right]$, which makes excessive reformulations necessary to find the forecast errors independently of individual endogenous state variables.

Plugging (2.24) and the relevant parts of the guess for the extended state law of motion, (2.21), back into (2.25) yields:

$$\begin{bmatrix}\Gamma^{(0:\bar{o})}_{t|jt}\\ X_{jt|jt}\end{bmatrix} = (M_1 - \mathscr{K}\underline{G}_1)\begin{bmatrix}\Gamma^{(0:\bar{o})}_{t-1|j,t-1}\\ X_{j,t-1|j,t-1}\end{bmatrix} + \mathscr{K}\underline{G}_1\begin{bmatrix}\Gamma^{(0:\bar{o})}_{t-1}\\ X_{j,t-1}\end{bmatrix} + \mathscr{K}\underline{H}\begin{bmatrix}s_t\\ s_{jt}\end{bmatrix}, \tag{2.26}$$

$$\text{where} \quad \mathscr{K} = \left[\mathscr{K}'_{ej\Gamma}\ \mathscr{K}'_{ejxj}\right]', \quad \underline{G}_1 = \left[\underline{G}_{e\Gamma}\ \underline{G}_{xj}\right],$$

$$M_1 = \begin{bmatrix}M^{e\Gamma}_{e\Gamma} & 0\\ 0 & M^{xj}_{xj}\end{bmatrix}, \quad N_1 = \begin{bmatrix}N_{e\Gamma}\\ N_{xj}\end{bmatrix}.$$

The individual expectation of the hierarchy of expectation as well as endogenous state variables cancel out. This is logical as they are taken conditional on the estimate of the state of the economy and thus cannot contribute to estimate it.

Equation (2.26) confirms the part of the guess, (2.21), that refers to the individual expectation of the hierarchy of expectation and the exogenous idiosyncratic state variables. It remains to identify the transition matrix and the impact matrix of the hierarchy of expectations, $M^{e\Gamma}_{e\Gamma}$ and $N_{e\Gamma}$. I find these matrices by aggregating the state law of motion of $\Gamma^{(0:\bar{o})}_{t|jt}$ and by amending it with the state law of motion of the aggregate non-expectational variables (2.22).

$$M^{e\Gamma}_{e\Gamma} = \begin{bmatrix}M^{e\Gamma}_{\Gamma}\\ M^{e\Gamma}_{ej\Gamma} + M^{ej\Gamma}_{ej\Gamma}T_{\bar{o}}\end{bmatrix} \qquad N_{e\Gamma} = \begin{bmatrix}N_{\Gamma}\\ N_{ej\Gamma}T_S\end{bmatrix} \tag{2.27}$$

Here, $T_{\bar{o}}$ is a matrix that shifts the state space at the beginning of the hierarchy by one order of hierarchy and truncates the last one. T_S sets the entries of the idiosyncratic innovations equal to zero.

This concludes the derivation of the extended state law of motion. The solution of the model is defined as the fixed point between the state law of motion of the extended state space, the identification of ξ^Z_{fj} as well as the Kalman gain and the mean square error (MSE).

The Kalman gain matrix can be computed as follows:

$$\mathscr{K} = \left(M_1\hat{P}\underline{G}'_1 + N_1\underline{H}'\right)\left(\underline{G}_1\hat{P}\underline{G}'_1 + \underline{H}\underline{H}'\right)^{-1}; \tag{2.28}$$

and the MSE is defined by the Riccati equation:

$$\hat{P} = M_1 \hat{P} M_1' + N_1 N_1' - \left(M_1 \hat{P} \underline{G}_1' + N_1 \underline{H}'\right) \left(\underline{G}_1 \hat{P} \underline{G}_1' + \underline{H}\underline{H}'\right)^{-1} \left(\underline{G}_1 \hat{P} M_1' + \underline{H} N_1'\right). \quad (2.29)$$

2.4.3 Reduction of dimensionality

In the previous subsection, I showed how to solve the model for the extended state law of motion including individual and aggregate state variables. However, this is computationally inefficient.

The solution is defined by the fixed point between the guess for the policy function of the individual and aggregate forward looking variables, the guess for the extended state law of motion and the Kalman gain. However, the signal extraction problem, presented in Section 2.4.2 includes only the aggregate hierarchy of expectation and exogenous idiosyncratic state variables. Thus, it is sufficient to solve for the policy function of the aggregate state variables, because the idiosyncratic exogenous state variable does not interact with the fixed point of the aggregate policy functions.

This means that one can reduce the problem to the fixed point solution between the aggregate part of the extended state law of motion, the aggregate guess for aggregate forward looking variables and the Kalman gain. Nevertheless, the discussion of the individual decision problem is important as only this way I can define which variables stay inside and which outside of the expectation operator when determining the policy function for forward looking variables.

2.5 The model

The model is a New Keynesian Model with incomplete markets following Lorenzoni (2009), extended by capital and a constant relative risk aversion (CRRA) utility function.

The economy is assumed to be separated in an infinite number islands indicated by j, on which there is each one household, one final goods producing firm, a continuum of capital producers in perfect competition and a continuum of firms producing heterogeneous intermediary goods. The agents on island j share the same information set Ω_{jt} available to them in time period t. The firms of the wholesale sector hire labour and rent capital from the household on their island. The final goods producing firm buys a random subset of goods produced by the firms of the wholesale sectors of other islands. The household consumes the bundle of goods produced by the final goods producer on their island, provides labour and rents capital to the wholesale sector only on their island. In addition, it invests in a one period discounted bond and accumulates capital. The capital producers use the final composite good as material input to produce new capital. Thereby, capital is island specific and it is sold only to the household on its island.[7] In this setting the agents on island j do not know all quantities and prices of the goods in the economy and they also do not know aggregate productivity. Hence, they need to form their expectations about the aggregate state of the economy.

2.5.1 The household's problem

The household on island j chooses its consumption of the composite good, C_{jt}, produced on their island by the final goods producer, labour supply in terms of hours worked, H_{jft}, to firm f on its island, tomorrow's bond holdings, $B_{j,t+1}$ and tomorrow's capital stock, $K_{j,t+1}$, such that it maximizes its expected discounted utility subject to its budget constraint and its individual capital state law of motion. The utility function exhibits CRRA preferences in consumption and a power form in hours worked, where ψ is a scaling factor between the utility of consumption and the disutility of labour. The subjective discount factor is denoted by β, σ represents the rate of intertemporal substitution and γ determines the Frisch elasticity of labour supply.

$$\max_{\{C_{jt},H_{jft},B_{j,t+1},K_{j,t+1}\}} U = E_{jt}\left[\sum_{t=0}^{\infty}\beta^t\left(\frac{C_{jt}^{1-\sigma}}{1-\sigma}-\frac{\psi}{1+\gamma}\int H_{jft}^{1+\gamma}\,df\right)\right] \tag{2.30}$$

s.t.

$$\frac{B_{j,t+1}}{R_t^n}+\bar{P}_{jt}C_{jt}+Q_{jt}^nY_{jt}^{k_d}=B_{jt}+W_{jt}^n\int H_{jft}\,df+Z_{jt}^nK_{jt}+\int_0^1\Pi_{jft}\,df+\int_0^1\Pi_{jlt}^k\,dl \tag{2.31}$$

$$Y_{jt}^{k_d}=K_{j,t+1}-(1-\delta)K_{jt} \tag{2.32}$$

[7]This assumption is made for simplicity. One could imagine that there is an integrated market for capital across the whole economy in which the agents potentially need to form expectations about the aggregate capital stock due to this reason. This could happen if there is a competitive advantage for the capital producer on one island, i.e. a lower price level or lower marginal costs and non-linear adjustment costs. In a linearised model, the latter condition is not satisfied. Hence, any capital producer could just produce enough to cover the demand on their island and no trade would happen.

On the expenditure side, the household can either save by investing in the one period risk free bond $B_{j,t+1}$ at the nominal return R_t^n, invest in its capital stock $Y_{jt}^{k_d}$ at the price Q_{jt} and it can consume the amount C_{jt} of the composite final good at the price $\bar{P}_{jt}$. The income of the household consists of the payout from last period's investment in the one period risk free bond, the income from labour and capital and, as the household owns the firms on the same island, the profits of the firms.

The maximization problem yields standard first order conditions, with Λ_{jt} being the Lagrangian multiplier to the budget constraint.

$$0 = C_{jt}^{-\sigma} - \Lambda_{jt}\bar{P}_{jt} \tag{2.33}$$

$$0 = \Lambda_{jt}W_{jt}^n - \theta H_{jt}^{\gamma} \tag{2.34}$$

$$0 = \frac{\Lambda_{jt}}{R_t^n} - \beta E_{jt}\left[\Lambda_{j,t+1}\right] \tag{2.35}$$

$$0 = \Lambda_{jt}Q_{jt}^n - \beta E_{jt}\left[\Lambda_{j,t+1}\left((1-\delta)Q_{j,t+1}^n + Z_{j,t+1}^n\right)\right] \tag{2.36}$$

Choice variables in time period t of the household on island j are part of the information set Ω_{jt}. Further, I make use of Proposition 1. It implies prices and quantities of the markets agents interact in to be part of the information set.

With heterogeneous information, the individual budget constraint matters for the dynamics of the model and cannot simply be aggregated. On the aggregate level bond holdings are traded in net zero supply, but the household on island j might have positive or negative bond holdings. The fact that bond holdings can be negative makes the log-linearisation of the budget constraint non-trivial. To overcome the issue, I follow Lorenzoni (2009). One can combine the budget constraint, (2.31), with the first order conditions for consumption, (2.33), and bond holdings, (2.35), as well as the definition of the profits of the firms in the wholesale and capital producing sector, to find the budget constraint in linear terms as:[8]

$$\beta b_{j,t+1} + C_*^{1-\sigma}\hat{c}_{jt} + \frac{X_*}{C_*^{\sigma}}\hat{x}_{jt} = b_{jt} + \frac{Y_*}{C_*^{\sigma}}\left(\hat{y}_{jt} + \hat{p}_{jt} - \hat{\bar{p}}_{jt}\right), \tag{2.37}$$

where $b_{j,t+\tau} = E_{jt}\left[\frac{B_{j,t+\tau}}{C_{j,t+\tau}^{\sigma}\bar{P}_{j,t+\tau}}\right]$. The budget constraint in this paper differs in two aspects from the version in Lorenzoni (2009). First, I allow for a more general utility function.

[8] Details on the derivation can be found in Appendix B.1.1.

Instead of log-utility in consumption, I allow for a more general CRRA utiltiy function. Second, my model includes capital.

2.5.2 The firm's problem

There are three types of firms on an island - the wholesale sector with a continuum of intermediate goods producing firms, a final goods producing firm and a continuum of capital producing firms. The intermediate goods producing firms produce differentiated consumption goods. They hire labour and rent capital from the household on its island as factor inputs. The final goods producer purchases differentiated goods from a random subset of the wholesale sectors of other islands and produces a composite final good. Parts of the final goods are sold to to the capital producers as material input and the remaining part is sold the household for consumption. The capital producers produce homogeneous capital goods and sell it to the household on their island.

The wholesale sector

The intermediate goods producing firm f on island j chooses labour demand, N_{jft}, capital demand, K_{jft}, and sets its price P_{jft}, that maximize its discounted profits subject to a its production technology and the demand function for its good.

It is common in the literature on new Keynesian models that capital is owned by households and rented to the firms of the wholesale sector. In this case, the intermediate goods producing firms choose first labour and capital input that minimize marginal costs first, and second they choose its price that maximizes profits conditional on marginal the costs.

The production technology is a Cobb-Douglas function with constant returns to scale. The production factors are composite productivity, labour and capital, where α is the output elasticity of labour. Composite log-productivity is assumed to consist of an aggregate and an idiosyncratic component $a_{jt} = a_t + \omega_{jt}$ Aggregate and idiosyncratic log-productivity are persistent and follow the processes $a_t = \rho_a a_{t-1} + \nu_t$ and $\omega_{jt} = \rho_\omega \omega_{j,t-1} + \varepsilon_{jt}$. The two innovations are assumed to be i.i.d. $N(0, \sigma_\nu^2)$ and $N(0, \sigma_{\varepsilon j}^2)$, respectively. Moreover, integrating idiosyncratic innovations over all islands yields zero, $\int \varepsilon_{jt}\, dj = 0$.

The firm's cost minimization problem reads:

$$\min_{\{N_{jft},K_{jft}\}} E_{jt}\left[\frac{W_{jt}^{n}}{P_{jt}}N_{jft}+\frac{Z_{jt}^{n}}{P_{jt}}K_{jft}\right]$$
$$s.t. \quad Y_{jft}=A_{jt}N_{jft}^{\alpha}K_{jft}^{1-\alpha}. \tag{2.38}$$

From which the first order conditions are:

$$0=\frac{W_{jt}^{n}}{P_{jt}}-\alpha\mu_{jft}\frac{Y_{jft}}{N_{jft}} \quad \text{and} \tag{2.39}$$
$$0=\frac{Z_{jt}^{n}}{P_{jt}}-(1-\alpha)\mu_{jft}\frac{Y_{jft}}{K_{jft}}. \tag{2.40}$$

Combining the first order conditions, I find real marginal costs, $\mu_{jt}=\mu_{jft}$ as:

$$\mu_{jt}=\left(\frac{1}{P_{jt}A_{jt}}\right)\left(\frac{W_{jt}^{n}}{\alpha}\right)^{\alpha}\left(\frac{Z_{jt}^{n}}{1-\alpha}\right)^{1-\alpha}.$$

The nominal marginal costs are defined as the product of the real marginal costs times the price index of the goods produced on island j, $\mu_{jt}^{n}=\mu_{jt}P_{jt}$.

For the optimal price setting, I apply the nominal stochastic discount factor,

$$M_{jt,t+\tau}^{n}=\beta^{\tau}\frac{C_{j,t+\tau}^{-\sigma}}{C_{jt}^{-\sigma}}\frac{\bar{P}_{jt}}{\bar{P}_{j,t+\tau}},$$

as the firms are owned by the households and profits are formulated in nominal terms. Price stickiness is implemented by assuming Calvo pricing. Under this assumption firms can only infrequently adjust prices at the rate $1-\theta$. This implies that in a given period the price on an island consists of a fraction of optimally chosen and lagged prices, $P_{jt}=\left[(1-\theta)P_{jt}^{*1-\varepsilon}+\theta P_{j,t-1}^{1-\varepsilon}\right]^{1-\varepsilon}$. All firms that can change the price in a given period choose the same price $P_{jf,t+\tau}=P_{jt}^{*}$ that maximizes its expected discounted profits subject to the demand for its product:

$$\max_{\{P_{jf,t+\tau}\}} \Pi_{jft}=E_{jt}\left[\sum_{\tau=0}^{\infty}\theta^{\tau}M_{jt,t+\tau}^{n}\left(P_{jf,t+\tau}-\mu_{jt}^{n}\right)Y_{jf,t+\tau}\right] \tag{2.41}$$
$$s.t. \quad Y_{jf,t+\tau}=\int_{\Theta_{j,t+\tau}}\left(\frac{P_{jf,t+\tau}}{\bar{P}_{i,t+\tau}}\right)^{-\varepsilon}Y_{i,t+\tau}\,di. \tag{2.42}$$

The total demand for the firm's product is defined by the demand of the final goods producers of a random subset of other islands Θ_{jt} and the relative price of the good to the price index of each of these final goods producers. The first order condition with respect to the individual price, $P_{jf,t+\tau}$, yields the non-linear version of the new Keynesian Phillips curve:

$$0 = E_{jt}\left[\sum_{\tau=0}^{\infty}\theta^{\tau}M^{n}_{jt,t+\tau}\left((1-\varepsilon)P^{*-\varepsilon}_{jt}+\varepsilon\mu^{n}_{jt}P^{*-(1+\varepsilon)}_{jt}\right)\int_{\Theta_{j,t+\tau}}\bar{P}^{\varepsilon}_{i,t+\tau}Y_{i,t+\tau}\,di\right]. \tag{2.43}$$

From this equation, I can derive an island specific linearised version of the new Keynesian Phillips curve:[9]

$$\pi_{jt} = \kappa\mu_{jt} + \beta E_{jt}\left[\pi_{j,t+1}\right], \tag{2.44}$$

where $\kappa = \frac{(1-\beta\theta)(1-\theta)}{\theta}$. The island specific new Keynesian Phillips curve leads to the classic aggregate new Keynesian Phillips curve, when integrated, with the difference that the function includes the aggregate expectation operator and not the full information operator.

Final goods sector

The final goods producing firm on island j produces the final good by bundling intermediate goods from a random subset, Ξ_{jt}, of the heterogeneous goods produced by the wholesale sector of other islands. The production function of the final goods producing firm on island j is defined as follows:

$$Y_{jt} = \left(\int_{\Xi_{jt}}\int_{0}^{1}Y_{jimt}^{\frac{\varepsilon-1}{\varepsilon}}\,dm\,di\right)^{\frac{\varepsilon}{\varepsilon-1}}. \tag{2.45}$$

The final goods producing firm chooses the components of the production bundle that minimizes the cost of the bundle subject to its production function:

$$\min_{\{Y_{jift}\}}\int_{\Xi_{jt}}\int_{0}^{1}P_{jimt}Y_{jimt}\,dm\,di \tag{2.46}$$

$$s.t. \qquad Y_{jt} = \left(\int_{\Xi_{jt}}\int_{0}^{1}Y_{jimt}^{\frac{\varepsilon-1}{\varepsilon}}\,dm\,di\right)^{\frac{\varepsilon}{\varepsilon-1}}. \tag{2.47}$$

[9]For details, see Appendix B.1.2.

From this, I find that the demand function for the good produced by firm m on island i by the final goods producing firm on island j and the price index of the final good, $\bar{P}_{jt}$:

$$Y_{jimt} = \left(\frac{P_{imt}}{\bar{P}_{jt}}\right)^{-\varepsilon} Y_{jt} \quad \text{and} \quad \bar{P}_{jt} = \left(\int_{\Xi_{jt}} P_{it}^{1-\varepsilon}\, di\right)^{\frac{1}{1-\varepsilon}}. \tag{2.48}$$

The capital producing sector

The capital producer l on island j produces new capital goods, $Y_{jlt}^{k_s}$, according to the following production technology:

$$Y_{jlt}^{k_s} = \phi\left(\frac{X_{jlt}}{K_{jlt}}\right) K_{jlt}. \tag{2.49}$$

The production technology has the properties $\phi(\delta) = \delta$, $\phi'(\delta) = 1$ and $\phi''\left(\frac{X_{jlt}}{K_{jlt}}\right) < 0$ and hence is convex. The capital producer buys material inputs X_{jlt} at the price $\bar{P}_{jt}$ from the final goods producing firm. Capital is sold at the nominal price Q_{jt}^{n} on the island specific market and maximises its profits:

$$\max_{\{X_{jlt},K_{jl,t+1}\}} \Pi_{jlt}^{k} = E_{jt}\left[Q_{jt}^{n}\phi\left(\frac{X_{jlt}}{K_{jlt}}\right) K_{jlt} - \bar{P}_{jt} X_{jlt}\right] \tag{2.50}$$

From the first order condition, I derive the nominal price of capital:

$$0 = Q_{jt}^{n}\phi'\left(\frac{X_{jlt}}{K_{jlt}}\right) - \bar{P}_{jt}. \tag{2.51}$$

Compared to the full information problem, the capital producing firms on different islands will not choose the same investment capital ratios with heterogeneous information as there is different productivity and prices on different islands. However, the capital producers within an island act identically such that I drop the subscript l without loss of generality.

2.5.3 The central bank

The central bank follows the Taylor rule:

$$r_t^n = (1-\rho_r) r_*^n + \rho_r r_{t-1}^n + \varphi \tilde{\pi}_t, \tag{2.52}$$

where the central bank responds to a noisy measure of inflation $\tilde{\pi}_t = \pi_t + \omega_t$. The innovation is assumed to be i.i.d. $\omega_t \sim N(0, \sigma_\omega)$.

2.5.4 Aggregate variables and market clearing

Households on island j supply labour to all firms f on the same island. And all firms set their individual labour demand, their capital stock and their prices to produce their differentiated good. All these variables are aggregated on the island level, $\Upsilon_{jt} = \int \Upsilon_{jft}\, df$. Further, the price index of the goods produced on an island is defined as:

$$P_{jt} = \left(\int_0^1 P_{jft}^{1-\varepsilon}\, df \right)^{\frac{1}{1-\varepsilon}}. \tag{2.53}$$

The equilibrium dynamics of the island specific variables,

$$\Upsilon_{jt} = \Big\{ B_{jt}, K_{jt}, C_{jt}, \Lambda_{jt}, H_{jt}, \bar{P}_{jt}, P_{jt}, W_{jt}^n, Z_{jt}^n, N_{jt}, \\ \mu_{jt}, X_{jt}, Q_{jt}^n, P_{jt}, A_{jt}, \omega_{jt}, Y_{jt}^{ks}, Y_{jt}^{kd}, Y_{jt}, \Pi_{jt}, \Pi_{jt}^k \Big\},$$

are fully described by the equations (2.31), (2.32), (2.33), (2.34), (2.35), (2.36), (2.38), (2.39), (2.40), (2.41), (2.42), (2.43), (2.49), (2.50), (2.51), (2.54), as well as the functional form of the capital adjustment function, the definition of composite productivity, a_{jt}, idiosyncratic productivity ω_{jt}, the definitions of the price indices and nominal marginal costs, and the market clearing condition for labour and capital.

Aggregate variables are defined by the integral over the realizations of idiosyncratic variables of all islands. In other words, for any variable in the set Υ_{jt}, $\Upsilon_t = \int \Upsilon_{jt}\, dj$. The equilibrium dynamics of the aggregate variables,

$$\Upsilon_{jt} = \{ K_t, C_t, \Lambda_t^n, H_t, P_t, W_t^n, Z_t^n, N_t, \\ \mu_t, X_t, Q_t^n, P_t, A_t, \omega_t, Y_t^{ks}, Y_t^{kd}, Y_t, \Pi_t, \Pi_t^k, R_t^n \Big\},$$

are defined by the individual equilibrium equations being aggregated as well as the definition of the Taylor rule (2.52) and the state law of motion of aggregate productivity.

The aggregate price index is defined as:

$$P_t = \left(\int_0^1 P_{jt}^{1-\varepsilon} \, dj \right)^{\frac{1}{1-\varepsilon}}. \tag{2.54}$$

Labour market clearing requires $H_{jt} = N_{jt}$ for all islands j as the household provides labour only to the firms on their island. The capital market also clears locally at the nominal price for capital Q^n_{jt} and the bonds market clears globally in zero net supply at the return R^n_t. Goods market clearing leads to the well known aggregate identity $Y_t = C_t + X_t$.

2.5.5 Calibration

For the numerical evaluation of the model, I calibrate the model as summarized in Table 2.1. Specifically, I set the intertemporal rate of substitution equal to one and the Frisch elasticity of labour supply to 2 in the baseline calibration. This allows me to directly compare the model dynamics of the new Keynesian model with capital to the one without, presented by Lorenzoni (2009). The discount factor is assumed to be 0.99 which corresponds to an annual risk free rate of 4% and the depreciation rate is set to 10% annually. The output elasticity is set to $2/3$ and so is the frequency of price adjustment. The latter is corresponding to an average duration of 6 months in which a firm does not change prices. The convexity of the capital adjustment function is set to approx. 0.23. All these values are standard in the literature.

Less common is the calibration of the exogenous processes and noise terms, because they are, besides the process for aggregate productivity, not relevant for full information models. Here, I also follow Lorenzoni (2009), who chooses the signal to noise ratios such that the effects of noise shocks are maximized. The standard deviation of the noise term is assumed to be four times as large as the standard deviation of the innovation to aggregate productivity. The measurement error of inflation instead is only 8% its size. The standard deviation of innovations to idiosyncratic variables are instead 10 to 20 times as large as the one to aggregate productivity.

Table 2.1 Calibration - structural parameters

Parameter	Values	Description
σ	1	Intertemporal rate of substitution
β	0.99	Discount factor
γ	0.5	Inverse Frisch elasticity
α	2/3	Output elasticity of labour
δ	0.025	Depreciation rate
η	1/4.3	Convexity of capital adjustment function
θ	2/3	Frequency of price adjustment
ρ_a	0.98	Auto-correlation parameter of aggregate productivity
ρ_ω	0.00	Auto-correlation parameter of idiosyncratic productivity
σ_v	0.077	Std of innovation to aggregate productivity
σ_{ε_j}	20 σ_v	Std of innovation to idiosyncratic productivity
σ_ω	0.08 σ_v	Std of measurement error in inflation
σ_ε	4 σ_v	Std of noise
$\sigma_{\varepsilon_{j1}}$	10 σ_v	Std of idiosyncratic supply draw
$\sigma_{\varepsilon_{j2}}$	10 σ_v	Std of idiosyncratic price draw
$\bar{H}$	0.3	Steady state labour share

2.6 Model dynamics

The model that I presented in Section 2.5 allows me to analyse the concluding remarks of Lorenzoni (2009).[10] He first conjectures that relaxing the assumption that productivity follows a random walk would lead to a higher consumption response than under the random walk assumption. Second, the idiosyncratic innovation could be assumed to follow an autoregressive process, too. He conjectures that this might either diminish the effects of noise shocks as the agent's ability to forecast idiosyncratic productivity increases or the serial correlation could induce stronger effects due to slower learning about aggregate variables. Third, "adding capital may help to generate larger demand responses following a noise shock".

[10]Lorenzoni (2009) modifies the method of Townsend (1983) to solve the hierarchy of expectations. Smaller differences may arise due to the used methodology. However, the methodology is sensitive to the convergence criteria. For details, see Appendix B.4.

2.6.1 New Keynesian model without capital

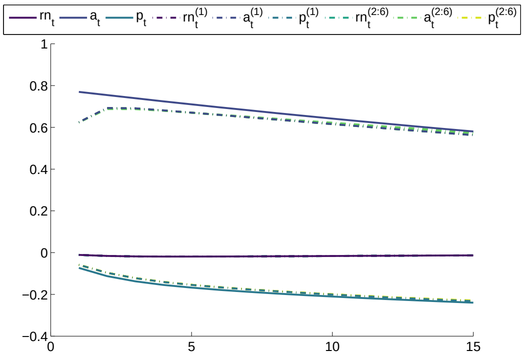
In this subsection, I first discuss the model dynamics of the new Keynesian model without capital for different hierarchies of expectations. Then, in the second step, I evaluate the first two conjectures stated above.

To understand the effect of the truncation of the higher order expectations on the model dynamics, I plotted the impulse response functions of the hierarchy of expectations of aggregate productivity, a_t, the nominal interest rate, r_t^n, and the aggregate price level, p_t for two different dimensions of the expanded state space in Figure 2.1. Thereby, I truncate the hierarchy of expectations at two degrees of higher order expectation. First, I truncate the hierarchy of expectations at $\overline{o} = 8$, which is the choice of Nimark (2014). Second, I truncate it at $\overline{o} = 50$, which is the choice of Lorenzoni (2009).

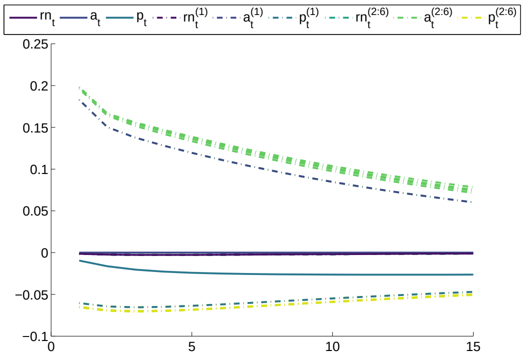
The graphs (a) and (b) show the impulse responses to an innovation to a productivity shock and a noise shock, when I truncate the hierarchy of expectations at $\overline{o} = 8$. The solid lines represent the realisations of the state variables, the dashed line of the same colour the first order expectation and the higher order expectations are shown as dashed lines in an alternative colour. The graphs show that the hierarchy of expectations is very narrow for all variables. The nominal interest rate does not exhibit a hierarchy of expectation as the realisation is known to the agents. The plots also show that the hierarchy of expectation is clearly ordered.

The next two graphs (c) and (d) show the hierarchy of expectations, when I truncate the hierarchy of expectations at $\overline{o} = 50$. As the implied dynamics conditional on the two dimensions of the hierarchy of expectation are very similar, I proceed the analysis with the smaller dimension.

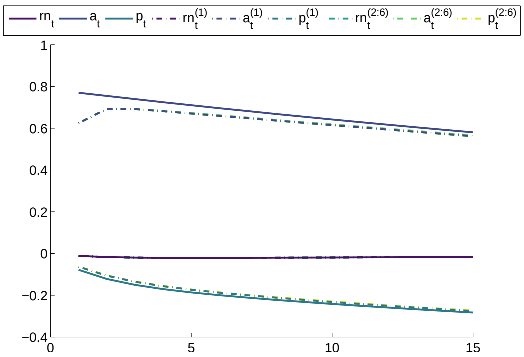
In the next step, I discuss the conjectures stated above. The first conjecture was that lower persistence of aggregate productivity would increase the initial response of consumption compared to the case in which productivity changes are permanent. Figure 2.2 shows the impulse responses for the two cases. It shows that the opposite is the case. If the persistence of the aggregate process is lower, than the response of consumption becomes weaker. The reason is the same as under full information. If productivity increases, agents do not consume all the gains they make in the current period. They save it instead for the coming periods in which productivity will be lower. The agents tend to smooth consumption.



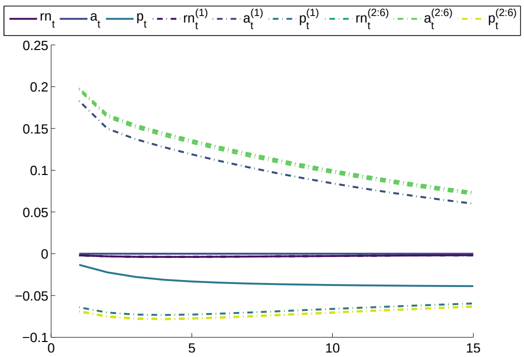
(a) Standard deviation shock to ν_t. $\overline{o} = 8$.



(b) Standard deviation shock to ε_t. $\overline{o} = 8$.



(c) Standard deviation shock to ν_t. $\overline{o} = 50$.



(d) Standard deviation shock to ε_t. $\overline{o} = 50$.

Fig. 2.1 Impulse responses of hierarchy of expectation. Model without capital.

The graphs (a) and (b) show the impulse response functions of the hierarchy of expectations to productivity and noise shocks when $\overline{o} = 8$. The solid lines represents the impulse responses of the realized state variables and the dashed lines the hierarchy of expectations of the state variables. The graphs (c) and (d) show the same variables with the only difference that the model is solved with $\overline{o} = 50$.

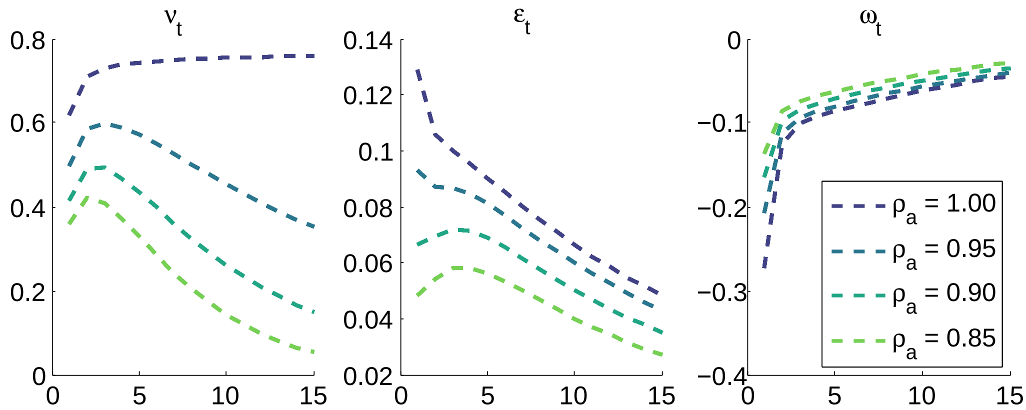


Fig. 2.2 Impulse responses of consumption for different values of ρ_a.

The graphs show the impulse responses of consumption to a productivity shock ν_t, a noise shock ε_t and a shock to inflation ω_t for different values of the persistence of the aggregate productivity process, ρ_a.

Next, introducing persistence to idiosyncratic productivity does not change the aggregate model dynamics much. The reasons are the following. First, the agents attribute most of the movements in composite productivity to the idiosyncratic process already when it is transitory. The persistence in the idiosyncratic productivity process increases the Kalman gain even more and closer to one, but in effect this does not change the overall picture. The same effect one achieves by increasing the standard deviation of the idiosyncratic innovation without introducing persistence. Second, beyond the first point, the individual expectation of the agents about their idiosyncratic productivity process has no impact on aggregate dynamics.

2.6.2 New Keynesian model with capital

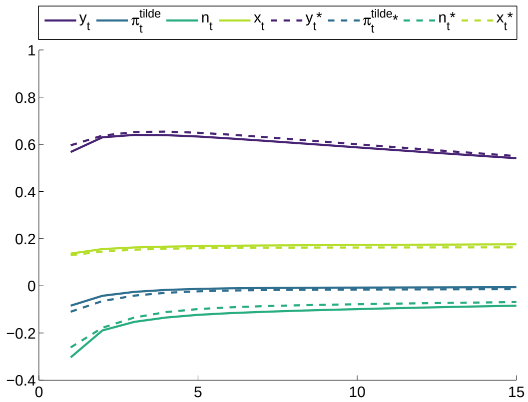
In this subsection, I analyse the equilibrium dynamics of the new Keynesian model with capital and heterogeneous information. I first compare the model dynamics of the full information new Keynesian model with the ones of the heterogeneous information model. Then, I discuss the third conjecture stated above.

Figure 2.3 shows that the impulse responses to a productivity shock under heterogeneous information is not too different from the ones under full information. Nevertheless, there are also differences when comparing the impulse responses to productivity shocks under heterogeneous and full information. It is most evident for the price of capital. It is 0.05 percent higher and the gap is persistently growing over the horizon. This result corresponds with the observed higher investment and lower consumption under heterogeneous information compared to the full information model. The reason is that the agents believe that the shock to be lower than under full information. Related to their consumption smoothing incentive they consume less and invest relatively more than under full information. The main differences appear in response to noise shocks as these shocks do not exist under full information.

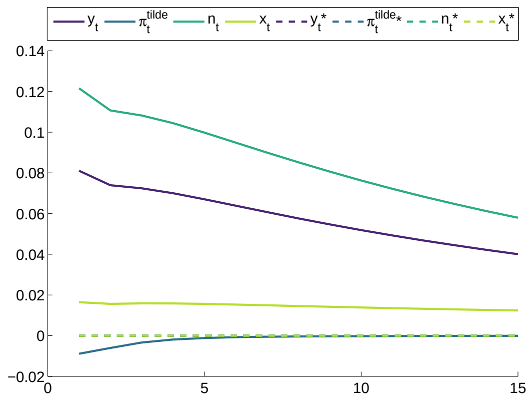
Now, I turn to the conjecture of Lorenzoni (2009). His conjecture was that capital might prolong the effects of noise shocks on demand. Figure 2.4 shows that relative to the model without capital, the responses of output to a productivity shock are lower. With respect to the noise shock the initial response of output is also lower which stands in contrast to Lorenzoni's conjecture.

2.7 Conclusion

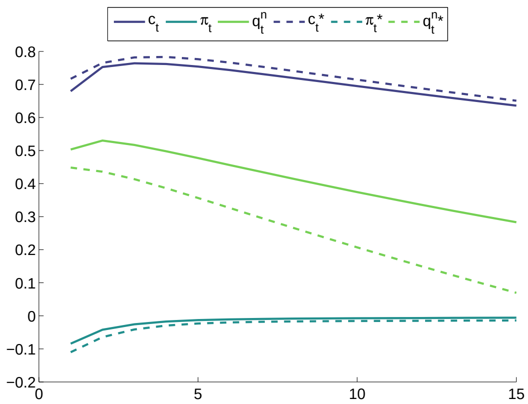
In this paper, I derived a solution algorithm that solves HI-DSGE models of a general form. I illustrated the power of the methodology at the example of a new Keynesian model with capital that has not been studied before. Based on this model, I discussed the effects of persistent aggregate and idiosyncratic productivity processes, and the effect of capital on the model dynamics compared to the full information model and the heterogeneous information new Keynesian model without capital. Altogether, the model with capital does not seem to be capable of creating more noise driven fluctuations than the model without, which, as stated by Lorenzoni (2009), does not generate sufficiently much fluctuation as indicated in the data. For future research it might be interesting to implement mechanisms that induce either higher strategic complementarity in prices or in other parts of the model in order to increase the effects of heterogeneous information. One way could be to impose firm specific capital in the veins of Sveen and Weinke (2005).



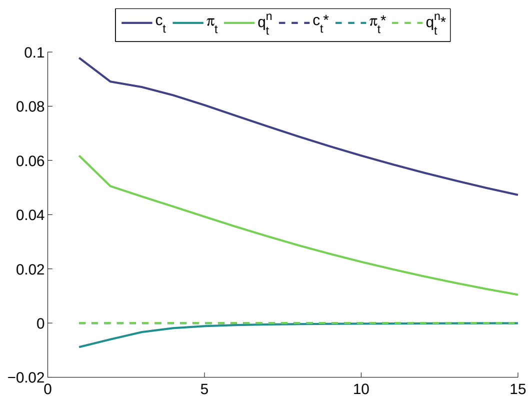
(a) Standard deviation shock to v_t. Contemporaneous variables.



(b) Standard deviation shock to ε_t. Forward looking variables.



(c) Standard deviation shock to v_t. Contemporaneous variables.



(d) Standard deviation shock to ε_t. Forward looking variables.

Fig. 2.3 Impulse responses of selected variables.

The graphs (a) and (b) show the impulse response functions of selected jump variables to a productivity shock and a noise shocks. The solid lines represents the impulse responses of the of the variables under heterogeneous information and the dashed lines the ones under full information. The graphs (c) and (d) show the the forward looking variables in the same way.

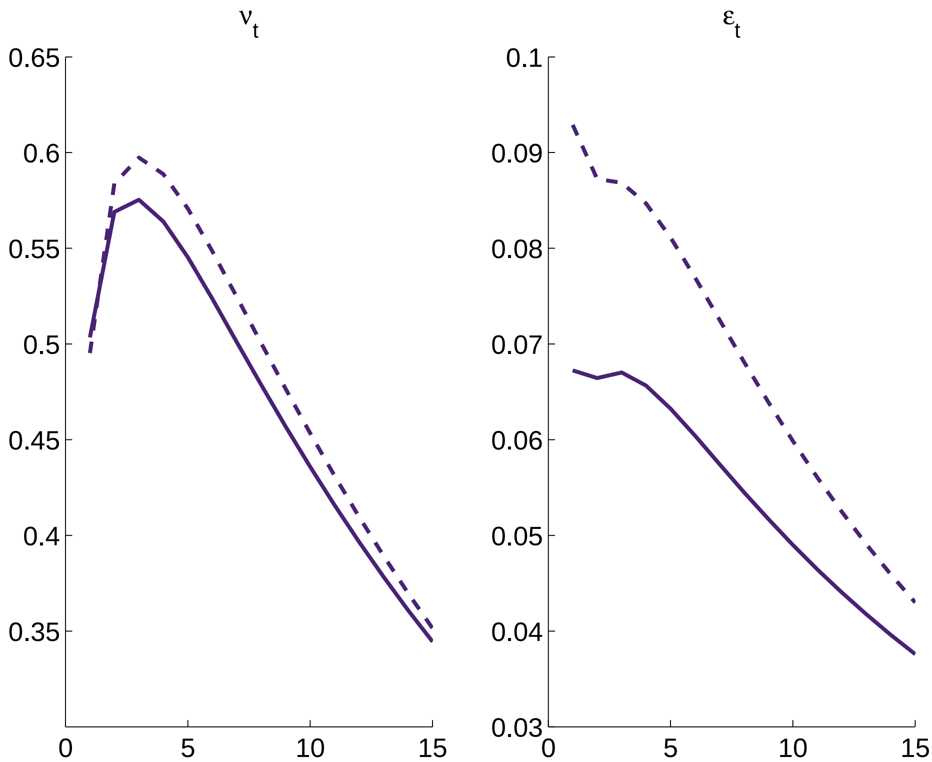


Fig. 2.4 Impulse responses of output.

The graphs show the impulse responses of output to a shock to aggregate productivity and a noise shock in the model without capital (dashed line) and capital (solid line).

Chapter 3

Asset Pricing Implications of a Heterogeneous Information New Keynesian Model

3.1 Introduction

The literature has identified a number of asset pricing puzzles in economics. I will limit the analysis in this paper to the two best known. As layed out by Kocherlakota (1996), the equity premium puzzle, documented by Mehra and Prescott (1985), states that there is not a plausible pair of the subjective discount factor and the measure of relative risk aversion in a representative agent framework that can match both the empirical values of the annual real interest rate and the equity premium. The stated problem is that the discount factor is bounded by one from above and thus the agents need to be very averse to consumption risk in order to generate asset pricing implications comparable to historical data, which stands in contrast to empirical evidence. In addition, with high risk aversion, although one can generate the equity premium, the risk free rate becomes significantly larger than in the data. This phenomenon is pointed out by Weil (1989) and it is generally known as the risk free rate puzzle. He is also showing that one can select a higher intertemporal rate of substitution to achieve a lower risk free rate with Epstein and Zin (1989) preferences. However, also in this case, one still needs high risk aversion in order to generate the empirically observable equity premium.[1]

[1]The utility function allows to calibrate the parameter for the risk aversion and the ratio of intertemporal substitution independently.

The subsequent work by Abel (1990) and Constantinides (1990) show in an endowment economy that habit formation can resolve the equity premium and the risk free rate puzzle. However, under habit formation the volatility of the risk free rate becomes implausibly high. In order to match the moments of the asset prices better, Campbell and Cochrane (1999) propose a sensitivity function which is time varying and that creates higher habit formation in bad than in good times. This improves the fit of their endowment economy with the data, especially the high volatility.

These ideas also have been brought forward to macroeconomic models. Lettau and Uhlig (2000) investigate Campbell and Cochrane (1999)'s mechanism in a business cycle model and find that under these conditions the volatility in consumption is ten percent of what is found in the data. Moreover, they find that habit formation in consumption in combination with habit formation in leisure resolves this issue. Uhlig (2007) proceeds in this vein and shows that sticky wages help to match business cycle facts and asset pricing implications.

Furthermore, Tallarini Jr. (2000) applies Epstein and Zin (1989) preferences and calibrates the relative risk aversion parameter and the rate of intertemporal substitution independently. In doing so, he finds that changes in the relative risk aversion has little impact on macroeconomic dynamics. However, he also shows that this does not resolve the equity premium and risk free rate puzzle, because relative risk aversion still needs to be implausibly high.[2]

The literature mentioned above discusses asset pricing implications in real economies. Instead De Paoli et al. (2010) compute the asset pricing implications of a new Keynesian model and find that nominal rigidities decrease risk premia when the model is mainly driven by productivity shocks and increase them when driven by demand shocks. In a standard New Keynesian model demand shocks appear only in the form of monetary policy shocks, while heterogeneous information models are shown to provide a broader theory of demand shocks. Lorenzoni (2009), in particular, shows that noise shocks can account for up to 72 percent of demand driven business cycle volatility. On a different note Heer et al. (2012) point out that the model by De Paoli et al. is not capturing fundamental labour market facts and propose to include wage stickiness, following Erceg et al. (2000).

As heterogeneous information models currently do not allow for higher order approximations, I will follow the approach of Jermann (1998) to derive the asset pricing implications of

[2]The relative risk aversion required to match the asset pricing facts in the production economy is $(\chi + \theta)/(1+\theta) = 25.8$.

the model. He shows that one can use the log-linear solution of a macroeconomic model and combine it with log-linear asset pricing formulae. With regard to heterogeneous information, Barillas and Nimark (2013) derive asset pricing formulae for a term structure model, but they do not decompose the rates in the veins of Jermann. Moreover, they do not derive the asset pricing formula for equity. It is also noteworthy that they formulate the stochastic discount factor exogenously. Thus, the contribution of this paper is twofold. First, I generalize the methodology of Jermann (1998) to assess asset pricing implications of a DSGE model with heterogeneous information. Second, I extend the model by Heer et al. (2012) to allow for heterogeneous information and assess its asset pricing implications which defines the stochastic discount factor endogenously.

I find that the heterogeneous information model mitigates if not resolves the equity premium and the risk free rate puzzle. Thereby, heterogeneous information especially affect the risk free rate. Due to the uncertainty about the aggregate state of the economy, precautionary savings are higher than under full information and hence, the risk free rate is lower. At the same time the return to equity remains relatively unchanged which implies a higher risk premium. Another helpful effect of heterogeneous information is that, while it decreases the risk free rate, it does not increase the standard deviation thereof.

Finally, in contrast to the results in the literature, higher nominal rigidities increase the equity premium when the heterogeneous information model is mostly driven by supply shocks and reduces the risk premium when it is driven by demand shocks.

The remainder of the paper is organized as follows. In Section 3.2, I derive the log-normal asset pricing formulae under heterogeneous information. In Section 3.3, I present the model. Then, I compute the asset pricing implications in in Section 3.4. Section 3.5 concludes.

3.2 Asset pricing formulae

Barillas and Nimark (2013) point out that the stochastic discount factor of the agents needs to be individual with heterogeneous information sets. This condition holds true in the model at hand, in which I derive the stochastic discount factor endogenously as part of the DSGE model, which has not been done before.

I derive log-linear asset pricing formulae in the line of Jermann (1998) to provide insights into the forces at play that differentiate asset pricing formulae under heterogeneous information from the ones under full information. Jermann (1998) decomposes the non-linear asset pricing formulae into expected values, variances and covariances. In doing so, he finds a closed form solution for the risk free rate, but not for the return to equity. To gain deeper insights into the return to equity, he splits the equity return into the infinite sum of the return to strips. Strips are future one time payments of dividends. I proceed similarly.

The general asset pricing formula under heterogeneous information looks as follows:

$$P_t = \beta \frac{1}{\Lambda_{jt}} E_{jt} \left[\Lambda_{j,t+1} \left(P_{t+1} + D_{t+1} \right) \right]. \tag{3.1}$$

The price of an asset is equal to the present discounted value of the next periods payout. In the general case, the payout is equal to next periods price plus a dividend or coupon payment. In the case of the risk free rate, the payout is one with certainty, $(P_{t+1} + D_{t+1}) = 1$.

The asset pricing formula needs to hold for all agents. When forward substituting, I therefore apply the expectation operator of agent $i \neq j$.[3] Iterating forward once yields:

$$\begin{aligned} P_t = {} & \beta^2 \frac{1}{\Lambda_{jt}} E_{jt} \left[\frac{\Lambda_{j,t+1}}{\Lambda_{i,t+1}} E_{i,t+1} \left[\Lambda_{i,t+2} P_{t+2} \right] \right] \\ & + \beta^2 \frac{1}{\Lambda_{jt}} E_{jt} \left[\frac{\Lambda_{j,t+1}}{\Lambda_{i,t+1}} E_{i,t+1} \left[\Lambda_{i,t+2} D_{t+2} \right] \right] + \beta \frac{1}{\Lambda_{jt}} E_{jt} \left[\Lambda_{j,t+1} D_{t+1} \right]. \end{aligned} \tag{3.2}$$

As private information is by definition private to the agent holding it, the best expectation that an agent can form about the expectation of another agent is the average expectation, i.e. $E_{jt}\left[E_{it}\left[\,\cdot\,\right]\right] = E_{jt}\left[\overline{E}_t\left[\,\cdot\,\right]\right]$. Taking this fact into account and integrating over j[4], I find the following expression when iterating forward K times:

$$P_t = \frac{1}{\Lambda_t} \beta^K \overline{E}_t \left[\Lambda_{t+K} P_{t+K} \right] + \ldots + \frac{1}{\Lambda_t} \sum_{k=1}^{K} \beta^k \overline{E}_t \left[\overline{E}_{t+1} \left[\ldots \overline{E}_{t+k-1} \left[\Lambda_{t+k} D_{t+k} \right] \right] \right]. \tag{3.3}$$

For $K \to \infty$ the first term vanishes and one finds the well known representation of the price as the present discounted sum of an infinite income stream. However, the present

[3] In this regard I follow Barillas and Nimark (2013).

[4] This is a short cut for solving the problem. Strictly speaking, one should aggregate over the approximated version of the equation. Up to a first order approximation the results are the same without loss of generality. However, the implications for the variance may differ. Nevertheless, I leave this issue open for future research.

discounted value of each term does not coincide with a strip as under full information.[5]

In the following, I derive the log-normal asset pricing formulae for the risk free rate and the the return to equity, by exploiting the characteristics of log-linearly distributed random variables. First, the expected value of a log-normally distributed random variable Υ_t is equal to the expected value plus one half its variance, $E_{jt}\left[\Upsilon_t\right] = \exp\left\{E_{jt}\left[\ln\Upsilon_t\right] + \frac{1}{2}Var_{jt}\left[\ln\Upsilon_t\right]\right\}$. Second, the standard deviation of a log-normally distributed random variable is defined as:

$$Std_{jt}\left[\Upsilon_t\right] = E_{jt}\left[\Upsilon_t\right]\sqrt{\exp\left\{Var_{jt}\left[\ln\Upsilon_t\right] - 1\right\}}.$$

3.2.1 Risk free rate

The real price of the one period zero coupon bond, which defines the risk free rate, is equal to the stochastic discount factor:

$$Q_t^f = \beta\frac{1}{\Lambda_{jt}}E_{jt}\left[\Lambda_{j,t+1}\right]. \tag{3.4}$$

Integrating over j and inverting the price formula for the zero coupon bond yields the formula for the risk free rate:

$$R_t^f = \frac{1}{\beta}\Lambda_t\overline{E}_t\left[\frac{1}{\Lambda_{t+1}}\right]. \tag{3.5}$$

Applying the transformations following the log-linearity assumption, the log-normal asset pricing formula for the risk free rate reads:

$$R_t^f = \frac{1}{\beta}\exp\left\{-\left(\overline{E}_t\left[\lambda_{t+1}\right] - \lambda_t\right) - \frac{1}{2}\overline{V}ar_t\left[\lambda_{t+1}\right]\right\}. \tag{3.6}$$

Further, taking the unconditional expectation of the risk free rate, the resulting expression under heterogeneous information reads in closed form:[6]

$$E\left[R_t^f\right] = \frac{1}{\beta}\exp\left\{\frac{1}{2}\left(Var\left[\overline{E}_t\left[\lambda_{t+1}\right] - \lambda_t\right] - E\left[\overline{V}ar_t\left[\lambda_{t+1}\right]\right]\right)\right\}.$$

[5]For details see Appendix C.1.1.
[6]For details see Appendix C.1.2.

As in the full information case, the unconditional expectation of the risk free rate depends on the steady state value of the risk free rate and the variance of the expected growth rate of marginal utility, but it deviates with respect to the third term. With heterogeneous information, the unconditional expectation of the risk free rate depends on the variance of marginal utility and not on the variance of the difference between marginal utility and its expectation. This means that the precautionary savings effect under heterogeneous information is typically larger under heterogeneous information than under full information. The additional uncertainty stems from the fact that aggregate marginal utility is unknown at any point in time. Further differences may arise due to the variance of the expected growth rate of marginal utility, which depends on the dynamics of marginal utility. In this respect, I cannot derive any theoretical insights. Hence, I will come back to this point in Section 3.4.2 when conducting the numerical assessment.

The standard deviation of the risk free rate could differ under heterogeneous information relative to full information, too. First, it depends on the unconditional expectation of the risk free rate. And second, it also depends on the variance of the expected growth rate of marginal utility:

$$Std\left[R_t^f\right] = E\left[R_t^f\right]\sqrt{\exp\left\{Var\left[\overline{E}_t\left[\lambda_{t+1}\right] - \lambda_t\right]\right\} - 1}. \tag{3.7}$$

Differences in the unconditional expectation of the risk free rate were discussed above and so was the variance of expected marginal utility growth.

3.2.2 Equity price

Assume that equity pays the dividend D_{t+k} in period k. Above, I showed that one can express the price of equity as the infinite sum of present discounted dividend stream:

$$V_t = \beta \frac{1}{\Lambda_t}\overline{E}_t\left[\Lambda_{t+1}\left(V_{t+1} + D_{t+1}\right)\right] = \sum_{k=1}^{\infty} V_t[D_{t+k}], \quad \text{where} \tag{3.8}$$

$$V_t[D_{t+k}] = \beta^k \frac{1}{\Lambda_t}\overline{E}_t\left[\overline{E}_{t+1}\left[\ldots\overline{E}_{t+k-1}\left[\Lambda_{t+k}D_{t+k}\right]\right]\right].$$

For the equity price one cannot derive a closed form solution neither for the unconditional expectation nor the standard deviation. Instead one needs to simulate them. However, following the approach of Jermann (1998), one can gain further insights to the drivers of the return to equity by analysing the terms of the infinite sum:

$$R_{t,t+1} = \frac{V_{t+1}}{V_t} = \sum_{k=1}^{\infty} \frac{V_{t+1}[D_{t+k}]}{V_t} = \sum_{k=1}^{\infty} W[D_{t+k}] R_{t,t+1}[D_{t+k}], \quad \text{where} \tag{3.9}$$

$$R_{t,t+1}[D_{t+k}] = \frac{V_{t+1}[D_{t+k}]}{V_t[D_{t+k}]} \quad \text{and} \quad W[D_{t+k}] = \frac{V_t[D_{t+k}]}{V_t}.$$

I call $R_{t,t+1}[D_{t+k}]$ the return to the strip of a single dividend payment D_{t+k} in period k. However, it is actually not exactly the same as the return to a strip under full information, as the law of iterated expectations does not hold with heterogeneous information. While this is important to point out, it does not affect the derivation of the details below. The return to equity is neverthelsess an infinite weighted sum of that term and I can use it to derive further insights from it. The weight between the return of a strip and the return to equity is defined by $W[D_{t+k}]$.

For convenience, I define the iterative higher expectations for an arbitrary variable Υ_t as:

$$\overline{E}_t^{(k-1)}\left[\Upsilon_{t+k}\right] = \overline{E}_t\left[\ldots \overline{E}_{t+k-1}\left[\Upsilon_{t+k}\right]\right]. \tag{3.10}$$

Then, the return to the strip can be written in log-linear form as:

$$R_{t,t+1}[D_{t+k}] = \frac{1}{\beta} \exp\left\{ \overline{E}_{t+1}^{(k-1)}\left[\lambda_{t+k} + d_{t+k}\right] - \lambda_{t+1} - \left(\overline{E}_t^{(k-1)}\left[\lambda_{t+k} + d_{t+k}\right] - \lambda_t \right) \right.$$
$$\left. + \frac{1}{2}\overline{V}ar_{t+1}^{(k-1)}\left[\lambda_{t+k} + d_{t+k}\right] - \frac{1}{2}\overline{V}ar_t^{(k-1)}\left[\lambda_{t+k} + d_{t+k}\right] \right\}, \tag{3.11}$$

where $\overline{V}ar_t^{(k-1)}\left[\Upsilon_{t+k}\right]$ defines the variance corresponding to the higher order expectations:

$$\overline{V}ar_t^{(k-1)}\left[\Upsilon_{t+k}\right] = \overline{E}_t\left[\left(\overline{E}_{t+1}^{(k-1)}\left[\Upsilon_{t+k}\right] - \overline{E}_t^{(k-1)}\left[\Upsilon_{t+k}\right]\right)\left(\overline{E}_{t+1}^{(k-1)}\left[\Upsilon_{t+k}\right] - \overline{E}_t^{(k-1)}\left[\Upsilon_{t+k}\right]\right)'\right].$$

Further, to find the expected return, I apply the expectation operator another time. Then, the expected return to the strip reads:

$$\begin{aligned}\overline{E}_t\left[R_{t,t+1}[D_{t+k}]\right] = \frac{1}{\beta}\exp\Big\{&\overline{E}_t\left[\overline{E}_{t+1}^{(k-1)}\left[\lambda_{t+k}+d_{t+k}\right]-\lambda_{t+1}-\left(\overline{E}_t^{(k-1)}\left[\lambda_{t+k}+d_{t+k}\right]-\lambda_t\right)\right] \\ &+\frac{1}{2}\overline{E}_t\left[\overline{V}ar_{t+1}^{(k-1)}\left[\lambda_{t+k}+d_{t+k}\right]\right]-\frac{1}{2}\overline{E}_t\left[\overline{V}ar_t^{(k-1)}\left[\lambda_{t+k}+d_{t+k}\right]\right] \\ &+\frac{1}{2}\overline{V}ar_t\left[\overline{E}_{t+1}^{(k-1)}\left[\lambda_{t+k}+d_{t+k}\right]-\lambda_{t+1}-\left(\overline{E}_t^{(k-1)}\left[\lambda_{t+k}+d_{t+k}\right]-\lambda_t\right)\right]\Big\}.\end{aligned}$$

Note that the variance of a log-normally distributed random variable is constant.

Under full information the first two terms with regard to the expectations of variables in time $t+k$ cancel out and only one part, the negative expected growth of marginal utility, remains. Here instead, all three terms stay part of the formula. Moreover, the conditional variance of contemporaneous variables is zero under full information. The other variance terms instead can be rewritten to find an expression consisting of the conditional variance of marginal utility and the covariance between the expectation of the payout in time period $t+k$ and tomorrows marginal utility. With heterogeneous information I arrive at a similar expression, which is different only due to the uncertainty about the current state of the economy.

$$\begin{aligned}\overline{E}_t\left[R_{t,t+1}[D_{t+k}]\right] = \frac{1}{\beta}\exp\Big\{&\overline{E}_t\left[\overline{E}_{t+1}^{(k-1)}\left[\lambda_{t+k}+d_{t+k}\right]-\lambda_{t+1}-\left(\overline{E}_t^{(k-1)}\left[\lambda_{t+k}+d_{t+k}\right]-\lambda_t\right)\right] \\ &+\frac{1}{2}\overline{V}ar_t\left[\overline{E}_{t+1}^{(k-1)}\left[\lambda_{t+k}+d_{t+k}\right]-\overline{E}_t^{(k-1)}\left[\lambda_{t+k}+d_{t+k}\right]\right] \\ &+\frac{1}{2}\overline{E}_t\left[\overline{V}ar_{t+1}^{(k-1)}\left[\lambda_{t+k}+d_{t+k}\right]\right]-\frac{1}{2}\overline{E}_t\left[\overline{V}ar_t^{(k-1)}\left[\lambda_{t+k}+d_{t+k}\right]\right] \\ &+\frac{1}{2}\overline{V}ar_t\left[\lambda_{t+1}-\lambda_t\right] \\ &-\overline{C}ov_t\left[\overline{E}_{t+1}^{(k-1)}\left[\lambda_{t+k}+d_{t+k}\right]-\overline{E}_t^{(k-1)}\left[\lambda_{t+k}+d_{t+k}\right],\lambda_{t+1}-\lambda_t\right]\Big\}.\end{aligned}$$

After further transformations, I find:

$$\begin{aligned}\overline{E}_t\left[R_{t,t+1}[D_{t+k}]\right] = \frac{1}{\beta}\exp\Bigg\{&\overline{E}_t\left[\overline{E}_{t+1}^{(k-1)}\left[\lambda_{t+k}+d_{t+k}\right]-\lambda_{t+1}-\left(\overline{E}_t^{(k-1)}\left[\lambda_{t+k}+d_{t+k}\right]-\lambda_t\right)\right]\\ &+\Im\left[\lambda_{t+k}+d_{t+k}\right]-\frac{1}{2}\overline{V}ar_t\left[\lambda_{t+1}-\lambda_t\right]\\ &-\overline{C}ov_t\left[\overline{E}_{t+1}^{(k-1)}\left[\lambda_{t+k}\right]-\overline{E}_t^{(k-1)}\left[\lambda_{t+k}\right]-\left(\lambda_{t+1}-\lambda_t\right),\lambda_{t+1}-\lambda_t\right]\\ &-\overline{C}ov_t\left[\overline{E}_{t+1}^{(k-1)}\left[d_{t+k}\right]-\overline{E}_t^{(k-1)}\left[d_{t+k}\right],\lambda_{t+1}-\lambda_t\right]\Bigg\}. \end{aligned} \tag{3.12}$$

In this formula, I summarised the variance terms that cancel out under full information as $\Im[\lambda_{t+k}+d_{t+k}]$:

$$\begin{aligned}\Im\left[\lambda_{t+k}+d_{t+k}\right] = &\frac{1}{2}\overline{V}ar_t\left[\overline{E}_{t+1}^{(k-1)}\left[\lambda_{t+k}+d_{t+k}\right]-\overline{E}_t^{(k-1)}\left[\lambda_{t+k}+d_{t+k}\right]\right]\\ &+\frac{1}{2}\overline{E}_t\left[\overline{V}ar_{t+1}^{(k-1)}\left[\lambda_{t+k}+d_{t+k}\right]\right]-\frac{1}{2}\overline{E}_t\left[\overline{V}ar_t^{(k-1)}\left[\lambda_{t+k}+d_{t+k}\right]\right].\end{aligned}$$

Equation (3.12) is ultimately very similar to the full information one. In addition to the full information expression it includes two expectational terms and an additional variance term, $\Im[\lambda_{t+k}+d_{t+k}]$. The covariance terms are structurally the same as under full information, with the difference that all terms are expressed relative to the current state of the world. This makes sense as the current state of the world is not directly observed and hence the expectation about it induces uncertainty.

3.3 The model

The model is a New Keynesian Model with incomplete markets following Lorenzoni (2009) using the transformation of signals introduced by Nimark (2014). The differences of the model at hand are that I include capital into the model and that price stickiness results from intangible costs in the veins of Rotemberg (1982) instead of staggered pricing as in Calvo (1983). Moreover, I allow for habit formation and sticky wages. These are the modelling aspects that have been proven to be useful to match asset pricing facts in macroeconomic models. Under full information the model corresponds to the model of Heer et al. (2012).

The economy is assumed to be geographically separated in an infinite number of regions (hereafter islands), indicated by j, in which there is each one household, one final goods producing firm, a continuum of capital producers and firms producing differentiated intermediary goods. The agents on island j share the same information set Ω_{jt} available to them in period t. The firms of the wholesale sector accumulate capital and hire labour, where capital and labour are traded only regionally. Moreover, the intermediate goods producing firms have monopolistic power and set prices subject to their demand. The final goods producing firm buys a random subset of goods produced by the firms of the wholesale sectors of other islands. The household consumes the bundle of goods produced by the final goods producer on their island, provides labour to the wholesale sector on their island and invests in a one period discounted bond. Moreover, it can only infrequently adjust wages with the firms it is providing labour to. The capital producers also use the composite final good as material input to produce new capital goods.

In this setting the agents on island j do not know all quantities and prices of the goods in the economy and they also do not know aggregate productivity. Hence, they need to form expectations about the aggregate state of the economy. Further, the aggregate state of the economy is relevant for the agents decision as price setting is complementary.

3.3.1 The household's maximization problem

The household on island j chooses consumption of the composite goods produced on their island by the final goods producer, C_{jt}, nominal wages of firm f on its island, W^n_{jft}, to which it provides a specific type of labour and tomorrow's bond holdings, $B_{j,t+1}$. It chooses these variables such that it maximizes its expected discounted utility subject to its budget constraint and the demand functions for each type of labour provided by the household. It is assumed that the household has a continuum of members such that they provide heterogeneous types of labour to all firms on their island and that they share the aggregate income. Households can negotiate wages only with probability $1-\varphi$, i.e. I apply a Calvo-type of wage setting. This type of wage setting has been advocated by Erceg et al. (2000) and Gali (2011). The utility function exhibits CRRA preferences in consumption and in hours worked, where θ is a scaling factor between the utility of consumption and and the disutility of labour. σ represents the rate of intertemporal substitution and γ determines the Frisch elasticity of labour supply. Furthermore, agents exhibit external habit formation, $\vartheta_{jt} = \chi^c C_{j,t-1}$, where χ^c defines the degree of the habit. The firms demand for any type of labour is defined by a

CES demand function.

$$\max_{\left\{C_{j,t+\tau},W_{jf,t+\tau},B_{j,t+\tau+1},S_{j,t+\tau+1}\right\}} E_{jt}\left[\sum_{\tau=0}^{\infty}(\varphi\beta)^{\tau}\left(\frac{\left(C_{j,t+\tau}-\vartheta_{j,t+\tau}\right)^{1-\sigma}}{1-\sigma}-\frac{\theta}{1+\gamma}\int H_{jf,t+\tau}^{1+\gamma}\,df\right)\right]$$

$$\text{s.t.}\quad \frac{B_{j,t+\tau+1}}{R_{t+\tau}^{n}}+\bar{P}_{j,t+\tau}C_{j,t+\tau}=B_{j,t+\tau}+\int\left(W_{jf,t+\tau}^{n}H_{jf,t+\tau}+\Pi_{jf,t+\tau}\right)\,df+\Pi_{j,t+\tau}^{k} \tag{3.13}$$

$$H_{jf,t+\tau}=\left(\frac{W_{jf,t+\tau}^{n}}{W_{j,t+\tau}^{n}}\right)^{-\varepsilon_w}H_{j,t+\tau} \tag{3.14}$$

On the expenditure side, the household can either save by investing in the one period risk free bond B_{jt} at the nominal return R_t^n and it can consume the amount C_{jt} of the composite final good at the price $\bar{P}_{jt}$. The income of the household consists of the payout from last period's investment in the one period risk free bond, the income from labour and the profits from the intermediary firms and the capital producers.

The maximization problem of the households yields a standard Euler equation for consumption and an optimal wage setting equation for the agents that can adjust their wages. All agents of the household that are selected to update wages in a given period set the same nominal wage $W_{j*,t}^n = W_{jf,t+\tau}^n$:

$$\frac{\Lambda_{jt}}{\bar{P}_{jt}}=\beta R_t^n E_{jt}\left[\frac{\Lambda_{j,t+1}}{\bar{P}_{j,t+1}}\right], \tag{3.15}$$

$$W_{j*,t}^{n}=\left[\theta\frac{\varepsilon_w}{\varepsilon_w-1}\frac{E_{jt}\left\{\sum_{\tau=0}^{\infty}(\varphi\beta)^{\tau}\left(W_{j,t+\tau}^{n}\right)^{(1+\gamma)\varepsilon_w}H_{j,t+\tau}^{1+\gamma}\right\}}{E_{jt}\left\{\sum_{\tau=0}^{\infty}(\varphi\beta)^{\tau}\Lambda_{j,t+\tau}\left(W_{j,t+\tau}^{n}\right)^{\varepsilon_w}H_{j,t+\tau}\right\}}\right]^{\frac{1}{1+\varepsilon_w\gamma}}, \tag{3.16}$$

where $\Lambda_{jt}=\left(C_{j,t+\tau}-\vartheta_{j,t+\tau}\right)^{-\sigma}$ is the marginal utility. As only a fraction of agents can adjust their nominal wages, the wage index on island j is equal to the weighted average of the agents adjusting wages and the previous periods wage index:

$$\left(W_{jt}^{n}\right)^{1-\varepsilon_w}=(1-\varphi)\left(W_{j*,t}^{n}\right)^{1-\varepsilon_w}+\varphi\pi\left(W_{j,t-1}^{n}\right)^{1-\varepsilon_w}. \tag{3.17}$$

When log-linearising the equations (3.16) and (3.17), I find the island specific wage Phillips-curve:

$$\pi_{jt}^{w} = \kappa^{w}\left(w_{jt} + \lambda_{jt} - \gamma h_{jt}\right) + \beta E_{jt}\left[\pi_{j,t+1}^{w}\right], \tag{3.18}$$

where $\kappa^{w} = -\frac{(1-\beta\varphi)(1-\varphi)}{\varphi(1+\varepsilon_{w}\varphi)}$.

3.3.2 The firm's problem

There are three types of firms on an island: a continuum of intermediate goods producing firms acting in monopolistic competition, a continuum of capital producers in perfect competition and a final goods producing firm. The intermediate goods producing firms produce differentiated consumption goods. They hire labour from the household on their island and they accumulate capital as factor inputs. They buy capital from the capital producing firms on their island. The final goods producer purchases the differentiated goods from a random subset of the wholesale sectors of other islands and produces a composite final good. Parts of the final goods are sold to to the capital producers as material input and the remaining part is sold the household for consumption.

Final goods sector

The final goods producing firm on island j produces the final good by bundling intermediate goods from a random subset, Ξ_{jt}, of the heterogeneous goods produced by the wholesale sector of other islands. The production function of the final goods producing firm is defined as follows:

$$Y_{jt} = \left(\int_{\Xi_{jt}}\int_{0}^{1} Y_{jimt}^{\frac{\varepsilon-1}{\varepsilon}}\, dm\, di\right)^{\frac{\varepsilon}{\varepsilon-1}}. \tag{3.19}$$

Choosing the combination of goods that maximises profits of the final goods producing firm leads to the following demand function of the final goods producer on island j for the good of firm m on island i:

$$Y_{jimt} = \left(\frac{P_{imt}}{\bar{P}_{jt}}\right)^{-\varepsilon} Y_{jt} \quad \text{and} \quad \bar{P}_{jt} = \left(\int_{\Xi_{jt}} P_{it}^{1-\varepsilon}\, di\right)^{\frac{1}{1-\varepsilon}}. \tag{3.20}$$

The price index of the bundle of goods is defined as $\bar{P}_{jt}$.

The capital producing sector

The capital producer l on island j chooses investments, X_{jlt}, to maximize its profits, Π^k_{jlt}, according to the following production technology:

$$Y^{k_s}_{jlt} = \phi\left(\frac{X_{jlt}}{K_{jlt}}\right) K_{jlt}, \tag{3.21}$$

where $\phi\left(\frac{X_{jlt}}{K_{jlt}}\right)$ is convex, i.e. $\phi''\left(\frac{X_{jlt}}{K_{jlt}}\right) < 0$. As it is assumed that the capital producers on an island to act in perfect competition, one can drop the subscript l without loss of generality. Thus, one finds the nominal price of capital on island j to be:

$$Q^n_{jt} = \frac{\bar{P}_{jt}}{\phi'\left(\frac{X_{jt}}{K_{jt}}\right)}. \tag{3.22}$$

The wholesale sector

The intermediate goods producing firm f on island j chooses labour demand, N_{jft}, next periods capital stock, $K_{jf,t+1}$, and sets its price P_{jft}, that maximize its discounted profits subject to a Cobb-Douglas type of production technology, the capital state law of motion, the demand function for its good and the adjustment cost functions for prices. I apply the nominal stochastic discount factor, as the firms are owned by the households and the profits are formulated in nominal terms.

$$\max_{\{N_{jf,t+\tau}, K_{jf,t+\tau+1}, P_{jf,t+\tau}\}} E_{jt}\left[\sum_{\tau=0}^{\infty} \beta^{\tau} \frac{\Lambda_{j,t+\tau}}{\Lambda_{jt}} \frac{\bar{P}_{jt}}{\bar{P}_{j,t+\tau}} \left(\Pi_{jf,t+\tau} - P^{adj}_{jf,t+\tau}\right)\right] \tag{3.23}$$

$$s.t. \quad \Pi_{jf,t+\tau} = P_{jf,t+\tau} Y_{jf,t+\tau} - W^n_{jf,t+\tau} N_{jf,t+\tau} - Q^n_{j,t+\tau} Y^{k_d}_{jf,t+\tau} \tag{3.24}$$

$$Y_{jf,t+\tau} = A_{j,t+\tau} K^{\alpha}_{jf,t+\tau} N^{1-\alpha}_{jf,t+\tau} \tag{3.25}$$

$$K_{jf,t+\tau+1} = (1-\delta) K_{jf,t+\tau} + Y^{k_d}_{jft} \tag{3.26}$$

$$Y_{jf,t+\tau} = \int_{\Theta_{j,t+\tau}} \left(\frac{P_{jf,t+\tau}}{\bar{P}_{i,t+\tau}}\right)^{-\varepsilon} Y_{i,t+\tau}\, di$$

$$P^{adj}_{jf,t+\tau} = \frac{\chi^p}{2}\left(\frac{P_{jf,t+\tau}}{\bar{\pi} P_{jf,t+\tau-1}} - 1\right)^2 Y_{j,t+\tau} P_{j,t+\tau}$$

The profits, Π_{jft}, of firm f on island j, are defined as the revenues from production less the costs for labour and expenditures for investment. The firm pays the nominal price for capital, $Q^n_t = Q_{jt}\bar{P}_{jt}$ times its capital demand, $Y^{k_d}_{jft}$. The firm produces its differentiated good Y_{jft} with the island specific composite productivity A_{jt} as well as labour and capital as factor inputs, where α is the output elasticity of capital. The capital state law of motion is fairly standard, where δ stands for the depreciation rate and $Y^{k_d}_{jft}$ is the capital that the firm demands from the capital producer. The firm's demand function depends on the demand of the final goods producers that are randomly selected to buy the goods from island j, Θ_{jt}. It is assumed that price adjustment costs are intangible an not tangible. This implies that they are not affecting the profits and hence the dividend payout.[7] The overall costs are determined by the adjustment parameter χ^p.

Composite productivity is assumed to consist of an aggregate and an idiosyncratic component $A_{jt} = A_t \exp\{\varepsilon_{jt}\}$, where aggregate productivity is persistent $A_t = A^{\rho_a}_{t-1}\exp\{\nu_t\}$. The two types of innovations are assumed to be i.i.d. $N(0,\sigma^2_{\varepsilon j})$ and $N(0,\sigma^2_{\nu})$, respectively. Moreover, integrating idiosyncratic innovations over all islands yields zero, $\int \varepsilon_{jt}\, dj = 0$.

The first order conditions yield the optimality condition for labour demand, the Euler equation of capital, and the non-linear version of the new Keynesian Phillips curve.

$$W^n_{jt} = (1-\alpha)\mu^n_{jt}\frac{Y_{jft}}{N_{jft}} \tag{3.27}$$

$$Q_{jt}\Lambda_{jt} = \beta E_{jt}\left[\Lambda_{j,t+1}\Big((1-\delta)Q_{j,t+1} + \alpha\frac{Y_{jf,t+1}}{K_{jf,t+1}}\frac{\mu^n_{j,t+1}}{\bar{P}_{j,t+1}}\Big)\right] \tag{3.28}$$

$$\begin{aligned}\left[(1-\varepsilon)+\varepsilon\mu_{jt}\right]P^{-\varepsilon}_{jft}\int_{\Theta_{jt}}\bar{P}^{\varepsilon}_{it}Y_{it}\,di - \chi^p Y_{jt}\frac{P_{jft}}{\bar{\pi}P_{jf,t-1}}\left(\frac{P_{jft}}{\bar{\pi}P_{jf,t-1}}-1\right) = \\ +\chi^p\beta E_{jt}\left[\frac{\Lambda_{j,t+1}}{\Lambda_{jt}}\frac{\bar{P}_{jt}}{\bar{P}_{j,t+\tau}}Y_{j,t+1}\left(\frac{P^2_{jf,t+1}}{\bar{\pi}P^2_{jft}}\right)\left(\frac{P_{jf,t+1}}{\bar{\pi}P_{jft}}-1\right)\right]\end{aligned} \tag{3.29}$$

Log-linearising equation (3.29) around the steady state and recognising that all firms on the same island choose the same prices, yields the island specific Phillips-curve:

$$\pi_{jt} = \kappa\mu_{jt} + \beta E_{jt}\left[\pi_{j,t+1}\right], \tag{3.30}$$

where $\kappa = \frac{\varepsilon-1}{\chi^p}$. The Phillips-curve derived from Calvo price setting and Rotembergs menu costs is the same for a given choice of χ^p, as under full information.

[7] The reason is, following the arguments of De Paoli et al. (2010), to focus on broader modelling aspects and not on the modelling details of price adjustment.

3.3.3 The central bank

The central bank follows the Taylor rule:

$$R_t^n = (R_*^n)^{1-\rho_r} \left(R_{t-1}^n\right)^{\rho_r} \left(\frac{\tilde{\pi}_t}{\pi_*}\right)^{\rho_\pi} \exp\{\xi_t\}, \tag{3.31}$$

where the central bank responds to a noisy measure of inflation $\tilde{\pi}_t = \pi_t \exp\{\omega_t\}$ and $\xi_t \sim N(0, \sigma_\xi^2)$ is a monetary policy shock.

3.3.4 Aggregate variables and market clearing

Households on island j supply labour to all firms f on the same island. And all firms set their individual labour demand, their investment and their prices to produce their differentiated good. All these variables are aggregated on the island level by $\Upsilon_{jt} = \int \Upsilon_{jft}\, df$. Further, the price index of the goods produced on an island is defined as:

$$P_{jt} = \left(\int_0^1 P_{jft}^{1-\varepsilon}\, df\right)^{\frac{1}{1-\varepsilon}}. \tag{3.32}$$

The equilibrium dynamics of the island specific variables:

$$\Upsilon_{jt} = \Big\{B_{j,t+1}, \Lambda_{jt}, W_{j*,t}^n, W_{jt}^n, \bar{P}_{jt}, Y_{jt}^{ks}, \phi'\left(\frac{X_{jt}}{K_{jt}}\right), D_{jt}, Y_{jt}, K_{j,t+1}, N_{jt},$$
$$Q_{jt}^n, P_{jt}, C_{jt}, A_{jt}, H_{jt}, Y_{jt}^{kd}\Big\}$$

are described by the equations (3.13), (3.15), (3.16), (3.17), (3.20), (3.21), (3.22), (3.24), (3.25), (3.26), (3.27), (3.28), (3.29) as well as the definition of marginal utility, composite productivity, A_{jt}, and the market clearing condition for labour and capital.

Aggregate variables are defined by the integral over the realizations of idiosyncratic variables of all islands. In other words, for any variable in the set Υ_{jt}, $\Upsilon_t = \int \Upsilon_{jt}\, dj$. The equilibrium dynamics of the aggregate variables:

$$\Upsilon_t = \left\{\Lambda_t, V_t, W_{*,t}^n, W_t^n, Y_t^{ks}, \phi'\left(\frac{X_t}{K_t}\right), D_t, Y_t, K_{t+1}, N_t, Q_t^n, P_t, C_t, A_t, H_t, Y_t^{kd}, R_t^f\right\}$$

are defined by the same equations being aggregated as well as the Taylor rule. Moreover, the aggregate price index is defined as:

$$P_t = \left(\int_0^1 P_{jt}^{1-\varepsilon} \, dj \right)^{\frac{1}{1-\varepsilon}} . \tag{3.33}$$

Labour market clearing requires $H_{jt} = N_{jt}$ for all islands j as the household provides labour only to the firms on their island and so does capital $Y_{jt}^{ks} = Y_{jt}^{kd}$. The bonds market clears in zero net supply.

Goods market clearing leads to the identity $Y_t = C_t + X_t$.

3.4 Asset pricing implications

Throughout the analysis, I assume that dividends are equal to the aggregate real profits of the firms in the wholesale and capital producing sectors.

3.4.1 Calibration

For the numerical evaluation of the model, I calibrate the model as summarized in Table 3.1.

In the literature, the parameters that I choose to define the decision rules of the households are mostly standard. The subjective discount factor is set equal to 0.99, which corresponds to an annual steady state interest rate of 4 %. Further, the Frisch elasticity is assumed to be equal to 2 which is in line with the estimate of Smets and Wouters (2007). The relative risk aversion is set to 5, which is in the middle ground between the log-utility assumption and the upper bound of 10 stated by Mehra and Prescott (1985). Moreover, it corresponds to the asset pricing literature such as Jermann (1998) and De Paoli et al. (2010). Habit formation is set to 0.6 which lies in the range of empirical estimates of Bartolomeo et al. (2011) and it is only slightly lower than the estimate of Smets and Wouters (2007). Not as common is the choice of nominal wage rigidity. I choose the Calvo-parameter to be 0.2 which leads to an average wage adjustment frequency of 5 months. Typically, the average wage adjustment duration is more similar to the average price adjustment duration o four quarters. However, the choice does not significantly affect the results.[8] The last parameter relevant for the households is the

[8]The algorithm to solve the heterogeneous information model seems to become sensitive to the parametrisation with the given number of state variables. It will be up to future research to find the reason for this.

Table 3.1 Baseline calibration - structural parameters

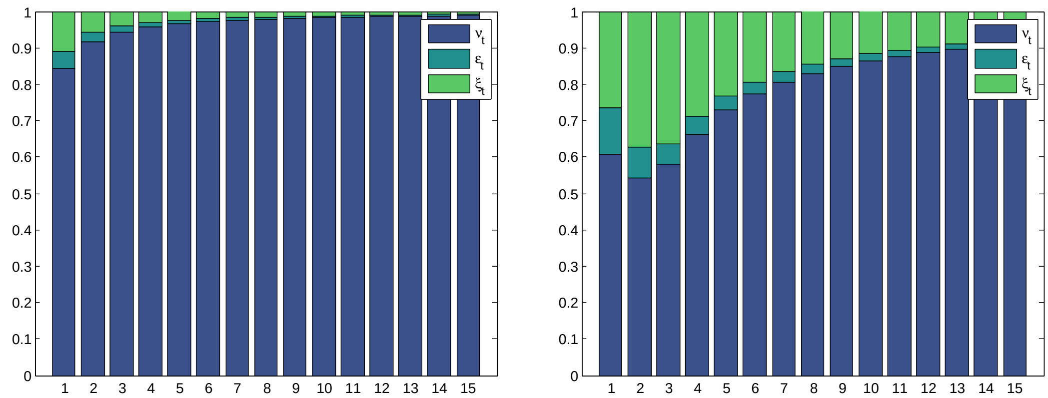
Parameter	Values	Description
β	0.99	Subjective discount factor
γ	0.5	Inverse Frisch elasticity
σ	5	Relative risk aversion
χ^c	0.6	Habit formation
ϕ	0.2	Calvo parameter, wages
ε_w	4	Wage elasticity of labour demand
α	1/3	Output elasticity of capital
χ^p	80	Menu costs of price setting
η	7.5	Price elasticity of demand
δ	0.025	Depreciation rate
ξ^k	4.3	Capital adjustment cost
ρ_r	0.90	Interest rate adjustment, Taylor rule
ρ_π	1.5	Inflation adjustment, Taylor rule
ρ_a	0.95	Auto-correlation of productivity
σ_v	0.77	Std of aggregate productivity innovation
σ_ω	$0\sigma_v$	Std of measurement error in inflation
σ_ε	$0.4\sigma_v$	Std of noise shock
σ_ξ	$0.4\sigma_v$	Std of monetary policy shock
$\sigma_{\varepsilon j}$	$20\sigma_v$	Std of idiosyncratic productivity innovation
$\sigma_{\varepsilon j}^1$	$10\sigma_v$	Std of idiosyncratic price draw
$\sigma_{\varepsilon j}^2$	$20\sigma_v$	Std of idiosyncratic supply draw
$\bar{H}$	0.3	Steady state labour share

wage elasticity of labour demand. Here, I choose the parameter as in Heer et al. (2012) who borrow the calibration from Erceg et al. (2000).

With regard to the firms decision rules, I calibrate the output elasticity of capital equal to 1/3. Furthermore, the price stickiness is calibrated to an average price duration of 4 quarters and a mark-up of 15 %. Capital depreciates at a rate of 10 % annually and the capital adjustment cost parameter is calibrated to be 4.3 as in Jermann (1998). The remaining parameters of the capital adjustment cost function can be derived from the assumption that the function is equal to the investment to capital ratio in steady state and that the first derivative of the function is equal to one in steady state.

The parameters for the Taylor rule are borrowed from Lorenzoni (2009). The reason for this decision is, as pointed out by Nimark (2014), that the noise shocks are only demand shocks if the adjustment parameter for output in the Taylor rule is sufficiently low. As, I want to evaluate in how far the additional demand shocks induced by heterogeneous information affect the asset pricing implications, I keep the calibration of Lorenzoni.

It remains to calibrate the stochastic processes. With regard to these parameters, I also mostly follow Lorenzoni (2009). Additionally, it is to be noted that the standard deviations of all innovations will be rescaled for computing the asset pricing implications such that the growth rate of output is equal to 0.01 as in Jermann (1998). Further, for the signal extraction problem of the agents only the relative standard deviations of the innovations matter. I choose the standard deviation of the productivity innovation to be equal to 0.77, which is ten times larger than empirical estimates but it facilitates the solution algorithm. The standard deviation of the noise shock is 0.4 times the standard deviation of the technology shock. This is chosen to maximize forecast error variance contribution of the noise shock to overall volatility. Generally, on the one hand, if the standard deviation of the noise shock is too low, the contribution is low as agents receive relatively precise signals on the state of the economy. If they are too high, on the other hand, then agents disregard the signal and the contribution decreases. In the benchmark calibration, I select the standard deviation of the monetary policy innovation also to be equal to 40 percent of the standard deviation of the technology innovation. The idiosyncratic innovations are chosen as in Lorenzoni (2009). Finally, the auto-correlation of the productivity process is set equal to 0.95, which is the one selected by De Paoli et al. (2010).



(a) Forecast error variance decomposition of marginal utility.

(b) Forecast error variance decomposition of dividend payments.

Fig. 3.1 Forecast error variance decomposition.

Graph (a) shows the forecast error variance decomposition of marginal utility for the technology shock ν_t, the noise shock, ε_t, and the monetary policy shock ξ_t. Graph (b) shows the same forecast error variance decomposition of the dividend payments.

3.4.2 Numerical results

In this subsection, I first illustrate the variance contribution of the noise shock to the overall variance. Additionally, I establish the noise shock to be a demand shock in the model at hand. Second, I compute the asset pricing implications for the full and the heterogeneous information model. Third, I close the analysis by dissecting the asset pricing formulas to get a better understanding of the differences between the asset pricing implications of the full and heterogeneous information version of the model.

Figure 3.1 shows the variance decomposition of marginal utility and the dividend payments. These two variables are the ones that are critical for asset prices. In the short run marginal utility is mostly driven by technology shocks, followed by monetary policy shocks and lastly by noise shocks. Noise shocks contribute barely five percent to total variance. Moreover, the contribution of both types of demand shocks diminish over time. The picture of the variance decomposition of the dividend payments is quantitatively different. In the short run technology shocks contribute the most and noise shock the least, but overall demand shocks play a significantly larger role. The noise shock contributes initially more than ten percent and the monetary policy shock up to 35 percent to total variance.

Table 3.2 Asset pricing implications

	spec.	p. recal.	$Var\left[\overline{E}_t\left[\lambda_{t+1}\right]-\lambda_t\right]$	$E\left[\overline{V}ar_t\left[\lambda_{t+1}\right]\right]$	$E\left[r_t^f\right]$	$Std\left[r_t^f\right]$	$E\left[r_{t,t+1}-r_t^f\right]$	$Std\left[r_{t,t+1}-r_t^f\right]$
	benchmark		3.56e-4	0.006	2.85	3.77	1.03	13.96
		$\chi_c = 0.3$	2.10e-4	0.004	3.31	2.90	0.62	10.66
		$\chi_p = 20$	3.31e-4	0.006	2.79	3.64	1.16	14.34
Full information	$\sigma_\xi = 0.2\sigma_v$	benchmark	2.07e-4	0.006	2.86	2.88	1.10	13.36
		$\chi_c = 0.3$	1.18e-4	0.003	3.32	2.17	0.66	10.23
		$\chi_p = 20$	2.51e-4	0.006	2.79	3.17	1.12	14.16
	$\sigma_\xi = 1.5\sigma_v$	benchmark	0.002	0.008	2.81	8.12	1.13	17.68
		$\chi_c = 0.3$	0.001	0.005	3.26	6.38	0.72	13.74
		$\chi_p = 20$	0.002	0.008	2.77	7.68	1.16	17.41
	benchmark		3.68e-4	0.016	0.69	3.84	3.36	14.85
		$\chi_c = 0.3$	2.15e-4	0.012	1.69	2.93	2.36	11.30
		$\chi_p = 20$	3.29e-4	0.016	0.88	3.63	3.10	14.94
	$\sigma_\xi = 0.2\sigma_v$	benchmark	2.15e-4	0.016	0.82	2.93	3.25	14.31
Het information		$\chi_c = 0.3$	1.22e-4	0.011	1.79	2.20	2.20	10.87
		$\chi_p = 20$	2.56e-4	0.011	1.82	3.20	2.29	14.64
	$\sigma_\xi = 1.5\sigma_v$	benchmark	0.002	0.016	1.19	8.18	2.92	18.10
		$\chi_c = 0.3$	0.001	0.011	2.03	6.42	1.94	14.24
		$\chi_p = 20$	0.002	0.020	0.31	7.69	3.75	18.11
	$\sigma_\varepsilon = 0.1\sigma_v$	benchmark	3.57e-4	0.007	2.67	3.78	1.40	14.21
		$\chi_c = 0.3$	2.11e-4	0.004	3.17	2.90	0.89	10.95
		$\chi_p = 20$	3.30e-4	0.007	2.56	3.63	1.43	14.52

The column spec. indicates the specifications for under full and under heterogeneous information. For the alternative specifications the column p. recal. shows the parameters that are recalibrated. The next two columns display the numerical values for two closed form expressions which define the risk free rate and the standard deviation of the risk free rate. The last four columns provide the asset pricing implications of the log of the risk free rate as well as the log of the risk premium. The risk free rate is computed using the closed form solution, while the statistics for return to equity is computed as the average of 100 simulations of 200 periods for the present discounted value of dividend payments for 1000 periods. For each specification the total standard deviation is rescaled such that the standard deviation of output growth is equal to 0.01.

Equity premium and the risk free rate puzzle

The asset pricing implications of the full information model and the heterogeneous information model are summarised in Table 3.2. To evaluate the effects from real and nominal rigidities conditional on the type of shock, I vary the menu costs for the price setting, representative for nominal rigidities and I vary the habit formation parameter representative for real rigidities.

The first difference between the benchmark calibration of the full and the heterogeneous information model is to recognise that the risk free rate is significantly lower under heterogeneous information. If one is comparing the two terms that define the risk free rate then it becomes evident that the unconditional variance of the expected marginal utility growth rate is very similar between the two models. However, the unconditional expectation of the one period ahead forecast error is significantly larger under heterogeneous information. The reason is that there is additional uncertainty stemming from the fact that the underlying aggregate state of the economy is unknown.

A second interesting finding is that the standard deviation of the risk free rate are very similar between the two models. The reason is that the standard deviation of the risk free rate depends on the variance of the expected growth rate of marginal utility. And this statistic is very similar between the two classes of models.

The implications for the equity premium stem almost exclusively from the lower risk free rate, leading to higher equity premia when considering heterogeneous information. The standard deviation is also for the equity premium comparable between the two models.

Altogether, heterogeneous information may provide a crucial component to explaining the equity premium puzzle and the risk free rate puzzle. The reason is that nominal and real rigidities can be weaker to arrive the low risk free rate. This diminishes the effect that the risk free rate becomes overly volatile. Additionally, one can find higher equity premia with fewer constraints on the parameter space.

Supply vs. demand shocks

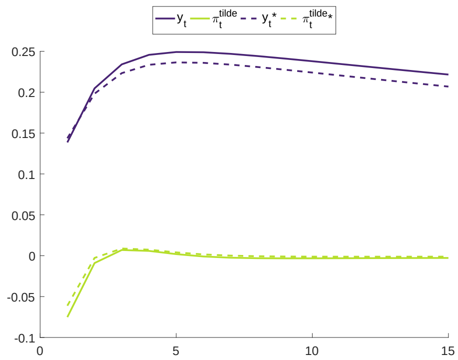
In this subsection, I bear upon the results of De Paoli et al. (2010) who found out that nominal rigidities reduce increase the equity premium if the economy is mostly driven by demand shock and decrease them if they are mostly driven by supply shocks.

My hypothesis is that the effects might be more pronounced in the presence of heterogeneous information as they, shown by Lorenzoni (2009) provide a theory of demand shocks which go beyond monetary policy shocks. In order to investigate the effects, I first group the shocks of the model into supply and demand shocks.

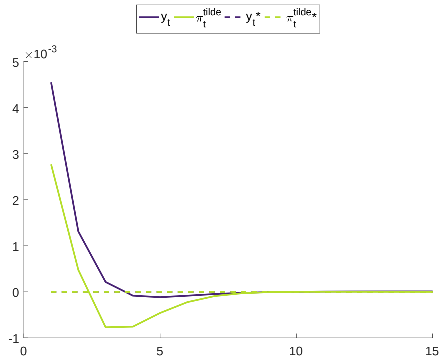
The model includes two structural shocks, namely a technology shock and a monetary policy shock. In addition it includes two noise shocks - inflation noise and noisy information on technology. However, for the sake of clarity, I abstain from inflation noise as it affects the Taylor rule in the same way as the monetary policy shock does. On the one hand, the technology shock defines the supply shock. Then, on the other hand, I group the monetary policy and the noise shock as demand shocks, as they drive output and prices in the same direction. Figure 3.2 shows the impulse responses of output and inflation to these three shocks.

Referring to Table 3.2, in the full information benchmark case, lower habit formation reduces the risk premium and lower price stickiness increases the risk premium. In contrast to the findings of De Paoli et al. (2010) this pattern is independent of the relative contribution of supply and demand shocks. However from their paper it is not clear if their findings only apply if the economy is only driven by demand or supply shocks, respectively.

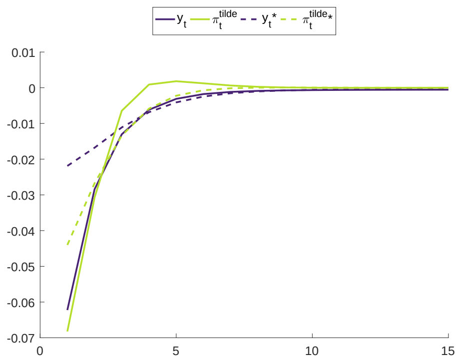
The results for the heterogeneous information specifications are a little bit more versatile. In the benchmark specification one finds a similar pattern as under full information. Lower habit formation decreases the risk premium, but lower price stickiness does not seem to increase it either. Moreover, if the model is mostly driven by supply shocks, both, lower habit formation and lower price stickiness clearly, reduce the risk premium. In the other case, in which the model is mostly driven by demand shocks, lower habit formation significantly decreases the risk premium and lower price stickiness significantly increases it. Altogether, this shows exactly the opposite of what De Paoli et al. (2010) found for full information models, when looking at heterogeneous information models. If the model is mostly driven by productivity shocks, then higher nominal rigidities lead to a higher risk premium. If instead the model is mostly driven by demand shocks, then higher nominal rigidities decrease the risk premium.



(a) Standard deviation supply shock (technology). Output and inflation.



(b) Standard deviation demand shocks (noise). Output and inflation.



(c) Standard deviation demand shock (monetary policy). Output and inflation.

Fig. 3.2 Impulse responses. Supply and demand shocks.

Graph (a) shows the impulse response functions of output and inflation to an aggregate technology shock. The solid lines represents the impulse responses of variables under heterogeneous information and the dashed lines the ones under full information. Graph (b) shows the same impulse responses to a noise shock and Graph (c) to a monetary policy shock.

3.5 Conclusion

In this paper, I derived the log-linear asset pricing equations under the assumption of heterogeneous information. Moreover, I adapted the model of Heer et al. (2012) to heterogeneous information, which is a new Keynesian model with sticky wages, habit formation and capital adjustment costs and has proven to be capable of capturing many relevant macroeconomic dynamics.

Using the log-linear asset pricing formulae under heterogeneous information and the solution of the model, I analysed first, if heterogeneous information can be helpful in explaining the equity premium and the risk free rate puzzle. Furthermore, I take on the results of De Paoli et al. (2010) who find that rigidities might have different impacts on the risk premium depending on the shocks that are driving the model.

I find that the heterogeneous information model mitigates if not resolves the equity premium and the risk free rate puzzle. Thereby, heterogeneous information especially affect the risk free rate. Due to the uncertainty about the aggregate state of the economy, precautionary savings are higher than under full information and hence, the risk free rate is lower. At the same time the return to equity remains relatively unchanged which implies a higher risk premium. Another helpful effect of heterogeneous information is that, while it decreases the risk free rate, it does not increase the standard deviation thereof.

Finally, in contrast to the results in the literature, higher nominal rigidities increase the equity premium when the heterogeneous information model is mostly driven by supply shocks and reduces the risk premium when it is driven by demand shocks.

References

Abel, A. B. (1990). Asset prices under habit formation and catching up with the joneses. *American Economic Review 80*(2), 38–42.

Angeletos, G.-M. and J. La'O (2009). Noisy business cycles. In *NBER Macroeconomics Annual 2009, Volume 24*, NBER Chapters, pp. 319–378. National Bureau of Economic Research, Inc.

Barillas, F. and K. P. Nimark (2013). Speculation, risk premia and expectations in the yield curve. Technical report, CEPR. Heterogenous information, Speculation, Survey data and Term structure of interest rates.

Bartolomeo, G. D., L. Rossi, and M. Tancioni (2011). Monetary policy, rule-of-thumb consumers and external habits: a G7 comparison. *Applied Economics 43*(21), 2721–2738.

Baxter, B., L. Graham, and S. Wright (2011). Invertible and non-invertible information sets in linear rational expectations models. *Journal of Economic Dynamics and Control 35*(3), 295–311.

Blanchard, O. J. and C. M. Kahn (1980). The solution of linear difference models under rational expectations. *Econometrica 48*(5), 1305–11.

Blanchard, O. J., J.-P. L'Huillier, and G. Lorenzoni (2013). News, noise, and fluctuations: An empirical exploration. *American Economic Review 103*(7), 3045–70.

Calvo, G. A. (1983). Staggered prices in a utility-maximizing framework. *Journal of Monetary Economics 12*(3), 383–398.

Campbell, J. Y. and J. Cochrane (1999). Force of habit: A consumption-based explanation of aggregate stock market behavior. *Journal of Political Economy 107*(2), 205–251.

Collard, F. and H. Dellas (2010). Monetary misperceptions, output, and inflation dynamics. *Journal of Money, Credit and Banking 42*(2-3), 483–502.

Constantinides, G. M. (1990). Habit formation: A resolution of the equity premium puzzle. *Journal of Political Economy 98*(3), 519–43.

De Paoli, B., A. Scott, and O. Weeken (2010). Asset pricing implications of a new keynesian model. *Journal of Economic Dynamics and Control 34*(10), 2056 – 2073.

Epstein, L. G. and S. E. Zin (1989). Substitution, risk aversion, and the temporal behavior of consumption and asset returns: A theoretical framework. *Econometrica 57*(4), 937–69.

Erceg, C. J., D. W. Henderson, and A. T. Levin (2000). Optimal monetary policy with staggered wage and price contracts. *Journal of Monetary Economics 46*(2), 281–313.

Gali, J. (2011). The return of the wage phillips curve. *Journal of the European Economic Association 9*(3), 436–461.

Graham, L. and S. Wright (2010). Information, heterogeneity and market incompleteness. *Journal of Monetary Economics 57*(2), 164–174.

Hamilton, J. (1994). *Time series analysis*. Princeton, NJ: Princeton Univ. Press.

Hassan, T. A. and T. M. Mertens (2014). Information Aggregation in a Dynamic Stochastic General Equilibrium Model. In *NBER Macroeconomics Annual 2014, Volume 29*, NBER Chapters, pp. 159–207. National Bureau of Economic Research, Inc.

Heer, B., T. Klarl, and A. Maussner (2012). Asset pricing implications of a new keynesian model: A note. CESifo Working Paper Series 4041, CESifo Group Munich.

Hellwig, C. and V. Venkateswaran (2009). Setting the right prices for the wrong reasons. *Journal of Monetary Economics 56*, S57 – S77.

Jermann, U. J. (1998). Asset pricing in production economies. *Journal of Monetary Economics 41*(2), 257–275.

Kasa, K. (2000, October). Forecasting the forecasts of others in the frequency domain. *Review of Economic Dynamics 3*(4), 726–756.

Klein, P. (2000). Using the generalized schur form to solve a multivariate linear rational expectations model. *Journal of Economic Dynamics and Control 24*(10), 1405 – 1423.

Kocherlakota, N. R. (1996). The equity premium: It's still a puzzle. *Journal of Economic Literature 34*(1), 42–71.

Lettau, M. and H. Uhlig (2000). Can habit formation be reconciled with business cycle facts? *Review of Economic Dynamics 3*(1), 79–99.

Levine, P., J. Pearlman, G. Perendia, and B. Yang (2012). Endogenous persistence in an estimated dsge model under imperfect information. *The Economic Journal 122*(565), 1287–1312.

Lorenzoni, G. (2009). A theory of demand shocks. *American Economic Review 99*(5), 2050–84.

Lorenzoni, G. (2010). Optimal monetary policy with uncertain fundamentals and dispersed information. *The Review of Economic Studies 77*(1), 305–338.

Lucas, Robert E, J. (1973, June). Some international evidence on output-inflation tradeoffs. *American Economic Review 63*(3), 326–34.

Mackowiak, B. and M. Wiederholt (2011). Business cycle dynamics under rational inattention. Working Paper Series 1331, European Central Bank.

Mankiw, N. G. and R. Reis (2002). Sticky information versus sticky prices: A proposal to replace the new keynesian phillips curve. *The Quarterly Journal of Economics 117*(4), 1295–1328.

Mehra, R. and E. C. Prescott (1985). The equity premium: A puzzle. *Journal of Monetary Economics 15*(2), 145–161.

Melosi, L. (2014). Estimating models with dispersed information. *American Economic Journal: Macroeconomics 6*(1), 1–31.

Monfort, A., J.-P. Renne, and G. Roussellet (2015). A quadratic kalman filter. *Journal of Econometrics 187*(1), 43 – 56.

Morris, S. and H. S. Shin (2002). Social value of public information. *American Economic Review 92*(5), 1521–1534.

Nimark, K. (2011). Dynamic higher order expectations. Economics Working Papers 1118, Department of Economics and Business, Universitat Pompeu Fabra.

Nimark, K. P. (2014). Man-bites-dog business cycles. *American Economic Review 104*(8), 2320–67.

Pearlman, J., D. Currie, and P. Levine (1986). Rational expectations models with partial information. *Economic Modelling 3*(2), 90 – 105.

Pearlman, J. G. (1992). Reputational and nonreputational policies under partial information. *Journal of Economic Dynamics and Control 16*(2), 339 – 357.

Phelps, E. S. (1970). *Microeconomic Foundations of Employment and Inflation Theory*, Chapter Introduction: The New Microeconomics in Employment and Inflation Theory. New York: Norton.

Radner, R. (1979). Rational expectations equilibrium: Generic existence and the information revealed by prices. *Econometrica 47*(3), 655–78.

Rondina, G. and T. Walker (2017). Confounding Dynamics. 2017 Meeting Papers 525, Society for Economic Dynamics.

Rondina, G. and T. B. Walker (2011). Information equilibria in dynamic economies. Technical report, CREI.

Rotemberg, J. J. (1982). Sticky prices in the united states. *Journal of Political Economy 90*(6), 1187–1211.

Schaefer, S. (2019a). Incomplete information in the real business cycle model.

Schaefer, S. (2019b). Solving heterogeneous information dynamic stochastic general equilibrium models easily.

Sims, C. A. (2002). Solving linear rational expectations models. *Computational Economics 20*(1-2), 1–20.

Sims, C. A. (2003). Implications of rational inattention. *Journal of Monetary Economics 50*(3), 665–690.

Smets, F. and R. Wouters (2007). Shocks and frictions in us business cycles: A bayesian dsge approach. *American Economic Review 97*(3), 586–606.

Sveen, T. and L. Weinke (2005). New perspectives on capital, sticky prices, and the taylor principle. *Journal of Economic Theory 123*(1), 21 – 39.

Svensson, L. E. and M. Woodford (2003). Indicator variables for optimal policy. *Journal of Monetary Economics 50*(3), 691 – 720.

Svensson, L. E. and M. Woodford (2004). Indicator variables for optimal policy under asymmetric information. *Journal of Economic Dynamics and Control 28*(4), 661 – 690.

Tallarini Jr., T. D. (2000). Risk-sensitive real business cycles. *Journal of Monetary Economics 45*(3), 507–532.

Townsend, R. M. (1983). Forecasting the forecasts of others. *Journal of Political Economy 91*(4), 546–88.

Uhlig, H. (2007). Explaining asset prices with external habits and wage rigidities in a dsge model. *American Economic Review 97*(2), 239–243.

Weil, P. (1989). The equity premium puzzle and the risk-free rate puzzle. *Journal of Monetary Economics 24*(3), 401–421.

Woodford, M. (2003). Imperfect common knowledge and the effects of monetary policy. In P. Aghion, R. Frydman, J. Stiglitz, and M. Woodford (Eds.), *Knowledge, Information, and Expectations in Modern Macroeconomics: In Honor of Edmund S. Phelps*. Princeton Univ. Press.

Appendix A

Chapter 1

A.1 The model

A.1.1 Equilibrium dynamics

The equilibrium dynamics of $\{K_{jt}, z_{jt}, Y_{jt}, C_{jt}, I_{jt}, H_{jt}, J_{jt}, N_{jt}, R_t^k, W_{jt}\}$ are fully described by by the equations (1.2), (1.3), (1.4), (1.5), (1.7), (1.8), (1.9), the net return to capital as well as the definition of z_{jt} and the market clearing condition for labour. Moreover, I present the equation system for the case in which market clearing is assured.

$$C_{jt} + I_{jt} = R_t^k K_{jt} + W_{jt} H_{jt} \tag{A.1}$$

$$K_{j,t+1} = (1-\delta)K_{jt} + I_{jt} \tag{A.2}$$

$$\frac{1}{C_{jt}} = \beta E\left[\frac{1}{C_{j,t+1}} R_{t+1} \middle| \Omega_{jt}\right] \tag{A.3}$$

$$\frac{W_{jt}}{C_{jt}} = \theta(1-H_{jt})^{\gamma} \tag{A.4}$$

$$Y_{jt} = J_{jt}^{\alpha}\left(Z_{jt}N_{jt}\right)^{1-\alpha} \tag{A.5}$$

$$R_t^k = \alpha \frac{Y_{jt}}{J_{jt}} \tag{A.6}$$

$$W_{jt} = (1-\alpha)\frac{Y_{jt}}{N_{jt}} \tag{A.7}$$

$$R_t^k = R_t - \left(1-\delta\right) \tag{A.8}$$

$$H_{jt} = N_{jt} \tag{A.9}$$

$$z_{jt} = z_t + \varepsilon_{jt} \tag{A.10}$$

$$a_{t+1} = \rho_a a_t + \nu_{t+1} \tag{A.11}$$

$$z_t = a_t + \varepsilon_t \tag{A.12}$$

A.1.2 Steady state

The steady state values of the system can be derived conditional on the model's deep parameters $\Theta = \{\alpha, \beta, \delta, \gamma, \theta, \rho_a\}$. Idiosyncratic and aggregate productivity are normalized to one in steady state and I target labour supply to be $H = \bar{H}$.

First, to compute the steady state values, drop the time subscripts of equations (A.1) - (A.9) and summarize.

$$C + I = R^k K + WH \tag{A.13}$$

$$I = \delta K \tag{A.14}$$

$$R = \frac{1}{\beta} \tag{A.15}$$

$$W = \theta(1-H)^{\gamma} C \tag{A.16}$$

$$Y = K^{\alpha} N^{1-\alpha} \tag{A.17}$$

$$R^k = \alpha \frac{Y}{K} \tag{A.18}$$

$$W = \left(1-\alpha\right)\frac{Y}{N} \tag{A.19}$$

$$R^k = R - (1-\delta) \tag{A.20}$$

$$H = N \tag{A.21}$$

Second, solve the system of non-linear equations to find the steady state values.

A.1.3 Log-linearisation around the steady state

Apply log-linearisation around the steady state to the equations (A.1) - (A.9) where $\hat{\Upsilon}_{jt} = \ln\left(\frac{\Upsilon_{jt}}{\Upsilon}\right)$ denotes log deviations from steady state and lower case letters denote logs.

$$0 = C\hat{c}_{jt} + I\hat{i}_{jt} - \alpha Y\left(r_t^k + \hat{k}_{jt}\right) - \left(1-\alpha\right)Y\left(\hat{w}_{jt} + \hat{h}_{jt}\right) \tag{A.22}$$

$$0 = \left(1+g\right)\hat{k}_{j,t+1} - \left(1-\delta\right)\hat{k}_{jt} - \frac{I}{K}\hat{i}_{jt} \tag{A.23}$$

$$0 = E\left[\hat{c}_{j,t+1} - r_{t+1} \middle| \Omega_{jt}\right] - \hat{c}_{jt} \tag{A.24}$$

$$0 = \hat{w}_{jt} - \gamma\frac{H}{1-H}\hat{h}_{jt} - \hat{c}_{jt} \tag{A.25}$$

$$0 = \hat{y}_{jt} - \alpha\hat{j}_{jt} - \left(1-\alpha\right)\left(z_{jt} + \hat{n}_{jt}\right) \tag{A.26}$$

$$0 = r_t^k - \hat{y}_{jt} + \hat{j}_{jt} \tag{A.27}$$

$$0 = \hat{w}_{jt} - \hat{y}_{jt} + \hat{n}_{jt} \tag{A.28}$$

$$0 = r_t^k - \frac{R}{R^k} r_t \tag{A.29}$$

$$0 = \hat{n}_{jt} - \hat{h}_{jt} \tag{A.30}$$

A.1.4 State space representation

Contemporaneous variables

To find individual production, (A.26), as a function of individual capital, composite productivity, individual consumption and the return to capital, set (A.25) equal to (A.28) and make use of (A.27) and (A.30):

$$\hat{y}_{jt} = -\frac{\alpha\left(1+\xi\right)}{\left(1-\alpha\right)} r_t^k + \left(1+\xi\right) z_{jt} - \xi \hat{c}_{jt}, \tag{A.31}$$

where $\xi = \frac{1-H}{\gamma H}$.

I use the expression of individual production to compute individual labour and wages, using (A.25) and (A.28):

$$\hat{n}_{jt} = -\frac{\alpha}{\left(1-\alpha\right)} \xi r_t^k + \xi z_{jt} - \xi \hat{c}_{jt}, \quad \text{and} \tag{A.32}$$

$$\hat{w}_{jt} = -\frac{\alpha}{\left(1-\alpha\right)} r_t^k + z_{jt}. \tag{A.33}$$

Plug (A.31), (A.32) and (A.33) in (A.22):

$$\hat{i}_{jt} = \alpha \frac{Y}{I} \hat{k}_{jt} + \left(1-\alpha\right)\left(1+\xi\right) \frac{Y}{I} z_{jt} - \left(\frac{C}{I} + \left(1-\alpha\right) \xi \frac{Y}{I}\right) \hat{c}_{jt} - \alpha \xi \frac{Y}{I} r_t^k. \tag{A.34}$$

Individual capital demand follows from equation (A.27) in combination with (A.31):

$$\hat{j}_{jt} = -\frac{1+\alpha\xi}{1-\alpha} r_t^k + \left(1+\xi\right) z_{jt} - \xi \hat{c}_{jt}. \tag{A.35}$$

I find the aggregate contemporaneous variables by integrating over the individual conditions. At this point I also make use of the market clearing condition on the capital market.

Aggregating (A.35) yields the solution to the return to capital:

$$r_t^k = -\frac{1-\alpha}{1+\alpha\xi}\hat{k}_t + \frac{(1+\xi)(1-\alpha)}{1+\alpha\xi}z_t - \frac{(1-\alpha)\xi}{1+\alpha\xi}\hat{c}_t. \tag{A.36}$$

The remaining aggregate contemporaneous variables read:

$$\hat{y}_t = -\frac{\alpha(1+\xi)}{(1-\alpha)}r_t^k + (1+\xi)z_t - \xi\hat{c}_t \tag{A.37}$$

$$\hat{n}_t = -\frac{\alpha}{(1-\alpha)}\xi r_t^k + \xi z_t - \xi\hat{c}_t \tag{A.38}$$

$$\hat{w}_t = -\frac{\alpha}{(1-\alpha)}r_t^k + z_t \tag{A.39}$$

$$\hat{i}_t = \alpha\frac{Y}{I}\hat{k}_t + (1-\alpha)(1+\xi)\frac{Y}{I}z_t - \left(\frac{C}{I} + (1-\alpha)\xi\frac{Y}{I}\right)\hat{c}_t - \alpha\xi\frac{Y}{I}r_t^k \tag{A.40}$$

Substitute r_t^k, (A.36), to find contemporaneous variables as functions of state and forward looking variables only. Then, I cast the parameters in the matrices of equation (1.11):

$$\begin{bmatrix} \Upsilon_{jt} \\ \Upsilon_t \end{bmatrix} = \begin{bmatrix} G_{cj} & G_c \end{bmatrix} \begin{bmatrix} X_{jt}^c \\ X_t^c \end{bmatrix} + \begin{bmatrix} G_{xj} & G_x \end{bmatrix} \begin{bmatrix} X_{jt} \\ X_t \end{bmatrix} + \begin{bmatrix} G_{fj} & G_f \end{bmatrix} \begin{bmatrix} F_{jt} \\ F_t \end{bmatrix}. \tag{A.41}$$

State variables

Combine (A.23) with (A.34) to find the state law of motions of individual capital and aggregate it to find the state law of motion of aggregate capital:

$$\begin{aligned} \hat{k}_{j,t+1} &= \left(1-\delta+(1-\alpha)\frac{Y}{K}\right)\hat{k}_{jt} + (1-\alpha)(1+\xi)\frac{Y}{K}z_{jt} \\ &\quad - \left(\frac{C}{K} + (1-\alpha)\xi\frac{Y}{K}\right)\hat{c}_{jt} - \alpha\xi\frac{Y}{K}r_t^k; \end{aligned} \tag{A.42}$$

$$\begin{aligned} \hat{k}_{t+1} &= \left(1-\delta+(1-\alpha)\frac{Y}{K}\right)\hat{k}_t + (1-\alpha)(1+\xi)\frac{Y}{K}z_t \\ &\quad - \left(\frac{C}{K} + (1-\alpha)\xi\frac{Y}{K}\right)\hat{c}_t - \alpha\xi\frac{Y}{K}r_t^k. \end{aligned} \tag{A.43}$$

I cast the equation of individual and aggregate capital as well as aggregate and idiosyncratic productivity in the corresponding matrices of equation (1.10):

$$\begin{bmatrix} X^c_{j,t+1} \\ X_{j,t+1} \\ X^c_{t+1} \\ X_{t+1} \end{bmatrix} = \begin{bmatrix} M^{cj}_{cj} & 0 & 0 & 0 \\ 0 & M^{xj}_{xj} & 0 & 0 \\ 0 & 0 & M^c_c & M^x_c \\ 0 & 0 & 0 & M^x_x \end{bmatrix} \begin{bmatrix} X^c_{jt} \\ X_{jt} \\ X^c_t \\ X_t \end{bmatrix} + \begin{bmatrix} M^{\Upsilon j}_{cj} & M^{\Upsilon}_{cj} \\ 0 & 0 \\ 0 & M^{\Upsilon}_c \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \underline{\Upsilon}_{jt} \\ \underline{\Upsilon}_t \end{bmatrix} + \begin{bmatrix} M^{fj}_{cj} & 0 \\ 0 & 0 \\ 0 & M^f_c \\ 0 & 0 \end{bmatrix} \begin{bmatrix} F_{jt} \\ F_t \end{bmatrix} + \begin{bmatrix} 0 \\ N_{xj} \\ 0 \\ N_x \end{bmatrix} \begin{bmatrix} s_{jt} \\ s_t \end{bmatrix}. \tag{A.44}$$

As outlined in the main text, the individual predetermined state variable can be written in terms of individual predetermined state variables, individual forward looking variables and the signals that insure market clearing. However, when I aggregate the individual predetermined state variables private signals generally do not aggregate to public signals and instead need to be directly attributed to aggregate state variables.

Euler equation

Individual consumption needs to satisfy the Euler equation (A.24):

$$\hat{c}_{jt} = E\left[\hat{c}_{j,t+1} - r_{t+1} \middle| \Omega_{jt}\right]. \tag{A.45}$$

The parameters of this equation can be directly cast in the matrices of the Euler equation, (1.12), in the main text:

$$R^0_c \begin{bmatrix} X^c_{jt} \\ E_{jt} X^c_t \end{bmatrix} + R^0_{\Upsilon} \begin{bmatrix} \underline{\Upsilon}_{jt} \\ \underline{\Upsilon}_t \end{bmatrix} + R^0_f \begin{bmatrix} F_{jt} \\ E_{jt} F_t \end{bmatrix} = E_{jt} \left\{ R^1_c \begin{bmatrix} X^c_{j,t+1} \\ X^c_{t+1} \end{bmatrix} + R^1_{\Upsilon} \begin{bmatrix} \underline{\Upsilon}_{j,t+1} \\ \underline{\Upsilon}_{t+1} \end{bmatrix} + R^1_f \begin{bmatrix} F_{j,t+1} \\ F_{t+1} \end{bmatrix} \right\}.$$

Substitute $\begin{bmatrix} \underline{\Upsilon}_{j,t+1} \\ \underline{\Upsilon}_{t+1} \end{bmatrix}$ using (A.41):

$$R_c^0 \begin{bmatrix} X_{jt}^c \\ E_{jt} X_t^c \end{bmatrix} + R_\Upsilon^0 \begin{bmatrix} \underline{\Upsilon}_{jt} \\ \underline{\Upsilon}_t \end{bmatrix} + R_f^0 \begin{bmatrix} F_{jt} \\ E_{jt} F_t \end{bmatrix} =$$
$$E_{jt} \left\{ \left(R_c^1 + R_\Upsilon^1 \begin{bmatrix} G_{cj} & G_c \end{bmatrix} \right) \begin{bmatrix} X_{j,t+1}^c \\ X_{t+1}^c \end{bmatrix} + R_\Upsilon^1 \begin{bmatrix} G_{xj} & G_x \end{bmatrix} \begin{bmatrix} X_{j,t+1} \\ X_{t+1} \end{bmatrix} + \left(R_f^1 + R_\Upsilon^1 \begin{bmatrix} G_{fj} & G_f \end{bmatrix} \right) \begin{bmatrix} F_{j,t+1} \\ F_{t+1} \end{bmatrix} \right\}.$$

Substitute $\begin{bmatrix} X_{j,t+1} \\ X_{t+1} \end{bmatrix}$ using (A.44):

$$R_c^0 \begin{bmatrix} X_{jt}^c \\ E_{jt} X_t^c \end{bmatrix} + R_\Upsilon^0 \begin{bmatrix} \underline{\Upsilon}_{jt} \\ \underline{\Upsilon}_t \end{bmatrix} + R_f^0 \begin{bmatrix} F_{jt} \\ E_{jt} F_t \end{bmatrix} - R_\Upsilon^1 \begin{bmatrix} G_{xj} & G_x \end{bmatrix} \begin{bmatrix} M_{xj}^{xj} & 0 \\ 0 & M_x^x \end{bmatrix} E_{jt} \begin{bmatrix} X_{jt} \\ X_t \end{bmatrix} =$$
$$E_{jt} \left\{ \left(R_c^1 + R_\Upsilon^1 \begin{bmatrix} G_{cj} & G_c \end{bmatrix} \right) \begin{bmatrix} X_{j,t+1}^c \\ X_{t+1}^c \end{bmatrix} + \left(R_f^1 + R_\Upsilon^1 \begin{bmatrix} G_{fj} & G_f \end{bmatrix} \right) \begin{bmatrix} F_{j,t+1} \\ F_{t+1} \end{bmatrix} \right\}.$$

Substitute $\begin{bmatrix} X_{j,t+1}^c \\ X_{t+1}^c \end{bmatrix}$ using (A.44):

$$\left(R_c^0 - \left(R_c^1 + R_\Upsilon^1 \begin{bmatrix} G_{cj} & G_c \end{bmatrix} \right) \begin{bmatrix} M_{cj}^{cj} & 0 \\ 0 & M_c^c \end{bmatrix} \right) \begin{bmatrix} X_{jt}^c \\ E_{jt} X_t^c \end{bmatrix}$$
$$+ \left(R_\Upsilon^0 - \left(R_c^1 + R_\Upsilon^1 \begin{bmatrix} G_{cj} & G_c \end{bmatrix} \right) \begin{bmatrix} M_{cj}^{\Upsilon j} & M_{cj}^\Upsilon \\ 0 & M_c^\Upsilon \end{bmatrix} \right) \begin{bmatrix} \underline{\Upsilon}_{jt} \\ \underline{\Upsilon}_t \end{bmatrix}$$
$$+ \left(R_f^0 - \left(R_c^1 + R_\Upsilon^1 \begin{bmatrix} G_{cj} & G_c \end{bmatrix} \right) \begin{bmatrix} M_{cj}^{fj} & 0 \\ 0 & M_c^f \end{bmatrix} \right) \begin{bmatrix} F_{jt} \\ E_{jt} F_t \end{bmatrix}$$
$$- \left(R_\Upsilon^1 \begin{bmatrix} G_{xj} & G_x \end{bmatrix} \begin{bmatrix} M_{xj}^{xj} & 0 \\ 0 & M_x^x \end{bmatrix} + \left(R_c^1 + R_\Upsilon^1 \begin{bmatrix} G_{cj} & G_c \end{bmatrix} \right) \begin{bmatrix} 0 & 0 \\ 0 & M_c^x \end{bmatrix} \right) E_{jt} \begin{bmatrix} X_{jt} \\ X_t \end{bmatrix} =$$
$$E_{jt} \left\{ \left(R_f^1 + R_\Upsilon^1 \begin{bmatrix} G_{fj} & G_f \end{bmatrix} \right) \begin{bmatrix} F_{j,t+1} \\ F_{t+1} \end{bmatrix} \right\}.$$

This can eventually be written as equation (1.13) in the main text:

$$Q_c^0 \begin{bmatrix} X_{jt}^c \\ E_{jt} X_t^c \end{bmatrix} + Q_x^0 E_{jt} \begin{bmatrix} X_{jt} \\ X_t \end{bmatrix} + Q_\Upsilon^0 \begin{bmatrix} \underline{\Upsilon}_{jt} \\ \underline{\Upsilon}_t \end{bmatrix} + Q_f^0 \begin{bmatrix} F_{jt} \\ E_{jt} F_t \end{bmatrix} = Q_f^1 E_{jt} \begin{bmatrix} F_{j,t+1} \\ F_{t+1} \end{bmatrix}. \tag{A.46}$$

A.2 Proofs

A.2.1 Market clearing

Proposition 1. *Goods market clearing requires that the return to capital, individual wages and idiosyncratic productivity are part of the information set* Ω_{jt}.

Proof. Goods market clearing follows from the budget constraint of households, (A.22), and the firm's first order conditions for capital, (A.27), and labour, (A.28), individual production, (A.26), as well as market clearing on the labour $\hat{h}_{jt} = \hat{n}_{jt}$ and capital market $\hat{j}_t = \hat{k}_t$. There are three groups of variables that move outside of the expectation operator. First, choice variables are outside the expectation operator because they are chosen conditional on their expectation about the state of the world. Second, prices and quantities of the markets on which the agents interact are part of their information set. Third, idiosyncratic composite productivity has to be revealed.

$$\begin{aligned} 0 &= C\hat{c}_{jt} + I\hat{i}_{jt} - \alpha Y\left(r_t^k + \hat{k}_{jt}\right) - \left(1-\alpha\right)Y\left(\hat{w}_{jt} + \hat{h}_{jt}\right) \\ r_{t|jt}^k &= \hat{y}_{jt|jt} - \hat{j}_{jt} \\ \hat{w}_{jt|jt} &= \hat{y}_{jt|jt} - \hat{n}_{jt} \\ \hat{y}_{jt|jt} &= \alpha\hat{j}_{jt} + \left(1-\alpha\right)\left(z_{jt|jt} + \hat{n}_{jt}\right) \end{aligned}$$

The reasons are the following. If the return to capital was not revealed, $r_{t|jt}^k \neq r_t^k$, and or the individual wage, $w_{jt|jt} \neq w_{jt}$, the first order conditions for capital and labour do not meet the budget constraint. Therefore, the prices have to be part of the information set, which is also in line with the statement above about the interaction of agents, $r_t^k = \left(1-\alpha\right)\left(z_{jt|jt} + \hat{n}_{jt} - \hat{j}_{jt}\right)$ and $w_{jt} = \alpha\left(\hat{j}_{jt} - \hat{n}_{jt}\right) + \left(1-\alpha\right)z_{jt|jt}$. Now, assume that the firms choose the capital demand $\hat{j}_{jt}$ and their labour demand $\hat{n}_{jt}$ conditional on their expectation about idiosyncratic composite productivity, the return to capital and the individual wages in such as way that they satisfy the first order conditions. If I plug these equations into the budget constraint, I find:

$$0 = C\hat{c}_{jt} + I\hat{i}_{jt} - Y\left(\alpha\hat{k}_{jt} + \left(1-\alpha\right)\left(z_{jt|jt} + \hat{h}_{jt}\right)\right).$$

Using the labour market and the capital market clearing conditions, it is easy to show that the goods market does not clear. After inserting the mentioned equations in the budget constraint of the households and aggregating, I find $C\hat{c}_t + I\hat{i}_t = Y\hat{y}_{t|t}$, with $\hat{y}_{t|t} = \left(\alpha\hat{k}_t + \left(1-\alpha\right)\left(z_{t|t} + \hat{h}_t\right)\right)$. This does not generate market clearing as $z_{t|t} \neq z_t$. Only when, next to the return to capital and the individual wages, idiosyncratic composite productivity is revealed, then goods market clear. In this case I find the identity $C\hat{c}_t + I\hat{i}_t = Y\hat{y}_t$. □

A.2.2 The choice of signals

Proposition 2. *The prices on the capital and labour market, being part of the information set, Ω_{jt}, are sufficient to guarantee market clearing on the goods market.*

Proof. The proposition holds true for the partial and for the heterogeneous information model. I start with the proof for the partial information model. To proceed, recall three facts with regard to the partial information case. First, all agents behave alike such that all individual variables are equal to aggregate ones, including capital and consumption. Therefore, one can drop the subscript j. Second, as stated in Proposition 3, capital, as the predetermined state variable, is known to the agents. The proposition also shows that, third, the forward looking variable is a function of capital, composite productivity and the expectation of the components thereof, $c_t = c^p\left(k_t, z_t, a_{t|t}, \varepsilon_{t|t}\right)$. I write the return to capital, (A.36), and the wages, (A.39), as a function of variables that are part of the information set:

$$r_t^k = r^k\left(\hat{k}_t, z_t, \hat{c}_t\right), \tag{A.47}$$

$$\hat{w}_t = w\left(\hat{k}_t, z_t, \hat{c}_t\right). \tag{A.48}$$

Plugging consumption into these equations yields:

$$r_t^k = r^k\left(\hat{k}_t, z_t, a_{t|t}, \varepsilon_{t|t}\right) \quad \text{and} \tag{A.49}$$

$$\hat{w}_t = w\left(\hat{k}_t, z_t, a_{t|t}, \varepsilon_{t|t}\right). \tag{A.50}$$

Finally, I know from equation (1.22) that the expectation of the state variables does not carry any informational content for the signal extraction problem as they are the variables to be identified themselves. Therefore, the only unknown variable on the right hand side of both equations is composite productivity. Concluding, any of the two variables reveals z_t and z_t reveals the other two variables. Thus, with partial information, wages or the return to capital are sufficient to guarantee market clearing on the goods market.

Similarly, I proceed with the proof for the heterogeneous information model. Recall three facts with regard to the heterogeneous information case. First, as stated in Proposition 4, individual capital as the predetermined state variables is known to the agents. Second, individual consumption as the individual forward looking variable is a function of individual capital, idiosyncratic composite productivity and the individual expectation of the hierarchy of expectations and the individual expectation of the idiosyncratic exogenous state variables, $c_{jt} = c_j\big(k_{jt}, z_{jt}, r^k_t, k^{(0:\infty)}_{t|jt}, a^{(0:\infty)}_{t|jt}, \varepsilon^{(0:\infty)}_{t|jt}, \varepsilon_{jt|jt}\big)$. Inserting the return to capital into individual consumption and then aggregating gives the policy function of aggregate consumption, $c_t = c\big(k^{(0:\infty)}_t, a^{(0:\infty)}_t, \varepsilon^{(0:\infty)}_t\big)$. I write the return to capital as a function of aggregate state variables and aggregate forward looking variables, (A.36). Further, I write individual wages in terms of observables, (A.33):

$$r^k_t = r^k\big(\hat{k}_t, a_t, \varepsilon_t, \hat{c}_t\big) \quad \text{and} \tag{A.51}$$

$$\hat{w}_{jt} = w_j\big(r^k_t, z_{jt}\big). \tag{A.52}$$

Plugging individual and aggregate consumption into the equations above yields:

$$r^k_t = r^k\big(k^{(0:\infty)}_t, a^{(0:\infty)}_t, \varepsilon^{(0:\infty)}_t\big) \quad \text{and} \tag{A.53}$$

$$\hat{w}_{jt} = w_j\big(z_{jt}, r^k_t\big). \tag{A.54}$$

Finally, I know from equation (1.35) that the individual expectation of the state variables do not carry any informational content for the signal extraction problem as they are the variables to be identified themselves. The unknown variables of the return to capital are manyfold, i.e. the complete hierarchy of expectations of the aggregate state variables. To reveal the return to capital, subsets of those variables need to be observable or, most straight forward, the return to capital itself. If the return to capital is revealed, individual wages are subject only to one unknown variable, namely idiosyncratic composite productivity. Concluding, the two variables $\hat{w}_{jt}$ and z_{jt} reveal each other. Thus, with heterogeneous information, individual wages and the return to capital are sufficient to satisfy market clearing on the goods market. □

A.2.3 Capital state law of motion

Proposition 3. *Capital is known to the agents in the model with partial information.*

Proof. Recall the following model assumptions. First, agents behave identically with partial information. Therefore, one can look at the aggregate dynamics only. Further, agents are ex-ante identical and know the initial capital stock k_0. Second, one can write the capital state law of motion as a function of today's state variables, signals and consumption, $k_{t+1} = k\big(k_t, z_t, r_t^k, c_t\big)$, (A.43). Third, agents observe at minimum one signal to ensure market clearing on the goods market. For example, the return to capital $r_t^k = r^k\big(k_t, z_t, c_t\big)$, (A.36).

For the proof I analyse two cases: on the one hand, the assumption that capital is unknown to the agents and, on the other hand, the assumption that capital is indeed known to the agent.

1. Capital is unknown to the agent.

If capital is unknown to the agents, then it stays within the expectation operator. In line with the assumption that capital is not known to the agent, the guess for the functional form of consumption is:

$$c_t = c^p\left(k_{t|t}, z_t, r_t^k, a_{t|t}, \varepsilon_{t|t}\right). \tag{A.55}$$

If I apply this guess, the return to capital becomes $r_t^k = r\big(k_t, z_t, k_{t|t}, a_{t|t}, \varepsilon_{t|t}\big)$. Further recall, that the effect of expectational variables do not carry any informational content for the signal extraction problem as they do not contribute to the forecast error of the signals.

Assume now agents to be located in time period $t = 0$. In time period 0, k_0 is known and so is $k_{0|0} = k_0$. Further, agents receive the signal $r_0^k = r\big(k_0, z_0, a_{0|0}, \varepsilon_{0|0}\big)$ which reveals, z_0 as agents estimate $a_{0|0}$ and $\varepsilon_{0|0}$ in the signal extraction problem. This is sufficient information to make the consumption choice, $c_0 = c\big(k_0, z_0, r_0^k, a_{0|0}, \varepsilon_{0|0}\big)$, and the the capital choice, $k_1 = k\big(k_0, z_0, r_0^k, c_0\big)$. Which shows that capital in the next period is also known to the agents. In period one, agents receive a new signal r_1^k and the procedure repeats, which proofs that capital is not unknown to the agents in the case with partial information in any time period and hence is never part of the signal extraction problem.

2. Capital is known to the agent.

The guess for the functional form of consumption then becomes

$$c_t = c\big(k_t, z_t, r_t^k, a_{t|t}, \varepsilon_{t|t}\big). \tag{A.56}$$

If I plug this guess for consumption in the return to capital, it becomes $r_t^k = r\big(k_t, z_t, a_{t|t}, \varepsilon_{t|t}\big)$. From here, the logic follows the case in which capital is assumed not to be known and arrives at the same conclusion. The correct guess is that capital is indeed known to the agents. □

Proposition 4. *Individual capital is known to the agents in a model with heterogeneous information.*

Proof. Recall the following model assumption. First, agents are ex-ante identical and know the initial capital stock k_0. Second, agents choose tomorrows capital $k_{j,t+1}$ as a function of today's individual capital stock, individual consumption, idiosyncratic composite productivity and the return to capital, $k_{j,t+1} = k_j\big(k_{jt}, z_{jt}, r_t^k, c_{jt}\big)$, (A.42). Aggregate capital follows the process, $k_{t+1} = k\big(k_t, z_t, r_t^k, c_t\big)$, (A.43). Third, agents observe signals that ensure goods market clearing, e.g. the return to capital $r_t^k = r^k\big(k_t, z_t, c_t\big)$, (A.36), and individual wages $w_{jt} = w_j\big(z_{jt}, r_t^k\big)$, (A.33).

For the proof I analyse again two cases: on the one hand, the assumption that individual capital is unknown to the agents and, on the other hand, the assumption that individual capital is indeed known to the agent.

1. Individual capital is unknown to the agent.

If individual capital is unknown to the agents, then it stays within the expectation operator. In line with the assumption that individual capital is unknown to the agent, the guess for the functional form of individual consumption and aggregate consumption are:

$$
\begin{aligned}
c_{jt} &= c^j\left(k_{jt|jt}, k_{t|jt}^{(0:\infty)'}, a_{t|jt}^{(0:\infty)'}, \varepsilon_{t|jt}^{(0:\infty)'}, \varepsilon'_{jt|jt}, z_{jt}, r_t^k\right) \\
c_t &= c\big(k_t^{(1:\infty)'}, a_t^{(0:\infty)'}, \varepsilon_t^{(0:\infty)'}, r_t^k\big).
\end{aligned}
$$

If I apply this guess of consumption, the state law of motion of individual capital reads $k_{j,t+1} = k_j\left(k_{jt|jt}, z_{jt}, r_t^k, k_{jt|jt}, k_{t|jt}^{(0:\infty)'}, a_{t|jt}^{(0:\infty)'}, \varepsilon_{t|jt}^{(0:\infty)'}, \varepsilon_{jt|jt}\right)$ and aggregate capital reads $k_{t+1} = k\big(k_t^{(1:\infty)'}, a_t^{(0:\infty)'}, \varepsilon_t^{(0:\infty)'}, r_t^k\big)$. The signals become $r_t = r\big(k_t^{(0:\infty)'}, a_t^{(0:\infty)'}, \varepsilon_t^{(0:\infty)'}\big)$ and $w_{jt} = w_j\big(z_{jt}, r_t^k\big)$.

Assume now that that the agents are located in time period $t = 0$. In time period 0, $k_0 = k_{j0}$ is known to the agents and so is $k_0^{(0:\infty)'} = k_0$. Further, the agents receive the signals $r_0^k = r\big(k_0, a_0^{(0:\infty)'}, \varepsilon_0^{(0:\infty)'}\big)$ and $w_{j0} = w_j\big(z_{j0}, r_0^k\big)$. The latter reveals z_{j0}, while the former does not reveal specific state variables. The innovation to the return to capital can potentially come from any of the stochastic processes or the hierarchy of expectations. Nevertheless, this is sufficient information to make a consumption choice $c_{j0} = c\big(k_0, a_{0|j0}^{(0:\infty)'}, \varepsilon_{0|j0}^{(0:\infty)'}, \varepsilon_{j0|j0}, z_{j0}, r_0^k\big)$ that is market clearing. Moreover, one can disregard the expectational variables, as I established that they do not carry any informational content for the signal extraction problem.

Hence, given the information in period $t = 0$, k_{j1} becomes known, $k_{j1} = k^j\big(k_{j0}, z_{j0}, r_0^k, c_{j0}\big)$: k_{j0} is known as the initial condition, z_{j0} via individual wages, w_{j0}, r_0^k is a public signal and c_{j0} is known as it is an individual choice variable.

2. Individual capital is known to the agent.

The guess for the functional form of consumption then becomes:

$$\begin{aligned} c_{jt} &= c^j\big(k_{jt}, k_{t|jt}^{(0:\infty)'}, a_{t|jt}^{(0:\infty)'}, \varepsilon_{t|jt}^{(0:\infty)'}, \varepsilon'_{jt|jt}, z_{jt}, r_t^k\big) \\ c_t &= c\big(k_t^{(0:\infty)'}, a_t^{(0:\infty)'}, \varepsilon_t^{(0:\infty)'}, r_t^k\big). \end{aligned}$$

Following the same steps as in the previous case, we find the result that individual capital is indeed known to the agents and is never part of the signal extraction problem. □

Proposition 5. *There is a fundamental solution to the heterogeneous information model in which aggregate capital is known to the agents and a non-fundamental solution in which aggregate capital is unknown to the agents.*

Proof. I distinguish two cases. One in which the agents use individual wages only to update their knowledge of z_{jt} and the return to capital to infer the aggregate states and one in which the private signal is also used to infer information about the aggregate state.

1. Assume that the agents use only the public signal, $r_t^k = r^k\big(k_t, z_t, c_t\big)$ for the signal extraction problem of aggregate capital, $k_{t+1} = k\big(k_t, z_t, c_t, r_t^k\big)$ and the private signal $w_{jt} = w_j\big(z_{jt}, r_t^k\big)$ to infer z_{jt}, with $c_t = c\big(k_t^{(0:\infty)'}, a_t^{(1:\infty)'}, \varepsilon_t^{(1:\infty)'}, z_t, r_t^k\big)$.

Next, assume that the agents are located in time period $t = 0$. In time period 0, $k_0 = k_{j0}$ is known to the agents and so is $k_0^{(0:\infty)'} = k_0$. In addition realize that higher order expectations only arise with heterogeneous information. If the agents use only the public signals to estimate aggregate state variables, higher order expectations do not arise. Hence, we can write aggregate consumption as $c_0 = c\big(k_0, a_{0|0}, \varepsilon_{0|0}, z_0, r_0^k\big)$ and the return to capital as $r_0^k = r^k\big(k_0, z_0, a_{0|0}, \varepsilon_{0|0}\big)$, which is identical to the formulation of aggregate capital with partial information in Proposition 3. As expectational variables do not carry any informational content and the initial capital stock is known, the return to capital reveals aggregate productivity z_0, which is sufficient to compute next periods aggregate capital, $k_1 = k\big(k_0, c_0, z_0, r_0^k\big)$.[1]

[1] Graham and Wright (2010) argue in the proof to their Proposition 1 that the solution in which agents use the return to capital only to forecast capital is unstable, because they keep capital as part of their filtering problem, i.e. they do not consider capital separately as a predetermined endogenous state variable.

2. Assume that agents use both private and public signals for the signal extraction problem of aggregate state variables.

Again, the signals can be written as $r_t^k = r^k\left(k_t, z_t, k_t^{(1:\infty)'}, a_t^{(1:\infty)'}, \varepsilon_t^{(1:\infty)'}\right)$ and $w_{jt} = w_j\left(z_{jt}, r_t^k\right)$.

In the initial period $t = 0$, aggregate capital is known, $k_0^{(0:\infty)'} = k_0$. Consequently, aggregate consumption can be written as $c_0 = c\left(k_0, a_0^{(1:\infty)'}, \varepsilon_0^{(1:\infty)'}, z_0, r_0^k\right)$ and the return to capital as $r_0^k = r^k\left(k_0, z_0, a_0^{(1:\infty)'}, \varepsilon_0^{(1:\infty)'}\right)$. Further, aggregate capital can be written as $k_1 = k\left(k_0, a_0^{(1:\infty)'}, \varepsilon_0^{(1:\infty)'}, z_0, r_0^k\right)$. In the previous case, the return to capital revealed composite productivity. In this case, the return to capital is a signal about composite productivity as well as the hierarchy of expectations of its elements. This information reveal next periods capital, if aggregate capital depends on these composite productivity and the hierarchy of expectations in the same way as the return to capital. As the hierarchy of expectation is entering these equations via the forward looking variable, the variables would depend on it in the same way, if they depend in the same way on composite productivity and consumption. Plugging (A.36) into (A.43), I find:

$$\hat{k}_{t+1} = \left(1 - \delta + (1-\alpha)\frac{Y}{K}\right)\hat{k}_t + \frac{(1-\alpha)(1+\xi)}{1+\alpha\xi}\frac{Y}{K}z_t - \left(\frac{C}{K} + \frac{(1-\alpha)\xi}{1+\alpha\xi}\frac{Y}{K}\right)\hat{c}_t.$$

The return to capital, (A.36), instead reads:

$$r_t^k = -\frac{1-\alpha}{1+\alpha\xi}\hat{k}_t + \frac{(1+\xi)(1-\alpha)}{1+\alpha\xi}z_t - \frac{(1-\alpha)\xi}{1+\alpha\xi}\hat{c}_t.$$

Capital and the return to capital would only depend on composite productivity in the same way if the steady state ratio of output to capital was equal to one. In addition, the steady state ratio of consumption to capital had to be equal to zero. Especially, the latter is not feasible under any reasonable assumption.

As only the joint combination for composite productivity and the hierarchy of expectation of the components is observable from the return to capital, and the combination is different to the one that enters the capital state law of motion, the individual agents will not be able to infer aggregate capital in the second period. Thus, aggregate capital is unknown to the agents if the agents use both, individual wages and the return to capital jointly to form their expectation about the state variables. □

A.3 Solution algorithm

A.3.1 Full information solution

I guess that the forward looking variables are a function of the state variables, i.e.

$$F_t = \eta^* \begin{bmatrix} X_t^{c\prime} & X_t^{\prime} \end{bmatrix}, \quad \text{where} \quad \eta^* = \begin{bmatrix} \eta_c^* & \eta_x^* \end{bmatrix}. \tag{A.57}$$

Then, in combination with (1.10) and (1.11), I write the state law of motion and the contemporaneous variables as:

$$\begin{bmatrix} X_{t+1}^c \\ X_{t+1} \end{bmatrix} = M^*(\eta^*) \begin{bmatrix} X_t^c \\ X_t \end{bmatrix} + N^* s_{t+1} \quad \text{and} \quad \Upsilon_t = G(\eta^*) \begin{bmatrix} X_t^c \\ X_t \end{bmatrix}, \tag{A.58}$$

where

$$M^*(\eta^*) = \left(\begin{bmatrix} M_c^c & M_c^x \\ 0 & M_x^x \end{bmatrix} + \begin{bmatrix} M_c^{\Upsilon} \\ 0 \end{bmatrix} \begin{bmatrix} G_c & G_x \end{bmatrix} + \begin{bmatrix} M_c^f + M_c^{\Upsilon} G_f \\ 0 \end{bmatrix} \eta^* \right), \quad N^* = \begin{bmatrix} 0 \\ N_x \end{bmatrix} \text{ and}$$

$$G^*(\eta^*) = \left(\begin{bmatrix} G_c & G_x \end{bmatrix} + G_f \eta^* \right).$$

I substitute observable variables, (1.11), and the guess for the forward looking variables in the Euler equations of the forward looking variables (A.46),

$$\left(\begin{bmatrix} Q_c^0 & Q_x^0 \end{bmatrix} + Q_{\Upsilon}^0 \begin{bmatrix} G_c & G_x \end{bmatrix} \right) \begin{bmatrix} X_t^c & X_t \end{bmatrix} + (Q_{\Upsilon}^0 G_f + Q_f^0) F_t = Q_f^1 E_t[F_{t+1}]. \tag{A.59}$$

Plug in the guess for forward looking variables and use the state law of motion. One finds the expression (1.15) in the main text after equating parameters of identical variables:

$$\begin{aligned} \eta^* &= C^0 + C^1(\eta^*) M^*(\eta^*), \quad \text{where} \\ C^0 &= -\left(Q_f^0 + Q_{\Upsilon}^0 G_f \right)^{-1} \left(\begin{bmatrix} Q_c^0 & Q_x^0 \end{bmatrix} + Q_{\Upsilon}^0 \begin{bmatrix} G_c & G_x \end{bmatrix} \right) \\ C^1(\eta^*) &= \left(Q_f^0 + Q_{\Upsilon}^0 G_f \right)^{-1} Q_f^1 \eta^*. \end{aligned}$$

A.3.2 Partial information solution

I formulate a guess about the functional form of the forward looking variables. I guess that they are a function of the predetermined state and exogenous state variables and the expectation of the exogenous state variables:

$$F_t = \begin{bmatrix} \eta_c^p & \eta_x^p & \eta_e^p \end{bmatrix} \begin{bmatrix} X_t^{c\prime} & X_t^{\prime} & X_{t|t}^{\prime} \end{bmatrix}^{\prime}. \tag{A.60}$$

This implies an expanded state law of motion, which includes the expectation of exogenous state variables $X_{t|t}$. Contemporaneous variables, (1.11), can be expressed as:

$$\Upsilon_t = G^p(\eta^p) \begin{bmatrix} X_t^{c\prime} & X_t^{\prime} & X_{t|t}^{\prime} \end{bmatrix}^{\prime}, \quad \text{where} \quad G^p(\eta^p) = \begin{bmatrix} G_c + G_f \eta_c^p & G_x + G_f \eta_x^p & G_f \eta_e^p \end{bmatrix}. \tag{A.61}$$

Then, I substitute the contemporaneous variables, (A.61), into the state law of motion, (1.10),

$$\begin{bmatrix} X_{t+1}^c \\ X_{t+1} \\ X_{t+1|t+1} \end{bmatrix} = \left(\begin{bmatrix} M_c^c + M_c^{\Upsilon} G_c & M_c^{\Upsilon} G_x & 0 \\ 0 & M_x^x & 0 \\ 0 & M_e^x(\mathscr{K}, \eta_x^p) & M_e^e(\mathscr{K}, \eta_x^p) \end{bmatrix} + \begin{bmatrix} M_c^f + M_c^{\Upsilon} G_f \\ 0 \\ 0 \end{bmatrix} \eta \right) \begin{bmatrix} X_t^c \\ X_t \\ X_{t|t} \end{bmatrix} + \begin{bmatrix} 0 \\ N_x \\ N_e(\mathscr{K}, \eta_x^p) \end{bmatrix} s_{t+1}, \tag{A.62}$$

where $M_e^x(\mathscr{K}, \eta_x^p)$, $M_e^e(\mathscr{K}, \eta_x^p)$ and $N_e(\mathscr{K}, \eta_x^p)$ remain to be defined. They are part of the solution of the fixed point problem between the state law of motion, the matrix mapping the forward looking variables into the state variables η^p and the Kalman gain, $\mathscr{K}$.

I confirm the guess of the forward looking variables analogue to the full information case:

$$\left(\begin{bmatrix} Q_c^0 & Q_x^0 & 0 \end{bmatrix} + Q_{\Upsilon}^0 \begin{bmatrix} G_c & G_x & 0 \end{bmatrix} \right) \begin{bmatrix} X_t^c \\ X_t \\ X_{t|t} \end{bmatrix} + \left(Q_f^0 + Q_{\Upsilon}^0 G_f \right) \eta^p \begin{bmatrix} X_t^c \\ X_t \\ X_{t|t} \end{bmatrix} = Q_f^1 \eta^p E_t \begin{bmatrix} X_{t+1}^c \\ X_{t+1} \\ X_{t+1|t+1} \end{bmatrix}. \tag{A.63}$$

Reorganize and use the transition matrix of the state law of motion to find the expression (1.19) in the main text:

$$\begin{aligned}
\eta^p &= C^0 + C^1(\eta^p)\left(T_k M^p(\mathscr{K},\eta^p) + T_{\neq k} M^p(\mathscr{K},\eta^p) T_e\right), \quad \text{where} \\
C^0 &= -\left(Q_f^0 + Q_\Upsilon^0 G_f\right)^{-1}\left(\begin{bmatrix} Q_c^0 & Q_x^0 & 0 \end{bmatrix} + Q_\Upsilon^0 \begin{bmatrix} G_c & G_x & 0 \end{bmatrix}\right) \\
C^1(\eta^p) &= \left(Q_f^0 + Q_\Upsilon^0 G_f\right)^{-1} Q_f^1 \eta^p \\
T_k &= \begin{bmatrix} I & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad T_{\neq k} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \quad \text{and} \quad T_e = \begin{bmatrix} I & 0 & 0 \\ 0 & 0 & I \\ 0 & 0 & I \end{bmatrix}.
\end{aligned}$$

Kalman Updating

The Kalman updating equation is defined as follows:

$$X_{t+1|t+1} - X_{t+1|t} = \mathscr{K}\left(\underline{\Upsilon}_{t+1} - \underline{\Upsilon}_{t+t|t}\right). \tag{A.64}$$

The forecast error of the signals $\underline{\Upsilon}_{t+1} - \underline{\Upsilon}_{t+1|t}$ is defined as:

$$\underline{\Upsilon}_{t+1} - \underline{\Upsilon}_{t+1|t} = \underline{G}(\eta^p)\left(\begin{bmatrix} X_{t+1}^c \\ X_{t+1} \\ X_{t+1|t+1} \end{bmatrix} - \begin{bmatrix} X_{t+1|t}^c \\ X_{t+1|t} \\ X_{t+1|t} \end{bmatrix}\right), \tag{A.65}$$

where predetermined choice variables cancel out. Plugging (A.64) into (A.65) we find:

$$\underline{\Upsilon}_{t+1} - \underline{\Upsilon}_{t+1|t} = J^{-1}\underline{G}_x(\eta_x^p)\left(X_{t+1} - X_{t+1|t}\right), \tag{A.66}$$

where $J = \left(I - \underline{G}_e(\eta_e^p)\mathscr{K}\right)$. I plug this expression back into the Kalman updating equation (A.64) to find:

$$X_{t+1|t+1} = \left(I - \mathscr{K} J^{-1}\underline{G}_x(\eta_x^p)\right) X_{t+1|t} + \mathscr{K} J^{-1}\underline{G}_x(\eta_x^p) X_{t+1}. \tag{A.67}$$

If I use the state law of motion for X_{t+1} from (A.62), I derive the state law of motion of the expectation of the state variables $X_{t+1|t+1}$:

$$X_{t+1|t+1} = \begin{bmatrix} 0 & \mathscr{K} J^{-1} \underline{G}_x(\eta_x^p) M_x^x & \left(I - K J^{-1} \underline{G}_x(\eta_x^p)\right) M_x^x \end{bmatrix} \begin{bmatrix} X_t^c \\ X_t \\ X_{t|t} \end{bmatrix}$$
$$+ \mathscr{K} J^{-1} \underline{G}_x(\eta_x^p) N_x s_{t+1}, \tag{A.68}$$

which verifies my guess and identifies $M_e^x(\mathscr{K}, \eta^p)$, $M_e^e(\mathscr{K}, \eta^p)$ and $N_e(\mathscr{K}, \eta^p)$.

Kalman Gain

It remains to solve for the Kalman gain $\mathscr{K}$ as well as the corresponding mean square error (MSE) and the variance-covariance matrix of the one period ahead forecast error.

First, define the variance-covariance matrix of the one period ahead forecast error as:

$$\begin{aligned} P &= E\left[\left(X_{t+1} - X_{t+1|t}\right)\left(X_{t+1} - X_{t+1|t}\right)'\right] \\ &= M_x^x \hat{P} M_x^{x\prime} + N_x N_x' \end{aligned} \tag{A.69}$$

and the MSE as:

$$\begin{aligned} \hat{P} &= E\left[\left(X_t - X_{t|t}\right)\left(X_t - X_{t|t}\right)'\right] \\ &= \left(I - \mathscr{K} J^{-1} \underline{G}_x(\eta_x^p)\right) P \end{aligned} \tag{A.70}$$

The Kalman gain can be computed using equations (A.66) and (A.69):

$$\mathscr{K} = \left\{ E\left[\left(X_{t+1} - X_{t+1|t}\right)\left(\underline{\Upsilon}_{t+1} - \underline{\Upsilon}_{t+1|t}\right)'\right]\right\} \tag{A.71}$$
$$\times \left\{ E\left[\left(\underline{\Upsilon}_{t+1} - \underline{\Upsilon}_{t+1|t}\right)\left(\underline{\Upsilon}_{t+1} - \underline{\Upsilon}_{t+1|t}\right)'\right]\right\}^{-1},$$
$$= \left\{ P \underline{G}_x(\eta_x^p)' (J^{-1})' \right\} \left\{ J^{-1} \left(\underline{G}_x(\eta_x^p) P \underline{G}_x(\eta_x^p)' \right) (J^{-1})' \right\}^{-1} \tag{A.72}$$
$$= \left\{ P \underline{G}_x(\eta_x^p)' \right\} \left\{ \underline{G}_x(\eta_x^p) P \underline{G}_x(\eta_x^p)' \right\}^{-1} J. \tag{A.73}$$

A.3.3 Heterogeneous information solution

Define $\Gamma_t = [X_t^{c\prime}\, X_t^{\prime}]^{\prime}$ and $\Gamma_{jt} = \left[X_{jt}^{c\prime}\, X_{jt}^{\prime}\right]^{\prime}$.

State space representation

Formulate a guess about the functional form of the forward looking variables:

$$F_{jt} = \eta_{fj}\left[\Gamma_t^{(0:\bar{o})\prime} \quad \Gamma_{t|jt}^{(0:\bar{o})\prime} \quad \Gamma_{jt}^{\prime} \quad X_{jt|jt}^{\prime}\right]^{\prime}, \text{ where } \eta_{fj} = \left[\eta_{fj}^{e\Gamma} \quad \eta_{fj}^{ej\Gamma} \quad \eta_{fj}^{\Gamma j} \quad \eta_{fj}^{ejxj}\right]. \quad \text{(A.74)}$$

I find the functional form of the aggregate forward looking variables by integrating over the individual ones of all islands,

$$F_t = \eta_f\left[\Gamma_t^{(0:\bar{o})\prime} \quad \Gamma_{t|jt}^{(0:\bar{o})\prime} \quad \Gamma_{jt}^{\prime} \quad X_{jt|jt}^{\prime}\right]^{\prime}, \text{ where } \eta_f = \eta_{fj}T_e. \quad \text{(A.75)}$$

The matrix T_e maps the idiosyncratic and individual state variables and the individual expectation of the hierarchy of expectation to the hierarchy of expectation. The idiosyncratic exogenous state variables and the expectation thereof cancel out as these variables aggregate to zero by assumption.

This implies an expanded state law of motion, which also includes the expectation of exogenous state variables $X_{t|t}$. Contemporaneous variables, (1.11), can be expressed as:

$$\begin{aligned} \Upsilon_t &= G(\eta)\left[\Gamma_t^{(0:\bar{o})\prime} \quad \Gamma_{t|jt}^{(0:\bar{o})\prime} \quad \Gamma_{jt}^{\prime} \quad X_{jt|jt}^{\prime}\right]^{\prime}, \quad \text{where} \quad \text{(A.76)} \\ G(\eta) &= \left[\left[G_{\Gamma} \quad 0\right] \quad 0 \quad G_{\Gamma j} \quad 0\right] + \left[G_{fj} \quad G_f\right]\eta \\ &= \left[G_{e\Gamma}(\eta^{e\Gamma}) \quad G_{ej\Gamma}(\eta^{ej\Gamma}) \quad G_{\Gamma j}(\eta^{\Gamma j}) \quad G_{ejxj}(\eta^{ejxj})\right]. \end{aligned}$$

If I plug the guess for the forward looking variables into the state law of motion of non-expectational state variables, (1.10), the state law of motion becomes:

$$\begin{bmatrix} X^c_{t+1} \\ X_{t+1} \\ X^c_{j,t+1} \\ X_{j,t+1} \end{bmatrix} = \begin{bmatrix} M^c_c + M^\Upsilon_c G_c & M^x_c + M^\Upsilon_c G_x & 0 & 0 \\ 0 & M^x_x & 0 & 0 \\ (M^{\Upsilon j}_{cj} + M^\Upsilon_{cj})G_c & (M^{\Upsilon j}_{cj} + M^\Upsilon_{cj})G_x & M^{cj}_{cj} + M^{\Upsilon j}_{cj} G_{cj} & M^{\Upsilon j}_{cj} G_{xj} \\ 0 & 0 & 0 & M^{xj}_{xj} \end{bmatrix} \begin{bmatrix} X^c_t \\ X_t \\ X^c_{jt} \\ X_{jt} \end{bmatrix}$$

$$+ \begin{bmatrix} 0 & M^f_c + M^\Upsilon_c G_f \\ 0 & 0 \\ M^{fj}_{cj} + M^{\Upsilon j}_{cj} G_{fj} & (M^{\Upsilon j}_{cj} + M^\Upsilon_{cj})G_f \\ 0 & 0 \end{bmatrix} \eta \begin{bmatrix} \Gamma^{(0:\bar{o})}_t \\ \Gamma^{(0:\bar{o})}_{t|jt} \\ \Gamma_{jt} \\ \Gamma_{jt|jt} \end{bmatrix} + \begin{bmatrix} 0 \\ N_{xj} \\ 0 \\ N_x \end{bmatrix} \begin{bmatrix} s_{jt} \\ s_t \end{bmatrix}. \tag{A.77}$$

I summarize the parameter matrices and define:

$$\begin{bmatrix} \Gamma_{t+1} \\ \Gamma_{j,t+1} \end{bmatrix} = \begin{bmatrix} M_\Gamma \\ M_{\Gamma j} \end{bmatrix} \begin{bmatrix} \Gamma^{(0:\bar{o})}_t \\ \Gamma^{(0:\bar{o})}_{t|jt} \\ \Gamma_{jt} \\ \Gamma_{jt|jt} \end{bmatrix} + \begin{bmatrix} N_\Gamma \\ N_{\Gamma j} \end{bmatrix} \begin{bmatrix} s_{jt} \\ s_t \end{bmatrix}, \quad \text{where} \tag{A.78}$$

$$\begin{aligned} M_\Gamma &= \left[\begin{bmatrix} M^c_c + M^\Upsilon_c G_c & M^x_c + M^\Upsilon_c G_x & 0 \\ 0 & M^x_x & 0 \end{bmatrix} \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} \right] + \begin{bmatrix} 0 & M^f_c + M^\Upsilon_c G_f \\ 0 & 0 \end{bmatrix} \eta \\ M_{\Gamma j} &= \left[\begin{bmatrix} (M^{\Upsilon j}_{cj} + M^\Upsilon_{cj})G_c & (M^{\Upsilon j}_{cj} + M^\Upsilon_{cj})G_x & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{matrix} 0 & M^{cj}_{cj} + M^{\Upsilon j}_{cj} G_{cj} & M^{\Upsilon j}_{cj} G_{xj} \\ 0 & 0 & M^{xj}_{xj} \end{matrix} \right] \\ &+ \begin{bmatrix} M^{fj}_{cj} + M^{\Upsilon j}_{cj} G_{fj} & (M^{\Upsilon j}_{cj} + M^\Upsilon_{cj})G_f \\ 0 & 0 \end{bmatrix} \eta \\ N_\Gamma &= \begin{bmatrix} 0 \\ N_x \end{bmatrix} \quad N_{\Gamma j} = \begin{bmatrix} 0 \\ N_{xj} \end{bmatrix}. \end{aligned} \tag{A.79}$$

The guess for the extended state law of motion looks then as follows, while the matrices of the expectational state variables will be defined in the course of the signal extraction problem.

$$\begin{bmatrix} \Gamma_{t+1}^{(0:\overline{o})} \\ \Gamma_{t+1|j,t+1}^{(0:\overline{o})} \\ \Gamma_{j,t+1} \\ \Gamma_{j,t+1|j,t+1} \end{bmatrix} = \left(\begin{bmatrix} M_{e\Gamma}^{e\Gamma} & 0 & 0 & 0 \\ M_{ej\Gamma}^{e\Gamma} & M_{ej\Gamma}^{ej\Gamma} & M_{ej\Gamma}^{\Gamma j} & M_{ej\Gamma}^{ej\Gamma j} \\ 0 & 0 & 0 & 0 \\ M_{ej\Gamma j}^{e\Gamma} & M_{ej\Gamma j}^{ej\Gamma} & M_{ej\Gamma j}^{\Gamma j} & M_{ej\Gamma j}^{ej\Gamma j} \end{bmatrix} + \begin{bmatrix} \begin{bmatrix} M_{\Gamma} \\ 0 \end{bmatrix} \\ 0 \\ M_{\Gamma j} \\ 0 \end{bmatrix} \right) \begin{bmatrix} \Gamma_{t}^{(0:\overline{o})} \\ \Gamma_{t|jt}^{(0:\overline{o})} \\ \Gamma_{jt} \\ \Gamma_{jt|jt} \end{bmatrix} + \begin{bmatrix} \begin{bmatrix} N_{\Gamma} \\ N_{e\Gamma} \end{bmatrix} \\ N_{ej\Gamma} \\ N_{\Gamma j} \\ N_{ej\Gamma j} \end{bmatrix} \begin{bmatrix} s_{j,t+1} \\ s_{t+1} \end{bmatrix} \tag{A.80}$$

The matrices of the Euler equation of forward looking variables are derived in the same way. Recall (A.46) and plug in contemporaneous variables as well as the guess for forward looking variables.

$$\left[\left(\left[\begin{bmatrix} Q_{\Upsilon}^{0} G_{\Gamma} & 0 \end{bmatrix} \quad 0 \quad Q_{\Gamma j}^{0} + Q_{\Upsilon}^{0} G_{\Gamma j} \quad 0 \right] + Q_{\Upsilon}^{0} G_{f} \eta_{f} \right) + \left(\left[\begin{bmatrix} Q_{ej\Gamma}^{0} & 0 \end{bmatrix} \quad 0 \quad Q_{ej\Gamma j}^{0} \quad 0 \right] + Q_{f}^{0} \eta_{f} \right) T_{ej} \right] \begin{bmatrix} \Gamma_{t}^{(0:\overline{o})} \\ \Gamma_{t|jt}^{(0:\overline{o})} \\ \Gamma_{jt} \\ \Gamma_{jt|jt} \end{bmatrix}$$
$$+ (Q_{fj}^{0} + Q_{\Upsilon}^{0} G_{fj}) \eta_{fj} \begin{bmatrix} \Gamma_{t}^{(0:\overline{o})} \\ \Gamma_{t|jt}^{(0:\overline{o})} \\ \Gamma_{jt} \\ \Gamma_{jt|jt} \end{bmatrix} = Q_{F}^{1} \eta E_{jt} \begin{bmatrix} \Gamma_{t+1}^{(0:\overline{o})} \\ \Gamma_{t+1|j,t+1}^{(0:\overline{o})} \\ \Gamma_{j,t+1} \\ \Gamma_{j,t+1|j,t+1} \end{bmatrix} \tag{A.81}$$

Reorganizing and using the state law of motion for the extended state space I find the expression in the main text, (1.33):

$$\eta_{fj} = C^{0}\left(\eta_{f}\right) + C^{1}\left(\eta\right)\left(T_{kj} M + T_{\neq kj} M T_{ej}\right) \quad \text{where} \tag{A.82}$$
$$C^{0}\left(\eta_{f}\right) = -(Q_{fj}^{0} + Q_{\Upsilon}^{0} G_{fj})^{-1} \left[\left(\left[\begin{bmatrix} Q_{\Upsilon}^{0} G_{\Gamma} & 0 \end{bmatrix} \quad 0 \quad Q_{\Gamma j}^{0} + Q_{\Upsilon}^{0} G_{\Gamma j} \quad 0 \right] + Q_{\Upsilon}^{0} G_{f} \eta_{f} \right) \right.$$
$$\left. + \left(\left[\begin{bmatrix} Q_{ej\Gamma}^{0} & 0 \end{bmatrix} \quad 0 \quad Q_{ej\Gamma j}^{0} \quad 0 \right] + Q_{f}^{0} \eta_{f} \right) T_{ej} \right]$$
$$C^{1}\left(\eta\right) = (Q_{fj}^{0} + Q_{\Upsilon}^{0} G_{fj})^{-1} Q_{F}^{1} \eta$$
$$T_{ej} = \begin{bmatrix} 0 & I & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & 0 & I \\ 0 & 0 & 0 & I \end{bmatrix}$$

Kalman Updating

The Kalman updating equation is defined as follows:

$$\begin{bmatrix}\Gamma^{(0:\bar{o})}_{t+1|j,t+1}\\ X_{j,t+1|j,t+1}\end{bmatrix}-\begin{bmatrix}\Gamma^{(0:\bar{o})}_{t+1|jt}\\ X_{j,t+1|jt}\end{bmatrix}=\begin{bmatrix}\mathscr{K}_{ej\Gamma}\\ \mathscr{K}_{ejxj}\end{bmatrix}\left(\begin{bmatrix}\underline{\Upsilon}_{t+1}\\ \underline{\Upsilon}_{j,t+1}\end{bmatrix}-\begin{bmatrix}\underline{\Upsilon}_{t+1|jt}\\ \underline{\Upsilon}_{j,t+1|jt}\end{bmatrix}\right). \tag{A.83}$$

In addition, we can compute the forecast error of the signals as:

$$\begin{bmatrix}\underline{\Upsilon}_{t+1}\\ \underline{\Upsilon}_{j,t+1}\end{bmatrix}-\begin{bmatrix}\underline{\Upsilon}_{t+1|jt}\\ \underline{\Upsilon}_{j,t+1|jt}\end{bmatrix}=\underline{G}(\eta)\left(\begin{bmatrix}\Gamma^{(0:\bar{o})}_{t+1}\\ \Gamma^{(0:\bar{o})}_{t+1|j,t+1}\\ \Gamma_{j,t+1}\\ X_{j,t+1|j,t+1}\end{bmatrix}-\begin{bmatrix}\Gamma^{(0:\bar{o})}_{t+1|jt}\\ \Gamma^{(0:\bar{o})}_{t+1|jt}\\ \Gamma_{j,t+1|jt}\\ X_{j,t+1|jt}\end{bmatrix}\right), \tag{A.84}$$

which simplifies, using (1.34), to:

$$\begin{bmatrix}\underline{\Upsilon}_{t+1}\\ \underline{\Upsilon}_{j,t+1}\end{bmatrix}-\begin{bmatrix}\underline{\Upsilon}_{t+1|jt}\\ \underline{\Upsilon}_{j,t+1|jt}\end{bmatrix}=J^{-1}\underline{G}_1\left(\begin{bmatrix}\Gamma^{(0:\bar{o})}_{t+1}\\ X_{j,t+1}\end{bmatrix}-\begin{bmatrix}\Gamma^{(0:\bar{o})}_{t+1|jt}\\ X_{j,t+1|jt}\end{bmatrix}\right), \tag{A.85}$$

where $J=\left(I-\begin{bmatrix}\underline{G}_{ej\Gamma} & \underline{G}_{ejxj}\end{bmatrix}\mathscr{K}\right)$, $\underline{G}_1=\begin{bmatrix}\underline{G}_{e\Gamma} & \underline{G}_{xj}\end{bmatrix}$ and $\mathscr{K}=\begin{bmatrix}\mathscr{K}'_{ej\Gamma} & \mathscr{K}'_{ejxj}\end{bmatrix}'$.

Plug (A.85) back into the Kalman updating equation (A.83) to find:

$$\begin{aligned}\begin{bmatrix}\Gamma^{(0:\bar{o})}_{t+1|j,t+1}\\ X_{j,t+1|j,t+1}\end{bmatrix}&=\begin{bmatrix}\Gamma^{(0:\bar{o})}_{t+1|jt}\\ X_{j,t+1|jt}\end{bmatrix}+KJ^{-1}\underline{G}_1\left(\begin{bmatrix}\Gamma^{(0:\bar{o})}_{t+1}\\ X_{j,t+1}\end{bmatrix}-\begin{bmatrix}\Gamma^{(0:\bar{o})}_{t+1|jt}\\ X_{j,t+1|jt}\end{bmatrix}\right)\\ &=\left(I-KJ^{-1}\underline{G}_1\right)\begin{bmatrix}\Gamma^{(0:\bar{o})}_{t+1|jt}\\ X_{j,t+1|jt}\end{bmatrix}+KJ^{-1}\underline{G}_1\begin{bmatrix}\Gamma^{(0:\bar{o})}_{t+1}\\ X_{j,t+1}\end{bmatrix}.\end{aligned} \tag{A.86}$$

Now, I make use of the state law of motion (1.32) and define:

$$A=\begin{bmatrix}A_{e\Gamma} & A_{xj}\end{bmatrix}=\begin{bmatrix}M^{e\Gamma}_{e\Gamma} & 0\\ 0 & M^{xj}_{xj}\end{bmatrix}\quad\text{and}\quad B=\begin{bmatrix}N_{e\Gamma}\\ N_{xj}\end{bmatrix}.$$

to find

$$\begin{bmatrix} \Gamma^{(0:\bar{o})}_{t+1|j,t+1} \\ X_{j,t+1|j,t+1} \end{bmatrix} = \begin{bmatrix} \left(I - \mathscr{K}_{ej\Gamma} J^{-1} \underline{G}_1\right) A_{e\Gamma} & \left(I - \mathscr{K}_{ej\Gamma} J^{-1} \underline{G}_1\right) A_{xj} \\ \left(I - \mathscr{K}_{ejxj} J^{-1} \underline{G}_1\right) A_{e\Gamma} & \left(I - \mathscr{K}_{ejxj} J^{-1} \underline{G}_1\right) A_{xj} \end{bmatrix} \begin{bmatrix} \Gamma^{(0:\bar{o})}_{t|jt} \\ X_{jt|jt} \end{bmatrix} + \begin{bmatrix} \mathscr{K}_{ej\Gamma} J^{-1} \underline{G}_1 A_{e\Gamma} & \mathscr{K}_{ej\Gamma} J^{-1} \underline{G}_1 A_{xj} \\ \mathscr{K}_{ejxj} J^{-1} \underline{G}_1 A_{e\Gamma} & \mathscr{K}_{ejxj} J^{-1} \underline{G}_1 A_{xj} \end{bmatrix} \begin{bmatrix} \Gamma^{(0:\bar{o})}_{t} \\ X_{jt} \end{bmatrix} + \begin{bmatrix} \mathscr{K}_{ej\Gamma} J^{-1} \underline{G}_1 B \\ \mathscr{K}_{ejxj} J^{-1} \underline{G}_1 B \end{bmatrix} \cdot \begin{bmatrix} s_{jt} \\ s_t \end{bmatrix}, \tag{A.87}$$

which defines the parameter matrices of the guess of the individual expectation of the hierarchy of expectation and the idiosyncratic state variables in the extended state law of motion, (1.32).

To conclude the guess for the state law of motion, I only need to find the expression for $M^{e\Gamma}_{e\Gamma}$. To find the state law of motion of the aggregate hierarchy of expectations, I aggregate over (A.87):

$$\Gamma^{(1:\bar{o})}_{t+1} = M^{ej\Gamma}_{ej\Gamma} \Gamma^{(1:\bar{o})}_{t} + M^{e\Gamma}_{ej\Gamma} \Gamma^{(0:\bar{o})}_{t} + N_{ej\Gamma} T_s \begin{bmatrix} s_{j,t+1} \\ s_{t+1} \end{bmatrix},$$

where T_s is a matrix that sets the parameters of idiosyncratic innovations equal to zero. Finally, I amend the hierarchy of expectation with the state law of motion of aggregate state variables, (A.78):

$$M^{e\Gamma}_{e\Gamma} = \begin{bmatrix} M^{e\Gamma}_{\Gamma} \\ M^{e\Gamma}_{ej\Gamma} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & M^{ej\Gamma}_{ej\Gamma} \end{bmatrix}, \quad \text{and} \quad N_e = \begin{bmatrix} N_{\Gamma} \\ N_{ej\Gamma} T_s \end{bmatrix}.$$

Kalman Gain

The Kalman filter reads:[2]

$$\begin{bmatrix} K_{ej\Gamma} \\ K_{ejxj} \end{bmatrix} = \left\{ E\left[\left(\begin{bmatrix} \Gamma^{(0:\bar{o})}_{t+1} \\ X_{j,t+1} \end{bmatrix} - \begin{bmatrix} \Gamma^{(0:\bar{o})}_{t+1|jt} \\ X_{j,t+1|jt} \end{bmatrix} \right) \left(\begin{bmatrix} \underline{\Upsilon}_{t+1} \\ \underline{\Upsilon}_{j,t+1} \end{bmatrix} - \begin{bmatrix} \underline{Y}_{t+1|jt} \\ \underline{\Upsilon}_{j,t+1|jt} \end{bmatrix} \right)' \right] \right\}$$
$$\times \left\{ E\left[\left(\begin{bmatrix} \underline{\Upsilon}_{t+1} \\ \underline{\Upsilon}_{j,t+1} \end{bmatrix} - \begin{bmatrix} \underline{\Upsilon}_{t+1|jt} \\ \underline{\Upsilon}_{j,t+1|jt} \end{bmatrix} \right) \left(\begin{bmatrix} \underline{\Upsilon}_{t+1} \\ \underline{\Upsilon}_{j,t+1} \end{bmatrix} - \begin{bmatrix} \underline{\Upsilon}_{t+1|jt} \\ \underline{\Upsilon}_{j,t+1|jt} \end{bmatrix} \right)' \right] \right\}^{-1}$$
$$= \{PG_1'\}\{G_1PG_1'\}^{-1}J. \tag{A.88}$$

The variance-covariance matrix of the one period ahead forecast error is defined as:

$$P = E\left[\left(\begin{bmatrix} \Gamma^{(0:\bar{o})}_{t+1} \\ X_{j,t+1} \end{bmatrix} - \begin{bmatrix} \Gamma^{(0:\bar{o})}_{t+1|jt} \\ X_{j,t+1|jt} \end{bmatrix} \right) \left(\begin{bmatrix} \Gamma^{(0:\bar{o})}_{t+1} \\ X_{j,t+1} \end{bmatrix} - \begin{bmatrix} \Gamma^{(0:\bar{o})}_{t+1|jt} \\ X_{j,t+1|jt} \end{bmatrix} \right)' \right]$$
$$= \begin{bmatrix} M^{e\Gamma}_{e\Gamma} & 0 \\ 0 & M^{xj}_{xj} \end{bmatrix} \hat{P} \begin{bmatrix} M^{e\Gamma}_{e\Gamma} & 0 \\ 0 & M^{xj}_{xj} \end{bmatrix}' + \begin{bmatrix} N_e \\ N_{xj} \end{bmatrix} \begin{bmatrix} N_{e\Gamma} \\ N_{xj} \end{bmatrix}'. \tag{A.89}$$

The MSE is defined as:

$$\hat{P} = \left(I - KJ^{-1}\underline{G}_1\right) P. \tag{A.90}$$

As with partial information, I can substitute out J from the Kalman gain, the variance-covariance of the one period ahead forecast error, the MSE and the state law of motion, by defining:

$$\tilde{K} = \{PG_1'\}\{G_1PG_1'\}^{-1} = KJ^{-1}. \tag{A.91}$$

[2]Recall that for symmetric matrices, $A = (J^{-1})G_1PG_1'(J^{-1})'$, it holds that $A = A'$. Further, for a non-singular matrix J^{-1} it holds that $(J')^{-1} = (J^{-1})'$.

Appendix B

Chapter 2

B.1 The model

B.1.1 The household's problem

The household's maximization problem reads

$$\max_{\{C_{jt},H_{jft},B_{j,t+1},K_{j,t+1}\}} E_{jt}\left[\sum_{t=0}^{\infty}\beta^t\left(\frac{C_{jt}^{1-\sigma}}{1-\sigma}-\frac{\psi}{1+\gamma}\int H_{jft}^{1+\gamma}\,df\right)\right] \tag{B.1}$$

s.t.

$$\frac{B_{j,t+1}}{R_t^n}+\bar{P}_{jt}C_{jt}+Q_{jt}^n Y_{jt}^{k_d}=B_{jt}+W_{jt}^n\int H_{jft}\,df+Z_{jt}^n K_{jt}+\int_0^1 \Pi_{jft}\,df+\Pi_{jt}^k \tag{B.2}$$

$$Y_{jt}^{k_d}=K_{j,t+1}-(1-\delta)K_{jt} \tag{B.3}$$

The maximization problem yields standard first order conditions,

$$0=C_{jt}^{-\sigma}-\Lambda_{jt}\bar{P}_{jt} \tag{B.4}$$

$$0=\Lambda_{jt}W_{jt}^n-\psi H_{jt}^{\gamma} \tag{B.5}$$

$$0=\frac{\Lambda_{jt}}{R_t^n}-\beta E_{jt}\left[\Lambda_{j,t+1}\right] \tag{B.6}$$

$$0=\Lambda_{jt}Q_t^n-\beta E_{jt}\left[\Lambda_{j,t+1}\left((1-\delta)Q_{j,t+1}^n+Z_{j,t+1}^n\right)\right], \tag{B.7}$$

where Λ_{jt} is the Lagrangian multiplier of the budget constraint, i.e. real marginal utility. Due to the nature of the problem, the individual budget constraint matters for the dynamics of the model and cannot be ignored. On the aggregate level bond holdings are traded in net zero supply, but the household on island j might have positive or negative bond holdings. As the bond holdings can be negative makes the linearisation of the budget constraint non-trivial. To overcome the issue, I follow Lorenzoni (2009).

Combine (2.33) with (2.35) to find the individual Euler equation for consumption of the form

$$\frac{1}{R_t^n}=\beta E_{jt}\left[\frac{C_{j,t+1}^{-\sigma}\bar{P}_{jt}}{C_{jt}^{-\sigma}\bar{P}_{j,t+1}}\right] \tag{B.8}$$

Further, collect the expressions for the profits of the wholesale sector (2.41) and the capital sector (2.50),

$$\Pi_{jft} = \left(1-\mu_{jt}\right) P_{jft} Y_{jft}$$
$$\Pi^k_{jt} = Q^n_{jt} Y^{k_s}_{jt} - \bar{P}_{jt} X_{jt}$$

as well as the real marginal cost of labour (2.39) and capital (2.40)

$$\frac{W^n_{jt}}{P_{jt}} = \alpha \mu_{jt} \frac{Y_{jft}}{N_{jft}}$$
$$\frac{Z^n_{jt}}{P_{jt}} = (1-\alpha) \mu_{jt} \frac{Y_{jft}}{K_{jft}}$$

and recall that labour market clearing requires $\int N_{jft}\, df = \int H_{jft}\, df$ and that $\mu^n_{jt} = \mu_{jt} P_{jt}$.

Plug these expressions into the budget constraint (2.31).

$$\beta E_{jt}\left[\frac{C^{\sigma}_{jt}\bar{P}_{jt}}{C^{\sigma}_{j,t+1}\bar{P}_{j,t+1}}\right] B_{j,t+1} + \bar{P}_{jt} C_{jt} + Q^n_t Y^{k_d}_{jt} =$$
$$B_{jt} + \alpha \mu^n_{jt} Y_{jt} + (1-\alpha)\mu^n_{jt} Y_{jt} + \left(P_{jt} - \mu^n_{jt}\right) Y_{jt} + Q^n_{jt} Y^{k_s}_{jt} - \bar{P}_{jt} X_{jt}$$

Collect terms and define $b_{j,t+\tau} = E_{jt}\left[\frac{B_{j,t+\tau}}{C^{\sigma}_{j,t+\tau}\bar{P}_{j,t+\tau}}\right]$.

$$\beta b_{j,t+1} - b_{jt} = -C^{1-\sigma}_{jt} + \frac{P_{jt} Y_{jt} - \bar{P}_{jt} X_{jt}}{C^{\sigma}_{jt}\bar{P}_{jt}}$$

I do not log-linearise the expressions of the bond holdings. However, the remaining variables can be log-linearised normally.

$$\beta b_{j,t+1} - b_{jt} = -C^{1-\sigma}_{*} \hat{c}_{jt} + \frac{Y_*}{C^{\sigma}_*}\left(\hat{y}_{jt} + \hat{p}_{jt} - \hat{\bar{p}}_{jt}\right) - \frac{X_*}{C^{\sigma}_*}\hat{x}_{jt} \qquad \text{(B.9)}$$

B.1.2 The firm's problem

The wholesale sector

The cost minimization problem of the intermediate goods producing firms reads:

$$\min_{\{N_{jft},K_{jft}\}} E_{jt}\left[\frac{W^n_{jt}}{P_{jt}}N_{jft}+\frac{Z^n_{jt}}{P_{jt}}K_{jft}\right], \tag{B.10}$$

$$s.t. \quad Y_{jft}=A_{jt}N^{\alpha}_{jft}K^{1-\alpha}_{jft}. \tag{B.11}$$

The corresponding Lagrangian is:

$$\min_{\{N_{jft},K_{jft}\}} \mathscr{L}_{jft}=E_{jt}\left[\frac{W^n_{jt}}{P_{jt}}N_{jft}+\frac{Z^n_{jt}}{P_{jt}}K_{jft}-\mu_{jt}\left(A_{jt}N^{\alpha}_{jft}K^{1-\alpha}_{jft}-Y_{jft}\right)\right]. \tag{B.12}$$

The first order conditions are:

$$0=E_{jt}\left[\frac{W^n_{jt}}{P_{jt}}-\alpha\mu_{jt}\frac{Y_{jft}}{N_{jft}}\right] \quad \text{and} \tag{B.13}$$

$$0=E_{jt}\left[\frac{W^n_{jt}}{P_{jt}}-(1-\alpha)\mu_{jt}\frac{Y_{jft}}{K_{jft}}\right]. \tag{B.14}$$

The first order conditions can be combined with the production function to find:

$$\mu_{jt}=\left(\frac{1}{P_{jt}A_{jt}}\right)\left(\frac{W^n_{jt}}{\alpha}\right)^{\alpha}\left(\frac{Z^n_{jt}}{1-\alpha}\right)^{1-\alpha}. \tag{B.15}$$

Conditional on the nominal marginal costs $\mu^n_{jt}=\mu_{jt}P_{jt}$, firms that can change prices in a given period choose all the same path of prices, $P_{jf,t+\tau}=P^*_{jt}$, that maximize profits. Marginal costs, and the price level on the island P_{jt} are taken as given. For the optimal price setting, I apply the nominal stochastic discount factor $M^n_{jt,t+\tau}$ as the firms are owned by the households and profits are formulated in nominal terms.

$$\max_{\{P_{jf,t+\tau}\}} E_{jt}\left[\sum_{\tau=0}^{\infty}\theta^{\tau}M^n_{jt,t+\tau}\left(P_{jf,t+\tau}-\mu^n_{jt}\right)Y_{jf,t+\tau}\right] \tag{B.16}$$

$$s.t. \quad Y_{jf,t+\tau}=\int_{\Theta_{j,t+\tau}}\left(\frac{P_{jf,t+\tau}}{\bar{P}_{i,t+\tau}}\right)^{-\varepsilon}Y_{i,t+\tau}\,di \tag{B.17}$$

I substitute the demand function into the objective function to find:

$$\max_{\{P_{jf,t+\tau}\}} E_{jt}\left[\sum_{\tau=0}^{\infty}\theta^{\tau}M^{n}_{jt,t+\tau}\left(P^{1-\varepsilon}_{jf,t+\tau}-\mu^{n}_{jt}P^{-\varepsilon}_{jf,t+\tau}\right)\int_{\Theta_{j,t+\tau}}\bar{P}^{\varepsilon}_{i,t+\tau}Y_{i,t+\tau}\,di\right]. \tag{B.18}$$

The first order condition yields the non-linear version of the new Keynesian Phillips curve:

$$0=E_{jt}\left[\sum_{\tau=0}^{\infty}\theta^{\tau}M^{n}_{jt,t+\tau}\left((1-\varepsilon)P^{*-\varepsilon}_{jt}+\varepsilon\mu^{n}_{jt}P^{*-(1+\varepsilon)}_{jt}\right)\int_{\Theta_{j,t+\tau}}\bar{P}^{\varepsilon}_{i,t+\tau}Y_{i,t+\tau}\,di\right]. \tag{B.19}$$

At this point, I substitute the stochastic discount factor, realize that P^{*}_{jt} and P_{jt} are independent of τ and hence can be pulled out of the sum, and rearrange, to find

$$P^{*}_{jt}=\frac{\varepsilon}{\varepsilon-1}\frac{E_{jt}\left[\sum_{\tau=0}^{\infty}(\beta\theta)^{\tau}\frac{\lambda^{n}_{j,t+\tau}}{\lambda^{n}_{jt}}\frac{1}{\bar{P}_{j,t+\tau}}\mu^{n}_{jt}\int_{\Theta_{j,t+\tau}}\bar{P}^{\varepsilon}_{i,t+\tau}Y_{i,t+\tau}\,di\right]}{E_{jt}\left[\sum_{\tau=0}^{\infty}(\beta\theta)^{\tau}\frac{\lambda^{n}_{j,t+\tau}}{\lambda^{n}_{jt}}\frac{1}{\bar{P}_{j,t+\tau}}\int_{\Theta_{j,t+\tau}}\bar{P}^{\varepsilon}_{i,t+\tau}Y_{i,t+\tau}\,di\right]}. \tag{B.20}$$

By assumption, $\int_{\Theta_{j,t+\tau}}P^{\varepsilon}_{it}Y_{it}\,di=\varepsilon\bar{p}_{t}+y_{t}+\varepsilon^{2}_{jt}$. Then, the log-linear approximation of (B.20) reads:

$$\begin{aligned}p^{*}_{jt}&=E_{jt}\left[\sum_{\tau=0}^{\infty}(\beta\theta)^{\tau}\left(\lambda^{n}_{j,t+\tau}-\lambda^{n}_{jt}-\bar{p}_{j,t+\tau}+\mu^{n}_{jt}+\varepsilon\bar{p}_{i,t+\tau}+y_{i,t+\tau}+\varepsilon^{2}_{jt}\right)\right]\\&\quad-E_{jt}\left[\sum_{\tau=0}^{\infty}(\beta\theta)^{\tau}\left(\lambda^{n}_{j,t+\tau}-\lambda^{n}_{jt}-\bar{p}_{j,t+\tau}+\varepsilon\bar{p}_{i,t+\tau}+y_{i,t+\tau}+\varepsilon^{2}_{jt}\right)\right]\\&=E_{jt}\left[\sum_{\tau=0}^{\infty}(\beta\theta)^{\tau}\mu^{n}_{jt}\right].\end{aligned} \tag{B.21}$$

Write the infinite sum as a Bellman equation:

$$p^{*}_{jt}=(1-\beta\theta)\mu^{n}_{jt}+\beta\theta E_{jt}\left[p^{*}_{j,t+1}\right]. \tag{B.22}$$

Next, rewrite the log-linear version of the price state law of motion to become $p_{jt}=\frac{1}{1-\theta}p^{*}_{jt}+\frac{\theta}{1-\theta}p_{j,t-1}$, plug it into (B.22) and substitute $\mu^{n}_{jt}=\mu_{jt}+p_{jt}$. After rearranging, I find the island specific linear new Keynesian Phillips curve:

$$\pi_{jt}=\frac{(1-\beta\theta)(1-\theta)}{\theta}\mu_{jt}+\beta E_{jt}\left[\pi_{j,t+1}\right]. \tag{B.23}$$

Final goods sector

The households choose the consumption goods that minimize the costs of the consumption bundle subject to the Dixit-Stiglitz type of aggregator of the consumption bundle.

$$\min_{\{Y_{jimt}\}} \int_{\Xi_{jt}} \int_0^1 P_{imt} Y_{jimt} \, di \, dm$$
$$\text{s.t.} \quad Y_{jt} = \left(\int_{\Xi_{jt}} \int_0^1 Y_{jimt}^{\frac{\varepsilon-1}{\varepsilon}} \, dm \, di \right)^{\frac{\varepsilon}{\varepsilon-1}} \tag{B.24}$$

The Lagrangian reads:

$$\min_{\{Y_{jimt}\}} \mathscr{L}_{jt} = \int_{\Xi_{jt}} \int_0^1 P_{imt} Y_{jimt} \, dm \, di - \bar{P}_{jt} \left(Y_{jt} - \left(\int_{\Xi_{jt}} \int_0^1 Y_{jimt}^{\frac{\varepsilon-1}{\varepsilon}} \, dm \, di \right)^{\frac{\varepsilon}{\varepsilon-1}} \right). \tag{B.25}$$

The first order condition is:

$$\frac{\partial \mathscr{L}_{jt}}{\partial Y_{jimt}} = P_{imt} - \bar{P}_{jt} \left(Y_{jimt}^{-\frac{1}{\varepsilon}} \left(\int_{\Xi_{jt}} \int_0^1 Y_{jimt}^{\frac{\varepsilon-1}{\varepsilon}} \, dm \, di \right)^{-\frac{1}{1-\varepsilon}} \right) = 0, \tag{B.26}$$

which I can reformulate to find:

$$Y_{jimt} = \left(\frac{P_{imt}}{\bar{P}_{jt}} \right)^{-\varepsilon} Y_{jt}. \tag{B.27}$$

I determine the price index by plugging the demand function (B.27) in the consumption aggregator:

$$Y_{jt} = \left(\int_{\Xi_{jt}} \int_0^1 Y_{jimt}^{\frac{\varepsilon-1}{\varepsilon}} \, dm \, di \right)^{\frac{\varepsilon}{\varepsilon-1}} \tag{B.28}$$
$$= \left(\int_{\Xi_{jt}} \int_0^1 \left(\left(\frac{P_{imt}}{\bar{P}_{jt}} \right)^{-\varepsilon} Y_{jt} \right)^{\frac{\varepsilon-1}{\varepsilon}} \, dm \, di \right)^{\frac{\varepsilon}{\varepsilon-1}}, \tag{B.29}$$

which can be simplified to:

$$\bar{P}_{jt}^{1-\varepsilon} = \int_{\Xi_{jt}} \int_0^1 P_{imt}^{1-\varepsilon} \, dm \, di. \tag{B.30}$$

Capital goods sector

The capital goods firm on island j chooses X_{jt} to maximize its profits

$$\max_{\{X_{jt}\}} \Pi_{jt}^{k} = E_{jt}\left[Q_{jt}^{n}\phi\left(\frac{X_{jt}}{K_{jt}}\right)K_{jt} - \bar{P}_{jt}X_{jt}\right] \tag{B.31}$$

The first order condition yields the equation for the nominal price of capital,

$$Q_{jt}^{n} = \frac{\bar{P}_{jt}}{\phi'\left(\frac{X_{jt}}{K_{jt}}\right)}. \tag{B.32}$$

B.1.3 Equilibrium dynamics

The individual equilibrium dynamics of

$$\begin{aligned}\Upsilon_{jt} = \{&B_{jt}, K_{jt}, C_{jt}, \Lambda_{jt}^{n}, H_{jt}, \bar{P}_{jt}, P_{jt}, W_{jt}^{n}, Z_{jt}^{n}, N_{jt},\\ &\mu_{jt}, X_{jt}, Q_{jt}^{n}, P_{jt}, A_{jt}, \omega_{jt}, Y_{jt}^{ks}, Y_{jt}^{kd}, Y_{jt}, \Pi_{jt}, \Pi_{jt}^{k}\}\end{aligned}$$

are fully described by by the equations (2.31), (2.32), (2.33), (2.34), (2.35), (2.36), (2.38), (2.39), (2.40), (2.41), (2.42), (2.43), (2.49), (2.50), (2.51), (2.54), as well as the functional form of the capital adjustment function, the definition of composite productivity, a_{jt}, idiosyncratic productivity ω_{jt}, the definitions of the price indices and nominal marginal costs, and the market clearing condition for labour and capital. The model is completely specified by the aggregate equations for $\{R_t^n, a_t\}$ which are (2.52) and the state law of motion of aggregate productivity.

$$\begin{aligned}0 &= \frac{B_{j,t+1}}{R_t^n} + \bar{P}_{jt}C_{jt} + Q_{jt}^{n}Y_{jt}^{k_d} - B_{jt} - W_{jt}^{n}\int H_{jft}\,df \qquad (B.33)\\ &\quad - Z_{jt}^{n}K_{jt} - \int_0^1 \Pi_{jft}\,df - \Pi_{jt}^{k}\\ 0 &= Y_{jt}^{k_d} - K_{j,t+1} + (1-\delta)K_{jt} \qquad (B.34)\\ 0 &= C_{jt}^{-\sigma} - \Lambda_{jt}^{n}\bar{P}_{jt} \qquad (B.35)\\ 0 &= \Lambda_{jt}W_{jt}^{n} - \psi H_{jt}^{\gamma} \qquad (B.36)\\ 0 &= \frac{\Lambda_{jt}}{R_t^n} - \beta E_{jt}\left[\Lambda_{j,t+1}\right] \qquad (B.37)\end{aligned}$$

$$0 = \Lambda_{jt} Q^n_{jt} - \beta E_{jt}\left[\Lambda_{j,t+1}\left((1-\delta)Q^n_{j,t+1} + Z^n_{j,t+1}\right)\right] \tag{B.38}$$

$$0 = Y_{jt} - A_{jt} N^{\alpha}_{jt} K^{1-\alpha}_{jt} \tag{B.39}$$

$$0 = \frac{W^n_{jt}}{P_{jt}} - \alpha\mu_{jt}\frac{Y_{jt}}{N_{jt}} \tag{B.40}$$

$$0 = \frac{Z^n_{jt}}{P_{jt}} - (1-\alpha)\mu_{jt}\frac{Y_{jt}}{K_{jt}} \tag{B.41}$$

$$0 = \Pi_{jt} - \left(P_{j,t+\tau} - \mu^n_{jt}\right) Y_{j,t+\tau} \tag{B.42}$$

$$0 = Y_{jt} - \int_{\Theta_{jt}} \left(\frac{P_{jt}}{\bar{P}_{it}}\right)^{-\varepsilon} Y_{it}\, di \tag{B.43}$$

$$0 = E_{jt}\left[\sum_{\tau=0}^{\infty}(\theta\beta)^{\tau}\frac{\Lambda_{j,t+\tau}}{\Lambda_{jt}}\left((1-\varepsilon)P^{*-\varepsilon}_{jt} + \varepsilon\mu^n_{jt}P^{*-(1+\varepsilon)}_{jt}\right)\int_{\Theta_{j,t+\tau}} \bar{P}^{\varepsilon}_{i,t+\tau} Y_{i,t+\tau}\, di\right] \tag{B.44}$$

$$0 = Y^{k_s}_{jt} - \phi\left(\frac{X_{jt}}{K_{jt}}\right) K_{jt} \tag{B.45}$$

$$0 = \Pi^k_{jt} - Q^n_t \phi\left(\frac{X_{jt}}{K_{jt}}\right) K_{jt} + \bar{P}_{jt} X_{jt} \tag{B.46}$$

$$0 = Q^n_{jt}\phi'\left(\frac{X_{jt}}{K_{jt}}\right) - \bar{P}_{jt} \tag{B.47}$$

$$0 = \phi_{jt} - \left(\frac{\omega_1}{1-\frac{1}{\eta}}\left(\frac{X_{jt}}{K_{jt}}\right)^{1-\frac{1}{\eta}} + \omega_2\right) \tag{B.48}$$

$$0 = \phi'_{jt} - \omega_1\left(\frac{X_{jt}}{K_{jt}}\right)^{-\frac{1}{\eta}} \tag{B.49}$$

$$0 = a_{jt} - a_t - \omega_{jt} \tag{B.50}$$

$$0 = \omega_{jt} - \rho_{\omega}\omega_{j,t-1} - \varepsilon_{jt} \tag{B.51}$$

$$0 = \mu^n_{jt} - \mu_{jt} P_{jt} \tag{B.52}$$

$$0 = P_{jt} - \left[(1-\theta)P^{*1-\varepsilon}_{jt} + \theta P^{1-\varepsilon}_{j,t-1}\right]^{1-\varepsilon} \tag{B.53}$$

$$0 = \bar{P}_{jt} - \left(\int_{\Xi_{jt}} P^{1-\varepsilon}_{it}\, di\right)^{\frac{1}{1-\varepsilon}} \tag{B.54}$$

$$0 = N_{jt} - H_{jt} \tag{B.55}$$

$$0 = Y^{k_s}_{jt} - Y^{k_d}_{jt} \tag{B.56}$$

$$0 = r^n_t - (1-\rho_r)r^n_* - \rho_r r^n_{t-1} - \varphi\tilde{\pi}_t \tag{B.57}$$

$$0 = a_t - \rho_a a_{t-1} - \nu_t \tag{B.58}$$

B.1.4 Steady state

The steady state values of the system can then be derived conditional on the model's deep parameters $\Theta = \{\sigma, \alpha, \beta, \delta, \gamma, \theta, \varepsilon, \rho_a, \varphi, \rho_r, \psi, \eta, \omega_1, \omega_2\}$. Idiosyncratic and aggregate productivity as well as the price level are normalized to one in steady state and I target labour supply to be $H = \bar{H}$. Moreover, recall that $\phi\left(\frac{X_*}{K_*}\right) = \delta$ and $\phi'\left(\frac{X_*}{K_*}\right) = 1$ by assumption, which implies that $\frac{X_*}{K_*} = \delta$.

To compute the steady state values, first, drop the time subscripts. From (B.37), (B.44) and (B.47) immediately follows that:

$$R_* = \frac{1}{\beta}, \tag{B.59}$$

$$\mu_* = \frac{\varepsilon - 1}{\varepsilon} \quad \text{and} \tag{B.60}$$

$$Q_* = 1. \tag{B.61}$$

Next, I use (B.37) in (B.38):

$$Z_* = \frac{1}{\beta} - (1 - \delta), \tag{B.62}$$

which I combine with (B.39) and (B.43) to find the steady state capital stock,

$$K_* = \left(\frac{Z_*}{(1-\alpha)\mu_* A_* N_*^{\alpha}}\right)^{-\frac{1}{\alpha}}. \tag{B.63}$$

The steady state capital stock directly identifies production, (B.39), and in a second step, capital demand, (B.35), wages, (B.40) and the profits of the wholesale sector, (B.42).

$$Y_* = A_* N_*^{\alpha} K_*^{1-\alpha} \tag{B.64}$$

$$X_* = \delta K_* \tag{B.65}$$

$$W_* = \alpha \mu_* \frac{Y_*}{N_*} \tag{B.66}$$

$$\Pi_* = (1 - \mu_*) Y_* \tag{B.67}$$

$$\tag{B.68}$$

It remains to solve for consumption and the weight, ψ, that determines the impact of labour disutility to the household. To do so, realize that $X_* = Y_*^{k_s} = Y_*^{k_d}$ and $H_* = N_* = \bar{H}$.

This implies, as expected, that the capital producers generate zero profits in steady state. Inserting all relevant steady state variables into (B.33), I find :

$$C_* = Y_* - X_*. \tag{B.69}$$

The steady state value for consumption defines Λ_* via (B.35), which can be used to define ψ:

$$\psi = \Lambda_* W_* H_*^{-\gamma}. \tag{B.70}$$

B.1.5 Log-linearisation around the steady state

Apply log-linearisation around the steady state to the equilibrium dynamics where $\hat{\Upsilon}_{jt} = \ln\left(\frac{\Upsilon_{jt}}{\Upsilon_*}\right)$ denotes log deviations from steady state and lower case letters denote logs with the exception of $b_{j,t+\tau}$. Furthermore, it is assumed that Θ_{jt} draws a random idiosyncratic sample around the aggregate values. The draws follow the normal distribution $\varepsilon^1_{jt} \sim N(0, \sigma_{\varepsilon_{j1}})$. Ξ_{jt} also draws a random sample around the aggregate values, with error terms being $\varepsilon^2_{jt} \sim N(0, \sigma_{\varepsilon_{j2}})$.

$$0 = \beta b_{j,t+1} - b_{jt} + C_*^{1-\sigma}\hat{c}_{jt} - \frac{Y_*}{C_*^{\sigma}}\left(\hat{y}_{jt} + \hat{p}_{jt} - \hat{\hat{p}}_{jt}\right) + \frac{X_*}{C_*^{\sigma}}\hat{x}_{jt} \tag{B.71}$$

$$0 = \delta\hat{y}^{k_d}_{jt} - \hat{k}_{j,t+1} + (1-\delta)\hat{k}_{jt} \tag{B.72}$$

$$0 = -\sigma\hat{c}_{jt} - \hat{\lambda}_{jt} - \hat{\hat{p}}_{jt} \tag{B.73}$$

$$0 = \hat{\lambda}_{jt} + \hat{w}^n_{jt} - \gamma\hat{h}_{jt} \tag{B.74}$$

$$0 = \hat{\lambda}_{jt} - r^n_t - E_{jt}\left[\hat{\lambda}_{j,t+1}\right] \tag{B.75}$$

$$0 = \hat{\lambda}^n_{jt} + \hat{q}^n_t - E_{jt}\left[\hat{\lambda}^n_{j,t+1} + \beta(1-\delta)\hat{q}^n_{t+1} + \beta Z_*\hat{z}^n_{j,t+1}\right] \tag{B.76}$$

$$0 = \hat{y}_{jt} - a_{jt} - \alpha\hat{n}_{jt} - (1-\alpha)\hat{k}_{jt} \tag{B.77}$$

$$0 = \hat{w}^n_{jt} - \hat{p}_{jt} - \hat{\mu}_{jt} - \hat{y}_{jt} + \hat{n}_{jt} \tag{B.78}$$

$$0 = \hat{z}^n_{jt} - \hat{p}_{jt} - \hat{\mu}_{jt} - \hat{y}_{jt} + \hat{k}_{jt} \tag{B.79}$$

$$0 = \hat{y}_{jt} - \hat{d}_{jt} + \varepsilon \hat{p}_{jt} \tag{B.80}$$
$$0 = d_{jt} - y_t - \varepsilon p_t - \varepsilon^1_{jt} \tag{B.81}$$
$$0 = \pi_{jt} - \kappa \mu_{jt} - \beta E_{jt}\left[\pi_{j,t+1}\right] \tag{B.82}$$
$$0 = \pi_{jt} - p_{jt} + p_{j,t-1} \tag{B.83}$$
$$0 = \hat{y}^{k_s}_{jt} - \hat{\phi}_{jt} - \hat{k}_{jt} \tag{B.84}$$
$$0 = \hat{q}^n_{jt} + \hat{\phi}'_{jt} - \hat{\bar{p}}_{jt} \tag{B.85}$$
$$0 = \hat{\phi}_{jt} - \hat{x}_{jt} + \hat{k}_{jt} \tag{B.86}$$
$$0 = \hat{\phi}'_{jt} + \frac{1}{\eta}\left(\hat{x}_{jt} - \hat{k}_{jt}\right) \tag{B.87}$$
$$0 = \hat{\mu}^n_{jt} - \hat{\mu}_{jt} - \hat{p}_{jt} \tag{B.88}$$
$$0 = \hat{n}_{jt} - \hat{h}_{jt} \tag{B.89}$$
$$0 = \hat{\bar{p}}_{jt} - \hat{p}_t - \varepsilon^2_{jt} \tag{B.90}$$
$$0 = \hat{y}^{k_s}_{jt} - \hat{y}^{k_d}_{jt} \tag{B.91}$$
$$0 = a_{jt} - a_t - \varepsilon_{jt} \tag{B.92}$$
$$0 = s_t - a_t - \varepsilon_t \tag{B.93}$$
$$0 = r^n_t - (1-\rho_r) r^n_* - \rho_r r^n_{t-1} - \varphi \tilde{\pi}_t \tag{B.94}$$
$$0 = a_t - \rho_a a_{t-1} - \nu_t \tag{B.95}$$

B.1.6 State space representation

The idiosyncratic signals are $\underline{\Upsilon}_{jt} = \left[a_{jt}\ d_{jt}\ \bar{p}_{jt}\right]'$. The aggregate signals are $\underline{\Upsilon}_t = \left[s_t\ r^n_t\ \tilde{\pi}_t\right]'$. Given these signals, the equilibrium conditions can be written in the state space system as presented in Section 2.3.

The individual endogenous predetermined state variables are $X^c_{j,t+1} = \left[k_{j,t+1}\, b_{j,t+1}\right]'$. The individual endogenous contemporaneous state variable is $X^n_{jt} = \left[p_{jt}\right]$. The individual forward looking variables are $F_{jt} = \left[c^n_{jt}\ \pi_{jt}\right]'$.

Contemporaneous variables

Write contemporaneous variables that appear in state variables or forward looking variables in terms of signals, individual endogenous state variables, individual forward looking variables.

First, combine (B.85) and (B.87).

$$\hat{x}_{jt} = \hat{k}_{jt} + \eta \hat{q}^{n}_{jt} - \eta \hat{\bar{p}}_{jt} \tag{B.96}$$

Second, combine (B.77) and (B.80).

$$\hat{n}_{jt} = \frac{1}{\alpha}\left(\hat{d}_{jt} - \varepsilon \hat{p}_{jt} - a_{jt} - (1-\alpha)\hat{k}_{jt}\right) \tag{B.97}$$

Third, combine (B.73), (B.74), (B.78), (B.80) and (B.97).

$$\begin{aligned} \hat{\mu}_{jt} = {} & \frac{1+\gamma-\alpha}{\alpha} d_{jt} - \frac{1+\gamma}{\alpha} a_{jt} - \frac{(1+\gamma)(1-\alpha)}{\alpha} k_{jt} \\ & - \left(1 - \varepsilon + \frac{(1+\gamma)\varepsilon}{\alpha}\right) p_{jt} + \sigma c_{jt} + \bar{p}_{jt} \end{aligned} \tag{B.98}$$

Finally, combine (B.73) and (B.98).

$$\hat{z}^{n}_{jt} = \frac{1+\gamma}{\alpha}\hat{d}_{jt} - \frac{1+\gamma}{\alpha} a_{jt} - \frac{1+(1-\alpha)\gamma}{\alpha}\hat{k}_{jt} - \frac{(1+\gamma)\varepsilon}{\alpha}\hat{p}_{jt} + \sigma \hat{c}_{jt} + \hat{\bar{p}}_{jt} \tag{B.99}$$

The remaining contemporaneous variables do not need to be transformed to formulate state and forward looking variables in terms of signals, state and forward looking variables in the model at hand. The aggregate corresponding contemporaneous variables can be found by integrating over the idiosyncratic contemporaneous variables. All contemporaneous variables can be cast in the form (2.5).

$$\begin{bmatrix} \Upsilon_t \\ \Upsilon_{jt} \end{bmatrix} = \begin{bmatrix} B^{\Gamma 1}_{\Upsilon} & 0 \\ B^{\Gamma 1}_{\Upsilon j} & B^{\Gamma j 1}_{\Upsilon j} \end{bmatrix} \begin{bmatrix} \Gamma_t \\ \Gamma_{jt} \end{bmatrix} + \begin{bmatrix} B^{\Gamma 0}_{\Upsilon} & 0 \\ B^{\Gamma 0}_{\Upsilon j} & B^{\Gamma j 0}_{\Upsilon j} \end{bmatrix} \begin{bmatrix} \Gamma_{t-1} \\ \Gamma_{j,t-1} \end{bmatrix} + \begin{bmatrix} B^{f}_{\Upsilon j} & 0 \\ B^{f}_{\Upsilon j} & B^{fj}_{\Upsilon j} \end{bmatrix} \begin{bmatrix} F_t \\ F_{jt} \end{bmatrix} + \begin{bmatrix} B^{s}_{\Upsilon} & 0 \\ B^{s}_{\Upsilon j} & B^{sj}_{\Upsilon j} \end{bmatrix} \begin{bmatrix} s_t \\ s_{jt} \end{bmatrix}. \tag{B.100}$$

State variables

Individual state variables need to be written in terms of individual state and forward looking variables as well as signals. To do so, I make use of the derivations in Section B.1.6. Inserting and summarizing yields the following expressions.

$$\beta b_{j,t+1} - \frac{(1-\varepsilon)Y_*}{C_*^{\sigma}}\hat{p}_{jt} = b_{jt} - C_*^{1-\sigma}\hat{c}_{jt} + \frac{Y_*}{C_*^{\sigma}}\hat{d}_{jt} - \frac{Y_* - X_*\varepsilon}{C_*^{\sigma}}\hat{\hat{p}}_{jt} - \frac{X_*}{C_*^{\sigma}}\hat{k}_{jt} - \frac{X_*\eta}{C_*^{\sigma}}\hat{q}_{jt} \tag{B.101}$$

$$\hat{k}_{j,t+1} = \hat{k}_{jt} + \delta\eta\hat{q}_{jt} - \delta\eta\hat{\hat{p}}_{jt} \tag{B.102}$$

$$\hat{p}_{jt} = \hat{p}_{j,t-1} + \pi_{jt} \tag{B.103}$$

$$\omega_{jt} = \rho_{\omega}\omega_{j,t-1} + \varepsilon_{jt} \tag{B.104}$$

Aggregate state variables are mostly defined by the integral over the individual state variables.

$$\hat{k}_{t+1} = \hat{k}_t + \delta\eta\hat{q}_t - \delta\eta\hat{\hat{p}}_t \tag{B.105}$$

$$\hat{p}_t = \hat{p}_{t-1} + \pi_t \tag{B.106}$$

Exceptions are aggregate decision rules such as the Taylor rule:

$$r_t^n = (1-\rho_r)r_*^n + \rho_r r_{t-1}^n + \varphi\tilde{\pi}_t. \tag{B.107}$$

These equations can be cast directly in form (2.4):

$$\begin{bmatrix} A_{\Gamma}^{\Gamma 1} & 0 \\ 0 & A_{\Gamma j}^{\Gamma j 1} \end{bmatrix} \begin{bmatrix} \Gamma_t \\ \Gamma_{jt} \end{bmatrix} = \begin{bmatrix} A_{\Gamma}^{\Gamma 0} & 0 \\ 0 & A_{\Gamma j}^{\Gamma j 0} \end{bmatrix} \begin{bmatrix} \Gamma_{t-1} \\ \Gamma_{j,t-1} \end{bmatrix} + \begin{bmatrix} A_{\Gamma j}^{f} & 0 \\ 0 & A_{\Gamma j}^{fj} \end{bmatrix} \begin{bmatrix} F_t \\ F_{jt} \end{bmatrix} \\ + \begin{bmatrix} A_{\Gamma}^{\Upsilon} & 0 \\ A_{\Gamma j}^{\Upsilon} & A_{\Gamma j}^{\Upsilon j} \end{bmatrix} \begin{bmatrix} \underline{\Upsilon}_t \\ \underline{\Upsilon}_{jt} \end{bmatrix} + \begin{bmatrix} A_{\Gamma}^{s} & 0 \\ 0 & A_{\Gamma j}^{sj} \end{bmatrix} \begin{bmatrix} s_t \\ s_{jt} \end{bmatrix}. \tag{B.108}$$

Euler equation

Individual forward looking variables need to be written in terms of individual state and forward looking variables as well as signals. To do so, I make use of the derivations in

Section B.1.6. Inserting and summarizing yields the following expressions.

$$0 = r_t^n + \sigma \hat{c}_{jt} + \hat{\tilde{p}}_{jt} - \sigma E_{jt}\left[\hat{c}_{j,t+1}\right] - E_{jt}\left[\hat{\tilde{p}}_{j,t+1}\right] \tag{B.109}$$

$$\begin{aligned} 0 = &- \hat{q}_{jt} + \sigma \hat{c}_{jt} + \hat{\tilde{p}}_{jt} + \beta(1-\delta) E_{jt}\left[\hat{q}^n_{j,t+1}\right] - (1-\beta Z_*)\sigma\sigma E_{jt}\left[\hat{c}_{j,t+1}\right] \\ &- (1-\beta Z_*) E_{jt}\left[\hat{\tilde{p}}_{j,t+1}\right] + \beta Z_* \frac{1+\gamma}{\alpha} E_{jt}\left[\hat{d}_{j,t+1}\right] - \beta Z_* \frac{1+\gamma}{\alpha} E_{jt}\left[a_{j,t+1}\right] \\ &- \beta Z_* \frac{(1+\gamma)\varepsilon}{\alpha} E_{jt}\left[\hat{p}_{j,t+1}\right] - \beta Z_* \frac{1+(1-\alpha)\gamma}{\alpha} E_{jt}\left[\hat{k}_{j,t+1}\right] \end{aligned} \tag{B.110}$$

$$\begin{aligned} 0 = &\pi_{jt} - \kappa\sigma\hat{c}_{jt} - \kappa\hat{\tilde{p}}_{jt} - \kappa\frac{1+\gamma-\alpha}{\alpha}\hat{d}_{jt} + \kappa\left(1-\varepsilon+\frac{(1+\gamma)\varepsilon}{\alpha}\right)\hat{p}_{jt} + \kappa\frac{1+\gamma}{\alpha}a_{jt} \\ &+ \kappa\frac{(1+\gamma)(1-\alpha)}{\alpha}\hat{k}_{jt} - \beta E_{jt}\left[\pi_{j,t+1}\right] \end{aligned} \tag{B.111}$$

These equations can be cast directly in form (2.6). Aggregate state variables that are theoretically revealed like the nominal interest rate are written inside the expectation operator. In the Matlab code these variables are specifically indicated and moved out of it. Without this procedure the attribution of variables would become unclear. Therefore, I decided to do as stated.

$$\begin{aligned} &\begin{bmatrix} 0 & C^{fj}_{fj0} \end{bmatrix} \begin{bmatrix} F_t \\ F_{jt} \end{bmatrix} + \begin{bmatrix} 0 & C^{\Gamma j1}_{fj0} \end{bmatrix} \begin{bmatrix} \Gamma_t \\ \Gamma_{jt} \end{bmatrix} + \begin{bmatrix} 0 & C^{\Gamma j0}_{fj0} \end{bmatrix} \begin{bmatrix} \Gamma_{t-1} \\ \Gamma_{j,t-1} \end{bmatrix} + C^{\Upsilon}_{fj0} \begin{bmatrix} \underline{\Upsilon}_t \\ \underline{\Upsilon}_{jt} \end{bmatrix} + \\ &E_{jt}\left\{ C^{F}_{fj1} \begin{bmatrix} F_{t+1} \\ F_{j,t+1} \end{bmatrix} + C^{\Upsilon}_{fj1} \begin{bmatrix} \underline{\Upsilon}_{t+1} \\ \underline{\Upsilon}_{j,t+1} \end{bmatrix} + C^{\Gamma}_{fj1} \begin{bmatrix} \Gamma_{t+1} \\ \Gamma_{j,t+1} \end{bmatrix} + \begin{bmatrix} C^{\Gamma 1}_{fj0} & 0 \end{bmatrix} \begin{bmatrix} \Gamma_t \\ \Gamma_{jt} \end{bmatrix} + \begin{bmatrix} C^{\Gamma 0}_{fj0} & 0 \end{bmatrix} \begin{bmatrix} \Gamma_{t-1} \\ \Gamma_{j,t-1} \end{bmatrix} \right\} = 0. \end{aligned} \tag{B.112}$$

B.2 Proofs

B.2.1 Market clearing

Proposition 1. *For all markets to clear it is necessary that agents take their choices based on variables that are part of their information set only.*

Proof. Define the budget constraint as the sum of income and sum of expenditures. Understand that any term of the budget constraint consists of a factor of price and quantity. On the markets in which the agent is price taker, he chooses his individual quantity or the factors that determine the quantity. Moreover, individual quantities may depend on exogenous processes. Next, realise that the price is the outcome of trading decisions of the agents participating in the market. Hence, the agents that participate in the market knows the price of that market. This is true for integrated markets over all islands, markets on a subsection of islands and single islands. When agents have market power they choose the price additional to the quantity. Concluding, the individual quantities and the prices of the markets in which they interact are known to the agents.

Write the individual budget constraints as a function f of variables that are part of the information set. Distinguish signals between endogenous (prices) and exogenous signals:

$$f(X_{jt}^{c}, X_{jt}^{n}, F_{jt}, \{\underline{\Upsilon}_{t}, \underline{\Upsilon}_{jt}\}, \{\underline{\Upsilon}_{t}^{ex}, \underline{\Upsilon}_{jt}^{ex}\}) = 0. \tag{B.113}$$

Then, I find market clearing by aggregating:

$$f(X_{t}^{c}, X_{t}^{n}, F_{t}, \{X_{t}^{c}, X_{t}^{n}, F_{t}\}, \{X_{t}\}) = 0. \tag{B.114}$$

Assume instead that agents take their choices based on the expectation of the components of exogenous signals and realise that by definition exogenous variables cannot be affected by the agents expectation or choice:

$$f(X_{jt}^{c}, X_{jt}^{n}, F_{jt}, \{\underline{\Upsilon}_{t}, \underline{\Upsilon}_{jt}\}, \{X_{t|jt}, X_{jt|jt}\}), \tag{B.115}$$

which reads when aggregated:

$$f(X_{t}^{c}, X_{t}^{n}, F_{t}, \{X_{t}^{c}, X_{t}^{n}, F_{t}\}, \{X_{t|t}\}). \tag{B.116}$$

All but one variable are the same. Consequently, under the given assumption market clearing is satisfied if and only if $X_{t|t} = X_t$. □

B.2.2 Invariant parameter matrices

Proposition 2. *The solution to the quadratic equation system that identifies* ξ_{fj}^{nj} *and* ξ_{fj}^{cj} *coincides with the parameters of the full information solution.*

Proof. Individual endogenous state variables are part of the information set. Hence, they will not be part of ξ_{fj}^{Z}.Through the state law of motion of individual endogenous state variables all the weight is passed through to ξ_{fj}^{nj} and ξ_{fj}^{cj}, the same way as under full information.

This might not be true for idiosyncratic exogenous state variables. However, there is no problem in identifying their root. □

B.3 Solution algorithm

The guess for forward looking variables reads:

$$\begin{bmatrix} F_t \\ F_{jt} \end{bmatrix} = \xi^{\Gamma} \begin{bmatrix} \Gamma_{t-1} \\ \Gamma_{j,t-1} \end{bmatrix} + \xi^{\Upsilon} \begin{bmatrix} \underline{\Upsilon}_t \\ \underline{\Upsilon}_{jt} \end{bmatrix} + \xi^{Z} \begin{bmatrix} Z_t \\ Z_{jt} \end{bmatrix} + \xi^{S} \begin{bmatrix} s_t \\ s_{jt} \end{bmatrix}. \tag{B.117}$$

Further, the state law of motion reads:

$$\begin{aligned} \begin{bmatrix} A_{\Gamma}^{\Gamma 1} & 0 \\ 0 & A_{\Gamma j}^{\Gamma j1} \end{bmatrix} \begin{bmatrix} \Gamma_t \\ \Gamma_{jt} \end{bmatrix} = \begin{bmatrix} A_{\Gamma}^{\Gamma 0} & 0 \\ 0 & A_{\Gamma j}^{\Gamma j0} \end{bmatrix} \begin{bmatrix} \Gamma_{t-1} \\ \Gamma_{j,t-1} \end{bmatrix} + \begin{bmatrix} A_{\Gamma j}^{f} & 0 \\ 0 & A_{\Gamma j}^{fj} \end{bmatrix} \begin{bmatrix} F_t \\ F_{jt} \end{bmatrix} \\ + \begin{bmatrix} A_{\Gamma}^{\Upsilon} & 0 \\ A_{\Gamma j}^{\Upsilon} & A_{\Gamma j}^{\Upsilon j} \end{bmatrix} \begin{bmatrix} \underline{\Upsilon}_t \\ \underline{\Upsilon}_{jt} \end{bmatrix} + \begin{bmatrix} A_{\Gamma}^{s} & 0 \\ 0 & A_{\Gamma j}^{sj} \end{bmatrix} \begin{bmatrix} s_t \\ s_{jt} \end{bmatrix}. \end{aligned} \tag{B.118}$$

In addition, I cast contemporaneous variables in the form:

$$\begin{aligned} \begin{bmatrix} B_{\Upsilon}^{\Upsilon} & 0 \\ B_{\Upsilon j}^{\Upsilon} & B_{\Upsilon j}^{\Upsilon j} \end{bmatrix} \begin{bmatrix} \Upsilon_t \\ \Upsilon_{jt} \end{bmatrix} = \begin{bmatrix} B_{\Upsilon}^{\Gamma 1} & 0 \\ B_{\Upsilon j}^{\Gamma 1} & B_{\Upsilon j}^{\Gamma j1} \end{bmatrix} \begin{bmatrix} \Gamma_t \\ \Gamma_{jt} \end{bmatrix} + \begin{bmatrix} B_{\Upsilon}^{\Gamma 0} & 0 \\ B_{\Upsilon j}^{\Gamma 0} & B_{\Upsilon j}^{\Gamma j0} \end{bmatrix} \begin{bmatrix} \Gamma_{t-1} \\ \Gamma_{j,t-1} \end{bmatrix} + \begin{bmatrix} B_{\Upsilon j}^{f} & 0 \\ B_{\Upsilon j}^{f} & B_{\Upsilon j}^{fj} \end{bmatrix} \begin{bmatrix} F_t \\ F_{jt} \end{bmatrix} \\ + \begin{bmatrix} B_{\Upsilon}^{s} & 0 \\ B_{\Upsilon j}^{s} & B_{\Upsilon j}^{sj} \end{bmatrix} \begin{bmatrix} s_t \\ s_{jt} \end{bmatrix}. \end{aligned} \tag{B.119}$$

B.3.1 Invariant solution

First, I combine the state law of motion (B.118) and contemporaneous variables (B.119) to find the expressions in the main text (2.11) and (2.12).

Multiply (B.119) with the inverse of the left hand side matrix:

$$\begin{aligned}\begin{bmatrix}\Upsilon_t\\ \Upsilon_{jt}\end{bmatrix} = &\begin{bmatrix}B_{\Upsilon}^{\Upsilon} & 0\\ B_{\Upsilon j}^{\Upsilon} & B_{\Upsilon j}^{\Upsilon j}\end{bmatrix}^{-1}\begin{bmatrix}B_{\Upsilon}^{\Gamma 1} & 0\\ B_{\Upsilon j}^{\Gamma 1} & B_{\Upsilon j}^{\Gamma j1}\end{bmatrix}\begin{bmatrix}\Gamma_t\\ \Gamma_{jt}\end{bmatrix} + \begin{bmatrix}B_{\Upsilon}^{\Upsilon} & 0\\ B_{\Upsilon j}^{\Upsilon} & B_{\Upsilon j}^{\Upsilon j}\end{bmatrix}^{-1}\begin{bmatrix}B_{\Upsilon}^{\Gamma 0} & 0\\ B_{\Upsilon j}^{\Gamma 0} & B_{\Upsilon j}^{\Gamma j0}\end{bmatrix}\begin{bmatrix}\Gamma_{t-1}\\ \Gamma_{j,t-1}\end{bmatrix}\\ &+\begin{bmatrix}B_{\Upsilon}^{\Upsilon} & 0\\ B_{\Upsilon j}^{\Upsilon} & B_{\Upsilon j}^{\Upsilon j}\end{bmatrix}^{-1}\begin{bmatrix}B_{\Upsilon j}^{f} & 0\\ B_{\Upsilon j}^{f} & B_{\Upsilon j}^{fj}\end{bmatrix}\begin{bmatrix}F_t\\ F_{jt}\end{bmatrix} + \begin{bmatrix}B_{\Upsilon}^{\Upsilon} & 0\\ B_{\Upsilon j}^{\Upsilon} & B_{\Upsilon j}^{\Upsilon j}\end{bmatrix}^{-1}\begin{bmatrix}B_{\Upsilon}^{s} & 0\\ B_{\Upsilon j}^{s} & B_{\Upsilon j}^{sj}\end{bmatrix}\begin{bmatrix}s_t\\ s_{jt}\end{bmatrix}.\end{aligned}\tag{B.120}$$

Plug the result into (B.118):

$$\begin{aligned}&\left(\begin{bmatrix}A_{\Gamma}^{\Gamma 1} & 0\\ 0 & A_{\Gamma j}^{\Gamma j1}\end{bmatrix} - \begin{bmatrix}A_{\Gamma}^{\Upsilon} & 0\\ A_{\Gamma j}^{\Upsilon} & A_{\Gamma j}^{\Upsilon j}\end{bmatrix}\begin{bmatrix}B_{\Upsilon}^{\Upsilon} & 0\\ B_{\Upsilon j}^{\Upsilon} & B_{\Upsilon j}^{\Upsilon j}\end{bmatrix}^{-1}\begin{bmatrix}B_{\Upsilon}^{\Gamma 1} & 0\\ B_{\Upsilon j}^{\Gamma 1} & B_{\Upsilon j}^{\Gamma j1}\end{bmatrix}\right)\begin{bmatrix}\Gamma_t\\ \Gamma_{jt}\end{bmatrix} =\\ &\left(\begin{bmatrix}A_{\Gamma}^{\Gamma 0} & 0\\ 0 & A_{\Gamma j}^{\Gamma j0}\end{bmatrix} + \begin{bmatrix}A_{\Gamma}^{\Upsilon} & 0\\ A_{\Gamma j}^{\Upsilon} & A_{\Gamma j}^{\Upsilon j}\end{bmatrix}\begin{bmatrix}B_{\Upsilon}^{\Upsilon} & 0\\ B_{\Upsilon j}^{\Upsilon} & B_{\Upsilon j}^{\Upsilon j}\end{bmatrix}^{-1}\begin{bmatrix}B_{\Upsilon}^{\Gamma 0} & 0\\ B_{\Upsilon j}^{\Gamma 0} & B_{\Upsilon j}^{\Gamma j0}\end{bmatrix}\right)\begin{bmatrix}\Gamma_{t-1}\\ \Gamma_{j,t-1}\end{bmatrix} +\\ &\left(\begin{bmatrix}A_{\Gamma j}^{f} & 0\\ 0 & A_{\Gamma j}^{fj}\end{bmatrix} + \begin{bmatrix}A_{\Gamma}^{\Upsilon} & 0\\ A_{\Gamma j}^{\Upsilon} & A_{\Gamma j}^{\Upsilon j}\end{bmatrix}\begin{bmatrix}B_{\Upsilon}^{\Upsilon} & 0\\ B_{\Upsilon j}^{\Upsilon} & B_{\Upsilon j}^{\Upsilon j}\end{bmatrix}^{-1}\begin{bmatrix}B_{\Upsilon j}^{f} & 0\\ B_{\Upsilon j}^{f} & B_{\Upsilon j}^{fj}\end{bmatrix}\right)\begin{bmatrix}F_t\\ F_{jt}\end{bmatrix} +\\ &\left(\begin{bmatrix}A_{\Gamma}^{s} & 0\\ 0 & A_{\Gamma j}^{sj}\end{bmatrix} + \begin{bmatrix}A_{\Gamma}^{\Upsilon} & 0\\ A_{\Gamma j}^{\Upsilon} & A_{\Gamma j}^{\Upsilon j}\end{bmatrix}\begin{bmatrix}B_{\Upsilon}^{\Upsilon} & 0\\ B_{\Upsilon j}^{\Upsilon} & B_{\Upsilon j}^{\Upsilon j}\end{bmatrix}^{-1}\begin{bmatrix}B_{\Upsilon}^{s} & 0\\ B_{\Upsilon j}^{s} & B_{\Upsilon j}^{sj}\end{bmatrix}\right)\begin{bmatrix}s_t\\ s_{jt}\end{bmatrix}.\end{aligned}\tag{B.121}$$

Invert the left hand side of (B.121) to find (2.11) and plug it back into (B.120) and rearrange to find (2.12).

$$\begin{bmatrix}\Gamma_t\\ \Gamma_{jt}\end{bmatrix} = A^{\Gamma}\begin{bmatrix}\Gamma_{t-1}\\ \Gamma_{j,t-1}\end{bmatrix} + A^{F}\begin{bmatrix}F_t\\ F_{jt}\end{bmatrix} + A^{S}\begin{bmatrix}s_t\\ s_{jt}\end{bmatrix}\tag{B.122}$$

$$\begin{bmatrix}\Upsilon_t\\ \Upsilon_{jt}\end{bmatrix} = B^{\Gamma}\begin{bmatrix}\Gamma_{t-1}\\ \Gamma_{j,t-1}\end{bmatrix} + B^{F}\begin{bmatrix}F_t\\ F_{jt}\end{bmatrix} + B^{S}\begin{bmatrix}s_t\\ s_{jt}\end{bmatrix}\tag{B.123}$$

Plug (B.117) into (B.122) and (B.123) to find the non-expectational part of the state space system.

$$\begin{bmatrix}\Gamma_t\\ \Gamma_{jt}\end{bmatrix} = \left(A^\Gamma + A^F\xi^\Gamma\right)\begin{bmatrix}\Gamma_{t-1}\\ \Gamma_{j,t-1}\end{bmatrix} + \left(A^S + A^F\xi^S\right)\begin{bmatrix}s_t\\ s_{jt}\end{bmatrix} + A^F\xi^Z\begin{bmatrix}Z_t\\ Z_{jt}\end{bmatrix} + + A^F\xi^\Upsilon\begin{bmatrix}\underline{\Upsilon}_t\\ \underline{\Upsilon}_{jt}\end{bmatrix}$$
$$\begin{bmatrix}\Upsilon_t\\ \Upsilon_{jt}\end{bmatrix} = \left(B^\Gamma + B^F\xi^\Gamma\right)\begin{bmatrix}\Gamma_{t-1}\\ \Gamma_{j,t-1}\end{bmatrix} + \left(B^S + B^F\xi^S\right)\begin{bmatrix}s_t\\ s_{jt}\end{bmatrix} + B^F\xi^Z\begin{bmatrix}Z_t\\ Z_{jt}\end{bmatrix} + B^F\xi^\Upsilon\begin{bmatrix}\underline{\Upsilon}_t\\ \underline{\Upsilon}_{jt}\end{bmatrix}$$

Proceed as before and rewrite the two equations in such a way that the states and the contemporaneous variables are independent from one another. This way, I find the equations (2.17) and (2.18) in the main text.

$$\begin{bmatrix}\Gamma_t\\ \Gamma_{jt}\end{bmatrix} = M^\Gamma\begin{bmatrix}\Gamma_{t-1}\\ \Gamma_{j,t-1}\end{bmatrix} + M^S\begin{bmatrix}s_t\\ s_{jt}\end{bmatrix} + M^F\xi^Z\begin{bmatrix}Z_t\\ Z_{jt}\end{bmatrix} \tag{B.124}$$
$$\begin{bmatrix}\Upsilon_t\\ \Upsilon_{jt}\end{bmatrix} = G^\Gamma\begin{bmatrix}\Gamma_{t-1}\\ \Gamma_{j,t-1}\end{bmatrix} + G^S\begin{bmatrix}s_t\\ s_{jt}\end{bmatrix} + G^F\xi^Z\begin{bmatrix}Z_t\\ Z_{jt}\end{bmatrix} \tag{B.125}$$

B.3.2 Signal extraction problem

The guess for extended spate space is given by equation (2.19) in compact form

$$\begin{bmatrix}Z_t\\ Z_{jt}\end{bmatrix} = M\begin{bmatrix}Z_{t-1}\\ Z_{j,t-1}\end{bmatrix} + N\begin{bmatrix}s_t\\ s_{jt}\end{bmatrix} \quad \text{and} \quad \begin{bmatrix}\Upsilon_t\\ \Upsilon_{jt}\end{bmatrix} = G\begin{bmatrix}Z_{t-1}\\ Z_{j,t-1}\end{bmatrix} + H\begin{bmatrix}s_t\\ s_{jt}\end{bmatrix} \tag{B.126}$$

and in the extensive form by equation (2.21)

$$\begin{bmatrix}\Gamma_t^{(0:\infty)}\\ \Gamma_{t|jt}^{(0:\infty)}\\ \Gamma_{jt}\\ X_{jt|jt}\end{bmatrix} = \begin{bmatrix}M_{e\Gamma}^{e\Gamma} & 0 & 0 & 0\\ M_{ej\Gamma}^{e\Gamma} & M_{ej\Gamma}^{ej\Gamma} & M_{ej\Gamma}^{\Gamma j} & M_{ej\Gamma}^{ejxj}\\ M_{\Gamma j}^{e\Gamma} & M_{\Gamma j}^{ej\Gamma} & M_{\Gamma j}^{\Gamma j} & M_{\Gamma j}^{ejxj}\\ M_{ejxj}^{e\Gamma} & M_{ejxj}^{ej\Gamma} & M_{ejxj}^{\Gamma j} & M_{ejxj}^{ejxj}\end{bmatrix}\begin{bmatrix}\Gamma_{t-1}^{(0:\infty)}\\ \Gamma_{t-1|j,t-1}^{(0:\infty)}\\ \Gamma_{j,t-1}\\ X_{j,t-1|j,t-1}\end{bmatrix} + \begin{bmatrix}N_{e\Gamma}\\ N_{ej\Gamma}\\ N_{\Gamma j}\\ N_{ejxj}\end{bmatrix}\begin{bmatrix}s_t\\ s_{jt}\end{bmatrix}. \tag{B.127}$$

I verify the guess combining the guess (B.126) with (B.124) and (B.125).

$$\begin{bmatrix} \Gamma_t \\ \Gamma_{jt} \end{bmatrix} = \left(M^\Gamma + M^F \xi^Z M\right) \begin{bmatrix} \Gamma_{t-1} \\ \Gamma_{j,t-1} \end{bmatrix} + \left(M^S + M^F \xi^Z N\right) \begin{bmatrix} s_t \\ s_{jt} \end{bmatrix} \tag{B.128}$$

$$\begin{bmatrix} \Upsilon_t \\ \Upsilon_{jt} \end{bmatrix} = \left(G^\Gamma + G^F \xi^Z M\right) \begin{bmatrix} \Gamma_{t-1} \\ \Gamma_{j,t-1} \end{bmatrix} + \left(G^S + G^F \xi^Z N\right) \begin{bmatrix} s_t \\ s_{jt} \end{bmatrix} \tag{B.129}$$

which has the form of (2.22) and (2.23) in the main text.

B.4 Miscellaneous

B.4.1 Convergence accuracy

Lorenzoni (2009) chooses changes in the impulse responses between different iterations of the fixed point as a convergence criteria. The tolerance chosen is $1e^{-4}$ for the Kalman filter and $1e^{-3}$ for the difference in the impulse responses. If one decreases the tolerance to lower levels such as $1e^{-7}$ then the Matlab code provided as part of the publication does not converge to a stable solution. The algorithm that I present works at an arbitrary tolerance which comes only at the costs of speed. For the computations that I present within this paper, I choose a tolerance of $1e^{-7}$.

The reason for the difference between the two algorithm is located in the solution to the hierarchy of expectations. Lorenzoni (2009) chooses to solve the hierarchy of expectation under the assumption that the state variables become revealed after T periods, in the veins of Townsend (1983). However, he is using a formulation which does not require to write the hierarchy of expectation in a fundamental moving average (MA) representation.

The state space system in his paper is written as $Z_t = AZ_{t-1} + Bs_t$ and $\Upsilon_t = GZ_t + Hs_{jt}$. Consequently, he writes the Kalman updating equation after aggregation as:

$$Z_{t|t} = (I - KG)AZ_{t-1|t-1} + KGZ_t \tag{B.130}$$

and guesses that $Z_{t|t} = \Xi Z_t$. Such that one finds:

$$\Xi Z_t = (I - KG)A\Xi Z_{t-1} + KGZ_t \tag{B.131}$$

and when one defines a matrix which shifts the time subscript by one period H, then one can equate the variables and finds $\Xi = (I - KG)A\Xi H + KG$. This is different from the original setting in Townsend (1983) and this solution is not feasible with aggregate predetermined state variables as in the paper at hand. Here, the state law of motion reads $Z_t = AZ_{t-1} + Bs_t$ and $\Upsilon_t = GZ_{t-1} + H_s s_t + H_{sj} s_{jt}$. Then, the Kalman updating equation becomes:

$$Z_{t|t} = (A - KG)Z_{t-1|t-1} + KGZ_{t-1} + Hs_t. \tag{B.132}$$

The guess for the state law of motion of the hierarchy of expectation could be $Z_{t|t} = \Xi_A Z_{t-1} + \Xi_B s_t$, which leads to the expression:

$$\Xi_A Z_{t-1} + \Xi_B s_t = (A - KG)\Xi_A Z_{t-2} + \Xi_B s_{t-1} + KGZ_{t-1} + H_s s_t, \tag{B.133}$$

which is not verifying the guess. The correct guess includes the history of shocks of the signals which can be formulated as an MA component.

The important insight of the difference between the two approaches is not located in the general representation. Instead, it is to be found in the specific model formulation. In the new Keynesian model as presented in Lorenzoni (2009), the nominal interest rate, denoted as i_t is known to the agents. Assume i_t be one of the state variables in Z_t. If one looks at equation (B.132), then the right hand side depends only on A as $Z_{t-1|t-1} = Z_{t-1}$ for i_{t-1}. This means that the updating equation depends only on the nominal interest rate, when the state variable, the agents take their expectation of, depends on it and not through the signal.

In the formulation of (B.130) this is not true as $Z_{t-1|t-1}$ and Z_t are written in different time periods. In effect one finds that the state law of motion of the hierarchy of expectation in the code of Lorenzoni (2009) includes expectations of i_t while they theoretically should not exist. In my eyes this is the strongest candidate as a reason why the solution is sensitive to the convergence accuracy.

Appendix C

Chapter 3

C.1 The model

C.1.1 Households asset pricing choice

The households choose their asset portfolio as part of their maximization problem. For the reason of conciseness, I write down only the part of the households problem that is relevant to the portfolio choice. The household chooses the amount of zero coupon bonds $B_{j,t+1}$ for the real price Q_t^f, the amount of equity $S_{j,t+1}$ at the real price V_t and the amount of strips for each of k different horizons $Z_{j,t+k}[D_{t+k}]$ at the real price of each of these strips $V_t[D_{t+k}]$.

$$\max_{\{C_{jt},B_{j,t+1},S_{j,t+1},Z_{j,t+k}[D_{t+k}]\}} E_{jt}\left[\sum_{t=0}^{\infty}\beta^t U\left(C_{jt},\cdot\right)\right] \tag{C.1}$$

$$\text{s.t.} \quad Q_t^f B_{j,t+1} + P_t S_{j,t+1} + \sum_{k=1}^{K} V_t[D_{t+k}]Z_{j,t+k}[D_{t+k}] + C_{jt}$$

$$= B_{jt} + S_{jt}\left(P_t + D_t\right) + \sum_{k=1}^{K} Z_{jt}[D_{t+k}]D_t \tag{C.2}$$

The first order condition for consumption yields the expression for the real marginal utility, $\Lambda_{jt} = U_c(C_{jt},\cdot)$, and the asset pricing formulae for the real price of the one period zero coupon bond, equity and strips:

$$Q_t^f = \beta E_{jt}\left[\frac{\Lambda_{j,t+1}}{\Lambda_{jt}}\right] \quad \text{and} \tag{C.3}$$

$$V_t = \beta E_{jt}\left[\frac{\Lambda_{j,t+1}}{\Lambda_{jt}}\left(V_{t+1} + D_{t+1}\right)\right] \tag{C.4}$$

$$V_t[D_{t+k}] = \beta E_{jt}\left[\frac{\Lambda_{j,t+k}}{\Lambda_{jt}} D_{t+k}\right] \tag{C.5}$$

C.1.2 The risk free rate

Integrating and inverting the price formula for the zero coupon bond yields the formula for the real risk free rate:

$$R_t^f = \frac{1}{\beta}\Lambda_t \overline{E}_t\left[\frac{1}{\Lambda_{t+1}}\right]. \tag{C.6}$$

The corresponding log-normal asset pricing formula reads:

$$R_t^f = \frac{1}{\beta}\exp\left\{-\left(\overline{E}_t\left[\lambda_{t+1}\right]-\lambda_t\right)-\frac{1}{2}\overline{V}ar_t\left[\lambda_{t+1}\right]\right\}. \tag{C.7}$$

Recall that the variance of the log-normal distribution is constant and hence its variance is equal to zero. The unconditional expectation of the expected growth rate of marginal utility is also equal to zero and hence drops out, too. This is the same under full and under heterogeneous information. However, a difference arises for the unconditional expectation of the variance of the marginal utility, $E\left[\overline{V}ar_t\left[\lambda_{t+1}\right]\right]$. Under full information this expression is equivalent to $Var\left[\lambda_{t+1}-\overline{E}_t\left[\lambda_{t+1}\right]\right]$, while there is no simplifying representation under heterogeneous information. The reason is that the aggregate state variables are known contemporaneously under full information and hence the conditional variance is equal to zero, while this does not hold under heterogeneous information.

Further, the unconditional expectation of the risk free rate can be written as:

$$E\left[R_t^f\right] = \frac{1}{\beta}\exp\left\{E\left[\lambda_t-\overline{E}_t\left[\lambda_{t+1}\right]-\frac{1}{2}\overline{V}ar_t\left[\lambda_{t+1}\right]\right]+\frac{1}{2}Var\left[\lambda_t-\overline{E}_t\left[\lambda_{t+1}\right]-\frac{1}{2}\overline{V}ar_t\left[\lambda_{t+1}\right]\right]\right\} \tag{C.8}$$

$$=\frac{1}{\beta}\exp\left\{\frac{1}{2}\left(Var\left[\overline{E}_t\left[\lambda_{t+1}\right]-\lambda_t\right]-E\left[\overline{V}ar_t\left[\lambda_{t+1}\right]\right]\right)\right\}.$$

Additionally, the standard deviation of the risk free rate can also be derived in closed form:

$$Std\left[R_t^f\right] = E\left[R_t^f\right]\sqrt{\exp\left\{Var\left[\overline{E}_t\left[\lambda_{t+1}\right]-\lambda_t\right]\right\}-1}. \tag{C.9}$$

Using the log-linear solution of the HI-DSGE model, the unconditional expectation and the standard deviation are computed as follows. First, note that the model solution can be represented in state space form with X_t including the aggregate state variables of the model as well as the higher order expectation thereof. Y_t includes all the jump variables.

$$X_t = MX_{t-1} + Ns_t,$$

$$Y_t = GX_{t-1} + Hs_t,$$

where $s_t \sim N(0,I)$. Next, I define G_λ and G_d to correspond to the rows in the matrix G that correspond to the marginal utility and the dividends. The same notation is used for H.

Then, on the one hand, I compute the variance of the expected growth rate of marginal utility as:

$$Var\left[\overline{E}_t\left[\lambda_{t+1}\right]\right] = \left(\left(G_\lambda + G_d\right)\left(T_K M - I\right)\right) Var[X_t] \left(\left(G_\lambda + G_d\right)\left(T_K M - I\right)\right)' \\ + \left(\left(G_\lambda + G_d\right) T_K N - H_\lambda + H_d\right)\left(\left(G_\lambda + G_d\right) T_K N - H_\lambda + H_d\right)'$$

The matrix T_K shifts the order of expectation one level lower, i.e. $\overline{E}_t^{(k-1)}\left[X_t\right] = T_K \overline{E}_t^{(k)}\left[X_t\right]$. Moreover, the unconditional variance is the solution to a simply Riccati equation.

On the other hand, I compute the unconditional expectation of the conditional variance of next periods marginal utility as:

$$E\left[\overline{V}ar_t\left[\lambda_{t+1}\right]\right] = \left(G_\lambda + G_d\right)\overline{V}ar_t[X_t]\left(G_\lambda + G_d\right)' + \left(H_\lambda + H_d\right)\left(H_\lambda + H_d\right)'.$$

In this equation, the conditional variance in time period t of the state vector in the same period is defined by the mean square error also used for the Kalman filter. These two expressions are sufficient to compute the unconditional expectation as well as the standard deviation of the risk free rate.

C.1.3 Simulating the return to equity

For the return to capital, I first calculate the price for strips of a dividend payment in k periods of time. For $k = 1$ the price of a strip reads:

$$V_t[D_{t+k}] = \beta^k \exp\Big\{ \left(G_\lambda + G_d\right)\left(T_K M\right)^{(k-1)} X_t + \log(D_*) - \lambda_t \\ + \frac{1}{2}\left(G_\lambda + G_d\right)\overline{V}ar_t[X_t]\left(G_\lambda + G_d\right)' + \left(H_\lambda + H_d\right)\left(H_\lambda + H_d\right)' \Big\},$$

where D_* is the steady state value for the dividend payout. For $k > 1$ the price of a strip is defined as:

$$V_t[D_{t+k}] = \beta^k \exp\Bigg\{ (G_\lambda + G_d)(T_K M)^{(k-1)} X_t + \log(D_*) - \lambda_t + \frac{1}{2}\Big((G_\lambda + G_d)(T_K M)^{(k)}\Big) \overline{V}ar_t[X_t] \Big((G_\lambda + G_d)(T_K M)^{(k)}\Big)' + \frac{1}{2}\Big((G_\lambda + G_d)(T_K M)^{(k-1)} T_K N\Big)\Big((G_\lambda + G_d)(T_K M)^{(k-1)} T_K N\Big)' \Bigg\}.$$

I simulate X_t, λ_t and d_t $J = 100$ times for $T = 200$ periods and compute in each period the price of a strip up to $K = 1000$ periods ahead. The price of equity, V_t, is then the sum over all K strips as shown in equation (3.8) of the main text. The unconditional expectation of the return to equity can then be computed as the average over all periods of all samples of the return to equity which is defined as the next periods payout over today's period price:

$$E\left[R_{t,t+1}\right] = \frac{1}{J}\frac{1}{T}\sum^{J}\sum_{t=1}^{T} \frac{V_{t+1} + \exp\left\{d_{t+1}\right\} D_*}{V_t}.$$

Analogously, I compute the standard deviation of the return to equity as the average of the standard deviations of each sample:

$$Std\left[R_{t,t+1}\right] = \frac{1}{J}\sum^{J} Std\left[R_{t_j,t_j+1}\right].$$

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