Quantization of the Hitchin system from loop operator expectation values

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Abstract

The subject of this thesis is exact results on $\mathcal{N} = 2$ supersymmetric quantum field theories. We restrict our attention to a certain class of quantum field theories in the literature referred to as class S. For these theories, we study partition functions on a deformed four-sphere in the presence of extra defect observables called loop and surface operators. In a degeneration limit of the four-sphere, we find that these partition functions are related to the eigenfunctions of the conserved quantities in certain quantum integrable models. We establish a direct correspondence between parameters classifying the eigenfunctions in the relevant integrable models and the charge labels of the loop operators. Furthermore, the problem to classify the eigenvalues is mapped to the mathematical problem of classifying what are called projective structures with real monodromy. We exhibit various points of contact between the mathematical description of these projective structures and known as well as new exact results on class S theories. Apart from providing new non-perturbative results on the theories of class S, one may obtain further support of the S-duality conjectures from the results presented here.

Zusammenfassung

In dieser Dissertation werden exakte Ergebnisse der $\mathcal{N} = 2$ supersymmetrischen Quantenfeldtheorien diskutiert. Wir fokussieren uns hierbei auf eine spezifische Klasse der Quantenfeldtheorien, die in der Literatur als Klasse S-Feldtheorien bezeichnet werden. Für diese Theorien wird die Zustandssumme auf einer deformierten Vier-Sphäre in Anwesenheit von Loop- und Oberflächendefekten betrachtet. Es wird gezeigt, dass in einem Degenerationslimit der Vier-Sphäre diese Zustandssummen mit den Eigenfunktionen der Erhaltungsgrößen in bestimmten quantenintegrierbaren Modellen verwandt sind. Es wird ein direkter Zusammenhang hergestellt zwischen Parametern, die Eigenfunktionen in den relevanten integrierbaren Modellen klassifizieren, und Ladungskennzeichnungen der Loopdefekten. Zudem bilden wir die Problematik der Klassifizierung der Eigenwerte auf das mathematische Problem der Klassifizierung von sogenannten projektiven Strukturen mit reeller Monodromie ab. Wir zeigen zahlreiche Kontaktpunkte zwischen der mathematischen Beschreibung dieser projektiven Strukturen und bekannten sowie neuen exakten Ergebnissen zu Klasse S-Feldtheorien. Abgesehen von der Bereitstellung neuer, nicht perturbativer Ergebnisse zu den Theorien der Klasse S kann man aus den hier vorgestellten Ergebnissen weitere bekräftigende Indizien für die S-Dualitäts-Vermutungen erhalten.

Contents

Part I

Background and motivation

1 Introduction

1.1 Gauge theories at low energy scales

From the point of view of quantum field theory, we describe forces acting between particles by gauge theories. In theoretical particle physics, gauge theories play an important role in describing the interactions between matter.

Due to an effect known as asymptotic freedom, the strength of these interactions goes to zero for high energy scales. On the other hand, in the low energy region these interactions become very strong and are much less understood. Although phenomena such as confinement, describing the binding of quarks together to form hadrons, have been experimentally observed, a mathematical description remains elusive.

For high energy scales we may describe the physics of gauge theories using the perturbative approach, however, for low energy scales the calculations quickly get out of hand. Furthermore, perturbation theory cannot access all phenomena observed at these energy scales. For example, instantons, which are non-trivial solutions to the Euclidean equations of motion, may lead to exponentially suppressed contributions. Describing the gauge theories for such strong coupling would therefore require a complete understanding of both the perturbative approach as well as the non-perturbative effects.

There have been many different ways in which physicists have tried to make progress, but we will focus on the introduction of what is known as supersymmetry. This is an extension of the algebra containing the Poincaré symmetries and the internal symmetries present in a gauge theory. The supersymmetry algebra includes a symmetry mixing bosonic and fermionic degrees of freedom.

Although such a symmetry has not been observed experimentally, the introduction of this type of extension makes certain calculations in quantum field theories tractable. For example, it had been long believed that instantons play a crucial role in gauge theories at strong coupling. This is precisely what is observed in the realm of supersymmetry. One may hope that the lessons learnt by introducing supersymmetry, survive if we study gauge theories without supersymmetry.

1.2 Dualities and relations to $\mathcal{N} = 2$ gauge theories

If we consider gauge theories with eight real supersymmetry generators, which we name $\mathcal{N} = 2$ theories, there have been various advances in the exact calculation of partition functions. It turns out that theories with $\mathcal{N} = 2$ supersymmetry are highly constrained at the low energy scale. In [1, 2] a method was developed which under certain assumptions allows one to calculate the prepotential, a function uniquely defining the low energy effective action.

Later on, it has been shown that the partition function can be calculated by taking into account all quantum corrections in a suitable regularization scheme called the Ω-background. The prepotential as defined by Seiberg and Witten, can be recovered in a suitable limit [3, 4]. See also [5, 6] for alternative approaches.

Various profound relations between the partition function in the Ω-background and other physical models and systems have been observed. We refer to the relevant work when discussing these relations and dualities, but should take note of that fact that the history on $\mathcal{N} = 2$ supersymmetric gauge theories is long and we can hardly refer to the entire body of work. For completeness we refer to two reviews [7, 8] which contain a concise overview of the original work and many of the corresponding references.

Relations to conformal field theories In [9] dualities of a class of four-dimensional $\mathcal{N} = 2$ supersymmetric field theories now known as theories of class S, have been related to changes of pair of pants decompositions of the Riemann surfaces labeling these theories. It had been observed later that the partition function in the Ω-background for these theories bears resemblance to conformal blocks in the Liouville conformal field theory. This correspondence has been set up in [10] and now goes by the name of the AGT-duality. Since then much work has been done on this duality and many natural objects we can introduce on the side of the gauge theories have found natural homes on the conformal field theory side.

Relations to integrable systems The ideas used by Seiberg and Witten in the construction of the prepotential [1, 2], have led to the understanding that we may describe the prepotential in terms of a classically integrable system. This leads to an intimate relation between integrable systems and $\mathcal{N} = 2$ gauge theories [11, 12, 13]. Connections between the deformations of the prepotential provided by the Ω-backgrounds and the quantization of the integrable systems related to Seiberg-Witten theory, have been discovered in [14]. If we consider theories of class S, we find that the associated family of integrable systems is known as the Hitchin integrable systems [15, 16]. The quantization of these integrable systems plays a central role in this work.

How do our results fit in the larger picture? The combinations of the above conjectures and proposals leads to highly non-trivial predictions. The results we present in this thesis fit into the literature as a check of this conjectural picture. Combining the extension of the AGT-duality to include loop operators and surface defects, we arrive at a statement relating single-valued eigenfunctions of the quantum Hitchin Hamiltonians to loop operators. In this thesis we provide the mathematical background to make this relation precise.

Originally used in the context of the Bethe ansatz in [17], a Yang-Yang function describes a given set of quantization conditions for an integrable system. Although it is not clear if such a function exists for any integrable system and set of quantization conditions, through the discovery of Nekrasov and Shatashvili [14] and additional work done in [18, 19], we may reexpress the single-valuedness condition in terms of a Yang-Yang function. This has been proposed before in [20]. We will name the quantum numbers introduced in this way the *Bethe quantum numbers*.

Using the work done by Pestun in [21] and later by Hama and Hosomichi in [22], we may consider what is now known as the Nekrasov-Shatashvili limit from [14] in the context of partition functions on a compactified spacetime taking the form of a squashed four-sphere. Upon introduction of loop operators in the gauge theory, the expectation values of the loop operators precisely lead to the Bethe quantum numbers in the Nekrasov-Shatashvili limit.

A priori, it is unclear how loop operators give rise to different sets of Bethe quantum numbers except in some simple cases. By comparing with the description of single-valued eigenfunctions of the quantum Hitchin Hamiltonians, we conjecture a simple relation between a set of parameters known as the Dehn-Thurston parameters, which turn out to classify the single-valued eigenfunctions, and the Bethe quantum numbers appearing from the Yang-Yang function. This expresses a profound relation between various topics in integrable models, hyperbolic geometry and supersymmetric gauge theories.

1.3 Integrable systems from $\mathcal{N} = 2$ gauge theories

Proposal by Nekrasov and Shatashvili The construction of Seiberg and Witten [1, 2] recovers the prepotential using an auxiliary space now known as the Seiberg-Witten curve. This curve realizes the structure of special geometry, defining an integrable model with base space the Coulomb branch of the $\mathcal{N} = 2$ theory. Although it is unknown whether an $\mathcal{N} = 2$ gauge theory always has an associated Seiberg-Witten curve, it is true that we can always associate a classically integrable model to the gauge theory [23].

In [14] Nekrasov and Shatashvili found relations between a quantization of this classically integrable system and the partition function of the corresponding $\mathcal{N} = 2$ gauge theory in what is known as the Ω-background, denoted by $\mathbb{R}^4_{\epsilon_1,\epsilon_2}$. The Ω-background plays the role of regularizing the IR-divergences and comes with two parameters ϵ_1 and ϵ_2, each associated to a two-dimensional subspace $\mathbb{R}^2_{\epsilon_1} \subset \mathbb{R}^4_{\epsilon_1,\epsilon_2}$ and $\mathbb{R}^2_{\epsilon_2} \subset \mathbb{R}^4_{\epsilon_1,\epsilon_2}$ of the four-dimensional theory. The partition function $Z(\mathbb{R}^4_{\epsilon_1,\epsilon_2})$ has been calculated through localization by Nekrasov in [3] for quiver gauge theories and is shown to relate to the prepotential in the limit $\epsilon_1, \epsilon_2 \to 0$.

By only considering the limit $\epsilon_2 \to 0$, the role of the prepotential is taken over by a function known as the twisted effective superpotential. It was argued on physical grounds by relating to an effective two-dimensional $\mathcal{N} = (2, 2)$ theory in this limit, that the twisted effective superpotential plays the role of a Yang-Yang function $\mathcal{Y}_{\epsilon_1}(\mathbf{a}, \boldsymbol{\mu}, \mathbf{t})$. We will denote this function simply by $\mathcal{Y}(\mathbf{a}, \boldsymbol{\mu}, \mathbf{t})$ omitting the dependence on ϵ_1. For our current discussion, it suffices to mention that the Yang-Yang function depends on vectors of parameters $\mathbf{a} = (a_1, ..., a_h)$, $\boldsymbol{\mu} = (\mu_1, ..., \mu_n)$ and $\mathbf{t} = (\tau_1, ..., \tau_h)$. We will discuss these parameters in more detail at various points of this thesis.

In general a Yang-Yang function encodes the quantization conditions of the integrable model. The quantization conditions considered in [14], lead to the following holomorphic equations

$$\frac{\partial \mathcal{Y}(\mathbf{a}, \boldsymbol{\mu}, \mathbf{t})}{\partial a_r} = 2\pi i n_r \tag{1.1}$$

for a set of integers n_r. If we denote by $\mathbf{a} = \mathbf{a}_*$ a solution to the above equation, we recover the spectrum of the Hamiltonians by

$$\frac{\partial \mathcal{Y}(\mathbf{a}_*, \boldsymbol{\mu}, \mathbf{t})}{\partial \tau_r} = E_r$$

It should be noted that the models of our interest are variants of spin chains associated to representations of non-compact groups. These are much more complicated than the usual spin chain models including the usual Gaudin model. In the absence of a reference state one can in particular not apply the Bethe ansatz.

A striking aspect of the work of Nekrasov and Shatashvili is that it leads to the proposal that the quantization conditions can be written in a form resembling the Bethe ansatz equations even if the Bethe ansatz is not applicable. In order to do this one just needs to replace the function introduced by Yang and Yang in the Bethe ansatz framework by another function. However, the function taking the role of the Yang-Yang function in our case has much more complicated analytical properties than the functions occurring in models which can be solved by the Bethe ansatz.

Although a powerful paradigm, it remains a question which quantization conditions precisely appear through the Yang-Yang function, since there is no unique way to impose quantization conditions on an integrable model. Different quantization conditions for an integrable model can be physically interesting for different reasons.

The quantization conditions from equation (1.1) are holomorphic in nature. We will explore a different type of quantization condition which does not have this holomorphic nature.

Hitchin systems appearing from theories of class S If we compactify the theories of class S on S^1, the associated integrable system is known as the Hitchin integrable system [24]. This is a famous family of exactly integrable systems associated to Riemann surfaces. Many known integrable models like the Gaudin model or the Calogero-Moser models are contained in this family of integrable systems [25, 26]. Furthermore, it has been found that the Seiberg-Witten curve of these theories is precisely the spectral curve of the Hitchin system.

The quantization of the Hitchin system is interesting not only from the side of gauge theories. First of all, many classically integrable systems are understood as special cases of Hitchin systems. The quantization of the Hitchin system will therefore lead to a better understanding of the quantization of these integrable systems. Secondly, from a more mathematical point of view, the Hitchin system plays a prominent role in the geometric Langlands conjecture as studied by Beilinson and Drinfeld [27]. Finding the eigenfunctions to the quantized Hitchin Hamiltonians provides a concrete realization of the conjecture.

This division of these different topics is not so strict as appears at first sight. For example, conformal field theories appear both in the relation to AGT-duality and the statement of the geometric Langlands conjecture by Beilinson and Drinfeld in [27]. In [28] Kapustin and Witten have interpreted the geometric Langlands program in terms of $\mathcal{N} = 4$ super Yang-Mills theory. Some of these relations have been explored in [29].

In this context, the proposal by Nekrasov and Shatashvili gives us a handle on the quantization of the Hitchin system from the perspective of supersymmetric gauge theories.

1.4 AGT-duality and relations to conformal field theory

AGT-duality and the introduction of defects in the gauge theory In [10] Alday, Gaiotto and Tachikawa proposed a remarkable relation between quantities in theories of class S and conformal field theories. This duality is physically motivated by the proposed existence of a famous six-dimensional $\mathcal{N} = (2,0)$ theory of type A_1 and relates SU(2)-gauge theories in the Ω-background $\mathbb{R}^4_{\epsilon_1,\epsilon_2}$ with Liouville conformal field theory on a Riemann surface X. The duality is now known under the name AGT-duality and has since been extended in various directions. For our purposes, we consider the generalizations by the introduction of surface defects and loop operators in the theory of class S.

From the point of view of the six-dimensional theory living on the product $\mathbb{R}^4_{\epsilon_1,\epsilon_2} \times X$, we can set up the following table of defects (from [30]):

These defects do not exhaust all possible defects, but are the only ones we will consider in this thesis.

6D theory	$\mathbb{R}^4_{\epsilon_1,\epsilon_2}$	X	**interpretation in 4D**
co-dimension four defect	2D	0D	surface defect
co-dimension four defect	1D	1D	loop operators
co-dimension two defect	2D	2D	surface defect

Table 1.1: Defects in the 6D theory

In table 1.1, there are two types of surface defects: We can either introduce a co-dimension two surface operator or a co-dimension four surface operator. The co-dimension two surface operator wraps the Riemann surface X, while the co-dimension four surface operator is pointlike on X. From the side of the gauge theory, we additionally require that the surface defects lie either completely in $\mathbb{R}^2_{\epsilon_1}$ or in $\mathbb{R}^2_{\epsilon_2}$, ensuring that these defects do not break too much supersymmetry.

In [30] the co-dimension two surface operator was studied. It was concluded that the introduction of such a surface operator on the side of the gauge theory, has the effect of changing the conformal field theory on X from the Liouville theory to the Wess-Zumino-Novikov-Witten (WZNW) theory with $\mathfrak{sl}(2,\mathbb{C})$-affine Kac-Moody symmetry. The extension of this proposal for arbitrary $\mathfrak{sl}(N,\mathbb{C})$ symmetry has been worked out in [31].

In [32] the co-dimension four surface operator was considered. This has the effect of introducing degenerate fields in the conformal blocks and correlation functions on the side of the Liouville theory. The proposal relating co-dimension four operators to degenerate fields has been checked extensively and many of the corresponding references may be found in [8].

Both the WZNW theory and the Liouville theory with the introduction of denegerate fields, come with sets of differential equations that the conformal blocks and correlation functions must satisfy. These differential equations come from the Ward identities and lead to Knizhnik-Zamolodchikov-Bernard equations (KZB-equations, in [33, 34, 35]) on the side of the WZNW theory and to Belavin-Polyakov-Zamolodchikov equations (BPZ-equations, in [36]) on the side of Liouville theory with degenerate fields. A proof of the fact that the instanton partition functions in the presence of co-dimension two surface operators satisfy the KZ-equations was anounced in [37].

Using a version of the separation of variables method, it was shown in [38], using the observation by Stoyanovsky in [39] (from unpublished work with Feigin and Frenkel), that solutions of one set of equations can be mapped to solutions of the other set of equations through an integral transform. In the context of $\mathcal{N} = 2$

gauge theories, this approach leads to a duality between the two types of surface defects in the IR-limit as shown in [40].

The third type of defect, the loop operators, has been studied in [32, 41] among others. On the side of the gauge theory, we can introduce the loop operators in two configurations: Either contained completely in $\mathbb{R}^2_{\epsilon_1}$ or completely in $\mathbb{R}^2_{\epsilon_2}$. These loop operators take the form of Verlinde loop operators on the conformal field theory side.

The loop operators in the gauge theory have been classified in terms of their allowed electric and magnetic charges in [42], which precisely matches a famous theorem by Dehn (theorem 10.1, see [43, 44]). This theorem classifies the space of non-self-intersecting multicurves on X up to homotopy through a set of integer parameters known as the Dehn-Thurston parameters $(\mathbf{p}, \mathbf{q})$, depending on a pants decomposition of X.

Extending the results to $S^4_{\epsilon_1,\epsilon_2}$ Many of the results in the context of the AGT-duality for gauge theories in the Ω-background, have been adjusted in a natural way when instead we regularize the four-dimensional $\mathcal{N} = 2$ gauge theory by compactification on $S^4_{\epsilon_1,\epsilon_2}$, a squashed version of the four-sphere with the following form

$$S^4_{\epsilon_1,\epsilon_2} := \{(x_0, ..., x_4) \in \mathbb{R}^5 | x_0^2 + \epsilon_1^2(x_1^2 + x_2^2) + \epsilon_2^2(x_3^2 + x_4^2) = 1\}$$

The partition function $Z(S^4_{\epsilon_1,\epsilon_2})$ was calculated using the localization formalism by Pestun in [21] and has been shown to relate to the partition function $Z(\mathbb{R}^4_{\epsilon_1,\epsilon_2})$ calculated by Nekrasov in [3] as follows

$$Z(S^4_{\epsilon_1,\epsilon_2}) = \int d\mathbf{a} |Z(\mathbb{R}^4_{\epsilon_1,\epsilon_2})|^2 \tag{1.2}$$

The parameters $\mathbf{a}$ are parameters describing the Higgs breaking of the gauge group SU(2) to U(1) in the infrared regime and give coordinates for the Coulomb branch of the gauge theory. It is known that the partition function $Z(\mathbb{R}^4_{\epsilon_1,\epsilon_2})$ depends holomorphically on the complexified gauge couplings of the theory. Note that this is not the case for $Z(S^4_{\epsilon_1,\epsilon_2})$, which has a mixed dependence on the complexified gauge couplings.

This factorized form of the partition function $Z(S^4_{\epsilon_1,\epsilon_2})$ is suggestive of the factorized form of correlation functions in conformal field theory in terms of conformal blocks. Indeed, it was already shown in [10] that the combination $\int d\mathbf{a}|Z(\mathbb{R}^4_{\epsilon_1,\epsilon_2})|^2$ maps to the Liouville correlation function.

Defects have been included in this extended set-up in various different references. Let us discuss some of the results from the literature.

The expectation value of some simple loop operators was already calculated by Pestun in [21]. These calculations have been extended in [32, 41] by a proposal through the AGT-duality including general types of loop operators in the context of partition functions on $S^4_{\epsilon_1,\epsilon_2}$. There it has been shown that we may calculate the expectation value $\langle \mathcal{L} \rangle_{S^4_{\epsilon_1,\epsilon_2}}$ through a difference operator $\mathcal{D}_{\mathcal{L}}$ acting on the parameters a by

$$\langle \mathcal{L} \rangle_{S^4_{\epsilon_1,\epsilon_2}} = \int d\mathbf{a}\overline{Z(\mathbb{R}^4_{\epsilon_1,\epsilon_2})}\mathcal{D}_{\mathcal{L}} \cdot Z(\mathbb{R}^4_{\epsilon_1,\epsilon_2}) \tag{1.3}$$

where the precise form of the difference operator has been calculated. Through the classification in [42], we may write these difference operators as $\mathcal{D}_{(\mathbf{p},\mathbf{q})}$.

We assume that the loop operators in the gauge theory, can be inserted either on the circle $\mathcal{C}_1 = \{(x_1, x_2)|\epsilon_1^2(x_1^2 + x_2^2) = 1\}$ or on $\mathcal{C}_2 = \{(x_3, x_4)|\epsilon_2^2(x_3^2 + x_4^2) = 1\}$. Exchanging the positions of the loop operators on the side of the gauge theory, leads to a dual description in terms of the conformal field theory.

Furthermore, if we insert co-dimension four surface operators in $S^4_{\epsilon_1,\epsilon_2}$, the extended AGT-duality tells us that the partition function $Z(S^4_{\epsilon_1,\epsilon_2})$ is mapped to a Liouville correlator with extra insertions of degenerate fields. See [32].

The co-dimension two surface operators on $S^4_{\epsilon_1,\epsilon_2}$ have been investigated by Nawata in [45]. One expects to find a relation between $Z(S^4_{\epsilon_1,\epsilon_2})$ and the correlation functions of the so-called H_3^+-WZNW model classically defined by evaluating the WZNW action on maps from X to the space of hermitian metric with unit determinant rather than on maps from X to SU(2).

Combining the AGT-duality with the Nekrasov-Shatashvili proposal

By combining the known relations between theories of class S and conformal field theories through the AGT-duality on one side, and between theories of class S and (quantum) integrable models on the other side, we can mathematically characterize the Yang-Yang function of the quantum Hitchin system for some quantization conditions in terms of the symplectic geometry of the moduli space of flat connections on X. This was proposed independently in [18] and [19].

By applying the limit $\epsilon_2 \to 0$, this result can be obtained by considering the semiclassical nature of Liouville theory under the AGT-duality.

Remark 1.1. In the case of the Gaudin model the situation is easy to clarify: We are led to consider a variant of the Gaudin model in which the representations of SU(2) considered in the ordinary Gaudin model are replaced by principal series representations of the group $\mathrm{SL}(2,\mathbb{C})$.

1.5 Quantization conditions from $Z(S^4_{\epsilon_1,\epsilon_2})$

Introducing a different set of quantization conditions Although the quantization conditions proposed originally in [14] are interesting in their own right, we will consider a different set of quantization conditions proposed in [20], which take the form

$$\begin{aligned} \mathfrak{Re}(a_r) &= \pi n_r \\ \mathfrak{Re}\left(\frac{\partial \mathcal{Y}(\mathbf{a},\mathbf{t})}{\partial a_r}\right) &= \pi m_r \end{aligned}$$

where $\mathcal{Y}(\mathbf{a},\mathbf{t})$ denotes the Yang-Yang function for these quantization conditions with $\mathbf{a} = (a_1, ..., a_h)$ playing the role of the auxiliary parameters and $\mathbf{t} = (\tau_1, ..., \tau_h)$ a set of complex moduli for the Riemann surface X. The integer h depends on the topology of the surface X.

If we find a solution $\mathbf{a} = \mathbf{a}_{(\mathbf{n},\mathbf{m})}$ to these equations, for $\mathbf{n} = (n_1, ..., n_h)$ and $\mathbf{m} = (m_1, ..., m_h)$, we recover the eigenvalues of the quantum Hitchin Hamiltonians by

$$4\pi i E_{r,(\mathbf{n},\mathbf{m})} = \left.\frac{\partial \mathcal{Y}(\mathbf{a},\mathbf{t})}{\partial \tau_r}\right|_{\mathbf{a}=\mathbf{a}_{(\mathbf{n},\mathbf{m})}}$$

We will call the integers $(\mathbf{n},\mathbf{m})$ Bethe quantum numbers.

These quantization conditions appeared in [20] in the context of finding single-valued eigenfunctions $\Psi_{\mathbf{E}}(\mathbf{x},\bar{\mathbf{x}})$ of both the holomorphic and anti-holomorphic quantum Hitchin Hamiltonians

$$\hat{H}_r \Psi_{\mathbf{E}}(\mathbf{x},\bar{\mathbf{x}}) = E_r \Psi_{\mathbf{E}}(\mathbf{x},\bar{\mathbf{x}}) \qquad \hat{\bar{H}}_r \Psi_{\mathbf{E}}(\mathbf{x},\bar{\mathbf{x}}) = \bar{E}_r \Psi_{\mathbf{E}}(\mathbf{x},\bar{\mathbf{x}})$$

and it was noted that any single-valued eigenfunction $\Psi_{\mathbf{E}}(\mathbf{x},\bar{\mathbf{x}})$ must give rise to a set of integers $(\mathbf{n},\mathbf{m})$.

Quantization conditions in the presence of loop operators The proposal by Nekrasov and Shatashvili implies that we may identify the partition function $Z(\mathbb{R}^4_{\epsilon_1,\epsilon_2})$

$$Z(\mathbb{R}^4_{\epsilon_1,\epsilon_2}) = \exp\left(-\frac{1}{\epsilon_2}\mathcal{Y}(\mathbf{a},\mathbf{t})\right)(1+\mathcal{O}(\epsilon_2))$$

in the limit $\epsilon_2 \to 0$, where $\mathcal{Y}(\mathbf{a},\mathbf{t})$ is the Yang-Yang function from the point of view of the Hitchin integrable system. By plugging in this expression into the factorized form of $Z(S^4_{\epsilon_1,\epsilon_2})$, we can apply a saddlepoint approximation to solve the integral. The integral will be dominated by the values $\mathbf{a} = \mathbf{a}_*$ such that

$$\mathfrak{Re}\left(\frac{\partial \mathcal{Y}(\mathbf{a},\mathbf{t})}{\partial a_r}\right)\Bigg|_{\mathbf{a}=\mathbf{a}_*} = 0$$

This gives rise to a similar type of quantization condition as above.

We will show that the introduction of loop operators can affect the Bethe quantum numbers $(\mathbf{n},\mathbf{m})$ appearing in the comparison with the quantization conditions in [20].

We will ask ourselves the question

Which loop operators give rise to which sets of Bethe quantum numbers?

Loop operators and single-valued eigenfunctions of the Hitchin Hamiltonians The key player for us to compare the quantization condition in [20] to the introduction of loop operators, is the co-dimension two surface operator. As suggested in [30], we can use these surface operators to construct eigenfunctions of the Hitchin Hamiltonians.

We will consider a set-up in which we only introduce a co-dimension two surface operator in the ϵ_1-plane $\mathbb{R}^2_{\epsilon_1}$ and use it as a probe to find the next-to-leading order of the saddlepoint approximation in the limit of small ϵ_2. By carefully doing the analysis, we find that this next-to-leading order term is a single-valued eigenfunction of the quantum Hitchin Hamiltonians. We propose that in this way we can also find and probe single-valued eigenfunctions in the next-to-leading order term upon introduction of loop operators.

This suggest the following question

Does there exist a correspondence between loop operators and single-valued eigenfunctions of the quantum Hitchin Hamiltonians?

1.6 Approach to the problem and new results

In the previous section we raised two important questions. Let us first discuss the second question.

Classification of single-valued eigenfunctions through non-self-intersecting multicurves To make the proposed relation between loop operators in theories of class S and single-valued eigenfunctions to the quantum Hitchin Hamiltonians precise, we will use the fact that the single-valued eigenfunctions are classified by the same parameters as used for the classification of loop operators in class S theories, as was shown by Drukker et al. in [42].

It will be shown that single-valued eigenfunctions are in one-to-one correspondence with differential operators of the form $\partial_u^2 + t(u)$ with real monodromy. This claim was first proposed in [20]. A detailed proof which had not been given in [20] will be presented in subsection 3.6.

It was proved by Goldman in [46] that opers with real monodromy have a classification in terms of an operation known as grafting. This operation takes as its input a non-self-intersecting multicurve together with a complex structure X. However, there are various subtleties preventing us from directly applying the results of [46] to our problem.

Introducing Dehn-Thurston coordinates classifying single-valued eigenfunctions Goldman's theorem implies a classification of opers with real monodromy in terms of multicurves. In order to get a concrete set of quantum numbers for single-valued eigenfunctions it turns out to be very convenient to use the set of parameters $(\mathbf{p}, \mathbf{q})$ for multicurves introduced by Dehn and Thurston. We denote the corresponding single-valued eigenfunctions by $\Psi_{(\mathbf{p},\mathbf{q})}$

Since both the loop operators in the theory of class S on $S^4_{\epsilon_1,\epsilon_2}$ and the single-valued eigenfunctions to the quantum Hitchin Hamiltonians are parameterized by Dehn-Thurston parameters and by the fact that every loop operator gives rise to a single-valued eigenfunction, we propose there does indeed exist a direct correspondence between these two different objects.

However, there is an important subtlety. In theorem 8.6 we show that the subset of multicurves which have even intersection index with any other curve, classify a subset of the single-valued eigenfunctions in a one-to-one correspondence. This is related to the fact that only a subset of the complex projective structures with real monodromy have monodromy in $\mathrm{PSL}(2,\mathbb{R})$, as we discuss later in section §9.

It has been stressed by Aharony et al. in [47], on the other hand, that the set of loop operators has to be mutually local. This leads to important corrections to the classification proposed by Drukker et al. in [42]. We will find a one-to-one correspondence between loop operators that are mutually local to any other loop operator and single-valued eigenfunctions, as suggested by our considerations in theorem 8.6. This provides some evidence for the proposal we stated above.

To clarify this relation further, we work out the restrictions on Dehn-Thurston parameters describing single-valued eigenfunctions explicitly. Although this gives a constructive way to determine the restrictions for any Riemann surface, the calculations are based on Penner's formulas [48, 49] which are highly complex.

Nonetheless, for a closed Riemann surface of genus two we may explicitly determine the restrictions on the Dehn-Thurston parameters in theorem 11.5. This is arguably the simplest, non-trivial example we can apply our algorithm to.

Relations to the Yang-Yang function and the Bethe quantum numbers The considerations in subsection 1.5 suggest an alternative classification in terms of a function $\mathcal{Y}$ called the Yang-Yang function following [14], and integer parameters $(\mathbf{n},\mathbf{m})$. Understanding the relation between the classification of single-valued eigenfunctions in terms of $(\mathbf{p},\mathbf{q})$ and $(\mathbf{n},\mathbf{m})$ turns out to be quite non-trivial. The main difficulty in relating the single-valued eigenfunctions and the loop operators in the gauge theory on $S^4_{\epsilon_1,\epsilon_2}$ to the Bethe quantum numbers $(\mathbf{n},\mathbf{m})$, lies in the fact that we do not know precisely how the Yang-Yang function must depend on the variables $\mathbf{a}$.

Through the relation between the Yang-Yang function and the symplectic structure of the moduli space of flat connections from [18] and [19], the variables $\mathbf{a}$ take the role of parameterizing the monodromy of the space of opers over a Riemann surface X. On the other hand, the space of opers over a fixed Riemann surface is itself parameterized by a set of coordinates $\mathbf{E} = (E_1, ..., E_h)$. We use the same notation $\mathbf{E}$ as for the eigenvalues of the quantum Hitchin Hamiltonian, since these different sets of coordinates agree under the separation of variables. We denote by $\mathbf{E}_{(\mathbf{p},\mathbf{q})}$ the coordinates corresponding to opers with real monodromy leading to the single-valued eigenfunctions $\Psi_{(\mathbf{p},\mathbf{q})}$.

It turns out we may analytically continue $\mathbf{a}$ as functions of $\mathbf{E}$ away from certain loci where the dependence becomes non-analytical. If we analytically continue $\mathbf{a}$ along a path starting at the oper with coordinates $\mathbf{E}_{(\mathbf{0},\mathbf{0})}$ to the oper with coordinates $\mathbf{E}_{(\mathbf{p},\mathbf{q})}$, the very definition of the parameters $(\mathbf{n},\mathbf{m})$ is done by analytical continuation of the function $\mathcal{Y}(\mathbf{a},\mathbf{t})$. This raises subtle issues about the existence of a path of analytical continuation relating $(\mathbf{n},\mathbf{m})$ to $(\mathbf{p},\mathbf{q})$.

We arrive at the following question:

Can we find a relation between the integers $(\mathbf{p},\mathbf{q})$ *and* $(\mathbf{n},\mathbf{m})$ *by picking the analytical continuation of* $\mathbf{a}$ *as a function of* $\mathbf{E}$ *'correctly'?*

This question is stated loosely right now, but will be made more precise.

We use the example where X is a four-punctured sphere with real punctures as a guide to explore the answer to this question. A subset of opers with real monodromy over X had already been classified in 1918 by Smirnov in his thesis (published later in [50]).

However, it turns out that the opers with real monodromy classified by Smirnov do not exhaust all opers of real monodromy relevant for us. We are going to demonstrate that all opers with real holonomy can be generated from the subset classified by Smirnov by the natural mapping class group action.

The advantage of this analysis is that this example provides a much more direct approach to the problem of classifying opers with real monodromy than the grafting operation. Indeed, we only have a single parameter τ and one variable a and E. In this case, we can make the analytical depence of $a(E)$ on E quite explicit.

Let us denote by the Dehn-Thurston parameters $(p,0)$ the multicurve which corresponds to p 't Hooft loops with respect to the pants decomposition defining the Dehn-Thurston coordinates. We show that we can construct a path between $E_{(0,0)}$ and $E_{(p,0)}$ such that the analytical continuation of a along this path leads to

$$\begin{aligned} \mathfrak{Re}(a) &= \pi p \\ \mathfrak{Re}\left(\frac{\partial \mathcal{Y}(a,\tau)}{\partial a}\right) &= 0 \end{aligned}$$

Using the results by McMullen in [51], we can also construct a path between

$E_{(0,0)}$ and $E_{(0,q)}$ leading to

$$\begin{aligned}\mathfrak{Re}(a) &= 0\\ \mathfrak{Re}\left(\frac{\partial\mathcal{Y}(a,\tau)}{\partial a}\right) &= \pi q\end{aligned}$$

The Dehn-Thurston coordinates $(0,q)$ correspond to q Wilson loops. For these type of loop operators, we do not necessarily need this formalism to determine which integers (n,m) arise, since their effect can be described explicitly in the language of the class S theory by introduction of the operator

$$\mathcal{D}_{(0,q)} = 2\cosh\left(\frac{2\pi a}{\epsilon_2}\right)$$

Nonetheless, the matching of the results coming from the grafting procedure and those coming from the gauge theory, is interesting in its own right.

We end the mathematical discussion with a conjecture giving an answer to the question we posed above.

Conjecture. *Let a possibly punctured Riemann surface X be given. Consider a pants decomposition and specify Dehn-Thurston parameters $(\mathbf{p},\mathbf{q})$. There exists a path in the space of opers over X starting at $\mathbf{E}_{(\mathbf{0},\mathbf{0})}$ and ending at $\mathbf{E}_{(\mathbf{p},\mathbf{q})}$ for any set of parameters $(\mathbf{p},\mathbf{q})$ such that analytical continuation of $\mathbf{a}=\mathbf{a}(\mathbf{E})$ as a function of $\mathbf{E}$ leads to the equations*

$$\begin{aligned}\mathfrak{Re}(a_r) &= \pi p_r\\ \mathfrak{Re}\left(\frac{\partial\mathcal{Y}(\mathbf{a},\mathbf{t})}{\partial a_r}\right) &= \pi q_r\end{aligned}$$

2 Physics and gauge theory background

2.1 Structure of gauge theories with $\mathcal{N} = 2$ supersymmetry

Quantum field theories with $\mathcal{N} = 2$ supersymmetry In this thesis we will consider quantum field theories with supersymmetry. More precisely, we shall focus our attention on gauge theories with $\mathcal{N} = 2$ supersymmetry in four dimensions. If we require the spin of our field content to not exceed one, we can introduce two supersymmetric multiplets known as the vector multiplet and hypermultiplet. The vector multiplet $(A, \lambda, \tilde{\lambda}, \phi)$, contains a gauge field A, two Weyl fermions λ and $\tilde{\lambda}$ and a complex scalar field ϕ. Since the gauge field transforms in the adjoint of the gauge group, so must all the other fields in this multiplet. The hypermultiplet $(\tilde{\psi}^\dagger, \tilde{Q}^\dagger, Q, \psi)$ contains two complex scalars Q and $\tilde{Q}^\dagger$ and two Weyl fermions ψ and $\tilde{\psi}^\dagger$. These fields can transform in any representation of the gauge group, but we shall only consider the fundamental or adjoint representation for our purposes. Additionally, the gauge group we choose will often be SU(2).

For a theory with $\mathcal{N} = 2$ supersymmetry, there often exists an extra symmetry of the theory which does not commute with the supercharges. This $\mathrm{SU}(2)_R$-symmetry is known as the R-symmetry. The group $\mathrm{SU}(2)_R$ rotates the two Weyl fermions λ and $\tilde{\lambda}$ into each other as well as the complex scalars $\tilde{Q}^\dagger$ and Q. The $\mathrm{U}(1)_r$ part which classically combines with $\mathrm{SU}(2)_R$ to form the full U(2)-group rotating the supercharges into each other, breaks down to a discrete symmetry in the quantum theory.

Due to asymptotic freedom, these gauge theories become strongly-coupled in the low energy regime. It is here that non-perturbative effects play an important role and bound states can arise. Nonetheless, the $\mathcal{N} = 2$ supersymmetry constrains the theory in such a way that the IR effective action is uniquely specified by a function known as the prepotential.

When we calculate the IR effective action, the Higgs mechanism generically breaks the gauge group SU(2) to U(1) and the complex scalar ϕ in the vector multiplet obtains a vacuum expectation value

$$\langle \phi \rangle_0 = \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix}$$

If we consider several vector multiplets labeled by r, the complex numbers $u^{(r)} := \langle \mathrm{tr}((\phi^{(r)})^2)\rangle_0 = 2(a_r)^2$ parametrize the subspace of the moduli space of vacua known as the Coulomb branch of the theory. This branch is specified by the vanishing of the vacuum expectation values of the complex scalars in the hypermultiplets. The Higgs branch is defined by a non-trivial vacuum expectation value of the scalars in the hypermultiplet and vanishing vacuum expectation value of the scalar in the vector multiplet. From the point of view of integrable systems, our interest will be in the Coulomb branch.

The prepotential specifying the theory in the IR is denoted by $\mathcal{F} = \mathcal{F}(\mathbf{a}, \boldsymbol{\mu}, \mathbf{t})$, which is a function of the expectation values a_r, the masses μ_k of the hypermultiplets and the gauge couplings $\tau_r = \frac{4\pi i}{g_r^2} + \frac{\theta_r}{2\pi}$. The coupling constants g_r are the standard Yang-Mills couplings, while θ_r are known as the theta-angles describing a topological change to the theory. For example, for the pure Yang-Mills theory we may set $r = 1$ and $a = a_1$. We recover the bosonic effective action by

$$S^{eff}_{boson} = \frac{1}{4\pi}\int d^4x(\mathfrak{Im}(\tau(a))\partial_\mu \bar{a}\partial^\mu a + \frac{1}{2}\mathfrak{Im}(\tau(a))F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}\mathfrak{Re}(\tau(a))F_{\mu\nu}\tilde{F}^{\mu\nu})$$

where

$$\tau(a) = \frac{\partial^2 \mathcal{F}}{\partial a^2}$$

is the coupling constant in the IR, dependent on a. The fermionic part of this action is uniquely determined by requiring supersymmetry invariance.

Seiberg-Witten theory and integrable systems In [1, 2] Seiberg and Witten found the prepotential using an auxiliary Riemann surface Σ_{SW} which is now called the Seiberg-Witten curve. If we pick a basis $B = \{\alpha_1, ..., \alpha_h, \beta_1, ..., \beta_h\}$ of $\mathrm{H}_1(\Sigma_{SW}, \mathbb{Z})$ with h the genus of Σ_{SW}, we may normalize the basis such that the intersection form $\hat{\imath}$ satisfies $\hat{\imath}(\alpha_r, \beta_s) = \delta_{rs}$ and $\hat{\imath}(\alpha_r, \alpha_s) = \hat{\imath}(\beta_r, \beta_s) = 0$. We recover the coordinates a_r and $a_D^r = \frac{\partial \mathcal{F}}{\partial a_r}$ as

$$\begin{aligned} a_r &= \int_{\alpha_r} \lambda_{SW} \\ a_D^r &= \int_{\beta_r} \lambda_{SW} \end{aligned}$$

The one-form λ_{SW} is the canonical one-form obtained by realizing the curve Σ_{SW} as an algebraic variety $P(x, y) = 0$. We may then set $\lambda_{SW} = ydx$.

Due to the Riemann bilinear relations, the functions a_D^r may be locally integrated with respect to the variables a_r to recover the prepotential $\mathcal{F}$. In doing so, we see that the data $(\Sigma_{SW}, \lambda_{SW}, B)$ of the Seiberg-Witten curve, the canonical one-form and the basis of $\mathrm{H}_1(\Sigma_{SW}, \mathbb{Z})$ is enough to recover $\mathcal{F}$. Different bases of $\mathrm{H}_1(\Sigma_{SW}, \mathbb{Z})$ are related by $\mathrm{Sp}(2h, \mathbb{Z})$ transformations. In the low energy physics, these lead to different prepotentials describing the same theory. We may therefore consider this a form of an IR duality of the theory.

Once we have the set of coordinates $(a_1, ..., a_h)$ for the Coulomb branch, we may introduce a torus fibration $\mathbb{C}^h/(\mathbb{Z}^h \oplus \tau\mathbb{Z}^h)$ over the Coulomb branch and consider coordinates for the torus fibres. Here τ is defined precisely through the prepotential as the matrix

$$\tau^{rs}(a) = \frac{\partial^2 \mathcal{F}}{\partial a_r \partial a_s}$$

This gives the space the structure of an integrable system. The data specifying the integrable system is precisely what is needed to specify the prepotential.

The prepotential can be constructed from the data of special geometry [23]. Given these data, there is a canonical way to put torus fibers over the base as above. Conversely, an integrable system is by definition a fibration by complex tori, more precisely abelian varieties. One can prove that such abelian varieties can always be represented in the form $\mathbb{C}^g/(\mathbb{Z}^g \oplus \tau\mathbb{Z}^g)$, allowing one to recover the special geometry structure on the base, and in particular the prepotential.

As we have seen, if we can find a Seiberg-Witten curve, we can construct the prepotential explicitly by integrating along cycles of this curve. Nonetheless, the statement in [23] is of a more general nature as its construction does not involve Seiberg-Witten curves. In fact, it is unknown whether we can always construct a Seiberg-Witten curve for a theory with $\mathcal{N} = 2$ supersymmetry.

The relation between the Seiberg-Witten construction of the prepotential and integrable systems had been understood only after the original work by Seiberg and Witten [11, 12, 13].

Later on, we shall restrict our attention even further to theories of class S. See [9]. For such theories, a Seiberg-Witten curve has been found and has a nice interpretation in terms of the integrable structure of the Hitchin system [24].

Regularizing IR divergences To calculate the partition function of a gauge theory, we first have to cure certain IR divergences by introducing a regularization scheme. This can in principle be done in many different ways, but in this thesis we will focus on two.

First of all, we may introduce an Ω-background [3], which depends on two IR regulators ϵ_1 and ϵ_2 associated to the planes $\mathbb{R}^2_{\epsilon_1}$ and $\mathbb{R}^2_{\epsilon_2}$ together making up $\mathbb{R}^4$. Such a deformation introduces a Lorentz-symmetry breaking deformation of the Lagrangian, but is such that we preserve a certain amount of supersymmetry. Effectively, the IR regulators will mix the rotation in the physical spacetime with the rotation of the supercharges generated by the R-symmetry.

A second way in which we may regularize our theories, is by compactifying the spacetime. Since the IR divergences grow with the volume of our spacetime, compactifying the spacetime makes these contributions finite. We will come back to a special type of such a regularization scheme later.

Localization formalism Furthermore, when we have a certain amount of supersymmetry in our theory, it becomes possible to apply the localization formalism in some interesting cases. This is a powerful tool to make the path integral over the infinite-dimensional space of field configurations well-defined by rewriting it as an integral over a finite-dimensional space. There are many reviews clarifying the localization formalism [52]. For example, localization in the context of $\mathcal{N} = 2$ supersymmetric theories is covered in [7]. Many other aspects of localization are covered in the special volume [53].

Consider the generator Q of a supersymmetry which squares to $Q^2 = P$ a generator of a bosonic symmetry. If the action functional S is invariant under the supersymmetry, we write $QS = 0$. Let us now introduce V a fermionic functional satisfying $PV = 0$. We may consider deforming the action to $S+tQV$ with t a real parameter. The expectation value of an observable now has an explicit dependence on the parameter t, which we may write as

$$\langle \mathcal{O} \rangle_t := \int [\mathcal{D}\Phi] \exp(-S - tQV)\mathcal{O}$$

Here we assume the existence of a supersymmetry invariant measure $\int [\mathcal{D}\Phi](...)$ of the path integral.

Let us now also assume $\mathcal{O}$ is invariant under the supersymmetry generator Q so that $Q\mathcal{O} = 0$. In this case, we may calculate

$$\begin{aligned} \frac{d}{dt}\langle \mathcal{O} \rangle_t &= \int [\mathcal{D}\Phi] \exp(-S - tQV)(-Q(V)\mathcal{O}) \\ &= -\int [\mathcal{D}\Phi] \exp(-S - tQV)Q(V\mathcal{O}) \\ &= 0 \end{aligned}$$

by the supersymmetry invariance of the measure. This shows that the expectation value of $\mathcal{O}$ actually does not depend on the parameter t!

If we furthermore assume QV has a positive semi-definite bosonic part, the limit $t \to \infty$ makes the saddlepoint approximation exact and localizes the integral at the path configurations such that $QV = 0$. If the space of solutions $\mathcal{M} = \{\Phi | QV(\Phi) = 0\}$ is finite-dimensional, the path-integral may be expressed as an ordinary, finite-dimensional integral over $\mathcal{M}$.

Derivation of the prepotential from the partition function in the Ω-background Pioneering work on the proposal by Seiberg and Witten was done in [3, 4, 5, 6]. The proposal was derived by Nekrasov and others by an honest calculation of the partition function in the Ω-background using the method of localization. There are many reviews on these topics such as [54] for the work by Seiberg and Witten and [55] for subsequent works. The localization formalism has been succesfully applied for quiver gauge theories in [56, 57].

Further important work on the study of these partition functions in the presence of defects was performed in Nekrasov's BPS/CFT-series [58, 59, 60, 61, 37].

If we denote the partition function by $\mathcal{Z}_{\epsilon_1,\epsilon_2}(\mathbf{a}, \boldsymbol{\mu}, \mathbf{t})$, depending on the expectation values $\mathbf{a}$ of the scalar fields, mass parameters $\boldsymbol{\mu}$ and the gauge couplings $\mathbf{t}$, the existence of $\mathcal{N} = 2$ supersymmetry implies the partition function splits as a product of three contributions

$$\mathcal{Z}_{\epsilon_1,\epsilon_2} = \mathcal{Z}^{\text{tree}}_{\epsilon_1,\epsilon_2} \mathcal{Z}^{1-\text{loop}}_{\epsilon_1,\epsilon_2} \mathcal{Z}^{\text{inst}}_{\epsilon_1,\epsilon_2}$$

where $\mathcal{Z}^{\text{tree}}_{\epsilon_1,\epsilon_2}$ is the tree-level contribution, $\mathcal{Z}^{1-\text{loop}}_{\epsilon_1,\epsilon_2}$ the one-loop correction and $\mathcal{Z}^{\text{inst}}_{\epsilon_1,\epsilon_2}$ the instanton corrections. This is a result of the fact that the partition function must depend holomorphically on the gauge coupling parameters $\mathbf{t}$, which implies the higher-loop corrections have to vanish. Let us denote

$$Z(\mathbb{R}^4_{\epsilon_1,\epsilon_2}) := \mathcal{Z}_{\epsilon_1,\epsilon_2}(\mathbf{a}, \boldsymbol{\mu}, \mathbf{t})$$

It turns out that there exists a functional V with respect to which we can apply the localization formalism. The instanton partition function $Z(\mathbb{R}^4_{\epsilon_1,\epsilon_2})$ localizes to the self-dual instanton configurations. A direct expression of the moduli space of such solutions was obtained through the ADHM construction [62]. This formalism shows in particular that this moduli space is finite dimensional.

By carefully taking the limit, we may show that

$$\lim_{\epsilon_1,\epsilon_2\to 0}(-\epsilon_1\epsilon_2 \log \mathcal{Z}_{\epsilon_1,\epsilon_2}(\mathbf{a},\boldsymbol{\mu},\mathbf{t})) = \mathcal{F}(\mathbf{a},\boldsymbol{\mu},\mathbf{t})$$

recovering the prepotential.

Localization on the four-sphere Let us now regularize the IR-divergences by compactifying our space time. Instead of $\mathbb{R}^4$, we now consider the squashed four-sphere $S^4_{\epsilon_1,\epsilon_2}$ defined by the coordinate parametrization

$$S^4_{\epsilon_1,\epsilon_2} := \{(x_0,...,x_4)|x_0^2 + \epsilon_1^2(x_1^2 + x_2^2) + \epsilon_2^2(x_3^2 + x_4^2) = 1\}$$

The space $S^4_{\epsilon_1,\epsilon_2}$ admits a supersymmetry which squares to a combination of a spacetime symmetry and an internal symmetry. The localization formalism has been applied originally to quiver gauge theories on the unsquashed four-sphere [21] for which $\epsilon_1 = \epsilon_2 = 1$ and later for ellipsoids in [22], leading to the calculation of the partition function on $S^4_{\epsilon_1,\epsilon_2}$.

The path integral localizes to the field configurations for which the scalar fields have constant values and the other fields vanish. The partition function $Z(S^4_{\epsilon_1,\epsilon_2})$ turns out to have the following structure

$$Z(S^4_{\epsilon_1,\epsilon_2}) = \int d\mathbf{a}|Z(\mathbb{R}_{\epsilon_1,\epsilon_2})|^2 \tag{2.1}$$

We will denote by $\int d\mathbf{a}(...)$ the integral over all scalar fields $a_1,...,a_h$.

Furthermore, it turns out that certain loop observables on $S^4_{\epsilon_1,\epsilon_2}$ preserve enough supersymmetry to be calculated as well. We pick the support of our loop observables to either be given by $\mathcal{C}_1 := \{(x_1,x_2)|x_1^2 + x_2^2 = \epsilon_1^{-2}\}$ or by $\mathcal{C}_2 := \{(x_3,x_4)|x_3^2 + x_4^2 = \epsilon_2^{-2}\}$.

From the localization on $S^4_{\epsilon_1,\epsilon_2}$ it follows that the expectation value of a loop operator $\langle\mathcal{L}\rangle_{S^4_{\epsilon_1,\epsilon_2}}$ is given by

$$\langle\mathcal{L}\rangle_{S^4_{\epsilon_1,\epsilon_2}} = \int d\mathbf{a}\overline{Z(\mathbb{R}_{\epsilon_1,\epsilon_2})}\mathcal{D}_\mathcal{L}\cdot Z(\mathbb{R}_{\epsilon_1,\epsilon_2})$$

where $\mathcal{D}_\mathcal{L}$ is a difference operator acting on the scalar zero mode variables $\mathbf{a}$. Wilson loops have already been studied in [21, 22]. On the squashed four-sphere these can be shown to correspond to pure multiplication operators

$$2\cosh\left(\frac{2\pi a_r}{\epsilon_j}\right)$$

where the parameter j in ϵ_j equals the parameter j of the support $\mathcal{C}_j$ of the loop operator. The action of 't Hooft loops has been studied in [63] on the unsquashed four-sphere. Such loop operators act by difference operators on $Z(\mathbb{R}_{\epsilon_1,\epsilon_2})$ that are not pure multiplications.

2.2 Theories of class S

Quiver construction theories class S In the class of $\mathcal{N}=2$ SU(2) quiver gauge theories, there is a family of theories known as theories of class S [9]. As quiver gauge theories, these theories may be represented by quivers built up from vector multiplets and matter transforming either in the fundamental, bifundamental, trifundamental or adjoint representation of the gauge fields it is charged under. It turns out that these quivers are in one-to-one correspondence with undirected, trivalent, planar graphs which are allowed to have multiple edges between two vertices and edges connecting a vertex to itself. On the other hand, the pants decompositions of a punctured Riemann surface X, i.e. decompositions of X into three-holed spheres (possibly replacing some holes by punctures), are also in one-to-one correspondence with such graphs. We refer to theorem 16.5 for a statement of this kind. The tubes connecting different pants in a decomposition are associated to the vector multiplets, while the punctures define the matter of the theory.

Consider as an example the theory with four flavour groups transforming in the fundamental representation of a single gauge group. This theory is known as the $N_F=4$ theory and can be defined by a quiver of the form shown in figure 2.1.

The blocks correspond to SU(2) flavour groups, while the circles define SU(2) gauge groups. Under the one-to-one correspondence with pants decompositions of punctured Riemann surfaces, this quiver defines a pants decomposition of the four-punctured sphere.

It turns out that this correspondence is more than a curiosity: Each pair (P,Γ) consisting of a pants decomposition P of a Riemann surface X and a gluing pattern Γ understood as a graph embedded in X, defines a different action functional for the theory.

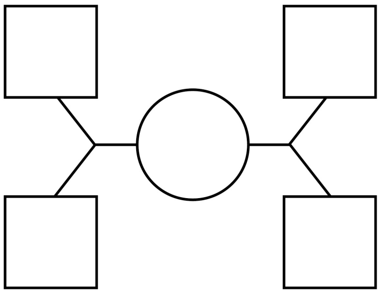


Figure 2.1: Quiver describing the $N_F = 4$ theory

The exactly marginal gauge coupling parameters τ_r for $r = 1, ..., 3g - 3 + n$ are related to the gluing parameters q_r defining the complex structure of X by the relation

$$q_r = e^{2\pi i \tau_r}$$

Although $\tau_r \to \tau_r + 1$ has a trivial effect on the complex structure X, it changes the gluing pattern Γ in a non-trivial way. The gluing pattern therefore contains information about the branch of the equation $2\pi i \tau_r = \log(q_r)$ we are working with. The moduli space $\mathcal{T}_{g,n}$ of these theories is therefore in one-to-one correspondence with the Teichmüller space $\mathcal{T}(S)$ classifying complex structures X together with a marking given by the gluing pattern Γ on a smooth surface S of genus g and with n punctures.

The perturbative limit of these theories can be recovered by letting all $\tau_r \to \infty$. In this region, the Yang-Mills coupling constants become small. From the point of view of the Riemann surface, the tubes gluing together the different pants become elongated. Depending on which pants decomposition and gluing pattern we pick, we can find many different perturbative descriptions of the same theory. This is an example of an S-duality of the theory. Indeed, it is believed that the group of S-dualities for these theories is isomorphic to the mapping class group MCG(S). See section §16 for a discussion of the mapping class group.

Sewing theories of class S together Since Riemann surfaces can be obtained by gluing pairs of pants together, we may look for a similar construction for theories of class S. We refer to [9] for this construction, but we will shortly discuss it here as well.

Our starting point is the trinion theory for flavour group SU(2). The trinion theory contains matter in the SU(2) trifundamental half-hypermultiplet, unlike any of the other theories containing matter in the fundamental, bifundamental or adjoint representations. This theory will play the role of the three-punctured sphere.

We can 'glue' two trinion theories together by gauging a diagonal combination of two of the flavour groups that belong to different pairs of pants. This leads to the moduli space of theories $\mathcal{T}_{0,4}$ containing an SU(2) gauge group coupled to four flavour groups in the fundamental representation. If the two flavour groups belong to the same pair of pants, they combine to form an adjoint and a singlet representation of SU(2). The singlet representation is not charged under the gauge group and therefore is free. We may gauge the former representation such that the remaining flavour group transforms in the adjoint of the gauge group. The moduli space of theories $\mathcal{T}_{1,1}$ then consists of a gauge group and a flavour group in the adjoint plus a free hypermultiplet. We can apply these operations consecutively to find a theory in $\mathcal{T}_{g,n}$.

Theories of class S from M-theory The theories of class S have a well-known construction based on M-theory. This is discussed for example in [24] and the reviews [7, 8]. Consider an M5-brane embedded in the eleven-dimensional spacetime wrapped along the Riemann surface X. We assume the M5-brane is embedded as $\mathbb{R}^4 \times \Sigma$, where $\Sigma \to X$ is a double cover. This defines a six-dimensional, strongly-interacting theory of type A_1 with $\mathcal{N} = (2,0)$ supersymmetry. To construct the theory of class S, we holomorphically twist along the Riemann surface X. This preserves $\mathcal{N} = 2$ supersymmetry orthogonal to X.

The twisted compactification along X has the property that the Coulomb branch geometry is independent of the area of X. At small areas, we recover the description of a theory of class S.

The six-dimensional theory of type A_1 has a single Coulomb branch operator which behaves upon twisting as a holomorphic quadratic differential φ on X. The dimension of the Coulomb branch is therefore given by $3g - 3 + n$ for g the genus of X and n the number of punctures. The positions of the branes from the point of view of the surface X are determined by the roots of the polynomial equation $v^2 = \varphi(u)$ defining the double cover $\Sigma = \{(v, u) \in \mathrm{T}^*X | v^2 = \varphi(u)\}$.

Theories of class S are closely related to Hitchin systems. By compactifying on S^1, the low energy effective action is described by a three-dimensional $\mathcal{N} = 4$ sigma model with target space the total space of a Hitchin integrable model [24]. We also refer to [64] for a description.

Relations to conformal field theory The sewing of these theories is reminiscent of the gluing of conformal blocks in conformal field theories. Indeed, in [10] it has been conjectured that there exists a map between the partition functions of the class S theories and objects known as conformal blocks in Liouville theory. The developments initiated by this proposal have led to considerable amounts of evidence and in some cases even proofs of profound relations between the partition functions introduced before and correlation functions of certain conformal field theories.

Let us first state the duality before diving into the details of conformal field theories. Some basic notions have to be known to understand this duality. We refer to subsection 2.3 for whatever is unclear to the reader at this stage.

The parameters of the Nekrasov partition function $Z(\mathbb{R}^4_{\epsilon_1,\epsilon_2})$ have the following interpretation in the context of conformal field theories

- The parameter $b^2 = \epsilon_1/\epsilon_2$ specifies the central charge c of the Virasoro algebra by $c = 1 + 6(b + b^{-1})^2$.
- The variables $\mathbf{t}$ parameterize the complex structure of the Riemann surface X on which the conformal field theory is studied.
- The variables $\boldsymbol{\mu}$ are parameters of vertex operator insertions.
- Finally, the variables $\mathbf{a}$ are the parameters of representations propagating in intermediate channels.

The AGT-duality states that there exists a one-to-one correspondence between partition functions of theories of class S and conformal blocks (solutions of the conformal Ward identities) and correlators in Liouville theory. The correspondence is as denoted in table 2.1.

It should be noted that conformal blocks do not form the physical correlators of Liouville theory. Generically, the conformal blocks are not single-valued on the Riemann surface X. To define a correlator, we have to integrate over the intermediate momenta $\mathbf{a}$. This is precisely how we arrive at the expression of the partition function $Z(S^4_{\epsilon_1,\epsilon_2})$.

Gauge theory	**Liouville theory**
Ω-background parameters ϵ_1, ϵ_2	Coupling constant $b^2 = \frac{\epsilon_1}{\epsilon_2}$ Central charge $c = 1 + 6Q^2$ Background charge $Q = b + b^{-1}$
Trinion theory	Three-punctured sphere
Mass parameter μ_k associated to an SU(2) flavour	Insertion of a vertex operator V_{μ_k}
An SU(2) gauge group with UV coupling parameter τ_r	A cutting curve with sewing parameter $q_r = e^{2\pi i \tau_r}$
Vacuum expectation value a_r of scalar in vector multiplet	Insertion of V_{α_r} for $\alpha_r = Q/2 + a_r$
$\mathcal{Z}^{\text{inst}}_{\epsilon_1,\epsilon_2}(\mathbf{a}, \boldsymbol{\mu}, \mathbf{t})$	Liouville conformal blocks
$Z(S^4_{\epsilon_1,\epsilon_2}) = \int d\mathbf{a} \lvert \mathcal{Z}_{\epsilon_1,\epsilon_2}(\mathbf{a}, \boldsymbol{\mu}, \mathbf{t})\rvert^2$	Liouville correlator

Table 2.1: The correspondence of the duality discovered in [10]

2.3 A quick overview on conformal field theories

Before we discuss the precise relation between conformal field theories and supersymmetric field theories in the context of the AGT-duality and its extensions, we will give a quick overview of two conformal field theories that will be important for us: the Liouville theory and the WZNW theory. An overview of a mathematical approach on two-dimensional CFTs may be found in [65]. We will follow these notes. We also refer to [40] for a discussion in the context of surface defects.

It should be noted that we can define two-dimensional conformal field theories over any Riemann surface, but for our purposes it suffices to clarify the main points for the sphere.

State-operator correspondence and vertex operators To define a conformal field theory, we first have to introduce a Hilbert space $\mathcal{H}$ which is a unitary representation of $\text{Vir}_c \times \overline{\text{Vir}}_c$ with generators L_k and $\bar{L}_k$.

The algebra Vir_c is known as the Virasoro algebra with generators $L_{k\in\mathbb{Z}}$ satisfying

$$[L_k, L_m] = (k - m)L_{k+m} + \frac{c}{12}(k^3 - k)\delta_{k+m,0}$$

and c the central charge of the algebra. We parametrize the central charge by a number b as $c = 1 + 6Q^2$ where $Q = b + b^{-1}$.

The Hilbert space will be constructed as

$$\mathcal{H} = \bigotimes_{R',R''} M_{R',R''} \otimes R' \otimes R''$$

where R' is a representation of Vir_c and R'' of $\overline{\mathrm{Vir}}_c$. The factor $M_{R',R''}$ describes the multiplicity and transforms trivially under $\mathrm{Vir}_c \times \overline{\mathrm{Vir}}_c$. For a physical theory, we require that R' and R'' are highest-weight representations. We will often assume the existence of a vacuum vector $|0\rangle$ invariant under L_k and $\bar{L}_k$ for $k = -1, 0, 1$. These six generators should correspond to the translations, rotations, dilatations and special conformal transformations.

Furthermore, a quantum field theory on X must contain a set of fields $\Phi_v(w, \bar{w})$ defined on X. We use v as a label for the set of all fields. Additionally, a quantum field theory contains an inner product such that we can define single-valued correlators

$$Z_{\mathbf{v}}(\mathbf{w}, \bar{\mathbf{w}}) = \langle \prod_{r=1}^{n} \Phi_{v_r}(w_r, \bar{w}_r) \rangle$$

An important property of a conformal field theory is the state-operator correspondence. This is an isomorphism $v \mapsto \Phi_v$ between $\mathcal{H}$ and the space of fields. This correspondence allows us to express the correlator Z as depending on the positions of the insertions $\mathbf{w}$ and $\bar{\mathbf{w}}$ and the vectors $\mathbf{v} = (v_1, ..., v_n)$.

If we consider X to be the complex plane, the state-operator correspondence is described by

$$\lim_{w,\bar{w}\to 0} \Phi_v(w, \bar{w})|0\rangle = v$$

The above discussion gives us some of the main ingredients that a physical conformal field theory should possess.

Let us now introduce the stress-energy tensor. In a two-dimensional conformal field theory, the stress-energy tensor splits into a holomorphic and an anti-holomorphic part defined as

$$\begin{aligned} T(y) &:= \sum_{k\in\mathbb{Z}} L_k y^{-k-2} \\ \bar{T}(\bar{y}) &:= \sum_{k\in\mathbb{Z}} \bar{L}_k \bar{y}^{-k-2} \end{aligned}$$

On the level of the correlation functions, the conformal symmetry expresses itself through the fact that the expectation value

$$\langle T(y) \prod_{r=1}^{n} \Phi_{v_r}(w_r, \bar{w}_r) \rangle$$

is holomorphic away from $y = w_r$ for $r = 1, ..., n$. A similar identity holds for the field $\bar{T}(\bar{y})$.

Due to the split of these identities into a holomorphic and anti-holomorphic component, we look for expansions of the correlators of the following form

$$Z_{\mathbf{v}}(\mathbf{w}, \bar{\mathbf{w}}) = \sum_{\beta', \beta''} C_{\beta', \beta'', \mathbf{m}} \mathcal{F}_{\beta', \mathbf{v}'}(\mathbf{w}) \mathcal{F}_{\beta'', \mathbf{v}''}(\bar{\mathbf{w}})$$

Here β' and β'' label basis elements for a space of solutions to the conformal Ward identities and the integers $\mathbf{m}$ label the multiplicities of the vectors $\mathbf{v}$ by $v_r = m_r \otimes v'_r \otimes v''_r$. The functions $\mathcal{F}_{\beta', \mathbf{v}'}(\mathbf{w})$ are known as conformal blocks.

Verma modules and Ward identities We look for representations of the Virasoro algebra Vir_c which have a highest-weight vector. Such representations must come from a quotient of a Verma module for the Virasoro algebra, which is a module $\mathcal{V}_\alpha$ constructed from a highest-weight vector v_α on which the operators $L_{-k<0}$ act as lowering operators, while $L_{k>0}$ annihilate the vector v_α. Furthermore, we have $(L_0 - \Delta_\alpha) v_\alpha = 0$ with $\Delta_\alpha := \alpha(Q - \alpha)$.

Although these modules are freely generated by the operators $L_{-k<0}$, for some values of α it is possible that the Verma module contains null vectors, i.e. vectors v which behave as highest-weight vectors themselves. These null vectors generate a submodule which we quotient the Verma module by. For example, we can show that $(L_{-2} - b^{-2} L_{-1}) v_{-b/2}$ in the Verma module $\mathcal{V}_{-b/2}$ defines a highest-weight vector.

Let us consider $X = \mathbb{C}P^1 \backslash \{z_1, ..., z_n\}$. To each puncture z_r we associate a representation R_r coming from the quotient of a Verma module by the submodule generated by the null vectors. Using this notation, we will rephrase the Ward identities:

We define an action of the Lie algebra $\mathrm{Vect}(X)$ on $\mathcal{R} = \bigotimes_{r=1}^{n} R_r$ by

$$T_\xi = \sum_{r=1}^{n} \sum_{k \in \mathbb{Z}} \xi_k^{(r)} L_k^{(r)}$$

where $L_{-1}^{(r)}$ acts by L_{-1} only on the component of the tensor product in $\mathcal{V}_{\alpha_r}$ and $\xi = \xi(z)\frac{\partial}{\partial z}$ is expanded as

$$\xi(z) = \sum_{k\in\mathbb{Z}} \xi_k^{(r)}(z - z_r)^{k+1}$$

If we denote by $f_X \in \mathrm{Lin}(\mathcal{R}, \mathbb{C})$ a linear functional, the Ward identities are given by

$$f_X(T_\xi v) = 0$$

for all $\xi \in \mathrm{Vect}(X)$ and $v \in \mathcal{R}$. If a functional f_X satisfies these identities, we call it a conformal block. Furthermore, if $v = v_{\alpha_1} \otimes ... \otimes v_{\alpha_n}$ is the tensor product of highest-weight vectors v_{α_r}, we denote

$$\mathcal{Z}_{f_X}(z_1, ..., z_n) := f_X(v)$$

for f_X a conformal block. This makes the dependence on the positions explicit.

Basis of solutions from pants decompositions To understand the appearance of the intermediate labels $\mathbf{a}$, we need to introduce a gluing construction of conformal blocks. This construction takes as its input conformal blocks on two Riemann surfaces with punctures and produces a conformal block on the Riemann surface obtained by gluing these two surfaces together along a given puncture. This creates large families of conformal blocks, which in some cases are known to generate bases for the space (or some subspace) of the conformal blocks. This requires that the punctures we glue together have the same representation associated to them. Such a construction allows us to describe conformal blocks by gluing together three-punctured spheres over which the form of the conformal block is known. These gluing patterns come with labels $\mathbf{a}$ denoting the the representations associated to the punctures we glue together, i.e. the variables $\mathbf{a}$ label the intermediate representations.

Defining conformal blocks over the complex moduli space In our formulation of conformal blocks so far, we have not yet put any restrictions on $\mathcal{Z}_{f_X}(z_1, ..., z_n)$ as a function of its parameters $\mathbf{z} := (z_1, ..., z_n)$. However, from the point of view of physics it is natural to require some form of regularity of $\mathcal{Z}_{f_X}(\mathbf{z})$ in $\mathbf{z}$. We therefore also impose the following axiom:

$$\frac{\partial}{\partial z_r}\mathcal{Z}_{f_X}(\mathbf{z}) := f_X(L_{-1}^{(r)}v) \tag{2.2}$$

We may physically understand this as stating that the operator L_{-1} must behave as the generator of translations in the holomorphic direction.

The action by $L_{-1}^{(r)}$ defines an automorphism on the space of conformal blocks, which allows us to interpret equation (2.2) as a connection on the configuration space of punctures

$$\mathcal{M}_{0,n} = \{(z_1, ..., z_n) | z_r \neq z_s \text{ if } r \neq s\}/\mathrm{PSL}(2, \mathbb{C})$$

In good cases, we may integrate the connection from equation (2.2) and define bases for the horizontal sections over neighbourhoods in $\mathcal{M}_{0,n}$. The corresponding functions $\mathcal{Z}_{f_X}$, for f_X being elements of such a basis, characterize the conformal blocks f_X uniquely through the Ward identities together with equation (2.2).

For a higher genus surface it is also possible to define the space of conformal blocks on which the generators of the Virasoro algebra act as differential operators. This implies that the stress-energy tensor on the one hand defines an automorphism on the space of conformal blocks and on the other hand a differential operator. However, the conformal symmetry and the Ward identities together are usually not enough to describe the conformal blocks as a finite-dimensional vector space over a given Riemann surface. To cut down the dimensionality of this vector space, we need additional relations. One could think for example of introducing null vectors leading to relations between Virasoro generators, or extending the Virasoro algebra to a larger symmetry algebra. We will consider both of these options.

Introducing degenerate representations If we assume some of the representations associated to the punctures are degenerate, we can use the geometrical description of conformal blocks over the complex moduli space to construct differential equations the conformal blocks have to satisfy. Let us consider only the case where the highest-weight vector is given by $v_{\lambda_{n+1}} = v_{-b/2}$ and we set $y = z_{n+1}$ for $X = \mathbb{C}P^1 \backslash \{z_1, ..., z_{n+1}\}$. A conformal block $f_X(v)$ defined on the tensor product of highest-weight vectors, satisfies

$$f_X(L_{-2}^{(n+1)} v) = \sum_{r=1}^{n} \left(\frac{\Delta_r}{(y - z_r)^2} f_X(v) + \frac{1}{y - z_r} f_X(L_{-1}^{(r)} v) \right)$$

where $\Delta_r = \alpha_r(Q - \alpha_r)$. The relation $(L_{-2} - b^{-2}L_{-1}^2)v_{-b/2} = 0$ implies a relation of the form

$$b^{-2}\frac{\partial^2}{\partial y^2}\mathcal{Z}_{f_X}(\mathbf{z}, y) = \sum_{r=1}^{n}\left(\frac{\Delta_r}{(y-z_r)^2} + \frac{1}{y-z_r}\frac{\partial}{\partial z_r}\right)\mathcal{Z}_{f_X}(\mathbf{z}, y)$$

These types of equations are known as the BPZ-equations [36].

Remark that if we introduce a total of $n-3$ variables $\mathbf{y} = (y_1, ..., y_{n-3})$, we find a total of $n-3$ BPZ-type equations.

Extending the symmetry to affine Kac-Moody symmetry Let us now consider the case where we extend the Virasoro symmetry to an affine Kac-Moody symmetry. The affine Kac-Moody algebra is a central extension of the loop algebra of a semi-simple Lie algebra $\mathfrak{g}$. If we let J^a_m be generators of the affine Kac-Moody algebra with a the Lie algebra index and m the loop algebra index, the algebra is defined by

$$[J^a_m, J^b_p] = \sum_{c=1}^{\dim(\mathfrak{g})} f^{ab}_c J^c_{k+m} + mK\delta_{m+p,0}\kappa^{ab}$$

where K is the central extension term, κ the (non-degenerate) Killing form on $\mathfrak{g}$ and f^{ab}_c the structure constants of $\mathfrak{g}$. We denote the affine Kac-Moody algebra by $\hat{\mathfrak{g}}_K$.

A large class of representations for the affine Kac-Moody algebra can be constructed starting from a representation of $\mathfrak{g}$. The Lie algebra $\mathfrak{g}$ naturally embeds into $\hat{\mathfrak{g}}_K$ as the zero mode subalgebra of the enveloping algebra of the Kac-Moody algebra. Starting from a representation R of $\mathfrak{g}$, we may extend R to a representation $\tilde{R}$ of $\hat{\mathfrak{g}}_K$ by assuming

$$\begin{aligned} J^a_{n>0}v &= 0 \\ J^a_0 v &= R(t^a)v \end{aligned}$$

for all $v \in R$. The element t^a is a generator of $\mathfrak{g}$.

The Ward identities in the context of affine Kac-Moody algebras take a form similar to the Ward identities of the Virasoso algebra. In this case we call the linear functional f_X, defined over the Riemann surface X, a conformal block for the affine Kac-Moody algebra if the identity

$$f_X(J_\eta v) = 0$$

holds for all $v \in \tilde{\mathcal{R}} := \bigotimes_{r=1}^{n} \tilde{R}_r$ and all $\eta \in \mathfrak{g} \otimes \mathbb{C}[X]$, meromorphic, $\mathfrak{g}$-valued functions on X. The action is defined by

$$J_\eta = \sum_{r=1}^{n} \sum_{k \in \mathbb{Z}} \sum_{a=1}^{\dim(\mathfrak{g})} \eta_{k,a}^{(r)} J_k^{a,(r)}$$

where $\eta = \eta_a(y) t^a$ has the expansion

$$\eta_a(y) = \sum \eta_{k,a}^{(r)} (y - z_r)^k$$

To show that a theory with affine Kac-Moody symmetry does define a conformal field theory, we use the Sugawara construction [66]. This construction proves that the Virasoro algebra is actually a subalgebra of the affine Kac-Moody algebra. Indeed, if we define

$$L_k := \frac{1}{2(K + \check{h})} \sum_{m \in \mathbb{Z}} \sum_{a,b=1}^{\dim(\mathfrak{g})} \kappa_{ab} : J_{k-m}^a J_m^b :$$

where $\check{h}$ is the dual Coxeter number of $\mathfrak{g}$, it can be checked that L_k generates the Virasoro algebra with central charge

$$c = \frac{K \dim(\mathfrak{g})}{K + \check{h}}$$

Moreover, the conformal blocks for the affine Kac-Moody algebra also define conformal blocks for the Virasoro algebra.

The notation $: J_{k-m}^a J_m^b :$ is called the normal-ordering, which means we place all the raising operators to the right.

Let us consider a vector $v = v_{R_1} \otimes ... \otimes v_{R_n} \in \tilde{\mathcal{R}}$ for $v_{R_r} \in \tilde{R}_r$ coming from $v_{R_r} \in R_r$. We denote a conformal block $f_X(v)$ of this type by

$$\mathcal{Z}_{f_X}(\mathbf{z}|v) := f_X(v)$$

Using the Ward identities and replacing the action of $L_{-1}^{(r)}$ by the derivative $\frac{\partial}{\partial z_r}$, we find the KZ-equations

$$(K + \check{h}) \frac{\partial}{\partial z_r} \mathcal{Z}_{f_X}(\mathbf{z}|v) + \hat{H}_r \mathcal{Z}_{f_X}(\mathbf{z}|v) = 0$$

where

$$\hat{H}_r = \sum_{s \neq r} \frac{\kappa_{ab} R_r(t^a) R_s(t^b)}{z_r - z_s}$$

and $R_r(t^a)$ only acts on the vector v_{R_r}. It can be checked that the operators $\hat{H}_r$ are mutually commuting operators.

We restrict to $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$ and assume the representations $\tilde{R}_r$ come from principal series representations $R_r = \mathcal{P}_{j_r}$ with Casimir element $j_r(j_r + 1)$. The generators of the zero mode subalgebra are represented by differential operators as follows:

$$\begin{aligned} R_r(t^-) &= -\frac{\partial}{\partial x_r} \\ R_r(t^0) &= -x_r \frac{\partial}{\partial x_r} + j_r \\ R_r(t^+) &= x_r^2 \frac{\partial}{\partial x_r} - 2 j_r x_r \end{aligned}$$

Elements in the principal series representation $\mathcal{P}_{j_r}$ are non-holomorphic functions of the variables x_r and $\bar{x}_r$. In this case, the operators $\hat{H}_r$ take the form

$$\hat{H}_r = -\sum_{s \neq r} \frac{1}{z_r - z_s} \left((x_r - x_s)^2 \frac{\partial^2}{\partial x_r \partial x_s} + 2(x_r - x_s) \left(j_s \frac{\partial}{\partial x_r} - j_r \frac{\partial}{\partial x_s} \right) - 2 j_r j_s \right) \tag{2.3}$$

and the conformal block $\mathcal{Z}_{f_X}(\mathbf{z}|\mathbf{x})$ depends on the positions $\mathbf{z} = (z_1, ..., z_n)$ and the parameters $\mathbf{x} = (x_1, ..., x_n)$. The parameters $\mathbf{x}$ define conformal blocks twisted by a holomorphic bundle. We will not go into this construction in more detail and refer instead to [19, 67].

2.4 Introducing defects in the gauge theory

We may define more general partition functions in the presence of certain supersymmetric defects. The type of defects that can occur in the context of AGT and its generalizations, can be classified from an M-theoretical point of view. We follow the discussion in [30]. For a classification of defects and the construction of theories of class S through an M-theoretical approach, we again refer to the reviews [7, 8].

From the M-theory point of view, we may introduce several different defects in our theories. These will come from two different brane set-ups: Either we introduce an M2-brane which has a boundary on the stack of M5-branes or we introduce another stack of M5-branes and intersect our original set-up with those. Depending on where these branes live precisely, we can get different defects.

The former type of set-up gives rise to two-dimensional defects from the point of view of the six-dimensional theory. The second set-up leads to a four dimensional defect. There are three types of defects that will be important for us. We may schematically describe these defects as in table 1.1.

If we intersect an M2-brane with the stack of M5-branes, we can either intersect the M2-brane completely with the four-dimensional theory such that it lives at a point with respect to the Riemann surface X, or we can intersect the M2-brane with a one-dimensional subspace of both the four-dimensional space and the Riemann surface X. The former set-up constructs a surface defect in the four-dimensional theory, while the latter defines loop operators.

If we take another stack of M5-branes and intersect the original stack of M5-branes, there is only one defect of interest for us, namely the one that defines a surface defect in the four-dimensional theory.

Each of these surface defects in the Ω-background $\mathbb{R}^4_{\epsilon_1,\epsilon_2}$ and on $S^4_{\epsilon_1,\epsilon_2}$ finds a natural place in the context of generalizations of the AGT-duality.

Co-dimension four surface operators To preserve supersymmetry there are two ways in which we can introduce co-dimension four operators as surface operators in the four-dimensional theory, depending on whether we embed the surface operator completely in the plane associated to the regulator ϵ_1 or to the regulator ϵ_2. The surface operator couples the four-dimensional theory to a two-dimensional gauged linear sigma model supported on the surface operator. This introduces a discrete label $\mathbf{s} = (s_1, ..., s_d)$ for the different types of surface operators, classified by the vacuum states of the sigma model, and Fayet-Iliopoulos parameters $\mathbf{y} = (y_1, ..., y_d)$. The partition functions $Z(\mathbb{R}^4_{\epsilon_1,\epsilon_2})$ are generalized to functions $\mathcal{Z}_{\epsilon_1,\epsilon_2}(\mathbf{a}, \boldsymbol{\mu}, \mathbf{s}, \mathbf{t}, \mathbf{y})$. The partition functions $Z(S^4_{\epsilon_1,\epsilon_2})$ generalize to

$$Z(S^4_{\epsilon_1,\epsilon_2}) = \sum_{\mathbf{s}} \int d\mathbf{a} |\mathcal{Z}_{\epsilon_1,\epsilon_2}(\mathbf{a}, \boldsymbol{\mu}, \mathbf{s}, \mathbf{t}, \mathbf{y})|^2$$

including a sum over the vacua of the sigma model. Details of this construction and an interpretation in terms of branes can be found in [68].

From the viewpoint of the AGT-duality, the partition functions $\mathcal{Z}_{\epsilon_1,\epsilon_2}(\mathbf{a}, \boldsymbol{\mu}, \mathbf{s}, \mathbf{t}, \mathbf{y})$ turn out to map to conformal blocks with extra insertions of degenerate fields at the positions $\mathbf{y}$. This was found in [69] using the method of topological recursion, generalizing the results from [32] which only considered the case of a single insertion of a surface operator. An important check was performed in [70].

The articles cited here form first steps in understanding this conjecture, but many others followed that we have not mentioned here. We once again refer to the review [8] for a more complete picture of the history.

The conformal blocks with extra insertions of degenerate fields satisfy BPZ-type partial differential equations with respect to each of the positions y_r. Under the duality the variables s_r label elements in a basis of solutions to these BPZ-equations on the conformal field theory side. The insertion of a surface operator in the ϵ_1-plane corresponds to the insertion of a degenerate field $V_{-b/2}$, while inserting a surface operator in the ϵ_2-plane leads to the insertion of a degenerate field $V_{-1/(2b)}$.

If we consider $X = \mathbb{C}P^1 \backslash \{z_1, ..., z_n\}$ to be the punctured sphere and describe the complex moduli $\mathbf{t}$ in terms of the positions $\mathbf{z} = (z_1, ..., z_n)$, the insertion of a co-dimension four surface operator with position y on the sphere induces a BPZ-type equation on the partition function $\mathcal{Z}_{\epsilon_1,\epsilon_2}(\mathbf{a}, \boldsymbol{\mu}, s, \mathbf{z}, y)$ through the extended AGT-duality. This differential equation takes the form

$$\left(\frac{\epsilon_2}{\epsilon_1}\partial_y^2 + \sum_{r=1}^{n}\left(\frac{1}{y-z_r}\partial_{z_r} + \frac{\Delta_r}{(y-z_r)^2}\right)\right)\mathcal{Z}_{\epsilon_1,\epsilon_2}(\mathbf{a}, \boldsymbol{\mu}, s, \mathbf{z}, y) = 0$$

The exchange $\epsilon_1 \leftrightarrow \epsilon_2$ corresponds to the exchange $b \leftrightarrow b^{-1}$.

Co-dimension two surface operators The surface operators coming from co-dimension two operators in the six-dimensional theory are introduced by imposing singular behaviour along the support of the defects in the gauge fields. The possible types of singular behaviour are characterized by continuous parameters $\mathbf{x} = (x_1, ..., x_h)$. The partition functions $Z(\mathbb{R}^4_{\epsilon_1,\epsilon_2})$ generalize to functions $\mathcal{Z}_{\epsilon_1,\epsilon_2}(\mathbf{a}, \boldsymbol{\mu}, \mathbf{t}, \mathbf{x})$. The four-sphere partition functions take the form

$$Z(S^4_{\epsilon_1,\epsilon_2}) = \int d\mathbf{a} |\mathcal{Z}_{\epsilon_1,\epsilon_2}(\mathbf{a}, \boldsymbol{\mu}, \mathbf{t}, \mathbf{x})|^2$$

On the side of the conformal field theory, the Liouville conformal block is upgraded to a conformal block of the affine Kac-Moody algebra $\hat{\mathfrak{sl}}_{2,K}$ [30]. The extension of this proposal to arbitrary $\mathfrak{sl}(N, \mathbb{C})$ was determined in [31].

The parameters $\mathbf{x}$ now parametrize the choice of a holomorphic bundle on the Riemann surface X. The central charge term K can be recovered in terms of ϵ_1

and ϵ_2 through the relation

$$K + 2 = -\frac{\epsilon_2}{\epsilon_1}$$

Through the AGT-duality, the KZB-equation for conformal blocks leads to a KZB-type equation for the partition functions $\mathcal{Z}_{\epsilon_1,\epsilon_2}(\mathbf{a}, \boldsymbol{\mu}, \mathbf{t}, \mathbf{x})$. Let us consider $X = \mathbb{C}P^1 \backslash \{z_1, ..., z_n\}$ to be the punctured sphere once again and relate the complex moduli $\mathbf{t}$ to the punctures $\mathbf{z} = (z_1, ..., z_n)$. In this case, we find the following relation on the partition function

$$-\frac{\epsilon_2}{\epsilon_1}\frac{\partial}{\partial z_r}\mathcal{Z}_{\epsilon_1,\epsilon_2}(\mathbf{a}, \boldsymbol{\mu}, \mathbf{z}, \mathbf{x}) = \hat{H}_r \mathcal{Z}_{\epsilon_1,\epsilon_2}(\mathbf{a}, \boldsymbol{\mu}, \mathbf{z}, \mathbf{x})$$

if we insert the surface operator in the ϵ_1-plane. For a surface operator in the ϵ_2-plane, the prefactor $-\frac{\epsilon_2}{\epsilon_1}$ must be replaced by $-\frac{\epsilon_1}{\epsilon_2}$. This replaces $K + 2$ by $(K + 2)^{-1}$. In [37] it was stated that a proof of the fact that the partition function satisfies the KZ-equation had been found for the four-punctured sphere case.

The differential operators $\hat{H}_r$ are the quantum Hitchin Hamiltonians on X taking the form of equation (2.3). The Hamiltonians can be generalized to other Riemann surfaces as well.

Relations between co-dimension two and co-dimension four surface operators Considerations from M-theory suggest an IR duality between surface operators of co-dimension two and particular systems of surface operators of co-dimension four [40]. The basic mechanism underlying this duality is a variant of the Hanany-Witten effect.

In the context of the AGT-correspondence, such a duality would predict relations between the conformal blocks of the Virasoro algebra and of the affine algebra $\hat{\mathfrak{sl}}_{2,K}$. It has been pointed out in [40] that the relations between conformal blocks underlying the relations between correlation functions of Liouville theory and of the H_3^+-WZNW model fit very well to the relations between partition functions expected to follow from the Hanany-Witten effect. The abovementioned relations between conformal blocks generalize the integral representation constructed using the separation of variables method of Sklyanin [71] for the eigenfunctions of the Gaudin model.

Introducing loop operators The expectation value of Wilson loop operators on $S^4_{\epsilon_1,\epsilon_2}$ was calculated in [21, 22, 63]. Let us recall from subsection 1.4 that these calculations have been extended in [32, 41] and give rise to equation (1.3).

The loop operators in the gauge theory can be inserted either on the circle $\mathcal{C}_1 = \{(x_1, x_2)|\epsilon_1^2(x_1^2 + x_2^2) = 1\}$ or on $\mathcal{C}_2 = \{(x_3, x_4)|\epsilon_2^2(x_3^2 + x_4^2) = 1\}$. This leads to the insertion of degenerate fields in the correlators of the conformal field theory on the other side of the AGT-duality.

2.5 Relations to integrable models

Relations between field theories in the Ω-background and integrable models In [14] Nekrasov and Shatashvili have proposed a profound relation between partition functions $Z(\mathbb{R}^4_{\epsilon_1,\epsilon_2})$ and integrable models deforming the relations between Seiberg-Witten theory and (classically) integrable models to a relation between a decompactification limit of supersymmetric gauge theories in the Ω-background.

The relation they discuss, appears in the limit $\epsilon_2 \to 0$. It is argued that in this decompactification limit, the four-dimensional $\mathcal{N} = 2$ theory gets effectively represented by a two-dimensional $\mathcal{N} = (2, 2)$ supersymmetric theory. The effective action of this theory appears on the plane $\mathbb{R}^2 \subset \mathbb{R}^4_{\epsilon_1,0}$ with vanishing Ω-deformation by integrating out the fluctuations with masses determined by the remaining parameter ϵ_1. The free energy of this $\mathcal{N} = (2, 2)$ theory defined on $\mathbb{R}^2$, can be obtained as the limit

$$\mathcal{Y}(\mathbf{a}, \boldsymbol{\mu}, \mathbf{t}) = \lim_{\epsilon_2 \to 0} \left(-\epsilon_2 \log \mathcal{Z}_{\epsilon_1,\epsilon_2}(\mathbf{a}, \boldsymbol{\mu}, \mathbf{t})\right)$$

Since the supersymmetric vacua are determined by the minimization of (a shifted version of) the free energy, it follows that there exists a relation between the extrema of $\mathcal{Y}(\mathbf{a}, \boldsymbol{\mu}, \mathbf{t}) - 2\pi \sum_{r=1}^{d} n_r a_r$ called the (shifted) twisted effective superpotential and the vacua of the $\mathcal{N} = (2, 2)$ theory. Here d is the dimension of the Coulomb branch and $(n_1, ..., n_d)$ a set of integers.

The space of vacua is acted upon by the twisted chiral ring generating a commutative family of observables. Nekrasov and Shatashvili propose to identify the generators of this chiral ring with the Hamiltonians of the quantum integrable system obtained by quantizing the integrable system appearing on the Coulomb branch through the relationship in [23]. The eigenstates of the generators of the chiral ring are associated to states describing the spectrum of the quantum integrable system.

All of these facts together amount to the prediction that the vacua of the effective two-dimensional theory, characterized by equation (1.1), are in one-to-one correspondence with the eigenstates in the spectrum of the quantum integrable model.

The fact that we can characterize the eigenstates and spectrum of many quantum integrable models in terms of conditions as in equation (1.1), is a highly non-trivial prediction by the work of Nekrasov and Shatashvili. For theories which allow for a solution by the Bethe ansatz, such descriptions have been well-known following the work of Yang and Yang in [17]. The role of the parameters a_r is played by the auxiliary variables in this case and extremization with respect to this variable leads to the Bethe ansatz equations.

The integrable systems appearing in the context of supersymmetric field theories, cannot in general be solved by the Bethe ansatz. Famous examples include the closed Toda chain and the Calogero-Moser model. For this class of integrable systems, it had not been known that there exist functions as in equation (1.1) allowing us to represent the quantization conditions of the quantum integrable model. The parameters m and τ take the role of parameters of the integrable model, while ϵ_1 becomes Planck's constant. Functions $\mathcal{Y}(\mathbf{a},\boldsymbol{\mu},\mathbf{t})$ describing the quantization conditions in this way, are often called Yang-Yang functions.

Through the relation to supersymmetric field theories, we furthermore have a concrete mathematical description of the functions $\mathcal{Y}(\mathbf{a},\boldsymbol{\mu},\mathbf{t})$. By carefully analyzing the limit $\epsilon_2 \to 0$ of the known series representations for $\mathcal{Z}_{\epsilon_1,\epsilon_2}(\mathbf{a},\boldsymbol{\mu},\mathbf{t})$, it was found in [14] that $\mathcal{Y}(\mathbf{a},\boldsymbol{\mu},\mathbf{t})$ may be calculated from the solutions to certain nonlinear integral equations similar to the equations describing the thermodynamic Bethe ansatz. For the case of the closed Toda chain, it was shown in [72] that the quantization conditions derived previously by [73, 74, 75], can be rewritten in the form of equation (1.1) with $\mathcal{Y}(\mathbf{a},\boldsymbol{\mu},\mathbf{t})$ being exactly the function defined using thermal Bethe ansatz type integral equations in [14].

Relations between the AGT-duality and quantum integrable models By interpreting the proposal by Nekrasov and Shatashvili through the lens of the AGT-duality, we get an alternative mathematical characterization of $\mathcal{Y}(\mathbf{a},\boldsymbol{\mu},\mathbf{t})$. By using the relations between partition functions $Z(\mathbb{R}^4_{\epsilon_1,\epsilon_2})$ in the Ω-background and conformal blocks, we may calculate the limit $\epsilon_2 \to 0$ from the point of view of conformal field theory. This limit leads to the characterization of $\mathcal{Y}(\mathbf{a},\boldsymbol{\mu},\mathbf{t})$ in terms of the symplectic geometry of the moduli space of flat connections on the Riemann surface X. For further discussions on this topic, see [18, 19].

From the point of view of integrable models one thereby gets a very interesting alternative, more geometric picture of the Yang-Yang function characterizing the quantization conditions of some quantum integrable systems. Of particular interest in our context is the limit $\epsilon_2 \to 0$ of partition functions in the presence of surface operators of co-dimension two. As already suggested in [30] we get a physical realization of the eigenfunctions associated to the solutions of the quantization conditions equation (1.1).

2.6 Open questions

Although these developments lead to profound relations between gauge theories, conformal field theories and integrable systems, there are some basic questions left unanswered. Let us consider two questions that will take precedent in our context.

First of all, many algebraically integrable systems admit more than one possible quantization condition. This comes from the fact that these integrable models may admit more than one reality condition allowing us to define real phase space coordinates from an algebraically integrable model with complex phase space coordinates. Some simple examples where this happens are provided by spin chains with representations of a complex semi-simple group attached to each lattice site. The quantization condition will depend on the real form under consideration, like SU(2) or $\mathrm{SL}(2,\mathbb{R})$ in $\mathrm{SL}(2,\mathbb{C})$, and the representations in which we find eigenfunctions of the quantum Hamiltonians. For example, we might consider discrete series representations, highest-weight representations or principal continuous series representations, which in general leads to different quantization conditions.

Even if an integrable model appears in relation to gauge theories as in the context of [14], it is not a priori clear which quantization conditions can be expressed in the form of equation (1.1) for a suitable Yang-Yang function $\mathcal{Y}(\mathbf{a}, \boldsymbol{\mu}, \mathbf{t})$. On top of that, if some set of quantization conditions does appear, it is not clear if other quantization conditions admit a similar representation ideally using the same Yang-Yang function.

The relations discovered by Nekrasov and Shatashvili could initiate a new way of looking for the solution of a spectral problem in quantum integrable systems, but the effectiveness of this approach depends sensitively on whether the answers to the above questions are known.

Secondly, it seems natural to expect that the partition functions on $S^4_{\epsilon_1,\epsilon_2}$ should also have an interesting limit $\epsilon_2 \to 0$ from the point of view of quantum integrable models. We have shown already in subsection 1.5 that we should expect a change in Bethe quantum numbers if we introduce loop operators. Although these equations resemble equation (1.1), it is a priori not clear whether they have an interpretation in the quantization of the integrable model.

2.7 Loop operators and single-valued eigenfunctions

In this section we will expand on the questions posed in 1.5.

Bethe quantum numbers and loop operators Using the proposal by Nekrasov and Shatashvili, we find

$$Z(\mathbb{R}^4_{\epsilon_1,\epsilon_2}) = \exp\left(-\frac{1}{\epsilon_2}\mathcal{Y}(\mathbf{a},\mathbf{t})\right)(1+\mathcal{O}(\epsilon_2))$$

in the limit $\epsilon_2 \to 0$. Here $\mathcal{Y}(\mathbf{a},\mathbf{t})$ is the Yang-Yang function for the Hitchin integrable system.

By plugging in this expression into the factorized relation of equation (1.2) found by Pestun, we obtain an integral expression

$$Z(S^4_{\epsilon_1,\epsilon_2}) = \int d\mathbf{a}\exp\left(-\frac{2}{\epsilon_2}\mathfrak{Re}(\mathcal{Y}(\mathbf{a},\mathbf{t}))\right)(1+\mathcal{O}(\epsilon_2))$$

which may be calculated by the saddlepoint approximation, becoming exact in the limit $\epsilon_2 \to 0$. This implies that the partition function $Z(S^4_{\epsilon_1,\epsilon_2})$ is dominated by the values $\mathbf{a} = \mathbf{a}_*$ where

$$\mathfrak{Re}\left(\frac{\partial\mathcal{Y}(\mathbf{a},\mathbf{t})}{\partial a_r}\right)\bigg|_{\mathbf{a}=\mathbf{a}_*} = 0$$

and the parameters $\mathbf{a}$ must satisfy

$$\mathfrak{Re}(a_r)|_{\mathbf{a}=\mathbf{a}_*} = 0$$

Comparing to the results in [20], we see that such a set-up leads to values $(\mathbf{n},\mathbf{m}) = (\mathbf{0},\mathbf{0})$ for the Bethe quantum numbers (as defined in subsection 1.5).

Let us now introduce loop operators on $\mathcal{C}_2$ in the decompactification limit $\epsilon_2 \to 0$. These loop operators will modify the asymptotic behaviour of the integrand of the partition function $Z(S^4_{\epsilon_1,\epsilon_2})$. One of the simplest cases is given by the insertion of a Wilson loop operator, which may be characterized by parameters $p_s = 0$ and $q_s = \delta_{rs}$ for a given value of r. An analysis along the lines of [21, 22] shows that such loop operators lead to difference operators acting by a simple multiplication by a factor of the form

$$2\cosh\left(\frac{2\pi a_r}{\epsilon_2}\right)$$

In the limit $\epsilon_2 \to 0$, it is clear that this changes the saddlepoint approximation to

$$\mathfrak{Re}\left(\frac{\partial \mathcal{Y}(\mathbf{a},\mathbf{t})}{\partial a_r}\right)\Bigg|_{\mathbf{a}=\mathbf{a}_*} = \pi$$

From this perspective, we see that the introduction of loop operators can affect the Bethe quantum numbers $(\mathbf{n},\mathbf{m})$ appearing in the comparison with the quantization conditions in [20].

We can easily generalize this example to the case where we insert multiple Wilson loops. Such a set-up defines a loop operator with Dehn-Thurston parameters $(\mathbf{0},\mathbf{q})$. The difference operator takes the form

$$\mathcal{D}_{(\mathbf{0},\mathbf{q})} = \prod_r \left(2\cosh\left(\frac{2\pi a_r}{\epsilon_2}\right)\right)^{q_r}$$

Therefore, we find

$$\mathfrak{Re}\left(\frac{\partial \mathcal{Y}(\mathbf{a},\mathbf{t})}{\partial a_r}\right)\Bigg|_{\mathbf{a}=\mathbf{a}_*} = \pi q_r$$

showing that we may identify $\mathbf{q}$ and $\mathbf{m}$. Hence, the Wilson loop operators have a clear interpretation in terms of the Bethe quantum numbers.

A similar calculation for a general dyonic loop operator, however, is not so simple, since a general dyonic loop can lead to shifts in the variables $\mathbf{a}$. Although a Wilson loop does not affect the integer $\mathbf{n}$, a general dyonic loop operator therefore could have such an effect.

The partition function $Z(S^4_{\epsilon_1,\epsilon_2})$ and single-valued eigenfunctions We will use the co-dimension two surface operator, inserted in the ϵ_1-plane, as a probe to find single-valued eigenfunctions to the quantum Hitchin Hamiltonians from the partition function $Z(S^4_{\epsilon_1,\epsilon_2})$.

Let us first go back to the holomorphic story: If we introduce such a surface operator in $\mathbb{R}^4_{\epsilon_1,\epsilon_2}$, we must find an expression of the form

$$Z(\mathbb{R}^4_{\epsilon_1,\epsilon_2}) = \exp\left(-\frac{1}{\epsilon_2}\mathcal{Y}(\mathbf{a},\mathbf{t})\right)\psi(\mathbf{a},\boldsymbol{\mu},\mathbf{t},\mathbf{x})\,(1+\mathcal{O}(\epsilon_2))$$

The partition function $Z(\mathbb{R}^4_{\epsilon_1,\epsilon_2})$ develops a dependence on parameters $\mathbf{x}$, which are the twisting parameters defining a holomorphic bundle over the Riemann surface X and coordinates for $\mathrm{Bun}_{\mathrm{SL}(2,\mathbb{C})}(X)$. The generalization of the AGT-duality proposed by Alday and Tachikawa relates the holomorphic functions $\mathcal{Z}_{\epsilon_1,\epsilon_2}(\mathbf{a},\boldsymbol{\mu},\mathbf{t},\mathbf{x})$ to conformal blocks of the affine Kac-Moody algebra $\hat{\mathfrak{sl}}_{2,k}$ [30, 37].

The relation to the WZNW model implies that $\mathcal{Z}_{\epsilon_1,\epsilon_2}(\mathbf{a},\boldsymbol{\mu},\mathbf{t},\mathbf{x})$ satisfies the KZB-equation

$$-\frac{\epsilon_2}{\epsilon_1}\frac{\partial}{\partial t_r}\mathcal{Z}_{\epsilon_1,\epsilon_2}(\mathbf{a},\boldsymbol{\mu},\mathbf{t},\mathbf{x}) = \hat{H}_r\mathcal{Z}_{\epsilon_1,\epsilon_2}(\mathbf{a},\boldsymbol{\mu},\mathbf{t},\mathbf{x})$$

where $\hat{H}_r$ is a differential operator acting on the variables $\mathbf{x}$ coming from the quantization of the Hitchin Hamiltonians.

The partition function $Z(S^4_{\epsilon_1,\epsilon_2})$ is mapped to

$$G(\boldsymbol{\mu},\mathbf{t},\bar{\mathbf{t}},\mathbf{x},\bar{\mathbf{x}}) = \exp\left(-\frac{2}{\epsilon_2}\mathfrak{Re}(\mathcal{Y}(\mathbf{a}_*,\mathbf{t}))\right)\Psi(\boldsymbol{\mu},\mathbf{t},\bar{\mathbf{t}},\mathbf{x},\bar{\mathbf{x}})\,(1+\mathcal{O}(\epsilon_2))$$

where the functions $G(\boldsymbol{\mu},\mathbf{t},\bar{\mathbf{t}},\mathbf{x},\bar{\mathbf{x}})$ are correlation functions for the H_3^+ WZNW model. If we set

$$E_r = \frac{1}{\epsilon_1}\frac{\partial\mathcal{Y}(\mathbf{a}_*,\mathbf{t})}{\partial\tau_r} \qquad \bar{E}_r = \frac{1}{\epsilon_1}\frac{\partial\bar{\mathcal{Y}}(\mathbf{a}_*,\bar{\mathbf{t}})}{\partial\bar{\tau}_r}$$

the functions Ψ must satisfy the equations

$$\hat{H}_r\Psi = E_r\Psi \qquad \hat{\bar{H}}_r\Psi = \bar{E}_r\Psi$$

and $\mathbf{a}_*$ solving

$$\mathfrak{Re}\left(\frac{\partial\mathcal{Y}(\mathbf{a},\mathbf{t})}{\partial a_r}\right)\Bigg|_{\mathbf{a}=\mathbf{a}_*} = 0$$

It turns out that Ψ is single-valued as we shall see. Before we go into detail, let us note that we only recover a single, special eigenfunction from $Z(S^4_{\epsilon_1,\epsilon_2})$ in this way. There is no hope to construct other single-valued eigenfunctions from the partition function.

The separation of variables for Gaudin models We will sketch the main ideas very briefly, since the separation of variables for Gaudin models will be discussed in greater detail in subsection 3.4.

Starting from the correlation function G, we may apply an integral transformation of the form

$$G(\boldsymbol{\mu}, \mathbf{t}, \bar{\mathbf{t}}, \mathbf{x}, \bar{\mathbf{x}}) = \int d^2\mathbf{u}\, \mathcal{K}(\mathbf{x}, \bar{\mathbf{x}}, \mathbf{u}, \bar{\mathbf{u}}) H(\boldsymbol{\mu}, \mathbf{t}, \bar{\mathbf{t}}, \mathbf{u}, \bar{\mathbf{u}})$$

where $\mathcal{K}(\mathbf{x}, \bar{\mathbf{x}}, \mathbf{u}, \bar{\mathbf{u}})$ is explicitly known for theories of class S associated to genus zero surfaces [40]. The function $H(\boldsymbol{\mu}, \mathbf{t}, \bar{\mathbf{t}}, \mathbf{u}, \bar{\mathbf{u}})$ is a correlation function of Liouville theory in the presence of a certain number of degenerate fields. This integral transformation relates the KZB-equations satisfied by G to the BPZ-equations satisfied by H.

In the Nekrasov-Shatashvili limit $\epsilon_2 \to 0$, we obtain a relation of the form

$$\Psi(\boldsymbol{\mu}, \mathbf{t}, \bar{\mathbf{t}}, \mathbf{x}, \bar{\mathbf{x}}) = \int d^2\mathbf{u}\, \mathcal{K}(\mathbf{x}, \bar{\mathbf{x}}, \mathbf{u}, \bar{\mathbf{u}}) \prod_k \chi(u_k, \bar{u}_k)$$

with $\chi(u_k, \bar{u}_k)$ satisfying the relevant limit of the BPZ-equations

$$(\partial_u^2 + t(u))\chi(u, \bar{u}) = 0 \qquad\qquad (\partial_{\bar{u}}^2 + t(\bar{u}))\chi(u, \bar{u}) = 0$$

where

$$t(u) = \sum_{r=1}^{n} \left(\frac{\delta_r}{(u - z_r)^2} + \frac{E_r}{u - z_r} \right)$$

for the case of genus zero. In this case, the complex moduli $\mathbf{t}$ may be expressed in terms of the positions of the punctures $\mathbf{z} = (z_1, ..., z_n)$. The residues E_r are obtained by

$$\frac{1}{\epsilon_1} \frac{\partial}{\partial z_r} \mathcal{Y}(\mathbf{a}_*, \boldsymbol{\mu}, \mathbf{z}) = E_r$$

Single-valuedness of the eigenfunction Ψ now follows from single-valuedness of the function χ.

Introducing loop operators If we insert a loop operator $\mathcal{L}$ in the ϵ_2-plane, we would expect a representation of the form

$$\langle \mathcal{L} \rangle_{S^4_{\epsilon_1, \epsilon_2}} = \int d\mathbf{a} \overline{\mathcal{Z}_{\epsilon_1, \epsilon_2}(\mathbf{a}, \boldsymbol{\mu}, \mathbf{t}, \mathbf{x})} \mathcal{D}_{\mathcal{L}} \cdot \mathcal{Z}_{\epsilon_1, \epsilon_2}(\mathbf{a}, \boldsymbol{\mu}, \mathbf{t}, \mathbf{x})$$

generalizing the factorized form that is known in the case without surface operators. If we now take the $\epsilon_2 \to 0$ limit, we expect the same construction of a single-valued eigenfunction Ψ to hold as for the case $Z(S^4_{\epsilon_1,\epsilon_2})$, but possibly leading to different eigenvalues E_r and $\bar{E}_r$ due to the adjustment of the Bethe quantum numbers. In any case, we may conclude that the expectation of a loop operator in the ϵ_2-plane in the presence of a surface operator of co-dimension two living in the ϵ_1-plane, gives rise to a single-valued eigenfunction Ψ of the quantum Hitchin Hamiltonians.

3 Quantization of the Hitchin system

3.1 Classical and quantum integrable systems

The Liouville-Arnold theorem is a keystone in defining integrability for a Hamiltonian dynamical system in a classical sense. Given a Poisson bracket on a smooth manifold (known as the phase space) $\mathcal{P}$, the Liouville-Arnold theorem states that if we find $\frac{1}{2}\dim(\mathcal{P})$ functions in involution, we can introduce action-angle coordinates for our integrable system. If the fibres of the map fixing the values of these functions are compact, they will be diffeomorphic to tori. The motion of the integrable system becomes linearized on this torus fibre and can be solved by integration.

When we quantize our integrable system, we replace the functions on the phase space by self-adjoint operators on a Hilbert space and the Poisson bracket is replaced by a commutator of operators. The problem we now address is finding the spectrum of a self-adjoint operator for eigenvectors in the given Hilbert space. The choice of a Hilbert space might restrict the spectrum of the operator, often times ensuring the spectrum becomes discrete. Two examples of conditions we might require are square-integrability and single-valuedness.

The notion of quantum integrability is less understood than in the classical case due to the lack of the Arnold-Liouville theorem. Although we might construct many operators in involution, integrability is not yet guaranteed. Two well-known techniques that can be used to solve quantum integrable systems which admit what is known as a Lax formulation, are the Bethe ansatz and the separation of variables method. The Bethe ansatz requires the introduction of a reference state in the Hilbert space from which we produce all other vectors by applying lowering operators defined through the Lax matrix. Although this technique can be applied widely, a reference state is needed. When such a state cannot be found or does not exist, we cannot apply the Bethe ansatz.

The separation of variables describes a representation of the Hilbert space for which the eigenfunctions can be represented as the product of one and the same function evaluated at different values of the argument. The function appearing in this product representation can be characterized as the solution of difference or differential equations called Baxter equations.

Hitchin integrable system The focus of this thesis lies on the quantization of the Hitchin system defined in [15, 16]. This is a family of complex integrable systems with input a Lie group and a Riemann surface on which the spectral parameter lives that gives rise to many interesting examples. The Gaudin model and the elliptic Calogero-Moser model can both be obtained from this construction if we assume the Riemann surface is a punctured sphere and once-punctured torus respectively. To describe much of the machinery we discuss in more concrete terms, we will often look at the Gaudin model as a hands-on example. The techniques that can be applied to this model, have a natural generalization to Riemann surfaces of higher genus.

It was shown by Frenkel in [76] that for the SU(2)-Gaudin model the eigenfunctions of the Hamiltonians constructed from the Bethe ansatz, have trivial monodromy as functions living on the punctured Riemann sphere. In this case, the Hilbert space in which the solutions live is a tensor product over finite-dimensional representations. More generally, we may replace these finite-dimensional representations of SU(2) by principal series representations of $\mathrm{SL}(2,\mathbb{C})$. We will consider this in more detail later, but note for now that the Bethe ansatz is not applicable here due to the lack of a reference state. Therefore, we should approach the quantization problem using separation of variables.

3.2 Introduction to the Hitchin system

We give a lightning review of the notions introduced in the definition of the Hitchin system, realizing its structure as an algebraically integrable model.

Stable Higgs bundles Let us consider a rank two holomorphic bundle $\mathcal{E}$ with trivial determinant over a possibly punctured Riemann surface X with $2g-2+n>0$. We introduce a global section $\theta \in \mathrm{H}^0(X, \mathrm{End}(\mathcal{E}) \otimes K_X)$ which we call a Higgs field. The bundle K_X is the canonical bundle over X. We require that near a puncture z_*, the Higgs field has an expansion of the form

$$\theta(z) = \frac{\theta_r(z)}{z-z_*} + \mathcal{O}((z-z_*)^0)$$

The pair $(\mathcal{E}, \theta)$ defines a Higgs bundle.

We will only consider pairs $(\mathcal{E}, \theta)$ that are stable, i.e. pairs that satisfy $2\deg(\mathcal{L}) < \deg(\mathcal{E})$ for every subline bundle $\mathcal{L} \subset \mathcal{E}$ such that $\theta(\mathcal{L}) \subset \mathcal{L} \otimes K_X$. This is a technical condition allowing us to find a harmonic metric compatible with the Higgs pair. We call the set of stable Higgs bundles up to gauge transformation the Higgs bundle moduli space $\mathcal{M}_H$.

Symplectic structure on moduli space Note that for the trivial Higgs field $\theta = 0$, the stability condition reduces to the stability condition of the underlying holomorphic bundle and that the stability of a holomorphic bundle $\mathcal{E}$ automatically implies the stability of the Higgs pair $(\mathcal{E}, \theta)$ for any θ. The space $\mathrm{Bun}_{\mathrm{SL}(2,\mathbb{C})}(X)$ of stable bundles with parabolic structures at the punctures can be embedded in $\mathcal{M}_H$ as the slice of stable Higgs bundles $(\mathcal{E}, 0)$. Its tangent space at a given $\mathcal{E} \in \mathrm{Bun}_{\mathrm{SL}(2,\mathbb{C})}(X)$ is identified with $\mathrm{H}^1(X, \mathrm{End}(\mathcal{E}))$, so that we may identify its cotangent fibre with the space of Higgs fields $\mathrm{H}^0(X, \mathrm{End}(\mathcal{E}) \otimes K_X)$ through the Serre duality. It turns out that $\mathrm{T}^*\mathrm{Bun}_{\mathrm{SL}(2,\mathbb{C})}$ is dense in $\mathcal{M}_H$ and we may induce a symplectic structure on $\mathcal{M}_H$ from the canonical symplectic structure on $\mathrm{T}^*\mathrm{Bun}_{\mathrm{SL}(2,\mathbb{C})}$.

Complex integrable structure We define a map $h : \mathcal{M}_H \to \mathrm{H}^0(X, K_X^2)$ sending $(\mathcal{E}, \theta) \mapsto \mathrm{tr}(\theta^2)$. By picking a basis a_r with $r = 1, ..., 3g - 3 + n$ for the space of quadratic differentials with at most a double pole at the punctures and with fixed behaviour of the double order pole, we may write

$$\mathrm{tr}(\theta^2) = \sum_{r=1}^{3g-3+n} H_r a_r$$

The map h therefore induces $3g-3+n$ coordinate functions H_r on $\mathcal{M}_H$. It has been shown in [15] that these coordinate functions H_r are mutually commuting on $\mathcal{M}_H$ with respect to the symplectic structure induced by the canonical symplectic structure on $\mathrm{T}^*\mathrm{Bun}_{\mathrm{SL}(2,\mathbb{C})}$. Since $\dim_{\mathbb{C}}(\mathrm{Bun}_{\mathrm{SL}(2,\mathbb{C})}(X)) = 3g - 3 + n$, we find $\dim_{\mathbb{C}}(\mathcal{M}_H) = 6g - 6 + 2n$.

This system has the structure of a complex integrable system, which can be understood by identifying the fibres of h with complex tori.

Fixing the value of a global quadratic differential $a(u)$ defines the equation $v^2 - a(u) = 0$. A quadratic differential therefore defines a curve $\Sigma := \{(u, v) \in \mathrm{T}^*X | v^2 - a(u) = 0\}$. We call the curve coming from the Higgs field θ by $a(u) = \frac{1}{2}\mathrm{tr}(\theta(u)^2)$ the spectral curve.

If we consider a pair (u, v) in the spectral curve for which $\mathrm{tr}(\theta(u)^2) \neq 0$, the Higgs field can be diagonalized with eigenvalues $\pm\sqrt{\mathrm{tr}(\theta^2)}$. Since these eigenvalues are distinct, we may introduce the eigenvector bundle at each such point (u, v). We may extend this construction to the definition of a line bundle over the complete spectral curve. The moduli space of such line bundles is called the Prym variety.

Conversely, given a line bundle $\mathcal{F}$ over Σ, we may consider its direct image along the two-to-one map $\pi : \Sigma \to C$ sending $(u, v) \mapsto u$. The direct image construction defines a rank two holomorphic bundle $\mathcal{E}$ with Higgs field θ. This turns out to be a stable Higgs pair and $\det(\mathcal{E}) \simeq \mathcal{O}_{\mathbb{C}}$, hence inverting the map we defined above.

We note that the Jacobian $\mathrm{Jac}(\Sigma)$, the moduli space of line bundles over Σ, is known to be isomorphic to a complex torus. The fibres of the fibration $h : \mathcal{M}_H \to \mathrm{H}^0(X, K_X^2)$ can be identified with the Prym variety, defining the subset of line bundles in $\mathrm{Jac}(\Sigma)$ giving rise to bundles $\mathcal{E}$ with $\det(\mathcal{E}) \simeq \mathcal{O}_{\mathbb{C}}$. This finally realizes the structure of the Hitchin system as a complex integrable system.

3.3 Quantization of the Gaudin model

As an illustrative example, we will now realize the $\mathrm{SL}(2,\mathbb{C})$-Gaudin model as a special case of a Hitchin system. For a discussion on the appearance of the Gaudin model from the Hitchin Hamiltonians, we refer to [76]. The Gaudin model has also been considered in [19, 20]. More recently, the Gaudin model has been discussed in the context of [77].

Classical Gaudin model from the Hitchin system Let us consider the space $X = \mathbb{C}P^1 \backslash \{z_1, ..., z_n\}$. The space $\mathrm{Bun}_{\mathrm{SL}(2,\mathbb{C})}(X)$ is characterized completely by the parabolic structures at the punctures. A parabolic structure is introduced by defining a flag $0 \subset F \subset \mathbb{C}^2$ at each puncture together with a set of weights. Such a flag defines a Borel subgroup $B \subset \mathrm{SL}(2,\mathbb{C})$ by considering all transformations keeping F invariant. Without loss of generality, we may assume B is the space of lower-diagonal matrices and describe $\mathcal{E} \in \mathrm{Bun}_{\mathrm{SL}(2,\mathbb{C})}(X)$ through a covering of $\mathbb{C}P^1$ consisting of $\{\mathbb{C}P^1 \backslash \{z_1, ..., z_n\}, D_1, ..., D_n\}$ and transition functions, with the $D_1, ..., D_n$ mutually non-intersecting small discs around each puncture $z_1, ..., z_n$.

We may identify the space of flags $0 \subset F \subset \mathbb{C}^2$ with $\mathrm{SL}(2,\mathbb{C})/B \simeq \mathbb{C}P^1$ and introduce coordinates $(x_1, ..., x_n) \in (\mathbb{C}P^1)^n$ for $\mathrm{Bun}_{\mathrm{SL}(2,\mathbb{C})}(X)$. The transition functions between X and each of the discs D_r then take the form

$$f_r = \begin{pmatrix} 1 & x_r \\ 0 & 1 \end{pmatrix}$$

Remark 3.1. Strictly speaking we only characterize the open dense subset of $\text{Bun}_{\text{SL}(2,\mathbb{C})}(X)$ in this way coming from the trivial SL(2, ℂ)-bundle over $\mathbb{C}P^1$. See also section 7 of [77].

The cotangent fibre may now be understood as the subspace in $\text{H}^0(X, \text{End}(\mathcal{E}) \otimes K_X)$ preserving the flag structure. Therefore, we can parametrize the cotangent fibre by the matrices

$$t_r = \begin{pmatrix} -l_r & 0 \\ p_r & l_r \end{pmatrix}$$

The numbers l_r define the weight at each puncture.

The space of Higgs fields may therefore be parametrized on $\mathbb{C}P^1 \backslash \{z_1, ..., z_n\}$ by

$$\theta(y) = \sum_{r=1}^{n} \frac{\theta_r}{y - z_r}$$

where

$$\begin{aligned} \theta_r &= f_r \cdot t_r \cdot f_r^{-1} \\ &= \begin{pmatrix} x_r p_r - l_r & -x_r^2 p_r + 2l_r x_r \\ p_r & -x_r p_r + l_r \end{pmatrix} \end{aligned}$$

The constraint that the point at infinity does not produce a pole in the Higgs field, leads to the equation

$$\sum_{r=1}^{n} \theta_r = 0$$

We may define a moment map $\mu = \text{Res}_{z=\infty}(\theta)$ such that the physically relevant space is defined by $\mu^{-1}(0)$. This is equivalent to the following three constraints

$$\begin{aligned} \sum_{r=1}^{n} p_r &= 0 \\ \sum_{r=1}^{n} (x_r p_r - l_r) &= 0 \\ \sum_{r=1}^{n} (x_r^2 p_r - 2l_r x_r) &= 0 \end{aligned}$$

The space $\mu^{-1}(0)$ has a residual $\mathfrak{sl}(2, \mathbb{C})$-action on the coordinates $(x_1, ..., x_n)$ generated by the constraints, which exponentiates to the Möbius transformations. We have to quotient by this action to get the physical phase space $\mu^{-1}(0) /\!\!/ \text{PSL}(2, \mathbb{C})$.

Effectively, we may set $x_1 = 0$, $x_{n-1} = 1$ and $x_n = \infty$ to pick a representative of the PSL$(2,\mathbb{C})$-orbits. Then the above equations allow us to express the coordinates p_1, p_{n-1} and p_n in terms of the other coordinates $(x_2, ..., x_{n-2}, p_2, ..., p_{n-2})$, reducing the system to a $2(n-3)$-complex dimensional system.

The Hamiltonians H_r of this system are defined as the residues at the punctures of the quadratic differential

$$\frac{1}{2}\mathrm{tr}(\theta(y)^2) = \sum_{r=1}^{n} \left(\frac{\delta_r}{(y-z_r)^2} + \frac{H_r}{y-z_r} \right)$$

As a function of the coordinates $(\mathbf{x}, \mathbf{p}) \in (\mathrm{T}^*\mathbb{C}P^1)^{n-3}$, we find $\delta_r = l_r^2$ and

$$H_r = -\sum_{s\neq r} \frac{1}{z_r - z_s} \left((x_r - x_s)^2 p_r p_s + 2(x_r - x_s)(p_r l_s - p_s l_r) - 2 l_r l_s \right)$$

Note that the lack of pole at infinity implies the constraints

$$\begin{aligned} \sum_{r=1}^{n} H_r &= 0 \\ \sum_{r=1}^{n} (z_r H_r + \delta_r) &= 0 \\ \sum_{r=1}^{n} (z_r^2 H_r + 2 z_r \delta_r) &= 0 \end{aligned}$$

Quantizing the Gaudin model The coordinates x_r and p_s are canonical coordinates for $\mathrm{T}^*\mathrm{Bun}_{\mathrm{SL}(2,\mathbb{C})}(X)$ satisfying $\{x_r, p_s\} = \delta_{rs}$ with respect to the standard Poisson bracket on $\mathrm{T}^*\mathrm{Bun}_{\mathrm{SL}(2,\mathbb{C})}(X)$. To quantize the Gaudin model, we may apply canonical quantization by setting $\hat{p}_r = -\epsilon_1 \frac{\partial}{\partial x_r}$ such that as operators $[\hat{x}_r, \hat{p}_s] = \epsilon_1 \delta_{rs}$. Although the parameter ϵ_1 derives its meaning from the gauge theoretical context of the Hitchin model, for the purpose of quantizating the Gaudin model, we may simply interpret ϵ_1 as a quantization parameter. After quantization we find operators

$$\hat{J}_r^- = -\epsilon_1 \frac{\partial}{\partial x_r} \tag{3.1}$$

$$\hat{J}_r^0 = -\epsilon_1 x_r \frac{\partial}{\partial x_r} + j_r \tag{3.2}$$

$$\hat{J}_r^+ = \epsilon_1 x_r^2 \frac{\partial}{\partial x_r} - 2 j_r x_r \tag{3.3}$$

representing the quantum analogues of θ_r^-, θ_r^0 and θ_r^+ respectively. Now the Casimir

$$\hat{J}_r^0\hat{J}_r^0 + \frac{1}{2}(\hat{J}_r^+\hat{J}_r^- + \hat{J}_r^-\hat{J}_r^+) = j_r(j_r+\epsilon_1)$$

The shift by $\epsilon_1 j_r$ is a result of the quantization.

Before applying the constraints, the Hamiltonians can simply be canonically quantized by taking

$$\hat{H}_r = -\sum_{s\neq r}\frac{1}{z_r-z_s}\left(\epsilon_1^2(x_r-x_s)^2\frac{\partial^2}{\partial x_r\partial x_s} + 2\epsilon_1(x_r-x_s)\left(j_s\frac{\partial}{\partial x_r} - j_r\frac{\partial}{\partial x_s}\right) - 2j_rj_s\right) \tag{3.4}$$

In the classical case, the constraints mix the coordinates $\mathbf{x}$ and $\mathbf{p}$. We will therefore need to carefully take these constraints into consideration. When $n=4$, the quantization of the Gaudin model corresponds to solving a single ordinary differential equation.

Remark 3.2. The equation (3.4) is equal to equation (2.3). This indeed shows that the quantum Gaudin Hamiltonians are equal to the operators $\hat{H}_r$ in the KZB-equation.

Gaudin model corresponding to the four-punctured sphere To quantize the Gaudin model, we will need to look for holomorphic solutions $\psi_{\mathbf{E}}(\mathbf{x})$ satisfying

$$\hat{H}_r\psi_{\mathbf{E}}(\mathbf{x}) = E_r\psi_{\mathbf{E}}(\mathbf{x})$$

with $\hat{H}_r$ defined by equation (3.4). For simplicity we may set $\epsilon_1 = 1$.

The constraints on the coordinates $\mathbf{p}$ translate to three constraints on the function $\psi_{\mathbf{E}}(\mathbf{x})$ of the form

$$\begin{aligned}\sum_{r=1}^n \hat{J}_r^-\psi_{\mathbf{E}}(\mathbf{x}) &= 0\\ \sum_{r=1}^n \hat{J}_r^0\psi_{\mathbf{E}}(\mathbf{x}) &= 0\\ \sum_{r=1}^n \hat{J}_r^+\psi_{\mathbf{E}}(\mathbf{x}) &= 0\end{aligned}$$

In the case where $X = \mathbb{C}P^1\backslash\{0,z,1,\infty\}$ and $j_r = -\frac{1}{2}$ for all $r = 1, ..., 4$, we may set $x_4 = \infty$ and completely remove $\frac{\partial}{\partial x_4}$ from the above Hamiltonian. This leaves us with two differential equations of the form

$$\begin{aligned}\left(\frac{\partial}{\partial x_1}+\frac{\partial}{\partial x_2}+\frac{\partial}{\partial x_3}\right)\psi_{\mathbf{E}}(\mathbf{x}) &= 0\\ \left(x_1\frac{\partial}{\partial x_1}+x_2\frac{\partial}{\partial x_2}+x_3\frac{\partial}{\partial x_3}+1\right)\psi_{\mathbf{E}}(\mathbf{x}) &= 0\end{aligned}$$

Using these properties, we may remove $\frac{\partial}{\partial x_1}$ and $\frac{\partial}{\partial x_3}$ from the equation as well. Assuming $\mathbf{x} = (0, x, 1, \infty)$ such that $\psi_{\mathbf{E}}(x) := \psi_{\mathbf{E}}(\mathbf{x})$ only depends on the crossratio x, it can be shown that the Hamiltonian $\hat{H}_2$ acts on $\psi_{\mathbf{E}}(x)$ by

$$\hat{H}_2\psi_{\mathbf{E}}(x) = \frac{x(x-z)(x-1)}{z(z-1)}\psi''_{\mathbf{E}}(x)+\frac{3x^2-2xz-2x+z}{z(z-1)}\psi'_{\mathbf{E}}(x)+\frac{x-\frac{1}{2}}{z(z-1)}\psi_{\mathbf{E}}(x)$$

By reparametrizing E_2 in terms of a parameter λ, the differential equation $\hat{H}_2\psi_{\mathbf{E}}(x) = E_2\psi_{\mathbf{E}}(x)$ becomes a differential equation of Sturm-Liouville type which we may write as

$$\frac{d}{dx}\left(x(x-z)(x-1)\psi'_\lambda(x)\right)+(x+\lambda)\psi_\lambda(x) = 0 \tag{3.5}$$

where the dependence on the parameter λ is made explicit in the form $\psi_\lambda(x) := \psi_{\mathbf{E}}(x)$.

At this stage, we should note that the choice $j_r = -\frac{1}{2}$ will become important for us due to the relation between equation (3.5) and the geometry of the four-punctured sphere (and more precisely the uniformization theorem). In section §4 we will discuss these statements in greater detail.

Remark 3.3. For the four-punctured sphere $X = \mathbb{C}P^1\backslash\{0, z, 1, \infty\}$, the differential equation (3.5) has regular solutions on the subspace $\mathbb{C}P^1\backslash\{0, z, 1, \infty\} \subset \mathrm{Bun}_{\mathrm{SL}(2,\mathbb{C})}(X)$. The fact that this subspace is isomorphic to X is unique to the four-punctured sphere!

Setting up a quantization problem Solving a quantum problem requires us to find elements in a Hilbert space which act as eigenfunctions of the operators defined in equation (3.4). Usually a quantization of the Gaudin model requires us to search for eigenfunctions to the Gaudin Hamiltonians which have holomorphic dependence on the auxiliary variable on the punctured sphere. This is not the type of quantization we will be looking at. Instead, we will look at both holomorphic and anti-holomorphic eigenvalue equations.

To describe an interesting subset of functions, we will consider the Hilbert space to be a direct product $\bigotimes_{r=1}^n \mathcal{P}_{j_r,\bar{j}_r}$ with $\mathcal{P}_{j_r,\bar{j}_r}$ the principal series representation associated to $\mathfrak{sl}(2,\mathbb{C})_{\mathbb{C}} \simeq \mathfrak{sl}(2,\mathbb{C}) \oplus \mathfrak{sl}(2,\mathbb{C})$. We may realize these representations as principal series representations of $\mathfrak{sl}(2,\mathbb{C})$ on which the operators $\hat{J}_r^-$, $\hat{J}_r^0$ and $\hat{J}_r^+$ act as holomorphic differential operators as in equation (3.1), equation (3.2) and equation (3.3) respectively, while $\bar{\hat{J}}_r^-$,$\bar{\hat{J}}_r^0$ and $\bar{\hat{J}}_r^+$ are realized as the anti-holomorphic variants of these operators with j_r replaced by $\bar{j}_r$.

From a quantum mechanical point of view, it is natural to require the existence of a single-valued solution $\Psi_{\mathbf{E}}(\mathbf{x}, \bar{\mathbf{x}})$ of

$$\hat{H}_r \Psi_{\mathbf{E}}(\mathbf{x}, \bar{\mathbf{x}}) = E_r \Psi_{\mathbf{E}}(\mathbf{x}, \bar{\mathbf{x}}) \qquad \bar{\hat{H}}_r \Psi_{\mathbf{E}}(\mathbf{x}, \bar{\mathbf{x}}) = \bar{E}_r \Psi_{\mathbf{E}}(\mathbf{x}, \bar{\mathbf{x}}) \tag{3.6}$$

with $\bar{H}_r$ the complex conjugate of H_r. This sets up a quantum problem that cannot be solved for generic values of E_r and $\bar{E}_r$. Square integrability of the wave function implies that $\bar{E}_r$ is the complex conjugate of E_r [77].

We will show that this quantization problem picks out a discrete set of allowed eigenvalues which we will consider to solve our quantum problem.

From the point of view of single-valuedness, it is clear why we have to give up holomorphicity of the eigenfunctions. A single-valued holomorphic function on the punctured sphere is necessarily a quotient of polynomials in the parameters $\mathbf{x}$. Such solutions have been constructed through the Bethe ansatz in [76].

If we consider any quantum mechanical problem, single-valuedness is one of the conditions that a wavefunction has to satisfy from a physical point of view. Usually, single-valuedness is only one of the properties we require the wavefunction to have, another being square-integrability with respect to a natural scalar product. In principle, it could happen that only a subset of the set of all single-valued wavefunctions is square-integrable. In [77] a scalar product is constructed and it is shown for the four-punctured sphere that the single-valued wavefunctions are precisely the square-integrable wavefunctions. This shows that the condition of single-valuedness is strong enough to characterize the allowed wavefunctions.

3.4 Separation of variables for the Gaudin model

Although we obtain an ordinary second order differential equation for the eigenfunction $\psi_{\mathbf{E}}(x)$ when $n = 4$, in general the differential equations will mix different coordinates x_r. We will describe the separation of variables for the Gaudin model leading to a set of $n-3$ ordinary differential equations in terms of a new set of variables $\mathbf{u} = (u_1, ..., u_{n-3})$. It turns out that such a transformation can be found both on a classical level as well as a quantum level.

Bethe ansatz for the Gaudin model Before we apply the separation of variables, we note that the Bethe ansatz does provide a way to quantize the SU(2)-Gaudin model as we noted before. The Bethe ansatz allows us to find solutions of our problem in finite-dimensional, highest-weight representations $\bigotimes_{r=1}^{n-1} V_{j_r}$. In this case, the spins must satisfy $j_r \in \frac{1}{2}\mathbb{Z}_{\geq 0}$. In particular, a finite-dimensional space implies the space of solutions we are looking for are polynomials in the variables x_r. Such solutions can never develop non-trivial monodromy in $\mathrm{PSL}(2, \mathbb{C})$.

The problem we set out to solve contains the uniformization problem as a solution and always excludes trivial monodromy. The solutions obtained for the SU(2)-Gaudin model through the Bethe ansatz therefore cannot be used in our case.

Classical separation of variables for the Gaudin model For the separation of variables, we will study the zeroes of the function $\theta^-(t) = \sum_{r=1}^{n-1} \frac{p_r}{t-z_r}$. This function is defined as the lower-left element in the matrix-form of the Higgs field

$$\theta(t) = \begin{pmatrix} \theta^0(t) & \theta^+(t) \\ \theta^-(t) & -\theta^0(t) \end{pmatrix}$$

In the context of the separation of variables we will use the notation t instead of y for the parameter living on the punctured sphere.

Let us define new variables u_0 and $\mathbf{u} = (u_1, ..., u_{n-3})$ by setting

$$\sum_{r=1}^{n-1} \frac{p_r}{t - z_r} = u_0 \frac{\prod_{k=1}^{n-3}(t - u_k)}{\prod_{r=1}^{n-1}(t - z_r)}$$

where we have sent $z_n \to \infty$ by using the invariance under complex Möbius transformations of this system. The coordinates $\mathbf{u}$ form one half of the new set of coordinates.

We can explicitly represent

$$\begin{aligned} p_r(\mathbf{u}) &= \mathrm{Res}_{t=z_r} \left(u_0 \frac{\prod_{k=1}^{n-3}(t - u_k)}{\prod_{r=1}^{n-1}(t - z_r)} \right) \\ &= u_0 \frac{\prod_{k=1}^{n-3}(z_r - u_k)}{\prod_{s \neq r}(z_r - z_s)} \end{aligned}$$

The other half of our new coordinates is given by $\mathbf{v} = (v_1, ..., v_{n-3})$ defined by

$$v_k = -\theta^0(u_k)$$

If we let

$$\begin{aligned} \vartheta(t) &= \frac{1}{2}\mathrm{tr}(\theta(t)^2) \\ &= (\theta^0(t))^2 + \frac{1}{2}(\theta^+(t)\theta^-(t) + \theta^-(t)\theta^+(t)) \end{aligned}$$

we find $v_k^2 = \vartheta(u_k)$. Therefore, the pair of coordinates (u_k, v_k) defines a point on the spectral curve

$$\Sigma = \{(u, v) \in \mathrm{T}^*X | v^2 = \vartheta(u)\}$$

and the change of coordinates $(\mathbf{x}, \mathbf{p}) \to (\mathbf{u}, \mathbf{v})$ defines a map $\mathrm{T}^*\mathrm{Bun}_{\mathrm{SL}(2,\mathbb{C})}(X) \to (\mathrm{T}^*X)^{[n-3]}$. This map is symplectic as the following result shows. See for example [40] for a discussion.

Proposition 3.4. *The coordinates* $(\mathbf{u}, \mathbf{v})$ *are Darboux coordinates for the canonical symplectic structure* $\sum_{r=1}^{n-1} dx_r \wedge dp_r$ *on* $\mathrm{T}^*\mathrm{Bun}_{\mathrm{SL}(2,\mathbb{C})}(X)$*.*

Proof. Consider the function

$$\mathcal{G}(\mathbf{x}, \mathbf{u}) = \kappa \log\left(\sum_{r=1}^{n-1} x_r \frac{\prod_{k=1}^{n-3}(z_r - u_k)}{\prod_{s\neq r}(z_r - z_s)}\right) + \sum_{r=1}^{n-1} l_r \log\left(\frac{\prod_{s\neq r}(z_r - z_s)}{\prod_{k=1}^{n-3}(z_r - u_k)}\right)$$

where $\kappa = -l_n + \sum_{r=1}^{n-1} l_r$.

One easily checks that

$$\mathcal{G}(\mathbf{x}, \mathbf{u}) = \kappa \log\left[\left(\sum_{r=1}^{n-1} x_r p_r(\mathbf{u})\right)\right] - \sum_{r=1}^{n-1} l_r \log(p_r(\mathbf{u})) + l_n \log(u_0)$$

This implies

$$\frac{\partial}{\partial x_r}\mathcal{G}(\mathbf{x}, \mathbf{u}) = \kappa \frac{p_r}{\sum_{r'=1}^{n-1} x_{r'} p_{r'}}$$

By using the constraint $\sum_{r'=1}^{n-1}(x_{r'}p_{r'} - l_{r'}) = -l_n$, we may replace $\sum_{r'=1}^{n-1} x_{r'}p_{r'} = \kappa$ to find

$$\frac{\partial}{\partial x_r}\mathcal{G}(\mathbf{x}, \mathbf{u}) = p_r$$

On the other hand

$$\begin{aligned}\frac{\partial}{\partial u_k}\mathcal{G}(\mathbf{x},\mathbf{u}) &= \kappa \log\left[\left(\sum_{r=1}^{n-1} x_r p_r(\mathbf{u})\right)\right] - \sum_{r=1}^{n-1} l_r \log(p_r(\mathbf{u})) + l_n \log(u_0) \\ &= \sum_{r=1}^{n-1} \frac{x_r p_r - l_r}{u_k - z_r} \\ &= \theta^0(u_k) \\ &= -v_k\end{aligned}$$

The function $\mathcal{G}(\mathbf{x},\mathbf{u})$ therefore describes a generating function for the change of coordinates $(\mathbf{x},\mathbf{p})$ to $(\mathbf{u},\mathbf{v})$.

The existence of such a generating function implies that

$$d\mathcal{G}(\mathbf{x},\mathbf{u}) = \sum_{r=1}^{n-1} p_r dx_r - \sum_{k=1}^{n-3} v_k du_k$$

Therefore,

$$\sum_{r=1}^{n-1} dx_r \wedge dp_r = \sum_{k=1}^{n-3} du_k \wedge dv_k$$

implying the canonical symplectic structure generated by $\mathbf{x}$ and $\mathbf{p}$ on $\mathrm{T}^*\mathrm{Bun}_{\mathrm{SL}(2,\mathbb{C})}(X)$ coincides with the canonical symplectic structure generated on $\mathrm{Sym}^{n-3}(\mathrm{T}^*X)$.

□

Quantum separation of variables for the Gaudin model Once we quantize the Gaudin model, the functions $\theta^a(t)$ are replaced by operators

$$\hat{J}^a(t) = \sum_{r=1}^{n-1} \frac{\hat{J}^a_r}{t - z_r}$$

where $a \in \{-,0,+\}$ and $\hat{J}^a_r$ are defined by equation (3.1), equation (3.2) and equation (3.3). Our goal is to find single-valued functions $\Psi_{\mathbf{E}}(\mathbf{x},\bar{\mathbf{x}})$ simultaneously solving the holomorphic and anti-holomorphic equalities in equation (3.6).

It turns out that we can still apply the separation of variables to the quantum Gaudin model. See [71, 76, 20]. To do so, we first we apply a (variant of the) Fourier transformation of the form

$$\tilde{\Psi}_{\mathbf{E}}(\mathbf{m},\bar{\mathbf{m}}) = \prod_{r=1}^{n-1}\left(\frac{|\mu_r|^{2j_r+2}}{\pi}\int d^2x_r e^{\mu_r x_r - \bar{\mu}_r \bar{x}_r}\right)\Psi_{\mathbf{E}}(\mathbf{x},\bar{\mathbf{x}})$$

where $\mathbf{m} = (\mu_1, ..., \mu_{n-1})$. We have to pick the minus-sign in front of the conjugate variables to ensure that the conjugate operators $\hat{\bar{J}}_r^a$ are mapped to the complex conjugate operators of $\hat{J}_r^a$.

The operators $\hat{J}_r^a$ transform to operators $\hat{D}_r^a$ of the form

$$\begin{aligned}
\hat{D}_r^- &= \mu_r \\
\hat{D}_r^0 &= \mu_r \frac{\partial}{\partial \mu_r} \\
\hat{D}_r^+ &= -\mu_r \left(\frac{\partial}{\partial \mu_r} \right)^2 + \frac{j_r(j_r+1)}{\mu_r}
\end{aligned}$$

In this basis, the Hamiltonians take the form

$$\hat{H}_r = -\sum_{s \neq r} \frac{1}{z_r - z_s} \left(\mu_r \mu_s \left(\frac{\partial}{\partial \mu_r} - \frac{\partial}{\partial \mu_r} \right)^2 - \mu_s \frac{j_r(j_r+1)}{\mu_r} - \mu_r \frac{j_s(j_s+1)}{\mu_s} \right)$$

If we set $\hat{D}^a(t) = \sum \frac{\hat{D}_r^a}{t - z_r}$, the operator

$$S(t) := \left(\hat{D}^0(t) \right)^2 + \partial_t \hat{D}^0(t) + \hat{D}^-(t) \hat{D}^+(t)$$

may be expressed as

$$S(t) = \sum_{r=1}^{n-1} \left(\frac{j_r(j_r+1)}{(t - z_r)^2} + \frac{\hat{H}_r}{t - z_r} \right)$$

We may apply the separation of variables in the same way as we did before by considering the zeroes of $\hat{D}^-(t)$. Let us therefore write

$$\sum_{r=1}^{n-1} \frac{\mu_r}{t - z_r} = u_0 \frac{\prod_{k=1}^{n-3} (t - u_k)}{\prod_{r=1}^{n-1} (t - z_r)}$$

Once again, we may express

$$\mu_r(\mathbf{u}) = u_0 \frac{\prod_{k=1}^{n-3} (z_r - u_k)}{\prod_{s \neq r} (z_r - z_s)}$$

such that

$$\begin{aligned}
\frac{\partial}{\partial u_k} &= \sum_{r=1}^{n-1} \frac{\partial \mu_r}{\partial u_k} \frac{\partial}{\partial \mu_r} \\
&= \hat{D}^0(u_k)
\end{aligned}$$

As operators we find

$$\begin{aligned}\left(\frac{\partial}{\partial u_k}\right)^2 &= \frac{\partial}{\partial u_k}\cdot\left(\hat{D}^0(t)|_{t=u_k}\right)\\ &= \left(\partial_t\hat{D}^0(t)\right)|_{t=u_k} + \left(\frac{\partial}{\partial u_k}\hat{D}^0(t)\right)|_{t=u_k}\end{aligned}$$

The change of variables $\mathbf{m}\to\mathbf{u}$ transforms $\tilde{\Psi}_{\mathbf{E}}(\mathbf{m},\bar{\mathbf{m}})\to\Phi_{\mathbf{E}}(\mathbf{u},\bar{\mathbf{u}})$. It is easy to check that

$$\begin{aligned}S(t)|_{t=u_k}\Phi_{\mathbf{E}}(\mathbf{u},\bar{\mathbf{u}}) &= \left[\left(\partial_t\hat{D}^0(t)\right)|_{t=u_k} + \left(\frac{\partial}{\partial u_k}\hat{D}^0(t)\right)|_{t=u_k}\right]\Phi_{\mathbf{E}}(\mathbf{u},\bar{\mathbf{u}})\\ &= \frac{\partial^2}{\partial u_k^2}\Phi_{\mathbf{E}}(\mathbf{u},\bar{\mathbf{u}})\end{aligned}$$

On the other hand, by definition

$$S(t)|_{t=u_k}\Phi_{\mathbf{E}}(\mathbf{u},\bar{\mathbf{u}}) = \sum_{r=1}^{n-1}\left(\frac{j_r(j_r+1)}{(u_k-z_r)^2}+\frac{E_r}{u_k-z_r}\right)\Phi_{\mathbf{E}}(\mathbf{u},\bar{\mathbf{u}})$$

The change of variables $\mathbf{m}\to\mathbf{u}$ therefore leads to a set of equations of the form

$$\left(\frac{\partial^2}{\partial u_k^2}-\sum_{r=1}^{n-1}\left(\frac{j_r(j_r+1)}{(u_k-z_r)^2}+\frac{E_r}{u_k-z_r}\right)\right)\Phi_{\mathbf{E}}(\mathbf{u},\bar{\mathbf{u}}) = 0 \qquad (3.7)$$

$$\left(\frac{\partial^2}{\partial \bar{u}_k^2}-\sum_{r=1}^{n-1}\left(\frac{\bar{j}_r(\bar{j}_r+1)}{(\bar{u}_k-\bar{z}_r)^2}+\frac{\bar{E}_r}{\bar{u}_k-\bar{z}_r}\right)\right)\Phi_{\mathbf{E}}(\mathbf{u},\bar{\mathbf{u}}) = 0 \qquad (3.8)$$

for $k=1,...,n-3$. Each of these equations defines an ordinary differential operator in one variable.

We can pass back to the original functions $\Psi_{\mathbf{E}}(\mathbf{x},\bar{\mathbf{x}})$ using the integration kernel

$$\mathcal{K}(\mathbf{x},\mathbf{u}) = \left(\sum_{r=1}^{n-1}x_r\frac{\prod_{k=1}^{n-3}(z_r-u_k)}{\prod_{s\neq r}(z_r-z_s)}\right)^J\prod_{r=1}^{n-1}\left(\frac{\prod_{s\neq r}(z_r-z_s)}{\prod_{k=1}^{n-3}(z_r-u_k)}\right)^{j_r+1}\prod_{k<l}^{n-3}(u_k-u_l)$$

such that

$$\Psi_{\mathbf{E}}(\mathbf{x},\bar{\mathbf{x}}) = N_J\int d^2u_1...d^2u_{n-3}|\mathcal{K}(\mathbf{x},\mathbf{u})|^2\Phi_{\mathbf{E}}(\mathbf{u},\bar{\mathbf{u}})$$

for some constant N_J only depending on the complex structure and the parameters j_r and $\bar{j}_r$. The parameter J is the quantum analogue of κ, defined as $J=-j_n+\sum_{r=1}^{n-1}j_r$.

3.5 Quantization of the Hitchin system for higher genus

The quantization of the Hitchin system is closely related to the geometric Langlands conjecture and conformal field theories, specifically those with an affine Kac-Moody symmetry. The Ward identities in conformal field theories allow us to express conformal blocks as vector bundles over moduli spaces. The stress-energy tensor in the field theory defines a connection on the bundle of conformal blocks over the moduli space of curves.

Upon a renormalization, we may define a limit known as the critical level limit for theories with affine Kac-Moody symmetry. This realizes the renormalized stress-energy tensor on X as a function that transforms as

$$t(u) \to (w'(u))^2 t(w(u)) + \frac{1}{2}\{w(u), u\} \tag{3.9}$$

under coordinate transformations. The function $\{w(u), u\} = \frac{w'''(u)}{w'(u)} - \frac{3}{2}\left(\frac{w''(u)}{w'(u)}\right)^2$ is called the Schwarzian derivative of $w(u)$.

It turns out that in this limit, the stress-energy tensor generates the center of the chiral algebra for $\mathfrak{sl}(2,\mathbb{C})$-affine Kac moody symmetry. For higher rank, the center generalizes to the classical $\mathcal{W}$-algebra.

In more generality, let us consider G to be a simple Lie group. The conformal blocks of $\mathfrak{g}$-affine Kac Moody algebras are defined as sheaves over both the moduli space of complex structures and the moduli space $\mathrm{Bun}_G(X)$ of holomorphic bundles. The Ward identities for the affine algebra relate the variations of the holomorphic bundle to the current algebra action on the spaces of conformal blocks. This leads to a relation between the differential operators on the line bundle $K^{1/2}_{\mathrm{Bun}_G(X)}$ and the affine algebra action on conformal blocks. The bundle $K^{1/2}_{\mathrm{Bun}_G(X)}$ is defined by a choice of square root of the bundle $K_{\mathrm{Bun}_G(X)}$. It is shown in [27] that elements in the center map to global differential operators represented diagonally on conformal blocks. In particular, the symbol of these operators corresponds to the classical Hitchin Hamiltonians.

In this way, we have constructed a quantization of the Hitchin Hamiltonians, realized as differential operators acting on the line bundle $K^{1/2}_{\mathrm{Bun}_G(X)}$. Restricting to the case $G = \mathrm{SL}(2,\mathbb{C})$ again, we see that the space of such differential operators is parameterized by the choice of a global function $t(u)$ transforming as in equation (3.9).

We may apply separation of variables in a similar way as for the Gaudin model. It is known that generically a stable holomorphic bundle $\mathcal{E}$ up to the tensor product by a line bundle, fits in a short exact sequence of the form

$$0 \to \mathcal{O}_{\mathbb{C}} \to \mathcal{E} \to \mathcal{L} \to 0$$

with $\deg(\mathcal{L}) > 2g-2$. The space of extension of the above form may be classified by $\mathrm{H}^1(X, \mathcal{L}^{-1})$. Since scaling an extension class by a constant preserves the vector bundle $\mathcal{E}$, it suffices to consider the space $\mathbb{P}\mathrm{H}^1(X, \mathcal{L}^{-1})$ for a classification of vector bundles. Assuming $\deg(\mathcal{L}) = 2g-1$, we find $\dim\left(\mathbb{P}\mathrm{H}^1(X, \mathcal{L}^{-1})\right) = 3g-3$. We may therefore introduce a total of $3g-3$ coordinates $\mathbf{x}$and represent the quantum Hitchin Hamiltonians in terms of these variables. We will find solutions $\Psi_{\mathbf{E}}(\mathbf{x}, \bar{\mathbf{x}})$ as before, which are now sections of the bundle $\left(K^{1/2}_{\mathrm{Bun}_{\mathrm{SL}(2,\mathbb{C})}(X)} \otimes \overline{K^{1/2}_{\mathrm{Bun}_{\mathrm{SL}(2,\mathbb{C})}(X)}}\right)^{\otimes(3g-3)}$.

The function $\theta^-(t)$ defines an element of $\mathrm{H}^0(X, \mathcal{L} \otimes K_X)$ which has a total of $\deg(\mathcal{L}) + \deg(K_X) = 4g-3$ zeroes. If we require $\det(\mathcal{E}) \simeq \mathcal{O}_{\mathbb{C}}$, we find g constraints on the positions of these zeroes, leaving us with a total of $3g-3$ free parameters. We may use these parameters to introduce coordinates $\mathbf{u} = (u_1, ..., u_{3g-3})$ in $\mathrm{Sym}^{3g-3}(X)$ and describe the generalization of the separation of variables to equation (3.7) and equation (3.8) by a Whittaker model as in [20], representing the eigenfunctions as

$$\left(\partial^2_{u_k} + t(u_k)\right) \Phi_{\mathbf{E}}(\mathbf{u}, \bar{\mathbf{u}}) = 0 \qquad \left(\partial^2_{\bar{u}_k} + \bar{t}(\bar{u}_k)\right) \Phi_{\mathbf{E}}(\mathbf{u}, \bar{\mathbf{u}}) = 0 \tag{3.10}$$

The functions $\Phi_{\mathbf{E}}(\mathbf{u}, \bar{\mathbf{u}})$ are sections of the bundle $(K_X^{-1/2} \otimes \overline{K_X^{-1/2}})^{\otimes(3g-3)}$. If we allow punctures, we require a simple pole in the Higgs field, leading to sections of $(K_X^{-1/2} \otimes \overline{K_X^{-1/2}})^{\otimes(3g-3+n)}$.

3.6 Opers with real monodromy and single-valued solutions

After the application of the separation of variables, we may note that the single-valuedness of the function $\Psi_{\mathbf{E}}(\mathbf{x}, \bar{\mathbf{x}})$ can be expressed through the single-valuedness of $\Phi_{\mathbf{E}}(\mathbf{u}, \bar{\mathbf{u}})$. Since equation (3.10) separates the dependence on the variables, we may look for a solution of the form

$$\Phi_{\mathbf{E}}(\mathbf{u}, \bar{\mathbf{u}}) = \prod_{k=1}^{3g-3+n} \phi(u_k, \bar{u}_k)$$

The single-valuedness of $\Phi_{\mathbf{E}}(\mathbf{u}, \bar{\mathbf{u}})$ reduces to the single-valuedness of the solution $\phi(u, \bar{u})$ to the differential equation $(\partial_u^2 + t(u))\phi(u, \bar{u}) = 0$ and its conjugate. In this context, these differential equations play the role of the Baxter equation.

We may express $\phi(u, \bar{u})$ in a basis of solutions $\chi(u) = (\chi_1(u), \chi_2(u))$ and $\chi^\dagger(\bar{u})$ to the holomorphic differential equation

$$\chi''(u) + t(u)\chi(u) = 0 \tag{3.11}$$

and its conjugate respectively.

We assume in doing so that $\bar{E}_r$ is the complex conjugate of E_r. This is justified by the results of [77].

The classification of single-valued solutions $\Phi_{\mathbf{E}}(\mathbf{u}, \bar{\mathbf{u}})$ may be obtained from a classification of second order differential equations of the form of equation (3.11) through the following theorem proposed in [20].

Theorem 3.5. *A single-valued function $\phi(u, \bar{u})$ solving*

$$\left(\partial_u^2 + t(u)\right) \phi(u, \bar{u}) = 0 \qquad\qquad \left(\partial_{\bar{u}}^2 + \bar{t}(\bar{u})\right) \phi(u, \bar{u}) = 0$$

exists if and only if there exists a vector of solutions $\chi(u)$ to equation (3.11) *transforming as $\chi(\gamma \cdot u) = \chi(u) \cdot \rho(\gamma)$ for $\rho \in \mathrm{Hom}(\pi_1(X), \mathrm{SL}(2, \mathbb{R}))$.*

If we analytically continue the solution $\chi(u)$ along a path $\gamma \in \pi_1(X)$, we find $\chi(\gamma \cdot u) = \chi(u) \cdot \rho(\gamma)$ where $\rho \in \mathrm{Hom}(\pi_1(X), \mathrm{GL}(2, \mathbb{C}))$. Similarly, $\chi^\dagger(\gamma \cdot \bar{u}) = \rho(\gamma)^\dagger \cdot \chi^\dagger(\bar{u})$. Let us assume $\rho \in \mathrm{Hom}(\pi_1(X), \mathrm{SL}(2, \mathbb{R}))$. Then we construct a function $\phi(u, \bar{u}) = \chi_1(u)\bar{\chi}_2(\bar{u}) - \bar{\chi}_1(\bar{u})\chi_2(u)$ which solves the differential equations in the statement of the theorem. Furthermore, we have find $\rho(\gamma) = \overline{\rho(\gamma)}$ and $\det(\rho(\gamma)) = 1$. This implies $\phi(u, \bar{u})$ is invariant under the action of the Deck transformations, i.e. $\phi(\gamma \cdot u, \gamma \cdot \bar{u}) = \phi(u, \bar{u})$ for $\gamma \in \pi_1(X)$, proving one direction of this theorem.

For the converse, let us introduce the two-by-two matrix C composed of complex constants such that

$$\phi(u, \bar{u}) = \chi(u) \cdot C \cdot \chi^\dagger(\bar{u})$$

This is the most general form of a solution to the pair of holomorphic and anti-holomorphic equations.

Single-valuedness of $\phi(u,\bar{u})$ implies the following property

$$\rho(\gamma)\cdot C\cdot\rho(\gamma)^{\dagger}=C \qquad \forall\gamma\in\pi_1(X) \tag{3.12}$$

As we will show, by changing the basis of solutions $\chi(u)$ appropriately, we may simplify C significantly.

Lemma 3.6. *By a change of basis in the space of solutions $\chi(u)$, we can always bring C to diagonal form.*

Proof. Consider a change of basis $\tilde{\chi}(u)\cdot N=\chi(u)$ for a two-by-two matrix N. The new matrix $\tilde{C}$ is defined by the equation $\tilde{\chi}(u)\cdot\tilde{C}\cdot\tilde{\chi}^{\dagger}(\bar{u})=\chi(u)\cdot C\cdot\chi^{\dagger}(\bar{u})$ which sets $\tilde{C}=NCN^{\dagger}$.

Let us consider this transformation component-wise

$$\begin{aligned}\tilde{C}_{12} &= N_{11}(C_{11}\overline{N_{21}}+C_{12}\overline{N_{22}})+N_{12}(C_{21}\overline{N_{21}}+C_{22}\overline{N_{22}})\\ \tilde{C}_{21} &= \overline{N_{11}}(C_{11}N_{21}+C_{21}N_{22})+\overline{N_{12}}(C_{12}N_{21}+C_{22}N_{22})\end{aligned}$$

We can set both to zero if we can find N_{11} and N_{12} such that

$$\begin{aligned}N_{11}(C_{11}\overline{N_{21}}+C_{12}\overline{N_{22}}) &= -N_{12}(C_{21}\overline{N_{21}}+C_{22}\overline{N_{22}})\\ \overline{N}_{11}(C_{11}N_{21}+C_{21}N_{22}) &= -\overline{N}_{12}(C_{12}N_{21}+C_{22}N_{22})\end{aligned}$$

Compatibility of these two equations implies

$$\begin{aligned}(N_{21})^2(\overline{C_{21}}C_{11}-C_{12}\overline{C_{11}})+(N_{22})^2(\overline{C_{22}}C_{21}-C_{22}\overline{C_{12}})+&\\ +N_{21}N_{22}(|C_{21}|^2-|C_{12}|^2+C_{11}\overline{C_{22}}-C_{22}\overline{C_{11}}) &= 0\end{aligned}$$

For a given matrix C we may always find N_{21} and N_{22} satisfying this equation and not simultaneously vanishing. □

We may always scale the matrix C by a constant. This is equivalent to scaling the solution $\phi(u,\bar{u})$ by a constant and clearly does not change the monodromy representation of $\phi(u,\bar{u})$. After rescaling, the solution $\phi(u,\bar{u})$ is still single-valued. Without loss of generality, we may therefore set

$$C=\begin{pmatrix}1 & 0\\ 0 & C_{22}\end{pmatrix}$$

Definition 3.7. The monodromy ρ of equation (3.11) is called exceptional if ρ has a set of generators consisting of diagonal elements or elements with vanishing diagonal.

Lemma 3.8. *If the monodromy ρ of equation* (3.11) *is not exceptional, single-valuedness of $\phi(u,\bar{u})$ implies $C_{22} \in \mathbb{R}$.*

Proof. Let us assume ρ is not exceptional and let us assume $\mathfrak{Im}(C_{22}) \neq 0$. We consider any element $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in the image of ρ. Single-valuedness implies

$$\begin{pmatrix} |a|^2 + |b|^2 C_{22} & a\bar{c} + b\bar{d}C_{22} \\ \bar{a}c + \bar{b}dC_{22} & |c|^2 + |d|^2 C_{22} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & C_{22} \end{pmatrix}$$

This can only be satisfied if

$$\mathfrak{Im}(C_{22})\bar{d}c = \mathfrak{Im}(C_{22})a\bar{b} = 0$$

Since $ad - bc = 1$, $a = c = 0$ or $b = d = 0$ are both excluded. The options $b = c = 0$ and $a = d = 0$ lead to a diagonal and completely off-diagonal matrix respectively. Since this has to be true for all elements in the image of ρ, the monodromy is exceptional, which is a contradiction. □

We conclude that $\mathfrak{Im}(C_{22}) = 0$. Finally, we may apply a transformation of the form $\tilde{C} = NCN^\dagger$ again, where now $N_{11} = 1$. If we assume C is diagonal with $C_{11} = 1$, this equation is trivial for all components except $\tilde{C}_{22} = |N_{22}|^2 C_{22}$. If $C_{22} \neq 0$, we may pick $N_{22} = (C_{22})^{-1/2}$ and bring $\tilde{C}_{22}$ to $+1$ or -1. This proves that without loss of generality, C is one of three cases where $C_{11} = 1$, $C_{12} = C_{21} = 0$ and $C_{22} \in \{-1, 0, 1\}$. We will now always assume C takes this form.

Lemma 3.9. *The following statements are true:*

- *If $C_{22} = 1$, the representation ρ is unitary, i.e. has image in* $\mathrm{SU}(2) \subset \mathrm{GL}(2,\mathbb{C})$
- *If $C_{22} = 0$, the representation ρ is reducible in* $\mathrm{GL}(2,\mathbb{C})$
- *If $C_{22} = -1$, the representation ρ has image in* $\mathrm{SU}(1,1) \subset \mathrm{GL}(2,\mathbb{C})$

Proof. Assume $C_{22} = 1$. Then equation (3.12) reduces to

$$\rho(\gamma) \cdot \rho(\gamma)^\dagger = 1 \qquad \forall \gamma \in \pi_1(X)$$

This implies the image of ρ lies in SU(2).

Let us now assume $C_{22} = 0$. The equation (3.12) reduces to

$$\begin{aligned} |\,(\rho(\gamma))_{11}\,|^2 &= 1 \\ |\,(\rho(\gamma))_{21}\,|^2 &= 0 \end{aligned}$$

for all $\gamma \in \pi_1(X)$. This implies that the solution $\chi_1(u)$ transforms under a representation of U(1), i.e. does not mix with $\chi_2(u)$. In other words, the representation ρ is reducible.

Let us finally assume $C_{22} = -1$. The equation (3.12) reduces to

$$\rho(\gamma) \cdot \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cdot \rho(\gamma)^\dagger = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \forall \gamma \in \pi_1(X)$$

This implies the image of ρ lies in SU(1, 1). □

To finish the proof of the theorem, we will need to prove that we can never find the cases $C_{22} = 1$ and $C_{22} = 0$ for equation (3.11). To put the proofs of these statements into context, we will first make a detour.

Oper differential equations A differential equation of the form of equation (3.11) is called an oper. The two solutions $\chi_1(u)$ and $\chi_2(u)$ define global sections of $K_X^{-1/2}$, which under a coordinate transformation $u \to w(u)$ transform as

$$\chi(u) \to (w'(u))^{-1/2}\chi(w(u))$$

This is equivalent to a transformation of the function $t(u)$ of the form of equation (3.9). We will also refer to a function $t(u)$ transforming in this way as an oper. It will be clear from the context which object we refer to. On Riemann surfaces with punctures, we require that the oper has regular singularities at the punctures. This means we can expand

$$t(u) = \frac{\delta_*}{(u-u_*)^2} + \frac{c_*}{u-u_*} + \mathcal{O}((u-u_*)^0)$$

near a puncture at $u = u_*$. The value of the parameters δ_* at each puncture, is an extra piece of data that comes with the definition of the oper.

To describe the space all opers, we note that for a given reference oper $t_0(u)$, the difference $t(u) - t_0(u) = a(u)$ must transform as a quadratic differential

$a \in \mathrm{H}^0(X, K_X^2)$ which has prescribed value at the poles. In other words, if we denote by $\mathcal{P}(X)$ the space of opers over X, this space is an affine vector space modeled over $\mathrm{H}^0(X, K_X^2)$. The dimensions of these spaces must be equal and can be calculated by the Riemann-Roch formula to be $\dim_{\mathbb{C}}(\mathcal{P}(X)) = 3g - 3 + n$. Hence, if we pick a basis of $\mathrm{H}^0(X, K_X^2)$, we may expand $a(u)$ in $3g - 3 + n$ complex numbers, which are also called accessory parameters.

Comparing to equation (3.7) and equation (3.8), we see that the role of the accessory parameters is played by the eigenvalues $\mathbf{E}$ in the context of the quantized Hitchin Hamiltonians.

Lemma 3.10. *The monodromy representation $\rho \in \mathrm{Hom}(\pi_1(X), \mathrm{GL}(2, \mathbb{C}))$ of any vector of solutions $\chi(u)$ must be reducible to $\mathrm{SL}(2, \mathbb{C})$. Furthermore, two linearly independent solutions $\chi_1(u)$ and $\chi_2(u)$ can never vanish simultaneously.*

Proof. Let us define the Wronskian

$$W(u) := \chi_1'(u)\chi_2(u) - \chi_1(u)\chi_2'(u)$$

We may show that this function satisfies the differential equation

$$\begin{aligned} W'(u) &= \chi_1''(u)\chi_2(u) - \chi_1(u)\chi_2''(u) \\ &= 0 \end{aligned}$$

Therefore the Wronskian $W(u) = W_0$ is a constant on all of X. If $\chi_1(u)$ and $\chi_2(u)$ both vanish at $u = u_0$, we find $W(u_0) = 0$ implying the Wronskian vanishes everywhere. This implies $\chi_2(u) \sim \chi_1(u)$ so that these solutions are not linearly independent, leading to a contradiction. This proves the final statement.

For the first statement, we note that $\chi(\gamma \cdot u) = \chi(u) \cdot \rho(\gamma)$. In matrix form, we may write the Wronskian as

$$W(u) = \det \begin{pmatrix} \chi_1'(u) & \chi_2'(u) \\ \chi_1(u) & \chi_2(u) \end{pmatrix}$$

The matrix $\begin{pmatrix} \chi_1'(u) & \chi_2'(u) \\ \chi_1(u) & \chi_2(u) \end{pmatrix}$ transforms as

$$\begin{pmatrix} \chi_1'(\gamma \cdot u) & \chi_2'(\gamma \cdot u) \\ \chi_1(\gamma \cdot u) & \chi_2(\gamma \cdot u) \end{pmatrix} = \begin{pmatrix} \chi_1'(u) & \chi_2'(u) \\ \chi_1(u) & \chi_2(u) \end{pmatrix} \cdot \rho(\gamma)$$

proving that $W(\gamma \cdot u) = \det(\rho(\gamma))W(u)$. Since $W(u) = W_0$ is a constant, we find $\det(\rho(\gamma)) = 1$, proving the lemma. □

Since the two linearly independent solutions $\chi_1(u)$ and $\chi_2(u)$ do not vanish simultaneously, we may construct a map $A : \tilde{X} \to \mathbb{C}P^1$ by $A(u) = [\chi_1(u) : \chi_2(u)]$. We call the map A the developing map.

It is easily seen that the Deck transformations act on the developing map A by

$$A(\gamma \cdot u) = \frac{\rho(\gamma)_{11} A(u) + \rho(\gamma)_{12}}{\rho(\gamma)_{21} A(u) + \rho(\gamma)_{22}}$$

The action of the monodromy ρ on $A(u)$ factors through to an element in $\mathrm{Hom}(\pi_1(X), \mathrm{PSL}(2,\mathbb{C}))$.

By changing the basis of solutions by an element $M \in \mathrm{SL}(2,\mathbb{C})$, the matrix M defines an action of the form

$$\begin{aligned} M \cdot A(u) &= \frac{M_{11} A(u) + M_{12}}{M_{21} A(u) + M_{22}} \\ M \cdot \rho(\gamma) &= M \rho(\gamma) M^{-1} \end{aligned}$$

This action factors through to an action of $\mathrm{PSL}(2,\mathbb{C})$ on the pair (A,ρ). We denote by $[A,\rho]$ the equivalence class $(A,\rho) \sim (M \cdot A, M \cdot \rho)$. It turns out that we can invert this construction. See for example [78] for a proof of the following proposition.

Proposition 3.11. *An oper $t(z)$ over X is equivalent to the pair $[A,\rho]$ of an immersive map $A : \tilde{X} \to \mathbb{C}P^1$ and a monodromy representation $\rho \in \mathrm{Hom}(\pi_1(S), \mathrm{PSL}(2,\mathbb{C}))$ satisfying the property*

$$A(\gamma \cdot u) = \frac{\rho(\gamma)_{11} A(u) + \rho(\gamma)_{12}}{\rho(\gamma)_{21} A(u) + \rho(\gamma)_{22}}$$

Proof. We have shown how to construct a developing map from an oper. It remains to prove that the map A is immersive. The function $A(u)$ is represented as

$$A(u) = \frac{\chi_1(u)}{\chi_2(u)}$$

in a chart defined by $\chi_2(u) \neq 0$. Therefore,

$$A'(u) = \frac{W_0}{\chi_2(u)^2}$$

For $\chi_1(u) \neq 0$, we pick the representation

$$A(u) = \frac{\chi_2(u)}{\chi_1(u)}$$

which shows that for both charts, the function $A'(u)$ remains finite and non-zero. Therefore, A defines an immersive map.

We now describe the inverse construction. Given a pair $[A, \rho]$ for which A is immersive, we may define a function $t(u) := \frac{1}{2}\{A(u), u\}$. Since the Schwarzian is invariant under Möbius transformations, the Deck transformation $t(\gamma \cdot u) = t(u)$ does not affect the function $t(u)$. It therefore factors through to a function $t(u)$ on X. Furthermore, under a coordinate transformation $u \to w(u)$, we find $A(u) \to A(w(u))$. The Schwarzian has the following chain rule property

$$\{A(w(u)), u\} = (w'(u))^2\{A(w), w\} + \{w(u), u\}$$

This implies the function $t(u)$ transforms according to equation (3.9) and defines an oper.

Finally, we have to check that these constructions are inverses. Let us start from two solutions $\chi_1(u)$ and $\chi_2(u)$ to equation (3.11). The Schwarzian derivative satisfies

$$\begin{aligned} \left\{\frac{\chi_1(u)}{\chi_2(u)}, u\right\} &= -2\frac{\chi_2''(u)}{\chi_2(u)} \\ &= 2t(u) \end{aligned}$$

This shows we recover $t(u)$.

For the other direction, assume we start from a pair $[A, \rho]$. We construct a new pair $[\tilde{A}, \tilde{\rho}]$ from the differential equation equation (3.11) with $t(u) = \frac{1}{2}\{A(u), u\}$. By construction,

$$\{A(u), u\} = \{\tilde{A}(u), u\}$$

This implies $A(u)$ and $\tilde{A}(u)$ differ by a Möbius transformation $\tilde{A}(u) = M \cdot A(u)$. By the intertwining property

$$A(\gamma \cdot u) = \frac{\rho(\gamma)_{11}A(u) + \rho(\gamma)_{12}}{\rho(\gamma)_{21}A(u) + \rho(\gamma)_{22}}$$

the representations ρ and $\tilde{\rho}$ must be related by $\tilde{\rho}(\gamma) = M \cdot \rho(\gamma)$.

This proves the two constructions are inverse to each other. □

Remark 3.12. The pair $[A, \rho]$ is sometimes also called a development-holonomy pair. The monodromy of such a pair is valued in $\mathrm{PSL}(2, \mathbb{C})$ while the monodromy of an oper is valued in $\mathrm{SL}(2, \mathbb{C})$.

Although a well-known result, we refer to lemma 4.1.7 in [79] for a proof of the following lemma

Lemma 3.13. *If ρ is the monodromy representation of an oper, it cannot have image in* SU(2).

Proof. Assume the monodromy representation is unitary. We may construct a developing map $A : \tilde{X} \to \mathbb{C}P^1$ where $A(u) = [\chi_1(u) : \chi_2(u)]$. The map A allows us to pull back the metric $(1 + |z|^2)^{-2} dz d\bar{z}$ on $\mathbb{C}P^1$ with constant curvature $+1$ to $\tilde{X}$. Since the isometries of this metric are precisely given by Möbius transformations in PU(2) and the Deck transformations are realized by PU(2) transformations acting on A, we can define a metric of constant curvature $+1$ on X. However, by assumption X has negative Euler characteristic, which is in contradiction with the Gauss-Bonnet theorem. Therefore the monodromy representation cannot be unitary. □

Lemma 3.14. *If ρ is the monodromy representation of an oper, it cannot be reducible.*

Proof. Assume the monodromy representation is reducible. Without loss of generality, we may assume $\chi_1(u)$ is the solution on which the representation acts by U(1). To have monodromy in U(1), the function $\chi_1(u)$ must have an expression of the form $\chi_1(u) = \chi_1(u_0) \exp(-\int_{u_0}^{u} s(u')du')$ for holomorphic $s(u)$, showing that $\chi_1(u)$ does not vanish anywhere. We may then construct a map $A : \tilde{X} \to \mathbb{C}$ by setting $A(u) = \chi_2(u)/\chi_1(u)$. With this map, we can pull back the metric $dz d\bar{z}$ on $\mathbb{C}$ with vanishing curvature to $\tilde{X}$. The isometries of this metric are precisely the reducible Möbius transformations generated by scaling and translations. Since the Deck transformations act by reducible transformations, the metric $dz d\bar{z}$ defines a metric of vanishing curvature on X. By assumption X has negative Euler characteristic, which again leads to a contradiction with the Gauss-Bonnet theorem, showing that the monodromy cannot be reducible. □

The only possibility we are left with, is that the monodromy representation of an oper has image in $\mathrm{SU}(1,1)$. Since this group is isomorphic to $\mathrm{SL}(2,\mathbb{R})$, we conclude the proof of theorem 3.5.

Remark 3.15. Although the monodromy of an oper $t(u)$ might not lie in $\mathrm{SL}(2,\mathbb{R})$, the monodromy of the development-holonomy pair $[A,\rho]$ can still be real! This happens when the monodromy of $t(u)$ projects to an action of $\mathrm{PGL}(2,\mathbb{R})$ on $A(u)$ not reducible to $\mathrm{PSL}(2,\mathbb{R})$. Classifying the pairs $[A,\rho]$ with real monodromy $\rho \in \mathrm{Hom}(\pi_1(X), \mathrm{PGL}(2,\mathbb{R}))$ is therefore not enough: We need to know which ρ are reducible to $\mathrm{PSL}(2,\mathbb{R})$.

Part II

Quantization on the four-punctured sphere

If we assume our Riemann surface X is topologically a four-punctured sphere, the single-valued solution $\Psi_{\mathbf{E}}(\mathbf{x}, \bar{\mathbf{x}})$ to equation (3.6), is constructed from a combination of solutions to equation (3.5) and its complex conjugate. These functions only depend on a single variable x and the complex conjugate $\bar{x}$. If we furthermore assume that the crossratio z lies in the interval $(0, 1)$ and the accessory parameter λ is real, the differential equation reduces to a real ordinary differential equation on the four real arcs connecting the punctures.

In this case, we can analyze the quantization of the Hitchin system directly. Indeed, the reality condition becomes a condition on the existence of a solution with certain regularity properties (see theorem 4.5). We will find the monodromy matrices explicitly and draw the image of the four real arcs under the developing map.

Recently the problem of finding accessory parameters, for which the oper on X has real monodromy, was discussed by Takhtajan in [80]. However, the problem itself already goes back to [81, 82, 83]. A complete solution was found by Smirnov using the theory of real ordinary differential equations of Sturm-Liouville type. This solution was originally published in 1918 (and much later collected in [50]) and afterwards in [84, 85].

By assuming the reality of the accessory parameter, we have restricted to a special series of parameters for which we find opers with real monodromy. An example of holomorphicity conditions not corresponding to real accessory parameters was constructed in [77]. In section §6 we show that we can act by the mapping class group to generate other series of accessory parameters leading to real monodromy. We show that all opers with real monodromy can be constructed in this way, but defer the proof to section §9.

4 Real monodromy and functional behaviour

4.1 Holomorphic solutions from real monodromy

The starting point of our analysis is equation (3.5). Two linearly independent solutions $\psi_\lambda^{(1)}$ and $\psi_\lambda^{(2)}$ can be combined into the function

$$\Psi_{\mathbf{E}}(x,\bar{x}) = \psi_\lambda^{(1)}(x)\bar{\psi}_\lambda^{(2)}(\bar{x}) - \bar{\psi}_\lambda^{(1)}(\bar{x})\psi_\lambda^{(2)}(x)$$

on the space $\mathbb{C}P^1\backslash\{0,z,1,\infty\}$ satisfying

$$\hat{H}_2\Psi_{\mathbf{E}}(x,\bar{x}) = E_2\Psi_{\mathbf{E}}(x,\bar{x}) \qquad \hat{\bar{H}}_2\Psi_{\mathbf{E}}(x,\bar{x}) = \bar{E}_2\Psi_{\mathbf{E}}(x,\bar{x})$$

Restricting to real punctures and accessory parameters Let us set $z \in (0,1)$ and $\lambda \in \mathbb{R}$. The Sturm-Liouville equation reduces to an ordinary differential equation on the real axis $x \in \mathbb{R}\backslash\{0,z,1\}$. The discussion mostly follows [82, 83] from this point on, albeit in a slightly different form.

Around each puncture we can find a basis of two solutions: One solution is holomorphic at the puncture, while the other solution has a logarithmic singularity. Since the differential equation is real, we can look for a basis of solutions which is real on the real axis. For this purpose, we introduce branch cuts around each puncture as in figure 4.1.

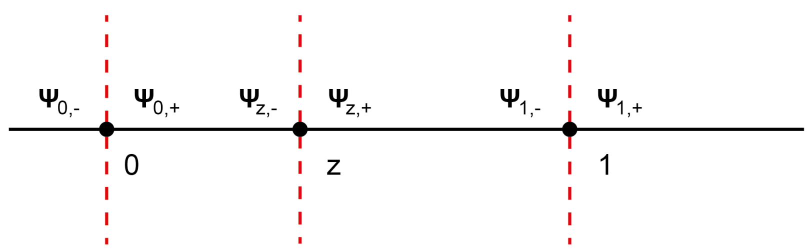


Figure 4.1: Branch cuts on $X = \mathbb{C}P^1\backslash\{0,z,1,\infty\}$

For notational simplicity, we will forget about the accessory parameter for now. We denote the two linearly independent real solutions in a neighbourhood of the puncture $* \in \{0, z, 1\}$ by $\psi_{*,+}^{(1)}$ and $\psi_{*,+}^{(2)}$ if they are defined to the right of the branch cut and by $\psi_{*,-}^{(1)}$ and $\psi_{*,-}^{(2)}$ if they are defined to the left. We will also assume that the solutions $\psi_{*,+}^{(1)}$ and $\psi_{*,-}^{(1)}$ are holomorphic at $*$. Therefore, we will simply denote this solution by $\psi_*^{(1)}$. The other solutions may be expressed as $\psi_{*,\pm}^{(2)} = \psi_*^{(1)} \log(\pm(x - *)) + f_*(x)$ for a function f_* holomorphic at $*$.

Transfer matrices Let us write $\vec{\psi}_{*,\pm} = (\psi_*^{(1)}, \psi_{*,\pm}^{(2)})$. To calculate the monodromy of $\vec{\psi}_{*,\pm}$ around each puncture, we break up the problem in several analytical continuations. In figure 4.2 we show which paths we analytically continue along.

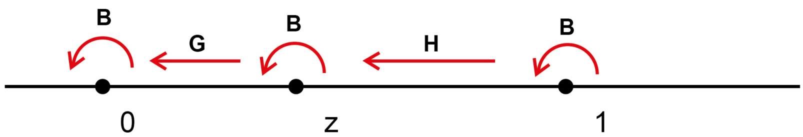


Figure 4.2: The paths along which we analytically continue in $\mathbb{C}P^1 \backslash \{0, z, 1, \infty\}$

We define $\vec{\psi}_{*,\pm}(B \cdot (* + x))$ to be obtained by analytically continuing the basis of solutions along the curve B. We may then set $B \cdot (* + x) = * - x$ and write for real values of x

$$\begin{aligned} \psi_{*,+}^{(2)}(* - x) &= \psi_*^{(1)} \log(* - x) + \pi i \psi_*^{(1)}(* - x) + f_*(* - x) \\ &= \psi_{*,-}^{(2)}(* - x) + \pi i \psi_*^{(1)}(* - x) \end{aligned}$$

The analytical continuation of our function therefore leads to a relation of the form

$$\vec{\psi}_{*,+} = \vec{\psi}_{*,-} \cdot B$$

in the domain of definition $x < *$. We understand B as a path as well as a matrix $\begin{pmatrix} 1 & \pi i \\ 0 & 1 \end{pmatrix}$ depending on the context.

Similarly, we can relate $\vec{\psi}_{0,+}$ and $\vec{\psi}_{z,-}$ through a matrix G by

$$\vec{\psi}_{z,-} = \vec{\psi}_{0,+} \cdot G$$

for $0 < x < z$ and $\vec{\psi}_{z,+}$ and $\vec{\psi}_{1,-}$ by

$$\vec{\psi}_{1,-} = \vec{\psi}_{z,+} \cdot H$$

for $z < x < 1$.

The matrices G and H are not dependent on a choice of branch cut, since the functions on the left- and right-hand side can be analytically extended to the same domain along the real axis. Because the matrices G and H appear through an analytical continuation of real functions, both of these matrices are completely real and invertible. We call the matrices B, G and H transfer matrices.

Through equation (3.5) the matrices G and H implicitly become functions of the accessory parameter λ. Moreover, due to the real analyticity of the differential equation both in x and in λ, the solutions $\vec{\psi}_{*,\pm}(x)$ are real analytical in x and λ and the transfer matrices $G = G(\lambda)$ and $H = H(\lambda)$ real analytical in λ.

Reality condition of monodromies The monodromy M_* of the solution $\vec{\psi}_{*,\pm}$ around the puncture $*$ is given by

$$\vec{\psi}_{*,\pm}(\gamma_* \cdot x) = \vec{\psi}_{*,\pm}(x) \cdot B^2$$

with γ_* a small loop going counter-clockwise around the puncture $*$. Note that for our choices $\gamma_\infty \gamma_1 \gamma_z \gamma_0 = 1$.

The basis $\vec{\psi}_{*,\pm}$ brings the monodromy to the form $M_* = B^2$ of the form

$$M_* = \begin{pmatrix} 1 & 2\pi i \\ 0 & 1 \end{pmatrix}$$

Since the traces of monodromies are invariant under the chosen basis, we may choose any basis to calculate $\mathrm{tr}(M_z M_0)$. For example, we may calculate this trace using the equality

$$\vec{\psi}_{z,-}((\gamma_z \gamma_0) \cdot x) = \vec{\psi}_{z,-}(x) \cdot (M_0 M_z)$$

as follows:

$$\begin{aligned}
\vec{\psi}_{z,-}(\gamma_z \cdot (\gamma_0 \cdot x)) &= \vec{\psi}_{z,-}(\gamma_0 \cdot x) \cdot B^2 \\
&= \vec{\psi}_{0,+}(\gamma_0 \cdot x) \cdot (GB^2) \\
&= \vec{\psi}_{0,+}(x) \cdot (B^2 G^{-1} B^2) \\
&= \vec{\psi}_{z,-}(x) \cdot (G^{-1} B^2 G B^2)
\end{aligned}$$

We therefore identify

$$\begin{aligned}\operatorname{tr}(M_0 M_z) &= \operatorname{tr}(G^{-1}B^2 G B^2)\\ &= 2+4\pi^2\frac{(G_{21})^2}{\det(G)}\end{aligned}$$

We furthermore note that a left action by Deck transformations on the space $\mathbb{C}P^1\backslash\{0,z,1,\infty\}$ turns into a right action of the monodromy matrices on the solutions $\vec{\psi}_{*,\pm}$ as evidenced by our calculation above.

Fact 4.1. *It is sufficient to calculate the seven values* $\operatorname{tr}(M_0)$, $\operatorname{tr}(M_z)$, $\operatorname{tr}(M_1)$, $\operatorname{tr}(M_0 M_z)$, $\operatorname{tr}(M_z M_1)$, $\operatorname{tr}(M_0 M_1)$ *and* $\operatorname{tr}(M_0 M_z M_1)$ *to uniquely determine the set* $(M_0, M_z, M_1) \in \mathrm{SL}(2,\mathbb{C})$ *up to conjugation. Since* $\operatorname{tr}(M_0) = \operatorname{tr}(M_z) = \operatorname{tr}(M_1) = 2$*, the non-trivial information is found in the other four traces. A proof can be found in [86], but we also briefly come back to this statement in section §12.*

We may calculate in a similar way as before

$$\begin{aligned}\operatorname{tr}(M_z M_1) &= 2+4\pi^2\frac{(H_{21})^2}{\det(H)}\\ \operatorname{tr}(M_0 M_1) &= 2+4\pi^2\left(\frac{((GH)_{21})^2-\pi^2(G_{21})^2(H_{21})^2}{\det(G)\det(H)}+2\pi i\frac{G_{21}H_{21}(GH)_{21}}{\det(G)\det(H)}\right)\\ \operatorname{tr}(M_0 M_z M_1) &= 2+4\pi^2\left(\frac{(G_{21})^2}{\det(G)}+\frac{(H_{21})^2}{\det(H)}+\frac{((GH)_{21})^2+\pi^2(G_{21})^2(H_{21})^2}{\det(G)\det(H)}\right)\end{aligned}$$

Only the trace $\operatorname{tr}(M_0 M_1)$ has a non-trivial imaginary part. To ensure all traces are real, we must set $G_{21}H_{21}(GH)_{21} = 0$. Therefore, at least one of these three factors must vanish. We will refer to these conditions as holomorphicity or quantization conditions.

Lemma 4.2. *If any two of the three components* G_{21}*,* H_{21} *and* $(GH)_{21}$ *vanish, all three must vanish.*

Proof. We may write out $(GH)_{21} = G_{21}H_{11} + G_{22}H_{21}$. If both $G_{21} = H_{21} = 0$, clearly $(GH)_{21} = 0$. On the other hand, if $G_{21} = (GH)_{21} = 0$, the combination $G_{22}H_{21} = 0$. Since $\det(G) = G_{11}G_{22}$ is non-vanishing, $G_{22} \neq 0$. Therefore, also $H_{21} = 0$. Finally, $H_{21} = (GH)_{21} = 0$ implies in the same way that $G_{21} = 0$. □

Lemma 4.3. *There exists no value of* λ *such that both* $G_{21}(\lambda) = 0$ *and* $H_{21}(\lambda) = 0$*.*

Proof. Assume there exists such a value of λ. The conditions $G_{21} = 0$ and $H_{21} = 0$ imply that $\psi_0^{(1)} = \psi_z^{(1)} G_{11} = \psi_1^{(1)} G_{11} H_{11}$ is a function holomorphic at the punctures 0, z and 1. Therefore, it must be holomorphic at ∞. Since the function is holomorphic everywhere on $\mathbb{C}P^1$, it must be a constant and the only constant solving equation (3.5) is $\psi_0^{(1)} = 0$. Hence, we cannot find a non-trivial solution for such a value of λ leading to a contradiction. □

From these lemmas we conclude that at most one of the conditions $G_{21} = 0$, $H_{21} = 0$ or $(GH)_{21} = 0$ can hold. In fact, no where did we need to assume the reality of λ. This result is therefore also valid for $\lambda \in \mathbb{C}$.

Definition 4.4. The real analytical continuation of a function $\vec{\psi}_{*,-}$ along a puncture $*$ is defined by $\vec{\psi}_{*,+}$. Similarly, we also call $\vec{\psi}_{*,-}$ the real analytical continuation of $\vec{\psi}_{*,+}$.

The vanishing of G_{21} and H_{21} implies that $\psi_z^{(1)}$ is regular at $x = 0$, respectively $x = 1$. By noting that the real analytical continuation of $\psi_1^{(1)}$ to $x = 0$ along $x = z$ is given by $\psi_0^{(2)} (GH)_{21} + \psi_0^{(1)} (GH)_{11}$, the condition $(GH)_{21} = 0$ is equivalent to the regularity of $\psi_1^{(1)}$ at $x = 0$ after real analytical continuation along $x = z$. We conclude the above discussion with the following theorem originally due to Klein and Hilbert [82, 83].

Theorem 4.5. *An accessory parameter $\lambda \in \mathbb{R}$ leads to real monodromy of equation* (3.5) *if and only if $G_{21}(\lambda) = 0$, $H_{21}(\lambda) = 0$ or $(GH)_{21}(\lambda) = 0$. Equivalently, a solution $\psi_\lambda(x)$ to the differential equation exists such that precisely one of the three following conditions holds*

1. *The solution $\psi_\lambda(x)$ is regular at both $x = 0$ and $x = z$*
2. *The solution $\psi_\lambda(x)$ is regular at both $x = z$ and $x = 1$*
3. *The solution $\psi_\lambda(x)$ is regular at $x = 0$ and at $x = 1$ after the real analytical continuation along $x = z$*

Hyperelliptic involutions on the four-punctured sphere There exist biholomorphic maps on the four-punctured sphere exchanging the positions of the punctures. These maps must be given by complex Möbius transformations. Indeed, we have the following lemma

Lemma 4.6. *The only biholomorphic maps $C \to C$ are given by the identity and three hyperelliptic involutions j_1, j_2 and j_3 defined by*

$$\begin{aligned} j_1(x) &= \frac{z-x}{1-x} \\ j_2(x) &= \frac{z}{x} \\ j_3(x) &= \frac{x-1}{z^{-1}x-1} \end{aligned}$$

Proof. Every biholomorphic map $h : C \to C$ must take the form of a Möbius transformation.

Let us first assume h fixes one of the punctures. Without loss of generality, we may assume this puncture is ∞. Then the map h is given by $h(x) = \alpha x + \beta$ for real α and β. To ensure we map punctures to punctures, we need to set $h(z) = z$. Hence $\beta = (1-z)z$. However, now $h(0) = (1-z)z$, which is not a puncture. This leads to a contradiction.

Therefore, we assume ∞ is mapped to 0. Then $h(x) = (\gamma x + \delta)^{-1}$. Now $h(0) = \delta^{-1}$ so that $\delta^{-1} \in \{z, 1, \infty\}$. The choices $\delta^{-1} \in \{z, 1\}$ lead to Möbius transformations which do not map punctures to punctures. We must set $\delta = 0$ and find $h(x) = \gamma^{-1}x^{-1}$. Noting that $h(1) = \gamma^{-1}$ and using the fact that h must switch the punctures a and 1, we find $h(x) = zx^{-1}$.

By assuming ∞ is mapped to a, we find the map $h(x) = \frac{x-1}{z^{-1}x-1}$ and by assuming ∞ is mapped to 1, we find the map $h(x) = \frac{z-x}{1-x}$ in a similar way.

Hence, j_1, j_2 and j_3 are the only non-trivial biholomorphic maps $C \to C$. □

Remark 4.7. These involutions form the group $\mathbb{Z}_2 \times \mathbb{Z}_2$ since $j_1 j_2 = j_2 j_1 = j_3$ and $j_1^2 = j_2^2 = 1$.

Since these involutions are Möbius transformations exchanging the punctures, the oper $t(j_k(x)) = t(x)$ for $k = 1, 2, 3$. Therefore the functions $\psi_\lambda(j_k(x))$ are solutions of the same differential equation. Hence, we find

Theorem 4.8. *The conditions from theorem 4.5 have the following alternate formulations*

- There exists a solution regular at $x = 0$ and at $x = z$ if and only if there exists a solution regular at $x = 1$ and at $x = \infty$.

- There exists a solution regular at $x = z$ and at $x = 1$ if and only if there exists a solution regular at $x = 0$ and at $x = \infty$.
- The following four statements are equivalent:
  - There exists a solution regular at $x = 0$ and at $x = 1$ after real analytical continuation along $x = z$.
  - There exists a solution regular at $x = z$ and at $x = \infty$ after real analytical continuation along $x = 1$.
  - There exists a solution regular at $x = 1$ and at $x = 0$ after real analytical continuation along $x = \infty$.
  - There exists a solution regular at $x = \infty$ and at $x = z$ after real analytical continuation along $x = 0$.

Each of these different formulations involve the puncture $x = \infty$. We may therefore restrict our attention to the behaviour of the problem near the punctures at $x = 0$, $x = z$ and $x = 1$.

4.2 Finding the monodromy matrices

We will treat the three holomorphicity conditions separately. For each of these conditions we will construct a developing map $A(x)$ defined as the quotient of two solutions to equation (3.5) and show that it has monodromy in $\mathrm{PSL}(2, \mathbb{R})$. The existence of such a developing map is guaranteed by the holomorphicity conditions.

To ensure the monodromy $\mathrm{tr}(M_\infty) = 2$, we must constrain the values of G_{21}, H_{21} and $(GH)_{21}$. This is equivalent to setting $\mathrm{tr}(M_0 M_z M_1) = 2$, i.e. to the equation

$$\det(H)(G_{21})^2 + \det(G)(H_{21})^2 + ((GH)_{21})^2 + \pi^2 (G_{21})^2 (H_{21})^2 = 0 \qquad (4.1)$$

We show the following

Lemma 4.9. *If we consider* $G_{21} = 0$*, equation* (4.1) *reduces to* $\mathrm{tr}(G) = 0$*. Similarly, if* $H_{21} = 0$*, we find* $\mathrm{tr}(H) = 0$*. Finally, for* $(GH)_{21} = 0$*, the equation* $\mathrm{tr}(GH) = \pi^2 G_{21} H_{21}$ *must hold.*

Proof. To obtain each of these cases, we only have to simplify equation (4.1).

For $G_{21} = 0$, we find $\det(G) = G_{11}G_{22}$ and $(GH)_{21} = G_{22}H_{21}$. Therefore, $G_{11} + G_{22} = 0$. Similarly, we find $H_{11} + H_{22} = 0$ for $H_{21} = 0$.

If we set $(GH)_{21} = 0$, we find $G_{21}H_{11} + G_{22}H_{21} = 0$ and

$$-((GH)_{11} + (GH)_{22})G_{21}H_{21} + \pi^2(G_{21})^2(H_{21})^2 = 0$$

Since neither G_{21} nor H_{21} can vanish, we may rewrite this as $\mathrm{tr}(GH) = \pi^2 G_{21}H_{21}$.

□

Let us now consider the case $G_{21} = 0$. We may construct a non-trivial developing map by

$$A(x) = \frac{1}{2\pi i G_{22}H_{21}} \frac{\psi_1^{(1)}(x)}{\psi_0^{(1)}(x)}$$

We calculate the action of the monodromy by first calculating

$$\begin{aligned}
\psi_1^{(1)}(\gamma_0 \cdot x) &= \sum_{j=1,2} \psi_0^{(j)}(x) \cdot (B^2GBH)_{j1} \\
&= \sum_{j=1,2} \psi_0^{(j)}(x) \cdot (GBH)_{j1} + 2\pi i \psi_0^{(1)}(z) \cdot (GBH)_{21} \\
&= \psi_1^{(1)}(x) + 2\pi i G_{22}H_{21}\psi_0^{(1)}(x)
\end{aligned}$$

Then

$$\begin{aligned}
A(\gamma_0 \cdot x) &= \frac{1}{2\pi i G_{22}H_{21}} \frac{\psi_1^{(1)}(\gamma_0 \cdot x)}{\psi_0^{(1)}(\gamma_0 \cdot x)} \\
&= \frac{1}{2\pi i G_{22}H_{21}} \frac{\psi_1^{(1)}(x) + 2\pi i G_{22}H_{21}\psi_0^{(1)}(x)}{\psi_0^{(1)}(x)} \\
&= A(x) + 1
\end{aligned}$$

We may calculate the other monodromies in the same way and find the following result

Theorem 4.10. *If we consider the developing map* $A(x) = \frac{1}{2\pi i (GBH)_{21}} \frac{\psi_1^{(1)}(x)}{\psi_0^{(1)}(x)}$, *we will always find* $A(\gamma_0 \cdot x) = A(x) + 1$.

Let us for simplicity set

$$\begin{aligned}
X &= 2 + 4\pi^2 \frac{(G_{21})^2}{\det(G)} \\
Y &= 2 + 4\pi^2 \frac{(H_{21})^2}{\det(H)}
\end{aligned}$$

We find the following monodromy matrices:

- *Consider* $G_{21} = 0$. *Then*

$$\begin{aligned} A(\gamma_z \cdot x) &= A(x) - 1 \\ A(\gamma_1 \cdot x) &= \frac{A(x)}{1 - (Y-2)A(x)} \end{aligned}$$

- *Consider* $H_{21} = 0$. *Then*

$$\begin{aligned} A(\gamma_z \cdot x) &= \frac{A(x)}{1 + (X-2)A(x)} \\ A(\gamma_1 \cdot x) &= \frac{A(x)}{1 - (X-2)A(x)} \end{aligned}$$

For the case $(GH)_{21} = 0$, *we consider the developing map* $A(x) = \frac{1}{2\pi i G_{21}} \frac{\psi_z^{(1)}(x)}{\psi_0^{(1)}(x)}$. *Then*

$$\begin{aligned} A(\gamma_0 \cdot x) &= A(x) + 1 \\ A(\gamma_z \cdot x) &= \frac{A(x)}{1 + (X-2)A(x)} \\ A(\gamma_1 \cdot x) &= \frac{\left(1 - 2\frac{X-2}{X+2}\right) A(x) - 1 + \frac{X-2}{X+2}}{\left(1 + 2\frac{X-2}{X+2}\right) + \frac{(X-2)^2}{X+2} A(x)} \end{aligned}$$

Proof. We may find these results by an explicit computation. Note also that from $A(\gamma_0 \cdot x)$, $A(\gamma_z \cdot x)$ and $A(\gamma_1 \cdot x)$ we may uniquely calculate $A(\gamma_\infty \cdot x)$. Nonetheless, for the proof of this theorem we may reverse the situation: It suffices to show that the trace coordinates have the correct values, since the monodromy $A(\gamma_0 \cdot x) = A(x) + 1$ uniquely fixes the conjugacy class of the monodromy representation. This is trivial for the cases $G_{21} = 0$ and $H_{21} = 0$. Let us do this in detail for the more complicated case $(GH)_{21} = 0$.

Clearly $\mathrm{tr}(M_0) = \mathrm{tr}(M_z) = \mathrm{tr}(M_1) = 2$. Indeed, if we represent M_0, M_z and M_1 as matrices acting on $A(x)$ by Möbius transformations, we find

$$\begin{aligned} M_0 &= \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \\ M_z &= \begin{pmatrix} 1 & 0 \\ X-2 & 1 \end{pmatrix} \\ M_1 &= \begin{pmatrix} 1 - 2\frac{X-2}{X+2} & -1 + \frac{X-2}{X+2} \\ \frac{(X-2)^2}{X+2} & 1 + 2\frac{X-2}{X+2} \end{pmatrix} \end{aligned}$$

These matrices define a monodromy representation of the four-punctured sphere satisfying

$$\mathrm{tr}(M_0 M_z M_1) = 2$$

We can also easily check that

$$\begin{aligned}
\mathrm{tr}(M_0 M_z) &= X \\
\mathrm{tr}(M_z M_1) &= 2 - (X-2) + \frac{(X-2)^2}{X+2} \\
\mathrm{tr}(M_0 M_1) &= 2 + \frac{(X-2)^2}{X+2}
\end{aligned}$$

The result $\mathrm{tr}(M_0 M_z) = X$ is obviously correct. From equation (4.1) we find the relation $XY + 2X + 2Y = 12$ implying

$$Y = -2 + \frac{16}{X+2}$$

This allows us to express

$$\begin{aligned}
\mathrm{tr}(M_z M_1) &= 4 - (X-2) + \frac{(X-2)^2}{X+2} \\
&= -2 + \frac{16}{X+2} \\
&= Y
\end{aligned}$$

Finally, we may rewrite using equation (4.1)

$$\begin{aligned}
\mathrm{tr}(M_0 M_1) &= 2 - 4\pi^4 \frac{(G_{21})^2 (H_{21})^2}{\det(G)\det(H)} \\
&= 2 - 4\pi^2 \left(\frac{(G_{21})^2}{\det(G)} + \frac{(H_{21})^2}{\det(H)} \right) \\
&= X + Y - 2
\end{aligned}$$

Indeed, we find

$$\frac{(X-2)^2}{X+2} = X + Y - 4$$

which proves that $\mathrm{tr}(M_0 M_1)$ also produces the correct result. □

4.3 Drawing fundamental domains

For each of these developing maps A, we may consider its image in $\mathbb{C}P^1$. By continuity, it suffices to calculate the image of $\mathbb{R}P^1$ under the map A to describe the image of this domain. We label each of the four arcs separately. See

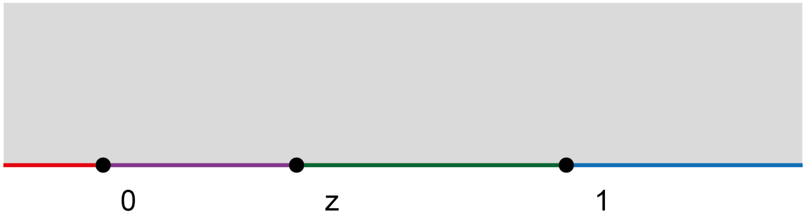


Figure 4.3: Labeling of four arcs in $\mathbb{R}P^1\backslash\{0,z,1,\infty\}$

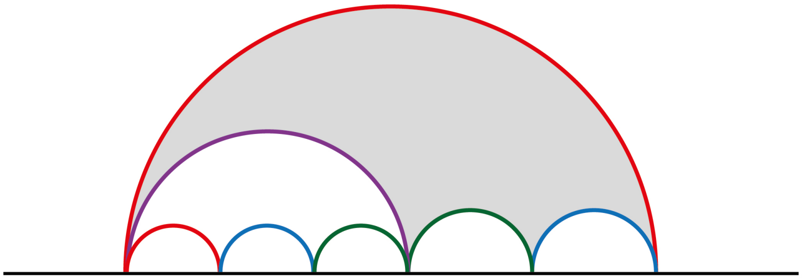


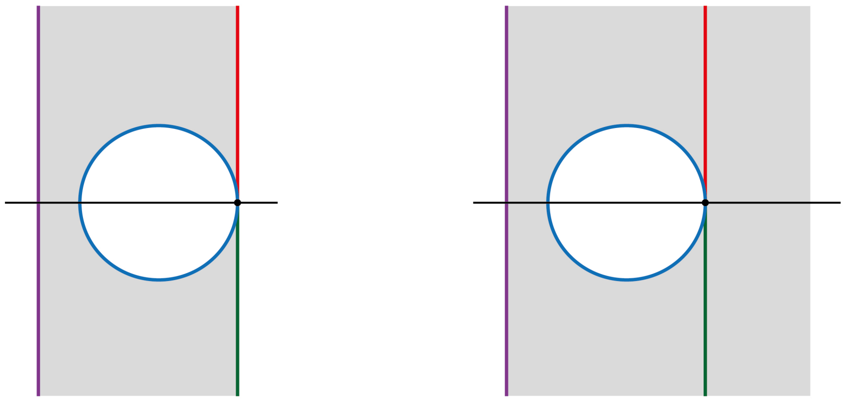
Figure 4.4: Example of a fundamental domain for $\mathbb{C}P^1\backslash\{0,z,1,\infty\}$

figure 4.3.

This will define a polygon with four corners in the image, bounded by circles or straight lines with fixed real part. The corners of this polygon are the images of the four punctures. By gluing together two copies of such polygons, we define a pillow case gluing of the four-punctured sphere, schematically drawn as in figure 4.4.

We will only consider the cases $G_{21} = 0$ and $(GH)_{21} = 0$ in detail and comment on $H_{21} = 0$ at the end.

First, let us set $G_{21} = 0$. The fixed points of the isometries defined by the monodromies along the paths γ_* for $* \in \{0, z, 1, \infty\}$, are the images of the points $*$ under the developing map. In this way, we find the fixed points $A(0) = A(z) = \infty$ and $A(1) = A(\infty) = 0$.



(a) $A(\mathbb{R}P^1)$ without zeroes of $\psi_0^{(1)}$ (b) $A(\mathbb{R}P^1)$ with zeroes of $\psi_0^{(1)}$

Figure 4.5: Image $A(\mathbb{R}P^1)$ for the condition $G_{21} = 0$

Let us assume $\psi_0^{(1)}$, which is regular at both $x = 0$ and $x = z$, has m zeroes on the interval $(0, z)$ and consider the combination

$$\eta(x) = \frac{\psi_0^{(1)}(x)}{\psi_{0,+}^{(2)}(x)}$$

It is clear that $\eta(0) = \eta(z) = 0$. Additionally, at each zero of $\psi_0^{(1)}$, η has a zero. Since $\text{Im}(\eta) \subset \mathbb{R}P^1$ and $\eta'(x) \neq 0$, the image $\eta([0, z])$ must cover $\mathbb{R}P^1$ a total of $m + 1$ times. If we apply a Möbius transformation, this covering behaviour will still appear for the image of $\mathbb{R}P^1$ under this Möbius transformation.

By the existence of the three involutions j_1, j_2 and j_3, we can show that the existence of a solution $\psi_0^{(1)}$ of this type is equivalent to the existence of a solution $\psi_1^{(1)}$ regular at $x = 1$ and $x = \infty$ and with m zeroes on the interval $(1, \infty)$. The image $A([1, \infty])$ must therefore also cover itself a total of $m + 1$ times.

It turns out that $G_{21} = 0$ implies that the solution to equation (3.5) cannot have zeroes on the intervals $(-\infty, 0)$ and $(z, 1)$. This is proved in lemma 5.3.

If we assume $\psi_0^{(1)}$ has no zeroes on the interval $(0, z)$, the image $A(\mathbb{R}P^1)$ takes the form of figure 4.5a. If we do assume zeroes, we will find covering behaviour as in figure 4.5b.

Let us now consider $(GH)_{21} = 0$. In this case, calculating the fixed points of

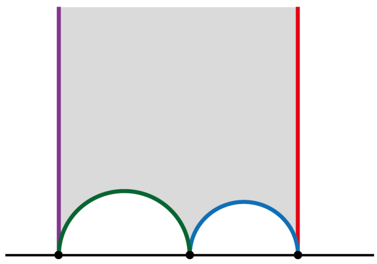


Figure 4.6: $A(\mathbb{R}P^1)$ for $(GH)_{21} = 0$ without zeroes of the solution $\psi_0^{(1)}$

the isometries leads to the results

$$\begin{array}{rclrcl} A(0) & = & \infty & A(z) & = & 0 \\ A(1) & = & \frac{2}{2-X} & A(\infty) & = & \frac{1}{2} \end{array}$$

We must necessarily find $X < -2$ to ensure $0 < \frac{2}{2-X} < \frac{1}{2}$.

The same result holds as before, relating the number of zeroes to the number of covers when we consider the solution on the intervals $(0, z)$ or $(z, 1)$. However, in this case we can have zeroes on either $(0, z)$ or $(z, 1)$, but not in both simultaneously. We refer to remark 5.4.

If we assume $\psi_0^{(1)}$ has no zeroes on either $(0, z)$ or $(z, 1)$, the image $A(\mathbb{R}P^1)$ takes the form of figure 4.6.

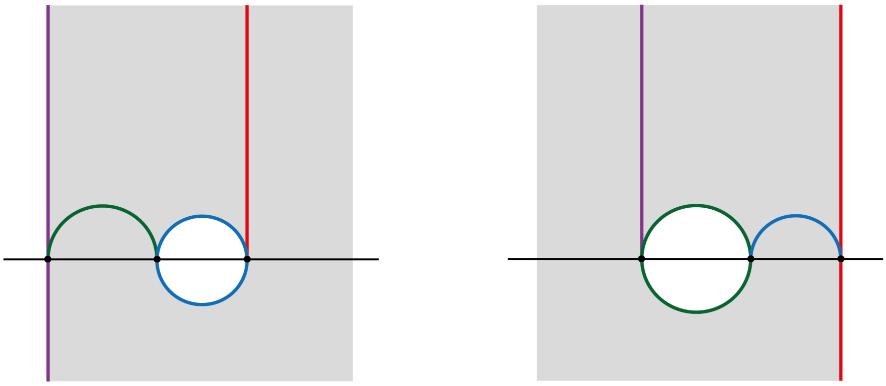
This developing map realizes $\mathbb{C}P^1 \backslash \{0, z, 1, \infty\}$ as a quotient of $\mathbb{H}$ by a Fuchsian group.

If we consider the solution $\psi_0^{(1)}$ to have $m > 0$ zeroes on the interval $(0, z)$, we find figure 4.7a, while for $m > 0$ zeroes on $(z, 1)$, we find figure 4.7b.

The case $H_{21} = 0$ leads to the same picture as in figure 4.7 with the difference being the images of the punctures under the developing map A: While for $G_{21} = 0$ we found $A(0) = A(z) = \infty$ and $A(1) = A(\infty) = 0$, for $H_{21} = 0$ we will find $A(0) = A(\infty) = \infty$ and $A(z) = A(1) = 0$.

4.4 Describing the single-valued solutions

We will now interpret the results from the previous subsections in terms of the construction of the single-valued functions $\Psi_{\mathbf{E}}(x, \bar{x})$ from the solutions to



(a) $A(\mathbb{R}P^1)$ with zeroes on $(0,z)$ (b) $A(\mathbb{R}P^1)$ with zeroes on $(z,1)$

Figure 4.7: Image $A(\mathbb{R}P^1)$ for $(GH)_{21}=0$ with zeroes of the solution $\psi_0^{(1)}$

equation (3.5). Since all of the monodromy matrices M_0, M_z, M_1 and M_∞ lie in $\mathrm{SL}(2,\mathbb{R})$, the monodromy of $A(x)$ lies in $\mathrm{PSL}(2,\mathbb{R})$. From the developing map $A(x) = \frac{1}{2\pi i (GBH)_{21}} \frac{\psi_1^{(1)}(x)}{\psi_0^{(1)}(x)}$, we may construct

$$\Psi_{\mathbf{E}}(x,\bar{x}) = \frac{2\mathfrak{Im}(A(x))}{|A'(x)|}$$

which is invariant under the action of real Möbius transformations on $A(x)$

The Wronskian is necessarily of the form

$$\begin{aligned} W(x) &:= (\psi_1^{(1)}(x))'\psi_0^{(1)}(x) - (\psi_0^{(1)}(x))'\psi_1^{(1)}(x) \\ &= \frac{W_0}{x(x-z)(x-1)} \end{aligned}$$

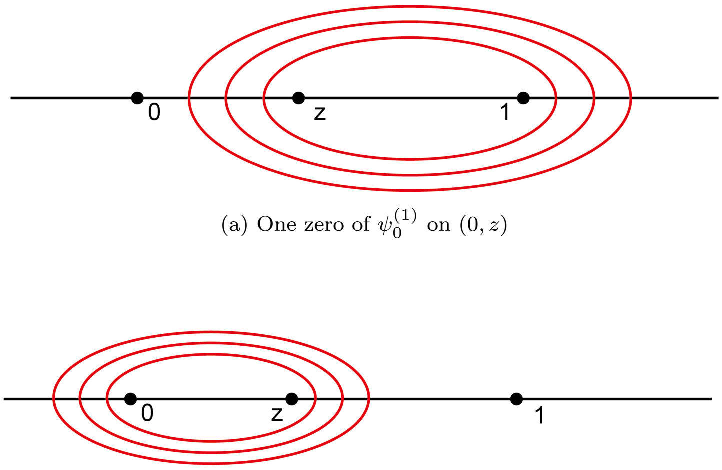
for a constant $W_0 \neq 0$. Scaling the constant W_0 scales the function $\Psi_{\mathbf{E}}(x,\bar{x})$. We may therefore set without loss of generality $W_0 = 1$.

If $G_{21} = 0$, this sets

$$\Psi_{\mathbf{E}}(x,\bar{x}) = -|x(x-z)(x-1)| \left(\psi_1^{(1)}(x)\bar{\psi}_0^{(1)}(\bar{x}) + \bar{\psi}_1^{(1)}(\bar{x})\psi_0^{(1)}(x)\right)$$

Remark 4.11. Near each puncture $*$, the function has at most a singularity of the form $|x-*|\log(x-*)\log(\bar{x}-\bar{*})$. Therefore $\Psi_{\mathbf{E}}(x,\bar{x})$ can be continuously extended to all of $\mathbb{C}P^1$, but it will not be differentiable at the punctures.

The function $\Psi_{\mathbf{E}}(x,\bar{x})$ vanishes precisely on the subvariety V where $\mathfrak{Im}(A(x)) =$



(a) One zero of $\psi_0^{(1)}$ on $(0, z)$

(b) One zero of $\psi_1^{(1)}$ on $(z, 1)$

Figure 4.8: Curves defined by $\mathfrak{Im}(A(x)) = 0$ for $m = 1$ zero

0. This subvariety $V \subset \mathbb{C}P^1 \backslash \{0, z, 1, \infty\}$ is given by the analytical equation

$$\psi_1^{(1)}(x)\bar{\psi}_0^{(1)}(\bar{x}) + \bar{\psi}_1^{(1)}(\bar{x})\psi_0^{(1)}(x) = 0$$

Remark 4.12. Since the equation (3.5) is invariant under taking the complex conjugation, the curves defined by the variety V must be invariant under complex conjugation. This means $V = \bar{V}$ considered as a subset $V \subset \mathbb{C}P^1 \backslash \{0, z, 1, \infty\}$. For $\lambda \notin \mathbb{R}$ or $z \notin \mathbb{R}P^1 \backslash \{0, 1, \infty\}$, we can also find developing maps with real monodromy, but the variety $V \neq \bar{V}$.

If the function $\psi_0^{(1)}$ has m zeroes on the interval $(0, z)$, it can be clearly seen from figure 4.5 that we find a total of $2m+1$ points in V intersecting $(0, z)$. We also find another $2m+1$ points lying in both V and $(1, \infty)$. The other intervals do not intersect V. The shape of the real curves can be deduced from figure 4.7 as well, so that on the four-punctured sphere these real curves take the shape as in figure 4.8a.

When we consider $H_{21} = 0$, the story is completely analogous. This leads to the real curves in figure 4.8b. The curves drawn in figure 4.8 are drawn for $m = 1$.

The analysis for the condition $(GH)_{21} = 0$ can be carried out as we did for $G_{21} = 0$. However, it should be noted that now we can only produce an even number of real curves, while for $G_{21} = 0$ and $H_{21} = 0$ we could only find an odd number of real curves!

Finally, we may check that the sign of $\Psi_{\mathbf{E}}(x, \bar{x}) \in \mathbb{R}$ changes every time we cross the variety V. This is a consequence of the fact that for any solution $\psi_\lambda(x)$ such that $\psi_\lambda(x_0) = 0$, we must find $\psi'_\lambda(x_0) \neq 0$. See lemma 5.2 for a proof.

5 Holomorphicity and accessory parameters

We have described in general terms which holomorphicity conditions lead to developing maps with real monodromy. The question that remains, is how to relate these conditions to values of the accessory parameter λ. The matrices $G(\lambda)$ and $H(\lambda)$ are analytic functions of the parameter λ and as such we are looking for the discrete values such that one of the three conditions $G_{21} = 0$, $H_{21} = 0$ or $(GH)_{21} = 0$ is satisfied. The important results can be formulated in theorem 5.1 originally proved by Smirnov. The discussion in this section follows the work done by Smirnov in [50].

Theorem 5.1. *There exist values λ_k of the accessory parameter for $k \in \mathbb{Z}$ such that*

- $G_{21}(\lambda_k) = 0$ *if* $k \in 2\mathbb{Z}_{\geq 0} + 1$
- $H_{21}(\lambda_k) = 0$ *if* $k \in 2\mathbb{Z}_{\leq 0} - 1$
- $(GH)_{21}(\lambda_k) = 0$ *if* $k \in 2\mathbb{Z}$.

Furthermore, these values are ordered as

$$-\infty \leftarrow ... < \lambda_{-2} < \lambda_{-1} < \lambda_0 < \lambda_1 < \lambda_2 < ... \rightarrow \infty$$

with $\lambda_{-1} < -z < \lambda_1$.

Finally,

- *If* $\lambda = \lambda_k$ *such that* $G_{21}(\lambda_k) = 0$, *the solution* $\psi_0^{(1)}(x)$ *regular at* $x = 0$ *and at* $x = z$ *has a total of* $\frac{k-1}{2}$ *zeroes on the interval* $(0, z)$.
- *If* $\lambda = \lambda_k$ *such that* $H_{21}(\lambda_k) = 0$, *the solution* $\psi_1^{(1)}(x)$ *regular at* $x = z$ *and at* $x = 1$ *has a total of* $\frac{-k-1}{2}$ *zeroes on the interval* $(z, 1)$.
- *If* $\lambda = \lambda_k$ *such that* $(GH)_{21}(\lambda_k) = 0$, *the solution* $\psi_0^{(1)}(x)$ *regular at* $x = 0$ *and at* $x = 1$ *after real analytical continuation along* $x = z$ *has a total of* $\frac{k}{2}$ *zeroes on the interval* $(0, z)$ *if* $k \geq 0$ *and* $\frac{-k}{2}$ *zeroes on the interval* $(z, 1)$ *if* $k \leq 0$.

These results may be compared to the search for suitable wavefunctions in quantum mechanics. If we have a bounding potential, the energy eigenvalues for which the wavefunctions can be normalized, form a discrete set. In particular, if the quantum mechanics takes place in one dimension, these wavefunctions are classified by the number of nodes. We will find a similar classification by the number of nodes for our problem.

To prove the theorem, we will take a step away from the calculations in the previous section and study the real equation (3.5) itself in more detail. In particular we will study the number of zeroes of its solutions. Much of the discussion in this section revolves around the comparison of the signs of different real quantities to obtain contradictions. These proofs can be found in [50].

Lemma 5.2. *If x_0 is a zero of a solution ψ_λ to equation* (3.5)*, we must find $\psi'_\lambda(x_0) \neq 0$. Additionally, we must find $\psi_\lambda(*) \neq 0$ for $* \in \{0, z, 1\}$ and all solutions ψ_λ.*

Proof. All of these statements follow from the same idea. Assume x_0 is a zero of ψ_λ and we have $\psi'_\lambda(x_0) = 0$. Any other solution $\tilde{\psi}_\lambda(x)$ can be combined with $\psi_\lambda(x)$ into the Wronskian

$$W(x) = \psi'_\lambda(x)\tilde{\psi}_\lambda(x) - \tilde{\psi}'_\lambda(x)\psi_\lambda(x)$$

which satisfies $W(x) = \frac{W_0}{x(x-z)(x-1)}$. Under our assumption $W(x_0) = 0$ which implies $W_0 = 0$. However, this implies that any other solution is linearly dependent on ψ_λ, which is a clear contradiction.

Similarly, if we can find a solution such that $\psi_\lambda(*) = 0$, the Wronskian will remain regular at $x = *$ implying $W_0 = 0$. This leads to a contradiction again. □

We will now prove another important lemma which tells us we cannot have zeroes in both the intervals $(0, z)$ and $(z, 1)$ simultaneously. More precisely

Lemma 5.3. *Denote by $\psi_*^{(1)}$ the solution to equation* (3.5) *regular at $x = *$. Then*

- *If $\lambda \leq -z$, the solutions $\psi_0^{(1)}$ and $\psi_z^{(1)}$ cannot have zeroes on $(0, z)$*
- *If $\lambda \geq -z$, the solutions $\psi_1^{(1)}$ and $\psi_z^{(1)}$ cannot have zeroes on $(z, 1)$*

Proof. We first note that $x(x-z)(x-1) \geq 0$ on $[0, z]$. If $\lambda \leq -z$ and $\psi_0^{(1)}$ has a zero x_0 on $(0, z)$ such that $(0, x_0)$ contains no other zeroes, it must satisfy

$$x_0(x_0 - z)(x_0 - 1)(\psi_0^{(1)})'(x_0) = -\int_0^{x_0} (x + \lambda)\psi_0^{(1)}(x)dx$$

This follows directly from equation (3.5). If we assume a normalization such that $\psi_0^{(1)}(0) = 1$, we find $(\psi_0^{(1)})'(x_0) < 0$ so that the left-hand side of the equation is negative, while the right-hand side is positive. This is a contradiction, proving that the existence of a zero implies $\lambda > -z$.

We prove the result for $\psi_1^{(1)}$ in a similar way. If $\lambda \geq -z$ and $\psi_1^{(1)}$ has a zero x_0 on $(z, 1)$ such that $(x_0, 1)$ contains no other zeroes, we find

$$x_0(x_0 - z)(x_0 - 1)(\psi_1^{(1)})'(x_0) = \int_{x_0}^{1} (x + \lambda)\psi_1^{(1)}(x)dx$$

Assuming a normalization such that $\psi_1^{(1)}(1) = 1$, we find $(\psi_0^{(1)})'(x_0) > 0$. Then the left-hand side is negative, while the right-hand side is positive, which is a contradiction.

The statement for $\psi_a^{(1)}$ is completely analogous to these cases. □

Remark 5.4. If we tune $\lambda = \lambda_k$ such that $(GH)_{21}(\lambda) = 0$, the real analytical continuation of $\psi_0^{(1)}$ to $(z, 1)$ is proportional to $\psi_1^{(1)}$. Therefore, it can have zeroes in $(0, z)$ only if $\lambda > -z$ and zeroes in $(z, 1)$ only if $\lambda < -z$. Clearly, these cases mutually exclude each other proving that we cannot have zeroes on both intervals simultaneously.

If we do have a zero x_0 of a solution, we wish to find out how this zero behaves when changing the accessory parameter. The position of the zero x_0 is as an analytical function of the accessory parameter λ denoted by $x_0 = x_0(\lambda)$. If λ' is a value of λ such that the zero x_0 exists in the first place, we can make the following statements in a neighbourhood around λ':

Lemma 5.5. *If x_0 is a zero of $\psi_0^{(1)}$ on $(0, z)$ or a zero of $\psi_1^{(1)}$ on $(z, 1)$, it must satisfy $\frac{dx_0}{d\lambda} < 0$ in a neighbourhood of λ'.*

Proof. Let us start by assuming x_0 is a zero of $\psi_0^{(1)}$ on $(0, z)$.

The quantity $\psi_{0,\lambda}^{(1)}(x) := \left(\frac{d}{d\lambda}\psi_0^{(1)}\right)(x)$ satisfies a modified form of equation (3.5) of the form

$$\frac{d}{dx}\left(x(x - z)(x - 1)(\psi_{0,\lambda}^{(1)}(x))'\right) + (x + \lambda)\psi_{0,\lambda}^{(1)}(x) = -\psi_0^{(1)}(x)$$

Therefore

$$\frac{d}{dx}\left(x(x - z)(x - 1)P(x)\right) = (\psi_0^{(1)}(x))^2$$

where $P(x) = \psi_{0,\lambda}^{(1)}(x)(\psi_0^{(1)})'(x) - \psi_0^{(1)}(x)(\psi_{0,\lambda}^{(1)})'(x)$. Now by assumption $\frac{d}{d\lambda}\left(\psi_0^{(1)}(x_0(\lambda))\right) = 0$. If we differentiate this result, we find

$$\psi_{0,\lambda}^{(1)}(x_0) + (\psi_0^{(1)})'(x_0)\frac{dx_0}{d\lambda} = 0$$

Plugging this into $P(x_0) = \psi_{0,\lambda}^{(1)}(x_0)(\psi_0^{(1)})'(x_0)$, the result is $P(x_0) = -\left((\psi_0^{(1)})'(x_0)\right)^2 \frac{dx_0}{d\lambda}$.

We may finally integrate and obtain

$$x_0(x_0 - z)(x_0 - 1)P(x_0) = \int_0^{x_0} (\psi_0^{(1)}(x))^2 dx$$

Since $x_0(x_0 - z)(x_0 - 1) > 0$, $\int_0^{x_0}(\psi_0^{(1)}(x))^2 dx > 0$ and $\left((\psi_0^{(1)})'(x_0)\right)^2 > 0$, we must find $\frac{dx_0}{d\lambda} < 0$.

A similar proof can be applied to $\psi_1^{(1)}$ to find $\frac{dx_0}{d\lambda} < 0$. □

This means that if a zero $x_0 \in (0, z)$ of $\psi_0^{(1)}$ exists for a given $\lambda' > -z$, increasing the value of $\lambda > \lambda'$ will move the zero to the left. However, since we normalize $\psi_0^{(1)}(0) = 1$, the zero cannot disappear from the interval. It must therefore exist for all such $\lambda > \lambda'$ implying that the number of zeroes can only increase in this way. Similarly, if $x_0 \in (z, 1)$ is a zero of $\psi_1^{(1)}$ and exists for some $\lambda' < -z$, decreasing the value $\lambda < \lambda'$ moves the zero to the right. Since we normalize $\psi_1^{(1)}(1) = 1$, it cannot disappear if we decrease λ.

The values λ_k of the accessory parameter such that $G_{21}(\lambda_k) = 0$ define solutions regular on the interval $[0, z]$. By standard results from Sturm-Liouville theory, it is known that for each such λ_k a solution exists with $\frac{k-1}{2}$ zeroes on $[0, z]$. Additionally, both the series $-z < \lambda_1 < \lambda_3 < \lambda_5 < ...$ and the number of zeroes on any subinterval $I \subset [0, z]$, are known to increase without bound.

Before we continue, let us first remark

Remark 5.6. The sign of $\det(G(\lambda))$ must always be negative. If λ_k is such that $G_{21}(\lambda_k) = 0$, we find $\det(G(\lambda_k)) = G_{11}(\lambda_k)G_{22}(\lambda_k)$. By lemma 4.9 we have $G_{11}(\lambda_k) = -G_{22}(\lambda_k)$, implying $\det(G(\lambda_k)) < 0$. By analyticity in λ and the fact that $G(\lambda)$ is real for real λ, we must have $\det(G(\lambda)) < 0$ for all $\lambda \in \mathbb{R}$. A similar argument can be applied to prove that $\det(H(\lambda)) < 0$ for all $\lambda \in \mathbb{R}$.

We will now show that the values λ_k for $k \in 2\mathbb{Z}_{\geq 0} + 1$ are precisely where the function $\psi_0^{(1)}$ develops an additional zero. More precisely,

Lemma 5.7. *The function $\psi_0^{(1)}$ develops a new zero if and only if $G_{21}(\lambda)$ changes sign. Analogously, $\psi_1^{(1)}$ can only develop a new zero if and only if $H_{21}(\lambda)$ changes sign.*

Proof. We will only prove the statement for $\psi_0^{(1)}$. We may expand

$$\psi_0^{(1)} = \psi_z^{(1)}(G^{-1})_{11} + \psi_{z,-}^{(2)}(G^{-1})_{21}$$

The local behaviour of $\psi_0^{(1)}$ near $x = z$ is determined by the $\psi_{z,-}^{(2)}$ component as this has a divergence of the form $\log(z - x)$. The sign of $(G^{-1})_{21}$ therefore determines whether $\psi_0^{(1)}$ diverges to $+\infty$ or $-\infty$. Writing out $(G^{-1})_{21} = -G_{21}\det(G)^{-1}$ and noting that $\det(G(\lambda)) < 0$ by remark 5.6, we get $\operatorname{sgn}((G^{-1})_{21}) = \operatorname{sgn}(G_{21})$.

By the fact that zeroes cannot disappear if we increase λ by lemma 5.5, a change in sign of G_{21} must produce a new zero. Conversely, assume $\psi_0^{(1)}$ develops a zero x_0 by increasing λ and G_{21} does not change sign. Then at this zero we find $\psi_0^{(1)}(x_0) = (\psi_0^{(1)})'(x_0) = 0$ which is in contradiction with lemma 5.2. Therefore a new zero of $\psi_0^{(1)}$ implies a change in sign of G_{21}. □

Lemma 5.8. *If λ is a value of the accessory parameter such that $\lambda_k < \lambda < \lambda_{k+2}$ for $k \in 2\mathbb{Z}_{\geq 0}+1$, the solution $\psi_0^{(1)}$ has a total of $\frac{k+1}{2}$ zeroes in the interval $(0, z)$. Similarly, if $\lambda_{k-2} < \lambda < \lambda_k$ for $k \in 2\mathbb{Z}_{\leq 0} - 1$, the solution $\psi_1^{(1)}$ has a total of $\frac{-k-1}{2}$ zeroes in $(z, 1)$. If $\lambda_{-1} < \lambda < \lambda_1$, neither $\psi_0^{(1)}$ nor $\psi_1^{(1)}$ develop zeroes in $(0, z)$ and $(z, 1)$ respectively.*

Proof. Again, we will only prove the result for $\psi_0^{(1)}$. We first note that for $\lambda = \lambda_k$ and $k \in 2\mathbb{Z}_{\geq 0} + 1$, the solution $\psi_0^{(1)}$ has a total of $\frac{k-1}{2}$ zeroes on $(0, a)$. By lemma 5.7, the only way $\psi_0^{(1)}$ can develop a new zero by increasing λ, is when G_{21} changes sign. However, this is only possible if λ increases past the value λ_k. Therefore the solutions with $\lambda \in (\lambda_k, \lambda_{k+2})$ must all have $\frac{k+1}{2}$ zeroes on $(0, z)$. In particular, for $\lambda < \lambda_1$ the solution $\psi_0^{(1)}$ does not have zeroes on $(0, z)$. □

Finally, we show that a unique value of λ such that $(GH)_{21}(\lambda) = 0$ exists in the interval $(\lambda_k, \lambda_{k+2})$ for $k \in 2\mathbb{Z}_{\geq 0} + 1$. We call this value $\lambda = \lambda_{k+1}$.

Lemma 5.9. *There exists a unique value λ_k of $\lambda \in (\lambda_{k-1}, \lambda_{k+1})$ such that $(GH)_{21}(\lambda_{k+1}) = 0$, where $k \in 2\mathbb{Z}$.*

Proof. Let us consider $\psi_0^{(1)}$ again. We extend this function by real analytical continuation to the interval $(z,1)$ so that its behaviour near $x=1$ is given by

$$\psi_0^{(1)} = \psi_1^{(1)}((GH)^{-1})_{11} + \psi_{1,-}^{(2)}((GH)^{-1})_{21}$$

We relate $((GH)^{-1})_{21} = -(GH)_{21}\det(GH)^{-1}$. Note that $\det(G) < 0$ and $\det(H) < 0$ for all $\lambda \in \mathbb{R}$ by 5.6. The divergent behaviour at $x=1$ is regulated by $((GH)^{-1})_{21}$ so that $\psi_0^{(1)}$ diverges to $\mathrm{sgn}((GH)_{21})\infty$.

Let us now consider the special value $\lambda = \lambda_k$ with $k \in 2\mathbb{Z}_{\geq 0}+1$. The solution $\psi_0^{(1)}$ now regular at $x=a$, must have $\frac{k-1}{2}$ zeroes on $(0,z)$. Since $\psi_0^{(1)}$ is regular at $x=z$, it must be proportional to $\psi_z^{(1)}$. By lemma 5.3 the function $\psi_z^{(1)}$ cannot have a zero on the interval $(z,1)$, which means neither can $\psi_0^{(1)}$.

Depending on the number of zeroes, either $\psi_0^{(1)}(z) > 0$ or $\psi_0^{(1)}(z) < 0$. If we normalize $\psi_0^{(1)}(0) = 1$, we find $\mathrm{sgn}(\psi_0^{(1)}(z)) = (-1)^{\frac{k-1}{2}}$. Since $\psi_0^{(1)}$ does not have zeroes on $(z,1)$, $\mathrm{sgn}(\psi_0^{(1)}(x)) = (-1)^{\frac{k-1}{2}}$ for all $x \in [z,1)$. Since the leading divergent behaviour of $\psi_0^{(1)}$ goes as $-(GH)_{21}\log(1-x)$ near $x=1$, we find $\mathrm{sgn}((GH)_{21}(\lambda_k)) = (-1)^{\frac{k-1}{2}}$.

Since $(GH)_{21}$ is an analytical function of λ which changes sign when we let λ run over the interval $(\lambda_k, \lambda_{k+2})$, there must exist a value $\lambda_k < \lambda_{k+1} < \lambda_{k+2}$ such that $(GH)_{21}(\lambda_k) = 0$. Using arguments similar to the ones above, we may show that this value λ_{k+1} is unique in the interval $(\lambda_k, \lambda_{k+2})$. This finishes the proof of the lemma. □

Putting all the results together, we conclude the proof of theorem 5.1 at the beginning of this section.

6 Generating solutions with real monodromy

We have given a complete description of the eigenvalues $\lambda \in \mathbb{R}$ for which we find real monodromy on the surface $X = \mathbb{C}P^1\backslash\{0, z, 1, \infty\}$ with $0 < z < 1$. We may try to generalize this result by letting λ become complex.

To prove the reality of the monodromy, it was essential for us to assume the reality of the accessory parameter. Nonetheless, we are able to express the solutions $\lambda = \lambda_k$ as smooth functions of the crossratio $z \in \mathbb{C}P^1\backslash\{0, 1, \infty\}$, because the set of accessory parameters λ for which the monodromy is real, is discrete and does not bifurcate.

By moving the point z around a path such that the complex structure of the surface at the starting point is the same as at the end point, we may construct new series of accessory parameters leading to real monodromy. Generically, the conditions $G_{21} = 0$, $H_{21} = 0$ and $(GH)_{21} = 0$ are not preserved under such transformations and will look quite different. The new accessory parameters therefore need not be real, although the monodromy they describe remains real. See section 10.5 of [77] for a discussion of such new quantization conditions.

An interesting question is whether we can describe all accessory parameters leading to opers with real monodromy through transformations of this kind on the series $... < \lambda_{-1} < \lambda_0 < \lambda_1 < ....$ Indeed, this turns out to be the case, although highly non-trivially through the classification of opers with real monodromy discussed in section §9.

6.1 Describing the mapping class group action

All transformations starting and ending at the same complex structure by moving z along a path, form what is known as the mapping class group of the four-punctured sphere MCG(S), where S is the topological surface underlying X. See section §16 for a short treatment of the mapping class group.

It is known that the mapping class group of the four-punctured sphere is generated by the braid group on four strands on the sphere. The group MCG(S)

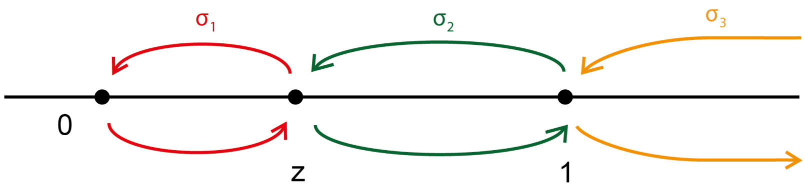


Figure 6.1: Action of braiding on the punctures of $\mathbb{C}P^1\backslash\{0,z,1,\infty\}$

has a presentation as follows

$$\mathrm{MCG}(S) \simeq \left\langle \sigma_1,\sigma_2,\sigma_3 \;\middle|\; \begin{array}{ccc} \sigma_1\sigma_2\sigma_1 & = & \sigma_2\sigma_1\sigma_2 \\ \sigma_3\sigma_2\sigma_3 & = & \sigma_2\sigma_3\sigma_2 \\ \sigma_1\sigma_3 & = & \sigma_3\sigma_1 \\ (\sigma_1\sigma_2\sigma_3)^4 & = & 1 \\ \sigma_1\sigma_2\sigma_3 & = & \sigma_1^{-1}\sigma_2^{-1}\sigma_3^{-1} \end{array} \right\rangle$$

The elements σ_1, σ_2 and σ_3 act on the punctures by braiding them as in figure 6.1.

By theorem 16.10 we may rewrite this presentation in terms of the two braiding elements σ_1 and σ_2, braiding the punctures 0 and z, respectively z and 1, and the three hyperelliptic involutions j_1, j_2 and j_3 represented by Möbius transformations.

If we let $0 < z < 1$ denote the initial crossratio, the braiding by σ_1 keeps the crossratio invariant, but sends z to $-z(1-z)^{-1}$. In the same way, σ_2 sends z to z^{-1}. Recall that the involutions j_1, j_2 and j_3 preserve the quantization conditions. Therefore, we may restrict our attention to σ_1 and σ_2.

Mapping class group action on holomorphicity conditions If we act by the mapping class group, we can easily check that the intervals $[0,z]$ and $[z,1]$ transform as in figure 6.2.

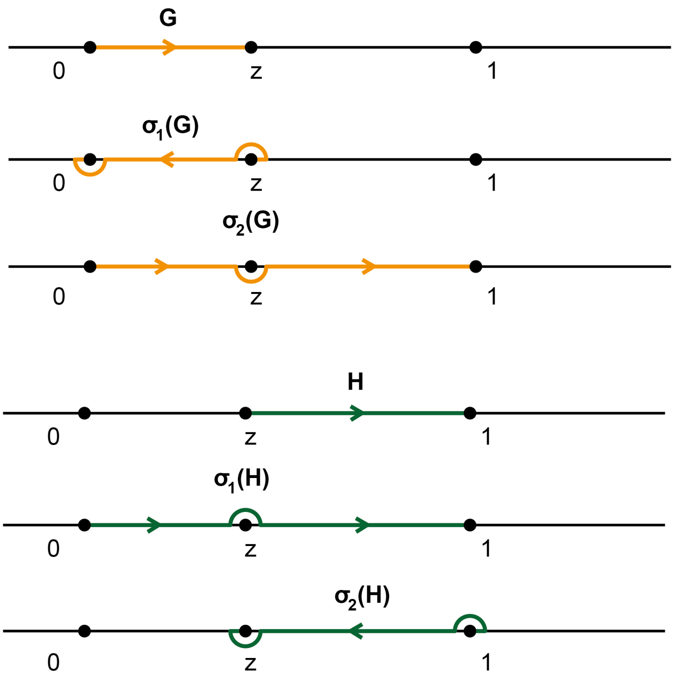


Figure 6.2: Action of braiding on the transfer matrices G and H

By the definition of the braidings, we may check that

$$\begin{aligned}\sigma_1(G) &= BG^{-1}B^{-1}\\ \sigma_1(H) &= GB^{-1}H\\ \sigma_2(G) &= GBH\\ \sigma_2(H) &= BH^{-1}B^{-1}\end{aligned}$$

and

$$\sigma_1(B) = \sigma_2(B) = B$$

Additionally, it is easily seen that if we move along the intervals in the opposite direction, we find

$$\begin{aligned}\sigma_1(G^{-1}) &= BGB^{-1}\\ \sigma_1(H^{-1}) &= H^{-1}BG^{-1}\\ \sigma_2(G^{-1}) &= H^{-1}B^{-1}G^{-1}\\ \sigma_2(H^{-1}) &= BHB^{-1}\end{aligned}$$

A general path $P = P(G, H, B)$ is a product of transfer matrices G and H, braidings B and their inverses G^{-1}, H^{-1} and B^{-1}. It transforms as

$$\mu(P) = P(\mu(G), \mu(H), \mu(B))$$

for μ a word in σ_1 and σ_2. This defines an action of the group generated by σ_1 and σ_2 on the set of paths of the form of P.

Furthermore, by restricting to the components P_{21}, we find an action of the form

$$\mu \cdot (P_{21}) := (\mu(P))_{21}$$

In this way, we can transform the holomorphicity conditions into other conditions that do not appear for real λ.

In this paragraph, we will prove the following theorem:

Theorem 6.1. *The action by elements σ_1 and σ_2 acting on P_{21} as above, generates the action of the mapping class group* $\mathrm{MCG}(S)$.

To prove that this is an action of the mapping class group, we need to prove the following two equalities:

$$\begin{aligned}(\sigma_1\sigma_2\sigma_1) \cdot (P_{21}) &= (\sigma_2\sigma_1\sigma_2) \cdot (P_{21}) \\ (\sigma_1\sigma_2)^3 \cdot (P_{21}) &= P_{21}\end{aligned}$$

for all paths P.

Lemma 6.2. *The transformations σ_1 and σ_2 satisfy $(\sigma_1\sigma_2\sigma_1)(P) = (\sigma_2\sigma_1\sigma_2)(P)$ on a path $P = P(G, H, B)$.*

Proof. We only have to check these relations on the transfer matrices G and H. By the transformation properties under the action of the group generated by σ_1 and σ_2, this proves the statement for any P.

We note that

$$\begin{aligned}(\sigma_1\sigma_2)(G) &= H \\ (\sigma_2\sigma_1)(H) &= G\end{aligned}$$

It now becomes straightforward to check,

$$\begin{aligned}(\sigma_1\sigma_2\sigma_1)(G) &= (\sigma_1\sigma_2)(BG^{-1}B^{-1}) \\ &= BH^{-1}B^{-1} \\ &= \sigma_2(H) \\ &= (\sigma_2\sigma_1\sigma_2)(G)\end{aligned}$$

Similarly,

$$\begin{aligned}(\sigma_2\sigma_1\sigma_2)(H) &= (\sigma_2\sigma_1)(BH^{-1}B^{-1}) \\ &= BG^{-1}B^{-1} \\ &= \sigma_1(G) \\ &= (\sigma_1\sigma_2\sigma_1)(H)\end{aligned}$$

This proves the result. □

Lemma 6.3. *The transformation* $(\sigma_1\sigma_2)^3$ *acts as* $(\sigma_1\sigma_2)^3(P) = B^2PB^{-2}$. *Therefore,* $P_{21} = ((\sigma_1\sigma_2)^3(P))_{21}$.

Proof. Using the fact that $(\sigma_1\sigma_2\sigma_1)(P) = (\sigma_2\sigma_1\sigma_2)(P)$ on any product P, we may write

$$(\sigma_1\sigma_2)^3(P) = (\sigma_1\sigma_2\sigma_1)^2(P)$$

The results from the previous lemma show that

$$\begin{aligned}(\sigma_1\sigma_2)^3(G) &= B^2GB^{-2} \\ (\sigma_1\sigma_2)^3(H) &= B^2HB^{-2}\end{aligned}$$

Therefore, for a general P, we find

$$(\sigma_1\sigma_2)^3(P) = B^2PB^{-2}$$

In particular, since $B_{21} = 0$, we find $(B^2PB^{-2})_{21} = P_{21}$. This proves the result.

Indeed, as an element of the mapping class group, the word $(\sigma_1\sigma_2)^3$ rotates the punctures by an angle 2π. This results in a conjugation by a full rotation around the point at infinity, which is given by B^2. □

This completes the proof of the fact that the mapping class group defines an action on the components P_{21} of a product P of transfer matrices and braiding matrices.

6.2 Surjectivity of the mapping class group action

Although we will prove the full classification theorem 8.2 later, currently it suffices to extract from it the statement that *opers with real monodromy are classified precisely by the real curves they induce.*

It is known that the action of the mapping class group on the space of simple closed curves on $\mathbb{C}P^1\backslash\{0, z, 1, \infty\}$ is transitive. We prove this fact in proposition 16.3. This implies that we can use the mapping class group to generate any oper with real monodromy from the opers corresponding to the conditions $G_{21} = 0$, $H_{21} = 0$ and $(GH)_{21} = 0$.

We may state the surjectivity of this action as follows

Theorem 6.4. *All projective structures with real monodromy on* $X = \mathbb{C}P^1\backslash\{0, z, 1, \infty\}$ *can be constructed by acting with elements in the mapping class group on the holomorphicity conditions* $G_{21} = 0$*,* $H_{21} = 0$ *and* $(GH)_{21} = 0$

Remark 6.5. It actually suffices to only consider the series

$$\lambda_0 < \lambda_1 < \lambda_2 < ...$$

The element $\sigma_1\sigma_2\sigma_1$ acts by

$$\begin{aligned} (\sigma_1\sigma_2\sigma_1)(G) &= BH^{-1}B^{-1} \\ (\sigma_1\sigma_2\sigma_1)(H) &= BG^{-1}B^{-1} \\ (\sigma_1\sigma_2\sigma_1)(GH) &= B(GH)^{-1}B^{-1} \end{aligned}$$

This action exchanges the conditions $G_{21} = 0$ and $H_{21} = 0$. The solution $(GH)_{21} = 0$ is invariant under the action by this element. However, if we consider the solution $\psi_0^{(1)}$ regular at 0 and at 1 after real analytical continuation along z, it will have a given number of zeroes on either the interval $[0, z]$ or on $[z, 1]$. The action by $\sigma_1\sigma_2\sigma_1$ exchanges the intervals on which this solution has zeroes. Therefore, $\sigma_1\sigma_2\sigma_1$ acts by exchanging $\lambda_k \leftrightarrow \lambda_{-k}$ for $k \in \mathbb{Z}$. The uniformizing solution corresponding to λ_0 is invariant under this action.

Part III

Classification of opers with real monodromy

In this part we will extend the analysis done in Part II to other Riemann surfaces. Although we do not perform a direct analysis on a general Riemann surface X as was done for the four-punctured sphere, by following the discussion in [80] and in [46] we show that the opers with real monodromy are in one-to-one correspondence with the set of hyperbolic metrics which are allowed to be singular along a finite union of circles on X. This finite union of circles is precisely the set where the developing map of the oper becomes real.

It has been shown in [46] that we can construct all opers with real monodromy from a special oper by an operation known as (half-integer) grafting. The input of this procedure is a marked Riemann surface X together with a non-self-intersecting multicurve μ up to homotopy. This classification relies on theorem 9.4 presented by Thurston [87] (see [88]).

Originally a theorem by Dehn from the 1920s and later reproved by Thurston in the 1970s, there exist Dehn-Thurston parameters $(\mathbf{p}, \mathbf{q})$ in one-to-one correspondence with the non-self-intersecting multicurves on a topological surface up to homotopy. These results can be found in [43, 44]. If we fix the Riemann surface, we may therefore use the Dehn-Thurston parameters as a first set of quantum numbers for the single-valued solutions $\Psi_{\mathbf{E}}(\mathbf{x}, \bar{\mathbf{x}})$.

It turns out that grafting along certain multicurves does not lead to monodromy valued in $\mathrm{PSL}(2, \mathbb{R})$, but more generally in $\mathrm{PGL}(2, \mathbb{R})$. Although not originally found in [46], this statement is clarified in [89]. The parameters $(\mathbf{p}, \mathbf{q})$ corresponding to these multicurves through Dehn's theorem, do not lead to single-valued solutions and we must therefore remove these parameters from the set of allowed quantum numbers. We analyze this restriction explicitly in the case of a closed surface of genus two and provide an algorithm based on formulas by Penner [48, 49]. In principle, this leads to a complete classification of opers with real monodromy through the Dehn-Thurston parameters, although the conditions can be hard to calculate in practice due to the complexity of Penner's formulas.

7 Constructing the uniformizing oper

The topics discussed in this section have been mathematically well-established for a long time. A basic reference on much of this section is given by [90].

7.1 Hyperbolic geometry

Hyperbolic geometry plays an important role in the uniformization problem and as a result in the classification of single-valued eigenfunctions of the quantum Hitchin Hamiltonians. Let us therefore first recall some basic facts about hyperbolic geometry.

We define the upper half plane $\mathbb{H} \subset \mathbb{C}$ by $\mathbb{H} := \{z \in \mathbb{C} | \mathfrak{Im}(z) > 0\}$. This space can be given a metric of constant curvature -4 of the form $g_{\text{hyp}} = \frac{1}{4}\mathfrak{Im}(z)^{-2}dzd\bar{z}$. The metric g_{hyp} is known as the hyperbolic metric and the metric space $(\mathbb{H}, g_{\text{hyp}})$ as the hyperbolic plane. We will study this space in more detail.

Geodesics of the hyperbolic metric In terms of coordinates $z = x+iy$, we may write the hyperbolic plane as $\mathbb{H} = \{(x, y) \in \mathbb{R}^2 | y > 0\}$ with the hyperbolic metric

$$g_{\text{hyp}} = \frac{dx^2 + dy^2}{y^2}$$

It can be shown that geodesics with respect to this metric are defined by one of two equations $(x - x_0)^2 + y^2 = R^2$ or $x = x_0$. The variable $x_0 \in \mathbb{R}$ and $R > 0$ are parameters specifying the geodesic. Therefore, the geodesics are either half-circles with origin $(x_0, 0)$ and radius R or straight lines defined by $x = x_0$ and starting at $(x_0, 0)$.

The boundary $\partial\mathbb{H}$ is given by the space $\mathbb{R}P^1 = \{(x, y) \in \mathbb{R}^2 | y = 0\} \cup \{\infty\}$. The hyperbolic metric blows up when we approach $\mathbb{R}P^1$ so that each of these geodesics has infinite length in the hyperbolic space.

Isometries of the hyperbolic plane It can easily be checked that the isometries of the metric g_{hyp} are precisely given by the group of real Möbius transformations. By identifying the real Möbius transformations with $\text{PGL}(2, \mathbb{R})$, the orientation-preserving isometries are given by $\text{Isom}^+(\mathbb{H}) \simeq \text{PSL}(2, \mathbb{R})$. The constant curvature metric g_{hyp} is the only metric up to scaling by a real number that is invariant under all these isometries.

If we consider the real Möbius transformations as a subset of the complex Möbius transformations, the set of isometries $\text{Isom}^+(\mathbb{H})$ splits into three types, depending on which points are fixed by the isometry:

- Elliptic transformations fix one point in $\mathbb{H}$
- Parabolic transformations fix one point on $\partial\mathbb{H}$
- Hyperbolic transformations fix two points on $\partial\mathbb{H}$

Assume a real Möbius transformations is of the form

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{PSL}(2, \mathbb{R})$$

acting on z by

$$z \mapsto \frac{az+b}{cz+d}$$

The matrix form M is only well-defined up to scaling by a real number. We may use this freedom to set $\det(M) = 1$ and consider M as an element in $\text{PSL}(2, \mathbb{R}) \simeq \text{SL}(2, \mathbb{R})/\mathbb{Z}_2$, where $\mathbb{Z}_2 = \{-1, +1\}$ is the center of $\text{SL}(2, \mathbb{R})$. Therefore, M is a matrix in $\text{SL}(2, \mathbb{R})$ well-defined up to multiplication by -1.

The fixed points of this transformation are found by solving $z = \frac{az+b}{cz+d}$. Rewriting, we find a quadratic polynomial $cz^2 + (d-a)z - b = 0$ which has solutions

$$\begin{aligned} z_\pm &= \frac{a-d}{2c} \pm \frac{\sqrt{(a-d)^2 + 4cb}}{2c} \\ &= \frac{a-d}{2c} \pm \frac{\sqrt{(a+d)^2 - 4(ad-bc)}}{2c} \end{aligned}$$

The type of transformation is specified by the discriminant $\text{tr}(M)^2 - 4$. Elliptic transformations satisfy $|\text{tr}(M)| < 2$ so that the fixed points satisfy $\overline{z_+} = z_-$. One fixed point lies in $\mathbb{H}$ while the other lies in the complex conjugate $\overline{\mathbb{H}}$. Parabolic transformations satisfy $|\text{tr}(M)| = 2$ with $z_+ = z_- \in \mathbb{R}P^1$. Finally, hyperbolic transformations satisfy $|\text{tr}(M)| > 2$ with both $z_+, z_- \in \mathbb{R}P^1$.

Lemma 7.1. *If an isometry is hyperbolic, it has a unique geodesic in $\mathbb{H}$ which is invariant under the isometry.*

Proof. Without loss of generality, we may assume the isometry fixes the points $\{0, \infty\} \subset \partial\mathbb{H}$. We can arrange for this situation by conjugating by a Möbius transformation preserving the metric structure of the hyperbolic plane. The invariant geodesic is then given by the imaginary axis. An isometry M with fixed points $\{0, \infty\}$ must satisfy $b = c = 0$. This means the transformation is a dilatation $z \mapsto a^2 z$, which only keeps straight lines passing through the origin fixed. The only geodesic in this set is the straight line lying along the imaginary axis. Therefore, there exists a unique invariant geodesic. □

Hyperbolic length Given a geodesic on a hyperbolic surface, we may associate a hyperbolic length to it. Let us assume the cover of the geodesic lies along the imaginary axis. The point i on this geodesic is identified with $a^2 i$ under the action of the Möbius transformation M. The hyperbolic distance is then defined as

$$\int_1^{a^2} \frac{dy}{y} = 2\log(a)$$

We set $2\log(a) = l$ where l is the hyperbolic length of the geodesic so that we find

$$\mathrm{tr}(M) = 2\cosh(l/2)$$

7.2 Uniformization theorem

If we consider an orientable topological surface, it admits a unique compatible smooth structure S. We define a conformal structure on S to be a metric defined up to scaling by a non-vanishing function called the conformal factor. The uniformization theorem states that any conformal structure on a simply connected surface is conformally equivalent to one of the following three conformal structures

- The Riemann sphere $\mathbb{C}P^1$ with constant curvature metric of curvature $+1$
- The Euclidean plane $\mathbb{C}$ with constant curvature metric of curvature 0
- The hyperbolic plane $\mathbb{H}$ with constant curvature metric of curvature -1

Any conformal structure on a surface can therefore be obtained as a quotient of one of these three surfaces by a free action of a discrete subgroup of the isometry groups of these three conformal structures. This implies in particular the existence of a constant curvature metric on any smooth surface. We call the pair of the smooth surface together with a constant curvature metric covered by the hyperbolic plane a hyperbolic surface.

For a genus g closed surface we know that the Euler characteristic $\chi(S) = 2-2g$. Therefore, the Gauss-Bonnet theorem implies a classification of closed surfaces based on the sign of $\chi(S)$ by

$$R\mathrm{Vol}(S) = 2\pi\chi(S)$$

where R is the curvature of a constant curvature metric on S. Since $\mathrm{Vol}(S) > 0$, the sign of $\chi(S)$ must correspond to the sign of R. Therefore, all closed Riemann surfaces except for $g = 1$ and $g = 0$ are covered by the hyperbolic plane.

We can introduce punctures or boundaries on S without drastically altering the above story. If we have a total of n punctures and boundaries together, the hyperbolic surfaces are classified by the set of tuples (g, n) of the genus g and the number of boundary components n satisfying $2g - 2 + n > 0$.

Pants decomposition of hyperbolic surface The three-holed sphere satisfies $\chi(S) = -1$ and is the simplest surface of negative Euler characteristic we can consider, in the sense that we may glue three-holed spheres together to obtain any other surface with negative Euler characteristic. In this context, we call the three-holed sphere a pair of pants and a decomposition of a hyperbolic surface in three-holed spheres a pants decomposition.

Unique geodesic representative If X is a hyperbolic surface, it admits a set of geodesics coming from the hyperbolic plane realization. The metric embeds the group $\pi_1(X)$ as a subgroup of $\mathrm{Isom}^+(\mathbb{H})$. Each element in $\pi_1(X)$ is in this way represented by a hyperbolic isometry. It turns out that every conjugacy class in $\pi_1(X)$ has a realization by a unique oriented, closed geodesic representative. Indeed, any curve representing the same element in $\pi_1(X)$ must lift to an arc in the universal cover $\mathbb{H}$ with the same fixed points as the fixed points of the hyperbolic isometry. Between any two points on $\partial\mathbb{H}$, we can find a unique geodesic connecting these points. This is precisely the unique geodesic representative of an element in $\pi_1(X)$. See proposition 1.3 in [91] for a proof of these statements.

7.3 Teichmüller space

We have seen above that every smooth surface S with $\chi(S) < 0$ can be given a hyperbolic metric such that the surface locally looks like the hyperbolic plane. The moduli space of such metrics defined up to isometries is called $\mathcal{M}(S)$. If instead we quotient the space of metrics by the action of isometries which can be connected to the identity, we define what is known as the Teichmüller space $\mathcal{T}(S)$. More precisely, we give the following definition

Definition 7.2. Consider the space of pairs (X, f) where X is a fixed hyperbolic surface modeled over the smooth surface S and $f : S \to X$ a diffeomorphism. We will say that two such pairs (X, f) and (Y, g) are equivalent if and only if $g \circ f^{-1} : X \to Y$ is isotopic to an isometry which can be connected to the identity. Let us write such pairs up to equivalence by $[X, f]$. Teichmüller space $\mathcal{T}(S)$ is the space of all pairs $[X, f]$. We call such a pair a marked hyperbolic surface where the diffeomorphism f is called the marking.

The moduli space $\mathcal{M}(S)$ is obtained from $\mathcal{T}(S)$ by taking the quotient by the mapping class group $\mathrm{MCG}(S) = \mathrm{Isom}(S)/\mathrm{Isom}_0(S)$ defined as the space of isometries modulo the isometries connected to the identity.

Complex versus conformal structure It turns out that on a smooth orientable surface, the choice of a complex structure and conformal structure are equivalent. Through the complex structure, one has a canonical notion of angle on the tangent bundle. If we are given a Riemann surface X with a local coordinate z, we can choose our conformal structure to be given by the class of metrics of the form $c(z, \bar{z}) dz d\bar{z}$ with $c : X \to \mathbb{R}_{>0}$ a conformal factor.

The converse construction is more involved. Given any Riemannian metric on a smooth surface S, we can show we can always locally bring it to the form $c(z, \bar{z}) dz d\bar{z}$ by solving the Beltrami equation

$$\frac{\partial w(z, \bar{z})}{\partial \bar{z}} = \mu(z, \bar{z}) \frac{\partial w(z, \bar{z})}{\partial z}$$

for $|\mu(z, \bar{z})| < 1$. We call the coordinates $(w, \bar{w})$ isothermal coordinates and it is easily shown that if we have two pairs of isothermal coordinates on an open neighbourhood, the transition function must be biholomorphic. This defines a complex structure X on our surface S, which is in bijection with the conformal structure defined by $c(z, \bar{z}) dz d\bar{z}$.

Since we will focus our attention to surfaces, we use these notions interchangeably. In particular, the Teichmüller space can equivalently be understood as classifying marked hyperbolic surfaces or marked Riemann surfaces.

Marking and fundamental domain It should be noted that equation (3.11) we obtain from the conditiosingle-valuedness of $\Psi_{\mathbf{E}}(\mathbf{x}, \bar{\mathbf{x}})$ does not have any reference to a marking. This equation is defined over a fixed Riemann surface. Nonetheless, to determine a monodromy representation, we need to introduce a marking. Indeed, picking a marking for an element in Teichmüller space is equivalent to picking a set of generators for the fundamental group $\pi_1(S)$. This statement is known as the Dehn-Nielsen-Baer theorem in the literature. In theorem 16.7 we give the precise statement.

The mapping class group has a natural action on the development-holonomy pairs $[A, \rho]$ by precomposing the developing map $A : \tilde{X} \to \mathbb{C}P^1$ and the monodromy representation $\rho : \mathrm{Hom}(\pi_1(S), \mathrm{PSL}(2, \mathbb{C}))$ by an element of the mapping class group represented as an isometry.

In section §9 we classify the development-holonomy pairs with real monodromy. To relate this classification to the classification of opers with real monodromy, we need to take the quotient by the mapping class group.

7.4 Uniformizing oper

For each marked hyperbolic surface X modeled over the topological surface S, the uniformization theorem defines an isometry $A : \tilde{X} \to \mathbb{H}$. This map embeds the action of the Deck transformations $\pi_1(S)$ in the group of orientation-preserving isometries $\mathrm{Isom}^+(\mathbb{H}) \simeq \mathrm{PSL}(2, \mathbb{R})$ defining a monodromy representation $\rho \in \mathrm{Hom}(\pi_1(S), \mathrm{PSL}(2, \mathbb{R}))$. The pair (A, ρ) has the intertwining property

$$A(\gamma \cdot u) = \frac{\rho(\gamma)_{11} A(u) + \rho(\gamma)_{12}}{\rho(\gamma)_{21} A(u) + \rho(\gamma)_{22}}$$

Since the image Γ of ρ is a discrete subgroup, it defines a Fuchsian group. Two marked hyperbolic surfaces define the same point in $\mathcal{T}(S)$ if and only if the monodromy representations up to conjugation are the same Fuchsian representation. This implies that the space of Fuchsian representations is also isomorphic to $\mathcal{T}(S)$. See for example proposition 10.2 in [91] for a proof of this statement.

Let us denote the uniformizing developing map by A_0 and the monodromy by ρ_0 for future convenience. By 3.11, the pair $[A_0, \rho_0]$ defines an oper $t_0(z) = \frac{1}{2}\{A_0(z), z\}$. Since the monodromy ρ_0 has image in $\mathrm{PSL}(2, \mathbb{R})$, it comes from an oper with monodromy in $\mathrm{SL}(2, \mathbb{R})$. Therefore, the uniformizing oper defines a single-valued eigenfunction of the Hitchin Hamiltonians by theorem 3.5.

Affine bundle structure of opers A dicussion of the following construction can be found in [78]. Let us denote the space of all opers over the topological surface S by $\mathcal{P}(S) := \bigsqcup_{X \in \mathcal{T}(S)} \mathcal{P}(X)$. Recall that we use the notation $\mathcal{P}(X)$ for the space of opers over a fixed Riemann surface X. The fibres can be patched together to define a smooth bundle $\mathcal{P}(S) \to \mathcal{T}(S)$ with fibres $\mathcal{P}(X)$. It turns out that the uniformizing opers patch together to a smooth section $\sigma_0 : \mathcal{T}(S) \to \mathcal{P}(S)$. The bundle $\mathcal{P}(S)$ is an affine bundle with each section $\sigma(X) - \sigma_0(X)$ living in $\mathrm{H}^0(X, K_X^2)$.

It can be shown by Kodaira-Spencer deformation theory, that the tangent bundle fibre of $\mathrm{T}_X\mathcal{T}(S)$ is isomorphic to $\mathrm{H}^1(X, K_X^{-1})$. Through the Serre duality, we can identify the cotangent fibres of $\mathrm{T}^*_X\mathcal{T}(S)$ with $\mathrm{H}^0(X, K_X^2)$. The bundle $\mathcal{P}(S) \to \mathcal{T}(S)$ has the structure of an affine bundle modeled over $\mathrm{T}^*\mathcal{T}(S) \to \mathcal{T}(S)$.

The identification between $\mathrm{T}^*\mathcal{T}(S)$ and $\mathcal{P}(S)$ allows us to carry over the geometrical structures from the former to the latter. Indeed, there exists a natural complex structure on $\mathcal{P}(S)$ coming from the complex structure on $\mathrm{T}^*\mathcal{T}(S)$ in terms of complex moduli $\mathbf{t}$ for the base $\mathcal{T}(S)$ and complex coordinates $\mathbf{E}$ for the fibres. Furthermore, the cotangent bundle admits a canonical symplectic structure

$$\Omega = \sum_{r=1}^{3g-3+n} dt_r \wedge dE_r$$

which carries over to a symplectic structure Ω on $\mathcal{P}(S)$.

8 Eigenfunctions and singular metrics

By the uniformization theorem, we know that $\mathbb{H}$ admits a unique metric g_{hyp} of constant curvature -4. Since $\text{Isom}(\mathbb{H}) \simeq \text{PGL}(2,\mathbb{R})$, the pair $[A_0, \rho_0]$ embeds the Deck transformations into the isometries of $\mathbb{H}$. We may therefore use A_0 to pull back g_{hyp} to a metric on X. Locally, the induced metric takes the form

$$A_0^*(g_{\text{hyp}}) = \frac{|A_0'(z)|^2}{4\mathfrak{Im}(A_0(z))^2} dz d\bar{z}$$

Since $\mathfrak{Im}(A_0(z)) > 0$ everywhere, this metric is well-defined on all of X.

Let us write the conformal factor as $e^{2\varphi_0(z,\bar{z})}$ for $\varphi_0 : X \to \mathbb{R}$ a globally defined function. Then

$$\varphi_0(z,\bar{z}) = \log\left(\frac{1}{2}\frac{|A_0'(z)|}{\mathfrak{Im}(A_0(z))}\right)$$

The function $\varphi_0(z,\bar{z})$ can be shown to satisfy the Liouville equation

$$\partial_z \partial_{\bar{z}} \varphi_0(z,\bar{z}) = e^{2\varphi_0(z,\bar{z})}$$

Furthermore, we can show by the definition of φ_0 that $\frac{1}{2}\{A_0(z), z\} = \partial_z^2 \varphi_0 - (\partial_z \varphi_0)^2$. This identifies the uniformizing oper by

$$t_0(z) = \partial_z^2 \varphi_0 - (\partial_z \varphi_0)^2$$

Although a priori not obvious, the combination $\partial_z^2 \varphi_0 - (\partial_z \varphi_0)^2$ is holomorphic as a consequence of the Liouville equation. Finding a solution to the Liouville equation is therefore equivalent to the construction of the uniformizing oper.

8.1 Singular hyperbolic metrics and real monodromy

Real decomposition of a surface To define the real decomposition of a surface, we follow the discussion in [46]. We also refer to the notion of real decompositions in [92].

If the pair $[A, \rho]$ defines a development-holonomy pair over $X \in \mathcal{T}(S)$ with monodromy representation ρ valued in $\text{PSL}(2,\mathbb{R})$, we may consider the decomposition $\mathbb{C}P^1 = \mathbb{H} \cup \mathbb{R}P^1 \cup \overline{\mathbb{H}}$. Here we denote by $\overline{\mathbb{H}}$ the space

$$\overline{\mathbb{H}} := \{z \in \mathbb{C} | \mathfrak{Im}(z) < 0\}$$

We can pull back this decomposition through the developing map and define $\tilde{X}_+ := A^{-1}(\mathbb{H})$, $\tilde{X}_{\mathbb{R}} := A^{-1}(\mathbb{R}P^1)$ and $\tilde{X}_- := A^{-1}(\overline{\mathbb{H}})$. Since each of the subsets $\mathbb{H}$, $\mathbb{R}P^1$ and $\overline{\mathbb{H}}$ is invariant under Möbius transformations valued in $\mathrm{PSL}(2,\mathbb{R})$, we find $A(\gamma \cdot \tilde{X}_\pm) = A(\tilde{X}_\pm)$ and $A(\gamma \cdot \tilde{X}_{\mathbb{R}}) = A(\tilde{X}_{\mathbb{R}})$ for any $\gamma \in \pi_1(X)$. The decomposition is therefore preserved under the action of the Deck transformations as long as the monodromy of ρ is valued in $\mathrm{PSL}(2,\mathbb{R})$. This allows us to define a decomposition of our surface $X = X_+ \cup X_{\mathbb{R}} \cup X_-$.

Remark 8.1. If the monodromy of ρ is valued in $\mathrm{PGL}(2,\mathbb{R})$, we still find $A(\gamma \cdot \tilde{X}_{\mathbb{R}}) = A(\tilde{X}_{\mathbb{R}})$. Although we cannot define a real decomposition $X = X_+ \cup X_{\mathbb{R}} \cup X_-$, the variety where A is real, is still invariant under the action of the Deck transformations.

On the space $\mathbb{H}$ we have the constant curvature metric g_{hyp}. Since the spaces $\mathbb{H}$ and $\overline{\mathbb{H}}$ are related by complex conjugation, we make $\overline{\mathbb{H}}$ into a copy of the hyperbolic plane by introducing g_{hyp}. We can pull back these metrics through the developing map and define a metric $A^*(g_{\mathrm{hyp}})$ on $X \backslash X_{\mathbb{R}}$. Each connected component of X_+ or X_- has a natural structure of a complete hyperbolic surface such that the ideal boundary, the boundary at infinite distance, is given by a union of components in $X_{\mathbb{R}}$.

Let us denote by $e^{2\varphi(z,\bar{z})}$ the conformal factor of this metric on $X \backslash X_{\mathbb{R}}$, i.e.

$$A^*(g_{\mathrm{hyp}}) = e^{2\varphi(z,\bar{z})} dz d\bar{z}$$

The conformal factor blows up when we approach $X_{\mathbb{R}}$, since the metric diverges as $\mathfrak{Im}(A(z))^{-2}$.

Each connected component Y of X_+ and X_- can be understood as consisting of a convex core $Y_{\mathrm{core}} \subset Y$ with boundaries of Y_{core} consisting of closed geodesics and a complement $Y \backslash Y_{\mathrm{core}}$ consisting of annuli of infinite area. In this way, the annuli of infinite area are bounded by a geodesic and a component of the ideal boundary.

Schwarz function of singular curves Let us describe in more detail how $\varphi(z, \bar{z})$ blows up when we approach $X_{\mathbb{R}}$ following [80]. By [92], it is known that the space $X_{\mathbb{R}}$ is a finite union of circles. Let $C \subset X_{\mathbb{R}}$ be such a circle. We pick a simply-connected neighbourhood $U \supset C$ and a local coordinate z. Then we may describe the intersection $C \cap U$ through the analytical equation

$$A(z) = \overline{A(z)}$$

We may rewrite this equation in terms of the Schwarz function $S_C(z) := \overline{A^{-1}(\overline{A(z)})}$ as

$$\bar{z} = S_C(z)$$

Applying the chain rule to the equation $A(\overline{S_C(z)}) = \overline{A(z)}$ shows that $\overline{A'(z)} = A'(\overline{S_C(z)})\overline{S'_C(z)}$. Therefore, we may express the conformal factor in terms of the developing map as

$$\begin{aligned} e^{2\varphi(z,\bar{z})} &= \frac{1}{4}\frac{|A'(z)|^2}{(\mathfrak{Im}(A(z)))^2} \\ &= -\frac{A'(z)A'(\overline{S_C(z)})\overline{S'_C(z)}}{(A(z)-A(\overline{S_C(z)}))^2} \end{aligned}$$

Since the developing map A is nowhere singular, the oper $t(z)$ does not become singular near the curve C. In terms of the conformal factor, we may show

$$t(z) = \partial_z^2\varphi - (\partial_z\varphi)^2$$

If we locally identify the coordinate z with the coordinate obtained by pulling back the standard coordinate on $\mathbb{C}P^1$, we may simplify

$$e^{2\varphi(z,\bar{z})} = \frac{1}{4}\mathfrak{Im}(z)^{-2}$$

Singular solutions to the Liouville equation We may summarize the above by stating that the problem of identifying development-holonomy pairs with real monodromy can be expressed in terms of a variant of the Liouville problem. This problem has been stated in section four of [80] before:

Theorem 8.2. *A development-holonomy pair $[A,\rho]$ over $X \in \mathcal{T}(S)$ has real monodromy representation ρ if and only if there exist simple closed analytic curves $C_1, ..., C_m$ such that $\bigsqcup_{k=1}^m C_k = X_{\mathbb{R}}$ and a function φ such that $\partial_z\partial_{\bar{z}}\varphi = e^{2\varphi}$ on $X\backslash X_{\mathbb{R}}$ with singularities along C_k of the form*

$$\varphi(z,\bar{z}) = \log\left(\frac{\sqrt{-A'(z)A'(\overline{S_{C_k}(z)})\overline{S'_{C_k}(z)}}}{A(z)-A(\overline{S_{C_k}(z)})}\right) + \mathcal{O}((A(z)-A(\overline{S_{C_k}(z)}))^0)$$

Furthermore, we may clarify theorem 8.2 further and state

Theorem 8.3. *Development-holonomy pairs over X with real monodromy are in one-to-one correspondence to the choice of elements $\sigma_1, ..., \sigma_m \in \pi_1(X)$, not necessarily distinct. Furthermore for each choice of elements $\sigma_1, ..., \sigma_m$, there* *exist unique representative curves* $C_1, \ldots, C_m$ *such that* $X_{\mathbb{R}} = \bigsqcup_{k=1}^m C_k$ *and a solution* $\varphi(z, \bar{z})$ *exists with singular behaviour as in theorem 8.2.*

We will prove this theorem in section §9. More precisely, proposition 9.7 gives a direct proof of this result.

Remark 8.4. Any set of $3g-3+n$ mutually non-intersecting simple closed curves on a surface X of genus g and with n boundary components leads to a pants decomposition. There exist no other curves not homotopic to the cutting curves and not intersecting any of the cutting curves. Therefore, the number of distinct elements in $\sigma_1, \ldots, \sigma_m$ is always bounded from above by the number of cutting curves $3g-3+n$.

Remark 8.5. The uniformizing oper is recovered by a solution of the Liouville equation on all of X. This is the unique solution for which $X_{\mathbb{R}} = \emptyset$.

8.2 Constructing single-valued functions

The conformal factor $e^{2\varphi(z,\bar{z})}$ of the metric $A^*(g_{\mathrm{hyp}})$ on $X \backslash X_{\mathbb{R}}$ allows us to define a single-valued and real function $\phi(z, \bar{z}) := e^{-\varphi(z,\bar{z})}$ on $X \backslash X_{\mathbb{R}}$ which behaves as

$$\phi(z, \bar{z}) = \frac{A(z) - A(\overline{S_{C_k}(z)})}{\sqrt{-A'(z)A'(\overline{S_{C_k}(z)})\overline{S'_{C_k}(z)}}} \exp\left(\mathcal{O}\left((A(z) - A(\overline{S_{C_k}(z)}))^0\right)\right)$$

along each curve C_k. Since the behaviour of $\phi(z, \bar{z})$ along C_k is regular, the function $\phi(z, \bar{z})$ can be extended to all of X and satisfies

$$\begin{aligned} \partial_z^2 \phi(z, \bar{z}) + t(z)\phi(z, \bar{z}) &= 0 \\ \partial_{\bar{z}}^2 \phi(z, \bar{z}) + \overline{t(z)}\phi(z, \bar{z}) &= 0 \end{aligned}$$

where $t(z) = (\partial_z^2 \varphi) - (\partial_z \varphi)^2$. By construction, it is a symmetric global section of the bundle $K_X^{-1/2} \otimes \overline{K_X}^{-1/2}$.

Although $\phi(z, \bar{z})$ is single-valued on $X \backslash X_{\mathbb{R}}$, it need not be when we extend it to X. The function ϕ can only vanish along the curves C_k. By the chain rule $S'_{C_k}(z) \neq 0$ on a neighbourhood $U \supset C_k$, so that we may always apply the inverse function theorem to define a coordinate z for which $S_{C_k}(z) = z$. Then

$$\phi(z, \bar{z}) = 2|A'(z)|^{-1} \mathfrak{Im}(A(z))$$

Since the curve $X_{\mathbb{R}}$ is defined by $\mathfrak{Im}(A(z)) = 0$ and $\mathfrak{Im}(A(z))$ is positive and negative on X_+ and X_- respectively, we find $\phi(z, \bar{z}) > 0$ on X_+ and $\phi(z, \bar{z}) < 0$

on X_-. The function $\phi(z,\bar{z})$ therefore changes sign when we cross $X_\mathbb{R}$. If we analytically continue $\phi(z,\bar{z})$ along any simple closed geodesic, we find

$$\phi(\gamma \cdot z, \gamma \cdot \bar{z}) = (-1)^{i(\gamma, X_\mathbb{R})}\phi(z,\bar{z})$$

where $i(\gamma, X_\mathbb{R})$ is the number of intersections between γ and the variety $X_\mathbb{R}$. Comparing with theorem 3.5, we conclude

Theorem 8.6. *If ρ is a real monodromy representation of a pair $[A,\rho]$, it is valued in* $\mathrm{PSL}(2,\mathbb{R})$ *if and only if*

$$i(\gamma, X_\mathbb{R}) = 0 \mod 2$$

for any simple closed geodesic γ.

A similar statement has been made by Goldman in the context of closed surfaces in [89]. See remark 11.3.

Remark 8.7. For an explicit example of the construction in this section on the four-punctured sphere, we refer back to subsection 4.4.

9 Grafting complex projective structures

Before we describe the grafting surgery, we will change our point of view slightly. To keep mathematical expressions more concise, we introduce the following notation

$$C_G(S) := \mathrm{Hom}(\pi_1(S), G)/G$$

for future notational simplicity. An oper $t(z)$ on a Riemann surface $X \in \mathcal{T}(S)$ is equivalent to a development-holonomy pair, which is a pair $[A, \rho]$ of a developing map $A : \tilde{X} \to \mathbb{C}P^1$ and a monodromy representation $\rho \in C_{\mathrm{PSL}(2,\mathbb{C})}(S)$ satisfying the intertwining property

$$A(\gamma \cdot z) = \frac{\rho(\gamma)_{11} A(z) + \rho(\gamma)_{12}}{\rho(\gamma)_{21} A(z) + \rho(\gamma)_{22}}$$

for $\gamma \in \pi_1(S)$. See 3.11 for a proof of this equivalence.

Starting from a pair $[A, \rho]$, we do not actually need to specify the complex structure $X \in \mathcal{T}(S)$. Instead, we may define $A : \tilde{S} \to \mathbb{C}P^1$ to simply be an immersion without reference to the complex structure. We may recover a complex structure on $\tilde{S}$ by pulling back the complex structure on $\mathbb{C}P^1$, i.e. we introduce a complex structure on $\tilde{S}$ such that A becomes holomorphic.

Complex projective structures A complex projective structure is a maximal atlas of charts on a smooth surface S valued in $\mathbb{C}P^1$ such that the transition functions are Möbius transformations on $\mathbb{C}P^1$. It is clear how we can map a development-holonomy pair $[A, \rho]$ to a complex projective structure. Conversely, we can also construct a development-holonomy pair from a maximal atlas of charts. We follow the explanation in [78].

Given a chart on an open neighbourhood $U \subset S$, we may analytically continue the map $U \to \mathbb{C}P^1$ to a map $\tilde{S} \to \mathbb{C}P^1$. This can be done uniquely, since $\tilde{S}$ is simply-connected. Since the transition functions are Möbius transformations, an analytical continuation in this way defines a monodromy representation in $C_{\mathrm{PSL}(2,\mathbb{C})}(S)$. This defines a pair (A, ρ) from the complex projective structure. Composing by a Möbius transformation M has the effect of sending a pair (A, ρ) to $(M \cdot A, M \cdot \rho)$. In this way, we have associated a development-holonomy pair $[A, \rho]$ to the complex projective structure.

We will switch freely between these two notions when we discuss the grafting surgery.

9.1 Defining the grafting surgery

Standard form of simple closed geodesic in uniformizing projective structure Let us consider the uniformizing development-holonomy pair $[A_0, \rho_0]$ with $A_0 : \tilde{S} \to \mathbb{H}$ an isomorphism and $\Gamma = \mathrm{im}(\rho_0)$ a Fuchsian group. We recover a hyperbolic structure $X \in \mathcal{T}(S)$ on S through the quotient $X \simeq \mathbb{H}/\Gamma$.

Let us consider a simple closed geodesic γ with monodromy realized by a hyperbolic isometry $\rho_0(\gamma)$. By lemma 7.1 we can find an invariant geodesic of this isometry in $\mathbb{H}$ with fixed points $\{z_1, z_2\}$ and $z_2 > z_1$ on $\mathbb{R}P^1$. By conjugating the monodromy ρ_0 with a Möbius transformation valued in $\mathrm{PSL}(2, \mathbb{R})$ of the form

$$z \mapsto \frac{z - z_2}{z - z_1}$$

we may map the fixed points of this isometry to $\{0, \infty\}$. This brings the action of the Deck transformation of γ on $\tilde{X}$ to the form

$$A_0(\gamma \cdot z) = e^{l_\gamma} A_0(z)$$

where l_γ is the geodesic length of γ.

Let us consider a tubular neighbourhood $U \supset \gamma$ defined by

$$U := \{x \in X | d_{\mathrm{hyp}}(\gamma, x) < \epsilon\}$$

We denote by $d_{\mathrm{hyp}}(\gamma, x)$ the hyperbolic distance between the point x and the geodesic γ. If we pick ϵ small enough, the neighbourhood U does not contain any other simple closed geodesic and we may realize U as the domain

$$U \simeq \{e^w \in \mathbb{C}^\times | -\epsilon < \mathfrak{Re}(w) < \epsilon\}$$

for a coordinate $w = w(z)$ of the universal cover $\tilde{U}$ of U

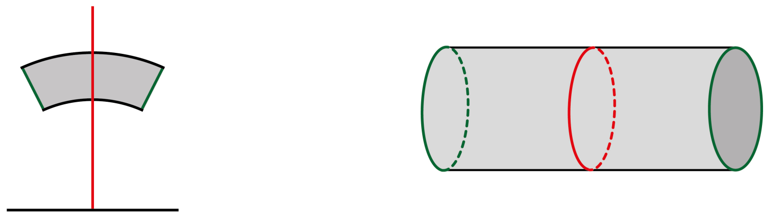
$$\tilde{U} \simeq \{w \in \mathbb{C} | -\epsilon < \mathfrak{Re}(w) < \epsilon\}$$

such that $w(\gamma \cdot z) = w(z) + 2\pi i$.

The projective structure on U induced by the pair (A_0, ρ_0) is recovered by the exponential map $A_0(z) = ie^{l_\gamma w/(2\pi i)}$ for which indeed $A_0(\gamma \cdot z) = e^{l_\gamma} A_0(z)$.

The developing map A_0 maps the universal cover $\tilde{U}$ to the set

$$A_0(\tilde{U}) = \{s \in \mathbb{C}^\times \subset \mathbb{C}P^1 | -\frac{l_\gamma \epsilon}{2\pi} + \frac{\pi}{2} < \arg(s) < \frac{l_\gamma \epsilon}{2\pi} + \frac{\pi}{2}\}$$



(a) Fundamental domain (b) Copy of U with geodesic

Figure 9.1: Standard form of a projective structure around a hyperbolic isometry

A fundamental domain for the action of the Deck transformation $A_0(\gamma \cdot z) = e^{l_\gamma} A_0(z)$ on the space $A_0(\tilde{U})$ is sketched in figure 9.1. We call this form the standard form of the projective structure around the geodesic γ.

Remark 9.1. For the uniformizing projective structure all simple closed geodesics have hyperbolic isometries as their monodromy. We may therefore bring a neighbourhood around any simple closed geodesic to the standard form.

Constructing a projective structure on the annulus Let us consider an annulus $\mathbb{A} = \mathbb{R} \times \gamma$ with γ diffeomorphic to a circle. We may put a complex structure on $\mathbb{A}$ through the diffeomorphism $\mathbb{A} \simeq \mathbb{C}^\times$ and let z be the corresponding complex coordinate. Let w be a coordinate of the universal cover $\tilde{\mathbb{A}}$ such that $w(\gamma \cdot z) = w(z) + 2\pi i$ once again. We may define a projective structure on $\mathbb{A}$ through the map $w \mapsto A(z) = ie^{l_\gamma w/(2\pi i)}$. At this point, the number $l_\gamma \in \mathbb{R}_{>0}$ is simply a parameter suggestively denoted by l_γ.

This is the same situation as for the uniformizing projective structure. In the open neighbourhood $(-\epsilon, \epsilon) \times \gamma$, this projective structure is clearly isomorphic to the uniformizing projective structure if we let l_γ be the length of the geodesic γ.

Grafting the uniformizing projective structure If we cut open the uniformizing projective structure along a geodesic γ with tubular neighbourhood U, the neighbourhood U splits into two neighbourhoods U_1 and U_2 around the two new boundary components which are copies of γ. We may glue the annular projective structure on $[0, t] \times \gamma \subset \mathbb{A}$ in a neighbourhood of $\{0\} \times \gamma$ to the uniformizing projective structure in the neighbourhood U_1.

Since any other neighbourhood of the form $(t - \epsilon, t + \epsilon) \times \gamma$ can be brought back to the open neighbourhood $(-\epsilon, \epsilon) \times \gamma$ by mapping $w \mapsto w - t$, equivalently

$A(z) \mapsto e^{il_\gamma (t/2\pi)} A(z)$, we can glue the projective structure in the neighbourhood of $\{t\} \times \gamma$ to the uniformizing projective structure in the neighbourhood U_2. This implies that if z is a local coordinate on U_1, crossing the annulus to the neighbourhood U_2 maps $A(z) \mapsto e^{il_\gamma (t/2\pi)} A(z)$.

It is important to note that for t large enough, this projective structure covers itself multiple times. The grafted projective structure is therefore not necessarily injective! Indeed, if we graft with parameter $t \in \frac{4\pi^2}{l_\gamma}\mathbb{Z}$, we introduce a copy of $\mathbb{C}^\times \simeq \mathbb{C}P^1 \backslash \{0, \infty\}$ into the image. Furthermore, for these values of t, the monodromy of the projective structure is preserved under the grafting.

Remark 9.2. Grafting does not necessarily preserve the underlying complex structure of $[A, \rho]$. See lemma 9.8.

Extending grafting to the space of measured laminations It is easy to see that we may extend the definition of grafting along a simple closed geodesic to grafting along multiple non-intersecting geodesics. We may extend this definition even further to the space of measured laminations $\mathcal{ML}(S)$. For our purposes of classifying opers with real monodromy, we do not need to consider this space in detail. The important properties of this space are that it can be given a topological structure and that it is the completion of the space of multiple non-intersecting geodesics.

Let us introduce the grafting angle $\theta := \frac{l_\gamma t}{2\pi}$. We may pick a number of non-intersecting simple closed geodesics $\{P_1, ..., P_{3g-3+n}\}$ and graft with angles $\theta_1, ..., \theta_{3g-3+n}$. These parameters determine the Möbius transformations $A(z) \to e^{i\theta_k} A(z)$ when we cross the annulus grafted along γ_k. A measured lamination $\lambda \in \mathcal{ML}(S)$ of this form, will be written as the formal sum $\lambda = \sum_{k=1}^{3g-3+n} \theta_k P_k$.

9.2 Action of grafting on monodromy representation

We have described how the developing map changes by grafting. Additionally, we have seen that when we cross the annulus grafted along a simple closed geodesic γ, the developing map changes by $A(z) \to e^{i\theta} A(z)$.

An insertion of such a transformations implies that we must adjust the monodromy of the uniformizing oper by

$$D_{i\theta} = \begin{pmatrix} e^{i\theta/2} & 0 \\ 0 & e^{-i\theta/2} \end{pmatrix}$$

or by its inverse whenever we cross γ. The matrix D_s for a given s is a representative of the dilatation $z \mapsto e^s z$ in $\mathrm{SL}(2,\mathbb{C})$.

There are now two distinct cases to consider: Either cutting along γ separates the surface X into two subsurfaces X_1 and X_2 or cutting along γ keeps the surface connected.

If γ separates the surface X into X_1 and X_2, we may describe $\pi_1(X)$ in terms of $\pi_1(X_1)$ and $\pi_1(X_2)$.

If $\sigma \in \pi_1(X_1) \cup \pi_1(X_2)$, the grafted monodromy representation denoted by $\rho_{i\theta\gamma}$will take the form

$$\rho_{i\theta\gamma}(\sigma) = \begin{cases} \rho(\sigma) & \text{if } \sigma \in \pi_1(X_1) \\ D_{-i\theta}\rho(\sigma)D_{i\theta} & \text{if } \sigma \in \pi_1(X_2) \end{cases}$$

All other elements $\sigma \in \pi_1(S)$ can be built from these elements.

If γ is non-separating, we may assume it is embedded in a one-holed torus. A representation ρ will reduce to a representation of this one-holed torus. If κ is the curve homotopic to the boundary and η a curve intersecting γ once, we may define a presentation of the fundamental group of the one-holed torus by $\langle \gamma, \eta, \kappa | \gamma\eta\gamma^{-1}\eta^{-1} = \kappa \rangle$.

The grafted representation $\rho_{i\theta\gamma}$ is defined by

$$\begin{aligned} \rho_{i\theta\gamma}(\gamma) &= \rho(\gamma) \\ \rho_{i\theta\gamma}(\eta) &= \rho(\eta)D_{i\theta} \\ \rho_{i\theta\gamma}(\kappa) &= \rho(\kappa) \end{aligned}$$

Since $\rho(\gamma)D_{i\theta} = D_{i\theta}\rho(\gamma)$, we find $\rho_{i\theta\gamma}(\gamma\eta\gamma^{-1}\eta^{-1}) = \rho_{i\theta\gamma}(\kappa)$ showing that $\rho_{i\theta\gamma}$ is also a representation of the one-holed torus. It is straightforward to construct the grafted representation $\rho_{i\theta\gamma}$ on the entire surface X from these results.

Reality of the monodromy and real curves The way the monodromy representation of the oper transforms if we apply the grafting surgery, implies the following proposition:

Proposition 9.3. *If we graft a projective structure along a simple closed geodesic γ, its monodromy representation is real, i.e. valued in $\mathrm{PGL}(2,\mathbb{R})$, up to conjugation if and only if $\theta \in \pi\mathbb{Z}_{\geq 0}$. Furthermore*

- *If γ is non-separating, the cases $\theta \in 2\pi\mathbb{Z}_{\geq 0}$ lead to representations valued in* $\mathrm{PSL}(2,\mathbb{R})$ *while for $\theta \in 2\pi\mathbb{Z}_{\geq 0}+\pi$ we find elements in* $\mathrm{PGL}(2,\mathbb{R})$ *which do not lie in* $\mathrm{PSL}(2,\mathbb{R})$.

- *If γ is separating, all of the cases $\theta \in \pi\mathbb{Z}_{\geq 0}$ lead to representations valued in* $\mathrm{PSL}(2,\mathbb{R})$.

It therefore suffices to consider $\theta \in \pi\mathbb{Z}_{\geq 0}$ if we require the reality of the monodromy of the development-holonomy pair.

9.3 Classifying opers with real monodromy

Surjectivity of the grafting operation The key result encouraging us to use the grafting procedure, is the following unpublished theorem by Thurston [87]. See also [88].

Theorem 9.4. *The grafting map* $\mathrm{Gr} : \mathcal{T}(S) \times \mathcal{ML}(S) \to \mathcal{P}(S)$ *from the space of marked hyperbolic structures and measured laminations to the space of complex projective structures over S defines a homeomorphism.*

In the statement of this theorem, we have identified $\mathcal{T}(S)$ with the image of the smooth section $\sigma_0(\mathcal{T}(S)) \subset \mathcal{P}(S)$, which associates to each $X \in \mathcal{T}(S)$ the uniformizing projective structure.

By theorem 9.4, any element in $\mathcal{P}(S)$ may be constructed by grafting a uniformizing projective structure. The measured laminations describing projective structures with real monodromy, are precisely those of the form $\lambda = \sum_{k=1}^{3g-3+n} m_k \pi P_k$ for a pants decomposition $\{P_1, ..., P_{3g-3+n}\}$ and grafting angles $m_k\pi$ where we allow $m_k \in \mathbb{Z}_{\geq 0}$. See proposition 9.3. Moreover, the construction of opers with real monodromy through the grafting surgery shows that the image of the monodromy map intersects the space $C_{\mathrm{PSL}(2,\mathbb{R})}(S)$ of real monodromy representations transversally.

We call such measured laminations half-integer measured laminations and denote the space of the half-integer measured laminations by $\mathcal{ML}_{\frac{1}{2}\mathbb{Z}}(S)$. Grafting along the half-integer measured lamination λ can be understood as simultaneously applying the grafting procedure for simple closed geodesics with angles $m_k\pi$ to all the curves P_k. See [46] for the classification of projective structures with real monodromy.

Quadratic differentials from measured laminations We will now discuss the relation between measured laminations and quadratic differentials. This discussion follows [78]. Our end goal is to classify the quadratic differentials over a fixed marked Riemann surface X leading to opers with real monodromy. Recall that by the uniformization theorem, we construct a smooth section $\sigma_0 : \mathcal{T}(S) \to \mathcal{P}(S)$ of the affine bundle $\pi : \mathcal{P}(S) \to \mathcal{T}(S)$ which associates to each $X \in \mathcal{T}(S)$ the corresponding uniformizing oper. This allows us to describe the fibres of $\mathcal{P}(S)$ as quadratic differentials.

The key result allowing us to relate quadratic differentials and measured laminations is given by the following theorem proved in [93]

Theorem 9.5. *For each $\lambda \in \mathcal{ML}(S)$ the map $\mathrm{gr}_\lambda : \mathcal{T}(S) \to \mathcal{T}(S)$, defined by $\mathrm{gr}_\lambda := \pi \circ \mathrm{Gr}_\lambda$, is a diffeomorphism.*

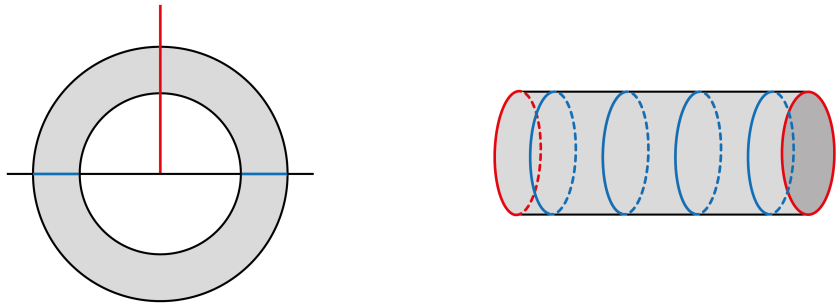
Since gr_λ is a diffeomorphism, we may invert it and define a smooth map $\sigma_\lambda(X) := \mathrm{Gr}_\lambda(\mathrm{gr}_\lambda^{-1}(X))$. By definition $\pi(\sigma_\lambda(X)) = X$ so that σ_λ defines a smooth section of $\pi : \mathcal{P}(S) \to \mathcal{T}(S)$ for each $\lambda \in \mathcal{ML}(S)$. There exists a unique element $0 \in \mathcal{ML}(S)$ which leads to a section σ_0 associating the uniformizing projective structure $\sigma_0(X)$ to X. This is compatible with the previous definition of σ_0: The uniformizing oper corresponds to the zero element in the measured laminations.

By theorem 9.4, there exists a unique pair $(\mathrm{gr}_\lambda^{-1}(X), \lambda)$ as the inverse image of Gr_λ for every oper in the fibre $\mathcal{P}(X)$. Therefore, we find a homeomorphism between the space of quadratic differentials $\mathrm{H}^0(X, K_X^2)$ and the space of measured laminations $\mathcal{ML}(S)$. Let us denote the accessory parameters of the opers in $\mathcal{P}(X)$ obtained by grafting along λ by $\mathbf{E}_\lambda$.

Corollary 9.6. *For every $\lambda \in \mathcal{ML}_{\frac{1}{2}\mathbb{Z}}(S)$, there exists a set of accessory parameters $\mathbf{E}_\lambda$ defining an oper over X with real monodromy. This oper is obtained from the projective structure $\mathrm{Gr}_\lambda(\mathrm{gr}_\lambda^{-1}(X))$. Conversely, every oper with real monodromy can be found in this way.*

Grafting and real curves By corollary 9.6, the opers with real monodromy are classified by half-integer measured laminations. To finish the proofs of theorem 8.2 and theorem 8.3, we want to clarify the relation between these measured laminations $\lambda \in \mathcal{ML}_{\frac{1}{2}\mathbb{Z}}(S)$ and the space $X_\mathbb{R}$ in the real decomposition of $\mathrm{Gr}_\lambda(\mathrm{gr}_\lambda^{-1}(X))$.

If we graft along a simple closed geodesic γ by angle $\theta = \pi m_0$ for $m_0 \in \mathbb{Z}_{\geq 0}$, the projective structure on $\mathbb{A}$ defined by $A(z) = ie^{l_\gamma w/(2\pi i)}$ becomes real along



(a) Fundamental domain

(b) Real curves on grafted tube

Figure 9.2: Real curves from grafting along a simple closed geodesic

a total of m_0 straight lines. Indeed, our domain is bounded by $0 \leq \mathfrak{Re}(w) \leq t = \frac{2\pi\theta}{l_\gamma}$ so these lines are precisely those for which $\mathfrak{Re}(w) = \frac{4\pi^2 m}{l_\gamma} - \frac{2\pi^2}{l_\gamma}$ and $m = 1, ..., m_0$. All of these lines map to copies of the curve γ. Since the developing map of the uniformizing projective structure has image contained in $\mathbb{H}$, it never becomes real. Therefore, the only real curves that are introduced by grafting along a simple closed geodesic are the m_0 copies of γ. See figure 9.2 for an example where we graft along a simple closed geodesic.

By extending this analysis to $\lambda \in \mathcal{ML}_{\frac{1}{2}\mathbb{Z}}(S)$, we learn that for $\lambda = \sum_{k=1}^{3g-3+n} m_k \pi P_k$, the grafting procedure introduces a total of m_k copies of the geodesic P_k in $X_\mathbb{R}$. If μ is the non-self-intersecting multicurve consisting of m_k copies of the curves P_k, we will write μ as the formal sum $\mu = \sum_{k=1}^{3g-3+n} m_k P_k$. This analysis shows that $X_\mathbb{R} \simeq \mu$ as submanifolds of X. Conversely, any set of non-self-intersecting curves up to homotopy on X can be understood as defining the space $X_\mathbb{R}$ of some projective structure with real monodromy.

We may write these observations as follows

Proposition 9.7. *Let $\mathcal{C}(S)$ be the space of non-self-intersecting multicurves μ up to homotopy. Then the map $\mathcal{C}(S) \to \mathcal{ML}_{\frac{1}{2}\mathbb{Z}}(S)$ defined by $\mu \mapsto \pi\mu$ is a one-to-one correspondence between non-self-intersecting multicurves and opers with real monodromy having the property that the space $X_\mathbb{R}$ is homotopic to the image of the multicurve μ.*

If we fix the underlying complex structure X, there can exist only one projective structure with $X_\mathbb{R}$ homotopic to the multicurve μ which is precisely the one given by $\mathrm{Gr}_{\pi\mu}(\mathrm{gr}_{\pi\mu}^{-1}(X))$. Therefore, a unique choice of curves $\{C_k^{(i_k)}\}$ with $k = 1, ..., 3g-3+n$ and $i_k = 1, ..., m_k$ exists such that $\mathrm{Gr}_{\pi\mu}(\mathrm{gr}_{\pi\mu}^{-1}(X))$ satisfies $X_\mathbb{R} = \bigsqcup_{k=1}^{3g-3+n} \bigsqcup_{i_k=1}^{m_k} C_k^{(i_k)}$. This proves theorem 8.3.

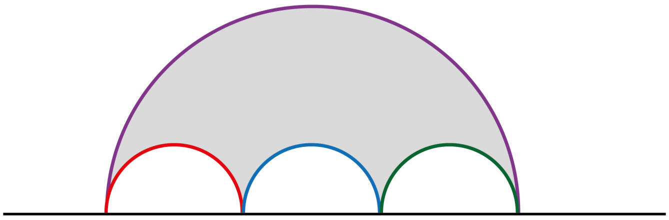


Figure 9.3: Uniformizing domain of $\mathbb{C}P^1 \backslash \{0, z, 1, \infty\}$ in standard form

9.4 Comparison to the quantization on the four-punctured sphere

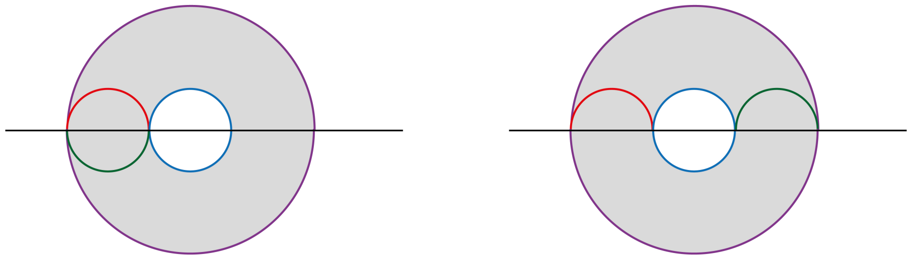
It is illuminating to clarify the grafting surgery in terms of the example of the four-punctured sphere we worked out explicitly in section §4. For the four-punctured sphere, any non-self-intersecting multicurve must be a number of parallel copies of a simple closed curve. Any other curve on the four-punctured sphere either has to intersect this simple closed curve or is homotopic to this curve or to the punctures.

Let us consider $\lambda \in \mathbb{R}$ for the four-punctured sphere. If we consider the conditions $G_{21} = 0$, $H_{21} = 0$ and $(GH)_{21} = 0$, the real curves we found, are shown in figure 4.8. By the classification of opers with real monodromy, we may therefore construct the values λ_k for $k \in \mathbb{Z}$ by grafting the uniformizing projective structure with accessory parameter λ_0 along the curve surrounding the interval $(0, z)$ or along the curve surrounding $(z, 1)$.

Indeed, we can see this explicitly by applying a Möbius transformation to figure 4.6 and bringing it to a more suggestive form. See figure 9.3

Applying the same Möbius transformation to figure 4.5b and figure 4.7a leads to the pictures in figure 9.4.

It is instructive to compare these pictures to figure 9.2 and conclude that we have introduced grafting annuli in the developing map.



(a) Transformation of figure 4.5b (b) Transformation of figure 4.7a

Figure 9.4: Domains of figure 4.5b and figure 4.7a in standard form

Changing the complex structure by grafting Although grafting by angles which are multiples of 2π preserves the monodromy, we can prove that this does not preserve the complex structure. More precisely, we will show that if the monodromy of two opers is equal up to conjugation over the surface $X = \mathbb{C}P^1 \backslash \{0, z, 1, \infty\}$, the accessory parameters of the opers must be equal. This is a special case of the Riemann-Hilbert correspondence discussed in section §17. A proof of the following lemma can be found in [50].

Lemma 9.8. *Let $z \in \mathbb{C}P^1 \backslash \{0, 1, \infty\}$ not necessarily real and consider two opers over $\mathbb{C}P^1 \backslash \{0, z, 1, \infty\}$. If their monodromy representations are equal up to conjugation, their accessory parameters must coincide.*

Proof. Let κ_1 and κ_2 be two accessory parameters for opers over $X = \mathbb{C}P^1 \backslash \{0, z, 1, \infty\}$ with the same monodromy representation up to conjugation. By our assumption, we may use a basis transformation to bring $\vec{\psi}_{0,+}(x, \kappa_1)$ and $\vec{\psi}_{0,+}(x, \kappa_2)$ to a form such that their monodromies are precisely equal, not just up to conjugation. Let us then consider the combination

$$V(x) = \psi_0^{(1)}(x, \kappa_1)\psi_{0,+}^{(2)}(x, \kappa_2) - \psi_0^{(1)}(x, \kappa_2)\psi_{0,+}^{(2)}(x, \kappa_1)$$

The function $V(x)$ can at most blow up as $\log(x - *)$ or $(\log(x - *))^2$ near each puncture at $x = *$. Furthermore, it is single-valued at each puncture. The function $V(x)$ must therefore have an expansion in terms of a Laurent series around each puncture. However, we know that $V(x)$ can blow up with at most log-type singularities implying that $V(x)$ is a constant.

For large values of x, the equation (3.5) becomes

$$x^3(1 + \mathcal{O}(x^{-1}))(\psi_\lambda)''(x) + 3x^2(1 + \mathcal{O}(x^{-1}))(\psi_\lambda)'(x) + x(1 + \mathcal{O}(x^{-1}))\psi_\lambda(x) = 0$$

This shows that both solutions of the differential equation have the behaviour $x^{-1}(1+\mathcal{O}(x^{-1}))$ for large x. Therefore

$$\lim_{x\to\infty} V(x) = 0$$

Since $\lim_{x\to\infty} V(x) = 0$, we find $V(x) = 0$ everywhere. This implies the relation

$$\frac{\psi^{(2)}_{0,+}(x,\kappa_1)}{\psi^{(1)}_0(x,\kappa_1)} = c_0(\kappa_1,\kappa_2)\frac{\psi^{(2)}_{0,+}(x,\kappa_2)}{\psi^{(1)}_0(x,\kappa_2)}$$

for some constant c_0 depending on κ_1 and κ_2. From this result, it is easily checked that

$$\left\{\frac{\psi^{(2)}_{0,+}(x,\kappa_1)}{\psi^{(1)}_0(x,\kappa_1)}, x\right\} = \left\{\frac{\psi^{(2)}_{0,+}(x,\kappa_2)}{\psi^{(1)}_0(x,\kappa_2)}, x\right\}$$

Recall that the Wronskian satisfies

$$W(x,\lambda) = \frac{W_0(\lambda)}{x(x-z)(x-1)}$$

for $W_0(\lambda)$ independent of x. Since

$$\left\{\frac{\psi^{(2)}_{0,+}(x,\lambda)}{\psi^{(1)}_0(x,\lambda)}, x\right\} = \frac{1}{2x^2} + \frac{1}{2(x-z)^2} + \frac{1}{2(x-1)^2} + \frac{z-x+1+2\lambda}{x(x-z)(x-1)}$$

we conclude with the equality $\kappa_1 = \kappa_2$. $\square$

10 Dehn-Thurston coordinates

10.1 Introducing Dehn-Thurston coordinates

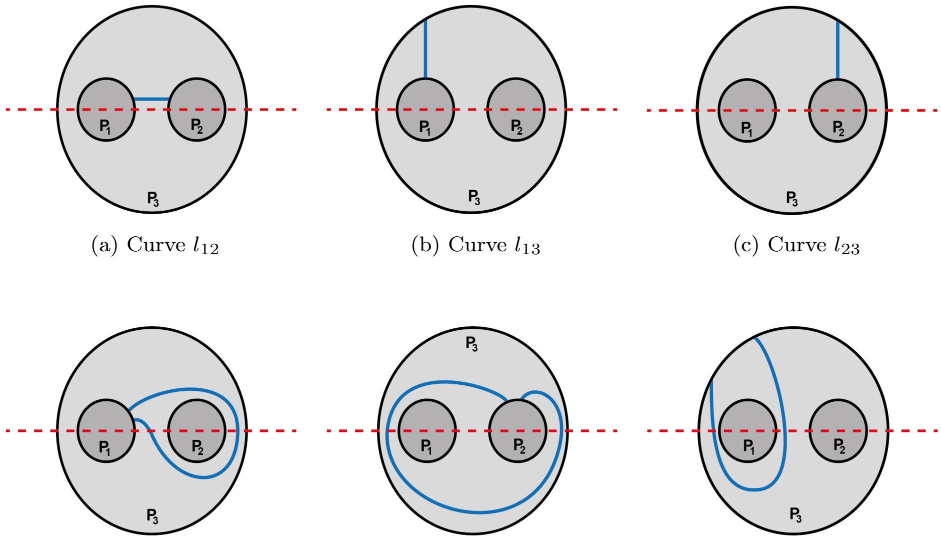
We denote by $\mathcal{C}(S)$ the space of non-self-intersecting multicurves on the topological surface S up to homotopy. We allow these multicurves to start and end at punctures or boundary components. It turns out there exists a bijection between the elements of this set and the so-called Dehn-Thurston parameters. In this section, we will clarify this statement. We refer to [42] for a clear exposition on the definition of the Dehn-Thurston coordinates.

To specify the Dehn-Thurston parameters, we first need to introduce a reference pants decomposition. If our surface is of genus g and has n boundary components, this pants decomposition consists of $3g-3+n$ cutting curves and $2g-2+n$ pairs of pants. Let us denote the pants decomposition by $\{P_1, ..., P_{3g-3+n}\}$.

Let $\mu \in \mathcal{C}(S)$. Then one half of the set of parameters will be defined as $p_k = i(\mu, P_k)$, where $i : \mathcal{C}(S) \times \mathcal{C}(S) \to \mathbb{Z}_{\geq 0}$ counts the minimal number of intersections between representatives of the two elements in $\mathcal{C}(S)$. Before we discuss the other half of the coordinates, we have to understand which multicurves can appear on pairs of pants.

Elementary multicurves on a pair of pants Without loss of generality, we may assume P_1, P_2 and P_3 bound a single pair of pants. We can build up multicurves on a pair of pants from six simple curves. We call these six curves in figure 10.1 elementary curves. An elementary curve satisfies the property that each intersection with the boundary lies in the top half of the pair of pants, i.e. above the red dotted line in figure 10.1. If a multicurve is built from elementary curves, we call its form elementary.

The number of intersections between an elementary multicurve μ and the boundary components is enough to specify the elementary multicurve uniquely. The intersection points must be connected in such a way that the multicurve does not intersect itself. We may therefore use three parameters $(p_1, p_2, p_3) \in (\mathbb{Z}_{\geq 0})^3$ to classify all elementary multicurves on the pair of pants. These numbers must satisfy $p_1+p_2+p_3 \in 2\mathbb{Z}$ since the multicurve must start and end at the boundary components.



(a) Curve l_{12} (b) Curve l_{13} (c) Curve l_{23}

(d) Curve l_{11} (e) Curve l_{22} (f) Curve l_{33}

Figure 10.1: Six elementary curves on the pair of pants

First consider the case $p_i > p_j + p_k$ for given i, j, k all distinct from one another. It is then impossible to find $p_j > p_i + p_k$ or $p_k > p_i + p_j$, since this would imply $p_k < 0$ or $p_j < 0$ respectively. We easily check that the multicurve μ takes the form

$$\mu = \frac{1}{2}(p_i - p_j - p_k)l_{ii} + p_j l_{ij} + p_k l_{ik}$$

The other case is given by $p_i \leq p_j + p_k$ for all i, j, k different from each other. The multicurve μ now takes the form

$$\mu = \frac{1}{2}(p_1 + p_2 - p_3)l_{12} + \frac{1}{2}(p_1 + p_3 - p_2)l_{13} + \frac{1}{2}(p_2 + p_3 - p_1)l_{23}$$

Gluing pairs of pants The other half of the coordinates is introduced when we glue together two different pairs of pants. Consider a four-holed sphere obtained by gluing two pairs of pants along a cutting curve P. By cutting along P, we obtain a boundary component isotopic to P for each pair of pants. Any multicurve on the four-holed sphere defines a multicurve on each pair of pants. Without loss of generality, we may assume that the multicurve on one pair of pants has elementary form. The multicurve on the other pair of pants can be brought back to elementary form by twisting the boundary curve P.

Consider two pairs of pants on which we have elementary multicurves. If the multicurves have the same number of intersections along one of the boundary

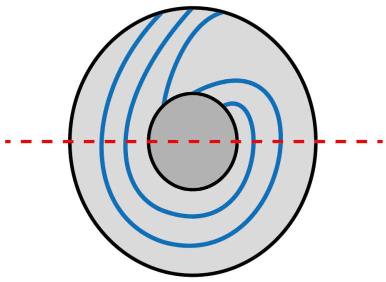


Figure 10.2: Twisting for $(p, q) = (3, 2)$

components, we can glue the pairs of pants together and define a multicurve on a four-holed sphere. However, before doing so, we may introduce an annulus which contains a certain amount of twisting. Any multicurve on the four-holed sphere can be described in this way.

More precisely, if we consider an annulus with p intersection points, we say we twist by $q > 0$ if we apply $\lfloor q/p \rfloor$ Dehn twists to the right and connect the first intersection point from the left on the inner annulus to the $(q \mod p)$-th intersection point on the outer annulus, also counted from the left. See figure 10.2 for an illustrative example.

If $q < 0$, we apply $\lfloor -q/p \rfloor$ Dehn twists to the left and connect the first intersection point on the inner annulus to the $(-q \mod p)$-th intersection point on the outer annulus, both counted from the right this time! For $p = 0$, we will always choose to set $q \geq 0$.

Note that due to the canonical ordering requiring the multicurves to start and end above the red dotted line, there is a unique way to connect these intersections without crossing the red dotted line. We set $q = 0$ for this untwisted configuration.

We therefore introduce an additional coordinate q_k for each cutting curve P_k which takes into account the twisting. For our purposes, it suffices to describe multicurves not intersecting the boundary components of S. By describing all such multicurves in the way described above, we find the following theorem originally due to Dehn in the 1920s and rediscovered by Thurston in the 1970s. Both articles have been published much later in [43, 44].

Theorem 10.1. *The set of integers* $(p_1, ..., p_{3g-3+n}, q_1, ..., q_{3g-3+n}) \in \mathbb{Z}_{\geq 0}^{3g-3+n} \times \mathbb{Z}^{3g-3+n}$ *restricted by* $p_i + p_j + p_k \in 2\mathbb{Z}_{\geq 0}$ *if* $P_i \cup P_j \cup P_k$ *bounds a pair of pants and* $q_m \geq 0$ *if* $p_m = 0$ *for* $i, j, k, m = 1, ..., 3g-3+n$ *is in one to one correspondence with the set* $\mathcal{C}(S)$ *of non-self-intersecting curves up to homotopy.*

Remark 10.2. If our surface has boundary components, we may assume the multicurves have components intersecting the boundary. We can introduce twisting along these boundary components in the same way as we twist along cutting curves. This leads to the set of coordinates

$$(p_{3g-3+n+1}, ..., p_{3g-3+2n}, q_{3g-3+n+1}, ..., q_{3g-3+2n})$$

If we have a puncture on our surface, twisting along the puncture does not change the multicurve. Therefore, if we wish to change a boundary component into a puncture, we can simply remove the corresponding twist parameter.

Dehn-Thurston parameters as quantum numbers By theorem 10.1 we get a classification of non-self-intersecting multicurves on S in terms of Dehn-Thurston parameters. If we put a hyperbolic structure $X \in \mathcal{T}(S)$ on the surface S, we may combine this result with corollary 9.6 to find a classification of opers on X with real monodromy in terms of the Dehn-Thurston parameters. The accessory parameters of these opers will be denoted by

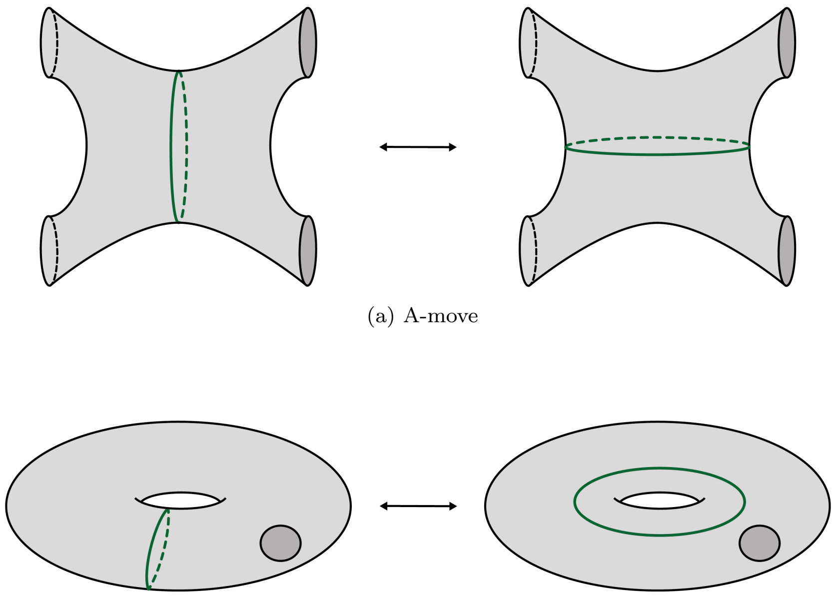
$$\mathbf{E}_{(\mathbf{p},\mathbf{q})} = (E_{1,(\mathbf{p},\mathbf{q})}, ..., E_{3g-3+n,(\mathbf{p},\mathbf{q})})$$

In this notation, the uniformizing oper is recovered from the set of accessory parameters $\mathbf{E}_{(\mathbf{0},\mathbf{0})}$.

This leads to a first set of quantum numbers for the opers with real monodromy.

By theorem 3.5 and the fact that each $\phi_k(u_k, \bar{u}_k)$ in the decomposition $\Phi_{\mathbf{E}}(\mathbf{u}, \bar{\mathbf{u}}) = \prod_{k=1}^{3g-3+n} \phi_k(u_k, \bar{u}_k)$ satisfies the same oper differential equation, we find a set of quantum numbers labeling the single-valued solutions $\Psi_{(\mathbf{p},\mathbf{q})} := \Psi_{\mathbf{E}_{(\mathbf{p},\mathbf{q})}}$.

Remark 10.3. Although all single-valued solutions are labeled in this way, not every function $\Psi_{(\mathbf{p},\mathbf{q})}$ is single-valued! To find the subset of single-valued functions, we have to apply theorem 8.6. This will be done in section §11.



(a) A-move

(b) S-move

Figure 10.3: Depiction of elementary moves on pants decomposition

10.2 Changing the pants decomposition

The Dehn-Thurston parameters were defined by gluing different pairs of pants together into the surface S and therefore depend on a pants decomposition of S. It is natural to ask ourselves how the Dehn-Thurston parameters transform under a change of pants decomposition. For this purpose, we will note the following proposition originally proved in the appendix of [94].

Proposition 10.4. *Any two pants decompositions of S are related by one another by a finite number of elementary moves.*

Although originally proved for closed surfaces, these results have been extended for punctured surfaces as well. See for example [95].

Furthermore, in [94] it is noted that two elementary moves suffice to generate all pants decompositions. These moves are named the A- and the S-move. See figure 10.3.

These two moves only change the pants decomposition in the subsurfaces homeomorphic to the four-holed sphere or one-holed torus inside S. Therefore, only the Dehn-Thurston coordinates associated to these subsurfaces are transformed. These moves act by piecewise integral linear transformations on the Dehn-Thurston parameters. Full formulas can be found in the original paper [48]. For a general overview, we also refer to [49].

We will finish this section by giving some examples of calculations we can perform using these S- and A-moves.

Example: Torus with one puncture Let us consider the example of the S-move acting on a once-punctured torus. A pants decomposition is defined by a single cutting curve so that we may introduce Dehn-Thurston parameters (p, p_0, q) for the number of intersections with the cutting curve, the number of intersections with the puncture and the number of twists along the cutting curve respectively. After the S-move, we find a new set of coordinates (p', p_0', q'). We first note that $p_0 = p_0' \in 2\mathbb{Z}_{\geq 0}$. The new coordinates can be calculated as follows

$$(p', q') = \begin{cases} \left(\frac{p_0}{2} - p + |q|, -q\right) & \text{if } p_0 > 2p \text{ and } p > |q| \\ \left(\frac{p_0}{2} - p + |q|, -\mathrm{sgn}(q)p\right) & \text{if } p_0 > 2p \text{ and } p \leq |q| \\ \left(|q|, -\mathrm{sgn}(q)\left(p - \frac{p_0}{2} + |q|\right)\right) & \text{if } p_0 \leq 2p \text{ and } p_0 > 2|q| \\ \left(|q|, -\mathrm{sgn}(q)p\right) & \text{if } p_0 \leq 2p \text{ and } p_0 \leq 2|q| \end{cases}$$

We refer to [42] for a proof of this statement.

From these transformations, we observe

Corollary 10.5. *For the one-holed torus, the number of intersections p' after the S-move between the new cutting curve and the multicurve, satisfies*

$$p' = \begin{cases} p_0/2 - p + |q| & \text{if } p_0 > 2p \\ |q| & \text{if } p_0 \leq 2p \end{cases}$$

Proof. By noting that the twist around the hole of the one-holed torus can be localized in a neighbourhood of the boundary, this lemma is an obvious corollary from the transformation of the Dehn-Thurston parameters under the action of an S-move on the once-punctured torus. □

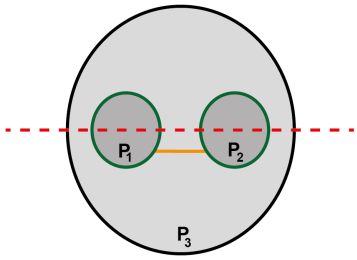


Figure 10.4: Two different cutting curves on one-holed torus

Nonetheless, we do not need the full formulas to prove this corollary. Indeed, we only need to know the expression for p'. We will now give a different proof that we can easily apply in different cases. We describe a pants decomposition of the one-holed torus as in figure 10.4.

Proof. On the pair of pants defined by cutting along the curve defining the coordinate p, there are two possible configurations of multicurves possible.

Case 1. If we let P_1 be the boundary of the one-holed torus and identify P_2 and P_3 in figure 10.4, we define a pants decomposition of the one-holed torus. For $p_0 > 2p$, we find the multicurve $(\frac{p_0}{2} - p)l_{11} + pl_{12} + pl_{13}$. See figure 10.1. Each copy of l_{11} intersects the orange curve once. The curves l_{12} and l_{13} do not intersect this curve. Additionally, if we insert an annular neighbourhood around the cutting curve with a number of twists, we find a total of $|q|$ more intersection points. We therefore find $p' = \frac{p_0}{2} - p + |q|$.

Case 2. On the other hand, for $p_0 \leq 2p$, we find the multicurve $\frac{p_0}{2}l_{12} + \frac{p_0}{2}l_{13} + (p - \frac{p_0}{2})l_{23}$. None of the curves l_{12}, l_{13} or l_{23} intersect the new cutting curve. All of the intersections are therefore contained in an annular neighbourhood in which the twisting defining q takes place. We therefore find $p' = |q|$.

□

Example: Connected components of multicurves on the once-punctured torus We may conclude the example of the S-move on the once-punctured torus by calculating the number of connected components of multicurves on the once-punctured torus not intersecting the puncture. Such multicurves consist of a number of parallel copies of a simple closed curve γ.

Let us pick any cutting curve to define a pants decomposition of the once-punctured torus. Since we set $p_0 = 0$, we may classify the multicurves by two parameters (p, q) corresponding to the number of intersections p between the multicurve and the cutting curve, and the amount of twisting q along the cutting curve.

We first note that Dehn twisting does not change the number of parallel copies of the multicurve (p, q). Therefore, by Dehn twisting the multicurve, we may bring it to a multicurve with parameters (p, q_1) with the same number of connected components and $0 < q_1 < p$.

After applying an S-move, we find a new set of coordinates $(p_1', q') = (q_1, -p)$. Once again, we may Dehn twist to bring the curve $(q_1, -p)$ to the form (q_1, p_1) where $0 < p_1 < q_1$.

If we continue this procedure of S-moves and Dehn twists, we can bring the curve back to the form $(0, Q)$ for some $Q \geq 0$. The algorithm we have defined, is the same as the Euclidean algorithm determining the greatest common divisor $\gcd(p, q)$ of the integers p and q. We therefore identify $Q = \gcd(p, q)$ and conclude that the curve with Dehn-Thurston parameters (p, q) has a total of $\gcd(p, q)$ connected components.

We may note that any simple closed geodesic intersects the multicurve $(0, Q)$ a multiple of Q times. Conversely, we can always construct a simple closed geodesic intersecting the multicurve $(0, Q)$ a total of Q times. We must therefore set $Q = 0 \mod 2$ according to theorem 8.6.

This is in agreement with 9.3 stating that we may only graft a simple closed geodesic along angles $2\pi\mathbb{Z}$ if we want the monodromy of the oper defined from the grafting to remain valued in $\mathrm{PSL}(2, \mathbb{R})$.

Example: Four-punctured sphere Finally, to continue our analysis for the four-punctured sphere, it will be useful to have the formula of the A-move at hand. We will only consider multicurves not starting or ending at the punctures. Such multicurves may be parameterized by two integers (p, q) where p is the number of intersections with a cutting curve and q the amount of twisting.

After the A-move, new parameters (p', q') are given in a similar fashion as for the once-punctured torus

$$(p', q') = (2|q|, -\text{sgn}(q)p/2)$$

The validity of this formula can easily be shown by an explicit calculation. For example, see [42].

Together with the braiding along the cutting curve

$$(p', q') = (p, q + \frac{p}{2})$$

which squares to a Dehn twist, these transformations generate all of $\mathrm{SL}(2, \mathbb{Z})$.

11 Reducibility of monodromy to $\mathrm{PSL}(2,\mathbb{R})$

11.1 Reducing to a finite number of calculations

We will describe how to produce restrictions on the allowed Dehn-Thurston coordinates through theorem 8.6. We have introduced the intersection number $i : \mathcal{C}(S) \times \mathcal{C}(S) \to \mathbb{Z}_{\geq 0}$ by

$$i(\lambda_1, \lambda_2) = \min_{C_1, C_2} \#(C_1 \cap C_2)$$

where C_1 and C_2 are representatives of λ_1 and λ_2 respectively. The notation $C_1 \cap C_2$ should be understood as the set of points in the intersection $\mathrm{im}(C_1) \cap \mathrm{im}(C_2)$ of the images of the multicurves C_1 and C_2. We minimize this number over all representatives C_1 and C_2 of the homotopy classes λ_1 and λ_2. By assuming C_1 and C_2 intersect transversally, we see that $i(\lambda_1, \lambda_2)$ must be finite. Proposition 1.7 in [91] shows how to algorithmically pick two representatives C_1 and C_2 minimizing the number $\#(C_1 \cap C_2)$.

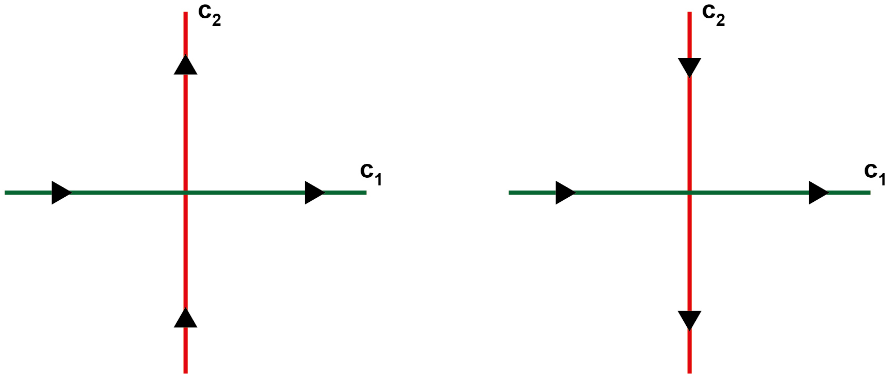
Algebraic intersection number We define a new space $\mathcal{C}_{(\mathrm{or.})}(S)$ to denote the space of elements in $\mathcal{C}(S)$ together with an orientation. We may extend the notion of the intersection number to $\mathcal{C}_{(\mathrm{or.})}(S)$. On the other hand, we may also define the algebraic intersection $\hat{i} : \mathcal{C}_{(\mathrm{or.})}(S) \times \mathcal{C}_{(\mathrm{or.})}(S) \to \mathbb{Z}$ by

$$\hat{i}(\lambda_1, \lambda_2) = \sum_{x \in C_1 \cap C_2} \mathrm{sgn}(x)$$

for C_1 and C_2 representatives of λ_1 and λ_2. We define $\mathrm{sgn}(x)$ by the right-hand rule. See figure 11.1.

A priori, one might expect the number produced by $\hat{i}$ to depend on the representatives C_1 and C_2. By proposition 1.7 in [91], two curves minimize i if and only if they do not have an embedded bigon as in figure 11.2. However, when calculating the number of intersections with sign, we see that a bigon does not change the algebraic intersection number $\hat{i}$.

In general, the numbers produced by i and $\hat{i}$ are different from each other. For example, consider a bigon as in figure 11.2. If we assume this bigon surrounds a



(a) Intersection with sign $+1$ (b) Intersection with sign -1

Figure 11.1: Algebraic intersection numbers of different configurations

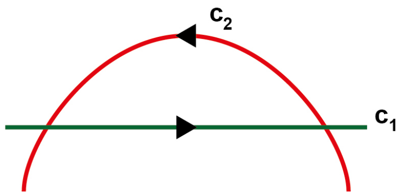


Figure 11.2: Bigon with trivial algebraic intersection

boundary component, we cannot in general reduce the number of intersections. Nonetheless, the algebraic intersection form does not pick up on this boundary component as it only counts signed intersections. The number of intersections counted by i must therefore be larger than the number of intersections counted with sign by $\hat{i}$.

For our purposes, it suffices to note that

$$i(\lambda_1, \lambda_2) = \hat{i}(\lambda_1, \lambda_2) \mod 2$$

as is easily seen from the definitions.

The algebraic intersection number $\hat{i}$ turns out to be easier to work with than i, because of the following well-known result

Proposition 11.1. *The algebraic intersection number $\hat{i}(-,-)$ which counts the number of intersections of two multicurves with orientation, only depends on the homology classes of the curves in $\mathcal{C}_{(\mathrm{or.})}(S)$.*

Proof. Let us consider two oriented multicurves λ_1 and λ_2 in $\mathcal{C}_{(\mathrm{or.})}(S)$. We will check that the map $\hat{\imath}(\lambda_1, -)$ defines a group homomorphism, i.e. $\hat{\imath}(\lambda_1, \mu_1+\mu_2) = \hat{\imath}(\lambda_1, \mu_1) + \hat{\imath}(\lambda_1, \mu_2)$ for $\mu_1 + \mu_2$ the formal sum of the multicurves μ_1 and μ_2.

Let C_1 be a representative of λ_1 and D_1 and D_2 representatives of μ_1 and μ_2 respectively. Then

$$\begin{aligned}\hat{\imath}(\lambda_1, \mu_1 + \mu_2) &= \sum_{x\in C_1\cap(D_1\cup D_2)} \operatorname{sgn}(x) \\ &= \sum_{x\in(C_1\cap D_1)\cup(C_1\cap D_2)} \operatorname{sgn}(x) \\ &= \sum_{x\in C_1\cap D_1} \operatorname{sgn}(x) + \sum_{x\in C_1\cap D_2} \operatorname{sgn}(x) \\ &= \hat{\imath}(\lambda, \mu_1) + \hat{\imath}(\lambda, \mu_2)\end{aligned}$$

Since the fundamental group of a Riemann surface is free of torsion, i.e. has no elements of finite order, this implies $\hat{\imath}(\lambda_1, -)$ factors through to the homology $\mathrm{H}_1(S, \mathbb{Z})$. Applying the same argument to λ_2 shows that $\hat{\imath}(\lambda_1, \lambda_2)$ only depends on the homology classes $[\lambda_1]$ and $[\lambda_2]$ and defines a bilinear map $\hat{\imath} : \mathrm{H}_1(S, \mathbb{Z}) \times \mathrm{H}_1(S, \mathbb{Z}) \to \mathbb{Z}$. □

Restrictions in terms of homology basis If S is closed, the homology $\mathrm{H}_1(S, \mathbb{Z})$ has a basis $\{[a_1], ..., [a_g], [b_1], ..., [b_g]\}$. If S has n boundary components, we can find a basis in terms of cycles $\{[a_1], ..., [a_g], [b_1], ..., [b_g], [B_1], ..., [B_{n-1}]\}$. These cycles come from simple closed geodesic curves which can be used as generators of the presentation

$$\pi_1(S) = \langle a_1, b_1, ..., a_g, b_g, B_1, ..., B_n | \prod_{k=1}^{g} a_k b_k a_k^{-1} b_k^{-1} = B_1 ... B_n\rangle$$

Note that for $n > 0$, $\pi_1(S) \simeq F_{2g+n-1}$ where F_{2g+n-1} is the free group in $2g + n - 1$ generators. We can remove B_n from the presentation by using the relation $\prod_{k=1}^{g} a_k b_k a_k^{-1} b_k^{-1} = B_1 ... B_n$. In homology this leads to $\sum_{k=1}^{n} [B_k] = 0$. For S closed, we find no such relation on the generators.

We can always choose the generators such that

$$\hat{\imath}([a_k], [b_l]) = \delta_{kl}$$

holds and $\hat{\imath}([a_k], [a_l]) = \hat{\imath}([b_k], [b_l]) = 0$. Furthermore, since the curves $\{a_1, ..., a_g, b_1, ..., b_g\}$ do not end on the boundary components, we must find $\hat{\imath}([a_k], [B_l] = \hat{\imath}([b_k], [B_l]) = 0$ by the bigon relation. We can extend these relations to all elements of $\mathrm{H}_1(S, \mathbb{Z})$

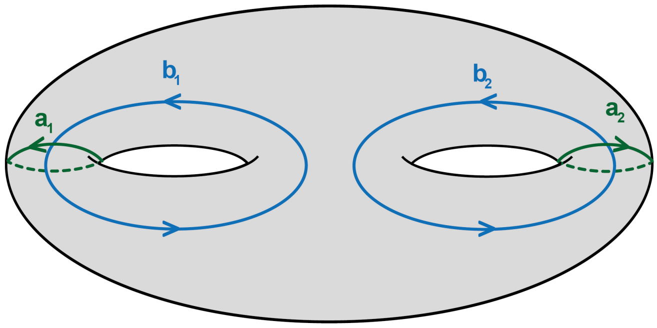


Figure 11.3: Choice of cycles on genus two surface forming a basis of $\mathrm{H}_1(S, \mathbb{Z})$

by linearity. For $g = 2$ and $n = 0$, these generators can be chosen as in figure 11.3.

We can reduce the statement in theorem 8.6 to a finite number of calculations by noting that

Proposition 11.2. *A multicurve μ intersects any other simple closed geodesic an even number of times if and only if it intersects each element in*

$$\{a_1, ..., a_g, b_1, ..., b_g, B_1, ..., B_{n-1}\}$$

an even number of times.

Proof. By applying 11.1, we can reduce this to a question in homology classes. Let μ be a multicurve intersecting elements in

$$\{a_1, ..., a_g, b_1, ..., b_g, B_1, ..., B_{n-1}\}$$

an even number of times. We expand as a homology class

$$[\mu] = \sum_{k=1}^{g} \left(\mu_a^{(k)} [a_k] + \mu_b^{(k)} [b_k] \right) + \sum_{l=1}^{n} \mu_B^{(l)} [B_l]$$

We find

$$\begin{aligned} \hat{i}([\mu], [a_k]) &= -\mu_b^{(k)} \\ \hat{i}([\mu], [b_k]) &= \mu_a^{(k)} \end{aligned}$$

implying

$$\begin{aligned}\mu_a^{(k)} &= 0 \mod 2\\ \mu_b^{(k)} &= 0 \mod 2\end{aligned}$$

Since any simple closed geodesic γ projects to an element $[\gamma]$ in $\mathrm{H}_1(S,\mathbb{Z})$, it can also be written as a sum of elements

$$[\gamma] = \sum_{k=1}^{g} \left(\gamma_a^{(k)}[a_k] + \gamma_b^{(k)}[b_k]\right) + \sum_{l=1}^{n} \gamma_B^{(l)}[B_l]$$

We may now calculate

$$\hat{\imath}([\mu],[\gamma]) = \sum_{k=1}^{g} \left(\mu_a^{(k)}\gamma_b^{(k)} - \mu_b^{(k)}\gamma_a^{(k)}\right)$$

Since both $\mu_a^{(k)} = \mu_b^{(k)} = 0 \mod 2$, we find the result

$$\hat{\imath}([\mu],[\gamma]) = 0 \mod 2$$

for any simple closed geodesic γ. Note that we did not need to assume γ is a simple closed geodesic. All of the above holds just as well for multicurves γ. □

Remark 11.3. For a closed surface S, there exists a reformulation of the above statements through the Poincaré duality. This duality states that $\hat{\imath}(-,-)$ defines an isomorphism $\mathrm{H}_1(S,\mathbb{Z}) \simeq \mathrm{H}^1(S,\mathbb{Z})$. If μ is an element such that $\hat{\imath}([\mu],-) = 0 \mod 2$, the Poincaré duality implies $[\mu] = 0 \in \mathrm{H}_1(S,\mathbb{Z}_2)$. This is not true for surfaces with boundary for which the Poincaré duality does not hold in this way. See [89] for a statement of this kind.

11.2 Restrictions on the Dehn-Thurston coordinates

Invariance under mapping class group action The mapping class group has a natural action on each multicurve $\mu \subset S$ by restricting the action of an element in $m \in \mathrm{MCG}(S)$ to the multicurve μ. This sends $\mu \mapsto m(\mu)$.

If a multicurve μ intersects any other element in $\mathcal{C}(S)$ an even number of times, acting by m does not change this fact.

Remark 11.4. This can be made more precise by considering the linearization of the action of the mapping class group. It turns out that the action of $\mathrm{MCG}(S)$ on $\mathcal{C}_{(\mathrm{or.})}(S)$ descends to an action on $\mathrm{H}_1(S,\mathbb{Z})$ preserving $\hat{\imath}$. The mapping class group acts by the group $\mathrm{Sp}(2g,\mathbb{Z}) \subset \mathrm{GL}(2g,\mathbb{Z})$ on $\{[a_1],...[a_g],[b_1],...,[b_g]\}$ and trivially on $\{[B_1],...,[B_{n-1}]\}$. If $[\mu]$ can be expanded in terms of $\mu_a^{(k)},\mu_b^{(k)} \in 2\mathbb{Z}$, it must remain true that $m(\mu)_a^{(k)}, m(\mu)_b^{(k)} \in 2\mathbb{Z}$. See for example chapter 6 in [91].

In other words, if $\mathcal{B}(S) \subset \mathcal{C}(S)$ denotes the elements $\mu \in \mathcal{C}(S)$ which intersect every other multicurve an even number of times, $m(\mu) \in \mathcal{B}(S)$ for every $m \in \mathrm{MCG}(S)$.

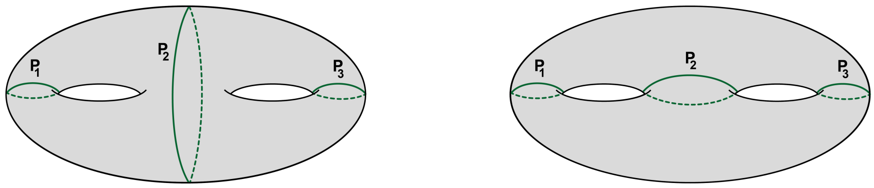
Homeomorphism classes of pants decompositions Given a pants decomposition and Dehn-Thurston parameters $(\mathbf{p},\mathbf{q})$ with respect to this pants decomposition, we want to find the subset of parameters corresponding to the subset $\mathcal{B}(S)$. Since this subset is invariant under the mapping class group, the restrictions on the Dehn-Thurston parameters must also be invariant. As such, the restrictions will only depend on the homeomorphism classes of the pants decompositions.

For each surface S, there exists a finite number of such homeomorphism classes as shown in theorem 16.5. We may calculate the restrictions for each pants decomposition by picking a single pants decomposition and repeatedly acting by A- and S-moves. This defines an algorithm which terminates after a finite number of steps.

Let us finish this section by applying the above discussion to two simple examples. We will discuss the more complicated example of the genus two surface in the next section.

Example: Once-punctured torus Let S be the once-punctured torus. There exists only a single pants decomposition for the once-punctured torus up to homeomorphism. We have already calculated the space $\mathcal{B}(S)$ of multicurves not intersecting the puncture before and found $(p,q) \in (2\mathbb{Z})^2$ as the parameters defining the subset $\mathcal{B}(S) \subset \mathcal{C}(S)$. We can use the above machinery to reprove this result, shining some light on the algorithm we use.

The homology $\mathrm{H}_1(S,\mathbb{Z}) \simeq \mathbb{Z}^2$ is generated by two curves a and b defining cycles $[a]$ and $[b]$ such that $\hat{\imath}([a],[b]) = 1$. See figure 10.4. If we pick the pants decomposition to be defined by a cutting curve homotopic to a, we find that p describes the number of intersections between a given multicurve $\mu \in \mathcal{C}(S)$ and a. Therefore, we must set $p \in 2\mathbb{Z}$.



(a) Separating curve P_2 (b) Non-separating curve P_2

Figure 11.4: Two distinct pants decompositions on S up to homeomorphism

If we apply an S-move, we can make b the cutting curve of the pants decomposition. In the new parametrization (p', q'), the number of intersections of $\mu \in \mathcal{C}(S)$ with b is given by p'. Hence, $p' \in 2\mathbb{Z}$.

The S-move expresses the parameters (p', q') in terms of (p, q) by

$$(p', q') = (|q|, -\mathrm{sgn}(q)p)$$

proving $q \in 2\mathbb{Z}$. This result may be compared to corollary 10.5.

Example: Sphere with boundary components If we consider S to be a sphere with n boundary components, the generators of the fundamental group are given by the set curves $\{B_1, ..., B_{n-1}\}$. Therefore, we will find no additional restrictions on the allowed multicurves and can identify $\mathcal{B}(S) = \mathcal{C}(S)$. In this sense, the four-punctured sphere we discussed before is too simple to find restrictions on the Dehn-Thurston parameters through theorem 8.6.

11.3 Restrictions for the closed surface of genus two

Let us set S to be the closed surface of genus two for the rest of this section. It is known that up to homeomorphism, S has two pants decompositions. See theorem 16.5 for a proof. For both we may introduce Dehn-Thurston parameters $(p_1, p_2, p_3, q_1, q_2, q_3)$. We have depicted these two pants decompositions in figure 11.4. These pants decompositions distinguish themselves by whether the curve P_2 is separating, i.e. cuts S into two one-holed tori, or not.

Multicurves on S along which we may graft and keep the monodromy valued in $\mathrm{PSL}(2, \mathbb{R})$, may be quite complicated. See figure 11.5 for two examples of such multicurves on S. Both of these multicurves intersect the curves in the set

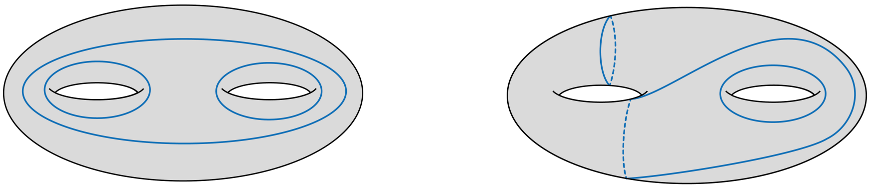


Figure 11.5: Two examples of multicurves on the genus two surface

$\{a_1, b_1, a_2, b_2\}$ an even number of times and therefore by 11.2 intersect all other curves an even number of times.

Let us first consider the pants decomposition in figure 11.4a.

The curves P_1 and P_3 correspond to the curves a_1 and a_2 in our chosen basis from figure 11.3. Therefore, we have to set $p_1, p_3 \in 2\mathbb{Z}$ to ensure the multicurve passes a_1 and a_2 an even number of times. Note that two copies of the curve P_1 and one copy of P_2 bound a pair of paints. Therefore we also know that any multicurve satisfies $p_2 \in 2\mathbb{Z}$.

We now wish to classify the restrictions on the Dehn-Thurston parameters coming from the fact that the multicurve must also intersect b_1 and b_2 an even number of times. To find the number of intersections of the multicurve with b_1 in terms of Dehn-Thurston parameters, we may restrict to the one-holed torus containing a_1 and b_1 defined by cutting along P_2. We may restrict our discussion to this one-holed torus with Dehn-Thurston parameters (p_1, p_2, q_1, q_2). The parameters p_2 and q_2 correspond to the number of intersections with the boundary component and the amount of twisting of the boundary.

If we act by the S-move, we find $p_2' = p_2$. By 10.5, the S-move sets

$$p_1' = \begin{cases} p_2/2 - p_1 + |q_1| & \text{if } p_2 > 2p_1 \\ |q_1| & \text{if } p_2 \leq 2p_1 \end{cases}$$

The number p_1' counts the number of intersections between the multicurve and b_1. We must therefore set $p_1' \in 2\mathbb{Z}$. This implies $p_2/2 + q_1 \in 2\mathbb{Z}$ if $p_2 > 2p_1$ and $q_1 \in 2\mathbb{Z}$ if $p_2 \leq 2p_1$. Therefore,

$$q_1 \in \begin{cases} 2\mathbb{Z} & \text{if } p_2 \leq 2p_1 \\ 2\mathbb{Z} + \frac{p_2}{2} & \text{if } p_2 > 2p_1 \end{cases}$$

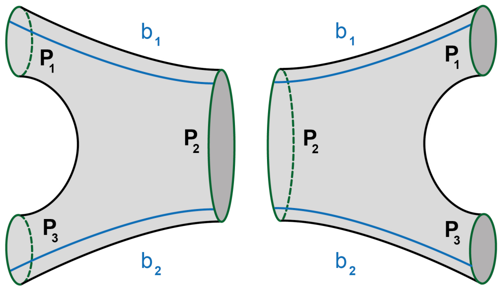


Figure 11.6: Pants decomposition including cycles in the homology basis $\mathrm{H}_1(S, \mathbb{Z})$

Applying the same argument to the one-holed torus containing a_2 and b_2, we find

$$q_3 \in \begin{cases} 2\mathbb{Z} & \text{if } p_2 \leq 2p_3 \\ 2\mathbb{Z} + \frac{p_2}{2} & \text{if } p_2 > 2p_3 \end{cases}$$

Let us now consider the other pair of pants decomposition figure 11.4b.

As before, we introduce Dehn-Thurston coordinates $(p_1, p_2, p_3, q_1, q_2, q_3)$ classifying non-self-intersecting multicurves with respect to the pants decomposition $\{P_1, P_2, P_3\}$. The curves P_1 and P_3 coincide with a_1 and a_2 again, implying that $p_1, p_3 \in 2\mathbb{Z}$. Since the curves $\{P_1, P_2, P_3\}$ define the cutting curves of a pair of pants, we immediately find $p_1 + p_2 + p_3 \in 2\mathbb{Z}$. This implies $p_2 \in 2\mathbb{Z}$ as well.

We now have to find restrictions on the Dehn-Thurston parameters following from the even number of intersections of the multicurve with the curves b_1 and b_2. Let us cut along the curves P_1 and P_3 to obtain a four-holed sphere. The pair of pants decomposition then takes the form as in figure 11.6, where we included the b_1 and b_2 curves.

We will first calculate the number of intersections between our multicurve and the curve b_1. There are two distinct cases: Either $p_3 > p_1 + p_2$ or $p_3 \leq p_1 + p_2$.

Consider the first case. Comparing to the elementary curves in figure 10.1, we find a total of $\frac{1}{2}(p_3 - p_2 - p_1)$ curves intersecting b_1 on the left pair of pants and $\frac{1}{2}(p_3 - p_2 - p_1)$ on the right. In total, there are $p_3 - p_2 - p_1 \in 2\mathbb{Z}$ intersections with the b_1 curve.

In the second case, there are no intersections between the b_1 curve and the multicurve.

Gluing the pairs of pants together and twisting introduces the q_2 coordinate. This leads to a total of $|q_2|$ additional intersections between the multicurve and b_1. Twisting along P_1 introduces an additional $|q_1|$ intersections.

Adding all of this together, the number of intersections between the multicurve and b_1 is given by $p_3 - p_2 - p_1 + |q_1| + |q_2|$. Therefore, an even number of intersections between the multicurves and b_1, implies $q_1 + q_2 \in 2\mathbb{Z}$.

Similarly, we can show that an even number of intersections with b_2 implies $q_2 + q_3 \in 2\mathbb{Z}$. We conclude $q_1, q_3 \in q_2 + 2\mathbb{Z}$.

Putting everything together, we may state the following theorem

Theorem 11.5. *If S is a closed surface of genus two, up to homeomorphism there are two distinct pants decompositions as in figure 11.4. Consider Dehn-Thurston parameters $(p_1, p_2, p_3, q_1, q_2, q_3)$ for these pants decompositions. The elements in $\mathcal{B}(S)$ are in one-to-one correspondence to the sets of parameters satisfying the following constraints:*

- *Consider the pants decomposition as in figure 11.4a. Then $(p_1, p_2, p_3) \in (2\mathbb{Z})^3$ and*
$$q_1 \in \begin{cases} 2\mathbb{Z} & \text{if } p_2 \leq 2p_1 \\ 2\mathbb{Z} + \frac{p_2}{2} & \text{if } p_2 > 2p_1 \end{cases}$$
and
$$q_3 \in \begin{cases} 2\mathbb{Z} & \text{if } p_2 \leq 2p_3 \\ 2\mathbb{Z} + \frac{p_2}{2} & \text{if } p_2 > 2p_3 \end{cases}$$
- *Consider the pants decomposition as in figure 11.4b. Then $(p_1, p_2, p_3) \in (2\mathbb{Z})^3$ and $q_1, q_3 \in q_2 + 2\mathbb{Z}$*

Remark 11.6. The multicurve on the left of figure 11.5 has Dehn-Thurston parameters $(2, 2, 2, 0, 0, 0)$ for both pants decompositions. On the other hand, the multicurve on the right of figure 11.5 is specified by the parameters $(0, 2, 2, 1, 0, 0)$ and $(0, 2, 2, 1, 1, 1)$ with respect to the pants decompositions in figure 11.4a and figure 11.4b respectively. Both of these multicurves satisfy the restrictions from theorem 11.5, proving that grafting along them indeed leads to a projective structure with monodromy in $\mathrm{PSL}(2, \mathbb{R})$.

Extension to general surfaces The pants decomposition in figure 11.4a can easily be generalized for a closed surface S of genus $g > 2$. With respect to this pants decomposition $\{P_1, ..., P_{3g-3}\}$, we introduce Dehn-Thurston parameters $(p_1, ..., p_{3g-3}, q_1, ..., q_{3g-3})$. Let us denote by $\{P_1, ..., P_g\}$ the curves such that cutting along them leads to a sphere with $2g$ boundary components and by $P_{g+1}, ..., P_{2g}$ the curves that bound $P_1, ..., P_g$ in a one-holed torus respectively. Since we only need to calculate the restrictions coming from the curves $\{a_1, ..., a_g, b_1, ..., b_g\}$ as we did in the case of $g = 2$, we immediately find the restrictions

$$(p_1, ..., p_{3g-3}) \in (2\mathbb{Z})^{3g-3}$$

and

$$q_k \in \begin{cases} 2\mathbb{Z} & \text{if } p_{k+g} \leq 2p_k \\ 2\mathbb{Z} + \frac{p_{k+g}}{2} & \text{if } p_{k+g} > 2p_k \end{cases}$$

for $k = 1, ..., g$.

Any set of parameters satisfying these restrictions, defines an element in $\mathcal{B}(S)$.

Since the number of pants decompositions of a (not necessarily closed) surface is finite, we may construct the restrictions with respect to any other pants decompositions by repeatedly acting with the A- and S-moves.

We can introduce n boundary components by adding the parameters

$$(p_{3g-3+1}, ..., p_{3g-3+n}, q_{3g-3+1}, ..., q_{3g-3+n})$$

to the Dehn-Thurston parameters for a closed surface of genus g. If we assume the multicurves do not start or and at boundary components, adding these parameters does not affect the result. Therefore, the restrictions we found are also valid if we introduce n boundary components in S and we may compute the restrictions with respect to any pants decomposition once again by applying a finite number of A- and S-moves.

Part IV

Quantization from the Yang-Yang functional

In this part our goal is to relate the Dehn-Thurston parameters $(\mathbf{p}, \mathbf{q})$, interpreted as quantum numbers for the classification of single-valued solutions $\Psi_{(\mathbf{p},\mathbf{q})}$, to another set of parameters $(\mathbf{n}, \mathbf{m})$ called the Bethe quantum numbers with $\mathbf{n} = (n_1, ..., n_{3g-3+n})$ and $\mathbf{m} = (m_1, ..., m_{3g-3+n})$. This will allow us to express the quantization conditions in terms of a Yang-Yang function $\mathcal{Y}(\mathbf{a}, \mathbf{t})$, resembling the way it was originally introduced by Yang and Yang in [17]. This function depends on the complex moduli $\mathbf{t}$ and parameters $\mathbf{a} = (a_1, ..., a_{3g-3+n})$ parametrizing, up to conjugation, the monodromies associated to the cutting curves of a pair of pants. The function $\mathcal{Y}(\mathbf{a}, \mathbf{t})$ explicitly depends on a choice of pants decomposition of our surface S.

The parameters $\mathbf{a}$ are defined through the complexification of a set of real coordinates $(\mathbf{l}, \mathbf{k})$ on $\mathcal{T}(S)$ known as Fenchel-Nielsen coordinates. On the character variety $C_{\mathrm{PSL}(2,\mathbb{C})}(S)$ which contains $\mathcal{T}(S)$, we can introduce a set of coordinates known as the trace coordinates. It turns out that the Fenchel-Nielsen coordinates depend in a real analytical way on the trace coordinates restricted to the Teichmüller locus $\mathcal{T}(S)$. The discussion on trace coordinates and their relation to Fenchel-Nielsen coordinates mostly follows [86].

The monodromy of opers in a fibre $\mathcal{P}(X)$ of the bundle $\mathcal{P}(S) \to \mathcal{T}(S)$ depends on the accessory parameters $\mathbf{E}$. Both spaces $\mathcal{P}(S)$ and $C_{\mathrm{PSL}(2,\mathbb{C})}(S)$ admit a natural complex structure. It was originally shown in [96] that the monodromy map $\mathcal{P}(S) \to C_{\mathrm{PSL}(2,\mathbb{C})}(S)$ is locally homeomorphic and later in [97, 98] that this map is in fact locally biholomorphic. We may therefore describe the complexified Fenchel-Nielsen coordinates over a fixed Riemann surface (at least locally) by analytically continuing with respect to the accessory parameters $\mathbf{E}$. This was also noted in [20].

On top of being locally biholomorphic, the map $\mathcal{P}(S) \to C_{\mathrm{PSL}(2,\mathbb{C})}(S)$ relates two different symplectic structures with Darboux coordinates $(\mathbf{t}, \mathbf{E})$ and $(\mathbf{l}, \mathbf{k})$ [99, 100, 101]. Therefore, a generating function $\mathcal{W}(\mathbf{a}, \mathbf{t})$ must exist for these different sets of Darboux coordinates. The function $\mathcal{W}(\mathbf{a}, \mathbf{t})$ can be understood geometrically as the generating function for the Lagrangian variety of opers. It had been observed before in [18] using the AGT-duality and in parallel in [19] that the Yang-Yang functional $\mathcal{Y}(\mathbf{a}, \mathbf{t})$ must be closely related to the generating function $\mathcal{W}(\mathbf{a}, \mathbf{t})$.

The precise relation between the generating function $\mathcal{W}(\mathbf{a}, \mathbf{t})$ and the quantization condition of single-valuedness of the solution $\Psi_{\mathbf{E}}$ was clarified in [20]. Indeed, this shows that we may also realize the generating function as a Yang-Yang function $\mathcal{Y}(\mathbf{a}, \mathbf{t})$, but for a different quantization condition than originally considered in [14] and used in [18].

We will consider the example of the four-punctured sphere in which we can derive some results directly through the comparison with the results from [50] and clarify the meaning of the integers $(\mathbf{n}, \mathbf{m})$ in this context.

Motivated by the example of the four-punctured sphere, we conjecture the following

Conjecture. *Let a possibly punctured Riemann surface $X \in \mathcal{T}(S)$ be given. There exists a path $\eta_{(\mathbf{p},\mathbf{q})} : [0,1] \to \mathcal{P}(X)$ for each set of parameters $(\mathbf{p}, \mathbf{q})$ such that*

$$\eta_{(\mathbf{p},\mathbf{q})}(0) = \mathbf{E}_{(\mathbf{0},\mathbf{0})} \qquad\qquad \eta_{(\mathbf{p},\mathbf{q})}(1) = \mathbf{E}_{(\mathbf{p},\mathbf{q})}$$

and analytical continuation of $(\mathbf{l}, \mathbf{k})$ along these paths, implies

$$\begin{aligned} \mathfrak{Re}(a_r(\eta_{(\mathbf{p},\mathbf{q})}(1))) &= \pi p_r \\ \mathfrak{Re}\left(\frac{\partial \mathcal{Y}(\mathbf{a}(\eta_{(\mathbf{p},\mathbf{q})}(1)), \mathbf{t})}{\partial a_r}\right) &= \pi q_r \end{aligned}$$

Afterwards we return to the physical side of the story. We use the Dehn-Thurston parametrization of the single-valued eigenfunctions $\Psi_{(\mathbf{p},\mathbf{q})}$ to give evidence for a direct relation between single-valued eigenfunctions $\Psi_{(\mathbf{p},\mathbf{q})}$ and loop operators $\mathcal{L}_{(\mathbf{p},\mathbf{q})}$. The restrictions obtained through theorem 8.6 have an interesting counterpart in physics relating to the mutual locality of the loop operators.

The above conjecture identifies the Dehn-Thurston parameters of the single-valued eigenfunctions $\Psi_{(\mathbf{p},\mathbf{q})}$ with the Bethe quantum numbers $(\mathbf{n}, \mathbf{m})$. We will also discuss the interpretation of this result from a physical point of view.

Finally, we discuss some possible future directions and interesting topics that could be studied from the perspective we have set up in this thesis.

12 Geometry of the character variety

12.1 Character variety and trace coordinates

For the topics discussed in this section, we will mainly follow the exposition in [86].

There exists a natural set of coordinates on the character variety $C_{\mathrm{SL}(2,\mathbb{C})}(S)$ defined through the traces $\mathrm{tr}(\rho(\gamma))$ of the representations $\rho \in C_{\mathrm{SL}(2,\mathbb{C})}(S)$ acting on an element $\gamma \in \pi_1(S)$. Using the isomorphism $\mathrm{SL}(2,\mathbb{C})/\mathbb{Z}_2 \simeq \mathrm{PSL}(2,\mathbb{C})$, we may relate the character varieties $C_{\mathrm{SL}(2,\mathbb{C})}(S)$ and $C_{\mathrm{PSL}(2,\mathbb{C})}(S)$ to each other.

Although a priori it might seem we need an infinite number of traces to describe the character variety, it should be noted that for $X, Y \in \mathrm{SL}(2,\mathbb{C})$ the following trace relation holds

$$\mathrm{tr}(XY) + \mathrm{tr}(XY^{-1}) = \mathrm{tr}(X)\mathrm{tr}(Y)$$

By using this relation, we may reduce the trace of any word with multiple occurrences of some elements to a polynomial in traces of words with only a single occurrence of these elements.

Character variety of pair of pants Let S be a pair of pants. The fundamental group of the pair of pants has a presentation of the form $\langle \gamma_1, \gamma_2, \gamma_3 | \gamma_1\gamma_2\gamma_3 = 1\rangle$, which is isomorphic to the free group in two generators. Using the trace relation, we may describe $C_{\mathrm{SL}(2,\mathbb{C})}(S) \simeq \mathbb{C}[L_1, L_2, L_3]$ where $L_k = \mathrm{tr}(\rho(\gamma_k))$, $k = 1, 2, 3$. By requiring the monodromies to be non-elliptic, we set $|L_k| \geq 2$.

To describe the real components of $C_{\mathrm{PSL}(2,\mathbb{C})}(S)$, we may pick without loss of generality a representation such that $L_1 \geq 2$ and $L_2 \geq 2$. We now either find $L_3 \geq 2$ or $L_3 \leq -2$. Only in the latter case do we find Fuchsian representations of the pair of pants.

Character variety of four-holed sphere Let S be the four-holed sphere. The fundamental group of the four-holed sphere has a presentation of the form $\langle \gamma_1, \gamma_2, \gamma_3, \gamma_4 | \gamma_1\gamma_2\gamma_3\gamma_4 = 1\rangle$. This group is isomorphic to the free group in three generators. We find more relations than the trace relation alone. Let us set

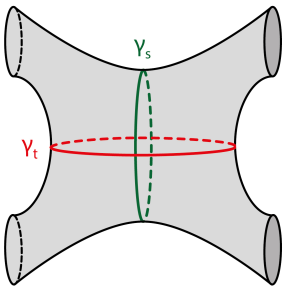


Figure 12.1: Curves γ_s and γ_t on a four-holed sphere

$L_k = \text{tr}(\rho(\gamma_k))$, $k = 1, 2, 3, 4$ and $\gamma_s = \gamma_1\gamma_2$, $\gamma_t = \gamma_1\gamma_3$, $\gamma_u = \gamma_2\gamma_3$ such that

$$\begin{aligned} L_s &= \text{tr}(\rho(\gamma_s)) \\ L_t &= \text{tr}(\rho(\gamma_t)) \\ L_u &= \text{tr}(\rho(\gamma_u)) \end{aligned}$$

See figure 12.1 for the configurations of the curves γ_s and γ_t on the four-holed sphere.

We find a relation of the form

$$P(L_s, L_t, L_u) = 0$$

where

$$\begin{aligned} P(L_s, L_t, L_u) = L_sL_tL_u + L_s^2 + L_t^2 + L_u^2 - (L_1L_2 + L_3L_4)L_s - (L_1L_3 + L_2L_4)L_t \\ - (L_1L_4 + L_2L_3)L_u + L_1L_2L_3L_4 + L_1^2 + L_2^2 + L_3^2 + L_4^2 \end{aligned} \quad (12.1)$$

This gives the character variety of the four-holed sphere the algebraic structure

$$C_{\text{SL}(2,\mathbb{C})}(S) \simeq \mathbb{C}[L_1, L_2, L_3, L_4, L_s, L_t, L_u]/P$$

Character variety of one-holed torus Let S be the one-holed torus. The fundamental group of the one-holed torus has a presentation of the form $\langle \gamma_a, \gamma_b, \gamma_K | \gamma_K = \gamma_a\gamma_b\gamma_a^{-1}\gamma_b^{-1} \rangle$. The curves γ_a and γ_b go around the two cycles of the torus while γ_K surrounds the boundary.

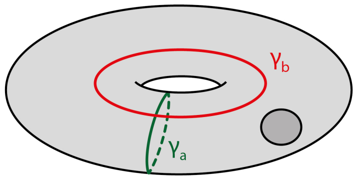


Figure 12.2: Curves γ_a and γ_b on a one-holed torus

We set

$$\begin{aligned} L_x &= \operatorname{tr}(\rho(\gamma_a)) \\ L_y &= \operatorname{tr}(\rho(\gamma_b)) \\ L_z &= \operatorname{tr}(\rho(\gamma_a\gamma_b)) \\ L_K &= \operatorname{tr}(\rho(\gamma_K)) \end{aligned}$$

See figure 12.2 for the configuration of curves γ_a and γ_b on the one-holed torus.

As in the case of the four-holed sphere, we find a polynomial equation relating the different trace coordinates. For the one-holed torus, this equation takes the form

$$P(L_x, L_y, L_z) = L_x^2 + L_y^2 + L_z^2 - L_x L_y L_z - 2 - L_K \qquad (12.2)$$

This gives $C_{\mathrm{SL}(2,\mathbb{C})}(S)$ the algebraic structure

$$C_{\mathrm{SL}(2,\mathbb{C})}(S) \simeq \mathbb{C}[L_x, L_y, L_z, L_K]/P$$

12.2 Fenchel-Nielsen coordinates

Introducing Fenchel-Nielsen coordinates Given a surface S, the Teichmüller space is embedded in $C_{\mathrm{PSL}(2,\mathbb{R})}(S)$ as the connected component containing the Fuchsian representations. There exists a set of coordinates on Teichmüller space called the Fenchel-Nielsen coordinates. For a full discussion we refer to section 3.2 in [90].

To define the Fenchel-Nielsen coordinates, we need a pants decomposition of S. A surface of genus g and with n boundary components has a pants decomposition into $2g-2+n$ pairs of pants along a total of $3g-3+n$ cutting curves.

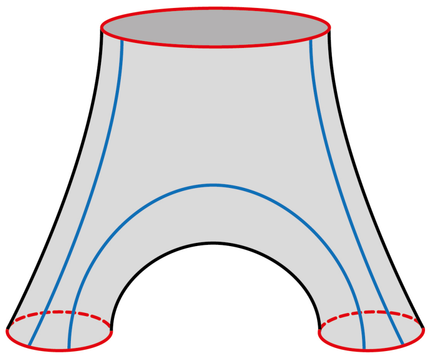


Figure 12.3: Pair of pants with perpendicular curves between the boundaries

We define one part of the coordinates as the hyperbolic lengths of the geodesics associated to the boundaries of the pairs of pants. If the traces of the boundary monodromies are given by L_r for $r = 1, ..., 3g - 3 + n$, the hyperbolic lengths are defined as

$$L_r = 2\cosh(l_r/2)$$

See for example subsection 7.1.

Since $\dim_{\mathbb{R}}(\mathcal{T}(S)) = 6g - 6 + 2n$, we need to add another $3g - 3 + n$ real coordinates to complete the set of Fenchel-Nielsen coordinates.

For a pair of pants, fixing the lengths of boundary geodesics fixes the monodromy representation up to conjugation uniquely. Between any two non-intersecting geodesics, we can always find a unique geodesic intersecting both geodesics perpendicularly. We define the distance between the non-intersecting geodesics as the length of this unique geodesic between the intersection points. Fixing the boundary lengths therefore fixes all of the geodesics as in figure 12.3.

By cutting open the pair of pants along the geodesics, we end up with two hexagons. We have drawn one copy in figure 12.4.

We may summarize these statements as follows

Proposition 12.1. *Let the geodesic lengths* $(l_\alpha, l_\beta, l_\gamma) \in \mathbb{R}^3_{\geq 0}$ *of the boundary geodesics* α*,* β *and* γ *of a pair of pants be given. We can construct a hexagonal*

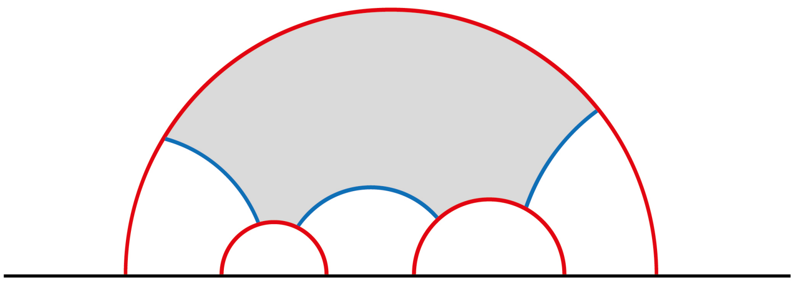


Figure 12.4: Copy of a hexagon obtained by cutting a pair of pants into simply-connected domains

tiling of $\mathbb{H}$ *by cutting along the perpendicular geodesics as above. Once we fix the boundary lengths, the hexagon is specified uniquely.*

For a proof, see theorem 3.4 in [90].

Remark 12.2. If the geodesic length goes to zero, the monodromy around the boundary is parabolic and we replace the boundary component by a puncture. When the geodesic length is strictly positive, the boundary components develop to geodesics that do not intersect each other.

The perpendicular geodesics give a natural way to glue two pairs of pants together, namely in such a way that the perpendicular geodesics fit together precisely to form a geodesic on the four-holed sphere. Let us start from a cutting curve γ with geodesic length l_γ. We may always bring the developing map to a form where analytical continuation along the cutting curve acts by $A(\gamma \cdot z) = e^{l_\gamma} A(z)$ for z a coordinate of X. This is the standard form we introduced when discussing the grafting operation.

We cut open the surface along the geodesic, multiply the coordinate on one side of the geodesic by e^{k_γ} and reglue. The coordinate k_γ defined in this way, is called the twisting coordinate because one may imagine this surgery as cutting open the surface, rotating one boundary component by an angle of $2\pi k_\gamma / l_\gamma$ and regluing. By attaching twist coordinates k_r for $r = 1, ..., 3g-3+n$ to each of the $3g-3+n$ cutting curves, we find coordinates $(\mathbf{l}, \mathbf{k})$ on $\mathcal{T}(S)$ with $\mathbf{l} = (l_1, ..., l_{3g-3+n})$ and $\mathbf{k} = (k_1, ..., k_{3g-3+n})$ realizing Teichmüller space as the domain

$$\mathcal{T}(S) \simeq (\mathbb{R}_{>0})^{3g-3+n} \times \mathbb{R}^{3g-3+n}$$

This set of coordinates is known as the set of Fenchel-Nielsen coordinates.

Furthermore, these coordinates turn out to form a set of Darboux coordinates for the Weil-Petersson symplectic structure on $\mathcal{T}(S)$ as was shown in [102, 103, 104]. In particular, since the Weil-Petersson symplectic structure is defined without any reference to a pants decomposition, we find that the equality

$$\omega_{WP} = \sum_{r=1}^{3g-3+n} dl_r \wedge dk_r$$

is actually independent of the choice of pants decomposition.

12.3 Analytically continuing Fenchel-Nielsen coordinates

Relating trace coordinates and Fenchel-Nielsen coordinates By the definition of the Fenchel-Nielsen coordinates and the fact that the Teichmüller space $\mathcal{T}(S)$ defines a connected component of $C_{\mathrm{PSL}(2,\mathbb{R})}(S)$, we expect a relation between trace coordinates and the Fenchel-Nielsen coordinates. For simplicity we will only describe this relation explicitly for the one-holed torus. This exposition follows section 4.5 from [86]. formulas relating trace coordinates and Fenchel-Nielsen coordinates can also be found in [105, 20].

Let us define a pants decomposition by cutting along the curve γ_a. See figure 12.2 for the configuration of curves. For the one-holed torus, we find one length coordinate l and one twist coordinate k.

By conjugation, we may always assume

$$\rho(\gamma_a) = \begin{pmatrix} e^{l/2} & 0 \\ 0 & e^{-l/2} \end{pmatrix}$$

is diagonal. The length coordinate l is the length of the geodesic γ_a.

Now

$$L_x = 2\cosh(l/2) \tag{12.3}$$

defines the trace coordinate L_x in terms of the hyperbolic length l.

The geodesic γ_a maps to a geodesic in $\mathbb{H}$ running between the fixed points $\{0, \infty\}$ in the upper-half plane. If we normalize the twist coordinate $k = 0$ such that the geodesic of $\rho(\gamma_b)$ is orthogonal to $\rho(\gamma_a)$, we must make sure γ_b lifts to a geodesic in $\mathbb{H}$ with fixed points $\{-s, s\}$ for some $s \in \mathbb{R}_{>0}$. By scaling, we may always assume $s = 1$. This implies we must set

$$\rho(\gamma_b) = \begin{pmatrix} \cosh(\mu/2) & \sinh(\mu/2) \\ \sinh(\mu/2) & \cosh(\mu/2) \end{pmatrix}$$

Twisting defines a new representation $\rho_{k\gamma_a}$ such that $\rho_{k\gamma_a}(\gamma_b)$ is related to $\rho(\gamma_b)$ by

$$\rho_{k\gamma_a}(\gamma_b) = \rho(\gamma_b)D_k$$

where D_k is the dilatation operator representing $z \mapsto e^k z$ in $\mathrm{SL}(2,\mathbb{C})$ as defined in subsection 9.2.

In terms of the parameters μ and k we find

$$\begin{aligned} \mathrm{tr}(\rho_{k\gamma_a}(\gamma_b)) &= L_y \\ &= 2\cosh(\mu/2)\cosh(k/2) \end{aligned}$$

Similarly,

$$L_z = 2\cosh(\mu/2)\cosh((l+k)/2)$$

The relation $P(L_x, L_y, L_z) = 0$ with $P(L_x, L_y, L_z)$ defined by equation (12.2) implies we can express $\cosh(\mu/2)$ in terms of the boundary trace L_K and the hyperbolic length l. After substituting $\cosh(\mu/2)$, we end up with the equations

$$L_y\sqrt{L_x^2 - 4} = 2\sqrt{L_x^2 - L_K + 2}\cosh(k/2) \tag{12.4}$$

$$L_z\sqrt{L_x^2 - 4} = 2\sqrt{L_x^2 - L_K + 2}\cosh((l+k)/2) \tag{12.5}$$

We assume $L_x > 0$ and $L_y > 0$ for these equations to hold. Changing the sign of these quantities simply amounts to changing the above equations accordingly.

We can compute the Fenchel-Nielsen coordinates for the four-holed sphere in a similar way. If we consider the curve γ_s to be the cutting curve for our pants decomposition, we find the following set of relations

$$L_s = 2\cosh(l/2) \tag{12.6}$$

$$\begin{aligned} L_t(L_s^2 - 4) &= 2(L_2L_3 + L_1L_4) + L_s(L_1L_3 + L_2L_4) \\ &\quad +2\cosh(k)\sqrt{c_{12}(L_s)c_{34}(L_s)} \end{aligned} \tag{12.7}$$

$$\begin{aligned} L_u(L_s^2 - 4) &= L_s(L_2L_3 + L_1L_4) + 2(L_1L_3 + L_2L_4) \\ &\quad +2\cosh(l/2 + k)\sqrt{c_{12}(L_s)c_{34}(L_s)} \end{aligned} \tag{12.8}$$

where $c_{ij}(L_s) = L_i^2 + L_j^2 + L_s^2 + L_iL_jL_s - 4$. For these equations to hold, we assume $L_s > 0$, $L_t > 0$ and $L_u > 0$.

Fenchel-Nielsen coordinates as functions of the accessory parameters We may invert equation (12.3) and equation (12.4) to express the Fenchel-Nielsen coordinates (l, k) as a function of the trace coordinates by

$$\begin{aligned} l &= 2 \operatorname{arccosh}(L_x/2) \\ k &= 2 \operatorname{arccosh}\left(\frac{L_y}{2}\frac{\sqrt{L_x^2-4}}{\sqrt{L_x^2-L_K+2}}\right) \end{aligned}$$

The Fenchel-Nielsen coordinates are analytical functions of L_x everywhere outside of the points where $|\cosh(l/2)| = 1$ or $|\cosh(k/2)| = 1$.

Similarly, for the four-holed sphere, the non-analytical behaviour appears for $|\cosh(l/2)| = 1$ and $|\cosh(k)| = 1$.

Therefore, we can extend the definition of the Fenchel-Nielsen coordinates analytically as a function of the trace coordinates away from these loci. We call these coordinates the complexified Fenchel-Nielsen coordinates.

As we stated in the introduction to Part IV, the monodromy map $\mathcal{P}(S) \to C_{\mathrm{PSL}(2,\mathbb{C})}(S)$ has been shown to be locally homeomorphic in [96] and later to be locally biholomorphic in [97, 98]. This implies that we may describe the trace coordinates themselves as analytical functions of the accesory parameters $\mathbf{E}$ and the complex moduli $\mathbf{t}$. The space of opers $\mathcal{P}(X)$ over a fixed point $X \in \mathcal{T}(S)$ is defined by fixing $\mathbf{t}$, so that the complexified Fenchel-Nielsen coordinates depend analytically on the accessory parameters $\mathbf{E}$ away from the non-analytical loci described above.

Since the functions $(\mathbf{l}(\mathbf{E}), \mathbf{k}(\mathbf{E}))$ are not analytical everywhere, different paths between the same points in the space of accessory parameters may lead to different values of the complexified Fenchel-Nielsen coordinates. If we wish to make sense of the complexified Fenchel-Nielsen coordinates, we must therefore pick a path to analytically continue along.

13 Quantization and Fenchel-Nielsen coordinates

13.1 Real representations in terms of Fenchel-Nielsen coordinates

Let a monodromy representation $\rho \in C_{\mathrm{SL}(2,\mathbb{C})}(S)$ of an oper be given. This representation defines a representation in $C_{\mathrm{PSL}(2,\mathbb{C})}(S)$ which is real if and only if for each simple closed curve $\gamma \in \pi_1(S)$, the image $\rho(\gamma)$ maps to an element which is completely real or completely imaginary. The real and imaginary matrices project to elements in $\mathrm{PGL}(2,\mathbb{R})$ with determinant $+1$ and -1 respectively.

We may describe this reality property more concretely in terms of Fenchel-Nielsen coordinates. Let us therefore introduce a pair of pants decomposition on the surface X with cutting curves $\{P_1, ..., P_{3g-3+n}\}$ and associate Fenchel-Nielsen coordinates (l_r, k_r) to each cutting curve P_r. Using equation (12.3) to equation (12.8), depending on whether P_r bounds a four-holed sphere or one-holed torus, we may express the trace coordinates away from the Teichmüller locus in terms of complexified Fenchel-Nielsen coordinates.

The reality of the monodromy ρ implies the following statement from [20].

Theorem 13.1. *For each single-valued eigenfunction $\Psi_{(\mathbf{p},\mathbf{q})}$ to the quantized Hitchin Hamiltonians, there exist tuples of integers $\mathbf{n} = (n_1, ..., n_{3g-3+n})$ and $\mathbf{m} = (m_1, ..., m_{3g-3+n})$ such that analytically continuing the Fenchel-Nielsen coordinates away from the uniformizing projective structure to the oper with accessory parameters $\mathbf{E}_{(\mathbf{p},\mathbf{q})}$ implies*

$$
\begin{aligned}
\mathfrak{Im}(\mathbf{l}(\mathbf{E}_{(\mathbf{p},\mathbf{q})})) &= \pi\mathbf{n} \\
\mathfrak{Im}(\mathbf{k}(\mathbf{E}_{(\mathbf{p},\mathbf{q})})) &= \pi\mathbf{m}
\end{aligned}
$$

Remark 13.2. The integers $(\mathbf{n}, \mathbf{m})$ could in principle depend on the precise analytical continuation of the complexified Fenchel-Nielsen coordinates in $\mathbf{E}$.

Remark 13.3. We have chosen to normalize the integers $(\mathbf{n}, \mathbf{m})$ differently from [20]. In this normalization, it is clear that some integers can never appear as coming from single-valued eigenfunctions. For example, the integers n_r must always be even in our case.

It is important to mention that it could be possible for $\cosh(l_r/2)$ to become real even if $\mathfrak{Im}(l_r)$ is not an integer multiple of π. This could lead to an off-set of the integers $\mathbf{n}$. Although we would expect the same classification result to hold with this off-set, this situation does not appear for the case of the four-punctured sphere with real punctures.

13.2 Quantization conditions and the generating function

From the discussion in the introduction to Part IV, we know that the monodromy map $\mathcal{P}(S) \to C_{\mathrm{SL}(2,\mathbb{C})}(S)$ is locally biholomorphic and relates the holomorphic symplectic structures ω_{WP} and Ω, defined in subsections 7.4 and 12.2, by

$$-4\pi i\Omega = \omega_{WP}$$

See [99, 100, 101]. The fact that $(\mathbf{t}, \mathbf{E})$ are Darboux coordinates for Ω and $(\mathbf{l}, \mathbf{k})$ for ω_{WP}, implies the existence of a generating function $\mathcal{W}(\mathbf{l}, \mathbf{t})$ satisfying

$$\begin{aligned} -4\pi i \frac{\partial}{\partial l_r}\mathcal{W}(\mathbf{l}, \mathbf{t}) &= k_r(\mathbf{l}, \mathbf{t}) \\ \frac{\partial}{\partial t_r}\mathcal{W}(\mathbf{l}, \mathbf{t}) &= E_r(\mathbf{l}, \mathbf{t}) \end{aligned}$$

Remark 13.4. The space of opers $\mathcal{P}(X)$ over a fixed point $X \in \mathcal{T}(S)$ is defined by fixing $\mathbf{t}$. Varying only $\mathbf{l}$ and defining $\mathbf{k}$ by the above equation, the generating function $\mathcal{W}(\mathbf{l}, \mathbf{t})$ generates the Lagrangian subspace $\mathcal{P}(X)$ in $\mathcal{P}(S)$.

The quantization conditions may be rewritten as

$$\begin{aligned} \mathfrak{Re}(a_r) &= \pi n_r \\ \mathfrak{Re}\left(4\pi i \frac{\partial}{\partial a_r}\mathcal{W}(\mathbf{a}, \mathbf{t})\right) &= \pi m_r \end{aligned}$$

where we have set $\mathbf{a} = i\mathbf{l}$. If we also set $\mathcal{Y} = 4\pi i\mathcal{W}$, we can express these relations as

$$\mathfrak{Re}(a_r) = \pi n_r \tag{13.1}$$

$$\mathfrak{Re}\left(\frac{\partial \mathcal{Y}(\mathbf{a}, \mathbf{t})}{\partial a_r}\right) = \pi m_r \tag{13.2}$$

The parameters $\mathbf{a}$ behave as the auxiliary parameters of the Yang-Yang function $\mathcal{Y}$.

This relation between the Yang-Yang function for the quantization conditions of the $\mathrm{SL}(2, \mathbb{C})$-Gaudin model and the generating function of opers has been observed before in [19]. In parallel, it was proposed in [18] that the quantization conditions for the Hitchin system are naturally formulated in terms of $\mathcal{W}$.

If we denote by $\mathbf{a}_{(\mathbf{n},\mathbf{m})}$ for fixed $\mathbf{t}$ a solution to equation (13.1) and equation (13.2), we can express

$$E_{r,(\mathbf{n},\mathbf{m})} = \frac{\partial \mathcal{W}(\mathbf{a}_{(\mathbf{n},\mathbf{m})}, \mathbf{t})}{\partial t_r} \tag{13.3}$$

Remark 13.5. There is no guarantee that the solutions $\mathbf{a}_{(\mathbf{n},\mathbf{m})}$ are unique. In principle, it could be possible that multiple solutions $\Psi_{(\mathbf{p},\mathbf{q})}$ lead to the same set of quantum numbers $(\mathbf{n}, \mathbf{m})$.

In light of remark 13.5, we will look for a way to define the analytical continuation of $(\mathbf{l}, \mathbf{k})$ such that the solutions are unique. In this case, we must find $\mathbf{E}_{(\mathbf{n},\mathbf{m})} = \mathbf{E}_{(\mathbf{p},\mathbf{q})}$ by the definition of $\Psi_{(\mathbf{p},\mathbf{q})}$.

If we find a unique solution $\mathbf{a}_{(\mathbf{n},\mathbf{m})}$ for each $(\mathbf{n}, \mathbf{m})$, we are able to describe all quantization conditions in terms of equation (13.1) and equation (13.2), but it remains a question which sets of integers $(\mathbf{n}, \mathbf{m})$ actually appear as quantum numbers for single-valued eigenfunctions $\Psi_{(\mathbf{p},\mathbf{q})}$.

Nonetheless, we remark that if a one-to-one-correspondence between the Dehn-Thurston parameters $(\mathbf{p}, \mathbf{q})$ and a subset of the Bethe quantum numbers $(\mathbf{n}, \mathbf{m})$ does indeed exist, we find restrictions on the allowed integers $(\mathbf{n}, \mathbf{m})$ through theorem 10.1 and theorem 8.6.

13.3 Relation between grafting and the twist coordinate

We can already give a partial answer to the question of how to relate the parameters $(\mathbf{p}, \mathbf{q})$ and the integers $(\mathbf{n}, \mathbf{m})$. By comparing the description of grafting to the twisting used to define the coordinates $\mathbf{k}$, we see that grafting can be understood as a twisting along the imaginary axis.

To make this more precise, let us denote by $\mathcal{ML}_H(S)$ the space

$$\mathcal{ML}_H(S) = \{(s, \lambda) \in H \times \mathcal{ML}(S) | (ts, \lambda) \sim (s, t\lambda) \text{ for } t \in \mathbb{R}_{>0}\}$$

for $H = \{z \in \mathbb{C} | \mathfrak{Im}(z) \geq 0\}$ and by $\mathcal{ML}_{\mathbb{R}}(S)$ the space

$$\mathcal{ML}_{\mathbb{R}}(S) = \{(s, \lambda) \in \mathbb{R} \times \mathcal{ML}(S) | (ts, \lambda) \sim (s, t\lambda) \text{ for } t \in \mathbb{R}_{>0}\}$$

We have the inclusions

$$\mathcal{ML}(S) \subset \mathcal{ML}_{\mathbb{R}}(S) \subset \mathcal{ML}_H(S)$$

Recall the notation $\mathrm{Gr}_\lambda(X)$ for the grafted projective structure constructed from the pair $(\lambda, X) \in \mathcal{ML}(S) \times \mathcal{T}(S)$. If we denote by $\mathrm{Tw}_\lambda(X)$ the twisted projective structure, it is constructed from a pair $(\lambda, X) \in \mathcal{ML}_{\mathbb{R}}(S) \times \mathcal{T}(S)$, since we may twist both clockwise and counter clockwise we have to pick a measured lamination λ valued in $\mathcal{ML}_{\mathbb{R}}(S)$ instead of $\mathcal{ML}(S)$. The twisting map Tw_λ gives rise to a map $\mathrm{tw}_\lambda : \mathcal{T}(S) \to \mathcal{T}(S)$ by concatenation with the projection $\mathcal{P}(S) \to \mathcal{T}(S)$.

We may define for $\lambda \in \mathcal{ML}_H(S)$ the map $\mathrm{Eq}_\lambda(X) := \mathrm{Gr}_{\mathfrak{Im}(\lambda)}(\mathrm{tw}_{\mathfrak{Re}(\lambda)}(X))$. Proposition 2.6 from [51] clarifies the interaction between grafting and twisting:

Theorem 13.6. *The map* $\mathrm{Eq}_{t\lambda}(X)$ *varies holomorphically with respect to* $t \in H$.

It is important to note that by holomorphicity, this result is independent of the path we pick in H to arrive at the result. The imaginary component of the twist coordinates k_r may therefore be understood as the grafting angle, while the real part of k_r corresponds to the twisting angle.

Expression in terms of quantization conditions Let us pick a pants decomposition and define Dehn-Thurston parameters $(\mathbf{p}, \mathbf{q})$ and the Fenchel-Nielsen coordinates $(\mathbf{l}, \mathbf{k})$. Let us graft along a multicurve μ with Dehn-Thurston parameters $(\mathbf{0}, \mathbf{q})$. Then theorem 13.6 implies that any path $\eta_{(\mathbf{0},\mathbf{q})} : [0, 1] \to \mathcal{P}(X)$ starting at $\mathbf{E}_{(\mathbf{0},\mathbf{0})}$ and ending at $\mathbf{E}_{(\mathbf{0},\mathbf{q})}$ must satisfy

$$\mathfrak{Im}(\mathbf{k}(\eta_{(\mathbf{0},\mathbf{q})}(1))) = \pi \mathbf{q}$$

More concretely, we may pick the path path $\mathrm{Gr}_{t\pi\mu}(\mathrm{gr}^{-1}_{t\pi\mu}(X))$ for $t \in [0, 1]$ which goes from $\mathbf{E}_{(\mathbf{0},\mathbf{0})}$ to $\mathbf{E}_{(\mathbf{0},\mathbf{q})}$ and satisfies $\mathfrak{Im}(\mathbf{k}(\mathbf{E}_{(\mathbf{0},\mathbf{q})})) = \pi\mathbf{q}$. Additionally, we know that the grafting procedure $\mathrm{Gr}_{t\pi\mu}(X)$ does not affect the hyperbolic lengths of the components of the multi curve μ. This means that changing the underlying complex structure from X to $\mathrm{gr}^{-1}_{t\pi\mu}(X)$ can only change the real part of the hyperbolic lengths $\mathbf{l}$. We may therefore conclude that

$$\mathfrak{Im}(\mathbf{l}(\mathbf{E}_{(\mathbf{0},\mathbf{q})})) = 0$$

along this path and identify the parameter $(\mathbf{0}, \mathbf{q})$ with the integers $(\mathbf{0}, \mathbf{m})$ for this special case.

It should be noted that $\mathrm{Gr}_{t\pi\mu}(\mathrm{gr}^{-1}_{t\pi\mu}(X))$ is only a single path for which $\mathfrak{Im}(\mathbf{l}(\mathbf{E}_{(\mathbf{0},\mathbf{q})})) = 0$. Many other paths $\eta_{(\mathbf{0},\mathbf{q})}$ with this property exist.

14 Revisiting the four-punctured sphere

The four-punctured sphere provides a hands-on example for which we can put the mathematical machinery developed in this thesis in practice. We will prove a special, but non-trivial case of the conjecture formulated before, relating Dehn-Thurston parameters to the set of Bethe quantum numbers.

14.1 Fenchel-Nielsen coordinates and the accessory parameter

Let us consider a monodromy representation $\rho \in C_{\mathrm{PSL}(2,\mathbb{C})}(S)$ for S the four-punctured sphere. If we require the boundary components to be punctures, we have to set $L_k = 2$ for $k = 1, ..., 4$ in equation (12.6) to equation (12.8). This simplifies the equations considerably to

$$\begin{aligned} L_s &= 2\cosh(l/2) \\ L_t(L_s - 2) &= 8 + 2\cosh(k)(L_s + 2) \\ L_u(L_s - 2) &= 8 + 2\cosh(k + l/2)(L_s + 2) \end{aligned}$$

if $L_s \neq -2$.

We may rewrite this relation as

$$\cosh(k) = \frac{L_t}{2}\frac{L_s - 2}{L_s + 2} - \frac{4}{L_s + 2}$$

Let us now specialize to the four-punctured sphere with real punctures. In this case, we may rewrite the above relations in terms of $G_{21}(\lambda)$, $H_{21}(\lambda)$ and $(GH)_{21}(\lambda)$ introduced in section §4 and which are all analytical functions of the accessory parameter λ. Let us for simplicity set $\det(G) = \det(H) = -1$ as is possible by remark 5.6. We may then note

Lemma 14.1. *The Fenchel-Nielsen coordinates (l, k) with respect to a pants decomposition defined by cutting along γ_s, may be expressed as a function of the accessory parameter λ in the form*

$$\begin{aligned} \cosh(l/2) &= -1 + 2\pi^2 (G_{21}(\lambda))^2 \\ \cosh(k) &= 1 - 2\left(\frac{(GH)_{21}(\lambda)}{G_{21}(\lambda)}\right)^2 \end{aligned}$$

Proof. The expression for $\cosh(l/2)$ follows immediately by comparing with equation (12.6) and the formulas for the trace coordinates in section §4. The second equation follows from equation (12.7) and equation (4.1). Indeed,

$$\cosh(k) = -1 + 2\pi^2 (H_{21})^2 - 2\left(\frac{H_{21}}{G_{21}}\right)^2$$

Using equation (4.1), we rewrite

$$(H_{21})^2 = \frac{((GH)_{21})^2 - (G_{21})^2}{1 - \pi^2 (G_{21})^2}$$

A simple calculation shows

$$-1 + 2\pi^2 (H_{21})^2 - 2\left(\frac{H_{21}}{G_{21}}\right)^2 = 1 - 2\left(\frac{(GH)_{21}}{G_{21}}\right)^2$$

proving the result. □

The change of sign with respect to the equations in section §4 comes from the fact that we assumed L_s and L_t to be positive when defining the Fenchel-Nielsen coordinates of the four-holed sphere.

The Fenchel-Nielsen coordinates are analytical away from the loci $|\cosh(l/2)| = 1$ and $|\cosh(k)| = 1$. From the first of these two equations, we extract that $l(\lambda)$ is analytical away from the accessory parameters λ for which $G_{21}(\lambda) \in \{-\pi^{-1}, 0, \pi^{-1}\}$. In combination with equation (4.1), we deduce from the second equation that $k(\lambda)$ is analytical away from the values λ for which $(GH)_{21}(\lambda) = 0$, $G_{21}(\lambda) \in \{-\pi^{-1}, 0, \pi^{-1}\}$ or $H_{21}(\lambda) = 0$.

14.2 Partial identification of Bethe and Dehn-Thurston parameters

In this section, we will construct a path $\eta_{(p,0)} : [0,1] \to \mathcal{P}(X)$ for $X = \mathbb{C}P^1 \backslash \{0, z, 1, \infty\}$ and $0 < z < 1$ satisfying the conditions in the conjecture proposed in the introduction to Part IV.

Non-analytical loci If we assume $\lambda \in \mathbb{R}$, we find from theorem 5.1 a series of parameters $... < \lambda_{-1} < \lambda_0 < \lambda_1 < ...$ for which $G_{21}(\lambda_k) = 0$, $H_{21}(\lambda_k) = 0$ or $(GH)_{21}(\lambda_k) = 0$. Let us consider $k \geq 0$. The other loci where the analyticity of the Fenchel-Nielsen coordinates breaks down, are given by $\lambda = \nu_k^{(j)}$ for $j = 1, ..., N_k$ with $N_k > 0$ satisfying

$$(G_{21}(\nu_k^{(j)}))^2 = \pi^{-2}$$

The integer N_k counts the number of these solutions in the interval $(\lambda_k, \lambda_{k+1})$. Its precise value is unimportant for our calculations as it is only important for us to know that $(G_{21}(\lambda))^2$ is a decreasing respectively increasing function at $\lambda = \nu_0^{(j)}$ for j odd respectively even.

Including these values $\nu_k^{(j)}$, we find the series

$$\lambda_0 < \nu_0^{(1)} < ... < \nu_0^{(N_0)} < \lambda_1 < \nu_1^{(1)} < ... < \nu_1^{(N_1)} < \lambda_2 < ...$$

Since the monodromy of the oper is Fuchsian for $\lambda = \lambda_0$, all curves have positive geodesic length for this value of the accessory parameter. In particular, we must find $(G_{21}(\lambda_0))^2 > \pi^{-2}$. For $\lambda = \lambda_1$ we have $G_{21}(\lambda_1) = 0$. From this we conclude that the integer N_0 must be odd. For the same reason we find N_k odd for all $k \geq 0$.

Defining the path We now consider a path $\eta_{(p,0)}$ for $p \in 2\mathbb{Z}_{\geq 0}$ starting at λ_0 and ending at $\lambda_{p/2}$, behaving as follows:

1. The path $\eta_{(p,0)}$ starts at λ_0 and increases its imaginary part until we reach $\lambda_0 + i\delta$ for some infinitesimally small $\delta \ll 1$.
2. We continue on a half circle parametrized by $\lambda = \lambda_0 - \delta e^{i\theta}$ for $\theta \in [\pi/2, \pi]$.
3. We analytically continue along the positive direction on the real axis until $|\lambda - \tilde{\lambda}| = \delta$ for

$$\tilde{\lambda} \in \{\nu_0^{(1)}, ..., \nu_0^{(N_0)}, \lambda_1, \nu_1^{(1)}, ..., \nu_1^{(N_1)}, \lambda_2, \nu_2^{(1)}, ..., \nu_2^{(N_2)}, ...\}$$

   and move along a half-circle parameterized by $\lambda = \tilde{\lambda} - \delta e^{i\theta}$ for $\theta \in [0, \pi]$.
4. We continue in this way until we reach the point $\lambda_{p/2} + i\delta$. We then decrease the imaginary component until we reach $\lambda_{p/2}$.

We may draw this path as in figure 14.1.

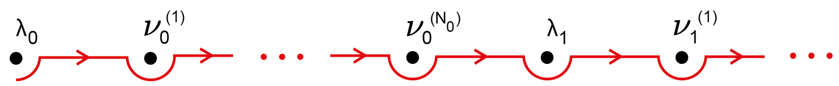


Figure 14.1: Image of the path $\eta_{(p,0)}$

Since G and H are both real analytical and map real values to real values, we know that the Fenchel-Nielsen coordinates cannot move away from the real axis as long as we are on a segment of the path $\eta_{(p,0)}$ which lies on the real axis. Therefore, we may deduce the entire analytical behaviour of the Fenchel-Nielsen coordinates l and k as a function of the accessory parameter by studying its analytical behaviour on the half-circles at distance δ away from non-analytical points

$$\{\nu_0^{(1)}, ..., \nu_0^{(N_0)}, \lambda_1, \nu_1^{(1)}, ..., \nu_1^{(N_1)}, \lambda_2, \nu_2^{(1)}, ..., \nu_2^{(N_2)}, ...\}$$

We will only describe the behaviour of $\eta_{(p,0)}$ for the values $\lambda_0 \leq \lambda \leq \lambda_2$ to avoid unnecessary complications.

Calculating values of the transfer matrix and derivative at non-analytical points Before we describe the analytical continuation of the Fenchel-Nielsen coordinates, we first calculate some of the values of G_{21}, G_{21}', $(GH)_{21}$ and $(GH)_{21}'$ at the non-analytical points. We note several facts

- By definition we find

$$(GH)_{21}(\lambda_0) = (GH)_{21}(\lambda_2) = G_{21}(\lambda_1) = 0$$

- Normalizing $\psi_0^{(1)}(0) = 1$, we must find $G_{21}(\lambda) > 0$ for $\lambda < \lambda_1$. If it were the case that $G_{21}(\lambda) < 0$ for $\lambda < \lambda_1$, we would find a zero of $\psi_0^{(1)}$ in the interval $[0, z]$. However, the first zero only appears for $\lambda > \lambda_1$ by lemma 5.7. Therefore,

$$G_{21}(\nu_0^{(j)}) = \pi^{-1} \qquad\qquad G_{21}(\nu_1^{(j)}) = -\pi^{-1}$$

- Since the value of $(G_{21}(\nu_0^{(1)}))^2$ is decreasing at $\nu_0^{(1)}$ and $G_{21}(\nu_0^{(j)}) = \pi^{-1}$, we must find $G_{21}'(\nu_0^{(1)}) < 0$. If we pass through $G_{21} = \pi^{-1}$ again for $\nu_0^{(2)}$, the value of $(G_{21}(\nu_0^{(2)}))^2$ must be increasing so that $G_{21}'(\nu_0^{(2)}) > 0$. Repeating this argument for $\nu_0^{(j)}$ and $\nu_1^{(j)}$, we find

$$\text{sgn}(G_{21}'(\nu_0^{(j)})) = (-1)^j \qquad\qquad \text{sgn}(G_{21}'(\nu_1^{(j)})) = (-1)^j$$

Value of λ	λ_0	$\nu_0^{(j)}$	λ_1	$\nu_1^{(j)}$	λ_2
Value of G_{21}		π^{-1}	0	$-\pi^{-1}$	
Sign of G_{21}'		$(-1)^j$		$(-1)^j$	
Value of $(GH)_{21}$	0	π^{-1}		π^{-1}	0
Sign of $(GH)_{21}'$		$(-1)^{j+1}$		$(-1)^j$	

Table 14.1: Values and signs of G_{21}, $(GH)_{21}$ and their derivatives

- Since $\psi_0^{(1)}$ cannot have zeroes in the interval $[z, 1]$ at the value $\lambda = \lambda_1$ and the sign of $(GH)_{21}$ is constant on the interval (λ_0, λ_2), we must find $(GH)_{21} > 0$ for all $\lambda \in (\lambda_0, \lambda_2)$. At the values $\nu_0^{(j)}$ and $\nu_1^{(j)}$, equation (4.1) implies
$$(GH)_{21}(\nu_0^{(j)}) = (GH)_{21}(\nu_1^{(j)}) = \pi^{-1}$$

- Finally, we determine the sign of $(GH)_{21}'(\nu_k^{(j)})$. Using equation (4.1), we find
$$\begin{aligned}\pi(GH)_{21}'(\nu_0^{(j)})(GH)_{21}(\nu_0^{(j)}) &= G_{21}'(\nu_0^{(j)})(1 - \pi^2 H_{21}(\nu_0^{(j)})^2)\\ \pi(GH)_{21}'(\nu_1^{(j)})(GH)_{21}(\nu_1^{(j)}) &= -G_{21}'(\nu_1^{(j)})(1 - \pi^2 H_{21}(\nu_1^{(j)})^2)\end{aligned}$$
If $\lambda \geq \lambda_0$, grafting appears along the curve γ_t. This implies that the monodromy along the curve γ_t must stay hyperbolic. This implies $H_{21}(\lambda)^2 \geq \pi^{-2}$ for any $\lambda \geq \lambda_0$. Therefore, we find
$$\mathrm{sgn}((GH)_{21}'(\nu_0^{(j)})) = (-1)^{j+1} \qquad \mathrm{sgn}((GH)_{21}'(\nu_1^{(j)})) = (-1)^j$$

Putting everything together we find table 14.1.

Analytically continuing the length coordinate along $\eta_{(p,0)}$ Let us consider the equation
$$\cosh(l(\lambda)/2) = -1 + 2\pi^2(G_{21}(\lambda))^2$$
and $\lambda = \nu_0^{(j)} - \delta e^{i\theta}$. This sets
$$\cosh(l(\lambda)/2) = 1 - 4\pi\delta e^{i\theta} G_{21}'(\nu_0^{(j)}) + \mathcal{O}(\delta^2)$$

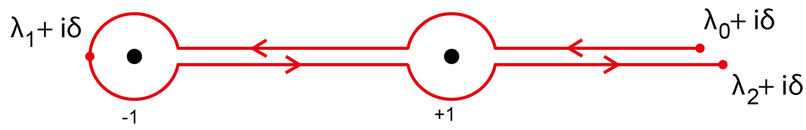


Figure 14.2: Image of $\cosh(l(\lambda)/2) \subset \mathbb{C}^\times$

On the half-circle centered at λ_1 and parameterized by $\lambda = \lambda_1 - \delta e^{i\theta}$, we find

$$\cosh(l(\lambda)/2) = -1 + 2\pi^2\delta^2 e^{2i\theta}(G'_{21}(\lambda_1))^2 + \mathcal{O}(\delta^3)$$

Finally, for the half-circle $\lambda = \nu_1^{(j)} - \delta e^{i\theta}$ centered at $\nu_1^{(j)}$, we find

$$\cosh(l(\lambda)/2) = 1 + 4\pi\delta e^{i\theta} G'_{21}(\nu_1^{(j)}) + \mathcal{O}(\delta^2)$$

Furthermore, the coordinate $l(\lambda)$ is analytic in λ at the points λ_0 and λ_2.

Putting all of this together and combining with the results in table 14.1, we see that the image of $\cosh(l/2)$ along $\eta_{(p,0)}$ between λ_0 and λ_2 looks like figure 14.2.

Furthermore, we can deduce that at $\lambda = \lambda_1 + i\delta$ we must have $\cosh(l(\lambda)/2) < -1$. This implies $\mathfrak{Im}(l(\lambda_1 + i\delta)) = 2\pi$. By following the path, we end up at $\lambda = \lambda_2 + i\delta$ for which $\cosh(l(\lambda)/2) = 4\pi + \mathcal{O}(\delta)$. By continuing the argument for $\lambda > \lambda_2$, we deduce

$$\mathfrak{Im}(l(\lambda_{p/2} + i\delta)) = \pi p + \mathcal{O}(\delta)$$

Closing the path, we find in full generality

$$\mathfrak{Im}(l(\lambda_{p/2})) = \pi p$$

Analytically continuing the twist coordinate along $\eta_{(p,0)}$ Let us now expand

$$\cosh(k(\lambda)) = 1 - 2\left(\frac{(GH)_{21}(\lambda)}{G_{21}(\lambda)}\right)^2$$

near each of the non-analytical points in the interval $[\lambda_0, \lambda_2]$.

On the half-circles at $\lambda = \lambda_0 - \delta e^{i\theta}$ and $\lambda = \lambda_2 - \delta e^{i\theta}$, we find

$$\begin{aligned}
\cosh(k(\lambda)) &= 1 - 2\delta^2 e^{2i\theta}((GH)'_{21}(\lambda_0))^2 (G_{21}(\lambda_0))^{-2} + \mathcal{O}(\delta^3) \\
\cosh(k(\lambda)) &= 1 - 2\delta^2 e^{2i\theta}((GH)'_{21}(\lambda_2))^2 (G_{21}(\lambda_2))^{-2} + \mathcal{O}(\delta^3)
\end{aligned}$$

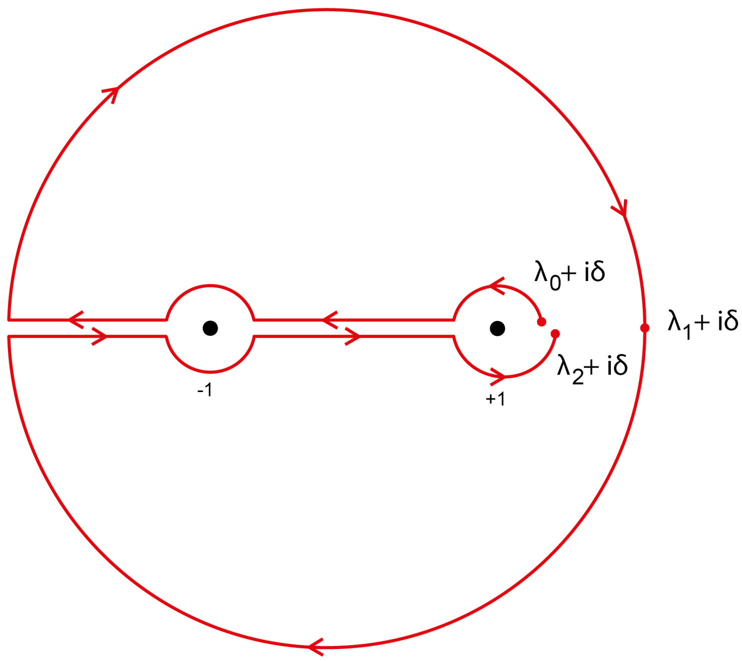


Figure 14.3: Image of $\cosh(k(\lambda)) \subset \mathbb{C}^\times$

respectively. On the half-circle $\lambda = \lambda_1 - \delta e^{i\theta}$, we find

$$\cosh(k(\lambda)) = -2\delta^{-2} e^{-2i\theta}((GH)_{21}(\lambda_1))^2(G'_{21}(\lambda_1))^{-2} + \mathcal{O}(\delta^{-1})$$

Finally, on the half-circles at $\lambda = \nu_0^{(j)} - \delta e^{i\theta}$ and $\lambda = \nu_1^{(j)} - \delta e^{i\theta}$, obtain the expressions

$$\begin{aligned} \cosh(k(\lambda)) &= -1 - 4\pi\delta e^{i\theta}(-(GH)'_{21}(\nu_0^{(j)}) + G'_{21}(\nu_0^{(j)})) + \mathcal{O}(\delta^2) \\ \cosh(k(\lambda)) &= -1 + 4\pi\delta e^{i\theta}((GH)'_{21}(\nu_1^{(j)}) + G'_{21}(\nu_1^{(j)})) + \mathcal{O}(\delta^2) \end{aligned}$$

respectively. By using table 14.1, we determine

$$\begin{aligned} \mathrm{sgn}(-(GH)'_{21}(\nu_0^{(j)}) + G'_{21}(\nu_0^{(j)})) &= (-1)^j \\ \mathrm{sgn}((GH)'_{21}(\nu_1^{(j)}) + G'_{21}(\nu_1^{(j)})) &= (-1)^j \end{aligned}$$

Once again combining these results with table 14.1, we see that the image of $\cosh(k)$ along $\eta_{(p,0)}$ looks like figure 14.3.

We determine that along the path $\eta_{(p,0)}$ defined as before, we must find

$$\mathfrak{Im}(k(\lambda_{p/2})) = 0$$

Relation to the conjecture We conclude by noting that we have found a constructive proof of the following theorem:

Theorem 14.2. *There exists a path $\eta_{(p,0)} : [0,1] \to \mathcal{P}(X)$ for $X = \mathbb{C}P^1 \backslash \{0, z, 1, \infty\}$ such that defining the Fenchel-Nielsen coordinates by analytically continuing along this path sets*

$$\begin{aligned} \mathfrak{Im}(l(\eta_{(p,0)}(1))) &= \pi p \\ \mathfrak{Im}(k(\eta_{(p,0)}(1))) &= 0 \end{aligned}$$

We may write these relations in terms of the Yang-Yang function by

$$\begin{aligned} \mathfrak{Re}(a) &= \pi p \\ \mathfrak{Re}\left(\frac{\partial \mathcal{Y}(a,t)}{\partial a}\right) &= 0 \end{aligned}$$

Since the oper with accessory parameter $\lambda_{p/2}$ has real curves with Dehn-Thurston coordinates $(p, 0)$, we may identify the Dehn-Thurston coordinates of these real curves with the integers $(n, 0)$. This provides a counterpart to the result extracted from [51], which identifies the integers $(0, q)$ with $(0, m)$ along a path $\eta_{(0,q)}$.

14.3 Extending the result to arbitrary integers?

Although we have only proved the conjecture for the parameters $(n, 0)$ and $(0, m)$, it is suggestive to conjecture that real curves with Dehn-Thurston parameters (p, q) can be identified with integers (n, m) by analytically continuing along a path $\eta_{(p,q)}$.

One interesting consequence of this conjecture is that the parameters (p, q) and (n, m) must transform in exactly the same way under changes of pants decompositions. Indeed, we have seen before that under an A-move, we find the relation

$$(p', q') = (2|q|, -\mathrm{sgn}(q)p/2)$$

If we would be able to identify the parameters (p, q) with the integers (n, m), the integers must transform in the same way under a change of pants decomposition.

This identification between parameters would imply that the Fenchel-Nielsen coordinates (which also depend on a choice of pants decomposition) must transform under an A-move in such a way that the integers (n, m) change as

$$(n', m') = (2|m|, -\mathrm{sgn}(m)n/2)$$

Whether this is the case in full generality is an open question, but with some extra work we can indeed confirm from our calculations that this transformation rule holds when either $m = 0$ or $n = 0$.

One way we might prove the conjecture for the four-punctured sphere, is by grafting projective structures that are not uniformizing. In principle this is possible, as long as we graft along geodesics which have hyperbolic monodromy. If we graft the opers coming from Dehn-Thurston parameters $(p, 0)$ along the cutting curves of the pants decomposition, it might be possible to apply the result from [51] to extend the identification of (p, q) with (n, m) to all integers.

Based on these results, we propose the following conjecture for a more general punctured Riemann surface, as we stated in the introduction to Part IV.

Conjecture 14.3. *Let a punctured Riemann surface $X \in \mathcal{T}(S)$ be given. There exists a path $\eta_{(\mathbf{p},\mathbf{q})} : [0, 1] \to \mathcal{P}(X)$ for each set of Dehn-Thurston parameters $(\mathbf{p}, \mathbf{q})$ such that*

$$\eta_{(\mathbf{p},\mathbf{q})}(0) = \mathbf{E}_{(\mathbf{0},\mathbf{0})} \qquad\qquad \eta_{(\mathbf{p},\mathbf{q})}(1) = \mathbf{E}_{(\mathbf{p},\mathbf{q})}$$

and analytical continuation of $(\mathbf{l}, \mathbf{k})$ along these paths, implies

$$\begin{aligned} \mathfrak{Re}(a_r(\eta_{(\mathbf{p},\mathbf{q})}(1))) &= \pi p_r \\ \mathfrak{Re}\left(\frac{\partial \mathcal{Y}(\mathbf{a}(\eta_{(\mathbf{p},\mathbf{q})}(1)), \mathbf{t})}{\partial a_r}\right) &= \pi q_r \end{aligned}$$

Since our interest lies in Dehn-Thurston parameters which define single-valued eigenfunctions $\Psi_{(\mathbf{p},\mathbf{q})}$, one might consider weakening the conjecture by only requiring this identification to hold between integers $(\mathbf{n}, \mathbf{m})$ and parameters $(\mathbf{p}, \mathbf{q})$ for which we can construct such a single-valued eigenfunction $\Psi_{(\mathbf{p},\mathbf{q})}$. Whether this is necessary, is not clear at this point.

15 Concluding remarks and open questions

Correspondence between loop operators and single-valued eigenfunctions? We had shown at the beginning of this thesis in subsection 2.7 that any loop operator in the presence of a co-dimension two surface defect, gives rise to a single-valued eigenfunction. The classification of single-valued eigenfunctions $\Psi_{(\mathbf{p},\mathbf{q})}$ in terms of Dehn-Thurston parameters therefore suggests that we could find a direct relation between the two objects.

It should be noted that the Dehn-Thurston parameters $(\mathbf{p}, \mathbf{q})$ are not simply remnants of the classification of the single-valued eigenfunctions: Applying the separation of variables, we find an expression of the form

$$\Psi_{(\mathbf{p},\mathbf{q})}(\mathbf{x}, \bar{\mathbf{x}}) = \int d\mathbf{u} |\mathcal{K}(\mathbf{x}, \mathbf{u})|^2 \prod_{r=1}^{3g-3+n} \phi_{(\mathbf{p},\mathbf{q})}(u_r, \bar{u}_r)$$

where $\phi_{(\mathbf{p},\mathbf{q})}$ is the single-valued eigenfunction constructed in subsection 8.2 coming from the oper with accesory parameters $\mathbf{E}_{(\mathbf{p},\mathbf{q})}$. We have seen that the vanishing locus $X_{\mathbb{R}} = \{(u, \bar{u}) | \phi_{(\mathbf{p},\mathbf{q})}(u, \bar{u}) = 0\}$ is precisely homeomorphic to the multicurve defined by Dehn-Thurston parameters $(\mathbf{p}, \mathbf{q})$.

From this point of view, it seems natural to expect that the loop operator $\mathcal{L}_{(\mathbf{p},\mathbf{q})}$ with support homeomorphic to $X_{\mathbb{R}}$ on X from the perspective of the six-dimensional $\mathcal{N} = (2, 0)$ theory, maps to the single-valued eigenfunction $\Psi_{(\mathbf{p},\mathbf{q})}$. Nevertheless, a subtle point remains, since not all parameters appear in the classification of the single-valued eigenfunction according to theorem 8.6. Let us clarify this point.

On the side of the gauge theory, the loop operators had been classified in terms of Dehn-Thurston parameters in [42]. However, later work by Aharony, Seiberg and Tachikawa [47] showed that this is not quite correct. The loop operators must be mutually local in any physical theory. Mutual locality holds in theories of class S if the curves labeling the loop operators always have even intersection index. Picking a maximal subset of curves satisfying this constraint amounts to choosing additional discrete data classifying theories of class S as discussed in [106].

From this point of view, we may interpret the meaning of theorem 8.6 in the following way:

The loop operators giving rise to single-valued eigenfunctions $\Psi_{(\mathbf{p},\mathbf{q})}$*, must exist* *in any theory of class S, independent of the choice of discrete data picking a specific one according to [47].*

Bethe quantum numbers and loop operators Let us now come back to the physical interpretation of the Bethe quantum numbers. We had seen before that the Bethe quantum numbers make an appearance when considering the Nekrasov-Shatashvili limit $\epsilon_2 \to 0$ of the expectation value of loop operators on $S^4_{\epsilon_1,\epsilon_2}$. The question we raised was which Bethe quantum numbers appear for which loop operators.

The effect of Wilson loops is easily calculated in the gauge theoretical picture and leads to a shift in the parameters $\mathbf{m}$. However, if we consider 't Hooft loops or other dyonic loop operators, the question of classification becomes much harder, because the difference operators shift the variables $\mathbf{a}$ so that we can no longer read off from the saddlepoint approximation which Bethe quantum numbers appear.

For the four-punctured sphere with real punctures, we have seen by theorem 14.2 that we can make sense of the Bethe quantum numbers $(n, 0)$ as coming from the Dehn-Thurston parameters of the single-valued eigenfunctions $\Psi_{(p,0)}$, where $n = p \in 2\mathbb{Z}$. This identifies the Bethe quantum numbers with the Dehn-Thurston parameters in this special case. Upon combination with the correspondence between single-valued eigenfunctions and loop operators, this leads to the statement that such a set-up would correspond to the insertion of $n/2$ 't Hooft loops.

Our observations for the four-punctured sphere lead us to consider conjecture 14.3, proposing that there exists a definition of the Yang-Yang function $\mathcal{Y}(\mathbf{a}, \mathbf{t})$ realizing the Bethe quantum numbers as the Dehn-Thurston parameters of the single-valued eigenfunctions. We may also relate this to the Dehn-Thurston parameters of the loop operators under the above mentioned correspondence.

Assuming the validity of the conjecture, we may give a physical interpretation of the following form:

In the Nekrasov-Shatashvili limit $\epsilon_2 \to 0$, the expectation value $\langle \mathcal{L}_{(\mathbf{p},\mathbf{q})} \rangle_{S^4_{\epsilon_1,\epsilon_2}}$ is dominated by the values $\mathbf{a} = \mathbf{a}_{(\mathbf{n},\mathbf{m})}$ solving the equations

$$\begin{aligned} \mathfrak{Re}(a_r) &= \pi n_r \\ \mathfrak{Re}\left(\frac{\partial \mathcal{Y}(\mathbf{a},\mathbf{t})}{\partial a_r}\right) &= \pi m_r \end{aligned}$$

which are, for a proper choice of $\mathcal{Y}(\mathbf{a},\mathbf{t})$, precisely the equations defining the Bethe quantum numbers for the single-valued eigenfunctions $\Psi_{(\mathbf{n},\mathbf{m})}$ of the quantum Hitchin Hamiltonians.

Outlook and open questions Although we have given a clear indication of the physical and mathematical relations between the Bethe quantum numbers and the Dehn-Thurston parameters, as well as between loop operators and single-valued eigenfunctions, work still has to be done to make this more rigorous. We could hope for a complete proof of conjecture 14.3, but should keep in mind that a full problem of this form could prove difficult to solve.

Furthermore, there is still work to be done on the classification between loop operators and single-valued eigenfunction to the quantum Hitchin Hamiltonians. It could prove interesting to deepen out this relation more than through a matching of parameters as we have shown in this thesis. The construction of the single-valued eigenfunctions shows in particular that we may use all sets of Dehn-Thurston parameters $(\mathbf{p},\mathbf{q})$ to define functions $\Psi_{(\mathbf{p},\mathbf{q})}$ which can at most have a sign change under analytical continuation. We might expect to extend the correspondence between mutually local loop operators and single-valued eigenfunctions to a correspondence involving multi-valued functions $\Psi_{(\mathbf{p},\mathbf{q})}$ which at most change by a sign and a larger set of loop operators. It should be noted that such multi-valued functions will still define single-valued probability densities when we consider the quantum mechanical problem at hand.

Lastly, we might try to generalize the set-up of the defects. By introducing loop operators on top of co-dimension four surface defects in the ϵ_2-plane, we might give rise to more exotic and singular functions Ψ. If these functions remain single-valued, we may still use theorem 3.5 to prove that they must come from an oper with real monodromy. However, this oper must now have singularities at the positions of the surface operators on the side of the Riemann surface. Such a set-up might prove meaningful in generalizing the notion of real geometric Langlands which first appeared in [20].

Part V

Appendices

16 Mapping class group

16.1 Action of the mapping class group on projective structures

Definition of the mapping class group Recall that Teichmüller space was defined to be the space of pairs $[X, f]$ where $(X, f) \sim (Y, g)$ iff $g \circ f^{-1} : X \to Y$ defines a biholomorphism isotopic to the identity. We can recast this in the conformal world by assuming X and Y are hyperbolic structures and $g \circ f^{-1}$ defines an isometry isotopic to the identity. We can therefore describe Teichmüller space of the surface S as the quotient $\mathcal{T}(S) = \mathrm{Conf}(S)/\mathrm{Diff}_0^+(S)$ of conformal structures on S modulo the action of diffeomorphisms isotopic to the identity. We do not always write the marking f explicitly for notational simplicity.

To obtain the moduli space $\mathcal{M}(S)$ of conformal structures on S, we have to take the quotient by the full diffeomorphism group $\mathrm{Diff}^+(S)$. This leads to the description as a quotient $\mathcal{M}(S) = \mathrm{Conf}(S)/\mathrm{Diff}^+(S)$. We can rewrite this as $\mathcal{M}(S) = \mathcal{T}(S)/\mathrm{MCG}^+(S)$ where

$$\mathrm{MCG}^+(S) := \mathrm{Diff}^+(S)/\mathrm{Diff}_0^+(S)$$

is known as the (oriented) mapping class group of the surface S.

We refer to [91] for a basic reference on mapping class groups and the rest of section §16.

We may equivalently define the mapping class group

$$\mathrm{MCG}^+(S) := \mathrm{Homeo}^+(S)/\mathrm{Homeo}_0^+(S)$$

since every homeomorphism on a compact smooth surface is isotopic to a diffeomorphism and every topological surface has a unique smooth structure making it into a smooth surface. We will switch between these two notions freely. For example, for the definition of Dehn twists as elements in the mapping class group it is more practical to take the definition as a quotient of homeomorphisms

The mapping class group acts on elements $[X, f]$ by $[m] \cdot [X, f] = [X, f \circ m^{-1}]$, where $[m]$ is the equivalence class of a diffeomorphism $m : S \to S$. This definition is well-defined, as one can easily check: If m' is another diffeomorphism such that $[m'] = [m]$, then $[m'] \cdot [X, f] = [X, f \circ m'^{-1}]$. Since $[m'] = [m]$, the diffeomorphism $m^{-1} \circ m'$ is isotopic to the identity and indeed, the function $f \circ (m^{-1} \circ m') \circ f^{-1}$ defines a biholomorphism isotopic to the identity. Therefore, $[X, f \circ m^{-1}] = [X, f \circ m'^{-1}]$.

Action on the space of measured laminations If we represent the mapping class group by the action of diffeomorphisms, we have a natural action on a pair $(X, \lambda) \in \mathcal{T}(S) \times \mathcal{ML}(S)$. We have described above how the mapping class group acts on a marked Riemann surface. On the other hand, if we are given any simple closed geodesic $\gamma \subset X$, the mapping class group acts naturally by a diffeomorphism acting on γ as a subset of X. This sends the geodesic γ to another curve in $\pi_1(S)$ with a unique geodesic representative. By continuity, we may extend this action to all of $\mathcal{ML}(S)$ and define an action of $m \in \mathrm{MCG}^+(S)$

$$m \cdot (X, \lambda) = (m \cdot X, m \cdot \lambda)$$

Since a projective structure is nothing but a choice of charts and transition functions on a marked Riemann surface such that the charts are given by Möbius transformations, the mapping class group also acts naturally on a projective structure. In fact, these two actions coincide and we may describe the action on the space of projective structures by the action on the pairs in $\mathcal{T}(S) \times \mathcal{ML}(S)$.

16.2 Change of coordinate principle

The classification of surfaces allows us to state some non-trivial facts about the the mapping class group. Let us start from the famous classification result

Theorem 16.1. *Let a compact connected surface S possibly with finite number of boundary components and marked points be given. The surface S is defined up to homeomorphism by the surface data (g, b, n) of its genus g, finite number of boundary components b and finite number of marked points n.*

Note that we can always find a homeomorphism of a connected surface switching any two boundary components or any two marked points.

Non-separating curves Let us assume γ is a non-separating curve on S, i.e. γ does not cut S into two connected components. Let a surface S be given specified by the data (g, b, n) of the genus, number of boundary components and number of marked points on S.

Cutting along γ produces a new surface S_γ with two more boundary components and one less handle. It therefore has surface data $(g-1, b+2, n)$. We can reproduce S by gluing together the two newly obtained boundary components in S_γ leading to a continuous gluing map $S_\gamma \to S$. Cutting along any other non-separating curve γ' defines a surface $S_{\gamma'}$ with the same data and another gluing map $S_{\gamma'} \to S$. By the classification of surfaces there must exist a homeomorphism $S_\gamma \to S_{\gamma'}$ which in particular sends the boundary components obtained by cutting along γ to the boundary components obtained by cutting along γ' while fixing the other boundary components.

We conclude by noting that there must exist a homeomorphism of S sending γ to γ' showing that all non-separating curves can be mapped to one another through homeomorphisms of S. More precisely

Proposition 16.2. *Given a compact connected surface S possibly with boundary and marked points. Let consider two non-separating curves α and β on S. We can always find a homeomorphism $\phi : S \to S$ such that $\phi(\alpha) = \beta$.*

Since a pants decomposition of the one-holed torus is defined by a single cutting curve not isotopic to the boundary, this result tells us that up to homeomorphism, there is only one pants decomposition of the one-holed torus.

Separating curves If we consider γ to be separating, the situation complicates. Indeed, if we are given a disjoint union of surface $S_{\gamma,-}$ and $S_{\gamma,+}$, we can only find a homeomorphism to another disjoint union of surfaces $S_{\gamma',-}$ and $S_{\gamma',+}$ if the surface data match up of the different surfaces. Even if the data do match up, it is still possible that we do not find a homeomorphism respecting the choice of boundary components. To illuminate this last statement, let us consider a simple example.

Proposition 16.3. *Let a four-holed sphere be given. A closed curve always separates the four-holed sphere into a pair of three-holed spheres. Up to homeomorphisms fixing the boundary, we have three distinct curves pairing the boundary components in the three-holed spheres. If we do not require the boundaries to be fixed, any closed curve can be mapped to any other closed curve.*

Proof. If a four-holed sphere S with boundary components $B_1, ..., B_4$ is given, any curve γ cuts it into two three-holed spheres. Each of these three-holed spheres has one boundary component coming from the cutting along γ and two boundary components in the set $\{B_1, B_2, B_3, B_4\}$. Since each subsurface is a three-holed sphere, we can always find a homeomorphism mapping the curve γ to any other closed curve on S. However, it is important to note that this homeomorphism might not fix the boundary components.

Requiring that the boundary components are fixed under the homeomorphism, cuts down the image of the curve γ in the space of all curves. The curve γ will separate the boundary components from each other pairwise after cutting along it: Two boundary components will be part of one three-holed sphere and the other two will belong to the other three-holed sphere. If cutting along γ' leads to different pairs of boundary components, it can never be homeomorphic to γ if we require the boundary components to be fixed.

Conversely, if two curves α and β separate the same boundary components from each other, the change of coordinate principle tells us that there must exist a homeomorphism $\phi : S \to S$ such that $\phi(\alpha) = \beta$. This follows from the fact that both curves cut S into two three-holed spheres with the same boundary components. □

This example illuminates the difference between what is known as the mapping class group and the pure mapping class group. The pure mapping class group consists of homeomorphisms keeping the boundary components and marked points fixed, while the full mapping class group allows these to be exchanged.

For completeness we will also consider an example without boundary components. This illuminates the first requirement we stated before: The curves have to separate the surface into subsurfaces with the same surface data.

Proposition 16.4. *Let a surface S of genus g without boundary components nor marked points be given. A separating curve γ will separate S into $S_{\gamma,-}$ and $S_{\gamma,+}$ which we may choose to have surface data $(k, 1, 0)$ and $(g - k, 1, 0)$ respectively for $0 < k \leq g/2$. The integer k is specified uniquely by the curve γ. Conversely, if two curves α and β specify the same integer k, there exists a homeomorphism $\phi : S \to S$ such that $\phi(\alpha) = \beta$.*

Proof. The idea is the same as before. The separating curve γ separates S into surfaces $S_{\gamma,-}$ and $S_{\gamma,+}$ which are up to homeomorphism specified by the data of their genus, number of boundary components and marked points. Since a separating curve introduces a boundary component on both $S_{\gamma,-}$ and $S_{\gamma,+}$ and does not change the genus, we find two triples of data $(k, 1, 0)$ and $(g - k, 1, 0)$ corresponding to $S_{\gamma,-}$ and $S_{\gamma,+}$ respectively, which specify these surfaces up to homeomorphism. □

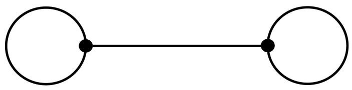
Application to pair of pants decompositions A pair of pants decomposition consists of $3g - 3 + n$ cutting curves and $2g - 2 + n$ pairs of pants, where we set n equal to the sum of the number of boundary components and marked points. Since the pairs of pants are all three-holed spheres, the question of which pants decompositions are homeomorphic to each other, becomes a combinatorial problem relating cutting curves to the boundary components of pairs of pants. We have a finite number of pairs of pants and a finite number of cutting curves which must be the boundary components of the pairs of pants. In particular, this implies that we find a finite number of pants decompositions up to homeomorphism.

It is interesting to note that we can associate a trivalent graph to the combinatorial problem described above. For each pair of pants we introduce a vertex and for each cutting curve an edge. For each boundary curve shared by two pairs of pants, we introduce an edge running between the two corresponding vertices. In particular, this means the graph can have multiple edges between the same vertices.

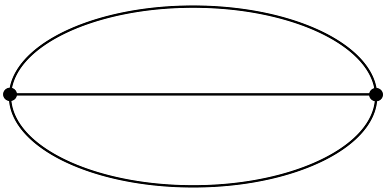
We also allow for the possibility of a pair of pants sharing a boundary curve with itself, meaning that in the graph an edge may start and end at the same vertex. This happens when the pair of pants is introduced by cutting a one-holed torus.

Finally, when we consider surfaces with boundary, we assume that these boundary components define vertices as well. If one of these boundary components also is the boundary component of a given pair of pants, we introduce an edge between the vertex of the correspoding pair of pants and the vertex of the boundary component.

By defining the graph as above, we find a total of $V = 2g - 2 + 2n$ vertices and $E = 3g - 3 + 2n$ edges. The number of inscribed loops or faces F satisfies the equation $F = g + 1$ (counting also the face bounded by the outer edges). The Euler characteristic is found to be $V - E + F = 2$. This implies we can embed this graph on a sphere, i.e. it is a planar graph. This leads us to conclude the following



(a) Graph for decomposition 11.4a



(b) Graph for decomposition 11.4b

Figure 16.1: Two non-isomorphic, trivalent graphs

Theorem 16.5. *There exists a one-to-one correspondence between trivalent, planar graphs (possibly with multiple edges between the same vertices or edges going from a vertex back to itself) up to isomorphism and pair of pants decompositions up to homeomorphism.*

In the case of the genus two surface without boundary, we find the two graphs in figure 16.1.

16.3 Different notions of marking

The Dehn-Nielsen-Baer theorem allows us to relate two different ways of defining a marking on a Riemann surface. Let us work with closed Riemann surfaces, which are specified precisely by their genus.

Let us introduce a different notion of marking on a Riemann surface than before:

Definition 16.6. A marking of a Riemann surface is a preferred set of generators of $\pi_1(X)$. More precisely, if our surface X has genus g, a presentation of $\pi_1(X)$ is of the following form:

$$\pi_1(X) = \langle \alpha_1, ..., \alpha_g, \beta_1, ..., \beta_g | \prod_{j=1}^{g} [\alpha_j, \beta_j] \rangle$$

The generators α_k and β_l for $k, l = 1, ..., g$ define a preferred set of generators which we may denote by $\Sigma = \{\alpha_1, ..., \alpha_g, \beta_1, ..., \beta_g\}$. We say that two markings are equivalent if they differ by an inner automorphism. A diffeomorphism f preserves the marking if we have two pairs (X, Σ) and (Y, Σ') such that $f : X \to Y$ defines a biholomorphism and $f_*(\Sigma) = \Sigma'$.

The Dehn-Nielsen-Baer theorem provides a link between these two ways of defining a marking:

Theorem 16.7. *If we are given a closed surface S of genus $g \geq 1$, we have the following isomorphism as groups*

$$\mathrm{MCG}^{+}(S) \simeq \mathrm{Out}^{+}(\pi_1(S))$$

where $\mathrm{Out}^{+}(\pi_1(S))$ defines the orientation-preserving outer automorphisms of the fundamental group. More generally without reference to the orientation, we find

$$\mathrm{MCG}(S) \simeq \mathrm{Out}(\pi_1(S))$$

From a geometrical perspective, any diffeomorphism of the surface S, defines an automorphism of $\pi_1(S)$. In particular, if the diffeomorphism is not isotopic to the identity, this automorphism cannot be an inner automorphism. The non-trivial statement is that any outer automorphism can be realized as coming from a diffeomorphism of the surface not isotopic to the identity. In this way, we may start with a marking defined by a diffeomorphism $f : S \to X$ and construct a preferred set of generators by pulling back the generators of $\pi_1(X)$ along f. This gives a marking $[X, \Sigma]$. Any other marking $[X, f']$ is obtained from an element in the mapping class group and in fact realizes a change in marking $[X, \Sigma']$. Conversely, any change in marking $[X, \Sigma']$ also realizes an associated diffeomorphism in the mapping class group so that we obtain a new marking $[X, f']$. We may therefore freely exchange the two notions of marking.

16.4 Braid groups as mapping class groups

Some simple example of mapping class groups are given by the mapping class groups of punctured spheres. If we let $X = \mathbb{C}P^1 \backslash \{z_1, ..., z_n\}$ and S the underlying topological surface, it can be shown that the mapping class group has a presentation in terms of braiding elements. The intuition behind this presentation is that the complex structure of X is completely determined by the positions of the punctures. By moving the punctures around each other, we can define a map from the Riemann surface X back to itself which changes the marking. Such movements are generated by the braiding elements.

The mapping class group has a presentation of the following form

$$\mathrm{MCG}(S) \simeq \Big\langle \sigma_1, ..., \sigma_{n-1} \;\Big|\; \begin{array}{rcll} \sigma_i\sigma_j\sigma_i &=& \sigma_j\sigma_i\sigma_j & \text{if } |i-j|=1 \\ \sigma_i\sigma_j &=& \sigma_j\sigma_i & \text{if } |i-j|>1 \\ (\sigma_1...\sigma_{n-1})^n &=& 1 & \\ \sigma_1...\sigma_{n-1} &=& \sigma_1^{-1}...\sigma_{n-1}^{-1} & \end{array} \Big\rangle$$

For example, for $n=3$, we recover

$$\begin{aligned} \mathrm{MCG}(S) &\simeq \Big\langle \sigma_1, \sigma_2 \;\Big|\; \begin{array}{rcl} \sigma_1\sigma_2\sigma_1 &=& \sigma_2\sigma_1\sigma_2 \\ (\sigma_1\sigma_2)^3 &=& 1 \\ \sigma_1\sigma_2 &=& \sigma_1^{-1}\sigma_2^{-1} \end{array} \Big\rangle \\ &\simeq S_3 \end{aligned}$$

where S_3 is the symmetric group on the set of three elements. For a larger amount of punctures, the groups become increasingly more complicated.

Mapping class group of the four-punctured sphere To compare with the results in Part II, we will explicitly describe the mapping class group of the four-punctured sphere. If we set $X = \mathbb{C}P^1 \backslash \{0, z, 1, \infty\}$ and let S denote the underlying topological surface, we will prove that the mapping class group $\mathrm{MCG}(S)$ is a semi-direct product

$$\mathrm{MCG}(S) \simeq \mathrm{PSL}(2, \mathbb{Z}) \rtimes (\mathbb{Z}_2 \times \mathbb{Z}_2)$$

with $\mathrm{PSL}(2, \mathbb{Z})$ generated by braiding elements σ_1 and σ_2 and the group $\mathbb{Z}_2 \times \mathbb{Z}_2$ containing the elements j_1, j_2 and j_3 defined as

$$j_1 = \sigma_3\sigma_1^{-1} \qquad j_2 = (\sigma_2\sigma_1\sigma_3)^2 \qquad j_3 = (\sigma_1\sigma_2\sigma_3)^2$$

We will prove this result in several steps starting from the explicit description of $\mathrm{MCG}(S)$ as the braiding group

$$\mathrm{MCG}(S) \simeq \Big\langle \sigma_1, \sigma_2, \sigma_3 \;\Big|\; \begin{array}{rcl} \sigma_1\sigma_2\sigma_1 &=& \sigma_2\sigma_1\sigma_2 \\ \sigma_3\sigma_2\sigma_3 &=& \sigma_2\sigma_3\sigma_2 \\ \sigma_1\sigma_3 &=& \sigma_3\sigma_1 \\ (\sigma_1\sigma_2\sigma_3)^4 &=& 1 \\ \sigma_1\sigma_2\sigma_3 &=& \sigma_1^{-1}\sigma_2^{-1}\sigma_3^{-1} \end{array} \Big\rangle$$

We first note

Lemma 16.8. *The mapping class group* MCG(S) *can be presented as follows*

$$\mathrm{MCG}(S) \simeq \left\langle \sigma_1, \sigma_2, \sigma_3 \;\middle|\; \begin{array}{rcl} \sigma_1\sigma_2\sigma_1 & = & \sigma_2\sigma_1\sigma_2 \\ \sigma_3\sigma_2\sigma_3 & = & \sigma_2\sigma_3\sigma_2 \\ \sigma_1\sigma_3 & = & \sigma_3\sigma_1 \\ (\sigma_1\sigma_2\sigma_3)^4 & = & 1 \\ \sigma_1^3 & = & \sigma_3^2 \\ (\sigma_1\sigma_2)^3 & = & 1 \\ (\sigma_2\sigma_3)^3 & = & 1 \end{array} \right\rangle$$

Proof. We have to show that the relation $\sigma_1\sigma_2\sigma_3^2\sigma_2\sigma_1 = 1$ is equivalent to the three relations $\sigma_1^2 = \sigma_3^2$, $(\sigma_1\sigma_2)^3 = 1$ and $(\sigma_2\sigma_3)^3 = 1$. Proving that the latter three relations imply the first relation, is easy to see:

$$\begin{aligned} \sigma_1\sigma_2\sigma_3^2\sigma_2\sigma_1 &= (\sigma_1\sigma_2\sigma_1)(\sigma_1^{-1}\sigma_3^2\sigma_1^{-1})(\sigma_1\sigma_2\sigma_1) \\ &= (\sigma_1\sigma_2\sigma_1)^2 \\ &= (\sigma_1\sigma_2)^3 \\ &= 1 \end{aligned}$$

The converse is more intricate. Assume $\sigma_1\sigma_2\sigma_3^2\sigma_2\sigma_1 = 1$. Then we may write $\sigma_3^{-2} = \sigma_2\sigma_1^2\sigma_2$. Hence, $\sigma_1^2\sigma_3^{-2} = (\sigma_1\sigma_2\sigma_1)^2 = (\sigma_1\sigma_2)^3$. Similarly, $\sigma_3^2\sigma_1^{-2} = (\sigma_2\sigma_3)^3$.

Let us now rewrite

$$\begin{aligned} (\sigma_1\sigma_2\sigma_3)^2 &= (\sigma_1\sigma_2\sigma_3)(\sigma_1\sigma_2\sigma_3) \\ &= (\sigma_1\sigma_2\sigma_1)(\sigma_3\sigma_2\sigma_3) \\ &= (\sigma_1\sigma_2\sigma_1)(\sigma_2\sigma_3\sigma_2) \\ &= \sigma_1^2\sigma_2\sigma_3\sigma_1\sigma_2 \end{aligned}$$

Therefore,

$$\begin{aligned} (\sigma_1\sigma_2\sigma_3)^4 &= (\sigma_1^2\sigma_2\sigma_3\sigma_1\sigma_2)^2 \\ &= \sigma_1^2\sigma_2\sigma_1\sigma_3\sigma_2\sigma_1^2\sigma_2\sigma_3\sigma_1\sigma_2 \\ &= \sigma_1^2\sigma_2\sigma_1^2\sigma_2 \\ &= (\sigma_1\sigma_2)^3 \end{aligned}$$

This implies

$$(\sigma_1\sigma_2)^3 = \sigma_1^2\sigma_3^{-2} = (\sigma_2\sigma_3)^{-3} = 1$$

proving the claim. □

The first interesting observation from this presentation, is that we find a subgroup $\mathrm{PSL}(2,\mathbb{Z}) < \mathrm{MCG}(S)$ of the form

$$\mathrm{PSL}(2,\mathbb{Z}) \simeq \left\langle \sigma_1, \sigma_2 \;\middle|\; \begin{array}{rcl} \sigma_1\sigma_2\sigma_1 & = & \sigma_2\sigma_1\sigma_2 \\ (\sigma_1\sigma_2)^3 & = & 1 \end{array} \right\rangle$$

Using this presentation, we easily check that $j_1^2 = j_2^2 = j_3^2 = 1$. Additionally, we see that

$$\begin{aligned} j_1 j_2 &= \sigma_3\sigma_1^{-1}(\sigma_2\sigma_1\sigma_3)^2 \\ &= \sigma_1^{-1}\sigma_3\sigma_2\sigma_3\sigma_1\sigma_2\sigma_1\sigma_3 \\ &= \sigma_1^{-1}(\sigma_3\sigma_2)^2\sigma_1\sigma_2\sigma_3 \\ &= \sigma_1^{-1}\sigma_2^{-1}\sigma_3^{-1}\sigma_3\sigma_2\sigma_1 \\ &= (\sigma_1\sigma_2\sigma_3)^2 \\ &= j_3 \end{aligned}$$

The elements j_1, j_2 and j_3 therefore generate the subgroup $\mathbb{Z}_2 \times \mathbb{Z}_2$. As noted before in lemma 4.6, these elements have realizations as Möbius transformations on X.

We continue by calculating the commutation relations for the elements j_1, j_2 and j_3 with any of the braiding elements σ_1, σ_2 and σ_3.

Lemma 16.9. *The elements j_1, j_2 and j_3 satisfy the following commutation relations with the generators σ_1, σ_2 and σ_3:*

$$\sigma_i j_1 \sigma_i^{-1} = \begin{cases} j_1 & i = 1,3 \\ j_3 & i = 2 \end{cases}$$

$$\sigma_i j_2 \sigma_i^{-1} = \begin{cases} j_3 & i = 1,3 \\ j_2 & i = 2 \end{cases}$$

$$\sigma_i j_3 \sigma_i^{-1} = \begin{cases} j_2 & i = 1,3 \\ j_1 & i = 2 \end{cases}$$

Proof. We will only need to prove the first two sets of commutation relations, since the third one follows from these through the relation $j_1 j_2 = j_3$. Let us now consider the first two sets of relations:

It is clear that $\sigma_1 j_1 \sigma_1^{-1} = \sigma_3 j_1 \sigma_3^{-1} = j_1$ as follows from $\sigma_1\sigma_3 = \sigma_3\sigma_1$. Now

$$\begin{aligned}
\sigma_2 j_1 \sigma_2^{-1} &= \sigma_2\sigma_1^{-1}\sigma_3\sigma_2^{-1} \\
&= \sigma_2\sigma_1\sigma_2\sigma_3\sigma_3\sigma_2\sigma_3\sigma_2^{-1} \\
&= \sigma_2\sigma_1\sigma_2\sigma_3\sigma_2\sigma_3 \\
&= (\sigma_1\sigma_2\sigma_3)^2 \\
&= j_3
\end{aligned}$$

This proves the first set of relations. Next

$$\begin{aligned}
\sigma_1 j_2 \sigma_1^{-1} &= \sigma_1(\sigma_2\sigma_1\sigma_3)^2\sigma_1^{-1} \\
&= \sigma_1\sigma_2\sigma_1\sigma_3\sigma_2\sigma_3 \\
&= (\sigma_1\sigma_2\sigma_3)^2 \\
&= j_3
\end{aligned}$$

and

$$\begin{aligned}
\sigma_3 j_2 \sigma_3^{-1} &= \sigma_3\sigma_1^{-1}\sigma_1 j_2 \sigma_1^{-1}(\sigma_3\sigma_1^{-1})^{-1} \\
&= j_1\sigma_1 j_2 \sigma_1^{-1} j_1 \\
&= j_3
\end{aligned}$$

Finally we show

$$\begin{aligned}
\sigma_2^{-1} j_2 \sigma_2 &= \sigma_2^{-1}(\sigma_2\sigma_1\sigma_3)^2\sigma_2 \\
&= \sigma_1\sigma_3\sigma_2\sigma_1\sigma_3\sigma_2 \\
&= \sigma_1\sigma_2\sigma_3\sigma_2\sigma_1\sigma_2 \\
&= \sigma_1\sigma_2\sigma_3\sigma_1\sigma_2\sigma_1 \\
&= \sigma_1\sigma_2\sigma_3\sigma_1\sigma_2\sigma_3\sigma_3^{-1}\sigma_1 \\
&= (\sigma_1\sigma_2\sigma_3)^2(\sigma_3\sigma_1^{-1})^{-1} \\
&= j_1 j_3 \\
&= j_2
\end{aligned}$$

This proves all commutation relations. □

Theorem 16.10. *The mapping class group* MCG(S) *is isomorphic to the semi-*

direct product $\mathrm{PSL}(2,\mathbb{Z}) \rtimes (\mathbb{Z}_2 \times \mathbb{Z}_2)$ *in the presentation*

$$\begin{aligned}\mathrm{MCG}(S) \simeq \langle \sigma_1, \sigma_2, j_1, j_2 | \quad & \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2 \\ & (\sigma_1\sigma_2)^3 = 1 \\ & j_1^2 = j_2^2 = 1 \\ & j_1 j_2 = j_2 j_1 \\ & \sigma_1 j_1 = j_1 \sigma_1 \\ & \sigma_2 j_1 = j_1 j_2 \sigma_2 \\ & \sigma_1 j_2 = j_1 j_2 \sigma_1 \\ & \sigma_2 j_2 = j_2 \sigma_2 \rangle\end{aligned}$$

where

$$\mathrm{PSL}(2,\mathbb{Z}) \simeq \left\langle \sigma_1, \sigma_2 \;\middle|\; \begin{array}{ccc} \sigma_1\sigma_2\sigma_1 & = & \sigma_2\sigma_1\sigma_2 \\ (\sigma_1\sigma_2)^3 & = & 1 \end{array} \right\rangle$$

and

$$\mathbb{Z}_2 \times \mathbb{Z}_2 \simeq \left\langle j_1, j_2 \;\middle|\; \begin{array}{ccc} j_1^2 & = & 1 \\ j_2^2 & = & 1 \\ j_1 j_2 & = & j_2 j_1 \end{array} \right\rangle$$

Proof. We already derived all of these relations in our previous lemmas. To prove the equivalence of this presentation to our previous presentations, it is enough to reintroduce $\sigma_3 = \sigma_1 j_1$ and prove the relations

$$\begin{aligned} \sigma_1\sigma_3 &= \sigma_3\sigma_1 \\ \sigma_2\sigma_3\sigma_2 &= \sigma_3\sigma_2\sigma_3 \\ (\sigma_1\sigma_2\sigma_3)^4 &= 1 \\ \sigma_1\sigma_2\sigma_3^2\sigma_2\sigma_1 &= 1 \end{aligned}$$

The first relation is trivial. The second relation follows from

$$\begin{aligned} \sigma_2\sigma_3\sigma_2 &= \sigma_2 j_1 \sigma_1 \sigma_2 \\ &= j_1 j_2 \sigma_2 \sigma_1 \sigma_2 \end{aligned}$$

and

$$\begin{aligned}\sigma_3\sigma_2\sigma_3 &= j_1\sigma_1\sigma_2 j_1\sigma_1 \\ &= \sigma_1 j_2\sigma_2\sigma_1 \\ &= j_1 j_2\sigma_1\sigma_2\sigma_1 \\ &= j_1 j_2\sigma_2\sigma_1\sigma_2\end{aligned}$$

The third relations follows from

$$\begin{aligned}\sigma_1\sigma_2\sigma_3 &= \sigma_1\sigma_2\sigma_1 j_1 \\ &= j_2\sigma_1\sigma_2\sigma_1\end{aligned}$$

Hence $(\sigma_1\sigma_2\sigma_3)^2 = j_2 j_1$ and $(\sigma_1\sigma_2\sigma_3)^4 = j_1^2 j_2^2 = 1$.

Finally, the fourth relation can be written as

$$\begin{aligned}\sigma_1\sigma_2\sigma_3^2\sigma_2\sigma_1 &= \sigma_1\sigma_2\sigma_1 j_1^2\sigma_1\sigma_2\sigma_1 \\ &= (\sigma_1\sigma_2)^3 \\ &= 1\end{aligned}$$

We have therefore proved the equivalence of this presentation to the previous presentations. □

By an explicit calculation using the semi-direct product form of the mapping class group, we find

Corollary 16.11. *The only elements in* MCG(S) *of order two and not fixing any of the punctures, must be equal to* j_1, j_2 *or* j_3.

17 The Riemann-Hilbert correspondence

Hilbert's twenty-first problem Historically, the Riemann-Hilbert correspondence was first discussed from the point of view of Hilbert's twenty-first problem. This problem asked for the existence of a Fuchsian differential equation on the complex plane with punctures $X = \mathbb{C}\backslash\{z_1, ..., z_n\}$ realizing a given representation (up to conjugation) of the free group $F_n \to \mathrm{GL}(N, \mathbb{C})$. We will set $N = 2$ and only consider representations that reduce to $\mathrm{SL}(2, \mathbb{C})$. The Fuchsian differential equations we consider are assumed to have regular singularities at the punctures. This means the differential equations take the form of equation (3.11), where

$$t(z) = \sum_{r=1}^{n} \left(\frac{\delta_r}{(z - z_r)^2} + \frac{E_r}{z - z_r} \right)$$

Furthermore, we require $\sum_{r=1}^{n} E_r = 0$ to ensure regularity of the solution when we let $z \to \infty$.

It turns out that the space of regular Fuchsian differential equations is not large enough to accommodate all monodromy representations in $C_{\mathrm{SL}(2,\mathbb{C})}(S)$ where S is the topological surface underlying X. Indeed, a simple calculation shows that we will need $3n - 3$ complex parameters to describe the monodromy representations, coming from three parameters per generator mapped to a matrix in $\mathrm{SL}(2, \mathbb{C})$ and one constraint by the overall action of conjugation by $\mathrm{SL}(2, \mathbb{C})$. On the other hand, regular Fuchsian differential equations have the freedom of the parameters $\{\delta_r, E_r\}$ together with one constraint, leading to a total of $2n-1$ complex parameters.

Introducing apparent singularities To find the missing $n - 2$ parameters, we may note that any monodromy can be realized by integrating a holomorphic connection of the form

$$\nabla' = \partial_z + A(z)$$

where

$$A(z) = \begin{pmatrix} \alpha(z) & \beta(z) \\ \gamma(z) & -\alpha(z) \end{pmatrix}$$

If we furthermore assume the monodromy is irreducible, the lower-left element $\gamma(z)$ is not identically zero.

By applying a gauge transformation $A(z) \to g(z)A(z)g(z)^{-1} + g(z)\partial_z(g(z)^{-1})$

with

$$g(z) = \left(\begin{array}{cc} \sqrt{\gamma(z)} & \frac{1}{\sqrt{\gamma(z)}} \left(-\alpha(z) + \frac{\gamma'(z)}{\gamma(z)} \right) \\ 0 & \frac{1}{\sqrt{\gamma(z)}} \end{array} \right)$$

we may bring this connection to the form

$$\nabla' = \partial_z + \left(\begin{array}{cc} 0 & -t(z) \\ 1 & 0 \end{array} \right)$$

for

$$t(z) = -\alpha(z)^2 - \beta(z)\gamma(z) - \alpha'(z) + \alpha(z)\frac{\gamma'(z)}{\gamma(z)} + \frac{1}{2}\left(\frac{\gamma''(z)}{\gamma(z)} - \frac{3}{2}\left(\frac{\gamma'(z)}{\gamma(z)}\right)^2 \right)$$

We may locally write

$$\begin{aligned} \left(\begin{array}{c} \eta'(z) \\ \chi'(z) \end{array} \right) &= \left(\begin{array}{cc} 0 & t(z) \\ -1 & 0 \end{array} \right) \left(\begin{array}{c} \eta(z) \\ \chi(z) \end{array} \right) \\ &= \left(\begin{array}{c} t(z)\chi(z) \\ -\eta(z) \end{array} \right) \end{aligned}$$

This sets $\eta(z) = -\chi'(z)$ and $\chi''(z) + t(z)\chi(z) = 0$.

The function $t(z)$ has a number of singularities defined by the zeroes of $\gamma(z)$. If we require $t(z)$ to have regular singularities at the punctures, we must assume $A(z)$ only has simple poles at the punctures. Since $\gamma(z)$ is regular at infinity, it generically has a total of $n-2$ simple zeroes.

These simple zeroes lead to singularities in $t(z)$, i.e. points u_k for $k = 1, ..., n-2$ around which

$$t(z) = \frac{3}{4(z-u_k)^2} + \frac{v_k}{z-u_k} + \check{t}_k(z)$$

and $\check{t}_k(z)$ is defined as the regular part of $t(z)$ at u_k.

By a simple calculation, we may check that the residue v_k satisfies the equation $v_k^2 + \check{t}_k(u_k) = 0$. This is precisely the equation needed to have trivial monodromy around the singularity $z = u_k$. Indeed, the missing $n-2$ parameters are precisely made up by the positions of the $n-2$ parameters introduced in this way.

Solutions unique up to gauge transformation For a given holomorphic connection ∇' on a Riemann surface X, we say that $\phi(z)$ is a solution to the Riemann-Hilbert problem if $\nabla'\phi = 0$. In a local frame, we may rewrite this equation as $\phi'(z) + A(z)\phi(z) = 0$.

Let us now assume we have found a solution $\phi(z)$ to the Riemann-Hilbert problem. We do not need to assume X is of genus $g = 0$! We find the following proposition:

Proposition 17.1. *If $\phi_1(z)$ and $\phi_2(z)$ have the same monodromy up to conjugation, the corresponding gauge connections $A_1(z)$ and $A_2(z)$ making $\phi_1(z)$ and $\phi_2(z)$ into solutions to a Riemann-Hilbert problem, must be gauge equivalent.*

Proof. Let us assume we have solutions $\phi_1(z)$ and $\phi_2(z)$ to the Riemann-Hilbert problems

$$\begin{aligned} \phi_1'(z) + A_1(z)\phi_1(z) &= 0 \\ \phi_2'(z) + A_2(z)\phi_2(z) &= 0 \end{aligned}$$

with conjugate monodromy on some Riemann surface X. We may assume $\phi_1(z)$ and $\phi_2(z)$ have equal monodromy by applying a gauge transformation to one of them. Let us now combine the solutions $\phi_1(z)$ and $\phi_2(z)$ into matrices

$$\begin{aligned} \Phi_1(z) &= (\phi_1(z), \phi_1'(z)) \\ \Phi_2(z) &= (\phi_2(z), \phi_2'(z)) \end{aligned}$$

If the monodromy acts on the right, the combination $\Phi_1(z)(\Phi_2(z))^{-1}$ is single-valued. As such, we may write

$$\Phi_1(z) = g(z)\Phi_2(z)$$

for some single-valued matrix-valued function $g(z)$ on X. Therefore,

$$\phi_2(z) = g(z)\phi_1(z)$$

and we identify

$$A_2(z) = g(z)A_1(z)g(z)^{-1} + g(z)\partial_z(g(z)^{-1})$$

Hence, $A_2(z)$ is a gauge transformation of $A_1(z)$ by the function $g(z)$. □

Apparent singularities on higher genus surfaces We can generalize the set-up from above to other marked Riemann surfaces and ask ourselves how many apparent singularities are needed to accommodate all the degrees of freedom of the monodromy representation. It turns out that we need at most $3g - 3 + n$ apparent singularities [107, 108] on a surface of genus g and with

n punctures to describe a dense subset of the space of all representations in $C_{\mathrm{SL}(2,\mathbb{C})}(S)$. In particular, applying 17.1 to the holomorphic connections ∇_1' and ∇_2' of the form

$$\nabla_k' = \partial_z + \begin{pmatrix} 0 & -t_k(z) \\ 1 & 0 \end{pmatrix}$$

for $k = 1, 2$, shows that two opers $t_1(z)$ and $t_2(z)$ without apparent singularities and with different accessory parameters cannot have the same monodromy up to conjugation. The monodromy map restricted to the fibre $\mathcal{P}(X)$ must be injective. A special case of this statement was proved for the four-punctured sphere in lemma 9.8.

Changing the complex structure of the surface Another way to introduce more parameters is by allowing the underlying marked complex structure of our surface to change. The space of opers on a fixed Riemann surface has complex dimension $3g - 3 + n$, which is precisely the complex dimension of Teichmüller space. By allowing ourselves to change the underlying complex structure, we can describe a large set in the space of monodromy representations $C_{\mathrm{PSL}(2,\mathbb{C})}(S)$.

To be more precise, assume $n = 0$ and note that the space $C_{\mathrm{PSL}(2,\mathbb{C})}(S)$ consists of two connected components. If we denote by $(C_{\mathrm{PSL}(2,\mathbb{C})}(S))^0$ the connected component to the identity, it was shown in [109] that the monodromy of $\mathcal{P}(S)$ lies in $(C_{\mathrm{PSL}(2,\mathbb{C})}(S))^0$ and the image of the monodromy map

$$\mathcal{P}(S) \to (C_{\mathrm{PSL}(2,\mathbb{C})}(S))^0$$

is dense in $(C_{\mathrm{PSL}(2,\mathbb{C})}(S))^0$.

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