Daniël Konstantin Thung

Symmetries of deformed c-map metrics and the HK/QK correspondence

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1. Gutachter: Prof. Dr. Vicente Cortés
2. Gutachter: Dr. Habil. Andriy Haydys

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Dedicated to all my teachers

Abstract

In this dissertation, we study the symmetry properties of quaternionic Kähler manifolds that arise from the c-map. The c-map is a differential-geometric method to construct quaternionic Kähler manifolds of negative scalar curvature out of projective special Kähler manifolds. It can not only be used to recover previously known examples of explicit quaternionic Kähler manifolds, including all known homogeneous examples with exception of the quaternionic hyperbolic spaces, but also gives rise to many new and interesting examples, including manifolds which are not locally homogeneous.

Given a projective special Kähler manifold, its image under the c-map can be constructed as follows. The first step is to canonically construct an associated pseudo-hyper-Kähler manifold, endowed with a distinguished circle action. Then, we apply the hyper-Kähler/quaternionic Kähler (HK/QK) correspondence, which assigns a quaternionic Kähler manifold to each such pseudo-hyper-Kähler manifold.

The resulting quaternionic Kähler metric depends on a choice of Hamiltonian function for the circle action, so in fact we obtain a one-parameter family of quaternionic Kähler manifolds $(\bar{N}, g^c)$, $c \geq 0$, which has been shown to be complete under natural assumptions on the initial projective special Kähler metric. The case $c = 0$ corresponds to a distinguished choice of Hamiltonian and g^0 is known as the undeformed c-map metric while the other members of this family are called (one-loop) deformed c-map metrics. Because of the simple interpretation of the deformation parameter on the hyper-Kähler side, we can use the HK/QK correspondence to study the entire family of quaternionic Kähler metrics simultaneously.

In our study of the symmetry properties of (deformed) c-map metrics, we first ask how automorphisms of the initial projective special Kähler manifold $\bar{M}$ are reflected in the quaternionic Kähler metric of its image $(\bar{N}, g^c)$ under the deformed c-map. We show that the (identity component of the) automorphism group of $\bar{M}$ injects into the isometry group of $(\bar{N}, g^c)$ for every $c \geq 0$. Consequently, if $\operatorname{Aut} \bar{M}$ acts with co-homogeneity n, then $\operatorname{Isom}(\bar{N}, g^c)$ acts with co-homogeneity at most $n+1$ for $c > 0$, and n for $c = 0$.

Next, we study the behavior of the curvature tensor under the HK/QK correspondence and derive an elegant formula that expresses the curvature tensor of the deformed c-map metric in terms of data on the hyper-Kähler side of the correspondence. This formula can be used to place lower bounds on the co-homogeneity of c-map metrics by computing non-constant curvature invariants. We demonstrate this for a series of examples $(\bar{N}_n, g_n^c)$, $n \in \mathbb{N}$, $c \geq 0$, which arise from $\mathbb{C}\mathrm{H}^n$, regarded as a homogeneous projective special Kähler manifold. The undeformed c-map metric g_n^0 on $\bar{N}_n \cong \mathbb{R}^{4n+3} \times S^1$ is (locally) symmetric, but for $c > 0$ we find a non-constant curvature invariant, proving that the deformed c-map metrics on these manifolds are of co-homogeneity one.

Finally, we discuss discrete subgroups of the groups of isometries of c-map metrics constructed by the above methods. We outline a method to construct quotients of c-map spaces which are diffeomorphic to $K \times \mathbb{R}$, where K is a compact, locally homogeneous manifold obtained by dividing out an appropriate discrete group of isometries of the c-map metric. We work out the details in a simple example, using input from the theory of quaternion algebras and Fuchsian groups.

Zusammenfassung

Die vorliegende Dissertation beschäftigt sich mit den Symmetrieeigenschaften quaternionischer Kählermannigfaltigkeiten, die sich durch die c-Abbildung ergeben. Die c-Abbildung ist eine differentialgeometrische Konstruktion, die jeder projektiven speziellen Kähler-Mannigfaltigkeit eine quaternionische Kähler-Mannigfaltigkeit negativer Skalarkrümmung zuordnet. Durch die c-Abbildung ergeben sich nicht nur bisher bekannte Beispiele quaternionischer Kähler-Mannigfaltigkeiten, wie zum Beispiel alle bekannten homogenen Beispiele mit Ausnahme der quaternionischen hyperbolischen Räume, sondern auch viele interessante neue Beispiele. Dazu gehören insbesondere Mannigfaltigkeiten, die nicht lokal homogen sind.

Gegeben sei eine projektive spezielle Kähler-Mannigfaltigkeit. Ihr Bild unter der c-Abbildung wird wie folgt konstruiert: Der erste Schritt besteht darin, eine kanonisch assoziierte Pseudo-Hyper-Kähler-Mannigfaltigkeit, die mit einer kanonischen Kreiswirkung ausgestattet ist, zu konstruieren. Darauf folgt die Anwendung der Hyper-Kähler/quaternionisch Kähler (HK/QK) Korrespondenz, die jeder solcher Hyper-Kähler-Mannigfaltigkeit eine quaternionische Kähler-Mannigfaltigkeit zuordnet.

Die daraus resultierende quaternionische Kähler-Metrik hängt von der Wahl einer Hamilton-Funktion für die Kreiswirkung ab, sodass man eine Ein-Parameter-Familie quaternionischer Kähler-Mannigfaltigkeiten $(\bar{N}, g^c)$, $c \geq 0$, erhält. Unter natürlichen Annahmen an die projektive spezielle Kähler-Metrik wurde bewiesen, dass die quaternionischen Kähler-Metriken vollständig sind. Der Fall $c = 0$ entspricht einer ausgezeichneten Wahl der Hamilton-Funktion und g^0 wird als undeformierte c-Abbildungs-Metrik bezeichnet. Die Metriken g^c, $c > 0$, werden als (Ein-Schleifen-)deformierte c-Abbildungs-Metriken bezeichnet. Dank der einfachen Interpretation des Deformationsparameters auf der Hyper-Kähler Seite der Korrespondenz, können wir die HK/QK Korrespondenz nutzen, um die gesamte Familie $(\bar{N}, g^c)$, $c \geq 0$, gleichzeitig zu untersuchen.

Wir untersuchen zuerst, wie sich Automorphismen der anfänglichen projektiven speziellen Kähler-Mannigfaltigkeit $\bar{M}$ in ihrem Bild $(\bar{N}, g^c)$ unter der (deformierten) c-Abbildung widerspiegeln. Wir beweisen für jedes $c \geq 0$ die Existenz einer injektiven Abbildung von der (Identitätskomponente der) Automorphismengruppe von $\bar{M}$ in die Isometriegruppe von $(\bar{N}, g^c)$. Es sei $n \in \mathbb{N}$ die Kohomogenität von $\bar{M}$ unter der Wirkung von $\mathrm{Aut}\,\bar{M}$. Aus unseren Ergebnissen folgt jetzt, dass die Kohomogenität von $(\bar{N}, g^0)$ höchstens n und die Kohomogenität von $(\bar{N}, g^c)$, $c > 0$, höchstens $n + 1$ ist.

Als Nächstes analysieren wir das Verhalten des Krümmungstensors unter der HK/QK Korrespondenz. Wir erhalten eine elegante Gleichung, die beschreibt, wie der Krümmungstensor der deformierten c-Abbildungs-Metrik aus der entsprechenden Hyper-Kähler-Mannigfaltigkeit bestimmt wird. Mittels dieser Gleichung ist es möglich, nichtkonstante Krümmungsinvarianten zu berechnen, womit die Kohomogenität der c-Abbil-

dungs-Metriken nach unten beschränkt wird. Dies zeigen wir an einer Folge von Beispielen $(\bar{N}_n, g_n^c)$, $n \in \mathbb{N}$, $c \geq 0$, die sich aus der homogenen projektiven speziellen Kähler-Mannigfaltigkeit $\mathbb{C}\mathrm{H}^n$ ergeben. Die undeformierte c-Abbildungs-Metrik g_n^0 auf $\bar{N}_n \cong \mathbb{R}^{4n+3} \times S^1$ ist (lokal) symmetrisch, aber für $c \geq 0$ bestimmen wir eine nicht-konstante Krümmungsinvariante, was beweist, dass die deformierten c-Abbildungs-Metriken von Kohomogenität eins sind.

Abschließend beschäftigen wir uns mit diskreten Untergruppen von c-Abbildungs-Metriken. Wir skizzieren eine Methode zur Konstruktion von Quotienten von c-Abbildungs-Räumen, die diffeomorph zu $K \times \mathbb{R}$ sind, wobei K kompakt und lokal homogen ist. Wir erarbeiten die Details in einem einfachen Beispiel unter Verwendung der Theorie der Quaternion-Algebren und der Fuchsschen Gruppen.

Contents

1 Introduction

1.1 Holonomy and quaternionic Kähler manifolds

For any Riemannian manifold (M, g), we can use the canonically associated Levi-Civita connection to define the notion of parallel transport along smooth paths in M. For any such path, this produces a linear isometry between the tangent spaces at the starting and end points. In particular, for a closed loop based at $x \in M$, we obtain a linear isometry $T_xM \to T_xM$. Since we can always invert this isometry by considering parallel transport in the opposite direction along the same loop, these transformations generate a subgroup of $\mathrm{O}(T_xM, g_x)$. We call this the holonomy group. Implicitly assuming that M is connected—as we will do throughout this work—we can connect any two point in M by a smooth path, and parallel transport along such a path induces an isomorphism of the corresponding groups, so we may speak about the holonomy group without reference to any specific point.

Given a Riemannian manifold (M, g), we may now ask what its holonomy group is. More generally, one can ask which subgroups of $\mathrm{O}(n)$ can occur as the holonomy group of a Riemannian manifold of dimension n. A celebrated theorem due to De Rham reduces this question to the case where the holonomy group acts irreducibly on the tangent space, in which case we call the manifold irreducible:

Theorem 1.1 (De Rham decomposition theorem). *Every simply connected and complete Riemannian manifold is isometric to a Riemannian product of simply connected, complete and irreducible Riemannian manifolds with a Euclidean space. This decomposition is unique up to ordering.* □

Early in the twentieth century, Élie Cartan classified the irreducible, simply connected symmetric spaces and computed their holonomy groups. These manifolds are given by a so-called symmetric pair (G, H), where G is simply connected and H is connected, such that $M \cong G/H$ and the metric on M is G-invariant. In this case, the holonomy group turns out to be equal to the isotropy group H. This determines the holonomy groups of symmetric spaces. The classification of holonomy groups of manifolds which are not symmetric was settled by Berger:

Theorem 1.2 (Berger, [Ber55]). *Let (M, g) be a complete, simply connected and irreducible Riemannian manifold which is not symmetric. Then its holonomy group is one of the following:*

dim M	*Holonomy group*
n	$\mathrm{SO}(n)$
$n = 2m,\ m \geq 2$	$\mathrm{U}(m)$
$n = 2m,\ m \geq 2$	$\mathrm{SU}(m)$
$n = 4m,\ m \geq 2$	$\mathrm{Sp}(m) \cdot \mathrm{Sp}(1)$
$n = 4m,\ m \geq 2$	$\mathrm{Sp}(m)$
$n = 7$	G_2
$n = 8$	$\mathrm{Spin}(7)$

What is most remarkable about Berger's list is how short it is: It shows that only a handful of holonomy groups are allowed in each dimension. One can think of it as providing a short list of possible flavors of Riemannian geometry, each of which has its own specific features.

Of course, not much can be said when the holonomy group is $\mathrm{SO}(n)$, which is the case for a generic Riemannian manifold. However, it turns out that all the other holonomy groups impose strong restrictions on the Riemannian metric. Generally, the stringency of these restrictions is proportional to the co-dimension of the holonomy group in $\mathrm{SO}(n)$. Correspondingly, the largest class of manifolds with special holonomy (i.e. holonomy group different from $\mathrm{SO}(n)$) corresponds to the case $\mathrm{U}(m)$, which is equivalent to the requirement that (M, g) is Kähler. There is a wealth of examples of Kähler manifolds, coming from complex algebraic geometry.

For the other groups appearing in Berger's list, however, it has turned out considerably more difficult to obtain examples, and their construction remains one of the most important motivating questions in Riemannian geometry to this day. Broadly speaking, this dissertation should be seen as a contribution to this program. More precisely, this work concerns the construction of complete manifolds whose holonomy group is $\mathrm{Sp}(m)\,\mathrm{Sp}(1)$.

Definition 1.3. A complete Riemannian manifold (M, g) of dimension $4n$, $n \geq 2$ is called quaternionic Kähler if its holonomy group is contained in $\mathrm{Sp}(n)\,\mathrm{Sp}(1) \subset \mathrm{SO}(4n)$, and hyper-Kähler if its holonomy group is contained in $\mathrm{Sp}(n) \subset \mathrm{SO}(4n)$.

Remark 1.4. The group $\mathrm{Sp}(n)\,\mathrm{Sp}(1) \cong (\mathrm{Sp}(n) \times \mathrm{Sp}(n))/\mathbb{Z}_2$ is defined as follows: Given $A \in \mathrm{Sp}(n)$ and $q \in \mathrm{Sp}(1)$, $(\pm A, \pm q) \in \mathrm{Sp}(n)\,\mathrm{Sp}(1)$ acts on $\mathbb{R}^{4n}$ via $v \mapsto Av\bar{q}$.

Note that the hyper-Kähler condition is a strengthening of the quaternionic Kähler condition, so hyper-Kähler manifolds are in particular quaternionic Kähler. Therefore, some authors choose to reserve the name quaternionic Kähler for manifolds which are quaternionic Kähler but not (locally) hyper-Kähler. As we will see, these are precisely the quaternionic Kähler manifolds of non-vanishing scalar curvature; we will refer to them as strict quaternionic Kähler manifolds.

Remark 1.5. In dimension four, the above definitions need to be modified. To see this, note for example that $\mathrm{Sp}(1)\,\mathrm{Sp}(1) = \mathrm{SO}(4)$, so any simply connected four-manifold has holonomy contained in $\mathrm{Sp}(1)\,\mathrm{Sp}(1)$. Instead of discussing this (degenerate) case now, we postpone the corresponding definitions for four-manifolds to chapter 2.

Let us introduce a useful reformulation of the quaternionic Kähler condition. There is a general procedure to translate a restriction on the holonomy group of (M, g) into the existence of parallel geometric structures on M (see e.g. [Bes08]). This correspondence between holonomy and parallel structures is often referred to as the holonomy principle. Picking an arbitrary point $x \in M$, we have the holonomy representation on T_xM, and may use linear algebra to find structures invariant under this representation. These then lead to parallel structures on M. For instance, if α_x is an invariant tensor, let $y \in M$ be arbitrary and let γ be a curve connecting x to y. Denote the induced parallel transport map by τ_γ. Now, setting $\alpha(y) = \tau_\gamma(\alpha_x)$ yields a well-defined tensor field by invariance of α_x.

In the quaternionic Kähler case, we need a slight variation of this idea. The tangent space $T_xM \cong \mathbb{R}^{4n}$ admits a hyper-Hermitian structure, determined by three endomorphisms I_1, I_2, I_3 satisfying $I_1^2 = I_2^2 = -\,\mathrm{id}$ and $I_1I_2 = -I_2I_1 = I_3$. They are $\mathrm{Sp}(n)$-invariant but none of them is preserved by the right action of $\mathrm{Sp}(1)$, which permutes them. Nevertheless, the three-dimensional linear subspace $\langle I_1, I_2, I_3\rangle \subset \mathrm{End}(T_xM)$ is $\mathrm{Sp}(1)$- and hence $\mathrm{Sp}(n)\,\mathrm{Sp}(1)$-invariant. Thus, though parallel transport along paths does not necessarily yield globally well-defined parallel almost complex structures, a quaternionic Kähler manifold admits a rank three bundle $\mathcal{Q}$ which admits local frames consisting of a quaternionic triple of almost complex structures compatible with the metric. Moreover, $\mathcal{Q}$ is parallel in the sense that $\nabla\Gamma(\mathcal{Q}) \subset \Gamma(T^*M \otimes \mathcal{Q})$. Note that if M is actually hyper-Kähler, the $\mathrm{Sp}(n)$-invariance of the three almost complex structures implies the existence of a triple of parallel (hence integrable) almost complex structures.

The advantage of this approach is twofold. Firstly, it helps to have concrete geometric structures to work with, rather than an abstract condition on the holonomy, in order to prove theorems. Secondly, this formulation lends itself well to generalizations in various directions. For instance, dropping the condition that $\mathcal{Q}$ is parallel yields the broader class of almost quaternionic-Hermitian manifolds, while a change in signature of the metric is also easily accommodated, allowing us to define pseudo-Riemannian versions of quaternionic Kähler and hyper-Kähler geometry. Therefore, this is the point of view that we take in this dissertation.

1.2 The c-map and the HK/QK correspondence

Despite the great interest in the construction of complete quaternionic Kähler manifolds, there has historically been a shortage of examples. As we will see later on, quaternionic

Kähler manifolds are automatically Einstein and therefore have constant scalar curvature. Excluding the case where it vanishes, which corresponds to hyper-Kähler geometry, the theory naturally splits into two parts according to the sign of the scalar curvature. Besides a few families of symmetric examples, there are no known quaternionic Kähler manifolds of positive scalar curvature. Their non-existence is in fact a famous conjecture of LeBrun and Salamon [LS94].

In the case of negative scalar curvature, non-locally symmetric examples are known to exist. Indeed, Alekseevsky constructed the first non-locally symmetric examples in his classification of quaternionic Kähler manifolds homogeneous under a solvable group, which was completed by Cortés [Ale75; Cor96b]. Moreover, a result of LeBrun [LeB91] shows that there is an infinite-dimensional moduli space of complete quaternionic Kähler metrics on $\mathbb{R}^{4n}$, suggesting that it ought to be possible to construct many examples. Until recently, however, explicit examples besides the Alekseevsky spaces were exceedingly rare.

This changed with the advent of the c-map construction [CFG89; FS90; dWvP92], first discovered in the context of string theory and supergravity, which was further developed by Cortés and collaborators as a global geometric construction [CHM12; ACM13; ACDM15; CDS17; CDJL17]. It provides a powerful method to construct quaternionic Kähler manifolds of negative scalar curvature out of so-called projective special Kähler (PSK) manifolds. The c-map and the closely related r-map, which can be used to construct PSK manifolds, are shining examples of the deep connections between theoretical physics and differential geometry. Using these constructions, many new complete quaternionic Kähler manifolds have been constructed, including explicit examples which are not locally homogeneous [CDJL17].

The quaternionic Kähler metrics that arise from the c-map can be embedded in a one-parameter family of quaternionic Kähler metrics, known as (one-loop) deformed c-map metrics, first discovered by physicists [RSV06]. This family of metrics has a natural interpretation in terms of a hyper-Kähler manifold which can be canonically associated to it by means of the so-called hyper-Kähler/quaternionic Kähler (HK/QK) correspondence [Hay08; ACDM15; MS15]. This hyper-Kähler manifold is equipped with a canonical circle action which is Hamiltonian with respect to one of the complex structures, and each member of the one-parameter family of metrics on the dual quaternionic Kähler side corresponds to a choice of Hamiltonian function for this circle action. As a consequence, the HK/QK correspondence can be used to study the entire family of deformed c-map metrics simultaneously.

In this dissertation, we exploit this fact and use the HK/QK correspondence to study the c-map and its one-loop deformation. Our focus is on understanding how the HK/QK correspondence and the c-map interact with symmetries. As we will see, such questions lead us to consider a rich theory, involving various types of geometric structures and the often surprising connections that exist between them.

1.3 Outline

In chapter 2, we review the general theory of quaternionic Kähler and hyper-Kähler manifolds, including relevant quotient construction and natural bundles over these quaternionic geometries. In particular, we introduce the Swann bundle over a strict quaternionic Kähler manifold, which connects the worlds of (strict) quaternionic Kähler and hyper-Kähler geometry.

Our next step is to introduce the HK/QK correspondence which combines the Swann bundle construction and the hyper-Kähler quotient to construct a duality between hyper-Kähler and quaternionic Kähler manifolds. We give an account of this correspondence, first discovered by Haydys [Hay08], in chapter 3. After introducing the model case, which relates $T^*\mathbb{C}\mathrm{P}^n$, endowed with the Calabi metric, to $\mathbb{H}\mathrm{P}^n$, we discuss how the HK/QK correspondence can be naturally viewed as a variation on Swann's twist construction [Swa10], following the elegant treatment of Macia and Swann [MS15].

In this thesis, we are mainly concerned with applying the HK/QK correspondence to the c-map construction, which is reviewed in chapter 4. We start with an introduction to special Kähler geometry, and then present the c-map, which assigns a quaternionic Kähler manifold of negative scalar curvature to every PSK manifold. We outline its two equivalent formulations: One version provides a direct construction of the c-map space as a bundle [CHM12; CDS17], while the other makes essential use of the HK/QK correspondence [ACM13; ACDM15; MS15]. We call them the direct approach and the twist approach. The interesting interplay between these equivalent but rather different points of view leads to a rich theory. For instance, it is obvious from the twist approach that the c-map produces a quaternionic Kähler metric, while this is a highly non-trivial theorem if one uses the direct approach. On the other hand, the direct construction is more effective in studying, for instance, the completeness of c-map metrics. We also introduce the (one-loop) deformed c-map, which extends the c-map by embedding the resulting metric in a one-parameter family of quaternionic Kähler metrics.

Using the tools developed in the preceding chapters, we study the properties of the HK/QK correspondence and the (deformed) c-map in chapter 5. We first investigate the following question: If the initial PSK manifold possesses symmetries, then are these reflected in the metrics produced by the deformed c-map construction? We answer this question by proving that every infinitesimal automorphism of the PSK manifold leads to a Killing field for the corresponding deformed c-map metric. More generally, we show that the algebra of infinitesimal automorphisms induces a subalgebra of the Killing algebra of $\bar{N}$ (see theorem 5.15). If the initial PSK manifold is homogeneous, it follows from our results that the deformed c-map metric is of co-homogeneity at most one.

The second question that we study is: How does curvature behave under the HK/QK correspondence? We prove a formula which expresses the curvature of the quaternionic Kähler metric in terms of data on the corresponding hyper-Kähler manifold, providing a complete answer, in theorem 5.27. We can view our formula as a refinement of a well-known curvature decomposition theorem due to Alekseevsky. Whereas our results on the c-map in the presence of additional symmetries give lower bounds on the degree of symmetry of (deformed) c-map metrics, this curvature formula can be used to bound it from above. For instance, the existence of a non-constant curvature invariant implies that a manifold is not locally homogeneous. Such invariants can be computed using our curvature formula. We demonstrate this by computing a non-trivial curvature invariant for the series of examples introduced in chapter 4. In these examples, the undeformed c-map metric is the symmetric metric on the non-compact quaternionic Kähler symmetric space $\frac{\mathrm{SU}(n,2)}{\mathrm{S}(\mathrm{U}(n)\times\mathrm{U}(2))}$ but our results show that the deformed c-map metrics are not locally homogeneous. Combining this with our other results, we deduce that they are of co-homogeneity one (see theorem 5.37). We conjecture that the same holds true for deformed c-map metrics on arbitrary c-map spaces that arise from homogeneous PSK manifolds.

In chapter 6 we study discrete subgroups of the groups of isometries constructed in chapter 5, which can be used to construct examples of quaternionic Kähler manifolds with non-trivial topology by taking quotients. We are in particular interested in co-compact lattices, which can be constructed by using input from the theory of arithmetic groups. We restrict ourselves to a case study of a relatively simple example, leaving a more general discussion for future studies. In this case, we describe how quaternion algebras and the theory of Fuchsian groups can be applied to construct examples of complete, eight-dimensional quaternionic manifolds Kähler diffeomorphic to $K \times \mathbb{R}$, where K is a compact, locally homogeneous space.

As explained above, chapters 2 to 4 are mostly of expository nature. Besides some minor new lemmata, this is a synthesis of known results contained in the cited sources. One exception is the final part of chapter 3, starting from definition 3.25, which extends results from [CST20a]. The results contained in chapters 5 and 6 are original. Chapter 5 is based on (and extends) work in collaboration with Vicente Cortés and Arpan Saha, published in the research articles [CST20a; CST20b]. Chapter 6 is a report on unpublished work in progress with Vicente Cortés and Markus Röser.

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2 Quaternionic geometry

Smooth manifolds are locally modeled on real vector spaces. Endowing such a vector space with additional structures leads to local models for restricted classes of manifolds. In this chapter, we study manifolds modeled on quaternionic vector spaces.

Definition 2.1. A $4n$-dimensional manifold M is called almost quaternionic if it is endowed with a rank three subbundle $\mathcal{Q} \subset \operatorname{End} TM$ such that every $x \in M$ admits a neighborhood U such that $\mathcal{Q}|_U$ admits a trivialization by almost complex structures $\{I_1, I_2, I_3\}$ such that $I_1 I_2 = -I_2 I_1 = I_3$.

Though $\mathcal{Q}$ admits local frames consisting of (local) almost complex structures, there is no guarantee that these can be used to construct globally non-vanishing sections. In particular, an almost quaternionic manifold is not necessarily almost complex.

Definition 2.2. An almost quaternionic-Hermitian manifold $(M, g, \mathcal{Q})$ is a Riemannian almost quaternionic manifold such that the local frames $\{I_1, I_2, I_3\}$ for $\mathcal{Q}$ can be chosen to consist of skew-symmetric almost complex structures.

Any almost quaternionic manifold admits an almost quaternionic-Hermitian structure, since an arbitrary Riemannian metric g can be used to produce a compatible metric $\tilde{g}$ by setting

$$\tilde{g}(X, Y) = \frac{1}{4}\big(g(X, Y) + g(IX, IY) + g(JX, JY) + g(KX, KY)\big)$$

On an arbitrary Riemannian manifold (M, g), a natural bundle metric on $\operatorname{End}(TM)$ such that id_{TM} has unit norm is given by

$$\langle A, B\rangle = \frac{1}{\dim M} \operatorname{tr}(AB^*)$$

where B^* denotes the adjoint of B with respect to g. Up to the factor $\dim M$, this is just the inner product naturally induced on $\operatorname{End}(TM)$ by g. Using the quaternionic algebra we see that, on an almost quaternionic-Hermitian manifold $(M, g, \mathcal{Q})$, the local almost complex structures $\{I_k\}$ determine an orthonormal frame of $\mathcal{Q}$. From now on, we will always implicitly endow $\mathcal{Q}$ with this metric.

Now let $(M, \mathcal{Q}, g)$ be almost quaternionic-Hermitian. By contracting with the metric, we can identify any skew-symmetric endomorphism A of $\mathcal{Q}$ with the two-form $g(A\cdot, \cdot)$;

invariantly, what we have is an bundle isomorphism $\operatorname{Skew}\operatorname{End} TM \cong \bigwedge^2 T^*M$, which we declare to be an isometry. This map induces an isometric embedding $\mathcal{Q} \hookrightarrow \bigwedge^2 T^*M$, and we will henceforth switch freely between these two points of view. In particular, we will sometimes regard $\mathcal{Q}$ as a subbundle of $\bigwedge^2 T^*M$. The two-forms corresponding to a local orthonormal frame $\{I_k\}$ of $\mathcal{Q}$ will be denoted by $\{\omega_k\}$.

We can canonically associate certain tensor fields to an almost quaternionic-Hermitian structure. Given a local orthonormal frame $\{\omega_k\}$ of $\mathcal{Q}$, consider the locally defined tensor field $\Omega^{\otimes} = \sum_k \omega_k \otimes \omega_k$. In fact, this tensor field is globally defined: Indeed, the transition functions for the bundle $\mathcal{Q}$ can be assumed to take values in SO(3) and therefore leave this tensor field invariant. Anti-symmetrizing, the local expression $\Omega = \sum_k \omega_k \wedge \omega_k$ also yields a globally defined four-form, which is often called the fundamental four-form of the almost quaternionic-Hermitian structure.

Before restricting our attention to distinguished classes of quaternionic geometries, we introduce some general notation and an important computational trick. In certain computations, it will be useful to treat the metric on equal footing with a (local) frame $\{\omega_k\}$ of $\mathcal{Q} \subset \bigwedge^2 T^*M$. For this purpose, we introduce $I_0 = \mathrm{id}$ and $\omega_0 := g$. When we perform such computations, Greek indices will be understood to run from 0 to 3, while Latin indices run from 1 to 3.

To demonstrate the usefulness of considering g and $\{\omega_k\}$ simultaneously, consider an expression of the form

$$\sum_{\mu=0}^{3} \omega_\mu(X,Y)\omega_\mu(Z, I_\lambda W)$$

where X, Y, Z, W are arbitrary tangent vectors, and $\lambda \in \{0,1,2,3\}$. Then we may transfer I_λ to the first factor in the following manner. First, we rewrite the expression as

$$\sum_{\mu=0}^{3} g(I_\mu X, Y) g(I_\lambda^{-1} I_\mu Z, W)$$

Next, we note that the substitution $I_\mu \mapsto I_\lambda I_\mu$ leaves the expression invariant, since each I_μ appears twice (so all signs that might appear cancel out). Thus, we find that

$$\sum_{\mu=0}^{3} \omega_\mu(X,Y)\omega_\mu(Z, I_\lambda W) = \sum_{\mu=0}^{3} \omega_\mu(X, I_\lambda^{-1} Y)\omega_\mu(Z, W) \tag{2.1}$$

This identity, and its variants, will come in usefully on several occasions.

2.1 Quaternionic Kähler geometry

The most interesting kinds of almost quaternionic-Hermitian manifolds are those for which the quaternionic structure bundle is parallel in the appropriate sense.

Definition 2.3. An almost quaternionic-Hermitian manifold $(M, g, \mathcal{Q})$ of dimension at least 8 is said to be a quaternionic Kähler manifold if the quaternionic structure bundle $\mathcal{Q}$ is preserved by the Levi-Civita connection ∇, i.e. $\nabla\Gamma(\mathcal{Q}) \subset \Gamma(T^*M \otimes \mathcal{Q})$.

As discussed in the introductory chapter, this is equivalent to requiring that the holonomy of M is contained in $\mathrm{Sp}(n)\,\mathrm{Sp}(1) \subset \mathrm{SO}(4n)$. The four-dimensional case is exceptional, and a further requirement is necessary in this case:

Definition 2.4. An almost quaternionic-Hermitian four-manifold $(M, g, \mathcal{Q})$ is called quaternionic Kähler if $\nabla\Gamma(\mathcal{Q}) \subset \Gamma(T^*M \otimes \mathcal{Q})$ and every $J \in \Gamma(\mathcal{Q})$, acting as a derivation on the tensor algebra, annihilates the curvature tensor. Concretely, this means

$$g(R(JX, Y)Z, W) + g(R(X, JY)Z, W) + g(R(X, Y)JZ, W) + g(R(X, Y)Z, JW) = 0$$

for every $X, Y, Z, W \in \mathfrak{X}(M)$.

Remark 2.5. Though we will not show this, this is equivalent to asking (M, g) to be an (anti-)self-dual (depending on the choice of orientation) Einstein manifold, which means that it is an Einstein manifold for which the Weyl curvature tensor W is an (anti-)self-dual two-form, i.e. satisfies $\star W = \pm W$. The quaternionic structure bundle turns out to be nothing but the bundle of (anti-)self-dual two-forms.

Note that, despite the name, quaternionic Kähler manifolds do *not* constitute a class of Kähler manifolds (in the language of holonomy, this corresponds to the simple observation that $\mathrm{Sp}(n)\,\mathrm{Sp}(1) \not\subset \mathrm{U}(2n)$). In fact, quaternionic Kähler manifolds generally fail to admit any almost complex structure at all. Nevertheless, quaternionic Kähler manifolds can be viewed as a quaternionic analog of Kähler manifolds, in some sense.

Lemma 2.6. *On a quaternionic Kähler manifold, the tensor field $\Omega^{\otimes}$ is parallel. Conversely, if $\dim M \geq 8$, $(M, g, \mathcal{Q})$ is almost quaternionic-Hermitian such that $\nabla\Omega^{\otimes} = 0$, then it is a quaternionic Kähler manifold.*

Proof. Since we have to check a local property, we may work with respect to a local orthonormal frame $\{\omega_k\}$. We will check that $\nabla_X\Omega^{\otimes} \in \Gamma(\mathcal{Q} \otimes \mathcal{Q})$ is orthogonal to $\omega_j \otimes \omega_k$

for every vector field X and every $j, k \in \{1,2,3\}$:

$$\begin{aligned}\langle \nabla_X \Omega^{\otimes}, \omega_j \otimes \omega_k \rangle &= \sum_l \langle \nabla_X \omega_l \otimes \omega_l + \omega_l \otimes \nabla_X \omega_l, \omega_j \otimes \omega_k \rangle \\ &= \sum_l \langle \nabla_X \omega_l, \omega_j \rangle \langle \omega_l, \omega_k \rangle + \langle \omega_l, \omega_j \rangle \langle \omega_k, \nabla \omega_l \rangle \\ &= \sum_l \langle \nabla_X \omega_k, \omega_j \rangle + \langle \omega_k, \nabla \omega_j \rangle = X \langle \omega_k, \omega_j \rangle = 0\end{aligned}$$

where we used that Levi-Civita connection is compatible with the bundle metric.

For the converse, we observe that since the Riemannian metric and $\Omega^{\otimes}$ are parallel, so is the orthogonal projection operator $\mathcal{P}_{\mathcal{Q}} = \sum_l g(\omega_l, \cdot) \otimes \omega_l \in \Gamma\big(\operatorname{End}(\bigwedge^2 T^*M)\big)$. This means that $\mathcal{P}_{\mathcal{Q}}(\nabla_X \omega_k) = \nabla_X(\mathcal{P}_{\mathcal{Q}}(\omega_k)) = \nabla_X \omega_k$. But since $\mathcal{P}_{\mathcal{Q}}$ projects onto $\mathcal{Q}$, this means that $\nabla_X \omega_k$ is already a section of $\mathcal{Q}$. When $\dim M \geq 8$, this means that $(M, g, \mathcal{Q})$ is quaternionic Kähler. □

Corollary 2.7. *On a quaternionic Kähler manifold, the fundamental four-form is parallel.* □

In fact, the converse holds as well: Ω already determines the full quaternionic Kähler structure. An elegant way to show that $\mathcal{Q}$ is determined by Ω can be found in [ACD03]. The fundamental four-form plays a role which is analogous to the Kähler form on a Kähler manifold. In the latter case, the covariant derivative of the Kähler form is controlled by its exterior derivative. In his PhD thesis, Swann proved an analog for quaternionic Kähler manifolds:

Theorem 2.8 (Swann, [Swa91])**.** *An almost quaternionic-Hermitian manifold of dimension at least* 12 *is quaternionic Kähler if and only if its fundamental four-form is closed. In dimension* 8*, an almost quaternionic-Hermitian manifold is quaternionic Kähler if and only if its fundamental four-form is closed and the algebraic ideal generated by* $\mathcal{Q}$ *is a differential ideal.* □

The analogy between Kähler and quaternionic Kähler manifolds was pursued further by Kraines [Kra66] and Bonan [Bon67; Bon82], who investigated the topology of quaternionic Kähler manifolds. In particular, for compact quaternionic Kähler manifolds they deduced certain inequalities between the Betti numbers, and established a quaternionic analog of the Lefschetz decomposition of the cohomology ring of a compact Kähler manifold. We will not pursue these developments further in this work.

Just as most other geometries appearing in Berger's list, quaternionic Kähler manifolds have strongly restricted curvature tensors. We postpone a more extensive discussion of curvature properties of quaternionic Kähler manifolds to section 2.1.2, but already mention one of the main results:

Theorem 2.9 (Berger, [Ber66]). *Every quaternionic Kähler manifold is Einstein.* □

There are multiple proofs, for instance by direct computation or using representation theory (see, for instance, [Ale68; Sal82; Bes08]). We note some interesting facts that can be deduced immediately from the proof.

Corollary 2.10. *A quaternionic Kähler manifold is Ricci-flat if and only if it is locally hyper-Kähler.* □

Furthermore, the computations (presented in detail in [Bes08]) make it clear that, when the scalar curvature does not vanish, the local almost complex structures $\{I_k\}$ that span $\mathcal{Q}$ determine the $\mathfrak{sp}(1)$-factor of the holonomy algebra $\mathfrak{h} \subset \mathfrak{sp}(n) \oplus \mathfrak{sp}(1)$. In particular, $\mathfrak{h}$ always contains this $\mathfrak{sp}(1)$-factor. This may be used to show that strict quaternionic Kähler manifolds are always irreducible.

2.1.1 Examples of quaternionic Kähler manifolds

Since any quaternionic Kähler manifold is Einstein, the theory naturally splits into three cases according to the sign of the Einstein constant, or equivalently the scalar curvature. Since the case of vanishing scalar curvature corresponds to hyper-Kähler geometry, we will not discuss it here, focusing instead on strict quaternionic Kähler manifolds.

The archetypal example of a compact quaternionic Kähler manifold of positive scalar curvature is quaternionic projective space $\mathbb{HP}^n = (\mathbb{H}^{n+1} \setminus \{0\})/\mathbb{H}^*$. It can be viewed a symmetric space $\mathbb{HP}^n \cong \mathrm{Sp}(n+1)/\mathrm{Sp}(n) \times \mathrm{Sp}(1)$. When equipped with its symmetric metric, this manifold is quaternionic Kähler since the holonomy group of a symmetric space is given by its isotropy group.[1] Note, in particular, that $\mathbb{HP}^1 = S^4$ is indeed a self-dual Einstein manifold—in fact, its Weyl tensor vanishes identically—and every $\mathbb{HP}^n$, being an irreducible symmetric space, is Einstein. A classical theorem, easily proven via characteristic classes, asserts that S^{4k}, and in particular S^4, does not admit an almost complex structure. In fact, $\mathbb{HP}^n$ does not admit an almost complex structure for any $n \in \mathbb{N}$ (a proof can be found in [Mas62]; see also [Hir60]).

The non-compact dual to $\mathbb{HP}^n$ is quaternionic hyperbolic space $\mathrm{Sp}(n,1)/\mathrm{Sp}(n) \times \mathrm{Sp}(1)$, denoted by $\mathbb{HH}^n$; it is the most prominent example of a quaternionic Kähler manifold with negative scalar curvature. There are further examples of quaternionic Kähler symmetric spaces. These were classified by Wolf [Wol65] and therefore we call them Wolf spaces. Note that this terminology is not completely standard: Some authors prefer to refer only to the quaternionic Kähler symmetric spaces of positive scalar curvature as Wolf spaces.

[1] Note that the isotropy group really is $\mathrm{Sp}(n)\,\mathrm{Sp}(1)$, not $\mathrm{Sp}(n) \times \mathrm{Sp}(1)$. Indeed, since $\mathrm{Sp}(n+1)$ only acts nearly effectively on $\mathbb{HP}^n$, one could more precisely write $\mathbb{HP}^n \cong (\mathrm{Sp}(n+1)/\mathbb{Z}_2)/\mathrm{Sp}(n)\,\mathrm{Sp}(1)$.

Wolf showed that there is a bijective correspondence[2] between quaternionic Kähler symmetric spaces of positive scalar curvature and compact, simple Lie groups (excluding $SU(2)$). Their non-compact duals then give the quaternionic Kähler symmetric spaces of negative scalar curvature. They are listed in table 1.

$\dim M$	G	H
$4n$	$\mathrm{Sp}(n+1)$	$\mathrm{Sp}(n) \times \mathrm{Sp}(1)$
$4n$	$\mathrm{SU}(n+2)$	$\mathrm{S}(\mathrm{U}(n) \times \mathrm{U}(2))$
$4n$	$\mathrm{SO}(n+4)$	$\mathrm{S}(\mathrm{O}(n) \times \mathrm{O}(4))$
8	G_2	$\mathrm{SO}(4)$
28	F_4	$\mathrm{Sp}(3) \cdot \mathrm{Sp}(1)$
40	E_6	$\mathrm{SU}(6) \cdot \mathrm{Sp}(1)$
64	E_7	$\mathrm{Spin}(12) \cdot \mathrm{Sp}(1)$
112	E_8	$\mathrm{E}_7 \cdot \mathrm{Sp}(1)$

Table 1: The Wolf spaces of positive scalar curvature are of the form G/H, where the pairs (G, H) are given above. Note that G is allowed to act nearly effectively.

Note that the non-compact duals admit compact quotients (by a famous theorem of Borel [Bor63]), so compact, locally symmetric examples of negative scalar curvature are known as well. However, since one of the main distinguishing features of quaternionic Kähler geometry is its appearance in Berger's list, it is natural that there exists a particular interest in complete examples that are not locally symmetric. These are, under the assumption of completeness, Riemannian manifolds of holonomy precisely $\mathrm{Sp}(n)\,\mathrm{Sp}(1)$. The search for such examples has proven rather difficult. Let us first consider the case of positive scalar curvature.

A first place one might look for non-locally symmetric quaternionic Kähler manifolds is in the broader class of homogeneous spaces. However, in this direction there is the following no-go theorem:

Theorem 2.11 (Alekseevsky–Cortés, [AC97]). *Any strict quaternionic Kähler manifold which is homogeneous under a unimodular group is symmetric.* □

Since the isometry group of a compact manifold is compact and therefore unimodular, one easily deduces:

Corollary 2.12. *Every compact, homogeneous strict quaternionic Kähler manifold is symmetric.* □

[2]Because of accidental isomorphisms between low-dimensional Lie groups, not all Wolf spaces in table 1 are distinct. We have $\mathrm{Sp}(2)/\mathrm{Sp}(1) \times \mathrm{Sp}(1) \cong S^4 \cong \mathrm{SO}(5)/\mathrm{S}(\mathrm{O}(1) \times \mathrm{O}(4))$ and $\mathrm{SU}(4)/\mathrm{S}(\mathrm{U}(2) \times \mathrm{U}(2)) \cong \mathrm{SO}(6)/\mathrm{S}(\mathrm{O}(2) \times \mathrm{O}(4))$.

This rules out homogeneous and non-symmetric examples in the case of positive scalar curvature, since Myers' theorem implies such manifolds must be compact. In the early 1980's, a major breakthrough made it possible to attack the problem of existence of non-symmetric examples without assumptions on the isometry group. It became clear that quaternionic Kähler $4n$-manifolds of positive scalar curvature come with a canonically associated $4n+2$-dimensional Kähler–Einstein manifold of positive scalar curvature (i.e. a Fano manifold), called its twistor space, which has additional nice properties such as the existence of a holomorphic contact structure and an anti-linear involution. Crucially, the complex-geometric properties of the twistor space can be shown to determine the quaternionic Kähler metric completely (for more information, see section 2.4).

Ideas from complex and algebraic geometry can be used to show that twistor spaces are severely constrained, which in turn restricts the corresponding quaternionic Kähler manifolds. In low dimensions, these ideas suffice to show that there are no possibilities beyond the Wolf spaces. The four-dimensional case is due to Friedrich and Kurke and, independently, Hitchin [Hit81].[3]

Theorem 2.13 (Friedrich–Kurke, [FK82]; Hitchin, [Hit81]). *Let (M, g) be a self-dual Einstein four-manifold of positive scalar curvature. Then (M, g) is homothetic to either S^4 or $\mathbb{C}\mathrm{P}^2$, equipped with their standard metrics.* □

Remark 2.14. The theorem stated in the paper of Friedrich–Kurke is actually that (M, g) is either homothetic to S^4 or diffeomorphic to $\mathbb{C}\mathrm{P}^2$, but it was pointed out by Hitchin, in his review of the paper on MathSciNet, that their argument can be easily adapted to obtain the above version.

The corresponding result in dimension eight was proven by Poon and Salamon:

Theorem 2.15 (Poon–Salamon, [PS91]). *Any quaternionic Kähler 8-manifold of positive scalar curvature is a Wolf space. In other words, it is homothetic to $\mathbb{H}\mathrm{P}^2$, $\mathrm{Gr}_2(\mathbb{C}^4)$ or $\mathrm{G}_2 / \mathrm{SO}(4)$.* □

Not much later, these results were extended by LeBrun and Salamon, who proved the following rigidity result, valid in arbitrary dimensions:

Theorem 2.16 (LeBrun–Salamon, [LS94]). *Up to homothety, there are only finitely many $4n$-dimensional quaternionic Kähler manifolds of positive scalar curvature for any $n \in \mathbb{N}$.* □

[3] In fact, most people seem to credit only Hitchin with this theorem. Hitchin himself, in his MathSciNet review of the paper by Friedrich and Kurke, does not take credit (though he notes the strong similarity).

Furthermore, they proved strong constraints on the second homotopy group (the fundamental group is known to always vanish) as well as the Betti numbers. These result can all be viewed as strong supporting evidence for their famous conjecture, stated in the same paper:

Conjecture 2.17 (LeBrun–Salamon). *Every quaternionic Kähler manifold of positive scalar curvature is a Wolf space.*

As alluded to above, this can be translated to a statement about Fano manifolds endowed with a holomorphic contact structure. Very recently, advances in complex algebraic geometry have led to a proof for Fano manifolds of (real) dimension 14 and 18, which settles the conjecture for quaternionic Kähler manifolds of dimension 12 and 16:

Theorem 2.18 (Buczyński–Wiśniewski–Weber, [BWW18]). *The conjecture of LeBrun and Salamon holds true in dimensions* 12 *and* 16. □

It is widely believed that the LeBrun–Salamon conjecture is true in every dimension, but proving it remains one of the major open problems in quaternionic Kähler geometry.

Let us now give a brief (and incomplete) survey of the case of negative scalar curvature, which is rather different. In classifying quaternionic Kähler manifolds which are homogeneous under a solvable group, Alekseevsky [Ale75] exhibited two infinite series of homogeneous examples which are not locally symmetric, showing that the situation is not as dire as in the positive case. Not much progress was made until about fifteen years later, when multiple breakthroughs occurred in rapid succession.

Firstly, LeBrun applied techniques from deformation theory in complex algebraic geometry to the twistor space of $\mathbb{H}\mathrm{H}^n$ to prove the following striking result:

Theorem 2.19 (LeBrun, [LeB91]). *There exists an infinite-dimensional family of pairwise distinct deformations of the standard quaternionic Kähler metric on* $\mathbb{H}\mathrm{H}^n$*, each member of which is a complete quaternionic Kähler metric.*

In particular, the moduli space of quaternionic Kähler metrics on $\mathbb{R}^{4n}$ is infinite dimensional for every $n \in \mathbb{N}$. Heuristically, this result says that quaternionic Kähler metrics with negative scalar curvature occur in abundance. The twistor-theoretic approach to quaternionic Kähler geometry employed by LeBrun has one major drawback, however. Although the quaternionic Kähler metric is determined by the complex geometry of its twistor space in principle, it is typically not practically feasible to explicitly recover the metric defining the quaternionic Kähler structure. Nevertheless, LeBrun's theorem indicated that it must be possible to construct additional non-symmetric examples, besides the spaces considered by Alekseevsky.

Meanwhile, unbeknownst to most mathematicians at the time, a second major development was taking place in the community of theoretical physicists. A description of the historical background would lead us too far astray at the moment, so we postpone it to section 4.1. Suffice it to say that the physicists De Wit and Van Proeyen used physical reasoning to give strong evidence [dWvP92] that Alekseevsky's claimed classification of quaternionic Kähler manifolds, homogeneous under a solvable group (which we will henceforth call Alekseevsky spaces) was in fact incomplete. Their conjecture was rigorously proven via Lie-theoretic methods by Cortés [Cor96b], completing the classification of Alekseevsky spaces.

The physical construction used in the work of De Wit and Van Proeyen, and the mathematics that was inspired by it, has led to many further examples of complete quaternionic Kähler metrics. This class of examples is the main focus of this work, and will be discussed in detail in chapters 4 and 5. For now, we only note it includes manifolds which are not (even locally) homogeneous, and that the construction yields explicit expression for the resulting quaternionic Kähler metrics.

Let us name a few other sources of examples, such as the quaternionic Kähler quotient construction (see e.g. [Gal91]), which we review in section 2.3. In the four-dimensional case and assuming the existence of a Killing field, there is a local characterization in terms of solutions to the so-called $\mathrm{SU}(\infty)$ Toda equation [Tod97]. Work of Calderbank and Pedersen [CP02] in the four-dimensional case has been extended by physicists to give a local description of quaternionic Kähler $4n$-manifolds with $n+1$ commuting Killing fields, see [RVV10].

2.1.2 Curvature and Killing fields

One of the reasons for the great interest in manifolds with reduced holonomy is that they have interesting curvature properties. Indeed, we have seen that quaternionic Kähler manifolds are always Einstein, but not Ricci-flat unless they are locally hyper-Kähler. This distinguishes them among the flavors of geometry appearing in Berger's list, which are all Ricci-flat except for the cases $\mathrm{SO}(n)$ (generic) and $\mathrm{U}(n)$ (Kähler), which are so broad that not much can be said about their curvature properties.

Before we continue our discussion of the curvature of quaternionic Kähler manifolds, we introduce some notation to help simplify the following expressions. Firstly, on a $4n$-manifold M, we will call $\nu = \frac{\mathrm{scal}}{4n(n+2)}$ the reduced scalar curvature. Next, we introduce some tensor fields which behave algebraically like curvature tensors.

Definition 2.20. We say a $(0,4)$-tensor field on a manifold M is an algebraic curvature tensor if it possesses the algebraic symmetries of the (lowered) Riemann curvature tensor.

We shall occasionally apply this terminology to $(1, 3)$-tensor fields if the corresponding $(0, 4)$-tensor, obtained by contracting with a Riemannian metric, is an algebraic curvature tensor.

Definition 2.21. Given a real vector space V, we define two linear maps $(V^*)^{\otimes 4} \to (V^*)^{\otimes 4}$ as follows. For $\Phi \in (V^*)^{\otimes 4}$ and $X, Y, Z, W \in V$, we set

$$\Phi^{\owedge}(X, Y, Z, W) := \Phi(X, Z, Y, W) - \Phi(X, W, Y, Z) + \Phi(Y, W, X, Z) - \Phi(Y, Z, X, W)$$

and

$$\Phi^{\obar}(X, Y, Z, W) := \Phi^{\owedge}(X, Y, Z, W) + 2\Phi(X, Y, Z, W) + 2\Phi(Z, W, X, Y)$$

Taking V to be the tangent spaces of a manifold M, we obtain maps of $(0, 4)$-tensor fields on M. For $(0, 2)$-tensors α and β, we set $\alpha \owedge \beta := (\alpha \otimes \beta)^{\owedge}$ and define $\alpha \obar \beta$ in analogous fashion. Note that, in this way, the first map reduces to the Kulkarni–Nomizu product. It is well-known that this product produces an algebraic curvature tensor when it is applied to two symmetric bilinear forms. On two-forms, however, the resulting $(0, 4)$-tensor may fail to satisfy the Bianchi identity. The $\obar$-product remedies this by constructing an algebraic curvature tensor out of two arbitrary two-forms.

With this notation, we can introduce the simplest, yet most important, example.

Example 2.22. Consider $\mathbb{H}\mathrm{P}^n$, equipped with its symmetric metric with reduced scalar curvature 1. Then the Riemann curvature tensor is given by

$$g(R(X, Y)Z, W) = -\frac{1}{8}\left(g \owedge g + \sum_{k=1}^{3} \omega_k \obar \omega_k\right)(X, Y, Z, W) \tag{2.2}$$

where $\{\omega_k\}$ is an arbitrary (local) orthonormal frame of $\mathcal{Q} \subset \bigwedge^2 T^*\mathbb{H}\mathrm{P}^n$.

The paramount importance of this example derives from the following curvature decomposition theorem of Alekseevsky:

Theorem 2.23 (Alekseevsky, [Ale68])**.** *The Riemann curvature tensor R of a quaternionic Kähler manifold M is of the form $R = \nu R_0 + R_1$, where R_0 is formally the curvature tensor of $\mathbb{H}\mathrm{P}^n$, and R_1 is an algebraic curvature tensor of hyper-Kähler type. This means that R_1 is trace-free and commutes with every section of $\mathcal{Q}$.* □

Remark 2.24. R_1 is sometimes called the quaternionic Weyl curvature of M.

As a corollary, we observe that the Ricci tensor of a quaternionic Kähler manifold is determined by the Ricci tensor of quaternionic projective space. Since the latter is Einstein, this gives a simple argument why every quaternionic Kähler manifold must be Einstein. As a related application of this important theorem, we see that the curvature $\mathcal{R}$ of $\mathcal{Q}$ is completely determined by the curvature of $\mathbb{H}\mathrm{P}^n$:

Proposition 2.25. *Let $\{I_k\}$ be a local orthonormal frame for $\mathcal{Q}$. Then*

$$g(I_k, \mathcal{R}(X,Y)I_l) = g(I_k, [R(X,Y), I_l]) = \nu \sum_{m=1}^{3} \epsilon_{klm}\omega_m(X,Y)$$

where ϵ_{klm} denotes the (totally anti-symmetric) Levi-Civita symbol, and X, Y are arbitrary tangent vectors.

Proof. Since the quaternionic Weyl curvature commutes with every section of $\mathcal{Q}$, it suffices to compute $[R_0(X,Y), I_k]$, where R_0 is given by (2.2). Using this expression, we have

$$\begin{aligned}
-8g([R_0(X,Y), I_k]Z, W) &= \Big(g \owedge g + \sum_l \omega_l \varobar \omega_l\Big)(X, Y, I_kZ, W) \\
&\quad + \Big(g \owedge g + \sum_l \omega_l \varobar \omega_l\Big)(X, Y, Z, I_kW) \\
&= \sum_{\mu=0}^{3} \omega_\mu \varobar \omega_\mu (X \otimes Y \otimes I_kZ \otimes W + X \otimes Y \otimes Z \otimes I_kW) \\
&\quad - 4g(X,Y)(g(I_kZ, W) + g(Z, I_kW))
\end{aligned}$$

Note that the summation in the final expression runs from 0 to 3, and that $\omega_0 = g$. The second line clearly vanishes, so we are left with

$$\begin{aligned}
-4g([R_0(X,Y), I_k]Z, W) &= \sum_{\mu=0}^{3} \Big(\omega_\mu(X, I_kZ)\omega_\mu(Y, W) + \omega_\mu(X, Z)\omega_\mu(Y, I_kW) \\
&\quad - \omega_\mu(X, W)\omega_\mu(Y, I_kZ) - \omega_\mu(X, I_kW)\omega_\mu(Y, Z) \\
&\quad + 2\omega_\mu(X,Y)\big(\omega_\mu(I_kZ, W) + \omega_\mu(Z, I_kW)\big)\Big) \\
&= \sum_\mu \Big(- \omega_\mu(X, Z)\omega_\mu(Y, I_kW) + \omega_\mu(X, Z)\omega_\mu(Y, I_kW) \\
&\quad + \omega_\mu(X, I_kW)\omega_\mu(Y, Z) - \omega_\mu(X, I_kW)\omega_\mu(Y, Z) \\
&\quad + 2\omega_\mu(X,Y)\big(\omega_\mu(I_kZ, W) + \omega_\mu(Z, I_kW)\big)\Big) \\
&= 2\sum_\mu \omega_\mu(X,Y) g([I_\mu, I_k]Z, W) \\
&= 2\sum_{m=1}^{3} g([I_m, I_k]Z, W)\omega_m(X,Y)
\end{aligned}$$

where we used (2.1) on the first and third terms. We deduce that

$$[R_0(X,Y), I_k] = \frac{1}{2}\sum_m [I_k, I_m]\omega_m(X,Y)$$

Now, we use that $[I_k, I_m] = 2\sum_n \epsilon_{kmn} I_n$ and $\{I_k\}$ is an orthonormal frame and find

$$g(I_k, [R_0(X,Y), I_l]) = \sum_m \epsilon_{klm}\omega_m(X,Y)$$

which was our claim. □

Corollary 2.26. *For any quaternionic Kähler manifold M with non-vanishing scalar curvature, the curvature of $\mathcal{Q}$ induces an injective bundle map $\mathcal{Q} \to \bigwedge^2 T^*M \otimes \mathcal{Q}$.*

Proof. Recalling that the local two-forms $\{\omega_k\}$ are non-degenerate, this follows directly from the statement of the above proposition. □

Using canonical isometries between certain bundles on M, we obtain various other interpretations of $\mathcal{R}$ which appear in the literature. For instance, since $\mathcal{Q}$ is an oriented rank three vector bundle, there is a bundle isometry $\epsilon : \operatorname{Skew}\operatorname{End}\mathcal{Q} \to \mathcal{Q}$ which generalizes (minus) the cross product familiar from $\mathbb{R}^3$. We define ϵ by sending $A = \sum_{i,j} A_{ij} I_i^* \otimes I_j \in \Gamma(\operatorname{Skew}\operatorname{End}\mathcal{Q})$, to $-\frac{1}{2}\sum_{i,j,k} A_{ij}\epsilon_{ijk} I_k \in \Gamma(\mathcal{Q})$. Under this identification (note that it sends $I_2^* \otimes I_3 \mapsto -I_1$!) we find $\mathcal{R} = \nu \sum_k \omega_k \otimes I_k \in \Gamma\big(\bigwedge^2 T^*M \otimes \mathcal{Q}\big)$. Contracting with the metric, we may also write $\mathcal{R} = \nu\Omega^{\otimes}$. Finally, dualizing in one of the factors, we can interpret $\mathcal{R}$ as $\nu\mathcal{P}_{\mathcal{Q}}$, where $\mathcal{P}_{\mathcal{Q}} : \bigwedge^2 T^*M \to \mathcal{Q}$ is the orthogonal projection onto $\mathcal{Q} \subset \bigwedge^2 T^*M$ (which we previously encountered in the proof of lemma 2.6).

Another important fact, which we will make use of later on, is that the infinitesimal isometries of a quaternionic Kähler manifold are automatically compatible with the quaternionic structure in the following sense:

Proposition 2.27. *Any Killing field X on a complete quaternionic Kähler manifold of non-zero scalar curvature preserves the quaternionic structure bundle $\mathcal{Q}$.*

Proof. Using a local frame $\{\omega_k\}$ for $\mathcal{Q}$, define local one-forms $\{\alpha_{jk}\}$ via $\nabla\omega_j = \sum_k \alpha_{jk} \otimes \omega_k$. Applying Cartan's formula and writing exterior derivatives in terms of covariant derivatives, we have

$$L_X\omega_j = \sum_k \big(\alpha_{jk}(X)\omega_k - \alpha_{jk} \wedge \iota_X\omega_k\big) + \mathrm{d}\iota_X\omega_j$$

We rewrite the last term. Using $\nabla(\iota_X\omega_j) = \iota_{\nabla X}\omega_j + \sum_k \alpha_{jk} \otimes \iota_X\omega_k$ we see that

$$\mathrm{d}\iota_X\omega_j(U,V) = \omega_j(\nabla_U X, V) - \omega_j(\nabla_V X, U) + \sum_k (\alpha_{jk} \wedge \iota_X\omega_k)(U,V)$$

which means that

$$(L_X\omega_j)(U,V) = \sum_k \alpha_{jk}(X)\omega_k(U,V) + \omega_j(\nabla_U X, V) - \omega_j(\nabla_V X, U)$$
$$=: \sum_k \alpha_{jk}(X)\omega_k(U,V) + (\omega_j(\nabla X,\cdot))^{\mathrm{alt}}(U,V)$$

Thus, it suffices to prove that $(\omega_j(\nabla X,\cdot))^{\mathrm{alt}}$ is a section of $\mathcal{Q}$. Using the fact that X is a Killing field, we obtain

$$(\omega_j(\nabla X,\cdot))^{\mathrm{alt}}(U,V) = g(\nabla_{I_jV}X,U) - g(I_j\nabla_V X,U) = g([\nabla X, I_j]V,U)$$

This means that we have to show that $[\nabla X, I_j]$ is a section $\mathcal{Q}$—now regarded as a bundle of endomorphisms. In other words, ∇X must normalize the $\mathfrak{sp}(1)$-factor of the holonomy algebra.

Now, there are two different cases to consider. If our manifold is locally symmetric, its Killing algebra is explicitly known and the condition can be checked to hold in each case [ACDV03]. If this is not the case, we know that the holonomy algebra is $\mathfrak{sp}(n)\oplus\mathfrak{sp}(1)$. Then our claim follows from a more general statement due to Kostant:

Lemma 2.28 (Kostant, [Kos55]). *Let X be a Killing field on a Riemannian manifold. Then $\nabla X \in \Gamma(\operatorname{End} TM)$ normalizes the holonomy algebra $\mathfrak{h}$.*

Proof. First note that L_X commutes with the Levi-Civita connection on account of X being a Killing field. Now let Y, Z be arbitrary vector fields. Then

$$\nabla_Y(\nabla X)(Z) = [\nabla_Y, \nabla X]Z = [\nabla_Y, (\nabla_X - L_X)]Z = -R(X,Y)Z$$

By the Ambrose–Singer theorem [AS53], $R(X,Y)$ lies in $\mathfrak{h}$. Now consider the (pointwise) decomposition $\nabla X = (\nabla X)^{\mathfrak{h}} + (\nabla X)^{\perp}$, where the first part is contained in $\mathfrak{h}$ and the second term is orthogonal to $\mathfrak{h}$. Because parallel transport preserves $\mathfrak{h}$, it also preserves its orthogonal complement, which means that $\nabla_Y(\nabla X)^{\mathfrak{h}} \in \mathfrak{h}$ and $\nabla_Y(\nabla X)^{\perp} \in \mathfrak{h}^{\perp}$. But then our first identity shows that $(\nabla X)^{\perp}$ must in fact be parallel, i.e. commutes with $\mathfrak{h}$. Since $(\nabla X)^{\mathfrak{h}} \in \mathfrak{h}$ certainly normalizes $\mathfrak{h}$, we conclude that ∇X does too. □

Thus, ∇X actually normalizes the full holonomy algebra $\mathfrak{sp}(n)\oplus\mathfrak{sp}(1)$, completing our proof. □

2.2 Hyper-Kähler geometry

In the introduction, we discussed why a hyper-Kähler structure, defined in terms of holonomy, is equivalent to the following definition:

Definition 2.29. An almost quaternionic-Hermitian manifold (M, g) is called hyper-Kähler if $\mathcal{Q}$ admits a global trivialization $\{I_1, I_2, I_3\}$ by almost complex structures which satisfy $I_1 I_2 = -I_2 I_1 = I_3$ and are covariantly constant.

These almost complex structures are automatically integrable, by definition of the Nijenhuis tensor. In fact, it suffices to assume that the corresponding two-forms $\{\omega_k\}$ are only closed, rather than parallel, to obtain a hyper-Kähler structure. This is a well-known lemma due to Hitchin [Hit87]. Thus, g is a Kähler metric with respect to each of these complex structures, justifying the name hyper-Kähler. Moreover, as noted in our discussion of quaternionic Kähler manifolds, hyper-Kähler metrics are always Ricci-flat.

There is yet another point of view on hyper-Kähler geometry, which has been exploited to great effect by algebraic geometers. Given a hyper-Kähler structure on a manifold M, we can pick a preferred complex structure, say, I_1. Now notice that $\varpi = \omega_2 + i\omega_3 \in \bigwedge^2 T^*_{\mathbb{C}} M$ is of type $(2,0)$ with respect to I_1. Indeed, for anti-holomorphic $X \in \Gamma(T^{0,1}M)$ and arbitrary $Y \in \Gamma(T_{\mathbb{C}}M)$, we have $I_1 X = -iX$ and therefore

$$-\omega_3(X,Y) + i\omega_2(X,Y) = \omega_2(I_1X, Y) + i\omega_3(I_1X, Y) = -i\omega_2(X,Y) + \omega_3(X,Y)$$

Thus, $\iota_X \varpi = 0$ for every $X \in \Gamma(T^{0,1}M)$, as claimed. Since ϖ is clearly closed, the type decomposition shows that $\bar{\partial}\varpi = (\mathrm{d}\varpi)^{2,1} = 0$ and thus ϖ is holomorphic; it is moreover non-degenerate.

Definition 2.30. A complex manifold (M, I) endowed with a holomorphic, non-degenerate two-form is called holomorphic symplectic.

Thus, every hyper-Kähler manifold can be viewed as a Kähler manifold which is also holomorphic symplectic. The holomorphic symplectic structure is parallel, and conversely a Kähler manifold with parallel holomorphic symplectic structure is hyper-Kähler: On the level of holonomy groups this corresponds to the observation that $\mathrm{Sp}(n) = \mathrm{U}(2n) \cap \mathrm{Sp}(2n, \mathbb{C})$. The existence of a holomorphic symplectic structure gives another way to see that hyper-Kähler manifolds must be Ricci-flat, since its n-fold exterior power gives a trivialization of the canonical bundle, whose curvature is the Ricci form.

Under the additional assumption of compactness, the assumption that the holomorphic symplectic structure is parallel may be dropped:

Theorem 2.31 (Beauville, [Bea83]). *Let (M, I) be a compact complex manifold admitting a Kähler metric and a holomorphic symplectic structure. Then M admits a unique hyper-Kähler metric in every Kähler class.* □

The proof uses Yau's resolution of the Calabi conjecture to produce a Ricci-flat Kähler metric from triviality of the canonical bundle. The holomorphic symplectic structure is then shown to be parallel by a Bochner-type formula.

2.2.1 Examples of hyper-Kähler manifolds

In contrast to the situation for strict quaternionic Kähler manifolds, there exist no hyper-Kähler symmetric or indeed even homogeneous spaces except flat manifolds. This is a consequence of the general fact that the only compact and Ricci-flat homogeneous spaces are flat tori. In fact, a more general result holds:

Theorem 2.32 (Alekseevsky–Kimelfeld, [AK75]). *Every Ricci-flat homogeneous space is flat, i.e. the direct product of a Euclidean space and a flat torus.* □

Thus, the basic model for a hyper-Kähler manifold is just the vector space $\mathbb{H}^n$, equipped with its flat hyper-Kähler structure. Splitting $q \in \mathbb{H}^n$ into a pair (z, w) of complex vectors satisfying $q = z + wj$, the standard hyper-Kähler structure on this space is determined by the tensor fields

$$\begin{aligned} g &= \sum_l \mathrm{d}z_l \mathrm{d}\bar{z}_l + \mathrm{d}w_l \mathrm{d}\bar{w}_l \\ \omega_1 &= \frac{i}{2} \sum_l \big(\mathrm{d}z_l \wedge \mathrm{d}\bar{z}_l + \mathrm{d}w_l \wedge \mathrm{d}\bar{w}_l\big) \\ \omega_2 &= \frac{1}{2} \sum_l \big(\mathrm{d}z_l \wedge \mathrm{d}w_l + \mathrm{d}\bar{z}_l \wedge \mathrm{d}\bar{w}_l\big) \\ \omega_3 &= \frac{1}{2i} \sum_l \big(\mathrm{d}z_l \wedge \mathrm{d}w_l - \mathrm{d}\bar{z}_l \wedge \mathrm{d}\bar{w}_l\big) \end{aligned} \tag{2.3}$$

Accordingly, the holomorphic symplectic form with respect to the complex structure I_1 is $\varpi_1 = \sum \mathrm{d}z_l \wedge \mathrm{d}w_l$. This structure corresponds to viewing $\mathbb{H}^n$ as the cotangent bundle of $\mathbb{C}^n$.

A small number of compact examples is known. The K3 surface, which we may define—as a smooth manifold—as a smooth quartic hypersurface in $\mathbb{C}\mathrm{P}^3$, was the only known compact hyper-Kähler manifold until Beauville [Bea83] constructed examples in every dimension. An extreme scarcity has persisted however, as noted in [Bea11]. This review lists two infinite families, which yield one example in every dimension, and two additional examples (one each in dimensions 12 and 20), as the only compact examples up to deformation. In the same paper, it is conjectured that there are only finitely many compact hyper-Kähler manifolds (up to deformation) in every dimension.

The situation is rather different if one asks for manifolds which are only complete: Many examples of complete, non-compact hyper-Kähler manifolds are known. A rich source of such examples is the hyper-Kähler quotient construction, which we review in section 2.3. A representative sample is given in the expository paper [Hit92], in which the following families of examples are discussed (among others):

(i) ALE spaces, which are minimal resolutions of $\mathbb{C}^2/\Gamma$, where $\Gamma \subset \mathrm{SU}(2)$ is a finite group. The hyper-Kähler quotient construction for these examples is due to Kronheimer [Kro89].

(ii) Co-adjoint orbits of complex semi-simple Lie groups, originally constructed as hyper-Kähler quotients in the papers [Kro90; Kov96; Biq96].

(iii) The moduli space of stable Higgs bundles with structure group G on a Riemann surface Σ or, equivalently, the moduli space of irreducible $G_{\mathbb{C}}$-representations of $\pi_1(\Sigma)$. The resulting hyper-Kähler structure was originally described in [Hit87].

(iv) The space of based rational maps $f : \mathbb{C}\mathrm{P}^1 \to \mathbb{C}\mathrm{P}^1$ of fixed degree, as described in the book [AH88].

Further examples that admit a similar description include the quiver varieties first described by Nakajima [Nak94], which provide a generalization of Kronheimer's construction. We should mention that some of these examples can also be obtained by twistor-theoretic means.

It is remarkable that many of these interesting examples arise from quotients by linear actions on symplectic vector spaces (in some cases, these are infinite-dimensional). Another, perhaps even more important observation is that there are strong ties to mathematical physics. Indeed, in most cases the hyper-Kähler structure is found by realizing the manifold as the moduli space of solutions to certain differential equations with physical significance.

Here, a central role is played by the Yang–Mills equations and particularly their specialization to the (anti-)self-duality equations. For instance, the hyper-Kähler structure on third and fourth items in the above list were found by showing that the relevant spaces can be identified with the moduli spaces of solutions to Hitchin's equations and the Bogomolny equation, respectively. These equations arise naturally by considering solutions to the self-duality equations on $\mathbb{R}^4$ which are invariant under translations in two, respectively one, direction.

In certain simple settings, hyper-Kähler metrics can be constructed via other methods. Especially the question of existence of hyper-Kähler structures on cotangent bundles of complex manifolds has been studied in great detail. It is reasonable to hope for a hyper-Kähler structure on such spaces, since they naturally carry a holomorphic symplectic structure. The first positive result in this direction is due to Calabi, who constructed a hyper-Kähler metric on $T^*\mathbb{C}\mathrm{P}^n$, which is now commonly referred to as the Calabi metric.

Calabi's construction was generalized to the broader class of cotangent bundles of compact Hermitian symmetric spaces G/H by Biquard and Gauduchon [BG97]. They construct a complete hyper-Kähler metric which is invariant under the action of G and extends the symmetric Kähler metric on the base in the sense that it restricts to this metric on the zero section. They prove that these two properties uniquely characterizes the metric, and explicitly describe it in terms of a global Kähler potential (with respect to one of the complex structures). This formula generalizes the one given by Calabi for $T^*\mathbb{C}\mathrm{P}^n$.

Biquard and Gauduchon also considered the dual symmetric spaces of non-compact type, and proved that, in this case, there is once again a unique, G-invariant hyper-Kähler metric extending the base metric. The big difference is that, in this case, it can only be defined on an open neighborhood of the zero section, and is incomplete. This local construction was vastly generalized independently by both Feix and Kaledin:

Theorem 2.33 (Feix, [Fei01], Kaledin [Kal99]). *If (M, g, I) is a real-analytic Kähler manifold, then a neighborhood of the zero section of its cotangent bundle carries a unique (up to bundle automorphisms) hyper-Kähler metric h which restricts to g on the zero section and is invariant under the* $\mathrm{U}(1)$*-action given by multiplication by unit complex numbers in the fibers.* □

Remark 2.34. Kaledin's result originally appeared as part of the book [KV99].

Though the hyper-Kähler metric is generally incomplete and only locally defined, one recovers the metrics of Calabi and Biquard–Gauduchon when M is a Hermitian symmetric space of compact type. These constructions on cotangent bundles are closely related to the hyper-Kähler quotient constructions mentioned above. Indeed, Biquard and Gauduchon [BG] exhibit an explicit diffeomorphism between T^*M, where M is a compact Hermitian symmetric space, and an appropriate co-adjoint orbit, and prove that the hyper-Kähler metric they constructed coincides with the metric obtained on the orbit by the hyper-Kähler quotient construction.

Finally, we mention that if one allows for indefinite signature, there are further examples of pseudo-Kähler manifolds whose cotangent bundles carry a pseudo-hyper-Kähler structure. We will encounter these spaces in the next chapter.

2.2.2 Curvature and Killing fields

In the statement of Alekseevsky's decomposition theorem for the curvature tensor of quaternionic Kähler manifolds, we encountered the notion of a curvature tensor of hyper-Kähler type. Of course, this terminology is consistent: The curvature tensor of a hyper-Kähler manifold is of hyper-Kähler type. Indeed, hyper-Kähler manifolds are Ricci-flat, i.e. traceless, and the curvature endomorphisms generate the holonomy algebra $\mathfrak{h} \subset \mathfrak{sp}(n)$, and therefore commute with sections of the quaternionic structure bundle $\mathcal{Q}$.

Let us now consider a complete and irreducible hyper-Kähler manifold equipped with a Killing field X. To study its interaction with the quaternionic structure bundle $\mathcal{Q}$, we recall the proof of proposition 2.27. Kostant's lemma shows that the endomorphism field

∇X normalizes the holonomy algebra $\mathfrak{sp}(n)$ and must therefore lie in $\mathfrak{sp}(n) \oplus \mathfrak{sp}(1) \subset \mathfrak{so}(4n)$. In conclusion, X preserves $\mathcal{Q}$ or equivalently the fundamental four-form Ω, which in the hyper-Kähler case can be written globally as $\Omega = \sum_k \omega_k \wedge \omega_k$.

Given a strict quaternionic Kähler manifold, this is all one can ask for, but in the hyper-Kähler case a refinement is possible. Indeed, we now have a global frame for $\mathcal{Q}$ and can study the action of X on it. In particular, we may ask if X preserves each of the three complex structure I_k independently.

Definition 2.35. An isometric group action on a hyper-Kähler manifold (M, g, I_k) is called tri-holomorphic if it preserves each of the complex structures I_k.

But this is not the only possibility, as we now show. Viewing $\mathcal{Q}$ as a bundle of two-forms and using that each ω_k is parallel, we have $L_X \omega_k = -\nabla X \cdot \omega_k$, where $(\nabla X \cdot \omega_k)(Y, Z) = -\omega_k(\nabla_Y X, Z) - \omega_k(Y, \nabla_Z X)$. This means that the action of X on $\mathcal{Q}$ is $C^\infty(M)$-linear. Moreover, the quaternionic relations $I_k^2 = I_1 I_2 I_3 = -\operatorname{id}$ force it to be skew, i.e. $L_X \in \mathfrak{so}(\mathcal{Q}) \cong \mathfrak{so}(3)$. The action on $\mathcal{Q}$ it generates is therefore a one-parameter subgroup of $\mathrm{SO}(3)$. Such a subgroup is either trivial—this is the tri-holomorphic case—or fixes a single complex structure while rotating its orthogonal complement.

Definition 2.36. A Killing field X on a hyper-Kähler manifold (M, g, I_k) which fixes one of the complex structures and acts by an $\mathfrak{so}(2)$-transformation on its orthogonal complement is called rotating.

If we are dealing with only one such Killing field we may assume, possibly after picking a different frame for $\mathcal{Q}$ and rescaling X, that $L_X I_1 = 0$ and $L_X I_2 = I_3$ while $L_X I_3 = -I_2$. With this terminology, the above discussion can be summarized as follows:

Proposition 2.37. *A Killing field on a complete and irreducible hyper-Kähler manifold is either tri-holomorphic or rotating.* □

2.3 Group actions and quotient constructions

When it comes to the construction of interesting manifolds, group actions provide one of fundamental tools. The foundational result is that the quotient of a smooth manifold by a smooth group action with nice topological properties inherits a smooth structure. Moreover, if the original manifold is endowed with additional geometric structures and the action is compatible with them, these can usually be pushed down to the quotient manifold. In this section, we review how to make this general philosophy precise in a number of situations. In particular, we recall the hyper-Kähler and quaternionic Kähler quotient constructions, largely following the exposition of [HKLR87] and [GL88].

The most general quotient construction concerns a smooth manifold M, equipped with a smooth group action $G \curvearrowright M$, where G is a connected Lie group. The following is well-known:

Proposition 2.38. *If $G \curvearrowright M$ is a principal—free and proper, that is—action, then M/G admits a unique smooth structure such that the projection map $\pi : M \to M/G$ is a smooth submersion.* □

In light of this result, we will deal only with principal group actions $G \curvearrowright M$ in the remainder of this section.

If M comes equipped with a Riemannian metric, one may try to induce a Riemannian metric on M/G by lifting tangent vectors to M and using the given metric. Of course, one cannot reasonably expect this to work out unless this metric is independent of our choice of lifting, i.e. "constant" along the G-orbits in a suitable sense. Put more precisely, one must require that G acts isometrically. Then, everything works out as one might hope:

Proposition 2.39. *Let $G \curvearrowright (M, g)$ be a principal, isometric action on a Riemannian manifold. Then M/G admits a Riemannian metric such that the projection map $\pi : M \to M/G$ is a Riemannian submersion.*

Proof. By assumption, M has the structure of a principal G-bundle over M/G. At a point $x \in M$, the vertical subspace $\mathcal{V}_x \subset T_xM$ is spanned by the generating vector fields of the action, i.e. the image of $\mathfrak{g}$ under the differential (at the identity) of the orbit map $\varphi_x : G \to M$. Taking orthogonal complements at each point, we obtain a horizontal subbundle $\mathcal{H} \subset TM$ which is in fact a principal connection, because G acts isometrically.

Now consider tangent vectors $X, Y \in T_pG/M$, where $p \in \pi(x)$. Using the linear isomorphism $\mathrm{d}\pi_x$, these correspond uniquely to tangent vectors $\tilde{X}, \tilde{Y}$ in T_xM. Now we define a Riemannian metric $\hat{g}$ on M/G by setting $\hat{g}_p(X, Y) := g(\tilde{X}, \tilde{Y})$. This expression is independent of our choice of $x \in \pi^{-1}(p)$ by invariance of g, and $\hat{g}$ is positive-definite because g is. By construction, $\pi_* : (\mathcal{H}, g|_{\mathcal{H}}) \to (TM/G, \hat{g})$ is pointwise a linear isometry, which means that π is a Riemannian submersion. □

We also note that, in this situation, the Levi-Civita connection $\hat{\nabla}$ on the quotient is given by $\hat{\nabla}_X Y = (\pi_* \circ \mathcal{H})(\nabla_{\tilde{X}} \tilde{Y})$, where $\mathcal{H}$ denotes orthogonal projection onto $\mathcal{H}$, and $\tilde{X}, \tilde{Y}$ are horizontal lifts of the vector fields X, Y. Indeed, it is easy to check that this connection is metric-compatible and torsion-free.

Another, rather different type of structure is provided by the presence of a symplectic form on M. In this case, it is not possible to directly define an induced symplectic structure on the quotient manifold by lifting tangent vectors, even if $G \curvearrowright (M, \omega)$ is compatible with the symplectic structure. The simplest way to see this is by noticing that $\dim M/G = \dim M - \dim G$ is odd if G is odd-dimensional, so that M/G cannot possibly be symplectic. This problem was overcome by Marsden and Weinstein, who invented what is now called the symplectic quotient construction. Their crucial insight was to make use of certain functions naturally associated to the group action.

A group action is called symplectic if it acts by symplectomorphisms. This is the case if and only if for every generating vector field X, the one-form $\iota_X\omega$ is closed. Locally, this implies the existence of a function μ^X such that $\iota_X\omega = -\mathrm{d}\mu^X$.

Definition 2.40. We say that a symplectic group action $G \curvearrowright (M,\omega)$ is Hamiltonian if, for every generating vector field X, there exists a function $\mu^X : M \to \mathbb{R}$ which satisfies $\iota_X\omega = -\mathrm{d}\mu^X$. This function is said to be a Hamiltonian function for X.

Remark 2.41. If either $H^1_{\mathrm{dR}}(M;\mathbb{R}) = 0$ or $H^1(\mathfrak{g};\mathbb{R}) = 0$, any symplectic G-action on M is Hamiltonian. The former holds by definition of the De Rham cohomology groups. To see the latter, recall that $H^1(\mathfrak{g};\mathbb{R}) = 0$ if and only if $\mathfrak{g} = [\mathfrak{g},\mathfrak{g}]$ (this is the case, for instance, if $\mathfrak{g}$ is semi-simple). Now observe that, for generating vector fields X, Y, the following identity holds: $\iota_{[X,Y]}\omega = L_X(\iota_Y\omega) = \mathrm{d}(\iota_X\iota_Y\omega) + \iota_X\mathrm{d}(\iota_Y\omega) = \mathrm{d}(\iota_X\iota_Y\omega)$, since $L_Y\omega = 0$. Thus, any generating vector field of the form $[X,Y]$ admits a Hamiltonian function.

Example 2.42. Let (V,ω) be a real, symplectic vector space and consider the group $\mathrm{Sp}(V)$ of linear symplectomorphisms of V. We will give an expression for the corresponding Hamiltonian functions. The generating vector field corresponding to $X \in \mathfrak{g}$—which we will denote by the same symbol—is given by $X_v = \frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0}\exp(tX)\cdot v = X\cdot v$, where we have implicitly identified V and its tangent spaces. Now consider the function μ^X defined by $\mu^X(v) \coloneqq -\frac{1}{2}\omega(X\cdot v, v)$. Then

$$\mathrm{d}_v\mu^X(w) = -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0}\omega(X\cdot(v+tw), v+tw) = -\omega(X\cdot v, w)$$

where we used the fact that $X \in \mathfrak{sp}(V)$ is skew with respect to ω. Thus, μ^X is a Hamiltonian function for X.

The Hamiltonian functions corresponding to a Hamiltonian G-action can be conveniently assembled into a single map. Given $X \in \mathfrak{g}$, define $\mu : M \to \mathfrak{g}^*$ by setting $\mu(p)(X) \coloneqq \mu^X(p)$, where $p \in M$ is arbitrary.

Definition 2.43. We call μ a moment map for the G-action if it is equivariant with respect to the co-adjoint action $G \curvearrowright \mathfrak{g}^*$.

Remark 2.44. A moment map can be found for any Hamiltonian action of a compact and connected Lie group. To see this, recall that such a group is, up to finite covering, the product of a compact and connected semi-simple Lie group and a torus, so it suffices to check the claim for these two classes of Lie groups. On the one hand, the existence of a moment map for a given Hamiltonian G-action is guaranteed (see, for instance, [GS90]) if $H^2(\mathfrak{g};\mathbb{R}) = 0$, which holds for semi-simple Lie groups by Whitehead's lemma. On the other, let $T^n \curvearrowright (M,\omega)$ be a Hamiltonian action, denote the Hamiltonian function of X by κ^X and define $\kappa : M \to \mathfrak{t}^*$ by $\langle \kappa(p), X\rangle = \kappa^X(p)$ for all $X \in \mathfrak{t}$. Now define $\mu : M \to \mathfrak{g}^*$ by $\mu(p) := \int_{T^n}(g^*\kappa)(p)\mathrm{d}g$, where the integral is taken with respect to the invariant (Haar) measure on T^n. Then we have $\mu(p)(X) = \int_{T^n}(g^*\kappa^X)(p)\mathrm{d}g =: \mu^X(p)$ and consequently

$$\mathrm{d}\mu^X(p) = -\int_{T^n} g^*(\iota_X\omega)(p)\mathrm{d}g$$

But $g^*(\iota_X\omega) = \iota_X\omega$; let us verify this infinitesimally. If Y is a generating vector field, $L_Y\iota_X\omega = \iota_{[X,Y]}\omega$, but this vanishes since T^n is Abelian. We conclude that $\mathrm{d}\langle\mu(p), X\rangle = -\iota_X\omega$, and since μ is T^n-invariant by construction we are done.

Proposition 2.45 (Marsden–Weinstein, [MW74]). *Let $G \curvearrowright (M,\omega)$ be a principal, Hamiltonian group action with moment map μ. Then, for every ad^*-invariant element $\xi \in \mathfrak{g}^*$, $\mu^{-1}(\xi)/G$ is a symplectic manifold of dimension $\dim M - 2\dim G$, with symplectic form $\hat\omega$ determined by $\pi^*\hat\omega = \iota^*\omega$, where $\pi : \mu^{-1}(\xi) \to \mu^{-1}(\xi)/G$ is the projection map and $\iota : \mu^{-1}(\xi) \hookrightarrow M$ denotes inclusion.*

Proof. First, we check that μ has maximal rank, so that $\mu^{-1}(\xi)$ is always a submanifold. The kernel of the differential of μ at $x \in M$ consists of all $V \in T_xM$ such that $\omega(V, X) = 0$ for every generating vector field X. Since G acts freely and ω is non-degenerate, this subspace $(\mathfrak{g}\cdot x)^{\perp_\omega}$ has dimension $\dim M - \dim G$, and the rank-nullity theorem now implies that every x is a regular point for μ.

Because μ is equivariant and ξ is central, we see that G preserves $\mu^{-1}(\xi)$, so the quotient $\mu^{-1}(\xi)/G$ is well-defined and has dimension $\dim M - 2\dim G$. Proceeding as in the Riemannian setting, we define the two-form $\hat\omega$ as follows. Given $X, Y \in T_p\mu^{-1}(\xi)/G$, choose any lifts $\tilde X$, $\tilde Y$ to $T_x\mu^{-1}(\xi)$, where x lies in the fiber over p. Now set $\hat\omega(X,Y) := \omega(\tilde X, \tilde Y)$. Compared to the Riemannian case, we now do not have a distinguished choice of horizontal lift at each point, but that is no matter: Any two lifts $\tilde X$ differ by some element in the vertical tangent space $\mathcal{V}_x = (\mathfrak{g}\cdot x) \subset T_x\mu^{-1}(\xi) = (\mathfrak{g}\cdot x)^{\perp_\omega} \subset T_xM$, and therefore are evaluated identically by ω. Note the identity $\pi^*\hat\omega = \iota^*\omega$, which implies that $\hat\omega$ is a closed form.

Finally, we observe that $\hat\omega$ is non-degenerate. Indeed, if $\iota_X\hat\omega = 0$ then $\iota_{\tilde X}\omega|_{(\mathfrak{g}\cdot x)^{\perp_\omega}} = \pi^*(\iota_X\hat\omega) = 0$, hence $\tilde X \in ((\mathfrak{g}\cdot x)^{\perp_\omega})^{\perp_\omega} = (\mathfrak{g}\cdot x) \subset T_x\mu^{-1}(\xi)$, so that $\pi_*(\tilde X) = X = 0$. □

Definition 2.46. In the above setting, $\mu^{-1}(\xi)/G$ is called the symplectic quotient of M by G at level ξ, and denoted by $M/\!\!/G$ if no confusion can arise about the level.

Combining Riemannian and symplectic structures, one is naturally lead to consider Kähler manifolds and their quotients.

Proposition 2.47. *Let (M, g, ω) be a Kähler manifold and consider a principal, Hamiltonian action $G \curvearrowright (M, g, \omega)$ by isometries with moment map μ. Then the symplectic quotient $M /\!\!/ G$ inherits a natural Kähler structure.*

Proof. We take an ad^*-invariant element $\xi \in \mathfrak{g}^*$ and consider the submanifold $\mu^{-1}(\xi)$, with inclusion map $\iota : \mu^{-1}(\xi) \hookrightarrow M$, and projection map $\pi : \mu^{-1}(\xi) \to \mu^{-1}(\xi)/G$ and consider the Riemannian metric $\hat{g}$ and symplectic form $\hat{\omega}$ constructed in the preceding propositions. In particular, we identify the tangent spaces to $M /\!\!/ G$ with the horizontal subspaces of $T\mu^{-1}(\xi)$. It now suffices to show that the complex structure I on M descends to an orthogonal almost complex structure which is parallel with respect to the Levi-Civita connection of $\hat{g}$—its integrability then follows.

First, we check that I preserves the horizontal subbundle $\mathcal{H} \subset T\mu^{-1}(\xi)$. As a subbundle of TM, it is the orthogonal complement of $\mathfrak{g} \oplus \{\operatorname{grad} \mu^X \mid X \in \mathfrak{g}\} \subset TM$, and this latter bundle is a complex subbundle. Indeed, we have $g(\operatorname{grad} \mu^X, Y) = \mathrm{d}\mu^X(Y) = -g(IX, Y)$, i.e. $\operatorname{grad} \mu^X = -IX$. Thus, I descends to an almost complex structure $\hat{I}$ on $M /\!\!/ G$ satisfying $\hat{g}(\hat{I}X, Y) = \hat{\omega}(X, Y)$. Moreover, the Levi-Civita connection satisfies $\pi^*\hat{\nabla} = \mathcal{H}\nabla$, and I is ∇-parallel and commutes with $\mathcal{H}$. Therefore, $\hat{I}$ is $\hat{\nabla}$-parallel as well, and $(\hat{g}, \hat{I}, \hat{\omega})$ defines a Kähler triple on $M /\!\!/ G$. □

Specializing even further to the hyper-Kähler setting, we have a triple of complex structures which behave like the imaginary quaternions, and correspondingly a triplet of symplectic structures. If a group action is Hamiltonian with respect to each of them, we call it tri-Hamiltonian. If we additionally have three moment maps μ_i, $i = 1, 2, 3$, it is natural to bunch them together into a single map $\mu : M \to \mathfrak{g}^* \otimes \operatorname{Im} \mathbb{H}$, defined as $\mu = \mu_1 i + \mu_2 j + \mu_3 k = \mu_1 i + (\mu_2 + i\mu_3)j$. We call μ the hyper-Kähler moment map.

Proposition 2.48 (Hitchin et al., [HKLR87])**.** *Let $G \curvearrowright (M, g, \omega_1, \omega_2, \omega_3)$ be a principal, tri-Hamiltonian action by isometries on a hyper-Kähler manifold. Then, for any ad^*-invariant element $\xi \in \mathfrak{g}^*$ and imaginary quaternion q, $\mu^{-1}(\xi \otimes q)/G$ is a hyper-Kähler manifold.*

Proof. From the holomorphic symplectic point of view on hyper-Kähler geometry, it is natural to consider the moment map of the holomorphic symplectic form with respect to I_1, which is $\Omega_{I_1} = \omega_2 + i\omega_3$. Its complex moment map $\mu_+ = \mu_2 + i\mu_3$ is holomorphic—indeed, its differential is easily seen to be of type $(1, 0)$—and therefore its level set defines a complex submanifold of M. Regarding M as Kähler with respect to I_1, we have an induced Kähler structure on this level set, to which we can apply the Kähler quotient construction to obtain a Kähler structure on $\mu^{-1}(\xi \otimes q)/G$. Repeating this procedure for I_2 and I_3 leads to a triple of Kähler structures on $\mu^{-1}(\xi \otimes q)/G$ which satisfy the same algebraic conditions as on M: We have obtained a hyper-Kähler structure. □

The resulting hyper-Kähler manifold is known as the hyper-Kähler quotient of M by G (at level ξ), and denoted by $M /\!\!/\!\!/ G$.

On a hyper-Kähler manifold M, the bundle $\mathcal{Q} \subset \operatorname{End} TM$ defining the quaternionic structure is globally trivialized by the complex structures I_1, I_2 and I_3. We can therefore view the hyper-Kähler moment map as a section of the trivial bundle $\mathfrak{g}^* \otimes \mathcal{Q} \to M$, whose fibers are copies of $\mathfrak{g}^* \otimes \operatorname{Im} \mathbb{H}$. On a quaternionic Kähler manifold this point of view still makes sense, so though $\mathcal{Q}$ is no longer trivial, this reformulation suggests a generalization of the usual notion of moment map to the quaternionic Kähler setting.

Proposition 2.49 (Galicki–Lawson, [GL88]). *Let G act isometrically on a quaternionic Kähler manifold (M, g) with non-zero scalar curvature, and let $X \in \mathfrak{g}$. Then there exists a unique section μ^X of $\mathcal{Q} \subset \bigwedge^2(T^*M)$ which satisfies $\nabla \mu^X = -\sum_j \iota_X \omega_j \otimes \omega_j$, where $\{\omega_j\}_{i=1,2,3}$ is any local orthonormal frame for $\mathcal{Q}$.*

Proof. Firstly, note that the right-hand side of the defining equation for μ^X is indeed a global object, since the (locally defined) tensor field $\sum_j \omega_j \otimes \omega_j$ is invariant under change of frame. The strategy of proof is to assume the existence of a solution to $\nabla \mu^X = -\sum_j \iota_X \omega_j \otimes \omega_j$ and reconstruct μ^X. To do so, we exploit the fact that the curvature of $\mathcal{Q}$ is tensorial (rather than a differential operator). Indeed, applying d^∇ to both sides of the equation and writing $\nabla \omega_j = \sum_k \alpha_{jk} \omega_k$ for local connection one-forms $\{\alpha_{jk}\}$, we find

$$\mathcal{R}(\mu^X) = -\sum_j \left(\mathrm{d}\iota_X \omega_j - \sum_k \alpha_{jk} \wedge \iota_X \omega_k \right) \otimes \omega_j =: -\sum_j \beta_j \otimes \omega_j$$

Crucially, this is an algebraic equation for μ^X; our task is to solve it. Since the curvature is injective (in the sense of corollary 2.26), uniqueness is automatic once we prove existence. All that remains is to prove that the right-hand side lies in the image of the curvature tensor $\mathcal{R} : \mathcal{Q} \to \bigwedge^2 T^*M \otimes \mathcal{Q}$. Using proposition 2.25, we can characterize elements of $\operatorname{im} \mathcal{R}$ as those $\mathcal{Q}$-valued two-forms of the form $\sum_j F_j \otimes \omega_j$ where $F_j = \sum_k \gamma_{jk} \omega_k$ with $\gamma_{jk} = -\gamma_{kj}$, so we should prove that β_j is of that form. More invariantly, we need to show that $\beta_j = \sum_k \gamma_{jk} \omega_k \in \Gamma(\mathcal{Q}) \subset \Gamma\big(\bigwedge^2 T^*M\big)$ such that $\sum_j \beta_j \wedge \omega_j = 0$ and $\gamma_{jk} = -\gamma_{kj}$ (the latter is automatic if the dimension is at least eight).

Since X preserves $\mathcal{Q}$ (cf. proposition 2.27), we know that $L_X \omega_j \in \Gamma(\mathcal{Q})$. Moreover, we have

$$L_X \omega_j = \mathrm{d}\iota_X \omega_j + \iota_X \sum_k \alpha_{jk} \wedge \omega_k = \beta_j + \sum_k \alpha_{jk}(X) \omega_k = \beta_j + \nabla_X \omega_j \tag{2.4}$$

This shows that $\beta_j \in \Gamma(\mathcal{Q})$ and its coefficients γ_{jk} satisfy $\gamma_{jk} = -\gamma_{kj}$. Moreover, from $\sum_{j,k} \alpha_{jk}(X) \omega_k \wedge \omega_j = 0$ and $L_X\big(\sum_j \omega_j \wedge \omega_j\big) = 0$ one obtains

$$0 = \sum_j (L_X \omega_j) \wedge \omega_j = \sum_j \beta_j \wedge \omega_j$$

which is what we needed to show. □

Recalling the bundle isometry $\epsilon : \operatorname{Skew}\operatorname{End}\mathcal{Q} \to \mathcal{Q}$, which we introduced in section 2.1, we obtain a nice expression for μ^X:

Corollary 2.50. *In the above notation,* $\mu^X = \frac{1}{\nu}\epsilon(\nabla_X - L_X)$.

Proof. Expanding $\mu^X = \sum_j \mu_j^X \omega_j$, we have

$$\mathcal{R}(\mu^X) = \sum_{j,k} \mu_k^X \mathcal{R}_{kj} \otimes \omega_j = -\sum_j \beta_j \otimes \omega_j$$

where $\beta_j = (L_X - \nabla_X)\omega_j$ by (2.4). Now set $B = \frac{1}{\nu}(\nabla_X - L_X)|_{\mathcal{Q}}$. Using proposition 2.25 to write $\mathcal{R}_{kj} = \nu \sum_m \epsilon_{jkm}\omega_m$, we find $B_{ij} = -\sum_k \mu_k^X \epsilon_{ijk}$ and consequently $\epsilon(B) = \sum_k \mu_k^X \omega_k = \mu^X$, as claimed. □

Definition 2.51. Let $G \curvearrowright M$ be an isometric action on a quaternionic Kähler manifold of non-zero scalar curvature. Then the unique section μ of the bundle $\mathfrak{g}^* \otimes \mathcal{Q}$ which satisfies $\mu(p)(X) = \mu^X(p)$ for every $p \in M$ and $X \in \mathfrak{g}$ is called the quaternionic moment map for the action.

The uniqueness of the quaternionic moment map implies that it is always equivariant with respect to G.

As a section of the generally non-trivial vector bundle $\mathfrak{g}^* \otimes \mathcal{Q}$, the only level set of the quaternionic moment map which is well-defined is its vanishing locus $\mu^{-1}(0)$, so it is this submanifold that is used for the corresponding quotient construction.

Proposition 2.52 (Galicki–Lawson)**.** *Let $G \curvearrowright (M, g)$ be a principal, isometric action on a quaternionic Kähler manifold of non-vanishing scalar curvature. Then $\mu^{-1}(0)/G$ is naturally endowed with the structure of a quaternionic Kähler manifold.*

Proof. We will prove the statement for $\dim M \geq 8$. The case $\dim M = 4$ is discussed in detail in [GL88].

Using the same arguments as for symplectic and Kähler quotients, we see that μ is always transversal to the zero section, and that the metric on $\mu^{-1}(0)$ descends to the quotient. We have to check that $\mathcal{Q}$ descends as well. Around a point $x \in \mu^{-1}(0)$, we can trivialize $\mathcal{Q}$ with a local frame $\{\omega_k\}$. For $X \in \mathfrak{g}$, we can now write $\mu^{X_j} = \sum_k \mu_k^X \omega_k$ and, analogous to the (hyper-)Kähler case, have $\mathcal{H}^\perp = \mathfrak{g} \oplus \langle \operatorname{grad} \mu_k^X \mid X \in \mathfrak{g}\rangle$. This bundle is invariant under each I_k. Indeed, the moment map condition implies that $d\mu_k^X + \sum_j \mu_j^X \alpha_{jk} = -\iota_X \omega_k$, and the second term on the left-hand side vanishes on $\mu^{-1}(0)$. This means that, on $\mu^{-1}(0)$, the equation $g(I_j X, Y) = -g(\operatorname{grad} \mu_j^X, Y)$ holds for every $X \in \mathfrak{g}$.

We conclude that $\mathcal{Q}$ descends to $\mu^{-1}(0)/G$, turning it into an almost quaternionic-Hermitian manifold. Because $\mathcal{Q}$ is preserved by the Levi-Civita connection on M and commutes with the orthogonal projection onto the horizontal subbundle of $T\mu^{-1}(0)$, the quaternionic structure bundle is also preserved by the Levi-Civita connection on the quotient. This shows that we have a quaternionic Kähler structure. □

Of course, the manifold produced by this construction is called the quaternionic Kähler quotient of M with respect to G. We denote it by $M /\!\!/\!\!/ G$, just as in the case of a hyper-Kähler quotient, when no confusion can arise as to which of the two is meant.

We remark that all these quotient constructions generalize to the pseudo-Riemannian setting with no changes, under the additional assumption that the metric restricts to a non-degenerate bilinear form on the distribution $\mathfrak{g}$ of tangent vectors to the orbits.

2.4 Bundle constructions

We have already encountered twistor spaces in our survey of the known examples of quaternionic Kähler and hyper-Kähler manifolds. They provide a way to encode the quaternionic geometry in complex-geometric data on a bundle over the quaternionic base. Though they will not play an important role in the rest of this thesis, their importance in the general theory of quaternionic manifolds justifies discussing them at some length. Next, we will introduce the notion of a Swann bundle, which provides a bridge between quaternionic Kähler and hyper-Kähler geometry.

2.4.1 Twistor theory

Perhaps surprisingly, the initial conception of twistor theory is due to the physicist Penrose, who envisioned it as an approach to the quantization of Einstein's theory of general relativity. He argued that a spacetime containing a single graviton (the quantum particle corresponding to the gravitational field) should be (the complexification of) an (anti-)self-dual Einstein four-manifold of Lorentzian signature, and outlined how its metric properties can be encoded in the complex geometry of a bundle of this space, whose fiber over every point is a copy of $\mathbb{C}\mathrm{P}^1$ [Pen76] (see [War78] for a clear exposition). Techniques from algebraic geometry can then be applied to these curves, which we may call twistor lines, to prove theorems about the base manifold.

The idea was quickly taken up by a group of mathematicians, led by Atiyah, who formulated a Riemannian version of the twistor construction (also called the Penrose transform).

Example 2.53. We can think of the four-sphere as the quaternionic projective line $\mathbb{H}\mathrm{P}^1 = (\mathbb{H}^2 \setminus \{0\})/\mathbb{H}^*$. Identifying $\mathbb{H}^2 \cong \mathbb{C}^4$, we see that there is a natural fiber bundle $\mathbb{C}\mathrm{P}^3 \to \mathbb{H}\mathrm{P}^1$. Thinking of a quaternionic line $[q] \in \mathbb{H}\mathrm{P}^1$ as a linear subspace of complex dimension two, we see that the fiber over it, which is the space of complex lines contained in $[q]$, is just a copy of $\mathbb{C}\mathrm{P}^1$. This casts $\mathbb{C}\mathrm{P}^3$ as the twistor space of S^4.

Twistor theory was applied with great success in the study of self-dual four-manifolds [AW77; AHS78; ADHM78]. Naturally, a generalization to manifolds of higher dimension was highly sought-after. This was achieved by Simon Salamon, who showed that twistor theory could be generalized to quaternionic Kähler manifolds of any dimension. The crucial insight is that Penrose's twistor space of an Einstein self-dual four-manifold is nothing but the (total space of the) sphere bundle of the quaternionic structure bundle.

Let us clarify this remark, and show how it naturally leads to a complex structure. We first consider a more general situation. Let (M, g) be an oriented Riemannian manifold of even dimension. Fixing a point $x \in M$, the tangent space $T_xM \cong \mathbb{R}^{2n}$ admits many complex structures: The space of complex structures compatible with the given inner product and orientation is $\mathrm{SO}(2n)/\mathrm{U}(n)$. In this way, we may construct a bundle $\pi : E \to M$ with fiber $\mathrm{SO}(2n)/\mathrm{U}(n)$, canonically associated with the $\mathrm{SO}(2n)$-principal frame bundle of M.

The most important property of E is that it always carries an almost complex structure. The Levi-Civita connection of (M, g) induces a splitting $TE \cong \mathcal{V} \oplus \mathcal{H}$, where the vertical subbundle $\mathcal{V}$ consists of tangent vectors along the fiber, which is a copy of $\mathrm{SO}(2n)/\mathrm{U}(n)$, and $\mathcal{H} \cong \pi^*TM$. With respect to this splitting, we may define an almost complex structure as follows (see e.g. [Sal85] for more details). The fiber E_x is a copy of the Hermitian symmetric space $\mathrm{SO}(2n)/\mathrm{U}(n)$, and the complex structure on this space induces an almost complex structure $J_{\mathcal{V}}$ on the vertical tangent bundle. Given a point $J \in E_x$, think of it as a complex structure on T_xM. The linear isomorphism $\mathcal{H}_J \cong T_xM$ then induces a complex structure on $\mathcal{H}_J$, which is really just J itself. Doing this at every point yields a tautological almost complex structure $J_{\mathcal{H}}$ on $\mathcal{H}$, and $\mathcal{J} = J_{\mathcal{V}} \oplus J_{\mathcal{H}}$ now defines an almost complex structure on E. The almost complex manifold $(E, \mathcal{J})$ is called the twistor space of (M, g).

If we started with a surface Σ, each fiber of E consists of a single point and we recover the structure of a Riemann surface. In case M is a four-manifold, the fibers are copies of $\mathrm{SO}(4)/\mathrm{U}(2) \cong \mathbb{C}\mathrm{P}^1$, and the twistor space is a $\mathbb{C}\mathrm{P}^1$-bundle. Given this almost complex manifold, we may ask if it is actually complex: Is $\mathcal{J}$ integrable? For $\dim M = 2$, this question is of course trivial. It was already noticed in [AHS78] that the integrability of $\mathcal{J}$ is equivalent to the self-duality of the base manifold in the four-dimensional case. In higher dimensions the manifold must actually be conformally flat, i.e. its Weyl curvature vanishes [Sal86]. This very restrictive condition shows that it is too much to ask $(E, \mathcal{J})$ to be complex.

However, when the base manifold (M, g) is quaternionic Kähler, the frame bundle can be reduced to an $\mathrm{Sp}(n)\,\mathrm{Sp}(1)$-principal bundle, so we can restrict our attention to the bundle of complex structures on the tangent spaces which are compatible with the $\mathrm{Sp}(n)\,\mathrm{Sp}(1)$-structure. Let us check what the fiber of such a bundle is:

$$\frac{\mathrm{Sp}(n)\,\mathrm{Sp}(1) \cap \mathrm{SO}(4n)}{\mathrm{Sp}(n)\,\mathrm{Sp}(1) \cap \mathrm{U}(2n)} \cong \frac{\mathrm{Sp}(n)\,\mathrm{Sp}(1)}{\mathrm{Sp}(n)\,\mathrm{U}(1)} \cong \frac{\mathrm{Sp}(1)}{\mathrm{U}(1)} \cong \mathbb{C}\mathrm{P}^1$$

Its description as an associated bundle to the $\mathrm{Sp}(n)\,\mathrm{Sp}(1)$-principal frame bundle shows that it is nothing but the sphere bundle of the quaternionic structure bundle $\mathcal{Q}$.

Definition 2.54. Let $(M, g, \mathcal{Q})$ be a quaternionic Kähler manifold. Then we say that $Z := S(\mathcal{Q})$ is its twistor space.

The fundamental property of Z is that the natural almost complex structure constructed above is integrable.

Theorem 2.55 (Salamon, [Sal82]). *Let $(M, g, \mathcal{Q})$ be quaternionic Kähler. Then its twistor space (Z, J) is a complex manifold.*

The twistor space also arises naturally if one attempts to generalize the properties of Wolf spaces. Indeed, Wolf not only identified the quaternionic Kähler symmetric spaces, but observed that his classification matched the classification of compact, simply connected, homogeneous holomorphic contact manifolds by Boothby [Boo62].

Definition 2.56. Let X be a complex manifold of complex dimension $2n+1$. We say that X carries a holomorphic contact structure if there exists a codimension-one holomorphic distribution $D \subset T^{1,0}X$ which is maximally non-integrable.

Wolf's most important contribution in [Wol65] was to provide an a priori reason for the bijective correspondence between homogeneous holomorphic contact manifolds and Wolf spaces of positive scalar curvature by exhibiting the former as a fiber bundle over the latter, with fiber $\mathbb{C}\mathrm{P}^1$. With the benefit of hindsight, we can say that Wolf exhibited the twistor fibration for these symmetric examples.

Example 2.57. The twistor space of $\mathbb{H}\mathrm{P}^n \cong \frac{\mathrm{Sp}(n+1)}{\mathrm{Sp}(n)\times\mathrm{Sp}(1)}$ is $\frac{\mathrm{Sp}(n+1)}{\mathrm{Sp}(n)\times\mathrm{U}(1)} \cong \mathbb{C}\mathrm{P}^{2n+1}$. As in the case $n = 1$, the fiber over $[q] \in \mathbb{H}\mathrm{P}^n$ should be understood as the space of complex lines lying in $[q]$.

Boothby's results imply that, besides carrying a holomorphic contract structure, the twistor spaces of the compact Wolf spaces admit a Kähler metric. Salamon proved the following generalization of Boothby's results:

Theorem 2.58 (Salamon, [Sal82]). *Let $(M, g, \mathcal{Q})$ be a quaternionic Kähler manifold and denote its twistor space by Z.*

(i) *If (M, g) has non-vanishing scalar curvature, Z carries a holomorphic contact structure.*

(ii) *If the scalar curvature of M is positive, Z carries a canonical Kähler–Einstein metric $\hat{g}$ of positive scalar curvature such that $\pi : (Z, \hat{g}) \to (M, g)$ is a Riemannian submersion with totally geodesic fibers.*

□

Remark 2.59.

(i) In particular, the first Chern class of the twistor space of a quaternionic Kähler manifold of positive scalar curvature is positive. In other words, it is a Fano manifold. Moreover, a simple computation due to Kobayashi [Kob59] shows that the integral class $c_1(Z)$ is divisible by $n+1$, where $\dim_{\mathbb{C}} Z = 2n+1$. Such Fano manifolds are severely restricted, and this in turn restricts quaternionic Kähler manifolds since, by a theorem of LeBrun, every holomorphic contact manifold that admits a Kähler–Einstein metric is a twistor space [LeB95].

(ii) An inspection of the proof given in [Bes08] shows that, in case the scalar curvature of (M, g) is negative, one obtains a pseudo-Kähler–Einstein metric on Z.

Salamon further showed how to encode various aspects of the quaternionic geometry in the holomorphic data of its twistor space. In particular, the vector space of Killing fields of a strict quaternionic Kähler manifold is identified with the space of global holomorphic sections of the vertical tangent bundle of Z [Sal82]. As in the four-dimensional case, the twistor construction can always be inverted to recover the quaternionic geometry (highly non-explicitly). The precise statement, which we will not give here, is due to LeBrun [LeB89].

Let us now briefly consider the twistor construction for hyper-Kähler manifolds; more details can be found in [HKLR87]. Our definition for twistor spaces for quaternionic Kähler manifolds of course still applies, but in the hyper-Kähler case the twistor bundle is trivial and we have $Z = M \times S^2$ as a smooth manifold. Here, it turns out that it is advantageous to consider the projection onto the second factor, $p : Z \to \mathbb{C}\mathrm{P}^1$, rather than the previously considered bundle structure $\pi : Z \to M$. The projection $p : Z \to S^2$ casts Z as as a holomorphic fiber bundle. The fiber over $J \in \mathbb{C}\mathrm{P}^1$ is the Kähler manifold (M, J), and for each $x \in M$ we have a holomorphic section $J \mapsto (x, J)$ of p, whose image is called a twistor line. These twistor lines are invariant under the antipodal map on S^2, and all have the same normal bundle (namely $\mathbb{C}^{2n} \otimes p^*\mathcal{O}(1)$) in Z. If we want to be able to recover the hyper-Kähler manifold from Z, we must also keep track of the holomorphic symplectic structure on each fiber of the projection. This is encoded by a holomorphic section of $\pi^* \bigwedge^2 T * M \otimes p^*\mathcal{O}(2)$, non-degenerate on the fibers of p. Hitchin et al. [HKLR87] show that, for any complex manifold Z endowed with such holomorphic data, the space of sections invariant under the antipodal map naturally inherits the structure of a hyper-Kähler manifold for which Z is the twistor space.

2.4.2 Swann bundles

Swann discovered a way to connect hyper-Kähler and quaternionic Kähler geometry even more tightly. To each quaternionic Kähler manifold of non-zero scalar curvature, he assigns a conical (pseudo-)hyper-Kähler manifold [Swa90; Swa91]. In this section, we review this construction. We start by studying such conical manifolds.

Definition 2.60. A (pseudo-)hyper-Kähler manifold is called conical if it admits a vector field X which satisfies $\nabla X = \mathrm{id}_{TM}$.

On a conical hyper-Kähler manifold (M, g, I_k) (or any conical manifold, for that matter), the natural function $\kappa = \frac{1}{2}g(X, X)$ plays a distinguished role, since one easily checks that $\nabla \mathrm{d}\kappa = g$, i.e. the metric is given by the Hessian of κ. This implies that $\mathrm{d}\mathrm{d}^c_{I_k}\kappa = -\mathrm{d}I_k^*\mathrm{d}\kappa = \mathrm{d}(g(I_kX, \cdot)) = 2\omega_k$, where the final step follows from a short computation. Thus, κ is a (global) Kähler potential for every complex structure compatible with the hyper-Kähler structure, and is therefore called a hyper-Kähler potential. The existence of a hyper-Kähler potential, the Hessian condition on g and the requirement that the manifold be conical are in fact all equivalent (see [Ion19] for a clear exposition).

Let us now investigate the quaternionic span $\mathbb{H}X = \langle X, I_1X, I_2X, I_3X\rangle$. Since the complex structures I_k are parallel, $[X, I_kX] = 0$, and moreover $[I_jX, I_kX] = 2\sum_l \epsilon_{jkl}I_lX$, where ϵ_{jkl} is the Levi-Civita symbol. We conclude that $\mathbb{H}X$ determines a Lie algebra isomorphic to $\mathbb{R} \oplus \mathfrak{sp}(1)$. Let us also note that I_kX is ω_k-Hamiltonian with Hamiltonian κ. More generally, each I_kX is a Killing field and we have

$$\begin{aligned} L_{I_jX}\omega_k(Y, Z) &= \mathrm{d}\iota_{I_jX}\omega_k(Y, Z) = \mathrm{d}(g(I_kI_jX, \cdot))(Y, Z) = g(I_kI_jY, Z) - g(I_kI_jZ, Y) \\ &= 2\sum_l \epsilon_{kjl}\omega_l \end{aligned}$$

In particular, each of the generators of $\mathfrak{sp}(1)$ is a rotating Killing field (as per definition 2.36).[4] Identifying $\mathfrak{sp}(1)$ with $\mathrm{Im}\,\mathbb{H}$, we can summarize this by saying that $q \in \mathfrak{sp}(1)$ acts on the $\mathrm{Im}\,\mathbb{H}$-valued two-form $\omega = \omega_1 i + \omega_2 j + \omega_3 k$ by $L_q\omega = [\omega, q]$.

Definition 2.61. We say that an infinitesimal Sp(1)-action by Killing fields on a hyper-Kähler manifold (M, g, ω_k) permutes the complex structures, or is permuting, if $L_q\omega = [\omega, q]$ for every $q \in \mathfrak{sp}(1) \cong \mathrm{Im}\,\mathbb{H}$.

Thus, a conical hyper-Kähler manifold carries a infinitesimal Sp(1)-action which permutes the complex structures.

In fact, Swann [Swa91] shows that the existence of a hyper-Kähler potential is sufficient to construct this infinitesimal action. Conversely, assume we have a local Sp(1)-action generated by the vector fields $\{X_1, X_2, X_3\}$ such that $I_1X_1 = I_2X_2 = I_3X_3$ (clearly, each equals $-X$ in the conical case) and such that X_k is ω_k-Hamiltonian. Then Swann proves that (M, g, ω_k) admits a hyper-Kähler potential κ. Thus, modulo some technical

[4] Note that the normalization here is different from the one discussed below definition 2.36. They are related by a factor $-\frac{1}{2}$.

requirements, the existence of an infinitesimal permuting Sp(1)-action is equivalent to requiring the hyper-Kähler manifold to be conical.

Now let us consider a strict quaternionic Kähler manifold $(M, g, \mathcal{Q})$ and show how to construct a bundle over it with a conical hyper-Kähler structure. From the quaternionic Kähler data, we construct the principal SO(3)-bundle $P(\mathcal{Q})$ of oriented, orthonormal frames of the quaternionic structure bundle $\mathcal{Q}$. Now we may consider its Cartesian product with $\mathbb{R}_{>0}$; call the resulting space $\mathcal{U}(M)$. This is the smooth manifold on which Swann constructs a (pseudo-)hyper-Kähler structure. It is a fiber bundle over M whose fibers are copies of $\mathrm{SO}(3) \times \mathbb{R}_{>0} \cong \mathbb{H}^*/\mathbb{Z}_2$, and we can alternatively construct it as an associated bundle to $P(\mathcal{Q})$. For this, recall that for $q \in \mathrm{Sp}(1)$ and arbitrary $p \in \operatorname{Im}\mathbb{H}$, the assignment $q \mapsto (p \mapsto qpq^{-1})$ defines the universal covering map $\mathrm{Sp}(1) \to \mathrm{SO}(3)$, whose kernel is $\mathbb{Z}_2$. This yields the standard action on $\operatorname{Im}\mathbb{H} \cong \mathbb{R}^3$, which extends to an action on $\mathbb{H} \cong \mathbb{R} \oplus \operatorname{Im}\mathbb{H}$ by acting trivially on the first factor. It induces a well-defined action on the quotient $\mathbb{H}^*/\{\pm 1\}$. Denoting this action by ρ, we may view $\mathcal{U}(M)$ as the associated bundle $P(\mathcal{Q}) \times_\rho \mathbb{H}^*/\mathbb{Z}_2$.

Theorem 2.62 (Swann, [Swa91]). *Let $(M, g, \mathcal{Q})$ be quaternionic Kähler manifold with non-vanishing reduced scalar curvature ν. Then $\mathcal{U}(M) = P(\mathcal{Q}) \times_\rho \mathbb{H}^*/\mathbb{Z}_2$ carries a (pseudo-)hyper-Kähler metric h and has the structure of a (pseudo-)hyper-Kähler cone whose hyper-Kähler potential function is given by the norm-squared on the fibers.*

Sketch of Proof. We outline the proof given in [Ion19]. On M, we can re-encode the local connection one-forms $\{\alpha_{jk}\}$ of the Levi-Civita connection in the one-forms $\{\theta_k\}$ defined through $\nabla\omega_k = 2\sum_{l,m} \epsilon_{klm}\theta_l \otimes \omega_m$. We assemble these into an $\operatorname{Im}\mathbb{H}$-valued one-form θ and analogously build an $\operatorname{Im}\mathbb{H}$-valued two-form ω out of an oriented local orthonormal frame $\{\omega_k\}$ of $\mathcal{Q}$. Now, for $q \in \mathbb{H}^*$, we may locally construct the $\operatorname{Im}\mathbb{H}$-valued two-form

$$\tilde{\omega} = \frac{\nu}{2} q\omega\bar{q} + (\mathrm{d}q - q\theta) \wedge (\overline{\mathrm{d}q - q\theta})$$

on $M \times \mathbb{H}^*$. Since it is quadratic in q, this form descends to $M \times \mathbb{H}^*/\mathbb{Z}_2$. In fact, it yields a globally well-defined two-form on $\mathcal{U}(M)$. On a pair of overlapping trivializing open sets $U, V \subset M$, the local frames are related by an SO(3)-valued transition function, whose pointwise action can be written as $\omega_V \mapsto u\omega_U u^{-1}$, where $u \in \mathrm{Sp}(1)$ is a lift of the SO(3)-element. Since $\theta_V = u\theta_U u^{-1} + u\mathrm{d}u^{-1}$ and $q_V = \pm q_U u^{-1}$, $\tilde{\omega}$ is seen to be invariant, hence globally defined. Note that $|q|$ is also well-defined globally.

One may now verify, though we will not do so, that the components of $\tilde{\omega}$ define a (pseudo-)hyper-Kähler structure (i.e. they are closed and satisfy the necessary algebraic conditions). The metric can be written locally as

$$h = \frac{\nu}{2} g\,|q|^2 + |\mathrm{d}q - q\theta|^2$$

which reveals the necessity of the assumption that $\nu \neq 0$. This also makes it clear that h is positive-definite if $\nu > 0$ and is otherwise of signature $(4, 4n)$.

It is not surprising, in light of our earlier discussion, that the conical structure of $\mathcal{U}(M)$ derives from fiberwise left-multiplication by quaternions. In particular, the conical vector field corresponds simply to the fiberwise Euler vector field, whose corresponding hyper-Kähler potential is $|q|^2$. □

Definition 2.63. The bundle $\mathcal{U}(M)$, equipped with its (pseudo-)hyper-Kähler structure, is called the Swann bundle of $(M, g, \mathcal{Q})$.

Swann bundles are characterized among all hyper-Kähler manifolds, at least locally, by the existence of a permuting $\mathrm{Sp}(1)$-action:

Theorem 2.64 (Swann, [Swa91]). *Let (N, h, ω_k) be a hyper-Kähler manifold which carries a free and isometric $\mathrm{Sp}(1)$-action such that*

(i) *There is a finite subgroup $\Gamma \subset \mathrm{Sp}(1)$ such that $\mathrm{Sp}(1)/\Gamma$ acts freely.*

(ii) *$\mathrm{Sp}(1)$ acts transitively on the sphere of complex structures compatible with the hyper-Kähler structures.*

(iii) *If X_k denotes a generator of the $\mathrm{U}(1)$-subgroup preserving I_k, then X_k is ω_k-Hamiltonian and the rank one distribution spanned by $I_k X_k$ is k-independent.*

Then, if μ is a moment map for one of the generators X_k (with respect to ω_k). Then. for every $x \in \mathbb{R}$, $M = \mu^{-1}/(\mathrm{Sp}(1)/\Gamma)$ is a quaternionic Kähler manifold and (N, h, ω_k) is locally homothetic to the Swann bundle over M. □

Remark 2.65. This result holds in the pseudo-Riemannian case as well, under the assumption that the restriction of h to the tangent spaces to the $\mathrm{Sp}(1)$-orbits is non-degenerate. This case is relevant for quaternionic Kähler manifolds of negative scalar curvature.

Thus, the quaternionic Kähler structure on M can be recovered from the Swann bundle. The twistor space of M is recovered in similar fashion. It is obtained by taking the Kähler quotient with respect to the above-mentioned $\mathrm{U}(1)$-subgroup, rather than all of $\mathrm{Sp}(1)$. Its (pseudo-)Kähler structure is then immediately apparent. The equivalence of the Swann bundle to the quaternionic Kähler geometry manifests itself in various concrete situations. For instance, a Killing field of M lifts to a tri-Hamiltonian Killing field on $\mathcal{U}(M)$. More generally, a principal action by isometries on the quaternionic Kähler base leads to a tri-Hamiltonian principal action by isometries on the Swann bundle. A proof of this claim, along with an expression for the corresponding hyper-Kähler moment map, is given in [Swa91].

Returning to our recurring example $\mathbb{H}\mathrm{P}^n$, there is an obvious bundle with fiber $\mathbb{H}^*/\mathbb{Z}_2$ over it: $(\mathbb{H}^{n+1} \backslash \{0\})/\mathbb{Z}_2$. This is indeed the Swann bundle of $\mathbb{H}\mathrm{P}^n$; its hyper-Kähler structure descends from the flat hyper-Kähler structure on $\mathbb{H}^{n+1} \setminus \{0\}$, which $\mathbb{Z}_2$-invariant.

The norm-squared defines its hyper-Kähler potential and the local Sp(1)-action corresponds to quaternionic multiplication. Now consider the following sequence of bundle maps:

$$(\mathbb{H}^{n+1} \setminus \{0\})/\mathbb{Z}_2 \longrightarrow \mathbb{R}\mathrm{P}^{4n+3} \longrightarrow \mathbb{C}\mathrm{P}^{2n+1} \longrightarrow \mathbb{H}\mathrm{P}^n$$

We already encountered the final bundle map, which is the twistor bundle. The real projective space $\mathbb{R}\mathrm{P}^{4n+3}$ is the total space of the SO(3)-principal bundle $P(\mathcal{Q})$ over M; it carries a so-called 3-Sasakian structure. In fact, for any quaternionic Kähler manifold of positive scalar curvature, $P(\mathcal{Q})$ carries such a structure [BG08]. The map $\mathbb{R}\mathrm{P}^{4n+3} \to \mathbb{C}\mathrm{P}^{2n+1}$, which arises from the construction of the twistor space as the associated bundle $P(\mathcal{Q}) \times_{\mathrm{std}} S^2$, can be thought of as the $\mathbb{Z}_2$-quotient of the Hopf fibration $S^1 \to S^{4n+3} \to \mathbb{C}\mathrm{P}^{2n+1}$.

More generally, Swann [Swa91] gives an explicit Lie-theoretic description of the Swann bundles of the compact Wolf spaces as nilpotent co-adjoint orbits.

3 The hyper-Kähler/quaternionic Kähler correspondence

The hyper-Kähler/quaternionic Kähler correspondence (henceforth HK/QK correspondence) was discovered by Haydys [Hay08]. We saw that, given a quaternionic Kähler manifold M of dimension $4n$, the Swann bundle construction yields a $(4n+4)$-dimensional conical hyper-Kähler manifold $\mathcal{U}(M)$ which fibers over it. Moreover, isometric group actions on the base lift to actions which are compatible with the conical hyper-Kähler structure on the Swann bundle. In particular, an isometric circle action on M lifts to an isometric and tri-Hamiltonian circle action on $\mathcal{U}(M)$ which commutes with the conical $\mathbb{H}^*/\mathbb{Z}_2$-action.

Haydys [Hay08] considered this situation, and in particular the hyper-Kähler quotient $\mathcal{U}(M) /\!/\!/ S^1$, which is a second hyper-Kähler manifold whose dimension equals that of the original quaternionic Kähler manifold. His crucial insight was that the hyper-Kähler quotient can be in some sense inverted, allowing one to reconstruct (most of) the hyper-Kähler geometry "upstairs". The inverse of the Swann bundle construction then yields a quaternionic Kähler manifold. Before introducing the general procedure, we motivate it by introducing the model case of the HK/QK correspondence, which relates $T^*\mathbb{C}\mathrm{P}^n$ to the quaternionic projective space $\mathbb{H}\mathrm{P}^n$.

3.1 The model case

Consider the space $\mathbb{H}^{n+1}$ of $(n+1)$-tuples of quaternions. Thinking of it as the cotangent bundle of $\mathbb{C}^{n+1}$, we split $q \in \mathbb{H}^{n+1}$ into a pair (z, w) of complex vectors satisfying $q = z + wj$. The standard hyper-Kähler structure on this space is determined by the tensor fields given in equation (2.3), and the holomorphic symplectic form with respect to the complex structure I_1 is $\varpi_1 = \omega_2 + i\omega_3 = \sum \mathrm{d}z_l \wedge \mathrm{d}w_l$.

The diagonal action of $\mathrm{U}(1)$ on $\mathbb{C}^{n+1}$ naturally induces a circle action on $\mathbb{H}^{n+1}$ which sends $(z, w) \mapsto (e^{i\theta}z, e^{-i\theta}w)$. Alternatively, one can think of this action as arising from the standard action $\mathbb{H}^{n+1} \curvearrowleft S^1$ by multiplication from the right. Since this action is isometric and triholomorphic, we can apply the hyper-Kähler quotient construction.

Lemma 3.1. *The function $\mu : \mathbb{H}^{n+1} \to \operatorname{Im}\mathbb{H}$ given by $\mu(q) = \frac{1}{2}\sum_l q_l i \bar{q}_l$ is the hyper-Kähler moment map for the above circle action.*

Proof. The vector field generating the circle action is

$$X = i\sum_l \left(z_l \frac{\partial}{\partial z_l} - \bar{z}_l \frac{\partial}{\partial \bar{z}_l} - w_l \frac{\partial}{\partial w_l} + \bar{w}_l \frac{\partial}{\partial \bar{w}_l} \right)$$

and therefore

$$\iota_X \omega_1 = -\frac{1}{2}\sum_l (\bar{z}_l \mathrm{d}z_l + z_l \mathrm{d}\bar{z}_l - \bar{w}_l \mathrm{d}w_l - w_l \mathrm{d}\bar{w}_l) = -\frac{1}{2}\mathrm{d}(|z|^2 - |w|^2)$$

and

$$\iota_X(\omega_2 + i\omega_3) = i\sum_l (w_l \mathrm{d}z_l + z_l \mathrm{d}w_l) = \mathrm{d}\Big(i \sum_l z_l w_l \Big)$$

so the hyper-Kähler moment map is $\mu = \frac{1}{2}(|z|^2 - |w|^2)i - i\,\vec{z}\cdot\vec{w}\,j$. On the other hand, if $q = z + wj$ is a quaternion, the formulas $z = \frac{1}{2}(q - iqi)$ and $w = -\frac{1}{2}(q + iqi)j$ can be used to check that

$$qi\bar{q} = (|z|^2 - |w|^2)i - 2izwj$$

and the claim follows directly. Note that the moment map is equivariant (i.e. invariant, since S^1 is Abelian): $\mu(q \cdot e^{i\theta}) = qe^{i\theta}ie^{-i\theta}\bar{q} = qi\bar{q}$. □

We may use any level set of the moment map to perform the hyper-Kähler quotient. In fact, there are essentially only two distinct possibilities, namely zero and non-zero level sets. To see this, we make use of the standard $\mathbb{H}^*$-action on $\mathbb{H}^{n+1}$, by left multiplication. If we additionally equip $\operatorname{Im}\mathbb{H}$ with the $\mathbb{H}^*$-action given by $p \cdot q = pq\bar{p}$, it is easy to see that the hyper-Kähler moment map is $\mathbb{H}^*$-equivariant. Since the latter action is transitive on $\operatorname{Im}\mathbb{H} \setminus \{0\}$, any two nonzero level sets are related to each other by some $p \in \mathbb{H}^*$. Therefore, the fact that $\mathbb{H}^*$ commutes with the triholomorphic S^1-action and acts by homotheties implies that the hyper-Kähler metrics on the resulting hyper-Kähler quotients are homothetic. On the other hand, $0 \in \operatorname{Im}\mathbb{H}$ is a fixed point of the $\mathbb{H}^*$-action, and may therefore yield a different hyper-Kähler manifold.

Proposition 3.2. *The hyper-Kähler quotient $\mathbb{H}^{n+1}/\!\!/\!\!/S^1$, where S^1 acts as specified above and the quotient is taken with respect to a non-zero value of the moment map, is $T^*\mathbb{C}\mathrm{P}^n$ endowed with the Calabi metric.*

Proof. We start by recalling that $T\mathbb{C}\mathrm{P}^n \cong \mathrm{Hom}(\gamma, \underline{\mathbb{C}}^{n+1}/\gamma)$, where $\gamma \to \mathbb{C}\mathrm{P}^n$ is the tautological line bundle. Therefore, $T^*\mathbb{C}\mathrm{P}^n \cong \mathrm{Hom}(\underline{\mathbb{C}}^{n+1}/\gamma, \gamma)$, and we will use this description to exhibit it as $\mathbb{H}^{n+1}/\!\!/\!\!/S^1$.

Since we may choose a non-zero value for the moment map at our convenience, we use $P := \mu^{-1}\big(\frac{i}{2}\big)$. Given a point $(z, w) \in P$, first note that since $|z|^2 - |w|^2 = 1$, z is non-vanishing. Now consider $zw^T \in \mathrm{End}(\mathbb{C}^{n+1})$. Clearly, its image is the complex line $\langle z \rangle$ spanned by z, and the moment map imposes $w^T z = 0$, so its kernel contains $\langle z \rangle$. This means that it defines an element of $\mathrm{End}(\mathbb{C}^{n+1}/\langle z\rangle, \langle z\rangle)$, and thus we obtain a smooth map

$$\begin{aligned} P &\longrightarrow T^*\mathbb{C}\mathrm{P}^n \cong \mathrm{Hom}(\mathbb{C}^{n+1}/\gamma, \gamma) \\ (z,w) &\longmapsto (\langle z\rangle, zw^T) \end{aligned}$$

This map is moreover invariant under the S^1-action, hence descends to a smooth map $\mathbb{H}^{n+1}/\!\!/\!\!/S^1 = P/S^1 \to T^*\mathbb{C}\mathrm{P}^n$. To prove that it is in fact a diffeomorphism, we construct its inverse. Given $(z,w) \in P$, note that the Hilbert–Schmidt norm of $A = zw^T$ is $\|A\|^2 = |z|^2\,|w|^2 = |z|^2\,(|z|^2 - 1)$, which implies that $|z|^2 = \psi(A)^2$, where $\psi(A) := \frac{1}{2}\sqrt{1+\sqrt{1+4\,\|A\|^2}}$. Hence, our map is inverted by the smooth map

$$\begin{aligned} T^*\mathbb{C}\mathrm{P}^n &\longrightarrow P/S^1 \\ (\ell_{\hat{z}}, A = \hat{z}v^T) &\longrightarrow \big(\psi(A)\hat{z}, \sqrt{\psi(A)^2-1}\cdot v\big)\cdot S^1 \end{aligned}$$

where $\hat{z}$ is any unit vector spanning ℓ (unique up to U(1)-transformation). Note that a different choice of $\hat{z}$, say $\hat{z}' = e^{i\theta}\hat{z}$, forces one to replace v by $v' = e^{-i\theta}v$, correctly reproducing the circle action.

Regarding the induced metric, we see that, on the submanifold $\{(z,0) \subset P\} \cong S^{2n+1}$, this hyper-Kähler quotient reduces to the standard (Kähler) quotient construction of $\mathbb{C}\mathrm{P}^n$. Thus, the hyper-Kähler metric on $T^*\mathbb{C}\mathrm{P}^n$ reduces to the Fubini–Study metric on $\mathbb{C}\mathrm{P}^n$. Moreover, the hyper-Kähler metric on $\mathbb{H}^{n+1}$ is invariant under the diagonal Sp(1)-action induced by left-multiplication by a unit quaternion. Passing to the level set $P = \mu^{-1}(\frac{i}{2})$ of the hyper-Kähler moment map, equivariance of the moment map shows that it is preserved precisely by the U(1)-subgroup given by unit quaternions of the form $e^{i\theta}$, $\theta \in \mathbb{R}$. This action descends to $T^*\mathbb{C}\mathrm{P}^n$ as multiplication by unit complex numbers in the fiber, and the induced metric is invariant under it. Now the uniqueness theorem 2.33 of Feix–Kaledin implies that this must be the Calabi metric. □

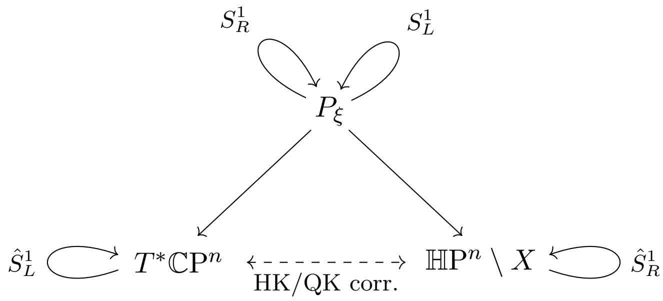
As we have already seen, $\mathbb{H}^{n+1}\setminus\{0\}$ is—up to a quotient by $\mathbb{Z}_2$, which we may disregard in this case[5]—also the Swann bundle of the archetypal quaternionic Kähler manifold $\mathbb{H}\mathrm{P}^n$, which is obtained upon dividing out the $\mathbb{H}^*$-action induced by left-multiplication in the quaternions. We will denote the canonical projection map by π. Since the tri-holomorphic circle action considered above is derived from right-multiplication, the two actions commute and it descends to $\mathbb{H}\mathrm{P}^n$. In fact, in this context we should really think of the action on $\mathbb{H}^{n+1}\setminus\{0\}$ as being the lift of this circle action on $\mathbb{H}\mathrm{P}^n$. In this sense, the full data which we used to construct $T^*\mathbb{C}\mathrm{P}^n$ was derived from the quaternionic Kähler manifold $\mathbb{H}\mathrm{P}^n$, equipped with an isometric circle action.

[5]This is because the $\mathrm{Sp}(n)\,\mathrm{Sp}(1)$-principal frame bundle of $\mathbb{H}\mathrm{P}^n$ lifts to a (doubly-covering) $\mathrm{Sp}(n)\times\mathrm{Sp}(1)$-principal frame bundle; this fails for general quaternionic Kähler manifolds

Haydys [Hay08] asked whether we can conversely recover the quaternionic Kähler geometry by reconstructing the Swann bundle structure from the induced data on $T^*\mathbb{C}\mathrm{P}^n$. In fact, one cannot quite hope to reconstruct the full Swann bundle: Recall that the $\mathbb{H}^*$-action $\mathbb{H}^{n+1}\setminus\{0\}$ acts transitively on the non-zero level sets of the hyper-Kähler moment map, but preserves the codimension-three submanifold $\mu^{-1}(0)$. This means that $T^*\mathbb{C}\mathrm{P}^n$, which arises from the hyper-Kähler quotient at a non-zero level, cannot be used to recover this part of $\mathbb{H}^{n+1}\setminus\{0\}$. Nevertheless, if P is any non-zero level set of μ, then we have $\mathbb{H}^*\cdot P = \mathbb{H}^{n+1}\setminus\mu^{-1}(0)$ and therefore $(\mathbb{H}^{n+1}\setminus\mu^{-1}(0))/\mathbb{H}^* \cong P/\operatorname{Stab}_{\mathbb{H}^*} P$, where $\operatorname{Stab}_{\mathbb{H}^*} P = \{q\in\mathbb{H}^* \mid q\cdot P = P\}$, so if one is able to recover a level set of the moment map together with the appropriate stabilizer, one may hope to reconstruct $\mathbb{H}\mathrm{P}^n\setminus X$, where $X=\pi(\mu^{-1}(0))$ is the space of quaternionic lines in $\mu^{-1}(0)$.

Example 3.3. Let us check that, in the case $n=1$, the "missing piece" $X=\pi(\mu^{-1}(0))$ of $\mathbb{H}\mathrm{P}^1\cong S^4$ is diffeomorphic to S^1. Since $\mu((q_1,q_2)) = q_1 i\bar{q}_1 + q_2 i\bar{q}_2$, we see that $\mu^{-1}(0)\subset\mathbb{H}^2\setminus\{0\}$ consists only of points with $q_1,q_2\neq 0$. Consequently, we have $\mu^{-1}(0)/\mathbb{H}^* = \{[1:q]\in\mathbb{H}\mathrm{P}^1 \mid i+qi\bar{q}=0\}$. In other words, X is the space of quaternions which satisfy $qi\bar{q}=-i$, which is just the circle group whose elements are of the form $e^{i\theta}j$. In particular, $X\cong S^1$, and so $\pi(\mathbb{H}^2\setminus\mu^{-1}(0))\cong S^4\setminus S^1$.

Let us consider this situation in more detail. We know that the level set $P_\xi := \mu^{-1}(\xi)$ is a principal circle bundle over $T^*\mathbb{C}\mathrm{P}^n$; let us denote the principal circle action $q\mapsto qe^{i\theta}$ by S^1_R (where R stands for right-multiplication). On the other hand, in order to obtain (an open dense subset of) $\mathbb{H}\mathrm{P}^n$ we must divide out the action of $\operatorname{Stab}_{\mathbb{H}^*} P_\xi = \{q\in\mathbb{H}^* \mid q\xi\bar{q}=\xi\}\cong S^1$, which acts by left-multiplication and is therefore denoted by S^1_L. The two circle actions commute, so S^1_L descends to $T^*\mathbb{C}\mathrm{P}^n$ and S^1_R descends to $\mathbb{H}\mathrm{P}^n$. In summary, P_ξ has a double fibration structure, summarized in the following diagram:



(3.1)

We already saw in the proof of proposition 3.2 that $\hat{S}^1_L$ has the interpretation of multiplication by unit complex numbers in the fibers of $T^*\mathbb{C}\mathrm{P}^n$. This circle action (or, more precisely, the corresponding Killing field) is rotating, rather than tri-holomorphic. Indeed, the canonical holomorphic symplectic structure on $T^*\mathbb{C}\mathrm{P}^n$, which controls two out of its three Kähler structures and can be locally expressed as $\varpi = \sum \mathrm{d}z_k\wedge\mathrm{d}w_k$, where $\{z_k\}$ are local coordinates on $\mathbb{C}\mathrm{P}^n$ and $\{w_k\}$ fiber coordinates, is clearly transformed to $e^{i\theta}\varpi$ under this action. The central question now becomes: Given only the hyper-Kähler structure and the rotating circle action on $T^*\mathbb{C}\mathrm{P}^n$, can we reconstruct the above diagram completely?

The answer is, of course, positive; this sets up a correspondence between $T^*\mathbb{C}\mathrm{P}^n$ and (an open and dense subset of) $\mathbb{H}\mathrm{P}^n$, providing the first instance of the so-called HK/QK correspondence. Haydys abstracted this situation and showed that it yields a general duality between hyper-Kähler manifolds endowed with a rotating circle action and quaternionic Kähler manifolds equipped with an isometric circle action. Instead of discussing Haydys' original method of proof [Hay08], we will study this construction as part of a broader framework, known as the twist construction.

3.2 The twist construction

Let us consider a smooth manifold M, endowed with a group action $G \curvearrowright M$ and some further, interesting geometric structure which is invariant under said action. Then, we may ask the following natural, but very broad, question: How can one use this data to generate further examples of manifolds with G-action and the same, or a closely related, type of G-invariant geometric structure? One possible answer was provided by Joyce [Joy92], who devised the following strategy: First, construct a principal G-bundle $P \to M$. Second, lift the original G-action to P in such a way that it commutes with the principal G-action, and the tangent spaces to the orbits are transverse to the vertical subspaces of TP. Finally, divide out the lifted action, and study the geometry of the quotient space $\bar{M} = P/G$. The principal G-action on P will descend to a G-action on $\bar{M}$, and if the original G-action and its lift are sufficiently well-behaved, this will be a manifold and geometric structures on M will induce "twisted" structures on $\bar{M}$.

This general method was taken up by Swann, who studied the case where G is a torus and used it to construct examples of various types of (hyper-)complex and (hyper-)Hermitian geometries [Swa10]. Swann's paper contains a clear exposition of the general theory of the twist construction, upon which we base our discussion. We restrict to the case $G = S^1$ throughout, since this is sufficient for our purposes.

Let M be a manifold endowed with a circle action, generated by a vector field Z. Take any principal S^1-bundle $\pi : P \to M$, endowed with a principal S^1-connection $\eta \in \Omega^1(P)$; denote the curvature of η by ω, and the generator of the principal S^1-action by V_P. As explained above, we aim to lift the circle action to P. We furthermore require that the lift preserves η and commutes with the principal circle action. The first step is to lift the generating vector field Z.

Lemma 3.4. *A lift Z_P of Z satisfying $L_{Z_P}\eta = 0$ exists if and only if Z is ω-Hamiltonian, i.e. there exists a function f such that $\iota_Z\omega = -\mathrm{d}f$. Such a lift automatically commutes with V_P.*

Proof. Denote the horizontal lift of Z with respect to η by $\tilde{Z}$. Then $Z_P = \tilde{Z} + \varphi V_P$, where φ is a function on P. Since $L_{Z_P}\eta = 0$, we find

$$0 = \iota_{Z_P}\pi^*\omega + \mathrm{d}\iota_{Z_P}\eta = \pi^*(\iota_Z\omega) + \mathrm{d}\varphi$$

This equation implies that φ is constant along the fibers, hence the $\varphi = \pi^* f$ for some $f \in C^\infty(M)$, which is then a Hamiltonian for Z with respect to ω. Let us check that this lift commutes with V_P. It is clear that the horizontal part of $[Z_P, V_P]$ vanishes, by naturality of the Lie bracket applied to π_*. Thus, it suffices to check its vertical part, which is measured by η. It vanishes as a consequence of the fact that $L_{Z_P}\eta = 0$:

$$\eta([Z_P, V_P]) = Z_P(\eta(V_P)) - (L_{Z_P}\eta)(V_P) = 0$$

Thus, $Z_P = \tilde{Z} + \pi^* f V_P$, is a lift with the required properties. □

Using techniques from algebraic topology it can be shown that, by appropriately choosing f (which is only determined up to an additive constant), we may arrange matter such that the lifted vector field Z_P generates a circle action. The precise statement is the following:

Lemma 3.5 ([Swa10]). *Suppose M admits an ω-Hamiltonian circle action, where ω is a closed, integral two-form (i.e. ω has integral periods). Then there is a principal bundle $\pi : P \to M$ such that*

(i) *There exists a circle action on P (generated by Z_P) covering the circle action on M and commuting with the principal circle action (generated by V_P).*

(ii) *P admits a principal S^1-connection $\eta \in \Omega^1(P)$ whose curvature is ω and which is invariant under the lifted circle action.*

Moreover, if $Z_P^0 = \tilde{Z} + \pi^* f V_P$ generates such a lifted circle action, then, for every $k \in \mathbb{Z}$, the vector field $Z_P^k = \tilde{Z} + \pi^*(f + k)V_P$, $k \in \mathbb{Z}$ does too. □

In summary, the input required for the twist construction is a triple (Z, ω, f), where:

(i) Z is a vector field which generates a circle action.

(ii) The two-form ω is closed and has integral periods.

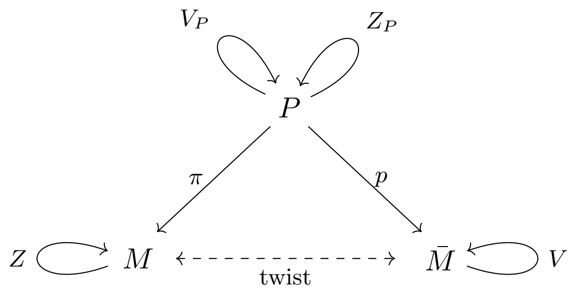
(iii) The function f is an ω-Hamiltonian for Z.

We will call such a triple a collection of twist data for the circle action. For every collection of twist data, we obtain a lift of the circle action generated by Z to an action on an S^1-principal bundle P endowed with an invariant connection η whose curvature is ω. Moreover, by adding a constant to f so that it satisfies the "quantization condition" given in lemma 3.5, we can ensure that we have a circle action on P.

We now divide out the lifted action and study the resulting quotient space. If the actions generated by Z and Z_P are free and proper, then this space is a smooth manifold. From now on, we will always assume that this is the case. In particular, this assumption is satisfied if the circle action on M is free and the Hamiltonian function f is chosen such that we have a circle action on P. For reasons that will be come apparent shortly, we will also require f to be invertible, i.e. nowhere-vanishing.

Definition 3.6. Given a smooth manifold M endowed with twist data (Z, ω, f) as above, let $\pi : P \to M$ be the associated circle bundle and Z_P be the generator of the lifted action. Then the manifold $\bar{M} := P/\langle Z_P \rangle$ is called the twist of M with respect to the twist data.

We now have a double fibration structure on P (compare 3.1):



(3.2)

Here, V_P generates the principal circle action with respect to the projection $\pi : P \to M$. Z_P fulfills the same role for the other projection, and correspondingly $V := p_*(V_P)$ generates a "twisted" circle action on $\bar{M}$, dual to the original action generated by Z. Indeed, the twist construction is most properly thought of as a duality between manifolds with circle actions. Before we can make this precise, however, we need explain how to carry over geometric structures from M to $\bar{M}$. This is the crucial feature that the twist construction derives most of its power from.

The key observation is that the double fibration structure on P allows us to identify the tangent spaces of M and $\bar{M}$ by using the connection η. Indeed, $\ker \eta$ defines the horizontal subbundle $\mathcal{H} \subset TP$, which is transverse to both V_P and Z_P (here, we use the fact that the Hamiltonian f is nowhere-vanishing), and the projection maps π and p induce linear isomorphisms $T_{\pi(x)}M \cong \mathcal{H}_x \cong T_{p(x)}\bar{M}$ for every $x \in P$. This allows us to identify a tensor field T on M with a tensor field T' on $\bar{M}$ if T is carried over to T' under these isomorphisms. Let us make this more precise.

Definition 3.7. We say that the functions $\phi \in C^\infty(M)$ and $\psi \in C^\infty(\bar{M})$ are $\mathcal{H}$-related, and write $\phi \sim_{\mathcal{H}} \psi$, if $\pi^*\phi = p^*\psi$. We call vector fields $X \in \mathfrak{X}(M)$ and $Y \in \mathfrak{X}(\bar{M})$ $\mathcal{H}$-related and write $X \sim_{\mathcal{H}} Y$ if their η-horizontal lifts are equal: $\tilde{X} = \hat{Y}$.

To extend the notion of $\mathcal{H}$-relatedness to arbitrary tensor fields, we demand compatibility with tensor products and contractions.

Example 3.8. One-forms $\alpha \in \Omega^1(M)$, $\beta \in \Omega^1(\bar{M})$ are $\mathcal{H}$-related if and only if, for every $X \in \mathfrak{X}(M)$ and $Y \in \mathfrak{X}(\bar{M})$ such that $X \sim_{\mathcal{H}} Y$, we have $\pi^*\alpha(\tilde{X}) = p^*\beta(\hat{Y})$. In other words, $\pi^*\alpha|_{\mathcal{H}} = p^*\beta|_{\mathcal{H}}$. Equivalently, using the projection map $\mathcal{H} : TP \to \mathcal{H}$, $U \mapsto U - \eta(U)V_P$, we require that their "horizontal lifts" $\tilde{\alpha} := \pi^*\alpha \circ \mathcal{H}$, $\hat{\beta} := p^*\beta \circ \mathcal{H}$ coincide.

In this terminology, two tensor fields T, T' on M and $\bar{M}$, respectively, are $\mathcal{H}$-related precisely if their horizontal lifts to P agree. We will often call T' the twist of T (with respect to the twist data (Z, ω, f)); it is clear that T' is uniquely determined by T. However, not every tensor field can be twisted:

Lemma 3.9. *Assume that $T \sim_{\mathcal{H}} T'$ is a pair of $\mathcal{H}$-related tensors. Then T is invariant under the circle action on M.*

Proof. Because of the compatibility of the Lie derivative with contractions and the Leibniz rule for tensor products, it suffices to check this for the cases where T is a function or a vector field.

If T is a function ϕ, we know that $\pi^*\phi$ is Z_P-invariant because it descends to a well-defined function on $\bar{M}$ and hence must be constant along the fibers of $p : P \to \bar{M}$. Since Z_P is a lift of Z, we then have $\pi^*(L_Z\phi) = L_{Z_P}\pi^*\phi = 0$, showing Z-invariance.

Now assume that T is a vector field X. Then $\widetilde{[Z,X]} = [\tilde{Z}, \tilde{X}] + \pi^*(\omega(Z,X))V_P$. We know that $\tilde{Z} = Z_P - \pi^* f V_P$, and that $\tilde{X}$ is V_P- and Z_P-invariant. Using that $\iota_Z\omega = -\mathrm{d}f$, this implies $\widetilde{[Z,X]} = \pi^*(\mathrm{d}f(X))V_P - \pi^*(\mathrm{d}f(X))V_P = 0$, which means that $[Z, X] = 0$. □

Conversely, it is clear that every Z-invariant tensor field does admit a twist. It is not difficult to give explicit formulas for the twists of specific types of tensor fields; here we record two important examples for later reference.

Example 3.10. Let $\alpha \in \Omega^k(M)$ be Z-invariant. Then the unique $\mathcal{H}$-related k-form β on $\bar{M}$ is determined by the equation

$$p^*\beta = \pi^*\alpha - \eta \wedge \pi^*(f^{-1}\iota_Z\alpha) \tag{3.3}$$

Indeed, by construction $p^*\beta = \pi^*\alpha + \eta \wedge \gamma$, where $\gamma \in \Omega^{k-1}(\mathcal{H})$. Moreover, we have $\iota_{Z_P} p^*\beta = 0$, which we can expand to find

$$0 = \iota_{Z_P}\pi^*\alpha + \pi^* f\gamma - \eta \wedge \iota_Z\gamma = \pi^*(\iota_Z\alpha) + \pi^* f\gamma - \eta \wedge \iota_Z\gamma$$

We see that γ is obtained by restricting this expression to $\mathcal{H}$, where we find $\pi^*(\iota_Z\alpha) + \pi^* f\gamma = 0$. This implies our claim.

Example 3.11. The twist h of a Z-invariant symmetric, bilinear form g is determined by

$$p^*h = \pi^*g - 2\eta \vee \pi^*(f^{-1}\iota_Z g) + \eta^2\pi^*(f^{-2}g(Z,Z)) \tag{3.4}$$

where $\alpha \vee \beta := \frac{1}{2}(\alpha \otimes \beta + \beta \otimes \alpha)$ is the symmetrized tensor product. The proof is analogous to the previous example, i.e. proceeds by using a decomposition into horizontal and vertical parts.

We can also twist the twist data (Z, ω, f) themselves.

Definition 3.12. Let (Z, ω, f) be twist data on M, and let $\bar{M}$ be the twist manifold. If (Z', ω', f') denote the twists of (Z, ω, f), then we say that $(-\frac{1}{f'}Z', \frac{1}{f'}\omega', \frac{1}{f'})$ are the dual twist data.

To show that the dual twist data on $\bar{M}$ deserve their name, let us show how to use them to recover M from $\bar{M}$. The lifted vector field Z_P is now interpreted as generating the principal circle action, and a natural choice of principal S^1-connection is given by $\eta' = \pi^*(f^{-1})\eta$. Its curvature is given by

$$\mathrm{d}\eta' = \pi^* f^{-1}(\pi^*\omega - \eta \wedge \pi^*(f^{-1}\omega)) = p^*(f'^{-1}\omega')$$

where we used equation (3.3). In order to recover M, we look for a circle action on $\bar{M}$ which is Hamiltonian with respect to $\frac{1}{f'}\omega'$, and such that its lift to P (using the prescription of lemma 3.4) yields V_P. A natural candidate is the action generated by $V = p_*(V_P)$, where V_P is the generator of the principal circle action on P viewed as a bundle over M. Using $Z_P = \tilde{Z} + p^*f'V_P$, we see that $V = -\frac{1}{f'}Z'$. This vector field is indeed Hamiltonian: $\iota_V \frac{1}{f'}\omega' = -\mathrm{d}\frac{1}{f'}$. Note that its horizontal lift is $-\pi^*f^{-1}\tilde{Z}$, so when we lift to P we obtain $-p^*f'^{-1}\tilde{Z} + p^*f'^{-1}Z_P = \pi^*f^{-1}(Z_P - \tilde{Z}) = V_P$, as required.

In summary, the twist construction is a duality between manifolds with circle action which moreover induces a isomorphism of the respective algebras of invariant tensor fields. Because its compatibility with tensor products and contractions, the twist construction preserves the algebraic properties of Z-invariant tensor fields, such as algebraic symmetries and non-degeneracy conditions. For instance, the twist of a pseudo-Riemannian metric is again a pseudo-Riemannian metric with the same signature.

The interaction of the twist construction with *differential* conditions is much more delicate. In particular, integrability of geometric structures (such as a complex structure) is typically lost upon twisting. Nevertheless, the failure to preserve differential conditions is controlled and can be understood completely in terms of the twist data. We collect several important examples:

Example 3.13. Let $X, Y \in \mathfrak{X}(M)$, $A, B \in \mathfrak{X}(\bar{M})$ be $\mathcal{H}$-related. Then $[A, B] \sim_{\mathcal{H}} [X, Y] + f^{-1}\omega(X, Y)Z$. We prove the equivalent statement that

$$\widehat{[A, B]} = \widetilde{[X, Y]} + \pi^*(f^{-1}\omega(X, Y))\tilde{Z}$$

where tildes and hats denote horizontal lifts from M and $\bar{M}$, respectively. We may write $\widehat{[A, B]} = [\hat{A}, \hat{B}] + \varphi Z_P$, where φ is a smooth function. By $\mathcal{H}$-relatedness, we can rewrite this as

$$\widehat{[A, B]} = [\tilde{X}, \tilde{Y}] + \varphi Z_P = \widetilde{[X, Y]} - \pi^*(\omega(X, Y))V_P + \varphi Z_P$$

which yields $\widehat{[A, B]} - \widetilde{[X, Y]} = \varphi(\tilde{Z} + \pi^* f V_P) - \pi^*(\omega(X, Y))V_P$. Since the left-hand side is horizontal, we conclude that $\varphi = \pi^*(f^{-1}\omega(X, Y))$, and consequently $\widehat{[A, B]} - \widetilde{[X, Y]} = \pi^*(f^{-1}\omega(X, Y))\tilde{Z}$, as claimed.

Example 3.14. Let $X \sim_{\mathcal{H}} A$ be vector fields and $g \sim_{\mathcal{H}} h$ be symmetric bilinear forms. Then $L_A h \sim_{\mathcal{H}} L_X g - 2f^{-1}\iota_X\omega \vee \iota_Z h$. To prove this, let $Y \sim_{\mathcal{H}} B$ and $W \sim_{\mathcal{H}} C$ be auxiliary vector fields. The Leibniz rule for the Lie derivative yields

$$\pi^*((L_X g)(Y, W)) = L_{\tilde{X}}(\pi^* g(\tilde{Y}, \tilde{W})) - \pi^* g(\widetilde{[X, Y]}, \tilde{W}) - \pi^* g(\tilde{Y}, \widetilde{[X, W]})$$

Now, plugging in formula for the commutator of vector fields given in the previous example, one obtains

$$\pi^*((L_X g)(Y, W)) = p^*((L_A h)(B, C)) + 2\pi^*(f^{-1}\iota_X\omega \vee \iota_Z g)(\tilde{Y}, \tilde{W})$$

and bringing the second term to the left-hand side yields the claimed relation.

Example 3.15. Let (M, I) be a complex manifold, i.e. the almost complex structure I is integrable. If $I \sim_{\mathcal{H}} J$, where J is an almost complex structure on the twist manifold, then J is integrable if and only if the twisting form ω is of type $(1, 1)$ with respect to I. Once again applying our formula for the twist of a commutator of vector fields, we find that the Nijenhuis tensor N_J of J is $\mathcal{H}$-related to the $(2, 1)$-tensor field $\mathcal{N}_I$ determined by

$$\mathcal{N}_I(X, Y) = N_I(X, Y) + f^{-1}\big[\omega(IX, IY) - \omega(X, Y) - I(\omega(IX, Y) + \omega(X, IY))\big]Z$$

where N_I is the Nijenhuis tensor of I. Clearly, if $N_I = 0$ then $\mathcal{N}_I$ vanishes if and only $\omega(IX, IY) = \omega(X, Y)$.

Example 3.16. The exterior derivative of differential forms also receives a correction upon twisting. Let $\alpha \sim_{\mathcal{H}} \beta$ be differential forms. Then $\mathrm{d}\beta \sim_{\mathcal{H}} \mathrm{d}\alpha - f^{-1}\omega \wedge \iota_Z\alpha =: \gamma$.

Indeed, we have

$$\begin{aligned} p^*\mathrm{d}\beta &= \mathrm{d}(\pi^*\alpha - \eta \wedge \pi^*(f^{-1}\iota_Z\alpha)) \\ &= \pi^*\mathrm{d}\alpha - \pi^*(f^{-1}\omega \wedge \iota_Z\alpha) + \eta \wedge \pi^*(f^{-2}\iota_Z\omega \wedge \iota_Z\alpha - f^{-1}\iota_Z\mathrm{d}\alpha) \\ &= \pi^*\gamma - \eta \wedge \pi^*(f^{-1}\iota_Z(\mathrm{d}\alpha - f^{-1}\omega \wedge \iota_Z\alpha)) \\ &= \pi^*\gamma - \eta \wedge \pi^*(f^{-1}\iota_Z\gamma) \end{aligned}$$

where we used Z-invariance of α and the fact that $\mathrm{d}f = -\iota_Z\omega$ in passing to the second line.

Besides tensor fields, we can also twist related objects such as connections. This will be important in our discussion of curvature.

Definition 3.17. We say that a connection ∇ on M is $\mathcal{H}$-related to a connection ∇' on the twist manifold $\bar{M}$ if, for all Z-invariant vector fields X, Y (and their twists A, B), we have $\nabla_X Y \sim_{\mathcal{H}} \nabla'_A B$.

Example 3.18. Let $g \sim_{\mathcal{H}} h$ be Riemannian metrics, and denote their respective Levi-Civita connections by ∇ and ∇'. Then, if $X, Y, W \sim_{\mathcal{H}} A, B, C$ are vector fields, we have $\nabla'_A B \sim_{\mathcal{H}} \nabla_X Y + S_X Y$, where the $(1,2)$-tensor field S is given by

$$2g(S_X Y, W) = \frac{1}{f}\big(g(Z,W)\omega(X,Y) - g(Z,X)\omega(Y,W) - g(Z,Y)\omega(X,W)\big)$$

The proof is very simple: Consider the Koszul formula for ∇' and apply the expression for the twist of a commutator of vector fields (as in example 3.14).

3.3 The HK/QK correspondence as a twist

Building on Swann's work, Macia and Swann [MS15] showed how the HK/QK correspondence arises as a variation on the twist construction. Let us recall the setting of the HK/QK correspondence. We are given a hyper-Kähler manifold (M, g, ω_k), endowed with an isometric circle action which fixes one of the complex structures and rotates the other two. Without loss of generality, we may assume that its generating vector field Z acts as follows on the Kähler forms: $L_Z\omega_1 = 0$, $L_Z\omega_2 = \omega_3$, $L_Z\omega_3 = -\omega_2$. Let us assume that Z is furthermore ω_1-Hamiltonian, with Hamiltonian function f_Z. Our aim is to construct a quaternionic Kähler manifold from these data.

Since the circle action preserves the fundamental four-form $\Omega_{\mathrm{HK}} = \sum_k \omega_k \wedge \omega_k$ which determines the quaternionic structure bundle $\mathcal{Q}$ as well as the metric, the twist manifold will inherit the structure of an almost quaternionic-Hermitian manifold for any choice of two-form and Hamiltonian to complete the twist data. That said, in the current setup there is an obvious such choice, namely (Z, ω_1, f_Z). What makes matters interesting is that these choices do *not* yield a quaternionic Kähler structure on the twist manifold.

In light of the preceding discussion, this is only mildly surprising. Both quaternionic Kähler and hyper-Kähler geometries are determined by almost quaternionic-Hermitian structures satisfying an integrability condition: The vanishing of the covariant derivative of the fundamental four-form and the three Kähler forms, respectively. Of course, the fundamental four-form Ω_{HK} of a hyper-Kähler manifold is also parallel.The twist of Ω_{HK} is the fundamental four-form on the twist manifold, and therefore asking the twist manifold to be quaternionic Kähler boils down to requiring this form to be parallel. Because of theorem 2.8, it suffices to check that it is closed (at least when $\dim M \geq 12$). But since Ω_{HK} is closed, we cannot expect its twist to be (cf. example 3.16). Consequently, the twist of a hyper-Kähler metric will not be quaternionic Kähler metric.

The remedy for this problem was found by Macia and Swann [MS15], informed by the quaternionic Kähler metrics given by Haydys and others [Hay08; ACM13; ACDM15]. Their approach is to deform the almost quaternionic-Hermitian structure in a controlled manner, using the notion of elementary deformations.

Definition 3.19. Let (M, g, ω_k, Z) be a hyper-Kähler manifold endowed with a rotating Killing field, i.e. $L_Z g = 0$, $L_Z\omega_1 = 0$, $L_Z\omega_2 = \omega_3$ and $L_Z\omega_3 = -\omega_2$. We say that a pseudo-Riemannian metric h is an elementary deformation of the hyper-Kähler metric if it is of the form

$$h = ag + b\sum_{\mu=0}^{3} \alpha_\mu \otimes \alpha_\mu$$

where $a, b \in C^\infty(M)$ are nowhere-vanishing functions, and we have set $\omega_0 := g$ and $\alpha_\mu := \iota_Z\omega_\mu$, $\mu = 0, 1, 2, 3$.

An elementary deformation is best thought of as the result of an overall conformal scaling, composed with an independent scaling along the quaternionic span $\mathbb{H}Z = \langle Z, I_1Z, I_2Z, I_3Z\rangle$ of Z. Indeed, a short computation shows that $g_\alpha := \sum_\mu \alpha_\mu^2 = g(Z,Z)g|_{\mathbb{H}Z}$. Note, in particular, that every elementary deformation defines a new almost quaternionic-Hermitian structure compatible with the given almost quaternionic structure.

Macia and Swann considered the following question: Do there exist elementary deformations of the hyper-Kähler metric and choices of twist data such that the twist of the elementary deformation is a quaternionic Kähler metric? They gave a complete answer:

Theorem 3.20 (Macia–Swann [MS15])**.** *Let (M, g, ω_k, Z, f_Z) be a pseudo-hyper-Kähler manifold endowed with an ω_1-Hamiltonian and rotating Killing field, and assume that f_Z is nowhere-vanishing and Z is not null. Then the following combinations of elementary* *deformation* g_{H} *and choices of twist data* $(Z, \omega_{\mathrm{H}}, f_{\mathrm{H}})$ *yield a pseudo-quaternionic Kähler structure on the twist manifold:*

$$\begin{gathered} g_{\mathrm{H}} = \frac{\kappa}{f_Z} g + \frac{\kappa}{f_Z^2} g_\alpha \\ \omega_{\mathrm{H}} = \lambda(\omega_1 + \mathrm{d}\alpha_0) \qquad\qquad f_{\mathrm{H}} = \lambda(f_Z + \alpha_0(Z)) \end{gathered} \tag{3.5}$$

where $\kappa, \lambda \in \mathbb{R} \setminus \{0\}$ *are arbitrary constants. Moreover, these are the only combinations for which the twist manifolds are (pseudo-)quaternionic Kähler.* □

The above formulas are not surprising. Haydys [Hay08] gave an explicit expression for the pullback of the quaternionic Kähler metric g_{Q} to the total space of the circle bundle $\pi : P \to M$, which agrees with the twist formula equation (3.4) applied to g_{H}. Though Haydys worked exclusively with positive-definite hyper-Kähler metrics, his methods were extended to allow for indefinite signature by Alekseevsky, Cortés and Mohaupt [ACM13]. They proved that the signature and scalar curvature of the resulting quaternionic Kähler metric only depends on the signature of the hyper-Kähler metric and the signs of f_Z and f_{H} (which are nowhere-vanishing). As a corollary, they discovered that the resulting quaternionic Kähler metric may be positive-definite even if the initial hyper-Kähler metric is not. This observation is crucial in applications of the HK/QK correspondence to the so-called c-map construction, which we will discuss at length in chapter 4 (see also [ACDM15]).

The contribution of Macia and Swann is twofold. Firstly, their uniqueness statement shows that, within the class of almost quaternionic-Hermitian metrics arising as twists of elementary deformations, the formulas of Haydys and Alekseevsky–Cortés–Dyckmanns–Mohaupt give the only quaternionic Kähler metrics. Secondly, their re-interpretation allows us to use the powerful machinery of the twist formalism in the study of the HK/QK correspondence. As we shall see, this leads to various new insights.

We can summarize the above discussion by means of the following statement, which combines the results of the papers [Hay08; ACM13; ACDM15; MS15]:

Theorem 3.21 (HK/QK correspondence). *Let* (M, g, ω_k) *be a pseudo-hyper-Kähler manifold endowed with an* ω_1*-Hamiltonian, rotating Killing field* Z *which is not null, with nowhere-vanishing Hamiltonian function* f_Z*. Then the twist construction applied to the data* (3.5) *yields a pseudo-quaternionic Kähler manifold* $(\bar{M}, g_{\mathrm{Q}}, \mathcal{Q})$ *equipped with a nowhere-vanishing Killing field* V*. The quaternionic Kähler metric is positive-definite of positive scalar curvature if and only if* (M, g) *is Riemannian and* $f_{\mathrm{H}} < 0$*. It is positive-definite of negative scalar curvature if and only if either* (M, g) *is Riemannian and* $f_Z > 0$*, or its signature is* $(4k, 4)$ *and* $f_Z > 0$ *while* $f_{\mathrm{H}} < 0$*.* □

Remark 3.22.

(i) The constant κ which appears in equation (3.5) only scales the metric g_{H}. We will set $\kappa = 1$ from now on. The constant λ, which scales the curvature of the auxiliary circle bundle, has no impact on the local twist geometry: Different values of λ correspond to coverings of the twist manifold. We will also set this constant to 1 in the following.

(ii) Though we only mention the case where the quaternionic Kähler manifold is Riemannian in the theorem statement, the signature and sign of the scalar curvature of g_{Q} are controlled by the initial hyper-Kähler data in every case. For the details, see [ACM13].

(iii) Since the twist construction is a duality, there is also an inverse construction to the HK/QK correspondence, where the input data is a quaternionic Kähler manifold endowed with a Killing field. This point of view is explored in great detail in [Sah20].

The Hamiltonian function f_Z plays a prominent role in the HK/QK correspondence, as witnessed by its appearance in the expression for g_{H}. Since it is only determined up to an additive constant, there is an implicit one-parameter freedom in the HK/QK correspondence. Considering all possible choices of Hamiltonian simultaneously, we obtain a one-parameter family of quaternionic Kähler metrics on the twist manifold. One may ask if this family is trivial, i.e. if every pair of metrics in this family is isometric. As we will see later, the answer is negative in general.

Naturally, there is much more interest in complete quaternionic Kähler manifolds than in incomplete ones. Regarding the completeness of the quaternionic Kähler metrics constructed via the HK/QK correspondence, we remark the following. Even in the model case of the correspondence, where the input is $T^*\mathbb{C}\mathrm{P}^n$ with its Calabi metric, we saw that the failure to recover the full Swann bundle $\mathbb{H}^{n+1} \setminus \{0\}$ led to an incomplete picture of $\mathbb{H}\mathrm{P}^n$. More precisely, we were able to recover the complement of the codimension-three submanifold consisting of quaternionic lines contained in $\mu^{-1}(0)$. The result is an incomplete manifold, which however can be completed to a compact quaternionic Kähler manifold.

In fact, we could not have hoped to obtain all of $\mathbb{H}\mathrm{P}^n$. The easiest way to understand this is to note that the S^1-invariant complex structure I_1 on the hyper-Kähler side certainly twists to an almost complex structure compatible with the $\mathrm{Sp}(n)\,\mathrm{Sp}(1)$-structure, i.e. a global section of the quaternionic bundle $\mathcal{Q}$ on the quaternionic Kähler side. But, as mentioned in the previous chapter, $\mathbb{H}\mathrm{P}^n$ admits no almost complex structure at all. This observation generalizes to arbitrary quaternionic Kähler manifolds with positive scalar curvature:

Theorem 3.23 (Alekseevsky–Marchiafava–Pontecorvo, [AMP98]). *No complete quaternionic Kähler manifold of positive scalar curvature admits an almost complex structure compatible with its quaternionic Kähler structure.* □

Note that the complex Grassmannian $\mathrm{Gr}_2(\mathbb{C}^{n+2})$ certainly admits an almost complex structure since is even a smooth projective variety (via the Plücker embedding). However, the induced Kähler structure is not compatible with the quaternionic Kähler structure. With this in mind, one may ask if any other quaternionic Kähler manifolds of positive scalar curvature admit an almost complex structure. The answer is negative:

Theorem 3.24 (Gauduchon–Moroianu–Semmelmann, [GMS11]). *The only complete quaternionic Kähler manifolds of positive scalar curvature that admit an almost complex structure are the complex Grassmannians* $\mathrm{Gr}_2(\mathbb{C}^{n+2})$. □

Returning to the setting of the HK/QK correspondence, we can say more about the twist of I_1. To do so, we must first study the relationship between the hyper-Kähler metric g and the twisting form ω_{H}.

Definition 3.25. Let (M, g, I_k, Z) be a hyper-Kähler manifold endowed with a rotating Killing field. We define the endomorphism field I_{H} by $I_{\mathrm{H}} := I_1 + 2\nabla Z$.

Because Z is Killing, I_{H} is skew with respect to g.

Lemma 3.26. *Let Z be a rotating Killing field on a hyper-Kähler manifold $(M, g = \omega_0, \omega_k)$. The one-forms $\alpha_\mu = \iota_Z\omega_\mu$ satisfy*

$$\begin{aligned}(\mathrm{d}\alpha_0)(X,Y) &= 2g(\nabla_X Z, Y)\\ (\mathrm{d}\alpha_k)(X,Y) &= (L_Z\omega_k)(X,Y)\\ &= \omega_k(\nabla_X Z, Y) + \omega_k(X, \nabla_Y Z)\end{aligned} \tag{3.6}$$

Proof. These identities follow from Cartan's formula and the fact that $(L_X\alpha_\mu)(Y) = (\nabla_X\alpha_\mu)(Y) + \alpha_\mu(\nabla_Y X)$. For instance:

$$\begin{aligned}\mathrm{d}\alpha_0(X,Y) &= (L_X\alpha_0)(Y) - Y(g(Z,X))\\ &= g(\nabla_X Z, Y) + g(Z, \nabla_Y X) - g(\nabla_Y Z, X) - g(Z, \nabla_Y X)\\ &= 2g(\nabla_X Z, Y)\end{aligned}$$

where the final step uses the Killing equation, which says that ∇Z is skew with respect to g. The other identities follow analogously. □

The first identity shows that I_{H} is the endomorphism field connecting g to the twisting form ω_{H} which is used in the HK/QK correspondence (cf. theorem 3.21) via the identity $\omega_{\mathrm{H}}(X,Y) = g(I_{\mathrm{H}}X, Y)$ for all vector fields X and Y.

Proposition 3.27. *Let (M, g, I_k, Z) be a hyper-Kähler manifold endowed with a rotating Killing field. Then I_{H} commutes with each I_μ, $\mu = 0, 1, 2, 3$.*

Proof. Using $L_Z\omega_1 = 0$, $L_Z\omega_2 = \omega_3$ and $L_Z\omega_3 = -\omega_2$ and the second line of (3.6), one obtains

$$\begin{aligned}
2\omega_1(\nabla_X Z, Y) + 2\omega_1(X, \nabla_Y Z) &= 0 = -\omega_1(I_1X, Y) + \omega_1(I_1Y, X)\\
2\omega_2(\nabla_X Z, Y) + 2\omega_2(X, \nabla_Y Z) &= 2\omega_3(X, Y) = \omega_1(I_2X, Y) - \omega_1(I_2Y, X)\\
2\omega_3(\nabla_X Z, Y) + 2\omega_3(X, \nabla_Y Z) &= -2\omega_2(X, Y) = \omega_1(I_3X, Y) - \omega_1(I_3Y, X)
\end{aligned}$$

After rearranging terms, we can write this compactly as $g(I_kI_{\mathrm{H}}X, Y) = g(I_kI_{\mathrm{H}}Y, X)$ for $k = 1, 2, 3$. We also have $g(I_0I_{\mathrm{H}}X, Y) = -g(I_0I_{\mathrm{H}}Y, X)$, and in summary find

$$g(I_\mu I_{\mathrm{H}}X, Y) = -g(I_\mu^{-1}I_{\mathrm{H}}Y, X) \qquad \mu = 0, 1, 2, 3$$

This implies that

$$g(I_\mu I_{\mathrm{H}}X, Y) = -g(I_\mu^{-1}I_{\mathrm{H}}Y, X) = -g(I_{\mathrm{H}}Y, I_\mu X) = g(Y, I_{\mathrm{H}}I_\mu X) = g(I_{\mathrm{H}}I_\mu X, Y)$$

which implies the claim. □

Remark 3.28. In fact, what we have shown is that a vector field Z on a hyper-Kähler manifold (M, g, I_k) is a rotating Killing field if and only if $I_1 + 2\nabla Z$ defines a skew-symmetric endomorphism with respect to g which commutes with every I_μ.

Corollary 3.29. *The twisting form ω_{H} used in the HK/QK correspondence is of type $(1, 1)$ with respect to each complex structure I_k.*

Proof. This follows from the following short computation:

$$\omega_{\mathrm{H}}(I_kX, I_kY) = g(I_{\mathrm{H}}I_kX, I_kY) = g(I_kI_{\mathrm{H}}X, I_kY) = g(I_{\mathrm{H}}X, Y) = \omega_{\mathrm{H}}(X, Y)$$

□

The fact that ω_{H} is of type $(1, 1)$ with respect to any complex structure compatible with the hyper-Kähler structure was first noted by Hitchin [Hit13], who used it as the starting point for a twistor-theoretic formulation of the HK/QK correspondence (see also [Hit14]).

We can now finally fulfill our promise and relate this to the twist of I_1:

Theorem 3.30. *Every quaternionic Kähler manifold $(\bar{M}, g_{\mathrm{Q}}, \mathcal{Q})$ that arises from the HK/QK correspondence carries an integrable complex structure which is compatible with the quaternionic structure.*

Proof. The almost complex structure I_1 is integrable and Z-invariant, and ω_{H} is of type $(1,1)$ with respect to it. Therefore, its twist is also an integrable almost complex structure (cf. example 3.15). □

Remark 3.31. Though this result has not appeared in this form in the literature, it is (non-trivially) equivalent to the discussion of [Sal99, Sec. 7], which predates the HK/QK correspondence and is formulated purely on the quaternionic Kähler manifold. Salamon observes that, given the quaternionic moment map $\mu \in \Gamma(\mathcal{Q})$ associated with a non-vanishing Killing field V on $\bar{M}$, its normalization $\mu/|\mu|$ is an integrable complex structure on $\{\mu \neq 0\} \subset \bar{M}$ (see also [Hay08, Thm. 14] for a closely related result).

In our setting, the Killing field V is the twist of $-\frac{1}{f_{\mathrm{H}}}Z$ and the quaternionic moment map is the twist of $\frac{1}{f_{\mathrm{H}}}I_1$ (for details, see [Dyc15; Sah20]), so that the normalized moment map is nothing but the twist of I_1. Thus, our point of view gives a new interpretation of this complex structure as arising from the dual hyper-Kähler geometry via the HK/QK correspondence.

We also note that this result was proven via different methods in [CDS17] in the special case where $(\bar{M}, g_{\mathrm{Q}}, \mathcal{Q})$ arises from the c-map construction (see chapter 4).

In conclusion, the HK/QK correspondence produces quaternionic Kähler manifolds which simultaneously carry a compatible complex structure. In particular, it will not provide us with examples of complete quaternionic Kähler manifolds of positive scalar curvature. However, the situation is rather different when $\mathrm{scal} < 0$: As we will see in the next chapter, the correspondence can be used to produce many complete examples in this case.

4 Special Kähler geometry and the c-map

Before giving a precise formulation of special Kähler geometry and the construction known as the c-map, we give a brief account of the historical development of these concepts, which first arose in theoretical physics. We will only briefly sketch the relevant physical results, and make no claim of completeness. Nevertheless, we hope that this discussion provides the reader with some motivation for what will follow, and highlights the importance of sustained collaborations between mathematicians and physicists.

4.1 Digression into the physical origins of the c-map

Classical (as opposed to quantum) physical theories are usually based on the concept of a physical field, defined on a pseudo-Riemannian manifold (X, h) which is interpreted as spacetime; basic examples include the familiar electric and magnetic fields. Such fields typically have a natural geometric interpretation. For instance, the electric and magnetic field can be described in terms of a connection on a $\mathrm{U}(1)$-bundle over X; other fields may be described as sections of certain vector bundles over spacetime. Perhaps the simplest case is that of a so-called scalar field, which geometrically corresponds to a map from spacetime to some other pseudo-Riemannian manifold (M, g), which is called its target space.

Given a set of fields, the dynamics of a given physical situation are determined by a functional on the space of all possible field configurations, which is known as the action of the theory. The most famous example is the Yang–Mills functional on the space of principal connections on a G-bundle over a four-manifold X:

$$S_{\mathrm{YM}}(A) = \int_X F_A \wedge \star F_A = \int_X \langle F_A, F_A \rangle \,\mathrm{vol}_h$$

where F_A is the curvature of the connection A. This functional determines the dynamics of a so-called gauge field with gauge group G (generalizing the electromagnetic field, which corresponds to $G = \mathrm{U}(1)$) in a vacuum. The integrand, $\langle F_A, F_A \rangle$ in this case, is called the Lagrangian of the theory. The situation can be made more complicated by adding ("coupling to", in the physics terminology) additional fields, which can be used to model, for instance, the presence of matter.

In many cases, the physical theory one is interested in possesses certain symmetries; these must be reflected in the Lagrangian, and hence impose constraints on it. In some cases, these constraints are so stringent that the most general form of the Lagrangian which conforms to the physical requirements can be written down explicitly. Now, if the physical theory contains scalar fields, it is reasonable to expect that such symmetry requirements lead to restrictions on the possible target spaces of the scalar fields. This philosophy has been extremely successful, in particular when the theory is invariant under so-called supersymmetries.

For us, the relevant instances of this general principle occur in the context of a theoretical model known as four-dimensional $\mathcal{N} = 2$ supergravity. It was first argued by Bagger and Witten [BW83] that $\mathcal{N} = 2$ supergravity coupled to so-called (massless) hypermultiplets contains scalar fields—known as hypermultiplet scalars—whose target space must be a quaternionic Kähler manifold of negative scalar curvature in order to preserve supersymmetry. Not much later, De Wit and Van Proeyen [dWvP84] studied the coupling of $\mathcal{N} = 2$ supergravity to another set of fields known as vector multiplets. They found that the target space of the scalars was a Kähler manifold which satisfied certain additional conditions. Rigorously formulating these additional restrictions remained an open problem for a considerable amount of time, until the issue was resolved by Freed [Fre99], who provided a precise definition and introduced the name projective special Kähler manifolds for these manifolds. They fit in the broader theoretical framework of special Kähler geometry, which can be used to describe target spaces in related physical situations as well. We will review these concepts from a mathematical point of view in section 4.2.

Another important development occurred in the late 1980's, when Cecotti, Ferrara and Girardello showed that the relationship of four-dimensional $\mathcal{N} = 2$ supergravity models to certain ten-dimensional theories of superstrings, known as type IIA and type IIB superstring theory, implies a relationship between the target spaces of hypermultiplet and vector multiplet scalars [CFG89]. Physical arguments indicate that there exists a duality between the type IIA and type IIB theories which, on the level of supergravity theories, serves to interchange hypermultiplets with vector multiplets. In particular, the target spaces of hypermultiplet scalars are determined by the geometry of the target space of the dually related vector multiplet scalars. Concretely, this indicates that there must be a method to construct a quaternionic Kähler manifold out of any projective special Kähler manifold. This construction came to be known as the *c-map* (though it does not correspond to a map between manifolds).

Although the c-map is motivated by stringy dualities in [CFG89], it is most easily be formulated purely in terms of supergravity (see e.g. [dWvP92]). Starting from $\mathcal{N} = 2$ supergravity theory coupled to vector multiplets in four dimensions, dimensional reduction to three dimensions yields $\mathcal{N} = 4$ supergravity coupled to hypermultiplets; the target space geometry of the scalar fields in this theory is also known to be quaternionic Kähler, so we have obtained a quaternionic Kähler manifold from a projective special Kähler manifold again. It turns out that this construction precisely reproduces the c-map.

This point of view was exploited by De Wit and Van Proeyen [dWvP92], who studied examples of homogeneous quaternionic Kähler manifolds arising from the c-map. Their startling conclusion was that the c-map gave rise to quaternionic Kähler metrics which are homogeneous under a solvable group, but did not appear in Alekseevsky's claimed classification of such manifolds [Ale75]. Their claims were proven to be correct by Cortés [Cor96b], who completed Alekseevsky's classification by rigorous methods.

Despite this striking demonstration of the potential of the c-map for constructing new examples of quaternionic Kähler manifolds, it remained largely unknown in mathematical circles for a long time. This is due, in part, to the large differences in notation and terminology between theoretical physicists and differential geometers.

The first exposition of the c-map construction in standard differential-geometric terms was given by Hitchin [Hit09]. The properties of the resulting quaternionic Kähler metrics, whose local expressions had first been given by Ferrara and Sabharwal [FS90], were subsequently investigated by several authors including Alekseevsky, Cortés and Mohaupt [CHM12; CDS17; ACM13; ACDM15]. In these works, they also studied the so-called one-loop deformed c-map metrics, which had been discovered by physicists through the consideration of certain quantum effects. We will review their main results in section 4.4.

Over the past decade, the c-map has been transformed into a well-understood and rigorous differential-geometric construction that has led to many new examples of complete quaternionic Kähler manifolds of negative scalar curvature. As such, it is a shining example of the deep mathematics uncovered in theoretical physics and string theory in particular.

4.2 Special Kähler geometry: affine, conical and projective

We launch straight into the definitions, relying on the historical context provided by the preceding section for motivation.

Definition 4.1. An (affine) special Kähler manifold is a pseudo-Kähler manifold (M, g, J) endowed with a torsion-free connection ∇ which is flat and symplectic and furthermore satisfies $(\nabla_X J)Y = (\nabla_Y J)X$ for every $X, Y \in \mathfrak{X}(M)$.

Remark 4.2. Note that we will not be using the prefix pseudo- even when we discuss special Kähler manifolds of indefinite signature.

In this section, we will denote Levi-Civita connections by ∇^{LC} in order to avoid confusion with the flat connection on a special Kähler manifold.

Lemma 4.3. *If (M, g, J, ∇) is an affine special Kähler, the tensor field ∇g is totally symmetric.*

Proof. It suffices to check that $(\nabla_X g)(Y,Z) = (\nabla_Z g)(Y,X)$ for arbitrary vector fields X, Y and Z. We compute

$$(\nabla_X g)(Y,Z) = \nabla_X(g(Y,Z)) - g(\nabla_X Y, Z) - g(Y, \nabla_X Z) = \omega(Y, (\nabla_X J)Z)$$

which is manifestly symmetric under exchange of X and Z. □

Definition 4.4. A conical affine special Kähler (CASK) manifold is an affine special Kähler manifold (M, g, J, ∇) endowed with a principal $\mathbb{C}^*$-action generated by vector fields $\{\xi, J\xi\}$ such that $\nabla\xi = \nabla^{\mathrm{LC}}\xi = \mathrm{id}_{TM}$, and such that g is negative-definite when restricted to $\mathcal{D} = \langle \xi, J\xi \rangle$, and positive-definite on its orthogonal complement $\mathcal{D}^\perp$.

In particular, the signature of a CASK manifold of (real) dimension $2n$ is $(2n-2, 2)$. The vector field ξ is sometimes called the Euler field. We will often denote a CASK manifold by a tuple (M, g, J, ∇, ξ), keeping the principal $\mathbb{C}^*$-action generated by $\{\xi, J\xi\}$ implicit.

Let us collect some well-known facts concerning the interaction between the vector fields $\xi, J\xi$ and the affine special Kähler structure (see e.g. [CM09]) We begin by disregarding the flat connection and focus purely on the conical Kähler structure.

Lemma 4.5. *Let (M, g, J, ξ) be a conical pseudo-Kähler manifold, i.e. a pseudo-Kähler manifold such that $\nabla^{\mathrm{LC}}\xi = \mathrm{id}_{TM}$. Then ξ and $J\xi$ are holomorphic. Moreover, $J\xi$ is Killing while ξ is homothetic.*

Proof. These results follow from short computations which are rather similar. Let X be an arbitrary vector field on M. Then

$$(L_\xi J)(X) = \nabla^{\mathrm{LC}}_\xi (JX) - \nabla^{\mathrm{LC}}_{JX}\xi - J(\nabla^{\mathrm{LC}}_\xi X - \nabla^{\mathrm{LC}}_X \xi) = 0$$

where we used that ∇^{LC} is torsion-free and commutes with J, plus the relation $\nabla^{\mathrm{LC}}\xi = \mathrm{id}$. The holomorphicity of $J\xi$ follows from the same exact proof. Exploiting $\nabla^{\mathrm{LC}}\omega = 0$, we also obtain

$$(L_\xi \omega)(X,Y) = \omega(\nabla^{\mathrm{LC}}_X \xi, Y) + \omega(X, \nabla^{\mathrm{LC}}_Y \xi) = 2\omega(X,Y)$$

and since $L_\xi J = 0$ this implies $L_\xi g = 2g$, i.e. ξ is homothetic. Using $J\xi$ instead, we find

$$(L_{J\xi}\omega)(X,Y) = \omega(JX, Y) + \omega(X, JY) = 0$$

This proves that $J\xi$ is Killing. □

The following lemma corresponds, in the Kähler setting, to what we already noticed for conical hyper-Kähler manifolds in our discussion of Swann bundles.

Lemma 4.6. *On a conical pseudo-Kähler manifold (M, g, J, ξ), the function $\kappa = \frac{1}{2}g(\xi,\xi)$ is a Kähler potential for g as well as a moment map for $J\xi$.*

Proof. The second claim has the shortest proof: $\mathrm{d}\kappa = g(\xi,\cdot) = -\omega(J\xi,\cdot)$. For the first claim, we compute $\mathrm{d}\mathrm{d}^c\kappa = -\mathrm{d}J^*\mathrm{d}\kappa = \mathrm{d}(\omega(\xi,\cdot))$. Now since $\mathrm{d}\alpha(X,Y) = (\nabla\alpha)(X,Y) - (\nabla\alpha)(Y,X)$ for any one-form α, the facts that $\nabla^{\mathrm{LC}}\omega = 0$ and $\nabla^{\mathrm{LC}}\xi = \mathrm{id}$ together imply that $\frac{1}{2}\mathrm{d}\mathrm{d}^c\kappa = \omega$, as claimed. □

Now we consider the flat connection ∇, which relates to the distinguished vector fields $\xi, J\xi$ as follows:

Lemma 4.7. *Let (M, g, J, ∇, ξ) be a CASK manifold. Then ξ is ∇-affine, i.e. $[\xi, \nabla_X Y] = \nabla_{[\xi,X]}Y + \nabla_X[\xi, Y]$ for arbitrary $X, Y \in \mathfrak{X}(M)$. $J\xi$ is ∇-affine if and only if $\nabla = \nabla^{LC}$.*

Proof. Since ∇ is flat, there exist local frames of ∇-parallel vector fields, and so we may compute with such. Let $X, Y \in \mathfrak{X}(M)$ be ∇-parallel. Then

$$[\xi, \nabla_X Y] - \nabla_{[\xi,X]}Y - \nabla_X[\xi, Y] = -\nabla_X[\xi, Y] = \nabla_X(\nabla_Y\xi) = \nabla_X Y = 0$$

Similarly

$$\begin{aligned}[J\xi, \nabla_X Y] - \nabla_{[J\xi,X]}Y - \nabla_X[J\xi, Y] &= \nabla_X(\nabla_Y J\xi) = \nabla_X((\nabla_Y J)\xi + J\nabla_Y\xi) \\ &= \nabla_X((\nabla_\xi J)Y + JY) = (\nabla_X J)Y\end{aligned}$$

where we used that ∇J is symmetric and the fact that $(\nabla_\xi J)Y = 0$, as follows from the following computation:

$$(\nabla_\xi J)Y = \nabla_\xi(JY) = L_\xi(JY) + \nabla_{JY}\xi = JL_\xi Y + JY = J\nabla_Y\xi + JY = 0$$

This shows that $J\xi$ is ∇-affine if and only if $\nabla J = 0$. Since ∇ is symplectic, this happens if and only if ∇ is metric-compatible, which means that $\nabla = \nabla^{\mathrm{LC}}$ since ∇ is torsion-free. □

The natural notion of symmetry for CASK manifolds is the following:

Definition 4.8. Given a CASK manifold (M, g, J, ∇, ξ), a diffeomorphism $\varphi : M \to M$ is called a CASK automorphism if it preserves the Kähler structure and the flat connection ∇, and commutes with the $\mathbb{C}^*$-action generated by ξ and $J\xi$.

Lemma 4.9. *Every one-parameter group of CASK automorphisms is Hamiltonian.*

Proof. Let X be the vector field generating a given one-parameter group of CASK automorphisms. We know that $\kappa = \frac{1}{2}g(\xi,\xi)$ satisfies $\mathrm{d}\mathrm{d}^c\kappa = 2\omega$, and now claim that $\frac{1}{2}\mathrm{d}^c\kappa(X)$ is a Hamiltonian function for X. Since $\mathrm{d}\iota_X\mathrm{d}^c\kappa = L_X\mathrm{d}^c\kappa - \iota_X\mathrm{d}\mathrm{d}^c\kappa = L_X\mathrm{d}^c\kappa - 2\iota_X\omega$, it suffices to check that the first term vanishes. But $\mathrm{d}^c\kappa = g(J\xi,\cdot)$ so this follows from the fact that X, by assumption, preserves g, J and ξ. □

Since any CASK manifold has the structure of a principal bundle with fiber $\mathbb{C}^*$, we can naturally associate another manifold to it.

Definition 4.10. Given a CASK manifold (M, g, J, ∇, ξ), the manifold $\bar{M} := M/\mathbb{C}^*$ obtained by dividing out the principal $\mathbb{C}^*$-action is called a projective special Kähler (PSK) manifold.

Remark 4.11. Note that the relationship between PSK and CASK manifolds is closely analogous to that between quaternionic Kähler manifolds and their Swann bundles.

Though this extrinsic definition of PSK structures is standard, it may seem a little unnatural. Finding a satisfactory, intrinsic characterization of these structures is a current topic of research. A promising proposal can be found in recent work of Mantegazza [Man19].

A PSK manifold is indeed always Kähler, since it can be viewed as a Kähler quotient of the corresponding CASK manifold. Indeed, the Euler field ξ generates an $\mathbb{R}_{>0}$-action whose orbits intersect the non-zero level sets of the function $f = -\frac{1}{2}g(\xi, \xi)$ exactly once. Since f is the moment map for the circle action generated by $-J\xi$,[6] we see that $\bar{M} = f^{-1}\big(\frac{1}{2}\big)/S^1 = M /\!\!/ S^1$. Moreover, our assumptions imply that a PSK manifold is positive-definite.

The following definition formalizes the natural notion of symmetry for PSK structures:

Definition 4.12. Let $\bar{M} = M /\!\!/ S^1$ be a PSK manifold, presented as the Kähler quotient of its associated CASK manifold (M, g, J, ∇, ξ). An automorphism of $\bar{M}$ is a diffeomorphism $\varphi : \bar{M} \to \bar{M}$ which is induced by a CASK automorphism of $\bar{M}$.

A CASK automorphism covers the identity map on the corresponding PSK manifold if and only if it preserves the fibers of the $\mathbb{C}^*$-bundle structure. If we have a one-parameter group of such automorphisms, we know that its generating vector field is pointwise a linear combination of ξ and $J\xi$. The Euler field ξ is never an infinitesimal CASK automorphism but, when the special connection coincides with the Levi-Civita connection, $J\xi$ is. Therefore, the following result is not surprising:

Lemma 4.13. *Let (M, g, J, ∇, ξ) be a CASK manifold and $(\bar{M}, \bar{g}, \bar{\omega})$ the corresponding PSK manifold such that we have a $\mathbb{C}^*$-principal bundle $\pi : M \to \bar{M}$. Consider the natural map $\pi_* : \mathfrak{aut}(M) \to \mathfrak{aut}(\bar{M})$ which maps an infinitesimal CASK automorphism to an infinitesimal PSK automorphism.*

(i) *If $\nabla \neq \nabla^{LC}$, π_* is a linear isomorphism.*

[6] The reason why we emphasize $-J\xi$ rather than $J\xi$ will become clear later, when we consider the pseudo-hyper-Kähler structure on T^*M.

(ii) *If $\nabla = \nabla^{LC}$, π_* induces an isomorphism $\mathfrak{aut}(M)/(\mathbb{R} \cdot J\xi) \cong \mathfrak{aut}(\bar{M})$.*

In particular, in both cases there exists an injective, linear map $\mathfrak{aut}\,\bar{M} \to \mathfrak{aut}\,M$.

Proof. The fact that the $\mathbb{C}^*$-action on M is homothetic implies that the distribution $\mathcal{D}^\perp$ of tangent vectors perpendicular to the vertical subbundle $\mathcal{D} = \langle \xi, J\xi \rangle$ is a principal $\mathbb{C}^*$-connection in the bundle $\pi : M \to \bar{M}$.

Let X be an infinitesimal CASK automorphism of M. Because $0 = X(g(\xi,\xi)) = 2g(\xi, X)$, we may write $X = X_{\mathcal{H}} + f_X J\xi$, where $X_{\mathcal{H}}$ is the horizontal lift of $\bar{X} = \pi_* X$. Because X commutes with ξ and $J\xi$, the function f_X must be constant along the fibers, and therefore $f_X = \pi^* h_X$ for some $h_X \in C^\infty(\bar{M})$.

All that remains is to determine h_X. If $\pi_* X = \pi_* X'$, where $X' = X_{\mathcal{H}} + h'_X J\xi$, then $X - X' = \pi^*(h_X - h'_X) J\xi \in \mathfrak{aut}(M)$, which means in particular that it is a Killing field. Since $L_{\varphi V} g = \varphi L_V g + 2\mathrm{d}\varphi \vee g(V, \cdot)$ for any smooth function φ and vector field V, the fact that $J\xi$ is a Killing field now implies that $h_X - h'_X$ is constant. Furthermore, as an infinitesimal CASK automorphism $X - X'$ must be ∇-affine. When $\nabla \neq \nabla^{\mathrm{LC}}$, this means that it vanishes (by lemma 4.7).

In general, we claim that h_X is, up to a factor, a Hamiltonian function for $\bar{X}$ with respect to $\bar{\omega}$. Once again using the function $\kappa = \frac{1}{2} g(\xi, \xi)$, we know $L_X \mathrm{d}^c \kappa = L_X g(J\xi, \cdot) = 0$ and $\mathrm{d}\mathrm{d}^c \kappa = 2\omega$, whence

$$0 = \mathrm{d}(g(J\xi, X)) + 2\iota_X \omega = \pi^* \mathrm{d}h_X + 2\iota_X \omega$$

Restricting this formula to the level set $S = \{g(\xi,\xi) = -1\} \subset M$, used to construct $\bar{M}$ as a Kähler quotient, we have $\iota^* \omega = \pi_S^* \bar{\omega}$ (ι the natural inclusion map of $S \subset M$ and $\pi_S : S \to \bar{M}$ the projection map), so our formula becomes $\pi_S^*(\mathrm{d}h_X + 2\iota_{\bar{X}} \bar{\omega}) = 0$. This proves that $\frac{1}{2} h_X$ is a Hamiltonian for $\bar{X}$. In the case where $\nabla = \nabla^{\mathrm{LC}}$ this Hamiltonian is not uniquely determined, and the lifts to M corresponding to different choices differ by a constant multiple of $J\xi$, which therefore spans the kernel of $\pi_* : \mathfrak{aut}(M) \to \mathfrak{aut}(\bar{M})$. □

4.3 Examples and extrinsic special Kähler geometry

The simplest affine special Kähler manifolds are open subsets of complex vector spaces, endowed with the standard, flat Kähler structure of signature (p, q). If we choose a $\mathbb{C}^*$-invariant subset (and appropriately choose the signature), we obtain a CASK structure. In this case, the Levi-Civita connection itself satisfies the conditions required of the distinguished flat connection ∇, so we trivially obtain a special Kähler structure, and in the $\mathbb{C}^*$-invariant case we find corresponding PSK manifolds.

Example 4.14. Consider $\mathbb{C}^{n+1}$, endowed with its Kähler structure of signature $(n, 1)$, determined by the Hermitian form $h = -\mathrm{d}z_0 \otimes \mathrm{d}\bar{z}_0 + \sum_{j=1}^n \mathrm{d}z_j \otimes \mathrm{d}\bar{z}_j$. Let ξ be the standard Euler field $\xi = \sum_{j=0}^n z_j \frac{\partial}{\partial z_j} + \bar{z}_j \frac{\partial}{\partial \bar{z}_j}$. Now consider the open subset $M = \{z \in$

$\mathbb{C}^{n+1} \mid h(z,z) < 0\}$. Then the above structures cast M as a CASK manifold. It is well known that, upon taking the Kähler quotient by the natural U(1)-action of the unit complex numbers, one obtains complex hyperbolic space $\mathbb{C}\mathrm{H}^n$, which is therefore a PSK manifold. Indeed, the set $\bar{M} = \{[z] \in \mathbb{C}\mathrm{P}^n \mid h(z,z) < 0\}$ is transformed, after applying the standard chart $\varphi_0 : \{[z] \in \mathbb{C}\mathrm{P}^n \mid z_0 \neq 0\} \to \mathbb{C}^n$, $\varphi_0([z]) = \left(\frac{z_1}{z_0}, \ldots, \frac{z_n}{z_0}\right)$, into the unit ball model of hyperbolic space. Let us explicitly work out the Kähler metric. First, we must pass to a level set of the moment map $f = -\frac{1}{2}g(\xi,\xi) = -\frac{1}{2}h(z,z)$; our conventions dictate that we consider $f^{-1}\left(\frac{1}{2}\right)$. The projection map $\pi : f^{-1}\left(\frac{1}{2}\right) \to \bar{M}$ admits a global section $s : \bar{M} \to f^{-1}\left(\frac{1}{2}\right)$, given by

$$s([1 : X_1, \cdots : X_n]) = \frac{1}{\sqrt{1 - \sum_j |X_j|^2}}(1, X_1, \ldots, X_n)$$

The induced Kähler metric is then

$$\bar{g} = \frac{1}{1 - \sum_j |X_j|^2}\left(\sum_j |\mathrm{d}X_j|^2 + \frac{1}{1 - \sum_k |X_k|^2}\left|\sum_j X_j \mathrm{d}\bar{X}_j\right|^2\right)$$

where the coordinates $\{X_j\}$ parametrize the unit ball. This is the well-known Bergman metric on $\mathbb{C}\mathrm{H}^n$. It admits a natural Kähler potential given by

$$\mathcal{K} = \log\left(\frac{-h(z,z)}{|z_0|^2}\right) = \log\left(1 - \sum_j |X_j|^2\right)$$

In search of further examples, the following no-go theorem may appear problematic:

Theorem 4.15 (Lu, [Lu99]). *Every complete affine special Kähler manifold with positive-definite signature is flat.* □

Nevertheless, it turns out to be possible to construct many examples of complete PSK manifolds, which arise from indefinite affine special Kähler manifolds. In order to appreciate this, it is helpful to change our point of view and adopt an extrinsic perspective of special Kähler geometry, which was systematically developed by Alekseevsky, Cortés and Devchand [ACD02]; the following discussion is based on their work.

The starting point is the well-known fact that special Kähler manifolds admit certain distinguished sets of local coordinates.

Proposition 4.16. *Every affine special Kähler manifold (M, g, J, ∇) admits a local system of coordinates $\{x_i, y_i\}$ which are ∇-affine, Darboux with respect to the Kähler form, and given by $x_i = \operatorname{Re} z_i$, $y_i = \operatorname{Re} w_i$ for two sets of holomorphic coordinates $\{z_i\}$ and $\{w_i\}$ on M.*

Sketch of Proof. The existence of a flat connection implies that there exist local parallel frames $\{\alpha_i\}$ for T^*M which, after possibly restricting to a simply connected open subset, can be written as $\alpha_i = \mathrm{d}x_i$ for local coordinate functions x_i. Such coordinates (satisfying $\nabla \mathrm{d}x_i = 0$) are called affine. Since $\nabla\omega = 0$, these coordinates can simultaneously be chosen to be Darboux with respect to ω; we then call them special real coordinates.

Expressing the symmetry of ∇J equivalently as $\mathrm{d}^\nabla J = 0$, where we view J as a TM-valued one-form, this condition translates to the fact that $\mathrm{d}x_i(J\cdot)$ is a closed one-form for an affine coordinate function x_i. Given a special real coordinate system $\{x_i, y_i\}$ (such that $\omega = \sum_i \mathrm{d}x_i \wedge \mathrm{d}y_i$), the complex one-forms $\mathrm{d}x_i - i\mathrm{d}x_i(J\cdot)$ are now closed and of type $(1,0)$, hence holomorphic. This means that we can write $x_i = \operatorname{Re} z_i$ for a holomorphic function z_i. Similarly, $y_i = \operatorname{Re} w_i$ where w_i is holomorphic. One can show that we may choose the special real coordinates such that the above functions $\{z_i\}$ and $\{w_i\}$ each determine a system of holomorphic coordinates for M as a complex manifold. □

The next step is to show that such coordinates, and in fact all the ingredients of an affine special Kähler structure, are induced on certain immersed submanifolds of $T^*\mathbb{C}^n$. Equip $V = T^*\mathbb{C}^n$ with it standard complex symplectic form $\Omega = \sum_i \mathrm{d}z_i \wedge \mathrm{d}w_i$ and real structure τ. Then there is a natural induced Hermitian form $\gamma = \Omega(\cdot, \tau\cdot)$ of split (complex) signature.

Definition 4.17. Let (M, J) be a connected complex manifold of dimension n. A holomorphic immersion $\phi : M \to V$ is called non-degenerate if $\phi^*\gamma$ is non-degenerate, and Lagrangian if $\phi^*\Omega = 0$. It is called totally complex if $\phi_*(T_pM) \cap V^\tau = \varnothing$ for every p, where $V^\tau = T^*\mathbb{R}^n$ is the set of τ-invariant points.

One may prove that ϕ is totally complex if and only if its real part $\operatorname{Re}\phi : M \to T^*\mathbb{R}^n$ is an immersion. Under this assumption, the restrictions of the functions $x_i := \operatorname{Re} z_i$ and $y_i := \operatorname{Re} w_i$ define local coordinates on M, and by declaring them to be affine, we obtain a flat connection. They are automatically Darboux with respect to $\omega = \sum_i \mathrm{d}x_i \wedge \mathrm{d}y_i$, so we have obtained local special real coordinates which, by construction, extend to two sets of local holomorphic coordinates on M. The condition $\mathrm{d}^\nabla J = 0$ is also automatically satisfied.

Further assuming that ϕ is non-degenerate, $\operatorname{Re}\phi^*\gamma$ defines a possibly indefinite Kähler metric on M. Finally, assuming ϕ is Lagrangian, one may verify that this metric and the candidate Kähler form ω are indeed related through the given complex structure J; this means that we have an affine special Kähler structure. In summary, we have:

Proposition 4.18 (Alekseevsky–Cortés–Devchand, [ACD02]). *Any holomorphic immersion of a complex manifold (M, J) into $T^*\mathbb{C}^n$ which is totally complex, non-degenerate and Lagrangian induces an affine special Kähler structure on M.* □

In fact, assuming that ϕ is Lagrangian, one can check that it is non-degenerate if and only if it is totally complex, so we need only assume ϕ is non-degenerate and Lagrangian.

The final step consists of showing that every special Kähler structure arises in this fashion, at least locally.

Theorem 4.19 (Alekseevsky–Cortés–Devchand, [ACD02]). *Let (M, g, J, ∇) be a simply connected affine special Kähler manifold. Then there exists a holomorphic, non-degenerate and Lagrangian immersion $\phi : M \to T^*\mathbb{C}^n$ such that the special Kähler structure on M is induced in the above fashion.* □

Dropping the assumption that $\pi_1(M) = 0$, this yields a local characterization of arbitrary affine special Kähler manifolds. Let us study the local models in more depth; let (M, J) be an open subset of $\mathbb{C}^n$ so that ϕ is given by a holomorphic one-form $\phi(z) = \sum_i F_i(z)\mathrm{d}z_i$. It is a well-known fact from symplectic geometry that the image of a one-form is Lagrangian if and only if the one-form is closed. The condition that ϕ is non-degenerate also admits a simple reformulation in this case.

Lemma 4.20. *A holomorphic one-form on an open subset $U \subset \mathbb{C}^n$ defines a non-degenerate holomorphic immersion if and only if the matrix $\operatorname{Im} \frac{\partial F_i}{\partial z_j}$ is invertible.*

Proof. We have to check that the induced metric, or equivalently the Kähler form, is non-degenerate. Set $z_j = x_j + iu_j$ and $w_j = y_j + iv_j$. Then the Kähler form is given by

$$\begin{aligned}\phi^*\big(\sum \mathrm{d}x_i \wedge \mathrm{d}y_i\big) &= \sum \mathrm{d}x_i \wedge \mathrm{d}(\operatorname{Re} F_i)\\ &= \sum \operatorname{Re} \frac{\partial F_i}{\partial z_j}\mathrm{d}x_i \wedge \mathrm{d}x_j - \operatorname{Im} \frac{\partial F_i}{\partial z_j}\mathrm{d}x_i \wedge \mathrm{d}u_j\end{aligned}$$

This shows that it is non-degenerate if and only if the matrix $\operatorname{Im} \frac{\partial F_i}{\partial z_j}$ is. □

Such one-forms are called regular and so we may rephrase the above as follows:

Corollary 4.21. *A holomorphic one-form $\alpha = \sum_i F_i \mathrm{d}z_i$ on an open subset $U \subset \mathbb{C}^n$ defines an affine special Kähler structure on U if and only if α is regular and closed.* □

Since regularity is a generic condition, this means that closed, holomorphic one-forms on open subsets in $\mathbb{C}^n$ generically give rise to affine special Kähler manifolds. In this way, one obtains a plethora of examples.

Moreover, every affine special Kähler manifold locally arises in this way. In fact, by restricting to simply connected domains, we may assume that the defining holomorphic one-form is exact, i.e. $\alpha = \mathrm{d}F$ for a holomorphic function F. The regularity condition is the non-degeneracy of $\operatorname{Im} \frac{\partial^2 F}{\partial z_i \partial z_j}$, and we conclude that every affine special Kähler manifold of (complex) dimension n is locally determined by a non-degenerate holomorphic function in n variables. Such a function is called a holomorphic prepotential.

Definition 4.22. An affine special Kähler manifold (M, g, J, ∇) is called an affine special Kähler domain if its special Kähler structure is globally induced by a holomorphic prepotential F in the manner described above.

Our discussion can be summarized by the statement that every affine special Kähler manifold is locally an affine special Kähler domain.

Let us now specialize to CASK manifolds, where the following improvements can be made. The special holomorphic coordinate sets $\{z_i\}$, $\{w_i\}$, whose real parts define affine Darboux coordinates on (M, g, J, ∇, ξ), may now be chosen in such a way that the $\mathbb{C}^*$-action generated by $\{\xi, J\xi\}$ acts on them by complex multiplication (within their domain of definition). Such coordinates and indeed the entire CASK structure can once again be induced by an appropriate holomorphic immersion into $T^*\mathbb{C}^n$, which is now required to be conical, meaning that for every $p \in M$ and every neighborhood $U \ni p$, there exist neighborhoods $U_1 \subset \mathbb{C}^*$ of $1 \in \mathbb{C}^*$ and $U_p \subset M$ of p such that for every $\lambda \in U_1$, $\lambda \cdot \phi(U_p) \subset \phi(U)$. Heuristically, this means that the image of ϕ is locally $\mathbb{C}^*$-invariant. As before, every CASK manifold locally arises in this fashion:

Theorem 4.23 (Alekseevsky–Cortés–Devchand, [ACD02]). *Let (M, g, J, ∇, ξ) be a simply connected CASK manifold. Then there exists a holomorphic, non-degenerate and Lagrangian conical immersion $\phi : M \to T^*\mathbb{C}^n$ such that the special Kähler structure on M is induced in the above fashion.* □

In the model case, where ϕ is given by a local holomorphic one-form $\alpha = \sum_i F_i \mathrm{d}z_i$, the conical condition means that the coefficient functions F_i are locally homogeneous of degree one: $F_i(\lambda z) = \lambda F_i(z)$ for λ close to $1 \in \mathbb{C}^*$. When α is exact, this means that the holomorphic primitive function F must satisfy $F(\lambda z) = \lambda^2 F(z)$, i.e. is locally homogeneous of degree two. We call such functions conical, and define a CASK domain as a $\mathbb{C}^*$-invariant domain in $\mathbb{C}^n$ endowed with a CASK structure which is globally induced by a conical holomorphic prepotential. Then we have:

Corollary 4.24. *Every CASK manifold is locally a CASK domain.* □

The local structure theorem for CASK manifolds is reflected in the local structure of PSK manifolds. Indeed, the conical holomorphic immersion defining the CASK structure induces a holomorphic immersion $\varphi : \bar{M} \to \mathbb{P}(T^*\mathbb{C}^n)$ and so we see that PSK manifolds arise locally as open subsets of projective spaces. For technical convenience, we will from now on assume that all CASK domains are contained in the open subset $\{(z_0, \ldots, z_n) \mid z_0 \neq 0\} \subset \mathbb{C}^{n+1}$, so that we can consider PSK domains as open subsets of complex vector spaces by applying the standard chart that identifies $\{[z_0 : \cdots : z_n] \in \mathbb{CP}^n \mid z_0 \neq 0\}$ with an (affine) copy of $\mathbb{C}^n$. For future reference, we note that the function $\mathcal{K} = -\log\left(\frac{g(\xi,\xi)}{|z_0|^2}\right)$ provides a canonical choice of Kähler potential for the projective special Kähler structure under these assumptions. This naturally generalizes the potential given for the Bergman metric in example 4.14, which we now see exhibits $\mathbb{C}\mathrm{H}^n$ as a PSK domain.

This flexible construction leads to various complete examples. For instance, examples of homogeneous (hence complete) projective special Kähler manifolds were constructed in [AC00; Cor96a]. Many additional complete examples arise from the so-called r-map construction; see [CHM12; CDL14; CDJL17].

Special Kähler manifolds arise naturally in several other interesting contexts. In particular, there are interesting examples of moduli spaces that carry special Kähler structures, the most famous being the so-called Kuranishi moduli space of complex structures on a Calabi–Yau three-fold (see e.g. [Cor98]). In fact, this class of examples motivated much of the early work by physicists on special Kähler geometry (see e.g. [Str90]). Another prominent example was discovered by Hitchin [Hit99], who constructed a special Kähler structure on the moduli space of deformations of a complex Lagrangian submanifold of a complex symplectic manifold.

4.4 The c-map

We now have all the necessary ingredients to introduce the protagonist of our story, the c-map construction. There are two equivalent approaches to the c-map, which nevertheless look rather different at first glance. We call them the direct approach and the twist approach. Both methods have their advantages and drawbacks, and they are complementary in the sense that certain features of the c-map are obvious from the one point of view but highly non-trivial from the other. We will first introduce the twist approach, then discuss the direct construction and compare the two methods before introducing the one-loop deformation, which shows that the c-map naturally fits into a one-parameter family of constructions.

4.4.1 The twist approach

An important feature of special Kähler geometry, which we have neglected thus far, is that the cotangent bundle of an affine special Kähler manifold carries a canonical pseudo-hyper-Kähler structure. This structure naturally generalizes the flat pseudo-hyper-Kähler structure on quaternionic vector spaces given, in the case of positive-definite signature, by equation (2.3). More precisely, the construction is as follows. Consider a Kähler manifold (M, g_M, J_M) equipped with a connection ∇. On the cotangent bundle $\pi : T^*M \to M$, ∇ determines a horizontal distribution and hence a decomposition $T(T^*M) \cong \pi^*TM \oplus \pi^*T^*M$. With respect to this decomposition, we can define a metric and two almost complex structures as follows:

$$g = \begin{pmatrix} g_M & 0 \\ 0 & g_M^* \end{pmatrix} \qquad I_1 = \begin{pmatrix} J_M & 0 \\ 0 & J_M^* \end{pmatrix} \qquad I_2 = \begin{pmatrix} 0 & -\omega_M^{-1} \\ \omega_M & 0 \end{pmatrix} \tag{4.1}$$

where g_M^* denotes the induces metric on T^*M and similarly for J_M^*, and ω is regarded as an isomorphism $\omega : TM \to T^*M$. We have suppressed the pullback by π, which is implicitly applied to all instances of g_M, J_M and ω_M. It is easily checked that, in the case $M = \mathbb{C}^{p,q}$, equipped with its flat Kähler structure and $\nabla = \nabla^{\mathrm{LC}}$, this reproduces the standard hyper-Kähler structure on $T^*\mathbb{C}^{p,q} = \mathbb{H}^{p,q}$. It was noted early on by physicists that this construction also works in the more general case of an affine special Kähler manifold:

Theorem 4.25 (Cecotti–Ferrara–Girardello, [CFG89]). *If (M, g, J, ∇) is an affine special Kähler manifold, then the metric g and almost complex structures I_1, I_2 define a pseudo-hyper-Kähler structure on T^*M.* □

For a clear exposition, we recommend [Fre99] or [MS15], where it is explained that this result can be strengthened to an "if and only if"-statement. We will say that (T^*M, g, I_k) arises from the rigid c-map construction applied to (M, g, J, ∇). The name rigid c-map reflects the physical origin of this construction as an analog of the c-map for physical theories involving so-called rigid supersymmetry rather than supergravity.

If we started with a CASK manifold, we additionally have the vector fields $\{\xi, J\xi\}$ which generate the distinguished $\mathbb{C}^*$-action. As first proven in [ACM13], this leads to a rotating Killing field on T^*M.

Proposition 4.26. *Let $(M, g_M, J_M, \nabla, \xi)$ be a CASK manifold. Then the ∇-horizontal lift $Z = -\widetilde{J\xi} \in \mathfrak{X}(T^*M)$ is a rotating Killing field on (T^*M, g, I_k), and the function $f_Z = -\frac{1}{2}\pi^* g_M(\xi, \xi) = -\frac{1}{2} g(Z, Z)$ is a Hamiltonian for Z with respect to ω_1.*

Proof. This is most easily proven by using local coordinates $\{q^i\}$ adapted to the CASK structure on M. Following our discussion in section 4.3, we may take these coordinates to be ∇-affine and Darboux with respect to ω_M, as well as conical so that $\xi = \sum_j q^j \frac{\partial}{\partial q^j}$. If $\{q^j, p_j\}$ are the corresponding canonical coordinates on T^*M, we have

$$g = \sum_{j,k} g_{jk} \mathrm{d}q^j \mathrm{d}q^k + g^{jk} \mathrm{d}p_j \mathrm{d}p_k$$

where (g^{jk}) is the inverse matrix to (g_{jk}), which represents g_M with respect to these coordinates. Similarly $Z = -\sum_{j,k} J^j_k q^k \frac{\partial}{\partial q^j}$ and since $J_M \xi$ preserves g_M, we then have $L_Z g = \sum_{j,k} Z(g^{jk}) \mathrm{d}p_j \mathrm{d}p_k$; it now suffices to show that $Z(g_{jk}) = 0$.

To show this, we will use the defining features of the special Kähler structure, expressed with respect to these coordinates. Since the coordinates $\{q^j\}$ are ∇-affine, the coordinate vector fields $\{\frac{\partial}{\partial q^j}\}$ are parallel and we have $\left(\nabla_{\frac{\partial}{\partial q^i}} g_M\right)\left(\frac{\partial}{\partial q^j}, \frac{\partial}{\partial q^k}\right) = \frac{\partial g_{jk}}{\partial q^i}$. By lemma 4.3, this expression is symmetric in its indices. Moreover, with respect to these coordinates the equations $\sum_l \frac{\partial}{\partial q^i}(J^l_j g_{lk}) = 0$ and $\frac{\partial J^i_j}{\partial q^k} = \frac{\partial J^i_k}{\partial q^j}$ are equivalent to $\nabla\omega_M = 0$ and symmetry of ∇J_M. We then find

$$\begin{aligned}
-Z(g_{jk}) &= \sum_l J^i_l q^l \frac{\partial g_{jk}}{\partial q^i} = \sum_l q^l J^i_l \frac{\partial g_{ij}}{\partial q^k} = -\sum_l q^l g_{ij} \frac{\partial J^i_l}{\partial q^k} = -\sum_l q^l g_{ij} \frac{\partial J^i_k}{\partial q^l} \\
&= \sum_l J^i_k q^l \frac{\partial g_{ij}}{\partial q^l}
\end{aligned}$$

We recognize that $q^l \frac{\partial g_{ij}}{\partial q^l} = \xi(g_{ij})$ and since

$$L_\xi g_M = \sum \xi(g_{ij}) \mathrm{d}q^i \mathrm{d}q^j + 2g_{ij}\mathrm{d}(\xi(q^i))\mathrm{d}q^j = \sum \xi(g_{ij})\mathrm{d}q^i\mathrm{d}q^j + 2g_{ij}\mathrm{d}q^i\mathrm{d}q^j = 2g_M$$

where the final equality follows from lemma 4.5, we find $\xi(g_{ij}) = 0$, which proves that Z is a Killing field.

Next, we verify that Z is ω_1-Hamiltonian with the prescribed Hamiltonian function. This follows immediately from the fact that $Z = -\widetilde{J_M\xi}$ and $\omega_1|_{\pi^*TM} = \pi^*\omega_M$, once we recall that $-\frac{1}{2}g_M(\xi,\xi)$ is a Hamiltonian for $-J_M\xi$.

Finally, we check that $L_Z\omega_2 = \omega_3$. From (4.1), we see that

$$\omega_2 = \begin{pmatrix} 0 & -J_M^* \\ J_M & 0 \end{pmatrix} \qquad\qquad \omega_3 = \begin{pmatrix} 0 & -\mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}$$

and therefore $\omega_2 = \sum_{j,k} J^k_j \mathrm{d}q^j \wedge \mathrm{d}p_k$ and $\omega_3 = \sum_j \mathrm{d}q^j \wedge \mathrm{d}p_j$. Using the same computational tricks as below, we obtain

$$\begin{aligned}
L_Z\omega_2 &= \sum Z(J^k_j)\mathrm{d}q^j \wedge \mathrm{d}p_k + J^k_j \mathrm{d}(Z(q^j)) \wedge \mathrm{d}p_k \\
&= -\sum J^l_i q^i \frac{\partial J^k_j}{\partial q^l}\mathrm{d}q^j \wedge \mathrm{d}p_k + J^k_j \mathrm{d}(J^j_i q^i) \wedge \mathrm{d}p_k \\
&= -\sum J^l_i q^i \frac{\partial J^k_l}{\partial q^j}\mathrm{d}q^j \wedge \mathrm{d}p_k + J^k_j \left(\frac{\partial J^j_i}{\partial q^l} q^i \mathrm{d}q^l + J^j_i \mathrm{d}q^i\right) \wedge \mathrm{d}p_k \\
&= -\sum -q^i J^k_l \frac{\partial J^l_i}{\partial q^j}\mathrm{d}q^j \wedge \mathrm{d}p_k + \left(q^i J^k_l \frac{\partial J^l_i}{\partial q^j}\mathrm{d}q^j - \mathrm{d}q^k\right) \wedge \mathrm{d}p_k \\
&= \omega_3
\end{aligned}$$

as claimed. The fact that $L_Z\omega_3 = -\omega_2$ or equivalently $L_Z I_3 = -I_2$ now follows automatically: $L_Z I_3 = L_Z(I_1 I_2) = I_1 L_Z I_2 = -I_2$. □

Corollary 4.27. *The cotangent bundle of a CASK manifold of complex dimension n carries a canonical pseudo-hyper-Kähler structure of quaternionic signature $(n-1, 1)$,* *given by the rigid c-map construction. It is moreover endowed with a canonical rotating Killing field Z.* □

Given a CASK manifold M, we have now constructed a hyper-Kähler manifold $N = T^*M$ which satisfies the hypotheses of theorem 3.21 and we can therefore apply the HK/QK correspondence to it and obtain a (pseudo-)quaternionic Kähler manifold $(\bar{N}, g_{\mathrm{Q}})$. Since $g_M(\xi, \xi) < 0$ by definition of the CASK structure, we see that $f_Z > 0$ and $f_{\mathrm{H}} = f_Z + g(Z, Z) = \frac{1}{2}\pi^* g_M(\xi, \xi) < 0$. We conclude that $(\bar{N}, g_{\mathrm{Q}})$ has positive-definite signature and negative scalar curvature. Since the CASK manifold is equivalent to the corresponding PSK manifold, we can think of this as a construction of quaternionic Kähler manifolds with negative scalar curvature from PSK manifolds. This is the (supergravity) c-map construction. Briefly put, we have:

Theorem 4.28. *The c-map canonically associates a quaternionic Kähler manifold with negative scalar curvature to every PSK manifold.* □

The following diagram conveniently summarizes our discussion:

$$\begin{array}{ccc} M & \xrightarrow{\text{rigid } c\text{-map}} & N \\ {\scriptstyle \mathbb{C}^*}\big\downarrow & & \big\downarrow{\scriptstyle \text{HK/QK}} \\ \bar{M} & \xrightarrow[c\text{-map}]{} & \bar{N} \end{array}$$

In the above discussion, it might not seem natural to think of the PSK manifold, rather than the corresponding CASK manifold, as the starting point of the c-map construction. This emphasis is justified by the direct approach to the c-map, where the PSK manifold plays a central role.

4.4.2 The direct approach

The direct approach is best explained by first restricting to the case of a PSK domain. Thus, let $\bar{M} \subset \mathbb{C}\mathrm{P}^n$ be a PSK domain, with Kähler metric $\bar{g}$. The PSK structure is induced by the CASK structure on the corresponding CASK domain M, which is determined by the holomorphic prepotential F. Recall that F is homogeneous of degree two and that the real matrix defined by $N_{ij} := 2\,\mathrm{Im}\,\frac{\partial^2 F}{\partial z^i \partial z^j}$ is invertible, where $\{z_j\}$ is a set of holomorphic coordinates adapted to the CASK structure. We define two further real matrices $(\mathcal{R}_{ij})$ and $(\mathcal{I}_{ij})$ by setting

$$\mathcal{R}_{jk} + i\mathcal{I}_{jk} = \frac{\partial^2 \bar{F}}{\partial \bar{z}^j \partial \bar{z}^k} + i\frac{\sum_{m,n} N_{jm} z^m N_{kn} z^n}{\sum_{m,n} N_{mn} z^m z^n}$$

It is known (see e.g. [CHM12]) that the matrix $\mathcal{I}$ is positive-definite and in particular invertible. Note that the entries of these matrices are homogeneous of degree zero, hence induce well-defined expressions on $\bar{M}$.

Now consider the product manifold $\bar{N} = \bar{M} \times \mathbb{R}_{>0} \times S^1 \times \mathbb{R}^{2n+2}$ and let $\{\rho, \tilde{\phi}, \tilde{\zeta}_i, \zeta^i\}$ be standard coordinates on $\mathbb{R}_{>0} \times S^1 \times \mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$ (where $\tilde{\phi}$ is periodic). With respect to these coordinates consider the Riemannian metric $g_{\bar{N}}$ given by

$$\begin{aligned} g_{\bar{N}} &= \bar{g} + g_G \\ g_G &= \frac{1}{4\rho^2}\mathrm{d}\rho^2 + \frac{1}{4\rho^2}\Big(\mathrm{d}\tilde{\phi} + \sum_j \big(\zeta^j \mathrm{d}\tilde{\zeta}_j - \tilde{\zeta}_j \mathrm{d}\zeta^j\big)\Big)^2 + \frac{1}{2\rho}\sum_{j,k} \mathcal{I}_{jk}\mathrm{d}\zeta^j \mathrm{d}\zeta^k \\ &\quad + \frac{1}{2\rho}\sum_{j,k} (\mathcal{I}^{-1})_{jk}\big(\mathrm{d}\tilde{\zeta}_j + \mathcal{R}_{jm}\mathrm{d}\zeta^m\big)\big(\mathrm{d}\tilde{\zeta}_k + \mathcal{R}_{kn}\mathrm{d}\zeta^n\big) \end{aligned} \tag{4.2}$$

Note that, since the matrices $\mathcal{I}$ and $\mathcal{R}$ depend on the point in $\bar{M}$, this is not a direct product metric, though the projection to $\bar{M}$ is a Riemannian submersion. The above expressions superficially depend on the chosen set of adapted coordinates $\{z_j\}$ on $\bar{M}$. However, the extrinsic point of view on special Kähler geometry, which dictates that all such coordinate systems are interrelated via real symplectic transformations [ACD02], can be used to show that $g_{\bar{N}}$, which we call the undeformed c-map metric, is actually independent of this choice [CHM12, Thm. 9]. It is sometimes called the Ferrara–Sabharwal metric after the physicists who first explicitly described it [FS90].

If we drop the assumption that $\bar{M}$ is a PSK domain, we may nevertheless use a covering $\{U_\alpha\}$ of $\bar{M}$ such that each U_α is a PSK domain. Thus, we may carry the above construction out on such patches, and ask if the resulting manifolds $\bar{N}_\alpha$ can be patched together in a manner consistent with the local coordinate expressions for the Riemannian metric on every patch. If two patches have non-trivial overlap, we have two possibly distinct structures of a PSK domain on $U_\alpha \cap U_\beta$. Now, the fact that the metric patches to a globally well-defined metric boils down to the independence of the metric of the adapted coordinate set used to define it on $U_\alpha \cap U_\beta$, which we already established. Therefore, the above construction can be applied to arbitrary PSK manifolds, the only difference being that $\bar{N}$ is, in general, non-trivial as a bundle over $\bar{M}$. This completes our description of the direct approach to the c-map.

It is far from obvious that $(\bar{N}, g_{\bar{N}})$ is a quaternionic Kähler manifold; this fact was first proven by Hitchin [Hit09]. However, the direct c-map construction offers some distinct advantages, which can be traced back to the fact that $\bar{N}$ is constructed as a bundle $\pi : \bar{N} \to \bar{M}$. It is worthwhile to study the bundle structure in some more detail. The following fact was already well-known to physicists since the discovery of the c-map construction; the proof we give is essentially due to Cortés, Han and Mohaupt [CHM12].

Proposition 4.29. *Let $\bar{M}$ be a PSK manifold of complex dimension n. Then, for each $x \in \bar{M}$, the fiber $\bar{N}_x = \pi^{-1}(x)$ carries a nearly effective and isometric action of the*

$(2n+3)$-dimensional Heisenberg group Heis_{2n+3}*. Moreover, it is a locally homogeneous space.*

Proof. We will prove that the universal Riemannian covering $\widetilde{N_x} \cong \mathbb{R}_{>0} \times \mathbb{R}^{2n+3}$ of each fiber is a Lie group equipped with a left-invariant metric, containing Heis_{2n+3} as a subgroup. More precisely, $\widetilde{N_x}$ is a one-dimensional (solvable) extension of Heis_{2n+3} and the action of Heis_{2n+3} descends to $\bar{N}_x$.

First, note that $\widetilde{N_x}$ is obtained simply by regarding $\tilde{\phi}$ as a real coordinate, rather than periodic. Now consider the following group law on $\mathbb{R}_{>0} \times \mathbb{R} \times \mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$:

$$
\begin{aligned}
&(e^{\lambda}, \tilde{\phi}, \tilde{\zeta}_i, \zeta^i)\cdot(e^{\lambda'}, \tilde{\phi}', \tilde{\zeta}'_i, \zeta'^i) \\
&\qquad = \left(e^{\lambda+\lambda'}, \tilde{\phi} + e^{\lambda}\tilde{\phi}' + e^{\lambda/2}\left(\sum_i \tilde{\zeta}_i \zeta'^i - \zeta^i \tilde{\zeta}'_i\right), \tilde{\zeta}_i + e^{\lambda/2}\tilde{\zeta}'_i, \zeta^i + e^{\lambda/2}\zeta'^i\right)
\end{aligned}
\tag{4.3}
$$

This casts $\mathbb{R}_{>0} \times \mathbb{R}^{2n+3} \cong \widetilde{N_x}$ as the Iwasawa subgroup of $\mathrm{SU}(n+2,1)$, which we will denote by $G(n+2)$. The $(2n+3)$-dimensional Heisenberg group is parametrized by the coordinates $(\tilde{\phi}, \tilde{\zeta}_i, \zeta^i) \in \mathbb{R}^{2n+3}$. It is not hard to check explicitly that the metric g_G given in (4.2) is left-invariant with respect to the action of $G(n+2)$.

The fiber $\bar{N}_x$ is obtained by dividing out a cyclic subgroup of (integer) translations in $\tilde{\phi}$. This is a central subgroup of Heis_{2n+3}, so its action descends to a nearly effective, isometric action on $\bar{N}_x$. Passing to the corresponding discrete quotient $\mathrm{Heis}_{2n+3}/\mathbb{Z}$, the action is free. Note that the action of $G(n+2)$ does not descend, as the cyclic subgroup is not normal as a subgroup of $G(n+2)$. □

As the proof shows, the cyclic covering $\tilde{N}$, whose fibers are the universal coverings of the fibers of $\bar{N}$, is naturally a bundle of Lie groups equipped over $\bar{M}$ with a left-invariant metric on each fiber. This can be leveraged to effectively control the completeness of the c-map metric.

Theorem 4.30 (Cortés–Han–Mohaupt, [CHM12])**.** *Let $(\bar{M}, \bar{g})$ be a PSK manifold and $(\bar{N}, g_{\bar{N}})$ its image under the c-map. Then $(\bar{N}, g_{\bar{N}})$ is complete if and only if $(\bar{M}, g_{\bar{M}})$ is.*

Sketch of Proof. To study questions of completeness, we may work with $(\tilde{N}, g_{\tilde{N}})$ instead of $\bar{N}$, since the two manifolds have the same universal covering, and a Riemannian manifold is complete if and only if its universal covering is. We may moreover restrict to the case where $\tilde{N}$ is a trivial bundle: $\tilde{N} = \bar{M} \times \mathbb{R}_{>0} \times \mathbb{R}^{2n+3} \cong \bar{M} \times G(n+2)$.

For general product manifolds $M_1 \times M_2$ equipped with a metric $g = g_1 + g_2$, where g_1 is (the pullback of) a metric on M_1 and g_2 is a family of Riemannian metrics parametrized by M_1, one may prove the following completeness criterion. Assume that (M_1, g_1) is complete, and that for every compact subset $K \subset M_1$, there exists a complete metric g_K on M_2 such that $g_2(x) \geq \pi^* g_K$ for every $x \in K$. Then $(M_1 \times M_2, g)$ is complete.

To apply this criterion in our setup, we note (cf. the proof of the previous proposition) that, in our case, the family of metrics $g_2 = g_G$ is a family of left-invariant metrics on the Lie group $M_2 = G(n+2)$. Such metrics are in bijective correspondence with scalar products on $\mathrm{Lie}(G(n{+}2))$. Thus, let $K \subset M_1 = \bar{M}$ be a compact subset and consider the family $g_G(x)$, $x \in K$. The corresponding family of scalar products on the Lie algebra of $G(n+2)$ is uniformly bounded from below by some other scalar product, corresponding to a left-invariant and hence complete metric g_K. Left-invariance of the metrics $g_G(x)$ implies that $g_G(x) \geq \pi^* g_K$ and we may now apply the criterion to deduce that $(\tilde{N}, g_{\tilde{N}})$ is complete if $\bar{M}$ is. The converse is obvious. □

Thus, the direct approach to the c-map elucidates the role of the PSK manifold $\bar{M}$, uncovers the existence of a large group of isometries of the c-map metric and can be used to study global properties such as completeness of the c-map metric.

4.4.3 The one-loop deformation

As noted in section 3.3, there is an implicit one-parameter ambiguity in the HK/QK correspondence. Given a hyper-Kähler manifold (N, g, I_k) equipped with ω_1-Hamiltonian rotating Killing field Z, every choice of Hamiltonian f_Z for Z leads to a quaternionic Kähler manifold, and there is no reason to believe that these manifolds are pairwise isometric. Indeed, the fact that f_Z features explicitly in the expression for the elementary deformations g_{H} suggests otherwise.

In case the data (N, g, I_k, Z) arise from the rigid c-map construction applied to a CASK manifold M, we have a canonical choice of Hamiltonian, namely $-\frac{1}{2}g(Z,Z)$. Therefore, we call the resulting quaternionic Kähler metric the undeformed c-map metric and think of the family of quaternionic Kähler metrics arising from different choices $f_Z = -\frac{1}{2}g(Z,Z) - \frac{1}{2}c$, $c \in \mathbb{R}$, as deformations of this metric. They are often called one-loop deformed c-map metrics, owing to their physical interpretation as one-loop quantum corrections to the undeformed c-map metric [RSV06], but we will usually refer to them simply as deformed c-map metrics.

An explicit (local) expression for the one-parameter family of deformed c-map metrics g^c, $c \in \mathbb{R}$, can be given via the direct construction of the c-map. Assume $\bar{M}$ is a PSK domain, with PSK structure induced by the holomorphic prepotential F and adapted holomorphic coordinates $\{z_j\}$ on the corresponding CASK manifold. Let $X_j = \frac{z_j}{z_0}$ (recall that, by assumption, $\bar{M} \subset \mathbb{C}\mathrm{P}^n$ is contained in the open subset $\{z_0 \neq 0\}$) be coordinates on $\bar{M}$ and set $X_0 = 1$ (this will simplify the coming expressions). Moreover, let $\mathcal{K}$ be the canonical Kähler potential inducing the PSK metric $\bar{g}$. Then, in the notation employed in (4.2), the deformed c-map metric on $\bar{N} = \bar{M} \times \mathbb{R}_{>0} \times S^1 \times \mathbb{R}^{2n+2}$ is

$$g^c_{\bar{N}} = \frac{\rho + c}{\rho}\bar{g} + g^c_G$$

where

$$g_G^c = \frac{1}{4\rho^2}\frac{\rho+2c}{\rho+c}\mathrm{d}\rho^2 + \frac{1}{4\rho^2}\frac{\rho+c}{\rho+2c}\Big(\mathrm{d}\tilde{\phi} + \sum_j \big(\zeta^j \mathrm{d}\tilde{\zeta}_j - \tilde{\zeta}_j \mathrm{d}\zeta^j\big) + c\,\mathrm{d}^c\mathcal{K}\Big)^2$$
$$+ \frac{1}{2\rho}\sum_{j,k}\mathcal{I}_{jk}\mathrm{d}\zeta^j\mathrm{d}\zeta^k + \frac{1}{2\rho}\sum_{j,k}(\mathcal{I}^{-1})_{jk}\big(\mathrm{d}\tilde{\zeta}_j + \mathcal{R}_{jm}\mathrm{d}\zeta^m\big)\big(\mathrm{d}\tilde{\zeta}_k + \mathcal{R}_{kn}\mathrm{d}\zeta^n\big) \quad (4.4)$$
$$+ \frac{2c}{\rho^2}e^{\mathcal{K}}\Big|\sum_{j=0}^n X^j\mathrm{d}\tilde{\zeta}_j + F_j(X)\mathrm{d}\zeta^i\Big|^2$$

where the value of F_j at $X = [z_0 : \cdots : z_n] \in \bar{M}$ is determined by evaluating $\frac{\partial F}{\partial z^j}$ at $X = (1, z_1/z_0, \ldots, z_n/z_0)$. Note that this yields a well-defined function on $\bar{M}$ (i.e. is invariant under the choice of representative for X), since $\frac{\partial F}{\partial z^j}$ is homogeneous of degree one.

A first observation is that, for $c = 0$, we indeed recover the undeformed c-map metric. For non-zero c, however, the metric is significantly more complicated than the undeformed c-map metric. By inspecting the expression (4.4), it is not hard to see that the metric $g_{\bar{N}}^c$ is positive-definite if $c \geq 0$, but not necessarily so if $c < 0$, in which case the metric even degenerates for certain values of ρ. Even when restricting to a subset where the metric is non-degenerate and of constant signature, it is known that the resulting pseudo-Riemannian manifold is in general incomplete (see [ACDM15, App. A]). Therefore, we will restrict to $c \geq 0$ from now on.

Proposition 4.31 (Cortés–Dyckmanns–Suhr, [CDS17]). *Let $(\bar{M}, \bar{g})$ be a PSK domain. Then, for any $c_1, c_2 > 0$, the deformed c-map metrics $g_{\bar{N}}^{c_1}$, $g_{\bar{N}}^{c_2}$ on $\bar{N}$ are locally isometric to each other.*

Proof. We pass to the cyclic covering $\tilde{N}$, obtained by regarding $\tilde{\phi}$ as a real (rather than periodic) coordinate. Now, we claim that the lifted metrics $g_{\tilde{N}}^{c_1}$, $g_{\tilde{N}}^{c_2}$ are isometric. Indeed, recall the action of $G(n+2)$ on the fibers of $\tilde{N}$ from (4.3). In particular, $(e^\lambda, 0, 0, 0) \in G(n+2)$ induces a diffeomorphism $\varphi_\lambda : \tilde{N} \to \tilde{N}$ which is easily checked to satisfy $\varphi_\lambda^* g_{\tilde{N}}^c = g_{\tilde{N}}^{e^{-\lambda}c}$. Since $c_2/c_1 > 0$ we may choose λ such that $e^{-\lambda} = c_2/c_1$, proving our claim. This implies that $(\bar{N}, g_{\bar{N}}^{c_1})$ and $(\bar{N}, g_{\bar{N}}^{c_2})$ are locally isometric. ☐

Thus, the local geometry depends only on the sign of c.

In order to define the deformed c-map for arbitrary PSK manifolds, one needs to patch together local expressions, defined relative to a given PSK domain structure on a patch $U_\alpha \subset \bar{M}$, to a global Riemannian metric, analogous to the case of the undeformed c-map metric. This matter was studied in [CDS17, Thm. 12], where it was found that the local metrics can be consistently patched together precisely if the circle factor, parametrized by $\tilde{\phi}$, has length $2\pi c$, i.e. $\tilde{\phi} + 2\pi c = \tilde{\phi}$.[7] Then, we have a well-defined one-parameter family of Riemannian manifolds $(\bar{N}, g^c_{\bar{N}})$, $c \geq 0$. Once again, it is not at all obvious that these metrics are quaternionic Kähler. In fact, the only complete proof of this fact proceeds by showing that they coincide with the metrics obtained via the twist approach:

Theorem 4.32 (Alekseevsky et al., [ACDM15]). *Let $\bar{M}$ be an arbitrary PSK manifold and M the associated CASK manifold. Then, for every $c \geq 0$, the Riemannian manifold $(\bar{N}, g^c_{\bar{N}})$ is locally isometric to the quaternionic Kähler manifold obtained by applying the HK/QK correspondence to T^*M, endowed with its canonical pseudo-hyper-Kähler structure and rotating Killing field Z (cf. corollary 4.27), and ω_1-Hamiltonian function $f_Z = -\frac{1}{2}g(Z,Z) - \frac{1}{2}c$.* □

Corollary 4.33. *Each member of the one-parameter family $(\bar{N}, g^c_{\bar{N}})$, $c \geq 0$, is a quaternionic Kähler manifold.* □

Their proof proceeds by explicitly carrying out the twist approach under the assumption that $\bar{M}$ is a PSK domain, finally reproducing the expression (4.4) for the resulting metric.

Despite the increase in complexity, the deformed c-map metrics retain some of the good properties of the undeformed c-map metric, such as the isometric action of the Heisenberg group Heis_{2n+3}.

Proposition 4.34. *For each $c \geq 0$, $(\bar{N}, g^c_{\bar{N}})$ carries a nearly effective and isometric action of Heis_{2n+3} which preserves the fibers of $\bar{N}$, regarded as a bundle over $\bar{M}$.*

Proof. The action of Heis_{2n+3} is given, as in the undeformed case, by (4.3). Using (4.4), one can easily check that it is isometric. Note, however, that there is no longer an isometric action of $G(n+2)$ on the cyclic covering $\tilde{N}$, so the fibers are not necessarily locally homogeneous. □

There is no general theorem asserting completeness for deformed c-map metrics arising from complete PSK manifolds. This can be traced back to the fact that the fibers of $\bar{N}$ are now no longer homogeneous, which was used to control the completeness in the proof of theorem 4.30. Nevertheless, partial results that cover the most important known examples have been established:

Theorem 4.35 (Cortés–Dyckmanns–Suhr, [CDS17]). *Let $\bar{M}$ be a PSK manifold with regular boundary behavior, or a PSK manifold arising from the supergravity r-map applied to a complete projective special real manifold. Then the deformed c-map space $(\bar{N}, g^c_{\bar{N}})$ is a complete quaternionic Kähler manifold for every $c \geq 0$.* □

[7] This is why we have chosen to work with periodic $\tilde{\phi}$, rather than using the cyclic cover $\tilde{N}$ which we have encountered several times. Another reason is compatibility with the picture that emerges from the twist construction, where $\tilde{\phi}$ parametrizes the "principal circle" of the S^1-bundle $P \to T^*M$.

For the precise definition of PSK manifolds with regular boundary behavior, projective special real manifolds and the supergravity r-map, we refer the interested reader to [CDS17], as well as [CDL14; CDJL17]. Let us only mention two important (classes of) examples. Firstly, all Alekseevsky spaces, that is, all known examples of homogeneous quaternionic Kähler manifolds of negative scalar curvature, with exception of the quaternionic hyperbolic spaces $\mathbb{H}\mathrm{H}^n$, arise from such PSK manifolds. In fact, they arise from homogeneous PSK manifolds in the image of the supergravity r-map.

Corollary 4.36. *All Alekseevsky spaces except the quaternionic hyperbolic spaces admit a one-parameter deformation by complete quaternionic Kähler metrics of negative scalar curvature.* □

Secondly, it was shown in [CDJL17] that there are PSK manifolds that arise from the supergravity r-map applied to a complete projective special real manifold, for which the corresponding undeformed c-map metric is not locally homogeneous.

Though we have so far worked with the direct formulation of the c-map to study the properties of the deformed c-map, this will change in chapter 5, where we rely mostly on the twist formulation in our investigation of the properties of the c-map. The reason for this is as follows. On the hyper-Kähler side of the HK/QK correspondence, the simple interpretation of the deformation parameter c as a constant added to the moment map for the rotating Killing field Z means that there is hardly any reason to prefer any specific value of c over any other. Thus, results proven about the undeformed c-map in the twist formalism typically generalize easily to the deformed case. This goes in particular for the existence and properties of interesting tensor fields on c-map spaces, which can be studied by means of the twist construction; we will soon see concrete instances of this principle.

4.5 Example: Deformations of non-compact symmetric spaces

For the sake of concreteness, and to illustrate the equivalence of the two approaches to the c-map (summarized by theorem 4.32), we will now consider an important series of examples and sketch how to derive the explicit expression—provided by the direct approach—for the quaternionic Kähler metric through the twist approach. We will return to these examples in the next chapter.

Recall (cf. example 4.14) that complex hyperbolic space $\mathbb{C}\mathrm{H}^n$ admits the structure of a PSK domain. Using standard coordinate $\{X_j\}$ for the unit ball model of $\mathbb{C}\mathrm{H}^n$, its Kähler metric is

$$\bar{g} = \frac{1}{1-\sum_j |X_j|^2}\left(\sum_j |\mathrm{d}X_j|^2 + \frac{1}{1-\sum_k |X_k|^2}\left|\sum_j X_j \mathrm{d}\bar{X}_j\right|^2\right)$$

The corresponding CASK domain M_n is the open subset of $\mathbb{C}^{n+1}$ defined by the inequality $|z_0|^2 > \sum_{j=1}^n |z_j|^2$. Its holomorphic prepotential is a quadratic polynomial. The expression for the metric on $\bar{N}_n = \mathbb{C}\mathrm{H}^n \times \mathbb{R}_{>0} \times S^1 \times \mathbb{R}^{2n+2}$, given by the direct approach to the c-map construction, takes the following form after some rewriting (cf. [CDS17]):

$$
\begin{aligned}
g_n^c = {} & \frac{\rho+c}{\rho}\bar{g} + \frac{1}{4\rho^2}\frac{\rho+2c}{\rho+c}\mathrm{d}\rho^2 - \frac{2}{\rho}\Big(\mathrm{d}w_0\mathrm{d}\bar{w}_0 - \sum_{j=1}^n \mathrm{d}w_j\mathrm{d}\bar{w}_j\Big) \\
& + \frac{1}{4\rho^2}\frac{\rho+c}{\rho+2c}\Big(\mathrm{d}\tilde{\phi} - 4\,\mathrm{Im}\Big[\bar{w}_0\mathrm{d}w_0 - \sum_{j=1}^n \bar{w}_j\mathrm{d}w_j\Big] + \frac{2c}{1-\|X\|^2}\,\mathrm{Im}\sum_{j=1}^n \bar{X}^j\mathrm{d}X^j\Big)^2 \\
& + \frac{\rho+c}{\rho^2}\frac{4}{1-\|X\|^2}\Big|\mathrm{d}w_0 + \sum_{j=1}^n X_j\mathrm{d}w_j\Big|^2
\end{aligned}
\tag{4.5}
$$

where, besides $w_0 = \frac{1}{2}(\tilde{\zeta}_0 + i\zeta^0)$ and $w_j = \frac{1}{2}(\tilde{\zeta}_j - i\zeta^j)$ for $j = 1, \ldots, n$, we use the same coordinates as in (4.4).

It is well-known (e.g. Table 2 in [dWvP92]) that, in this case, the cyclic covering $\tilde{N}_n = \mathbb{C}\mathrm{H}^n \times \mathbb{R}_{>0} \times \mathbb{R}^{2n+3}$ is the non-compact Wolf space $\frac{\mathrm{SU}(n+1,2)}{\mathrm{S}(\mathrm{U}(n+1)\times\mathrm{U}(2))}$. These symmetric spaces can be characterized as the only quaternionic Kähler manifolds of negative scalar curvature which are simultaneously Kähler. Thus, the deformed c-map metrics determine a one-parameter deformation of a locally symmetric metric through complete quaternionic Kähler metrics of negative scalar curvature. It will follow from our results that this deformation is non-trivial. To derive the deformed c-map metric explicitly via the twist approach, we first apply the rigid c-map construction to M_n, obtaining a hyper-Kähler structure on $N_n = T^*M_n$.

Since M_n is an open subset of $\mathbb{C}^{n+1}$ equipped with the flat Kähler structure induced by the standard Hermitian form h of signature $(n, 1)$, its cotangent bundle is $M_n \times \mathbb{C}^{n+1}$ and the rigid c-map endows it with the standard, flat hyper-Kähler structure of quaternionic signature $(n, 1)$. Explicitly, we have

$$
\begin{aligned}
g &= -\mathrm{d}z_0\mathrm{d}\bar{z}_0 - \mathrm{d}w_0\mathrm{d}\bar{w}_0 + \sum_{j=1}^n(\mathrm{d}z_j\mathrm{d}\bar{z}_j + \mathrm{d}w_j\mathrm{d}\bar{w}_j) \\
\omega_1 &= \frac{i}{2}\Big(- \mathrm{d}z_0\wedge\mathrm{d}\bar{z}_0 - \mathrm{d}w_0\wedge\mathrm{d}\bar{w}_0 + \sum_{j=1}^n(\mathrm{d}z_j\wedge\mathrm{d}\bar{z}_j + \mathrm{d}w_j\wedge\mathrm{d}\bar{w}_j)\Big) \\
\omega_2 + i\omega_3 &= \sum_{j=0}^n \mathrm{d}z_j\wedge\mathrm{d}w_j
\end{aligned}
\tag{4.6}
$$

with respect to standard coordinates (z, w) on $N_n \cong M_n \times \mathbb{C}^{n+1}$.

The conical structure on M_n is determined by the standard Euler field. In coordinates, this is $\xi = \sum_{j=0}^{n} z_j \frac{\partial}{\partial z_j} + \bar{z}_j \frac{\partial}{\partial \bar{z}_j}$ and therefore the canonical rotating Killing field on N_n is

$$Z = -i \sum_{j=0}^{n} \left(z_j \frac{\partial}{\partial z_j} - \bar{z}_j \frac{\partial}{\partial \bar{z}_j} \right)$$

The Hamiltonian for Z with respect to ω_1 is $f_Z = -\frac{1}{2} g(Z, Z) - \frac{1}{2} c$, $c \geq 0$.

Now we assemble the twist data, i.e. determine the twisting form ω_{H} and the corresponding Hamiltonian f_{H}. Consulting theorem 3.21, we see that they are

$$\begin{aligned} \omega_{\mathrm{H}} &= \frac{i}{2} \left(\mathrm{d}z_0 \wedge \mathrm{d}\bar{z}_0 - \mathrm{d}w_0 \wedge \mathrm{d}\bar{w}_0 - \sum_{j=1}^{n} (\mathrm{d}z_j \wedge \mathrm{d}\bar{z}_j - \mathrm{d}w_j \wedge \mathrm{d}\bar{w}_j) \right) \\ f_{\mathrm{H}} &= \frac{1}{2} g(Z, Z) - \frac{1}{2} c = -\frac{1}{2} \left(|z_0|^2 - \sum_{j=1}^{n} |z_j|^2 + c \right) \end{aligned}$$

Since ω_{H} is exact, it can be realized as the curvature of a connection on the trivial bundle $P = N_n \times S^1$. Denoting the standard (local) coordinate on S^1 by s, which is periodic with period 2π, a natural choice for this connection is

$$\begin{aligned} \eta &= \mathrm{d}s + \frac{i}{4} \left(z_0 \mathrm{d}\bar{z}_0 - \bar{z}_0 \mathrm{d}z_0 - w_0 \mathrm{d}\bar{w}_0 + \bar{w}_0 \mathrm{d}w_0 - \sum_{j=1}^{n} (z_j \mathrm{d}\bar{z}_j - \bar{z}_j \mathrm{d}z_j - w_j \mathrm{d}\bar{w}_j + \bar{w}_j \mathrm{d}w_j) \right) \\ &= \mathrm{d}s + \frac{1}{2} \iota_{\Xi} \omega_{\mathrm{H}} \end{aligned}$$

where Ξ is the Euler field on N_n, i.e. $\Xi = \sum_{j=0}^{n} \left(z_j \frac{\partial}{\partial z_j} + \bar{z}_j \frac{\partial}{\partial \bar{z}_j} + w_j \frac{\partial}{\partial w_j} + \bar{w}_j \frac{\partial}{\partial \bar{w}_j} \right)$.

The lift of Z to P is

$$Z_P = \tilde{Z} + f_{\mathrm{H}} \frac{\partial}{\partial s} = Z + (f_{\mathrm{H}} - \eta(Z)) \frac{\partial}{\partial s} = Z - \frac{c}{2} \frac{\partial}{\partial s} \tag{4.7}$$

where we used

$$\eta(Z) = \frac{1}{2} \omega_{\mathrm{H}}(\Xi, Z) = -\frac{1}{2} \left(|z_0|^2 - \sum_{j=1}^{n} |z_j|^2 \right) = f_{\mathrm{H}} + \frac{c}{2}$$

For these examples, Z_P is easily integrated explicitly and we see that the flow is periodic precisely if $c \in 2\mathbb{Z}$. This is in accordance with lemma 3.5, which dictates that we obtain a lifted circle action only upon imposing an integrality condition on c. Moreover, it is clear that the circle action has the same period as the circle action on N_n precisely if $c = \pm 2$. Therefore, we implicitly assume $c = 2$ in the following. This is no problem, since we already know that the resulting metrics will be locally isometric for all strictly positive values of c. Moreover, it will be clear a posteriori that the resulting expressions determine a complete quaternionic Kähler metric for arbitrary $c \geq 0$.

Before we can carry out the twist construction, one more difficulty needs to be addressed. By definition, the twist manifold is a quotient space. In order to explicitly write down tensor fields on this manifold, we must choose a section for the projection map. With our implicit choice of $c = 2$, a global section is provided by the submanifold $\bar{N}_n = \{(z, w, s) \in M_n \times \mathbb{C}^{n+1} \times S^1 \mid \arg z_0 = 0\}$. This choice is compatible with the choice of section in example 4.14, to which it reduces on the submanifold $\{(z, 0, 0) \in P \mid h(z, z) = -1\}$.

All that remains in order to compute the quaternionic Kähler metric g_{Q} is to use (3.4) to determine $p^* g_{\mathrm{Q}}$, where $p : P \to \bar{N}_n$ is the projection map, and restrict this tensor field to $\bar{N}_n$. The resulting metric is best expressed in terms of the following new variables:

$$\begin{aligned} X_j &\coloneqq \frac{z_j}{z_0} \qquad\qquad j = 1, \dots, n \\ \rho &\coloneqq 2 f_Z = |z_0|^2 - \sum_{j=1}^{n} |z_j|^2 - c \\ \tilde{\phi} &\coloneqq 4s \end{aligned} \tag{4.8}$$

The functions $\{X_j, \rho, \tilde{\phi}, w_\mu\}$, $j = 1, \dots, n$, $\mu = 0, 1, \dots, n$, provide a set of global coordinates on $\bar{N}_n \cong \mathbb{C}\mathrm{H}^n \times \mathbb{R}_{>0} \times S^1 \times \mathbb{R}^{2n+2}$, extending the standard coordinates on $\mathbb{C}\mathrm{H}^n$. After a lengthy series of algebraic manipulations, the expression for g_{Q} dictated by the twist construction reproduces (4.5) with respect to these coordinates.

It is remarkable that, despite starting from extremely simple initial data, we obtain such interesting and non-trivial quaternionic Kähler metrics. Even in the case $n = 0$, when the PSK manifold is reduced to a point, we obtain a non-trivial family of metrics on $\mathbb{R}^4$. The undeformed c-map metric is the symmetric metric on $\mathrm{SU}(1, 2)/\,\mathrm{U}(2) = \mathbb{C}\mathrm{H}^2$; it is known as the universal hypermultiplet in the physics literature. The deformed metrics were studied by Cortés and Saha [CS18], who showed that they interpolate between $\mathbb{C}\mathrm{H}^2$ and the constant curvature metric of (real) hyperbolic four-space, to which they tend as $c \to \infty$. In the following chapter, the contrast in complexity between the hyper-Kähler and the quaternionic Kähler sides of the HK/QK correspondence will be a central motif, and we shall leverage it to prove theorems both about the HK/QK correspondence in general and for this particular series of examples.

5 Symmetry properties of the HK/QK correspondence and c-map

The c-map provides a powerful method to construct explicit, complete quaternionic Kähler manifolds of negative scalar curvature. As we have seen, all Alekseevsky spaces except the quaternionic hyperbolic spaces arise from the c-map, as well as a wealth of further interesting examples. Therefore, understanding the general properties which are shared by all manifolds constructed in this fashion is of paramount importance. In this chapter, we study the symmetry properties of (deformed) c-map metrics, following the articles [CST20b; CST20a]. We do this via two methods, which are in some sense complementary.

In the first part of this chapter, we study the HK/QK correspondence and the c-map in the presence of additional symmetries. More precisely, we study the isometry group of the deformed c-map spaces $(\bar{N}, g^c_{\mathrm{Q}})$ that arise by applying the c-map construction to a PSK manifold $\bar{M}$ that admits (continuous families of) non-trivial automorphisms. Using the twist formulation of the c-map, we show that every one-parameter group of automorphisms of the PSK manifolds can be lifted to a one-parameter group of isometries of the deformed c-map metric. Moreover, the Lie algebra $\mathfrak{aut}(\bar{M})$ of infinitesimal automorphisms of $\bar{M}$ lifts to a Lie algebra of Killing fields of $(\bar{N}, g^c_{\mathrm{Q}})$, which is a (possibly trivial) one-dimensional central extension of $\mathfrak{aut}(\bar{M})$. These results have several interesting corollaries, which include strong lower bounds on the dimension of the isometry group of c-map metrics. In particular, if $\bar{M}$ is homogeneous as a PSK manifold then we prove that the deformed c-map metrics g^c_{Q}, $c > 0$, are of co-homogeneity at most one.

The second part of this chapter consists of a study of the curvature tensors of quaternionic Kähler metrics that arise from the HK/QK correspondence. A long computation eventually yields elegant formulae expressing the Levi-Civita connection and the curvature tensor of the deformed c-map metrics g^c_{Q} in terms of the hyper-Kähler structure and twist data on the dual hyper-Kähler manifold. Our result refines Alekseevsky's decomposition theorem 2.23 by giving an explicit expression for the quaternionic Weyl curvature for quaternionic Kähler metrics arising from the HK/QK correspondence.

Finally, we demonstrate our results by applying them to the series of examples considered in section 4.5, which arise by applying the c-map to $\mathbb{C}\mathrm{H}^n$, viewed as a homogeneous PSK manifold. First, we give explicit expressions for the Killing fields obtained by lifting infinitesimal automorphisms of $\mathbb{C}\mathrm{H}^n$. Then we show how our description of the curvature can be used to study the symmetries of the deformed c-map metrics on these

c-map spaces. Concretely, we use our formulae to compute the Hilbert–Schmidt norm of the curvature, viewed as an operator $\mathcal{R} : \bigwedge^2 TM \to \bigwedge^2 TM$. For $c > 0$, this invariant turns out to be an injective function of precisely one of the coordinate functions on these manifolds, so we conclude that the isometry groups of the metrics g^c_{Q}, $c > 0$, preserve the level sets of this function. Combining this with our results from the first part, we deduce that deformed c-map metrics are of co-homogeneity precisely one. In particular, they define a non-trivial deformation of the symmetric space $\frac{\mathrm{SU}(n,2)}{\mathrm{S}(\mathrm{U}(n)\times\mathrm{U}(2))}$ through complete quaternionic Kähler metrics of co-homogeneity one and negative scalar curvature. We conjecture that the same phenomenon occurs for other c-map spaces that arise from homogeneous PSK manifolds.

5.1 The c-map in the presence of additional symmetries

Let $\bar{M}$ be a PSK manifold and assume that it admits a non-trivial automorphism. By definition, every automorphism of the PSK structure is induced by a diffeomorphism φ of the corresponding CASK manifold M which preserves the CASK data (cf. definition 4.12). Our first step is to lift this CASK automorphism to T^*M, which is to be thought of as a pseudo-hyper-Kähler manifold endowed with a free and isometric circle action, induced by the rotating Killing field Z. For any manifold M there is a canonical method of lifting self-diffeomorphisms of M to its cotangent bundle, using the pullback on one-forms. Indeed, for any diffeomorphism $\varphi : M \to M$ and any covector $\alpha_p \in T^*_pM$, $p \in M$, we know that $(\varphi^{-1})^*\alpha_p \in T^*_{\varphi(p)}M$, so the map $\Phi : T^*M \to T^*M$, $\alpha_p \mapsto (\varphi^{-1})^*\alpha_p$ is a bundle automorphism of T^*M covering φ. We call this the canonical lift of φ. The following lemma shows that canonically lifting is compatible with the rigid c-map construction.

Lemma 5.1. *Let (M, g_M, J_M, ∇) be an affine special Kähler manifold endowed with an automorphism φ. Equip its cotangent bundle $N = T^*M$ with the (pseudo-)hyper-Kähler structure induced by the rigid c-map. Then the canonical lift $\Phi : N \to N$ of φ respects this hyper-Kähler structure, i.e. is a tri-holomorphic isometry.*

Proof. Since φ preserves the special Kähler connection ∇, it preserves the splitting $TN = \pi^*TM \oplus \pi^*T^*M$ induced by ∇. Thus, we may work with respect to this splitting. But with respect to this splitting, (4.1) shows that g and ω_k, $k = 1, 2, 3$, are expressed in terms of the Kähler structure on M, which is invariant under φ. This means that g and ω_k are preserved by Φ. □

Now assume that M is CASK, so that $N = T^*M$ is endowed with a canonical Killing field which is moreover ω_1-Hamiltonian (see proposition 4.26).

Lemma 5.2. *Let φ be a CASK automorphism and Φ its canonical lift to $N = T^*M$. Then Φ preserves the Hamiltonian f_Z of Z with respect to ω_1.*

Proof. The Hamiltonian is $f_Z = -\frac{1}{2}\pi^* g_M(\xi,\xi) - \frac{1}{2}c$. Since φ preserves g_M and ξ, it preserves $g(\xi,\xi)$ and consequently Φ preserves f_Z. □

Functoriality of the pullback now implies that the automorphism group of the CASK manifold lifts to a subgroup of the group of tri-holomorphic isometries of $N = T^*M$ that preserve f_Z.

We can also formulate a canonical lifting procedure for the vector fields generating one-parameter groups of automorphisms, but we have to introduce some notation first. There is a canonical vector field on T^*M which is vertical with respect to the projection $\pi : T^*M \to M$ and restricts to the Euler field on each fiber; we denote it by η. Under the identification of the vertical tangent bundle of T^*M with π^*T^*M, it corresponds to the tautological one-form on T^*M.

Given a vector field $X \in \mathfrak{X}(M)$, $\nabla X \in \Gamma(\mathrm{End}\, TM)$ can be pulled back to an endomorphism of $\mathrm{End}(\pi^*TM)$, which we continue to denote by ∇X. First taking its adjoint and then pulling back, we obtain $(\nabla X)^* \in \Gamma(\mathrm{End}(\pi^*(T^*M))$, which we interpret as an endomorphism of the vertical tangent bundle of T^*M. In this notation, we have:

Lemma 5.3. *Let (M, g_M, J_M, ∇) be an affine special Kähler manifold and φ_t a one-parameter group of automorphisms of the affine special Kähler structure. Denote its generating vector field by X. Then, if Φ_t is the canonically lifted one-parameter group of tri-holomorphic isometries of N, endowed with its canonical hyper-Kähler structure, its generating vector field is $Y = \tilde{X} - (\nabla X)^*(\eta)$, where $\tilde{X}$ is the ∇-horizontal lift of X.*

Proof. It is a basic result from symplectic geometry that a diffeomorphism $T^*M \to T^*M$ is the canonical lift of a diffeomorphism of M if and only if it preserves the tautological one-form λ. Since Y is certainly a lift of the vector field X, it now suffices to check that $L_Y\lambda = 0$ to verify that it is the canonical lift.

The simplest way to go about this is to pick local ∇-affine coordinates $\{q^i\}$ on M, which exist because ∇ is flat. Then, with respect to the canonical coordinates $\{q^i, p_i\}$ on T^*M, we have

$$\lambda = \sum p_i \mathrm{d}q^i \qquad \eta = \sum p_i \frac{\partial}{\partial p_i} \qquad (\nabla X)^* = \sum \frac{\partial X^i}{\partial q^j} \mathrm{d}p_i \otimes \frac{\partial}{\partial p_j}$$

This implies that

$$Y = \sum \left(X^i \frac{\partial}{\partial q^i} - \frac{\partial X^i}{\partial q^j} p_i \frac{\partial}{\partial p_j} \right)$$

and consequently

$$L_Y\lambda = Y(p_i)\mathrm{d}q^i + p_i \mathrm{d}(Y(q^i)) = 0$$

which proves the claim. □

We already saw that $L_Y\omega_k = 0$ for $k = 1, 2, 3$ for any canonical lift Y of an infinitesimal automorphisms of an affine special Kähler manifold. In case M is a CASK manifold, the fact that every one-parameter group of automorphisms is Hamiltonian, proven in lemma 4.9, can be leveraged to say more.

Lemma 5.4. *Let $N = T^*M$ be the cotangent bundle of a CASK manifold M, equipped with its canonical pseudo-hyper-Kähler structure. Then, if $\Phi_t : N \to N$ is the canonical lift of a one-parameter group of CASK automorphisms $\varphi_t : M \to M$, it is ω_1- as well as ω_3-Hamiltonian.*

Proof. Let Y denote the generating vector field for Φ_t, and X the generating vector field for φ_t. With our conventions, ω_3 is the canonical symplectic structure, which exists on any cotangent bundle (cf. the proof of proposition 4.26). Therefore, $\omega_3 = -\mathrm{d}\lambda$, where λ is the tautological one-form. We already know that $L_Y\lambda = 0$ and therefore $-\lambda(Y)$ is a Hamiltonian with respect to ω_3.

The proof that the action is ω_1-Hamiltonian is more involved, and makes use of the special Kähler structure. We will find a local expression for the Hamiltonian function, and show that it is globally well-defined. Pick a set of local ∇-affine Darboux coordinates on M and consider the corresponding canonical coordinates $\{q^i, p_i\}$ on N. With respect to these coordinates we can write

$$\omega_1 = \sum (\omega_M)_{ij}\mathrm{d}q^i \wedge \mathrm{d}q^j + (\omega_M)^{ij}\mathrm{d}p_i \wedge \mathrm{d}p_j$$

where ω_M is the Kähler form on M; its coefficients are constant with respect to these coordinates, and $(\omega_M)^{ij}$ are the coefficients of the inverse matrix. Using the formula of lemma 5.3, we have

$$\iota_Y\omega_1 = \sum 2(\omega_M)_{ij}X^i\mathrm{d}q^j - 2(\omega_M)^{ij}\frac{\partial X^k}{\partial q^i}p_k\mathrm{d}p_j$$

Since X is ω_M-Hamiltonian by lemma 4.9, the first term equals $-\pi^*\mathrm{d}f_X$ for some $f_X \in C^\infty(M)$. Regarding the second term, the fact that X preserves the special Kähler connection ∇, i.e. is ∇-affine, implies that its coefficients with respect to the coordinates $\{q^i\}$ on M are affine functions. In particular, $\frac{\partial X^k}{\partial q^i}$ is constant, so after setting $S_Y^{jk} = \sum(\omega_M)^{ij}\frac{\partial X^k}{\partial q^i}$, and noting that each S_Y^{jk} is constant, we can write the second term as $-\mathrm{d}\big(\sum S_Y^{jk}p_jp_k\big)$.

We show how the coefficients S_Y^{jk} arise from an invariantly defined object on N. Let $\mathcal{V} \subset TN$ denote the vertical subbundle, and define $S_Y \in \mathrm{Sym}^2(\mathcal{V}^*)$ by $S_Y = \omega_1((\nabla X)^*\cdot, \cdot)|_{\mathcal{V}}$, where we once again view $(\nabla X)^*$ as a section of $\mathrm{End}(\mathcal{V})$. The symmetry of S_Y follows from the fact that $0 = L_X\omega_M(U, V) = \omega_M(\nabla_U X, V) - \omega_M(\nabla_V X, U)$, where we used that ω_M is ∇-parallel. Then we have

$$S_Y\left(\frac{\partial}{\partial p_i}, \frac{\partial}{\partial p_j}\right) = \omega_1\left(\sum \frac{\partial X^i}{\partial q^l}\frac{\partial}{\partial p_l}, \frac{\partial}{\partial p_j}\right) = \sum(\omega_M)^{lj}\frac{\partial X^i}{\partial q^l}$$

and therefore the (constant) functions S_Y^{jk} are nothing but the coefficient functions of S_Y with respect to these coordinates. Applying S_Y to the fiberwise Euler field η twice, we obtain

$$\iota_Y\omega_1 = -\mathrm{d}\big(\pi^* f_X + S_Y(\eta,\eta)\big)$$

and thus we have constructed a Hamiltonian for Y with respect to ω_1. □

Definition 5.5. Let (N, g, I_k, Z, f_Z) be a hyper-Kähler manifold endowed with a rotating Killing field Z with Hamiltonian f_Z with respect to ω_1. We denote the group of triholomorphic isometries which preserve f_Z and are ω_1-Hamiltonian by $\mathrm{Aut}_{\mathrm{Ham}}(N, f_Z)$.

This is the appropriate notion of symmetry in the context of the HK/QK correspondence. Note that elements of $\mathfrak{aut}_{\mathrm{Ham}}(N, f_Z)$ automatically commute with Z, since Z is ω_1-dual to $-\mathrm{d}f_Z$.

The above discussion can essentially be summarized as follows:

Theorem 5.6. *Let M be a CASK manifold endow $N = T^*M$ with its the canonical pseudo-hyper-Kähler structure and rotating Killing field. Then the CASK automorphism group* $\mathrm{Aut}\, M$ *lifts canonically to a subgroup of* $\mathrm{Aut}_{\mathrm{Ham}}(N, f_Z)$. □

The above depends crucially on the fact that $N = T^*M$ is naturally associated to M. When we apply the HK/QK correspondence to N, we make use of an auxiliary space, the circle bundle P, which is not canonically associated with N. We therefore should not expect a natural way to lift automorphisms of N to P or, by extension, twist them to automorphisms of the quaternionic Kähler twist manifold $\bar{N}$. Nevertheless, we can twist Z-invariant tensor fields on N, such as the Z-invariant vector fields that make up $\mathfrak{aut}_{\mathrm{Ham}}(N, f_Z)$, to tensor fields on $\bar{N}$. We will now show how to produce Killing fields on the quaternionic Kähler side of the correspondence out of these vector fields.

Lemma 5.7. *Let (N, g, ω_k, Z, f_Z) be a hyper-Kähler manifold equipped with ω_1-Hamiltonian rotating Killing field. Then every $X \in \mathfrak{aut}_{Ham}(N, f_Z)$ preserves the twist data $(Z, \omega_{\mathrm{H}}, f_{\mathrm{H}})$ and the elementary deformation g_{H} specified in theorem 3.20.*

Proof. By assumption, X preserves g, ω_k and f_Z, hence also Z. Therefore, it certainly preserves $g_{\mathrm{H}} = \frac{1}{f_Z}g + \frac{1}{f_Z^2}g_\alpha$ and the function $f_{\mathrm{H}} = f_Z + g(Z,Z)$. To see that it is ω_{H}-Hamiltonian, we use the assumption that there exists a Hamiltonian φ of X with respect to ω_1. Then $\iota_X\omega_{\mathrm{H}} = -\mathrm{d}\varphi + \iota_X \mathrm{d}\alpha_0 = -\mathrm{d}(\varphi + g(Z,X))$ by Cartan's formula. □

We will denote a (choice of) Hamiltonian function for $X \in \mathfrak{aut}_{\mathrm{Ham}}(N, f_Z)$ with respect to ω_{H} by f_X. Now, let us consider what happens when we twist such a vector field. We would like to produce a Killing field with respect to the quaternionic Kähler metric g_{Q}, which is $\mathcal{H}$-related to g_{H}. However, as explained in chapter 3, differential conditions are not preserved by the twist construction, and therefore Killing fields do not (usually) twist to Killing fields. Indeed, if X' denotes the twist of X, we have the relation $L_X g_{\mathrm{H}} - 2f_{\mathrm{H}}^{-1}\iota_X\omega_{\mathrm{H}} \vee \iota_Z g_{\mathrm{H}} \sim_{\mathcal{H}} L_{X'} g_{\mathrm{Q}}$, proven in example 3.14. Thus, an element $\mathfrak{aut}_{\mathrm{Ham}}(N, f_Z)$ needs to be modified if we are to produce a Killing field on the quaternionic Kähler side of the correspondence. Moreover, this modification needs to be Z-invariant, so that we can still twist the modified vector field.

In order to find an appropriate modification, it is helpful to consider the big picture. In the context of the twist construction, there is one distinguished vector field, which in this case is the rotating Killing field Z. Therefore, it is natural to modify the infinitesimal automorphism X by adding a term proportional to Z. The following lemma proves that this is the right approach:

Lemma 5.8. *Let $X \in \mathfrak{aut}_{\mathrm{Ham}}(N, f_Z)$ as above. Then there exists a Z-invariant function $\psi \in C^\infty(N)$ such that the twist X_{Q} of $X_{\mathrm{H}} := X + \psi Z$ is a Killing field on $(\bar{N}, g_{\mathrm{Q}})$.*

Proof. We use the ansatz $X_{\mathrm{H}} = X + \psi Z$ and attempt to solve the equation $L_{X_{\mathrm{Q}}} g_{\mathrm{Q}} = 0$, where X_{Q} is the twist of X_{H}. This is equivalent to solving $L_{X_{\mathrm{H}}} g_{\mathrm{H}} - 2f_{\mathrm{H}}^{-1}\iota_{X_{\mathrm{H}}}\omega_{\mathrm{H}} \vee \iota_Z g_{\mathrm{H}} = 0$ on N. Because $L_X g_{\mathrm{H}} = L_Z g_{\mathrm{H}} = 0$ and $\iota_X\omega_{\mathrm{H}} = -\mathrm{d}f_X$, $\iota_Z\omega_{\mathrm{H}} = -\mathrm{d}f_{\mathrm{H}}$, we obtain:

$$L_{X_{\mathrm{H}}} g_{\mathrm{H}} - 2f_{\mathrm{H}}^{-1}\iota_{X_{\mathrm{H}}}\omega_{\mathrm{H}} \vee \iota_Z g_{\mathrm{H}} = 2\Big(\mathrm{d}\psi - f_{\mathrm{H}}^{-1}(-\mathrm{d}f_X - \psi\mathrm{d}f_{\mathrm{H}})\Big) \vee \iota_Z g_{\mathrm{H}}$$

It is clearly sufficient to solve the following differential equation for ψ:

$$\mathrm{d}\psi = -\left(\frac{\mathrm{d}f_X}{f_{\mathrm{H}}} + \frac{\psi\mathrm{d}f_{\mathrm{H}}}{f_{\mathrm{H}}}\right)$$

which has the simple solution $\psi = -\frac{f_X}{f_{\mathrm{H}}}$. All that remains is to check that ψ is Z-invariant; this follows from the fact that both f_X and f_{H} are. Indeed, f_{H} is the Hamiltonian for Z itself with respect to ω_{H}, and $Z(f_X) = -\omega_{\mathrm{H}}(X, Z) = X(f_{\mathrm{H}}) = 0$ by lemma 5.7. □

Recall that, as a general feature of the twist construction, $\bar{N}$ comes equipped with a Killing field $V = p_*(V_P)$, where V_P is the vector field generating the principal circle action on P and p is the projection map to $\bar{N}$. Tensor fields that arise from twisting are automatically V-invariant. Denoting the algebra of Killing fields of $\bar{N}$ which commute with V by $\mathfrak{aut}(\bar{N}, V)$, the above construction therefore produces elements of $\mathfrak{aut}(\bar{N}, V)$ out of elements of $\mathfrak{aut}_{\mathrm{Ham}}(N, f_Z)$.

This procedure does not automatically induce a unique map $\mathfrak{aut}_{\mathrm{Ham}}(N, f_Z) \to \mathfrak{aut}(\bar{N}, V)$ since it involves a choice of Hamiltonian function on the hyper-Kähler side. To investigate this freedom, let us consider two choices f_X, f'_X of moment maps for $X \in \mathfrak{aut}_{\mathrm{Ham}}(N, f_Z)$ with respect to ω_{H}. Then we obtain two Killing fields X_{Q}, X'_{Q} on $\bar{N}$, which are the twists of $X_{\mathrm{H}} = X - \frac{f_X}{f_{\mathrm{H}}}Z$ and $X'_{\mathrm{H}} = X - \frac{f'_X}{f_{\mathrm{H}}}Z$, respectively. Since $f'_X = f_X + K$ for some $K \in \mathbb{R}$, we have $X'_{\mathrm{H}} - X_{\mathrm{H}} = -\frac{K}{f_{\mathrm{H}}}Z$, the twist of which is nothing but KV, since

$V = p_*(V_P)$ and $V_P = \frac{1}{\pi^* f_{\mathrm{H}}}(Z_P - \tilde{Z})$, i.e. V is the twist of $-\frac{1}{f_{\mathrm{H}}}Z$. Thus, a different choice of Hamiltonian corresponds, on the quaternionic Kähler side, to adding a constant multiple of V.

Proposition 5.9. *Let (N, g, ω_k, Z, f_Z) be a hyper-Kähler manifold with rotating, ω_1-Hamiltonian Killing field and let $\{X_j\}$, $j = 1, \dots, d$ be a basis of $\mathfrak{aut}_{\mathrm{Ham}}(N, f_Z)$, where $d = \dim \mathfrak{aut}_{\mathrm{Ham}}(N, f_Z)$. Then, for any set of choices of Hamiltonian functions $\{f_{X_j}\}$ with respect to ω_{H}, we have an injective linear map*

$$\begin{aligned} \varphi :\ \mathfrak{aut}_{\mathrm{Ham}}(N, f_Z) &\longrightarrow \mathfrak{aut}(\bar{N}, V) \\ X = \textstyle\sum_j \alpha_j X_j &\longmapsto X_{\mathrm{Q}} = \textstyle\sum_j \alpha_j X_j^{\mathrm{Q}} \end{aligned}$$

where X_j^{Q} is the twist of $X_j^{\mathrm{H}} := X_j - \frac{f_{X_j}}{f_{\mathrm{H}}}Z$.

Proof. It is clear that the above defines a linear map, so we need only verify injectivity. Since twisting a vector field is done by applying two pointwise linear isomorphisms, it introduces no kernel and $\varphi(X) = 0$ if and only if $X_{\mathrm{H}} = 0$. This means, in particular, that $X = \phi Z$ for some $\phi \in C^\infty(N)$. Since $X \in \mathfrak{aut}_{\mathrm{Ham}}(N, f_Z)$ and Z is a Killing field, we know $0 = L_{\phi Z} g = 2\mathrm{d}\phi \vee \iota_Z g$, so that ϕ must be constant. But Z is a rotating Killing field, so no non-zero multiple of it lies in $\mathfrak{aut}_{\mathrm{Ham}}(N, f_Z)$. Therefore, $X = 0$ and we deduce that φ is injective. □

Since both $\mathfrak{aut}_{\mathrm{Ham}}(N, f_Z)$ and $\mathfrak{aut}(\bar{N}, V)$ are Lie algebras, it is natural to ask whether the maps φ constructed above are Lie algebra homomorphisms. This is not quite the case in general:

Proposition 5.10. *Consider $\{X_j\}$, $\{f_{X_j}\}$ and $\{X_j^{\mathrm{Q}}\}$ as in proposition 5.9. Let $\{c_{jk}^l\}$ be the structure constants of $\mathfrak{aut}_{\mathrm{Ham}}(N, f_Z)$ with respect to the basis $\{X_j\}$, i.e. $[X_j, X_k] = \sum_l c_{jk}^l X_l$. Then*

$$[X_j^{\mathrm{Q}}, X_k^{\mathrm{Q}}] = \sum_l c_{jk}^l X_l^{\mathrm{Q}} + A_{jk} V$$

where $A_{jk} = \omega_{\mathrm{H}}(X_j, X_k) - \sum_l c_{jk}^l f_{X_l}$ are constants.

Proof. The commutators of the Killing fields $\{X_j^{\mathrm{Q}}\}$ are given in terms of those of their twists, $\{X_j^{\mathrm{H}}\}$, by example 3.13. We first compute the latter.

$$\begin{aligned} [X_j^{\mathrm{H}}, X_k^{\mathrm{H}}] &= \sum_l c_{jk}^l X_l - \left(\left[\frac{f_{X_j}}{f_{\mathrm{H}}} Z, X_k \right] + \left[X_j, \frac{f_{X_k}}{f_{\mathrm{H}}} Z \right] \right) + \left[\frac{f_{X_j}}{f_{\mathrm{H}}} Z, \frac{f_{X_k}}{f_{\mathrm{H}}} Z \right] \\ &= \sum_l c_{jk}^l X_l + \frac{1}{f_{\mathrm{H}}}(X_k(f_{X_j}) - X_j(f_{X_k}))Z \\ &= \sum_l c_{jk}^l X_l - \frac{2}{f_{\mathrm{H}}}\omega_{\mathrm{H}}(X_j, X_k) Z \end{aligned}$$

Now

$$[X_j^{\mathrm{Q}}, X_k^{\mathrm{Q}}] \sim_{\mathcal{H}} [X_j^{\mathrm{H}}, X_k^{\mathrm{H}}] + f_{\mathrm{H}}^{-1}\omega_{\mathrm{H}}(X_j^{\mathrm{H}}, X_k^{\mathrm{H}})Z = [X_j^{\mathrm{H}}, X_k^{\mathrm{H}}] + f_{\mathrm{H}}^{-1}\omega_{\mathrm{H}}(X_j, X_k)Z$$

which means that

$$\begin{aligned}[X_j^{\mathrm{Q}}, X_k^{\mathrm{Q}}] &\sim_{\mathcal{H}} \sum_l c_{jk}^l X_l - \frac{1}{f_{\mathrm{H}}}\omega_{\mathrm{H}}(X_j, X_k)Z \\ &= \sum_l c_{jk}^l X_l^{\mathrm{H}} - \frac{1}{f_{\mathrm{H}}}\left(\omega_{\mathrm{H}}(X_j, X_k) - \sum_l c_{jk}^l f_{X_l}\right)Z\end{aligned}$$

Denoting the twist by a prime, we can write this as

$$[X_j^{\mathrm{Q}}, X_k^{\mathrm{Q}}] = \sum_l c_{jk}^l X_l^{\mathrm{Q}} + \left(\omega_{\mathrm{H}}(X_j, X_k) - \sum_l c_{jk}^l f_{X_l}\right)' V$$

where we used that $-(f_{\mathrm{H}}^{-1}Z)'$ twists to V.

It now only remains to show that the parenthesized expression is constant. To this end, we compute $\mathrm{d}(\omega_{\mathrm{H}}(X_j, X_k))$, using Cartan's formula and $\mathrm{d}\omega_{\mathrm{H}} = 0$:

$$\mathrm{d}\iota_{X_k}\iota_{X_j}\omega_{\mathrm{H}} = L_{X_k}\iota_{X_j}\omega_{\mathrm{H}} - \iota_{X_k}L_{X_j}\omega_{\mathrm{H}} = \iota_{[X_k, X_j]}\omega_{\mathrm{H}}$$

where the last equality follows from the fact that elements of $\mathfrak{aut}_{\mathrm{Ham}}(N, f_Z)$ preserve ω_{H} (cf. lemma 5.7). Therefore, we find

$$\begin{aligned}\mathrm{d}\left(\omega_{\mathrm{H}}(X_j, X_k) - \sum_l c_{jk}^l f_{X_l}\right) &= \iota_{[X_k, X_j]}\omega_{\mathrm{H}} + \sum_l c_{jk}^l \iota_{X_l}\omega_{\mathrm{H}} \\ &= \iota_{[X_k, X_j]}\omega_{\mathrm{H}} + \iota_{[X_j, X_k]}\omega_{\mathrm{H}} = 0\end{aligned}$$

This finishes the proof. □

Corollary 5.11. *Any Lie subalgebra $\mathfrak{g} \subset \mathfrak{aut}_{\mathrm{Ham}}(N, f_Z)$ of dimension k induces a $(k+1)$-dimensional subalgebra $\mathfrak{g}_{\mathrm{Q}} \subset \mathfrak{aut}(\bar{N}, V)$ which is isomorphic to a (possibly trivial) central extension of $\mathfrak{g}$.* □

The constants A_{jk} define a two-cocycle $\alpha = \frac{1}{2}A_{jk}X_j^* \wedge X_k^* \in Z^2(\mathfrak{g})$, where $\{X_j^*\}$ is the dual basis of $\mathfrak{g}^*$. It is well-known that the vanishing of the cohomology class represented by this cocycle is equivalent to the triviality of the central extension defined by $\{A_{jk}\}$.

Corollary 5.12. *If $[\alpha] \in H^2(\mathfrak{g})$ is trivial, there exist new basis vectors $\hat{X}_j^{\mathrm{Q}} = X_j^{\mathrm{Q}} + \beta_j V$, $\beta_j \in \mathbb{R}$, such that $\hat{\mathfrak{g}} = \operatorname{span}\left\{\hat{X}_j^{\mathrm{Q}}\right\} \subset \mathfrak{g}_{\mathrm{Q}}$ is isomorphic to $\mathfrak{g}$ and $\mathfrak{g}_{\mathrm{Q}} \cong \hat{\mathfrak{g}} \oplus \mathbb{R}V$.* □

Corollary 5.13. *If $\mathfrak{g} \subset \mathfrak{aut}_{\mathrm{Ham}}(N, f_Z)$ is semi-simple, there exists a set of choices of Hamiltonian functions f_{X_j} such that the map φ from proposition 5.9 is an injective Lie algebra homomorphism.*

Proof. A basic result, known as Whitehead's lemma, asserts that $H^2(\mathfrak{g}) = 0$ if $\mathfrak{g}$ is semi-simple. Now note that adding a constant multiple of V to X_j^{Q} corresponds to picking a different Hamiltonian function for X_j. □

Now let us apply these results to the c-map.

Theorem 5.14. *Let M be a CASK manifold and $\bar{N}$ the quaternionic Kähler manifold associated to it by the composition of the rigid c-map and the HK/QK correspondence. Then there is a canonical injective, linear map $\mathfrak{aut}(M) \to \mathfrak{aut}(\bar{N}, V)$.*

Proof. Denoting the cotangent bundle of M, endowed with its canonical pseudo-hyper-Kähler structure and rotating Killing field, by (N, g, ω_k, Z, f_Z) as usual, we have already shown how to obtain a canonical, injective Lie group homomorphism $\mathrm{Aut}\, M \to \mathrm{Aut}_{\mathrm{Ham}}(N, f_Z)$. Its differential at the identity is of course an injective Lie algebra homomorphism.

Now we have obtained a subalgebra of $\mathfrak{aut}_{\mathrm{Ham}}(N, f_Z)$, whose elements moreover come with canonical choices of Hamiltonian functions with respect to ω_{H}. Indeed, let Y be the canonical lift of an infinitesimal CASK automorphism X. Then an inspection of the proofs of lemma 4.9, lemma 5.4 and lemma 5.7 yields the formula $f_Y = \frac{1}{2}\pi_M^* \mathrm{d}^c\kappa(X) + S(\eta, \eta) + g(Z, Y)$ for the Hamiltonian with respect to ω_{H}, where

(i) $\pi_M : N = T^*M \to M$ is the projection map and $\mathrm{d}^c\kappa = g_M(J_M\xi, \cdot)$.

(ii) $S_Y \in \mathrm{Sym}^2(\mathcal{V}^*)$, where $\mathcal{V} \subset TN$ is the vertical subbundle, is given by $S(A, B) = \omega_1((\nabla X)^* A, B)$.

(iii) $\eta \in \Gamma(\mathcal{V})$ is the fiberwise Euler field on N.

Since $\pi_M^* \mathrm{d}^c\kappa(X) = g(\widetilde{J_M\xi}, \tilde{X}) = -g(Z, Y)$, we can simplify to $f_Y = \frac{1}{2} g(Z, Y) + S_Y(\eta, \eta)$. This gives us a canonical choice of Hamiltonian for vector fields in the image of $\mathfrak{aut}(M)$ and therefore a canonical injective, linear map into $\mathfrak{aut}(\bar{N}, V)$. □

Theorem 5.15. *Let $\bar{M}$ be a PSK manifold and $(\bar{N}, g^c)$, $c \geq 0$, its image under the c-map. Then there exists an injective, linear map $\mathfrak{aut}(\bar{M}) \to \mathfrak{aut}(\bar{N}, V)$.*

Proof. We combine the preceding theorem with lemma 4.13, which yields an injective, linear map $\mathfrak{aut}(\bar{M}) \to \mathfrak{aut}(M)$, which is moreover a canonical isomorphism in case the special Kähler connection on M is not equal to its Levi-Civita connection. In this case, the map $\mathfrak{aut}(\bar{M}) \to \mathfrak{aut}(\bar{N}, V)$ is also canonical. □

Corollary 5.16. *Let $\bar{M}$ be a PSK manifold of dimension $2n$ and set $d = \dim \mathfrak{aut}\, \bar{M}$. Let $(\bar{N}, g^c)$, $c \geq 0$ be its image under the (one-loop deformed) c-map. Then $\mathfrak{isom}(\bar{N}, g^c)$ has dimension at least $d + 2n + 3$. For $c = 0$, it has dimension at least $d + 2n + 4$.*

Proof. By proposition 4.34, $(\bar{N}, g^c)$ carries an isometric and nearly effective action of Heis_{2n+3}, which covers the identity map on $\bar{M}$, for every $c \geq 0$. The vector fields generating this action are therefore linearly independent of those constructed via theorem 5.15, which cover non-trivial infinitesimal PSK automorphisms on $\bar{M}$. In case $c = 0$, there is an additional Killing field tangent to the fibers, which corresponds to the one-dimensional extension of Heis_{2n+3} discussed in proposition 4.29. □

If $(\bar{N}, g^c)$ is a complete manifold, its Killing fields are also complete and hence determine the Lie algebra of the isometry group, which is obtained upon integration.

Corollary 5.17. *Let $\bar{M}$ be a PSK manifold whose automorphism group acts with co-homogeneity k, and such that its image $(\bar{N}, g^c)$, $c \geq 0$, under the deformed c-map is complete. Then the isometry group of $(\bar{N}, g^c)$ acts with co-homogeneity at most k if $c = 0$ and co-homogeneity at most $k+1$ if $c > 0$.* □

Remark 5.18. For $c = 0$, these corollaries recover known results from [CDJL17].

Since Alekseevsky spaces and their deformations are complete and arise from homogeneous PSK manifolds, we also have:

Corollary 5.19. *The one-loop deformed c-map metrics g^c provide a one-parameter deformation through complete quaternionic Kähler metrics of co-homogeneity at most one of all Alekseevsky spaces, with exception of the quaternionic hyperbolic spaces.* □

Given these large isometry groups on quaternionic Kähler manifolds arising from PSK manifolds with a lot of symmetry, it is natural to study discrete subgroups of them. These give rise to quaternionic Kähler manifolds with interesting fundamental groups. This topic is discussed in chapter 6.

5.2 Curvature and the HK/QK correspondence

As announced at the start of the chapter, our next goal is to understand the behavior of the curvature tensor under the HK/QK correspondence. Along the way, we will also investigate this question for the Levi-Civita connection. The proofs of the two main theorems are postponed to the endings of the respective subsections in order to accommodate those who wish to skip these long computations. We warn the reader that, due to the size of some of the expressions involved, it was impossible to avoid page breaks occurring in the middle of a formula on some occasions.

5.2.1 Preliminaries and general twist formulae

As far as the metric is concerned, the HK/QK correspondence consists of two steps, namely elementary deformation and twisting. It is convenient to introduce an endomorphism field which relates the hyper-Kähler metric g with its elementary deformation $g_{\mathrm{H}} = \frac{1}{f_Z} g + \frac{1}{f_Z^2} g_\alpha$.

Definition 5.20. Given a hyper-Kähler manifold (N, g, I_k) endowed with an ω_1-Hamiltonian rotating Killing field (Z, f_Z), we define the endomorphism field $\mathcal{K} : TN \to TN$ by $g_{\mathrm{H}}(\mathcal{K}X, Y) = g(X, Y)$, where g_{H} is the elementary deformation of g specified in theorem 3.20, and X, Y are arbitrary vector fields on N.

Remark 5.21. Regarding g and g_{H} as isomorphisms $TN \to T^*N$, we may also write $\mathcal{K} = g_{\mathrm{H}}^{-1} \circ g$. Since $g_\alpha = g(Z, Z)|_{\mathbb{H}Z}$, we can write

$$g_{\mathrm{H}} = \frac{1}{f_Z} g|_{(\mathbb{H}Z)^\perp} + \frac{f_{\mathrm{H}}}{f_Z^2} g|_{\mathbb{H}Z}$$

In particular, $\mathcal{K}|_{(\mathbb{H}Z)^\perp} = f_Z$ and $\mathcal{K}|_{\mathbb{H}Z} = \frac{f_Z^2}{f_{\mathrm{H}}}$.

Lemma 5.22. *$\mathcal{K}$ is self-adjoint with respect to g and commutes with every I_μ, $\mu = 0, 1, 2, 3$.*

Proof. Self-adjointness of $\mathcal{K}$ follows from the fact that $\mathcal{K}$ is diagonal with respect to an appropriate choice of orthonormal basis for TN, namely any orthonormal basis compatible with the quaternionic structure, as is clear from the preceding remark. This remark also shows that

$$\mathcal{K}(X) = f_Z X - \frac{f_Z}{f_{\mathrm{H}}} \sum_{\lambda=0}^{3} \alpha_\lambda(X) I_\lambda Z$$

where, we recall, $\alpha_\lambda = \iota_Z \omega_\lambda$ and $\omega_0 = g$. Indeed, the right-hand side restricts to the correct expressions on $\mathbb{H}Z$ and $(\mathbb{H}Z)^\perp$. Now we check that $\mathcal{K}$ commutes with I_μ:

$$\begin{aligned}\mathcal{K}(I_\mu X) &= f_Z I_\mu X - \frac{f_Z}{f_{\mathrm{H}}} \sum_\lambda g(I_\lambda Z, I_\mu X) I_\lambda Z = f_Z I_\mu X - \frac{f_Z}{f_{\mathrm{H}}} \sum_\lambda g(I_\mu^{-1} I_\lambda Z, X) I_\lambda Z \\ &= I_\mu f_Z X - \frac{f_Z}{f_{\mathrm{H}}} \sum_\lambda g(I_\lambda Z, X) I_\mu I_\lambda Z = I_\mu \mathcal{K}(X)\end{aligned}$$

To arrive at the second line, we substituted I_λ for $I_\mu I_\lambda$ in the second term; there are no additional signs because I_λ appears twice. □

Before addressing the curvature, we will describe the Levi-Civita connection on the quaternionic Kähler manifold in terms of modifications of the Levi-Civita connection of the original hyper-Kähler metric.

Lemma 5.23. *Let (N, g, ω_k, Z, f_Z) be a pseudo-hyper-Kähler manifold with ω_1-Hamiltonian rotating Killing field and $\nabla, \nabla^{\mathrm{H}}$ the Levi-Civita connections of g and the elementary deformation g_{H}. Then $\nabla^{\mathrm{H}} = \nabla + S^{\mathrm{H}}$, where S^{H} is the $(1,2)$-tensor defined by*

$$2g_{\mathrm{H}}(S^{\mathrm{H}}_A B, C) = (\nabla_A g_{\mathrm{H}})(B, C) + (\nabla_B g_{\mathrm{H}})(C, A) - (\nabla_C g_{\mathrm{H}})(A, B)$$

for arbitrary vector fields A, B, C on N.

Proof. Since $\nabla^{\mathrm{H}} = \nabla + S^{\mathrm{H}}$ is compatible with g_{H}, we have

$$0 = (\nabla^{\mathrm{H}}_A g_{\mathrm{H}})(B, C) = (\nabla_A g_{\mathrm{H}})(B, C) - g_{\mathrm{H}}(S^{\mathrm{H}}_A B, C) - g_{\mathrm{H}}(B, S^{\mathrm{H}}_A C)$$

and therefore

$$g_{\mathrm{H}}(S^{\mathrm{H}}_A B, C) = (\nabla_A g_{\mathrm{H}})(B, C) - g_{\mathrm{H}}(B, S^{\mathrm{H}}_A C)$$

Consider this equation also for cyclic permutations of (A, B, C). Then the appropriate (signed) sum of these three equations yields, upon using $S^{\mathrm{H}}_A B = S^{\mathrm{H}}_B A$, the claimed identity. The symmetry of $S^{\mathrm{H}}_A B$ in (A, B) follows from the fact that both ∇ and ∇^{H} are torsion-free. □

We saw in example 3.18 that the Levi-Civita connection picks up an additional correction upon twisting. Let us denote the corresponding $(1,2)$-tensor by S^{Q}, so that the Levi-Civita connection ∇^{Q} of the quaternionic Kähler metric $g_{\mathrm{Q}} \sim_{\mathcal{H}} g_{\mathrm{H}}$ is itself $\mathcal{H}$-related to $\nabla^S \coloneqq \nabla + S^{\mathrm{H}} + S^{\mathrm{Q}}$.

At this point, it is not surprising that the curvature tensor R^{Q} of g_{Q} is not simply $\mathcal{H}$-related to the curvature tensor R^S of ∇^S, but picks up yet another correction term, controlled by the twist data:

Lemma 5.24. *Let (N, g) be a Riemannian manifold endowed with an isometric circle action generated by the vector field Z and twist data $(Z, \omega_{\mathrm{H}}, f_{\mathrm{H}})$. Let ∇ be the (Z-invariant) Levi-Civita connection of g and let $D = \nabla + S$ be another Z-invariant connection on N. If $D' \sim_{\mathcal{H}} D$ is its twist, then its curvature tensor $R^{D'}$ satisfies*

$$R^{D'}(A', B')C' \sim_{\mathcal{H}} R(A, B)C + T(A, B)C$$

for arbitrary Z-invariant vector fields $A, B, C \sim_{\mathcal{H}} A', B', C'$, where R is the curvature of ∇ and the $\mathrm{End}(TN)$-valued two-form T is given by

$$T(A, B)C \coloneqq (\nabla_A S)_B C - (\nabla_B S)_A C + [S_A, S_B]C - \frac{1}{f_{\mathrm{H}}}\omega_{\mathrm{H}}(A, B)(\nabla_C Z + S_Z C)$$

Proof. Our definition of $\mathcal{H}$-relatedness for connections (cf. example 3.18) implies that $[D_A, D_B]C \sim_{\mathcal{H}} [D'_{A'}, D'_{B'}]C'$, so the curvature formula $R(A, B)C = [\nabla_A, \nabla_B]C - \nabla_{[A,B]}C$

and the formula for twisting commutators of vector fields shows that

$$\begin{aligned} R^{D'}(A',B')C' &\sim_{\mathcal{H}} R^D(A,B)C + \frac{1}{f_{\mathrm{H}}}\omega_{\mathrm{H}}(A,B)D_Z C \\ &= R^D(A,B)C + \frac{1}{f_{\mathrm{H}}}\omega_{\mathrm{H}}(A,B)(\nabla_C Z + S_Z C) \end{aligned}$$

where we used torsion-freeness of ∇ and Z-invariance of C to rewrite the last term. The proof is finished by observing that $R^D - R$ reproduces the first three terms in our claimed expression for T. □

These formulae give an in-principle description of the Levi-Civita connection and curvature tensor of a quaternionic Kähler metric which arises from the HK/QK correspondence, in terms of the dual hyper-Kähler metric and twist data. In their current form, however, they are too complicated to allow for any useful computation. In particular the repeated appearance of $S = S^{\mathrm{H}} + S^{\mathrm{Q}}$, whose definition involves six terms, makes matters difficult. Surprisingly, it turns out that it is possible to tremendously simplify the expressions and obtain elegant formulae for both the Levi-Civita connection and the curvature tensor.

5.2.2 The Levi-Civita connection

We start by stating the main result of this section. To do this, we recall some pieces of previously introduced notation. In the following, (N, g, I_k, Z, f_Z) will always denote a pseudo-hyper-Kähler manifold, endowed with an isometric circle action generated by an ω_1-Hamiltonian rotating Killing field Z whose Hamiltonian function is f_Z. The HK/QK correspondence further involves the twist data $(Z, \omega_{\mathrm{H}}, f_{\mathrm{H}})$. We also recall (cf. section 3.3) that, in this setting, we have a canonical endomorphism field $I_{\mathrm{H}} = I_1 + 2\nabla Z$, where ∇ is the Levi-Civita connection of (N, g). This endomorphism field satisfies $g(I_{\mathrm{H}}\cdot, \cdot) = \omega_{\mathrm{H}}$ and commutes with I_μ, $\mu = 0, 1, 2, 3$.

Theorem 5.25. *Let (N, g, I_k, Z, f_Z) be as above, and denote its image under the HK/QK correspondence by $(\bar{N}, g_{\mathrm{Q}})$, with Levi-Civita connection ∇^{Q}. Then we have $\nabla^{\mathrm{Q}} \sim_{\mathcal{H}} \nabla + S$, where*

$$S_A B = \frac{1}{2}\sum_{\mu=0}^{3}\left(\frac{1}{f_{\mathrm{H}}}g(I_\mu I_{\mathrm{H}} A, B)I_\mu Z - \frac{1}{f_Z}\big(\alpha_\mu(A)I_\mu I_1 B + \alpha_\mu(B)I_\mu I_1 A\big)\right) \tag{5.1}$$

for arbitrary vector fields A, B on N.

Remark 5.26. Note that, though our description in the previous section involves writing S as the sum of two independent parts S^{H} and S^{Q}, which arise through elementary deformation and twisting respectively, the two contributions have been thoroughly mixed up to obtain the final result. No particularly interesting simplifications arise when one treats elementary deformations and twisting separately, so we will not show explicitly how to do so.

Proof. In the following, we use the notation $\omega_0 = g$, $I_0 = \mathrm{id}$ and $\alpha_\mu = \iota_Z\omega_\mu$, $\mu = 0, 1, 2, 3$, freely. Summations over Greek indices are understood to run from 0 to 3, while Latin indices run from 1 to 3.

Our first observation is that the two correction terms S^{H} and S^{Q} display a similar structure. Indeed, regarding $\iota_Z g_{\mathrm{H}} \otimes \omega_{\mathrm{H}}$ and ∇g_{H} as linear maps $\Gamma(TN^{\otimes 3}) \to C^\infty(N)$, we can write

$$2g_{\mathrm{H}}((S^{\mathrm{H}}_A B + S^{\mathrm{Q}}_A B, C) = \left(\frac{1}{f_{\mathrm{H}}}\iota_Z g_{\mathrm{H}} \otimes \omega_{\mathrm{H}} - \nabla g_{\mathrm{H}}\right)\Big(C\otimes A\otimes B - A\otimes B\otimes C - B\otimes A\otimes C\Big)$$

where A, B, C are arbitrary vector fields. Multiplying by f_Z^2 for convenience and recognizing that $\frac{f_Z^2}{f_{\mathrm{H}}}\iota_Z g_{\mathrm{H}} = \iota_Z g = \alpha_0$ (cf. remark 5.21), we have

$$\begin{aligned}
\frac{f_Z^2}{f_{\mathrm{H}}}\iota_Z g_{\mathrm{H}} \otimes \omega_{\mathrm{H}} - f_Z^2\nabla g_{\mathrm{H}} &= \alpha_0 \otimes \omega_{\mathrm{H}} - f_Z^2\nabla\Big(\frac{1}{f_Z}g + \frac{1}{f_Z^2}g_\alpha\Big)\\
&= \alpha_0 \otimes \omega_{\mathrm{H}} + \mathrm{d}f_Z \otimes g + 2\frac{\mathrm{d}f_Z}{f_Z} \otimes g_\alpha - \nabla g_\alpha\\
&= \alpha_0 \otimes \omega_1 + \alpha_0 \otimes \mathrm{d}\alpha_0 - \alpha_1 \otimes \omega_0 - \frac{2}{f_Z}\alpha_1 \otimes g_\alpha - \nabla g_\alpha\\
&= \alpha_0 \otimes \omega_1 - \alpha_1 \otimes \omega_0 - \frac{2}{f_Z}\alpha_1 \otimes g_\alpha + 2\alpha_0 \otimes \omega_0(\nabla Z, \cdot) - \nabla g_\alpha
\end{aligned}$$

where we re-ordered terms and used (3.6) in the last step. Now we focus on the final two terms, which still feature derivatives. We start with the final term.

$$\begin{aligned}
(\nabla g_\alpha)(A\otimes B\otimes C) &= \sum_\mu (\nabla_A\alpha_\mu)(B)\alpha_\mu(C) + \alpha_\mu(B)(\nabla_A\alpha_\mu)(C)\\
&= \Big(\sum_\mu \alpha_\mu \otimes \omega_\mu(\nabla Z, \cdot)\Big)\Big(C\otimes A\otimes B + B\otimes A\otimes C\Big)
\end{aligned}$$

and thus, using (3.6) and the fact that Z generates a rotating circle symmetry, i.e. $L_Z g = 0$, $L_Z\omega_1 = 0$ while $L_Z\omega_2 = \omega_3$ and $L_Z\omega_3 = -\omega_2$, we find

$$\begin{aligned}
&\big(2\alpha_0 \otimes \omega_0(\nabla Z, \cdot) - \nabla g_\alpha\big)\big(C\otimes A\otimes B - A\otimes B\otimes C - B\otimes A\otimes C\big)\\
&\quad= \big(\alpha_0 \otimes \omega_0(\nabla Z, \cdot)\big)\Big(2C\otimes A\otimes B - B\otimes C\otimes A - A\otimes C\otimes B\\
&\qquad\qquad - 2A\otimes B\otimes C + B\otimes A\otimes C + C\otimes A\otimes B\\
&\qquad\qquad - 2B\otimes A\otimes C + A\otimes B\otimes C + C\otimes B\otimes A\Big)
\end{aligned}$$

$$
\begin{aligned}
&- \Big(\sum_{k=1}^{3} \alpha_k \otimes \omega_k(\nabla Z, \cdot)\Big)\Big(B \otimes C \otimes A + A \otimes C \otimes B - B \otimes A \otimes C \\
&\qquad\qquad\qquad\qquad - C \otimes A \otimes B - A \otimes B \otimes C - C \otimes B \otimes A\Big) \\
&= 2\alpha_0(C)\omega_0(\nabla_A Z, B) + 2\alpha_1(C)\omega_1(\nabla_A Z, B) \\
&\quad + 2\alpha_2(C)\omega_2(\nabla_A Z, B) - \alpha_2(C)\omega_3(A, B) + \alpha_2(A)\omega_3(B, C) + \alpha_2(B)\omega_3(A, C) \\
&\quad + 2\alpha_3(C)\omega_3(\nabla_A Z, B) + \alpha_3(C)\omega_2(A, B) - \alpha_3(A)\omega_2(B, C) - \alpha_3(B)\omega_2(A, C) \\
&= 2\sum_\mu \alpha_\mu \otimes \omega_\mu(\nabla Z, \cdot)(C \otimes A \otimes B) \\
&\quad - (\alpha_2 \otimes \omega_3 - \alpha_3 \otimes \omega_2)\big(C \otimes A \otimes B - A \otimes B \otimes C - B \otimes A \otimes C\big)
\end{aligned}
$$

We can plug this into our expression for $2f_Z^2 g_{\mathrm{H}}(S_A B, C)$, where $S = S^{\mathrm{H}} + S^{\mathrm{Q}}$ as before. Doing so yields

$$
\begin{aligned}
&2f_Z^2 g_{\mathrm{H}}(S_A B, C) \\
&\quad = 2\Big(\sum_\mu \alpha_\mu \otimes g(I_\mu \nabla Z, \cdot)\Big)(C \otimes A \otimes B) \\
&\quad + \Big(\sum_\mu \alpha_\mu \otimes g(I_\mu I_1 \cdot, \cdot) - \frac{2}{f_Z}\alpha_1 \otimes g_\alpha\Big)\big(C \otimes A \otimes B - A \otimes B \otimes C - B \otimes A \otimes C\big)
\end{aligned}
$$

Now we use the endomorphism $I_{\mathrm{H}} = I_1 + 2\nabla Z$, which allows us to write this as

$$
\begin{aligned}
&\Big(\sum_\mu \alpha_\mu \otimes g(I_\mu I_{\mathrm{H}} \cdot, \cdot) - \frac{2}{f_Z}\alpha_1 \otimes g_\alpha\Big)(C \otimes A \otimes B) \\
&\quad - \Big(\sum_\mu \alpha_\mu \otimes g(I_\mu I_1 \cdot, \cdot) - \frac{2}{f_Z}\alpha_1 \otimes g_\alpha\Big)\big(A \otimes B \otimes C + B \otimes A \otimes C\big) \\
&\quad = \sum_\mu \Big(g(I_\mu I_{\mathrm{H}} A, B) g(I_\mu Z, C) - \frac{2}{f_Z}\alpha_\mu(A)\alpha_\mu(B) g(I_1 Z, C) \\
&\quad - \alpha_\mu(A) g(I_\mu I_1 B, C) - \alpha_\mu(B) g(I_\mu I_1 A, C) \\
&\quad + \frac{2}{f_Z}\big(\alpha_1(A)\alpha_\mu(B) + \alpha_1(B)\alpha_\mu(A)\big) g(I_\mu Z, C)\Big) \\
&\quad = g\Big(\sum_\mu \Big(g(I_\mu I_{\mathrm{H}} A, B) I_\mu Z - \frac{2}{f_Z}\alpha_\mu(A)\alpha_\mu(B) I_1 Z - \alpha_\mu(A) I_\mu I_1 B - \alpha_\mu(B) I_\mu I_1 A \\
&\quad + \frac{2}{f_Z}\big(\alpha_1(A)\alpha_\mu(B) + \alpha_1(B)\alpha_\mu(A)\big) I_\mu Z\Big), C\Big)
\end{aligned}
$$

Our equation is now of the form $g_{\mathrm{H}}(2f_Z^2 S_A B, C) = g(X, C)$, where X is given by a complicated expression. By definition of the endomorphism $\mathcal{K}$, $\mathcal{K}(X)$ now yields $2f_Z^2 S_A B$. To compute it, we use the explicit expression for $\mathcal{K}$ given in the proof of lemma 5.22.

This leads us to the equation

$$\begin{aligned}
S_A B = \frac{1}{2f_{\mathrm{H}}} \sum_{\mu} g(I_\mu I_{\mathrm{H}} A, B) I_\mu Z - f_Z \sum_{\mu} \big(\alpha_\mu(A) I_\mu I_1 B + \alpha_\mu(B) I_\mu I_1 A\big) \\
+ \frac{1}{f_Z f_{\mathrm{H}}} \sum_{\mu} \big(\alpha_1(A)\alpha_\mu(B) I_\mu Z + \alpha_\mu(A)\alpha_1(B) I_\mu Z - \alpha_\mu(A)\alpha_\mu(B) I_1 Z\big) \\
+ \frac{1}{2 f_Z f_{\mathrm{H}}} \sum_{\mu,\lambda} \big(\alpha_\mu(A)\alpha_\lambda(I_\mu I_1 B) + \alpha_\mu(B)\alpha_\lambda(I_\mu I_1 A)\big) I_\lambda Z
\end{aligned}$$

We focus on the last line of the right-hand side expression. After replacing I_λ by $I_\lambda I_1$ and swapping the indices μ and λ for later convenience, we can write it as

$$\begin{aligned}
&- \frac{1}{2 f_Z f_{\mathrm{H}}} \sum_{\mu,\lambda} \big(g(I_\lambda I_1 Z, A) g(I_\lambda^{-1} I_\mu Z, B) + g(I_\lambda I_1 Z, B) g(I_\lambda^{-1} I_\mu Z, A)\big) I_\mu Z \\
&= \frac{1}{2 f_Z f_{\mathrm{H}}} \sum_{\mu,\lambda} \big(g(I_\lambda I_1 Z, A) g(I_\lambda I_\mu Z, B) + g(I_\lambda I_1 Z, B) g(I_\lambda I_\mu Z, A)\big) I_\mu Z \\
&\quad - \frac{1}{f_Z f_{\mathrm{H}}} \sum_{\mu} \big(\alpha_1(A)\alpha_\mu(B) + \alpha_1(B)\alpha_\mu(A)\big) I_\mu Z
\end{aligned}$$

where the equality follows by using $I_\lambda^{-1} = -I_\lambda$ for $\lambda = 1, 2, 3$, while $I_0^{-1} = I_0$. The single-summation term cancels out in our formula for $S_A B$, which reduces to

$$\begin{aligned}
S_A B = \frac{1}{2f_{\mathrm{H}}} \sum_{\mu} g(I_\mu I_{\mathrm{H}} A, B) I_\mu Z - \frac{1}{2f_Z} \sum_{\mu} \big(\alpha_\mu(A) I_\mu I_1 B + \alpha_\mu(B) I_\mu I_1 A\big) \\
- \frac{1}{f_Z f_{\mathrm{H}}} \sum_{\mu} \alpha_\mu(A)\alpha_\mu(B) I_1 Z \\
+ \frac{1}{2 f_Z f_{\mathrm{H}}} \sum_{\mu,\lambda} \big(g(I_\lambda I_1 Z, A) g(I_\lambda I_\mu Z, B) + g(I_\lambda I_1 Z, B) g(I_\lambda I_\mu Z, A)\big) I_\mu Z
\end{aligned}$$

We continue analyzing the last line. First, we replace I_μ by $I_\mu I_1$, and then consider each term for fixed μ. If $\mu = 0$, the first term is $\sum_\lambda g(I_\lambda I_1 Z, A) g(I_\lambda I_1 Z, B)$, which is clearly symmetric in (A, B), but for $\mu = k \neq 0$ the substitution $I_\lambda \mapsto I_\lambda I_k$ yields $\sum_\lambda g(I_\lambda I_1 Z, A) g(I_\lambda I_k I_1 Z, B) = -\sum_\lambda g(I_\lambda I_k I_1 Z, A) g(I_\lambda I_1 Z, B)$, showing anti-symmetry in (A, B). Thus, in the final line only the term with $\mu = 0$ is non-vanishing. This term cancels with the second line, so we are left with

$$S_A B = \frac{1}{2} \sum_{\mu} \left(\frac{1}{f_{\mathrm{H}}} g(I_\mu I_{\mathrm{H}} A, B) I_\mu Z - \frac{1}{f_Z} \big(\alpha_\mu(A) I) I_\mu I_1 B + \alpha_\mu(B) I_\mu I_1 A\big) \right)$$

which completes the proof of the theorem. □

5.2.3 The curvature tensor

Now that we have a more manageable expression for the tensor field S, we are ready to start working on the expression for the curvature tensor of the quaternionic Kähler metric. As in the previous section, we start by introducing our main results and some interesting observations about it. Our expression for the curvature tensor involves the operations $\owedge$ and $\obar$, introduced in definition 2.21, which allow us to produce algebraic curvature tensors out of symmetric bilinear forms and two-forms, respectively. We will also continue to use the notation and objects used in the previous section, such as the endomorphism field I_{H} and twist data $(Z, \omega_{\mathrm{H}}, f_{\mathrm{H}})$.

Theorem 5.27. *The Riemann curvature of the quaternionic Kähler metric g_{Q} on $\bar{N}$ is $\mathcal{H}$-related to $\tilde{R}$, which is given by the expression*

$$\begin{aligned} g_{\mathrm{H}}(\tilde{R}(A,B)C,X) = {} & \frac{1}{f_Z} g(R(A,B)C,X) \\ & + \frac{1}{8}\Big(g_{\mathrm{H}} \owedge g_{\mathrm{H}} + \sum_k g_{\mathrm{H}}(I_k\cdot,\cdot) \obar g_{\mathrm{H}}(I_k\cdot,\cdot)\Big)(A,B,C,X) \\ & - \frac{1}{8 f_Z f_{\mathrm{H}}}\Big(\omega_{\mathrm{H}} \obar \omega_{\mathrm{H}} + \sum_k \omega_{\mathrm{H}}(I_k\cdot,\cdot) \owedge \omega_{\mathrm{H}}(I_k\cdot,\cdot)\Big)(A,B,C,X) \end{aligned} \tag{5.2}$$

where R is the curvature tensor of the pseudo-hyper-Kähler metric on N, and A, B, C, X are arbitrary vector fields.

The first observation is that each term in the given expression is itself an algebraic curvature tensor: This follows from the facts that both g_{H} and $\omega_{\mathrm{H}}(I_k\cdot,\cdot)$ are symmetric bilinear forms—the latter because I_k commutes with I_{H} (by proposition 3.27)—and ω_{H} and $g_{\mathrm{H}}(I_k\cdot,\cdot)$ are two-forms since g_{H} is compatible with the almost quaternionic-Hermitian structure.

Secondly, let us compare theorem 5.27 to Alekseevsky's curvature decomposition theorem 2.23, which is valid for arbitrary quaternionic Kähler manifolds. Recalling the explicit expression for the curvature tensor of $\mathbb{HP}^n$ given in example 2.22, we see that the $\mathbb{HP}^n$-part of the curvature corresponds to the second line of (5.2) and read off that the reduced scalar curvature is $\nu = -1$. The following lemma reconfirms this:

Lemma 5.28. *The algebraic curvature tensor R_1 on $\bar{N}$, determined by the relation*

$$\begin{aligned} g_{\mathrm{Q}}(R_1(A,B)C,X) \sim_{\mathcal{H}} {} & \frac{1}{f_Z} g(R(A,B)C,X) \\ & - \frac{1}{8 f_Z f_{\mathrm{H}}}\Big(\omega_{\mathrm{H}} \obar \omega_{\mathrm{H}} + \sum_k \omega_{\mathrm{H}}(I_k\cdot,\cdot) \owedge \omega_{\mathrm{H}}(I_k\cdot,\cdot)\Big)(A,B,C,X) \end{aligned}$$

is of hyper-Kähler type.

Proof. First, we check that R_1 is trace-free by computing its "Ricci tensor". We may do this on the hyper-Kähler side, since the twist construction is compatible with tensor products and contractions. It is clear that the hyper-Kähler curvature tensor R is trace-free, so we need only verify this for the remaining terms. We will do so with respect to a local orthonormal frame which is adapted to the quaternionic structure, which means it is of the form $\{I_\mu e_i\}$, $i = 1, \ldots, \dim_{\mathbb{H}} N$, $\mu = 0, 1, 2, 3$:

$$
\begin{aligned}
\mathrm{Ric}_{R_1}(X,Y) &= \sum_{\mu=0}^{3} \sum_{i=1}^{\dim_{\mathbb{H}} N} \Big(\omega_{\mathrm{H}} \owedge \omega_{\mathrm{H}} + \sum_k \omega_{\mathrm{H}}(I_k\cdot,\cdot) \owedge \omega_{\mathrm{H}}(I_k\cdot,\cdot)\Big)(X, I_\mu e_i, I_\mu e_i, Y) \\
&= \sum_{\mu,i} \Big(6\omega_{\mathrm{H}}(X, I_\mu e_i)\omega_{\mathrm{H}}(I_\mu e_i, Y) \\
&\qquad + 2\sum_k \big(\omega_{\mathrm{H}}(I_k X, I_\mu e_i)\omega_{\mathrm{H}}(I_k I_\mu e_i, Y) - \omega_{\mathrm{H}}(I_k X, Y)\omega_{\mathrm{H}}(I_k I_\mu e_i, I_\mu e_i)\big)\Big) \\
&= \sum_{\mu,i} \Big(6\omega_{\mathrm{H}}(X, I_\mu e_i)\omega_{\mathrm{H}}(I_\mu e_i, Y) \\
&\qquad\quad - 2\sum_k \big(\omega_{\mathrm{H}}(X, I_\mu e_i)\omega_{\mathrm{H}}(I_\mu e_i, Y) + \omega_{\mathrm{H}}(I_k X, Y)\omega_{\mathrm{H}}(I_k I_\mu e_i, I_\mu e_i)\big)\Big) \\
&= -2\sum_{\mu,i,k} g(I_{\mathrm{H}} I_k X, Y) g(I_{\mathrm{H}} I_k I_\mu e_i, I_\mu e_i)
\end{aligned}
$$

where we substituted $I_k I_\mu$ for I_μ in the second term, and noted that it cancels with the first. Now we show that the remainder vanishes. Since $\sum_{\mu,i} g(I_{\mathrm{H}} I_k I_\mu e_i, I_\mu e_i) = \mathrm{tr}(I_{\mathrm{H}} I_k)$, it suffices to prove that $\mathrm{tr}(I_{\mathrm{H}} I_k) = 0$ for every $k \in \{1,2,3\}$. We may always write $I_k = I_i I_j$ for some $i, j \in \{1,2,3\}$. Cyclicity of the trace and the fact that I_{H} commutes with each I_k then leads to $\mathrm{tr}(I_{\mathrm{H}} I_i I_j) = \mathrm{tr}(I_j I_{\mathrm{H}} I_i) = -\mathrm{tr}(I_{\mathrm{H}} I_k)$. This proves that R_1 is trace-free.

We now need to check that R_1 commutes with any section of $\mathcal{Q}$. Since $\mathcal{Q}$ is preserved by the twisted circle action on $\bar{N}$, it admits a local invariant frame. Since we are concerned with a pointwise condition, it suffices to check the assertion on such a frame, which we may transfer (using the inverse twist construction) to the hyper-Kähler side. Since the condition is frame-independent, we may also work with respect to any other frame on the hyper-Kähler side, so it suffices to prove that the $\mathcal{H}$-related tensor field on N commutes with every I_k.

This claim is also trivial for the first term, because it is (up to scaling) the curvature tensor of the hyper-Kähler metric g. The remaining terms satisfy the following identity:

$$
\Big(\omega_{\mathrm{H}} \owedge \omega_{\mathrm{H}} + \sum_k \omega_{\mathrm{H}}(I_k\cdot,\cdot) \owedge \omega_{\mathrm{H}}(I_k\cdot,\cdot)\Big)(A, B, I_j C, I_j X)
$$

$$
\begin{aligned}
&= 4\omega_{\mathrm{H}}(A,B)\omega_{\mathrm{H}}(I_jC,I_jX)\\
&\quad + 2\sum_{\mu=0}^{3}\Big(\omega_{\mathrm{H}}(I_\mu A,I_jC)\omega_{\mathrm{H}}(I_\mu B,I_jX) - \omega_{\mathrm{H}}(I_\mu A,I_jX)\omega_{\mathrm{H}}(I_\mu B,I_jC)\Big)\\
&= 4\omega_{\mathrm{H}}(A,B)\omega_{\mathrm{H}}(C,X)\\
&\quad + 2\sum_{\mu=0}^{3}\Big(\omega_{\mathrm{H}}(I_jI_\mu A,C)\omega_{\mathrm{H}}(I_jI_\mu B,X) - \omega_{\mathrm{H}}(I_jI_\mu A,X)\omega_{\mathrm{H}}(I_jI_\mu B,C)\Big)\\
&= \Big(\omega_{\mathrm{H}} \owedge \omega_{\mathrm{H}} + \sum_k \omega_{\mathrm{H}}(I_k\cdot,\cdot) \owedge \omega_{\mathrm{H}}(I_k\cdot,\cdot)\Big)(A,B,C,X)
\end{aligned}
$$

where $j \in \{1,2,3\}$ and we used that I_{H} and I_j commute. The final step follows from the substitution $I_\mu \mapsto I_j^{-1}I_\mu$ in the summation. This identity actually implies our claim, since for every endomorphism field E and vector fields X and Y, $g([E,I_k]X,I_kY) = g(EI_kX,I_kY) - g(EX,Y)$. □

In conclusion, theorem 5.27 provides a refinement of Alekseevsky's decomposition theorem for quaternionic Kähler metrics arising from the HK/QK correspondence. More precisely, it gives an explicit expression for the trace-free part of the curvature, i.e. the quaternionic Weyl curvature. Of course, the interpretation of the curvature tensor in terms of hyper-Kähler data via the twist construction is another important novelty.

Proof of Theorem 5.27. The formula of lemma 5.24 contains three types of terms, which we will first compute individually. Though none of the resulting three expressions is particularly simple, massive cancellations take place when combining them.

Lemma 5.29. *For S as given in theorem 5.25, we have:*

$$
\begin{aligned}
&(\nabla_A S)_B C - (\nabla_B S)_A C\\
&= \frac{1}{2}\sum_\mu \Bigg[\frac{1}{f_{\mathrm{H}}^2}\big(\omega_{\mathrm{H}}(Z,A)\omega_\mu(I_{\mathrm{H}}B,C)I_\mu Z + \frac{1}{f_Z^2}\alpha_1(A)\big(g(I_\mu I_1 Z,B)I_\mu C + g(I_\mu I_1 Z,C)I_\mu B\big)\\
&\qquad - \frac{1}{f_{\mathrm{H}}^2}\omega_{\mathrm{H}}(Z,B)\omega_\mu(I_{\mathrm{H}}A,C)I_\mu Z - \frac{1}{f_Z^2}\alpha_1(B)\big(g(I_\mu I_1 Z,A)I_\mu C + g(I_\mu I_1 Z,C)I_\mu A\big)\\
&\qquad + \frac{1}{f_{\mathrm{H}}}\big(\omega_\mu(I_{\mathrm{H}}B,C)I_\mu\nabla_A Z - \omega_\mu(I_{\mathrm{H}}A,C)I_\mu\nabla_B Z + 2\omega_\mu(R(A,B)Z,C)I_\mu Z\big)\\
&\qquad + \frac{1}{2f_Z}\Big(\big(g(I_\mu I_1 I_{\mathrm{H}}A,C) + g(I_\mu A,C)\big)I_\mu B - \big(g(I_\mu I_1 I_{\mathrm{H}}B,C) + g(I_\mu B,C)\big)I_\mu A\Big)\\
&\qquad + \frac{1}{2f_Z}\big(\omega_\mu(A,B) - \omega_\mu(B,A)\big)I_\mu C\Bigg] - \frac{1}{2f_Z}\omega_{\mathrm{H}}(A,B)I_1 C
\end{aligned}
$$

for arbitrary vector fields A,B,C on N.

Proof. Let us start by computing $(\nabla_A S)_B C$; we will anti-symmetrize in (A, B) afterwards.

$$\begin{aligned}
&(\nabla_A S)_B C \\
&= \frac{1}{2}\sum_\mu \Big(-\frac{\mathrm{d}f_\mathrm{H}(A)}{f_\mathrm{H}^2} g(I_\mu I_\mathrm{H} B, C) I_\mu Z + \frac{2}{f_\mathrm{H}} g(I_\mu(\nabla_A(\nabla_B Z) - \nabla_{\nabla_A B} Z), C) I_\mu Z \\
&\qquad + \frac{1}{f_\mathrm{H}} g(I_\mu I_\mathrm{H} B, C) I_\mu \nabla_A Z + \frac{\mathrm{d}f_Z(A)}{f_Z^2}(\alpha_\mu(B) I_\mu I_1 C + \alpha_\mu(C) I_\mu I_1 B) \\
&\qquad - \frac{1}{f_Z}(g(I_\mu \nabla_A Z, B) I_\mu I_1 C + g(I_\mu \nabla_A Z, C) I_\mu I_1 B) \Big) \\
&= \frac{1}{2}\sum_\mu \Big(\frac{1}{f_\mathrm{H}^2} \omega_\mathrm{H}(Z, A) \omega_\mu(I_\mathrm{H} B, C) I_\mu Z \\
&\qquad + \frac{1}{f_Z^2} \alpha_1(A)(g(I_\mu I_1 Z, B) I_\mu C + g(I_\mu I_1 Z, C) I_\mu B) \\
&\qquad + \frac{1}{f_\mathrm{H}}(\omega_\mu(I_\mathrm{H} B, C) I_\mu \nabla_A Z + 2\omega_\mu(\nabla_A(\nabla_B Z) - \nabla_{\nabla_A B} Z, C)) I_\mu Z \\
&\qquad - \frac{1}{f_Z}\Big(\omega_\mu(\nabla_A Z, B) I_\mu I_1 C - \frac{1}{2} g(I_\mu I_1 (I_\mathrm{H} - I_1) A, C) I_\mu B\Big)\Big)
\end{aligned}$$

The second expression results from a number of standard manipulations and grouping terms according to their pre-factors.

After anti-symmetrizing in (A, B), some of the terms involving ∇Z can be simplified considerably. Firstly,

$$\nabla_A(\nabla_B Z) - \nabla_{\nabla_A B} Z - \nabla_B(\nabla_A Z) + \nabla_{\nabla_B A} Z = R(A, B) Z$$

The second simplification takes a little more work. Starting from equation (3.6), one easily verifies that

$$\omega_\mu(\nabla_A Z, B) - \omega_\mu(\nabla_B Z, A) = -\frac{1}{2}(\omega_\mu(I_1 A, B) - \omega_\mu(I_1 B, A)) + \delta_{\mu 0} \omega_\mathrm{H}(A, B)$$

where δ is the Kronecker delta symbol. Summing over μ, this yields, after substituting $I_\mu I_1$ by I_μ in the summation, the identity

$$\begin{aligned}
&\sum_\mu \big(\omega_\mu(\nabla_A Z, B) - \omega_\mu(\nabla_B Z, A)\big) I_\mu I_1 C \\
&\quad = -\frac{1}{2}\sum_\mu \big(\omega_\mu(A, B) - \omega_\mu(B, A)\big) I_\mu C + \omega_\mathrm{H}(A, B) I_1 C
\end{aligned} \tag{5.3}$$

Upon anti-symmetrizing our expression for $(\nabla_A S)_B C$ in (A, B) and applying the identity (5.3) to the term that appears first on the final line, one is left with the claimed expression for $(\nabla_A S)_B C - (\nabla_B S)_A C$. □

Lemma 5.30. *For S as in theorem 5.25, we have:*

$$\begin{aligned}
[S_A, S_B]C = \frac{1}{4}\sum_{\mu,\lambda}\Bigg(&\frac{1}{f_{\mathrm{H}}^2}\big(g(I_\mu I_{\mathrm{H}}A, Z)g(I_\lambda I_{\mathrm{H}}B, C) - g(I_\mu I_{\mathrm{H}}B, Z)g(I_\lambda I_{\mathrm{H}}A, C)\big)I_\lambda I_\mu Z \\
&+ \frac{1}{f_Z^2}\Big[\big(g(I_\mu I_1 Z, A)g(I_\lambda I_1 Z, B) - g(I_\mu I_1 Z, B)g(I_\lambda I_1 Z, A)\big)I_\mu I_\lambda C \\
&+ g(I_\mu I_1 Z, B)g(I_\lambda I_1 Z, C)I_\lambda I_\mu A - g(I_\mu I_1 Z, A)g(I_\lambda I_1 Z, C)I_\lambda I_\mu B\Big]\Bigg) \\
+ \frac{1}{4 f_Z f_{\mathrm{H}}}\sum_\mu\Big(&2g(I_{\mathrm{H}}A, B)g(I_\mu I_1 Z, C)I_\mu Z \\
&- g(Z,Z)\big(g(I_\mu I_{\mathrm{H}}B, C)I_\mu I_1 A - g(I_\mu I_{\mathrm{H}}A, C)I_\mu I_1 B\big)\Big)
\end{aligned}$$

Proof. Replacing I_μ by $I_\mu I_1$ in the second part of our formula for S, one obtains

$$S_A B = \frac{1}{2}\sum_\mu\left(\frac{1}{f_{\mathrm{H}}}g(I_\mu I_{\mathrm{H}}A, B)I_\mu Z + \frac{1}{f_Z}\big(g(I_\mu I_1 Z, A)I_\mu B + g(I_\mu I_1 Z, B)I_\mu A\big)\right)$$

and therefore

$$\begin{aligned}
S_A(S_B C) &= \frac{1}{2}\sum_\mu\Bigg(\frac{1}{f_{\mathrm{H}}}g(I_\mu I_{\mathrm{H}}A, S_B C)I_\mu Z \\
&\qquad + \frac{1}{f_Z}\big(g(I_\mu I_1 Z, A)I_\mu S_B C + g(I_\mu I_1 Z, S_B C)I_\mu A\big)\Bigg) \\
&= \frac{1}{4}\sum_{\mu,\lambda}\Bigg(\frac{1}{f_{\mathrm{H}}^2}g(I_\mu I_{\mathrm{H}}A, I_\lambda Z)g(I_\lambda I_{\mathrm{H}}B, C)I_\mu Z \\
&\qquad + \frac{1}{f_Z f_{\mathrm{H}}}\Big[\big(g(I_\mu I_{\mathrm{H}}A, I_\lambda C)g(I_\lambda I_1 Z, B) + g(I_\mu I_{\mathrm{H}}A, I_\lambda B)g(I_\lambda I_1 Z, C)\big)I_\mu Z \\
&\qquad + g(I_\mu I_1 Z, A)g(I_\lambda I_{\mathrm{H}}B, C)I_\mu I_\lambda Z + g(I_\mu I_1 Z, I_\lambda Z)g(I_\lambda I_{\mathrm{H}}B, C)I_\mu A\Big] \\
&\qquad + \frac{1}{f_Z^2}\Big[g(I_\mu I_1 Z, A)\big(g(I_\lambda I_1 Z, B)I_\mu I_\lambda C + g(I_\lambda I_1 Z, C)I_\mu I_\lambda B\big) \\
&\qquad\quad + \big(g(I_\mu I_1 Z, I_\lambda C)g(I_\lambda I_1 Z, B) + g(I_\mu I_1 Z, I_\lambda B)g(I_\lambda I_1 Z, C)\big)I_\mu A\Big]\Bigg)
\end{aligned}$$

Whenever we have a term of the form $g(I_\mu X, I_\lambda Y)I_\mu W$, we can rewrite it as follows:

$$\sum_\mu g(X, I_\mu^{-1}I_\lambda Y)I_\mu W = \sum_\mu g(X, I_\mu^{-1}Y)I_\lambda I_\mu W = \sum_\mu g(I_\mu X, Y)I_\lambda I_\mu W$$

After consistently doing this (and sometimes exchanging the labels μ and λ) we find

$$\begin{aligned}S_A(S_B C) = \frac{1}{4}\sum_{\mu,\lambda}\Bigg(&\frac{1}{f_{\mathrm{H}}^2} g(I_\mu I_{\mathrm{H}} A, Z) g(I_\lambda I_{\mathrm{H}} B, C) I_\lambda I_\mu Z \\ &+ \frac{1}{f_Z^2}\Big[g(I_\mu I_1 Z, A) g(I_\lambda I_1 Z, B) I_\mu I_\lambda C + g(I_\lambda I_1 Z, A) g(I_\mu I_1 Z, C) I_\lambda I_\mu B \\ &+ \big(g(I_\mu I_1 Z, C) g(I_\lambda I_1 Z, B) + g(I_\mu I_1 Z, B) g(I_\lambda I_1 Z, C)\big) I_\lambda I_\mu A\Big] \\ &+ \frac{1}{f_Z f_{\mathrm{H}}}\Big[\big(g(I_\mu I_{\mathrm{H}} A, C) g(I_\lambda I_1 Z, B) + g(I_\mu I_{\mathrm{H}} A, B) g(I_\lambda I_1 Z, C)\big) I_\lambda I_\mu Z \\ &+ g(I_\lambda I_1 Z, A) g(I_\mu I_{\mathrm{H}} B, C) I_\lambda I_\mu Z + g(I_\mu I_1 Z, Z) g(I_\lambda I_{\mathrm{H}} B, C) I_\lambda I_\mu A\Big]\Bigg)\end{aligned}$$

Consider the terms proportional to f_Z^{-2}. The sum of the second and third terms is symmetric in (A, B), hence it will disappear upon anti-symmetrization. The same argument works to get rid of the first and third terms proportional to $(f_Z f_{\mathrm{H}})^{-1}$, so we obtain:

$$\begin{aligned}[S_A, S_B]C = \frac{1}{4}\sum_{\mu,\lambda}\Bigg(&\frac{1}{f_{\mathrm{H}}^2}\big(g(I_\mu I_{\mathrm{H}} A, Z) g(I_\lambda I_{\mathrm{H}} B, C) - g(I_\mu I_{\mathrm{H}} B, Z) g(I_\lambda I_{\mathrm{H}} A, C)\big) I_\lambda I_\mu Z \\ &+ \frac{1}{f_Z^2}\Big[\big(g(I_\mu I_1 Z, A) g(I_\lambda I_1 Z, B) - g(I_\mu I_1 Z, B) g(I_\lambda I_1 Z, A)\big) I_\mu I_\lambda C \\ &+ g(I_\mu I_1 Z, B) g(I_\lambda I_1 Z, C) I_\lambda I_\mu A - g(I_\mu I_1 Z, A) g(I_\lambda I_1 Z, C) I_\lambda I_\mu B\Big] \\ &+ \frac{1}{f_Z f_{\mathrm{H}}}\Big[\big(g(I_\mu I_{\mathrm{H}} A, B) - g(I_\mu I_{\mathrm{H}} B, A)\big) g(I_\lambda I_1 Z, C) I_\lambda I_\mu Z \\ &\quad + g(I_\mu I_1 Z, Z)\big(g(I_\lambda I_{\mathrm{H}} B, C) I_\lambda I_\mu A - g(I_\lambda I_{\mathrm{H}} A, C) I_\lambda I_\mu B\big)\Big]\Bigg)\end{aligned}$$

We focus on the terms proportional to $(f_Z f_{\mathrm{H}})^{-1}$. $g(I_\mu I_{\mathrm{H}} A, B)$ is symmetric in (A, B) for $\mu = k \neq 0$ but anti-symmetric for $\mu = 0$. This implies that only the latter term yields a non-zero contribution. Furthermore, $g(I_\mu I_1 Z, Z) = 0$ unless $\mu = 1$. This means that

$$\begin{aligned}[S_A, S_B]C = \frac{1}{4}\sum_{\mu,\lambda}\Bigg(&\frac{1}{f_{\mathrm{H}}^2}\big(g(I_\mu I_{\mathrm{H}} A, Z) g(I_\lambda I_{\mathrm{H}} B, C) - g(I_\mu I_{\mathrm{H}} B, Z) g(I_\lambda I_{\mathrm{H}} A, C)\big) I_\lambda I_\mu Z \\ &+ \frac{1}{f_Z^2}\Big[\big(g(I_\mu I_1 Z, A) g(I_\lambda I_1 Z, B) - g(I_\mu I_1 Z, B) g(I_\lambda I_1 Z, A)\big) I_\mu I_\lambda C \\ &+ g(I_\mu I_1 Z, B) g(I_\lambda I_1 Z, C) I_\lambda I_\mu A - g(I_\mu I_1 Z, A) g(I_\lambda I_1 Z, C) I_\lambda I_\mu B\Big]\Bigg) \\ + \frac{1}{4 f_Z f_{\mathrm{H}}}\sum_{\mu}\Big(&2 g(I_{\mathrm{H}} A, B) g(I_\mu I_1 Z, C) I_\mu Z \\ &- g(Z, Z)\big(g(I_\mu I_{\mathrm{H}} B, C) I_\mu I_1 A - g(I_\mu I_{\mathrm{H}} A, C) I_\mu I_1 B\big)\Big)\end{aligned}$$

which was the claim. □

These first two lemmata determine the curvature R^D of $D = \nabla + S$. The final term is rather simple in comparison:

Lemma 5.31. *For S as in theorem 5.25, the $(1,1)$-tensor field $\nabla Z + S_Z$ is given by the following expression:*

$$\nabla_A Z + S_Z A = \frac{1}{2}\left(I_{\mathrm{H}} - \frac{f_{\mathrm{H}}}{f_Z} I_1\right) A + \frac{1}{2}\sum_{\mu}\left(\frac{1}{f_{\mathrm{H}}} g(I_\mu I_{\mathrm{H}} Z, A) + \frac{1}{f_Z} g(I_\mu I_1 Z, A)\right) I_\mu Z$$

Proof. We directly compute:

$$\begin{aligned}
\nabla_A Z + S_Z A &= \frac{1}{2}(I_{\mathrm{H}} - I_1)A \\
&\quad + \frac{1}{2}\sum_{\mu}\left(\frac{1}{f_{\mathrm{H}}} g(I_\mu I_{\mathrm{H}} Z, A) I_\mu Z + \frac{1}{f_Z}\big(g(I_\mu I_1 Z, Z) I_\mu A + g(I_\mu I_1 Z, A) I_\mu Z\big)\right) \\
&= \frac{1}{2}(I_{\mathrm{H}} - I_1)A - \frac{g(Z,Z)}{2 f_Z} I_1 A \\
&\quad + \frac{1}{2}\sum_{\mu}\left(\frac{1}{f_{\mathrm{H}}} g(I_\mu I_{\mathrm{H}} Z, A) + \frac{1}{f_Z} g(I_\mu I_1 Z, A)\right) I_\mu Z \\
&= \frac{1}{2}\left(I_{\mathrm{H}} - \frac{f_{\mathrm{H}}}{f_Z} I_1\right) A + \frac{1}{2}\sum_{\mu}\left(\frac{1}{f_{\mathrm{H}}} g(I_\mu I_{\mathrm{H}} Z, A) + \frac{1}{f_Z} g(I_\mu I_1 Z, A)\right) I_\mu Z
\end{aligned}$$

The second step uses the fact that $g(I_\mu I_1 Z, Z)$ vanishes unless $\mu = 1$. To then arrive at the claimed expression, we use $f_{\mathrm{H}} = f_Z + g(Z,Z)$. □

Taking the tensor product with $-\frac{1}{f_{\mathrm{H}}}\omega_{\mathrm{H}}$, this determines the final term.

Now we put the results of lemmata 5.29 to 5.31 together. Grouping terms according to their pre-factors, the result (after a few immediate cancellations) is:

$$\begin{aligned}
&\tilde{R}(A,B)C - R(A,B)C \\
&\quad = (\nabla_A S)_B C - (\nabla_B S)_A C + [S_A, S_B]C - \frac{1}{f_{\mathrm{H}}}\omega_{\mathrm{H}}(A,B)(\nabla_C Z + S_Z C)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{4f_{\mathrm{H}}^2}\Bigg(2\sum_\mu \Big[\big(\omega_{\mathrm{H}}(Z,A)\omega_\mu(I_{\mathrm{H}}B,C) - \omega_{\mathrm{H}}(Z,B)\omega_\mu(I_{\mathrm{H}}A,C)\big)I_\mu Z \\
&\qquad\qquad - \omega_{\mathrm{H}}(A,B)g(I_\mu I_{\mathrm{H}}Z,C)I_\mu Z\Big] \\
&\qquad + \sum_{\mu,\lambda}\big(g(I_\mu I_{\mathrm{H}}A,Z)g(I_\lambda I_{\mathrm{H}}B,C) - g(I_\mu I_{\mathrm{H}}B,Z)g(I_\lambda I_{\mathrm{H}}A,C)\big)I_\lambda I_\mu Z\Bigg) \\
&\quad + \frac{1}{4f_Z^2}\Bigg(2\sum_\mu\Big[\alpha_1(A)\big(g(I_\mu I_1 Z,B)I_\mu C + g(I_\mu I_1 Z,C)I_\mu B\big) \\
&\qquad\qquad - \alpha_1(B)\big(g(I_\mu I_1 Z,A)I_\mu C + g(I_\mu I_1 Z,C)I_\mu A\big)\Big] \\
&\qquad + \sum_{\mu,\lambda}\Big[\big(g(I_\mu I_1 Z,A)g(I_\lambda I_1 Z,B) - g(I_\mu I_1 Z,B)g(I_\lambda I_1 Z,A)\big)I_\mu I_\lambda C \\
&\qquad\qquad + g(I_\mu I_1 Z,B)g(I_\lambda I_1 Z,C)I_\lambda I_\mu A - g(I_\mu I_1 Z,A)g(I_\lambda I_1 Z,C)I_\lambda I_\mu B\Big]\Bigg) \\
&\quad - \frac{1}{4f_Z f_{\mathrm{H}}}\Bigg(g(Z,Z)\sum_\mu\Big[g(I_\mu I_{\mathrm{H}}B,C)I_\mu I_1 A - g(I_\mu I_{\mathrm{H}}A,C)I_\mu I_1 B\Big]\Bigg) \\
&\quad + \frac{1}{2f_{\mathrm{H}}}\Bigg(\sum_\mu\Big[\omega_\mu(I_{\mathrm{H}}B,C)I_\mu\nabla_A Z - \omega_\mu(I_{\mathrm{H}}A,C)I_\mu\nabla_B Z + 2\omega_\mu(R(A,B)Z,C)I_\mu Z\Big] \\
&\qquad - \omega_{\mathrm{H}}(A,B)I_{\mathrm{H}}C\Bigg) \\
&\quad + \frac{1}{4f_Z}\Bigg(\sum_\mu\Big[\big(g(I_\mu I_1 I_{\mathrm{H}}A,C) + g(I_\mu A,C)\big)I_\mu B - \big(g(I_\mu I_1 I_{\mathrm{H}}B,C) + g(I_\mu B,C)\big)I_\mu A \\
&\qquad\qquad + \big(\omega_\mu(A,B) - \omega_\mu(B,A)\big)I_\mu C\Big]\Bigg)
\end{aligned}
$$

Using the identities $\frac{g(Z,Z)}{f_Z f_{\mathrm{H}}} = \frac{1}{f_Z} - \frac{1}{f_{\mathrm{H}}}$ and $I_1 = I_{\mathrm{H}} - 2\nabla Z$, we can rewrite

$$
-\frac{g(Z,Z)}{4f_Z f_{\mathrm{H}}}\sum_\mu\big(g(I_\mu I_{\mathrm{H}}B,C)I_\mu I_1 A - g(I_\mu I_{\mathrm{H}}A,C)I_\mu I_1 B\big)
$$

as

$$
\begin{aligned}
&\frac{1}{4f_{\mathrm{H}}}\sum_\mu\big(\omega_\mu(I_{\mathrm{H}}B,C)I_\mu(I_{\mathrm{H}}A - 2\nabla_A Z) - \omega_\mu(I_{\mathrm{H}}A,C)I_\mu(I_{\mathrm{H}}B - 2\nabla_B Z)\big) \\
&\quad + \frac{1}{4f_Z}\sum_\mu\big(g(I_\mu I_1 I_{\mathrm{H}}B,C)I_\mu A - g(I_\mu I_1 I_{\mathrm{H}}A,C)I_\mu B\big)
\end{aligned}
$$

Plugging this back into our main expression for $\tilde{R} - R$, this expressions is almost completely canceled by pre-existing terms, leaving us with

$$
\begin{aligned}
&\tilde{R}(A,B)C - R(A,B)C \\
&\quad= \frac{1}{4f_{\mathrm{H}}^2}\Bigg(2\sum_{\mu}\Big[\big(\omega_{\mathrm{H}}(Z,A)\omega_\mu(I_{\mathrm{H}}B,C) - \omega_{\mathrm{H}}(Z,B)\omega_\mu(I_{\mathrm{H}}A,C)\big)I_\mu Z \\
&\qquad\qquad - \omega_{\mathrm{H}}(A,B)g(I_\mu I_{\mathrm{H}}Z,C)I_\mu Z\Big] \\
&\qquad + \sum_{\mu,\lambda}\big(g(I_\mu I_{\mathrm{H}}A,Z)g(I_\lambda I_{\mathrm{H}}B,C) - g(I_\mu I_{\mathrm{H}}B,Z)g(I_\lambda I_{\mathrm{H}}A,C)\big)I_\lambda I_\mu Z\Bigg) \\
&\quad + \frac{1}{4f_Z^2}\Bigg(2\sum_{\mu}\Big[\alpha_1(A)\big(g(I_\mu I_1 Z,B)I_\mu C + g(I_\mu I_1 Z,C)I_\mu B\big) \\
&\qquad\qquad - \alpha_1(B)\big(g(I_\mu I_1 Z,A)I_\mu C + g(I_\mu I_1 Z,C)I_\mu A\big)\Big] \\
&\qquad + \sum_{\mu,\lambda}\Big[\big(g(I_\mu I_1 Z,A)g(I_\lambda I_1 Z,B) - g(I_\mu I_1 Z,B)g(I_\lambda I_1 Z,A)\big)I_\mu I_\lambda C \\
&\qquad\qquad + g(I_\mu I_1 Z,B)g(I_\lambda I_1 Z,C)I_\lambda I_\mu A - g(I_\mu I_1 Z,A)g(I_\lambda I_1 Z,C)I_\lambda I_\mu B\Big]\Bigg) \\
&\quad + \frac{1}{4f_{\mathrm{H}}}\Bigg(\sum_{\mu}\Big[\omega_\mu(I_{\mathrm{H}}B,C)I_\mu I_{\mathrm{H}}A - \omega_\mu(I_{\mathrm{H}}A,C)I_\mu I_{\mathrm{H}}B + 4\omega_\mu(R(A,B)Z,C)I_\mu Z\Big] \\
&\qquad - 2\omega_{\mathrm{H}}(A,B)I_{\mathrm{H}}C\Bigg) \\
&\quad + \frac{1}{4f_Z}\Bigg(\sum_{\mu}\Big[g(I_\mu A,C)I_\mu B - g(I_\mu B,C)I_\mu A + \big(\omega_\mu(A,B) - \omega_\mu(B,A)\big)I_\mu C\Big]\Bigg)
\end{aligned}
$$

Now we attend to the terms proportional to f_{H}^{-2}. Upon splitting off the case $\lambda = 0$ in the terms featuring double summation, a simplification occurs:

$$
\begin{aligned}
&2\sum_{\mu}\Big[\omega_{\mathrm{H}}(Z,A)\omega_\mu(I_{\mathrm{H}}B,C) - \omega_{\mathrm{H}}(Z,B)\omega_\mu(I_{\mathrm{H}}A,C) - \omega_{\mathrm{H}}(A,B)\omega_\mu(I_{\mathrm{H}}Z,C)\Big]I_\mu Z \\
&\quad + \sum_{\mu,\lambda}\big(g(I_\lambda I_{\mathrm{H}}A,Z)g(I_\mu I_{\mathrm{H}}B,C) - g(I_\lambda I_{\mathrm{H}}B,Z)g(I_\mu I_{\mathrm{H}}A,C)\big)I_\mu I_\lambda Z \\
&\quad = \sum_{\mu}\Big[\omega_{\mathrm{H}}(Z,A)\omega_\mu(I_{\mathrm{H}}B,C) - \omega_{\mathrm{H}}(Z,B)\omega_\mu(I_{\mathrm{H}}A,C) - 2\omega_{\mathrm{H}}(A,B)\omega_\mu(I_{\mathrm{H}}Z,C)\Big]I_\mu Z \\
&\qquad + \sum_{\mu}\sum_{k=1}^{3}\big(g(I_k I_{\mathrm{H}}Z,A)\omega_\mu(I_{\mathrm{H}}B,C) - g(I_k I_{\mathrm{H}}Z,B)\omega_\mu(I_{\mathrm{H}}A,C)\big)I_\mu I_k Z
\end{aligned}
$$

$$
\begin{aligned}
&= -2\sum_{\mu} \omega_{\mathrm{H}}(A,B)\omega_\mu(I_{\mathrm{H}}Z,C)I_\mu Z \\
&\quad + \sum_{\mu,\lambda} \big(\omega_\lambda(I_{\mathrm{H}}Z,A)\omega_\mu(I_{\mathrm{H}}B,C) - \omega_\lambda(I_{\mathrm{H}}Z,B)\omega_\mu(I_{\mathrm{H}}A,C)\big) I_\mu I_\lambda Z
\end{aligned}
$$

The second group of terms, consisting of those proportional to f_Z^{-2}, can be treated by the same methods to obtain

$$
\begin{aligned}
&2\sum_{\mu} \Big[\alpha_1(A)g(I_\mu I_1 Z,B) - \alpha_1(B)g(I_\mu I_1 Z,A)\big) I_\mu C \\
&\qquad + \alpha_1(A)g(I_\mu I_1 Z,C)I_\mu B - \alpha_1(B)g(I_\mu I_1 Z,C)I_\mu A\big)\Big] \\
&\quad + \sum_{\mu,\lambda} \Big[\big(g(I_\lambda I_1 Z,A)g(I_\mu I_1 Z,B) - g(I_\lambda I_1 Z,B)g(I_\mu I_1 Z,A)\big) I_\lambda I_\mu C \\
&\qquad + g(I_\lambda I_1 Z,B)g(I_\mu I_1 Z,C)I_\mu I_\lambda A - g(I_\lambda I_1 Z,A)g(I_\mu I_1 Z,C)I_\mu I_\lambda B\Big] \\
&\quad = \sum_{\mu,\lambda} \Big[\big(g(I_\mu I_1 Z,B)g(I_1 Z,I_\lambda A) - g(I_\mu I_1 Z,A)g(I_1 Z,I_\lambda B)\big) I_\mu I_\lambda C \\
&\qquad + g(I_1 Z,I_\lambda A)g(I_\mu I_1 Z,C)I_\mu I_\lambda B - g(I_1 Z,I_\lambda B)g(I_\mu I_1 Z,C)I_\mu I_\lambda A\Big]
\end{aligned}
$$

But further simplification of these terms can achieved. First substituting I_λ by I_λ^{-1} and then I_μ by $I_\mu I_\lambda$, we find

$$
\begin{aligned}
&\sum_{\mu,\lambda} \Big[g(I_\mu I_\lambda I_1 Z,B)g(I_1 Z,I_\lambda^{-1}A) - g(I_\mu I_\lambda I_1 Z,A)g(I_1 Z,I_\lambda^{-1}B)\big) I_\mu C \\
&\qquad + g(I_1 Z,I_\lambda^{-1}A)g(I_\mu I_\lambda I_1 Z,C)I_\mu B - g(I_1 Z,I_\lambda^{-1}B)g(I_\mu I_\lambda I_1 Z,C)I_\mu A\Big] \\
&\quad = \sum_{\mu,\lambda} \Big[g(I_\lambda I_1 Z,I_\mu^{-1}B)g(I_\lambda I_1 Z,A) - g(I_\lambda I_1 Z,I_\mu^{-1}A)g(I_\lambda I_1 Z,B)\big) I_\mu C \\
&\qquad + g(I_\lambda I_1 Z,A)g(I_\lambda I_1 Z,I_\mu^{-1}C)I_\mu B - g(I_\lambda I_1 Z,B)g(I_\lambda I_1 Z,I_\mu^{-1}C)I_\mu A\Big] \\
&= \sum_{\mu} \Big[\big(g_\alpha(I_\mu^{-1}B,A) - g_\alpha(I_\mu^{-1}A,B)\big) I_\mu C + g_\alpha(A,I_\mu^{-1}C)I_\mu B - g_\alpha(B,I_\mu^{-1}C)I_\mu A\Big]
\end{aligned}
$$

where the last step follows from the substitution $I_\lambda \mapsto I_\lambda I_1^{-1}$.

Returning to our main expression, we have now shown that

$$\begin{aligned}
&\tilde{R}(A,B)C - R(A,B)C \\
&\quad= \frac{1}{4f_{\mathrm{H}}^2}\Big(\sum_{\mu,\lambda}\big(\omega_\lambda(I_{\mathrm{H}}Z,A)\omega_\mu(I_{\mathrm{H}}B,C) - \omega_\lambda(I_{\mathrm{H}}Z,B)\omega_\mu(I_{\mathrm{H}}A,C)\big)I_\mu I_\lambda Z \\
&\qquad\qquad - 2\sum_\mu \omega_{\mathrm{H}}(A,B)\omega_\mu(I_{\mathrm{H}}Z,C)I_\mu Z\Big) \\
&\qquad + \frac{1}{4f_{\mathrm{H}}}\Big(\sum_\mu \Big[\omega_\mu(I_{\mathrm{H}}B,C)I_\mu I_{\mathrm{H}}A - \omega_\mu(I_{\mathrm{H}}A,C)I_\mu I_{\mathrm{H}}B + 4\omega_\mu(R(A,B)Z,C)I_\mu Z\Big] \\
&\qquad\qquad - 2\omega_{\mathrm{H}}(A,B)I_{\mathrm{H}}C\Big) \\
&\qquad + \frac{1}{4f_Z^2}\sum_\mu \Big[\big(g_\alpha(B,I_\mu A) - g_\alpha(A,I_\mu B)\big)I_\mu C + g_\alpha(I_\mu A,C)I_\mu B - g_\alpha(I_\mu B,C)I_\mu A\Big] \\
&\qquad + \frac{1}{4f_Z}\sum_\mu \Big[g(I_\mu A,C)I_\mu B - g(I_\mu B,C)I_\mu A + \big(\omega_\mu(A,B) - \omega_\mu(B,A)\big)I_\mu C\Big] \\
&\quad= \frac{1}{4f_{\mathrm{H}}^2}\Big(\sum_{\mu,\lambda}\big(\omega_\mu(I_{\mathrm{H}}A,C)\alpha_\lambda(I_{\mathrm{H}}B) - \omega_\mu(I_{\mathrm{H}}B,C)\alpha_\lambda(I_{\mathrm{H}}A)\big)I_\mu I_\lambda Z \\
&\qquad\qquad + 2\sum_\mu \omega_{\mathrm{H}}(A,B)\alpha_\mu(I_{\mathrm{H}}C)I_\mu Z\Big) \\
&\qquad + \frac{1}{4f_{\mathrm{H}}}\Big(\sum_\mu \Big[\omega_\mu(I_{\mathrm{H}}B,C)I_\mu I_{\mathrm{H}}A - \omega_\mu(I_{\mathrm{H}}A,C)I_\mu I_{\mathrm{H}}B - 4\alpha_\mu(R(A,B)C)I_\mu Z\Big] \\
&\qquad\qquad - 2\omega_{\mathrm{H}}(A,B)I_{\mathrm{H}}C\Big) \\
&\qquad + \frac{1}{4}\sum_\mu \Big[g_{\mathrm{H}}(I_\mu A,C)I_\mu B - g_{\mathrm{H}}(I_\mu B,C)I_\mu A + \big(g_{\mathrm{H}}(I_\mu A,B) - g_{\mathrm{H}}(I_\mu B,A)\big)I_\mu C\Big]
\end{aligned}$$

where we combined the terms proportional to f_Z^{-1} and f_Z^{-2}, and used that $R(A,B)$ and I_{H} both commute with each I_μ. Using $\frac{1}{f_Z}\mathcal{K}(X) = X - \frac{1}{f_{\mathrm{H}}}\sum_\mu \alpha_\mu(A)I_\mu Z$, this can be written as

$$\begin{aligned}
&\tilde{R}(A,B)C \\
&\quad= \frac{1}{f_Z}\mathcal{K}(R(A,B)C) - \frac{1}{2f_Z f_{\mathrm{H}}}\omega_{\mathrm{H}}(A,B)\mathcal{K}(I_{\mathrm{H}}C) \\
&\qquad + \frac{1}{4f_Z f_{\mathrm{H}}}\sum_\mu \big(\omega_\mu(I_{\mathrm{H}}B,C)\mathcal{K}(I_{\mathrm{H}}I_\mu A) - \omega_\mu(I_{\mathrm{H}}A,C)\mathcal{K}(I_{\mathrm{H}}I_\mu B)\big) \\
&\qquad + \frac{1}{4}\sum_\mu \Big[g_{\mathrm{H}}(I_\mu A,C)I_\mu B - g_{\mathrm{H}}(I_\mu B,C)I_\mu A + \big(g_{\mathrm{H}}(I_\mu A,B) - g_{\mathrm{H}}(I_\mu B,A)\big)I_\mu C\Big]
\end{aligned}$$

To finish the proof, we contract with another arbitrary vector field X, use the defining property of $\mathcal{K}$ and separate the case $\mu = 0$ in the terms featuring a summation:

$$\begin{aligned}
&g_{\mathrm{H}}(\tilde{R}(A,B)C,X) \\
&\quad= \frac{1}{f_Z} g(R(A,B)C,X) - \frac{1}{2f_Z f_{\mathrm{H}}} \omega_{\mathrm{H}}(A,B)\omega_{\mathrm{H}}(C,X) \\
&\qquad+ \frac{1}{4f_Z f_{\mathrm{H}}} \sum_{\mu=0}^{3} \Big[\omega_{\mathrm{H}}(I_\mu B,C)\omega_{\mathrm{H}}(I_\mu A,X) - \omega_{\mathrm{H}}(I_\mu A,C)\omega_{\mathrm{H}}(I_\mu B,X)\Big] \\
&\qquad+ \frac{1}{4} \sum_{\mu=0}^{3} \Big[g_{\mathrm{H}}(I_\mu A,C)g_{\mathrm{H}}(I_\mu B,X) - g_{\mathrm{H}}(I_\mu B,C)g_{\mathrm{H}}(I_\mu A,X)\Big] \\
&\qquad+ \frac{1}{2} \sum_{k=1}^{3} g_{\mathrm{H}}(I_k A,B)g_{\mathrm{H}}(I_k C,X) \\
&\quad= \frac{1}{f_Z} g(R(A,B)C,X) \\
&\qquad+ \frac{1}{4f_Z f_{\mathrm{H}}} \Big[\omega_{\mathrm{H}}(B,C)\omega_{\mathrm{H}}(A,X) - \omega_{\mathrm{H}}(A,C)\omega_{\mathrm{H}}(B,X) - 2\omega_{\mathrm{H}}(A,B)\omega_{\mathrm{H}}(C,X)\Big] \\
&\qquad+ \frac{1}{4f_Z f_{\mathrm{H}}} \sum_{k=1}^{3} \Big[\omega_{\mathrm{H}}(I_k B,C)\omega_{\mathrm{H}}(I_k A,X) - \omega_{\mathrm{H}}(I_k A,C)\omega_{\mathrm{H}}(I_k B,X)\Big] \\
&\qquad+ \frac{1}{4}\big(g_{\mathrm{H}}(A,C)g_{\mathrm{H}}(B,X) - g_{\mathrm{H}}(B,C)g_{\mathrm{H}}(A,X)\big) \\
&\qquad+ \frac{1}{4} \sum_{k=1}^{3} \Big[g_{\mathrm{H}}(I_k A,C)g_{\mathrm{H}}(I_k B,X) - g_{\mathrm{H}}(I_k B,C)g_{\mathrm{H}}(I_k A,X) + 2g_{\mathrm{H}}(I_k A,B)g_{\mathrm{H}}(I_k C,X)\Big] \\
&\quad= \frac{1}{f_Z} g(R(A,B)C,X) \\
&\qquad- \frac{1}{8f_Z f_{\mathrm{H}}} \Big(\omega_{\mathrm{H}} \varobar \omega_{\mathrm{H}} + \sum_k \omega_{\mathrm{H}}(I_k\cdot,\cdot) \owedge \omega_{\mathrm{H}}(I_k\cdot,\cdot)\Big)(A,B,C,X) \\
&\qquad+ \frac{1}{8}\Big(g_{\mathrm{H}} \owedge g_{\mathrm{H}} + \sum_k g_{\mathrm{H}}(I_k\cdot,\cdot) \varobar g_{\mathrm{H}}(I_k\cdot,\cdot)\Big)(A,B,C,X)
\end{aligned}$$

This, finally, is the sought-after expression given in the statement of the theorem. □

5.3 Application to examples

To illustrate our results, we return to the series of examples $(\bar{N}_n, g_n^c)$ considered in section 4.5. Applying the (deformed) c-map to $\mathbb{C}\mathrm{H}^n$ (and passing to the universal covering), we obtain a one-parameter family of metrics which we interpret as a deformation of the symmetric space $\frac{\mathrm{SU}(n+1,2)}{\mathrm{S}(\mathrm{U}(n+1)\times\mathrm{U}(2))}$. Since $\mathbb{C}\mathrm{H}^n$ itself is the symmetric space $\frac{\mathrm{SU}(n,1)}{\mathrm{U}(n)}$, we have a transitive action of $\mathrm{SU}(n,1)$ by isometries. In fact, $\mathrm{SU}(n,1)$ acts by PSK automorphisms. Indeed, the pseudo-Kähler structure on the corresponding CASK manifold M_n is given by restricting the constant pseudo-Kähler structure of signature $(n,1)$ on $\mathbb{C}^{n+1}$, and therefore preserved by $\mathrm{SU}(n,1)$. Moreover, the special Kähler connection coincides with the Levi-Civita connection in this case, so it is automatically preserved, and $\mathrm{SU}(n,1)$ certainly commutes with the standard $\mathbb{C}^*$-action. We conclude that $\mathbb{C}\mathrm{H}^n$ is a homogeneous PSK manifold.

5.3.1 Lifting automorphisms

Section 5.1 gives us a canonical procedure to turn elements of the algebra $\mathfrak{su}(n,1)$ of infinitesimal automorphisms into Killing fields on $(\bar{N}_n, g_n^c)$. We will now go through this procedure explicitly. First, let us set up some convenient notation.

Given $A \in \mathfrak{su}(n,1)$, the holomorphic part of the corresponding vector field X_A on $\mathbb{C}^{n+1}$ is $X_A^{1,0}(z) = \frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0} e^{tA} \cdot z = A \cdot z$, where z is regarded as a complex vector to which we can apply A, and we have identified $T_z^{1,0}\mathbb{C}^{n+1} \cong \mathbb{C}^{n+1}$. Under this identification, we think of $X^{1,0}$ as a $\mathbb{C}^{n+1}$-valued function. The (real) vector field X_A is of course recovered via the formula $X_A = X_A^{1,0} + \overline{X_A^{1,0}} = 2\,\mathrm{Re}(X_A^{1,0})$. Let $h = -\mathrm{d}z_0 \otimes \mathrm{d}\bar{z}_0 + \sum_j \mathrm{d}z_j \otimes \mathrm{d}\bar{z}_j$ be the Hermitian form which governs the pseudo-Kähler structure on M_n. For two arbitrary vector fields X, Y, we have $h(X,Y) = \langle X^{1,0}, Y^{1,0} \rangle$, where $\langle \cdot, \cdot \rangle$ is the corresponding Hermitian scalar product on $\mathbb{C}^{n+1}$. It will be convenient in the following computations to think in these terms, rather than in terms of tensor fields.

We know that the one-parameter group of automorphisms generated by $A \in \mathfrak{su}(n,1)$ is Hamiltonian with respect to the Kähler form on M_n, which corresponds to $-\mathrm{Im}\langle \cdot, \cdot \rangle$. In these terms, the Hamiltonian is $\mu^A(z) = \frac{1}{2}\,\mathrm{Im}\langle A \cdot z, z \rangle$ (compare with example 2.42). Translating back to the language of tensor fields, where we find $\mu^A = -\frac{1}{2}\omega_M(X_A, \xi)$, with ξ the Euler field, we also see that it is consistent with the expression derived in the proof of lemma 4.9.

Now we pass to the hyper-Kähler manifold $N_n = M_n \times \mathbb{C}^{n+1}$. The pseudo-Kähler structure defined by ω_1 is induced by the Hermitian form $h \oplus h$. We will denote the corresponding Hermitian scalar product of signature $(2n, 2)$ on $\mathbb{C}^{2n+2} \cong \mathbb{C}^{n+1} \oplus \mathbb{C}^{n+1}$ by $\langle \cdot, \cdot \rangle$, as well as its restrictions to each factor. Since they agree, no confusion should arise from this abuse of notation. We need to compute the twisting form $\omega_{\mathrm{H}} = \omega_1 + \mathrm{d}\alpha_0$. A first observation is that, under the identification $T_{(z,w)}^{1,0} N_n \cong \mathbb{C}^{n+1} \oplus \mathbb{C}^{n+1}$, $Z^{1,0} = (-iz, 0)$. For an arbitrary vector field $U = U^{1,0} + \overline{U^{1,0}}$, where we can think of $U^{1,0}$ as the $\mathbb{C}^{n+1} \oplus \mathbb{C}^{n+1}$-valued function $(U_z^{1,0}, U_w^{1,0})$, this formula for $Z^{1,0}$ yields $(\nabla_U Z)^{1,0} = (-iU_z^{1,0}, 0)$ and therefore we find

$$\mathrm{d}\alpha_0(U,V) = 2g(\nabla_U Z, V) = 2\,\mathrm{Re}(\langle (-iU_z^{1,0}, 0), (V_z^{1,0}, V_w^{1,0}) \rangle) = 2\,\mathrm{Im}\langle U_z^{1,0}, V_z^{1,0} \rangle$$

Since $\omega_1(U,V) = -\operatorname{Im}(\langle U_z^{1,0}, V_z^{1,0}\rangle + \langle U_w^{1,0}, V_w^{1,0}\rangle)$, we see that

$$\omega_{\mathrm{H}}(U,V) = \omega_1(U,V) + \mathrm{d}\alpha_0(U,V) = \operatorname{Im}\langle U_z^{1,0}, V_z^{1,0}\rangle - \operatorname{Im}\langle U_w^{1,0}, V_w^{1,0}\rangle \tag{5.4}$$

Using $g = \operatorname{Re}(h \oplus h)$, we can work with endomorphisms instead of the corresponding two-forms. We know that I_1 is the direct sum of two integrable almost complex structures (namely, the standard complex structures on each factor). What we have just shown is that I_{H} (first introduced in section 3.3) is obtained from it by changing the sign of the complex structure on the first factor. In particular, for this series of examples, I_{H} its itself an integrable almost complex structure on N_n. This fact has consequences on the quaternionic Kähler side of the correspondence:

Proposition 5.32. *The endomorphism field I_{H} is an integrable almost complex structure on N_n, whose twist is a complex structure on $(\bar{N}_n, g_n^c)$ which is not subordinate to the quaternionic structure on $\bar{N}_n$.*

Proof. We have already shown that I_{H} is an integrable almost complex structure. It is clear that I_{H} is not a section of $\mathcal{Q}$ on the hyper-Kähler side, so its twist will not be subordinate to the twisted quaternionic structure either. Since we know that I_{H} is integrable, it now suffices to check that ω_{H} is of type $(1,1)$ with respect to I_{H} (by example 3.15). This follows easily from the fact that I_{H} is skew with respect to g. $\square$

Let us now return to the lifting procedure for infinitesimal symmetries. Upon canonically lifting the infinitesimal automorphism X_A, we obtain the vector field Y_A on N_n, whose holomorphic part is given by

$$Y_A^{1,0}(z,w) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (e^{tA}z, (e^{-tA})^T w) = (A \cdot z, -A^T \cdot w)$$

where $\{z_j, w_j\}$ are holomorphic coordinates (with respect to I_1) on N_n. Correspondingly, the moment map for Y with respect to ω_{H} is

$$f_{Y_A} = -\frac{1}{2}\operatorname{Im}\left(\langle A \cdot z, z\rangle + \langle A^T \cdot w, w\rangle\right)$$

by (5.4). To compute $Y_A^{\mathrm{H}} = Y_A - \frac{f_{Y_A}}{f_{\mathrm{H}}} Z$, we only need f_{H}, which is given by $f_{\mathrm{H}} = \frac{1}{2}g(Z,Z) - \frac{1}{2}c = \frac{1}{2}\operatorname{Re}\langle -iz, -iz\rangle - \frac{1}{2}c = \frac{1}{2}\langle z, z\rangle - \frac{1}{2}c$. We conclude that

$$(Y_A^{\mathrm{H}})^{1,0} = (A \cdot z, -A^T \cdot w) + \frac{\operatorname{Im}(\langle A \cdot z, z\rangle + \langle A^T \cdot w, w\rangle)}{\langle z, z\rangle - c}(-iz, 0)$$

Now we are ready to twist. As explained in section 4.5, the auxiliary circle bundle $P = N_n \times S^1$ carries a natural connection with curvature ω_{H}, namely $\eta = \mathrm{d}s + \frac{1}{2}\iota_\Xi \omega_{\mathrm{H}}$, where Ξ is the Euler field on $\mathbb{C}^{2n+2}$. We now have to horizontally lift Y_A^{H} to P and then push it down to $\bar{N}_n$. Since P is trivial, we can think of Y_A^{H} as living on P, and thus the horizontal lift is simply $\tilde{Y}_A^{\mathrm{H}} = Y_A^{\mathrm{H}} - \eta(Y_A^{\mathrm{H}})\frac{\partial}{\partial s}$, where $s \in \mathbb{R}/2\pi\mathbb{Z}$ is the periodic coordinate on the circle. We compute the second term more explicitly:

$$\begin{aligned}\eta(Y_A^{\mathrm{H}}) &= \frac{1}{2}\omega_{\mathrm{H}}(\Xi, Y_A^{\mathrm{H}}) = \frac{1}{2}\operatorname{Im}\left(\langle z, A\cdot z\rangle + \langle w, A^T\cdot w\rangle + \left\langle z, i\frac{f_{Y_A}}{f_{\mathrm{H}}}z\right\rangle\right)\\ &= -\frac{1}{2}\operatorname{Im}\left(\langle A\cdot z, z\rangle + \langle A^T\cdot w, w\rangle + \left\langle i\frac{f_{Y_A}}{f_{\mathrm{H}}}z, z\right\rangle\right)\\ &= \frac{f_{Y_A}}{f_{\mathrm{H}}}\left(f_{\mathrm{H}} - \frac{1}{2}\langle z, z\rangle\right) = -\frac{f_{Y_A}}{f_{\mathrm{H}}}\frac{c}{2}\end{aligned}$$

This means that

$$\tilde{Y}_A^{\mathrm{H}} = Y_A^{\mathrm{H}} + \frac{f_{Y_A}}{f_{\mathrm{H}}}\frac{c}{2}\frac{\partial}{\partial s} = Y_A - \frac{f_{Y_A}}{f_{\mathrm{H}}}\left(Z - \frac{c}{2}\frac{\partial}{\partial s}\right) = Y_A - \frac{f_{Y_A}}{f_{\mathrm{H}}}Z_P$$

where we used the formula for Z_P derived in equation (4.7).

Now, all that remains is to push this vector field down to $\bar{N}_n = \{(z, w, s) \in M \times \mathbb{C}^{n+1} \times S^1 \mid \arg z_0 = 0\}$. Concretely, what this entails is to write $\tilde{Y}_A^{\mathrm{H}}$ in the form $\tilde{Y}_A^{\mathrm{H}} = Y_A^{\mathrm{Q}} - \eta_{\mathrm{Q}}(Y_A^{\mathrm{Q}})Z_P$, where Y_A^{Q} is tangent to $\bar{N}_n$ (at every point contained in $\bar{N}_n \subset P$) and $\eta_{\mathrm{Q}} = \frac{1}{f_{\mathrm{H}}}\eta$ is the natural dual (in the sense of dual twist data, cf. definition 3.12) principal S^1-connection on P, viewed as a bundle over $\bar{N}_n$. Then Y_A^{Q}, upon restriction to $\bar{N}_n$, yields the twist of Y_A^{H}.

Though existence and uniqueness of the twisted vector field are guaranteed by the general theory of the twist construction, it may not seem easy to determine Y_A^{Q} in practice. Here, the way we have expressed $\tilde{Y}_A^{\mathrm{H}}$ comes in handy. Indeed, since $\eta(Y_A) = f_{Y_A}$, we have $\tilde{Y}_A^{\mathrm{H}} = Y_A - \eta_{\mathrm{Q}}(Y_A)Z_P$. Any other vector field X satisfying $\tilde{Y}_A^{\mathrm{H}} = X - \eta_{\mathrm{Q}}(X)Z_P$ is of the form $X = Y + \phi Z_P$ for some function ϕ on P, so finding Y_A^{Q} is now reduced to finding the appropriate function ϕ such that $Y_A + \phi Z_P$ restricts to a vector field on $\bar{N}_n$.

A vector field X on P restricts to a vector field on $\bar{N}_n$ if and only if $\mathrm{d}\varphi(X)|_{\bar{N}_n} = 0$, where $\mathrm{d}\varphi = \mathrm{d}(\arg z_0) = -\frac{i}{2}\left(\frac{\mathrm{d}z_0}{z_0} - \frac{\mathrm{d}\bar{z}_0}{\bar{z}_0}\right)$. Since Z generates the (inverse of the) standard U(1)-action on M, it is no surprise that $\mathrm{d}\varphi(Z_P) = \mathrm{d}\varphi(Z) = -1$. Therefore, $\mathrm{d}\varphi(Y_A + \phi Z_P) = \mathrm{d}\varphi(Y_A) - \phi$ and we see that the choice $\phi = \mathrm{d}\varphi(Y_A)$ yields Y_A^{Q}. In conclusion, the vector field $Y_A^{\mathrm{Q}} := Y_A + \mathrm{d}\varphi(Y_A)Z_P$ restricts to the twist of Y_A^{H} on $\bar{N}_n$. Let us work out this second term in more detail: $\mathrm{d}\varphi(Y_A) = \operatorname{Im}\left(\frac{\mathrm{d}z_0}{z_0}\right)(Y_A^{1,0}) = \operatorname{Im}\left(\frac{(A\cdot z)_0}{z_0}\right)$, where $(A\cdot z)_0$ is the 0-component of $A\cdot z$. We can also express it as $(A\cdot z)_0 = \langle A\cdot z, e_0\rangle$, where e_0 is the corresponding unit vector. It is convenient to introduce the notation

$$A(X) := A\cdot\begin{pmatrix}1\\ X\end{pmatrix} \qquad\qquad X_j = \frac{z_j}{z_0},\ j = 1, \ldots, n$$

so that $\mathrm{d}\varphi(Y_A) = \operatorname{Im}\langle A(X), e_0\rangle$. Here, we already see the standard coordinates $\{X_j\}$

on $\bar{N}_n$ arising naturally. In summary, we see that $Y_A^{\mathrm{Q}} = Y_A + \operatorname{Im}\langle A(X), e_0\rangle Z_P$, when restricted to $\bar{N}_n$, yields the sought-after Killing field.

Now let us apply this entire procedure to a basis (as a real vector space) of $\mathfrak{su}(n,1)$. Let $E_{j,k}$, $j,k \in \{0,1,\dots,n\}$, denote the elementary matrix whose only non-zero entry, which equals one, has row number j and column number k. Then the following matrices provide a basis for $\mathfrak{su}(n,1)$:

$$\begin{aligned}
B_k^+ &= E_{0,k} + E_{k,0} && k \in \{1,\dots,n\}\\
B_k^- &= i(E_{0,k} - E_{k,0}) && k \in \{1,\dots,n\}\\
C_{j,k}^+ &= E_{j,k} - E_{k,j} && j<k,\ j,k \in \{1,\dots,n\}\\
C_{j,k}^- &= i(E_{j,k} + E_{k,j}) && j<k,\ j,k \in \{1,\dots,n\}\\
D_k &= i(E_{k-1,k-1} - E_{k,k}) && k \in \{1,\dots,n\}
\end{aligned}$$

The lifting procedure for the infinitesimal automorphisms $X_{C_{j,k}^{\pm}}$ is rather simple, as their canonical lifts $Y_{C_{j,k}^{\pm}}$ satisfy $\mathrm{d}\varphi(Y_{C_{j,k}^{\pm}}) = 0$. Indeed, $\langle C_{j,k}^{\pm}(X), e_0\rangle = 0$, and therefore $Y_{C_{j,k}^{\pm}}$, viewed as a vector field on the trivial bundle P, restricts to a Killing field for the quaternionic Kähler metric on the twist. For example, we have

$$Y^{\mathrm{Q}}_{C_{j,k}^-} = 2\operatorname{Re}\left[i\left(X_j\frac{\partial}{\partial X_k} + X_k\frac{\partial}{\partial X_j} - w_j\frac{\partial}{\partial w_k} - w_k\frac{\partial}{\partial w_j}\right)\right]$$

with respect to the standard coordinates $(X,\rho,\tilde{\phi},w)$ on $\bar{N}_n$, introduced in (4.8), in passing to the second line. The same simple procedure works for the diagonal basis elements D_k for $k>1$. The case $k=1$, however, is special. Indeed,

$$Y_{D_1} = 2\operatorname{Re}\left[i\left(z_0\frac{\partial}{\partial z_0} - z_1\frac{\partial}{\partial z_1} - w_0\frac{\partial}{\partial w_0} + w_1\frac{\partial}{\partial w_1}\right)\right]$$

and $\mathrm{d}\varphi(Y_{D_1}) = \operatorname{Im}\langle D_1(X), e_0\rangle = 1$. This means that, on P, we can write Y_{D_1} in the form $Y_{D_1} = \hat{Y}_{D_1} + \frac{\partial}{\partial\varphi}$ where $\hat{Y}_{D_1}$ is tangent to $\bar{N}_n$ (at each $x \in \bar{N}_n$). Adding $Z_P = -\frac{\partial}{\partial\varphi} - \frac{c}{2}\frac{\partial}{\partial s} = -\frac{\partial}{\partial\varphi} - 2c\frac{\partial}{\partial\tilde{\phi}}$, we obtain

$$Y^{\mathrm{Q}}_{D_1} = Y_{D_1} + Z_P = -2\operatorname{Re}\left[i\left(\sum_{j=1}^{n} X_j\frac{\partial}{\partial X_j} + X_1\frac{\partial}{\partial X_1} + w_0\frac{\partial}{\partial w_0} - w_1\frac{\partial}{\partial w_1}\right)\right] - 2c\frac{\partial}{\partial\tilde{\phi}}$$

We see that this Killing vector is c-dependent. However, the dependence on c is not essential: As $\frac{\partial}{\partial\tilde{\phi}}$ is itself a Killing field, the c-dependence can be removed by subtracting another Killing field.

The remaining basis elements $B_k^{\pm}$ are most interesting. The corresponding Killing fields are c-dependent in an essential way. On the hyper-Kähler manifold, we have

$$Y_{B_k^+} = 2\,\mathrm{Re}\left(z_0\frac{\partial}{\partial z_k} + z_k\frac{\partial}{\partial z_0} - w_0\frac{\partial}{\partial w_k} - w_k\frac{\partial}{\partial w_0}\right)$$

Since $\mathrm{Im}\langle B_k^+(X), e_0\rangle = \mathrm{Im}\, X_k$, we have

$$\begin{aligned} Y^{\mathrm{Q}}_{B_k^+} &= Y_{B_k^+} + \mathrm{Im}(X_k) Z_P \\ &= 2\,\mathrm{Re}\left(\frac{\partial}{\partial X_k} - X_k\sum_j X_j\frac{\partial}{\partial X_j} - w_0\frac{\partial}{\partial w_k} - w_k\frac{\partial}{\partial w_0} + icX_k\frac{\partial}{\partial\tilde{\phi}}\right) \end{aligned} \tag{5.5}$$

Starting from

$$Y_{B_k^-} = 2\,\mathrm{Re}\left[i\left(z_k\frac{\partial}{\partial z_0} - z_0\frac{\partial}{\partial z_k} + w_k\frac{\partial}{\partial w_0} - w_0\frac{\partial}{\partial w_k}\right)\right]$$

we analogously derive

$$\begin{aligned} Y^{\mathrm{Q}}_{B_k^-} &= Y_{B_k^-} + \mathrm{Re}(X_k) Z_P \\ &= 2\,\mathrm{Re}\left[i\left(-\frac{\partial}{\partial X_k} - X_k\sum_j X_j\frac{\partial}{\partial X_j} + w_k\frac{\partial}{\partial w_0} - w_0\frac{\partial}{\partial w_k} + icX_k\frac{\partial}{\partial\tilde{\phi}}\right)\right] \end{aligned} \tag{5.6}$$

This completes our description of the Killing fields on $\bar{N}_n$ derived from $\mathfrak{aut}(\mathbb{C}\mathrm{H}^n) \cong \mathfrak{su}(n,1)$. Besides these lifted Killing fields, the free and isometric action of Heis_{2n+3} on the fibers of $\bar{N}_n$ yields $2n+3$ additional Killing fields. Together, they generate a group of isometries whose principal orbits have dimension $4n+3 = \dim\bar{N}_n - 1$, so $\bar{N}_n$ is of co-homogeneity at most one. This is, of course, consistent with the results presented in section 5.1, in particular corollary 5.19.

5.3.2 A non-trivial curvature invariant

It is remarkable that the Killing fields constructed above, which arise from the underlying PSK manifold $\mathbb{C}\mathrm{H}^n$, all preserve the coordinate function ρ. This is no coincidence: Since $\rho = 2f_Z$, any vector field Y_A that arises as the canonical lift of an element $A \in \mathfrak{su}(n,1)$ lies in $\mathfrak{aut}_{\mathrm{Ham}}(N_n, f_Z)$, hence satisfies $\mathrm{d}\rho(Y_A) = 2Y_A(f_Z) = 0$. The vector field $\rho\frac{\partial}{\partial\rho}$ is a Killing field for the undeformed c-map metric, but ceases to be one for non-zero values of the deformation parameter c, when only the Heisenberg group Heis_{2n+3} acts by fiberwise isometries.

Thus, in the deformed case, we have obtained a group of isometries which preserves the level sets of ρ. On these level sets, however, the action is transitive. Indeed, the lifted Killing fields generate an action which covers the standard (projective) action of $\mathrm{SU}(n,1)$

on $\mathbb{C}\mathrm{H}^n$, which is transitive, and the action of Heis_{2n+3} is transitive on the constant-ρ slices of the fibers.

Our aim in this section is to prove that any isometry of the deformed c-map metric on $\bar{N}_n$ must in fact preserve the level sets of ρ. To do this, we will use theorem 5.27 to compute a curvature invariant of $\bar{N}_n$. Since the scalar curvature is always constant on a quaternionic Kähler manifold, the simplest non-trivial curvature invariant is the Hilbert–Schmidt norm of the curvature, regarded as a self-adjoint operator $\mathcal{R}^{\mathrm{Q}} : \bigwedge^2 T\bar{N}_n \to \bigwedge^2 T\bar{N}_n$. This quadratic curvature invariant is sometimes also called the Kretschmann scalar (up to a factor four). If $\{e_j\}$ is an orthonormal frame for $T\bar{N}_n$, we may compute this invariant as follows:

$$\left\|\mathcal{R}^{\mathrm{Q}}\right\|^2 = \sum g^c_n(\mathcal{R}^{\mathrm{Q}}(e_i \wedge e_j), e_m \wedge e_n) g^c_n(\mathcal{R}^{\mathrm{Q}}(e_m \wedge e_n), e_i \wedge e_j)$$

where we sum over the induced orthonormal basis of $\bigwedge^2 T\bar{N}_n$.

Compatibility of twisting with tensor products and contractions implies that we may compute this on the hyper-Kähler side instead, so we have

$$\left\|\mathcal{R}^{\mathrm{Q}}\right\|^2 = \sum g_{\mathrm{H}}(\tilde{\mathcal{R}}(e'_i \wedge e_j), e'_m \wedge e'_n) g_{\mathrm{H}}(\tilde{\mathcal{R}}(e'_m \wedge e_n), e'_i \wedge e'_j)$$

where $\tilde{\mathcal{R}}$ is the curvature operator associated to the algebraic curvature tensor $\tilde{R}$ from theorem 5.27, and $\{e'_j\}$ is an orthonormal frame with respect to g_{H}.

We will translate the curvature formula from theorem 5.27 to this setting, where we think of the curvature as an endomorphism. Using the metric g_{H}, we can extend the operations $\owedge$ and $\varowedge$ to apply to endomorphisms as follows. Let E, F be self-adjoint endomorphism fields with respect to g_{H}. Then $E \owedge_{g_{\mathrm{H}}} F : \bigwedge^2 TN_n \to \bigwedge^2 TN_n$ is defined by the relation

$$g_{\mathrm{H}}((E \owedge_{g_{\mathrm{H}}} F)(A \wedge B), C \wedge X) := \big(g_{\mathrm{H}}(E\cdot,\cdot) \owedge g_{\mathrm{H}}(F\cdot,\cdot)\big)(A, B, C, X)$$

Analogously, we define $K \varowedge_{g_{\mathrm{H}}} L : \bigwedge^2 TN_n \to \bigwedge^2 TN_n$, where $K, L \in \Gamma(\operatorname{End} TN_n)$ are skew-adjoint with respect to g_{H}. With this notation, our curvature formula translates to

$$\begin{aligned} &g_{\mathrm{H}}(\tilde{\mathcal{R}}(A \wedge B), C \wedge X) \\ &\quad = \frac{1}{f_Z} g(\mathcal{R}(A \wedge B), C \wedge X) \\ &\qquad + \frac{1}{8} g_{\mathrm{H}}\Big(\Big(\mathrm{id} \owedge_{g_{\mathrm{H}}} \mathrm{id} + \sum_k I_k \varowedge_{g_{\mathrm{H}}} I_k\Big)(A \wedge B), C \wedge X\Big) \\ &\qquad - \frac{1}{8 f_Z f_{\mathrm{H}}} g_{\mathrm{H}}\Big(\Big(\mathcal{K} I_{\mathrm{H}} \varowedge_{g_{\mathrm{H}}} \mathcal{K} I_{\mathrm{H}} + \sum_k \mathcal{K} I_{\mathrm{H}} I_k \owedge_{g_{\mathrm{H}}} \mathcal{K} I_{\mathrm{H}} I_k\Big)(A \wedge B), C \wedge X\Big) \end{aligned}$$

where, in the final line, we used that $\omega_{\mathrm{H}}(\cdot,\cdot) = g(I_{\mathrm{H}}\cdot,\cdot) = g_{\mathrm{H}}(\mathcal{K}I_{\mathrm{H}}\cdot,\cdot)$. Now we are in a position to compute the curvature norm.

Theorem 5.33. *Let $(\bar{N}_{n-1}, g^c_{n-1})$, $n \geq 1$ be the image of $\mathbb{C}\mathrm{H}^{n-1}$ under the deformed c-map, and denote its curvature operator by $\mathcal{R}^{\mathrm{Q}}$. Then*

$$\left\|\mathcal{R}^{\mathrm{Q}}\right\|^2 = n(5n+1) + 3\left(\frac{f_Z^3}{f_{\mathrm{H}}^3} + (n-1)\frac{f_Z}{f_{\mathrm{H}}}\right)^2 + 3\left(\frac{f_Z^6}{f_{\mathrm{H}}^6} + (n-1)\frac{f_Z^2}{f_{\mathrm{H}}^2}\right)$$

Proof. The norm of the curvature, which is equal to the norm of $\tilde{\mathcal{R}}$ on the hyper-Kähler side, is nothing but the trace of the $\tilde{\mathcal{R}}^2$. Since the initial hyper-Kähler metric is flat, its curvature operator $\mathcal{R}$ vanishes identically, and thus we have to compute

$$\left\|\mathcal{R}^{\mathrm{Q}}\right\|^2 = \frac{1}{64}\mathrm{tr}\left[\left(\mathrm{id} \owedge_{g_{\mathrm{H}}} \mathrm{id} + \sum_k I_k \obar_{g_{\mathrm{H}}} I_k - \frac{1}{f_Z f_{\mathrm{H}}}\left(\mathcal{K}I_{\mathrm{H}} \obar_{g_{\mathrm{H}}} \mathcal{K}I_{\mathrm{H}} + \sum_k \mathcal{K}I_{\mathrm{H}}I_k \owedge_{g_{\mathrm{H}}} \mathcal{K}I_{\mathrm{H}}I_k\right)\right)^2\right]$$

There are three different types of terms. The following lemma shows how to reduce their computation to determining the traces of the compositions of the relevant endomorphisms.

Lemma 5.34. *Let h be a (pseudo-)Riemannian metric on a smooth manifold N, and let $E, F, K, L \in \Gamma(\mathrm{End}(TN))$ be endomorphism fields such that E and F are self-adjoint with respect to h, while K and L are skew-adjoint with respect to h. Then the following identities hold:*

(i) $\mathrm{tr}\left((E \owedge_h E) \circ (F \owedge_h F)\right) = 2\big(\mathrm{tr}(E \circ F)\big)^2 - 2\,\mathrm{tr}\left((E \circ F)^2\right)$.

(ii) $\mathrm{tr}\left((K \obar_h K) \circ (L \obar_h L)\right) = 6\big(\mathrm{tr}(K \circ L)\big)^2 + 6\,\mathrm{tr}\left((K \circ L)^2\right)$.

(iii) $\mathrm{tr}\left((E \owedge_h E) \circ (K \obar_h K)\right) = 2\big(\mathrm{tr}(E \circ K)\big)^2 - 6\,\mathrm{tr}\left((E \circ K)^2\right)$.

Proof. Let $\{e_a\}$ be an orthonormal basis for h with $h(e_a, e_a) = \epsilon_a \in \{\pm 1\}$. In each case, the identity is derived by a straightforward computation with respect to this basis, using the definition of $\owedge_h$ and $\obar_h$. For the first identity, we have:

$$\begin{aligned}
&\mathrm{tr}\left(E \owedge_h E) \circ (F \owedge_h F)\right) \\
&\quad = \frac{1}{4}\sum_{a,b,c,d} \epsilon_a\epsilon_b\epsilon_c\epsilon_d h((E \owedge_h E)e_c \wedge e_d, e_a \wedge e_b)h((F \owedge_h F)e_a \wedge e_b, e_c \wedge e_d) \\
&\quad = \sum_{a,b,c,d} \epsilon_a\epsilon_b\epsilon_c\epsilon_d\big(h(Ee_c, e_a)h(Ee_d, e_b) - h(Ee_c, e_b)h(Ee_d, e_a)\big) \\
&\qquad\qquad\qquad \big(h(Fe_a, e_c)h(Fe_b, e_d) - h(Fe_a, e_d)h(Fe_b, e_c)\big) \\
&\quad = 2\big(\mathrm{tr}(E \circ F)\big)^2 - 2\,\mathrm{tr}\left((E \circ F)^2\right)
\end{aligned}$$

Note that this part of the lemma actually does not use any properties of E and F. We turn to the second identity.

$$
\begin{aligned}
&\operatorname{tr}\big((K \obar_h K) \circ (L \obar_h L)\big)\\
&\quad= \sum_{a,b,c,d} \epsilon_a\epsilon_b\epsilon_c\epsilon_d\\
&\qquad\quad \big(h(Ke_c,e_a)h(Ke_d,e_b) - h(Ke_c,e_b)h(Ke_d,e_a) + 2h(Ke_c,e_d)h(Ke_a,e_b)\big)\\
&\qquad\quad \big(h(Le_a,e_c)h(Le_b,e_d) - h(Le_a,e_d)h(Le_b,e_c) + 2h(Le_a,e_b)h(Le_c,e_d)\big)\\
&\quad= 2\Big(\operatorname{tr}(K\circ L)\Big)^2 - 2\operatorname{tr}\big((K\circ L)^2\big) + 8\operatorname{tr}\big((L\circ K)^2\big) + 4\Big(\operatorname{tr}(K\circ L)\Big)^2\Big)\\
&\quad= 6\Big(\operatorname{tr}(K\circ L)\Big)^2 + 6\operatorname{tr}\big((K\circ L)^2\big)
\end{aligned}
$$

In this case, we did use the fact that K and L are skew, namely in passing to the penultimate line. We now derive the final identity:

$$
\begin{aligned}
&\operatorname{tr}\big((E \oslash_h F) \circ (K \obar_h K)\big)\\
&\quad= \sum_{a,b,c,d} \epsilon_a\epsilon_b\epsilon_c\epsilon_d\big(h(Ee_c,e_a)h(Ee_d,e_b) - h(Ee_c,e_b)h(Ee_d,e_a)\big)\\
&\qquad\quad \big(h(Ke_a,e_c)h(Ke_b,e_d) - h(Ke_a,e_d)h(Ke_b,e_c) + 2h(Ke_a,e_b)h(Ke_c,e_d)\big)\\
&\quad= 2\Big(\operatorname{tr}(E\circ K)\Big)^2 - 2\operatorname{tr}\big((E\circ K)^2\big) - 4\operatorname{tr}\big((E\circ K)^2\big)\\
&\quad= 2\Big(\operatorname{tr}(E\circ K)\Big)^2 - 6\operatorname{tr}\big((E\circ K)^2\big)
\end{aligned}
$$

This computation uses the self-adjointness of E as well as the fact that K is skew-adjoint. ☐

In the case at hand, the relevant endomorphism fields are various compositions of id, $\mathcal{K}$, I_{H} and I_k. Since both $\mathcal{K}$ and I_{H} commute with each I_k by proposition 3.27 and lemma 5.22, and $I_{\mathrm{H}}^2 = I_k^2 = -\operatorname{id}$ (here, we use proposition 5.32), we can always reduce these traces to one of the four traces computed in the next lemma:

Lemma 5.35. *On N_{n-1}, the following trace identities hold for any non-negative integer $m \in \mathbb{N}_0$ and any $k \in \{1,2,3\}$:*

(i) $\operatorname{tr}(\mathcal{K}^m) = 4\Big((n-1)f_Z^m + \frac{f_Z^{2m}}{f_{\mathrm{H}}^m}\Big)$.

(ii) $\operatorname{tr}(\mathcal{K}^m I_k) = 0$.

(iii) $\operatorname{tr}(\mathcal{K}^m I_{\mathrm{H}}) = 0$.

(iv) $\operatorname{tr}(\mathcal{K}^m I_{\mathrm{H}} I_k) = 0$.

Proof. Recall that $\mathcal{K}|_{\mathbb{H}Z} = \frac{f_Z^2}{f_{\mathrm{H}}}\operatorname{id}|_{\mathbb{H}Z}$ while $\mathcal{K}|_{(\mathbb{H}Z)^\perp} = f_Z \operatorname{id}|_{(\mathbb{H}Z)^\perp}$. This immediately implies the first identity. For the second, note that $\mathcal{K}$ is self-adjoint with respect to g, while I_k is skew. Since they commute, this means that $\mathcal{K}^m I_k$ is skew with respect to g, so its trace must vanish. The same argument works to prove that $\operatorname{tr}(\mathcal{K}^m I_{\mathrm{H}}) = 0$. Finally, consider $\operatorname{tr}(\mathcal{K}^m I_{\mathrm{H}} I_k)$. Writing $I_k = I_i I_j$ for $i, j \in \{1, 2, 3\}$, the fact that the complex structures commute with $\mathcal{K}$ and I_{H} implies that $\operatorname{tr}(\mathcal{K}^m I_{\mathrm{H}} I_i I_j) = \operatorname{tr}(I_j \mathcal{K}^m I_{\mathrm{H}} I_i) = \operatorname{tr}(\mathcal{K}^m I_{\mathrm{H}} I_j I_i)$, but then $I_j I_i = -I_k$ shows that this trace vanishes. □

With this preliminary work, we have reduced the computation of $\left\|\mathcal{R}^{\mathrm{Q}}\right\|^2$ to an essentially mechanical process, though in some cases some work is required in reducing each term to one of the standard forms computed above. Therefore, we only give a few intermediate steps in this final computation:

$$
\begin{aligned}
32\left\|\mathcal{R}^{\mathrm{Q}}\right\|^2 &= \operatorname{tr}(\mathrm{id})^2 - \operatorname{tr}(\mathrm{id}) + 2\sum_k \left[\operatorname{tr}(I_k)^2 - 3\operatorname{tr}(I_k^2)\right] \\
&\quad - \frac{2}{f_Z f_{\mathrm{H}}}\Big(\operatorname{tr}(\mathcal{K} I_{\mathrm{H}})^2 - 3\operatorname{tr}\left((\mathcal{K} I_{\mathrm{H}})^2\right) + \sum_k \Big[\operatorname{tr}(\mathcal{K} I_{\mathrm{H}} I_k)^2 - \operatorname{tr}\left((\mathcal{K} I_{\mathrm{H}} I_k)^2\right)\Big]\Big) \\
&\quad + 3\sum_{j,k}\Big[\operatorname{tr}(I_j I_k)^2 + \operatorname{tr}\left((I_j I_k)^2\right)\Big] \\
&\quad - \frac{2}{f_Z f_{\mathrm{H}}}\Big(3\sum_k \Big[\operatorname{tr}(I_k \mathcal{K} I_{\mathrm{H}})^2 + \operatorname{tr}\left((I_k \mathcal{K} I_{\mathrm{H}})^2\right)\Big] \\
&\qquad\qquad + \sum_{j,k}\Big[\operatorname{tr}(I_k \mathcal{K} I_{\mathrm{H}} I_j)^2 - 3\operatorname{tr}\left((I_k \mathcal{K} I_{\mathrm{H}} I_j)^2\right)\Big]\Big) \\
&\quad + \frac{1}{f_Z^2 f_{\mathrm{H}}^2}\Big(3\Big[\operatorname{tr}\left((\mathcal{K} I_{\mathrm{H}})^2\right)^2 + \operatorname{tr}\left((\mathcal{K} I_{\mathrm{H}})^4\right)\Big] \\
&\qquad\qquad + 2\sum_k \Big[\operatorname{tr}(\mathcal{K} I_{\mathrm{H}} \mathcal{K} I_{\mathrm{H}} I_k)^2 - 3\operatorname{tr}\left((\mathcal{K} I_{\mathrm{H}} \mathcal{K} I_{\mathrm{H}} I_k)^2\right)\Big]\Big) \\
&\quad + \frac{1}{f_Z^2 f_{\mathrm{H}}^2}\sum_{j,k}\Big[\operatorname{tr}(\mathcal{K} I_{\mathrm{H}} I_j \mathcal{K} I_{\mathrm{H}} I_k)^2 - \operatorname{tr}\left((\mathcal{K} I_{\mathrm{H}} I_j \mathcal{K} I_{\mathrm{H}} I_k)^2\right)\Big] \\
&= \operatorname{tr}(\mathrm{id})^2 - \operatorname{tr}(\mathrm{id}) + 18\operatorname{tr}(\mathrm{id}) + 3\sum_{j,k}\Big[\delta_{jk}\operatorname{tr}(\mathrm{id})^2 - (-1)^{\delta_{jk}}\operatorname{tr}(\mathrm{id})\Big] \\
&\quad - \frac{6}{f_Z f_{\mathrm{H}}}\Big[\operatorname{tr}(\mathcal{K}^2) - \sum_{j,k}(-1)^{\delta_{jk}}\operatorname{tr}(\mathcal{K}^2)\Big] \\
&\quad + \frac{1}{f_Z^2 f_{\mathrm{H}}^2}\Big(3\Big[\operatorname{tr}(\mathcal{K}^2)^2 + \operatorname{tr}(\mathcal{K}^4)\Big] + 18\operatorname{tr}(\mathcal{K}^4) \\
&\qquad\qquad + \sum_{j,k}\Big[\delta_{jk}\operatorname{tr}(\mathcal{K}^2)^2 + (-1)^{\delta_{jk}}\operatorname{tr}(\mathcal{K}^4)\Big]\Big) \\
&= 160n^2 + 32n + \frac{6}{f_Z^2 f_{\mathrm{H}}^2}(\operatorname{tr}(\mathcal{K}^2)^2 + 4\operatorname{tr}(\mathcal{K}^4))
\end{aligned}
$$

Finally, we obtain:

$$\left\|\mathcal{R}^{\mathrm{Q}}\right\|^2 = n(5n+1) + 3\left((n-1)\frac{f_Z}{f_{\mathrm{H}}} + \frac{f_Z^3}{f_{\mathrm{H}}^3}\right)^2 + 3\left((n-1)\frac{f_Z^2}{f_{\mathrm{H}}^2} + \frac{f_Z^6}{f_{\mathrm{H}}^6}\right)$$

This is indeed the claimed expression for $\left\|\mathcal{R}^{\mathrm{Q}}\right\|^2$. □

The curvature norm in the case $n = 1$, which arises by applying the c-map to the trivial PSK manifold consisting of a single point, was worked out by Cortés and Saha [CS18], and we recover their formula for $\left\|\mathcal{R}^{\mathrm{Q}}\right\|^2$ in this case.

Corollary 5.36. *For every $c > 0$, any isometry of $(\bar{N}_{n-1}, g^c_{n-1})$ preserves the level sets of the coordinate function ρ.*

Proof. To interpret theorem 5.33 in terms of the coordinates on the quaternionic Kähler manifold $\bar{N}_n$, we use $f_Z = \frac{1}{2}\rho$ and $f_{\mathrm{H}} = -\frac{1}{2}(\rho + 2c)$. We obtain

$$\left\|\mathcal{R}^{\mathrm{Q}}\right\|^2 = n(5n+1) + 3\left((n-1)\frac{\rho}{\rho+2c} + \frac{\rho^3}{(\rho+2c)^3}\right)^2 + 3\left((n-1)\frac{\rho^2}{(\rho+2c)^2} + \frac{\rho^6}{(\rho+2c)^6}\right)$$

We see that $\left\|\mathcal{R}^{\mathrm{Q}}\right\|^2$ depends only on ρ. By invariance of the curvature under isometries, it now suffices to show that $\left\|\mathcal{R}^{\mathrm{Q}}\right\|^2 : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ is injective. Up to an additive constant, it is the composition of two functions $\Phi_c, \Psi : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$, defined by

$$\Phi_c(x) = \frac{x}{x+2c}$$
$$\Psi(x) = 3\big((x^3 + (n-1)x)^2 + x^6 + (n-1)x^2\big)$$

where $c > 0$. It is clear that Φ_c is injective for every $c > 0$.

Now assume that $\Psi(x_1) - \Psi(x_2) = 0$ for some $x_1, x_2 \in \mathbb{R}_{>0}$. By definition, this means that

$$(x_1^3 + (n-1)x_1)^2 + x_1^6 + (n-1)x_1^2 - (x_2^3 + (n-1)x_2)^2 - x_2^6 - (n-1)x_2^2 = 0$$

Clearly, this expression vanishes if $x_1 = x_2$, and since Ψ is even, it also vanishes if $x_1 = -x_2$. Factorizing, we find:

$$(x_1 - x_2)(x_1 + x_2)\Big(2(x_1^4 + x_2^4) + 2x_1^2x_2^2 + 2(x_1^2 + x_2^2)(n-1) + n(n-1)\Big) = 0$$

The second and third factor can never vanish as long as $x_1, x_2 \in \mathbb{R}_{>0}$, so we must have $x_1 = x_2$. □

Since we have already constructed a group of isometries which acts transitively on the level sets of ρ, we deduce:

Theorem 5.37. *The deformed c-map metrics g^c_n on $\bar{N}_n$ are of co-homogeneity one.* □

Thus, the one-loop deformations of these symmetric (and in particular homogeneous) spaces are of co-homogeneity one. This infinite series provides the first examples of this phenomenon in dimensions higher than eight. In dimension eight, it had previously been observed [CDS17, Ex. 28] the one-loop deformation of the symmetric space $\mathrm{G}_2^* / \mathrm{SO}(4)$ is not locally homogeneous; corollary 5.17 shows that it is also of co-homogeneity one. These results provide evidence for the following conjecture:

Conjecture 5.38. *Let $\bar{M}$ be a homogeneous PSK manifold, and let $(\bar{N}, g^c)$ denote its image under the (deformed) c-map. Then the metrics $g^{c>0}$ are of co-homogeneity one.*

Of course, we have already proven that the co-homogeneity is at most one in section 5.1, so it suffices to produce a non-constant curvature invariant, such as the Hilbert–Schmidt norm of the curvature operator. In order to do this for *c*-map spaces arising from arbitrary homogeneous PSK manifolds, a better understanding of the corresponding hyper-Kähler manifolds and their curvature tensors is required.

6 Discrete quotients of c-map spaces

In the preceding chapter, we have shown how to construct large groups of isometries of deformed c-map metrics. In particular, for any homogeneous PSK manifold, we obtain a group which acts with co-homogeneity one. It is natural to study discrete subgroups of these groups, since they can be used to construct further interesting examples of complete quaternionic Kähler manifolds via quotient constructions. Such quotients yield many examples of quaternionic Kähler manifolds with non-trivial fundamental group, and can be used to (partially) "compactify" the quaternionic Kähler manifolds that arise from the c-map. In this chapter, we initiate the study of these discrete subgroups, and show in some simple cases how they can be used to construct complete quaternionic Kähler manifolds of the form $K \times \mathbb{R}$, where K is compact.

The study of discrete subgroups of the isometry group of one-loop deformed c-map metrics is also of importance in theoretical physics. From a physical point of view, it is expected that so-called non-perturbative quantum corrections will give rise to quaternionic Kähler metrics on c-map spaces whose isometry groups are discrete subgroups of the isometry group of the (deformed) c-map metrics (see e.g. [Ale13; AMPP15; ABMP18] and references therein). The discrete subgroups considered in this chapter may therefore be of physical significance, though we will not pursue this question further.

6.1 Generalities

Recall (cf. proposition 4.34) that every c-map space $(\bar{N}, g^c)$, when viewed as a bundle over the corresponding PSK manifold $\bar{M}$, carries a proper, fiber-preserving and isometric action of a Heisenberg group. Furthermore, after passing to a cyclic Riemannian covering $(\tilde{N}, \tilde{g}^c)$, this action is free, so we have:

Lemma 6.1. *Let $\bar{M}$ be a complete and simply-connected PSK manifold of dimension $2n$. Then for every discrete subgroup $\Gamma \subset \mathrm{Heis}_{2n+3}$, the universal covering $(\tilde{N}, \tilde{g}^c)$, $c \geq 0$, of its image under the one-loop deformed c-map admits a smooth quotient $\tilde{N}/\Gamma$ which is a complete quaternionic Kähler manifold with fundamental group Γ.* □

When we make more assumptions about the PSK manifold $\bar{M}$ which underlies the c-map space, this story gets more interesting. Assuming completeness, the results of section 5.1 imply that the (identity component of the) automorphism group of $\bar{M}$ lifts to a connected group $G_{\bar{M}}$ of isometries of the corresponding c-map space $\bar{N}$.

Lemma 6.2. *Let $(\bar{N}, g^c)$, $c \geq 0$ lie in the image of the deformed c-map, and assume each g^c is complete. Let G be the group of isometries of g^c generated by $G_{\bar{M}}$ and Heis_{2n+3}. Then Heis_{2n+3} is a normal subgroup of G. If $\mathrm{Aut}(\bar{M})$ is semi-simple and $G_{\bar{M}}$ has compact center, then $G \cong G_{\bar{M}} \ltimes \mathrm{Heis}_{2n+3}$.*

Proof. Thinking of $\bar{N}$ as a bundle over $\bar{M}$, our first claim is that every element of G sends fibers to fibers, i.e. covers a well-defined map on $\bar{M}$. This is certainly true for the elements of Heis_{2n+3}, which cover the identity, so we need only prove this for elements of $G_{\bar{M}}$. Since $G_{\bar{M}}$ is connected and therefore generated by the image of its exponential map, it suffices to check that the Killing fields that generate the action of $G_{\bar{M}}$ cover well-defined vector fields on $\bar{M}$; the corresponding maps are obtained by integrating these vector fields. But it follows directly from our construction of these Killing fields that they cover (non-trivial) infinitesimal PSK automorphism on $\bar{M}$. We conclude that $G_{\bar{M}}$ sends fibers to fibers.

Now we prove that $\mathrm{Heis}_{2n+3} \subset G$ is normal. For any $\alpha \in G$ and $h \in \mathrm{Heis}_{2n+3}$, we must prove that $\alpha h \alpha^{-1} \in \mathrm{Heis}_{2n+3}$. First observe that the diffeomorphism $\alpha h \alpha^{-1} : \bar{N} \to \bar{N}$ covers the identity map on $\bar{M}$. Recall that, in standard coordinates $\{\rho, \tilde{\phi}, \tilde{\zeta}, \zeta)$ on the fibers, Heis_{2n+3} acts transitively on the level sets of ρ. Thus, it suffices to check that $\alpha h \alpha^{-1}$ preserves ρ. This will follow once we prove that all elements of $G_{\bar{M}}$ do so. Again, since $G_{\bar{M}}$ is generated by the image of its exponential map, it suffices to check that for any Killing field Y_A^{Q} generating $G_{\bar{M}}$, $L_{Y_A^{\mathrm{Q}}} \rho = 0$.

A close inspection of the proof of equivalence of the twist approach and the direct approach to the c-map, given in [ACDM15], shows that, up to a multiplicative factor, ρ is the twist of f_Z, where f_Z is the Hamiltonian for the rotating Killing field Z with respect to ω_1 on the corresponding hyper-Kähler manifold N. But then $Y_A^{\mathrm{Q}}(\rho) = Y_A^{\mathrm{H}}(f_Z) = 0$ because Y_A is an element of $\mathfrak{aut}_{\mathrm{Ham}}(N, f_Z)$ and therefore preserves f_Z. This proves that Heis_{2n+3} is a normal subgroup.

If $\mathrm{Aut}(\bar{M})$ is semi-simple our lifting procedure for infinitesimal automorphism yields a subalgebra $\mathfrak{g}_{\bar{M}}$ of Killing fields isomorphic to $\mathfrak{aut}(\bar{M})$ (cf. corollary 5.13) such that $\mathfrak{g} \cong \mathfrak{g}_{\bar{M}} \ltimes \mathfrak{heis}_{2n+3}$. This means that the normal subgroup $K = G_{\bar{M}} \cap \mathrm{Heis}_{2n+3}$ is discrete, hence central. In particular, K is contained in the center of both $G_{\bar{M}}$ and Heis_{2n+3}. Assuming that the center of $G_{\bar{M}}$ is compact, we see that K must be finite. But the center of Heis_{2n+3}, which is isomorphic to $\mathbb{R}$, contains no finite subgroups. In conclusion, $G_{\bar{M}} \cap \mathrm{Heis}_{2n+3} = \mathrm{id}$ and consequently $G \cong G_{\bar{M}} \ltimes \mathrm{Heis}_{2n+3}$. □

The group structure of G is therefore—at least in a good case—determined by the action of $G_{\bar{M}}$ on Heis_{2n+3}. This suggests the following strategy for constructing discrete subgroups of G. First, we construct a discrete subgroup $\Gamma_{\bar{M}}$ of $G_{\bar{M}}$, and then we look for a discrete subgroup Γ_{Heis} of Heis_{2n+3} which is invariant under the action of $\Gamma_{\bar{M}}$ on Heis_{2n+3}. Then $\Gamma_{\bar{M}}$ and Γ_{Heis} generate a discrete subgroup $\Gamma = \Gamma_{\bar{M}} \ltimes \Gamma_{\mathrm{Heis}} \subset G$.

Every Lie group admits a unique left-invariant volume form up to scaling, which defines the so-called (left) Haar measure on this group. By left-invariance, this volume form descends to the quotient of G by any discrete subgroup Γ. In the following, we will be particularly interested in discrete subgroups Γ such that the quotient G/Γ has finite volume, or is even compact.

Definition 6.3. We say that a discrete subgroup $\Gamma \subset G$ of a Lie group is a lattice if G/Γ has finite volume with respect to the induced G-invariant volume form. A lattice is called co-compact if G/Γ is compact.

As the name suggests, lattices in Lie groups generalize the familiar discrete subgroup $\mathbb{Z}^n \subset \mathbb{R}^n$. The study of lattices is a vast area of research, with deep connections to various fields of mathematics such as algebraic geometry and number theory. As we will see, it is non-trivial to construct lattices even for very simple choices of Lie group. For this reason, we will limit ourselves to a case study of certain simple examples and leave a more general study of these matters for a future investigation.

6.2 Case study: Lattices from quaternion algebras

Once again, we return to the series $(\bar{N}_n, g^c)$ of quaternionic Kähler manifolds obtained by applying the c-map to the homogeneous PSK manifold $\mathbb{C}\mathrm{H}^n$, and passing to the universal Riemannian cover (so that $(\bar{N}_n, g^0) \cong \frac{\mathrm{SU}(n+1,2)}{\mathrm{S}(\mathrm{U}(n)\times\mathrm{U}(2))}$). We have $\mathrm{Aut}(\mathbb{C}\mathrm{H}^n) = \mathrm{SU}(n,1)$, and $\mathrm{SU}(n,1)$ acts on the associated CASK manifold, which is an open and $\mathrm{SU}(n,1)$-invariant subset of $\mathbb{C}^{n+1}$, by its defining representation ρ_{std}. The induced action on $N_n = T^*M$ is also simple to describe. Indeed, our description of the canonical lifting procedure shows that $\mathrm{SU}(n,1)$ acts on the fibers of N_n via the dual representation, so on the level of representations we can describe N_n as the direct sum $\rho_{\mathrm{std}} \oplus \rho^*_{\mathrm{std}}$.

When we twist, moving from N_n to the quaternionic Kähler manifold $\bar{N}_n$, the induced action is c-dependent. This can be seen by inspecting the expressions (5.5), (5.6) for the corresponding Killing fields. Though the action preserves the level sets of ρ for every value of c, the coordinate function $\tilde{\phi}$ is also preserved for $c = 0$. In this case, the action restricts to an action on $\bar{M} \times \{\rho_0\} \times \{\tilde{\phi}_0\} \times \mathbb{R}^{2n+2}$, $\rho_0 \in \mathbb{R}_{>0}$, $\tilde{\phi}_0 \in S^1$, which is nothing but the standard action on $\mathbb{C}\mathrm{H}^n \times (\mathbb{C}^{n+1})^*$ induced by projectivizing the first factor of the action on N.

In the context of the c-map, we should think of the second factor as a symplectic vector space which, together with the coordinate $\tilde{\phi}$, defines the Heisenberg group Heis_{2n+3} acting on the fibers of $N_n = T^*M_n$ as specified by (4.3). Taking the convention that for a symplectic vector space (V, ω), the corresponding Heisenberg group is defined by the group law $(v,t)\cdot(v',t') = (v+v', t+t'+\frac{1}{2}\omega(v,v'))$ on $V \oplus R$, we read off from equation (4.3) that the symplectic structure in this case is given by the two-form $2\sum_{i=0}^n \mathrm{d}\tilde{\zeta}_i \wedge \mathrm{d}\zeta^i$ which is nothing but (a multiple of) the restriction of ω_1 to the fibers of N_n. Our description of the hyper-Kähler structure on N_n given in (4.1) shows that ω_1 restricts to the natural induced (dual) symplectic structure on $(\mathbb{C}^{n+1})^*$, which is invariant under the dual representation ρ_{std}^* of $\mathrm{SU}(n,1)$. We conclude that, in the case $c=0$, $\mathrm{SU}(n,1)$ acts on the Heisenberg group $\mathrm{Heis}_{2n+3} \cong \mathbb{C}^{n+1} \oplus \mathbb{R}$ by automorphisms in the standard way, namely via linear symplectomorphisms on $\mathbb{C}^{n+1}$ and trivially on the second factor.

For positive values of c, however, the lifted Killing fields do not preserve $\tilde{\phi}$, and explicitly integrating the infinitesimal action poses a significant problem. Therefore, we focus on the case $c=0$ in this work. Then, we know that $G_{\bar{M}} \cong \mathrm{SU}(n,1)$ and understand the action on Heis_{2n+3}; our next task is to construct (co-compact) lattices in $\mathrm{SU}(n,1)$ which preserve a lattice in Heis_{2n+3}.

In the case $n=0$, when we need only consider Heis_3, things are rather simple because it is not hard to construct co-compact lattices in Heis_3. Thinking of Heis_3 as $\mathbb{R}^3$, the most obvious example is the standard lattice $\mathbb{Z}^3 \subset \mathbb{R}^3$, which indeed forms a subgroup (for a suitable normalization of the symplectic form on $\mathbb{R}^2$). Since Heis_3 acts freely on $(\bar{N}_0, g^c)$ for every $c \geq 0$, any co-compact lattice Γ gives rise to a manifold $\bar{N}_0/\Gamma$ which is diffeomorphic to $K \times \mathbb{R}$, where $K = \mathrm{Heis}_3/\Gamma$ is compact. Many new complications arise in the first non-trivial case, $n=1$, which we will now study in detail.

Our first step is to construct a lattice in $\mathrm{SU}(1,1)$. It is well-known that $\mathrm{SU}(1,1)$ is isomorphic to $\mathrm{SL}(2,\mathbb{R})$. Therefore, we can equivalently study discrete subgroups of $\mathrm{SL}(2,\mathbb{R})$, which are called Fuchsian groups. The theory of Fuchsian groups is a classical topic in mathematics and much is known about the structure of these groups (see, for instance, [Kat92]). In particular, it is possible to construct many explicit examples of co-compact Fuchsian groups. One of the most important methods of construction of such co-compact Fuchsian groups uses the theory of quaternion algebras. We will outline this construction (following [Kat92] and [Voi20]) and show how it allows us to construct infinitely many examples of co-compact lattices in $\mathrm{SU}(1,1)$ which preserve a lattice in Heis_5.

In the following, F always denotes a field of characteristic different from two.

Definition 6.4. An algebra A over F is called a quaternion algebra if there exist elements $i, j \in A$ such that $\{1, i, j, ij\}$ is an F-basis for A, and such that

$$i^2 = a \qquad\qquad j^2 = b \qquad\qquad ij = -ji$$

for some $a, b \in F^\times$.

Remark 6.5. Alternatively, one can define quaternion algebras as four-dimensional algebras which contain no non-trivial nilpotent two-sided ideals and have center F. The above definition has the advantage of being more concrete, and making the reason for their name obvious.

The fourth basis element ij of a quaternion algebra is conventionally denoted by k, mimicking the standard notation for quaternions.

As a shorthand, we will often denote a quaternion algebra by $A = (a,b)_F$. Observe that $(-1,-1)_{\mathbb{R}}$ yields the usual quaternions, and that $(1,1)_{\mathbb{R}} \cong \mathrm{Mat}_2(\mathbb{R})$. Any quaternion algebra can be realized as a matrix algebra upon adjoining certain elements to F. For example, the map

$$\begin{aligned} \varphi : A &\longrightarrow \mathrm{Mat}_2(F(\sqrt{a},\sqrt{b})) \\ 1 &\longmapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ i &\longmapsto \begin{pmatrix} \sqrt{a} & 0 \\ 0 & -\sqrt{a} \end{pmatrix} \\ j &\longmapsto \begin{pmatrix} 0 & \sqrt{b} \\ \sqrt{b} & 0 \end{pmatrix} \\ k &\longmapsto \begin{pmatrix} 0 & \sqrt{a}\sqrt{b} \\ -\sqrt{a}\sqrt{b} & 0 \end{pmatrix} \end{aligned} \tag{6.1}$$

defines a convenient homomorphism into a matrix algebra, which is an isomorphism onto its image.

The structure constants a, b determine a quaternion algebra, but different choices of do not necessarily yield non-isomorphic algebras. A first observation is that interchanging i and j induces an isomorphism $(a,b)_F \cong (b,a)_F$. It is also not difficult to prove that, if $a \in (F^\times)^2$, $(a,b)_F \cong \mathrm{Mat}_2(F)$ and that for any $\lambda \in F^\times$, $(\lambda^2 a, b)_F \cong (a,b)_F$. This can be used to deduce that, over an algebraically closed field, every quaternion algebra is isomorphic to $\mathrm{Mat}_2(F)$, and that any quaternion algebra over $\mathbb{R}$ is isomorphic to either $\mathbb{H}$ or $\mathrm{Mat}_2(\mathbb{R})$.

Definition 6.6. Let A be a quaternion algebra over F. Then the conjugate of $x = x_0 + x_1 i + x_2 j + x_3 k \in A$ is $\bar{x} = x_0 - x_1 i - x_2 j - x_3 k$. Furthermore, $x + \bar{x} = 2x_0$ is called the reduced trace of x and $x\bar{x} = x_0^2 - x_1^2 a - x_2^2 b + x_3^2 ab$ is called the reduced norm of x.

We can give a natural interpretation for the reduced trace and norm by considering the embedding φ from (6.1). Indeed, the reduced trace and norm of x are $\mathrm{tr}(\varphi(x))$ and $\det(\varphi(x))$, respectively. Note that this shows that the reduced norm behaves multiplicatively.

Lemma 6.7. *A quaternion algebra A over F is a division algebra if and only if, for every non-zero $x \in A$, $x\bar{x} \neq 0$.*

Proof. If A is a division algebra, then for any non-zero x the multiplicativity of the reduced norm implies that $\det(\varphi(x)^{-1})\det(\varphi(x)) = 1$, so $\det(\varphi(x)) \neq 0$. Conversely, if $\det(\varphi(x)) \neq 0$ then $x^{-1} = \bar{x}/\det(\varphi(x))$. □

This lemma can be used to prove the following important generalization of the classification of quaternion algebras over $\mathbb{R}$:

Proposition 6.8. *Let A be a quaternion algebra over F. Then either A is isomorphic to $\mathrm{Mat}_2(F)$, or A is a division algebra.*

Proof. Assuming that $A = (a,b)_F$ is not isomorphic to $\mathrm{Mat}_2(F)$, we know that $a \notin (F^\times)^2$. Then $F(i)$ is a quadratic field extension of F and $F(i) \oplus F(i)j \cong A$. Additionally assuming that A is not a division algebra, we obtain a non-vanishing element x whose reduced norm vanishes, so that

$$0 = \det(\varphi(x)) = (x_0 + x_1 i)(x_0 - x_1 i) - (x_2 + x_3 i)(x_2 - x_3 i)b$$

If the second term vanishes, then so must the first. But the first term is the field norm of $x_0 + x_1 i \in F(i)$, which vanishes if and only if $x_0 + x_1 i = 0$. This would be a contradiction with the assumption that $x \neq 0$, so the second term cannot vanish. Setting $q_0 + q_1 i = \frac{x_0 + x_1 i}{x_2 + x_3 i} \in F(i)$, we may therefore rewrite the vanishing of the reduced norm as $b = q_0^2 - q_1^2 a$. But now the map

$$\begin{aligned} A &\longrightarrow \mathrm{Mat}_2(F) \\ i &\longmapsto \begin{pmatrix} 0 & 1 \\ a & 0 \end{pmatrix} \\ j &\longmapsto \begin{pmatrix} q_0 & -q_1 \\ aq_1 & -q_0 \end{pmatrix} \\ k &\longmapsto \begin{pmatrix} aq_1 & -q_0 \\ aq_0 & -aq_1 \end{pmatrix} \end{aligned}$$

provides an explicit isomorphism $A \cong \mathrm{Mat}_2(F)$, which is a contradiction. □

This yields an alternative criterion for checking whether a quaternion algebra is a division algebra. For instance, let F be a totally real algebraic number field of degree n (i.e. a field extension of $\mathbb{Q}$ of degree n, all of whose embeddings into $\mathbb{C}$ are already contained in $\mathbb{R}$). Then its n distinct embeddings into $\mathbb{R} \subset \mathbb{C}$, which we will denote by φ_i, $i = 1, \ldots, n$, yield new quaternion algebras $A_i = (\varphi_i(a), \varphi_i(b))_{\varphi_i(F)}$. Taking a tensor product with $\mathbb{R}$ yields real quaternion algebras $A_i^{\mathbb{R}}$. We know that $A_i^{\mathbb{R}}$ is isomorphic either to $\mathbb{H}$, in which case we say that A is ramified at φ_i, or to $\mathrm{Mat}_2(\mathbb{R})$, in which case we say A is unramified at φ_i.

Proposition 6.9. *Let F be a totally real algebraic number field of degree $n \geq 2$. If A is a quaternion algebra over F which is unramified at one φ_i, $i = 1, \ldots, n$, and ramified at all others, then A is a division algebra.*

Proof. If A were not a division algebra, then $A \cong \mathrm{Mat}_2(F) \cong (1,1)_F$ and therefore $A_i^{\mathbb{R}} \cong (1,1)_{\varphi_i(F)} \otimes \mathbb{R} \cong (1,1)_{\mathbb{R}} \cong \mathrm{Mat}_2(\mathbb{R})$. This means that A would be unramified at every φ_i. □

This produces many examples of division quaternion algebras over totally real algebraic number fields of degree $n > 1$.

For $F = \mathbb{Q}$, it is also possible to produce infinitely many examples of division quaternion algebras, using lemma 6.7 directly:

Proposition 6.10. *Let $a, b \in \mathbb{N}$ such that b is prime and a is a quadratic non-residue modulo b, i.e. the equation $x^2 \equiv a \mod b$ has no solution in the integers. Then the quaternion algebra $A = (a,b)_{\mathbb{Q}}$ is a division algebra.*

Proof. If $(a,b)_{\mathbb{Q}}$ is not a division algebra, we have a non-zero element $x = x_0 = x_1 i + x_2 j + x_3 k$ with vanishing reduced norm. We may assume that its coefficients have no common factor. Since $x\bar{x} = x_0^2 - x_1^2 a - (x_2^2 - x_3^2 a)b = 0$, we see that $x_0^2 \equiv x_1^2 a \mod b$. If the prime b does not divide x_1, then by the fundamental theorem of arithmetic it also does not divide x_1^2. But then $\left(\frac{x_0}{x_1}\right)^2 \equiv a \mod b$, contradicting the assumption that a is a quadratic non-residue modulo b. We conclude that b must divide x_1, hence also x_0. Writing $x_0 = y_0 b$ and $x_1 = y_1 b$, we now have $(y_0^2 b - y_1^2 b - x_2^2 + x_3^2 a)b = 0$ and thus $x_2^2 \equiv x_3^2 a \mod b$. Going through the same argument as before, we deduce that b divides x_2 and x_3 too. This means that the x_i share b as a common factor, contradicting our assumption. □

At this point, the reader might wonder why we choose to place so much emphasis on the existence of division quaternion algebras. Before providing a justification, we will show how to derive Fuchsian groups from quaternion algebras. To do this, we need to introduce a way to discretize a quaternion algebra.

Definition 6.11. Given an algebraic number field F, the ring of integers $\mathcal{O}_F$ is the ring of all elements of F which are integral elements, i.e. roots of a monic polynomial $x^n + a_1 x^{n-1} + \cdots + a_{n-1}x + a_n$ with integer coefficients $a_1, \ldots, a_n$.

Definition 6.12. An order $\mathcal{O}$ in an algebra A over an algebraic number field F is a subring of A (containing 1) which is a finitely generated $\mathcal{O}_F$-module and which generates A over F.

The concept of an order provides an analog of the notion of a lattice in the context of (quaternion) algebras over a field. A simple yet important example of an order is the following: Suppose $A = (a,b)_F$, where $a, b \in \mathcal{O}_F^{\times}$. Then

$$\mathcal{O}_{\mathrm{std}} = \{x = x_0 + x_1 i + x_2 j + x_3 k \mid x_0, x_1, x_2, x_3 \in \mathcal{O}_F\}$$

is an order in A, which we will call the standard order.

Our aim is to produce a Fuchsian group from a quaternion algebra. We have already introduced a notion of discretization, so the next step is to obtain a group. Fortunately, every order $\mathcal{O}$ in a quaternion algebra contains a natural group, namely the group of units of reduced norm one: $\mathcal{O}^1 = \{x \in \mathcal{O}^\times \mid \det \varphi(x) = 1\}$.

In the case where $A = (a,b)_F$, $a, b \in \mathcal{O}_F^\times$ and $\mathcal{O} = \mathcal{O}_{\mathrm{std}}$, every $x \in \mathcal{O}$ of reduced norm one is invertible. Indeed, the inverse of x is $\bar{x} = (x + \bar{x}) - x$, which lies in $\mathcal{O}_{\mathrm{std}}$ since $x + \bar{x} = 2x_0 \in \mathcal{O}_F$. Therefore, $\mathcal{O}^1_{\mathrm{std}} = \{x \in \mathcal{O} \mid \det \varphi(x) = 1\}$.

Next, we need an embedding of our quaternion algebra into $\mathrm{Mat}_2(\mathbb{R})$. When F is a totally real algebraic number field, we can think of elements of F as real numbers (using the "identity embedding" $F \hookrightarrow \mathbb{R}$). Then, if $a, b \in F^\times$ are positive, we have $A = (a,b)_F \otimes_F \mathbb{R} = (a,b)_{\mathbb{R}} \cong \mathrm{Mat}_2(\mathbb{R})$. Concretely, we use (6.1) to view A as a subalgebra of $\mathrm{Mat}_2(\mathbb{R})$, and observe that, for dimension reasons, it generates all of $\mathrm{Mat}_2(\mathbb{R})$ over $\mathbb{R}$. Restricting the embedding φ to the group $\mathcal{O}^1$, the fact that the reduced norm in A coincides with the determinant of the corresponding matrices implies that we obtain a subgroup of $\mathrm{SL}(2,\mathbb{R})$. Since it is determined by the quaternion algebra A and the order $\mathcal{O}$, it is denoted by $\Gamma(A, \mathcal{O})$.

Theorem 6.13. *Let F be a totally real algebraic number field, and $A = (a,b)_F$, where $a, b > 0$. Then, for any choice of order $\mathcal{O}$, the group $\Gamma(A,\mathcal{O}) \subset \mathrm{SL}(2,\mathbb{R})$ is a Fuchsian group.* □

For a complete proof, see e.g. [Voi20, Ch. 38]. Any Fuchsian group $\Gamma(A, \mathcal{O})$ that arises in this fashion is said to be derived from a quaternion algebra. We are now in a position to state the theorem that justifies our emphasis on division quaternion algebras (for a proof, see [Mor15, Ch. 6]).

Theorem 6.14. *Let $\Gamma(A,\mathcal{O})$ be a Fuchsian group derived from a division quaternion algebra. Then $\Gamma(A,\mathcal{O})$ is a co-compact lattice in $\mathrm{SL}(2,\mathbb{R})$.* □

Remark 6.15. The co-compact lattices $\Gamma(A,\mathcal{O})$ are examples of so-called arithmetic subgroups (in this case of $\mathrm{SL}(2,\mathbb{R}) \cong \mathrm{SU}(1,1)$), and we expect that arithmetic subgroups of $\mathrm{SU}(n,1)$ will play an important role in generalizations of our results to the higher-dimensional members of our family of examples.

Thus, after passing from $\mathrm{SL}(2,\mathbb{R})$ to $\mathrm{SU}(1,1)$, the constructions of propositions 6.9 and 6.10 give us many examples of co-compact lattices in $\mathrm{SU}(1,1)$. The isomorphism we will use to identify $\mathrm{SL}(2,\mathbb{R})$ and $\mathrm{SU}(1,1)$ is given by conjugation by the matrix

$$R = \begin{pmatrix} i & 1 \\ 1 & i \end{pmatrix}$$

The next question is whether the co-compact lattices in $\mathrm{SU}(1,1)$ thus obtained, which we will also denote by $\Gamma(A,\mathcal{O})$, preserve a lattice in Heis_5.

Proposition 6.16. *Let $A=(a,b)_{\mathbb{Q}}$, where $b\in\mathbb{N}$ is prime and $a\in\mathbb{N}$ a quadratic non-residue modulo b. Then the co-compact lattice $\Gamma(A,\mathcal{O}_{\mathrm{std}})\subset\mathrm{SU}(1,1)$ preserves a co-compact lattice in* Heis_5.

Proof. To construct an invariant lattice in Heis_5, we think of Heis_5 as $\mathbb{C}^2\oplus\mathbb{R}$, with group law defined by the symplectic form $\omega=-\operatorname{Im}\langle\cdot,\cdot\rangle$, where $\langle\cdot,\cdot\rangle$ is the standard Hermitian form of signature $(1,1)$ on $\mathbb{C}^2$. Conjugation by R, which we used to define the isomorphism $\mathrm{SL}(2,\mathbb{R})\to\mathrm{SU}(1,1)$, extends to an embedding $\mathrm{Mat}_2(\mathbb{R})\to\mathrm{Mat}_2(\mathbb{C})$, so we can apply it to $\varphi(\mathcal{O}_{\mathrm{std}})\subset\mathrm{Mat}_2(\mathbb{R})$ to realize $\mathcal{O}_{\mathrm{std}}$ as a subring of $\mathrm{Mat}_2(\mathbb{C})$. Applying this procedure to the $\mathbb{Z}$-basis $\{1,i,j,k\}$ of $\mathcal{O}_{\mathrm{std}}$, we obtain the matrices

$$\mathbb{1}=\begin{pmatrix}1&0\\0&1\end{pmatrix}\qquad I=\sqrt{a}i\begin{pmatrix}0&1\\-1&0\end{pmatrix}$$
$$J=\sqrt{b}\begin{pmatrix}0&1\\1&0\end{pmatrix}\qquad K=\sqrt{ab}i\begin{pmatrix}1&0\\0&-1\end{pmatrix}$$

where $i=\sqrt{-1}$. Applying these complex matrices to $e_1=\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)\in\mathbb{C}^2$, we obtain a real basis for $\mathbb{C}^2\cong\mathbb{R}^4$. Taking integer linear combinations, we obtain a lattice $\mathcal{O}_{\mathrm{std}}\cdot e_1$ in $\mathbb{C}^2$. By construction, it is $\mathcal{O}_{\mathrm{std}}$-invariant, hence certainly invariant under the action of $\Gamma(A,\mathcal{O}_{\mathrm{std}})$.

The last step is to extend to Heis_5. In fact, the subgroup of Heis_5 generated by the given lattice in $\mathbb{C}^2$ is a lattice in Heis_5. Since the group law in $\mathrm{Heis}_5\cong\mathbb{C}^2\oplus\mathbb{R}$ is given by $(z,t)\cdot(z',t')=(z+z',t+t'+\frac{1}{2}\omega(z,z'))$ and $\mathrm{SU}(1,1)$ acts trivially on the last factor, it suffices to check that ω takes on a discrete set of values when evaluated on elements $z,z'\in\mathcal{O}_{\mathrm{std}}\cdot e_1$. By linearity, we only have to verify this for the generators e_1, $Ie_1=\sqrt{a}ie_2$, $Je_1=\sqrt{b}e_2$ and $Ke_1=\sqrt{ab}ie_1$. These satisfy

$$\omega(Ie_1,e_1)=0\qquad \omega(Je_1,e_1)=0\qquad \omega(Ke_1,e_1)=-\sqrt{ab}$$
$$\omega(Ie_1,Ke_1)=0\qquad \omega(Je_1,Ke_1)=0\qquad \omega(Ie_1,Je_1)=-\sqrt{ab}$$

and therefore ω takes values in $\sqrt{ab}\mathbb{Z}\subset\mathbb{R}$. This means the subgroup generated by the given lattice in $\mathbb{C}^2$ is a $\Gamma(A,\mathcal{O}_{\mathrm{std}})$-invariant co-compact lattice in Heis_5. In fact, it is nothing but a rescaling (independently in each orthogonal direction) of the standard lattice $\mathbb{Z}^5\subset\mathbb{R}^5$. $\square$

Proposition 6.16 yields infinitely many examples of lattices $\Lambda\subset\mathrm{SU}(1,1)\ltimes\mathrm{Heis}_5$, of the form $\Lambda=\Gamma(A,\mathcal{O}_{\mathrm{std}})\ltimes\Lambda_{\mathrm{Heis}_5}$. Now we return to our original question and consider the corresponding quotients of the quaternionic Kähler manifold $(\bar{N}_1,g^0)$.

Theorem 6.17. *Every co-compact lattice* $\Lambda \subset \mathrm{SU}(1,1) \ltimes \mathrm{Heis}_5$ *obtained by means of proposition 6.16 admits a co-compact sub-lattice* Γ *such that* $(\bar{N}_1, g^0)/\Gamma$ *is a smooth quaternionic Kähler manifold diffeomorphic to* $K \times \mathbb{R}$*, where* K *is a compact, locally homogeneous space.*

Proof. The co-compact lattice Λ does not necessarily act freely on $\bar{N}_1 = \frac{\mathrm{SU}(1,1)}{\mathrm{U}(1)} \times \mathbb{R}_{>0} \times \mathrm{Heis}_5$, but the intersection of Λ with the stabilizer of a point is a discrete and therefore finite subgroup of $\mathrm{U}(1)$. If we can find a co-compact sub-lattice that has trivial intersection with this finite group (i.e. acts freely), we will obtain a smooth quotient manifold with the claimed properties.

The existence of such a subgroup follows from a result known as Selberg's lemma [Mor15, Sec. 4.8]. Selberg's lemma asserts that every finitely generated subgroup of $\mathrm{GL}(n,\mathbb{C})$ admits a finite-index normal subgroup which is torsion-free. In particular, such a subgroup contains no non-trivial finite subgroups. We must check that Λ is indeed finitely generated, so that Selberg's lemma applies to it. But this a general feature of lattices in semi-simple Lie groups, which are in fact always finitely presented [Mor15, Sec. 4.7]. Thus, we obtain a finite-index subgroup $\Gamma \subset \Lambda$ which acts freely on $\bar{N}_1$. Since every finite-index subgroup of a co-compact lattice is itself a co-compact lattice, Γ has all the required properties. □

In summary, we have constructed infinitely many co-compact lattices $\Lambda \subset G = \mathrm{SU}(1,1) \ltimes \mathrm{Heis}_5$, each of which gives rise to a smooth quotient of $(\bar{N}_1, g^0)$ of the form $K \times \mathbb{R}$, where K is a compact, locally homogeneous space whose fundamental group is a finite-index normal subgroup of Λ. The matter of extending these methods to obtain similar results for $(\bar{N}_n, g^c)$ for all $n \in \mathbb{N}$ and $c \geq 0$ is work in progress together with Vicente Cortés and Markus Röser.

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Publication list

This dissertation is based in part on the following pre-prints, co-authored by me:

V. Cortés, A. Saha and D. Thung. "Symmetries of quaternionic Kähler manifolds with S^1-symmetry". arXiv:2001.10026

V. Cortés, A. Saha and D. Thung. "Curvature of quaternionic Kähler manifolds with S^1-symmetry". arXiv:2001.10032

During my time as a doctoral student I co-authored one further article, based on work done as part of my MSc. thesis:

D. Kotschick and D. Thung. "The complex geometry of two exceptional flag manifolds". *Annali di Matematica Pura ed Applicata* (2020). DOI: 10.1007/s10231-020-00965-8.

This article is unrelated to this dissertation.

Eidesstattliche Erklärung

Hiermit erkläre ich an Eides statt, dass ich die vorliegende Dissertationsschrift selbst verfasst und keine anderen als die angegebenen Quellen und Hilfsmittel benutzt habe.

Daniël Konstantin Thung
Hamburg, 17.07.2020