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Modified traces and monadic cointegrals for quasi-Hopf algebras

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Hamburg, 28. Januar 2021 Johannes Berger

Abstract

We introduce the notion of $\boldsymbol{\gamma}$-*symmetrized cointegrals* for a finite-dimensional pivotal quasi-Hopf algebra H over a field k, where $\boldsymbol{\gamma}$ is the modulus of H. In case H is unimodular and k is algebraically closed, we give explicit bijections relating them to non-degenerate left and right modified traces on the tensor ideal of projective H-modules in the (finite tensor) category of finite-dimensional left H-modules, generalizing previous Hopf-algebraic results from [BBGa].

Then we introduce monadic cointegrals in (pivotal) finite tensor categories. For a pivotal finite tensor category $\mathcal{C}$, four versions $A_1, ..., A_4$ of the so-called central Hopf monad exist. A monadic cointegral for A_i is a morphism of A_i-modules $\mathbf{1} \to A_i(D)$, where D is the distinguished invertible object of $\mathcal{C}$; we relate them to Shimizu's categorical cointegral [Sh4], and in the braided case to the integral of Lyubashenko's Hopf algebra $\int^{X\in\mathcal{C}} X^\vee \otimes X$ [Ly1]. If $\mathcal{C}$ is the category of modules over a pivotal Hopf algebra H, then one easily sees that the four monadic cointegrals are given by four notions of cointegrals for H, including $\boldsymbol{\gamma}$-symmetrized cointegrals. We show that this relation, up to non-trivial isomorphisms, remains true if H is a quasi-Hopf algebra, i.e. we relate the cointegrals of Hausser and Nill [HN2] and the $\boldsymbol{\gamma}$-symmetrized cointegrals above to monadic cointegrals for the category of H-modules.

Finally, for a modular tensor category $\mathcal{C}$, we concern ourselves with the projective $SL(2,\mathbb{Z})$-actions (on certain Hom-spaces in $\mathcal{C}$) constructed by Lyubashenko [Ly2]. In the case that $\mathcal{C}$ is the category of modules over a factorizable ribbon quasi-Hopf algebra H, we derive a simple expression for the action of the S and T-generators on the center of H using the monadic cointegral. Let now H be the quasi-Hopf algebra modification of the restricted quantum group of $\mathfrak{sl}_2$ at a primitive $2p$th root of unity as defined in [CGR], for an integer $p \geq 2$. We show that Lyubashenko's action on the center of H agrees projectively with the $SL(2,\mathbb{Z})$-action on the center of the (original) restricted quantum group, as constructed in [FGST1].

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Introduction

"Spinning, twisting, circling on."
— Spiral Architect

Fix an algebraically closed field k; all linear structures are over it.

In this thesis we will first describe modified traces for quasi-Hopf algebras explicitly, and then define, for general (pivotal) finite tensor categories, the notion of monadic cointegrals, which we also specialize to quasi-Hopf algebras. These words will be explained in due course within this introduction. One major motivation for carrying out this research is the application to non-semisimple (generalizations of) topological field theories (TFTs). While TFTs themselves do not appear much in this thesis, let us nonetheless recall some of their basic aspects, and explain where our main objects of interest appear.

A 3-*dimensional topological field theory*, or 3d TFT for short, is a symmetric monoidal functor from a category of 3-dimensional cobordisms to the category of vector spaces. Let us explain what this means. By a category of 3d cobordisms we mean a category Cob whose objects are closed oriented surfaces, possibly with extra structure, and a morphism between such surfaces is a cobordism between them, possibly equipped with extra structure in a suitable sense. The operation of disjoint union is easily seen to equip Cob with the structure of a symmetric monoidal category. Thus a 3d TFT sends surfaces to vector spaces, and cobordisms to linear maps between the respective vector spaces; lastly, the words 'symmetric monoidal' mean that this assignment is compatible with the symmetric monoidal structures on both sides.

In [RT], Reshetikhin and Turaev first constructed a 3d TFT from the input datum of a certain (non-semisimple) Hopf algebra satisfying some conditions. The construction involves passing to a certain semisimple quotient of the category of representations of the input Hopf algebra. This quotient category has the structure of what is now called a *modular fusion category*—a notion we will presently explain—and Turaev [Tu] subsequently gave a construction of a 3d TFT, taking as input any modular fusion category $\mathcal{C}$. This construction is commonly referred to as the RT-construction. Here, the extra structure on Cob is given by certain (possibly empty) decorations of manifolds with objects and morphisms from $\mathcal{C}$. The rough idea is to represent 3-manifolds as framed links via the famous Lickorish-Wallace theorem, evaluate the links via certain canonical data of the category $\mathcal{C}$, and show that this can be extended to a TFT. The evaluation procedure uses the so-called *Kirby color* to decorate components of the surgery link, only to then apply the categorical trace to evaluate the closed string diagram as a number.

Let us now briefly explain what a modular fusion category is, starting more generally with the notion of a *modular tensor category* (for more details see Chapter 1). This is, first of all, a finite tensor category, i.e. a linear abelian category $\mathcal{C}$ which is equivalent to the category of finite-dimensional modules over a finite-dimensional algebra, and which is further equipped with a monoidal structure compatible with the linear structure. Moreover, it has a braiding and a ribbon twist. Finally, the braiding has to satisfy a certain non-degeneracy condition called *factorizability*, see e.g. [Sh3]. If $\mathcal{C}$ is in addition semisimple, then it is called a modular *fusion* category.

Naturally, one would like to generalize the RT-construction to any (not necessarily semisimple) modular tensor category. At first, Hennings [He] managed to construct invariants of 3-manifolds from certain non-semisimple Hopf algebras without using some semisimplification procedure. Lyubashenko generalized Hennings approach to modular tensor categories, using the coend $\mathcal{L} = \int^{X\in\mathcal{C}} X^\vee \otimes X$, and realizing that its integral $\Lambda : \mathbf{1} \to \mathcal{L}$ serves as a non-semisimple generalization of the Kirby color in the definition of non-semisimple invariants of 3-manifolds [Ly2].

However, it was not possible to directly use this to define 3d TFTs. The main reason, as we recall in Proposition 1.2.2, is that for non-semisimple categories the categorical trace vanishes on the class of projective objects. As a consequence, a 3d TFT directly extending Lyubashenko's manifold invariant would assign the 0-dimensional vector space to the 2-sphere, whence the TFT would be trivial [Ke].

To overcome this, the notion of *modified trace* was introduced in [GPV, GKP1]. It provides a generalization of the categorical trace, now only defined on (endomorphisms of) projective objects, and often having very nice non-degeneracy conditions [GKP3, GR3]. More details follow below, but let us mention here that

- the modified trace, and
- the integral of $\mathcal{L}$

play an indispensable role in the construction of a generalization of the RT-construction, taking as input any modular tensor category. The construction was pioneered for Hopf algebras in [DGP], and formulated for general modular tensor categories in [DGGPR]. The cobordism category here does not allow for arbitrary undecorated 3-manifolds, unlike the category used in the RT-construction. Instead, members of a certain subset of cobordisms (including all closed ones) have to be equipped with a ribbon graph with at least one edge colored by a projective object in $\mathcal{C}$, so that one can use the modified trace in the evaluation of the resulting graph. We refer to [DGGPR] for details.

In this thesis, we will investigate modified traces for quasi-Hopf algebras, and define and study the notion of *monadic cointegrals* for finite tensor categories, again specializing to quasi-Hopf algebras. This is interesting for the following reasons. We will see that monadic cointegrals, in the case of pivotal (quasi-)Hopf algebras, provide a uniform setting for both the integral of $\mathcal{L}$ and the modified trace—the two main ingredients in the non-semisimple generalization of the RT-construction in [DGGPR]. Thus one could hope that there is some interesting intrinsic relation between Lyubashenko's integral and the modified trace, which are both known to exist in a modular tensor category. Furthermore, this is potentially interesting from the viewpoint of quasi-Hopf algebras, because the new notion of cointegrals, coming from monadic cointegrals, may be easier to use than the conventional one of [HN2, BC1, BC2]. Finally, several important examples of modular tensor categories arise from factorizable ribbon quasi-Hopf algebras that are (conjecturally) related to some fundamental examples of logarithmic conformal field theories, see [GR1, FGR2, CGR, GLO, Ne].

We will now explain the setting and the results of the thesis at hand in more detail.

Modified traces for quasi-Hopf algebras

Let us first describe modified traces and our results more explicitly. Here we will only talk about *right modified traces*. An analogous statement for the left version is given in the main text.

Let $\mathcal{C}$ be a pivotal finite tensor category, i.e. a finite tensor category together with a fixed monoidal natural isomorphism between left and right duals. Denote by $\mathrm{Proj}(\mathcal{C})$ the full subcategory of $\mathcal{C}$ consisting of projective objects. In the present context, by a *modified trace* (on $\mathrm{Proj}(\mathcal{C})$), we mean a family of linear maps

$$\{\mathsf{t}_P\colon \operatorname{End}_{\mathcal{C}}(P) \to k\}_{P\in\mathrm{Proj}(\mathcal{C})} ,$$

satisfying certain natural conditions, namely cyclicity and compatibility with the categorical trace, see Section 1.2.2. The latter compatibility conditions ensures that certain invariants of framed links defined using the modified trace are well-defined. By [BBGa, Lem. 3.2], the family $\mathsf{t}_\bullet$ is completely and uniquely determined by its value on a projective generator G of $\mathcal{C}$. We therefore say that a linear map $t\colon \operatorname{End}_{\mathcal{C}}(G) \to k$ *extends* to a modified trace if there is a modified trace $\mathsf{t}_\bullet$ such that $t = \mathsf{t}_G$. Given a modified trace, one immediately gets pairings of Hom-spaces $\mathcal{C}(M,P)\times\mathcal{C}(P,M) \to k$, $(g,f) \mapsto \mathsf{t}_P(gf)$ for all projective objects P and all objects $M \in \mathcal{C}$. The modified trace $\mathsf{t}_\bullet$ is called *non-degenerate* iff all of these pairings are non-degenerate, and it is known that a non-degenerate modified trace exists if $\mathcal{C}$ is unimodular [GKP3]. Note that these non-degenerate pairings are in general quite different from those induced by the categorical trace: the latter are identically zero whenever one of the objects involved is projective, unless the category is semisimple; we recall this fact in Proposition 1.2.2.

Let now H be a finite-dimensional quasi-Hopf algebra. Thus, H is an algebra equipped with a counit ε and a comultiplication Δ that is coassociative up to conjugation with an invertible element $\Phi \in H^{\otimes 3}$. Moreover, it admits an antipode S, and evaluation and coevaluation elements $\boldsymbol{\alpha}, \boldsymbol{\beta} \in H$. These data have to satisfy some axioms, for which we refer to Chapter 2. It is well-known that H admits left and right (co)integrals [HN2]. A right cointegral is an element $\boldsymbol{\lambda} \in H^*$, which is, up to scalar, uniquely determined by a certain linear equation (we will soon discuss this in a bit more detail for Hopf algebras in the context of monadic cointegrals). A left integral is an element $\Lambda^l \in H$ such that $h\Lambda^l = \varepsilon(h)\Lambda^l$, for all $h \in H$, and right integrals are defined similarly. Moreover, there is an algebra map $\boldsymbol{\gamma} \in H^*$, called the *modulus* of H, which encodes the difference between left and right integrals.

Suppose now that $(H, \boldsymbol{g})$ is a unimodular pivotal quasi-Hopf algebra. Unimodularity means that $\boldsymbol{\gamma} = \varepsilon$, while pivotality means that $\boldsymbol{g} \in H$ satisfies some axioms which can be found in Section 2.2.1. We denote by ${}_H\mathcal{M}$ the category of finite-dimensional left H-modules—this is a pivotal finite tensor category, and the pivotal structure is governed by $\boldsymbol{g}$. For a linear form $\lambda \in H^*$, define $\widehat{\lambda} \in H^*$ by $\widehat{\lambda}(h) = \lambda(\boldsymbol{g}h)$ for $h \in H$. In Corollary 3.1.5, we show that $\boldsymbol{\lambda}$ is a right cointegral if and only if

$$(\widehat{\boldsymbol{\lambda}} \otimes \boldsymbol{g})\Big(q^R \Delta(h) p^R\Big) = \widehat{\boldsymbol{\lambda}}(h)\mathbf{1} \tag{1}$$

holds, for all $h \in H$. Here, $\mathbf{1} \in H$ is the unit of H, and $q^R, p^R \in H \otimes H$ are elements that are ubiquitous when dealing with quasi-Hopf algebras, see Section 2.1.6 for their definitions. We remark that, in fact, we prove a stronger version, Lemma 3.1.1, of this statement in the non-unimodular case, and the above is then simply the specialization to the unimodular case. A solution $\widehat{\boldsymbol{\lambda}}$ to (1) automatically induces a symmetric bilinear form on H. We therefore call $\widehat{\boldsymbol{\lambda}}$ a *right symmetrized cointegral*.

The aim now is to relate this to modified traces. The canonical choice of projective generator for ${}_H\mathcal{M}$ is the regular module H, and thus a modified trace is completely determined by an element of $H^* \cong (\mathrm{End}_H(H))^*$, subject to some conditions. Our first main result, Theorem 3.2.5 (1), says that we do not lose generality by requiring H to be unimodular:

Theorem 1. *Let H be a finite-dimensional pivotal quasi-Hopf algebra over k. Then a non-degenerate right modified trace on* $\mathrm{Proj}({}_H\mathcal{M})$ *exists iff H is unimodular.*

In Theorem 3.2.5 (2) and (3), we then obtain

Theorem 2. *Let H be as in Theorem 1 and unimodular. Then $\boldsymbol{\lambda} \in H^*$ is a right cointegral if, and only if, the linear map*

$$\mathrm{End}_H(H) \to k, \quad f \mapsto \widehat{\boldsymbol{\lambda}}\Big(f(\mathbf{1})\Big)$$

extends to a right modified trace on $\mathrm{Proj}({}_H\mathcal{M})$.

This in particular states that, in the unimodular case, such modified traces exist and are unique (up to scalar). Moreover, since a non-zero right cointegral provides a non-degenerate form on H, it immediately gives that non-zero right modified traces on $\mathrm{Proj}({}_H\mathcal{M})$ are non-degenerate.

Theorem 2 generalizes results on modified traces for Hopf algebras from [BBGa], and the proofs are essentially step-by-step generalizations of the corresponding proofs in [BBGa], although markedly more delicate because of the quasi-Hopf data involved. As mentioned before, the linear form $\widehat{\boldsymbol{\lambda}}$ from Theorem 2 is symmetric. Therefore the main point in the proof will be to show that the partial trace condition (i.e. the sensible compatibility condition between modified trace and categorical trace) boils down and is equivalent to (1), the equation defining the symmetrized right cointegral up to scalar.

We remark that the notion of a modified trace, and also the theorems above, do not require k to be algebraically closed; however, because it fits in better with our conventions for finite tensor categories, we prefer to keep k algebraically closed throughout.

Finally, we compute the modified trace in an example. In [FGR2], the family $\mathsf{Q}(N, \beta)$ of *symplectic fermion* quasi-Hopf algebras was introduced, where $N \in \mathbb{N}_{>0}$ and $\beta \in \mathbb{C}$ satisfy $\beta^4 = (-1)^N$. These are all non-semisimple factorizable ribbon quasi-Hopf algebras over $\mathbb{C}$, and in particular $\mathcal{C} = {}_{\mathsf{Q}(N,\beta)}\mathcal{M}$ is an example of a non-semisimple modular tensor category. In Section 3.3, we explicitly compute the left and right cointegral of $\mathsf{Q}(N, \beta)$, as well as the corresponding left and right modified trace on projective modules. Note that our result agrees with the computation of the modified trace in [GR3, Sec. 9], which was carried out using a different method. We remark that the method investigated in this part of this thesis seems to lend itself more readily to computations.

Monadic cointegrals (for quasi-Hopf algebras)

We now go more in depth on our definition of and results on monadic cointegrals, adhering mostly to the general outline of Chapter 4.

Monadic cointegrals. Let us begin by considering something easier than a quasi-Hopf algebra, namely a (finite-dimensional) Hopf algebra—i.e. a quasi-Hopf algebra H with $\Phi = \mathbf{1}^{\otimes 3}$ and $\boldsymbol{\alpha} = \mathbf{1} = \boldsymbol{\beta}$. As mentioned above, H possesses left and right cointegrals. These are elements $\boldsymbol{\lambda} \in H^*$ which satisfy, for all $h \in H$,

$$\begin{array}{lll} 2) & (\boldsymbol{\lambda} \otimes \mathrm{id}) \circ \Delta(h) = \boldsymbol{\lambda}(h)\mathbf{1} & \text{(right cointegral)}\ , \\ 3) & (\mathrm{id} \otimes \boldsymbol{\lambda}) \circ \Delta(h) = \boldsymbol{\lambda}(h)\mathbf{1} & \text{(left cointegral)}\ . \end{array}$$

The unusual numbering will be explained below.

In the coassociative case, a pivot (as mentioned above) for H is simply a grouplike element $\boldsymbol{g}$ satisfying $S^2(h) = \boldsymbol{g}h\boldsymbol{g}^{-1}$ for all $h \in H$. Recall that $\boldsymbol{\gamma} \in H^*$, the *modulus*, was a certain algebra morphism encoding the difference between left and right integrals. Given now a pivotal Hopf algebra H with pivot $\boldsymbol{g}$, one can introduce two more notions of cointegrals, so-called left and right $\boldsymbol{\gamma}$-symmetrized cointegrals [BBGa, FOG]. The defining equation for $\boldsymbol{\lambda} \in H^*$ to be a left/right $\boldsymbol{\gamma}$-symmetrized cointegral is

$$\begin{array}{lll} 1) & (\boldsymbol{\lambda} \otimes \mathrm{id}) \circ \Delta(h) = \boldsymbol{\lambda}(h)\boldsymbol{g}^{-1} & \text{(right } \boldsymbol{\gamma}\text{-symmetrized cointegral)}\ , \\ 4) & (\mathrm{id} \otimes \boldsymbol{\lambda}) \circ \Delta(h) = \boldsymbol{\lambda}(h)\boldsymbol{g} & \text{(left } \boldsymbol{\gamma}\text{-symmetrized cointegral)}\ . \end{array}$$

The first of these equations is the specialization of the right symmetrized cointegral equation (1) to a Hopf algebra. Note that for Hopf algebras, the equation defining e.g. a right $\boldsymbol{\gamma}$-symmetrized integral is in fact the same in the unimodular and the non-unimodular case—this is not true for quasi-Hopf algebras. We will see in Proposition 3.1.3 that a right $\boldsymbol{\gamma}$-symmetrized cointegral $\boldsymbol{\lambda}$ satisfies $\boldsymbol{\lambda}(ab) = \boldsymbol{\gamma}(b_{(1)})\boldsymbol{\lambda}(b_{(2)}a)$, for all $a, b \in H$. Here we used sumless Sweedler notation to express $\Delta(b) = b_{(1)} \otimes b_{(2)}$, see Notation 2.1.1. Thus

$\boldsymbol{\lambda}$ is symmetric up to something involving $\boldsymbol{\gamma}$, explaining the name. We remark that $\boldsymbol{\gamma}$-symmetrized cointegrals are an example of g-cointegrals for a group-like g as introduced in [Ra1].

As above, ${}_H\mathcal{M}$ is a finite tensor category in the sense of [EGNO]. It is even pivotal iff H is pivotal, see also [AAGTV]. A natural question is now: Can we describe the two (or four in the pivotal case) notions of cointegrals from above only in terms of the representation category ${}_H\mathcal{M}$, and can we do it in such a way that it readily generalizes to arbitrary (pivotal) finite tensor categories $\mathcal{C}$? The answer to this is 'yes', and it has been done for left cointegrals in [Sh4], using the language of Hopf comonads in $\mathcal{C}$. The main results of the second part of this thesis will concern the other (three) notions of cointegrals in the more general setting of (pivotal) quasi-Hopf algebras. Instead of working with Hopf comonads like [Sh4], we will use the dual notion of *Hopf monads*.

Hopf monads on a rigid monoidal category $\mathcal{C}$ were introduced in [BV1]. Recall first that a monad M on $\mathcal{C}$ is a monoid in the category of endofunctors of $\mathcal{C}$, which is monoidal under composition. A *Hopf monad* is a monad M on $\mathcal{C}$, together with the following extra data: firstly, the functor M is lax comonoidal, i.e. there are morphisms

$$\begin{aligned} &\text{(comultiplication)} \qquad && M(X \otimes Y) \to M(X) \otimes M(Y) \ , \qquad && \text{natural in } X, Y \ , \\ &\text{(counit)} && M(\mathbf{1}) \to \mathbf{1} \ , && \end{aligned}$$

satisfying certain natural conditions; moreover, the multiplication and the unit of the monad M are required to be comonoidal as well. Finally, M has (unique) left and right antipodes, again given by certain natural transformations, see Section 1.2.4 for details. A *module* over a monad M is an object $V \in \mathcal{C}$ together with an *action* $M(V) \to V$ of the monad. For a Hopf monad, the category $\mathcal{C}_M$ of M-modules is again a rigid monoidal category [BV1], hence the name. A simple example of a Hopf monad on Vect is given by tensoring with a (finite-dimensional) Hopf algebra.

Let now $\mathcal{C}$ be a finite tensor category. The four monads we are interested in are all given in terms of coends:

$$\begin{aligned} A_1(V) &= \int^{X \in \mathcal{C}} {}^{\vee}X \otimes (V \otimes X) \ , \qquad & A_2(V) &= \int^{X \in \mathcal{C}} X^{\vee} \otimes (V \otimes X) \ , \\ A_3(V) &= \int^{X \in \mathcal{C}} (X \otimes V) \otimes {}^{\vee}X \ , & A_4(V) &= \int^{X \in \mathcal{C}} (X \otimes V) \otimes X^{\vee} \ . \end{aligned}$$

Here ${}^{\vee}X$ denotes the right dual and $X^{\vee}$ the left dual of an object $X \in \mathcal{C}$ (see Chapter 1 below for our conventions on rigid categories). Since $\mathcal{C}$ is finite, these coends exist, see e.g. Proposition 1.2.6. Note that the index i on A_i is meant to suggest the "position" of the duality symbol $\vee$.

The above coends extend to endofunctors on $\mathcal{C}$, and it turns out that A_2 and A_3 are Hopf monads [BV2, Sec. 5.4], called *central monads*. These are isomorphic as Hopf monads. In Proposition 4.1.2, we prove that, for $\mathcal{C}$ pivotal, the functors A_1 and A_4 are Hopf monads as well, and moreover that all A_i are canonically isomorphic as Hopf monads. From now on it should be understood that, whenever we speak about A_1 and A_4, we assume that $\mathcal{C}$ is pivotal.

Now we come to the main definition of the second part of this thesis, namely the four types of *monadic cointegrals*. The monoidal unit $\mathbf{1} \in \mathcal{C}$ carries a natural A_i-action via the counit of the Hopf monad; this is the trivial module. Moreover, for any object $V \in \mathcal{C}$, the object $A_i(V)$ is canonically equipped with the free A_i-module structure induced by the multiplication of A_i. Let D be the distinguished invertible object of $\mathcal{C}$, see Section 1.1.3. Then a *monadic cointegral for* A_i is a morphism $\boldsymbol{\lambda}_i\colon \mathbf{1} \to A_i(D)$ of A_i-modules from the trivial module to the free module on D. Thus $\boldsymbol{\lambda}_i$ is a morphism in $\mathcal{C}$ such that

$$\begin{tikzcd} A_i(\mathbf{1}) \arrow[r, "A_i(\boldsymbol{\lambda}_i)"] \arrow[d, "\epsilon_i"'] & A_i^2(D) \arrow[d, "\mu_i(D)"] \\ \mathbf{1} \arrow[r, "\boldsymbol{\lambda}_i"'] & A_i(D) \end{tikzcd} \tag{2}$$

commutes, where ϵ_i and μ_i denote the counit and multiplication of A_i, respectively.

We remark that if $\mathcal{C}$ is unimodular (i.e. if $D \cong \mathbf{1}$), then this definition of a cointegral for a Hopf monad actually first appeared in [BV1, Sec. 6.3].

In Corollary 4.1.11 we show that a monadic cointegral for A_2 is related to the "categorical cointegral" of [Sh4] via an isomorphism. This in fact is a piece of the proof of our first major result of this part, concerning existence and uniqueness of all four types of monadic cointegrals (see Proposition 4.1.8).

Theorem 3. *For a finite tensor category $\mathcal{C}$, non-zero monadic cointegrals for A_2, A_3 (and for A_1, A_4 if $\mathcal{C}$ is pivotal) exist and are unique up to scalar multiples.*

To prove this, we exploit the fact that the central Hopf monad A_2 is left adjoint to the central Hopf comonad Z_4 used in [Sh4], which induces an isomorphism between the rigid categories of A_2-modules and Z_4-comodules; this implies the isomorphism in Corollary 4.1.11. The claim then follows from the existence and uniqueness statement in [Sh4], and the fact that all four Hopf monads are isomorphic as Hopf monads.

One might now object that it is not useful to keep all four versions of the central Hopf monads and all four versions of the respective cointegrals—we have, after all, just proved that they are all equivalent. However, there is the following good reason for doing so. Let H be a finite-dimensional Hopf algebra, and $\mathcal{C} = {}_H\mathcal{M}$. The distinguished invertible object D in ${}_H\mathcal{M}$ is the one-dimensional module with action given by the algebra morphism $\boldsymbol{\gamma}^{-1} = \boldsymbol{\gamma} \circ S \in H^*$, where $\boldsymbol{\gamma}$ is the modulus of H. Each Hopf monad A_i has a particularly natural realization in $\mathcal{C}$, see Example 4.1.4 (again, whenever $i = 1, 4$, we require H be in addition pivotal). It turns out that $A_i(D) = H^*$ as a vector space, and a monadic cointegral for A_i may therefore be thought of as an element of H^*, intertwining certain

H-actions and making the diagram in (2) commute. One finds that

$$\boldsymbol{\lambda} \text{ is a mon. coint. for } \begin{cases} A_1 \\ A_2 \\ A_3 \\ A_4 \end{cases} \Leftrightarrow \quad \boldsymbol{\lambda} \text{ is a } \begin{cases} \text{right } \boldsymbol{\gamma}\text{-sym.} \\ \text{right} \\ \text{left} \\ \text{left } \boldsymbol{\gamma}\text{-sym.} \end{cases} \text{coint. for } H \, , \tag{3}$$

explaining the numbering at the beginning of this section. This is more explicitly studied in Example 4.1.7.

Thus, keeping all four variants of A_i explicitly showcases a unified treatment of the four notions of cointegrals for Hopf algebras, while from the isomorphism between the different Hopf monads we get an isomorphism between the corresponding spaces of monadic cointegrals, and hence between the spaces of cointegrals.

The quasi-coassociative case. Let now H be a finite-dimensional quasi-Hopf algebra. As in the coassociative case, $A_i(D) = H^*$ as vector spaces, see Appendix A for a proof. More generally, our realizations of $A_i(V)$ for any H-module V can be found in Section 4.2.1. From this, one can describe the different monadic cointegrals for H via equations involving the quasi-Hopf data: one equation from the intertwining condition, and one equation from (2). We remark that for Hopf algebras, the latter implies the former, while the analogous statement for quasi-Hopf algebras remains to be shown.

The main result of this part of the thesis is the generalization of the relations in (3) to quasi-Hopf algebras, i.e. the precise relation between the four kinds of monadic cointegrals and the four types of cointegrals for quasi-Hopf algebras – left and right cointegrals from [HN2], and the $\boldsymbol{\gamma}$-symmetrized versions introduced in Chapter 3, see also [SS].

Theorem 4. *We have the bijections from the various types of cointegrals $\boldsymbol{\lambda} \in H^*$ for the finite-dimensional quasi-Hopf algebra H to the corresponding types of monadic cointegrals in ${}_H\mathcal{M}$ as shown in Table 1.*

This is the content of Theorems 4.3.1 and 4.3.3. We prove the relation for right cointegrals explicitly, a major step will be relating a comonad on the category of $H \otimes H^{\mathrm{op}}$-modules—which is used in the definition of cointegrals from [HN2]—to the central Hopf comonad Z_4 in a specific way. The details are beyond the scope of this introduction, so we refer to Section 4.3 and Appendix B.2 for more on the strategy, and the proof itself. The proofs for the other three versions are not explicit, but instead rely on the following strategy, which we roughly sketch for left cointegrals. Consider the diagram:

$$\begin{tikzcd} \{\text{right cointegrals}\} \arrow[rr, "(i)"] \arrow[d, "(a)"'] & & \{\text{monadic cointegrals for } A_2\} \arrow[d, "(b)"] \\ \{\text{left cointegrals}\} \arrow[r, "(ii)"] & {}_H\mathcal{M}\big(\mathbf{1}, A_3(D)\big) \supset & \{\text{monadic cointegrals for } A_3\} \end{tikzcd}$$

If the quasi-Hopf cointegral $\boldsymbol{\lambda}$ is ...	then the element of H^* given by ...	is a monadic cointegral for ...
right $\boldsymbol{\gamma}$-sym.	$\boldsymbol{\lambda}\Big(S(\boldsymbol{\beta})\,?\,S^{-1}(\boldsymbol{\vartheta})\Big)$	A_1
right	$\boldsymbol{\lambda}\Big(S(\boldsymbol{\beta})\,?\,S^{-1}(\boldsymbol{\xi})\Big)$	A_2
left	$\boldsymbol{\lambda}\Big(S^{-2}(\boldsymbol{\beta})\,?\,S(\hat{\boldsymbol{\xi}})\Big)$	A_3
left $\boldsymbol{\gamma}$-sym.	$\boldsymbol{\lambda}\Big(\boldsymbol{\beta}\,?\,S(\hat{\boldsymbol{\vartheta}})\Big)$	A_4

Table 1: The relations between the different notions of cointegrals. Here, " ? " is a placeholder, and $\boldsymbol{\vartheta}$, $\boldsymbol{\xi}$, $\hat{\boldsymbol{\xi}}$, $\hat{\boldsymbol{\vartheta}}$ are certain elements of H defined in Section 4.3. For cases 1 and 4, H is of course required to be pivotal. If H is a Hopf algebra, then $\boldsymbol{\beta} = \boldsymbol{\vartheta} = \boldsymbol{\xi} = \hat{\boldsymbol{\xi}} = \hat{\boldsymbol{\vartheta}} = \mathbf{1}$ and one recovers the simple relation in (3).

Here, (i) and (ii) are the maps from Table 1. For the bottom row we also use that (ii) indeed maps a left cointegral to the indicated Hom-space, as shall be proved later. The map (a) is an isomorphism between the spaces of left and right cointegrals from [BC2], while the map (b) is induced by the monad isomorphism $A_2 \cong A_3$. Since every space except for the Hom-space is known to be one-dimensional, we can conclude that (ii) maps left cointegrals to monadic cointegrals for A_3 once we have shown that the two paths in the diagram act identically on a right cointegral. This is a calculation, and the statements for the monadic cointegrals for A_1 and A_4 are shown similarly.

Monadic cointegrals for braided finite tensor categories. Let $\mathcal{C}$ be a braided finite tensor category. The coend $\mathcal{L} = \int^{X\in\mathcal{C}} X^\vee \otimes X$ is a Hopf algebra in $\mathcal{C}$, cf. [LM, Ly1] and also [FGR1]. One may now use the braiding of $\mathcal{C}$ and the universal property of coends to construct isomorphisms

$$\xi_V : A_2(V) \to \mathcal{L} \otimes V$$

natural in V, and it can further be shown that they provide an isomorphism of Hopf monads $A_2 \cong \mathcal{L}\otimes ?$.

A notion of (co)integrals for Hopf algebras H in braided categories exists: integrals are certain morphisms from the *object of integrals* $\operatorname{Int} H \in \mathcal{C}$ to H, while cointegrals go in the other direction, see [KL, Ch. 4]. We recall this in more detail in Section 4.5. In essence, an integral of H is a morphism of H-modules from the H-module $\operatorname{Int} H$ (on which, up to unitors, H acts by the counit) to the 'regular' H-module H (where the action is multiplication). However, here one needs to make the distinction between left or right H-modules, resulting in the definition of left or right integrals of H.

We show that the object of integrals of $\mathcal{L}$ is dual to the distinguished invertible object, i.e. $\operatorname{Int}\mathcal{L} \cong D^\vee \cong {}^\vee D$, and moreover relate monadic cointegrals for A_2 to both left and right integrals of $\mathcal{L}$ in Proposition 4.5.1. More precisely, we show for example

Theorem 5. *Let $\mathcal{C}$ be a braided finite tensor category. Then $\Lambda\colon {}^\vee D \to \mathcal{L}$ is a left integral for $\mathcal{L}$ if and only if*

$$\boldsymbol{\lambda} = \left[\mathbf{1} \xrightarrow{\widetilde{\mathrm{coev}}_D} {}^\vee D \otimes D \xrightarrow{\Lambda\otimes\mathrm{id}} \mathcal{L} \otimes D \xrightarrow{\xi_D^{-1}} A_2\left(D\right)\right]$$

is a monadic cointegral for A_2.

A corresponding formula exists for right integrals. The proof is chiefly a computation using the Hopf monad isomorphism ξ from above, and as a result we get a new proof of the fact that left and right integrals for $\mathcal{L}$ agree if $\mathcal{C}$ is unimodular, see Remark 4.5.2.

In Section 4.5.3, these results are specialized to the case $\mathcal{C} = {}_H\mathcal{M}$ for H a finite-dimensional quasi-triangular quasi-Hopf algebra, yielding explicit formulas. In particular, in the unimodular case the monadic cointegral for A_2 and the left (or right) integral for $\mathcal{L}$ are given by the same linear form on H.

Comparing $SL(2,\mathbb{Z})$-actions

In the third part we concern ourselves with projective $SL(2,\mathbb{Z})$-actions.

Review and simplification. A (not necessarily semisimple) ribbon tensor category is called *modular* if its braiding satisfies a certain non-degeneracy condition called *factorizability*, see Section 1.1.10. Given a modular tensor category $\mathcal{C}$, Lyubashenko [Ly2] constructed projective representations of the mapping class group of a genus g surface on the Hom-space $\mathcal{C}(\mathbf{1},\mathcal{L}^{\otimes g})$. The integral of $\mathcal{L}$ (a modular tensor category is automatically unimodular, so the integral is both left and right) is an essential ingredient in the construction. Of particular interest to us is the action of $SL(2,\mathbb{Z})$, the mapping class group of the torus, on $\mathcal{C}(\mathbf{1},\mathcal{L})$; this space can be shown to be isomorphic to the endomorphisms of the identity functor $\mathrm{id}_\mathcal{C}$ of $\mathcal{C}$. We recall the definition of the projective $SL(2,\mathbb{Z})$-action on $\mathcal{C}(\mathbf{1},\mathcal{L})$, $\mathcal{C}(\mathcal{L},\mathbf{1})$, and $\mathrm{End}(\mathrm{id}_\mathcal{C})$ in Section 5.1.

Specializing to $\mathcal{C} = {}_H\mathcal{M}$ for H a finite-dimensional factorizable ribbon quasi-Hopf algebra, in [FGR1] explicit expressions were derived for the $SL(2,\mathbb{Z})$-action on $Z(H) \cong \mathrm{End}(\mathrm{id}_\mathcal{C})$, the center of H. We simplify these expressions, and rewrite them in terms of monadic cointegrals for ${}_H\mathcal{M}$. For example, let $\mathbf{S}$ and $\mathbf{T}$ be the generators of $SL(2,\mathbb{Z})$, and set $\boldsymbol{\alpha}Z = \{\boldsymbol{\alpha}z \mid z \in Z(H)\}$, where $\boldsymbol{\alpha}$ was the evaluation element of H. One may show that $\boldsymbol{\alpha}Z \cong {}_H\mathcal{M}(\mathcal{L},\mathbf{1})$. We then find in Proposition 5.2.1 that the action on $\boldsymbol{\alpha}Z$ is given by the simple formulas

$$\mathbf{S}.(\boldsymbol{\alpha}z) = \boldsymbol{\lambda}\Big(\widehat{\omega}_1 z\Big)\,\widehat{\omega}_2 \quad \text{and} \quad \mathbf{T}.(\boldsymbol{\alpha}z) = \boldsymbol{v}^{-1}\boldsymbol{\alpha}z\ . \tag{4}$$

Here, $\boldsymbol{\lambda}$ is a monadic cointegral for A_2, $\widehat{\omega}_{1,2}$ are the components of a canonical Hopf-pairing $\omega\colon \mathcal{L}\otimes\mathcal{L} \to \mathbf{1}$, and $\boldsymbol{v}$ is the ribbon element of H.

Comparing two actions. Let $p \geq 2$ be an integer. It is well-known, see [CGR] and references therein, that $\overline{U}_q(\mathfrak{sl}_2)$, the restricted quantum group of $\mathfrak{sl}_2$ at q a primitive $2p$th root of unity, is not factorizable—in fact, it was shown in [KS, GR1] that it does not admit a quasi-triangular structure. Therefore, Lyubashenko's construction of the projective $SL(2,\mathbb{Z})$-action does not apply. Nevertheless, in [FGST1] an action on the center of $\overline{U}_q(\mathfrak{sl}_2)$ was constructed, using a trick: it was observed that $\overline{U}_q(\mathfrak{sl}_2)$ can be realized as a subalgebra of a quasi-triangular ribbon Hopf algebra, whose ribbon element actually lives in $\overline{U}_q(\mathfrak{sl}_2)$. Moreover, the larger Hopf algebra is not factorizable: its monodromy element is in $\overline{U}_q(\mathfrak{sl}_2) \otimes \overline{U}_q(\mathfrak{sl}_2)$, and in fact provides a non-degenerate copairing for $\overline{U}_q(\mathfrak{sl}_2)$. Thus, though not formally correct, one could think of $\overline{U}_q(\mathfrak{sl}_2)$ as factorizable and ribbon. Using these data, an $SL(2,\mathbb{Z})$-action on $Z(\overline{U}_q(\mathfrak{sl}_2))$ was constructed in [FGST1, Thm. 5.2].

The motivation for the construction in [FGST1] comes from mathematical physics, more precisely from the theory of vertex operator algebras (VOAs). A VOA is an algebraic gadget, which may informally be described as an infinite-dimensional $\mathbb{Z}$-graded vector space with a family of multiplications parameterized by $z \in \mathbb{C}$, plus some axioms. It admits a sensible notion of module, and if it moreover satisfies some finiteness conditions (see e.g. [GR2, Sec. 5]), then its modules are conjectured to form a modular tensor category $\operatorname{Rep} \mathcal{V}$. In fact, if $\mathcal{V}$ is a so-called *rational* VOA (and satisfies the above finiteness conditions), then it is known that $\operatorname{Rep} \mathcal{V}$ is a modular fusion category [Hu]. If $\mathcal{V}$ is a *logarithmic* VOA (+ finiteness conditions), then it is only known that $\operatorname{Rep} \mathcal{V}$ is finite abelian with a braided monoidal structure and simple tensor unit, see e.g. [GR2, Thm. 5.1] for a concise statement collecting multiple references.

In [Zh], Zhu introduced the space of so-called *torus 1-point functions* $C_1(\mathcal{V})$ and showed that it comes with a natural $SL(2,\mathbb{Z})$-action, see also [GR2, Sec. 4]. In the rational case above, the VOA-characters of simple $\mathcal{V}$-modules form a basis of $C_1(\mathcal{V})$, and Zhu's action agrees with the one obtained from the RT-construction for $\operatorname{Rep} \mathcal{V}$ [Hu]. This is no longer true for logarithmic VOAs. However, in this case a subspace of $C_1(\mathcal{V})$ was shown to be spanned by *pseudo-trace functions* [AN], and it is conjectured that $C_1(\mathcal{V})$ is isomorphic to the center of $\operatorname{End}_{\mathcal{V}}(G) \cong \operatorname{Rep} \mathcal{V}(\mathcal{L}, \mathbf{1})$, where G is a projective generator of $\operatorname{Rep} \mathcal{V}$ [GR2, Conj. 5.8]. In light of these conjectures, it makes sense to ask if the $SL(2,\mathbb{Z})$-action on the VOA side agrees with the one coming from Lyubashenko's construction for the modular tensor category $\operatorname{Rep} \mathcal{V}$.

The logarithmic VOA considered in [FGST1], called the triplet W-algebra $\mathcal{W}(p)$, has been studied extensively, see [CGR] and references therein. In [FGST1, Thm. 5.2] it is shown that the $SL(2,\mathbb{Z})$-action on the center of $\overline{U}_q(\mathfrak{sl}_2)$ constructed in the same paper is equivalent to the one on $\mathcal{W}(p)$-characters mentioned above. It is known that $\operatorname{Rep} \mathcal{W}(p)$ is equivalent to $\overline{U}_q(\mathfrak{sl}_2)$-mod as an abelian category. However, in light of the fact that $\overline{U}_q(\mathfrak{sl}_2)$ does not admit a quasi-triangular structure, there can be no such equivalence of modular tensor categories. In [CGR], a quasi-Hopf modification $\overline{U}_q^{\Phi}(\mathfrak{sl}_2)$ of $\overline{U}_q(\mathfrak{sl}_2)$ is introduced. It has same algebra structure as $\overline{U}_q(\mathfrak{sl}_2)$ but a modified coproduct. Thus its representation category agrees with that of $\overline{U}_q(\mathfrak{sl}_2)$ as a linear category, but not as a monoidal one. It was shown that $\overline{U}_q^{\Phi}(\mathfrak{sl}_2)$ is factorizable and ribbon, and we may thus use it as input to Lyubashenko's construction described earlier. Since in $\overline{U}_q(\mathfrak{sl}_2)$ the evaluation element satisfies $\boldsymbol{\alpha} = \mathbf{1}$, Lyubashenko's action on the center is given by (4).

It is further conjectured in [CGR, Conj. 1.2] that the above categories agree as ribbon categories. Theorem 5.3.5, the main result of this part, then, relates these two (projective) actions: the one coming from the modular transformation properties of VOA-characters, and the one coming from the (conjecturally) corresponding modular tensor category.

Theorem 6. *Denote by Z_{FGST} the $SL(2,\mathbb{Z})$-representation on the center of the algebra $\overline{U}_q(\mathfrak{sl}_2)$ constructed in [FGST1], and by $Z_{\mathcal{L}}$ the projective representation on the same space coming from Lyubashenko's construction applied to $\overline{U}_q^{\Phi}(\mathfrak{sl}_2)$. Let $\mathcal{S}_Z$ be the endomorphism by which the generator $\boldsymbol{S}$ acts on $Z_{\mathcal{L}}$. Then*

$$\mathcal{S}_Z \colon Z_{\mathcal{L}} \to Z_{\mathrm{FGST}} \tag{5}$$

is an isomorphism of projective $SL(2,\mathbb{Z})$-representation.

To prove this, we have to do some tedious calculations in $\overline{U}_q^{\Phi}(\mathfrak{sl}_2)$. Among other things, we compute the Drinfeld element $\boldsymbol{u}$ of $\overline{U}_q^{\Phi}(\mathfrak{sl}_2)$, and a canonical spanning set of the center. This spanning set consists of some elements for which general closed expressions were derived in [FGR1] and simplified in the first sections of this part. These elements are shown to coincide with corresponding ones from [FGST1]. From this we will infer Theorem 6.

Thus, assuming [CGR, Conj. 1.2], we have shown that the $SL(2,\mathbb{Z})$-actions on the categorical side agrees with the one on the VOA side.

Structure of the thesis

The top-level structure of the thesis is as follows. In Chapter 1, we will review some categorical notions as needed, largely following [EGNO]. This includes our conventions for finite tensor categories, along with conventions for additional structure (e.g. pivotal or ribbon) imposed on them. Some generalities on modified traces are stated, and after talking briefly about the notion of a (co)end, we finish by properly recalling the definition of a Hopf monad.

After that, in Chapter 2, we give a brief overview of quasi-Hopf algebras, following the content of e.g. [BCPO] and the conventions of [BGR1, BGR2, FGR1, FGR2]. We discuss cointegrals (in the sense of [HN2]) in some detail, and at the end give three examples of quasi-Hopf algebras.

From Chapter 3 onward, original results are presented. We start with the results on modified traces for (finite-dimensional pivotal unimodular) quasi-Hopf algebras as discussed above.

In Chapter 4, we recall the central Hopf monad, and use it to define monadic cointegrals for any (pivotal) finite-tensor categories. We specialize to the case of the category being representations of a finite-dimensional (pivotal) quasi-Hopf algebra, and give the main result of this part as stated above.

Then, in Chapter 5 we first review a (projective) $SL(2,\mathbb{Z})$-action defined by [Ly2] for any modular tensor category. We specialize this to the representation category of the quasi-Hopf algebra $\overline{U}_q^{\Phi}(\mathfrak{sl}_2)$ and compare it to the action obtained in [FGST1] as outlined above.

The first appendix, Appendix A, contains the proof that our realization of the central Hopf monad for quasi-Hopf algebras indeed satisfies the required universal property.

Appendix B contains various proofs of statements in Chapter 4, which were deemed too long or technical to include in the main texts. This comprises the finer details of the proof of Theorem 4.

Finally, in Appendix C, we provide the somewhat tedious proofs to multiple statements made in Chapter 5.

Chapter 1

Category theoretic preliminaries

In this chapter we review the necessary basics of finite tensor categories. Basic knowledge of abelian and monoidal categories is assumed. No original results are presented, and most of the contents can be found in the canonical reference [EGNO].

The first section contains a brief review of our conventions for finite tensor categories and their possible additional structures/properties (up to and including factorizability), as well as their graphical calculus. In the second section, we go on to discuss the notions of tensor ideals, modified traces, coends, and (Hopf) monads, as we will need them later on.

Throughout this thesis we fix k, an algebraically closed field. All linear structures will be considered over k, and, unless otherwise stated, all vector spaces will be finite-dimensional. By a module over an algebra we always mean a left module unless otherwise stated.

1.1 Conventions for finite tensor categories

We will first state the definition of a finite tensor category, then explain the meaning and our conventions for some words appearing in it, and then briefly discuss finite tensor categories with more structure (e.g. pivotal, ribbon).

1.1.1 The definition of a finite tensor category

Following [EGNO], by a *finite tensor category* we mean a linear abelian category that

- has finite-dimensional Hom-spaces, and every object is of finite length,
- possesses a finite set of isomorphism classes of simple objects,
- is rigid monoidal, such that the tensor product functor $\otimes$ is bilinear and the monoidal unit $\mathbf{1}$ is simple,
- has enough projectives.

Note that, because k is algebraically closed, the endomorphisms of a simple object automatically are one-dimensional.

We shall now unravel parts of this definition. Let $\mathcal{C}$ be a finite tensor category, and fix a set $\operatorname{Irr}\mathcal{C}$ of representatives of isomorphism classes of simple objects; we agree that $\mathbf{1} \in \operatorname{Irr}\mathcal{C}$.

1.1.2 Rigid structure

We denote the left and the right dual of an object X by $X^\vee$ and ${}^\vee X$, respectively. In particular, we fix (contravariant) duality endofunctors ${}^\vee ?, ?^\vee$ of $\mathcal{C}$. The corresponding evaluations and coevaluations are

$$\begin{aligned} &\mathrm{ev}_X \colon X^\vee \otimes X \to \mathbf{1} \ , \quad \mathrm{coev}_X \colon \mathbf{1} \to X \otimes X^\vee \ , \\ &\widetilde{\mathrm{ev}}_X \colon X \otimes {}^\vee X \to \mathbf{1} \ , \quad \widetilde{\mathrm{coev}}_X \colon \mathbf{1} \to {}^\vee X \otimes X \ , \end{aligned} \tag{1.1.1}$$

satisfying the familiar zig-zag equations. In view of our applications, we do not assume that $\mathcal{C}$ is strict monoidal, and (compositions of) coherence isomorphisms will therefore be indicated.

An object in $\mathcal{C}$ is *invertible* if its evaluations and coevaluations are isomorphisms. Intuitively, this means its left (and also its right) dual can be seen as inverses with respect to taking tensor product. It is easy to see that the left dual of an invertible object X is isomorphic to the right dual of X.

Note that by rigidity, $X^\vee \otimes ? \dashv X \otimes ? \dashv {}^\vee X \otimes ?$ for any fixed $X \in \mathcal{C}$. It follows that $\otimes$ is exact in each argument [EGNO, Prop. 4.2.1]; for simplicity we say that $\otimes$ is exact.

1.1.3 Projective objects and the distinguished invertible object

Recall that $P \in \mathcal{C}$ is projective iff the Hom-functor $\mathcal{C}(P, -)$ is exact; equivalently, every morphism out of P factors through every epimorphism. Exactness of $\otimes$ implies that duals of projective objects are projective, and hence projective and injective objects agree [EGNO, Prop. 6.1.3].

Since $\mathcal{C}$ has enough projectives, every object has a projective cover. The projective cover of $U \in \operatorname{Irr}\mathcal{C}$ is denoted $(P_U, p_U \colon P_U \to U)$, where U is the unique maximal semi-simple quotient of P_U. Projective covers of simple objects are the projective indecomposable objects in the category, so by the above remark, the unique maximal semi-simple subobject $\mathrm{soc}(P_U) \subset P_U$, the *socle* of P_U, is simple for all U.

One can show that the socle D of $P_{\mathbf{1}}$ is in fact invertible [EGNO, Sec. 6.4], and we call it the *distinguished invertible object of* $\mathcal{C}$.[1] Equivalently, one may define D as the simple object determined by

$$P_{\mathbf{1}}^\vee \cong P_{D^\vee}. \tag{1.1.2}$$

We call $\mathcal{C}$ *unimodular* if $D \cong \mathbf{1}$. This terminology will be explained later in Remark 2.3.9.

[1] This means that our D is in fact dual to the distinguished invertible object of [EGNO, Sec. 6.4]. However, our definition agrees with the one given in [ENO, Sec. 6].

Finally, a *projective generator* of $\mathcal{C}$ is a projective object G such that every object is a quotient of $G^{\oplus n}$ for some $n \in \mathbb{N}$. Thus projective covers of irreducibles have to occur at least once each as summands in G.

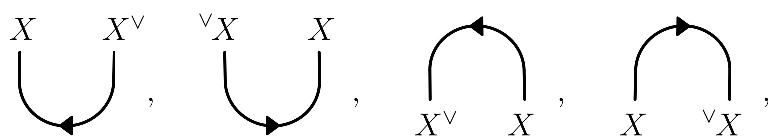
Remark 1.1.1. By the above, a good choice for a projective generator would be given by the minimal one, i.e. the one which is just the direct sum of the projective covers of simple objects.

In this thesis, however, we deal mostly with categories of modules over an algebra. In that case, the easiest choice for a projective generator would be the algebra itself, i.e. the direct sum of projective covers of simples, each with multiplicity the k-dimension of the corresponding simple. $\triangledown$

1.1.4 Graphical calculus

Our string diagrams are read from bottom to top, and coherence isomorphisms will usually not be drawn. Morphisms are as usual depicted as coupons, and for the identity morphism on an object we do not draw a coupon. A horizontal line with no label and multiple incoming and one single out-going line represents the identity morphism of the respective objects, i.e. we change from viewing them as separate entities to viewing them as one object. Similarly for multiple out-going and one incoming line.

The left and right coevaluation and evaluation morphisms for the object $X \in \mathcal{C}$ are drawn as



(1.1.3)

respectively, so that in our conventions for duals and string diagrams, arrows on the duality maps for left (right) duals point to the left (right).

1.1.5 (Co)monoidal functors

A *lax comonoidal functor* is a tuple (F, F_0, F_2), where $F\colon \mathcal{C} \to \mathcal{D}$ is a functor between monoidal categories together with a natural transformation F_2 and a morphism F_0,[2]

$$F_2(X,Y)\colon F(X \otimes Y) \to FX \otimes FY, \quad F_0\colon F\mathbf{1} \to \mathbf{1}, \tag{1.1.4}$$

satisfying certain coherence conditions which in particular imply that coalgebras in $\mathcal{C}$ are sent to coalgebras in $\mathcal{D}$, see e.g. [EGNO, Sec. 2.4]. For that reason we will sometimes refer to F_2 and F_0 as the comultiplication and the counit of the lax comonoidal functor F. If F_2 and F_0 are isomorphisms (identities) then F is called a strong (strict) comonoidal functor. We abbreviate a (lax) comonoidal functor (F, F_0, F_2) by just the symbol F.

[2]Here and below we abbreviate $F(X)$ by FX when applying functors to objects, and similarly for functors on morphisms.

Similarly, a functor $F\colon \mathcal{C} \to \mathcal{D}$ between monoidal categories is *lax/strong monoidal* if $F^{\mathrm{op}}\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{D}^{\mathrm{op}}$ is lax/strong comonoidal. Here $\mathcal{C}^{\mathrm{op}}$ is the category with the same objects of $\mathcal{C}$ but the arrows are reversed. Thus, it maps algebras in $\mathcal{C}$ to algebras in $\mathcal{D}$ and the corresponding natural transformation F_2 and the morphism F_0 are called the multiplication and the unit, respectively.

A natural transformation $\varphi\colon F \Rightarrow G$ between two comonoidal functors is called *comonoidal* if it commutes with the comonoidal structures. That is, if

$$G_2(X,Y) \circ \varphi_{X\otimes Y} = (\varphi_X \otimes \varphi_Y) \circ F_2(X,Y) \quad \text{and} \quad F_0 = G_0 \circ \varphi_{\mathbf{1}} \tag{1.1.5}$$

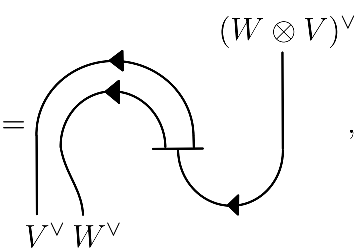
holds for all objects X, Y.

Monoidal natural transformations between monoidal functors are defined similarly.

1.1.6 Pivotal structure

In any rigid monoidal category one can define the isomorphism

$$\gamma_{V,W}\colon V^\vee \otimes W^\vee \to (W\otimes V)^\vee, \quad \gamma_{V,W} =$$



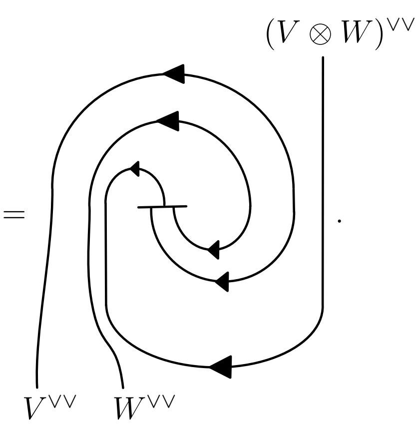
, (1.1.6)

natural in V and W (recall that the horizontal line is our graphical representation of $\mathrm{id}_{V\otimes W}$). Then the canonical monoidal structure of the double dual functor is given by

$$(?)^{\vee\vee}{}_2(V,W) = \left[V^{\vee\vee} \otimes W^{\vee\vee} \xrightarrow{\gamma_{V^\vee,W^\vee}} (W^\vee \otimes V^\vee)^\vee \xrightarrow{(\gamma^{-1}_{W,V})^\vee} (V\otimes W)^{\vee\vee} \right]. \tag{1.1.7}$$

Pictorially, this means

$$(?)^{\vee\vee}{}_2(V,W) =$$



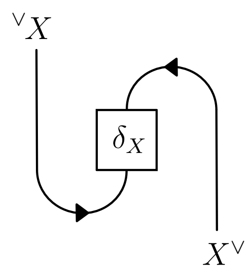
. (1.1.8)

A rigid category $\mathcal{C}$ is called *pivotal* if there is a monoidal natural isomorphism $\delta\colon \mathrm{id}_{\mathcal{C}} \Rightarrow (?)^{\vee\vee}$, that is, from the identity functor on $\mathcal{C}$ to the double dual functor. Here we regard the identity functor as monoidal with trivial monoidal structure. The requirement that δ be monoidal is precisely the commutativity of the diagram

$$\begin{tikzcd} VW \arrow[r, "\delta_V \otimes \delta_W"] \arrow[dr, "\delta_{VW}"'] & V^{\vee\vee} \otimes W^{\vee\vee} \arrow[d, "(?)^{\vee\vee}{}_2(V,W)"] \\ & (VW)^{\vee\vee} \end{tikzcd} \tag{1.1.9}$$

Here we omitted the tensor product symbol to shorten expressions. We will continue doing that from now on whenever possible, although sometimes, a tensor symbol has to be inserted to make an expression unambiguous.

Note that the existence of the pivotal structure δ is equivalent to requiring that the left and the right dual functor be isomorphic as monoidal functors. Indeed, given δ we can form the isomorphism



, (1.1.10)

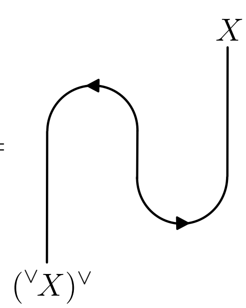
and one checks from the axioms above that this is indeed monoidal. Conversely, given a natural monoidal isomorphism $^\vee X \cong X^\vee$, we have

$$X^{\vee\vee} \cong ({}^\vee X)^\vee \cong X \tag{1.1.11}$$

where the second isomorphism is

$$\omega_X = \Big[({}^\vee X)^\vee \xrightarrow{\sim} ({}^\vee X)^\vee \otimes \mathbf{1} \xrightarrow{\mathrm{id} \otimes \widetilde{\mathrm{coev}}_X} ({}^\vee X)^\vee \otimes ({}^\vee X X) \xrightarrow{\sim} (({}^\vee X)^\vee \otimes {}^\vee X) X$$
$$\xrightarrow{\mathrm{ev}_{{}^\vee X} \otimes \mathrm{id}} \mathbf{1} X \xrightarrow{\sim} X \Big] . \tag{1.1.12}$$

As a string diagram (1.1.12) simply reads

$\omega_X =$ 

. (1.1.13)

We stress that we do not assume pivotal structures to be strict. Furthermore, while it is customary to assume that in a pivotal category the left and the right dual have the same underlying object, we do not do this, unless explicitly stated (e.g. in Chapter 3).

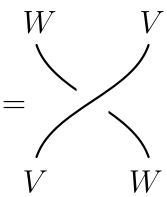
1.1.7 Braiding

Let $\mathcal{C}$ be a finite tensor category. A *braiding* is a family of isomorphisms

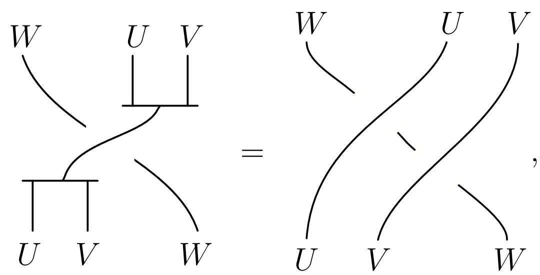
$$c_{V,W} \colon V \otimes W \to W \otimes V, \tag{1.1.14}$$

natural in V, W, depicted as

$$c_{V,W} = \tag{1.1.15}$$



satisfying the *left hexagon axiom*



(1.1.16)

and an analogous right hexagon axiom. The inverse of the braiding is drawn as the other crossing. If $\mathcal{C}$ admits a braiding we call it *braidable*, and $\mathcal{C}$ together with a choice of braiding is then a *braided finite tensor category*.

It follows from the hexagon axioms that

$$c_{\mathbf{1},V} = [\mathbf{1}V \xrightarrow{\sim} V \xrightarrow{\sim} V\mathbf{1}] \ ,$$

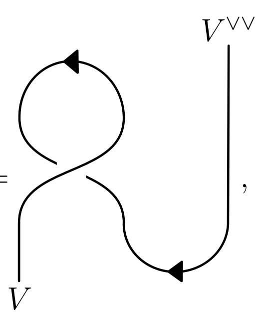
i.e. braiding with the tensor unit is expressed in terms of unitors. This implies $c_{\mathbf{1},V}^{-1} = c_{V,\mathbf{1}}$ for all $V \in \mathcal{C}$. In particular, the tensor unit is a transparent object: an object V in a braided monoidal category is *transparent* iff $c_{W,V} \circ c_{V,W} = \mathrm{id}_{VW}$ for any $W \in \mathcal{C}$

There also exists the notion of a *braided monoidal functor* between braided finite tensor categories. This is, of course, simply a monoidal functor commuting with the braidings involved.

An easy example of a braided finite tensor category is Vect, the category of finite-dimensional vector spaces. The braiding is given by the flip map τ, which, for all vector spaces V, W, simply acts as $\tau(v \otimes w) = w \otimes v$, for all $v \in V$, $w \in W$.

In any braided finite tensor category, one can define the natural isomorphism

$$u_V = \tag{1.1.17}$$



$, \quad V \in \mathcal{C}$

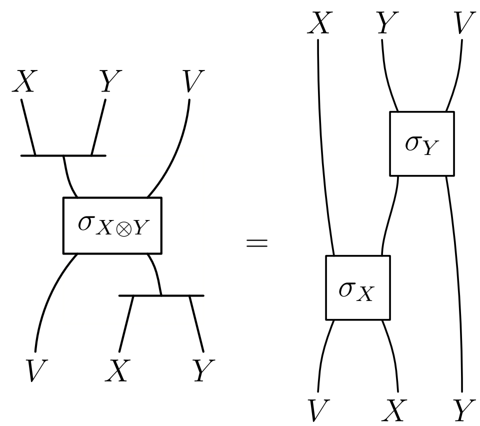
called the *Drinfeld morphism* $u\colon \mathrm{id}_{\mathcal{C}} \xRightarrow{\simeq} ?^{\vee\vee}$. In general, however, it will not be monoidal.

Lastly, to any braided finite tensor category we can associate another such category, sometimes denoted $\overline{\mathcal{C}}$ or $\mathcal{C}^{\mathrm{rev}}$, which agrees with $\mathcal{C}$ as finite abelian category, but has opposite tensor product and inverse braiding, i.e. $X \bar{\otimes} Y = Y \otimes X$ and $\bar{c} = c^{-1}$.

1.1.8 The Drinfeld center

A finite tensor category does not have to be braided, just as a monoid does not have to be commutative. However, for a monoid we have the notion of a center. In this subsection, we recall the definition of the Drinfeld center $\mathcal{Z}(\mathcal{C})$ of a finite tensor category $\mathcal{C}$.

Its objects are pairs $(V, \sigma\colon V \otimes ? \xRightarrow{\simeq} ? \otimes V)$, where $V \in \mathcal{C}$, and σ is a so-called *half-braiding*, meaning that it satisfies



(1.1.18)

Note that this condition is reminiscent of the (right) hexagon above.

A morphism $f\colon (V,\sigma) \to (W,\rho)$ in $\mathcal{Z}(\mathcal{C})$ is a morphism $f \in \mathcal{C}(V,W)$ that commutes with the half-braidings, i.e.

$$\mathrm{id} \otimes f \circ \sigma_X = \rho_X \circ f \otimes \mathrm{id} \quad \forall X \in \mathcal{C}\ .$$

It is well-known that $\mathcal{Z}(\mathcal{C})$ is a braided finite tensor category [EGNO, 7.13.8]. If $\mathcal{C}$ is braided with braiding c, then we have two canonical braided functors

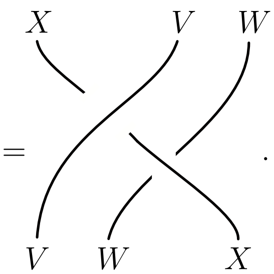
$$\begin{aligned} \mathcal{C} &\to \mathcal{Z}(\mathcal{C}) & \qquad \overline{\mathcal{C}} &\to \mathcal{Z}(\mathcal{C}) \\ V &\mapsto (V, c_{V,?}) & \qquad V &\mapsto (V, c^{-1}_{?,V}) \end{aligned} \tag{1.1.19}$$

which are both exact. Thus they give rise to one single functor

$$\mathcal{C} \boxtimes \overline{\mathcal{C}} \to \mathcal{Z}(\mathcal{C}), \quad V \boxtimes W \mapsto (V \otimes W, \sigma) \tag{1.1.20}$$

out of the enveloping category[3] of $\mathcal{C}$ into its Drinfeld center, where σ is the half-braiding given by

$$\sigma_X =$$



$$. \tag{1.1.21}$$

1.1.9 Ribbon

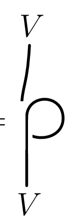
The last structural addition we will recall is that of a *ribbon structure*. Intuitively, this lets us think about strings in string diagrams as though they were elongated rectangles, and as such we should be able to twist them.

A braided finite tensor category $\mathcal{C}$ is *ribbon* if it is equipped with a natural isomorphism $\vartheta\colon \mathrm{id}_{\mathcal{C}} \Rightarrow \mathrm{id}_{\mathcal{C}}$ of the identity functor, satisfying the two axioms

$$\vartheta_{V\otimes W} = c_{W,V} \circ c_{V,W} \circ \vartheta_V \otimes \vartheta_W \quad \text{and} \quad (\vartheta_V)^\vee = \vartheta_{V^\vee}\ . \tag{1.1.22}$$

This isomorphism is called the *ribbon twist*, depicted as

$$\vartheta_V =$$



$$. \tag{1.1.23}$$

Note that the cups and caps in this picture are *not a priori* evaluations and coevaluations.

A ribbon category is automatically pivotal. Indeed, recall the Drinfeld morphism u from (1.1.17). Then a pivotal structure is given by $\delta := u\vartheta$, cf. [FGR2, Rem. 3.5] or [BK, Sec. 2.2] (this also justifies the pictorial representation of the twist above). For the converse, note that a braided pivotal finite tensor category will in general only be *balanced* (i.e. the twist one tries to define will only satisfy the first identity in (1.1.22)), and not necessarily ribbon.

1.1.10 Factorizable and modular finite tensor categories

We finish this section by recalling the definitions (and some equivalent characterizations) of two particularly nice classes of finite tensor categories: factorizable FTCs and modular FTCs.

[3] Here, $\mathcal{A}\boxtimes\mathcal{B}$ is the *Deligne product* of the finite tensor categories $\mathcal{A}$ and $\mathcal{B}$, see e.g. [EGNO, Sec. 1.11]. It is again a finite tensor category. Moreover, to specify uniquely a linear right exact functor out of it, it is enough to specify a bilinear in both variables right exact bifunctor out of $\mathcal{A}\times\mathcal{B}$, as we did above.

Let $\mathcal{C}$ be a braided finite tensor category. *Factorizability* is a certain non-degeneracy condition on the braiding, namely, that the embedding (1.1.20) from $\mathcal{C} \boxtimes \overline{\mathcal{C}}$ into $\mathcal{Z}(\mathcal{C})$ be a (braided monoidal) equivalence. Recently, Shimizu proved [Sh3] the following remarkable theorem, which we first state and then explain a bit; see also [FGR1] and [GR3, Sec. 2] for reviews.

Theorem 1.1.2 ([Sh3, Thm. 1.1])**.** *The following are equivalent.*

1. *$\mathcal{C}$ is factorizable.*
2. *Every transparent object is isomorphic to $\mathbf{1}^{\oplus n}$ for some $n \in \mathbb{N}$.*
3. *The canonical Hopf pairing of the Hopf algebra $\mathcal{L} \in \mathcal{C}$ is non-degenerate.*
4. *The Hopf pairing from the previous point gives rise to an isomorphism of Hom-spaces $\mathcal{C}(\mathbf{1}, \mathcal{L})$ and $\mathcal{C}(\mathcal{L}, \mathbf{1})$.*

For the second point, recall from Section 1.1.7 the notion of transparent object. The third item is more easily understood after reading Example 1.2.7 (and maybe Section 5.1), where the Hopf algebra $\mathcal{L}$ is given, and its Hopf pairing is defined via the double braiding of $\mathcal{C}$. The fourth point is a certain variant of the third, the direction from (3) to (4) being trivial.

Finally, by a *modular tensor category* we mean a factorizable finite tensor category which is ribbon. The qualifier *modular* is due to the fact that such categories give rise to a certain projective action of the modular group $SL(2, \mathbb{Z})$; see Section 5.1 for a brief review, or [FGR1, Sec. 5] for a full review, of this action.

1.2 Traces, coends, monads

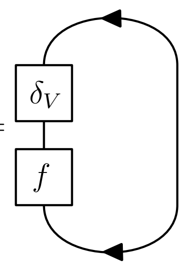
Now that we know what we mean by a finite tensor category (with or without some extra structure/properties), we will define some additional categorical notions needed later. We begin by reviewing traces; followed by the basis aspects of the theory of modified traces of [GPV, GPT, GKP1, GKP2, GKP3]; then give the definition of and an existence statement about coends; and finally, we introduce Hopf monads, following [BV1].

We remark that we do not necessarily present these things as general as possible, but rather in way that fits our categorical needs.

1.2.1 The categorical trace

Let $\mathcal{C}$ be a pivotal finite tensor category. To any endomorphism $f \in \mathrm{End}_{\mathcal{C}}(V)$ of any object $V \in \mathcal{C}$ we can associate a number called the *right trace* of f, defined by

$$\mathrm{tr}_V^{\mathcal{C},r}(f) =$$



$$\in \mathrm{End}_{\mathcal{C}}(\mathbf{1}) \cong k \; . \tag{1.2.1}$$

Analogously one defines a left trace. Left and right traces do not agree in general, but if they do, $\mathcal{C}$ is called *spherical*—the intuition being that by pulling the right hand vertical strand of the right trace past a point at infinity one obtains the left trace. In that case we simply write $\mathrm{tr}_V^{\mathcal{C}}(f)$.

The trace is linear and has some nice properties, most importantly *cyclicity*

$$\mathrm{tr}_V^{\mathcal{C},r}(V \xrightarrow{f} W \xrightarrow{g} V) = \mathrm{tr}_W^{\mathcal{C},r}(W \xrightarrow{g} V \xrightarrow{f} W) \tag{1.2.2}$$

and *multiplicativity*

$$\mathrm{tr}_{V\otimes W}^{\mathcal{C},r}(f \otimes g) = \mathrm{tr}_V^{\mathcal{C},r}(f) \cdot \mathrm{tr}_W^{\mathcal{C},r}(g) \; . \tag{1.2.3}$$

The name *trace* is justified by the fact that for $\mathcal{C} = \mathsf{Vect}$, $\mathrm{tr}_V^{\mathcal{C}}(f)$ is indeed simply the trace of the linear map $f : V \to V$.

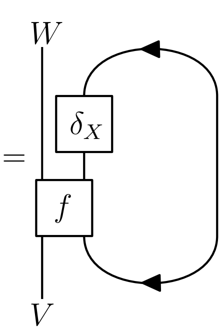
Remark 1.2.1.

1. In a braided finite tensor category one could try defining a ‘trace’ using the Drinfeld morphism (1.1.17) to replace the pivotal structure. But unless the braiding is symmetric, the resulting trace-candidate is in general not multiplicative.
2. It is well-known that any ribbon category is spherical, see [Ka, Sec. XIV.4].

▽

Given a morphism $f \in \mathcal{C}(VX, WX)$, we can form a new morphism $\mathrm{ptr}_X^{\mathcal{C},r}(f) \in \mathcal{C}(V, W)$, called its *right partial trace over* X, by

$$\mathrm{ptr}_X^{\mathcal{C},r}(f) =$$



$$. \tag{1.2.4}$$

Similarly, one may define left partial traces of morphisms in $\mathcal{C}(XV, XW)$. The existence of partial traces may be interpreted as a weaker version of multiplicativity.

We finish our discussion of the trace by recalling one of its more degenerate features in the following well-known proposition, see e.g. [GR3, Rem. 4.6].

Proposition 1.2.2. *Let $\mathcal{C}$ be a pivotal finite tensor category. If $\mathcal{C}$ is not semisimple, then*

$$\mathrm{tr}_P^{\mathcal{C},r}(f) = 0 = \mathrm{tr}_P^{\mathcal{C},l}(f)$$

for all projective P and all $f \in \mathrm{End}_{\mathcal{C}}(P)$.

Proof. We show the contrapositive, so without less of generality assume $\mathrm{tr}_P^{\mathcal{C},r}(f) = 1$. Then $P \otimes P^\vee \cong \mathbf{1} \oplus X$ for some $X \in \mathcal{C}$. But then $\mathbf{1}$ is projective, and so $\mathcal{C}$ is semisimple by exactness of $\otimes$. □

1.2.2 Modified traces on tensor ideals

Let $\mathcal{C}$ be a monoidal category. Following [GPT, GKP1, GKP2] a *right tensor* ideal I of $\mathcal{C}$ is a full subcategory of $\mathcal{C}$ satisfying

(Closure under tensor products)
If $X \in I$, and $V \in \mathcal{C}$, then $X \otimes V \in I$.

(Closure under retracts)
If $X \in I$, $V \in \mathcal{C}$, and there exist morphisms f, g such that

$$\mathrm{id}_V = V \xrightarrow{f} X \xrightarrow{g} V,$$

then $V \in I$.

Similarly one defines left and two-sided ideals. The latter will simply be called *ideals*. If I is an ideal, we write $I \leq \mathcal{C}$.

The axioms imply that an ideal is a *replete* subcategory, i.e. $X \in I$ and $f \colon X \xrightarrow{\sim} Y$ in $\mathcal{C}$ imply that both f and Y are in I.

Remark 1.2.3. In practice, we look at abelian categories. Closure under retracts then means that direct summands of objects in the ideal are again in the ideal. ▽

Example 1.2.4.
1. The easiest non-trivial example is given by $I = \mathcal{C}$. This is a two-sided ideal.
2. Let $\mathcal{C}$ be a finite tensor category. The example we will be most interested in is given by the *projective ideal* $\mathrm{Proj}(\mathcal{C})$ of $\mathcal{C}$, which is the (full) subcategory of projective objects in $\mathcal{C}$.[4] Closure under tensor products follows from exactness of $\otimes$—in

[4] This is in fact the smallest non-trivial ideal in $\mathcal{C}$. Indeed, any non-trivial ideal I contains $\mathrm{Proj}(\mathcal{C})$: $X \in I$ implies $Y = (X^\vee \otimes X) \otimes P \in I$ for any projective P. But P is a quotient of Y (since $\otimes$ is exact) and projective, thus $P \in I$.

particular, $\mathrm{Proj}(\mathcal{C})$ is two-sided. Closure under retracts is the familiar statement that direct summands of projectives are projective. Note that $\mathrm{Proj}(\mathcal{C}) = \mathcal{C}$ iff $\mathcal{C}$ is semisimple.

△

Let now $\mathcal{C}$ be a pivotal linear category, and I be a right tensor ideal of $\mathcal{C}$. A *right modified trace* on I is a family $\mathsf{t}_\bullet$ of linear functions

$$\{\mathsf{t}_X\colon \mathrm{End}_{\mathcal{C}}(X) \to k\}_{X\in I} \tag{1.2.5}$$

satisfying two conditions, *cyclicity* and *right partial trace property*, given as follows.

(Cyclicity)
If $X, X' \in I$, then for all $f\colon X \to X'$, $g\colon X' \to X$

$$\mathsf{t}_X(g \circ f) = \mathsf{t}_{X'}(f \circ g)\ . \tag{1.2.6}$$

(Right Partial Trace Property)
If $X \in I$ and $V \in \mathcal{C}$, then for all $f \in \mathrm{End}_{\mathcal{C}}(X \otimes V)$

$$\mathsf{t}_{X\otimes V}(f) = \mathsf{t}_X\left(\mathrm{ptr}_V^{\mathcal{C};r}(f)\right)\ . \tag{1.2.7}$$

Similarly one defines left modified traces on left ideals. A family of such functions which is both a left and a right modified trace on a two-sided ideal is simply called a *modified trace*.

A right (resp. left) modified trace $\mathsf{t}_\bullet$ is *non-degenerate* if the canonical pairings

$$\mathcal{C}(M, X) \times \mathcal{C}(X, M) \to k, \qquad (f, g) \mapsto \mathsf{t}_X(f \circ g)\ , \tag{1.2.8}$$

are non-degenerate for all $M \in \mathcal{C}, X \in I$.

Example 1.2.5. 1. Consider again $I = \mathcal{C}$. Then the usual categorical trace is a modified trace—but for $\mathcal{C}$ not semisimple certainly not non-degenerate, see Proposition 1.2.2.

2. If $\mathcal{C}$ is a factorizable finite tensor category over an algebraically closed field of characteristic 0, then it contains a simple projective object, which can be used to define a non-degenerate modified trace on $\mathrm{Proj}(\mathcal{C})$, see [GR3].

3. More generally, if k is algebraically closed and $\mathcal{C}$ is unimodular (cf. Section 1.1.3), then a corollary to [GKP3, Sec. 5.3] is that a non-degenerate modified trace on $\mathrm{Proj}(\mathcal{C})$ exists and is unique up to scalars. This significantly generalizes earlier existence and uniqueness results, see e.g. [GKP2, GR3].

△

From now on, we will only focus on modified traces on the ideal of projective objects. We will for simplicity just call them 'modified traces', dropping the qualifier 'on Proj($\mathcal{C}$)'.

In Chapter 3 we will assume $\mathcal{C} = H$-mod, the category of finite-dimensional modules over a pivotal unimodular quasi-Hopf algebra; see Chapter 2 for the definitions of these words. We will then obtain an explicit construction of a unique non-degenerate modified trace (without necessarily requiring k to be algebraically closed) in Theorem 3.2.5, extending the main result of [BBGa].

1.2.3 Ends and coends

We recall the notion of a (co)end, which is a certain type of (co)limit, following [Mac, KL], see also [FS, Sec. 4.2].

Let $\mathcal{C}$, $\mathcal{D}$ be categories, and let $F\colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{D}$ be a functor. As the name suggests, an end is a coend in the opposite category, and we shall only explain what a coend of F in $\mathcal{D}$ is.

Briefly, a *coend of* F is an object $A \in \mathcal{D}$ together with a universal dinatural transformation $j\colon F \ddot{\to} A$. That means j is a family of morphisms

$$j_X\colon F(X,X) \to A$$

indexed by objects in $\mathcal{C}$ and satisfying

$$j_X \circ F(f, \mathrm{id}_X) = j_Y \circ F(\mathrm{id}_Y, f) \quad \text{for } f \in \mathcal{C}(X,Y)\ , \tag{1.2.9}$$

which is universal with respect to this property, i.e. if $k\colon F \ddot{\to} W$ is a dinatural transformation as above, then there exists a unique morphism $m\colon A \to W$ such that $k_X = m \circ j_X$, for all $X \in \mathcal{C}$. Thus, if the outer square in

$$\begin{tikzcd}
F(Y,X) \arrow[r, "F(f,\mathrm{id})"] \arrow[d, "F(\mathrm{id},f)"'] & F(X,X) \arrow[d, "j_X"] \arrow[ddr, bend left, "k_X"] & \\
F(Y,Y) \arrow[r, "j_Y"] \arrow[drr, bend right, "k_Y"'] & A \arrow[dr, dashed, "\exists! m"] & \\
 & & W
\end{tikzcd} \tag{1.2.10}$$

commutes for all $X, Y \in \mathcal{C}$ and all $f\colon X \to Y$, then there is a unique (dashed) arrow making the entire diagram commute. It follows that a coend is unique up to unique isomorphism. Denoting by $\mathrm{Dinat}(F, W)$ the set of dinatural transformations from F to $W \in \mathcal{D}$, the universal property can be written suggestively as

$$\mathrm{Dinat}(F,W) \cong \mathcal{D}(A,W)\ . \tag{1.2.11}$$

In other words, we can *uniquely* specify morphisms out of a coend by simply giving a dinatural transformation from the functor to the desired target. This will be used extensively later in Chapter 4 when we discuss the central Hopf (co)monads.

It is customary to denote the coend of F as either

$$\int^{X\in\mathcal{C}} F(X,X) \quad \text{or simply} \quad \int^{X} F\ . \tag{1.2.12}$$

Dually, an end is denoted with the "integration variable" on the bottom.

As for other colimits, there are conditions which guarantee the existence of coends of certain functors. For later use we state an immediate corollary to [KL, Cor. 5.1.8].

Proposition 1.2.6 ([KL, Cor. 5.1.8])**.** *Let $\mathcal{C}$, $\mathcal{D}$ be finite tensor categories. If $F\colon \mathcal{C}^{\mathrm{op}}\times\mathcal{C}\to\mathcal{D}$ is bilinear, and exact in both variables, then $\int^X F$ exists.*

Example 1.2.7. Let $\mathcal{C}$ be a finite tensor category, and consider the functor

$$\mathcal{C}^{\mathrm{op}}\times\mathcal{C}\to\mathcal{C}, \quad (X,Y)\mapsto X^\vee\otimes Y\ . \tag{1.2.13}$$

This functor satisfies the assumptions of Proposition 1.2.6, thus its coend exists. It is a rather famous coend [LM, Ma, Ly1, Ly2], and we denote it by

$$\mathcal{L} = \int^{X\in\mathcal{C}} X^\vee\otimes X\ . \tag{1.2.14}$$

Lyubashenko used it in his definition of a certain modular functor and a projective action of mapping class groups of surfaces on certain Hom-spaces [Ly2], which we review briefly in Chapter 5.

Under some additional conditions on $\mathcal{C}$, one can in fact show—see [FGR1, Sec. 3.3] for a review—that $\mathcal{L}$ carries naturally the structure of a Hopf algebra, and that there is a Hopf pairing $\omega\colon \mathcal{L}\otimes\mathcal{L}\to\mathbf{1}$ defined in terms of the double braiding of $\mathcal{C}$. We describe it more explicitly in Section 5.1. The (left or right) dual of $\mathcal{L}$ can be described as a certain end which also carries a canonical Hopf algebra structure. *Factorizability* (as introduced in Section 1.1.10), then, is the requirement that the canonical morphism of Hopf algebras $\mathcal{L}\to\mathcal{L}^\vee$ induced by the Hopf pairing be an isomorphism.[5] △

Remark 1.2.8. We note that coend in Example 1.2.7 will in a way serve as a basis for the definition of the central Hopf monad, meaning that in the braided case, the central Hopf monad is given by tensoring with $\mathcal{L}$. ▽

1.2.4 Hopf monads

We first recall some basic notions from the theory of Hopf monads on rigid categories. As an initial motivation, a monad is an endofunctor M which admits a sensible notion of 'category of modules over M'. The broad idea of a Hopf monad on a rigid category is then that this category of modules is rigid in a nice way. Throughout, our conventions will closely follow [BV1].

[5] Here, for a second, we assume left and right duals to coincide via the pivotal structure.

Monads

Recall [Mac, Sec. VI] that a monad M on a category $\mathcal{C}$ is an algebra in $\mathrm{End}(\mathcal{C})$, the category of endofunctors of $\mathcal{C}$, which is a monoidal category under composition. Thus M is equipped with two natural transformations

$$\mu\colon M^2 \Rightarrow M, \quad \eta\colon \mathrm{id}_{\mathcal{C}} \Rightarrow M, \tag{1.2.15}$$

the *multiplication* and *unit* of M, respectively, satisfying associativity and unitality,

$$\mu \circ M\mu = \mu \circ \mu M \qquad \text{and} \qquad \mu \circ \eta M = M = \mu \circ M\eta \ . \tag{1.2.16}$$

This simply means that the diagrams

$$\begin{tikzcd} M^3V \arrow[r, "M\mu_V"] \arrow[d, "\mu_{MV}"'] & M^2V \arrow[d, "\mu_V"] \\ M^2V \arrow[r, "\mu_V"] & MV \end{tikzcd} \qquad \text{and} \qquad \begin{tikzcd} M^2V \arrow[dr, "\mu_V"'] & MV \arrow[l, "\eta_{MV}"'] \arrow[r, "M\eta_V"] \arrow[d, equal] & M^2V \arrow[dl, "\mu_V"] \\ & MV & \end{tikzcd} \ , \tag{1.2.17}$$

commute for each $V \in \mathcal{C}$.

A comonad on $\mathcal{C}$ is a monad on $\mathcal{C}^{\mathrm{op}}$, or, alternatively, a coalgebra in $\mathrm{End}(\mathcal{C})$.

Example 1.2.9. 1. If $\mathcal{C}$ is monoidal, and $A \in \mathcal{C}$ is an algebra, then $A\otimes ?$ is a monad.

2. Let $L\colon \mathcal{C} \to \mathcal{D}$ be a functor with right adjoint R. Then RL is a monad. Its unit is the unit of the adjunction, and its multiplication is $R\varepsilon L\colon RLRL \Rightarrow RL$, where ε is the counit of the adjunction. Similarly LR becomes a comonad.

$\triangle$

A *module*[6] over a monad M is a tuple (V, ρ), consisting of an object $V \in \mathcal{C}$ together with a morphism $\rho\colon MV \to V$, called the action, such that

$$\begin{tikzcd} M^2V \arrow[r, "M\rho"] \arrow[d, "\mu_V"] & MV \arrow[d, "\rho"] \\ MV \arrow[r, "\rho"] & V \end{tikzcd} \qquad \text{and} \qquad \begin{tikzcd} V \arrow[r, "\eta_V"] \arrow[dr, equal] & MV \arrow[d, "\rho"] \\ & V \end{tikzcd} \tag{1.2.18}$$

commute. A morphism $f\colon (V, \rho) \to (W, \sigma)$ of M-modules is a morphism of the underlying objects in $\mathcal{C}$ which commutes with the action, i.e.

$$\sigma \circ Mf = f \circ \rho. \tag{1.2.19}$$

The category of M-modules is denoted by $\mathcal{C}_M$. The forgetful functor from $\mathcal{C}_M$ to $\mathcal{C}$ has a left adjoint which sends an object $V \in \mathcal{C}$ to the free M-module (MV, μ_V). Indeed, if (B, ν)

[6] Sometimes such modules are called 'M-algebras', but we prefer the following point of view: An object $V \in \mathcal{C}$ is the same as a constant endofunctor $\hat{V}$. Then, V is a module over the monad M if and only if $\hat{V}$ is a (left) module over the algebra M in $\mathrm{End}\,\mathcal{C}$.

is an M-module and A any object in $\mathcal{C}$, then we easily see that the Hom-set adjunction $\mathcal{C}_M\big((MA, \mu_A), (B, \nu)\big) \cong \mathcal{C}(A, B)$ is given by sending an M-module morphism f to $f \circ \eta_A$, and conversely, a morphism $g\colon A \to B$ is sent to $\nu \circ Mg$.

A morphism of monads is a morphism $\phi\colon M \Rightarrow M'$ of algebras in $\operatorname{End}(\mathcal{C})$. It therefore induces a functor $\phi^*\colon \mathcal{C}_{M'} \to \mathcal{C}_M$ via pullback, cf. [BV1, Lem. 1.6].

The following proposition is well-known.

Proposition 1.2.10. *Let L, $R \in \operatorname{End}(\mathcal{C})$, and suppose that L is left adjoint to R. The set of monad (comonad) structures on R is in bijection with the set of comonad (monad) structures on L.*

Proof. The category $\operatorname{End}(\mathcal{C})$ is monoidal, thus it makes sense to speak about dual objects. An adjunction $L \dashv R$ is nothing more but the statement that L is a left dual object of R. The unit $\mathrm{id}_{\mathcal{C}} \Rightarrow RL$ and counit $LR \Rightarrow \mathrm{id}_{\mathcal{C}}$ of the adjunction serve as coevaluation and evaluation for the dual pair. It is well-known that the left (right) dual of an object is a coalgebra iff the object itself is an algebra. The proposition follows. □

Bimonads

If $\mathcal{C}$ is a monoidal category, then a *bimonad* on $\mathcal{C}$ is a monad M such that the functor M is lax comonoidal and the multiplication and unit of M are comonoidal natural transformations, see Section 1.1.5. A bicomonad on $\mathcal{C}$ is a bimonad on $\mathcal{C}^{\mathrm{op}}$.

Example 1.2.11. If $\mathcal{C}$ is braided monoidal, and $B \in \mathcal{C}$ is a bialgebra, then $B\otimes ?$ is a bimonad. △

The name "bimonad" is in analogy to algebras and bialgebras: The category of modules over a bimonad (M, M_0, M_2) is monoidal, and a lax comonoidal structure on M is the same as a monoidal structure on $\mathcal{C}_M$ such that the forgetful functor to $\mathcal{C}$ is strong monoidal, cf. [Moe, Thm. 7.1]. Given two M-modules $(V, \rho), (W, \sigma)$, their tensor product is defined by

$$(V, \rho) \otimes (W, \sigma) = (V \otimes W, (\rho \otimes \sigma) \circ M_2(V, W)), \tag{1.2.20}$$

and the monoidal unit of $\mathcal{C}_M$ is the M-module $(\mathbf{1}, M_0)$, which we will also denote by $\mathbf{1}$.

A morphism of bimonads is a comonoidal natural transformation which is a morphism of the underlying monads.

We will later need the following lemma.

Lemma 1.2.12 ([BV1, Lem. 2.7])**.** *Let M, M' be bimonads on $\mathcal{C}$. Then there is a one-to-one correspondence between morphisms $f\colon M \Rightarrow M'$ of bimonads and strict monoidal functors $F\colon \mathcal{C}_{M'} \to \mathcal{C}_M$ which are lifts of the identity functor on $\mathcal{C}$.*

Here by a *lift* of $F\colon \mathcal{C} \to \mathcal{C}$ we mean a functor $\tilde{F}\colon \mathcal{C}_{M'} \to \mathcal{C}_M$ such that $U_M\tilde{F} = FU_{M'}$, where $U_M, U_{M'}$ are the respective forgetful functors to $\mathcal{C}$.

Remark 1.2.13. For later use, let us quickly give a rough sketch of the proof. Given a bimonad morphism f, the corresponding functor determined via Lemma 1.2.12 is just the pullback of the M'-module structure along f, i.e. $F = f^*$. Conversely, the monad morphism $f\colon M \Rightarrow M'$ corresponding to the functor $F\colon \mathcal{C}_{M'} \to \mathcal{C}_M$ is given as follows. Let $(M'V, F\mu'_V) \in \mathcal{C}_M$ be the image of the free M'-module on V under F. Then $f_V = F\mu'_V \circ M\eta'_V$. In particular, $f^* = F$. $\triangledown$

The following proposition is folklore.

Proposition 1.2.14. *Let L, $R \in \operatorname{End}(\mathcal{C})$, and suppose that L is left adjoint to R. Then R is a bimonad (bicomonad) iff L is a bicomonad (bimonad).*

Proof. Assume L is a bimonad, i.e. a comonoidal monad, and denote by $\widetilde{\eta}$ the unit and by $\widetilde{\varepsilon}$ the counit of the adjunction. The other case is similar. By Proposition 1.2.10 we know that R is a comonad. Then R becomes monoidal with multiplication

$$\begin{aligned} R_2(V,W) \equiv \Big[& RV \otimes RW \xrightarrow{\widetilde{\eta}_{RV\otimes RW}} RL(RV \otimes RW) \\ & \xrightarrow{RL_2(RV,RW)} R(LRV \otimes LRW) \xrightarrow{R(\widetilde{\varepsilon}_V \otimes \widetilde{\varepsilon}_W)} R(V \otimes W) \Big] \\ = {} & R(\widetilde{\varepsilon}_V \otimes \widetilde{\varepsilon}_W) \circ RL_2(RV,RW) \circ \widetilde{\eta}_{RV\otimes RW} \end{aligned} \tag{1.2.21}$$

and unit

$$R_0 \equiv \Big[\mathbf{1} \xrightarrow{\widetilde{\eta}_{\mathbf{1}}} RL\mathbf{1} \xrightarrow{RL_0} R\mathbf{1} \Big] = RL_0 \circ \widetilde{\eta}_{\mathbf{1}} \tag{1.2.22}$$

Since it is not hard—albeit rather tedious—to verify with a direct equational proof the monoidality of this structure, we will leave it to the reader. We also remark that a very nice and much simpler proof of this proposition can be given using the diagrammatic language of monoidal functors and monoidal natural transformations, see e.g. [Wi]. $\square$

Hopf monads

A bimonad M on a rigid category $\mathcal{C}$ is called a *Hopf monad* if $\mathcal{C}_M$ is rigid, following again the familiar nomenclature of algebras, bialgebras, and Hopf algebras. For a Hopf algebra, the rigid structure of its category of modules is encoded in the antipode. For Hopf monads, the corresponding concept is as follows, cf. [BV1]. A natural transformation S^l with components

$$S^l_V\colon M\big((MV)^\vee\big) \to V^\vee \tag{1.2.23}$$

is called a *left antipode* for M if it satisfies

$$\begin{tikzcd}
M((MV)^\vee \otimes V) \arrow[r, "M_2"] \arrow[dd, "M(\eta_V^\vee \otimes \mathrm{id})"'] & M(MV)^\vee \otimes MV \arrow[r, "M\mu_V^\vee \otimes \mathrm{id}"] & M(M^2V)^\vee \otimes MV \arrow[d, "S^l_{MV} \otimes \mathrm{id}"] \\
 & & (MV)^\vee \otimes MV \arrow[d, "\mathrm{ev}_{MV}"] \\
M(V^\vee \otimes V) \arrow[r, "M\,\mathrm{ev}_V"'] & M\mathbf{1} \arrow[r, "M_0"'] & \mathbf{1}
\end{tikzcd} \tag{1.2.24}$$

and

$$\begin{tikzcd}
M\mathbf{1} \arrow[r, "M_0"] \arrow[d, "M(\mathrm{coev}_{MV})"'] & \mathbf{1} \arrow[r, "\mathrm{coev}_V"] & V \otimes V^\vee \arrow[d, "\eta_V \otimes \mathrm{id}"] \\
M(MV \otimes (MV)^\vee) \arrow[r, "M_2"'] & M^2V \otimes M(MV)^\vee \arrow[r, "\mu_V \otimes S^l_V"'] & MV \otimes V^\vee
\end{tikzcd} \; . \tag{1.2.25}$$

Given an M-module (V, ρ), the antipode allows us to define a morphism

$$\tilde{\rho} = \left[M(V^\vee) \xrightarrow{M(\rho^\vee)} M\big((MV)^\vee\big) \xrightarrow{S^l_V} V^\vee \right], \tag{1.2.26}$$

which turns $(V^\vee, \tilde{\rho})$ into an M-module [BV1, Thm. 3.8]. The evaluation and coevaluation are those in $\mathcal{C}$,

$$\mathrm{ev}_{(V,\rho)} = \mathrm{ev}_V, \quad \mathrm{coev}_{(V,\rho)} = \mathrm{coev}_V, \tag{1.2.27}$$

and that they are indeed M-module intertwiners is guaranteed by the two commuting diagrams (1.2.24) and (1.2.25). Right duals via the right antipode are defined similarly. It was also shown in [BV1, Thm. 3.8] that $\mathcal{C}_M$ is rigid if and only if the left and right antipodes exist, and that the antipodes are unique.

A morphism of Hopf monads is a morphism of the underlying bimonads. It automatically commutes with the antipodes, [BV1, Lem. 3.13].

A Hopf comonad on $\mathcal{C}$ is a Hopf monad on $\mathcal{C}^{\mathrm{op}}$.

Example 1.2.15. If $\mathcal{C}$ is braided rigid monoidal, and $H \in \mathcal{C}$ is a Hopf algebra with invertible antipode, then $H \otimes ?$ is a Hopf monad, see [BV1, Ex. 3.10]. This example will be important in Section 4.5. △

Chapter 2

Preliminaries on quasi-Hopf algebras

The ubiquity of Hopf algebras is ever-increasing. This also brings with it, as so often in mathematics, several natural generalizations, such as Hopf monads, weak Hopf algebras, Hopf group coalgebras, see e.g. [BV1, EGNO, Vi]. In this chapter, we will review another generalization, that of a *quasi-Hopf algebra* introduced by Drinfeld in [Dr].

Whereas Hopf algebras are equipped with a coassociative coalgebra structure, quasi-Hopf algebras lose, or rather loosen, the coassociativity—at the expense of some technical complications. From a categorical point of view, quasi-Hopf algebras are in some sense more natural than Hopf algebras: a Hopf algebra corresponds to a finite tensor category together with a strong monoidal functor to Vect, while a quasi-Hopf algebra corresponds only to such a category together with a multiplicative functor to Vect (i.e. the functor commutes with the respective tensor products without satisfying any coherence conditions), see [EGNO, Ch. 5].

In the first section, we will review the fundamental definitions and properties of quasi-Hopf algebras, along with glimpses of categorical interpretations. The second section will briefly survey some additional structures and properties one might want a quasi-Hopf algebra to have. Afterwards, we recall the theory of cointegrals from [HN2, BC1, BC2], and in the last section give some examples.

Much of the material presented in this chapter can be found in the (recent) monograph [BCPO], which is the first book entirely devoted to quasi-Hopf algebras.

2.1 The fundamentals

In this section we give our conventions for quasi-Hopf algebras, and state some important results. We follow the conventions in [BGR1, BGR2, FGR1, FGR2], which in some sense are opposite to those in [HN2, BC1, BC2, BCT], so a bit of care when comparing equations over different sources has to be imposed.

Recall that we agreed that all linear structures are over an algebraically closed field k, and finite (in the appropriate sense), unless otherwise stated. Note that mostly, in this and the next chapter, algebraic closedness is not actually needed; we keep it, however, to stay in line with our conventions for finite tensor categories.

2.1.1 The definition of a quasi-Hopf algebra

A *quasi-Hopf algebra* is a tuple $H = (H, \mu, \eta, \Delta, \varepsilon, S, \boldsymbol{\alpha}, \boldsymbol{\beta}, \Phi)$ together with some axioms which we will now explain in detail.

(H, μ, η) is an associative algebra H with multiplication μ and unit η, and we set $\mathbf{1} = \eta(1) \in H$. Multiplication will as usual be written as juxtaposition, i.e. for $a, b \in H$, $ab = \mu(a \otimes b)$.

The three linear maps

$$S\colon H \to H, \quad \Delta\colon H \to H \otimes H \quad \text{and} \quad \varepsilon\colon H \to k\ , \tag{2.1.1}$$

are called the *antipode*, the *coproduct*, and the *counit*, respectively. The antipode is required to be an anti-algebra morphism, while the two other maps are algebra morphisms.

Lastly, the elements $\boldsymbol{\alpha}$, $\boldsymbol{\beta} \in H$ are called the *evaluation* and *coevaluation element*, and $\Phi \in H^{\otimes 3}$ is the *coassociator*, which is invertible. We write $\Psi = \Phi^{-1}$.

The coassociator makes the coproduct *quasi-coassociative*[7], i.e.

$$(\mathrm{id} \otimes \Delta)(\Delta(h)) = \Psi \cdot (\Delta \otimes \mathrm{id})(\Delta(h)) \cdot \Phi\ , \tag{2.1.2}$$

for all $h \in H$. Thus, the Φ in [HN2, BC2] is our Ψ. It is also *counital* and satisfies a *3-cocycle condition*, that is

$$(\mathrm{id} \otimes \varepsilon \otimes \mathrm{id})(\Phi) = \mathbf{1} \otimes \mathbf{1} \tag{2.1.3}$$

and

$$\begin{aligned}(\Delta \otimes \mathrm{id} \otimes \mathrm{id})(\Phi) \cdot &(\mathrm{id} \otimes \mathrm{id} \otimes \Delta)(\Phi) \\ &= (\Phi \otimes \mathbf{1}) \cdot (\mathrm{id} \otimes \Delta \otimes \mathrm{id})(\Phi) \cdot (\mathbf{1} \otimes \Phi)\ .\end{aligned} \tag{2.1.4}$$

Before listing the rest of the defining axioms, we introduce some notation.

Notation 2.1.1. Two variants of the *sumless Sweedler notation* will be used. For an element $u \in H^{\otimes n}$ we write $u = u_1 \otimes \cdots \otimes u_n$. Note that this is just a notation useful for suppressing unwieldy sums—it does not imply that u is a simple tensor. The correct way to parse an expression like $u_1 u_2$ for $u \in H \otimes H$ is thus to read it as $\mu(u)$. We also write $u_{21} = \tau(u) = u_2 \otimes u_1$, where τ is the flip map in Vect, and this notation is extended to higher tensor powers in the obvious way, e.g. for $u \in H \otimes H$, we also define the notation $u_{13}, u_{31} \in H^{\otimes 3}$ etc, to mean $u_{13} = u_1 \otimes \mathbf{1} \otimes u_2$ and $u_{31} = u_2 \otimes \mathbf{1} \otimes u_1$.

[7] For simplicity, this axiom will later also be referred to as the *coassociativity* of the quasi-Hopf algebra H.

Because often expressions will involve multiple copies of the coassociator or its inverse, we will distinguish them by writing the first copy as

$$\Phi = X_1 \otimes X_2 \otimes X_3 \quad \text{and} \quad \Psi = x_1 \otimes x_2 \otimes x_3 \ , \tag{2.1.5}$$

and using letters like Y, y and Z, z for further copies. That is, the coassociator is expressed using capital Latin letters, while its inverse uses lowercase Latin letters.

The second Sweedler notation is for the coproduct and the iterated coproduct. We write

$$\Delta(h) = h_{(1)} \otimes h_{(2)} \quad \text{and e.g.} \quad (\mathrm{id} \otimes \Delta)(\Delta(h)) = h_{(1)} \otimes h_{(2,1)} \otimes h_{(2,2)} \ , \tag{2.1.6}$$

for $h \in H$. $\diamondsuit$

With this notation established, we can write the coassociativity (2.1.2) of H in its probably most convenient form as

$$X_1 h_{(1)} \otimes X_2 h_{(2,1)} \otimes X_3 h_{(2,2)} = h_{(1,1)} X_1 \otimes h_{(1,2)} X_2 \otimes h_{(2)} X_3 \ . \tag{2.1.7}$$

We are ready to state the antipode axioms. Any $h \in H$ satisfies

$$S(h_{(1)}) \boldsymbol{\alpha} h_{(2)} = \varepsilon(h) \boldsymbol{\alpha} \quad \text{and} \quad h_{(2)} \boldsymbol{\beta} S(h_{(2)}) = \varepsilon(h) \boldsymbol{\beta} \ , \tag{2.1.8}$$

and the coassociator satisfies either[8] of

$$S(X_1) \boldsymbol{\alpha} X_2 \boldsymbol{\beta} S(X_3) = \mathbf{1} \quad \text{and} \quad x_3 \boldsymbol{\beta} S(x_2) \boldsymbol{\alpha} x_1 = \mathbf{1} \ . \tag{2.1.9}$$

The latter pair of equations is referred to as the *zig-zag axioms*, and we will understand the reason for this after talking about the category of representation of H. Note that the zig-zag axioms imply that $\varepsilon(\boldsymbol{\alpha})$ and $\varepsilon(\boldsymbol{\beta})$ are non-zero. Thus without loss of generality we may (and will) always assume $\varepsilon(\boldsymbol{\alpha}) = 1 = \varepsilon(\boldsymbol{\beta})$.

A linear map $f\colon A \to B$ between quasi-Hopf algebras is a morphism of quasi-Hopf algebras if it is an algebra map that commutes with the coaction, the counit, and the antipode, and which sends the (co)evaluation elements and the coassociator of A to those of B.

Following e.g. [HN2], we use the *hook notation*

$$\begin{aligned} (h \rightharpoonup f)(a) &= f(ah), & f \rightharpoonup h &= h_{(1)} f(h_{(2)}) \ , \\ (f \leftharpoonup h)(a) &= f(ha), & h \leftharpoonup f &= f(h_{(1)}) h_{(2)} \ , \end{aligned} \tag{2.1.10}$$

for $h, a \in H$, $f \in H^*$. Note that the left column defines an action of H on H^*.

[8]Indeed, one of the two equations implies the other, [Dr, Prop. 1.3] or [BCPO, Prop. 3.2.2]. However, both are displayed for later use.

Example 2.1.2. Let H be a quasi-Hopf algebra. We get three additional quasi-Hopf algebras for free by considering the opposite multiplication $\mu^{\mathrm{op}} = \mu \circ \tau$ and comultiplication $\Delta^{\mathrm{cop}} = \tau \circ \Delta$. Namely, the opposite

$$H^{\mathrm{op}} = (H, \mu^{\mathrm{op}}, \eta, \Delta, \varepsilon, S^{-1}, S^{-1}(\boldsymbol{\beta}), S^{-1}(\boldsymbol{\alpha}), \Psi) \ ,$$

the coopposite

$$H^{\mathrm{cop}} = (H, \mu, \eta, \Delta^{\mathrm{cop}}, \varepsilon, S^{-1}, S^{-1}(\boldsymbol{\alpha}), S^{-1}(\boldsymbol{\beta}), \Psi_{321}) \ ,$$

and the op-coopposite

$$H^{\mathrm{op,cop}} = (H, \mu^{\mathrm{op}}, \eta, \Delta^{\mathrm{cop}}, \varepsilon, S, \boldsymbol{\beta}, \boldsymbol{\alpha}, \Phi_{321})$$

Furthermore, the tensor product of two quasi-Hopf algebras is naturally a quasi-Hopf algebra. △

For actual examples we refer to Section 2.4, the last section of this chapter.

Remark 2.1.3.
1. The antipode of a finite-dimensional quasi-Hopf algebra is automatically bijective [BC1, Thm. 2.2].
2. Any Hopf algebra is a quasi-Hopf algebra with $\boldsymbol{\alpha} = \boldsymbol{\beta} = \mathbf{1}$ and $\Phi = \mathbf{1}^{\otimes 3}$.
3. It is easy to see that counitality and the cocycle condition also imply

$$\varepsilon(X_1)X_2 \otimes X_3 = \mathbf{1} \otimes \mathbf{1} = \varepsilon(X_3)X_1 \otimes X_2 \ . \tag{2.1.11}$$

Similar equations also hold for the inverse coassociator.

▽

Remark 2.1.4.
1. A *(quasi-)bialgebra* is defined like a (quasi)-Hopf algebra without an antipode and (co)evaluation elements. It is easy to see that for a bialgebra, there is at most one antipode, so that being a Hopf algebra is a *property* of a bialgebra. In the strictly non-coassociative case this is no longer true. Instead, if the triple $(S, \boldsymbol{\alpha}, \boldsymbol{\beta})$ turns a quasi-bialgebra into a quasi-Hopf algebra, then so does $(S^u, u\boldsymbol{\alpha}, \boldsymbol{\beta}u^{-1})$ for any invertible element $u \in H$. Here S^u is the map $S^u(h) = uS(h)u^{-1}$. In fact, any two quasi-Hopf structures on a quasi-bialgebra are related in this way, via a unique u, cf. [Dr, Prop. 1.1]. This is more easily understood after Section 2.1.3: it comes from the fact that a rigid structure, if it exists, is unique up to unique isomorphism (in the appropriate sense), see e.g. [Sch1, Cor. 1.2.10].
2. Let A, H be quasi-Hopf algebras. As a consequence of the first point, a morphism $f\colon A \to H$ of quasi-bialgebras—i.e. a linear map preserving (co)multiplication, (co)unit, and coassociator—is not necessarily a morphism of quasi-Hopf algebras. This is in contrast to when A and H are strictly coassociative: a morphism of the underlying bialgebras automatically commutes with the antipodes.

3. The reader knowledgeable in Hopf algebras will also have notice that we *required* the antipode S to be an anti-algebra morphism. If H is a Hopf algebra, then this requirement is superfluous, see e.g. [Ka, Thm. III.3.4]. The argument uses two facts. Firstly, linear maps $H \otimes H \to H$ form an associative unital algebra under the convolution product

$$H \otimes H \xrightarrow{\mathrm{id} \otimes \tau \otimes \mathrm{id} \circ \Delta \otimes \Delta} H^{\otimes 4} \xrightarrow{\mu \otimes \mu} H \otimes H \xrightarrow{\mu} H \ . \tag{2.1.12}$$

   Secondly, the maps $\mu^{\mathrm{op}} \circ (S \otimes S)$ and $S \circ \mu$ are a left and right inverse of μ in the convolution algebra. Hence they necessarily agree. One can show (for a Hopf algebra) that S is an anti-coalgebra map in a very similar way.

   For a quasi-Hopf algebra, the space of linear maps as above does, in general, not form an associative algebra, so the above approach fails right away; hence the requirement. As for S being an anti-coalgebra map, this is not even true anymore, as we will review in Section 2.1.2.

$\triangledown$

2.1.2 Twist equivalence

Following e.g. [Dr, Ka, HN2], we recall that a *twist* is an invertible element $F \in H \otimes H$ which is normalized, i.e. satisfies

$$(\varepsilon \otimes \mathrm{id})(F) = \mathbf{1} = (\mathrm{id} \otimes \varepsilon)(F) \ . \tag{2.1.13}$$

A twist induces a *twist transformation*. This means that by setting

$$\begin{aligned} \Delta_F(h) &= F\Delta(h)F^{-1} \\ \Phi_F &= (F \otimes \mathbf{1}) \cdot (\Delta \otimes \mathrm{id})(F) \cdot \Phi \cdot (\mathrm{id} \otimes \Delta)(F^{-1}) \cdot (\mathbf{1} \otimes F^{-1}) \\ \boldsymbol{\alpha}_F &= S(F_1^{-1})\boldsymbol{\alpha} F_2^{-1}, \quad \boldsymbol{\beta}_F = F_1 \boldsymbol{\beta} S(F_2) \end{aligned} \tag{2.1.14}$$

we obtain a new quasi-Hopf algebra $H_F = (H, \mu, \eta, \Delta_F, \varepsilon, S, \boldsymbol{\alpha}_F, \boldsymbol{\beta}_F, \Phi_F)$. It is clear that if F' is a twist on H_F, then $F'F$ is a twist on H, and $(H_F)_{F'} = H_{F'F}$. Consequently $(H_F)_{F^{-1}} = H_{\mathbf{1}} = H$. Furthermore, it is clear that this also works on the level of quasi-bialgebras, disregarding (co)evaluation elements and the antipode.

This yields a notion of equivalence slightly weaker and more flexible than isomorphism. Namely, two quasi-bialgebras (resp. quasi-Hopf algebras) H and H' are said to be *twist equivalent* (resp. *strongly twist equivalent*) if there is an isomorphism $f\colon H' \to H_F$ of quasi-bialgebras (resp. quasi-Hopf algebras) for a twist F on H. In that case the first equation of (2.1.14) becomes

$$F\Delta(f(h'))F^{-1} = (f \otimes f)(\Delta'(h')) \tag{2.1.15}$$

for $h' \in H'$ and where Δ, Δ' are the coproducts of H, H', respectively. One checks that this indeed defines an equivalence relation on the set of quasi-bialgebras (resp. quasi-Hopf algebras).

Remark 2.1.5. Note that if $f\colon H' \to H_F$ exhibits a strong twist equivalence, then f also exhibits a twist equivalence of the underlying quasi-bialgebras; the converse need not hold. If H, H' are quasi-Hopf algebras, then we can transport the quasi-Hopf structure on H' along an isomorphism $f\colon H' \to H_F$ of quasi-bialgebras. The two quasi-Hopf structures on H_F are related via a unique invertible element as in Remark 2.1.4. $\triangledown$

In Section 2.1.5 we will see that twist equivalent quasi-bialgebras have the same representation theory.[9]

The most famous example of a strong twist equivalence is perhaps given by the Drinfeld twist, introduced by Drinfeld in [Dr]. To motivate it, recall from Remark 2.1.4 that for ordinary Hopf algebras the antipode is an anti-coalgebra morphism, and thus induces a Hopf algebra isomorphism $H \cong H^{\mathrm{op,cop}}$. In the quasi-setting, this is no longer true. However, the antipode *is* a quasi-Hopf algebra isomorphism between $H_{\boldsymbol{f}}$ and $H^{\mathrm{op,cop}}$, for a certain twist $\boldsymbol{f}$ on H called the *Drinfeld twist*. In particular, it satisfies

$$\boldsymbol{f}\Delta(S(h))\boldsymbol{f}^{-1} = (S \otimes S)(\Delta^{\mathrm{cop}}(h)) \ . \tag{2.1.16}$$

In Section 2.1.6, after we have started thinking about quasi-Hopf algebras in a categorical way, we will be able to easily give a closed form for $\boldsymbol{f}$. We also postpone until then the specialization of (2.1.14) to $F = \boldsymbol{f}$.

2.1.3 Modules over quasi-Hopf algebras

In this thesis *module* will mean *left module*, unless otherwise stated. Recall also crucially that all vector spaces are finite-dimensional.

A module over a quasi-Hopf algebra H is a module over the algebra underlying H. We therefore get a finite abelian category of H-modules, which we denote by ${}_H\mathcal{M}$ or H-mod. The additional and fairly complicated-looking axioms of a quasi-Hopf algebra equip ${}_H\mathcal{M}$ with a much richer structure, namely that of a non-strict finite tensor category.[10],[11] We will presently explain the structure of this category. To this end let us agree that we denote the action of $h \in H$ on $v \in V$ by $h.v$, for $V \in {}_H\mathcal{M}$.

The tensor product of two H-modules $V, W \in {}_H\mathcal{M}$ is given by the tensor product $V \otimes_k W$ equipped with the diagonal action, that is

$$h.(v \otimes w) = h_{(1)}.v \otimes h_{(2)}.w \tag{2.1.17}$$

for $h \in H$, $v \in V$, $w \in W$.

[9] More concretely, we will see that twist-equivalence produces a monoidal isomorphism between the respective categories of representations. Requiring *strong* twist equivalence puts extra (unnecessary) constraints on this isomorphism, which is why we want to make the distinction.

[10] In our convention, ${}_H\mathcal{M}$ has a non-trivial associator, but strict unitors, cf. Remark 2.1.9 later.

[11] More precisely, a quasi-Hopf algebra is an algebra with additional structure and property such that its category of modules is finite tensor, and such that the forgetful functor to vector spaces is multiplicative (i.e. it preserves tensor products but does not necessarily do so coherently).

As mentioned above the tensor product is not strict. The associator is given by acting with the coassociator of H,

$$\begin{aligned}\alpha_{U,V,W} : U \otimes (V \otimes W) &\to (U \otimes V) \otimes W \\ u \otimes v \otimes w &\mapsto \Phi \cdot (u \otimes v \otimes w) = X_1.u \otimes X_2.v \otimes X_3.w \ .\end{aligned} \tag{2.1.18}$$

This linear map intertwines the H-action thanks to coassociativity (2.1.7), and the pentagon axiom follows immediately from the cocycle condition (2.1.4). The tensor unit $\mathbf{1}$ is given by the ground field k equipped with the action by ε.

The regular module H plays a special role later, and we introduce some extra notation for the left regular action. For $h \in H$ we denote it by l_h, so that for all $a \in H$

$$l_h(a) = ha \ . \tag{2.1.19}$$

Similarly, the right regular action by h is denoted r_h.

We denote the canonical pairing between a vector space V and its dual V^* by angled brackets, i.e.

$$V^* \times V \to k, \quad (f, v) \mapsto \langle f \mid v \rangle := f(v) \ . \tag{2.1.20}$$

An H-module V has both a left and a right dual. They are given by the dual vector space V^*, and $h \in H$ acts on the

$$\text{left dual } V^\vee \qquad\qquad\qquad \text{right dual } {}^\vee V$$

by, for $v \in V$, $f \in V^*$,

$$(h.f)(v) = \langle f \mid S(h).v \rangle \qquad\qquad (h.f)(v) = \langle f \mid S^{-1}(h).v \rangle \ . \tag{2.1.21}$$

The corresponding evaluation is given by

$$\mathrm{ev}_V(f \otimes v) = \langle f \mid \boldsymbol{\alpha}.v \rangle \qquad\qquad \widetilde{\mathrm{ev}}_V(v \otimes f) = \langle f \mid S^{-1}(\boldsymbol{\alpha}).v \rangle \ , \tag{2.1.22}$$

and the coevaluation by

$$\mathrm{coev}_V(1) = \sum_{i=1}^{\dim V} \boldsymbol{\beta}.v_i \otimes v^i \qquad\qquad \widetilde{\mathrm{coev}}_V(1) = \sum_{i=1}^{\dim V} v^i \otimes S^{-1}(\boldsymbol{\beta}).v_i \ , \tag{2.1.23}$$

for a basis $\{v_i\}$ of V with corresponding dual basis $\{v^i\}$. By (2.1.8), these four maps are intertwiners, and the zig-zag axioms for ${}_H\mathcal{M}$ follow from the zig-zag axioms (2.1.9) for H.

For later use we extend the hook notation from (2.1.10) to dual vector spaces, so that for example the action on the left dual of the H-module V could then be written as $f \leftharpoondown S(h)$, since

$$\langle h.f \mid v \rangle = \langle f \mid S(h)v \rangle = \langle f \leftharpoondown S(h) \mid v \rangle \tag{2.1.24}$$

for all $h \in H$, $v \in V$, $f \in V^*$.

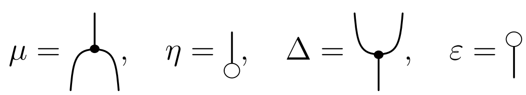
Note also that the canonical isomorphism (1.1.12) $V \cong ({}^\vee V)^\vee$ is the same linear map as in Vect: this follows from the zig-zag axioms (2.1.9).

Remark 2.1.6. We will later (in Section 2.2) define additional structure on H turning ${}_H\mathcal{M}$ into a pivotal category. As mentioned in Chapter 1, one then gets another candidate for right duals from left duals using the pivotal structure. Whenever 'right dual' is mentioned, however, one should always think of the canonical right dual as described above, unless we explicitly state otherwise. ▽

Remark 2.1.7. Let H be a quasi-bialgebra. Note that then, similarly, ${}_H\mathcal{M}$ is a monoidal category. If ${}_H\mathcal{M}$ admits a rigid structure compatible with taking duals in vector spaces, then H admits an antipode triple $(S, \boldsymbol{\alpha}, \boldsymbol{\beta})$, compare with Remark 2.1.4. A choice of antipode triple is then a choice of rigid structure, see [BCPO, Sec. 3.5]. ▽

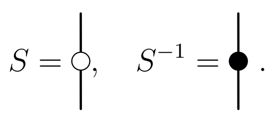
2.1.4 Structural morphisms and actions in pictures

We will later often draw the structural morphisms of a quasi-Hopf algebra H in a certain way, i.e. not with coupons labeled by, for example, Δ. We follow e.g. [BBGa, FSS, FGR1]. The multiplication, unit, comultiplication, and counit are



(2.1.25)

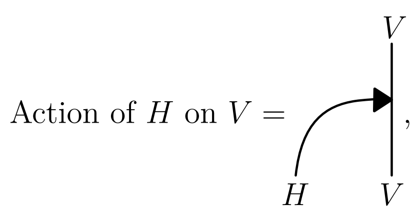
respectively, while for the antipode and its inverse we use



(2.1.26)

Note that these are, in general, only string diagrams in Vect.

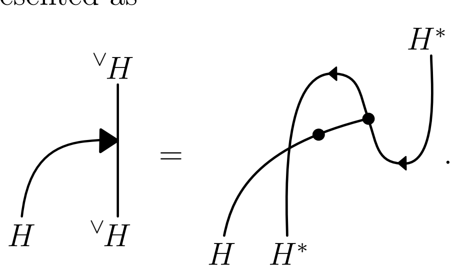
For $V \in {}_H\mathcal{M}$, we denote the action of H by



(2.1.27)

and note that, when regarding V in the source as a vector space with trivial H-action, this is a morphism in ${}_H\mathcal{M}$.

As an example putting all of this together, the action on the right dual ${}^\vee H$ of the regular module can be represented as



(2.1.28)

We stress again that these pictures (and in particular the rigid structure in the last picture) are in Vect.

2.1.5 The categorical meaning of twist equivalence

(Strong) twist equivalence as discussed in Section 2.1.2 has the following very nice well-known categorical interpretation, which we now recall from e.g. [BCPO, Dr, Ka]. We refer to [Ka, Sec. XV.3] for full proofs.

At first, let us just consider one quasi-Hopf algebra H and a twist F on H. By construction of H_F, the identity functor on ${}_H\mathcal{M}$ regarded as a functor $\mathrm{id} : {}_H\mathcal{M} \to {}_{H_F}\mathcal{M}$ is a (strong monoidal) isomorphism of finite tensor categories. Its unit is trivial, and the multiplication is the linear map

$$\mathrm{id}_2(V,W) : V \otimes W \to V \otimes W, \quad v \otimes w \mapsto F^{-1}.(v \otimes w) \ . \tag{2.1.29}$$

That this is an intertwiner and monoidal is precisely the content of the defining data (2.1.14) of H_F (as a quasi-bialgebra).[12]

More generally, consider now two quasi-Hopf algebras A and H. Denote the quasi-Hopf structures of A and H by the standard symbols with superscripts A and H, respectively. Let $f : A \to H$ be an isomorphism of the underlying algebras. The isomorphism immediately induces a (linear) pullback functor $f^* : {}_H\mathcal{M} \to {}_A\mathcal{M}$, i.e. we precompose the H-action with f to obtain an A-action. The pullback is an equivalence (of finite linear abelian categories); more precisely, it is even an isomorphism.

Now, on the one hand, suppose that A and H are twist-equivalent. Then there exists a twist F on H and a quasi-bialgebra isomorphism $f : A \to H_F$, which equip f^* with a strict monoidal structure. Precomposing with the equivalence from above, we obtain the strong monoidal functor $f^* : {}_H\mathcal{M} \to {}_A\mathcal{M}$ with strict unit and multiplication given by

$$f_2^*(V,W) : f^*(V) \otimes f^*(W) \to f^*(V \otimes W), \quad v \otimes w \mapsto F^{-1}.(v \otimes w) \ . \tag{2.1.30}$$

for all $V, W \in {}_H\mathcal{M}$. Indeed, f_2^* being an intertwinter amounts to

$$F^{-1} \cdot (f \otimes f)(\Delta^A(a)) = \Delta^H(f(a)) \cdot F^{-1} \ , \tag{2.1.31}$$

and after rearranging, the associativity coherence condition of the functor f^* is

$$f^{\otimes 3}(\Phi^A) = (F \otimes \mathbf{1}) \cdot (\Delta^H \otimes \mathrm{id})(F) \cdot \Phi^H \cdot (\mathrm{id} \otimes \Delta^H)(F^{-1}) \cdot (\mathbf{1} \otimes F^{-1}) \ . \tag{2.1.32}$$

Comparing with the definition (2.1.14) of the quasi-Hopf data on H_F, we see that this equation is equivalent to the true statement $f^{\otimes 3}(\Phi^A) = \Phi^{H_F}$. Thus ${}_H\mathcal{M} \cong {}_A\mathcal{M}$ as finite tensor categories.

One the other hand, suppose $G : {}_H\mathcal{M} \to {}_A\mathcal{M}$ is a strong monoidal equivalence whose unit is trivial, and which commutes with the canonical forgetful functors to Vect. Then it is given by the pullback along a unique algebra isomorphism $f : A \to H$, cf. [BV1, Lem. 1.6], so we can without loss assume $G = f^*$. Set $F^{-1} = f_2^*(H,H)(\mathbf{1} \otimes \mathbf{1}) \in H \otimes H$.

[12] 'Monoidal' means that it satisfies some coherence conditions which we did not give explicitly earlier in Section 1.1.5. The important condition in question here can be formulated as an obvious diagram involving the multiplication and the associativity isomorphisms of both categories.

Then f_2^* is completely determined by F^{-1}. Indeed, to see this, note first that for any $v \in V \in {}_H\mathcal{M}$ we get an H-intertwiner

$$\rho_v : H \to V, \quad h \mapsto h.v$$

from the regular module to V; moreover, the pullback doesn't do anything on morphisms. Together with naturality this implies the linear equation

$$f_2^*(V,W)(v \otimes w) = (\rho_v \otimes \rho_w)\Big(f_2^*(H,H)(\mathbf{1} \otimes \mathbf{1})\Big) = F_1^{-1}.v \otimes F_2^{-1}.w \ .$$

Clearly f_0^* being strict is equivalent to F being a twist. The equations that f_2^* satisfies—i.e. (2.1.31) and (2.1.32)—now show that $f : A \to H_F$ is an isomorphism of quasi-bialgebras, so that A and H are twist-equivalent.

Remark 2.1.8. Strong twist equivalence puts a further condition on the equivalence ${}_H\mathcal{M} \cong {}_A\mathcal{M}$. Namely, if $f : A \to H_F$ is a strong twist equivalence, then f^* commutes with the rigid structure on the nose. But this is unnecessary, since the rigid structure of a monoidal category is unique (up to unique isomorphism) if is exists [Sch1, Cor. 1.2.10]. $\triangledown$

Remark 2.1.9. One could have included in the definition of a quasi-Hopf algebra two additional elements $l, r \in H$, implementing left and right unitors of ${}_H\mathcal{M}$. However, this may be disregarded, since every quasi-Hopf algebra in that sense is twist-equivalent to a quasi-Hopf algebra as defined earlier (without unitors), see [BCPO, Cor. 3.11]. $\triangledown$

2.1.6 Special elements and their relations

When working with quasi-Hopf algebras it is convenient to introduce some special elements in tensor powers of H and their relations. Introducing them is what this section is about. We closely follow [BC2], but note again that our conventions are slightly different, i.e. our Φ is their Φ^{-1}.

We first introduce the four elements q^R, p^R, q^L, p^L in $H \otimes H$, given by

$$\begin{aligned} q^R &= x_1 \otimes S^{-1}(\boldsymbol{\alpha} x_3)x_2 \ , \qquad & p^R &= X_1 \otimes X_2 \boldsymbol{\beta} S(X_3) \ , \\ q^L &= S(X_1)\boldsymbol{\alpha} X_2 \otimes X_3 \ , \qquad & p^L &= x_2 S^{-1}(x_1 \boldsymbol{\beta}) \otimes x_3 \ . \end{aligned} \tag{2.1.33}$$

These satisfy the identities

$$\begin{aligned} \Delta(q_1^R)p^R[\mathbf{1} \otimes S(q_2^R)] &= \mathbf{1} \otimes \mathbf{1} \ , \qquad & [\mathbf{1} \otimes S^{-1}(p_2^R)]q^R\Delta(p_1^R) &= \mathbf{1} \otimes \mathbf{1} \ , \\ \Delta(q_2^L)p^L[S^{-1}(q_1^L) \otimes \mathbf{1}] &= \mathbf{1} \otimes \mathbf{1} \ , \qquad & [S(p_1^L) \otimes \mathbf{1}]q^L\Delta(p_2^L) &= \mathbf{1} \otimes \mathbf{1} \ , \end{aligned} \tag{2.1.34}$$

and, for all $a \in H$,

$$\begin{aligned} [\mathbf{1} \otimes S^{-1}(a_{(2)})]q^R\Delta(a_{(1)}) &= [a \otimes \mathbf{1}]q^R \ , \\ [S(a_{(1)}) \otimes \mathbf{1}]q^L\Delta(a_{(2)}) &= [\mathbf{1} \otimes a]q^L \ , \end{aligned}$$

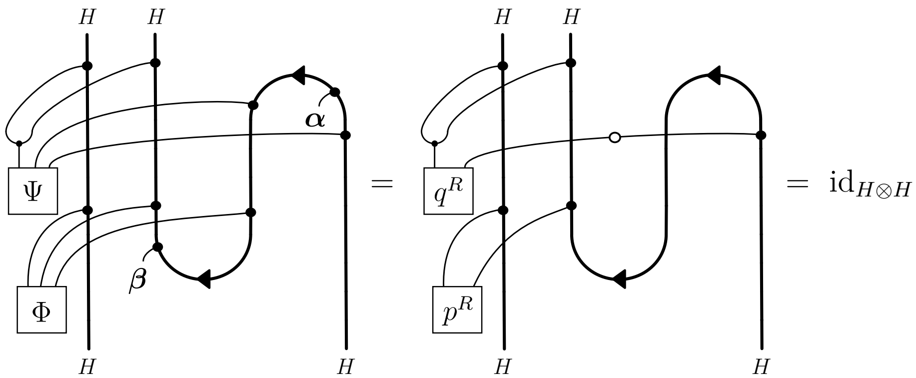


Figure 2.1: The equalities in (2.1.34) follow from the zig-zag identities for duals in H-mod, where both sides are tensored with the identity. We show this above for first equality in (2.1.34), where we have written out id_H times the zig-zag identity for H in H-mod as a string diagram in Vect.

$$\begin{aligned}\Delta(a_{(1)})p^R[\mathbf{1}\otimes S(a_{(2)})] &= p^R[a\otimes \mathbf{1}]\ ,\\ \Delta(a_{(2)})p^L[S^{-1}(a_{(1)})\otimes \mathbf{1}] &= p^L[\mathbf{1}\otimes a]\ .\end{aligned} \tag{2.1.35}$$

These elements and relations are well-known in the representation theory of quasi-Hopf algebras. They, in fact, have nice interpretations in H-mod, see Figure 2.1 for an example. More about this may be found in e.g. [HN1, Sec. 2].

We will now describe the Drinfeld twist already mentioned in Section 2.1.2 explicitly, and use insights from Sections 2.1.3 and 2.1.5 to help understand some of its properties. Recall the natural isomorphism $\gamma_{V,W}\colon V^\vee\otimes W^\vee \to (WV)^\vee$ from (1.1.6). There is a unique invertible element $\boldsymbol{f}\in H\otimes H$, called the *Drinfeld twist*, such that

$$\gamma_{V,W}(f\otimes g)(w\otimes v) = g(\boldsymbol{f}_1 w)f(\boldsymbol{f}_2 v) \tag{2.1.36}$$

for all $f\in V^*, v\in V$, $g\in W^*, w\in W$, see [Dr] and e.g. [FGR1, Lem. 6.7]. The statement that $\gamma_{V,W}$ be an intertwiner translates to the equation

$$\boldsymbol{f}\cdot\Delta(S(a))\cdot\boldsymbol{f}^{-1} = (S\otimes S)\left(\Delta^{\mathrm{cop}}(a)\right)\ . \tag{2.1.37}$$

We also find

$$(\varepsilon\otimes\mathrm{id})(\boldsymbol{f}) = \mathbf{1} = (\mathrm{id}\otimes\varepsilon)(\boldsymbol{f})\ , \tag{2.1.38}$$

so that $\boldsymbol{f}$ indeed is a twist as defined in Section 2.1.2. We may therefore define the quasi-Hopf algebra $H_{\boldsymbol{f}}$ and ask if, in light of (2.1.37), the antipode is an isomorphism of quasi-Hopf algebras $S\colon H^{\mathrm{op,cop}}\to H_{\boldsymbol{f}}$. The answer is *yes*, and a first proof can be found in [Dr, Sec. 1], a more detailed newer proof in [BCPO, Prop. 3.23].

Let us now state some technical properties of the Drinfeld twist which we will need later on; this is not even close to an exhaustive list, but it includes exactly those we need, for more see [BC2, Sec.2]. The Drinfeld twist satisfies the identity

$$(\mathbf{1} \otimes \boldsymbol{f}) \cdot (\mathrm{id} \otimes \Delta)(\boldsymbol{f}) \cdot \Psi = (S \otimes S \otimes S)(\Psi_{321}) \cdot (\boldsymbol{f} \otimes \mathbf{1}) \cdot (\Delta \otimes \mathrm{id})(\boldsymbol{f}) \ , \tag{2.1.39}$$

or, written in Sweedler-notation,

$$\begin{aligned} &\boldsymbol{f}_1 x_1 \otimes \widetilde{\boldsymbol{f}}_1 \boldsymbol{f}_{2(1)} x_2 \otimes \widetilde{\boldsymbol{f}}_2 \boldsymbol{f}_{2(2)} x_3 \\ &\qquad = S(x_3) \widetilde{\boldsymbol{f}}_1 \boldsymbol{f}_{1(1)} \otimes S(x_2) \widetilde{\boldsymbol{f}}_2 \boldsymbol{f}_{1(2)} \otimes S(x_1) \boldsymbol{f}_2 \ , \end{aligned} \tag{2.1.40}$$

where we use the symbol $\widetilde{\boldsymbol{f}}$ to denote another copy of the Drinfeld twist. This looks rather complicated, but note that this is just the specialization of (2.1.32) to the quasi-Hopf algebra isomorphism $S\colon H^{\mathrm{op,cop}} \to H_{\boldsymbol{f}}$.

The Drinfeld twist further satisfies

$$\boldsymbol{f}_1 \boldsymbol{\beta} S(\boldsymbol{f}_2) = S(\boldsymbol{\alpha}) \ , \quad S(\boldsymbol{\beta} \boldsymbol{f}_1) \boldsymbol{f}_2 = \boldsymbol{\alpha} \ , \tag{2.1.41}$$

which can be seen by either a direct computation, or simply an appeal to Sections 2.1.2 and 2.1.5.

Now we shall write out the closed form of the Drinfeld twist and its inverse, which may readily be read off from (2.1.36) and (1.1.6). To this end, we also define $\boldsymbol{\varepsilon}, \boldsymbol{\delta} \in H \otimes H$ by

$$\boldsymbol{\varepsilon} = S(x_2) q^L_1 x_{3(1)} \otimes S(x_1) \boldsymbol{\alpha} q^L_2 x_{3(2)} = S(q^R_2 X_{1(2)}) X_2 \otimes S(q^R_1 X_{1(1)}) \boldsymbol{\alpha} X_3 \tag{2.1.42}$$

and

$$\boldsymbol{\delta} = x_{1(1)} p^R_1 \boldsymbol{\beta} S(x_3) \otimes x_{1(2)} p^R_2 S(x_2) = X_1 \boldsymbol{\beta} S(X_{3(2)} p^L_2) \otimes X_2 S(X_{3(1)} p^L_1) \ . \tag{2.1.43}$$

These come from the canonical morphisms

$$(W^\vee \otimes V^\vee) \otimes (V \otimes W) \to \mathbf{1} \quad \text{and} \quad \mathbf{1} \to (V \otimes W) \otimes (W^\vee \otimes V^\vee)$$

for all $V, W \in {}_H\mathcal{M}$, respectively, and the two different expressions for $\boldsymbol{\varepsilon}$ and $\boldsymbol{\delta}$ on each line indeed agree by coherence. From this point of view, we easily see that they satisfy

$$\Delta(\boldsymbol{\alpha}) = \boldsymbol{f}^{-1} \boldsymbol{\varepsilon} \quad \text{and} \quad \Delta(\boldsymbol{\beta}) = \boldsymbol{\delta} \boldsymbol{f} \ . \tag{2.1.44}$$

The Drinfeld twist is now given explicitly by

$$\boldsymbol{f} = (S \otimes S)(\Delta^{\mathrm{cop}}(p^R_1)) \boldsymbol{\varepsilon} \Delta(p^R_2) \tag{2.1.45}$$

and its inverse is

$$\boldsymbol{f}^{-1} = \Delta(q^L_1) \boldsymbol{\delta} (S \otimes S)(\Delta^{\mathrm{cop}}(q^L_2)) \ , \tag{2.1.46}$$

see also [FGR1, Lem. 6.7].

We finish this section by mentioning the following variant of the Drinfeld twist. The natural isomorphism ${\gamma^r}_{V,W} \colon {}^\vee V \otimes {}^\vee W \to {}^\vee(WV)$, obtained by mirroring the diagram in (1.1.6) at the vertical axis, can be expressed by using what we call the *Drinfeld twist for right duals* $\boldsymbol{f}^r$:

$$
{\gamma^r}_{V,W}(f \otimes g)(w \otimes v) = g(\boldsymbol{f}^r_1 w) f(\boldsymbol{f}^r_2 v) \tag{2.1.47}
$$

for all $f \in V^*, v \in V$, $g \in W^*, w \in W$. The explicit form of the Drinfeld twist for right duals can similarly be given as

$$
\begin{aligned}
\boldsymbol{f}^r &= (S^{-1} \otimes S^{-1})(\boldsymbol{\varepsilon}_{21} \Delta^{\mathrm{cop}}(p^L_1))\Delta(p^L_2) \ , \\
(\boldsymbol{f}^r)^{-1} &= \Delta(q^R_2)(S^{-1} \otimes S^{-1})\Big(\Delta^{\mathrm{cop}}(q^R_1)\boldsymbol{\delta}_{21}\Big) \ .
\end{aligned} \tag{2.1.48}
$$

2.2 Additional structures and properties

We have seen that modules over the quasi-Hopf algebra H form a finite tensor category. We also saw in Chapter 1 that finite tensor categories can have some desirable additional structures (or properties). In this section we briefly address the natural question: How is additional structure on the category expressed as additional structure on H?

2.2.1 Pivotal structure

The following is well-known, see e.g. [BCT, Prop. 3.2]. Note, however, that we again do not get exactly the formulas there, but rather their ‘inverses’ due to our conventions.

Proposition 2.2.1. *There is a bijective correspondence between*

(a) pivotal structures on ${}_H\mathcal{M}$

(b) invertible elements $\boldsymbol{g} \in H$ satisfying

$$
S^2(h) = \boldsymbol{g} h \boldsymbol{g}^{-1} \quad \textit{and} \quad \Delta(\boldsymbol{g}) = \boldsymbol{f}^{-1} \cdot (S \otimes S)(\boldsymbol{f}_{21}) \cdot (\boldsymbol{g} \otimes \boldsymbol{g}) \ . \tag{2.2.1}
$$

An element $\boldsymbol{g} \in H$ satisfying (2.2.1) is called a *pivot*, and it induces the pivotal structure

$$
\delta_V \colon V \to V^{\vee\vee}, \quad v \mapsto \delta^{\mathsf{Vect}}_V(\boldsymbol{g}.v) \tag{2.2.2}
$$

for all $V \in {}_H\mathcal{M}$. Here δ^{Vect} is the canonical pivotal structure of Vect. The first equation in (2.2.1) then corresponds to δ_V being an intertwiner, while the second implements monoidality.

Remark 2.2.2. If $\boldsymbol{g}_1$ and $\boldsymbol{g}_2$ are two pivots of H, then $\boldsymbol{g}_2 = z\boldsymbol{g}_1$ for a unique invertible central element z satisfying $\Delta(z) = z \otimes z$. The element z corresponds to the monoidal automorphism $\delta_1^{-1} \circ \delta_2$ of the identity functor of ${}_H\mathcal{M}$, where δ_i is the pivotal structure implemented by $\boldsymbol{g}_i$ as in (2.2.2). We will indicate our choice of pivot by saying that $(H, \boldsymbol{g})$ is a pivotal quasi-Hopf algebra. $\triangledown$

A pivot satisfies [BT2, Prop. 3.12]

$$\varepsilon(\boldsymbol{g}) = 1 \quad \text{and} \quad S(\boldsymbol{g}) = \boldsymbol{g}^{-1} \; . \tag{2.2.3}$$

The first equality follows directly from the second equation in (2.2.1), while the second property stems from the more general fact that in every pivotal category we have the identity $\delta_{V^\vee}^{-1} = (\delta_V)^\vee$, see e.g. [Sch3, Prop. A.1].

Remark 2.2.3. If $(H, \boldsymbol{g})$ is pivotal then so is $(H^{\mathrm{cop}}, \boldsymbol{g}^{\mathrm{cop}})$ with $\boldsymbol{g}^{\mathrm{cop}} = \boldsymbol{g}^{-1}$. ▽

Lastly, recall from (1.1.10) the natural monoidal isomorphism $V^\vee \cong {}^\vee V$ induced by the pivotal structure. In the setting of quasi-Hopf algebras, it is very easy to describe. Indeed, it is simply given by the bijective linear map

$$V^* \to V^*, \quad f \mapsto (f \leftharpoonup \boldsymbol{g}) \; , \tag{2.2.4}$$

where we used the hook notation from (2.1.10). This follows from the zig-zag axioms (2.1.9).

2.2.2 Quasi-triangular structure

Recall Notation 2.1.1. An *R-matrix* for a quasi-Hopf algebra H is an invertible element $R \in H \otimes H$, with inverse denoted by $\overline{R}$, satisfying the hexagon axioms

$$(\Delta \otimes \mathrm{id})(R) = \Psi_{231} R_{13} \Phi_{132} R_{23} \Psi, \quad (\mathrm{id} \otimes \Delta)(R) = \Phi_{312} R_{13} \Psi_{213} R_{12} \Phi \; , \tag{2.2.5}$$

and relating the coproduct and the coopposite coproduct as

$$R\Delta(h)\overline{R} = \Delta^{\mathrm{cop}}(h) \tag{2.2.6}$$

for all $h \in H$.

A *quasi-triangular quasi-Hopf algebra* is then a quasi-Hopf algebra together with an R-matrix. Given R, we can define linear maps

$$c_{V,W} : V \otimes W \xrightarrow{\sim} W \otimes V, \quad v \otimes w \mapsto R_2.w \otimes R_1.v \; , \tag{2.2.7}$$

and the conditions above ensure not only that these maps are morphisms in ${}_H\mathcal{M}$, but also that they satisfy the hexagon axioms of a braided finite tensor category, cf. Section 1.1.7. Conversely, if ${}_H\mathcal{M}$ is braided with braiding c, then one sets $R_{21} = c_{H,H}(\mathbf{1} \otimes \mathbf{1})$.

Proposition 2.2.4 ([BCPO, Prop. 10.1])**.** *A quasi-Hopf algebra admits an R-matrix if and only if ${}_H\mathcal{M}$ is braidable. The correspondence between these structures is as described above.*

For the finite tensor categories we are interested in, the double braiding plays an important role, see Chapter 1. In the realm of (quasi-)Hopf algebras, this is implemented by (multiplication with) the *monodromy element* or *monodromy matrix*

$$M = R_{21} R \tag{2.2.8}$$

in $H \otimes H$. For more details in the same notation as used here see e.g. [FGR1, Sec. 6].

2.2.3 Ribbon structure

Let H be a quasi-triangular quasi-Hopf algebra. Following [So], a *ribbon element* is a non-zero central element $\boldsymbol{v} \in H$ such that

$$\Delta(\boldsymbol{v}) = M^{-1} \cdot \boldsymbol{v} \otimes \boldsymbol{v} \quad \text{and} \quad S(\boldsymbol{v}) = \boldsymbol{v} \ . \tag{2.2.9}$$

It satisfies $\varepsilon(\boldsymbol{v}) = 1$, and is invertible [So, Sec. 2.3].

These equations correspond precisely to (1.1.22), the ones imposed on a ribbon twist in a braided finite tensor category. Indeed, define for each H-module X a map ϑ_X as the action with $\boldsymbol{v}^{-1}$, i.e.

$$\vartheta_X(x) = \boldsymbol{v}^{-1}.x \tag{2.2.10}$$

for all $x \in X$. Then ϑ is natural in X since $\boldsymbol{v}$ is central. Moreover, it satisfies the first (second) equation of (1.1.22) iff $\boldsymbol{v}$ satisfies the first (second) equation of (2.2.9).

Remark 2.2.5. With applications in mind, we are interested in ribbon quasi-Hopf algebras only. If one now wants to compute the modified trace, then a pivot is needed, see Theorem 2. There is a canonical one, coming from the fact that ribbon categories are pivotal, see Section 1.1.9. From the description there we see that the canonical pivot of a ribbon quasi-Hopf algebra is

$$\boldsymbol{g} = \boldsymbol{u}\boldsymbol{v}^{-1} \ . \tag{2.2.11}$$

Here, $\boldsymbol{u} \in H$ is the invertible element corresponding to the Drinfeld morphism (1.1.17)—its closed form can easily be deduced, but we will not need it; see e.g. [FGR1, Sec. 6.3] or [BN] for an expression of $\boldsymbol{u}$ in terms of the defining data of a quasi-triangular quasi-Hopf algebra, along with additional properties. ▽

2.2.4 Factorizable quasi-Hopf algebras

By [BT1], a quasi-triangular quasi-Hopf algebra H is factorizable if and only if the linear map

$$\mathfrak{M}\colon H^* \to H, \quad f \mapsto (f \otimes \mathrm{id})\left(q_1^L x_1 M_1 p_1^R \otimes q^L_{2\,(1)} x_2 M_2 p_2^R S(q^L_{2\,(2)} x_3)\right) \tag{2.2.12}$$

is bijective, where M is the monodromy matrix of H from (2.2.8). In [FGR1, Sec. 7.3] it was shown that this is equivalent to ${}_H\mathcal{M}$ being factorizable: the map $\mathfrak{M}$ is precisely the Hopf algebra map $\mathcal{L} \to \mathcal{L}^\vee$ mentioned in Example 1.2.7.

In particular, ${}_H\mathcal{M}$ is a modular tensor category for H factorizable and ribbon.

2.3 Integrals and cointegrals

For finite-dimensional Hopf algebras, the definition of integrals and cointegrals is fairly straightforward, and very symmetric—cointegrals are not more difficult than integrals, see also Remark 4.1.12. For quasi-Hopf algebras, this story is much more involved. In this section we review without much detail the construction and definition of (co)integrals for quasi-Hopf algebras, as well as their properties, from [HN2].

Quasi-Hopf bimodules

The main idea behind the construction of [HN2] is to generalize the fundamental theorem of Hopf modules, see e.g. [Mon, Thm. 1.9.4], to the quasi-coassociative setting. To do this, one approach could be to find a category in which H is a coalgebra; it turns out that this works, as we will explain now.

Let $H^{\mathrm{e}} = H \otimes H^{\mathrm{op}}$ be the enveloping algebra of H. It is naturally a quasi-Hopf algebra, and its category of modules H^{e}-mod is, as a finite abelian category, equivalent to the category ${}_H\mathcal{M}_H$ of H-bimodules. Using the equivalence, we turn ${}_H\mathcal{M}_H$ into a rigid monoidal category—in particular, the tensor product is $\otimes_k$ and not $\otimes_H$.

In ${}_H\mathcal{M}_H$, the regular bimodule H is a coalgebra. Indeed, the category is not strict, so that an object C in ${}_H\mathcal{M}_H$ is a coalgebra with coproduct Δ_C only if

$$\begin{tikzcd} C \arrow[r, "(\mathrm{id}\otimes\Delta_C)\circ\Delta_C"] \arrow[dr, "(\Delta_C\otimes\mathrm{id})\circ\Delta_C"'] & C\otimes(C\otimes C) \arrow[d, "\Phi\cdot?\cdot\Psi"] \\ & (C\otimes C)\otimes C \end{tikzcd} \tag{2.3.1}$$

commutes. This is satisfied by the regular bimodule because of quasi-coassociativity (2.1.2).[13] Counitality is clear, and we can thus consider the comonad

$$\mathcal{Y}^r \colon B \mapsto B \otimes H \tag{2.3.2}$$

on ${}_H\mathcal{M}_H$, cf. Section 1.2.4 for a brief review of comonads. We denote its category of comodules by

$${}_H\mathcal{M}_H^H := ({}_H\mathcal{M}_H)^{\mathcal{Y}^r} \tag{2.3.3}$$

and call it the category of *right quasi-Hopf bimodules*. Note also that, by construction, ${}_H\mathcal{M}_H^H$ is the category of right comodules over the coalgebra H internal to ${}_H\mathcal{M}_H$.

Similarly, one defines *left quasi-Hopf bimodules* ${}_H^H\mathcal{M}_H$ as the comodules over the comonad $\mathcal{Y}^l \colon B \to H \otimes B$ on ${}_H\mathcal{M}_H$.[14]

[13] Note that by the same arguments, H will *not* be an algebra in ${}_H\mathcal{M}_H$, and thus this in particular does not exhibit H as a Hopf algebra internal to a category in the quasi-triangular case. But, by basic abstract nonsense, both the left and the right dual of H now carry naturally an algebra structure.

[14] Note that these are right quasi-Hopf bimodules over H^{cop}. Indeed, if B is a right quasi-Hopf bimodule

The fundamental theorem of quasi-Hopf bimodules

Consider the functor

$$?_\varepsilon \colon {}_H\mathcal{M} \to {}_H\mathcal{M}_H, \quad V \mapsto V_\varepsilon \tag{2.3.4}$$

sending a left H-module to the H-bimodule with trivial right action. By tensoring the result with H on the right, and imposing on the result the coaction coming from the coproduct, we obtain a canonical functor from left modules to right quasi-Hopf bimodules.

On the other hand, there is the functor of *right coinvariants*

$$?^{\mathsf{coH}} \colon {}_H\mathcal{M}_H^H \to {}_H\mathcal{M}, \quad B \mapsto B^{\mathsf{coH}} \ , \tag{2.3.5}$$

where B^{coH} is a linear subspace of B, constructed as the image of a certain linear map, and equipped with a certain left H-action. We do not need to characterize it any more at this point.

The *fundamental theorem of right quasi-Hopf bimodules* now states:

$$\begin{tikzcd} {}_H\mathcal{M} \arrow[r, bend left, "?_\varepsilon\otimes H"] \arrow[r, phantom, "\perp"] & {}_H\mathcal{M}_H^H \arrow[l, bend left, "?^{\mathsf{coH}}"] \end{tikzcd} \tag{2.3.6}$$

is an adjoint equivalence. This is shown in detail in [HN2, Sec. 3]. Note that it is in fact a monoidal equivalence, but we did not specify the monoidal structure on ${}_H\mathcal{M}_H^H$, as we will not need it.

A similar statement about left quasi-Hopf bimodules can be made, see e.g. [SS, Sec. 4]; in this case, the adjoint equivalence is

$$\begin{tikzcd} {}_H\mathcal{M} \arrow[r, bend left, "H\otimes ?_\varepsilon"] \arrow[r, phantom, "\perp"] & {}_H^H\mathcal{M}_H \arrow[l, bend left, "{}^{\mathsf{coH}}?"] \end{tikzcd} \tag{2.3.7}$$

where ${}^{\mathsf{coH}}?$, the functor of *left coinvariants*, is defined similarly to $?^{\mathsf{coH}}$ above.

Cointegrals and their properties

The right dual ${}^\vee H$ of the regular bimodule is in a very natural way a $\mathcal{Y}^r$-comodule, i.e. an object in ${}_H\mathcal{M}_H^H$. It is trivial to check (using string diagrams in ${}_H\mathcal{M}_H$) that

$$\rho^r \colon {}^\vee H \to \mathcal{Y}^r({}^\vee H),$$

over H^{cop} with coaction $\rho : B \to B\otimes H$, then one can check that with the coaction $\rho_{21} = \tau\circ\rho \colon B \to H\otimes B$, B becomes a left quasi-Hopf bimodule over H. Here τ again denotes the tensor flip in vector spaces. Thus statements about left quasi-Hopf bimodules over H can be obtained by 'copping' statements about right quasi-Hopf bimodules over H^{cop}, and then flipping tensor factors.

See also e.g. [BCPO, Prop. 8.10] for the relation between H^{cop}-modules and H-modules.

$$\rho^r = \Big[{}^\vee H \xrightarrow{\sim} \mathbf{1}{}^\vee H \xrightarrow{\widetilde{\mathrm{coev}}_H \otimes \mathrm{id}} ({}^\vee H H){}^\vee H \xrightarrow{(\mathrm{id} \otimes \Delta)\otimes \mathrm{id}} ({}^\vee H(HH)){}^\vee H \xrightarrow{\sim}$$
$$\xrightarrow{\sim} ({}^\vee H H)(H\, {}^\vee H) \xrightarrow{\mathrm{id} \otimes \mathrm{ev}_H} ({}^\vee H H)\mathbf{1} \xrightarrow{\sim} {}^\vee H H = \mathcal{Y}^r({}^\vee H) \Big] \tag{2.3.8}$$

turns ${}^\vee H$ into a right H-comodule. Similarly, the left dual $H^\vee$ carries a natural left H-comodule structure, i.e. it is a comodule of the comonad $\mathcal{Y}^l \colon B \mapsto H \otimes B$; we shall denote the corresponding coaction by ρ^l.

Definition 2.3.1. A left cointegral is an element of $\int_H^l = ({}^\vee H)^{\mathsf{coH}}$. A right cointegral is an element of $\int_H^r = {}^{\mathsf{coH}}(H^\vee)$.

Remark 2.3.2. By footnote 14, right cointegrals for H are left cointegrals for H^{cop}. Note also that, somewhat confusingly, right cointegrals are defined via $\mathcal{Y}^l$, while the definition of left cointegrals uses $\mathcal{Y}^r$. ▽

Having defined left and right cointegrals, we want to now present some of their properties. By the explicit construction in [HN2, Sec. 3], the fundamental theorem of quasi-Hopf bimodules supplies us with an isomorphism

$$(\textstyle\int_H^l)_\varepsilon \otimes H \xrightarrow{\sim} {}^\vee H \ , \quad \lambda \otimes h \mapsto \lambda(?S(h)) \tag{2.3.9}$$

of $\mathcal{Y}^r$-comodules, cf. [HN2, Thm. 4.3]. From dimensional considerations alone it is clear that $\int_H^l$ is one-dimensional. Thus, if $\boldsymbol{\lambda}^l \in \int_H^l$ is non-zero, the associative pairing $H \times H \to k$, $(a, h) \mapsto \boldsymbol{\lambda}^l(ah)$ is non-degenerate, and hence equips H with the structure of a Frobenius algebra.[15] Moreover, there exists an algebra morphism $\boldsymbol{\gamma} \colon H \to k$ such that the H-action on $\int_H^l$ is $h.\lambda = \boldsymbol{\gamma}(h)\lambda$, for $\lambda \in \int_H^l$. We call $\boldsymbol{\gamma}$ the *modulus* of H.

Similarly, $\int_H^r$ is one-dimensional, and its H-module structure is again given by $\boldsymbol{\gamma}$ [SS, Lem. 4.5].

For later reference we collect parts of the preceding discussion (and more) in the following proposition.

Proposition 2.3.3. *Let $\boldsymbol{\gamma}$ be the modulus of H.*

1. *Left (resp. right) cointegrals exist and are unique up to scalar.*

2. *Non-zero left (resp. right) cointegrals are non-degenerate forms on H. Thus H is a Frobenius algebra.*

3. *Let $\boldsymbol{\lambda}^l$ be a non-zero left cointegral. The Nakayama automorphism of $(H, \boldsymbol{\lambda}^l)$ is given by $\theta_{\boldsymbol{\lambda}^l}(a) = S(S(a) \leftharpoonup \boldsymbol{\gamma})$, for $a \in H$. In particular for all $a, b \in H$*

$$\boldsymbol{\lambda}^l(S^{-1}(a)b) = \boldsymbol{\lambda}^l(bS(a \leftharpoonup \boldsymbol{\gamma})) \ . \tag{2.3.10}$$

[15] Recall from e.g. [HN2] that a *Frobenius algebra* is a pair (A, ω) consisting of an algebra A and a non-degenerate linear map $\omega \colon A \to k$. A Frobenius algebra (A, ω) admits a unique (up to inner automorphism) algebra automorphism θ_ω satisfying $\omega(ab) = \omega(b\theta_\omega(a))$ for all $a, b \in A$. We refer to θ_ω as the *Nakayama automorphism* of (A, ω).

4. *Let $\boldsymbol{\lambda}^r$ be a non-zero right cointegral. The Nakayama automorphism of $(H, \boldsymbol{\lambda}^r)$ is given by $\theta_{\boldsymbol{\lambda}^l}(a) = S^{-1}(\boldsymbol{\gamma} \rightharpoonup S^{-1}(a))$, for $a \in H$. In particular for all $a, b \in H$*

$$\boldsymbol{\lambda}^r(S(a)b) = \boldsymbol{\lambda}^r\Big(bS^{-1}(\boldsymbol{\gamma} \rightharpoonup a)\Big) \ . \tag{2.3.11}$$

5. *Set*

$$\mathsf{u} = (\boldsymbol{\gamma} \otimes S^2)(\mathsf{V}) \quad \textit{and} \quad \mathsf{u}^{\text{cop}} = (\boldsymbol{\gamma} \otimes S^{-2})(\mathsf{V}^{\text{cop}}) \ , \tag{2.3.12}$$

and let $\boldsymbol{\lambda}^l$ and $\boldsymbol{\lambda}^r$ be a left and a right cointegral, respectively. Then u and u^{cop} are invertible, and

$$(\boldsymbol{\lambda}^r \leftharpoonup \mathsf{u}) \circ S \ \in \textstyle\int_H^l \quad \textit{and} \quad (\boldsymbol{\lambda}^l \leftharpoonup \mathsf{u}^{\text{cop}}) \circ S^{-1} \ \in \textstyle\int_H^r \ . \tag{2.3.13}$$

Proof. Points *1.* and *2.* were discussed above, and their proofs can be found in, or along the lines of, [HN2, Sec. 4]. Point *3.* is part of [HN2, Prop. 5.1], and *4.* then follows from *3.* for H^{cop}. The last point is [BC2, Prop. 4.3] for H and H^{cop}. □

Our definition of cointegrals, Definition 2.3.1, is quite sophisticated, and we now want to recall a more explicit (and computationally more useful) characterization from e.g. [BC2, Sec. 3]. We start by writing the coactions ρ^r and ρ^l as linear maps. To abbreviate the resulting expressions, we set

$$\mathsf{U} = \boldsymbol{f}^{-1}(S \otimes S)(q^R_{21}), \quad \text{and} \quad \mathsf{V} = (S^{-1} \otimes S^{-1})\left(\boldsymbol{f}_{21}p^R_{21}\right), \tag{2.3.14}$$

following [HN2, BC2]. Explicitly, then, ρ^r sends $f \in H^*$ to

$$\rho^r(f) = \sum_i f\Big(\mathsf{V}_2(e_i)_{(2)}\mathsf{U}_2\Big) \cdot e^i \otimes \mathsf{V}_1(e_i)_{(1)}\mathsf{U}_1 \ , \tag{2.3.15}$$

where we sum over a basis, see [BC2, Prop. 3.2] for a proof. To write out the linear map ρ^l, we again invoke footnote 14. To do so, we first remark that

$$(q^R)^{\text{cop}} = q^L_{21}, \quad (p^R)^{\text{cop}} = p^L_{21}, \quad (q^L)^{\text{cop}} = q^R_{21}, \quad \text{and} \quad (p^L)^{\text{cop}} = p^R_{21} \ . \tag{2.3.16}$$

Likewise one finds $\boldsymbol{f}^{\text{cop}} = (S^{-1} \otimes S^{-1})(\boldsymbol{f})$, cf. [BC2, Sec. 3], and therefore

$$\mathsf{U}^{\text{cop}} = (S^{-1} \otimes S^{-1})(q^L\boldsymbol{f}^{-1}) \ , \quad \text{and} \quad \mathsf{V}^{\text{cop}} = (S \otimes S)\left(p^L\right)\boldsymbol{f}_{21} \ . \tag{2.3.17}$$

By footnote 14, $\rho^l = (\rho^r)^{\text{cop}}_{21}$; this may also be verified using direct calculation. In particular,

$$\rho^l(f) = \sum_i f\Big(\mathsf{V}^{\text{cop}}_2(e_i)_{(1)}\mathsf{U}^{\text{cop}}_2\Big) \cdot \mathsf{V}^{\text{cop}}_1(e_i)_{(2)}\mathsf{U}^{\text{cop}}_1 \otimes e^i \in H \otimes H^* \tag{2.3.18}$$

for $f \in H^*$.

Then, as can be shown using definition 3.5 and corollary 3.9 in [HN2] (see also [BC1, Prop. 3.4] for this statement), the above definition of cointegrals is equivalent to

Definition 2.3.4. A *left cointegral* for H is an element $\boldsymbol{\lambda}^l \in H^*$ satisfying

$$(\mathrm{id} \otimes \boldsymbol{\lambda}^l)\,(\mathsf{V}\Delta(h)\mathsf{U}) = \boldsymbol{\gamma}(X_1)\boldsymbol{\lambda}^l(hS(X_2))X_3 \tag{2.3.19}$$

for all $h \in H$, and a *right cointegral* for H is a left cointegral for H^{cop}. More explicitly, this means it is an element $\boldsymbol{\lambda}^r \in H^*$ satisfying

$$(\mathrm{id} \otimes \boldsymbol{\lambda}^r)\Big(\mathsf{V}^{\mathrm{cop}}\Delta^{\mathrm{cop}}(h)\mathsf{U}^{\mathrm{cop}}\Big) = \boldsymbol{\gamma}(x_3)\boldsymbol{\lambda}^r(hS^{-1}(x_2))x_1. \tag{2.3.20}$$

Remark 2.3.5. We derive another characterization of cointegrals for *pivotal* quasi-Hopf algebras in Lemma 3.1.1, within the theory of $\boldsymbol{\gamma}$-*symmetrized cointegrals*, which we develop in Chapter 3. $\triangledown$

Finally, we want to slightly rewrite the equations in Definition 2.3.4 to bring them into a form which will be much more useful later on. Namely, with the coaction ρ^r given in (2.3.15), we can rewrite (2.3.19) as

$$\rho^r(\boldsymbol{\lambda}^l) = \boldsymbol{\gamma}(X_1)\boldsymbol{\lambda}^l.X_2 \otimes X_3 \ , \tag{2.3.21}$$

where by the dot we mean the right H-action on the right dual ${}^{\vee}H$, i.e. $(f.a)(h) = f(hS(a))$ for $f \in {}^{\vee}H$, $a \in H$. Similarly we obtain that $\boldsymbol{\lambda}^r \in H^*$ is a right cointegral if and only if

$$\rho^l(\boldsymbol{\lambda}^r) = \boldsymbol{\gamma}(x_3)x_1 \otimes \boldsymbol{\lambda}^r.x_2 \ , \tag{2.3.22}$$

and the dot here denotes the action on the left dual of the regular bimodule.

Integrals and their properties

Let $\mathrm{Alg}(H,k)$ be the set of algebra morphisms from H to k.

Proposition 2.3.6. *With multiplication* $(\nu \cdot \phi)(h) = \nu(h_{(1)})\phi(h_{(2)})$ *and unit* ε, $\mathrm{Alg}(H,k)$ *becomes a group. The inverse of* $\nu \in \mathrm{Alg}(H,k)$ *is* $\nu^{-1} = \nu \circ S = \nu \circ S^{-1}$.

The proof is elementary. For the expression for the inverse of an element note that e.g. $\nu(\boldsymbol{\alpha}) \neq 0$ for any $\nu \in \mathrm{Alg}(H,k)$.

Definition 2.3.7. For $\nu \in \mathrm{Alg}(H,k)$ define the sets of *left* and *right* ν-*integrals* as

$$L_\nu = \{l \in H \mid hl = \nu(h)l\} \quad \text{and} \quad R_\nu = \{r \in H \mid rh = \nu(h)r\} \ . \tag{2.3.23}$$

A *left* (resp. *right*) *integral*[16] in H is a left (resp. right) ε-integral.

There are some nice immediate and well-known properties, see e.g. [HN2, Sec. 4, 5].

[16] In [BBGa] this is called a left cointegral, and what we call cointegral here is called integral there (all in the case of Hopf algebras). We follow the naming conventions in e.g. [HN2, BC1, BC2].

Proposition 2.3.8. *1. Let $\nu \in H^*$ be any algebra morphism. Then L_ν is one-dimensional, and $S(L_\nu) = R_{\nu^{-1}}$.*

2. Let $\nu \in H^*$ be as above, and let $\boldsymbol{\lambda} \in H^*$ be a left or right cointegral which is non-zero. Then

$$L_\nu = R_{\nu \circ \theta_{\boldsymbol{\lambda}}^{-1}} , \tag{2.3.24}$$

where $\theta_{\boldsymbol{\lambda}}$ is the Nakayama automorphism of the Frobenius algebra $(H, \boldsymbol{\lambda})$, cf. footnote 15.

3. Let ν, $\boldsymbol{\lambda} \in H^*$ be as above, and recall the modulus $\boldsymbol{\gamma}$ of H. Then $\varepsilon \circ \theta_{\boldsymbol{\lambda}}^{-1} = \boldsymbol{\gamma}$. In particular, if $c^l \in H$ is a left integral, then

$$c^l h = \boldsymbol{\gamma}(h) c^l \tag{2.3.25}$$

for all $h \in H$.

Proofs of these properties can be found scattered about in [HN2, Sec. 4, 5]. We include them for completeness.

Proof. 1. By Proposition 2.3.3, $(H, \boldsymbol{\lambda})$ is a Frobenius algebra, for $\boldsymbol{\lambda} \in H^*$ a non-zero (left or right) cointegral. Then the map

$$L_\nu \to k\nu, \quad l \mapsto \boldsymbol{\lambda}(?l) = \boldsymbol{\lambda}(l) \cdot \nu \tag{2.3.26}$$

is, by non-degeneracy of $\boldsymbol{\lambda}$, an isomorphism. For $l \in L_\nu$, note that $S(l)h = \nu(S^{-1}(h))S(l)$ for all $h \in H$.

2. This follows immediately from $\boldsymbol{\lambda}(lh) = \boldsymbol{\lambda}(\theta_{\boldsymbol{\lambda}}^{-1}(h)l)$, for $l \in L_\nu$, $h \in H$.

3. This is proved in [HN2, Prop. 5.1].

$\square$

Because of Proposition 2.3.8.*3* we may think of the modulus $\boldsymbol{\gamma}$ of H as capturing the difference between left and right integrals.

Remark 2.3.9. By [EGNO, Prop. 6.5.5], the distinguished invertible object D of ${}_H\mathcal{M}$ is the one-dimensional module with action given by $\boldsymbol{\gamma}^{-1} = \boldsymbol{\gamma} \circ S$, and we have $\boldsymbol{\gamma}^{-1} = \boldsymbol{\gamma}^\vee$ as H-modules.

We call H *unimodular* if $\boldsymbol{\gamma} = \varepsilon$, or equivalently, if every left integral is also right. Note that H is unimodular if and only if ${}_H\mathcal{M}$ is unimodular, see Section 1.1.3. $\triangledown$

2.4 Examples

We now turn to some examples of quasi-Hopf algebras, and expand a bit on some of their structures and properties. These will be used throughout the rest of the thesis.

2.4.1 An 8-dimensional example

This is taken from examples 2.2 and 3.4 in [BC2]. Consider the unital $\mathbb{C}$-algebra generated by g and x, obeying relations $g^2 = \mathbf{1}$, $x^4 = 0$ and $gxg^{-1} = -x$. Define two orthogonal idempotents $p_\pm = \frac{1}{2}(\mathbf{1} \pm g)$. The comultiplication and counit are given on generators by

$$\begin{aligned} \Delta(g) &= g \otimes g, & \varepsilon(g) &= 1 \\ \Delta(x) &= x \otimes (p_+ \pm ip_-) + \mathbf{1} \otimes p_+ x + g \otimes p_- x, & \varepsilon(x) &= 0 \end{aligned} \tag{2.4.1}$$

and with $\Phi = \Psi = \mathbf{1} \otimes \mathbf{1} \otimes \mathbf{1} - 2p_- \otimes p_- \otimes p_-$ we obtain two 8-dimensional quasi-bialgebras, denoted $H_\pm(8)$, both of which admit an antipode $S(g) = g$, $S(x) = -x(p_+ \pm ip_-)$, and with evaluation and coevaluation element $\boldsymbol{\alpha} = g$ and $\boldsymbol{\beta} = 1$, respectively.

A right integral is given by $c^r = x^3 p_+$, while $c^l = p_+ x^3 = x^3 p_-$ is a left integral. One computes that the modulus of $H_\pm(8)$ is $\boldsymbol{\gamma}(x) = 0$, $\boldsymbol{\gamma}(g) = -1$. Thus, the quasi-Hopf algebra in this example is not unimodular. Note also that $\boldsymbol{\gamma} = \boldsymbol{\gamma}^{-1}$.

We postpone giving the cointegrals to Example 4.4.1, where we also give the monadic cointegrals that will be introduced and studied in Chapter 4.

2.4.2 Symplectic fermions

The next example is given by the so-called symplectic fermion quasi-Hopf algebras defined in [FGR2]. One reason that these quasi-Hopf algebras are of interest is their relation to a fundamental example of logarithmic two-dimensional conformal field theories, namely the symplectic fermion conformal field theory; see [FGR2] for more details and references.

The family of symplectic fermion ribbon quasi-Hopf algebras $\mathsf{Q} = \mathsf{Q}(N, \beta)$, where N is a non-zero natural number and $\beta \in \mathbb{C}$ satisfies $\beta^4 = (-1)^N$, is defined as follows [FGR2, Sec. 3]. As a $\mathbb{C}$-algebra, Q is a unital associative algebra generated by

$$\{\, \mathsf{K}, \mathsf{f}_i^\epsilon \mid 1 \le i \le N,\ \epsilon = \pm \,\} \ . \tag{2.4.2}$$

With the elements

$$\boldsymbol{e}_0 = \tfrac{1}{2}(\mathbf{1} + \mathsf{K}^2) \ , \qquad \boldsymbol{e}_1 = \tfrac{1}{2}(\mathbf{1} - \mathsf{K}^2) \tag{2.4.3}$$

we can write the defining relations for Q as

$$\{\mathsf{f}_i^\pm, \mathsf{K}\} = 0 \ , \quad \{\mathsf{f}_i^+, \mathsf{f}_j^-\} = \delta_{i,j}\boldsymbol{e}_1 \ , \quad \{\mathsf{f}_i^\pm, \mathsf{f}_j^\pm\} = 0 \ , \quad \mathsf{K}^4 = \mathbf{1} \ , \tag{2.4.4}$$

where $\{-,-\}$ is the anticommutator. Then $\boldsymbol{e}_0, \boldsymbol{e}_1$ are central orthogonal idempotents with $\boldsymbol{e}_0 + \boldsymbol{e}_1 = \mathbf{1}$. The dimension of Q is 2^{2N+2}.

It is enough to specify the quasi-Hopf algebra structure on generators. The coproduct is

$$\Delta(\mathsf{K}) = \mathsf{K} \otimes \mathsf{K} - (1 + (-1)^N)\ \boldsymbol{e}_1\mathsf{K} \otimes \boldsymbol{e}_1\mathsf{K} \ ,$$

$$\Delta(\mathsf{f}_i^{\pm}) = \mathsf{f}_i^{\pm} \otimes \mathbf{1} + \omega_{\pm} \otimes \mathsf{f}_i^{\pm} \ , \tag{2.4.5}$$

where $\omega_{\pm} = (\boldsymbol{e}_0 \pm i\boldsymbol{e}_1)\mathsf{K}$. The counit is

$$\varepsilon(\mathsf{K}) = 1 \ , \qquad \varepsilon(\mathsf{f}_i^{\pm}) = 0 \ . \tag{2.4.6}$$

We introduce

$$\boldsymbol{\beta}_{\pm} = \boldsymbol{e}_0 + \beta^2(\pm i\mathsf{K})^N \boldsymbol{e}_1 \tag{2.4.7}$$

to define the coassociator and its inverse as

$$\Phi^{\pm 1} = \mathbf{1} \otimes \mathbf{1} \otimes \mathbf{1} + \boldsymbol{e}_1 \otimes \boldsymbol{e}_1 \otimes \Big\{ \boldsymbol{e}_0(\mathsf{K}^N - \mathbf{1}) + \boldsymbol{e}_1(\boldsymbol{\beta}_{\pm} - \mathbf{1}) \Big\} \ . \tag{2.4.8}$$

Finally, the antipode S and the evaluation and coevaluation elements $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are given by

$$\begin{aligned} S(\mathsf{K}) &= \mathsf{K}^{(-1)^N} = (\boldsymbol{e}_0 + (-1)^N \boldsymbol{e}_1)\mathsf{K} \ , & \boldsymbol{\alpha} &= \mathbf{1} \ , \\ S(\mathsf{f}_i^{\pm}) &= \mathsf{f}_i^{\pm}(\boldsymbol{e}_0 \pm (-1)^N i\boldsymbol{e}_1)\mathsf{K} \ , & \boldsymbol{\beta} &= \boldsymbol{\beta}_+ \ . \end{aligned} \tag{2.4.9}$$

For convenience we also state the inverse antipode on generators:

$$S^{-1}(\mathsf{K}) = \mathsf{K}^{(-1)^N} \ , \qquad S^{-1}(\mathsf{f}_i^{\pm}) = \omega_{\pm}\mathsf{f}_i^{\pm} \ . \tag{2.4.10}$$

Note that $S(\boldsymbol{\beta}_{\pm}) = S^{-1}(\boldsymbol{\beta}_{\pm}) = \boldsymbol{\beta}_{\mp}$, and $\boldsymbol{\beta}_+\boldsymbol{\beta}_- = \mathbf{1}$.

From [FGR2, Eq. (3.35)] we know that the Drinfeld twist and its inverse are given by

$$\boldsymbol{f}^{\pm 1} = \boldsymbol{e}_0 \otimes \mathbf{1} + \boldsymbol{e}_1 \otimes \boldsymbol{e}_0\mathsf{K}^N + \boldsymbol{e}_1\boldsymbol{\beta}_{\mp} \otimes \boldsymbol{e}_1 \ . \tag{2.4.11}$$

A pivot of Q is[17]

$$\boldsymbol{g} = (\boldsymbol{e}_0 + (-i)^{N+1}\boldsymbol{e}_1\mathsf{K}^N)\mathsf{K} \ . \tag{2.4.12}$$

One furthermore computes

$$\begin{aligned} q^R &= \mathbf{1} \otimes \mathbf{1} + \boldsymbol{e}_1 \otimes \Big(\boldsymbol{e}_1(\boldsymbol{\beta} - \mathbf{1}) \Big) \ , \\ p^R &= \mathbf{1} \otimes \mathbf{1} + \boldsymbol{e}_0 \otimes \Big(\boldsymbol{e}_1(\boldsymbol{\beta} - \mathbf{1}) \Big) \ , \\ q^L &= \mathbf{1} \otimes \mathbf{1} + \boldsymbol{e}_1 \otimes \Big\{ \boldsymbol{e}_0(\mathsf{K}^N - \mathbf{1}) + \boldsymbol{e}_1(\boldsymbol{\beta} - \mathbf{1}) \Big\} \ , \\ p^L &= \boldsymbol{\beta}_- \otimes \mathbf{1} + \boldsymbol{e}_1\boldsymbol{\beta}_- \otimes \Big\{ \boldsymbol{e}_0(\mathsf{K}^N - \mathbf{1}) + \boldsymbol{e}_1(\boldsymbol{\beta}_- - \mathbf{1}) \Big\} \ . \end{aligned} \tag{2.4.13}$$

[17] The symbol $\boldsymbol{g}$ has a slightly different meaning in [FGR2], and so the expression for $\boldsymbol{g}$ stated there differs from the one given here.

For later use, we fix a basis of Q. The basis elements are

$$B_{\vec{a},\vec{b},i} := \left(\prod_{j=1}^{|\vec{a}|} \mathsf{f}^{+}_{a_j}\right)\left(\prod_{k=1}^{|\vec{b}|} \mathsf{f}^{-}_{b_k}\right)\mathsf{K}^i \ , \tag{2.4.14}$$

where $i \in \mathbb{Z}_4$, and $\vec{a}, \vec{b}$ are strictly ordered multi-indices of lengths $0 \le |\vec{a}|, |\vec{b}| \le N$. By "strictly ordered" we mean that for $\vec{a} = (a_1, a_2, \dots, a_{|\vec{a}|})$ we have $1 \le a_1 < \dots < a_{|\vec{a}|} \le N$, and similarly for $\vec{b}$. For the (non-commutative) product we use the convention

$$\prod_{i=1}^{n} a_i = a_1 \cdot \prod_{i=2}^{n} a_i \ . \tag{2.4.15}$$

The element corresponding to $B_{\vec{a},\vec{b},i}$ in the dual basis is denoted by

$$\left(B_{\vec{a},\vec{b},i}\right)^*. \tag{2.4.16}$$

We will use the shorthand

$$\vec{N} = (1, 2, \dots, N) \ . \tag{2.4.17}$$

Using this notation we can state that

$$\boldsymbol{\Lambda} = \sum_{j=0}^{3} B_{\vec{N},\vec{N},j} \tag{2.4.18}$$

is both a left and a right integral in Q [FGR2, Sec. 3.5]. In particular, Q is unimodular.

The quasi-Hopf algebra Q can be equipped with an R-matrix and a ribbon element, turning it into a ribbon quasi-Hopf algebra. In [FGR2, Prop. 3.2] it was shown that it is in fact a factorizable ribbon quasi-Hopf algebra. Factorisability implies unimodularity[18], giving another argument showing that Q is unimodular. A ribbon category is in particular pivotal. The pivot in (2.4.12) was obtained as $\boldsymbol{g} = \boldsymbol{v}^{-1}\boldsymbol{u}$, where $\boldsymbol{v}$ is the ribbon element and $\boldsymbol{u}$ is the Drinfeld element, cf. Remark 2.2.5.

Cointegrals for these quasi-Hopf algebras will be given in Section 3.3, after we have derived another (simpler) way of characterizing them in Lemma 3.1.1.

2.4.3 Restricted quantum group at $2p$th roots of unity

Lastly we consider the quasi-Hopf algebra modification $\overline{U}_q^{(\Phi)}\mathfrak{sl}(2)$ of the restricted quantum group $\overline{U}_q\mathfrak{sl}(2)$ of $\mathfrak{sl}(2)$ at roots of unity of even order, as introduced in [CGR]. This quasi-Hopf algebra is interesting since it turns out to be factorizable (by construction, in fact),

[18] See [BT1, Sec. 6] for the quasi-Hopf algebra case, or note that 'factorizability' there is equivalent to factorizability of the finite tensor category by [FGR1, Sec. 7.4], so that the statement follows from the more general [KL, Lem. 5.2.8].

while it is well-known that $\overline{U}_q\mathfrak{sl}(2)$ is not even braidable [KS]. Moreover, it is conjectured to be related to a certain vertex operator algebra, as remarked in the Introduction and also Chapter 5.

Fix an odd integer t, let $p \geq 2$ be an integer, and set $q = e^{i\pi/p}$. We use the notation

$$\{n\} = q^n - q^{-n}, \quad [n] = \frac{\{n\}}{\{1\}}, \quad [n]! = \prod_{i=1}^{n}[i] \tag{2.4.19}$$

for q-numbers and q-factorials, where $n \in \mathbb{N}$.

Define $\overline{U}_q^{(\Phi)}\mathfrak{sl}(2)$ to be the algebra over $\mathbb{C}$ generated by elements $E, F, K^{\pm 1}$, subject to the relations

$$KEK^{-1} = q^2 E, \qquad KFK^{-1} = q^{-2}F, \qquad [E,F] = \frac{K - K^{-1}}{q - q^{-1}}, \tag{2.4.20}$$

and

$$E^p = 0 = F^p, \qquad K^{2p} = \mathbf{1}. \tag{2.4.21}$$

A standard PBW-type basis is given by

$$\{E^i F^j K^k \mid 0 \leq i,j \leq p-1,\, 0 \leq k \leq 2p-1\}, \tag{2.4.22}$$

and so we see $\dim \overline{U}_q^{(\Phi)}\mathfrak{sl}(2) = 2p^3$.

With the two canonical orthogonal central idempotents

$$\boldsymbol{e}_0 = \tfrac{1}{2}(\mathbf{1} + K^p), \qquad \boldsymbol{e}_1 = \tfrac{1}{2}(\mathbf{1} - K^p)\ , \tag{2.4.23}$$

we define a coproduct and a counit by

$$\begin{aligned} \Delta_t(E) &= E \otimes K + (\boldsymbol{e}_0 + q^t \boldsymbol{e}_1)\mathbf{1} \otimes E, & \varepsilon(E) &= 0, \\ \Delta_t(F) &= F \otimes \mathbf{1} + (\boldsymbol{e}_0 + q^{-t}\boldsymbol{e}_1)K^{-1} \otimes F, & \varepsilon(F) &= 0, \\ \Delta_t(K) &= K \otimes K, & \varepsilon(K) &= 1. \end{aligned} \tag{2.4.24}$$

This coproduct is quasi-coassociative with coassociator

$$\Phi_t^{\pm 1} = \mathbf{1} \otimes \mathbf{1} \otimes \mathbf{1} + \boldsymbol{e}_1 \otimes \boldsymbol{e}_1 \otimes (K^{\mp t} - \mathbf{1})\ , \tag{2.4.25}$$

and an antipode is given by

$$\begin{aligned} S_t(E) &= -EK^{-1}(\boldsymbol{e}_0 + q^t \boldsymbol{e}_1), \\ S_t(F) &= -KF(\boldsymbol{e}_0 + q^{-t}\boldsymbol{e}_1), \\ S_t(K) &= K^{-1}, \end{aligned} \tag{2.4.26}$$

with evaluation and coevaluation elements

$$\boldsymbol{\alpha} = \mathbf{1}, \quad \boldsymbol{\beta}_t = \boldsymbol{e}_0 + K^{-t}\boldsymbol{e}_1 \tag{2.4.27}$$

respectively. By [CGR, Thm. 4.1], this data leads to a quasi-Hopf algebra, for each value of t.[19] In fact, the theorem *loc. cit.* also proves that, with

$$R_t = \frac{1}{4p}\sum_{n=0}^{p-1}\sum_{s,r=0}^{2p-1}\frac{\{1\}^n}{[n]!}q^{\frac{1}{2}n(n-1)-2sr}\left(1+q^{tr}+q^{-t(n+s)}+q^{t(\frac{1}{2}t+r-n-s)}\right) \times K^s E^n \otimes K^r F^n \tag{2.4.28}$$

and

$$\boldsymbol{v} = \frac{1-i}{2\sqrt{p}}\sum_{n=0}^{p-1}\sum_{j=0}^{2p-1}\frac{\{1\}^n}{[n]!}q^{n(j-\frac{1}{2})+\frac{1}{2}(j+p+1)^2}F^n E^n K^j \ , \tag{2.4.29}$$

$\overline{U}_q^{(\Phi)}\mathfrak{sl}(2)$ is a factorizable ribbon quasi-Hopf algebra. Thus it is unimodular.

In order to describe the pivotal structure of this quasi-Hopf algebra, we first compute its Drinfeld element.

Lemma 2.4.1. *The Drinfeld element $\boldsymbol{u}_t$ of $\overline{U}_q^{(\Phi)}\mathfrak{sl}(2)$ is given by*

$$\boldsymbol{u}_t = \frac{1-i}{2\sqrt{p}}\sum_{n,r=0}^{p-1}\frac{\{-1\}^n}{[n]!}(1+(-1)^{n-r}i^p)q^{n(r-\frac{3}{2})+\frac{1}{2}r^2} \times\left(\boldsymbol{e}_0 + q^{t(\frac{1}{2}t-r-n)}\boldsymbol{e}_1\right)F^n E^n K^r \tag{2.4.30}$$

Proposition 2.4.2. *The element*

$$\boldsymbol{g}_t = (\boldsymbol{e}_0 - \boldsymbol{e}_1 K^t)K \ . \tag{2.4.31}$$

is a pivot for $\overline{U}_q^{(\Phi)}\mathfrak{sl}(2)$. Moreover, it is the canonical pivot compatible with the ribbon structure of $\overline{U}_q^{(\Phi)}\mathfrak{sl}(2)$, meaning that it satisfies $\boldsymbol{g}_t = \boldsymbol{u}_t\boldsymbol{v}^{-1}$.

The proofs of these statements are not very illuminating and therefore given in Appendices C.2 and C.3, respectively.

As already mentioned in the introduction, in Section 5.3 we will use techniques developed in this thesis to show that two specific actions of $SL(2,\mathbb{Z})$ on the center of $\overline{U}_q^{(\Phi)}\mathfrak{sl}(2)$ agree. One of them is Lyubashenko's action defined for any modular tensor category.

Finally, we note that for $p = 2$ and $t = 1$, the Hopf algebra $\overline{U}_q^{\Phi}(\mathfrak{sl}_2)$ is isomorphic to $\mathsf{Q}(N = 1, \beta = i)$ via

$$E \mapsto \mathsf{f}_1^-\mathsf{K}, \quad F \mapsto i\mathsf{f}_1^+, \quad K \mapsto \mathsf{K} \ , \tag{2.4.32}$$

see [CGR, Rem. 4.3].

[19] Recall the notion of *twist-equivalence* from Section 2.1.2. Different values of t lead to twist-equivalent quasi-bialgebras, cf. [CGR, Thm. 4.1]. Explicitly, with $J(t',t) = \boldsymbol{e}_0 \otimes \mathbf{1} + \boldsymbol{e}_1 \otimes K^{(t'-t)/2}$ we get $J(t',t)\cdot\Delta_t(h)\cdot J(t',t)^{-1} = \Delta_{t'}(h)$ for all $h \in \overline{U}_q^{(\Phi)}\mathfrak{sl}(2)$. It is easy to see that $J(t',t)$ is normalized, so it is indeed a twist. The quasi-Hopf structure described here can then be obtained from the twisted one via Remark 2.1.4, with $u = \boldsymbol{e}_0 + \boldsymbol{e}_1 K^{(t'-t)/2}$.

Chapter 3

Modified traces for quasi-Hopf algebras

In this chapter, we extend the results of [BBGa] on modified traces for Hopf algebras to the realm of quasi-Hopf algebras.

Throughout this chapter, whenever $(H, \boldsymbol{g})$ is a pivotal quasi-Hopf algebra (see Section 2.2.1), we will—in contrast to the standard way explained in Section 2.1.3—assume that right duals are given via the pivotal structure. That is, we write both the left and the right dual of an object $V \in {}_H\mathcal{M}$ as $V^\vee$, so that the evaluations and coevaluations

$$\begin{aligned} \mathrm{ev}_V : V^\vee \otimes V \to \mathbf{1} \quad &, \qquad \mathrm{coev}_V : \mathbf{1} \to V \otimes V^\vee \\ \widetilde{\mathrm{ev}}_V : V \otimes V^\vee \to \mathbf{1} \quad &, \qquad \widetilde{\mathrm{coev}}_V : \mathbf{1} \to V^\vee \otimes V \ , \end{aligned} \tag{3.0.1}$$

are related by

$$\widetilde{\mathrm{ev}}_V = \mathrm{ev}_{V^\vee} \circ (\delta_V \otimes \mathrm{id}_{V^\vee}) \ , \qquad \widetilde{\mathrm{coev}}_V = \left(\mathrm{id}_{V^\vee} \otimes \delta_V^{-1}\right) \circ \mathrm{coev}_{V^\vee} \ . \tag{3.0.2}$$

Explicitly, for example,

$$\begin{aligned} \widetilde{\mathrm{ev}}_V(v \otimes f) &= \left\langle \boldsymbol{g}v \, , \, f \leftharpoonup S(\boldsymbol{\alpha}) \right\rangle \\ &= \left\langle S(\boldsymbol{\alpha})\boldsymbol{g}v \, , \, f \right\rangle = \left\langle \boldsymbol{g}S^{-1}(\boldsymbol{\alpha})v \, , \, f \right\rangle , \end{aligned} \tag{3.0.3}$$

where in the last step we used $S(h)\boldsymbol{g} = \boldsymbol{g}S^{-1}(h)$ for $h \in H$.

In the first section, we introduce a new version of cointegrals for pivotal quasi-Hopf algebras, called $\boldsymbol{\gamma}$-*symmetrized cointegrals*. Specializing to the unimodular case, i.e. $\boldsymbol{\gamma} = \varepsilon$, we obtain the natural quasi-Hopf generalization of the symmetrized cointegrals of [BBGa].

In the second section, we generalize some necessary results from [BBGa], which then enables us to prove the first main theorems, Theorems 1 and 2 from the introduction, relating the modified trace on the projective ideal to the symmetrized cointegrals. Note that in this chapter the projective ideal may be denoted by $\mathrm{Proj}({}_H\mathcal{M})$ or H-pmod.

Finally, we give an example in the third section. We compute the modified trace for the symplectic fermion quasi-Hopf algebras, which we reviewed in Section 2.4.2.

3.1 $\boldsymbol{\gamma}$-symmetrized cointegrals

We now introduce $\boldsymbol{\gamma}$-symmetrized cointegrals. But first, we give the following equivalent characterization of cointegrals, using Proposition 2.3.3.[20]

Lemma 3.1.1. *Suppose that $(H, \boldsymbol{g})$ is pivotal, Let $\boldsymbol{\lambda} \in H^*$ and set*

$$\widehat{\boldsymbol{\lambda}}^l = \boldsymbol{\lambda} \leftharpoonup \mathsf{u}^{\mathrm{cop}} \boldsymbol{g}^{-1} \quad \textit{and} \quad \widehat{\boldsymbol{\lambda}}^r = \boldsymbol{\lambda} \leftharpoonup \mathsf{u}\boldsymbol{g} \ . \tag{3.1.1}$$

Then

1. *$\boldsymbol{\lambda}$ is a left cointegral if and only if*
$$(\mathrm{id} \otimes \widehat{\boldsymbol{\lambda}}^l) \left(q^L \Delta(h) p^L\right) = \boldsymbol{\gamma}(x_3) \widehat{\boldsymbol{\lambda}}^l (x_2 h) \cdot \boldsymbol{g} S^{-1}(x_1) \tag{3.1.2}$$
2. *$\boldsymbol{\lambda}$ is a right cointegral if and only if*
$$(\widehat{\boldsymbol{\lambda}}^r \otimes \mathrm{id}) \left(q^R \Delta(h) p^R\right) = \boldsymbol{\gamma}(X_1) \widehat{\boldsymbol{\lambda}}^r (X_2 h) \cdot \boldsymbol{g}^{-1} S(X_3) \tag{3.1.3}$$

Proof. We will prove the second part, the first statement is completely analogous. Let $\boldsymbol{\lambda}^r \in H^*$ be a right cointegral. By Proposition 2.3.3 (5), we have

$$\widehat{\boldsymbol{\lambda}}^r = \boldsymbol{\lambda}^r \leftharpoonup \mathsf{u}\boldsymbol{g} = (\boldsymbol{\lambda}^l \circ S^{-1}) \leftharpoonup \boldsymbol{g} = (\boldsymbol{g}^{-1} \rightharpoonup \boldsymbol{\lambda}^l) \circ S^{-1}. \tag{3.1.4}$$

for a left cointegral $\boldsymbol{\lambda}^l$. Using this equality and evaluating the left cointegral equation (2.3.19) on $S^{-1}(h)\boldsymbol{g}^{-1}$ for $h \in H$ gives

$$(\mathrm{id} \otimes \boldsymbol{\lambda}^l)(\mathsf{V}\Delta(S^{-1}(h)\boldsymbol{g}^{-1})\mathsf{U}) = \boldsymbol{\gamma}(X_1)\widehat{\boldsymbol{\lambda}}^r(X_2 h) X_3. \tag{3.1.5}$$

We have

$$\begin{aligned} \Delta(\boldsymbol{g}^{-1})\mathsf{U} &= (\boldsymbol{g}^{-1} \otimes \boldsymbol{g}^{-1})(S \otimes S)(q^R_{21} \boldsymbol{f}^{-1}_{21}) \\ &= (S^{-1} \otimes S^{-1})(q^R_{21} \boldsymbol{f}^{-1}_{21})(\boldsymbol{g}^{-1} \otimes \boldsymbol{g}^{-1}). \end{aligned} \tag{3.1.6}$$

Using $\mathsf{V} = (S^{-1} \otimes S^{-1})(\boldsymbol{f}_{21} p^R_{21})$ and

$$\Delta(S^{-1}(h)) = (S^{-1} \otimes S^{-1})(\boldsymbol{f}_{21} \Delta^{\mathrm{cop}}(h) \boldsymbol{f}^{-1}_{21}) \tag{3.1.7}$$

we immediately simplify (3.1.5) to

$$\begin{aligned} \boldsymbol{\gamma}(X_1)\widehat{\boldsymbol{\lambda}}^r(X_2 h) X_3 &= (\mathrm{id} \otimes \boldsymbol{\lambda}^l) \left[(S^{-1} \otimes S^{-1}) \left(q^R_{21} \Delta^{\mathrm{cop}}(h) p^R_{21} \right) \cdot (\boldsymbol{g}^{-1} \otimes \boldsymbol{g}^{-1}) \right] \\ &= ((r_{\boldsymbol{g}^{-1}} \circ S^{-1}) \otimes \widehat{\boldsymbol{\lambda}}^r) \left(q^R_{21} \Delta^{\mathrm{cop}}(h) p^R_{21} \right) \\ &= ((S^{-1} \circ l_{\boldsymbol{g}^{-1}}) \otimes \widehat{\boldsymbol{\lambda}}^r) \left(q^R_{21} \Delta^{\mathrm{cop}}(h) p^R_{21} \right) \end{aligned} \tag{3.1.8}$$

Then, applying S on both sides and multiplying with $\boldsymbol{g}^{-1}$ on the left gives

$$(\widehat{\boldsymbol{\lambda}}^r \otimes \mathrm{id}) \left(q^R \Delta(h) p^R\right) = \boldsymbol{\gamma}(X_1) \widehat{\boldsymbol{\lambda}}^r (X_2 h) \cdot \boldsymbol{g}^{-1} S(X_3), \tag{3.1.9}$$

as desired. $\square$

[20] The shifted left cointegral $\widehat{\boldsymbol{\lambda}}^l$ also appears in [SS, Sec. 6.4] in relation to *$\boldsymbol{\gamma}$-twisted module traces.*

We give these 'shifted cointegrals' a special name.

Definition 3.1.2. Let $(H,\boldsymbol{g})$ be a finite-dimensional pivotal quasi-Hopf algebra. A linear form $\widehat{\boldsymbol{\lambda}}^l \in H^*$ is a *left $\boldsymbol{\gamma}$-symmetrized cointegral* if, and only if,

$$(\mathrm{id}\otimes\widehat{\boldsymbol{\lambda}}^l)(q^L\Delta(h)p^L) = \boldsymbol{\gamma}(x_3)\widehat{\boldsymbol{\lambda}}^l(x_2h)\cdot \boldsymbol{g}S^{-1}(x_1) \tag{3.1.10}$$

for all $h \in H$.

A linear form $\widehat{\boldsymbol{\lambda}}^r \in H^*$ is a *right $\boldsymbol{\gamma}$-symmetrized cointegral* if, and only if,

$$(\widehat{\boldsymbol{\lambda}}^r \otimes \mathrm{id})(q^R\Delta(h)p^R) = \boldsymbol{\gamma}(X_1)\widehat{\boldsymbol{\lambda}}^r(X_2h)\cdot \boldsymbol{g}^{-1}S(X_3) \tag{3.1.11}$$

for all $h \in H$.

We denote the spaces of left and right $\boldsymbol{\gamma}$-symmetrized cointegrals by

$$\textstyle\int_H^{r,\gamma} \quad \text{and} \quad \int_H^{l,\gamma}. \tag{3.1.12}$$

The adjective '$\boldsymbol{\gamma}$-symmetrized' is justified by the following proposition.

Proposition 3.1.3. *Let $\widehat{\boldsymbol{\lambda}}^l, \widehat{\boldsymbol{\lambda}}^r \in H^*$ be a left and a right $\boldsymbol{\gamma}$-symmetrized cointegral.*

1. *$\widehat{\boldsymbol{\lambda}}^l$ and $\widehat{\boldsymbol{\lambda}}^r$ are unique up to scalar. That is, $\int_H^{l,\gamma}$ and $\int_H^{r,\gamma}$ are one-dimensional. Furthermore, non-zero $\boldsymbol{\gamma}$-symmetrized cointegrals are non-degenerate.*

2. *$\widehat{\boldsymbol{\lambda}}^l$ and $\widehat{\boldsymbol{\lambda}}^r$ are 'symmetric up to $\boldsymbol{\gamma}$' in the sense that*

$$\widehat{\boldsymbol{\lambda}}^l(ab) = \widehat{\boldsymbol{\lambda}}^l((\boldsymbol{\gamma} \rightharpoonup b)a) \quad \textit{and} \quad \widehat{\boldsymbol{\lambda}}^r(ab) = \widehat{\boldsymbol{\lambda}}^r((b \leftharpoonup \boldsymbol{\gamma})a) \tag{3.1.13}$$

   for all $a, b \in H$.

Proof. For *1.* observe e.g. that by Lemma 3.1.1 we have an isomorphism between left cointegrals and left $\boldsymbol{\gamma}$-symmetrized cointegrals.

To see the first identity of point *2.*, use Lemma 3.1.1 and Proposition 2.3.3 (5) to obtain a right cointegral $\boldsymbol{\lambda}^r$ such that $\widehat{\boldsymbol{\lambda}}^l = (\boldsymbol{\lambda}^r \circ S) \leftharpoonup \boldsymbol{g}^{-1}$. Then, with Proposition 2.3.3 (4), we have

$$\begin{aligned}\widehat{\boldsymbol{\lambda}}^l(ab) &= \boldsymbol{\lambda}^r(S(ab)\boldsymbol{g}) = \boldsymbol{\lambda}^r(S(a)\boldsymbol{g}S^{-1}(\boldsymbol{\gamma} \rightharpoonup b)) \\ &= \boldsymbol{\lambda}^r(S(a)S(\boldsymbol{\gamma} \rightharpoonup b)\boldsymbol{g}) = \widehat{\boldsymbol{\lambda}}^l((\boldsymbol{\gamma} \rightharpoonup b)a).\end{aligned} \tag{3.1.14}$$

The second identity can be seen using points (3) and (5) of Proposition 2.3.3. □

Remark 3.1.4. Let $(H,\boldsymbol{g})$ be a pivotal quasi-Hopf algebra, and let $\widehat{\boldsymbol{\lambda}}^l$ be a left $\boldsymbol{\gamma}$-symmetrized cointegral. This linear form has the following interpretation. The category $\mathcal{B} = {}_H\mathcal{M}_H$ of H-bimodules from Section 2.3 is pivotal. In particular, then, the left and the right dual of the regular bimodule are isomorphic, and one easily sees that the isomorphism is given by

$$^{\vee}H \xrightarrow{\sim} H^{\vee}, \quad f \mapsto \boldsymbol{g}^{-1} \rightharpoonup f \leftharpoonup \boldsymbol{g}^{-1} \ , \tag{3.1.15}$$

see (1.1.10). As reviewed in Section 2.3, $H \in \mathcal{B}$ is a coalgebra, and we can thus consider its category of left comodules ${}^{H}\mathcal{B}$ in $\mathcal{B}$. By a fundamental theorem of [HN2], there is an equivalence

$$^{H}\mathcal{B} \cong {}_{H}\mathcal{M} \ , \tag{3.1.16}$$

sending $M \in {}^{H}\mathcal{B}$ to its coinvariants ${}^{\mathsf{coH}}M$. By Definition 2.3.1, a right cointegral is precisely an element of ${}^{\mathsf{coH}}(H^{\vee})$, where $H^{\vee}$ carries the canonical left H-comodule structure, see e.g. [BC2, Prop. 3.1]. Furthermore, ${}^{\vee}H$ becomes a left H-comodule via the isomorphism (3.1.15), we denote this comodule by $\overline{{}^{\vee}H}$. By (3.1.16) we get

$${}^{\mathsf{coH}}(\overline{{}^{\vee}H}) \cong {}^{\mathsf{coH}}(H^{\vee}) \ . \tag{3.1.17}$$

Using the explicit form of the isomorphism (3.1.15), (3.1.17) is equivalent to

$${}^{\mathsf{coH}}(\overline{{}^{\vee}H}) \ni \varphi \iff (\boldsymbol{g}^{-1} \rightharpoonup \varphi \leftharpoonup \boldsymbol{g}^{-1}) \text{ is a right cointegral.} \tag{3.1.18}$$

Inserting the element on the right hand side into the right cointegral equation (2.3.20), and using the isomorphism (2.3.13) between left and right cointegrals, one arrives precisely at the statement that $\widehat{\boldsymbol{\lambda}}^{l}$ satisfies equation (3.1.2) from Lemma 3.1.1. A similar interpretation exists for $\widehat{\boldsymbol{\lambda}}^{r}$. $\triangledown$

Symmetrized cointegrals

For the rest of this section we assume:

$$(H, \boldsymbol{g}) \text{ is pivotal and unimodular .}$$

Note that in this case the elements u and $\mathsf{u}^{\mathrm{cop}}$ from Proposition 2.3.3 are both equal to $\mathbf{1}$. The following immediate corollary to Lemma 3.1.1 will be useful when comparing to modified traces.

Corollary 3.1.5. *Let $\boldsymbol{\lambda} \in H^{*}$. Then*

1. *$\boldsymbol{\lambda}$ is a left cointegral if and only if*

$$\widehat{\boldsymbol{\lambda}}^{l}(h)\ \mathbf{1} = \left(\boldsymbol{g}^{-1} \otimes \widehat{\boldsymbol{\lambda}}^{l}\right)(q^{L}\Delta(h)p^{L}) \tag{3.1.19}$$

for all $h \in H$.

2. $\boldsymbol{\lambda}$ is a right cointegral if and only if

$$\widehat{\boldsymbol{\lambda}}^r(h)\ \mathbf{1} = \left(\widehat{\boldsymbol{\lambda}}^r \otimes \boldsymbol{g}\right)(q^R\Delta(h)p^R) \tag{3.1.20}$$

for all $h \in H$.

Following [BBGa, Sec. 4], we call $\widehat{\boldsymbol{\lambda}}^l$ and $\widehat{\boldsymbol{\lambda}}^r$ the *symmetrized left* and *right cointegral*, respectively. The adjective "symmetrized" is justified by the following corollary, which follows from Proposition 2.3.3 (2), equation (3.1.13), and the fact that here we assume H to be unimodular.

Corollary 3.1.6 ([BBGa, Prop. 4.4])**.** *The non-zero symmetrized left (resp. right) cointegrals are non-degenerate symmetric linear forms on H.*

Remark 3.1.7. Let H be a finite-dimensional Hopf algebra. Following [Ra1], for any grouplike element $g \in H$, one can define the left ideal $L_g \subseteq H^*$ of *left g-cointegrals* (called left g-integrals in [Ra1]) as

$$L_g = \{\varphi \in H^* \mid (\mathrm{id} \otimes \varphi)(\Delta(h)) = \varphi(h)g \quad \forall h \in H\}\ . \tag{3.1.21}$$

These ideals are all one-dimensional [Ra1, Prop. 3]. Indeed, note that L_1 is the space of left cointegrals. Then the linear isomorphism

$$L_g \ni \varphi \mapsto (\varphi \leftharpoonup h) \in L_{h^{-1}g} \quad \text{for all grouplike } g, h, \tag{3.1.22}$$

shows $L_g \cong L_1$. Similarly one may define the space R_g of *right g-cointegrals*. Thus, if $(H, \boldsymbol{g})$ is a pivotal Hopf algebra, Lemma 3.1.1 reduces to the statement

$$(\boldsymbol{\lambda}^l \leftharpoonup \boldsymbol{g}^{-1}) \in L_{\boldsymbol{g}} \quad \text{and} \quad (\boldsymbol{\lambda}^r \leftharpoonup \boldsymbol{g}) \in R_{\boldsymbol{g}^{-1}}\ . \tag{3.1.23}$$

For unimodular H, a symmetrized left cointegral is therefore a left $\boldsymbol{g}$-cointegral, and a symmetrized right cointegral is a right $\boldsymbol{g}^{-1}$-cointegral. $\triangledown$

3.2 Modified traces for quasi-Hopf algebras

Throughout this section H will be a finite-dimensional quasi-Hopf algebra over k.

Tensoring with the regular representation

Let $V \in H$-mod. We denote by ${}_{\varepsilon}V$ the vector space V with trivial H-module structure, i.e. $hv = \varepsilon(h)v$ for $h \in H, v \in V$. Recall the definition of the elements p^R, etc., from (2.1.33). We need the following generalization of [BBGa, Thm. 5.1] to quasi-Hopf algebras (see also [Sch2, Sec. 2.3] for this statement).

Proposition 3.2.1.

1. *The map*

$$\phi^r : H \otimes_\varepsilon V \to H \otimes V,$$
$$h \otimes v \mapsto \Big(\Delta(h)p^R\Big) \cdot (1 \otimes v) = h_{(1)}p_1^R \otimes h_{(2)}p_2^R v$$

is an isomorphism of H-modules, with inverse

$$\psi^r : H \otimes V \to H \otimes_\varepsilon V,$$
$$h \otimes v \mapsto \Big[(\mathrm{id} \otimes S)(q^R \Delta(h))\Big] \cdot (1 \otimes v)\ .$$

2. *The map*

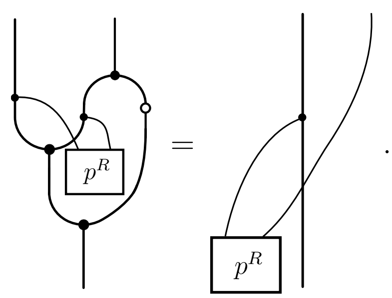
$$\phi^l : {}_\varepsilon V \otimes H \to V \otimes H, \quad v \otimes h \mapsto \Big(\Delta(h)p^L\Big) \cdot (v \otimes 1)$$

is an isomorphism of H-modules, with inverse

$$\psi^l : V \otimes H \to {}_\varepsilon V \otimes H,$$
$$v \otimes h \mapsto \Big[(S^{-1} \otimes \mathrm{id})(q^L \Delta(h))\Big] \cdot (v \otimes 1)\ .$$

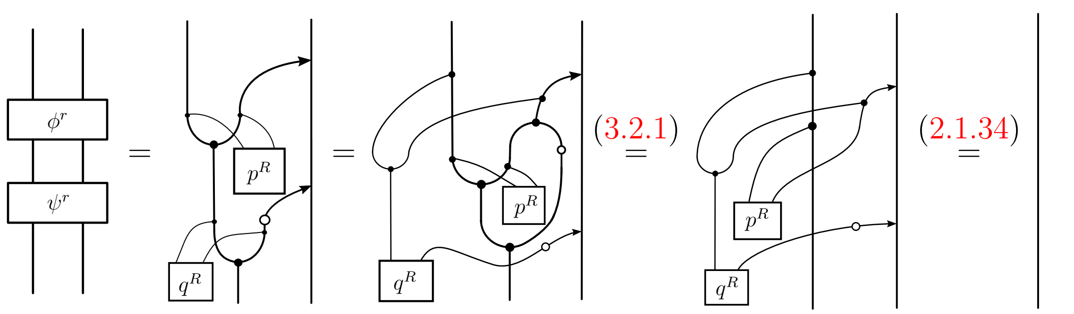
Proof. We only prove the first part, the second part is completely analogous. It is obvious that ϕ^r is an intertwiner, so we only need to show that ψ^r is a two-sided inverse.

Recall the second identity in (2.1.35), which can be graphically represented as



(3.2.1)

Using pictures we compute the composition $\phi^r \circ \psi^r$:



(3.2.2)

Similarly one shows $\psi^r \circ \phi^r = \mathrm{id}$. Since ϕ^r is an intertwiner and bijective, ψ^r necessarily is an intertwiner as well and we are done. $\square$

As a consequence of the previous considerations we have the following lemma.

Lemma 3.2.2. *Let W be an H-module. The map*

$$\begin{aligned}\Xi : H^{\mathrm{op}} \otimes_k \mathrm{End}_k(W) &\to \mathrm{End}_H(H \otimes W)\ ,\\ a \otimes m &\mapsto \phi^r \circ (r_a \otimes m) \circ \psi^r\end{aligned} \tag{3.2.3}$$

is an algebra isomorphism.

Proof. Since for any algebra A we have the algebra isomorphism

$$A^{\mathrm{op}} \cong \mathrm{End}_A(A), \quad a \mapsto r_a\ , \tag{3.2.4}$$

together with the isomorphism property from Proposition 3.2.1 we see that the prescription $\Xi : (a \otimes m) \mapsto \phi^r \circ (r_a \otimes m) \circ \psi^r$ is bijective, and thus the isomorphism is established. It remains to be shown that the isomorphism is one of algebras. The multiplication for the endomorphism algebras is just composition, and for $H^{\mathrm{op}} \otimes_k \mathrm{End}_k(W)$ we simply take the one induced by the tensor product of k-algebras. Then the calculation

$$\begin{aligned}\Xi(a \otimes m) \circ \Xi(b \otimes n) &= \phi^r \circ (r_a \otimes m) \circ \psi^r \circ \phi^r (r_b \otimes n) \circ \psi^r\\ &= \phi^r \circ ((r_a \circ r_b) \otimes (m \circ n)) \circ \psi^r\\ &= \Xi((a \otimes m) \cdot (b \otimes n))\end{aligned} \tag{3.2.5}$$

shows that Ξ indeed preserves the algebra structure. □

A similar result holds for $\mathrm{End}_H(W \otimes H)$.

The main theorem

We will need the following extension result for symmetric linear forms: Let A be a finite-dimensional unital k-algebra. By a *family of trace maps*[21] $\{\mathsf{t}_P : \mathrm{End}_A(P) \to k\}_{P \in A\text{-pmod}}$ we mean a family as in (1.2.5) which, however, only satisfies cyclicity and no partial trace property — they do not make sense in A-mod. We have [BBGa, Prop. 2.4], see also [GR3, Prop. 5.8]):

Proposition 3.2.3. *Let A be a finite-dimensional unital k-algebra. Then a symmetric linear form t on A extends uniquely to a family of trace maps $\{\mathsf{t}_P : \mathrm{End}_A(P) \to k\}_{P \in A\text{-pmod}}$. Moreover, we have*

$$\mathsf{t}_P(f) = \sum_{i=1}^{n} t((b_i \circ f \circ a_i)(1)), \quad f \in \mathrm{End}_A(P), \tag{3.2.6}$$

where n depends on P, and $a_i : A \to P$, $b_i : P \to A$ satisfy

$$\mathrm{id}_P = \sum_{i=1}^{n} a_i \circ b_i\ . \tag{3.2.7}$$

In particular

$$t_A(r_x) = t(x), \quad x \in A\ . \tag{3.2.8}$$

[21] as opposed to left/right modified traces

The next lemma is an instance of the Reduction Lemma [BBGa, Lem. 3.2] when one takes $\mathcal{C} = H$-mod and H as projective generator.

Lemma 3.2.4. *Let H be pivotal with pivot $\boldsymbol{g}$. A symmetric linear function t on H extends to a right modified trace on H-pmod if and only if*

$$\mathsf{t}_{H\otimes H}(f) = \mathsf{t}_H\left(\mathrm{tr}_H^{H\text{-mod},r}(f)\right) \tag{3.2.9}$$

holds for all $f \in \mathrm{End}_H(H \otimes H)$, where t_P is as in Proposition 3.2.3, for $P \in H$-pmod.

Similarly, t extends to a left modified trace on $\mathrm{Proj}({}_H\mathcal{M})$ if and only if

$$\mathsf{t}_{H\otimes H}(f) = \mathsf{t}_H\left(\mathrm{tr}_H^{H\text{-mod},l}(f)\right) \tag{3.2.10}$$

holds for all $f \in \mathrm{End}_H(H \otimes H)$.

We denote the subspace of symmetric forms $t \in H^*$ which extend to a right/left modified trace on H-pmod by

$$\mathsf{Sym}_{\mathrm{tr}}^{r/l} \ . \tag{3.2.11}$$

Given $t \in \mathsf{Sym}_{\mathrm{tr}}^{r/l}$, the corresponding modified trace $\mathsf{t}_\bullet$ takes the value

$$\mathrm{End}_H(H) \to k \quad , \quad f \mapsto \mathsf{t}_H(f) = t(f(\mathbf{1})) \tag{3.2.12}$$

on the left regular module H.

We can now state the main theorem of this chapter. Parts 2 and 3 generalize [BBGa, Thm. 1] to the setting of quasi-Hopf algebras. A stronger version of Part 1 was shown for Hopf algebras in [FOG, Cor. 6.1].

Theorem 3.2.5. *Let $(H, \boldsymbol{g})$ be a finite-dimensional pivotal quasi-Hopf algebra over k. We have:*

1. *A non-degenerate left (right) modified trace on H-pmod exists if and only if H is unimodular.*

Suppose now that H is in addition unimodular. Then:

2. *$\mathsf{Sym}_{\mathrm{tr}}^{r/l}$ is equal to the space of symmetrized right/left cointegrals. In particular, $\dim(\mathsf{Sym}_{\mathrm{tr}}^{r/l}) = 1$.*

3. *A non-zero element of $\mathsf{Sym}_{\mathrm{tr}}^{r/l}$ extends to a non-degenerate right/left modified trace on H-pmod.*

Proof. (1) If $\mathsf{t}_\bullet$ is a non-degenerate left or right modified trace, then $H \ni h \mapsto \mathsf{t}_H(r_h) \in k$ is a non-degenerate symmetric linear form on H. Unimodularity follows from [HN2, Prop. 5.6]. The converse direction amounts to parts 2 and 3.

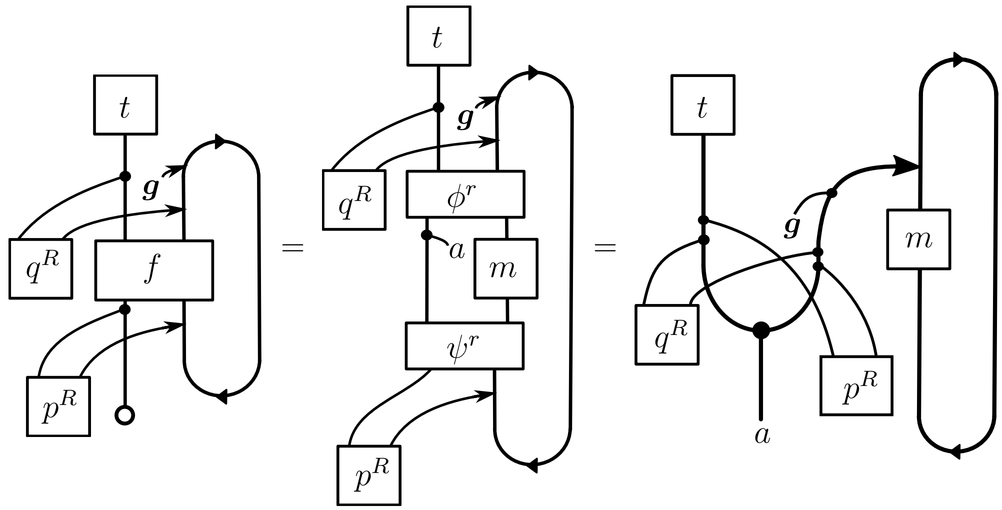


Figure 3.1: Calculating the left hand side of (3.2.15). The string diagrams are all in Vect. To arrive at the initial string diagram, recall the expression (3.0.3) for the right evaluation map. The first step is just substitution of f from (3.2.13). In the second step, we use that $\psi^r(p_1^R \otimes p_2^R w) = \mathbf{1} \otimes w$ for all $w \in W$ (which in turn is immediate from $\psi^r \circ \phi^r = \mathrm{id}$) and substitute the definition of ϕ^r.

(2) Suppose now that H is unimodular, and let $\mathsf{t}_\bullet$ be a family of trace maps on H-pmod (not necessarily left/right modified traces). Let $t \in H^*$ be the symmetric form on H which corresponds to $\mathsf{t}_\bullet$ via Proposition 3.2.3. We will now compute both sides of (3.2.9) in Lemma 3.2.4 separately and then use that lemma to prove the statement. To this end, let $W \in H$-mod and $f \in \mathrm{End}_H(H \otimes W)$.

$\underline{\mathsf{t}_{H\otimes W}(f)}$: By Lemma 3.2.2, every $f \in \mathrm{End}_H(H \otimes W)$ is of the form

$$f = \sum_{(a,m)} \phi^r \circ (r_a \otimes m) \circ \psi^r \ , \tag{3.2.13}$$

where $a \otimes m$ is a simple tensor in $H^{\mathrm{op}} \otimes \mathrm{End}_k(W)$. For simplicity and without loss of generality we will assume that f actually corresponds to the simple tensor $a \otimes m$. By cyclicity of $\mathsf{t}_\bullet$ we get

$$\begin{aligned} \mathsf{t}_{H\otimes W}(f) &= \mathsf{t}_{H\otimes W}(\phi^r \circ (r_a \otimes m) \circ \psi^r) \\ &= \mathsf{t}_{H\otimes_\varepsilon W}(r_a \otimes m) \\ &= \mathsf{tr}_k(m)\ t(a)\ , \end{aligned} \tag{3.2.14}$$

where $\mathsf{tr}_k(m)$ is the trace of the linear operator m. This can be seen by choosing any basis of W and considering a decomposition of $H \otimes_\varepsilon W$ into $(\dim W)$ copies of H.

$\underline{\mathsf{t}_H\left(\mathsf{tr}_W^r(f)\right)}$: Here we use that $\mathsf{t}_H\left(\mathsf{tr}_W^r(f)\right) = t\left(\mathsf{tr}_W^r(f)(\mathbf{1})\right)$ and then rewrite the resulting expression as in Figure 3.1. Altogether, this gives

$$\mathsf{t}_H\left(\mathsf{tr}_W^r(f)\right) = t(q_1^R a_{(1)} p_1^R)\mathsf{tr}_k\left(\rho(\boldsymbol{g} q_2^R a_{(2)} p_2^R \otimes -) \circ m\right)\ , \tag{3.2.15}$$

where $\rho : H \otimes_k W \to W$ is the action of H on W.

Since (3.2.14) and (3.2.15) hold in particular for $W = H$, the left regular module, and for all a, m, we can rephrase condition (3.2.9) in Lemma 3.2.4 as follows: the symmetric linear form t on H extends to a right modified trace on H-pmod if and only if

$$t(a)\mathbf{1} = (t \otimes \boldsymbol{g})\left(q^R \Delta(a) p^R\right) \ . \tag{3.2.16}$$

But this is just the defining equation (3.1.20) for a symmetrized right cointegral.

The left version of the proof is completely analogous and uses (3.1.19).

(3) By Corollary 3.1.6 the symmetrized right/left cointegrals are non-degenerate. It is shown in [BBGa, Thm. 2.6] that this implies that the corresponding right/left modified traces are non-degenerate in the sense of Section 1.2.2. □

3.3 Example: symplectic fermion quasi-Hopf algebra

In this section we will use Theorem 3.2.5 to compute the modified trace for the symplectic fermion quasi-Hopf algebras $\mathsf{Q} = \mathsf{Q}(N, \beta)$, which we reviewed in Section 2.4.2.

We will see that the spaces of left and right modified traces coincide for Q. To compute the modified trace explicitly, we first find the (also coinciding) left and right symmetrized cointegrals via Corollary 3.1.5. Then we employ Theorem 3.2.5 and the relation (3.2.12) to obtain the value of the modified trace on the projective generator Q, and the indecomposable projectives.

Proposition 3.3.1. *The linear form*

$$\widehat{\boldsymbol{\lambda}}^r = (\beta^2 + i) \cdot \left(B_{\vec{N},\vec{N},1}\right)^* + (\beta^2 - i) \cdot \left(B_{\vec{N},\vec{N},3}\right)^* \tag{3.3.1}$$

is simultaneously a left and a right symmetrized cointegral for Q.

Proof. We will verify that $\widehat{\boldsymbol{\lambda}}^r$ satisfies both conditions in Corollary 3.1.5. To this end, we first note that the coproduct takes the following form on elements of the basis chosen in Section 2.4.2:

$$\begin{aligned}\Delta(B_{\vec{l},\vec{a},i}) = \Big(& B_{\vec{l},\vec{a},i} \otimes \mathsf{K}^i + \omega_+^{|\vec{a}|}\omega_-^{|\vec{b}|}\mathsf{K}^i \otimes B_{\vec{l},\vec{a},i} + \text{ (lower terms)}\Big) \\ & \times \Big(\boldsymbol{e}_0 \otimes \boldsymbol{e}_0 + \boldsymbol{e}_0 \otimes \boldsymbol{e}_1 + \boldsymbol{e}_1 \otimes \boldsymbol{e}_0 + (-1)^{(N+1)i}\boldsymbol{e}_1 \otimes \boldsymbol{e}_1\Big) \ ,\end{aligned} \tag{3.3.2}$$

where in each tensor factor in "(lower terms)" the number of f^+'s is strictly less than $|\vec{a}|$, or the number of f^-'s is strictly less than $|\vec{b}|$, or both. Therefore, both sides of the two conditions in Corollary 3.1.5 vanish identically unless one chooses $h = B_{\vec{N},\vec{N},i}$, $i \in \{0, 1, 2, 3\}$. In these four cases a straightforward computation shows that the conditions in Corollary 3.1.5 hold. □

Note that because $\beta^4 = (-1)^N$, for odd N only one of the two summands in (3.3.1) is present, the other coefficient is zero. For N even, both summands are present.

Since the symmetrized cointegral is two-sided, so is the corresponding modified trace. By (3.2.12) the explicit value of the modified trace on $f \in \mathrm{End}_{\mathsf{Q}}(\mathsf{Q})$ is

$$\mathsf{t}_{\mathsf{Q}}(f) = \widehat{\boldsymbol{\lambda}}(f(\mathbf{1})) \ , \tag{3.3.3}$$

with $\widehat{\boldsymbol{\lambda}}$ as in (3.3.1).

The modified trace has also been computed by a different method in [GR3, Sec. 9], namely by using the existence of a simple projective object in Q-mod. There, the modified trace is given on the four indecomposable projectives. To relate the two computations, first note that the central idempotents of Q are

$$\boldsymbol{e}_0 \ , \qquad \boldsymbol{e}_1^{\pm} = \tfrac{1}{2}\boldsymbol{e}_1(\mathbf{1} \pm \beta^{-1}\boldsymbol{v}) = \tfrac{1}{2}\boldsymbol{e}_1\Big(\mathbf{1} \mp i\mathsf{K}\prod_{k=1}^{N}(\mathbf{1} - 2\mathsf{f}_k^+\mathsf{f}_k^-)\Big) \ , \tag{3.3.4}$$

see [FGR2, Sec. 3.6]. The decomposition of the right regular module Q is

$$\mathsf{Q} = P_{0+} \oplus P_{0-} \oplus X_{1+}^{\oplus 2^N} \oplus X_{1-}^{\oplus 2^N} \ , \tag{3.3.5}$$

where $P_{0\pm}$ are the projective covers of the two one-dimensional simple modules of Q and $X_{1\pm}$ are projective simple objects of dimension 2^N [FGR2, Sec. 3.7]. The projections to $P_{0\pm}$ are given by right-multiplication with the (non-central) idempotents $\boldsymbol{e}_0^{\pm} = \frac{1}{2}(\mathbf{1}\pm\mathsf{K})\boldsymbol{e}_0$. The central idempotents $\boldsymbol{e}_1^{\pm}$ project to the direct sums $X_{1\pm}^{\oplus 2^N}$. Set

$$x_{\pm} = \left(\prod_{j=1}^{N}\mathsf{f}_j^+\mathsf{f}_j^-\right)\boldsymbol{e}_0^{\pm} \ , \qquad y_{\pm} = \boldsymbol{e}_1^{\pm} \ . \tag{3.3.6}$$

Note that $x_{\pm}$ and $y_{\pm}$ are central in Q [FGR2, Sec. 3.6]. It is straightforward to compute the modified trace of $r_{x_{\pm}}, r_{y_{\pm}} \in \mathrm{End}_{\mathsf{Q}}(\mathsf{Q})$:

$$\mathsf{t}_{\mathsf{Q}}(r_{x_{\pm}}) = \pm\tfrac{1}{2}(-1)^{\frac{1}{2}N(N-1)}\beta^2 \ , \quad \mathsf{t}_{\mathsf{Q}}(r_{y_{\pm}}) = \pm\tfrac{1}{2}(-1)^{\frac{1}{2}N(N-1)}(-2)^N \ , \tag{3.3.7}$$

where r_h denotes the right multiplication with $h \in \mathsf{Q}$, cf. (2.1.19). This agrees with [GR3, Sec. 9] up to a normalization factor of $\frac{1}{2}(-1)^{\frac{1}{2}N(N+1)}$.

Since $\boldsymbol{g}$ is of order 2, the left and right cointegrals also agree. One can compute the cointegral for Q by shifting the symmetrized cointegral from Proposition 3.3.1 by $\boldsymbol{g}$. Similar to the symmetrized cointegral, it is non-vanishing only on the top components, and with

$$a_{\pm} = \beta^2 \pm \delta_{N,\mathrm{even}} \quad , \quad b_{\pm} = \pm i\delta_{N,\mathrm{odd}} \tag{3.3.8}$$

it can be expressed as

$$\boldsymbol{\lambda} = a_+\left(B_{\vec{N},\vec{N},0}\right)^* + b_+\left(B_{\vec{N},\vec{N},1}\right)^* + a_-\left(B_{\vec{N},\vec{N},2}\right)^* + b_-\left(B_{\vec{N},\vec{N},3}\right)^* \ . \tag{3.3.9}$$

Chapter 4

Monadic cointegrals and applications to quasi-Hopf algebras

In this chapter we introduce the new notion of *monadic cointegral*. These are two (or four, in the pivotal case) generalizations of Hopf algebra cointegrals to any finite tensor category.

In the first section, the central Hopf (co)monads are reviewed and the definitions of all four versions of monadic cointegrals are given.

These definitions are specialized to quasi-Hopf algebras in the second section.

In the third section, our main theorem relating the quasi-Hopf cointegrals from Chapter 2 to our monadic cointegrals is stated. The proof is given modulo technical details, which have been moved to Appendix B.

Some explicit examples of the main theorem in action are in Section 4.4

We end this chapter with a final section discussing the relation between monadic cointegrals and Lyubashenko's integral for the canonical coend $\mathcal{L}$ in case the category is braided, also specializing to quasi-Hopf algebras again.

4.1 Monadic cointegrals

This section contains the main definition of this chapter, namely that of the two (or four in the pivotal case) types of monadic cointegrals (Definition 4.1.5). To state the definition we first review the two (or four) versions of the central Hopf monad. At the end, we also realize monadic cointegrals via Hopf comonads, to establish existence and uniqueness via results in [Sh4].

4.1.1 The central Hopf monad

Throughout the rest of this section $\mathcal{C}$ will denote a finite tensor category. Recall the notion of a coend from Section 1.2.3. By the immediate corollary to [KL, Cor. 5.1.8] we gave in

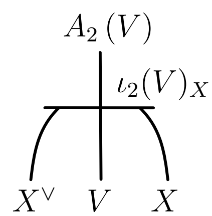
Proposition 1.2.6, the coends

$$A_1(V) = \int^{X\in\mathcal{C}} {}^{\vee}X(VX)\,, \qquad A_2(V) = \int^{X\in\mathcal{C}} X^{\vee}(VX)$$
$$A_3(V) = \int^{X\in\mathcal{C}} (XV)\,{}^{\vee}X, \qquad A_4(V) = \int^{X\in\mathcal{C}} (XV)\,X^{\vee} \tag{4.1.1}$$

exist for all $V \in \mathcal{C}$. A different and more detailed proof of existence is given in [Sh1, Thm. 3.6]. Note that the subscript i indicates the 'position' of the dual symbol $^{\vee}$. We write $\iota_i(V)$ for the universal dinatural transformation of $A_i(V)$ so that for example

$$\iota_2(V)_X : X^{\vee}(VX) \to A_2(V)\,. \tag{4.1.2}$$

In particular, $A_i : V \mapsto A_i(V)$ is an endofunctor, and the universal dinatural transformations $\iota_i(V)$ are natural in $V \in \mathcal{C}$. In our graphical notation the dinatural transformation $\iota_2(V)$ will be drawn as



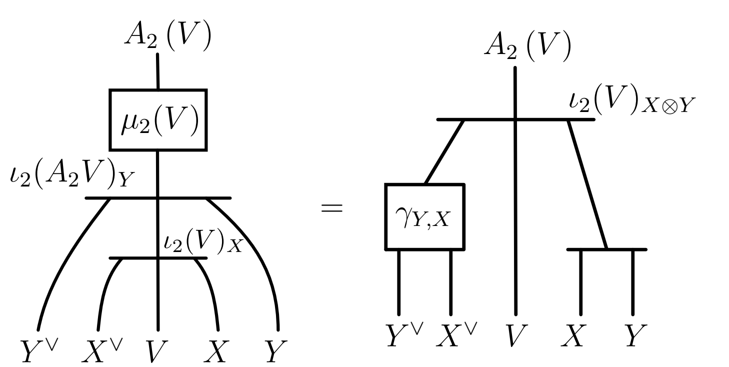
(4.1.3)

for all $V, X \in \mathcal{C}$.

Functors like A_i, and in particular the functor A_2, were already studied in e.g. [BV2]. The latter is known as the *central Hopf monad* [Sh1], see also Remark 4.1.3.

We will now describe the monad structures in more detail, restricting our exposition to the case $i = 2$. The monad structure is similar for all other cases.

Recall the natural isomorphism $\gamma_{X,Y} : (X^{\vee})(Y^{\vee}) \to (YX)^{\vee}$ from (1.1.6). The multiplication $\mu_2 : (A_2)^2 \Rightarrow A_2$ with components $\mu_2(V) : A_2A_2(V) \to A_2(V)$ is determined by the universal property of coends via



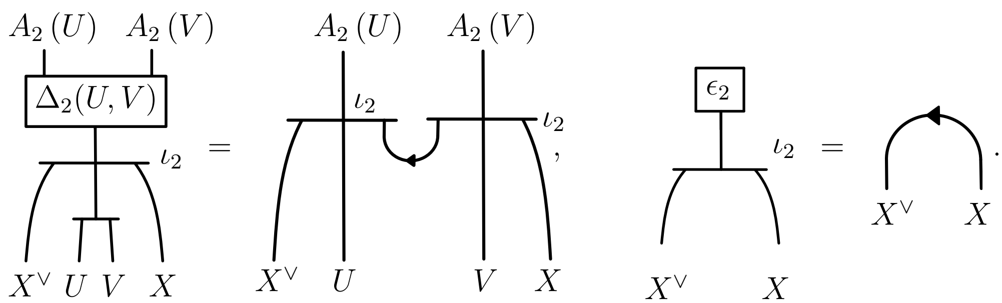
(4.1.4)

Here we used what is known as the 'Fubini theorem' for ends and coends, cf. [Mac, Sec. IX.8], see also [Lo, Rem. 1.9], to express the dinatural transformation of the iterated coend $A_2A_2(V)$ in terms of $\iota_2(V)$ and $\iota_2(A_2V)$.

The unit of A_2, i.e. the natural transformation $\eta_2 : \mathrm{id}_{\mathcal{C}} \Rightarrow A_2$, is defined as

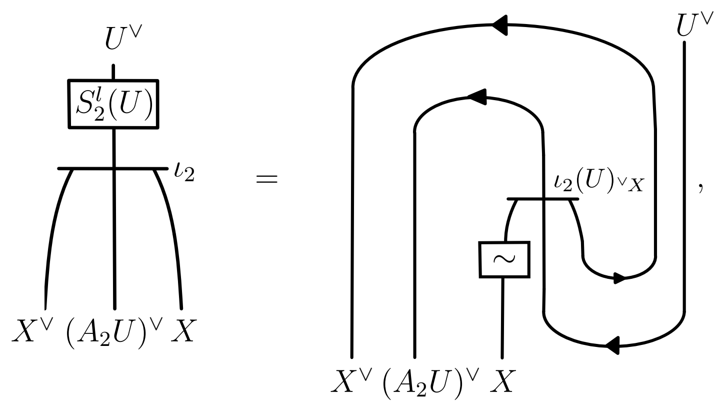
$$\eta_2(V) := \left[V \xrightarrow{\sim} \mathbf{1}^\vee (V\mathbf{1}) \xrightarrow{\iota_2(V)_{\mathbf{1}}} A_2(V) \right] . \tag{4.1.5}$$

For $i = 2, 3$, A_i is always a Hopf monad [BV2, Sec. 5.4]. As an example illustrating this fact, we again consider $i = 2$, the other case is similar. The lax comonoidal structure is defined by[22]



(4.1.6)

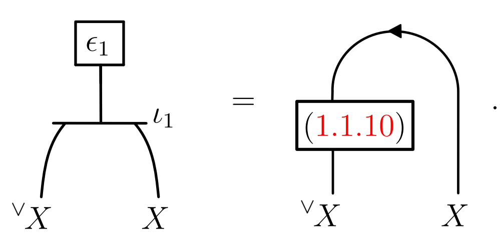
These are the comultiplication and counit of A_2. The left antipode of A_2 is defined by



(4.1.7)

following [BV2, Thm. 5.6]. Here, by $\sim$ we mean the canonical isomorphism $X \cong ({}^\vee X)^\vee$, defined similarly to $\omega_X : {}^\vee(X^\vee) \to X$ from (1.1.12). The right antipode is obtained analogously.

For $i = 1, 4$, the above definition of a bimonad structure on A_i does not work. If $\mathcal{C}$ is pivotal, however, the natural monoidal isomorphism $X^\vee \cong {}^\vee X$ from (1.1.10) can be used when the duals do not match up in the comultiplication and counit. For example, the counit of A_1 is given by



(4.1.8)

[22] Here and in similar places below, we often omit spelling out all components and arguments of the dinatural transformations, e.g. on the LHS we have $\iota_2(U \otimes V)_X$, etc.

One checks that in this way one obtains a Hopf monad structure on A_1 and A_4.

We summarize the preceding discussion in the following proposition for later use.

Proposition 4.1.1. *The functors A_2 and A_3 are Hopf monads. If $\mathcal{C}$ is pivotal, then A_1 and A_4 are also Hopf monads.*

The following proposition will now show that the canonical natural isomorphisms $\kappa_{i,j} : A_i \Rightarrow A_j$ defined by

$$(\kappa_{i,j})_V \circ \iota_i(V)_X = \iota_j(V)_{X'} \circ (\text{isomorphism of components}), \tag{4.1.9}$$

are isomorphisms of Hopf monads.[23] Here, X' stands for X, $X^\vee$ or ${}^\vee X$ as appropriate, and the 'isomorphisms of components' consist of coherence isomorphisms and the isomorphisms ${}^\vee(X^\vee) \cong X$ and, in the pivotal case, $X^\vee \cong {}^\vee X$. For example,

$$\begin{aligned} &\left[X^\vee (VX) \xrightarrow{\iota_2(V)_X} A_2(V) \xrightarrow{(\kappa_{2,3})_V} A_3(V)\right] \\ &= \left[X^\vee (VX) \xrightarrow{\sim} (X^\vee V)X \xrightarrow{\mathrm{id}\otimes\omega_X^{-1}} (X^\vee V)({}^\vee(X^\vee)) \xrightarrow{\iota_3(V)_{X^\vee}} A_3(V)\right]. \end{aligned} \tag{4.1.10}$$

Proposition 4.1.2. *The natural isomorphism $\kappa_{2,3}$ is an isomorphism of Hopf monads. If $\mathcal{C}$ is pivotal, then $\kappa_{i,j}$ are isomorphisms of Hopf monads for all i, j.*

Proof. We first claim that the pullback $F = (\kappa_{2,3})^* : \mathcal{C}_{A_3} \to \mathcal{C}_{A_2}$ is a well-defined functor. Namely, on an A_3-module (V, ρ) the functor acts as $F(V, \rho) = (V, F\rho)$, where we define $F\rho : A_2V \to V$ by

$$F\rho \circ \iota_2(V)_X = \left[X^\vee (VX) \xrightarrow{\sim} (X^\vee V)({}^\vee(X^\vee)) \xrightarrow{\iota_3(V)_{X^\vee}} A_3(V) \xrightarrow{\rho} V\right], \tag{4.1.11}$$

so that indeed $F\rho = \rho \circ (\kappa_{2,3})_V$ by (4.1.10). A quick calculation shows that $F\rho$ is an A_2-action.

Next we check the conditions in Lemma 1.2.12. As F is given by pullback, the underlying functor is the identity on $\mathcal{C}$. To verify strict monoidality, one checks that for (V, ρ), $(W, \sigma) \in \mathcal{C}_{A_3}$ one has

$$F(\rho \otimes \sigma \circ \Delta_3(V, W)) = \left(F\rho \otimes F\sigma\right) \circ \Delta_2(V, W) \tag{4.1.12}$$

and $F(\epsilon_3) = \epsilon_2$, which is easy to see.

Thus from Lemma 1.2.12 we obtain a morphism of bimonads (and hence of Hopf monads) $A_2 \Rightarrow A_3$. Since $F = (\kappa_{2,3})^*$, by Remark 1.2.13 this morphism is given by $\kappa_{2,3}$. As $\kappa_{2,3}$ is an isomorphism, we finally get $A_2 \cong A_3$.

[23] Alternatively, one could have introduced the Hopf monad structure only on A_2, and then used the natural isomorphisms $\kappa_{2,j}$ to transport the structure to the other functors A_j. However, each A_j comes with a canonical choice of Hopf monad structure defined using universal properties of coends, which is the structure we want to use throughout the remainder of the text. Thus we prefer to present the canonical structure in the four cases first and then establish the isomorphisms afterwards.

If $\mathcal{C}$ is pivotal, then e.g. the equivalence $G : \mathcal{C}_{A_2} \to \mathcal{C}_{A_1}$ is given by $G(V, \rho) = (V, G\rho)$ with

$$G\rho \circ \iota_1(V)_X = \left[{}^{\vee}X(VX) \xrightarrow{\sim} X^{\vee}(VX) \xrightarrow{\iota_2(V)_X} A_2\,(V) \xrightarrow{\rho} V \right], \tag{4.1.13}$$

where the first isomorphism is given by the inverse to the one in (1.1.10). It is straightforward to check that G is a well-defined functor satisfying $G = (\kappa_{1,2})^*$, and that it is strict monoidal and the identity on objects from $\mathcal{C}$. Hence also $A_2 \cong A_1$ as Hopf monads. □

Remark 4.1.3. Let us assume strictness for this remark. It is not hard to see that $\mathcal{C}_{A_2} \cong \mathcal{Z}(\mathcal{C}) \cong \mathcal{C}_{A_3}$ as monoidal categories, where $\mathcal{Z}(\mathcal{C})$ is the Drinfeld center of $\mathcal{C}$, cf. Section 1.1.8 and [BV1, Sec. 9.3]. Indeed, if $\rho : A_2\,(V) \to V$ is an A_2-action, then one quickly checks that the canonical natural transformation with components

$$V \otimes X \xrightarrow{\mathrm{coev}_X \otimes \mathrm{id}} X \otimes X^{\vee} \otimes V \otimes X \xrightarrow{\mathrm{id} \otimes \iota_2(V)_X} X \otimes A_2\,(V) \xrightarrow{\mathrm{id} \otimes \rho} X \otimes V \tag{4.1.14}$$

for $X \in \mathcal{C}$ is a lax half-braiding for V, i.e. a 'half-braiding' which is *a priori* not invertible but whose $\mathbf{1}$-component is the identity. However, lax half-braidings in rigid categories are invertible, cf. [BLV, Sec. 5]. This equivalence and the resulting new description of the center were the main reasons for introducing central monads, and also explains the name. ▽

Example 4.1.4. Let $\mathcal{C} = {}_H\mathcal{M}$ be the category of finite-dimensional modules over a finite-dimensional Hopf algebra H, and let $i = 2, 3$. As vector spaces, the $A_i(V)$ are isomorphic to $H^* \otimes V$, and we choose the module structures as follows. A proof that this data indeed realizes the coend(s) is given more generally for quasi-Hopf algebras in Appendix A. With $h \in H$, $f \in H^*$, and $v \in V$, the action $\overset{i}{\curvearrowright}$ of H on $A_i(V)$ is [24]

$$\begin{aligned} h \overset{2}{\curvearrowright} (f \otimes v) &= \langle f \mid S(h_{(1)})?h_{(3)} \rangle \otimes h_{(2)}v \;, \\ h \overset{3}{\curvearrowright} (f \otimes v) &= \langle f \mid S^{-1}(h_{(3)})?h_{(1)} \rangle \otimes h_{(2)}v \;. \end{aligned} \tag{4.1.15}$$

Here we use the strictly coassociative version

$$h_{(1)} \otimes h_{(2)} \otimes h_{(3)} := h_{(1,1)} \otimes h_{(1,2)} \otimes h_{(2)} = h_{(1)} \otimes h_{(2,1)} \otimes h_{(2,2)}$$

of the sumless Sweedler-notation for iterated coproducts, recall Notation 2.1.1.

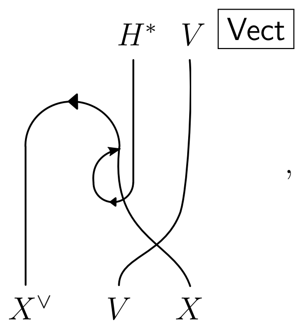
Note that $A_2\,(\mathbf{1})$ is the coadjoint representation of H, cf. [FGR1, Sec. 7]. The universal dinatural transformations are defined as

$$\begin{aligned} \iota_2(V)_X(f \otimes v \otimes x) &= \sum_i \langle f \mid e_i.x \rangle e^i \otimes v, \\ \iota_3(V)_X(x \otimes v \otimes f) &= \sum_i \langle f \mid e_i.x \rangle e^i \otimes v, \end{aligned} \tag{4.1.16}$$

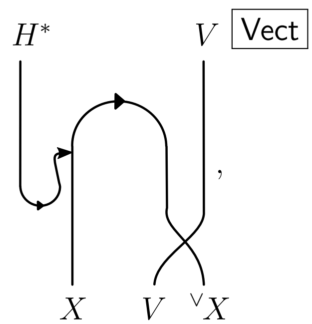
[24] We use $\langle ? \mid ? \rangle : V^* \otimes V \to k$ to denote the canonical pairing in Vect.

where $f \in X^*$, $v \in V$, $x \in X$, and $\{e_i\}$ is a basis of H with dual basis $\{e^i\}$. In string diagram notation, these can be depicted as

$\iota_2(V)_X =$



, $\iota_3(V)_X =$



,   (4.1.17)

where the boxed Vect signifies that this is to be read in the category of vector spaces. The actions in (4.1.15) are uniquely determined by requiring ι_2 and ι_3 to be morphisms in $\mathcal{C}$.

The units are

$$\eta_2(V) = \eta_3(V) = \varepsilon \otimes \mathrm{id}_V, \tag{4.1.18}$$

where ε is the counit of H, and with Δ the comultiplication of H, the multiplications are given by[25]

$$\mu_2(V) = (\Delta^{\mathrm{op}})^* \otimes \mathrm{id}_V, \quad \mu_3(V) = \Delta^* \otimes \mathrm{id}_V \ . \tag{4.1.19}$$

The comultiplication of A_i is given by linear maps

$$\Delta_i(V,W) : H^* \otimes V \otimes W \to H^* \otimes V \otimes H^* \otimes W \tag{4.1.20}$$

for all $V, W \in \mathcal{C}$ explicitly as follows. We have

$$\Delta_2(V,W)(f \otimes v \otimes w) = \sum_{i,j} \langle f \mid e_i \cdot e_j \rangle \, e^i \otimes v \otimes e^j \otimes w \tag{4.1.21}$$

and

$$\Delta_3(V,W)(f \otimes v \otimes w) = \sum_{i,j} \langle f \mid e_j \cdot e_i \rangle \, e^i \otimes v \otimes e^j \otimes w \tag{4.1.22}$$

for $f \in H^*$. The counits, being morphisms $A_i(\mathbf{1}) \to \mathbf{1}$, can be identified with elements in H, and we find that they are given by the unit of H,

$$\epsilon_2 = \epsilon_3 = \mathbf{1}. \tag{4.1.23}$$

The left antipode is given by linear maps

$$S_i^l(V) : H^* \otimes (H^* \otimes V)^* \to V^*, \tag{4.1.24}$$

[25] The convention we use for the dual map Δ^* on $H^* \otimes H^*$ is as follows: for $f, g \in H^*$ and $b \in H$ we set $(\Delta^*(f \otimes g))(b) := (f \otimes g)(\Delta(b))$. Ditto for $(\Delta^{\mathrm{op}})^*$.

for $V \in \mathcal{C}$, and, denoting by $\widetilde{S^l_i}(V)$ the corresponding canonical endomorphism of $H \otimes V$, we have

$$\widetilde{S^l_2}(V) = S \otimes \mathrm{id}_V \quad \text{and} \quad \widetilde{S^l_3}(V) = S^{-1} \otimes \mathrm{id}_V \ . \tag{4.1.25}$$

Assume now that H is a pivotal Hopf algebra, i.e. that it contains a grouplike element $\boldsymbol{g}$, called the *pivot*, satisfying $S^2(a) = \boldsymbol{g}a\boldsymbol{g}^{-1}$ for all $a \in H$, cf. Section 2.2.1 or [AAGTV, BBGa]. The two remaining actions on $A_i(V)$ for $i = 1, 4$ can be chosen as

$$\begin{aligned} h \overset{1}{\curvearrowright} (f \otimes v) &= \langle f \mid S^{-1}(h_{(1)})?h_{(3)}\rangle \otimes h_{(2)}v \\ h \overset{4}{\curvearrowright} (f \otimes v) &= \langle f \mid S(h_{(3)})?h_{(1)}\rangle \otimes h_{(2)}v \ . \end{aligned} \tag{4.1.26}$$

With this definition, the corresponding universal dinatural transformations are the same linear maps as before:

$$\iota_1(V)_X = \iota_2(V)_X \quad \text{and} \quad \iota_4(V)_X = \iota_3(V)_X \ . \tag{4.1.27}$$

The counits are

$$\epsilon_1 = \boldsymbol{g}^{-1} \quad \text{and} \quad \epsilon_4 = \boldsymbol{g}. \tag{4.1.28}$$

Rather than determining the Hopf monad structure on each A_i separately as stated before Proposition 4.1.1, it may be easier to work out only one, say A_2, and then to transport the structure via the isomorphisms κ_{ij}. By Proposition 4.1.2, this gives the same result. The κ_{ij} take a simple form in the Hopf case:

$$\begin{aligned} (\kappa_{12})_V(f \otimes v) &= \langle f \mid \boldsymbol{g}^{-1}\ ?\rangle \otimes v \ , \\ (\kappa_{23})_V(f \otimes v) &= \Big(f \circ S\Big) \otimes v \ , \\ (\kappa_{43})_V(f \otimes v) &= \langle f \mid \boldsymbol{g}\ ?\rangle \otimes v \ , \end{aligned} \tag{4.1.29}$$

for all $f \in H^*, v \in V$. Note that then e.g. $\epsilon_1 = \epsilon_2 \circ (\kappa_{12})_{\mathbf{1}}$. △

4.1.2 Monadic cointegrals for finite tensor categories

Recall the distinguished invertible object D from Section 1.1.3, and consider the free A_i-module $(A_i\,(D)\,, \mu_i(D))$.

Definition 4.1.5. For $i = 2$ (resp. $i = 3$), a morphism

$$\boldsymbol{\lambda}_i : \mathbf{1} \to A_i\,(D) \tag{4.1.30}$$

is called a *right* (resp. *left*) *monadic cointegral of* $\mathcal{C}$ if it intertwines the trivial A_i-action on $\mathbf{1}$ and the free action on $A_i\,(D)$. If $\mathcal{C}$ is pivotal, then for $i = 1$ (resp. $i = 4$) such a morphism is called a *right* (resp. *left*) *D-symmetrized monadic cointegral of* $\mathcal{C}$.

We denote the subspace of monadic cointegrals in $\mathcal{C}(\mathbf{1}, A_i\,(D))$ by:

$$\begin{array}{llll} i=1: & \int_{\mathcal{C}}^{r,D-\mathrm{sym}} & i=2: & \int_{\mathcal{C}}^{r,\mathrm{mon}} \\ i=4: & \int_{\mathcal{C}}^{l,D-\mathrm{sym}} & i=3: & \int_{\mathcal{C}}^{l,\mathrm{mon}} \end{array} \tag{4.1.31}$$

Remark 4.1.6. 1. The A_i-module intertwining condition from (4.1.30) for a morphism $\boldsymbol{\lambda}_i : \mathbf{1} \to A_i\,(D)$ in $\mathcal{C}$ is equivalent to the commutativity of the diagram

$$\begin{tikzcd} A_i\,(\mathbf{1}) \arrow[r, "A_i(\boldsymbol{\lambda}_i)"] \arrow[d, "\epsilon_i"'] & A_i^2(D) \arrow[d, "\mu_i(D)"] \\ \mathbf{1} \arrow[r, "\boldsymbol{\lambda}_i"'] & A_i\,(D) \end{tikzcd}, \tag{4.1.32}$$

or as an equation

$$\boldsymbol{\lambda}_i \circ \epsilon_i = \mu_i(D) \circ A_i(\boldsymbol{\lambda}_i)\ . \tag{4.1.33}$$

In [BV1, Eq. (45)], a cointegral of a bimonad T was defined as an intertwiner of T-modules from $(\mathbf{1}, T_0)$ to $(T(\mathbf{1}), \mu_{\mathbf{1}})$. Thus, if $\mathcal{C}$ is unimodular, a right (resp. left) monadic cointegral of $\mathcal{C}$ is just a cointegral of the bimonad A_2 (resp. A_3).

2. It follows immediately from Proposition 4.1.2 that $\boldsymbol{\lambda}_i$ is a monadic cointegral for A_i if and only if $(\kappa_{i,j})_D \circ \boldsymbol{\lambda}_i$ is a monadic cointegral for A_j.

$\triangledown$

The names for the monadic cointegrals are chosen because of the relation to cointegrals for Hopf algebras, as we will see in the following example.[26]

Example 4.1.7. Let $\mathcal{C} = {}_H\mathcal{M}$ be as in Example 4.1.4. By Remark 2.3.9, the distinguished invertible object D is just the ground field k with action given by the algebra morphism $\boldsymbol{\gamma}^{-1}$, where $\boldsymbol{\gamma}$ is the modulus of H. Thus, a morphism $\mathbf{1} \to A_i\,(D)$ is the same as an element in H^* intertwining some specific H-actions.

Let us first look at the linear condition coming from diagram (4.1.32). Using the Hopf monad structure as given in Example 4.1.4, we see that a right (resp. left) monadic cointegral is, as a linear form, a solution to

$$(\boldsymbol{\lambda}_2 \otimes \mathrm{id})(\Delta(h)) = \boldsymbol{\lambda}_2(h)\mathbf{1}, \quad \text{resp.} \quad (\mathrm{id} \otimes \boldsymbol{\lambda}_3)(\Delta(h)) = \boldsymbol{\lambda}_3(h)\mathbf{1}. \tag{4.1.34}$$

[26] According to our convention of calling the invariants under the regular actions of a Hopf algebra *integrals*, one could also call e.g. the right D-symmetrized monadic cointegral simply an integral for the Hopf monad A_1. This would follow more closely the nomenclature of [BV1] (who, however, call the invariants under the regular actions of a Hopf algebra "cointegrals", which is opposite to our convention). It would also fit to Corollary 4.1.11, which roughly states that monadic cointegrals are dual to the categorical cointegrals of [Sh4].

However, as we explain in Example 4.1.7 and Section 4.3, the reason for keeping these names is that the four versions of monadic cointegrals automatically correspond to the four versions of cointegrals for H if $\mathcal{C} = H$-mod for a pivotal (quasi) Hopf algebra H.

That is, it is a right (resp. left) cointegral for the Hopf algebra H in the usual sense, cf. [Ra2, Def. 10.1.2]. Conversely, for example, a solution to the first equation in (4.1.34) is a right monadic cointegral, provided it is in addition an intertwiner $\mathbf{1} \to A_2(D)$ of H-modules. However, by [Ra2, Thm. 10.5.4(e)] or simply (2.3.11), a right cointegral $\boldsymbol{\lambda}$ satisfies $\boldsymbol{\lambda}(aS^{-1}(b)) = \boldsymbol{\gamma}^{-1}(b_{(2)})\boldsymbol{\lambda}(S(b_{(1)})a)$. With the H-action on $A_2(V)$ from (4.1.15), we thus have

$$\boldsymbol{\gamma}^{-1}(h_{(2)})\boldsymbol{\lambda}(S(h_{(1)})ah_{(3)}) = \boldsymbol{\lambda}(ah_{(2)}S^{-1}(h_{(1)})) = \varepsilon(h)\boldsymbol{\lambda}(a) \ . \tag{4.1.35}$$

But this is simply the required intertwining property, so a right cointegral is automatically a right monadic cointegral.

If $(H, \boldsymbol{g})$ is a pivotal Hopf algebra, then diagram (4.1.32) can, as a linear equation, be evaluated for $i = 1, 4$, and it gives the equations

$$(\boldsymbol{\lambda}_1 \otimes \mathrm{id})(\Delta(h)) = \boldsymbol{\lambda}_1(h)\boldsymbol{g}^{-1}, \quad (\mathrm{id} \otimes \boldsymbol{\lambda}_4)(\Delta(h)) = \boldsymbol{\lambda}_4(h)\boldsymbol{g}. \tag{4.1.36}$$

According to [FOG, Sec. 4.4], solutions to these equations are precisely $\boldsymbol{\gamma}$-symmetrized cointegrals for H (where we regard H as a Hopf G-coalgebra for G the trivial group), see also [BBGa] for the unimodular case. As above, in the converse direction, solutions to e.g. the first equation in (4.1.36) are automatically intertwiners of H-modules from $\mathbf{1}$ to $A_1(D)$.[27] Let us also recall from Remark 3.1.7 that $\boldsymbol{\gamma}$-symmetrized cointegrals are an example of g-cointegrals for a group-like g as introduced in [Ra1].

Finally, let us stress a point already made in the introduction. As we just saw, via the very natural realization of each monad A_i given in Example 4.1.4, the monadic cointegrals for $A_1, \ldots, A_4$ reduce directly to four known versions of cointegrals for finite dimensional (pivotal) Hopf algebras. This is an important motivation for keeping all four of the A_i, even though they are all isomorphic. Indeed, also in the Hopf case one can easily give explicit isomorphisms between the four spaces of cointegrals, but in practice it is important to have all four notions available, rather than singling one out arbitrarily. △

The preceding example shows that for $\mathcal{C} = {}_H\mathcal{M}$ with H a finite-dimensional (pivotal) Hopf algebra, left/right (D-symmetrized) monadic cointegrals exist and are unique up to scalar. The next proposition states that this remains true for any (pivotal) finite tensor category.

Proposition 4.1.8. *Let $\mathcal{C}$ be a finite tensor category.*

1. *Non-zero left/right monadic cointegrals exist and are unique up to scalar multiples.*
2. *Suppose $\mathcal{C}$ is in addition pivotal. Then non-zero left/right D-symmetrized monadic cointegrals exist and are unique up to scalar multiples.*

The proof will follow from results in [Sh4], after we relate monadic cointegrals to the categorical cointegral of [Sh4], and is given at the end of the next subsection.

[27] This follows from results in the appendix. More precisely, note that by [FOG, Prop. 4.18] the linear form $\boldsymbol{\lambda}_1$ lies in the space $\mathcal{X}_1$ from (B.4.1). This space is isomorphic to $\mathcal{C}(\mathbf{1}, A_1(D))$, and the isomorphism (B.4.2) is the identity in the Hopf case.

4.1.3 Relation to the categorical cointegral

Define functors Z_i via the ends

$$Z_1(V) = \int_{X\in\mathcal{C}} {}^{\vee}X\,(VX)\,, \qquad Z_2(V) = \int_{X\in\mathcal{C}} X^{\vee}\,(VX)$$
$$Z_3(V) = \int_{X\in\mathcal{C}} (XV)\,{}^{\vee}X, \qquad Z_4(V) = \int_{X\in\mathcal{C}} (XV)\,X^{\vee} \tag{4.1.37}$$

with corresponding universal dinatural transformations $\pi_i(V)$, so that for example

$$\pi_4(V)_X : Z_4(V) \to (XV)\,X^{\vee}\,. \tag{4.1.38}$$

Below we will give an adjunction between Z_4 and A_2. One can formulate such adjunctions in the three other cases, too, but we will not need this and will only consider Z_4 in the following.

Similarly to how the A_i, $i = 2, 3$ became Hopf monads, Z_4 becomes a Hopf comonad and we denote the comultiplication, counit, multiplication, and unit by

$$\Delta^4(V) : Z_4(V) \to Z_4Z_4(V), \qquad \varepsilon^4(V) : Z_4(V) \to V,$$
$$\mu^4(V,W) : Z_4(V) \otimes Z_4(W) \to Z_4(V \otimes W)\,, \qquad u^4 \;: \mathbf{1} \to Z_4(\mathbf{1})\;, \tag{4.1.39}$$

respectively. Z_4 is precisely the central comonad of [Sh4], where also a detailed description of the structure maps (4.1.39) can be found.

With this, we can now recall the definition of the *categorical cointegral* from [Sh4, Def. 4.3]: It is a Z_4-comodule morphism

$$\boldsymbol{\lambda}^{\mathrm{Sh}} : (Z_4(D^{\vee}), \Delta^4(D^{\vee})) \to \mathbf{1} \tag{4.1.40}$$

from the cofree comodule on $D^{\vee}$ to the tensor unit considered as the trivial comodule.[28] 'Cofree comodule' is simply a technical term meaning the image under the right adjoint of the forgetful functor $\mathcal{C}^{Z_4} \to \mathcal{C}$.

To relate the two notions *categorical cointegral* and *monadic cointegral*, we observe that there is an adjunction $A_2 \dashv Z_4$, i.e. the central Hopf monad A_2 is left adjoint to Z_4. Indeed, the following simple argument shows this:

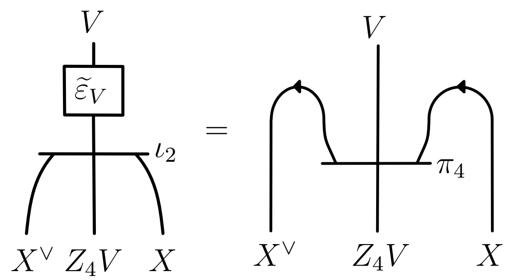
$$\begin{aligned}\mathcal{C}(A_2(V), W) &\cong \mathrm{Dinat}(-^{\vee}(V-), W)\\ &\cong \mathrm{Dinat}(V, (-W)-^{\vee}) \cong \mathcal{C}(V, Z_4(W)).\end{aligned} \tag{4.1.41}$$

We denote the counit and unit of this adjunction by

$$\widetilde{\varepsilon} : A_2Z_4 \Rightarrow \mathrm{id}_{\mathcal{C}}, \quad \widetilde{\eta} : \mathrm{id}_{\mathcal{C}} \Rightarrow Z_4A_2 \tag{4.1.42}$$

[28] Although Shimizu's definition is not explicitly stated this way, it is easy to see that [Sh4, Def. 4.3] and (4.1.40) are equivalent. This is also mentioned in the proof of [Sh4, Thm. 4.8].

respectively. They can easily be deduced from (4.1.41); for example



(4.1.43)

determines the counit. One can check that the bicomonad structure on Z_4 as described in [Sh4] is obtained from that of A_2 via the adjunction, as explained in Proposition 1.2.14.

For a comonad M on $\mathcal{C}$, the category of comodules will be denoted by $\mathcal{C}^M$.

Lemma 4.1.9. *The functor $F : \mathcal{C}^{Z_4} \to \mathcal{C}_{A_2}$, given on objects and morphisms by*

$$F(V, \rho) = (V, \widetilde{\varepsilon}_V \circ A_2\,(\rho)), \qquad Ff = f, \tag{4.1.44}$$

is an equivalence.

Proof. This statement follows immediately from the fact that A_2 is left adjoint to Z_4, since a left adjoint is nothing but a left dual in $\mathrm{End}(\mathcal{C})$. The inverse equivalence $G : \mathcal{C}_{A_2} \to \mathcal{C}^{Z_4}$ is given on objects and morphisms by

$$G(V, \nu) = (V, Z_4\,(\nu) \circ \widetilde{\eta}_V), \qquad Gf = f, \tag{4.1.45}$$

and a simple check using the adjunction triangles proves the claim. □

We make the following observation.

Proposition 4.1.10. *Let F be as in Lemma 4.1.9. There is an isomorphism*

$$(A_2\,(V)\,, \mu_2(V)) \cong \Big(F(Z_4\,({}^\vee V)\,, \Delta^4({}^\vee V))\Big)^\vee \tag{4.1.46}$$

of A_2-modules, natural in $V \in \mathcal{C}$.

Proof. Abbreviate

$$\widetilde{V} = Z_4\,({}^\vee V) \quad \text{and} \quad \rho_{\widetilde{V}} = \Delta^4({}^\vee V) : \widetilde{V} \to Z_4\big(\widetilde{V}\big). \tag{4.1.47}$$

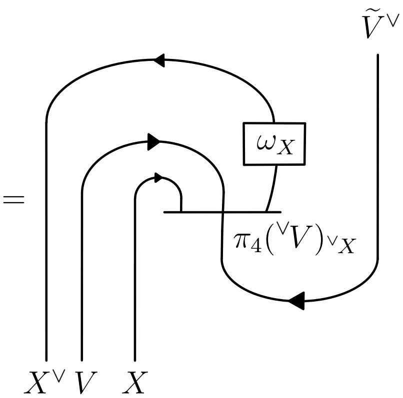
Under the equivalence from Lemma 4.1.9 we have

$$F(\widetilde{V}, \rho_{\widetilde{V}}) = (\widetilde{V}, \sigma_{\widetilde{V}}) \tag{4.1.48}$$

where $\sigma_{\widetilde{V}} = \widetilde{\varepsilon}_{\widetilde{V}} \circ A_2(\rho_{\widetilde{V}}) : A_2(\widetilde{V}) \to \widetilde{V}$ is the A_2-action corresponding to the cofree coaction.

Define the natural isomorphism $E_V : A_2(V) \to \widetilde{V}^\vee$ by

$$E_V \circ \iota_2(X)_V =$$



(4.1.49)

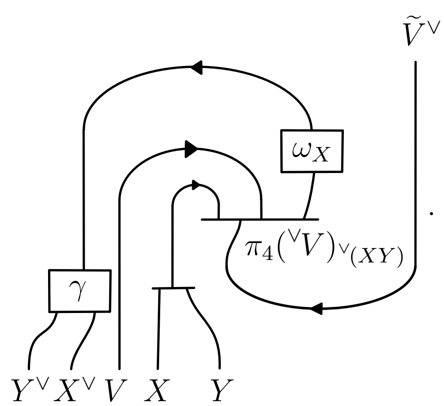
for $V \in \mathcal{C}$. Here $\omega_X : ({}^\vee X)^\vee \to X$ denotes the natural isomorphism from (1.1.12).

We want to show that E_V is an A_2-module map, that is

$$E_V \circ \mu_2(V) = S_2^l(\widetilde{V}) \circ A_2\left(\sigma_{\widetilde{V}}^\vee \circ E_V\right), \tag{4.1.50}$$

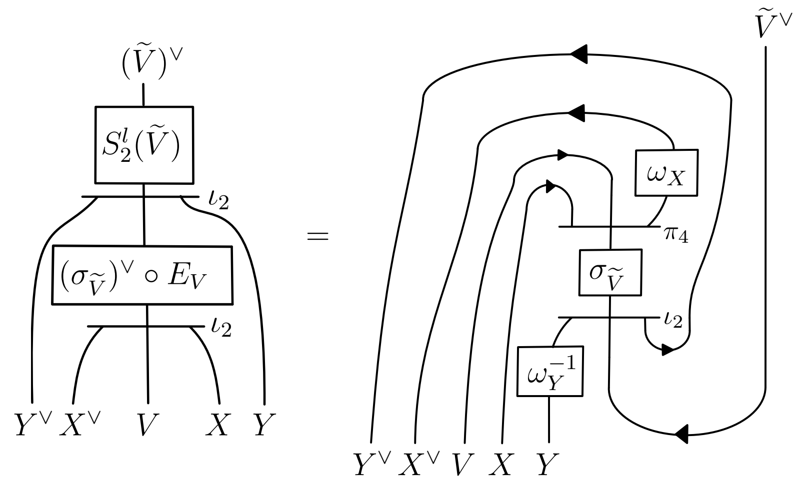
where we also used the action (1.2.26) on the dual A_2-module. To check that this equality holds we establish that both sides of (4.1.50) satisfy the same universal property for the iterated coend A_2A_2. For the left hand side of (4.1.50) we get

$$E_V \circ \mu_2(V) \circ \iota_2(A_2V)_Y \circ \left(\mathrm{id}_{Y^\vee} \otimes (\iota_2(V)_X \otimes \mathrm{id}_Y)\right) =$$



. (4.1.51)

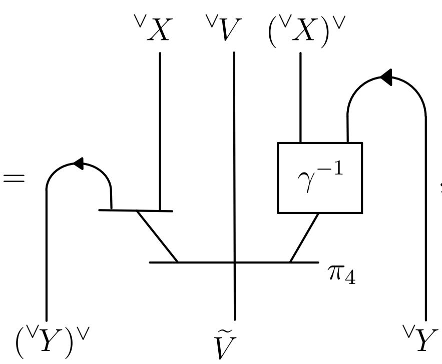
The right hand side of (4.1.50) composed with the same dinatural transformation immediately yields



(4.1.52)

A simple calculation shows

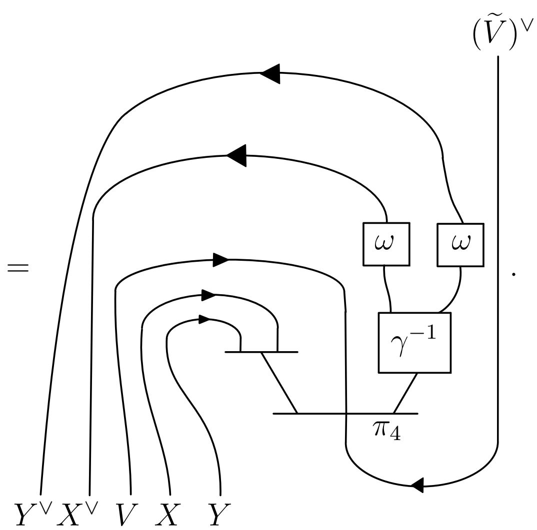
$$\pi_4({}^\vee V)_{{}^\vee X} \circ \sigma_{\widetilde{V}} \circ \iota_2(\widetilde{V})_{{}^\vee Y} =$$



, (4.1.53)

and we thus get

(4.1.52) =



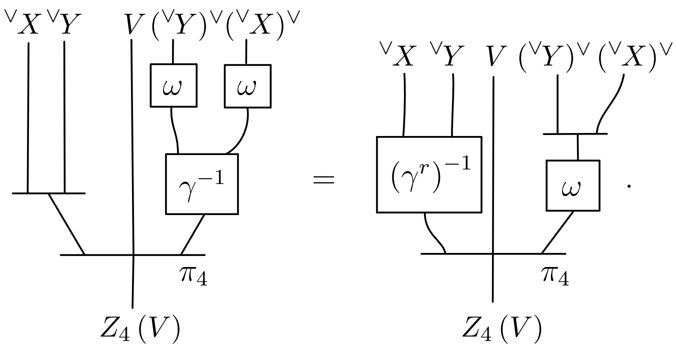
. (4.1.54)

Let γ^r be the analogue of γ (defined in (1.1.6)) for right duals. One quickly checks that the diagram

$$\begin{tikzcd}
({}^\vee X \otimes {}^\vee Y)^\vee \arrow[r, "\gamma^{-1}_{{}^\vee Y, {}^\vee X}"] \arrow[d, "\left((\gamma^r)^{-1}_{Y,X}\right)^\vee"'] & ({}^\vee Y)^\vee \otimes ({}^\vee X)^\vee \arrow[d, "\omega_Y \otimes \omega_X"] \\
({}^\vee (Y \otimes X))^\vee \arrow[r, "\omega_{Y \otimes X}"'] & Y \otimes X
\end{tikzcd} \tag{4.1.55}$$

commutes.

Dinaturality of π_4 then implies



. (4.1.56)

After plugging this into (4.1.54) and substituting the definitions of γ^r, γ, and ω, we see that this agrees with (4.1.51). □

Combining Lemma 4.1.9 and Proposition 4.1.10, and using ${}^\vee D \cong D^\vee$ (this holds for all invertible objects), we get

Corollary 4.1.11. *There is an isomorphism*

$$\mathcal{C}_{A_2}\Big(\mathbf{1}, (A_2(D), \mu_2(D))\Big) \cong \mathcal{C}^{Z_4}\Big((Z_4(D^\vee), \Delta^4(D^\vee)), \mathbf{1}\Big) . \tag{4.1.57}$$

After these preparations, we can show the existence and uniqueness (up to scalar) of monadic cointegrals.

Proof of Proposition 4.1.8. By the preceding corollary, the right monadic cointegral is equivalent to the categorical cointegral (4.1.40) of [Sh4]. Existence and uniqueness of categorical cointegrals were established in [Sh4, Thm. 4.8]. The claim then follows from Remark 4.1.6 (2). □

We now provide an interpretation of Corollary 4.1.11.

Remark 4.1.12. Recall that the definition of integrals and cointegrals in the Hopf case is symmetric under duality. More precisely, if H is a finite-dimensional Hopf algebra, then a left cointegral for H is the same as a morphism $\boldsymbol{\lambda}^l : H \to \mathbf{1}$ in the category of left H-comodules, where we regard H as the coregular comodule. Equivalently, we can consider the dual Hopf algebra H^* (with the structure given by transposition of that of H, cf. footnote 25). In this case, a left cointegral for H is a left integral for H^*, and thus a morphism $\boldsymbol{\lambda}^l : \mathbf{1} \to H^*$ in the category of left H^*-modules, where we regard H^* as the regular module.

Taking duals provides a (contravariant) equivalence,

$$H\text{-comod} \cong H^*\text{-mod} \tag{4.1.58}$$

and in particular we have an isomorphism

$$(H\text{-comod})(H, \mathbf{1}) \cong (H^*\text{-mod})(\mathbf{1}, H^*) \tag{4.1.59}$$

of vector spaces. We will need one more observation. Abbreviate $\mathcal{C} = H$-mod and recall the computation (4.1.35). This showed that there is an H-module structure on $H^* = A_3(D)$ and similarly on $H = Z_4(D^\vee)$ (which are not the (co)regular) such that:

$$\begin{array}{ccc} (H\text{-comod})(H, \mathbf{1}) & \cong & (H^*\text{-mod})(\mathbf{1}, H^*) \\ \cap & & \cap \\ \mathcal{C}(Z_4(D^\vee), \mathbf{1}) & & \mathcal{C}(\mathbf{1}, A_3(D)) \end{array} \tag{4.1.60}$$

For quasi-Hopf algebras H the corresponding line of reasoning to relate integrals and cointegrals fails at the outset, as H^* is not again a quasi-Hopf algebra. Instead, there is the following categorical version of it. We have a (contravariant) equivalence given by the composition of equivalences

$$\mathcal{C}^{Z_4} \overset{\text{Lem. 4.1.9}}{\cong} \mathcal{C}_{A_2} \overset{\text{dualizing}}{\cong} \mathcal{C}_{A_2} \overset{\text{Prop. 4.1.2}}{\cong} \mathcal{C}_{A_3}\ . \tag{4.1.61}$$

Corollary 4.1.11 then implies

$$\mathcal{C}^{Z_4}\Big((Z_4(D^\vee), \Delta^4(D^\vee)),\, \mathbf{1} \Big) \;\cong\; \mathcal{C}_{A_3}\Big(\mathbf{1},\, (A_3(D), \mu_3(D)) \Big)\ . \tag{4.1.62}$$

To relate the right hand side of (4.1.62) in the Hopf case to that of (4.1.60), one uses the explicit form of the monad multiplication in (4.1.19). For the left hand side, one correspondingly uses the coproduct of Z_4, we omit the details.

The categorical version (4.1.62) of the isomorphism (4.1.60) provides a more conceptual reason for the relation between the left monadic cointegral and the categorical cointegral of [Sh4]. $\triangledown$

4.1.4 Rewriting the monadic cointegral via Z_4

In Corollary 4.1.11 we saw one way to rewrite the definition of monadic cointegrals in terms of Hom-spaces in $\mathcal{C}^{Z_4}$. We will later need the more direct relation we present here.

Under the equivalence from Lemma 4.1.9, specifically (4.1.45), we can map the free A_2-module on any $U \in \mathcal{C}$ to its corresponding Z_4-comodule

$$(A_2(U), R_U) \text{ with } R_U = \left[A_2(U) \xrightarrow{\widetilde{\eta}_{A_2(U)}} Z_4 A_2^2(U) \xrightarrow{Z_4(\mu_2(U))} Z_4 A_2(U) \right]\ . \tag{4.1.63}$$

Note that this assignment is in fact natural in U, i.e. we have a natural transformation

$$R : A_2 \Rightarrow Z_4 A_2, \tag{4.1.64}$$

which we call the *categorical coaction*.[29] Since the equivalence (4.1.44) doesn't do anything on morphisms, we now see the equality

$$\mathcal{C}_{A_2}\Big(\mathbf{1},\ (A_2\,(D)\,, \mu_2(D)\,\Big)\ =\ \mathcal{C}^{Z_4}\Big(\mathbf{1}, (A_2\,(D)\,, R_D)\Big) \tag{4.1.65}$$

of subspaces of $\mathcal{C}\Big(\mathbf{1}, A_2\,(D)\Big)$. An element in the subspace on the left hand side is by definition a right monadic cointegral. Spelling out the condition to be in the subspace on the right hand side proves the following useful lemma (recall from (4.1.39) the notation u^4 for the unit of Z_4):

Lemma 4.1.13. *A morphism $\boldsymbol{\lambda} : \mathbf{1} \to A_2\,(D)$ in $\mathcal{C}$ is a right monadic cointegral if and only if*

$$R_D \circ \boldsymbol{\lambda} = Z_4\,(\boldsymbol{\lambda}) \circ u^4\ . \tag{4.1.66}$$

4.2 Cointegrals for quasi-Hopf algebras

In this section we specialize monadic cointegrals to the category of modules over a quasi-Hopf algebra. Then we briefly discuss an initial observation regarding the relation between monadic cointegrals and cointegrals as defined in Section 2.3. This observation will turn out to be very important in the proof of the main theorem(s) of this chapter.

Throughout this section H is a finite-dimensional quasi-Hopf algebra over k.

4.2.1 Monadic cointegrals for quasi-Hopf algebras

Recall from Section 2.3 that the modulus $\boldsymbol{\gamma} \in H^*$ of H is an algebra morphism which encodes the difference between left and right integrals, see e.g. Proposition 2.3.8. Recall also that the distinguished invertible object D of ${}_H\mathcal{M}$ is the one-dimensional module with action given by $\boldsymbol{\gamma}^{-1} = \boldsymbol{\gamma} \circ S$, and we have $\boldsymbol{\gamma}^{-1} = \boldsymbol{\gamma}^{\vee}$ as H-modules.

We now want to describe the monadic cointegrals for H. To this end, let us first give our realizations of the central Hopf monads A_i on the category ${}_H\mathcal{M}$. Consider the following four H-module structures on the vector space $H^* \otimes V$

$$\begin{aligned}
h \overset{1}{\curvearrowright} (f \otimes v) &= \langle f \mid S^{-1}(h_{(1)})?h_{(2,2)} \rangle \otimes h_{(2,1)}.v\ , \\
h \overset{2}{\curvearrowright} (f \otimes v) &= \langle f \mid S(h_{(1)})?h_{(2,2)} \rangle \otimes h_{(2,1)}.v\ , \\
h \overset{3}{\curvearrowright} (f \otimes v) &= \langle f \mid S^{-1}(h_{(2)})?h_{(1,1)} \rangle \otimes h_{(1,2)}.v\ ,
\end{aligned}$$

[29] The categorical coaction turns A_2 into a Z_4-comodule in $\mathrm{End}(\mathcal{C})$.

$$h \overset{4}{\curvearrowright} (f \otimes v) = \langle f \mid S(h_{(2)}) ? h_{(1,1)} \rangle \otimes h_{(1,2)}.v \ . \tag{4.2.1}$$

Define also the linear maps

$$\begin{aligned} \iota_1(V)_X = \iota_2(V)_X : X^* \otimes V \otimes X &\to H^* \otimes V \\ f \otimes v \otimes x &\mapsto \langle f \mid ?.x \rangle \otimes v, \\ \iota_3(V)_X = \iota_4(V)_X : X \otimes V \otimes X^* &\to H^* \otimes V \\ x \otimes v \otimes f &\mapsto \langle f \mid ?.x \rangle \otimes v \ . \end{aligned} \tag{4.2.2}$$

In Appendix A we prove the following proposition, namely that these data realize the coend(s). In particular, the linear maps ι_i are morphisms in ${}_H\mathcal{M}$.

Proposition 4.2.1. *The H-module $H^* \otimes V$ with action $\overset{i}{\curvearrowright}$ together with the dinatural transformation $\iota_i(V)$ realizes the coend $A_i(V)$.*

Let us also record here the Hopf monad isomorphisms from Proposition 4.1.2 for (pivotal) quasi-Hopf algebras, using our realizations of the monads. To this end, recall the hook notation from (2.1.10).

Proposition 4.2.2. *The canonical Hopf monad isomorphisms*

$$A_1(V) \xrightarrow{(\kappa_{1,2})_V} A_2(V) \xrightarrow{(\kappa_{2,3})_V} A_3(V) \xrightarrow{(\kappa_{3,4})_V} A_4(V) \tag{4.2.3}$$

from Proposition 4.1.2 (for the maps $(\kappa_{1,2})_V$, $(\kappa_{3,4})_V$, we require a pivot $\boldsymbol{g}$) are given by the linear maps

$$\begin{aligned} (\kappa_{1,2})_V(f \otimes v) &= (f \leftharpoonup \boldsymbol{g}^{-1}) \otimes v \ , \\ (\kappa_{2,3})_V(f \otimes v) &= \langle f \mid S(?X_1)X_3 \rangle \otimes X_2.v \ , \\ (\kappa_{3,4})_V(f \otimes v) &= (f \leftharpoonup \boldsymbol{g}^{-1}) \otimes v \ , \end{aligned} \tag{4.2.4}$$

for $f \in H^*$, $v \in V$.

Proof. The proof is a straightforward computation. For example, for $\kappa_{2,3}$ one needs to check the commutativity of

$$\begin{tikzcd} X^\vee \otimes (V \otimes X) \arrow[r, "\sim"] \arrow[d, "\iota_2(V)_X"] & (X^\vee \otimes V) \otimes X \arrow[r, "\sim"] & (X^\vee \otimes V) \otimes {}^\vee(X^\vee) \arrow[d, "\iota_3(V)_{X^\vee}"] \\ A_2(V) \arrow[rr, "(\kappa_{2,3})_V"] & & A_3(V) \end{tikzcd} \ . \tag{4.2.5}$$

Note here that the isomorphism $X \cong {}^\vee(X^\vee)$ is the same linear map as in Vect. □

Remark 4.2.3. Recall from Proposition 2.3.3 that left and right cointegrals for quasi-Hopf algebras can be related via a somewhat complicated element u. It is worth comparing the definition of u to that of $\kappa_{2,3}$ from Proposition 4.2.2, which only used a single coassociator. ▽

Define $\tau \in H^{\otimes 5}$ by

$$\tau = X_1 \otimes X_2 y_1 \otimes x_1 (X_{3(1)} y_2)_{(1)} \otimes x_2 (X_{3(1)} y_2)_{(2)} \otimes x_3 X_{3(2)} y_3 \tag{4.2.6}$$

The multiplication of A_2 can be computed explicitly from (4.1.4). For $f, g \in H^*$ and $v \in V$, the image $\mu_2(V)(f \otimes g \otimes v) \in H^* \otimes V$ under multiplication can be identified with the linear map

$$H \ni h \mapsto (g \otimes f)\Big((S \otimes S)(\tau_{21}) \boldsymbol{f} \Delta(h) \tau_{45}\Big)\ \tau_3.v \in V\ . \tag{4.2.7}$$

The counit $\epsilon_2 : A_2\,(\mathbf{1}) \to \mathbf{1}$ is easily computed from (4.1.6), and we identify it with the element $\epsilon_2 = \boldsymbol{\alpha} \in H$. Let us recall the right monadic cointegral equation from (4.1.33):

$$\boldsymbol{\lambda} \circ \epsilon_2 = \mu_2(D) \circ A_2\,(\boldsymbol{\lambda})\ . \tag{4.2.8}$$

This is an equality of (linear) endomorphisms of H^*, and evaluating it on $f \in H^*$, we immediately get

$$f(\boldsymbol{\alpha})\boldsymbol{\lambda} = \boldsymbol{\gamma}^{-1}(\tau_3)(\boldsymbol{\lambda} \otimes f)\Big((S \otimes S)(\tau_{21}) \boldsymbol{f} \Delta(e_i) \tau_{45}\Big) e^i. \tag{4.2.9}$$

This is clearly equivalent to

$$\boldsymbol{\lambda}(h)\boldsymbol{\alpha} = \boldsymbol{\gamma}^{-1}(\tau_3)(\boldsymbol{\lambda} \otimes \mathrm{id})\Big((S \otimes S)(\tau_{21}) \boldsymbol{f} \Delta(h) \tau_{45}\Big) \tag{4.2.10}$$

for all $h \in H$.

Altogether, an element $\boldsymbol{\lambda} \in H^*$ is a right monadic cointegral if and only if it satisfies (4.2.10) and is an H-module intertwiner $\mathbf{1} \to A_2\,(\boldsymbol{\gamma}^{-1})$.[30]

Similarly, with $\boldsymbol{f}^r$ the Drinfeld twist for right duals and $\sigma = (\tau^{\mathrm{cop}})_{54321}$ given explicitly by

$$\sigma = x_{1(1,1)} Y_1 X_1 \otimes x_{1(1,2)} Y_2 X_{2(1)} \otimes x_{1(2)} Y_3 X_{2(2)} \otimes x_2 X_3 \otimes x_3\ , \tag{4.2.11}$$

one obtains necessary conditions for the three remaining types of monadic cointegrals. Namely, if $\boldsymbol{\lambda} \in H^*$ is a

1. right D-symmetrized monadic cointegral then it satisfies

$$\boldsymbol{\lambda}(h)\boldsymbol{g}^{-1}\boldsymbol{\alpha} = \boldsymbol{\gamma}^{-1}(\tau_3)(\boldsymbol{\lambda} \otimes \mathrm{id})\Big((S^{-1} \otimes S^{-1})(\tau_{21}) \boldsymbol{f}^r \Delta(h) \tau_{45}\Big) \tag{4.2.12}$$

3. left monadic cointegral then it satisfies

$$\boldsymbol{\lambda}(h)S^{-1}(\boldsymbol{\alpha}) = \boldsymbol{\gamma}^{-1}(\sigma_3)(\mathrm{id} \otimes \boldsymbol{\lambda})\Big((S^{-1} \otimes S^{-1})(\sigma_{54}) \boldsymbol{f}^r \Delta(h) \sigma_{12}\Big) \tag{4.2.13}$$

4. left D-symmetrized monadic cointegral then it satisfies

$$\boldsymbol{\lambda}(h)\boldsymbol{g}S^{-1}(\boldsymbol{\alpha}) = \boldsymbol{\gamma}^{-1}(\sigma_3)(\mathrm{id} \otimes \boldsymbol{\lambda})\Big((S \otimes S)(\sigma_{54}) \boldsymbol{f} \Delta(h) \sigma_{12}\Big) \tag{4.2.14}$$

for all $h \in H$.

[30] The H-intertwiner condition, which is explicitly stated as a linear equation in (B.2.3), is automatic for monadic cointegrals for Hopf algebras, and an analogous condition is automatic for cointegrals of quasi-Hopf algebras as defined in Definition 2.3.4 below. The corresponding statement remains to be shown in the monadic setting for quasi-Hopf algebras; in all of the examples we looked at, and which are discussed in Section 4.4, the solution spaces to equation (4.2.10) turn out to be one-dimensional.

4.2.2 Relation to cointegrals via coactions

We now show that the comonads Z_4 on ${}_H\mathcal{M}$ (c.f. Section 4.1.3) and $\mathcal{Y}^l$ on ${}_H\mathcal{M}_H$ (which we introduced in Section 2.3) are related as follows. Consider the functor

$$\mathcal{A} : {}_H\mathcal{M}_H \to {}_H\mathcal{M} \tag{4.2.15}$$

sending a bimodule B to the vector space B with H-action

$$h \otimes b \mapsto h_{(1)}.b.S(h_{(2)} \leftharpoonup \boldsymbol{\gamma}^{-1}) = \boldsymbol{\gamma}^{-1}(h_{(2,1)})h_{(1)}.b.S(h_{(2,2)}), \tag{4.2.16}$$

for $h \in H$, $b \in B$. We claim that there is a natural isomorphism φ making the diagram

$$\begin{tikzcd} {}_H\mathcal{M}_H \arrow[r, "\mathcal{Y}^l"] \arrow[d, "\mathcal{A}"'] & {}_H\mathcal{M}_H \arrow[d, "\mathcal{A}"] \arrow[dl, Rightarrow, "\varphi"] \\ {}_H\mathcal{M} \arrow[r, "Z_4"'] & {}_H\mathcal{M} \end{tikzcd} \tag{4.2.17}$$

commute.

Before proving this, let us specialize the comonad Z_4 from [Sh4] to the case $\mathcal{C} = {}_H\mathcal{M}$. We choose the realization such that the objects $Z_4(V)$ are given by the underlying vector space $H \otimes V$ with actions

$$h \overset{4}{.} (a \otimes v) = h_{(1,1)}aS(h_{(2)}) \otimes h_{(1,2)}.v \ . \tag{4.2.18}$$

Lastly, let us also record here that the unit of Z_4 is given by the coevaluation element,

$$u^4 = \boldsymbol{\beta} \ . \tag{4.2.19}$$

We get the following explicit form of φ.

Proposition 4.2.4. *The family of maps*

$$\begin{aligned} \varphi_B : \mathcal{A}\mathcal{Y}^l(B) &\to Z_4\mathcal{A}(B), \\ h \otimes b &\mapsto \boldsymbol{\gamma}^{-1}(x_{3(1)}X_{2(1)}Y_1)\ x_1X_{1(1)}h\boldsymbol{f}_1^{-1}S(X_3Y_3) \\ &\qquad \otimes x_2X_{1(2)}.b.\boldsymbol{f}_2^{-1}S(x_{3(2)}X_{2(2)}Y_2), \end{aligned} \tag{4.2.20}$$

for $B \in {}_H\mathcal{M}_H$ defines a natural isomorphism as in (4.2.17).

Proof. That the above map is natural in $B \in {}_H\mathcal{M}_H$ is immediate, and proving that it intertwines the corresponding H-actions is a straightforward calculation. For convenience we state that the action on $\mathcal{A}\mathcal{Y}^l(B)$ is

$$h \otimes (a \otimes b) \mapsto h_{(1,1)}aS(h_{(2)} \leftharpoonup \boldsymbol{\gamma}^{-1})_{(1)} \otimes h_{(1,2)}.b.S(h_{(2)} \leftharpoonup \boldsymbol{\gamma}^{-1})_{(2)} \tag{4.2.21}$$

and the action on $Z_4\mathcal{A}(B)$ is

$$h \otimes (a \otimes b) \mapsto h_{(1,1)} a S(h_{(2)}) \otimes h_{(1,2,1)}.b.S(h_{(1,2,2)} \leftharpoonup \boldsymbol{\gamma}^{-1}) \tag{4.2.22}$$

for $a, h \in H$, $b \in B$. Here the dot denotes the action on the bimodule B.

Finally, the inverse of φ_B can be read off directly from the explicit expression in (4.2.20). We state it for convenience; it is given by

$$\begin{aligned} \varphi_B^{-1} : Z_4\mathcal{A}(B), &\to \mathcal{AY}^l(B) \\ h \otimes b &\mapsto \boldsymbol{\gamma}^{-1}(y_1 x_{2(1)} X_{3(1)}) x_{1(1)} X_1 h S(y_3 x_3) \boldsymbol{f}_1 \\ &\qquad \otimes x_{1(2)} X_2 .b.S(y_2 x_{2(2)} X_{3(2)}) \boldsymbol{f}_2 \ . \end{aligned} \tag{4.2.23}$$

The verification that this is indeed a (two-sided) inverse of φ is left as an easy exercise. □

4.3 Main Theorem

We are now ready to state our two main theorems of this chapter, which are Theorems 4.3.1 and 4.3.3 below.

4.3.1 Left and right monadic cointegrals

Theorem 4.3.1. *Let H be a finite-dimensional quasi-Hopf algebra with modulus $\boldsymbol{\gamma}$.*

1. *Define the linear map*

$$(?)^{\mathrm{mon}} : H^* \to H^*, \quad f^{\mathrm{mon}} = \langle f \mid S(\boldsymbol{\beta}) \ ? \ S^{-1}(\boldsymbol{\xi}) \rangle \ , \tag{4.3.1}$$

   where $\boldsymbol{\xi} = (\mathrm{id} \otimes \boldsymbol{\gamma})(\boldsymbol{f}^{-1})$. Then $\boldsymbol{\lambda}^{\mathrm{mon}}$ is a right monadic cointegral if and only if $\boldsymbol{\lambda} \in H^*$ is a right cointegral.

2. *Define the linear map*

$$^{\mathrm{mon}}(?) : H^* \to H^*, \quad {}^{\mathrm{mon}}f = \langle f \mid S^{-2}(\boldsymbol{\beta}) \ ? \ S(\hat{\boldsymbol{\xi}}) \rangle \ , \tag{4.3.2}$$

   where $\hat{\boldsymbol{\xi}} = \boldsymbol{\xi}^{\mathrm{cop}} = (S^{-1} \otimes \boldsymbol{\gamma}^{-1})(\boldsymbol{f}^{-1})$. Then ${}^{\mathrm{mon}}\boldsymbol{\lambda}$ is a left monadic cointegral if and only if $\boldsymbol{\lambda} \in H^*$ is a left cointegral,

Let us explain the main ideas in the proof. First, we specialize the equivalent characterization of (right) monadic cointegrals from Lemma 4.1.13 to the quasi-Hopf setting. The resulting equation resembles the right cointegral equation (2.3.22) we encountered in our discussion of quasi-Hopf algebras. Indeed, we find a nice relationship between the categorical coaction R_D from (4.1.63) and the left coaction from [BC2], cf. Proposition 4.3.2 below.

Using the relation between these two coactions we then show that the map (4.3.1) sends a right cointegral to a right monadic cointegral via a direct calculation. This establishes Part 1. Part 2 will then be inferred from Part 1 using the isomorphism $A_2 \cong A_3$.

The details follow below, with some technical steps deferred to Appendix B.2.

Relation to quasi-Hopf cointegrals

Let $\mathcal{C} = {}_H\mathcal{M}$ and recall our realization of the Hopf comonad Z_4 from (4.2.18) and (4.2.19). In this setting, the distinguished invertible object of $\mathcal{C}$ is $\boldsymbol{\gamma}^\vee$, and we can rewrite Equation (4.1.66) on right monadic cointegrals as the linear equation

$$R_{\gamma^\vee} \circ \boldsymbol{\lambda} = \boldsymbol{\beta} \otimes_k \boldsymbol{\lambda} \ , \tag{4.3.3}$$

where we identified morphisms from $\mathbf{1}$ to H with elements in H. We will show that this equation is equivalent to the right cointegral equation (2.3.22), which re recall here for convenience,

$$\rho^l(\boldsymbol{\lambda}^r) = \boldsymbol{\gamma}(x_3)x_1 \otimes \boldsymbol{\lambda}^r.x_2 \ , \tag{4.3.4}$$

where $\boldsymbol{\lambda}^r \in H^*$ satisfies $\boldsymbol{\lambda} = (\boldsymbol{\lambda}^r)^{\mathrm{mon}}$ with $(-)^{\mathrm{mon}}$ defined in (4.3.1).

We will first relate the categorical coaction $R_{\gamma^\vee}$ and the coaction ρ^l. To this end, recall the functor $\mathcal{A} : {}_H\mathcal{M}_H \to {}_H\mathcal{M}$ from (4.2.15). One can check that in our realization of the central Hopf monad we have the equality $\mathcal{A}(H^\vee) = A_2(\boldsymbol{\gamma}^\vee)$ of H-modules.

Proposition 4.3.2. *With the natural isomorphism $\varphi : \mathcal{A}\mathcal{Y}^l \Rightarrow Z_4\mathcal{A}$ from (4.2.17) and the left H-coaction $\rho^l : H^\vee \to \mathcal{Y}^l(H^\vee)$ from Section 2.3 we have that*

$$R_{\gamma^\vee} = \left[\mathcal{A}(H^\vee) \xrightarrow{\mathcal{A}(\rho^l)} \mathcal{A}\mathcal{Y}^l(H^\vee) \xrightarrow{\varphi_{H^\vee}} Z_4\mathcal{A}(H^\vee) \right] . \tag{4.3.5}$$

The proof of this proposition has been relegated to Appendix B.1.

Note that $\mathcal{A}$ does not do anything on morphisms; in particular, the linear maps $\mathcal{A}(\rho^l)$ and ρ^l are identical. Then this proposition together with (4.3.3), says that $\boldsymbol{\lambda}$ is a right monadic cointegral if and only if

$$(\varphi_{H^\vee} \circ \rho^l)(\boldsymbol{\lambda}) = \boldsymbol{\beta} \otimes_k \boldsymbol{\lambda} \ . \tag{4.3.6}$$

In Appendix B.2 we show that this is equivalent to the right cointegral equation (4.3.4) using the map (4.3.1). This finishes the proof of the first part of Theorem 4.3.1.

The second part is the same as the first part for H^{cop}, but we prefer not passing to a different quasi-Hopf algebra. Instead, we follow a more direct approach using the isomorphism $A_2 \cong A_3$ from Proposition 4.1.2 and Proposition 4.2.2, see Appendix B.3.

Namely, in the appendix, we show that the diagram

$$\begin{tikzcd} \int_H^r \arrow[r, "(4.3.1)"] \arrow[d, "(*)"'] & \int_{\mathcal{C}}^{r,\mathrm{mon}} \arrow[d, "(\kappa_{2,3})_D \,\circ\, ?"] \\ \int_H^l \arrow[r, "(4.3.2)"'] & \int_{\mathcal{C}}^{l,\mathrm{mon}} \end{tikzcd} \tag{4.3.7}$$

commutes, where $(*)$ is, up to a non-zero factor, the isomorphism between left and right cointegrals (cf. Proposition 2.3.3), and $\kappa_{2,3} : A_2 \Rightarrow A_3$ is the Hopf monad isomorphism from (4.2.4).

4.3.2 Left and right D-symmetrized monadic cointegrals

To prove an analogous result of Theorem 4.3.1 for D-symmetrized monadic cointegrals, we recall our notion of $\boldsymbol{\gamma}$-symmetrized cointegrals for H, cf. Section 3.1: a left (resp. right) $\boldsymbol{\gamma}$-symmetrized cointegral is a linear form $\widehat{\boldsymbol{\lambda}}^l$ (resp. $\widehat{\boldsymbol{\lambda}}^r$) on H such that

$$\widehat{\boldsymbol{\lambda}}^l = \boldsymbol{\lambda} \leftharpoonup \mathsf{u}^{\mathrm{cop}}\boldsymbol{g}^{-1} \quad \text{resp.} \quad \widehat{\boldsymbol{\lambda}}^r = \boldsymbol{\lambda} \leftharpoonup \mathsf{u}\boldsymbol{g} \;, \tag{4.3.8}$$

where $\boldsymbol{\lambda}$ is a left (resp. right) cointegral of H, and $\mathsf{u}, \mathsf{u}^{\mathrm{cop}}$ are defined in (2.3.12).

Now we can extend Theorem 4.3.1 to the pivotal case.

Theorem 4.3.3. *Let $(H, \boldsymbol{g})$ be a pivotal quasi-Hopf algebra.*

- *Consider the linear map*

$$(?)^{\gamma\text{-}sym} : H^* \to H^*, \quad f^{\gamma\text{-}sym} = \langle f \mid S(\boldsymbol{\beta}) \;?\; S^{-1}(\boldsymbol{\vartheta})\rangle \;, \tag{4.3.9}$$

where $\boldsymbol{\vartheta} = (\boldsymbol{\gamma}^{-1} \otimes S^{-1})(p^L)$. Then $\boldsymbol{\lambda}^{\gamma\text{-}sym}$ is a right D-symmetrized monadic cointegral if and only if $\boldsymbol{\lambda} \in H^*$ is a right $\boldsymbol{\gamma}$-symmetrized cointegral.

- *Consider the linear map*

$${}^{\gamma\text{-sym}}(?) : H^* \to H^*, \quad {}^{\gamma\text{-sym}}f = \langle f \mid \boldsymbol{\beta} \;?\; S(\hat{\boldsymbol{\vartheta}})\rangle \;, \tag{4.3.10}$$

where $\hat{\boldsymbol{\vartheta}} = \boldsymbol{\vartheta}^{\mathrm{cop}} = (S \otimes \boldsymbol{\gamma}^{-1})(p^R)$. Then ${}^{\gamma\text{-sym}}\boldsymbol{\lambda}$ is a left D-symmetrized monadic cointegral if and only if $\boldsymbol{\lambda} \in H^*$ is a left $\boldsymbol{\gamma}$-symmetrized cointegral.

The proof is via monad isomorphisms as in the second part of Theorem 4.3.1 and can be found in Appendix B.4.

4.4 Examples of monadic cointegrals

Here we give examples of quasi-Hopf algebras and their cointegrals. Our examples are mostly non-unimodular; some unimodular examples can be found e.g. in [BC2, Ex. 3.7] and Sections 2.4 and 3.3. All vector spaces below are considered over the complex numbers $\mathbb{C}$.

Example 4.4.1. Our first example is again given by the two 8-dimensional quasi-Hopf algebras $H_{\pm}(8)$ from Section 2.4.1. We want to give their (monadic) cointegrals.

Basis elements of $H_{\pm}(8)$ are of the form $B_{m,n} = g^m x^n$, $0 \le m \le 1$, $0 \le n \le 3$, and we denote the element dual to $B_{m,n}$ by $B^*_{m,n}$. The Drinfeld twist is given by

$$\boldsymbol{f}^{\pm 1} = 2p_+ \otimes p_+ - g \otimes g \tag{4.4.1}$$

and so we obtain the right monadic cointegral

$$\boldsymbol{\lambda}^{r,\mathrm{mon}} = B^*_{0,3} \pm i B^*_{1,3} \;. \tag{4.4.2}$$

Concretely, one solves (4.2.10) and finds that its solution space is one-dimensional, so its elements automatically are morphisms in ${}_H\mathcal{M}$. With the Hopf monad isomorphism $\kappa_{2,3}$ from Proposition 4.2.2 it is then easy to show that

$$\boldsymbol{\lambda}^{l,\mathrm{mon}} = B^*_{1,3} \tag{4.4.3}$$

is a non-zero left monadic cointegral. Using the isomorphisms from Theorem 4.3.1, we obtain the 'classical' right and left cointegrals,

$$\boldsymbol{\lambda}^r = B^*_{0,3} \mp iB^*_{1,3} \quad \text{and} \quad \boldsymbol{\lambda}^l = B^*_{0,3} \ . \tag{4.4.4}$$

The same expression for the left cointegral was also derived in [BC2, Ex. 3.9].

We remark that $H_\pm(8)$ is not pivotal. Indeed, one easily checks that already for the generator x, $S^2(x)h = hx$ implies $h = 0$, so that S^2 is not inner. △

Example 4.4.2. Fix $N \in \mathbb{N}$, $\beta \in \mathbb{C}$ satisfying $\beta^4 = (-1)^N$. This example is based on the symplectic fermion ribbon quasi-Hopf algebra $\mathsf{Q}(N,\beta)$, which we already discussed in Section 2.4.2. $\mathsf{Q}(N,\beta)$ is factorizable, so in particular unimodular, and its cointegrals were already discussed in Section 3.3. We now restrict to the sub-quasi-Hopf algebra $H(N,\beta) \subset \mathsf{Q}(N,\beta)$, which is defined as follows. As a unital $\mathbb{C}$-algebra, it is generated by K, f_i, $1 \leq i \leq N$, with defining relations

$$\{\mathsf{f}_i, \mathsf{K}\} = 0 \ , \quad \{\mathsf{f}_i, \mathsf{f}_j\} = 0 \ , \quad \mathsf{K}^4 = \mathbf{1} \ , \tag{4.4.5}$$

where $\{a, b\} = ab + ba$ is the anticommutator.

Recall our convention (2.4.15) for the order in product notation. A PBW-type basis of $H(N,\beta)$ is

$$\left\{ B_{\vec{j},l} = \left(\prod_{i=1}^{N} \mathsf{f}_i^{j_i}\right)\mathsf{K}^l \;\middle|\; \vec{j} \in \{0,1\}^N, \;\; 0 \leq l \leq 3 \right\} \ . \tag{4.4.6}$$

Elements in the corresponding dual basis are simply decorated with an asterisk.

Using the orthogonal central idempotents $\boldsymbol{e}_0 = \frac{1}{2}(\mathbf{1} + \mathsf{K}^2)$ and $\boldsymbol{e}_1 = \mathbf{1} - \boldsymbol{e}_0$, and setting $\omega = (\boldsymbol{e}_0 + i\boldsymbol{e}_1)\mathsf{K}$, the comultiplication and the counit are

$$\begin{aligned} \Delta(\mathsf{K}) &= \mathsf{K} \otimes \mathsf{K} - (1 + (-1)^N)\boldsymbol{e}_1\mathsf{K} \otimes \boldsymbol{e}_1\mathsf{K} \ , \qquad & \varepsilon(\mathsf{K}) &= 1 \ , \\ \Delta(\mathsf{f_i}) &= \mathsf{f}_i \otimes \mathbf{1} + \omega \otimes \mathsf{f}_i \ , & \varepsilon(\mathsf{f}_i) &= 0 \ . \end{aligned} \tag{4.4.7}$$

The coassociator and its inverse are

$$\Phi^{\pm 1} = \mathbf{1} \otimes \mathbf{1} \otimes \mathbf{1} + \boldsymbol{e}_1 \otimes \boldsymbol{e}_1 \otimes \{\boldsymbol{e}_0(\mathsf{K}^N - \mathbf{1}) + \boldsymbol{e}_1(\boldsymbol{\beta}_\pm - \mathbf{1})\} \ , \tag{4.4.8}$$

where $\boldsymbol{\beta}_\pm = \boldsymbol{e}_0 + \beta^2(\pm i\mathsf{K})^N\boldsymbol{e}_1$. The evaluation and coevaluation elements are $\boldsymbol{\alpha} = \mathbf{1}$, $\boldsymbol{\beta} = \boldsymbol{\beta}_+$, and the antipode is

$$S(\mathsf{K}) = \mathsf{K}^{(-1)^N} \ , \quad S(\mathsf{f}_i) = \mathsf{f}_i(\boldsymbol{e}_0 + (-1)^N i\boldsymbol{e}_1)\mathsf{K} \ . \tag{4.4.9}$$

Then, with $X = \mathbf{1} + \mathsf{K} + \mathsf{K}^2 + \mathsf{K}^3$, we see that

$$c^l = X \prod_{i=1}^{N} \mathsf{f}_i, \quad c^r = \left(\prod_{i=1}^{N} \mathsf{f}_i\right) X \tag{4.4.10}$$

are a left and a right integral, respectively. From this, one easily computes that the modulus is the algebra homomorphism given on generators by

$$\boldsymbol{\gamma}(\mathsf{K}) = (-1)^N, \quad \boldsymbol{\gamma}(\mathsf{f}_i) = 0. \tag{4.4.11}$$

In particular, $H(N,\beta)$ is unimodular if and only if N is even. Note that just as in the previous example $\boldsymbol{\gamma} = \boldsymbol{\gamma}^{-1}$.

Now we describe the cointegrals of $H(N,\beta)$. The Drinfeld twist and its inverse are given by

$$\boldsymbol{f}^{\pm 1} = \boldsymbol{e}_0 \otimes \mathbf{1} + \boldsymbol{e}_1 \otimes \boldsymbol{e}_0 \mathsf{K}^N + \boldsymbol{e}_1 \boldsymbol{\beta}_{\mp} \otimes \boldsymbol{e}_1 \ , \tag{4.4.12}$$

see also [FGR2, (3.35)]. We again find the right monadic cointegral via (4.2.10) and then obtain the left monadic cointegral via the isomorphism of Hopf monads from Proposition 4.2.2:

$$\boldsymbol{\lambda}^{r,\mathrm{mon}} = B^*_{\vec{N},0} \quad \text{and} \quad \boldsymbol{\lambda}^{l,\mathrm{mon}} = \delta_{N,\mathrm{even}} B^*_{\vec{N},0} + \delta_{N,\mathrm{odd}}(B^*_{\vec{N},1} - iB^*_{\vec{N},3}) \ , \tag{4.4.13}$$

where $\vec{N}$ is the multi-index consisting only of 1s. In particular, the left and the right monadic cointegral do not agree unless N is even.

With our main theorem, we obtain the right and the left quasi-Hopf cointegral

$$\begin{aligned}
\boldsymbol{\lambda}^r &= a^r_+ B^*_{\vec{N},0} + a^r_- B^*_{\vec{N},2} - \delta_{N,odd}(B^*_{\vec{N},1} - B^*_{\vec{N},3}), \\
\boldsymbol{\lambda}^l &= a^l_+ B^*_{\vec{N},0} + a^l_- B^*_{\vec{N},2} - \delta_{N,odd}(B^*_{\vec{N},1} + B^*_{\vec{N},3}),
\end{aligned} \tag{4.4.14}$$

where the coefficients are

$$a^r_{\pm} = \delta_{N,\mathrm{even}}(1 \pm \beta^2) + \delta_{N,\mathrm{odd}}\beta^2 i \quad \text{and} \quad a^l_{\pm} = \delta_{N,\mathrm{even}} \pm \beta^2 \ . \tag{4.4.15}$$

△

Example 4.4.3. Fix an odd integer t, let $p \geq 2$ be an integer, and set $q = e^{i\pi/p}$. We again consider the quasi Hopf modification $\overline{U}_q^{(\Phi)}\mathfrak{sl}(2)$ of the restricted quantum group $\overline{U}_q\mathfrak{sl}(2)$ reviewed in Section 2.4.3. This quasi-Hopf algebra is factorizable, and as in the previous example we will consider a non-unimodular sub-quasi-Hopf algebra U^-, namely the subalgebra generated by F and K. The defining defining relations and the quasi-Hopf structure are the same as for $\overline{U}_q^{(\Phi)}\mathfrak{sl}(2)$, but we restate them for convenience. We have

$$F^p = 0, \quad K^{2p} = \mathbf{1}, \quad \text{and} \quad KFK^{-1} = q^{-2}F,$$

and a natural choice of basis of U^- is therefore

$$\{B_{m,n} = F^m K^n \mid 0 \leq m \leq p-1, \quad 0 \leq n \leq 2p-1\} \ .$$

Using the two central idempotents $\boldsymbol{e}_0 = \frac{1}{2}(\mathbf{1} + K^p)$ and $\boldsymbol{e}_1 = \mathbf{1} - \boldsymbol{e}_0$, the quasi-Hopf structure is again given by

$$\begin{aligned} \Delta_t(F) &= F \otimes \mathbf{1} + (\boldsymbol{e}_0 + q^{-t}\boldsymbol{e}_1)K^{-1} \otimes F, \qquad & \varepsilon(F) &= 0, \\ \Delta_t(K) &= K \otimes K, & \varepsilon(K) &= 1 \end{aligned}$$

and

$$\begin{aligned} \Phi_t &= \mathbf{1} \otimes \mathbf{1} \otimes \mathbf{1} + \boldsymbol{e}_1 \otimes \boldsymbol{e}_1 \otimes (K^{-t} - \mathbf{1}) \\ S_t(F) &= -KF(\boldsymbol{e}_0 + q^{-t}\boldsymbol{e}_1), \\ S_t(K) &= K^{-1}, \\ \boldsymbol{\alpha} &= \mathbf{1}, \\ \boldsymbol{\beta}_t &= \boldsymbol{e}_0 + K^{-t}\boldsymbol{e}_1 \ . \end{aligned}$$

From these data one computes via (2.1.45) that the Drinfeld twist and its inverse are

$$\boldsymbol{f}^{\pm 1} = \boldsymbol{e}_0 \otimes \mathbf{1} + \boldsymbol{e}_1 \otimes \boldsymbol{e}_0 K^{\mp t} + \boldsymbol{e}_1 K^{\pm t} \otimes \boldsymbol{e}_1 \tag{4.4.16}$$

This quasi-Hopf algebra is pivotal, and the pivot we choose is

$$\boldsymbol{g}_t = \boldsymbol{e}_0 K - \boldsymbol{e}_1 K^{t+1} \ . \tag{4.4.17}$$

Set $X = \sum_{i=0}^{2p-1} K^i$. Then one can see that $c^r = F^{p-1}X$ and $c^l = XF^{p-1}$ are a right and a left integral for U^-, respectively. From

$$\begin{aligned} c^l F = 0 \ , \quad c^l K = q^{-2} c^l \ , \\ F c^r = 0 \ , \quad K c^r = q^2 c^r \end{aligned} \tag{4.4.18}$$

we can see that U^- is non-unimodular. The modulus is

$$\boldsymbol{\gamma}(F) = 0 \ , \quad \boldsymbol{\gamma}(K) = q^{-2} \ . \tag{4.4.19}$$

The order of $\boldsymbol{\gamma}$ is p, and in particular $\boldsymbol{\gamma} \neq \boldsymbol{\gamma}^{-1}$ if $p > 2$.

Using (4.2.10) one verifies that the space of right monadic cointegrals is

$$\textstyle\int_{\mathcal{C}}^{r,\mathrm{mon}} = \mathbb{C} B^*_{p-1,0} \ . \tag{4.4.20}$$

The left monadic cointegral can then be found via the isomorphism from Proposition 4.2.2. Normalizing the result, we obtain

$$\boldsymbol{\lambda}^{l,\mathrm{mon}} = (1 + q^{-t(p-1)}) B^*_{p-1,p-1} + (1 - q^{-t(p-1)}) B^*_{p-1,2p-1} \ . \tag{4.4.21}$$

Also from Proposition 4.2.2 we obtain the right D-symmetrized monadic cointegral

$$\boldsymbol{\lambda}^{r,D-\mathrm{sym}} = B^*_{p-1,p-1} + B^*_{p-1,2p-1} + q^{2t}\Big(B^*_{p-1,p-t-1} - B^*_{p-1,2p-t-1}\Big) \tag{4.4.22}$$

and the left D-symmetrized monadic cointegral

$$\boldsymbol{\lambda}^{l,D-\mathrm{sym}} = B^*_{p-1,0} + B^*_{p-1,p} + q^{-t(p+1)}\Big(B^*_{p-1,t} - B^*_{p-1,p+t}\Big)\ . \tag{4.4.23}$$

The 'classical' cointegrals (in the sense of [HN2]) of U^- can now be obtained using Theorem 4.3.1. The right and the left cointegral are found to be

$$\boldsymbol{\lambda}^{r} = B^*_{p-1,0} + B^*_{p-1,p} + B^*_{p-1,t} - B^*_{p-1,p+t} \tag{4.4.24}$$

and

$$\boldsymbol{\lambda}^{l} = B^*_{p-1,p-1} + B^*_{p-1,2p-1} + q^{-t(p-1)}\Big(B^*_{p-1,p-t-1} - B^*_{p-1,2p-t-1}\Big)\ , \tag{4.4.25}$$

respectively. Finally, we give the $\boldsymbol{\gamma}$-symmetrized cointegrals of U^- using the characterization (3.1.1). The right $\boldsymbol{\gamma}$-symmetrized cointegral is given by

$$\widehat{\boldsymbol{\lambda}}^{r} = B^*_{p-1,p-1}\ , \tag{4.4.26}$$

and the left $\boldsymbol{\gamma}$-symmetrized cointegral is

$$\widehat{\boldsymbol{\lambda}}^{l} = (1+q^{-t(p-1)})B^*_{p-1,0} + (1-q^{-t(p-1)})B^*_{p-1,p}\ . \tag{4.4.27}$$

For $p=2$ and $t=1$, U^- is, as a quasi-Hopf algebra, isomorphic to

$$H(N=1,\beta=\exp(i\tfrac{\pi}{4})) \tag{4.4.28}$$

from the previous example, by mapping generators according to

$$F \mapsto i\mathsf{f}, \quad K \mapsto \mathsf{K}, \tag{4.4.29}$$

cf. [CGR, Rem. 4.3(2)]. Note that, under this isomorphism, the cointegrals of U^- agree with those of $H(N=1,\beta=\exp(i\frac{\pi}{4}))$. △

4.5 Cointegrals for the coend in the braided case

In a braided category $\mathcal{C}$ there exist notions of integrals and cointegrals for Hopf algebras internal to $\mathcal{C}$. If $\mathcal{C}$ is in addition finite tensor, then the coend $\mathcal{L} = \int^{X\in\mathcal{C}} X^\vee \otimes X$ is an example of such a Hopf algebra, recall Example 1.2.7, and see [LM, Ly1, FGR1]. In this section we relate left and right integrals for $\mathcal{L}$ to right monadic cointegrals and consider quasi-triangular quasi-Hopf algebras as an example.

Fix a braided finite tensor category $\mathcal{C}$.

4.5.1 Integrals and cointegrals for Hopf algebras in $\mathcal{C}$

Let A be a Hopf algebra in $\mathcal{C}$ with invertible antipode, see e.g. [KL] or [FGR1, Sec. 2.2]. Then the notions of (left/right) integrals and (left/right) cointegrals are well-defined, see [KL, Prop. 4.2.4]. We repeat the definition of a left integral for A. It consists of an invertible object $\operatorname{Int} A$, the *object of integrals*, and a morphism $\Lambda_A : \operatorname{Int} A \to A$ making the diagram

$$\begin{tikzcd} A \otimes \operatorname{Int} A \arrow[rr, "\mathrm{id} \otimes \Lambda_A"] \arrow[d, "\varepsilon \otimes \mathrm{id}"'] & & A \otimes A \arrow[dd, "m"] \\ \mathbf{1} \otimes \operatorname{Int} A \arrow[d, "\sim"'] & & \\ \operatorname{Int} A \arrow[rr, "\Lambda_A"'] & & A \end{tikzcd} \tag{4.5.1}$$

commute. Here, m and ε are multiplication and counit of the Hopf algebra A, respectively. Right integrals are defined similarly (with the same object $\operatorname{Int} A$). It is known that non-zero (left/right) integrals Λ_A exists and are uniquely determined up to scalar [KL, Prop. 4.2.4]. Note that the above diagram is just the statement that a left integral for A is a morphism $\Lambda_A : \operatorname{Int} A \to A$ of left A-modules, where the A-actions are given be the left and right side of the diagram (4.5.1), respectively.

As remarked in [BV1, Ex. 3.10], tensoring with a Hopf algebra with invertible antipode in a braided category yields a Hopf monad. The category of modules over the Hopf algebra is then the same as the category of modules over the corresponding Hopf monad.

4.5.2 Central Hopf monad via the coend in the braided case

In the braided setting, the coend $\mathcal{L} = \int^{X \in \mathcal{C}} X^\vee \otimes X$ with universal dinatural transformation j becomes a Hopf algebra [LM, Ly1], see also [FS, FGR1] for a review. It is easy to see that A_2 is in fact isomorphic to the Hopf monad obtained by tensoring with $\mathcal{L}$. The isomorphism $\xi_V : A_2(V) \to \mathcal{L} \otimes V$ we choose is obtained via

$$\xi_V \circ \iota_2(V)_X = \left[X^\vee \,(V X) \xrightarrow{\mathrm{id} \otimes c^{-1}_{X,V}} X^\vee \,(X V) \xrightarrow{\sim} (X^\vee X)\, V \xrightarrow{j_X \otimes \mathrm{id}} \mathcal{L} V \right]. \tag{4.5.2}$$

The inverse of the braiding (as opposed to the braiding) appears to make ξ an isomorphism of bimonads, with the bimonad structure on $\mathcal{L}\otimes ?$ inherited from the bialgebra structure on $\mathcal{L}$ used in [FGR1, Sec. 3.3]. In the same way, $? \otimes \mathcal{L}$ becomes a bimonad and we get a bimonad isomorphism $\zeta : A_2 \Rightarrow (? \otimes \mathcal{L})$ via

$$\begin{aligned} \zeta_V \circ \iota_2(V)_X = \Big[X^\vee \,(V X) &\xrightarrow{\sim} (X^\vee V)\, X \xrightarrow{c^{-1}_{V,X^\vee} \otimes \mathrm{id}} (V X^\vee)\, X \\ &\xrightarrow{\sim} V\,(X^\vee X) \xrightarrow{\mathrm{id} \otimes j_X} V \mathcal{L} \Big] . \end{aligned} \tag{4.5.3}$$

Again the inverse braiding is required to make ζ a bimonad morphism.

In a finite tensor category, an invertible object has isomorphic left and right duals, and so in particular there is an up to scalars unique isomorphism $D^\vee \xrightarrow{\sim} {}^\vee D$. This fact is used in formulating the following proposition.

Proposition 4.5.1. *Let $\mathcal{C}$ be a braided finite tensor category.*

1. *The distinguished invertible object is dual to the object of integrals for $\mathcal{L}$,*

$$D \cong (\operatorname{Int}\mathcal{L})^\vee. \tag{4.5.4}$$

2. *Let $\Lambda_{\mathcal{L}} : {}^\vee D \to \mathcal{L}$ be non-zero. Then*

$$\Lambda_{\mathcal{L}} \text{ is a } \begin{cases} \text{left integral for } \mathcal{L} \text{ in the sense of (4.5.1), resp.} \\ \text{right integral for } \mathcal{L} \end{cases}$$

if and only if

$$\boldsymbol{\lambda} := \begin{cases} \left[\mathbf{1} \xrightarrow{\widetilde{\operatorname{coev}}_D} {}^\vee D \otimes D \xrightarrow{\Lambda_{\mathcal{L}}\otimes \mathrm{id}} \mathcal{L} \otimes D \xrightarrow{\xi_D^{-1}} A_2\left(D\right)\right] \text{, resp.} \\ \left[\mathbf{1} \xrightarrow{\operatorname{coev}_D} D \otimes D^\vee \xrightarrow{\sim} D \otimes {}^\vee D \xrightarrow{\mathrm{id}\otimes\Lambda_{\mathcal{L}}} D \otimes \mathcal{L} \xrightarrow{\zeta_D^{-1}} A_2\left(D\right)\right] \end{cases} \tag{4.5.5}$$

is a non-zero right monadic cointegral of $\mathcal{C}$.

The first statement was already observed in [Sh1, Thm. 6.8].

Proof. We will only treat the case of left integrals for $\mathcal{L}$ explicitly, the case of right integrals can be shown analogously.

Since left integrals for $\mathcal{L}$ exist, there is an object $\operatorname{Int}\mathcal{L}$ and a non-zero morphism $\Lambda_{\mathcal{L}} : X \to \mathcal{L}$ such that (4.5.1) is satisfied. Let us abbreviate $X = \operatorname{Int}\mathcal{L}$. We now define $\boldsymbol{\lambda}$ as in part (2), but with X instead of ${}^\vee D$:

$$\boldsymbol{\lambda} := \left[\mathbf{1} \xrightarrow{\operatorname{coev}_X} X \otimes X^\vee \xrightarrow{\Lambda_{\mathcal{L}}\otimes \mathrm{id}} \mathcal{L} \otimes X^\vee \xrightarrow{\xi_{X^\vee}^{-1}} A_2\left(X^\vee\right)\right]. \tag{4.5.6}$$

Note that $\boldsymbol{\lambda}$ is non-zero, too.

The somewhat lengthy computation below will establish that $\boldsymbol{\lambda}$ from (4.5.6) is an A_2-intertwiner. By [Sh4, Lem. 4.1] and Corollary 4.1.11 the distinguished invertible object D is the unique (up to isomorphism) invertible object such that the space of A_2-intertwiners from $\mathbf{1}$ to $A_2\left(D\right)$ is non-zero. Thus we must have $X^\vee \cong D$, proving part (1). Together with part (1), the fact that $\boldsymbol{\lambda}$ is an A_2-intertwiner implies that it is a right monadic cointegral, proving the direction $\Rightarrow$ of part (2). The direction $\Leftarrow$ of part (2) can be verified by an analogous computation, where a right monadic cointegral $\boldsymbol{\lambda}$ gets mapped to a left integral of $\mathcal{L}$ via

$$\Lambda_{\mathcal{L}} := \left[{}^\vee D \xrightarrow{\sim} \mathbf{1} \otimes {}^\vee D \xrightarrow{\boldsymbol{\lambda}\otimes\mathrm{id}} A_2\left(D\right) \otimes {}^\vee D \xrightarrow{\xi_D\otimes\mathrm{id}} (\mathcal{L}D){}^\vee D \xrightarrow{\sim} \mathcal{L}(D{}^\vee D)\right.$$

$$\xrightarrow{\mathrm{id}\otimes\widetilde{\mathrm{ev}}_D} \mathcal{L}\otimes\mathbf{1} \xrightarrow{\sim} \mathcal{L}\Big]. \tag{4.5.7}$$

Note that this assignment is indeed inverse to (4.5.5).

Let us now turn to the verification that $\boldsymbol{\lambda}$ in (4.5.6) is indeed an A_2-intertwiner. Note that since ξ is an isomorphism of bimonads, it satisfies

$$\begin{aligned}&\Big[\mathcal{L}(\mathcal{L}V) \xrightarrow{\sim} (\mathcal{L}\mathcal{L})V \xrightarrow{m\otimes\mathrm{id}_V} \mathcal{L}V \xrightarrow{\xi_V^{-1}} A_2\,(V)\Big]\\ &= \Big[\mathcal{L}(\mathcal{L}V) \xrightarrow{\xi_{\mathcal{L}V}^{-1}} A_2\,(\mathcal{L}V) \xrightarrow{A_2(\xi_V^{-1})} (A_2)^2(V) \xrightarrow{\mu_2(V)} A_2\,(V)\Big],\end{aligned} \tag{4.5.8}$$

and

$$\Big[\mathcal{L}\mathbf{1} \xrightarrow{\xi_{\mathbf{1}}^{-1}} A_2\,(\mathbf{1}) \xrightarrow{\epsilon_2} \mathbf{1}\Big] = \Big[\mathcal{L}\mathbf{1} \xrightarrow{\varepsilon\otimes\mathrm{id}_{\mathbf{1}}} \mathbf{1}\mathbf{1} \xrightarrow{\sim} \mathbf{1}\Big], \tag{4.5.9}$$

where m and ε are the multiplication and the counit of $\mathcal{L}$.

For the next calculation, let us explicitly denote components of the left unitor and the associator by

$$l_V : \mathbf{1}V \to V \quad \text{and} \quad \alpha_{U,V,W} : U(VW) \to (UV)W, \tag{4.5.10}$$

respectively, for $U, V, W \in \mathcal{C}$. Then

$$\begin{aligned}
&\mu_2(X^\vee) \circ A_2\,(\boldsymbol{\lambda})\\
&\overset{(4.5.6)}{=} \mu_2(X^\vee) \circ A_2\Big(\xi_{X^\vee}^{-1} \circ (\Lambda_{\mathcal{L}} \otimes \mathrm{id}_{X^\vee}) \circ \mathrm{coev}_X\Big)\\
&\overset{(4.5.8)}{=} \xi_{X^\vee}^{-1} \circ (m \otimes \mathrm{id}_{X^\vee}) \circ \alpha_{\mathcal{L},\mathcal{L},X^\vee} \circ \xi_{\mathcal{L}\otimes X^\vee} \circ A_2\Big((\Lambda_{\mathcal{L}} \otimes \mathrm{id}_{X^\vee}) \circ \mathrm{coev}_X\Big)\\
&\overset{\xi\,\text{nat.}}{=} \xi_{X^\vee}^{-1} \circ (m \otimes \mathrm{id}_{X^\vee}) \circ \alpha_{\mathcal{L},\mathcal{L},X^\vee} \circ (\mathrm{id}_{\mathcal{L}} \otimes (\Lambda_{\mathcal{L}} \otimes \mathrm{id}_{X^\vee})) \circ (\mathrm{id}_{\mathcal{L}} \otimes \mathrm{coev}_X) \circ \xi_{\mathbf{1}}\\
&\overset{\alpha\,\text{nat.}}{=} \xi_{X^\vee}^{-1} \circ \Big((m \circ (\mathrm{id}_{\mathcal{L}} \otimes \Lambda_{\mathcal{L}})) \otimes \mathrm{id}_{X^\vee}\Big) \circ \alpha_{\mathcal{L},X,X^\vee} \circ (\mathrm{id}_{\mathcal{L}} \otimes \mathrm{coev}_X) \circ \xi_{\mathbf{1}}\\
&\overset{(4.5.1)}{=} \xi_{X^\vee}^{-1} \circ \Big((\Lambda_{\mathcal{L}} \circ l_X \circ (\varepsilon \otimes \mathrm{id}_X)) \otimes \mathrm{id}_{X^\vee}\Big) \circ \alpha_{\mathcal{L},X,X^\vee} \circ (\mathrm{id}_{\mathcal{L}} \otimes \mathrm{coev}_X) \circ \xi_{\mathbf{1}}\\
&\overset{\alpha\,\text{nat.}}{=} \xi_{X^\vee}^{-1} \circ \Big((\Lambda_{\mathcal{L}} \circ l_X) \otimes \mathrm{id}_{X^\vee}\Big) \circ \alpha_{\mathbf{1},X,X^\vee} \circ (\mathrm{id}_{\mathbf{1}} \otimes \mathrm{coev}_X) \circ (\varepsilon \otimes \mathrm{id}_{\mathbf{1}}) \circ \xi_{\mathbf{1}}\\
&\overset{\text{coher.}}{=} \xi_{X^\vee}^{-1} \circ (\Lambda_{\mathcal{L}} \otimes \mathrm{id}_{X^\vee}) \circ l_{XX^\vee} \circ (\mathrm{id}_{\mathbf{1}} \otimes \mathrm{coev}_X) \circ (\varepsilon \otimes \mathrm{id}_{\mathbf{1}}) \circ \xi_{\mathbf{1}}\\
&\overset{l\,\text{nat.}}{=} \xi_{X^\vee}^{-1} \circ (\Lambda_{\mathcal{L}} \otimes \mathrm{id}_{X^\vee}) \circ \mathrm{coev}_X \circ l_{\mathbf{1}} \circ (\varepsilon \otimes \mathrm{id}_{\mathbf{1}}) \circ \xi_{\mathbf{1}}\\
&\overset{(4.5.9)}{=} \xi_{X^\vee}^{-1} \circ (\Lambda_{\mathcal{L}} \otimes \mathrm{id}_{X^\vee}) \circ \mathrm{coev}_X \circ \epsilon_2\\
&\overset{(4.5.6)}{=} \boldsymbol{\lambda} \circ \epsilon_2,
\end{aligned} \tag{4.5.11}$$

where "coher." invokes the coherence theorems. Thus $\boldsymbol{\lambda}$ is an A_2-intertwiner. □

By composing with $\kappa_{3,2}$ one obtains analogous statements to those in the above proposition for left monadic cointegrals.

Remark 4.5.2. Let $\mathcal{C}$ be a unimodular braided finite tensor category. Then $D = \mathbf{1}$, and by Proposition 4.5.1, the object of integrals of $\mathcal{L}$ is the tensor unit. It follows from the coherence of braided monoidal categories that

$$\xi_{\mathbf{1}} = \left[A_2(\mathbf{1}) \xrightarrow{\zeta_{\mathbf{1}}} \mathbf{1}\mathcal{L} \xrightarrow{\sim} \mathcal{L}\mathbf{1}\right] \tag{4.5.12}$$

for the bimonad isomorphisms ξ and ζ from (4.5.2) and (4.5.3). Now one can show that left and right integrals for $\mathcal{L}$ agree. Indeed, the composition

$$\{\text{right } \mathcal{L}\text{-integrals}\} \xrightarrow{(4.5.5)} \mathcal{C}_{A_2}(\mathbf{1}, A_2(\mathbf{1})) \xrightarrow{(4.5.7)} \{\text{left } \mathcal{L}\text{-integrals}\} \tag{4.5.13}$$

of isomorphisms is (non-trivially) proportional to the identity on the one-dimensional subspace of right $\mathcal{L}$-integrals of $\mathcal{C}(\mathbf{1}, \mathcal{L})$. Therefore, a right integral for $\mathcal{L}$ is also left and vice versa. This result has also been shown by different means in [Sh1, Thm. 6.9]. ▽

Remark 4.5.3. By the first item in Proposition 4.5.1, the distinguished invertible object in the prevailing convention of e.g. [EGNO] is precisely Lyubashenko's object of integrals. ▽

4.5.3 Quasi-triangular quasi-Hopf algebras

Let H be a finite-dimensional quasi-triangular quasi-Hopf algebra with universal R-matrix R; recall our brief discussion of what this means from Section 2.2.2. We again denote the multiplicative inverse of the R-matrix by $\overline{R}$. The category $\mathcal{C} = {}_H\mathcal{M}$ is a braided finite tensor category.

The coend $\mathcal{L}$ can be realized by H^* with the coadjoint action, see [FGR1, Sec. 7], and with our realization of the Hopf monad A_2 as in Section 4.2.1 we get the following formula for the Hopf monad isomorphism $A_2 \cong \mathcal{L}\otimes ?$ from (4.5.2).

Lemma 4.5.4. *The isomorphism $\xi_V : A_2(V) \to \mathcal{L} \otimes V$ from (4.5.2) is given by*

$$\xi_V(f \otimes v) = \langle f \mid S(X_1)?X_2\overline{R}_1\rangle \otimes X_3\overline{R}_2.v \tag{4.5.14}$$

for $V \in \mathcal{C}$, $f \in H^*$, $v \in V$.

The proof is a straightforward computation.

Next, we give the explicit formulas relating right monadic cointegrals and left integrals for the coend. As usual, we identify linear maps $k \to V$ with elements in V.

Lemma 4.5.5. *Let $\boldsymbol{\lambda} \in H^*$ be a right monadic cointegral. Then*

$$\Lambda_{\mathcal{L}} := \boldsymbol{\gamma}^{-1}(q_2^R X_3\overline{R}_2)\langle \boldsymbol{\lambda} \mid S(q^R_{1\,(1)}X_1)?q^R_{1\,(2)}X_2\overline{R}_1\rangle \tag{4.5.15}$$

is a left integral for the coend $\mathcal{L}$.

The proof amounts to evaluating (4.5.7) in ${}_H\mathcal{M}$ using Lemma 4.5.4. We arrive at the following corollary.

Corollary 4.5.6. *Let $\boldsymbol{\lambda}^r \in H^*$ be a right cointegral for the quasi-Hopf algebra H. Then*

$$\Lambda_{\mathcal{L}} := \boldsymbol{\gamma}^{-1}(q_2^R X_3 \overline{R}_2 S^{-1}(\boldsymbol{f}_2^{-1}))\langle \boldsymbol{\lambda}^r \mid S(q^R_{1\,(1)} X_1 \boldsymbol{\beta})?q^R_{1\,(2)} X_2 \overline{R}_1 S^{-1}(\boldsymbol{f}_1^{-1})\rangle \tag{4.5.16}$$

is a left integral for the coend $\mathcal{L}$.

Proof. Combining Theorem 4.3.1 with the previous lemma yields the formula. □

Using Proposition 4.5.1 (2), one can also write formulas similar to (4.5.15) or (4.5.16) for the relation between right integrals for $\mathcal{L}$ and right monadic cointegrals or right cointegrals for H. We will skip the details.

Remark 4.5.7. Let H be unimodular. Observe that then $D = \mathbf{1}$ and $A_2(\mathbf{1}) = \mathcal{L}$ as H-modules (in our preferred realizations). The relationship (4.5.15) between integrals for $\mathcal{L}$ and right monadic cointegrals is now particularly simple:

$$\Lambda_{\mathcal{L}} = \boldsymbol{\lambda}\ . \tag{4.5.17}$$

By Remark 4.5.2, left and right integrals for $\mathcal{L}$ coincide, and so (4.5.17) says that the right monadic cointegral and the left/right integral for $\mathcal{L}$ are given by the same linear form on H. The relation to the right cointegral $\boldsymbol{\lambda}^r$ of H also simplifies: $\Lambda_{\mathcal{L}} = \langle \boldsymbol{\lambda}^r \mid S(\boldsymbol{\beta})?\rangle$. ▽

Chapter 5

$SL(2,\mathbb{Z})$-action for modular tensor categories

In the first two sections of this chapter, we start by recalling the (projective) $SL(2,\mathbb{Z})$-action(s) constructed by Lyubashenko for every *modular* tensor category. Then we apply results from Chapter 4 to simplify some previous results from [FGR1] on these actions in the case $\mathcal{C} = {}_H\mathcal{M}$, for H a factorizable ribbon quasi-Hopf algebra. This is, for the most part, taken almost verbatim from my paper [BGR2] with Gainutdinov and Runkel.

The last section contains unpublished work. On the one hand, the restricted quantum group $H = \overline{U}_q(\mathfrak{sl}_2)$, where q is a primitive $2p$th root of unity, is known to not admit a braiding. In particular, ${}_H\mathcal{M}$ is not a modular tensor category, and so Lyubashenko's construction of the (projective) $SL(2,\mathbb{Z})$-action from above does not formally apply. Nevertheless, in [FGST1], an action on $Z(H)$ was constructed, and it was shown there that this action agrees with one coming from modular transformation properties of characters of the triplet VOA $\mathcal{W}(p)$. On the other hand, a quasi-Hopf algebra modification $H^\Phi = \overline{U}_q^\Phi(\mathfrak{sl}_2)$ of H was introduced in [CGR], and it turns out that H^Φ *is* a factorizable ribbon quasi-Hopf algebra. We show that Lyubashenko's action for ${}_{H^\Phi}\mathcal{M}$ agrees projectively with that from [FGST1].

5.1 $SL(2,\mathbb{Z})$-action for modular tensor categories

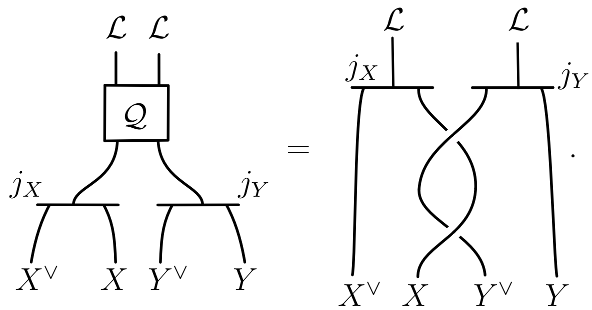
We briefly review Lyubashenko's construction [Ly1]. For a more detailed review, we refer the reader to [FGR1].

Recall from Example 1.2.7 that in a braided finite tensor category $\mathcal{C}$, the Hopf algebra $\mathcal{L} = \int^X X^\vee \otimes X$ admits a Hopf pairing

$$\omega\colon \mathcal{L} \otimes \mathcal{L} \to \mathbf{1}\ , \tag{5.1.1}$$

see [Ly1] or [FGR1, Sec. 3.3] for details. Recall from Section 1.1.10 the definitions of factorizable and modular tensor categories, and let for the rest of this section $\mathcal{C}$ be a modular tensor category with ribbon twist ϑ. Since $\mathcal{C}$ is factorizable it is in particular unimodular [KL, Lem. 5.2.8], and the Hopf algebra $\mathcal{L}$ has a two-sided integral $\Lambda_{\mathcal{L}} \colon \mathbf{1} \to \mathcal{L}$ by Remark 4.5.2, see also [KL, Cor. 5.2.11].

Define the morphism $\mathcal{Q} \colon \mathcal{L} \otimes \mathcal{L} \to \mathcal{L} \otimes \mathcal{L}$ by



(5.1.2)

This is related to the Hopf pairing ω via $\omega = (\varepsilon \otimes \varepsilon) \circ \mathcal{Q}$, denoting by ε the counit of $\mathcal{L}$. Next, define $\mathcal{S}, \mathcal{T} \in \operatorname{End}_{\mathcal{C}}(\mathcal{L})$ by

$$\mathcal{S} = (\varepsilon \otimes \mathrm{id}) \circ \mathcal{Q} \circ (\mathrm{id} \otimes \Lambda_{\mathcal{L}}) \quad \text{and} \quad \mathcal{T} \circ j_X = j_X \circ (\mathrm{id} \otimes \vartheta_X) \ . \tag{5.1.3}$$

These endomorphisms satisfy

$$(\mathcal{S}\mathcal{T})^3 = \lambda \mathcal{S}^2 = \lambda S_{\mathcal{L}}^{-1} \tag{5.1.4}$$

where λ is a non-zero constant and $S_{\mathcal{L}}$ is the antipode of $\mathcal{L}$ [Ly1], see also [FGR1, Sec. 3.3] for the Hopf algebra structure on $\mathcal{L}$. Recall that the special linear group $SL(2,\mathbb{Z})$ over $\mathbb{Z}$ consists of (2×2)-matrices with unit determinant. It is generated by $\mathbf{S} = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ and $\mathbf{T} = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$, which satisfy

$$(\mathbf{S}\mathbf{T})^3 = \mathbf{S}^2 \quad \text{and} \quad \mathbf{S}^4 = \mathrm{id} \, . \tag{5.1.5}$$

It was shown in [Ly1] that the k-vector space $\mathcal{C}(\mathbf{1}, \mathcal{L})$ carries a projective $SL(2,\mathbb{Z})$-action given by

$$\mathbf{S}.f = \mathcal{S} \circ f \quad \text{and} \quad \mathbf{T}.f = \mathcal{T} \circ f \tag{5.1.6}$$

for $f \colon \mathbf{1} \to \mathcal{L}$. Indeed, this follows easily from (5.1.4) and the fact that $S_{\mathcal{L}}^2 = \vartheta_{\mathcal{L}}$, see [FGR1, Sec. 5] for more details. Alternatively, there is also a projective action on $\mathcal{C}(\mathcal{L}, \mathbf{1})$, given by

$$\mathbf{S}.f = f \circ \mathcal{S} \quad \text{and} \quad \mathbf{T}.f = f \circ \mathcal{T} \tag{5.1.7}$$

for $f \colon \mathcal{L} \to \mathbf{1}$. The formulas (5.1.7) are obtained by transporting the action on $\mathcal{C}(\mathbf{1}, \mathcal{L})$ along a certain isomorphism, which we will briefly touch on in the next paragraph.

In order to prepare for the main result of this chapter, we recall a third $SL(2,\mathbb{Z})$-action associated to $\mathcal{C}$. Since we assumed $\mathcal{C}$ to be modular, there are linear isomorphisms

$$\begin{tikzcd} \operatorname{End}(\mathrm{id}_C) \arrow[r, "\psi"] & \mathcal{C}(\mathcal{L}, \mathbf{1}) \arrow[r, bend left, "\rho"] & \mathcal{C}(\mathbf{1}, \mathcal{L}) \arrow[l, bend left, "\Omega"] \end{tikzcd} \quad , \tag{5.1.8}$$

see [GR3, Sec. 2]. In fact, ψ and Ω are algebra isomorphisms, where the algebra structure on the morphism spaces comes from the bialgebra structure on $\mathcal{L}$. We will not need their precise definitions. One can show

$$(\rho \circ \Omega)(f) = \mathcal{S} \circ f \quad \text{and} \quad (\Omega \circ \rho)(g) = g \circ \mathcal{S} \tag{5.1.9}$$

for $f\colon \mathbf{1} \to \mathcal{L}$ and $g\colon \mathcal{L} \to \mathbf{1}$. Moreover, one can show that the action on $\mathcal{C}(\mathcal{L},\mathbf{1})$ described above is obtained from the one on $\mathcal{C}(\mathbf{1},\mathcal{L})$ by transport via ρ. Similarly, we can pull back the projective $SL(2,\mathbb{Z})$-action on $\mathcal{C}(\mathcal{L},\mathbf{1})$ to $\mathrm{End}(\mathrm{id}_{\mathcal{C}})$ along ψ. This action depends on the choice of coend $\mathcal{L}$ and integral Λ only up to a choice of sign of Λ, cf. [FGR1, Prop. 5.3]. Let us denote the linear endomorphisms of $\mathrm{End}(\mathrm{id}_{\mathcal{C}})$ implementing the action of the generators by

$$\mathcal{S}_{\mathcal{C}}(\nu) = \mathbf{S}.\nu \quad \text{and} \quad \mathcal{T}_{\mathcal{C}}(\nu) = \mathbf{T}.\nu \tag{5.1.10}$$

for a natural transformation ν.

Finally, we recall *internal characters*, see [FS, Sh2]. The internal character of an object $V \in \mathcal{C}$ is the morphism

$$\chi_V = \left[\mathbf{1} \xrightarrow{\widetilde{\mathrm{coev}}_V} {}^{\vee}V \otimes V \cong V^{\vee} \otimes V \xrightarrow{j_V} \mathcal{L}\right] , \tag{5.1.11}$$

where we used the natural isomorphism ${}^{\vee}V \cong V^{\vee}$ induced by the pivotal structure. Transporting the internal character to $\mathrm{End}(\mathrm{id}_{\mathcal{C}})$, we set

$$\phi_V = (\rho \circ \psi)^{-1}(\chi_V) . \tag{5.1.12}$$

Later we will need the following properties of the ϕ's, see [Sh2, FGR1, GR3] and references therein.

Proposition 5.1.1. *1. The set $\{\phi_U \mid U \in \mathrm{Irr}\,\mathcal{C}\}$ is linearly independent.*

2. *Let $V \in \mathcal{C}$. Then $\mathcal{S}_{\mathcal{C}}^2(\phi_V) = \phi_{V^\vee}$.*

3. *The assignments $V \mapsto \phi_V$ and $V \mapsto \mathcal{S}_{\mathcal{C}}(\phi_V)$ for $V \in \mathcal{C}$ factor through the Grothendieck ring $\mathrm{Gr}(\mathcal{C})$.[31] The induced linear maps $\mathrm{Gr}_k(\mathcal{C}) \to \mathrm{End}(\mathrm{id}_{\mathcal{C}})$, where $\mathrm{Gr}_k(\mathcal{C})$ is the linearized Grothendieck ring, are injective. Moreover, for $V \mapsto \mathcal{S}_{\mathcal{C}}(\phi_V)$, the induced linear map is an algebra homomorphism, and if k is of characteristic zero, the corresponding ring homomorphism $\mathrm{Gr}(\mathcal{C}) \to \mathrm{End}(\mathrm{id}_{\mathcal{C}})$ is injective.*

[31] Recall from e.g. [EGNO, Ch. 1] that the *Grothendieck ring* $\mathrm{Gr}(\mathcal{C})$ of a finite tensor category $\mathcal{C}$ is the free $\mathbb{Z}$-module on the objects of $\mathcal{C}$, modulo $Y = X + Z$ if there is a short exact sequence $0 \to X \to Y \to Z \to 0$ in $\mathcal{C}$. Thus it is in fact generated by $\mathrm{Irr}\,\mathcal{C}$: every object X can be expressed as a weighted sum over simples, the weight being the multiplicity of the corresponding simple in the composition series of X. The multiplication is $V \cdot W = V \otimes W$. The *linearized* Grothendieck ring is $\mathrm{Gr}_k(\mathcal{C}) = \mathrm{Gr}(\mathcal{C}) \otimes_{\mathbb{Z}} k$.

5.2 The case of factorizable quasi-Hopf algebras

Let now H be a finite-dimensional factorizable ribbon quasi-Hopf algebra with ribbon element $\boldsymbol{v}$ and R-matrix R. As mentioned before, H being factorizable implies that it is unimodular. Thus, by Remark 4.5.7, the left integral for $\mathcal{L}$ and the right monadic cointegral of ${}_H\mathcal{M}$ are given by the same linear form on H.

The linear injection

$$\boldsymbol{\alpha} Z = \{\boldsymbol{\alpha} z \mid z \in Z(H)\} \to \mathcal{C}(\mathcal{L}, \mathbf{1}), \quad \boldsymbol{\alpha} z \mapsto \delta_{\mathsf{Vect}}(\boldsymbol{\alpha} z) = \langle ? \mid \boldsymbol{\alpha} z\rangle \ , \tag{5.2.1}$$

where δ_{Vect} is the canonical pivotal structure of vector spaces, can be shown to be an isomorphism.[32] We get an action $\mathcal{S}_{\boldsymbol{\alpha} Z}$ of the $\mathbf{S}$-generator on $\boldsymbol{\alpha} Z$ by setting

$$\mathcal{S}_{\boldsymbol{\alpha} Z}(h) := \delta_{\mathsf{Vect}}^{-1}\Big(\mathbf{S}.\delta_{\mathsf{Vect}}(h)\Big) = \delta_{\mathsf{Vect}}^{-1}(\langle ? \mid h\rangle \circ \mathcal{S}), \tag{5.2.2}$$

for $h \in \boldsymbol{\alpha} Z$. Since $\mathcal{S} \in \mathrm{End}_{\mathcal{C}}(\mathcal{L})$ and $\mathcal{L} \cong_k H^*$, we can define $\hat{\mathcal{S}} \in \mathrm{End}_k(H)$ via

$$\langle f \mid \hat{\mathcal{S}}(h)\rangle = \langle \mathcal{S}(f) \mid h\rangle \tag{5.2.3}$$

for all $h \in H$, $f \in H^*$, and it is then immediate that

$$\mathcal{S}_{\boldsymbol{\alpha} Z} = \hat{\mathcal{S}}\Big|_{\boldsymbol{\alpha} Z}. \tag{5.2.4}$$

Similar statements hold for $\mathbf{T}$.

In the following proposition, we will express the Hopf pairing ω from (5.1.1) via an element $\widehat{\omega} \in H \otimes H$ such that

$$\omega(f \otimes g) = g(\widehat{\omega}_1) f(\widehat{\omega}_2) \tag{5.2.5}$$

for all $f, g \in H^*$. Its explicit form is given in [FGR1, Thm. 7.3].

Proposition 5.2.1. *Let $\boldsymbol{\lambda} \in H^*$ be the right monadic cointegral for ${}_H\mathcal{M}$. The $\mathbf{S}$- and $\mathbf{T}$-transformations on $\boldsymbol{\alpha} Z$ are given by the linear maps*

$$\begin{aligned} \mathcal{S}_{\boldsymbol{\alpha} Z}(\boldsymbol{\alpha} z) &= \langle \boldsymbol{\lambda} \mid \widehat{\omega}_1 z\rangle \ \widehat{\omega}_2 \\ \mathcal{T}_{\boldsymbol{\alpha} Z}(\boldsymbol{\alpha} z) &= \boldsymbol{v}^{-1} \boldsymbol{\alpha} z \end{aligned} \tag{5.2.6}$$

for $z \in Z$.

Proof. The action of $\mathcal{T}$ is immediate from [FGR1, Sec. 8]. For the action of $\mathcal{S}$ we use (5.2.4) and compute

$$\hat{\mathcal{S}}(\boldsymbol{\alpha} z) \overset{\text{(i)}}{=} \langle \boldsymbol{\lambda} \mid S(X_1)\widehat{\omega}_1 X_2 S(X_{3(1)} p_1^L) \boldsymbol{\alpha} z X_{3(2)} p_2^L\rangle \ \widehat{\omega}_2$$

[32]By general facts about modular categories (see e.g. [GR3, Sec. 2], as reviewed above), the center of H is isomorphic to $\mathcal{C}(\mathcal{L}, \mathbf{1})$. As observed in [FGR1, Sec. 8], the image of this isomorphism is precisely $\delta_{\mathsf{Vect}}(\boldsymbol{\alpha} Z)$. Note also that $\mathcal{C}(\mathbf{1}, \mathcal{L}) \cong \boldsymbol{\beta} Z$.

$$\overset{\text{(ii)}}{=} \quad \langle \boldsymbol{\lambda} \mid \widehat{\omega}_1 S(p_1^L)\boldsymbol{\alpha} p_2^L z\rangle \; \widehat{\omega}_2$$

$$\overset{\text{(iii)}}{=} \quad \langle \boldsymbol{\lambda} \mid \widehat{\omega}_1 z\rangle \; \widehat{\omega}_2 \; , \tag{5.2.7}$$

where in the first step (i) we used the form of $\hat{\mathcal{S}}$ as given in [FGR1, (8.15)], (ii) uses (2.1.8) for the underlined part and that Φ is normalized, and (iii) follows from the definition of p^L in (2.1.33) and the zig-zag axiom (2.1.9). □

We can thus express the action of $\mathcal{S}$ on $\boldsymbol{\alpha} Z$ using the right cointegral $\boldsymbol{\lambda}^r$ from (2.3.20) and Theorem 4.3.1 as

$$\mathcal{S}_{\boldsymbol{\alpha} Z}(\boldsymbol{\alpha} z) = \langle \boldsymbol{\lambda}^r \mid S(\boldsymbol{\beta})\widehat{\omega}_1 z\rangle \; \widehat{\omega}_2 \; . \tag{5.2.8}$$

Remark 5.2.2. One can also show that

$$\mathcal{S}_{\boldsymbol{\alpha} Z}(\boldsymbol{\alpha} z) = \widehat{\omega}_1 \langle \boldsymbol{\lambda} \mid \widehat{\omega}_2 z\rangle \; , \tag{5.2.9}$$

where $\boldsymbol{\lambda}$ is the right monadic cointegral. To see this, one checks $\omega \circ (\vartheta_{\mathcal{L}} \otimes \mathrm{id}) = \omega \circ c^{-1}_{\mathcal{L},\mathcal{L}}$ using $\vartheta_{\mathcal{L}} = (S_{\mathcal{L}})^2$, see [KL, Lem. 5.2.4]. This then readily implies

$$\omega \circ (f \otimes \mathrm{id}) = \omega \circ (\mathrm{id} \otimes f) \tag{5.2.10}$$

for any $f \in \mathcal{C}(\mathbf{1}, \mathcal{L})$, since $\vartheta_{\mathbf{1}} = 1$. The claim follows because $\langle \boldsymbol{\lambda} \mid ?z\rangle \in \mathcal{C}(\mathbf{1}, \mathcal{L})$ for z central. ▽

Finally, we specialize internal characters and the **S**-action on them to this setting, following [FGR1, Sec. 7.6]. Note that

$$\mathrm{End}(\mathrm{id}_{\mathcal{C}}) \to Z(H), \quad \nu \mapsto \nu_H(\mathbf{1}) \quad \text{and} \quad Z(H) \to \mathrm{End}(\mathrm{id}_{\mathcal{C}}), \quad z \mapsto z.? \tag{5.2.11}$$

are mutually inverse algebra maps. Recall that $\mathrm{End}(\mathrm{id}_{\mathcal{C}})$ carries a projective $SL(2,\mathbb{Z})$-action. We denote the induced action on $Z(H)$ by $\mathcal{S}_Z$. Define the central elements

$$\boldsymbol{\chi}_V = \mathcal{S}_{\mathcal{C}}(\phi_V)_H(\mathbf{1}) \quad \text{and} \quad \boldsymbol{\phi}_V = (\phi_V)_H(\mathbf{1}) \; . \tag{5.2.12}$$

Then, specializing Proposition 5.1.1, we have

$$\boldsymbol{\chi}_V = \mathcal{S}_Z(\boldsymbol{\phi}_V) \quad \text{and} \quad \mathcal{S}_Z^2(\boldsymbol{\phi}_V) = \boldsymbol{\phi}_{V^\vee} \; , \tag{5.2.13}$$

see [FGR1, Cor. 8.2].

Explicit expressions for these central elements were given in [FGR1, Sec. 7.6]. For later use, we simplify them in the following proposition.

Proposition 5.2.3. *Let $V \in \mathcal{C}$, and let c be the two-sided (non-zero) integral of $\mathcal{C}$ implementing the cointegral of $\mathcal{L}$, as in [FGR1, Prop. 7.8]. Then*

$$\boldsymbol{\chi}_V = (\mathrm{id} \otimes \mathrm{tr}_V \circ S)\Big(q^R M p^R \cdot (\mathbf{1} \otimes \boldsymbol{u}\boldsymbol{v}^{-1})\Big) \tag{5.2.14}$$

and

$$\boldsymbol{\phi}_V = (\mathrm{id} \otimes \mathrm{tr}_V \circ S)\Big(q^R \Delta(c) p^R \cdot (\mathbf{1} \otimes \boldsymbol{u}\boldsymbol{v}^{-1})\Big) \; , \tag{5.2.15}$$

where tr *is the trace in the category of vector spaces.*

Proof. The proof is elementary, and we only present the first part. We compute

$$\begin{aligned}\boldsymbol{\chi}_V &= x_1 M_1 X_1 \cdot \mathrm{tr}_V\Big(\boldsymbol{u}^{-1}\boldsymbol{v} S(x_2 M_2 X_2 \boldsymbol{\beta})\boldsymbol{\alpha} x_3 X_3\Big) \\ &= q_1^R M_1 X_1 \cdot \mathrm{tr}_V\Big(X_3 \boldsymbol{u}^{-1}\boldsymbol{v} S(q_2^R M_2 X_2 \boldsymbol{\beta})\Big) \\ &= q_1^R M_1 p_1^R \cdot \mathrm{tr}_V\Big(\boldsymbol{u}^{-1}\boldsymbol{v} S(q_2^R M_2 p_2^R)\Big)\ . \end{aligned} \tag{5.2.16}$$

The first line uses [FGR1, (7.42)], for the second line note that the trace is cyclic and recall the definition of q^R from (2.1.33). In the last line we use that $\boldsymbol{g} = \boldsymbol{u}\boldsymbol{v}^{-1}$ is a pivot, i.e. it satisfies $h\boldsymbol{g}^{-1} = \boldsymbol{g}^{-1}S^2(h)$ for all $h \in H$, and the definition of p^R from (2.1.33). The claim now follows from $S(\boldsymbol{g}) = \boldsymbol{g}^{-1}$.

The formula for $\boldsymbol{\phi}_V$ is obtained analogously, using [FGR1, (7.44)]. □

5.3 Comparing two specific $SL(2,\mathbb{Z})$-actions

Let now $\mathcal{C} = {}_{H^\Phi}\mathcal{M}$, where $H^\Phi = \overline{U}_q^{(\Phi)}\mathfrak{sl}(2)$ is the quasi-Hopf algebra from Section 2.4.3. In [FGST1], an $SL(2,\mathbb{Z})$-action on the center of the restricted quantum group $H = \overline{U}_q\mathfrak{sl}(2)$ at q a primitive $2p$th root of unity was constructed. We will show here that this action agrees with Lyubashenko's action on $\mathrm{End}(\mathrm{id}_\mathcal{C})$, which we reviewed previously in this chapter.

Below we will make use of q-numbers as recalled in Section 2.4.3, as well as the q-binomials

$$\begin{bmatrix} n \\ k \end{bmatrix} = \begin{cases} 0, & \text{if } k < 0 \text{ or } k > n \\ \frac{[n]!}{[k]!\cdot[n-k]!}, & \text{else.} \end{cases} \tag{5.3.1}$$

5.3.1 Simple modules

The underlying algebras of H and H^Φ are the same, and so their categories of representations coincide as linear abelian categories. In particular, the simple and projectives objects are the same. We now first recall the simple modules from [FGST1], and show that simple H^Φ-modules are self-dual.

Simple modules. We denote the irreducibles by $\mathcal{X}^\alpha(s)$, where $\alpha \in \{+,-\}$ is a sign and $1 \le s \le p$. The module $\mathcal{X}^\alpha(s)$ is highest-weight, and we write basis elements as $|s,n\rangle^\alpha$, where $0 \le n \le s-1$. Let us agree that $|s,s\rangle^\alpha = 0 = |s,-1\rangle^\alpha$. The H^Φ-action on $\mathcal{X}^\alpha(s)$ is

$$\begin{aligned} K|s,n\rangle^\alpha &= \alpha q^{s-1-2n}|s,n\rangle^\alpha \\ E|s,n\rangle^\alpha &= \alpha[n][s-n]|s,n-1\rangle^\alpha \\ F|s,n\rangle^\alpha &= |s,n+1\rangle^\alpha\ . \end{aligned} \tag{5.3.2}$$

The highest weight vector is $|s,0\rangle^\alpha$. By [FGST2, Thm. 1.7] this exhausts the simple modules. Note also that $\mathcal{X}^+(1)$ is the monoidal unit $\mathbf{1}$ in $\mathcal{C}$, and that $\dim_k \mathcal{X}^\alpha(s) = s$.

Self-duality of simple modules. Let us denote basis elements of the H^Φ-module $\mathcal{X}^\alpha(s) \otimes \mathcal{X}^\alpha(s)$ by $|s;n,m\rangle^\alpha$, for $0 \leq n,m \leq s-1$.

Lemma 5.3.1. *Let $\beta \in \{0,1\}$ such that $\alpha = (-1)^\beta$. The non-zero linear map*

$$\begin{aligned} &\mathbb{C} \to \mathcal{X}^\alpha(s) \otimes \mathcal{X}^\alpha(s) \\ &1 \mapsto \sum_{i=0}^{s-1} \left(\delta_{s+\beta p,\text{odd}} + \delta_{s+\beta p,\text{even}} q^{it}\right) (-1)^i \alpha^i q^{i(s-2-i)} |s;i,s-1-i\rangle^\alpha \end{aligned} \tag{5.3.3}$$

is an intertwiner of H^Φ-modules.

The proof, which is a calculation, can be found in Appendix C.5. As a consequence, we have

Corollary 5.3.2. *Simple H^Φ-modules are self-dual.*

Proof. By Lemma 5.3.1, the space of intertwiners from $(\mathcal{X}^\alpha(s))^\vee$ to $\mathcal{X}^\alpha(s)$ is non-zero. In a finite tensor category, the dual of a simple object is simple: short exact sequences $A \hookrightarrow U \twoheadrightarrow B$ are in one-to-one correspondence with short exact sequences $B^\vee \hookrightarrow U^\vee \twoheadrightarrow A^\vee$. Hence the result follows from Schur's lemma. □

5.3.2 The central elements $\boldsymbol{\phi}_V$ and $\boldsymbol{\chi}_V$

Recall now the elements $\boldsymbol{\chi}_V$ and $\boldsymbol{\phi}_V$ for $V \in \mathcal{C}$ from e.g. Proposition 5.2.3. The proofs of the following two lemmata can be found in Appendix C.4.

Lemma 5.3.3. *Let $V = \mathcal{X}^\alpha(s)$. The central element $\boldsymbol{\phi}_V \in Z(H^\Phi)$ agrees with the element $\widehat{\boldsymbol{\phi}}^\alpha(s)$ from [FGST1, (4.19)].*

Lemma 5.3.4. *Let $V = \mathcal{X}^\alpha(s)$. The central element $\boldsymbol{\chi}_V \in Z(H^\Phi)$ agrees with the element $\widehat{\boldsymbol{\chi}}^\alpha(s)$ from [FGST1, (4.6)].*

As a consequence, we get the main result of this chapter.

Theorem 5.3.5. *The two (projective) $SL(2,\mathbb{Z})$-actions on $Z(H)$—i.e. the one coming from Lyubashenko's construction, and the one constructed in [FGST1]—are projectively isomorphic.*

Proof. Combining Lemmas 5.3.3 and 5.3.4 with [FGST1, Thm. 5.2], we see that the center is spanned by $\{\boldsymbol{\phi}_V, \boldsymbol{\chi}_V\}_{V \in \mathrm{Irr}\,\mathcal{C}}$. Note that $\boldsymbol{\phi}_{V^\vee} = \boldsymbol{\phi}_V$, since $V \cong V^\vee$ by Corollary 5.3.2, and $\boldsymbol{\phi}_V$ depends only on the class of V in the Grothendieck ring (Proposition 5.1.1). Then, by (5.2.13), $\mathbf{S}$ acts on this spanning set via the endomorphism $\mathcal{S}_Z$ as

$$\mathcal{S}_Z(\boldsymbol{\phi}_V) = \boldsymbol{\chi}_V \quad \text{and} \quad \mathcal{S}_Z(\boldsymbol{\chi}_V) = \boldsymbol{\phi}_V \ . \tag{5.3.4}$$

In particular, $\mathcal{S}_Z^2 = \mathrm{id}_{Z(H)}$. Employing Lemmas 5.3.3 and 5.3.4, we see that in [FGST1, Sec. 5], $\mathbf{S}$ acts the same way, hence the underlying linear maps agree. Now the claim follows from the fact that $\mathcal{S}_Z$ intertwines the (projective) actions. Indeed, this is trivially true for the **S**-generator. For the **T**-generator, let us denote the action constructed in [FGST1] and Lyubashenko's action by $\triangleright$ and $\cdot$, respectively. By [FGST1, (5.1)] $\mathbf{T} \triangleright z = b\mathcal{S}_Z^{-1}(\boldsymbol{v}^{-1}\mathcal{S}_Z(z))$, where b is some non-zero constant. Since $\mathcal{S}_Z$ has order two, we have

$$\mathcal{S}_Z(\mathbf{T} \cdot z) = \mathcal{S}_Z(\boldsymbol{v}^{-1} z) = \frac{1}{b}\mathbf{T} \triangleright \mathcal{S}_Z(z) \ , \tag{5.3.5}$$

finishing the proof. $\square$

Appendix A

Proof of Proposition 4.2.1

In the main text, we worked with a specific realization of the Hopf monads A_i on $\mathcal{C} = {}_H\mathcal{M}$, the category of modules over a fixed (pivotal) quasi-Hopf algebra H. We did not, however, prove that this realization works.

In this short appendix, we give the proof for completeness, using a trick similar to that in [FSS, App. A]. We also remark that the proof was known to us when we wrote the paper [BGR2], where it was not included. We will here only treat the case of the central Hopf monad A_2, the other cases being completely similar.

Recall that in Chapter 4 we claimed that the vector space $A_2(V) := H^* \otimes V$ with H-action[33]

$$h \overset{2}{\curvearrowright} (f \otimes v) = \langle f \mid S(h_{(1)})?h_{(2,2)}\rangle \otimes h_{(2,1)}.v$$

together with the dinatural transformation

$$\iota_2(V)_X(f \otimes v \otimes x) = \langle f \mid ?.x\rangle \otimes v$$

is a coend for the functor

$$\mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{C}, (X, Y) \mapsto X^\vee(VY) \ .$$

First of all, it is easy to see that $\overset{2}{\curvearrowright}$ indeed defines an H-action, and that ι_2 is an intertwiner, and we leave it to the reader. Moreover,

$$\begin{aligned}\iota_2(V)_X(f \otimes v \otimes \phi(y)) &= \langle f \mid ?.\phi(y)\rangle \otimes v \\ &= \langle f \mid \phi(?.y)\rangle \otimes v = \iota_2(V)_Y(\phi^\vee(f) \otimes v \otimes y)\end{aligned} \tag{A.1}$$

for $\phi \in \mathcal{C}(Y, X)$ shows that ι_2 is dinatural.

We need to show that whenever there is an object $\tilde{A} \in \mathcal{C}$ and a family of morphisms $\nu_X : X^\vee(VX) \to \tilde{A}$ dinatural in X, then there is a unique morphism $\tilde{\nu} : A_2(V) \to \tilde{A}$ such that $\nu_X = \tilde{\nu} \circ \iota_2(V)_X$ for all $X \in \mathcal{C}$.

[33] Here, as in earlier incarnations of these equations, $f \in X^*$, $v \in V$, $x \in X$.

A useful identity. Let for $X \in \mathcal{C}$ and $x \in X$ the intertwiner $\rho_x \in \mathcal{C}(H, X)$ be given by $\rho_x(h) = h.x$. In particular $\rho_x(\mathbf{1}) = x$. We have

$$(\rho_x^\vee \otimes \mathrm{id}_V)(f \otimes v) = \langle f \,|?.x\rangle \otimes v = \iota_2(V)_X(f \otimes v \otimes x)\ , \tag{A.2}$$

for all $f \in H^*$, $v \in V$, so that the linear maps $\rho_x^\vee \otimes \mathrm{id}_V$ and $\iota_2(V)_X(? \otimes x)$ from $X^* \otimes V$ to $H^* \otimes V$ agree. Thus we have

$$\begin{aligned}\nu_X \circ \mathrm{id}_{X^\vee \otimes V} \otimes x &= \nu_X \circ (\mathrm{id}_{X^\vee \otimes V} \otimes \rho_x) \circ (\mathrm{id}_{X^\vee \otimes V} \otimes \mathbf{1})\\ &= \nu_H \circ (\rho_x^\vee \otimes \mathrm{id}_{V \otimes H}) \circ (\mathrm{id}_{X^\vee \otimes V} \otimes \mathbf{1})\\ &= \nu_H \circ \iota_2(V)_X \otimes \mathrm{id}_H \circ (\mathrm{id}_{X^\vee \otimes V} \otimes x \otimes \mathbf{1})\end{aligned} \tag{A.3}$$

for all $x \in X$, whence the equality

$$\nu_X = \nu_H \circ \iota_2(V)_X \otimes \mathbf{1} \tag{A.4}$$

of linear maps follows.

Defining $\tilde{\nu}$. Now define the linear map

$$\tilde{\nu} = \nu_H \circ \mathrm{id}_{H^* \otimes V} \otimes \mathbf{1}\ :\ H^* \otimes V \to \tilde{A}\ . \tag{A.5}$$

This is in fact in $\mathcal{C}(A_2(V), \tilde{A})$, since

$$\begin{aligned}\tilde{\nu}(h \overset{2}{\curvearrowright} (f \otimes v)) &= \tilde{\nu}(\langle f \mid S(h_{(1)})?h_{(2,2)}\rangle \otimes h_{(2,1)}.v)\\ &= \nu_H(\langle f \mid S(h_{(1)})?h_{(2,2)}\rangle \otimes h_{(2,1)}.v \otimes \mathbf{1})\\ &= \nu_H(\rho_{h_{(2,2)}}^\vee(\langle f \mid S(h_{(1)})?\rangle) \otimes h_{(2,1)}.v \otimes \mathbf{1})\\ &\overset{(A.2)}{=} (\nu_H \circ \iota_2(V)_H)(\langle f \mid S(h_{(1)})?\rangle \otimes h_{(2,1)}.v \otimes h_{(2,2)}.\mathbf{1})\\ &= (\nu_H \circ \iota_2(V)_H)(h.(f \otimes v \otimes \mathbf{1}))\\ &= h.(\nu_H \circ \iota_2(V)_H)(f \otimes v \otimes \mathbf{1})\\ &= h.\tilde{\nu}(f \otimes v)\ .\end{aligned} \tag{A.6}$$

Moreover, it satisfies the equality

$$\tilde{\nu} \circ \iota_2(V)_X = \nu_H \circ \iota_2(V)_X \otimes \mathbf{1} \overset{(A.4)}{=} \nu_X \tag{A.7}$$

of morphisms, as desired.

Uniqueness of $\tilde{\nu}$. The only thing left to check is that $\tilde{\nu}$ is unique. But this follows easily from the fact that the linear map $\iota_2(V)_H$ has a right inverse, namely $\mathrm{id}_{H^\vee} \otimes \mathrm{id}_V \otimes \mathbf{1}$. Indeed,

$$\iota_2(V)_H(\mathrm{id}_{H^\vee} \otimes \mathrm{id}_V \otimes \mathbf{1}) : f \otimes v \mapsto \langle f \,|?.\mathbf{1}\rangle \otimes v = f \otimes v\ . \tag{A.8}$$

Thus $\tilde{\nu}$ is (uniquely) determined by

$$\tilde{\nu} = (\tilde{\nu} \circ \iota_2(V)_H)(\mathrm{id}_{H^\vee} \otimes \mathrm{id}_V \otimes \mathbf{1}) = \nu_H(\mathrm{id}_{H^\vee} \otimes \mathrm{id}_V \otimes \mathbf{1})\ . \tag{A.9}$$

The proof is finished.

Appendix B

Proofs for Section 4.3

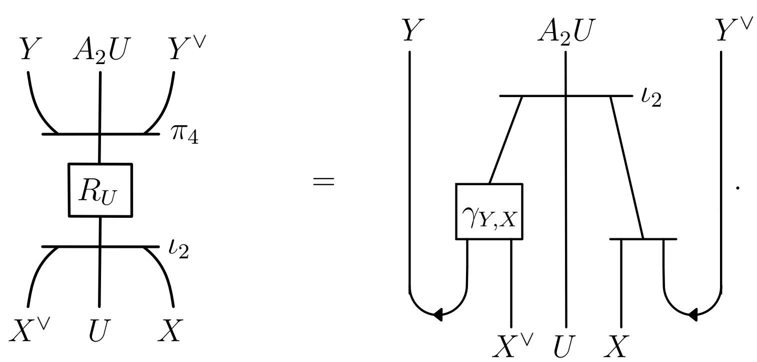
This appendix contains proofs of lemmata and technical steps used in the proof of the main theorem relating cointegrals for quasi-Hopf algebras to monadic cointegrals.

B.1 Proof of Proposition 4.3.2

Before giving the proof we need to show some intermediate results.

Using the explicit form of the unit and the counit of the adjunction, we can give the following simple characterization of the components of the categorical coaction R.

Lemma B.1.1. *The coactions R_U defined in (4.1.63) satisfy*

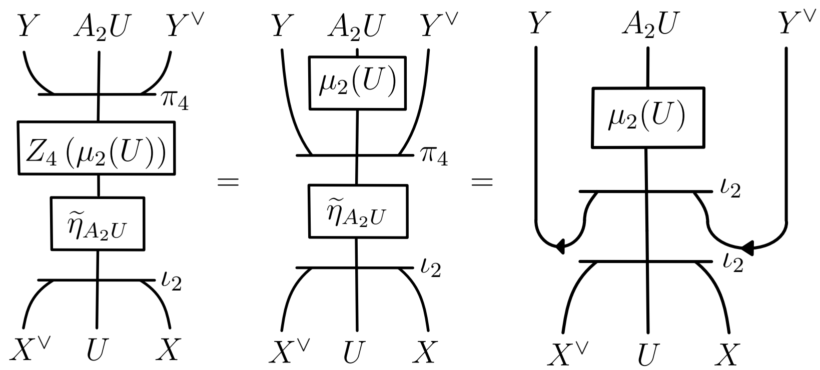


. (B.1.1)

Proof. By definition, we have the equality

$$(A_2(U), R_U) = (A_2(U), Z_4(\mu_2(U)) \circ \widetilde{\eta}_{A_2 U}) \tag{B.1.2}$$

of Z_4-comodules. Then



(B.1.3)

together with the definition (4.1.4) of the multiplication of A_2 proves the claim. $\square$

Lemma B.1.2. *Let $V \in {}_H\mathcal{M}$, $v \in V$, $h^* \in H^*$, and choose the realization of the central Hopf monad A_2 and the central Hopf comonad Z_4 as given in (4.2.1) and (4.2.18), respectively. Then*

$$\begin{aligned} R_V(h^* \otimes v) =& \langle h^* \mid S(x_{2(2)} p_2^L X_1) \boldsymbol{f}_1 \left[e_i x_{3(2)} X_{2(2)} Y_2 \right]_{(1)} p_1^R \rangle \\ & \times x_1 S(x_{2(1)} p_1^L) \boldsymbol{f}_2 \left[e_i x_{3(2)} X_{2(2)} Y_2 \right]_{(2)} p_2^R S(X_3 Y_3) \\ & \otimes e^i \otimes x_{3(1)} X_{2(1)} Y_1 . v, \end{aligned} \tag{B.1.4}$$

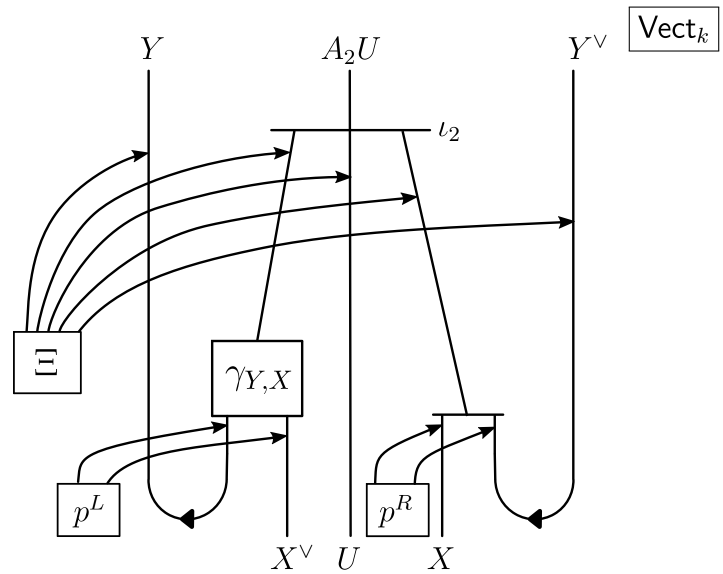
where $\{e_i\}$ is a basis of H with corresponding dual basis $\{e^i\}$, and summation over i is implied.

Proof. To abbreviate things, we first set

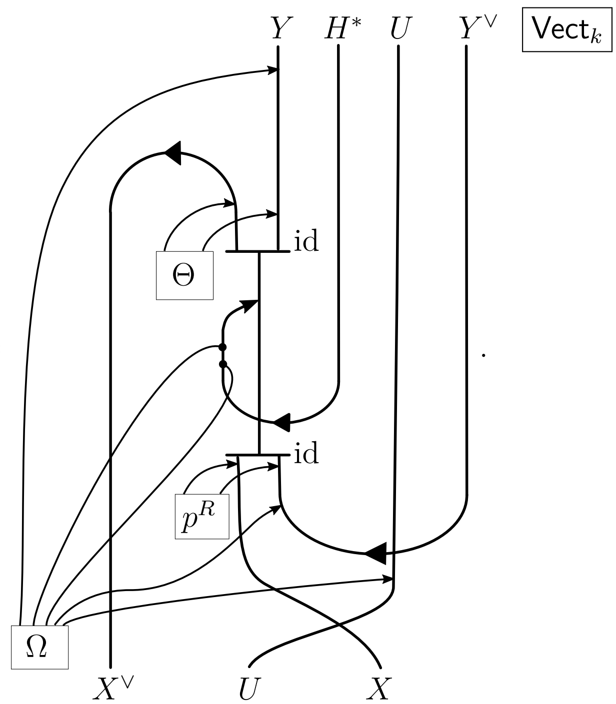
$$\begin{aligned} \Theta &= (S \otimes S)(p_{21}^L) \boldsymbol{f}, \\ \Xi &= (\mathrm{id} \otimes \mathrm{id} \otimes \Delta \otimes \mathrm{id})(\Psi \otimes \mathbf{1}) \cdot (\Delta \otimes \Delta \otimes \mathrm{id})(\Phi) \cdot (\mathbf{1} \otimes \mathbf{1} \otimes \Phi) \\ &= x_1 X_{1(1)} \otimes x_2 X_{1(2)} \otimes x_{3(1)} X_{2(1)} Y_1 \otimes x_{3(2)} X_{2(2)} Y_2 \otimes X_3 Y_3, \\ \Omega &= \Xi_1 \otimes S(\Xi_2) \otimes \Xi_4 \otimes S(\Xi_5) \otimes \Xi_3 \end{aligned} \tag{B.1.5}$$

Then from Lemma B.1.1 one computes the categorical coaction R as

$$\pi_4(A_2U)_Y \circ R_U \circ \iota_2(U)_X =$$



$$=$$



$$. \qquad \text{(B.1.6)}$$

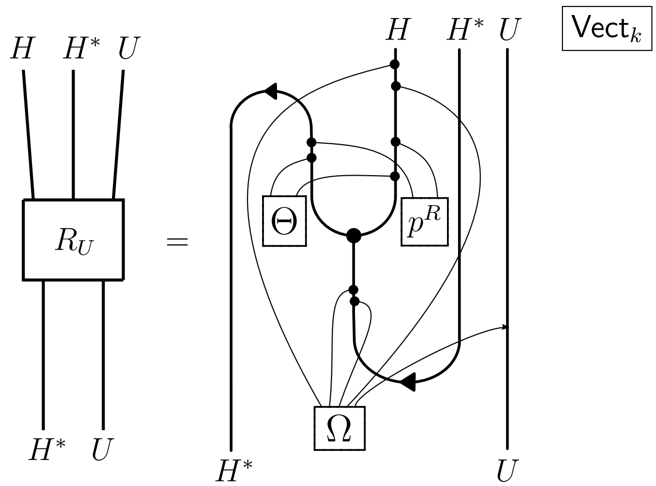
Recall that the box with Vect_k means that these pictures are to be understood as linear maps. Specializing X and Y to the regular left module H, we note that $\pi_4(V)_H$ has a left inverse

$$(H \otimes V) \otimes H^\vee \to H \otimes V, \quad h \otimes v \otimes f \mapsto f(\mathbf{1})h \otimes v \ , \tag{B.1.7}$$

while $\iota_4(V)_H$ has a right inverse

$$H^* \otimes V \to H^\vee \otimes (V \otimes H) \quad f \otimes v \mapsto f \otimes v \otimes \mathbf{1} \ . \tag{B.1.8}$$

Applying the inverses we obtain the explicit form of R_U,



(B.1.9)

From this we can read off that R_U is the linear map given by (to better see where we apply the changes from one line to the next we sometimes underline the relevant part)

$$
\begin{aligned}
R_U(h^* \otimes u) &= \langle h^* \otimes \mathrm{id} \mid (\mathbf{1} \otimes \Omega_1)\Theta\Delta(\Omega_2 e_i \Omega_3) p^R (\mathbf{1} \otimes \Omega_4)\rangle \otimes e^i \otimes \Omega_5.v \\
&= \langle h^* \otimes \mathrm{id} \mid (\mathbf{1} \otimes \Xi_1)\Theta\Delta(S(\Xi_2) e_i \Xi_4) p^R (\mathbf{1} \otimes S(\Xi_5))\rangle \otimes e^i \otimes \Xi_3.v \\
&= \langle h^* \mid S(p_2^L) \boldsymbol{f}_1 \, [S(\Xi_2) e_i \Xi_4]_{(1)} \, p_1^R \rangle \\
&\quad \times \Xi_1 S(p_1^L) \boldsymbol{f}_2 \, [S(\Xi_2) e_i \Xi_4]_{(2)} \, p_2^R S(\Xi_5) \otimes e^i \otimes \Xi_3.v \\
&\overset{\text{(B.1.5)}}{=} \langle h^* \mid S(p_2^L) \underline{\boldsymbol{f}}_1 \Big[\underline{S(x_2 X_{1(2)})} e_i x_{3(2)} X_{2(2)} Y_2\Big]_{(1)} p_1^R \rangle \\
&\quad \times x_1 X_{1(1)} S(p_1^L) \underline{\boldsymbol{f}}_2 \Big[\underline{S(x_2 X_{1(2)})} e_i x_{3(2)} X_{2(2)} Y_2\Big]_{(2)} p_2^R S(X_3 Y_3) \\
&\quad \otimes e^i \otimes x_{3(1)} X_{2(1)} Y_1.v \\
&\overset{\text{(2.1.37)}}{=} \langle h^* \mid S(x_{2(2)} \underline{X_{1(2,2)} p_2^L}) \boldsymbol{f}_1 \, \Big[e_i x_{3(2)} X_{2(2)} Y_2\Big]_{(1)} \, p_1^R \rangle \\
&\quad \times x_1 X_{1(1)} S(x_{2(1)} \underline{X_{1(2,1)} p_1^L}) \boldsymbol{f}_2 \, \Big[e_i x_{3(2)} X_{2(2)} Y_2\Big]_{(2)} \, p_2^R S(X_3 Y_3) \\
&\quad \otimes e^i \otimes x_{3(1)} X_{2(1)} Y_1.v \\
&\overset{\text{(2.1.35)}}{=} \langle h^* \mid S(x_{2(2)} p_2^L X_1) \boldsymbol{f}_1 \, \Big[e_i x_{3(2)} X_{2(2)} Y_2\Big]_{(1)} \, p_1^R \rangle \\
&\quad \times x_1 S(x_{2(1)} p_1^L) \boldsymbol{f}_2 \, \Big[e_i x_{3(2)} X_{2(2)} Y_2\Big]_{(2)} \, p_2^R S(X_3 Y_3) \\
&\quad \otimes e^i \otimes x_{3(1)} X_{2(1)} Y_1.v
\end{aligned}
\tag{B.1.10}
$$

for $h^* \in H^*$, $u \in U$. $\square$

With this we can now begin with the proof of Proposition 4.3.2, which relates the categorical coaction with a coaction in ${}_H\mathcal{M}_H$.

Proof of Proposition 4.3.2. We will need the identity

$$q_2^L[\boldsymbol{f}_2^{-1}]_{(2)} \otimes S(\boldsymbol{f}_1^{-1})q_1^L[\boldsymbol{f}_2^{-1}]_{(1)} = (S\otimes S)(p^R)\boldsymbol{f}_{21}\ , \tag{B.1.11}$$

which can be seen as follows:

$$\begin{aligned}
&q_2^L[\boldsymbol{f}_2^{-1}]_{(2)} \otimes S(\boldsymbol{f}_1^{-1})q_1^L[\boldsymbol{f}_2^{-1}]_{(1)}\\
\overset{(2.1.33)}{=}\ & X_3[\boldsymbol{f}_2^{-1}]_{(2)} \otimes S(X_1\boldsymbol{f}_1^{-1})\boldsymbol{\alpha}X_2[\boldsymbol{f}_2^{-1}]_{(1)}\\
\overset{(2.1.40)}{=}\ & \boldsymbol{f}_2^{-1}S(X_1)\boldsymbol{f}_2 \otimes S([\boldsymbol{f}_1^{-1}]_{(1)}\widetilde{\boldsymbol{f}}_1^{-1}S(X_3))\boldsymbol{\alpha}[\boldsymbol{f}_1^{-1}]_{(2)}\widetilde{\boldsymbol{f}}_2^{-1}S(X_2)\boldsymbol{f}_1\\
\overset{(2.1.8)}{=}\ & \varepsilon(\boldsymbol{f}_1^{-1})\boldsymbol{f}_2^{-1}S(X_1)\boldsymbol{f}_2 \otimes S(\widetilde{\boldsymbol{f}}_1^{-1}S(X_3))\boldsymbol{\alpha}\widetilde{\boldsymbol{f}}_2^{-1}S(X_2)\boldsymbol{f}_1\\
\overset{(*)}{=}\ & S(X_1)\boldsymbol{f}_2 \otimes S^2(X_3)S(\boldsymbol{\beta})S(X_2)\boldsymbol{f}_1\\
\overset{(2.1.33)}{=}\ & (S\otimes S)(p^R)\cdot \boldsymbol{f}_{21}\ .
\end{aligned} \tag{B.1.12}$$

In the step labeled $(*)$ one uses (2.1.41) (dashed underline) and that the counit applied to any leg of the inverse Drinfeld twist yields $\mathbf{1}$ (dotted underline). The identity (B.1.11) immediately implies

$$p^R = S^{-1}(q_2^L\boldsymbol{f}_2^{-1}{}_{(2)}\widetilde{\boldsymbol{f}}_2^{-1}) \otimes S^{-1}(q_1^L\boldsymbol{f}_2^{-1}{}_{(1)}\widetilde{\boldsymbol{f}}_1^{-1})\boldsymbol{f}_1^{-1}. \tag{B.1.13}$$

For the proof of the proposition, let now $h^* \in H^*$. Then

$$\begin{aligned}
&\Big(\varphi_{H^\vee}\circ\mathcal{A}(\rho)\Big)(h^*)\\
&\quad= \boldsymbol{\gamma}^{-1}(x_{3(1)}X_{2(1)}Y_1)\langle h^* \mid S(p_2^L)\boldsymbol{f}_1\,[e_i]_{(1)}\,S^{-1}(q_2^L\boldsymbol{f}_2^{-1})\rangle\\
&\qquad\times x_1X_{1(1)}S(p_1^L)\boldsymbol{f}_2\,[e_i]_{(2)}\,S^{-1}(q_1^L\boldsymbol{f}_1^{-1})\widetilde{\boldsymbol{f}}_1^{-1}S(X_3Y_3)\\
&\qquad\otimes x_2X_{1(2)}.e^i.\widetilde{\boldsymbol{f}}_2^{-1}S(x_{3(2)}X_{2(2)}Y_2)\\
&\overset{(2.1.10)}{=}\boldsymbol{\gamma}^{-1}(x_{3(1)}X_{2(1)}Y_1)\langle h^* \mid S(p_2^L)\boldsymbol{f}_1\,[e_i]_{(1)}\,S^{-1}(q_2^L\boldsymbol{f}_2^{-1})\rangle\\
&\qquad\times x_1X_{1(1)}S(p_1^L)\boldsymbol{f}_2\,[e_i]_{(2)}\,S^{-1}(q_1^L\boldsymbol{f}_1^{-1})\widetilde{\boldsymbol{f}}_1^{-1}S(X_3Y_3)\\
&\qquad\otimes x_{3(2)}X_{2(2)}Y_2S^{-1}(\widetilde{\boldsymbol{f}}_2^{-1}) \rightharpoonup e^i \leftharpoonup S(x_2X_{1(2)})\\
&\overset{(2.1.37)}{=}\ \boldsymbol{\gamma}^{-1}(x_{3(1)}X_{2(1)}Y_1)\langle h^* \mid S(p_2^L)\boldsymbol{f}_1\Big[e_ix_{3(2)}X_{2(2)}Y_2\Big]_{(1)}S^{-1}(q_2^L\boldsymbol{f}_2^{-1}{}_{(2)}\widetilde{\boldsymbol{f}}_2^{-1})\rangle\\
&\qquad\times x_1X_{1(1)}S(p_1^L)\boldsymbol{f}_2\Big[e_ix_{3(2)}X_{2(2)}Y_2\Big]_{(2)}S^{-1}(q_1^L\boldsymbol{f}_2^{-1}{}_{(1)}\widetilde{\boldsymbol{f}}_1^{-1})\widetilde{\boldsymbol{f}}_1^{-1}S(X_3Y_3)\\
&\qquad\otimes e^i \leftharpoonup S(x_2X_{1(2)})\\
&\overset{(B.1.13)}{=}\ \boldsymbol{\gamma}^{-1}(x_{3(1)}X_{2(1)}Y_1)\langle h^* \mid S(p_2^L)\boldsymbol{f}_1\Big[e_ix_{3(2)}X_{2(2)}Y_2\Big]_{(1)}p_1^R\rangle
\end{aligned}$$

$$\begin{aligned}
&\times x_1 X_{1(1)} S(p_1^L) \boldsymbol{f}_{\underline{2}} \left[e_i x_{3(2)} X_{2(2)} Y_2\right]_{(2)} p_2^R S(X_3 Y_3) \\
&\otimes e^i \leftharpoonup \underline{S(x_2 X_{1(2)})} \\
\overset{(2.1.37)}{=} \quad & \boldsymbol{\gamma}^{-1}(x_{3(1)} X_{2(1)} Y_1) \langle h^* \mid S(x_{2(2)} \underline{X_{1(2,2)}} p_2^L) \boldsymbol{f}_1 \left[e_i x_{3(2)} X_{2(2)} Y_2\right]_{(1)} p_1^R \rangle \\
&\times x_1 \underline{X_{1(1)}} S(x_{2(1)} \underline{X_{1(2,1)}} p_1^L) \boldsymbol{f}_2 \left[e_i x_{3(2)} X_{2(2)} Y_2\right]_{(2)} p_2^R S(X_3 Y_3) \otimes e^i \\
\overset{(2.1.35)}{=} \quad & \boldsymbol{\gamma}^{-1}(x_{3(1)} X_{2(1)} Y_1) \langle h^* \mid S(x_{2(2)} p_2^L X_1) \boldsymbol{f}_1 \left[e_i x_{3(2)} X_{2(2)} Y_2\right]_{(1)} p_1^R \rangle \\
&\times x_1 S(x_{2(1)} p_1^L) \boldsymbol{f}_2 \left[e_i x_{3(2)} X_{2(2)} Y_2\right]_{(2)} p_2^R S(X_3 Y_3) \otimes e^i .
\end{aligned} \tag{B.1.14}$$

From Lemma B.1.2 we obtain

$$\begin{aligned}
R_{\boldsymbol{\gamma}^\vee}(h^*) = \boldsymbol{\gamma}^{-1}(x_{3(1)} X_{2(1)} Y_1) \ &\langle h^* \mid S(x_{2(2)} p_2^L X_1) \boldsymbol{f}_1 \left[e_i x_{3(2)} X_{2(2)} Y_2\right]_{(1)} p_1^R \rangle \\
&\times x_1 S(x_{2(1)} p_1^L) \boldsymbol{f}_2 \left[e_i x_{3(2)} X_{2(2)} Y_2\right]_{(2)} p_2^R S(X_3 Y_3) \otimes e^i \ ,
\end{aligned} \tag{B.1.15}$$

so that

$$R_{\boldsymbol{\gamma}^\vee} = \varphi_{H^\vee} \circ \mathcal{A}(\rho) \tag{B.1.16}$$

indeed holds, finishing the proof. □

B.2 Proof of Theorem 4.3.1 (1)

The first step in the proof of Theorem 4.3.1 is to map cointegrals to a Hom-space containing monadic cointegrals. To do this, we define the space

$$\begin{aligned}
\mathcal{X}_2 &= \{ f \in H^* \mid f \leftharpoonup S(a) = S^{-1}(\boldsymbol{\gamma} \rightharpoonup a) \rightharpoonup f \quad \forall a \in H \} \\
&= \{ f \in H^\vee \in {}_H\mathcal{M}_H \mid a.f = f.(\boldsymbol{\gamma} \rightharpoonup a) \quad \forall a \in H \},
\end{aligned} \tag{B.2.1}$$

where the dot denotes the action on the left dual of the regular bimodule in ${}_H\mathcal{M}_H$, the category of $H \otimes H^{\mathrm{op}}$-modules as introduced in Section 2.3.

Note that right cointegrals are automatically contained in $\mathcal{X}_2$ by (2.3.11):

$$\textstyle\int_H^r \subset \mathcal{X}_2 \ . \tag{B.2.2}$$

By (4.2.1), we have

$$\begin{aligned}
&\mathcal{C}(\mathbf{1}, A_2(\boldsymbol{\gamma}^\vee)) \\
&= \{ f \in H^* \mid \varepsilon(h) f(a) = f(S(h_{(1)}) a (h_{(2)} \leftharpoonup \boldsymbol{\gamma}^{-1})) \quad \forall h, a \in H \} \\
&= \{ f \in H^\vee \in {}_H\mathcal{M}_H \mid \varepsilon(h) f = h_{(1)}.f.S(h_{(2)} \leftharpoonup \boldsymbol{\gamma}^{-1}) \quad \forall h \in H \},
\end{aligned} \tag{B.2.3}$$

where in the second line we again let the dot denote the action on the left dual of the regular bimodule.

We then have the following proposition.

Proposition B.2.1. *Let* $\boldsymbol{\xi} = (\mathrm{id} \otimes \boldsymbol{\gamma})(\boldsymbol{f}^{-1})$. *Then the map*

$$\mathbb{A}_2 : \mathcal{X}_2 \to \mathcal{C}\Big(\mathbf{1}, A_2(\boldsymbol{\gamma}^\vee)\Big), \quad f \mapsto \boldsymbol{\beta}.f.\boldsymbol{\xi}, \tag{B.2.4}$$

is a linear isomorphism.

Proof. Abbreviate $\mathcal{C}_2 := \mathcal{C}\Big(\mathbf{1}, A_2(\boldsymbol{\gamma}^\vee)\Big)$, and let us check that $\mathbb{A}_2(\mathcal{X}_2) \subset \mathcal{C}_2$. To this end, observe that the defining equation for $f \in H^*$ to be in $\mathcal{X}_2$ may be rewritten as

$$S(a).f = f.\boldsymbol{\xi}S(a \leftharpoondown \boldsymbol{\gamma}^{-1})\boldsymbol{\xi}^{-1} \tag{B.2.5}$$

by using the definition of the Drinfeld twist. Then we compute

$$\begin{aligned} h_{(1)}.\mathbb{A}_2(f).S(h_{(2)} \leftharpoondown \boldsymbol{\gamma}^{-1}) &= h_{(1)}\boldsymbol{\beta}.f.\boldsymbol{\xi}S(h_{(2)} \leftharpoondown \boldsymbol{\gamma}^{-1}) \\ &\overset{(B.2.5)}{=} h_{(1)}\boldsymbol{\beta}S(h_{(2)}).f.\boldsymbol{\xi} \\ &\overset{(2.1.8)}{=} \varepsilon(h)\boldsymbol{\beta}.f.\boldsymbol{\xi} \\ &= \varepsilon(h)\mathbb{A}_2(f). \end{aligned} \tag{B.2.6}$$

Next, we claim that the assignment

$$\mathbb{B}_2 : f \mapsto q_1^L.f.S(q_2^L \leftharpoondown \boldsymbol{\gamma}^{-1})\boldsymbol{\xi}^{-1} \tag{B.2.7}$$

is the two-sided inverse of $\mathbb{A}_2$. First of all, $\mathbb{B}_2(\mathcal{C}_2) \subset \mathcal{X}_2$. Indeed,

$$\begin{aligned} \mathbb{B}_2(f).\boldsymbol{\xi}S(a \leftharpoondown \boldsymbol{\gamma}^{-1})\boldsymbol{\xi}^{-1} &= q_1^L.f.S(q_2^L \leftharpoondown \boldsymbol{\gamma}^{-1})\boldsymbol{\xi}^{-1}\boldsymbol{\xi}S(a \leftharpoondown \boldsymbol{\gamma}^{-1})\boldsymbol{\xi}^{-1} \\ &= q_1^L.f.S((aq_2^L) \leftharpoondown \boldsymbol{\gamma}^{-1})\boldsymbol{\xi}^{-1} \\ &\overset{(2.1.35)}{=} S(a_{(1)})q_1^L a_{(2,1)}.f.S((q_2^L a_{(2,2)}) \leftharpoondown \boldsymbol{\gamma}^{-1})\boldsymbol{\xi}^{-1} \\ &\overset{(\star)}{=} S(a)q_1^L.f.S((q_2^L) \leftharpoondown \boldsymbol{\gamma}^{-1})\boldsymbol{\xi}^{-1} \\ &= S(a).\mathbb{B}_2(f). \end{aligned} \tag{B.2.8}$$

Here $(\star)$ uses that $f \in \mathcal{C}_2$.

It is not hard to see that $\mathbb{B}_2$ is a left inverse of $\mathbb{A}_2$:

$$\mathbb{B}_2\mathbb{A}_2(f) = \mathbb{B}_2(\boldsymbol{\beta}.f.\boldsymbol{\xi}) = q_1^L\boldsymbol{\beta}.f.\boldsymbol{\xi}S(q_2^L \leftharpoondown \boldsymbol{\gamma}^{-1})\boldsymbol{\xi}^{-1} = q_1^L\boldsymbol{\beta}S(q_2^L).f = f. \tag{B.2.9}$$

To see that $\mathbb{A}_2\mathbb{B}_2 = \mathrm{id}$ we need the fact that $\boldsymbol{\beta} = (S \otimes \varepsilon)(p^L)$, and the p^L, q^L-relation in (2.1.34). We compute

$$\begin{aligned} \mathbb{A}_2\mathbb{B}_2(f) &= \mathbb{A}_2(q_1^L.f.S(q_2^L \leftharpoondown \boldsymbol{\gamma}^{-1})\boldsymbol{\xi}^{-1}) \\ &= \boldsymbol{\beta}q_1^L.f.S(q_2^L \leftharpoondown \boldsymbol{\gamma}^{-1}) \\ &= S(p_1^L)q_1^L.\varepsilon(p_2^L)f.S(q_2^L \leftharpoondown \boldsymbol{\gamma}^{-1}) \\ &\overset{(\star)}{=} S(p_1^L)q_1^L p^L_{2\,(1)}.f.S((q_2^L p^L_{2\,(2)}) \leftharpoondown \boldsymbol{\gamma}^{-1}) \\ &\overset{(2.1.34)}{=} 1.f.S(1 \leftharpoondown \boldsymbol{\gamma}^{-1}) \\ &= f\ , \end{aligned} \tag{B.2.10}$$

using that $f \in \mathcal{C}_2$ in $(\star)$. □

We will need the following technical lemma.

Lemma B.2.2. *Let $f \in \mathcal{X}_2$. Then*

$$\boldsymbol{\gamma}(x_3)\ \varphi_{H^\vee}\Big(\boldsymbol{\beta}_{(1)} x_1 \boldsymbol{\xi}_{(1)} \otimes \boldsymbol{\beta}_{(2)}.f.x_2 \boldsymbol{\xi}_{(2)}\Big) = \boldsymbol{\beta} \otimes_k \boldsymbol{\beta}.f.\boldsymbol{\xi} \tag{B.2.11}$$

Proof.

$$\begin{aligned}
&\boldsymbol{\gamma}(x_3)\ \varphi_{H^\vee}\Big(\boldsymbol{\beta}_{(1)} x_1 \boldsymbol{\xi}_{(1)} \otimes \boldsymbol{\beta}_{(2)}.f.x_2 \boldsymbol{\xi}_{(2)}\Big) \\
\overset{(2.1.44)}{=}\ & \boldsymbol{\gamma}(x_3)\ \varphi_{H^\vee}\Big(\boldsymbol{\delta}_1 \boldsymbol{f}_1 x_1 \boldsymbol{\xi}_{(1)} \otimes \boldsymbol{\delta}_2 \boldsymbol{f}_2.f.x_2 \boldsymbol{\xi}_{(2)}\Big) \\
=\ & \boldsymbol{\gamma}^{-1}(y_{3(1)} X_{2(1)} Y_1) \boldsymbol{\gamma}(x_3)\ y_1 X_{1(1)} \boldsymbol{\delta}_1 \boldsymbol{f}_1 x_1 \boldsymbol{\xi}_{(1)} \boldsymbol{f}_1^{-1} S(X_3 Y_3) \\
& \otimes y_2 X_{1(2)} \boldsymbol{\delta}_2 \boldsymbol{f}_2.f.x_2 \boldsymbol{\xi}_{(2)} \boldsymbol{f}_2^{-1} S(y_{3(2)} X_{2(2)} Y_2) \\
=\ & \boldsymbol{\gamma}^{-1}(y_{3(1)} X_{2(1)} Y_1) \boldsymbol{\gamma}(x_3 \boldsymbol{F}_2^{-1})\ y_1 X_{1(1)} \boldsymbol{\delta}_1 \boldsymbol{f}_1 x_1 {\boldsymbol{F}_1^{-1}}_{(1)} \boldsymbol{f}_1^{-1} S(X_3 Y_3) \\
& \otimes y_2 X_{1(2)} \boldsymbol{\delta}_2 \boldsymbol{f}_2.f.x_2 {\boldsymbol{F}_1^{-1}}_{(2)} \boldsymbol{f}_2^{-1} S(y_{3(2)} X_{2(2)} Y_2) \\
\overset{(\star)}{=}\ & \boldsymbol{\gamma}^{-1}(y_{3(1)} X_{2(1)} Y_1) \boldsymbol{\gamma}(\boldsymbol{f}_{2(2)} x_3 \boldsymbol{F}_2^{-1})\ y_1 X_{1(1)} \boldsymbol{\delta}_1 \boldsymbol{f}_1 x_1 {\boldsymbol{F}_1^{-1}}_{(1)} \boldsymbol{f}_1^{-1} S(X_3 Y_3) \\
& \otimes y_2 X_{1(2)} \boldsymbol{\delta}_2.f.\boldsymbol{f}_{2(1)} x_2 {\boldsymbol{F}_1^{-1}}_{(2)} \boldsymbol{f}_2^{-1} S(y_{3(2)} X_{2(2)} Y_2) \\
\overset{(2.1.40)}{=}\ & \boldsymbol{\gamma}^{-1}(y_{3(1)} X_{2(1)} Y_1) \boldsymbol{\gamma}(\boldsymbol{f}_2^{-1} S(x_1))\ y_1 X_{1(1)} \boldsymbol{\delta}_1 S(x_3) S(X_3 Y_3) \\
& \otimes y_2 X_{1(2)} \boldsymbol{\delta}_2.f.\boldsymbol{f}_1^{-1} S(x_2) S(y_{3(2)} X_{2(2)} Y_2) \\
=\ & \boldsymbol{\gamma}^{-1}(y_{3(1)} X_{2(1)}) y_1 X_{1(1)} \boldsymbol{\delta}_1 S(X_3) \otimes y_2 X_{1(2)} \boldsymbol{\delta}_2.f.\boldsymbol{\xi} S(y_{3(2)} X_{2(2)}) \\
\overset{(2.1.43)}{=}\ & \boldsymbol{\gamma}^{-1}(y_{3(1)} X_{2(1)}) y_1 X_{1(1)} x_{1(1)} Y_1 \boldsymbol{\beta} S(x_3) S(X_3) \\
& \otimes y_2 X_{1(2)} x_{1(2)} Y_2 \boldsymbol{\beta} S(x_2 Y_3).f.\ \boldsymbol{\xi}\ S(y_{3(2)} X_{2(2)}) \\
=\ & \boldsymbol{\gamma}^{-1}(y_{3(1)} X_{2(1)} (x_2 Y_3)_{(1)}) y_1 X_{1(1)} x_{1(1)} Y_1 \boldsymbol{\beta} S(x_3) S(X_3) \\
& \otimes y_2 X_{1(2)} x_{1(2)} Y_2 \boldsymbol{\beta}.f.\boldsymbol{\xi} S((x_2 Y_3)_{(2)}) S(y_{3(2)} X_{2(2)}) \\
& = \boldsymbol{\beta} \otimes \boldsymbol{\beta}.f.\boldsymbol{\xi}\ ,
\end{aligned} \tag{B.2.12}$$

where in $(\star)$ we used that $f \in \mathcal{X}_2$. $\square$

Now we have all the necessary ingredients and can prove our main theorem.

Proof of Theorem 4.3.1 (1). By Proposition B.2.1, for any $\boldsymbol{\lambda}^{\mathcal{C}} \in \mathcal{C}(\mathbf{1}, A_2(D))$ there is a unique $\boldsymbol{\lambda} \in \mathcal{X}_2$ such that $\boldsymbol{\lambda}^{\mathcal{C}} = \mathbb{A}_2(\boldsymbol{\lambda}) = \boldsymbol{\beta}.\boldsymbol{\lambda}.\boldsymbol{\xi}$. Assume first that $\boldsymbol{\lambda}$ is a right cointegral. Then

$$(\varphi_{H^\vee} \circ \rho^l)(\boldsymbol{\lambda}^{\mathcal{C}}) \overset{(*)}{=} \boldsymbol{\gamma}(x_3)\varphi_{H^\vee}(\Delta(\boldsymbol{\beta}).(x_1 \otimes \boldsymbol{\lambda}.x_2).\Delta(\boldsymbol{\xi})) \overset{(**)}{=} \boldsymbol{\beta} \otimes \boldsymbol{\lambda}^{\mathcal{C}} \tag{B.2.13}$$

shows that $\boldsymbol{\lambda}^{\mathcal{C}}$ is a right monadic cointegral, using the equivalent characterization (4.3.6). Here $(*)$ uses that $\boldsymbol{\lambda}$ is a right cointegral, and $(**)$ uses Lemma B.2.2.

Conversely, assume that $\boldsymbol{\lambda}^{\mathcal{C}}$ is a right monadic cointegral. Note that for any $f \in \mathcal{X}_2$ we have

$$f = q_1^L \boldsymbol{\beta} S(q_2^L).f = q_1^L \boldsymbol{\beta}.f.(\boldsymbol{\gamma} \rightharpoonup S(q_2^L)), \tag{B.2.14}$$

where the first step is the zig-zag axiom (2.1.9), and the second step uses that $f \in \mathcal{X}_2$. Then

$$\begin{aligned}
\rho^l(\boldsymbol{\lambda}) &\overset{\text{(B.2.14)}}{=} \rho^l(q_1^L\boldsymbol{\beta}.\boldsymbol{\lambda}.\boldsymbol{\xi}\boldsymbol{\xi}^{-1}(\boldsymbol{\gamma} \rightharpoonup S(q_2^L)))\\
&\overset{(1)}{=} \Delta(q_1^L)\rho^l(\boldsymbol{\beta}.\boldsymbol{\lambda}.\boldsymbol{\xi})\Delta(\boldsymbol{\xi}^{-1}(\boldsymbol{\gamma} \rightharpoonup S(q_2^L))))\\
&\overset{(2)}{=} \Delta(q_1^L)\varphi_{H^\vee}^{-1}(\boldsymbol{\beta} \otimes \boldsymbol{\beta}.\boldsymbol{\lambda}.\boldsymbol{\xi})\Delta(\boldsymbol{\xi}^{-1}(\boldsymbol{\gamma} \rightharpoonup S(q_2^L))))\\
&\overset{(3)}{=} \boldsymbol{\gamma}(x_3)\ \Delta(q_1^L).\big(\boldsymbol{\beta}_{(1)}x_1\boldsymbol{\xi}_{(1)} \otimes \boldsymbol{\beta}_{(2)}.\boldsymbol{\lambda}.x_2\boldsymbol{\xi}_{(2)}\big).\Delta(\boldsymbol{\xi}^{-1}(\boldsymbol{\gamma} \rightharpoonup S(q_2^L)))\\
&= \boldsymbol{\gamma}(x_3)\ \big((q_1^L\boldsymbol{\beta})_{(1)}x_1 \otimes (q_1^L\boldsymbol{\beta})_{(2)}.\boldsymbol{\lambda}.x_2\big).\Delta(\boldsymbol{\gamma} \rightharpoonup S(q_2^L))\\
&\overset{(4)}{=} \boldsymbol{\gamma}((q_1^L\boldsymbol{\beta})_{(2,2)}x_3)\ \big((q_1^L\boldsymbol{\beta})_{(1)}x_1 \otimes \boldsymbol{\lambda}.(q_1^L\boldsymbol{\beta})_{(2,1)}x_2\big).\Delta(\boldsymbol{\gamma} \rightharpoonup S(q_2^L))\\
&\overset{(5)}{=} \boldsymbol{\gamma}(x_3)\ \big(x_1 \otimes \boldsymbol{\lambda}.x_2\big).\Delta(\boldsymbol{\gamma} \rightharpoonup (q_1^L\boldsymbol{\beta}))\Delta(\boldsymbol{\gamma} \rightharpoonup S(q_2^L))\\
&= \boldsymbol{\gamma}(x_3)\ x_1 \otimes \boldsymbol{\lambda}.x_2
\end{aligned} \tag{B.2.15}$$

shows that $\boldsymbol{\lambda} \in \mathcal{X}_2$ is a right cointegral in the sense of [BC2, HN2]. The step labeled (1) uses that ρ^l is a bimodule morphism, (2) is the fact that $\boldsymbol{\lambda}^{\mathcal{C}} = \boldsymbol{\beta}.\boldsymbol{\lambda}.\boldsymbol{\xi}$ is a monadic cointegral, (3) follows from Lemma B.2.2, (4) uses that $\boldsymbol{\lambda} \in \mathcal{X}_2$, and (5) is an application of quasi-coassociativity. $\square$

B.3 Proof of Theorem 4.3.1 (2)

Similarly to right cointegrals, left cointegrals for H are automatically contained in the space

$$\begin{aligned}
\mathcal{X}_3 &= \{f \in H^* \mid f \leftharpoonup S^{-1}(a) = S(a \leftharpoonup \boldsymbol{\gamma}) \rightharpoonup f\}\\
&= \{f \in {}^\vee H \in {}_H\mathcal{M}_H \mid a.f = f.(a \leftharpoonup \boldsymbol{\gamma})\},
\end{aligned} \tag{B.3.1}$$

see (2.3.10), and analogously to Proposition B.2.1 one can show that

$$\mathbb{A}_3 = \mathbb{A}_2^{\mathrm{cop}} : \mathcal{X}_3 \to \mathcal{C}(\mathbf{1}, A_3(\boldsymbol{\gamma}^\vee)), \quad \mathbb{A}_3(f) = S^{-1}(\beta).f.\hat{\boldsymbol{\xi}} \tag{B.3.2}$$

is an isomorphism. Note that here the dot denotes the action on the right dual of the regular $H \otimes H^{\mathrm{op}}$-module. The Hopf monads A_2 and A_3 are canonically isomorphic via $\kappa_{2,3} : A_2 \Rightarrow A_3$, see (4.1.11) and Proposition 4.2.2. This allows us to transport right monadic cointegrals to left monadic cointegrals. Thus, upon showing that

$$\begin{tikzcd}
\int^r_H \arrow[r, hook] \arrow[d, "(*)"'] & \mathcal{X}_2 \arrow[r, "\mathbb{A}_2"] & \mathcal{C}(\mathbf{1}, A_2(\boldsymbol{\gamma}^\vee)) \arrow[d, "(\kappa_{2,3})_{\boldsymbol{\gamma}^\vee} \circ ?"] & \int^{r,\mathrm{mon}}_{\mathcal{C}} \arrow[l, hook'] \arrow[d, "(\kappa_{2,3})_{\boldsymbol{\gamma}^\vee} \circ ?"] \\
\int^l_H \arrow[r, hook] & \mathcal{X}_3 \arrow[r, "\mathbb{A}_3"'] & \mathcal{C}(\mathbf{1}, A_3(\boldsymbol{\gamma}^\vee)) & \int^{l,\mathrm{mon}}_{\mathcal{C}} \arrow[l, hook']
\end{tikzcd} \tag{B.3.3}$$

commutes, we know that $\mathbb{A}_3$ maps left cointegrals to left monadic cointegrals. Here $(*)$ maps the right cointegral $\boldsymbol{\lambda}^r$ to[34]

$$\boldsymbol{\lambda}^l = \boldsymbol{\gamma}(\boldsymbol{\alpha} S(\boldsymbol{\beta}))^{-1} \cdot (\boldsymbol{\lambda}^r \circ S \leftharpoonup (\mathsf{u}^{\mathrm{cop}})^{-1}). \tag{B.3.4}$$

By [BC2, Prop. 4.3] this is a left cointegral, recall Proposition 2.3.3.

The right hand square in (B.3.3) commutes by construction. Using the explicit formula (4.2.4) for $\kappa_{2,3}$, one finds that the upper path of the left hand square is

$$\begin{aligned}
\boldsymbol{\lambda}^r \mapsto\ & \boldsymbol{\gamma}^{-1}(X_2)\langle \boldsymbol{\lambda}^r \mid S(\boldsymbol{\beta})S(?X_1)X_3S^{-1}(\boldsymbol{\xi})\rangle \\
=\ & \boldsymbol{\gamma}^{-1}(X_2)\langle \boldsymbol{\lambda}^r \circ S \mid S^{-2}(\boldsymbol{\xi})S^{-1}(X_3)?X_1\boldsymbol{\beta}\rangle \\
=\ & \boldsymbol{\gamma}(\boldsymbol{\alpha}S(\boldsymbol{\beta}))\boldsymbol{\gamma}^{-1}(X_2)\langle \boldsymbol{\lambda}^l \mid \mathsf{u}^{\mathrm{cop}}S^{-2}(\boldsymbol{\xi})S^{-1}(X_3)?X_1\boldsymbol{\beta}\rangle \\
\overset{(1)}{=}\ & \boldsymbol{\gamma}(\boldsymbol{\alpha}S(\boldsymbol{\beta}))\boldsymbol{\gamma}^{-1}(X_2p^L_1)\langle \boldsymbol{\lambda}^l \mid S^{-1}(X_3p^L_2)?X_1\boldsymbol{\beta}\rangle \\
\overset{(2)}{=}\ & \boldsymbol{\gamma}(\boldsymbol{\alpha}S(\boldsymbol{\beta}))\boldsymbol{\gamma}^{-1}(X_2p^L_1)\boldsymbol{\gamma}((X_3p^L_2)_{(1)})\langle \boldsymbol{\lambda}^l \mid ?X_1\boldsymbol{\beta}S((X_3p^L_2)_{(2)})\rangle,
\end{aligned} \tag{B.3.5}$$

the step marked (1) follows directly from the definition of u and $\boldsymbol{\xi}$, see (2.3.12) resp. Theorem 4.3.1, and step (2) uses $\boldsymbol{\lambda}^l \in \mathcal{X}_3$.

The lower path of the left square of (B.3.3) evaluates to

$$\begin{aligned}
\boldsymbol{\lambda}^r \mapsto\ & \langle \boldsymbol{\lambda}^l \mid S^{-2}(\boldsymbol{\beta})?S(\hat{\boldsymbol{\xi}})\rangle \\
\overset{(2.3.10)}{=}\ & \langle \boldsymbol{\lambda}^l \mid ?S((S^{-1}(\boldsymbol{\beta}) \leftharpoonup \boldsymbol{\gamma})\hat{\boldsymbol{\xi}})\rangle \\
\overset{(*)}{=}\ & \boldsymbol{\gamma}^{-1}(\boldsymbol{\beta}_{(2)}\boldsymbol{f}^{-1}_2)\langle \boldsymbol{\lambda}^l \mid ?\boldsymbol{\beta}_{(1)}\boldsymbol{f}^{-1}_1\rangle \\
\overset{(2.1.44)}{=}\ & \boldsymbol{\gamma}^{-1}(X_2)\boldsymbol{\gamma}((X_3)_{(1)}p^L_1)\langle \boldsymbol{\lambda}^l \mid ?X_1\boldsymbol{\beta}S((X_3)_{(2)}p^L_2)\rangle\ ,
\end{aligned} \tag{B.3.6}$$

where $(*)$ uses the definition of $\hat{\boldsymbol{\xi}}$ as in Theorem 4.3.1.

We have

$$\boldsymbol{\gamma}\Big(\boldsymbol{\alpha}S(\boldsymbol{\beta})S(p^L_1){p^L_2}_{(1)}\Big){p^L_2}_{(2)} \overset{(2.1.33)}{=} \boldsymbol{\gamma}\Big(\boldsymbol{\alpha}S(\boldsymbol{\beta})x_1\boldsymbol{\beta}S(x_2){x_3}_{(1)}\Big){x_3}_{(2)}$$

[34]The zig-zag axiom (2.1.9) implies that both $\boldsymbol{\gamma}(\boldsymbol{\alpha})$ and $\boldsymbol{\gamma}(\boldsymbol{\beta})$ are invertible in k. Therefore, the prefactor in (B.3.4) is well-defined.

$$
\begin{aligned}
&\overset{(2.1.4),(*)}{=} \boldsymbol{\gamma}\Big(\boldsymbol{\alpha} S(\boldsymbol{\beta}) x_1 \boldsymbol{\beta} S(y_1 x_2) y_2 x_3\Big) y_3 \\
&\overset{(2.1.9)}{=} \boldsymbol{\gamma}\Big(S(y_1 \boldsymbol{\beta}) y_2\Big) y_3 \\
&\overset{(**)}{=} \quad \boldsymbol{\gamma}(p_1^L) p_2^L
\end{aligned}
\tag{B.3.7}
$$

where $(*)$ uses $(\boldsymbol{\gamma} \otimes \boldsymbol{\gamma}^{-1}) \circ \Delta = \varepsilon$, and $(**)$ uses $\boldsymbol{\gamma} \circ S = \boldsymbol{\gamma} \circ S^{-1}$ and (2.1.33). Therefore the two expressions (B.3.5) and (B.3.6) are equal.

B.4 Proof of the pivotal case

The proof of Theorem 4.3.3 is similar to the proof of the second part above.

First, define the spaces

$$
\begin{aligned}
\mathcal{X}_1 &= \{ f \in H^* \mid f(ab) = f((b \leftharpoonup \boldsymbol{\gamma}) a) \} \ , \\
\mathcal{X}_4 &= \{ f \in H^* \mid f(ab) = f((\boldsymbol{\gamma} \rightharpoonup b) a) \} \ .
\end{aligned}
\tag{B.4.1}
$$

By (3.1.13) we have right (left) $\boldsymbol{\gamma}$-symmetrized cointegrals are automatically in $\mathcal{X}_1$ ($\mathcal{X}_4$), and similarly to Proposition B.2.1 one may show that

$$
\begin{aligned}
\mathbb{A}_1 : \mathcal{X}_1 \to \mathcal{C}(\mathbf{1}, A_1(\boldsymbol{\gamma}^\vee)) \ , &\quad f \mapsto \langle f \mid S^{-1}(\boldsymbol{\beta}) ? S(\boldsymbol{\vartheta}) \rangle \ , \\
\mathbb{A}_4 : \mathcal{X}_4 \to \mathcal{C}(\mathbf{1}, A_4(\boldsymbol{\gamma}^\vee)) \ , &\quad f \mapsto \langle f \mid \boldsymbol{\beta} ? S^{-1}(\hat{\boldsymbol{\vartheta}}) \rangle \ ,
\end{aligned}
\tag{B.4.2}
$$

with $\boldsymbol{\vartheta} = (\boldsymbol{\gamma}^{-1} \otimes S^{-1})(p^L)$ and $\hat{\boldsymbol{\vartheta}} = \boldsymbol{\vartheta}^{\mathrm{cop}}$, are linear isomorphisms.

Recall that we denote the one-dimensional spaces of right and left $\boldsymbol{\gamma}$-symmetrized cointegral by

$$
\textstyle\int_H^{r,\gamma} \quad \text{and} \quad \int_H^{l,\gamma} \ .
\tag{B.4.3}
$$

A simple computation shows that the diagram

$$
\begin{tikzcd}
\int_H^r \arrow[r, hook] \arrow[d, "(*)"'] & \mathcal{X}_2 \arrow[r, "\mathbb{A}_2"] & \mathcal{C}(\mathbf{1}, A_2(\boldsymbol{\gamma}^\vee)) \arrow[d, "\sim"] & \int_{\mathcal{C}}^{r,\mathrm{mon}} \arrow[l, hook'] \arrow[d, "\sim"] \\
\int_H^{r,\gamma} \arrow[r, hook] & \mathcal{X}_1 \arrow[r, "\mathbb{A}_1"'] & \mathcal{C}(\mathbf{1}, A_1(\boldsymbol{\gamma}^\vee)) & \int_{\mathcal{C}}^{r,D-\mathrm{sym}} \arrow[l, hook']
\end{tikzcd}
\tag{B.4.4}
$$

commutes, with $(*)$ sending a right cointegral $\boldsymbol{\lambda}^r$ to the right symmetrized cointegral $\boldsymbol{\lambda}^r \leftharpoonup \mathsf{u}\boldsymbol{g}$, and $\sim$ is induced by the isomorphism of Hopf monads from Proposition 4.1.2.

Indeed, a right cointegral $\boldsymbol{\lambda}^r$ gets mapped to the right D-symmetrized monadic cointegral

$$
\boldsymbol{\lambda}^{r,D-\mathrm{sym}} = \langle \boldsymbol{\lambda}^r \mid S(\boldsymbol{\beta}) \boldsymbol{g} ? S^{-1}(\boldsymbol{\xi}) \rangle
\tag{B.4.5}
$$

by the upper path, and to

$$(\boldsymbol{\lambda}^{r,D-\mathrm{sym}})' = \langle \boldsymbol{\lambda}^r \mid \mathsf{u}\boldsymbol{g}S^{-1}(\boldsymbol{\beta})?S(\boldsymbol{\vartheta})\rangle \tag{B.4.6}$$

by the lower path.

The upper path, evaluated on $S^{-1}(h)$, $h \in H$, yields

$$\begin{aligned}\boldsymbol{\lambda}^{r,D-\mathrm{sym}}(S^{-1}(h)) &= \langle \boldsymbol{\lambda}^r \mid S(\boldsymbol{\beta})\boldsymbol{g}S^{-1}(\boldsymbol{\xi}h)\rangle \\ &= \langle \boldsymbol{\lambda}^r \mid S(\boldsymbol{\xi}h\boldsymbol{\beta})\boldsymbol{g}\rangle \\ &\overset{(2.3.13)}{=} \langle \boldsymbol{\lambda}^l \mid \mathsf{u}^{\mathrm{cop}}\boldsymbol{g}^{-1}\boldsymbol{\xi}h\boldsymbol{\beta}\rangle \ .\end{aligned} \tag{B.4.7}$$

Evaluating the lower path on $S^{-1}(h)$, $h \in H$, we get

$$\begin{aligned}(\boldsymbol{\lambda}^{r,D-\mathrm{sym}})'(S^{-1}(h)) &= \langle \boldsymbol{\lambda}^r \mid \mathsf{u}\boldsymbol{g}S^{-1}(h\boldsymbol{\beta})S(\boldsymbol{\vartheta})\rangle \\ &\overset{(2.3.13)}{=} \langle \boldsymbol{\lambda}^l \circ S^{-1} \mid S(h\boldsymbol{\beta})\boldsymbol{g}S(\boldsymbol{\vartheta})\rangle \\ &= \langle \boldsymbol{\lambda}^l \mid \boldsymbol{\vartheta}\boldsymbol{g}^{-1}h\boldsymbol{\beta}\rangle \ .\end{aligned} \tag{B.4.8}$$

The claim then follows from

$$\begin{aligned}\mathsf{u}^{\mathrm{cop}}\boldsymbol{g}^{-1}\boldsymbol{\xi}\boldsymbol{g} &\overset{(2.3.12)}{=} \boldsymbol{\gamma}(\mathsf{V}_1^{\mathrm{cop}}\boldsymbol{f}_2^{-1})S^{-2}(\mathsf{V}_2^{\mathrm{cop}})\boldsymbol{g}^{-1}\boldsymbol{f}_1^{-1}\boldsymbol{g} \\ &= \boldsymbol{\gamma}(\mathsf{V}_1^{\mathrm{cop}}\boldsymbol{f}_2^{-1})\boldsymbol{g}^{-1}\mathsf{V}_2^{\mathrm{cop}}\boldsymbol{f}_1^{-1}\boldsymbol{g} \\ &\overset{(2.3.14)}{=} \boldsymbol{\gamma}\Big(S(p_1^L)\tilde{\boldsymbol{f}}_2\boldsymbol{f}_2^{-1}\Big)\boldsymbol{g}^{-1}S(p_2^L)\tilde{\boldsymbol{f}}_1\boldsymbol{f}_1^{-1}\boldsymbol{g} \\ &= \boldsymbol{\gamma}^{-1}(p_1^L)S^{-1}(p_2^L) \\ &= \boldsymbol{\vartheta}\end{aligned} \tag{B.4.9}$$

A similar diagram involving left cointegrals and their symmetrized version then finishes the proof of the theorem. □

Appendix C

Proofs for Chapter 5

Here we will give the proofs promised in Chapter 5.

C.1 Some preliminaries

From the formulas given in Section 2.1.6, one computes

$$\begin{aligned} q_t^R &= \boldsymbol{e}_0 \otimes \mathbf{1} + \boldsymbol{e}_1 \otimes \boldsymbol{e}_0 + \boldsymbol{e}_1 \otimes \boldsymbol{e}_1 K^{-t} \\ p_t^R &= \boldsymbol{e}_0 \otimes \boldsymbol{\beta} + \boldsymbol{e}_1 \otimes \mathbf{1} \\ q_t^L &= \boldsymbol{e}_0 \otimes \mathbf{1} + \boldsymbol{e}_1 \otimes K^{-t} \\ p_t^L &= \boldsymbol{e}_0 \otimes \mathbf{1} + \boldsymbol{e}_1 K^t \otimes K^t \ , \end{aligned} \tag{C.1.1}$$

and

$$\boldsymbol{f}_t^{\pm 1} = \boldsymbol{e}_0 \otimes \mathbf{1} + \boldsymbol{e}_1 \otimes \boldsymbol{e}_0 K^{\mp t} + \boldsymbol{e}_1 K^{\pm t} \otimes \boldsymbol{e}_1 \tag{C.1.2}$$

Before continuing, let us agree on the following, to avoid too cluttered a notation.

> **Convention:** We agree that we are allowed to drop subscript t's, e.g. we may write $\boldsymbol{f}$ instead of $\boldsymbol{f}_t$ in longer formulas.

This does not cause confusion, as t was fixed at the beginning, and we do not wish to switch between H^Φ at different values of t.

We also note the following general formula for the coproduct of an element in the PBW-basis.

Lemma C.1.1. *Let $0 \leq n, m \leq p-1$ and $0 \leq l \leq 2p-1$. Then*

$$\begin{aligned} \Delta(E^n F^m K^l) = \sum_{r=0}^{n} \sum_{s=0}^{m} \begin{bmatrix} m \\ s \end{bmatrix} \begin{bmatrix} n \\ r \end{bmatrix} q^{r(n-r)+s(m-s)-2(n-r)(m-s)} \\ \times \left(\boldsymbol{e}_0 + q^{t(r-(m-s))} \boldsymbol{e}_1 \right) E^{n-r} F^s K^{l-(m-s)} \otimes E^r F^{m-s} K^{n-r+l} \ . \end{aligned} \tag{C.1.3}$$

Proof. The coproduct is multiplicative, and so it suffices to compute $\Delta(E)^n$ and $\Delta(F)^m$ for $n, m \geq 0$. To do this, we note that by [Ka, IV.(1.9)] we have the q-binomial theorem

$$(x+y)^n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} q^{k(n-k)} x^k y^{n-k} \tag{C.1.4}$$

whenever $yx = q^2 xy$. Set

$$a = \boldsymbol{e}_0 + q^t \boldsymbol{e}_1 \quad \text{and} \quad b = \boldsymbol{e}_0 + q^{-t} \boldsymbol{e}_1 \ , \tag{C.1.5}$$

so that

$$\Delta(E) = E \otimes K + a \otimes E \quad \text{and} \quad \Delta(F) = F \otimes \mathbf{1} + bK^{-1} \otimes F \ . \tag{C.1.6}$$

Since $(E \otimes K) \cdot (\mathbf{1} \otimes E) = q^2 (\mathbf{1} \otimes E) \cdot (E \otimes K)$, we can use (C.1.4) to obtain

$$\Delta(E)^n = \sum_{r=0}^{n} \begin{bmatrix} n \\ r \end{bmatrix} q^{r(n-r)} a^r E^{n-r} \otimes E^r K^{n-r} \ . \tag{C.1.7}$$

Similarly, one uses $(K^{-1} \otimes F) \cdot (F \otimes \mathbf{1}) = q^2 (F \otimes \mathbf{1}) \cdot (K^{-1} \otimes F)$ to get

$$\Delta(F)^m = \sum_{s=0}^{m} \begin{bmatrix} m \\ s \end{bmatrix} q^{s(m-s)} b^{m-s} F^s K^{-(m-s)} \otimes F^{m-s} \ . \tag{C.1.8}$$

The desired formula now follows from the multiplicativity of Δ. $\square$

C.2 The Drinfeld element

Before we prove our explicit formula for the Drinfeld element, i.e. Lemma 2.4.1, we have the following lemma.

Lemma C.2.1. *Let $l \in \mathbb{Z}$. Then*

$$S_p(l) := \sum_{s=0}^{2p-1} q^{-2s(s+l)} = (1-i) \cdot \sqrt{p} \cdot (1 + (-1)^l i^p) \cdot q^{l^2/2} \ . \tag{C.2.1}$$

Proof. Let a, b, c be integers, with c non-zero. The (generalized) Gauss sum is

$$G(a,b,c) = \sum_{n=0}^{|c|-1} \exp\left(\pi i \cdot \frac{an^2 + bn}{c}\right) \ . \tag{C.2.2}$$

Thus we have

$$S_p(l) = \sum_{s=0}^{p-1} \left(q^{-2s^2 - 2sl} + q^{-2(s+p)^2 - 2(s+p)l}\right)$$

$$
\begin{aligned}
&= 2\sum_{s=0}^{p-1} q^{-2s^2-2sl} \\
&= 2G(-2,-2l,p) \ . && \text{(C.2.3)}
\end{aligned}
$$

The Gauss sums $G(a,b,c)$ satisfy the following well-known (see e.g. [BEW, Thm. 1.2.2]) reciprocity relation. Namely, if $ac \neq 0$ and $ac+b$ is even, then

$$
G(a,b,c) = \sqrt{\left|\frac{c}{a}\right|} \cdot \exp\left(\pi i \cdot \frac{\mathrm{sgn}(ac) - \frac{b^2}{ac}}{4}\right) \cdot G(-c,-b,a) \ , \tag{C.2.4}
$$

where $\mathrm{sgn}(x) = \frac{x}{|x|}$ for $x \neq 0$.

Certainly, $-2p \neq 0$ and $-2p-2l$ is even, so we get

$$
\begin{aligned}
S_p(l) &= 2\sqrt{\left|\frac{p}{-2}\right|} \cdot \exp\left(\pi i \cdot \frac{\mathrm{sgn}(-2p) - \frac{(-2l)^2}{-2p}}{4}\right) \cdot G(-p,2l,-2) \\
&= \sqrt{2p} \cdot \exp\left(\pi i \cdot \frac{\frac{2l^2}{p} - 1}{4}\right) \cdot G(-p,2l,-2) \\
&= \sqrt{2p} \cdot q^{\frac{l^2}{2}} \frac{1-i}{\sqrt{2}} \cdot G(-p,2l,-2) \ . && \text{(C.2.5)}
\end{aligned}
$$

Now

$$
G(-p,2l,-2) = \sum_{n=0}^{1} \exp\left(\pi i \frac{-pn^2+2ln}{-2}\right) = 1 + \exp\left(\tfrac{\pi i}{2}(p-2l)\right) \tag{C.2.6}
$$

and so the claim follows. □

Now we prove the formula for the Drinfeld element.

Proof of Lemma 2.4.1. By [FGR1, (6.38)], the Drinfeld element is given by

$$
\boldsymbol{u} = S(R_2 p_2^R)\boldsymbol{\alpha} R_1 p_1^R = S(R_2)R_1 \ , \tag{C.2.7}
$$

where the second equality follows from (C.1.1) and $\boldsymbol{\alpha} = \mathbf{1}$ in H^Φ. Thus, the explicit expression of the R-matrix from (2.4.28) gives

$$
\boldsymbol{u} = \frac{1}{4p} \sum_{n=0}^{p-1} \sum_{s,r=0}^{2p-1} \frac{\{1\}^n}{[n]!} q^{\frac{1}{2}n(n-1)-2sr} W_{n,s,r} \times S(K^r F^n) K^s E^n \ . \tag{C.2.8}
$$

where we have abbreviated

$$
W_{n,s,r} = 1 + q^{tr} + q^{-t(n+s)} + q^{t(\frac{1}{2}t+r-n-s)} \tag{C.2.9}
$$

We have $S(F^n) = (-1)^n q^{-n(n+1)} F^n K^n (\boldsymbol{e}_0 + q^{-tn}\boldsymbol{e}_1)$, so that

$$\begin{aligned}
\boldsymbol{u} &= \frac{1}{4p}\sum_{n=0}^{p-1}\sum_{s,r=0}^{2p-1} \frac{\{1\}^n}{[n]!} q^{\frac{1}{2}n(n-1)-2sr} W_{n,s,r} (-1)^n q^{-n(n+1)} (\boldsymbol{e}_0 + q^{-tn}\boldsymbol{e}_1) F^n K^{n+s-r} E^n \\
&= \frac{1}{4p}\sum_{n=0}^{p-1}\sum_{s,r=0}^{2p-1} \frac{\{-1\}^n}{[n]!} q^{-\frac{1}{2}n(n+3)-2sr} W_{n,s,r} (\boldsymbol{e}_0 + q^{-tn}\boldsymbol{e}_1) F^n K^{n+s-r} E^n \\
&= \frac{1}{4p}\sum_{n=0}^{p-1}\sum_{s,r=0}^{2p-1} \frac{\{-1\}^n}{[n]!} q^{-\frac{1}{2}n(n+3)+2rn} \widetilde{W}_{n,s,r} (\boldsymbol{e}_0 + q^{-tn}\boldsymbol{e}_1) F^n E^n K^r \ , \qquad \text{(C.2.10)}
\end{aligned}$$

where we substituted $r \to n+s-r$ in the last step, and set

$$\begin{aligned}
\widetilde{W}_{n,s,r} &= q^{-2s(n+s-r)} W_{n,s,n+s-r} \\
&= q^{-2s(n+s-r)} \left(1 + q^{t(n+s-r)} + q^{-t(n+s)} + q^{t(\frac{1}{2}t-r)}\right) \qquad \text{(C.2.11)}
\end{aligned}$$

Since $H^\Phi = \boldsymbol{e}_0 H^\Phi \oplus \boldsymbol{e}_1 H^\Phi$, where $\boldsymbol{e}_i H^\Phi$ has a basis $\{\boldsymbol{e}_i F^m E^n K^k \mid 0 \le m,n,k \le p-1\}$, we can write

$$\boldsymbol{e}_i \sum_{l=0}^{2p-1} a_l F^m E^n K^l = \sum_{l=0}^{p-1} (a_l + (-1)^i a_{l+p}) \boldsymbol{e}_i F^m E^n K^l \qquad \text{(C.2.12)}$$

in the corresponding basis. Since t is odd, we have

$$\widetilde{W}_{n,s,r} + (-1)^i \widetilde{W}_{n,s,r+p} = 2q^{-2s(n+s-r)} \cdot \begin{cases} 1 + q^{-t(n+s)} & i=0 \\ q^{t(n+s-r)} + q^{t(\frac{1}{2}t-r)} & i=1 \end{cases} . \qquad \text{(C.2.13)}$$

Therefore, the Drinfeld element is given on the 0 sector by

$$\begin{aligned}
\boldsymbol{e}_0 \boldsymbol{u} &= \frac{1}{2p}\sum_{n,r=0}^{p-1} \frac{\{-1\}^n}{[n]!} q^{-\frac{1}{2}n(n+3)+2rn} \left(\sum_{s=0}^{2p-1} q^{-2s(n+s-r)} (1 + q^{-t(n+s)})\right) \boldsymbol{e}_0 F^n E^n K^r \\
&= \frac{1}{2p}\sum_{n,r=0}^{p-1} \frac{\{-1\}^n}{[n]!} q^{-\frac{1}{2}n(n+3)+2rn} \left(\sum_{s=0}^{2p-1} q^{-2s(s+n-r)}\right) \boldsymbol{e}_0 F^n E^n K^r \\
&= \frac{1-i}{2\sqrt{p}}\sum_{n,r=0}^{p-1} \frac{\{-1\}^n}{[n]!} q^{n(r-\frac{3}{2})+\frac{1}{2}r^2} (1 + (-1)^{n-r} i^p) \boldsymbol{e}_0 F^n E^n K^r \ . \qquad \text{(C.2.14)}
\end{aligned}$$

The last line follows from Lemma C.2.1 and a simple algebraic manipulation, while for the first equality we use

$$\sum_{s=0}^{2p-1} q^{s(2s+x)} = \sum_{s=0}^{p-1} q^{s(2s+x)} + (-1)^x \sum_{s=0}^{p-1} q^{s(2s+x)} \ , \qquad \text{(C.2.15)}$$

with $x = t$ in our case.

On the other hand, on the 1 sector, we compute

$$\boldsymbol{e}_1\boldsymbol{u} = \frac{1}{2p}\sum_{n,r=0}^{p-1}\frac{\{-1\}^n}{[n]!}q^{-\frac{1}{2}n(n+3)+2rn-tn}\boldsymbol{e}_1F^nE^nK^r \\ \times \sum_{s=0}^{2p-1} q^{-2s(s+n-r)}\left(q^{t(n+s-r)}+q^{t(\frac{1}{2}t-r)}\right) . \tag{C.2.16}$$

By the argument from above, only the second term in the brackets in the sum over s contributes, and we can use Lemma C.2.1 and a simple algebraic manipulation to obtain

$$\boldsymbol{e}_1\boldsymbol{u} = \frac{1-i}{2\sqrt{p}}\sum_{n,r=0}^{p-1}\frac{\{-1\}^n}{[n]!}q^{n(r-\frac{3}{2})+\frac{1}{2}r^2+t(\frac{1}{2}t-r-n)}\boldsymbol{e}_1F^nE^nK^r \\ \times\left(1+(-1)^{n-r}i^p\right) . \tag{C.2.17}$$

Putting everything together we obtain

$$\boldsymbol{u} = \frac{1-i}{2\sqrt{p}}\sum_{n,r=0}^{p-1}\frac{\{-1\}^n}{[n]!}(1+(-1)^{n-r}i^p)q^{n(r-\frac{3}{2})+\frac{1}{2}r^2} \\ \times\left(\boldsymbol{e}_0+q^{t(\frac{1}{2}t-r-n)}\boldsymbol{e}_1\right)F^nE^nK^r , \tag{C.2.18}$$

which is what we claimed. □

C.3 The pivot

We now prove Proposition 2.4.2, which was the statement that the element $\boldsymbol{g}_t = \boldsymbol{e}_0K - \boldsymbol{e}_1K^{t+1}$ is a pivot compatible with the ribbon structure on H^Φ.

Proof of Proposition 2.4.2. The first part, i.e. $\boldsymbol{g}_t$ being a pivot, follows automatically from $\boldsymbol{g}_t = \boldsymbol{u}_t\boldsymbol{v}^{-1}$, which we now show. We again do so on the 0 and 1 sector separately.

To this end, recall first that the ribbon element $\boldsymbol{v}$ was given in (2.4.29). Then we have

$$\begin{aligned}\boldsymbol{e}_0\boldsymbol{g}\boldsymbol{v} &= \frac{1-i}{2\sqrt{p}}\sum_{n=0}^{p-1}\sum_{j=0}^{2p-1}\frac{\{1\}^n}{[n]!}q^{n(j-\frac{1}{2})+\frac{1}{2}(j+p+1)^2}\boldsymbol{e}_0F^nE^nK^{j+1} \\ &= \frac{1-i}{2\sqrt{p}}\sum_{n=0}^{p-1}\sum_{j=0}^{2p-1}\frac{\{1\}^n}{[n]!}q^{n(j-\frac{3}{2})+\frac{1}{2}(j+p)^2}\boldsymbol{e}_0F^nE^nK^{j} \\ &= \frac{1-i}{2\sqrt{p}}\sum_{n,j=0}^{p-1}\frac{\{1\}^n}{[n]!}\left(q^{n(j-\frac{3}{2})+\frac{1}{2}(j+p)^2}+q^{n(j+p-\frac{3}{2})+\frac{1}{2}j^2}\right)\boldsymbol{e}_0F^nE^nK^{j} \\ &= \frac{1-i}{2\sqrt{p}}\sum_{n,j=0}^{p-1}\frac{\{1\}^n}{[n]!}\left((-1)^ji^p+(-1)^n\right)q^{n(j-\frac{3}{2})+\frac{1}{2}j^2}\boldsymbol{e}_0F^nE^nK^{j}\end{aligned}$$

$$
\begin{aligned}
&= \frac{1-i}{2\sqrt{p}} \sum_{n,j=0}^{p-1} \frac{\{-1\}^n}{[n]!} \Big((-1)^{j+n} i^p + 1\Big) q^{n(j-\frac{3}{2})+\frac{1}{2}j^2} \boldsymbol{e}_0 F^n E^n K^j \\
&= \boldsymbol{e}_0 \boldsymbol{u} \qquad \text{(C.3.1)}
\end{aligned}
$$

and

$$
\begin{aligned}
\boldsymbol{e}_1 \boldsymbol{g} \boldsymbol{v} &= -\frac{1-i}{2\sqrt{p}} \sum_{n=0}^{p-1} \sum_{j=0}^{2p-1} \frac{\{1\}^n}{[n]!} q^{n(j-\frac{1}{2})+\frac{1}{2}(j+p+1)^2} \boldsymbol{e}_1 F^n E^n K^{j+t+1} \\
&= -\frac{1-i}{2\sqrt{p}} \sum_{n=0}^{p-1} \sum_{j=0}^{2p-1} \frac{\{1\}^n}{[n]!} q^{n(j-t-\frac{3}{2})+\frac{1}{2}(j-t+p)^2} \boldsymbol{e}_1 F^n E^n K^j \\
&= -\frac{1-i}{2\sqrt{p}} \sum_{n,j=0}^{p-1} \frac{\{1\}^n}{[n]!} \Big(q^{n(j-t-\frac{3}{2})+\frac{1}{2}(j-t+p)^2} - q^{n(j+p-t-\frac{3}{2})+\frac{1}{2}(j-t)^2} \Big) \boldsymbol{e}_1 F^n E^n K^j \\
&= -\frac{1-i}{2\sqrt{p}} \sum_{n,j=0}^{p-1} \frac{\{1\}^n}{[n]!} \Big((-1)^{j+t} i^p - (-1)^n\Big) q^{n(j-t-\frac{3}{2})+\frac{1}{2}(j-t)^2} \boldsymbol{e}_1 F^n E^n K^j \\
&= \frac{1-i}{2\sqrt{p}} \sum_{n,j=0}^{p-1} \frac{\{-1\}^n}{[n]!} \Big((-1)^{n+j} i^p + 1\Big) q^{n(j-\frac{3}{2})+\frac{1}{2}j^2+t(\frac{1}{2}t-n-j)} \boldsymbol{e}_1 F^n E^n K^j \\
&= \boldsymbol{e}_1 \boldsymbol{u} \ . \qquad \text{(C.3.2)}
\end{aligned}
$$

Thus $\boldsymbol{g}_t \boldsymbol{v} = \boldsymbol{u}_t$, which is what we wanted to show. □

C.4 The central elements $\boldsymbol{\chi}_V$ and $\boldsymbol{\phi}_V$

Before proving the statements regarding the $\boldsymbol{\chi}$'s and $\boldsymbol{\phi}$'s, we will prove two lemmata.

Lemma C.4.1. *Let $a, b \in H$ such that $[b, K] = 0$. The pivot $\boldsymbol{g}_t$ satisfies*

$$
q_t^R \cdot (a \otimes b) \cdot p_t^R \cdot (1 \otimes \boldsymbol{g}_t) = a \otimes bK^{p+1} \ , \qquad \text{(C.4.1)}
$$

Proof. From the explicit formulas in (C.1.1), one easily computes that the LHS of the equation in the statement of this lemma equals

$$
\boldsymbol{e}_0 a \otimes bK^{p+1} + \boldsymbol{e}_1 a \otimes \boldsymbol{e}_0 bK^{p+1} + \boldsymbol{e}_1 a \otimes \boldsymbol{e}_1 K^{-t} bK^{t+p+1} \ , \qquad \text{(C.4.2)}
$$

and so the claim follows. □

Note in particular that we can apply this lemma whenever b is of the form $E^i F^i K^j$.

Lemma C.4.2. *Let*

$$
c = \zeta E^{p-1} F^{p-1} \sum_{j \in \mathbb{Z}_{2p}} K^j \qquad \text{(C.4.3)}
$$

be the integral of H^Φ, where $\zeta \in \mathbb{C}$ is some non-zero normalization coefficient. Let $V = \mathfrak{X}^\alpha(s)$, and suppose $f : H \to H$ is generated by the linear maps S, 'multiplication with K', and their inverses.

Then

$$c_{(1)} \cdot \mathrm{tr}_V\Big(f(c_{(2)})\Big) = \zeta \sum_{j \in \mathbb{Z}_{2p}} \sum_{r=0}^{p-1} E^{p-1-r} F^{p-1-r} K^j \cdot \mathrm{tr}_V\Big(f(F^r E^r K^{p+j-1})\Big) \ . \tag{C.4.4}$$

Proof. Firstly, note that by Lemma C.1.1,

$$\begin{aligned}\Delta(E^m F^m) &= \sum_{r=0}^{m} \sum_{s=0}^{m} \begin{bmatrix} m \\ s \end{bmatrix} \begin{bmatrix} m \\ r \end{bmatrix} q^{r(m-r)+s(m-s)-2(m-r)(m-s)} \\ &\quad \times \Big(\boldsymbol{e}_0 + q^{t(r-(m-s))} \boldsymbol{e}_1\Big) E^{m-r} F^s K^{l-(m-s)} \otimes E^r F^{m-s} K^{m-r+l} \\ &\approx \sum_{s=0}^{m} \begin{bmatrix} m \\ s \end{bmatrix}^2 E^s F^s K^{l-(m-s)} \otimes E^{m-s} F^{m-s} K^{s+l} \ . \end{aligned} \tag{C.4.5}$$

where by $\approx$ we mean that only terms with E and F equipotent in the second tensor factor contribute, since we already know that we will be tracing it out. This knowledge gives a $\delta_{r,m-s}$, and using $\begin{bmatrix} m \\ s \end{bmatrix} = \begin{bmatrix} m \\ m-s \end{bmatrix}$ we get (C.4.5).

Thus we obtain

$$\begin{aligned}\Delta(c) &= \zeta \sum_{j \in \mathbb{Z}_{2p}} \Delta(E^{p-1} F^{p-1} K^j) \\ &\approx \zeta \sum_{j \in \mathbb{Z}_{2p}} \sum_{s=0}^{p-1} \begin{bmatrix} p-1 \\ s \end{bmatrix}^2 E^s F^s K^{l-(p-1-s)} \otimes E^{p-1-s} F^{p-1-s} K^{s+l} \\ &= \zeta \sum_{j \in \mathbb{Z}_{2p}} \sum_{s=0}^{p-1} \begin{bmatrix} p-1 \\ p-1-s \end{bmatrix}^2 E^{p-1-s} F^{p-1-s} K^{l-s} \otimes E^s F^s K^{p-1-s+l} \\ &= \zeta \sum_{j \in \mathbb{Z}_{2p}} \sum_{s=0}^{p-1} E^{p-1-s} F^{p-1-s} K^l \otimes E^s F^s K^{p-1+l} \ , \end{aligned} \tag{C.4.6}$$

where we substituted $s \to p-1-s$, $l \to l-s$, and used

$$\begin{bmatrix} p-1 \\ p-1-s \end{bmatrix} = \frac{[p-1] \cdot \ldots \cdot [p-s]}{[s]!} = 1 \ , \tag{C.4.7}$$

since $[p-k] = [k]$ in general. The claim follows. □

Recall now the *Cartan automorphism* w of the algebra H, given on generators by

$$w(E) = F, \quad w(F) = E, \quad w(K) = K^{-1} \ , \tag{C.4.8}$$

see e.g. [Ka, Lem. VI.1.2]. An indispensable observation in what follows is the next proposition.

Proposition C.4.3. *The Cartan automorphism defined in* (C.4.8) *restricts to the identity on the center* $Z(H)$*.*

Proof. This follows from [FGST1, App. D.1]. More precisely, somewhat explicit formulas for a canonical basis of the center are given there; one checks that they are invariant under the Cartan automorphism. □

Now we are ready to prove our main results in this section.

Proof of Lemma 5.3.3. By Proposition 5.2.3, the central elements $\boldsymbol{\phi}_V$ are given by

$$\boldsymbol{\phi}_V = q_1^R c_{(1)} p_1^R \cdot \mathrm{tr}_V\left(\boldsymbol{u}_t^{-1}\boldsymbol{v} S(q_2^R c_{(2)} p_2^R)\right) \ . \tag{C.4.9}$$

From $\boldsymbol{g}_t = \boldsymbol{u}_t \boldsymbol{v}^{-1}$, Lemma C.4.1, and Lemma C.4.2, we obtain

$$= \zeta \sum_{k=0}^{p-1} \sum_{l\in\mathbb{Z}_{2p}} E^{p-1-k} F^{p-1-k} K^l \cdot \mathrm{tr}_V\left(S(E^k F^k K^l)\right) \ . \tag{C.4.10}$$

Next we use $S(E^k F^k) = F^k E^k$, cyclicity of the trace, and the explicit expression for $\mathrm{tr}_{\mathcal{X}^\alpha(s)}(F^m E^m K^a)$ from [FGST1, (4.10)], which altogether yields

$$= \zeta \sum_{k=0}^{p-1} \sum_{l\in\mathbb{Z}_{2p}} \sum_{n=0}^{s-1} \alpha^{k-l}([k]!)^2 q^{-l(s-1-2n)} \begin{bmatrix} s-n+k-1 \\ s \end{bmatrix} \begin{bmatrix} n \\ k \end{bmatrix} E^{p-1-k} F^{p-1-k} K^l \cdot \ . \tag{C.4.11}$$

After rearranging and noting that $\begin{bmatrix} n \\ k \end{bmatrix} = 0$ for $k > n$, we get

$$\begin{aligned} \boldsymbol{\phi}_V = \zeta \sum_{n=0}^{s-1} \sum_{i=0}^{n} \sum_{j\in\mathbb{Z}_{2p}} \alpha^{i+j}([i]!)^2 q^{-j(s-1-2n)} \begin{bmatrix} s-n+i-1 \\ i \end{bmatrix} \begin{bmatrix} n \\ i \end{bmatrix} \\ \times E^{p-1-i} F^{p-1-i} K^j \ . \end{aligned} \tag{C.4.12}$$

Applying the Cartan automorphism to this expressions yields, after change of summation variable $-j \to j$,

$$\begin{aligned} w(\boldsymbol{\phi}_V) = \zeta \sum_{n=0}^{s-1} \sum_{i=0}^{s} \sum_{j\in\mathbb{Z}_{2p}} \alpha^{i+j}([i]!)^2 q^{j(s-1-2n)} \begin{bmatrix} s-n+i-1 \\ s \end{bmatrix} \begin{bmatrix} n \\ i \end{bmatrix} \\ \times F^{p-1-i} E^{p-1-i} K^j \cdot \ , \end{aligned} \tag{C.4.13}$$

which is exactly $\widehat{\boldsymbol{\phi}}^\alpha(s)$ as given in [FGST1, (4.19)]. The claim thus follows from Proposition C.4.3. □

Now we prove the corresponding statement for $\boldsymbol{\chi}_V$.

Proof of Lemma 5.3.4. By Proposition 5.2.3, the central elements $\boldsymbol{\chi}_V$ are given by

$$\boldsymbol{\chi}_V = q_1^R M_1 p_1^R \cdot \mathrm{tr}_V \left(\boldsymbol{u}_t^{-1}\boldsymbol{v} S(q_2^R M_2 p_2^R)\right) \ , \tag{C.4.14}$$

where M is the monodromy of H. From $\boldsymbol{g}_t = \boldsymbol{u}_t\boldsymbol{v}^{-1}$ and Lemma C.4.1 we immediately get

$$= M_1 \cdot \mathrm{tr}_V \left(S(M_2 K^{p+1})\right) \tag{C.4.15}$$

Now the explicit formula for the monodromy stated in [CGR, (4.13)] gives

$$\begin{aligned}= \frac{1}{2p}\sum_{m,n=0}^{p-1}\sum_{i,j=0}^{2p-1}\frac{\{1\}^{m+n}}{[m]![n]!}q^{\frac{1}{2}m(m-1)+\frac{1}{2}n(n-1)-m^2+m(j-i)-ij} \\ \times (\delta_{i+m,\mathrm{even}} + \delta_{i+m,\mathrm{odd}}q^{t(m-n)})K^jF^mE^n \cdot \mathrm{tr}_V \left(S(K^iE^mF^nK^{p+1})\right) \ .\end{aligned} \tag{C.4.16}$$

All terms with $m \neq n$ are zero, and using again $S(E^rF^r) = F^rE^r$ and cyclicity, we get

$$= \frac{1}{2p}\sum_{n=0}^{p-1}\sum_{i,j=0}^{2p-1}\frac{\{1\}^{2n}}{([n]!)^2}q^{n(j-i-1)-ij}K^jF^nE^n \cdot \mathrm{tr}_V \left(F^nE^nK^{p-i-1}\right) \ . \tag{C.4.17}$$

Next, we make the substitution $i \to p-i-1$

$$\begin{aligned}&= \frac{1}{2p}\sum_{n=0}^{p-1}\sum_{i,j=0}^{2p-1}\frac{\{1\}^{2n}}{([n]!)^2}q^{n(j+i+p)+(i+1+p)j}K^jF^nE^n \cdot \mathrm{tr}_V \left(F^nE^nK^{i}\right) \\ &= \frac{1}{2p}\sum_{n=0}^{p-1}\sum_{i,j=0}^{2p-1}(-1)^{n+j}\frac{\{1\}^{2n}}{([n]!)^2}q^{n(j+i)+(i+1)j}K^jF^nE^n \cdot \mathrm{tr}_V \left(F^nE^nK^{i}\right) \ ,\end{aligned} \tag{C.4.18}$$

and with the explicit formula [FGST1, (4.10)] for the trace we get

$$\begin{aligned}&= \frac{1}{2p}\sum_{n=0}^{p-1}\sum_{i,j=0}^{2p-1}\sum_{m=0}^{s-1}(-1)^{n+j}\alpha^{n+i}\{1\}^{2n}q^{n(j+i)+(i+1)j+i(s-1-2m)} \\ &\qquad \times \begin{bmatrix} s-m+n-1 \\ n \end{bmatrix}\begin{bmatrix} m \\ n \end{bmatrix} K^jF^nE^n \qquad \text{(C.4.19)} \\ &= \frac{1}{2p}\sum_{m=0}^{s-1}\sum_{n=0}^{m}\sum_{j=0}^{2p-1}(-1)^{n+j}\alpha^{n}\{1\}^{2n}q^{(n+1)j}\left(\sum_{i=0}^{2p-1}\alpha^iq^{i(s-1-2m+n+j)}\right) \\ &\qquad \times \begin{bmatrix} s-m+n-1 \\ n \end{bmatrix}\begin{bmatrix} m \\ n \end{bmatrix} K^jF^nE^n \ . \qquad \text{(C.4.20)}\end{aligned}$$

The binomial $\begin{bmatrix} m \\ n \end{bmatrix}$ restricts the sum over n. Let $\beta \in \{0,1\}$ such that $\alpha = (-1)^\beta$. Then in particular $\alpha = q^{p\beta}$, and we can explicitly calculate the sum over i. Namely, we have

$$\begin{aligned}\sum_{i=0}^{2p-1} \alpha^i q^{i(s-1-2m+n+j)} &= \sum_{i=0}^{2p-1} q^{i(j+s-1-2m+n+\beta p)} \\ &= 2p \cdot \delta(j \equiv \beta p + 2m + 1 - n - s \mod 2p)\end{aligned} \tag{C.4.21}$$

. Thus (recall that $\alpha = (-1)^\beta$),

$$\begin{aligned}\boldsymbol{\chi}_V = &\sum_{m=0}^{s-1} \sum_{n=0}^{m} (-1)^{n+\beta p+2m+1-n-s} \alpha^n \{1\}^{2n} q^{(n+1)(\beta p+2m+1-n-s)} \\ &\times \begin{bmatrix} s-m+n-1 \\ n \end{bmatrix} \begin{bmatrix} m \\ n \end{bmatrix} K^{\beta p+2m+1-n-s} F^n E^n\end{aligned} \tag{C.4.22}$$

$$\begin{aligned}=&(-1)^{s+1} \sum_{m=0}^{s-1} \sum_{n=0}^{m} (-1)^{\beta(n+p)+\beta(n+1)} \{1\}^{2n} q^{(n+1)(2m+1-n-s)} \\ &\times \begin{bmatrix} s-m+n-1 \\ n \end{bmatrix} \begin{bmatrix} m \\ n \end{bmatrix} K^{\beta p+2m+1-n-s} F^n E^n\end{aligned} \tag{C.4.23}$$

$$\begin{aligned}=&(-1)^{s+1} \alpha^{p+1} \sum_{m=0}^{s-1} \sum_{n=0}^{m} \{1\}^{2n} q^{(n+1)(2m+1-n-s)} \\ &\times \begin{bmatrix} s-m+n-1 \\ n \end{bmatrix} \begin{bmatrix} m \\ n \end{bmatrix} K^{\beta p+2m+1-n-s} F^n E^n\end{aligned} \tag{C.4.24}$$

$$\begin{aligned}=&\alpha^{p+1} (-1)^{s+1} \sum_{n=0}^{s-1} \sum_{m=0}^{n} \{1\}^{2m} q^{-(m+1)(m+s-1-2n)} \\ &\times \begin{bmatrix} s-n+m-1 \\ m \end{bmatrix} \begin{bmatrix} n \\ m \end{bmatrix} F^m E^m K^{-s+1+\beta p+2n-m} .\end{aligned} \tag{C.4.25}$$

The reason for the last rewrite is that one can now easily see that $w(\boldsymbol{\chi})$ yields the expression for $\widehat{\boldsymbol{\chi}}^\alpha(s)$ from [FGST1, Prop. 4.3.1]. By Proposition C.4.3, $\boldsymbol{\chi}_{\mathcal{X}^\alpha(s)} = \widehat{\boldsymbol{\chi}}^\alpha(s)$, and the proof is finished. □

C.5 A certain intertwiner

Recall that we denoted the basis of $\mathcal{X}^\alpha(s) \otimes \mathcal{X}^\alpha(s)$ by $|s; n, m\rangle^\alpha$, for $0 \leq n, m \leq s-1$. On this basis, H^Φ acts as

$$K|s;n,m\rangle^\alpha = q^{2(s-1-(n+m))}|s;n,m\rangle^\alpha \tag{C.5.1}$$

$$E|s;n,m\rangle^\alpha = [n][s-n]q^{s-1-2m}|s;n-1,m\rangle^\alpha$$

$$+\left(\delta_{s+\beta p,\mathrm{odd}}+\delta_{s+\beta p,\mathrm{even}}q^{t}\right)\alpha[m][s-m]|s;n,m-1\rangle^{\alpha} \tag{C.5.2}$$

and

$$\begin{aligned} F|s;n,m\rangle^{\alpha} &= |s;n+1,m\rangle^{\alpha} \\ &\quad +\left(\delta_{s+\beta p,\mathrm{odd}}+\delta_{s+\beta p,\mathrm{even}}q^{-t}\right)\alpha q^{-(s-1-2n)}|s;n,m+1\rangle^{\alpha}\ , \end{aligned} \tag{C.5.3}$$

where $\beta\in\{0,1\}$ is so that $\alpha=(-1)^{\beta}$.

Proof of Lemma 5.3.1. We want to show that

$$\begin{aligned} f:\mathbb{C}&\to\mathcal{X}^{\alpha}(s)\otimes\mathcal{X}^{\alpha}(s) \\ 1&\mapsto\sum_{i=0}^{s-1}\left(\delta_{s+\beta p,\mathrm{odd}}+\delta_{s+\beta p,\mathrm{even}}q^{it}\right)(-1)^{i}\alpha^{i}q^{i(s-2-i)}|s;i,s-1-i\rangle^{\alpha} \end{aligned} \tag{C.5.4}$$

is an intertwiner.

One easily sees $K\cdot f(1)=f(K\cdot 1)$. For the action of E, we compute

$$\begin{aligned} E\cdot f(1) &= \sum_{i=0}^{s-1}\left(\delta_{s+\beta p,\mathrm{odd}}+\delta_{s+\beta p,\mathrm{even}}q^{it}\right)(-1)^{i}\alpha^{i}q^{i(s-2-i)}E|s;i,s-1-i\rangle^{\alpha} \\ &= \sum_{i=1}^{s-1}\left(\delta_{s+\beta p,\mathrm{odd}}+\delta_{s+\beta p,\mathrm{even}}q^{it}\right)(-1)^{i}\alpha^{i}q^{i(s-2-i)-s+1+2i} \\ &\qquad\times[i][s-i]|s;i-1,s-1-i\rangle^{\alpha} \\ &\quad+\sum_{i=0}^{s-2}\left(\delta_{s+\beta p,\mathrm{odd}}+\delta_{s+\beta p,\mathrm{even}}q^{(i+1)t}\right)(-1)^{i}\alpha^{i+1}q^{i(s-2-i)} \\ &\qquad\times[s-1-i][1+i]|s;i,s-2-i\rangle^{\alpha} \\ &= \sum_{i=1}^{s-1}\left(\delta_{s+\beta p,\mathrm{odd}}+\delta_{s+\beta p,\mathrm{even}}q^{it}\right)(-1)^{i}\alpha^{i}q^{i(s-2-i)-s+1+2i} \\ &\qquad\times[i][s-i]|s;i-1,s-1-i\rangle^{\alpha} \\ &\quad-\sum_{i=1}^{s-1}\left(\delta_{s+\beta p,\mathrm{odd}}+\delta_{s+\beta p,\mathrm{even}}q^{it}\right)(-1)^{i}\alpha^{i}q^{(i-1)(s-1-i)} \\ &\qquad\times[s-i][i]|s;i-1,s-1-i\rangle^{\alpha} \\ &= 0\ , \end{aligned} \tag{C.5.5}$$

where we used that $[0]=0$ and $(i-1)(s-1-i)=i(s-2-i)-(s-1-2i)$, so f intertwines the action by E. For F, we compute

$$\begin{aligned} F\cdot f(1) &= \sum_{i=0}^{s-1}\left(\delta_{s+\beta p,\mathrm{odd}}+\delta_{s+\beta p,\mathrm{even}}q^{it}\right)(-1)^{i}\alpha^{i}q^{i(s-2-i)}F|s;i,s-1-i\rangle^{\alpha} \\ &= \sum_{i=0}^{s-2}\left(\delta_{s+\beta p,\mathrm{odd}}+\delta_{s+\beta p,\mathrm{even}}q^{it}\right)(-1)^{i}\alpha^{i}q^{i(s-2-i)}|s;i+1,s-1-i\rangle^{\alpha} \end{aligned}$$

$$
\begin{aligned}
&+ \sum_{i=1}^{s-1} \left(\delta_{s+\beta p,\mathrm{odd}} + \delta_{s+\beta p,\mathrm{even}} q^{(i-1)t} \right) (-1)^i \alpha^{i+1} q^{i(s-2-i)-(s-1-2i)} \\
&\qquad \times |s; i, s-i\rangle^\alpha \\
&= - \sum_{i=1}^{s-1} \left(\delta_{s+\beta p,\mathrm{odd}} + \delta_{s+\beta p,\mathrm{even}} q^{(i-1)t} \right) (-1)^i \alpha^{i-1} q^{(i-1)(s-1-i)} |s; i, s-i\rangle^\alpha \\
&+ \sum_{i=1}^{s-1} \left(\delta_{s+\beta p,\mathrm{odd}} + \delta_{s+\beta p,\mathrm{even}} q^{(i-1)t} \right) (-1)^i \alpha^{i+1} q^{i(s-2-i)-(s-1-2i)} \\
&\qquad \times |s; i, s-i\rangle^\alpha \\
&= 0 \;, \qquad \text{(C.5.6)}
\end{aligned}
$$

which shows that f intertwines the action by F as well. Thus, f is a morphism in $\mathcal{C}$. □

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Summary

In the first part of this thesis, we showed how to obtain the unique non-degenerate right modified trace of the (finite tensor) category of finite-dimensional representations of a pivotal unimodular quasi-Hopf algebra H with $\dim H < \infty$. More concretely, we showed explicitly how to relate it to the so-called right symmetrized cointegral, which itself is derived from the right cointegral of H—an element in the dual space important in the general theory of quasi-Hopf algebras. Analogous statements hold when 'right' is replaced with 'left'. This generalizes the results from [BBGa] to the quasi-coassociative setting.

The second part was focused generally on what we dubbed *monadic cointegrals* for finite tensor categories, and particularly on their realizations in representation categories of (pivotal) quasi-Hopf algebras. Monadic cointegrals serve, on the nose, as a direct categorical formulation of four different types of cointegrals for Hopf algebras. The main contribution here is then that we explicitly related the monadic cointegrals associated to the representation category of a *quasi*-Hopf algebra H to the four types of cointegrals of H, given them a novel categorical interpretation. An open question at this point, as mentioned in Section 4.2.1, is whether equation (4.2.10) is enough to uniquely characterize right monadic cointegrals.

We specialized to the braided case, and found that monadic cointegrals are basically the same as Lyubashenko's integral. Assume now the category is given by modules over a quasi-Hopf algebra with appropriate adjectives. In particular we can now see that, fundamentally, both Lyubashenko's integral and the modified trace are related to the integral of the underlying quasi-Hopf algebra. One hope therefore is that this is true in the general case, i.e. that for an arbitrary but, say, factorizable finite tensor category, the modified trace can be expressed in terms of Lyubashenko's integral (or simply in terms of the monadic cointegral), providing a unifying framework. This is an interesting take, since both Lyubashenko's integral and the modified trace are the main ingredients in the non-semisimple generalization of the RT-TFT proposed in [DGGPR]. Aside from this conjecture, the results themselves are already interesting, since recently numerous examples of factorizable quasi-Hopf algebras conjecturally related to logarithmic VOAs have appeared [CGR, GR1, FGR2].

Lastly, in a brief third part, we compared two (projective) $SL(2,\mathbb{Z})$-actions: one arising as the modular invariance properties of VOA-characters of the triplet VOA $\mathcal{W}(p)$ studied in e.g. [FGST1]; the other one arising from Lyubashenko's construction applied to a certain quasi-Hopf modification of the restricted quantum group of $\mathfrak{sl}_2$ at a $2p$th root of unity, as introduced in [CGR]. We found that these actions indeed agree.

Outlook. In addition to tackling the open questions posed above, I am computing some explicit examples of the renormalized 3-manifold invariants of [DGGPR], using as input data certain (quasi-)Hopf algebras. Both the modified trace and Lyubashenko's integral (which is easily computed from the monadic cointegral) enter here. I hope to see to what extent the invariants distinguish for example the lens spaces $L(n,1)$, or if and how they can be connected to the semisimple invariants coming from the RT-construction.

Zusammenfassung

Im ersten Teil meiner Arbeit haben wir eine konkrete Konstruktion der eindeutigen nicht-entarteten rechten modifizierten Spur der (endlichen Tensor-)Kategorie endlich-dimensionaler Moduln einer pivotalen und unimodularen quasi-Hopfalgebra H mit $\dim H < \infty$ gegeben und bewiesen. Genauer haben wir explizit gezeigt, wie die modifizierte Spur durch das sogenannte rechte symmetrisierte Kointegral bestimmt wird. Dieses wiederum ist abgeleitet vom rechten Kointegral von H—einem Element des Dualraums, welches in der allgemeinen Theorie der quasi-Hopfalgebren wohlbekannt ist. Analoge Aussagen gelten auch, wenn 'rechts' durch 'links' ersetzt wird. Damit wurden Ergebnisse aus [BBGa] in den lediglich quasi-koassoziativen Rahmen verallgemeinert.

Der zweite Teil hatte einen allgemeinen Fokus auf die von uns *monadische Kointegrale* genannten Entitäten, welche sich für jede endliche Tensorkategorie definieren lassen, und einen speziellen Fokus auf deren Realisierung für quasi-Hopfalgebren. Im Falle von Hopf-Algebren sieht man leicht, dass monadische Kointegrale eine direkte kategorielle Formulierung der vier verschiedenen Kointegrale liefern. Der Hauptbeitrag dieses Teils ist nun der Beweis der exakten Beziehung zwischen den vier Kointegralen einer quasi-Hopfalgebra H und den monadischen Kointegralen in der Kategorie der H-Moduln. Insbesondere bekommen dadurch quasi-Hopf-Kointegrale eine neuartige kategorielle Interpretation. Zum jetzigen Zeitpunkt ist offen, ob—wie bereits in Abschnitt 4.2.1 erwähnt—die Gleichung (4.2.10) rechte monadische Kointegral bereits eindeutig charakterisiert.

Dann haben wir den verzopften Fall betrachtet, und herausgefunden, dass monadische Kointegrale letztendlich das gleiche sind wie Lyubashenkos Integral. Betrachten wir nun die Darstellungskategorie einer quasi-Hopfalgebra mit allen nötigen Adjektiven. Wir sehen dann insbesondere, dass Lyubashenkos Integral und die modifizierte Spur einen gemeinsamen Nenner haben, nämlich das Kointegral der quasi-Hopfalgebra. Daher hoffen wir unter anderem, dass sich dies auch im allgemeinen Fall bewahrheitet, dass also die modifizierte Spur für eine beliebige aber z.B. faktorisierbare endliche Tensorkategorie über das Lyubashenko-Integral ausgedrückt werden kann (bzw. direkt über ein monadisches Kointegral). Dies ist eine interessante Vermutung, denn Lyubashenkos Integral und die modifizierte Spur sind die Hauptzutaten der nicht-halbeinfachen Verallgemeinerung der RT-TFT aus [DGGPR]. Von dieser Vermutung abgesehen sind die Resultate selbst interessant, u.A. da in jüngerer Zeit mehrere Beispiele faktorisierbarer quasi-Hopfalgebren aufgetaucht sind, welche (mutmaßlich) eine Rolle in der Beschreibung einiger logarithmischer konformer Feldtheorien spielen [CGR, GR1, FGR2].

In einem dritten Teil, haben wir zwei (projektive) $SL(2,\mathbb{Z})$-Wirkungen verglichen: die erste fällt aus dem Verhalten der VOA-Charactere der 'triplet VOA' $\mathcal{W}(p)$ unter modularen Transformationen heraus; die andere kommt aus Lyubashenkos Konstruktion, angewandt auf die (faktorisierbare) Darstellungskategorie der in [CGR] eingeführten quasi-Hopf-Modifikation der beschränkten Quantengruppe von $\mathfrak{sl}_2$ bei einer primitiven $2p$ten Einheitswurzel. Wir sahen, dass beide Wirkungen (projektiv) übereinstimmen.

Ausblick. Zum einen plane ich, die oben genannten offenen Fragen zu beantworten. Zum anderen berechne ich explizite Beispiele der renormierten 3-Mannigfaltigkeits-Invarianten von [DGGPR], ausgehend von verschiedenen (quasi-)Hopf-Algebren. Die Ergebnisse dieser Arbeit kommen dabei durch die Verwendung der modifizierten Spur und Lyubashenkos Integral (welches wir hier über das monadische Kointegral berechnen) ins Spiel. Ich möchte sehen, bis zu welchem Grad die resultierenden Invarianten z.B. die Linsenräume $L(n, 1)$ unterscheiden können, und ob bzw. wie wir sie mit den halbeinfachen Invarianten aus der RT-Konstruktion in Verbindung setzen können.

List of publications and contributions

This is the list of publications that resulted from this dissertation, i.e. they form its basis.

1. J. Berger, A. M. Gainutdinov, I. Runkel, *Modified traces for quasi-Hopf algebras*, J. Algebra **548** (2020) 96–119, [arXiv: 1812.10445 [math.QA]].

2. J. Berger, A. M. Gainutdinov, I. Runkel, *Monadic cointegrals and applications to quasi-Hopf algebras*, J. Pure Appl. Algebra **225** (2021), [arXiv: 2003.13307 [math.QA]]

The first paper is the basis for Chapter 3. Large parts of the paper were developed in collaboration with the coauthors. The proof of Lemma 3.1.1 was developed by me, after the referee of the journal made some suggestions on how to generalize our original proof from the unimodular setting to the non-unimodular setting. Most of the generalizations of auxiliary results from [BBGa] to the quasi-Hopf setting, as well as the proof of the main theorem, and the computation of the example, were done by me, and independently verified by the coauthors.

The second paper forms the basis of Chapter 4, as well as some parts of Chapter 5. Large parts were developed in collaboration with the coauthors. All examples were computed by me, and the proofs are for the most part based on my ideas as well; in particular those of the main theorem(s). The proof of the universality of our chosen realization of the coends $A_2(V)$ was done by me, and is not contained in the paper 2. The usefulness of the adjunction between the central Hopf monad and the central Hopf comonad was noticed by A. M. Gainutdinov and me independently on the same day. The trick for the proof of the left and the $\boldsymbol{\gamma}$-symmetrized versions of the main theorem of that part was devised by me.

The first two sections of Chapter 5 are based on the seventh section of the second paper, which was for the most part done independently by me. Of the things added to the chapter in this thesis (Proposition 5.2.3 and Section 5.3), most results were derived and computed by me; the proof of the main theorem of the chapter is based on my observations and was developed in discussion with A. M. Gainutdinov.

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