Integrable sigma models from affine Gaudin models

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Abstract

In this thesis we describe recent results obtained in the area of integrable field theories. In particular, we present the construction of two new broad classes of integrable sigma models in the framework of affine Gaudin models. Firstly, we focus on integrable deformations of a class of theories defined on the direct product of N copies of a Lie group. More precisely, for $N = 1$ the corresponding models coincide with the Yang-Baxter or λ-deformations of the principal chiral model, while for general N they consist of arbitrary combinations of these deformed models. We describe both the Hamiltonian and Lagrangian formulation of models with general N and give explicit expressions of their action and Lax connection. The second class of theories is defined on a coset of the direct product of N copies of a Lie group over some diagonal subgroup, generalising the well-known symmetric space sigma model corresponding to $N = 1$. Specifying the construction to the case of two copies of the group $SU(2)$, we obtain a new three-parametric integrable sigma model on the manifold $T^{1,1}$. We comment on the connection of our results with the ones existing in the literature.

Zusammenfassung

In dieser Arbeit beschreiben wir aktuelle Ergebnisse im Bereich integrierbarer Feldtheorien. Insbesondere präsentieren wir die Konstruktion von zwei neuen, breiten Klassen integrierbarer Sigma-Modelle im Rahmen affiner Gaudin-Modelle. Erstens konzentrieren wir uns auf integrierbare Deformationen einer Klasse von Theorien, die für das direkte Produkt von N Kopien einer Lie-Gruppe definiert sind. Genauer gesagt, für $N = 1$ stimmen die entsprechenden Modelle mit den Yang-Baxter- oder λ-Deformationen des chiralen Hauptmodells überein, während sie für allgemein N aus beliebigen Kombinationen dieser deformierten Modelle bestehen. Wir beschreiben sowohl die Hamilton- als auch die Lagrange-Formulierung von Modellen mit allgemeinem N und geben explizite Ausdrücke ihrer Wirkung und ihrer Lax-Verbindung. Die zweite Klasse von Theorien wird auf einer Nebenmenge des direkten Produkts von N Kopien einer Lie-Gruppe über eine diagonale Untergruppe definiert, wobei das bekannte, symmetrische Raum-Sigma-Modell verallgemeinert wird, welches $N = 1$ entspricht. Wenn wir die Konstruktion für zwei Kopien der Gruppe $SU(2)$ spezifizieren, erhalten wir ein neues, integrierbares Sigma-Modell mit drei Parametern auf der Mannigfaltigkeit $T^{1,1}$. Wir kommentieren den Zusammenhang unserer Ergebnisse mit den in der Literatur vorhandenen.

This thesis is based on the following articles, which I wrote together with my collaborators during my doctoral project:

- C. Bassi and S. Lacroix, *Integrable deformations of coupled sigma models*, *JHEP* **05** (2020) 059 [arXiv:1912.06157].
- G. Arutyunov, C. Bassi and S. Lacroix, *New integrable coset sigma models*, *JHEP* **03** (2021) 062 [arXiv:2010.05573]

Contents

Chapter 1

Introduction

The description of a great number of physical phenomena is formalised in the language of field theories. Notable examples of this fact include the Standard Model of particle physics, which describes the fundamental constituents of matter and their interactions, general relativity and also systems relevant to condensed matter and fluid theory. One of the main objectives of such description is to determine how the system under consideration evolves in time. In mathematical terms, this comes to the problem of solving certain partial differential equations for the fields which describe the system. In particular, most of the methods that we employ for the solution of this problem are based on perturbative expansions in the parameter controlling the interactions of the fields. However, despite the success of these methods in many contexts, numerous examples of field theories cannot be studied in this way. It occurs nonetheless that for some systems it is possible to go beyond the perturbative expansion and obtain an exact solution of the equations describing their evolution. In particular, it is the case that for some models the amount of symmetries they possess is sufficient to exactly solve their motion. We speak in this case of integrable systems. The systematic study of these systems has given rise to a great number of exact techniques which are vastly applied currently in theoretical physics. These techniques are known collectively under as theory of integrability.

The formalisation of this theory started early on with the systematic characterisation of integrable models in Hamiltonian mechanics by the theorem of Liouville [1] and its global extension by Arnol'd [2]. According to this theorem, if a system with $2n$ Hamiltonian degrees of freedom possesses n independent conserved quantities in involution, *i.e.* Poisson commuting one with another, then it can be solved (integrated) exactly by quadratures. The study of such integrable systems has been further advanced by the introduction of the notion of Lax pairs. In particular, the integrability of a mechanical system can be inferred from the existence of two matrices L and M, the Lax pair, which are functions of the phase space coordinates, such that the equations of motion of the model can be rewritten in the form of the so-called Lax equation and the components of the matrix L satisfy specific Poisson bracket relations which ensure the existence of conserved charges in involution.

A similar concept of Lax pair can be used to define integrability for classical two-dimensional field theories. In this context, we require that the equations of motion can be recast in the form of a zero curvature equation for a two-dimensional connection [3], called the Lax connection, whose spatial and temporal components are matrices depending on fields of the model and on an auxiliary complex parameter known as the spectral parameter. Given such a formulation, there is a canonical procedure to construct an infinite number of conserved charges from the spatial part of the connection, called the Lax matrix. Furthermore, integrability follows from the fact that the components of this Lax matrix satisfy certain Poisson bracket relations of the Sklyanin [4] or Maillet [5,6] type, which imply that the conserved charges we construct are in involution. This Lax formulation of integrable field theories stands at the basis of the so-called inverse scattering method [7,8], which brought to many developments in the theory of solitons, for example for the Korteweg-de Vries [9], non-linear Schrödinger [10] and Sine-Gordon [11] equations.

In this thesis we focus on a particular kind of classical integrable field theories known as integrable (non-linear) sigma models, which have shown important applications to various fields of theoretical physics. Prototypical examples of these models include the principal chiral model on a Lie group and the coset sigma model on a symmetric space. For example, an instance of the coset model, the so-called non-linear $O(3)$ model, is known in condensed matter theory as the continuum limit of certain spin systems which are useful for the description of the magnetic properties of some materials [12]. Integrable sigma models have also been found to have important applications relevant to high-energy physics and string theory and in particular in the domain of the AdS/CFT correspondence [13–15]. In this context, a crucial result was the discovery in [16] of the classical integrability of the Green-Schwarz superstring sigma model on the $AdS_5 \times S^5$ background [17] (see also the review [18]). Following this result, integrability techniques have been applied fruitfully to both the string model and its dual $\mathcal{N} = 4$ supersymmetric Yang-Mills theory (see the review [19]), allowing in particular the exact computation of certain physical observables of this 4d gauge theory.

Even more recently, integrable sigma models have attracted interest following the discovery of their integrable deformations. The latter deform the initial theories while preserving integrability and introducing a set of continuous parameters, hence providing whole new classes of integrable models. Main examples are given by the Yang-Baxter and the λ-deformation of the principal chiral model, which have been obtained in [20, 21] and [22] for the case of a generic Lie group, generalising the results found in [23] and [24] for the low-dimensional target space $SU(2)$. These deformations have then been extended in [25,26] and [27] to the $AdS_5 \times S^5$ superstring, where they allow to lift partially or completely its supersymmetries and where they have attracted recently a lot of attention concerning the nature of the corresponding supergravity backgrounds [28–41] (see the recent review [42] for a complete list of references).

These new developments stimulated further explorations into the general properties of integrable sigma models. Firstly, studying these theories in the Hamiltonian formulation, it was shown that the classical integrability structure of a very broad class of these models is controlled by a rational function of the spectral parameter known as the twist function, which characterises the Poisson bracket of the Lax matrix [43–46]. This class of theories was remarkably proven to include integrable models such as the principal chiral model [43], its deformations [47–51] and models relevant to the AdS/CFT correspondence [27, 46, 52, 53]. More recently, integrable field theories with a twist function have been shown to be particular realisations of classical Gaudin models associated with affine Kac-Moody algebras.

Gaudin models are integrable systems associated with Lie algebras. Historically, they were introduced for finite dimensional Lie algebras, in which case they describe classes of integrable spin systems [54, 55]. They were later generalised to the infinite dimensional setting of affine Kac-Moody algebras in [56, 57], where it was shown that they describe integrable field theories. In particular, the way classical fields are recovered in this context is by realising the underlying affine Kac-Moody algebra as a centrally extended current algebra on the circle. Moreover, as shown in [56, 57] affine Gaudin models provide a procedure to define integrable field theories in a systematic way. In particular, it was shown in [57] that this procedure allows to recover many known examples of integrable sigma models, including the principal chiral model and its deformations.

The consideration of more general examples of affine Gaudin models offers an unprecedented possibility to construct and study a new large panorama of integrable field theories. For instance, the scope of this panorama of models has been substantially widened by the methods developed recently in [58,59], where a procedure has been outlined to obtain more general affine Gaudin models by coupling together non-trivially basic instances of them. This procedure was applied in the same references to construct a theory coupling together an arbitrary number of principal chiral models on the same Lie group. Many possibilities offered by this method for the exploration of the panorama of integrable sigma models still remain to be studied. The present thesis intends to contribute precisely to this exploration.

Firstly, we deal with Yang-Baxter and λ-deformations of the coupled principal chiral models introduced in [58,59]. More precisely, after recalling briefly the main results about affine Gaudin models in chapter 2, we apply these methods to the construction of a class of models coupling together an arbitrary number N of Yang-Baxter and λ-deformations of the principal chiral model. In particular, we define these models in the Hamiltonian formulation, finding explicit expressions for their Hamiltonian and Lax connection. We perform subsequently the inverse Legendre transform and construct the Lagrangian formulation, obtaining the action of the models in terms of a certain number of free parameters. Furthermore, we show that taking particular limits in these parameters we recover the coupled models obtained recently in [60–63], which can thus be seen as particular instances of the theories constructed here. As another result, we describe the relation of the models constructed here with the 4d semi-holomorphic Chern-Simons theory introduced in [64] (see also [65–69]). It was shown in [70] that this 4d Chern-Simons theory can be used to generate broad classes of integrable two-dimensional field theories. Moreover, it was discussed in [71] that it is deeply related with the framework of affine Gaudin models. It is natural to search for a construction of the deformed coupled sigma models considered here from the 4d semi-holomorphic Chern-Simons theory. We present this construction explicitly and relate it to the affine Gaudin model approach. This result and the ones mentioned above are the subject of chapter 3 of the thesis.

Furthermore, by considering the general framework of dihedral affine Gaudin models presented in [57], we construct a new class of integrable sigma models on coset target spaces. These models are defined on the coset G^N/H given by the product of an arbitrary number N of copies of a Lie group G modulo the action of a diagonal gauge subgroup H. For $N = 1$ they coincide with the well-known symmetric space sigma model. We describe in detail how the gauge symmetry associated to H is implemented at the Hamiltonian level by imposing a first-class constraint on the phase space of the corresponding affine Gaudin models. By performing the inverse Legendre transform explicitly in the case of models with two copies, we obtain the action of these theories. We show that this action admits a recasting in terms of the so-called $\mathcal{R}$-matrix of the models, which allows us to find a generalisation to the case of an arbitrary number of copies depending on $3N-2$ free parameters. As a concrete application of the model with two copies, we specify our construction to the $SU(2)$ case and find a new integrable sigma model on the space $T^{1,1} = SU(2) \times SU(2)/U(1)$. In this context, we note that the presence of the B-field found from the affine Gaudin models construction is crucial for the integrability of the model, arguing moreover that any other choice of B-field would exhibit a non-integrable behaviour. These results are contained in chapter 4.

We end the thesis with some concluding remarks about general perspectives of the work presented in it.

Chapter 2

Affine Gaudin models

In this chapter we introduce the formalism of affine Gaudin models as a framework to construct very broad classes of classical integrable field theories with twist function. This formalism will be then applied in the rest of the thesis to construct integrable sigma models on product and coset manifolds built from a generic Lie group G. In order to fix notations, let us start by recalling some general notions about classical integrable field theories.

2.1 Classical integrable field theories

In this section we describe the notion of Lax formulation for field theories. We explain how this formulation allows one to find an infinite set of conserved charges from the so-called monodromy matrix. Subsequently, we introduce two other central notions in the study of the integrable models described in this thesis, namely the Maillet bracket and the $\mathcal{R}$-matrix, which emerge from the Hamiltonian analysis of integrable field theories.

2.1.1 Lax formalism

Let us consider a field theory described in terms of dynamical fields ϕ_i depending on the coordinates (x,t) of a two-dimensional Minkowski space-time $\mathbb{D}\times\mathbb{R}$, where we take either $\mathbb{D}=\mathbb{R}$ or $\mathbb{D}=S^1$. The time evolution of these fields is dictated by their equations of motion, namely a set of partial differential equations with properly chosen boundary conditions. We will show now that a Lax reformulation of these equations allows one to find an infinite set of conserved charges for the field theory.

Lax connection. Let us consider a one-form $\mathcal{L}_1(z)=\mathcal{M}(z,x,t)\mathrm{d}t+\mathcal{L}(z,x,t)\mathrm{d}x$ on the two-dimensional Minkowksi space-time $\mathbb{D}\times\mathbb{R}$, where z is an auxiliary complex parameter called the spectral parameter and $\mathcal{M}(z,x,t)$ and $\mathcal{L}(z,x,t)$ are valued in a complex Lie algebra $\mathfrak{g}^{\mathbb{C}}$ with Lie bracket $[\cdot,\cdot]$. The field theory under consideration is said to admit a Lax reformulation if the equations of motion for the fields ϕ_i can be rewritten in the form of the Lax equation [3]:

$$\partial_t\mathcal{L}(z,x,t)-\partial_x\mathcal{M}(z,x,t)+[\mathcal{M}(z,x,t),\mathcal{L}(z,x,t)]=0, \tag{2.1.1}$$

for all $z\in\mathbb{C}$. We will often refer to (2.1.1) as the zero curvature equation since it can be rewritten in terms of $\mathcal{L}_1$ as $\mathrm{d}\mathcal{L}_1+\mathcal{L}_1\wedge\mathcal{L}_1=0$, which is the requirement that the curvature of the two-dimensional connection $\nabla=\mathrm{d}+\mathcal{L}_1$ vanishes. For this reason the pair $(\mathcal{M}(z,x,t),\mathcal{L}(z,x,t))$ is often referred to as the Lax connection of the theory.

Monodromy matrix and integrability. Let us show how the Lax equation (2.1.1) implies the existence of an infinite set of conserved charges under the time evolution of the theory. Let $G^{\mathbb{C}}$ be a connected Lie group with Lie algebra $\mathfrak{g}^{\mathbb{C}}$. We define the transfer matrix of the connection ∇ between two points a and b as the following $G^{\mathbb{C}}$-valued path-ordered exponential:

$$T(z,a,b,t) = P\overleftarrow{\exp}\left(-\int_a^b \mathrm{d}x\ \mathcal{L}(z,x,t)\right). \tag{2.1.2}$$

As a consequence of the Lax equation (3.3.1), the transfer matrix satisfies the following important property (see for example [72]):

$$\partial_t T(z,a,b,t) = T(z,a,b,t)\mathcal{M}(z,b,t) - \mathcal{M}(z,a,t)T(z,a,b,t), \tag{2.1.3}$$

for all $z \in \mathbb{C}$.

Let us distinguish the two cases $\mathbb{D} = \mathbb{R}$ and $\mathbb{D} = S^1$. In the first one, we define the monodromy matrix as the following path-ordered exponential:

$$T(z,t) = T(z,-\infty,+\infty,t). \tag{2.1.4}$$

If we suppose that the field $\mathcal{M}(z,x,t)$ goes to zero sufficiently fast at $x = \pm\infty$, equation (2.1.3) becomes:

$$\partial_t T(z,t) = 0,\ \forall z \in \mathbb{C}. \tag{2.1.5}$$

The Lax equation (2.1.1) thus implies that the whole monodromy matrix $T(z,t)$ is conserved for every value of the spectral parameter z. Thanks to this arbitrariness, one gets in general an infinite number of conserved charges from $T(z,t)$ by varying z or by expanding in power series.

Let us consider the second possibility $\mathbb{D} = S^1 \simeq [0,2\pi]$. In this case, we define the monodromy matrix as:

$$T(z,t) = T(z,0,2\pi,t). \tag{2.1.6}$$

Demanding periodic boundary conditions $\mathcal{M}(z,0,t) = \mathcal{M}(z,2\pi,t)$ the expression in the right hand side of (2.1.3) becomes a commutator. It follows that not the whole monodromy matrix is conserved this time. However, the images under conjugacy invariant functions $\Phi : G^{\mathbb{C}} \to \mathbb{R}$ of this matrix remain conserved. Such functions can be constructed as traces of powers of the given group element. For example, in the case of a semi-simple Lie algebra $\mathfrak{g}^{\mathbb{C}}$ it is known that such choice of functions generates all conjugacy invariant functions on $G^{\mathbb{C}}$. In particular, it holds that (in a chosen representation):

$$\partial_t \operatorname{tr}\left(T(z,t)^n\right) = 0,\ \forall n \in \mathbb{Z}_+,\ \forall z \in \mathbb{C}. \tag{2.1.7}$$

The dependence on the spectral parameter then implies again the existence of an infinite set of conserved charges.

Since (2.1.7) holds also for $\mathbb{D} = \mathbb{R}$, we can summarise the concepts discussed so far by stating that the Lax equation (2.1.1) implies the conservation of the following set of charges:

$$\mathcal{Q}_n(z) = \operatorname{tr}\left(T(z,t)^n\right). \tag{2.1.8}$$

Finally, we note that the charges extracted from the monodromy matrix are in general non-local. However, a specific property of integrable models with twist function, which we will define later in this section, is that they possess an infinite set of local conserved charges independent from the ones extracted from the monodromy matrix. We shall see in the main text how these charges are constructed in the specific models described in later chapters.

2.1.2 Hamiltonian formulation and Maillet Poisson brackets

In this section, we define the concept of integrability for a field theory by requiring that it possesses an infinite set of conserved charges in involution in the Hamiltonian formulation. This will bring us to the important notion of the Maillet bracket for the spatial part of the Lax connection. Let us first briefly recall the Hamiltonian formulation of classical field theories.

Hamiltonian field theory. Let us consider the phase space of the theory as the Poisson manifold M which describes the field configurations of the fundamental dynamical fields of the model ϕ_i. Hence, the space of functionals $\mathcal{F}(M)$ over M is equipped with a Poisson bracket:

$$\begin{array}{rcl} \{\cdot,\cdot\}:\mathcal{F}(M)\times\mathcal{F}(M) & \longrightarrow & \mathcal{F}(M) \\ (f,g) & \mapsto & \{f,g\}, \end{array} \tag{2.1.9}$$

which is bilinear and skew-symmetric in its arguments. This bracket satisfies the following properties:

$$\{f_1, f_2 f_3\} = \{f_1, f_2\} f_3 + f_2\{f_1, f_3\}, \tag{2.1.10}$$

$$\{f_1, \{f_2, f_3\}\} + \{f_3, \{f_1, f_2\}\} + \{f_2, \{f_3, f_1\}\} = 0, \tag{2.1.11}$$

$\forall f_1, f_2, f_3 \in \mathcal{F}(M)$, known as the Leibniz rule and the Jacobi identity, respectively.

In the Hamiltonian formulation, the time evolution of the field theory is generated by a functional $\mathcal{H} \in \mathcal{F}(M)$ called the Hamiltonian. More precisely, for any functional $f \in \mathcal{F}(M)$ which does not depend explicitly on time, its time evolution is given by the following Hamiltonian flow:

$$\partial_t f = \{\mathcal{H}, f\}. \tag{2.1.12}$$

In particular, if f and $\mathcal{H}$ are in involution, *i.e.* $\{\mathcal{H}, f\} = 0$, f is conserved in time.

Lax matrix and involution of the charges. Working in the Hamiltonian formulation we consider now fields defined on an equal time slice. We will hence drop the t-dependence from their expressions. Supposing that the Lax connection $(\mathcal{M}(z,x), \mathcal{L}(z,x))$ depends on time only through the fields ϕ_i, the zero curvature equation (2.1.1) takes the following form in the Hamiltonian formalism:

$$\{\mathcal{H}, \mathcal{L}(z,x)\} - \partial_x \mathcal{M}(z,x) + [\mathcal{M}(z,x), \mathcal{L}(z,x)] = 0. \tag{2.1.13}$$

As a consequence, one can extract an infinite set of charges from the monodromy matrix $T(z)$ which are conserved under time evolution. The field theory is said to be integrable if these charges are in involution. Since they are constructed from the spatial part $\mathcal{L}(z,x)$ of the Lax connection, we will now describe two Poisson bracket relations for $\mathcal{L}(z,x)$ that ensure this property.

In order to do this in a compact way, let us first introduce some notation. For any $\mathfrak{g}^{\mathbb{C}}$-valued functional X, *i.e.* $X \in \mathfrak{g}^{\mathbb{C}} \otimes \mathcal{F}(M)$, we define

$$X_{\underline{\mathbf{1}}} = X \otimes \mathrm{Id} \quad \text{and} \quad X_{\underline{\mathbf{2}}} = \mathrm{Id} \otimes X, \tag{2.1.14}$$

belonging to $U(\mathfrak{g}^{\mathbb{C}}) \otimes U(\mathfrak{g}^{\mathbb{C}}) \otimes \mathcal{F}(M)$, where $U(\mathfrak{g}^{\mathbb{C}})$ is the universal enveloping algebra of $\mathfrak{g}^{\mathbb{C}}$. Let us consider a basis $\{I^a\}$, $a = 1, \dots, \dim \mathfrak{g}^{\mathbb{C}}$ of the Lie algebra $\mathfrak{g}^{\mathbb{C}}$ and decompose any two $\mathfrak{g}^{\mathbb{C}}$-valued functionals X and Y as $X = X_a I^a$ and $Y = Y_b I^b$. We can then write compactly their Poisson bracket as

$$\{X_{\underline{\mathbf{1}}}, Y_{\underline{\mathbf{2}}}\} = \{X_a, Y_b\}\, I^a \otimes I^b, \tag{2.1.15}$$

where the Poisson bracket on the right hand side is the one described at the beginning of this section.

It was found by Sklyanin in [4] (see also [73]) that a sufficient condition for the conserved charges constructed from the monodromy matrix to be in involution is that $\mathcal{L}(z,x)$ has a Poisson bracket of the form

$$\{\mathcal{L}_{\underline{\mathbf{1}}}(z,x), \mathcal{L}_{\underline{\mathbf{2}}}(w,y)\} = [\mathcal{R}_{\underline{\mathbf{12}}}(z,w), \mathcal{L}_{\underline{\mathbf{1}}}(z,x) + \mathcal{L}_{\underline{\mathbf{2}}}(w,x)]\delta_{xy}, \tag{2.1.16}$$

where we denoted the Dirac distribution $\delta(x-y)$ by δ_{xy} and $\mathcal{R}_{\underline{\mathbf{12}}}$ is a $\mathfrak{g}^{\mathbb{C}} \otimes \mathfrak{g}^{\mathbb{C}}$-valued matrix depending on the spectral parameters z and w. This matrix is known as the $\mathcal{R}$-matrix of the theory. In order for (2.1.16) to satisfy the properties of Poisson brackets, this matrix has to satisfy two constraints. Firstly, the antisymmetry of the Poisson bracket requires it to be antisymmetric:

$$\mathcal{R}_{\underline{\mathbf{12}}}(z,w) = -\mathcal{R}_{\underline{\mathbf{21}}}(w,z). \tag{2.1.17}$$

Let us now consider the classical Yang-Baxter equation (CYBE),

$$[\mathcal{R}_{\underline{\mathbf{12}}}(z_1,z_2), \mathcal{R}_{\underline{\mathbf{13}}}(z_1,z_3)] + [\mathcal{R}_{\underline{\mathbf{12}}}(z_1,z_2), \mathcal{R}_{\underline{\mathbf{23}}}(z_3,z_3)] + [\mathcal{R}_{\underline{\mathbf{32}}}(z_3,z_2), \mathcal{R}_{\underline{\mathbf{13}}}(z_1,z_3)] = 0, \tag{2.1.18}$$

which is an identity in $\mathfrak{g}^{\mathbb{C}} \otimes \mathfrak{g}^{\mathbb{C}} \otimes \mathfrak{g}^{\mathbb{C}}$. This equation provides a sufficient condition for the bracket (2.1.16) to satisfy the Jacobi identity. In the following, we will restrict to the case in which $\mathcal{R}_{\underline{\mathbf{12}}}$ is a solution of the CYBE. The reason for this is that equation (2.1.18) is algebraic and non-dynamical, and thus its solutions can be constructed following general schemes, as we shall explain in the next section.

The bracket (2.1.16) is said to be ultralocal since it only contains terms which are of order zero in the derivatives of the δ-distribution. However, the models we will consider in this thesis are characterised by the presence of higher order terms in these derivatives in the Poisson bracket of $\mathcal{L}(z,x)$. Such a non-ultralocal generalisation is obtained for the case of non-skew-symmetric $\mathcal{R}$-matrix and it takes the name of Maillet bracket [5,6]. It reads:

$$\begin{aligned}\{\mathcal{L}_{\underline{\mathbf{1}}}(z,x), \mathcal{L}_{\underline{\mathbf{2}}}(w,y)\} = [\mathcal{R}_{\underline{\mathbf{12}}}(z,w), \mathcal{L}_{\underline{\mathbf{1}}}(z,x)]\delta_{xy} - [\mathcal{R}_{\underline{\mathbf{21}}}(w,z), \mathcal{L}_{\underline{\mathbf{2}}}(w,x)]\delta_{xy}\\ - (\mathcal{R}_{\underline{\mathbf{12}}}(z,w) + \mathcal{R}_{\underline{\mathbf{21}}}(w,z))\delta'_{xy},\end{aligned} \tag{2.1.19}$$

where we denoted $\delta'_{xy} = \partial_x \delta(x-y)$. As stated for the Sklyanin bracket, in the following $\mathcal{R}_{\underline{\mathbf{12}}}$ will be taken to be a solution of the CYBE (2.1.18).

2.1.3 $\mathcal{R}$-matrices and twist function

In this section, we briefly discuss the standard $\mathcal{R}$-matrix solutions to the CYBE, which will be crucial to the description of the integrability of the models contained in the next chapters.

Split quadratic Casimir. Let us suppose that $\mathfrak{g}^{\mathbb{C}}$ is a Lie algebra with a non-degenerate invariant bilinear form κ. Let us introduce the split quadratic Casimir of $\mathfrak{g}^{\mathbb{C}}$ as the following symmetric combination of the elements of the basis $\{I^a\}$ of $\mathfrak{g}^{\mathbb{C}}$:

$$C_{\underline{\mathbf{12}}} = \kappa_{ab}\, I^a \otimes I^b, \tag{2.1.20}$$

For any $X \in \mathfrak{g}^{\mathbb{C}}$, one can check that it satisfies

$$\kappa_{\underline{\mathbf{2}}}(C_{\underline{\mathbf{12}}}, X_{\underline{\mathbf{2}}}) = X. \tag{2.1.21}$$

Moreover, from the ad-invariance of the bilinear form κ, it follows that

$$[C_{\underline{\mathbf{12}}}, X_{\underline{\mathbf{1}}} + X_{\underline{\mathbf{2}}}] = 0, \tag{2.1.22}$$

$\forall X \in \mathfrak{g}^{\mathbb{C}}$.

Standard untwisted $\mathcal{R}$-matrix and twist function. From (2.1.22), the split quadratic Casimir satisfies the following identity on $\mathfrak{g}^{\mathbb{C}} \otimes \mathfrak{g}^{\mathbb{C}} \otimes \mathfrak{g}^{\mathbb{C}}$:

$$[C_{\underline{\mathbf{12}}}, C_{\underline{\mathbf{23}}}] = [C_{\underline{\mathbf{32}}}, C_{\underline{\mathbf{13}}}] = -[C_{\underline{\mathbf{12}}}, C_{\underline{\mathbf{13}}}]. \tag{2.1.23}$$

Together with the circle lemma,

$$\frac{1}{z_2 - z_1}\frac{1}{z_3 - z_1} - \frac{1}{z_2 - z_1}\frac{1}{z_3 - z_2} - \frac{1}{z_2 - z_3}\frac{1}{z_3 - z_1} = 0, \tag{2.1.24}$$

identity (2.1.23) implies that a particular solution of the CYBE is given by

$$\mathcal{R}^0_{\underline{\mathbf{12}}}(z, w) = \frac{C_{\underline{\mathbf{12}}}}{w - z}, \tag{2.1.25}$$

which is known as standard untwisted $\mathcal{R}$-matrix. We note that since the split quadratic Casimir $C_{\underline{\mathbf{12}}}$ is symmetric $\mathcal{R}_{\underline{\mathbf{12}}}$ is skew-symmetric.

The standard untwisted $\mathcal{R}$-matrix (2.1.25) is part of an infinite family of solutions of the CYBE which is obtained by dividing this solution by an arbitrary complex function $\varphi(w)$ of the spectral parameter w. This complex function, known as the twist function of the theory [46] (see also [43, 45]), will be a crucial element in the rest of the thesis. Explicitly, the family of solution is given by

$$\mathcal{R}_{\underline{\mathbf{12}}}(z, w) = \mathcal{R}^0_{\underline{\mathbf{12}}}(z, w)\varphi(w)^{-1}. \tag{2.1.26}$$

We note that this solution is in general non-skew-symmetric.

Standard twisted $\mathcal{R}$-matrix and twist function. Let us suppose that $\mathfrak{g}^{\mathbb{C}}$ is semi-simple. Let us also suppose that on the Lie algebra $\mathfrak{g}^{\mathbb{C}}$ acts an automorphism σ of order $T \in \mathbb{Z}_{\geq 1}$. It is a standard result that this automorphism preserves the bilinear form κ. This fact is equivalent to the following identity for the split quadratic Casimir:

$$\sigma_{\underline{\mathbf{1}}}\sigma_{\underline{\mathbf{2}}}C_{\underline{\mathbf{12}}} = C_{\underline{\mathbf{12}}}. \tag{2.1.27}$$

Let us take ω a T-th rooth of unity:

$$\omega = \exp\left(\frac{2\pi i}{T}\right). \tag{2.1.28}$$

Using the identity (2.1.27) and the fact that σ is an automorphism of $\mathfrak{g}^{\mathbb{C}}$, one can construct another solution of the CYBE as

$$\mathcal{R}^0_{\underline{\mathbf{12}}}(z, w) = \frac{1}{T}\sum_{k=0}^{T-1} \frac{\sigma^k_{\underline{\mathbf{1}}} C_{\underline{\mathbf{12}}}}{w - \omega^{-k} z}. \tag{2.1.29}$$

This solution is known as the standard twisted $\mathcal{R}$-matrix. Again, by dividing by a twist function one finds an infinite family of solutions as in (2.1.26). Note that the twist function has nothing to do with the $\mathcal{R}$-matrix (2.1.29) being called twisted, although both involve the word twist.

2.2 AGMs as integrable field theories with twist function

In this section we review the construction of affine Gaudin models. These models were introduced by Feigin and Frenkel in [56], where they were obtained from affine Kac-Moody algebras.

They were further developed by Vicedo in [57]. We review here the main results of the construction contained in these references, following the conventions and notations of [59]. Starting with a brief motivation of the definition of affine Gaudin models (AGMs), we describe the Hamiltonian formulation of the models and discuss their space-time symmetries and integrability. We conclude this section by providing basic examples of this construction, showing how the formalism of AGMs can be used to recover known integrable field theories such as the principal chiral model and its coupled version recently introduced in [58, 59].

The models discussed in this section are the simplest examples of AGMs. More general models built on a Lie algebra $\mathfrak{g}^{\mathbb{C}}$ admitting a finite order automorphism, also known as dihedral AGMs, will be discussed in section 4.2.1.

2.2.1 Motivation

Let us start by considering an integrable field theory with Lax matrix $\mathcal{L}(z,x)$ and twist function $\varphi(z)$, as described in the previous section. Let us assume that it satisfies the Maillet bracket (2.1.19) with $\mathcal{R}$-matrix given by the standard untwisted $\mathcal{R}$-matrix (2.1.26). To better understand the structure of this theory, let us define the following $\mathfrak{g}^{\mathbb{C}}$-valued field from $\mathcal{L}(z,x)$ and $\varphi(z)$:

$$\Gamma(z,x) = \varphi(z)\mathcal{L}(z,x). \tag{2.2.1}$$

This new field is known as the Gaudin Lax matrix of the theory. Let us now suppose that both $\Gamma(z,x)$ and $\varphi(z)$ depend rationally on z and that they have only simple poles at $N \in \mathbb{Z}_+$ positions z_r, $r = 1, \dots, N$ in the complex plane, so that they can be generically written as[1]

$$\varphi(z) = \sum_{r=1}^{N} \frac{\ell_r}{z - z_r} - \ell^\infty \quad \text{and} \quad \Gamma(z,x) = \sum_{r=1}^{N} \frac{\mathcal{J}_r(x)}{z - z_r}, \tag{2.2.2}$$

where the ℓ_r and ℓ^∞ are complex numbers and the $\mathcal{J}_r(x)$ are $\mathfrak{g}^{\mathbb{C}}$-valued fields. From this assumption, one finds that the Maillet bracket for the Lax matrix $\mathcal{L}(z,x)$ can be resolved in terms of the fields $\mathcal{J}_r(x)$ provided they satisfy the following Poisson brackets:

$$\{\mathcal{J}_{r\underline{\mathbf{1}}}(x), \mathcal{J}_{s\underline{\mathbf{2}}}(y)\} = \delta_{rs}\left([C_{\underline{\mathbf{12}}}, \mathcal{J}_{r\underline{\mathbf{1}}}(x)]\delta_{xy} - \ell_r C_{\underline{\mathbf{12}}}\delta'_{xy}\right). \tag{2.2.3}$$

Hence, the Maillet bracket is inherited in this case from the fields $\mathcal{J}_r(x)$ being commuting Kac-Moody currents with levels ℓ_r.

As we shall explain now, in the framework of AGMs one applies the logic described above in reverse to construct integrable field theories from a similar set of currents.

2.2.2 Realisations of affine Gaudin models

In this section we define AGMs in the Hamiltonian formulation. We start by introducing Takiff currents $\mathcal{J}_{r,[p]}$ generalising the fields $\mathcal{J}_r$ seen in the previous section to treat the case of higher order poles at the positions z_r. We will see in the examples section below that this generalisation is necessary if one wants to recover from this construction basic instances of integrable field theories such as the principal chiral model on a Lie group.

[1] In principle, one could also add a constant term in the spectral parameter z to the Gaudin Lax matrix $\Gamma(z,x)$. However, one can see that in order to satisfy the Maillet bracket (2.1.19) this term would need to be non-dynamical, *i.e.* its Poisson bracket with any other observable should vanish. For this reason and since this term will not play a role in the rest, we will omit it from the discussion.

Takiff currents. Let us consider a set of N positions $z_r \in \mathbb{C}$ in the complex plane and a set of N integer numbers $m_r \in \mathbb{Z}_{\geq 1}$, which we respectively refer to as the sites and multiplicities (by a slight abuse of notation). To every site we associate m_r complex numbers $\ell_{r,p}$, $p = 1, \dots, m_r$, which we call the levels and m_r $\mathfrak{g}^{\mathbb{C}}$-valued currents $\mathcal{J}_{r,[p]}(x)$, called the Takiff currents. We suppose that these currents satisfy the following Poisson bracket relations:

$$\{\mathcal{J}_{r,[p]\underline{\mathbf{1}}}(x), \mathcal{J}_{s,[q]\underline{\mathbf{2}}}(y)\} = \delta_{rs} \begin{cases} [C_{\underline{\mathbf{12}}}, \mathcal{J}_{r,[p+q]\underline{\mathbf{1}}}(x)]\delta_{xy} - \ell_{r,p+q} C_{\underline{\mathbf{12}}} \delta'_{xy} & \text{if } p+q < m_r \\ 0 & \text{if } p+q \geq m_r \end{cases}. \tag{2.2.4}$$

These relations generalise to the case $m_r > 1$ the Poisson algebra (2.2.3) of Kac-Moody currents ($m_r = 1$). Together, the sites, the levels and the Takiff currents specify the defining data of an AGM.

Abstractly, we can consider the phase space of the AGM as given by the configurations of the Takiff currents $\mathcal{J}_{r,[p]}(x)$. In the rest of the thesis we will be interested in different phase spaces. Having a phase space M, we thus suppose that we can form combinations of the fields belonging to M that realise the Takiff algebra (2.2.4). We speak in this case of the Takiff realisation in M. In the following we will assume that the currents $\mathcal{J}_{r,[p]}(x)$ are realised in this way. This will allow us to apply directly the formalism developed here to chapters 3 and 4.

Gaudin Lax matrix and twist function. As anticipated in the previous section, two main quantities appearing in the study of AGMs are the Gaudin Lax matrix and the twist function. In the case of higher multiplicities $m_r > 1$, they are defined as the generalisation to higher order poles of the expressions (2.2.2) found in the previous section. In particular, the Gaudin Lax matrix for a realisation of an AGM is defined as the following $\mathfrak{g}^{\mathbb{C}}$-valued observable depending on the spectral parameter z:

$$\Gamma(z, x) = \sum_{r=1}^{N} \sum_{p=0}^{m_r - 1} \frac{\mathcal{J}_{r,[p]}(x)}{(z - z_r)^{p+1}}, \tag{2.2.5}$$

Similarly, the twist function is defined as follows:

$$\varphi(z) = \sum_{r=1}^{N} \sum_{p=0}^{m_r - 1} \frac{\ell_{r,p}}{(z - z_r)^{p+1}} - \ell^\infty. \tag{2.2.6}$$

Let us make a brief remark. The parameter ℓ^∞ could be seen as the level of a site with position ∞ and multiplicity 2. As explained in [57], this site at infinity is treated slightly differently from the others. In this thesis, it will not be necessary to give further details on this, instead we can treat ℓ^∞ more simply as an additional parameter without associating it to a site. Moreover, we assume in this chapter that it is non-zero.

From the Poisson brackets (2.2.4) for the currents $\mathcal{J}_{r,[p]}$, one finds that the Gaudin Lax matrix satisfies the following bracket controlled by the twist function:

$$\begin{aligned} \{\Gamma_{\underline{\mathbf{1}}}(z, x), \Gamma_{\underline{\mathbf{2}}}(w, y)\} = {} & [\mathcal{R}^0_{\underline{\mathbf{12}}}(z, w), \Gamma_{\underline{\mathbf{1}}}(z, x)]\delta_{xy} - [\mathcal{R}^0_{\underline{\mathbf{21}}}(w, z), \Gamma_{\underline{\mathbf{2}}}(w, x)]\delta_{xy} \\ & - \left(\mathcal{R}^0_{\underline{\mathbf{12}}}(z, w)\varphi(z) + \mathcal{R}^0_{\underline{\mathbf{21}}}(w, z)\varphi(w)\right) \delta'_{xy}, \end{aligned} \tag{2.2.7}$$

with $\mathcal{R}^0_{\underline{\mathbf{12}}}$ given by the standard untwisted $\mathcal{R}$-matrix (2.1.25).

Reality conditions. Since we want the models that we will construct from AGMs to be real, we need to impose some reality conditions on the currents $\mathcal{J}_{r,[p]}$ and the levels ℓ_r. There are two types of reality conditions, depending on the site z_r being real or complex. Let τ be an antilinear involutive automorphism of the Lie algebra $\mathfrak{g}^{\mathbb{C}}$ and let $\mathfrak{g}$ be the real form of $\mathfrak{g}^{\mathbb{C}}$ with respect to τ. In other words, $\mathfrak{g}$ is the subalgebra of fixed points under τ of $\mathfrak{g}^{\mathbb{C}}$. We describe the two types of reality conditions by means of this automorphism τ. In the case of a real site z_r, we suppose that the currents $\mathcal{J}_{r,[p]}$ are invariant under this automorphism (*i.e.* they are $\mathfrak{g}$-valued) and the corresponding levels are real:

$$\tau(\mathcal{J}_{r,[p]}) = \mathcal{J}_{r,[p]} \qquad \text{and} \qquad \overline{\ell_r} = \ell_r. \tag{2.2.8}$$

In the case of complex z_r instead, we make the assumption that we have a site also at the complex conjugate position $\overline{z_r}$. Moreover, we suppose that the currents at the two conjugate sites are related by the automorphism τ and that the levels are related by complex conjugation:

$$\tau(\mathcal{J}_{r,[p]}) = \mathcal{J}_{\bar{r},[p]} \qquad \text{and} \qquad \overline{\ell_r} = \ell_{\bar{r}}, \tag{2.2.9}$$

where we used the notation $\bar{r}$ to indicate quantities associated to the conjugate site $\overline{z_r}$.

As a consequence of the conditions above, one checks that the Gaudin Lax matrix and the twist function of the models satisfy the following equivariance properties with respect to the action of τ and the complex conjugation $z \to \bar{z}$:

$$\tau(\Gamma(z,x)) = \Gamma(\bar{z},x) \qquad \text{and} \qquad \overline{\varphi(z)} = \varphi(\bar{z}). \tag{2.2.10}$$

2.2.3 Hamiltonian and momentum

Zeroes of the twist function. In this section we will construct the Hamiltonian for realisations of AGMs. In order to describe its form, it will be useful to start by rewriting the twist function $\varphi(z)$ in terms of its zeroes. From (2.2.6) $\varphi(z)$ can be rewritten as the quotient of two polynomials of degree $M = \sum_{r=1}^{N} m_r$. Hence, it will have M complex zeroes ζ_i, with $i = 1, \dots, M$:

$$\varphi(z) = -\ell^\infty \frac{\prod_{i=1}^{M}(z-\zeta_i)}{\prod_{r=1}^{N}(z-z_r)^{m_r}}. \tag{2.2.11}$$

In the following, we will assume that these zeroes are real and simple.

Hamiltonian. Let us consider the following quantity:

$$\mathcal{Q}(z) = -\frac{1}{2\varphi(z)} \int_{\mathbb{D}} \mathrm{d}x \; \kappa(\Gamma(z,x),\Gamma(z,x)), \tag{2.2.12}$$

where the integration region $\mathbb{D}$ was defined in section 2.1 to be either $\mathbb{R}$ or S^1. We use $\mathcal{Q}(z)$ to extract local charges quadratic in the currents $\mathcal{J}_{r,[p]}$[2]

$$\mathcal{Q}_i = \operatorname*{res}_{z=\zeta_i} \mathcal{Q}(z)\mathrm{d}z, \tag{2.2.13}$$

or, more explicitly,

$$\mathcal{Q}_i = -\frac{1}{2\varphi'(\zeta_i)} \int_{\mathbb{D}} \mathrm{d}x \; \kappa(\Gamma(\zeta_i,x),\Gamma(\zeta_i,x)), \tag{2.2.14}$$

[2]We consider here the one-form $\mathcal{Q}(z)\mathrm{d}z$ rather than the function $\mathcal{Q}(z)$ since we will be interested here and in the following chapters in computing residues of this quantity at infinity, which are then more naturally defined for one-forms.

for $i = 1, \ldots, M$. As shown in [59], one can check that they Poisson commute with each other, *i.e.* $\{\mathcal{Q}_i, \mathcal{Q}_j\} = 0$, for all i and j. The Hamiltonian of realisations of AGMs is defined from these charges as their linear combination:

$$\mathcal{H} = \sum_{i=1}^{M} \epsilon_i \mathcal{Q}_i, \tag{2.2.15}$$

for some real numbers ϵ_i. The Hamiltonian $\mathcal{H}$ generates the time evolution of the models in the sense that

$$\partial_t f = \{\mathcal{H}, f\}, \tag{2.2.16}$$

for any $f \in \mathcal{F}(M)$. Under this time evolution, the charges $\mathcal{Q}_i$ are conserved by construction. Moreover, as a consequence of the equivariance properties (2.2.10), $\mathcal{H}$ is real.

Momentum. Let us consider the momentum of the models $\mathcal{P}$. It is defined as the generator of spatial translations:

$$\partial_x \phi(x) = \{\mathcal{P}, \phi(x)\}, \tag{2.2.17}$$

for any field $\phi(x)$ in M. In the following, we will make the assumption that it is given by the following expression:

$$\mathcal{P} = -\sum_{r=1}^{N} \left(\operatorname*{res}_{z=z_r} \mathcal{Q}(z)\mathrm{d}z \right). \tag{2.2.18}$$

To justify this assumption we first observe that for any of the $\mathcal{J}_{r,[p]}$

$$\left\{ -\operatorname*{res}_{z=z_s} \mathcal{Q}(z)\mathrm{d}z, \mathcal{J}_{r,[p]}(x) \right\} = \delta_{rs}\, \partial_x \mathcal{J}_{r,[p]}(x), \tag{2.2.19}$$

as one can check from (2.2.12) and (2.2.4). Thus, the right hand side of (2.2.18) generates the spatial derivative on the currents $\mathcal{J}_{r,[p]}$. Then the assumption (2.2.18) is equivalent to supposing that the Hamiltonian flow of (2.2.18) generates the spatial derivative on all fields in M. Note that this may not be the case if M contains fields that do not appear in the definitions of the currents $\mathcal{J}_{r,[p]}$. We will see that this assumption is explicitly verified for the models described in the next chapters.

From (2.2.12), $\mathcal{Q}(z)\mathrm{d}z$ has residues at the positions z_r and ζ_i only. Since the sum of the residues of $\mathcal{Q}(z)\mathrm{d}z$ on the Riemann sphere vanishes, the assumption (2.2.18) then implies that

$$\mathcal{P} = \sum_{i=1}^{M} \mathcal{Q}_i. \tag{2.2.20}$$

We note from equation (2.2.15) that the the Hamiltonian and the momentum of the theories thus differ only by a choice of the coefficients ϵ_i. This fact allows a particularly simple treatment of the symmetries associated to space and time translations of the models, as we shall now show.

2.2.4 Space-time symmetries

Energy-momentum tensor. In order to discuss further the space-time symmetries of the realisations of AGMs, let us describe their energy-momentum tensor. We will indicate the components of this tensor by $T^{\mu}{}_{\nu}$, where both indices can be equal either to 0 or 1, for time and space components, respectively. The components ${T^0}_0$ and ${T^0}_1$ correspond to the densities of the Hamiltonian and momentum of the models:

$$\mathcal{H} = \int_{\mathbb{D}} \mathrm{d}x \, {T^0}_0(x) \qquad \text{and} \qquad \mathcal{P} = \int_{\mathbb{D}} \mathrm{d}x \, {T^0}_1(x). \tag{2.2.21}$$

Let us consider the densities of the charges $\mathcal{Q}_i$, which we will denote by $q_i(x)$:

$$q_i(x) = -\frac{1}{2\varphi'(\zeta_i)} \kappa(\Gamma(\zeta_i, x), \Gamma(\zeta_i, x)). \tag{2.2.22}$$

From the expression (2.2.15) and (2.2.20) of the Hamiltonian and momentum, we can rewrite the components ${T^0}_0$ and ${T^0}_1$ in terms of the fields q_i as

$${T^0}_0 = \sum_{i=1}^{M} \epsilon_i \, q_i \qquad \text{and} \qquad {T^0}_1 = \sum_{i=1}^{M} q_i. \tag{2.2.23}$$

As $\mathcal{H}$ and $\mathcal{P}$ are conserved under time evolution, their densities should satisfy the following local conservation equations:

$$\partial_t {T^0}_\mu + \partial_x {T^1}_\mu = 0, \qquad \text{for } \mu = 0, 1, \tag{2.2.24}$$

which allow us to define the other two components of the energy-momentum tensor ${T^1}_\nu$. In order to compute these quantities, we need to calculate the time evolution of the expressions (2.2.23). A direct computation from the Poisson bracket (2.2.7) of the Gaudin Lax matrix $\Gamma(z, x)$ shows that

$$\big\{ q_i(x), q_j(y) \big\} = -\delta_{ij} \big(\partial_x q_i(x) \delta_{xy} + 2 q_i(x) \delta'_{xy} \big). \tag{2.2.25}$$

From the equation above, one obtains the evolution of the densities $q_i(x)$ under the Hamiltonian flow of the charges $\mathcal{Q}_j$, *i.e.* $\{\mathcal{Q}_j, q_i(x)\} = \delta_{ij} \partial_x q_i(x)$. Hence, the time evolution of the $q_i(x)$ reads:

$$\partial_t q_i = \epsilon_i \, \partial_x q_i. \tag{2.2.26}$$

By reinserting in the expressions (2.2.23) for the components ${T^0}_0$ and ${T^0}_1$, we arrive at:

$$\partial_t {T^0}_0 = \sum_{i=1}^{M} \epsilon_i^2 \, \partial_x q_i \qquad \text{and} \qquad \partial_t {T^0}_1 = \sum_{i=1}^{M} \epsilon_i \, \partial_x q_i. \tag{2.2.27}$$

Then, comparing with the conservation equation (2.2.24), we read the expressions for the other two components of the energy-momentum tensor:

$${T^1}_0 = -\sum_{i=1}^{M} \epsilon_i^2 \, q_i \quad \text{and} \quad {T^1}_1 = -\sum_{i=1}^{M} \epsilon_i \, q_i. \tag{2.2.28}$$

Classical scale invariance. We note from (2.2.23) and (2.2.28) that ${T^\mu}_\mu = 0$, *i.e.* the energy-stress tensor is traceless and the models then classically scale invariant. In general, this property is broken at the quantum level.

Relativistic invariance. Let us consider the two-dimensional Minkowski metric $\eta_{\mu\nu}$, with $\eta_{00} = -\eta_{11} = 1$ and $\eta_{01} = \eta_{10} = 0$. It is a standard result in field theory that a model is invariant under Lorentz transformations if the tensor $T_{\mu\nu} = \eta_{\mu\rho} T^{\rho}{}_{\nu}$ obtained by lowering one of the indices of the energy-momentum tensor is symmetric. Let us then consider its components T_{01} and T_{10}. From equations (2.2.23) and (2.2.28), one can check that

$$T_{01} = \sum_{i=1}^{M} q_i \qquad \text{and} \qquad T_{10} = \sum_{i=1}^{M} \epsilon_i^2 \, q_i. \tag{2.2.29}$$

Hence, we find the following simple condition for the relativistic invariance of the models[3]:

$$\epsilon_i = \pm 1, \qquad \forall\, i \in \{1, \dots, M\}. \tag{2.2.30}$$

Finally, since in this case the stress tensor is both traceless and symmetric, the models are also conformal invariant.

2.2.5 Integrability

In this section, we introduce the Lax formulation of the models. We then proceed to show that the Lax matrix has a Poisson bracket of the form of Maillet, hence proving that the models are integrable field theories with twist function as expected.

Lax formulation. In agreement with the considerations in section 2.2.1, we define the Lax matrix of the models as the following $\mathfrak{g}^{\mathbb{C}}$-valued field:

$$\mathcal{L}(z, x) = \frac{\Gamma(z, x)}{\varphi(z)}. \tag{2.2.31}$$

Since it will be useful later, let us briefly discuss its pole structure in z. Since $\Gamma(z)$ and $\varphi(z)$ have poles of the same order m_r at the same positions z_r, $\mathcal{L}(z)$ has only poles at the zeroes ζ_i of the twist function. Since we assumed that these zeroes are simple, the Lax matrix can then be rewritten as:

$$\mathcal{L}(z, x) = \sum_{i=1}^{M} \frac{1}{\varphi'(\zeta_i)} \frac{\Gamma(\zeta_i, x)}{z - \zeta_i}. \tag{2.2.32}$$

Let us study the dynamics of this field starting from its definition (2.2.31). From the Poisson bracket (2.2.7) of the Gaudin Lax matrix, a brief computation gives the following bracket of $\mathcal{Q}(z)$ with $\mathcal{L}(z, x)$:

$$\bigl\{ \mathcal{Q}(w), \mathcal{L}(z, x) \bigr\} = \bigl[\mathcal{L}(z, x), \mathcal{M}(w\,; z, x) \bigr] + \partial_x \mathcal{M}(w\,; z, x) - \partial_x \left(\frac{1}{\varphi(z)} \kappa_{\underline{\mathbf{2}}} \Bigl(\mathcal{R}^0_{\underline{\mathbf{21}}}(w, z), \Gamma_{\underline{\mathbf{2}}}(w, x) \Bigr) \right), \tag{2.2.33}$$

where we defined the field

$$\mathcal{M}(w\,; z, x) = -\frac{1}{\varphi(w)} \kappa_{\underline{\mathbf{2}}} \Bigl(\mathcal{R}^0_{\underline{\mathbf{12}}}(z, w), \Gamma_{\underline{\mathbf{2}}}(w, x) \Bigr) \tag{2.2.34}$$

and $\mathcal{R}^0$ is the untwisted standard $\mathcal{R}$-matrix (2.1.25). We take now the residue of (2.2.33) at $w = \zeta_i$, so that we find back from $\mathcal{Q}(z)$ the charges $\mathcal{Q}_i$ from which the Hamiltonian is defined.

[3] We expect this condition to be also necessary for relativistic invariance since there is in general no other apparent way to bring the difference $T_{01} - T_{10}$ in the form of a total derivative.

Moreover, when taking this residue, the last piece on the right hand side of (2.2.33) vanishes since both $\mathcal{R}^0_{\underline{\mathbf{21}}}(w,z)$ and $\Gamma(w,x)$ are regular at $w=\zeta_i$. Hence, we arrive at the following Poisson bracket:

$$\{\mathcal{Q}_i,\mathcal{L}(z,x)\}-\partial_x\mathcal{M}_i(z,x)+\big[\mathcal{M}_i(z,x),\mathcal{L}(z,x)\big]=0, \tag{2.2.35}$$

where $\mathcal{M}_i(z,x)$ is the residue of $\mathcal{M}(w;z,x)$ at $w=\zeta_i$:

$$\mathcal{M}_i(z,x)=\operatorname*{res}_{w=\zeta_i}\mathcal{M}(w;z,x)\mathrm{d}w=\frac{1}{\varphi'(\zeta_i)}\frac{\Gamma(\zeta_i,x)}{z-\zeta_i}, \tag{2.2.36}$$

where the second equality is obtained by direct computation. As expected, equation (2.2.35) is in the form of the zero curvature equation discussed in section 2.1. Multiplying by ϵ_i and summing over the index i we obtain the Lax equation:

$$\{\mathcal{H},\mathcal{L}(z,x)\}-\partial_x\mathcal{M}(z,x)+[\mathcal{M}(z,x),\mathcal{L}(z,x)]=0, \tag{2.2.37}$$

where the temporal part of the Lax connection is defined as

$$\mathcal{M}(z,x)=\sum_{i=1}^{M}\frac{\epsilon_i}{\varphi'(\zeta_i)}\frac{\Gamma(\zeta_i,x)}{z-\zeta_i}. \tag{2.2.38}$$

We note that as for $\mathcal{H}$ and $\mathcal{P}$ the definitions of $\mathcal{L}(z,x)$ and $\mathcal{M}(z,x)$ differ only by the choice of the parameters ϵ_i.

Maillet bracket. From the Poisson bracket (2.2.7) of the Gaudin Lax matrix and the form (2.2.32) of the Lax matrix, it is simple to prove that the Poisson bracket of $\mathcal{L}(z,x)$ is given by the Maillet bracket [5,6]:

$$\begin{aligned}\{\mathcal{L}_{\underline{\mathbf{1}}}(z,x),\mathcal{L}_{\underline{\mathbf{2}}}(w,y)\}=&\,[\mathcal{R}_{\underline{\mathbf{12}}}(z,w),\mathcal{L}_{\underline{\mathbf{1}}}(z,x)]\delta_{xy}-[\mathcal{R}_{\underline{\mathbf{21}}}(w,z),\mathcal{L}_{\underline{\mathbf{2}}}(w,x)]\delta_{xy}\\&-(\mathcal{R}_{\underline{\mathbf{12}}}(z,w)+\mathcal{R}_{\underline{\mathbf{21}}}(w,z))\delta'_{xy},\end{aligned} \tag{2.2.39}$$

where $\mathcal{R}_{\underline{\mathbf{12}}}(z,w)=\mathcal{R}^0_{\underline{\mathbf{12}}}(z,w)\varphi(w)^{-1}$ is the standard $\mathcal{R}$-matrix (2.1.25) twisted by the twist function. As a consequence of the Maillet bracket (2.2.39), one can show that the infinite set of conserved charges extracted from the monodromy matrix of the models is in involution and hence that realisations of AGMs are integrable.

Integrable local hierarchies. Note that the conserved charges extracted from the monodromy matrix of the models are in general non-local. It was shown in [74] (see also [75,76]) that models with twist function possess infinite integrable hierarchies of local conserved functions associated with the zeroes of the twist function. We already noted in section 2.2.3 that realisations of AGMs possess a set of local conserved quadratic charges $\mathcal{Q}_i$ which was given in (2.2.14). Similarly, one can associate to these zeroes the following higher degree local conserved charges:

$$\mathcal{Q}^d_i=-\frac{1}{(d+1)\varphi'(\zeta_i)}\int_{\mathbb{D}}\mathrm{d}x\ \Phi_d(\Gamma(\zeta_i,x)), \tag{2.2.40}$$

where we denoted by Φ_d well-chosen invariant polynomials on $\mathfrak{g}^{\mathbb{C}}$ of degree $d+1$. The index d is restricted to take specific values which depend on the underlying Lie algebra $\mathfrak{g}^{\mathbb{C}}$ and the zero under consideration. We refer to [74] for more details about the construction of these hierarchies. We note that for the case $d=1$ the corresponding quadratic polynomial is given by the bilinear form κ as shown in the expression (2.2.14) of the $\mathcal{Q}_i$.

2.2.6 Examples

We end this section by providing two illustrative examples of the framework outlined in the previous sections. In particular, we will first describe how to obtain from AGMs the prototypical example of an integrable field theory defined on a semi-simple real Lie group G, namely the principal chiral model (PCM). Our aim will be to show how the Hamiltonian of this model is recovered, which then allows to obtain the well-known PCM action. Later, we show how to construct a theory coupling together an arbitrary number of PCMs on the same Lie group. This example serves to show the power of AGMs and provides an intermediate logical step towards the models presented in the next chapters. We follow the construction described in the previous sections, starting by describing the Takiff realisation of a single PCM.

Canonical fields on T^*G. Let us consider canonical fields taking values in the cotangent bundle T^*G of the Lie group G. We note that the cotangent space T^*_pG at a point $p \in G$ can always be sent to the cotangent space $T^*_{\mathrm{Id}}G = \mathfrak{g}^*$ (the dual of the Lie algebra $\mathfrak{g}$ of G) at the identity by translating through multiplication by p^{-1}. As we supposed $\mathfrak{g}$ to be semi-simple, we have a canonical isomorphism between $\mathfrak{g}^*$ and $\mathfrak{g}$ through the bilinear form κ and thus $T^*G \simeq G \times \mathfrak{g}$. Hence, the canonical fields can be described by a pair of fields $(g, X) : \mathbb{D} \to G \times \mathfrak{g}$ which encode the coordinate and momentum fields, respectively.

The cotangent bundle T^*G is a symplectic manifold. In terms of the fields g and X, the corresponding Poisson bracket for canonical fields with values in this symplectic manifold is given by:

$$\{g_{\underline{\mathbf{1}}}(x), g_{\underline{\mathbf{2}}}(y)\} = 0, \tag{2.2.41a}$$
$$\{X_{\underline{\mathbf{1}}}(x), g_{\underline{\mathbf{2}}}(y)\} = g_{\underline{\mathbf{2}}}(x) C_{\underline{\mathbf{12}}} \delta_{xy}, \tag{2.2.41b}$$
$$\{X_{\underline{\mathbf{1}}}(x), X_{\underline{\mathbf{2}}}(y)\} = [C_{\underline{\mathbf{12}}}, X_{\underline{\mathbf{1}}}(x)] \delta_{xy}. \tag{2.2.41c}$$

Takiff realisation of the PCM. We will use now the fields g and X to construct the Takiff realisation of the PCM. Let us consider the following $\mathfrak{g}$-valued current:

$$j(x) = g^{-1}(x) \partial_x g(x). \tag{2.2.42}$$

Together with the field X, we use it to construct a pair of $\mathfrak{g}$-valued currents:

$$\mathcal{J}_{[0]}(x) = X(x) \tag{2.2.43a}$$
$$\mathcal{J}_{[1]}(x) = \ell\, j(x). \tag{2.2.43b}$$

As one can check from the brackets (2.2.41), these currents satisfy the following Poisson brackets:

$$\{\mathcal{J}_{[0]\underline{\mathbf{1}}}(x), \mathcal{J}_{[0]\underline{\mathbf{2}}}(y)\} = [C_{\underline{\mathbf{12}}}, \mathcal{J}_{[0]\underline{\mathbf{1}}}(x)] \delta_{xy} \tag{2.2.44a}$$
$$\{\mathcal{J}_{[0]\underline{\mathbf{1}}}(x), \mathcal{J}_{[1]\underline{\mathbf{2}}}(y)\} = [C_{\underline{\mathbf{12}}}, \mathcal{J}_{[1]\underline{\mathbf{1}}}(x)] \delta_{xy} - \ell\, C_{\underline{\mathbf{12}}} \delta'_{xy}, \tag{2.2.44b}$$
$$\{\mathcal{J}_{[1]\underline{\mathbf{1}}}(x), \mathcal{J}_{[1]\underline{\mathbf{2}}}(y)\} = 0. \tag{2.2.44c}$$

In the terminology of section 2.2.2, $\mathcal{J}_{[0]}$ and $\mathcal{J}_{[1]}$ are hence Takiff currents of multiplicity $m = 2$ with levels $\ell_0 = 0$ and $\ell_1 = \ell$. This realisation is usually referred to as the PCM realisation since it gives rise to the PCM model, as we shall now show.

PCM Gaudin Lax matrix and twist function. Let us use the Takiff currents (2.2.43) to build the PCM as an AGM. For that, we need to consider the Gaudin Lax matrix and the twist function of the model. Let us fix a real site z_1 in the complex plane. Then, the Gaudin Lax matrix is built by combining the Takiff currents as in (2.2.5). By using the currents (2.2.43) of the realisation, we get for this case the following explicit form:

$$\Gamma(z,x) = \frac{\ell\, j(x)}{(z-z_1)^2} + \frac{X(x)}{z-z_1}. \tag{2.2.45}$$

Similarly, from (2.2.6) we read the twist function of the model:

$$\varphi(z) = \frac{\ell}{(z-z_1)^2} - \ell^\infty. \tag{2.2.46}$$

The zeroes of this function are thus:

$$\zeta_{1,2} = z_1 \pm \frac{\ell}{K} \qquad \text{with} \qquad K = \sqrt{\ell\ell^\infty}, \tag{2.2.47}$$

where we introduced the reparametrisation of ℓ^∞ in terms of K for future convenience.

Hamiltonian and action of the PCM. In order to compute the Hamiltonian of the model, we need to calculate the residues of the charge $\mathcal{Q}(z)$ introduced in (2.2.12) at $\zeta_{1,2}$. A brief calculation shows that they are given by

$$\mathcal{Q}_{1,2} = \operatorname*{res}_{z=\zeta_{1,2}} \mathcal{Q}(z)\mathrm{d}z = \pm\frac{K}{4}\int_{\mathbb{D}} \mathrm{d}x\ \kappa\left(\frac{X(x)}{K} \pm j(x), \frac{X(x)}{K} \pm j(x)\right). \tag{2.2.48}$$

The Hamiltonian is defined as the following linear combination of the charges above:

$$\mathcal{H} = \epsilon_1\mathcal{Q}_1 + \epsilon_2\mathcal{Q}_2, \tag{2.2.49}$$

where $\epsilon_i \in \{+1,-1\}$ to have relativistic invariance, as seen in the general AGMs construction. In order to find the standard action of the PCM when passing from the Hamiltonian to the Lagrangian formulation, one takes the choice [57, 59] $\epsilon_1 = +1$, $\epsilon_2 = -1$[4]. With this choice, we find the following expression for the Hamiltonian of the model:

$$\mathcal{H} = \frac{1}{2}\int_{\mathbb{D}} \mathrm{d}x\ \frac{1}{K}\,\kappa\left(X(x), X(x)\right) + K\,\kappa\left(j(x), j(x)\right). \tag{2.2.50}$$

Passing from the Hamiltonian to the Lagrangian formulation of the model, we find the standard action of the PCM. In terms of the field $g(x,t)$ (in the Lagrangian formulation this field depends explicitly on time), this action is given by

$$S[g] = \frac{K}{2}\iint_{\mathbb{D}\times\mathbb{R}} \mathrm{d}x\,\mathrm{d}t\ \kappa(g^{-1}\partial_+ g, g^{-1}\partial_- g), \tag{2.2.51}$$

where we introduced the light-cone coordinates $x^\pm = (t\pm x)/2$ and the corresponding derivatives $\partial_\pm = \partial_t \pm \partial_x$. Since the computation of this action by inverse Legendre transform is of a technical nature we will leave it for the next chapters, where it will be discussed in detail for more general models.

[4]In order not to have trivial dynamics, one cannot take ϵ_1 and ϵ_2 to be of the same sign, which would give an Hamiltonian either equal to the momentum or its opposite. The choice between $\epsilon_1 = +1$, $\epsilon_2 = -1$ and $\epsilon_1 = -1$, $\epsilon_2 = +1$ is then taken to make the Hamiltonian positive, as described in [57, 59].

Coupled principal chiral models. Let us end this section by a discussion of how AGMs allow one to construct a theory coupling together an arbitrary number of the PCMs (2.2.51) on the same Lie group, as first described in [58]. Following the terminology of this section, let us start by considering an AGM with $N \in \mathbb{Z}_{\geq 1}$ real sites z_r, $r \in \{1, \dots, N\}$, of multiplicity two. Each site is associated with two constant numbers $\ell_{r,0} \in \mathbb{R}$ and $\ell_{r,1} \in \mathbb{R}^*$. For simplicity, we fix here $\ell_{r,0} = 0$ and rewrite $\ell_{r,1} = \ell_r$[5]. To each site are also attached two Takiff currents $\mathscr{J}_{r,[0]}$ and $\mathscr{J}_{r,[1]}$. We consider then the realisation of these Takiff currents in the phase space given by canonical fields on the cotangent bundle T^*G^N. This choice corresponds for the case $N = 1$ to the phase space of the PCM which we discussed earlier in this section. For general N, it is then described by N G-valued fields $g_1(x), \cdots, g_N(x)$ and N $\mathfrak{g}$-valued fields $X_1(x), \cdots, X_N(x)$, which are the equivalents of the fields $g(x)$ and $X(x)$ introduced for the PCM. Similarly to that case, we realise the currents $\mathscr{J}_{r,[0]}$ and $\mathscr{J}_{r,[1]}$ at each site z_r as in equation (2.2.43), with $j(x)$ and $X(x)$ replaced by $j_r(x)$ and $X_r(x)$, respectively.

We conclude the construction by defining the Hamiltonian of the model as outlined in the section 2.2.3. By passing to the Lagrangian formulation and computing the inverse Legendre transform, we arrive at the following form of the action of the model (here also we do not detail the computations as for the PCM):

$$S[g_1, \dots, g_r] = \iint_{\mathbb{D}\times\mathbb{R}} \mathrm{d}x \, \mathrm{d}t \sum_{r,s=1}^{N} \rho_{rs} \, \kappa(g_r^{-1}\partial_+ g_r, g_s^{-1}\partial_- g_s), \tag{2.2.52}$$

where the ρ_{rs} are scalar coefficients defined in terms of the twist function of the model coupling together the fields at different sites. In particular, as one can show, the action above depends on a total of $2N-1$ free parameters. Moreover, one can see that by taking for example the N-th site z_N to infinity in (2.2.52), one recovers the sum of an action $S[g_1, \dots, g_{N-1}]$ of the same form and a PCM action $S[g_N]$ for the the field g_N, uncoupled to the fields at other sites. The action (2.2.52) hence describes a model coupling non-trivially together an arbitrary number of PCMs in a way that preserves integrability, as it was anticipated.

[5] Leaving $\ell_{r,0}$ unfixed would result in the introduction of Wess-Zumino terms in the action of the model. Historically, this was the case in which these models were originally described in [58]. For simplicity we will leave the discussion of this case for the next chapters, where we will present the construction in details.

Chapter 3

Integrable deformations of coupled sigma models

3.1 Introduction

In section 2.2.6 of the previous chapter we have shown how the framework of AGMs provides a powerful way to construct basic instances of integrable field theories, such as the PCM and its more general coupled version. In this chapter we start exploring how the framework of AGMs can be used to construct more general integrable field theories. More precisely, we describe the construction of integrable deformations of the coupled PCMs. The work presented here is based on the article [77], which I wrote during my PhD in collaboration with S. Lacroix.

In order to introduce the concept of integrable deformations, let us start by considering the so-called Yang-Baxter model [20, 21]. It was shown by Klimčík in these references that the PCM (without Wess-Zumino term) admits a continuous integrable deformation, which is known as the Yang-Baxter model. This model depends on the choice of a skew-symmetric R-matrix on $\mathfrak{g}$, *i.e.* a linear operator $R : \mathfrak{g} \to \mathfrak{g}$ satisfying the modified classical Yang-Baxter equation $[RX, RY] - R[RX, Y] - R[X, RY] = -c^2[X, Y]$ for every $X, Y \in \mathfrak{g}$, with c equal to 1 or i. The action of this theory is given by

$$S_{\text{YB}}[g] = \rho \iint \mathrm{d}t\,\mathrm{d}x\ \kappa\left(g^{-1}\partial_+ g, \frac{1}{1 - \eta\, R_g} g^{-1}\partial_- g\right),$$

where η is the deformation parameter and $R_g = \mathrm{Ad}_g^{-1} \circ R \circ \mathrm{Ad}_g$. Note that in the limit where η goes to zero one recovers the action (2.2.51) of the PCM. Hence, the Yang-Baxter model constitutes a continuous deformation of the latter. Furthermore, it was shown this model possesses a Lax connection [21] satisfying a Maillet bracket with twist function [47].

Other examples of such deformed integrable sigma models are known. For instance, in [22][1] Sfetsos constructed a model which corresponds to a deformation of the so-called non-abelian T-dual of the PCM (without Wess-Zumino term), called the λ-model. The action of this theory is defined as

$$S_\lambda[g] = S_{\text{WZW},\Bbbk}[g] + \Bbbk \iint \mathrm{d}t\,\mathrm{d}x\ \kappa\left(\partial_+ g g^{-1}, \frac{1}{\lambda^{-1} - \mathrm{Ad}_g^{-1}} g^{-1}\partial_- g\right),$$

where $\Bbbk$ and λ are constant parameters and $S_{\text{WZW},\Bbbk}[g]$ is the action of the conformal Wess-Zumino-Witten model at level $\Bbbk$. The Lax connection of this model was shown to satisfy the Maillet bracket in [51] (see also [80] for first results).

[1] The reformulation of this model as a theory on $G{\times}G{\times}G$ is a special case of one that was originally considered in [78] and whose classical integrability was first proven in [79].

Following the formulation of these models as integrable field theories with twist function, both the Yang-Baxter and the λ-model were reinterpreted as realisations of affine Gaudin models in [57]. In particular, in the language of chapter 2, both models possess a twist function with two simple poles, which correspond to two sites of multiplicity one for the corresponding AGM. Recall from section 2.2.6 that the twist function of the PCM has a single pole with multiplicity two. This is also the case for the non-abelian T-dual of the PCM. From the point of view of the undeformed model the deformation corresponds then to splitting the double pole into a pair of simple ones. The distance between the simple poles will constitute the deformation parameter of the models. Moreover, following the formalism of chapter 2, the deformed models are described by two commuting Kac-Moody currents in T^*G rather than Takiff currents of multiplicity two in the same phase space as for the undeformed models.

A natural direction of exploration is thus to construct integrable deformations of the kind discussed above for the integrable coupled PCMs discussed in chapter 2. For instance, we can apply a Yang-Baxter deformation to any of the N copies of the model. Similarly, one could also consider a λ-deformation, which would be a deformation of the model where the corresponding copy of the PCM is replaced by its non-abelian T-dual. In this chapter, we will describe the explicit construction of the class of models coupling together an arbitrary number of these deformations.

Before discussing this construction in detail, let us summarise briefly its main results. Let us first consider the model with N copies of the principal chiral model with Wess-Zumino term, each subject to a Yang-Baxter deformation. It is defined by $4N-1$ parameters, which can be thought of as the $3N-1$ parameters of the undeformed model[2] together with N deformation parameters, and by the choice of N R-matrices R_r on $\mathfrak{g}$[3]. We find the action of this model to be

$$S[g_1,\dots,g_N] = \frac{1}{2}\iint \mathrm{d}t\,\mathrm{d}x \sum_{r,s=1}^{N} \kappa\Big(g_r^{-1}\partial_+ g_r,\, \mathcal{O}_{rs}(R_l)\; g_s^{-1}\partial_- g_s\Big) + \sum_{r=1}^{N} \Bbbk_r\, I_{\mathrm{WZ}}\big[g_r\big]. \tag{3.1.1}$$

In this expression, $\mathcal{O}$ is an operator on $\mathfrak{g}^N$ whose entries $\mathcal{O}_{rs}$ depend on the R-matrices R_r and the defining parameters of the model. Moreover, $I_{\mathrm{WZ}}\big[g_r\big]$ is the Wess-Zumino term for the field g_r. We prove that in the limit where the N deformation parameters are taken to 0 the entries $\mathcal{O}_{rs}$ of the operator $\mathcal{O}$ tend to $\rho_{rs}\mathrm{Id}$, where the coefficients ρ_{rs} were introduced in the expression (2.2.52) for the action of the coupled PCMs. Hence, in the undeformed limit we find back the action (2.2.52) (given for the case $\Bbbk_r = 0$) as expected.

Let us now consider the model coupling together N copies of the λ-model. This model possesses $3N-1$ defining parameters of which N can be thought of as deformation parameters. Its action takes the form:

$$S[g_1,\dots,g_N] = \sum_{r=1}^{N} S_{\mathrm{WZW},\Bbbk_r}\big[g_r\big] + \iint \mathrm{d}t\,\mathrm{d}x \sum_{r,s=1}^{N} \Bbbk_r\,\kappa\left(\partial_+ g_r g_r^{-1}, \left(\frac{1}{\mathcal{M}-\mathcal{D}^{-1}}\right)_{rs} g_s^{-1}\partial_- g_s\right), \tag{3.1.2}$$

where $\mathcal{M}$ and $\mathcal{D}$ are operators on $\mathfrak{g}^N$ with entries $\mathcal{M}_{rs} = \mu_{rs}\,\mathrm{Id}$ and $\mathcal{D}_{rs} = \mathrm{Ad}_{g_r}\,\delta_{rs}$ expressed in terms of the defining parameters of the models. We note that actions of this form were already considered in the articles [60–63]. We will show that these models can be obtained as particular limits of (3.1.2) where only $2N-2$ entries of $\mathcal{M}$ stay non-zero.

[2]We refer here to the model constructed in [58], which can be obtained by a similar construction to the one presented in section 2.2.6 where the parameters $\ell_{r,0}$ are not fixed to be zero. As anticipated before, this corresponds to the introduction of Wess-Zumino terms in the action of the model.

[3]The R-matrix R_r is assumed to satisfy the additional property $R_r^3 = c_r^2 R_r$, except if the r-th copy does not possess a Wess-Zumino term, *i.e.* if $\Bbbk_r = 0$.

The plan of the chapter is as follows. In Section 3.2, we explain the construction of the coupled deformed models in the Hamiltonian framework. More precisely, after recalling some definitions and notations in section 3.2.1, we describe in details in section 3.2.2 the Kac-Moody realisations in T^*G that serve as building blocks of the models. We then proceed to construct these models as realisations of affine Gaudin models in section 3.2.4. We go on to perform the inverse Legendre transform of these models in section 3.3, constructing in particular their action and their Lagrangian Lax connection. Subsequently, we study the models obtained from arbitrary combinations of Yang-Baxter realisations and λ-realisations in section 3.4: in particular, we find a form of the action of these field theories which mixes the expressions (3.1.1) and (3.1.2) above. Finally, in section 3.5, we explain the relation of this work with the approach of [70] based on 4d semi-holomorphic Chern-Simons theory. Some technical results are gathered in Appendices 3.A and 3.B.

3.2 Hamiltonian formulation

In this section, we apply the construction of affine Gaudin models (AGMs) to define the class of integrable field theories anticipated in the previous section. In particular, we will be interested in the class of realisations given by a pair of Kac-Moody currents in the phase space of canonical fields in T^*G introduced in section 2.2.6 of the previous chapter. We start this section by recalling the definition of this phase space. In sections 3.2.2 and 3.2.3, we proceed to describe in detail the particular Kac-Moody realisations that are the basic building blocks of the models we want to construct. We conclude by describing the construction of the models themselves in section 3.2.4.

3.2.1 Phase space of canonical fields in T^*G

All the Kac-Moody realisations that we shall consider in this section are constructed from the phase space of canonical fields in T^*G, which was described in section 2.2.6. Let us start by recalling a few background definitions.

Canonical fields on T^*G and the momentum. As we explained in section 2.2.6, canonical fields in T^*G can be described by a pair of fields $(g, X) : \mathbb{D} \to G \times \mathfrak{g}$ which encode the coordinate and momentum fields, respectively. Since T^*G is a cotangent bundle, the phase space of these fields is naturally equipped with the Poisson bracket (2.2.41).

As in section 2.2.6, one proceeds to define the following $\mathfrak{g}$-valued current:

$$j(x) = g^{-1}(x)\partial_x g(x), \tag{3.2.1}$$

which, from (2.2.41), satisfies the Poisson brackets

$$\{g_{\underline{\mathbf{1}}}(x), j_{\underline{\mathbf{2}}}(y)\} = 0, \tag{3.2.2a}$$
$$\{j_{\underline{\mathbf{1}}}(x), j_{\underline{\mathbf{2}}}(y)\} = 0, \tag{3.2.2b}$$
$$\{X_{\underline{\mathbf{1}}}(x), j_{\underline{\mathbf{2}}}(y)\} = [C_{\underline{\mathbf{12}}}, j_{\underline{\mathbf{1}}}(x)]\delta_{xy} - C_{\underline{\mathbf{12}}}\delta'_{xy}. \tag{3.2.2c}$$

Let us now consider the following quantity:

$$\mathcal{P}_G = \int_{\mathbb{D}} \mathrm{d}x \; \kappa(j(x), X(x)). \tag{3.2.3}$$

From (2.2.41) and (3.2.2), one can check that its Hamiltonian flow generates the spatial derivatives on both $g(x)$ and $X(x)$:

$$\{\mathcal{P}_G, g(x)\} = \partial_x g(x) \qquad \text{and} \qquad \{\mathcal{P}_G, X(x)\} = \partial_x X(x).$$

Hence, $\mathcal{P}_G$ is the momentum of the phase space of canonical fields in T^*G.

Wess-Zumino term and current $W(x)$. For this paragraph, let us consider the field g as depending explicitly also on a time coordinate $t \in \mathbb{R}$ (in the Hamiltonian formulation, this time dependence is implicitly defined by the choice of a Hamiltonian). Let us further extend the space-time $\mathbb{D} \times \mathbb{R}$ (with coordinates (x,t)) to a 3-dimensional manifold $\mathbb{B}$ with boundary $\partial\mathbb{B} = \mathbb{D} \times \mathbb{R}$ (parametrised by coordinates (x,t,ξ)) and let us consider an extension of the field g to $\mathbb{B}$ (which restricts to the initial field g on $\partial\mathbb{B}$). The Wess-Zumino term of g is then defined as [81–83]

$$I_{\text{WZ}}[g] = \iiint_{\mathbb{B}} \mathrm{d}x\, \mathrm{d}t\, \mathrm{d}\xi \; \kappa\Big(\big[g^{-1}\partial_x g, g^{-1}\partial_t g\big], g^{-1}\partial_\xi g \Big).$$

Up to the addition of a constant term, it does not depend on the choice of extension of g from $\mathbb{D} \times \mathbb{R}$ to $\mathbb{B}$. It is a standard result that the 3-form $\kappa\Big(\big[g^{-1}\partial_x g, g^{-1}\partial_t g\big], g^{-1}\partial_\xi g \Big)\, \mathrm{d}x \wedge \mathrm{d}t \wedge \mathrm{d}\xi$ is closed and thus locally exact. Therefore, the Wess-Zumino term can be rewritten, at least locally, as a 2-dimensional integral on $\partial\mathbb{B} = \mathbb{D} \times \mathbb{R}$, which takes the form

$$I_{\text{WZ}}[g] = \iint_{\mathbb{D}\times\mathbb{R}} \mathrm{d}x\, \mathrm{d}t \; \kappa(W, g^{-1}\partial_t g), \tag{3.2.4}$$

where W is a $\mathfrak{g}$-valued current depending on the coordinate fields in g and their spatial derivatives. We will not need here the precise definition of W and refer for instance to [59] for more details.

In the Hamiltonian formalism, this current can be seen as a $\mathfrak{g}$-valued local observable $W(x)$ on the phase space of canonical fields on T^*G. One can then show that it satisfies the following Poisson bracket with the fields g, X and j introduced above:

$$\{g_{\underline{\mathbf{1}}}(x), W_{\underline{\mathbf{2}}}(y)\} = 0, \qquad\qquad \{j_{\underline{\mathbf{1}}}(x), W_{\underline{\mathbf{2}}}(y)\} = 0 \tag{3.2.5a}$$

and

$$\{X_{\underline{\mathbf{1}}}(x), W_{\underline{\mathbf{2}}}(y)\} + \{W_{\underline{\mathbf{1}}}(x), X_{\underline{\mathbf{2}}}(y)\} = [C_{\underline{\mathbf{12}}}, W_{\underline{\mathbf{1}}}(x) - j_{\underline{\mathbf{1}}}(x)]\delta_{xy}. \tag{3.2.5b}$$

Moreover, let us note that it satisfies the following orthogonality property:

$$\kappa\big(j(x), W(x)\big) = 0. \tag{3.2.6}$$

3.2.2 Kac-Moody realisations in T^*G

Commuting Kac-Moody currents. We introduce now the Kac-Moody realisations which will be the basic building blocks for the construction of the integrable models in section 3.2.4. These realisations are characterised by two commuting Kac-Moody currents in the phase space of fields in T^*G, *i.e.* two $\mathfrak{g}^{\mathbb{C}}$-valued fields $\mathcal{J}_\pm(x)$ satisfying the Poisson brackets

$$\{\mathcal{J}_{\pm\underline{\mathbf{1}}}(x), \mathcal{J}_{\pm\underline{\mathbf{2}}}(y)\} = [C_{\underline{\mathbf{12}}}, \mathcal{J}_{\pm\underline{\mathbf{1}}}(x)]\delta_{xy} - \ell_\pm C_{\underline{\mathbf{12}}}\delta'_{xy}, \tag{3.2.7a}$$

$$\{\mathcal{J}_{\pm\underline{\mathbf{1}}}(x), \mathcal{J}_{\mp\underline{\mathbf{2}}}(y)\} = 0, \tag{3.2.7b}$$

where $\ell_\pm$ are constant numbers. Such currents have appeared in the study of integrable deformations of sigma models [47–49, 80] leading to examples of Kac-Moody realisations such as the Yang-Baxter realisation (with or without Wess-Zumino term) and the λ-realisation [59]. We will describe these examples more in detail in section 3.2.3. In order to keep the treatment as general and uniform as possible, we focus for the moment on aspects which are common to all the realisations we shall describe.

In all the examples we shall consider, the Kac-Moody currents $\mathcal{J}_\pm(x)$ are expressed as linear combinations of the $\mathfrak{g}$-valued currents $X(x)$, $j(x)$ and $W(x)$ introduced in section 3.2.1. Moreover, the currents $X(x)$ and $W(x)$ always appear through the unique combination

$$Y(x) = X(x) - \not{k}\, W(x),$$

for some real constant $\not{k}$ which depends on the choice of a particular realisation. As one can see from (3.2.4), the current W is related to the Wess-Zumino term of the corresponding field g. Because of this relation, and as we will see more precisely in section 3.3.2, the presence of the current W in the realisation, *i.e.* the non-vanishing of $\not{k}$, will lead to the presence of a corresponding Wess-Zumino term in the action of the model.

From now on, we will suppose that the Kac-Moody currents $\mathcal{J}_\pm(x)$ take the form

$$\mathcal{J}_\pm(x) = \mathcal{B}_\pm Y(x) + \mathcal{C}_\pm j(x), \tag{3.2.8}$$

where $\mathcal{B}_\pm, \mathcal{C}_\pm : \mathfrak{g}^{\mathbb{C}} \to \mathfrak{g}^{\mathbb{C}}$ are linear operators on the Lie algebra $\mathfrak{g}^{\mathbb{C}}$. We will allow these operators to be dynamical (and thus have non-trivial Poisson brackets with other quantities in the phase space), but will suppose them to depend only on the field g (that is, not on X or derivatives of g). As we shall see in section 3.2.3, both the Yang-Baxter realisation and the λ-realisation can be recovered in this formalism by making some specific choices for the operators $\mathcal{B}_\pm$ and $\mathcal{C}_\pm$.

Let us note that, in general, these operators cannot be arbitrary. Indeed, they should be chosen such that the currents (3.2.8) satisfy the brackets (3.2.7). We will not try to write here the most general conditions on $\mathcal{B}_\pm$ and $\mathcal{C}_\pm$ for these brackets to hold. However, as explained in details in appendix 3.A, one can already obtain some useful constraints on these operators by focusing on the non-ultralocal terms in the brackets (3.2.7), *i.e.* terms proportional to the derivative of the Dirac distribution. More precisely, one finds that $\mathcal{B}_\pm$ and $\mathcal{C}_\pm$ should satisfy the following identities:

$$\mathcal{B}_\pm {}^t\mathcal{C}_\pm + \mathcal{C}_\pm {}^t\mathcal{B}_\pm = \ell_\pm \mathrm{Id}, \tag{3.2.9a}$$
$$\mathcal{B}_\pm {}^t\mathcal{C}_\mp + \mathcal{C}_\pm {}^t\mathcal{B}_\mp = 0, \tag{3.2.9b}$$

where we have introduced the transpose ${}^t\mathcal{O}$ with respect to the form κ for an operator $\mathcal{O}$ on the Lie algebra $\mathfrak{g}^{\mathbb{C}}$.

Reality conditions. As explained in section 2.2.2 of the previous chapter, in order for the models that we will construct to be real one has to impose some reality conditions on both the currents $\mathcal{J}_\pm$ and the levels $\ell_\pm$ that define the realisations. There are two possible types of conditions that we shall consider. In the first case, we suppose that the currents are invariant under the antilinear involutive automorphism τ considered in 2.2.2 (*i.e.* we assume that they are $\mathfrak{g}$-valued) and the corresponding levels are real:

$$\tau(\mathcal{J}_\pm(x)) = \mathcal{J}_\pm(x) \qquad \text{and} \qquad \overline{\ell_\pm} = \ell_\pm. \tag{3.2.10}$$

In the second case, one requires the currents to be conjugate with respect to τ and the levels to be complex conjugate to each other:

$$\tau(\mathcal{J}_\pm(x)) = \mathcal{J}_\mp(x) \qquad \text{and} \qquad \overline{\ell_\pm} = \ell_\mp. \tag{3.2.11}$$

Momentum. Let us end this section by proving an identity about the momentum (3.2.3) that will become useful later. To start with, it is simple to check that from the relations (3.2.9) obeyed by the operators $\mathcal{B}_\pm$ and $\mathcal{C}_\pm$, one can derive the following additional identities:

$$ {}^t\mathcal{B}_+\mathcal{B}_+/\ell_+ + {}^t\mathcal{B}_-\mathcal{B}_-/\ell_- = 0, \tag{3.2.12a} $$

$$ {}^t\mathcal{C}_+\mathcal{C}_+/\ell_+ + {}^t\mathcal{C}_-\mathcal{C}_-/\ell_- = 0, \tag{3.2.12b} $$

$$ {}^t\mathcal{B}_+\mathcal{C}_+/\ell_+ + {}^t\mathcal{B}_-\mathcal{C}_-/\ell_- = \mathrm{Id}. \tag{3.2.12c} $$

Together with the definition of the currents (3.2.8) above and equation (3.2.6), these identities allow one to prove that the momentum (3.2.3) can be re-expressed as

$$ \mathcal{P}_G = \frac{1}{2\ell_+}\int_{\mathbb{D}} \mathrm{d}x\ \kappa(\mathcal{J}_+(x), \mathcal{J}_+(x)) + \frac{1}{2\ell_-}\int_{\mathbb{D}} \mathrm{d}x\ \kappa(\mathcal{J}_-(x), \mathcal{J}_-(x)). \tag{3.2.13} $$

3.2.3 Examples of realisations

We will now review some relevant examples of Kac-Moody realisations.

Inhomogeneous Yang-Baxter realisation without Wess-Zumino term. Let us start by considering a solution $R : \mathfrak{g} \to \mathfrak{g}$ of the modified classical Yang-Baxter equation (mCYBE):

$$ [RX, RY] - R([RX, Y] + [X, RY]) = -c^2[X, Y], \qquad \forall\, X, Y \in \mathfrak{g}, \tag{3.2.14} $$

with $c = 1$ (so-called split case) or $c = i$ (non-split case), which we suppose to be skew-symmetric with respect to the non-degenerate form κ:

$$ \kappa(RX, Y) = -\kappa(X, RY), \qquad \forall\, X, Y \in \mathfrak{g}. $$

R can be used to construct a Kac-Moody realisation which takes the name of inhomogeneous Yang-Baxter realisation without Wess-Zumino term. The adjective inhomogeneous refers to the fact that R is a solution of the mCYBE rather than the CYBE, corresponding to the case $c = 0$ (we will comment later on this homogeneous case). The Kac-Moody currents for this realisation read [47, 49, 59]:

$$ \mathcal{J}_\pm = \left(\frac{1}{2}\mathrm{Id} \mp \frac{1}{2c}R_g\right) X \pm \frac{1}{2c\gamma} j, \tag{3.2.15} $$

where γ is a real constant and

$$ R_g = \mathrm{Ad}_g^{-1} \circ R \circ \mathrm{Ad}_g. $$

The proof that these are Kac-Moody currents can be found in [47], where the levels are shown to be

$$ \ell_\pm = \pm\frac{1}{2c\gamma}. \tag{3.2.16} $$

Note in particular that the levels $\ell_\pm$ are opposite to one another.

The reality conditions discussed in section 3.2.2 are simple to verify. In particular, in the split case ($c = 1$) the currents $\mathcal{J}_\pm$ are $\mathfrak{g}$-valued and the levels $\ell_\pm$ are real, hence (3.2.10) is satisfied. In the non-split case ($c = i$), it is a simple check that the currents and the levels satisfy (3.2.11).

In the general language of section 3.2.2, we see that the current W does not appear in the expression (3.2.15), which means that for this realisation we take the coefficient $\Bbbk$ in (3.2.8) to be zero. According to what has been discussed in the previous section, this justifies the fact that the models constructed from this realisation will not contain the Wess-Zumino term of g. Finally, by comparing with (3.2.8) we read for the operators $\mathcal{B}_\pm$ and $\mathcal{C}_\pm$:

$$\mathcal{B}_\pm = \frac{1}{2}\text{Id} \mp \frac{1}{2c}R_g \qquad \text{and} \qquad \mathcal{C}_\pm = \ell_\pm \text{Id}. \tag{3.2.17}$$

One easily checks that these operators satisfy the identities (3.2.9) as expected.

Inhomogeneous Yang-Baxter realisation with Wess-Zumino term. The inhomogeneous Yang-Baxter realisation defined in the previous paragraph has no Wess-Zumino term, *i.e.* does not contain the current $W(x)$ (or equivalently has $\Bbbk = 0$). Following [48], one can generalise this construction to include the current $W(x)$ and thus a non-zero coefficient $\Bbbk$, at least when the R-matrix underlying the realisation satisfies the additional condition $R^3 = c^2 R$, with c as in the right-hand side of the mCYBE (3.2.14) (note in particular that the standard Drinfeld-Jimbo R-matrix satisfies this condition). The levels of this generalised realisation are given by

$$\ell_\pm = \pm\frac{1}{2c\gamma} - \Bbbk,$$

with γ a real constant. Comparing to the levels (3.2.16) of the realisation without Wess-Zumino term, one sees that turning on the coefficient $\Bbbk$ corresponds to relaxing the fact that the levels $\ell_\pm$ are opposite one to another.

The Kac-Moody currents of the inhomogeneous Yang-Baxter realisation with Wess-Zumino term can be computed from the results of [48, section 3], up to a few differences in the conventions[4]. In the present notations, they read as follows:

$$\mathcal{J}_\pm = \left(\frac{1}{2}\text{Id} \mp \frac{1}{2c}R_g \mp \frac{\delta}{2}\Pi_g\right) Y + \left(\left(\pm\frac{1}{2c\gamma} - \frac{\Bbbk}{2}\right)\text{Id} \mp \frac{\Bbbk}{2c}R_g \mp \frac{\Bbbk\delta}{2}\Pi_g\right) j,$$

where we recall that $Y = X - \Bbbk W$ and where we have defined the quantities

$$\Pi_g = 1 - \frac{R_g^2}{c^2} \qquad \text{and} \qquad \delta = \frac{1-\sqrt{1-4c^2\Bbbk^2\gamma^2}}{2c\Bbbk\gamma}.$$

Note that $\delta \to 0$ as $\Bbbk \to 0$ so that the currents $\mathcal{J}_\pm$ above tend to (3.2.15) in the limit.

It is simple to check that the reality conditions are satisfied for both the choices $c = 1$ and $c = i$, similarly to the case without Wess-Zumino term. From the form of the currents, one can make the following identifications comparing to equation (3.2.8):

$$\mathcal{B}_\pm = \frac{1}{2}\text{Id} \mp \frac{1}{2c}R_g \mp \frac{\delta}{2}\Pi_g \qquad \text{and} \qquad \mathcal{C}_\pm = \left(\ell_\pm + \frac{\Bbbk}{2}\right)\text{Id} \mp \frac{\Bbbk}{2c}R_g \mp \frac{\Bbbk\delta}{2}\Pi_g. \tag{3.2.18}$$

Let us note that, as expected, the identities (3.2.9) are again satisfied by these operators $\mathcal{B}_\pm$ and $\mathcal{C}_\pm$ (using the fact that we restrict here to R-matrices satisfying $R^3 = c^2 R$).

$\boldsymbol{\lambda}$-realisation. For the λ-realisation, the Kac-Moody currents are given by [49, 59, 80]:

$$\begin{aligned}\mathcal{J}_+ &= X - \Bbbk W - \Bbbk j = Y - \Bbbk j,\\ \mathcal{J}_- &= -\text{Ad}_g(X - \Bbbk W + \Bbbk j) = -\text{Ad}_g(Y + \Bbbk j).\end{aligned}$$

[4] For completeness, note that this reference only treats the non-split case $c = i$. The results generalise straightforwardly to the split case $c = 1$.

The levels are given by:

$$\ell_{\pm} = \mp 2\cancel{k}.$$

Note that, similarly to the inhomogeneous Yang-Baxter realisation without Wess-Zumino term, these levels $\ell_{\pm}$ are opposite one to another. In this case, the reality condition (3.2.10) is satisfied, since the currents $\mathcal{J}_{\pm}$ are $\mathfrak{g}$-valued and the levels $\ell_{\pm}$ are real.

Comparing to equation (3.2.8), one sees that for the λ-realisation the operators $\mathcal{B}_{\pm}$ and $\mathcal{C}_{\pm}$ have the following form:

$$\mathcal{B}_{+} = \mathrm{Id}, \qquad \mathcal{B}_{-} = -\mathrm{Ad}_g, \qquad \mathcal{C}_{+} = -\cancel{k}\,\mathrm{Id}, \qquad \mathcal{C}_{-} = -\cancel{k}\,\mathrm{Ad}_g, \tag{3.2.19}$$

and again one can check that the identities (3.2.9) are satisfied.

3.2.4 Affine Gaudin models construction

In this section, we proceed to construct the integrable field theories that we will consider in the rest of this chapter as realisations of AGM. We follow the procedure described in chapter 2 for general AGMs.

Sites, levels and twist function. Let us consider an AGM with $2N$ sites of multiplicity one, which for the purpose of this chapter we gather in pairs $(r,+)$ and $(r,-)$ with $r \in \{1, \cdots, N\}$[5]. The position of the site $(r,\pm)$ in the complex plane $\mathbb{C}$ will be denoted by $z_r^{\pm}$. Since each site $(r,\pm)$ is of multiplicity one, it is associated with one level, which is a non-zero constant number and which we will denote by $\ell_{r,\pm}$. Let us also fix a non-zero real number ℓ^{∞}. Altogether, this data specifies the twist function of the AGM. From (2.2.6), this is given by:

$$\varphi(z) = \sum_{r=1}^{N} \left(\frac{\ell_{r,+}}{z - z_r^{+}} + \frac{\ell_{r,-}}{z - z_r^{-}} \right) - \ell^{\infty}. \tag{3.2.20}$$

Kac-Moody currents and phase space. Following the discussion of chapter 2, with the sites fixed as above, the phase space of the model is characterised by N independent pairs of commuting Kac-Moody currents $(\mathcal{J}_{r,+}, \mathcal{J}_{r,-})$, $r \in \{1, \cdots, N\}$. The Poisson brackets of these fields are specified by the choice of levels $\ell_{r,\pm}$. More precisely, we have the following:

$$\left\{ \mathcal{J}_{r,\pm\underline{\mathbf{1}}}(x), \mathcal{J}_{s,\pm\underline{\mathbf{2}}}(y) \right\} = \delta_{rs} \left([C_{\underline{\mathbf{12}}}, \mathcal{J}_{r,\pm\underline{\mathbf{1}}}(x)]\delta_{xy} - \ell_{r,\pm} C_{\underline{\mathbf{12}}} \delta'_{xy} \right), \tag{3.2.21a}$$

$$\left\{ \mathcal{J}_{r,\pm\underline{\mathbf{1}}}(x), \mathcal{J}_{s,\mp\underline{\mathbf{2}}}(y) \right\} = 0. \tag{3.2.21b}$$

We have described in detail in section 3.2.2 how such a pair can be realised in the phase space of canonical fields on T^*G. A natural way to realise the $2N$ currents $\mathcal{J}_{r,\pm}$ is then to consider N independent realisations in T^*G of the type described in section 3.2.2. This means that we choose the phase space of the model to be the space of fields on the product T^*G^N, with the currents $\mathcal{J}_{r,\pm}$ belonging to the r^{th}-factor in T^*G^N.

This r^{th}- factor is described by a pair of canonical fields $g_r(x)$ and $X_r(x)$, valued respectively in the group G and the Lie algebra $\mathfrak{g}$, which are the equivalent of the fields $g(x)$ and $X(x)$ discussed in section 3.2.1 to describe one copy of T^*G. Similarly, one can define from these canonical fields the equivalent of the currents $j(x)$ and $W(x)$, which we shall denote by $j_r(x)$ and $W_r(x)$. Following the discussion above, we then also define the currents $\mathcal{J}_{r,\pm}$ as the analogues

[5]We note that due to this gathering of indices the notations of this chapter will differ slightly from the ones of the previous one.

in the r^{th}-factor of the Kac-Moody currents $\mathcal{J}_\pm$ described in section 3.2.2. Therefore, they take the form

$$\mathcal{J}_{r,\pm}(x) = \mathcal{B}_r^\pm Y_r(x) + \mathcal{C}_r^\pm j_r(x), \tag{3.2.22}$$

where

$$Y_r(x) = X_r(x) - \cancel{k}_r W_r(x) \tag{3.2.23}$$

and $\cancel{k}_r$ is a real constant number depending on the choice of realisation in the r^{th}-copy. The $\mathcal{B}_r^\pm$'s and $\mathcal{C}_r^\pm$'s are linear operators on the Lie algebra $\mathfrak{g}^{\mathbb{C}}$, which are the equivalent of the operators $\mathcal{B}_\pm$ and $\mathcal{C}_\pm$ introduced in section 3.2.2. In particular, they depend only on g_r and satisfy analogous identities to the ones of equation (3.2.9).

Gaudin Lax matrix. Let us consider the Gaudin Lax matrix of the model. From equation (2.2.5) of the previous chapter, it is defined from the currents (3.2.22) as the following $\mathfrak{g}^{\mathbb{C}}$-valued field:

$$\Gamma(z,x) = \sum_{r=1}^{N} \left(\frac{\mathcal{J}_{r,+}(x)}{z - z_r^+} + \frac{\mathcal{J}_{r,-}(x)}{z - z_r^-} \right). \tag{3.2.24}$$

Reality conditions. As we discussed in section 3.2.2, in order for the models which we construct in this chapter to be real, we have to impose some reality conditions. For each pair of sites $(r, \pm)$, there are two cases. In the first one, we suppose the positions of the two sites $z_r^\pm$ to be real and that the condition (3.2.10) on the currents $\mathcal{J}_{r,\pm}$ and the levels $\ell_{r,\pm}$ holds. In the second case, we assume instead that the the positions of the sites are complex conjugate to each other and that the currents and levels satisfy the condition (3.2.11).

One can check that these conditions imply the equivariance relations for the twist function and the Gaudin Lax matrix of the models that were discussed in section 2.2.2:

$$\tau(\Gamma(z,x)) = \Gamma(\bar{z},x) \qquad \text{and} \qquad \overline{\varphi(z)} = \varphi(\bar{z}).$$

3.2.5 Hamiltonian and momentum

Hamiltonian. Following chapter 2, we start by rewriting the twist function in terms of its zeroes ζ_i ($i \in \{1, \cdots, 2N\}$), which are supposed to be real and distinct:

$$\varphi(z) = -\ell^\infty \frac{\prod_{i=1}^{2N} (z - \zeta_i)}{\prod_{r=1}^{N} (z - z_r^+)(z - z_r^-)}. \tag{3.2.25}$$

As discussed in section 2.2.3, the Hamiltonian for realisations of AGMs is built from the following quadratic charges:

$$\mathcal{Q}_i = -\frac{1}{2\varphi'(\zeta_i)} \int_{\mathbb{D}} \mathrm{d}x \; \kappa(\Gamma(\zeta_i,x), \Gamma(\zeta_i,x)), \tag{3.2.26}$$

$i = 1, \ldots, 2N$. More precisely, we define the Hamiltonian of the model as the linear combination

$$\mathcal{H} = \sum_{i=1}^{2N} \epsilon_i \mathcal{Q}_i, \tag{3.2.27}$$

for some real numbers ϵ_i.

As explained in section 2.2.3, as a consequence of the reality conditions we introduced, $\mathcal{H}$ is real.

Momentum and relativistic invariance. Let us consider the momentum of the theory. Since the phase space is described by canonical fields on T^*G^N, this momentum is given by a sum over the copies $r = 1, \dots, N$ of the expression (3.2.3) for the corresponding copy. Denoting the momentum of the r^{th}-copy by $\mathcal{P}_r$ and re-expressing it through the identity in (3.2.13), one finds:

$$\mathcal{P} = \sum_{r=1}^{N} \mathcal{P}_r = \frac{1}{2\ell_{r,+}} \int_{\mathbb{D}} \mathrm{d}x \; \kappa(\mathcal{J}_{r,+}(x), \mathcal{J}_{r,+}(x)) + \frac{1}{2\ell_{r,-}} \int_{\mathbb{D}} \mathrm{d}x \; \kappa(\mathcal{J}_{r,-}(x), \mathcal{J}_{r,-}(x)). \tag{3.2.28}$$

It is simple to check that it can be rewritten as:

$$\mathcal{P} = -\sum_{r=1}^{N} \left(\operatorname*{res}_{z=z_r^+} \mathcal{Q}(z)\mathrm{d}z + \operatorname*{res}_{z=z_r^-} \mathcal{Q}(z)\mathrm{d}z \right), \tag{3.2.29}$$

where $\mathcal{Q}(z)$ was defined in (2.2.12). This shows explicitly that the assumption (2.2.18) made in section 2.2.3 for general realisations of AGMs is verified in this particular model. From the discussion in section 2.2.3, we then have:

$$\mathcal{P} = \sum_{i=1}^{2N} \mathcal{Q}_i. \tag{3.2.30}$$

As we explained in section 2.2.4 for general realisations of AGMs, this fact then allows us to find a simple condition for the relativistic invariance of the model. More precisely, requiring this invariance restricts the choice of the coefficients ϵ_i in the definition of $\mathcal{H}$ to

$$\epsilon_i = +1 \qquad \text{or} \qquad \epsilon_i = -1,$$

for every $i \in \{1, \cdots, 2N\}$. We then see that the indices $i \in \{1, \cdots, 2N\}$ labelling the zeroes ζ_i divide naturally into the sets $\mathcal{I}_\pm = \{i \mid \epsilon_i = \pm 1\}$. In the rest of this chapter, we will suppose that there are as many ϵ_i's equal to $+1$ as ϵ_i's equal to -1 (*i.e.* that the sets $\mathcal{I}_\pm$ are both of size $|\mathcal{I}_+| = |\mathcal{I}_-| = N$)[6].

3.2.6 Integrability

Lax connection and integrability. We proved in chapter 2 that realisations of AGMs admit a reformulation of the equations of motion in terms of the zero curvature equation for a Lax connection $(\mathcal{L}, \mathcal{M})$. From equations (2.2.32) and (2.2.38), the Lax connection for the models considered in this chapter is given by:

$$\mathcal{L}(z,x) = \sum_{i=1}^{2N} \frac{1}{\varphi'(\zeta_i)} \frac{\Gamma(\zeta_i, x)}{z - \zeta_i} \qquad \text{and} \qquad \mathcal{M}(z,x) = \sum_{i=1}^{2N} \frac{\epsilon_i}{\varphi'(\zeta_i)} \frac{\Gamma(\zeta_i, x)}{z - \zeta_i}. \tag{3.2.31}$$

As discussed in section 2.2.5, the integrability of the model then follows from the fact that the Lax matrix satisfies the Maillet non-ultralocal bracket (2.1.19). Moreover, the $\mathcal{R}$-matrix appearing in this bracket is given by the standard untwisted $\mathcal{R}$-matrix multiplied by the inverse of the twist function $\varphi(w)$:

$$\mathcal{R}_{\underline{\mathbf{12}}}(z,w) = \frac{C_{\underline{\mathbf{12}}}}{w - z} \varphi(w)^{-1}.$$

As explained in section 2.2.5, this bracket then implies that the infinite set of charges that can be extracted from the monodromy matrix of the model is in involution.

[6] As first observed in [59], the models obtained when choosing $\mathcal{I}_+$ and $\mathcal{I}_-$ of different sizes would not posses otherwise a well-defined inverse Legendre transform.

Lax connection in light-cone coordinates. As we will need this in section 3.3, let us briefly discuss the reparametrisation of the Lax connection in light-cone components. Let us firstly consider the light-cone coordinates $x^\pm = (t \pm x)/2$ and the corresponding derivatives $\partial_\pm = \partial_t \pm \partial_x$. The zero curvature equation can then be rewritten as

$$\partial_+ \mathcal{L}_-(z,x) - \partial_- \mathcal{L}_+(z,x) + [\mathcal{L}_+(z,x), \mathcal{L}_-(z,x)] = 0,$$

where we have introduced the light-cone Lax connection

$$\mathcal{L}_\pm(z,x) = \mathcal{M}(z,x) \pm \mathcal{L}(z,x). \tag{3.2.32}$$

Finally, from the expressions (3.2.31), one finds the following one for $\mathcal{L}_\pm(z,x)$:

$$\mathcal{L}_\pm(z,x) = \pm 2 \sum_{i\in\mathcal{I}_\pm} \frac{1}{\varphi'(\zeta_i)} \frac{\Gamma(\zeta_i,x)}{z-\zeta_i}, \tag{3.2.33}$$

where we note the appearance of the two sets $\mathcal{I}_\pm$ introduced in section 3.2.5.

3.2.7 Exploring the "space of models"

Gaudin parameters. Let us describe the "space of models" that we are considering in this chapter by summarising what are the defining parameters of the integrable field theories that we have constructed so far. As affine Gaudin models, these theories are characterised by the following quantities, that we shall refer to as Gaudin parameters:

- the positions $z_r^\pm$;
- the levels $\ell_{r,\pm}$;
- the constant term ℓ^∞ in the twist function ;
- the Kac-Moody realisations with levels $\ell_{r,\pm}$ attached to each pair of sites $(r,\pm)$.

As explained in [59, section 1.4.2], there exists a redundancy between the Gaudin parameters of the model, corresponding to the freedom of translating and dilating the spectral parameter. Indeed, the model with parameters $z_r^\pm$, $\ell_{r,\pm}$ and ℓ^∞ is invariant under the transformation

$$z_r^\pm \longmapsto a z_r^\pm + b \qquad \text{and} \qquad \ell^\infty \longmapsto a^{-1}\ell^\infty, \tag{3.2.34}$$

where a and b are real numbers with $a \neq 0$ and where we keep the levels $\ell_{r,\pm}$ and the Kac-Moody realisations fixed. Note that one can fix the dilation redundancy (corresponding to the parameter a in the transformation above) by setting the constant term ℓ^∞ to a specific value. Similarly, one can fix the translation redundancy (corresponding to the parameter b) by setting one of the positions $z_r^\pm$ to a specific point.

We note that the Gaudin parameters introduced above are in general not all real but should satisfy the reality conditions described in sections 3.2.2 and 3.2.4. Let us then discuss what are the real parameters of the models. Note first that the constant term ℓ^∞ is always assumed to be real. Moreover, recall that for each pair of sites $(r,\pm)$, there are two possible reality conditions: either the positions $z_r^\pm$ and the levels $\ell_{r,\pm}$ are real or they form pairs of complex conjugate numbers. We will encode the choice of reality condition for the sites $(r,\pm)$ by introducing a number c_r, which is defined to be 1 in the first case and i in the second one. In particular, $z_r^\pm$ and $\ell_{r,\pm}$ can then be written using the following parametrisation:

$$z_r^\pm = z_r \pm c_r \eta_r \qquad \text{and} \qquad \ell_{r,\pm} = \frac{\ell_{0,r}}{2} \pm \frac{\ell_{1,r}}{2c_r\eta_r}, \tag{3.2.35}$$

where the parameters z_r, η_r, $\ell_{0,r}$ and $\ell_{1,r}$ are real. As we shall see, this particular choice of parametrisation will also be convenient for the interpretation of the models as deformations in the next section. Note that it is equivalent to defining

$$z_r = \frac{z_r^+ + z_r^-}{2}, \qquad \eta_r = \frac{z_r^+ - z_r^-}{2c_r}, \qquad \ell_{0,r} = \ell_{r,+} + \ell_{r,-} \qquad \text{and} \qquad \ell_{1,r} = \frac{z_r^+ - z_r^-}{2}(\ell_{r,+} - \ell_{r,-}). \tag{3.2.36}$$

Choice of realisations. As explained in section 3.2.4, the choice of the Kac-Moody realisation attached to the sites $(r, \pm)$ corresponds to specifying the explicit form of the operators $\mathcal{B}_r^\pm$ and $\mathcal{C}_r^\pm$ and the value of the coefficient $\Bbbk_r$ appearing in equations (3.2.22) and (3.2.23). In particular, one can choose this realisation among the examples described in section 3.2.3.

For instance, if one takes the inhomogeneous Yang-Baxter realisation (with Wess-Zumino term), $\Bbbk_r$ is set to $-(\ell_{r,+}+\ell_{r,-})/2 = -\ell_{0,r}/2$ and the operators $\mathcal{B}_r^\pm$ and $\mathcal{C}_r^\pm$ are given by equation (3.2.18) (replacing g by g_r and c by the number $c_r \in \{1, i\}$ defined in the previous paragraph, which encodes the choice of reality conditions for the sites $(r, \pm)$).

Similarly, if one chooses the λ-realisation, the operators $\mathcal{B}_r^\pm$ and $\mathcal{C}_r^\pm$ are given by equation (3.2.19) (with g replaced by g_r), while $\Bbbk_r$ is given by $-\ell_{r,+}/2$. Note however that one can choose the λ-realisation only if the levels $\ell_{r,\pm}$ are real (*i.e.* $c_r = 1$ in the notations of the previous paragraph) and are such that

$$\ell_{0,r} = \ell_{r,+} + \ell_{r,-} = 0. \tag{3.2.37}$$

This is in contrast with the case of the inhomogeneous Yang-Baxter realisation with Wess-Zumino term considered above, where the levels $\ell_{r,\pm}$ are not subject to any constraints (other than the reality conditions).

Note that the choice of a Yang-Baxter realisation at the sites $(r, \pm)$ comes with the additional freedom of choosing a skew-symmetric R-matrix R_r, solution of the mCYBE (3.2.14). As explained in section 3.2.3, this operator should in general satisfy the additional property $R_r^3 = c_r^2 R_r$. However, if the levels $\ell_{r,\pm}$ satisfy the constraint (3.2.37), *i.e.* if one considers a Yang-Baxter realisation without Wess-Zumino term, one does not need to require this additional condition on R_r.

The space of models. The discussion above concerns the choice of realisation for one pair of sites $(r, \pm)$. One can then construct different models by considering different combinations of realisations for the N pairs $(1, \pm), \cdots, (N, \pm)$ describing the models. In particular, one can consider a model with N_1 copies of the Yang-Baxter realisation and N_2 copies of the λ-realisation, where $N_1 + N_2 = N$. Let us discuss what are the free parameters of this theory. As explained in the previous paragraphs, the model is described by the $4N+1$ Gaudin parameters $z_r^\pm$, $\ell_{r,\pm}$ and ℓ^∞, or equivalently by the $4N+1$ real parameters $z_r, \eta_r, \ell_{0,r}$, $\ell_{1,r}$ and ℓ^∞. Taking into account the translation and dilation redundancy (3.2.34) and the fact that the levels corresponding to the λ-realisations should satisfy the constraints (3.2.37), we arrive at the conclusion that this model is described by $3N + N_1 - 1$ free parameters. Note that in addition to these parameters, which specify its structure as an AGM, the model is also determined by the choice of N_1 R-matrices for the Yang-Baxter realisations (which do not need to be identical).

As was explained in [57], see also [59], the models with only one realisation, *i.e.* with $N = 1$, correspond to well-known integrable sigma models, which served as basis for defining the Yang-Baxter and λ-realisations. Indeed, the inhomogeneous Yang-Baxter realisation (without or with Wess-Zumino term) is defined in such a way that the AGM with one copy of this realisation, corresponding in the above paragraph to $N_1 = 1$ and $N_2 = 0$, coincides with the so-called Yang-Baxter sigma model, without [20, 21] or with [48] Wess-Zumino term. Similarly, the

AGM with one copy of the λ-realisation, *i.e.* with $N_1 = 0$ and $N_2 = 1$, yields the so-called λ-model [22]. The model defined above with arbitrary numbers N_1 and N_2 is thus a generalisation of these models. According to the general coupling procedure described in [59, section 2.3.3], it corresponds to coupling together N_1 copies of the Yang-Baxter model and N_2 copies of the λ-model in a non-trivial way which however ensures the integrability of this interacting model (as, by construction, it is a realisation of AGM).

Zeroes versus levels. Let us end this section with some remarks about a possible more convenient reparametrisation of the models that we are considering. Recall from sections 3.2.5 and 3.2.6 that in order to define the Hamiltonian and express the Lax connection of the models, one uses the zeroes ζ_i, $i \in \{1, \cdots, 2N\}$, of the twist function. These zeroes are related implicitly to the Gaudin parameters $z_r^\pm$, $\ell_{r,\pm}$ and ℓ^∞ through the equation $\varphi(\zeta_i) = 0$, with the twist function $\varphi(z)$ defined in terms of the Gaudin parameters as in (3.2.20). This equation is equivalent to a polynomial equation of degree $2N$ in ζ_i. Thus, it is in general impossible to give an explicit expression of the zeroes ζ_i in terms of the Gaudin parameters.

One way of bypassing this difficulty is to consider as defining parameters of the models the positions $z_r^\pm$, the zeroes ζ_i and the constant term ℓ^∞. One then defines the twist function of the model by equation (3.2.25) instead of equation (3.2.20) and the levels $\ell_{r,\pm}$ as the corresponding residues:

$$\ell_{r,\pm} = \operatorname*{res}_{z=z_r^\pm} \varphi(z)\, \mathrm{d}z = \mp \frac{\ell^\infty}{z_r^+ - z_r^-} \frac{\prod_{i=1}^{2N}(z_r^\pm - \zeta_i)}{\prod_{s=1, s\neq r}^{N}(z_r^\pm - z_s^\pm)(z_r^\pm - z_s^\mp)}.$$

The main advantage of this re-parametrisation is that all the relevant quantities that are used to describe the models, in particular the levels $\ell_{r,\pm}$ and the Hamiltonian $\mathcal{H}$, can be written as rational expressions of the parameters $z_r^\pm$, ζ_i and ℓ^∞. Note however that this parametrisation has a disadvantage when one wants to consider λ-realisations and/or Yang-Baxter realisations without Wess-Zumino terms. Indeed, for these realisations, the levels should satisfy the additional constraint (3.2.37), which translates in a rather complicated algebraic condition on the parameters $z_r^\pm$ and ζ_i, using the above expressions for the levels. Finally, let us note that the translation and dilation redundancy (3.2.34) among the Gaudin parameters can be re-expressed in terms of this new parametrisation as the invariance of the model under the transformation

$$z_r^\pm \longmapsto a z_r^\pm + b, \qquad \zeta_i \longmapsto a\zeta_i + b \qquad \text{and} \qquad \ell^\infty \longmapsto a^{-1}\ell^\infty.$$

3.2.8 Recovering undeformed models

In this section, following the results of [59], we discuss how the model defined above by taking N_1 Yang-Baxter realisations and N_2 λ-realisations can be interpreted as a deformation of a simpler model. This result generalises the well known facts that the Yang-Baxter model (with or without Wess-Zumino term) is a deformation of the principal chiral model (PCM, with or without Wess-Zumino term) and the λ-model is a deformation of the non-abelian T-dual of the PCM. In the present language, these correspond respectively to the cases $(N_1 = 1, N_2 = 0)$ and $(N_1 = 0, N_2 = 1)$. The undeformed limit of the model with arbitrary N_1 and N_2 corresponds to a theory coupling together N_1 copies of the PCM (with Wess-Zumino terms) and N_2 copies of its non-abelian T-dual. In particular, the model with only copies of the PCM without Wess-Zumino terms corresponds to the one introduced in the examples section of chapter 2.

This undeformed model is also defined as a realisation of AGM but possesses a slightly different sites structure. Indeed, in the language of chapter 2, instead of the $2N$ sites $(r, \pm)$ of multiplicity one, it possesses N sites (r) of multiplicity two. These sites correspond to double poles in the twist function and the Gaudin Lax matrix of the model and are associated with

Takiff realisations of multiplicity two. As we shall now explain, the site (r) of multiplicity two is obtained from the pair of sites $(r,\pm)$ in the deformed model by making their positions z_r^+ and z_r^- collide, while controlling the behaviour of the corresponding levels $\ell_{r,\pm}$.

Colliding two simple poles into a double pole. Let us focus here on one pair of sites $(r,\pm)$. In order to isolate the parts of the twist function and the Gaudin Lax matrix of the model corresponding to this pair, let us rewrite them as

$$\varphi(z) = \frac{\ell_{r,+}}{z - z_r^+} + \frac{\ell_{r,-}}{z - z_r^-} + \widetilde{\varphi}(z),$$
$$\Gamma(z) = \frac{\mathcal{J}_{r,+}}{z - z_r^+} + \frac{\mathcal{J}_{r,-}}{z - z_r^-} + \widetilde{\Gamma}(z),$$

where $\widetilde{\varphi}$ and $\widetilde{\Gamma}$ contain all the information related to the other sites. Using the parameters c_r, z_r, η_r, $\ell_{0,r}$ and $\ell_{1,r}$ introduced in the previous section (see equation (3.2.36)), one can rewrite the twist function as

$$\varphi(z) = \frac{\ell_{1,r}}{(z - z_r)^2 - c_r^2\eta_r^2} + \frac{\ell_{0,r}(z - z_r)}{(z - z_r)^2 - c_r^2\eta_r^2} + \widetilde{\varphi}(z). \tag{3.2.38}$$

As mentioned above, the undeformed limit corresponds to making the two positions z_r^+ and z_r^- collide at the point z_r and thus to taking $\eta_r \to 0$. In particular, this leads us to interpret η_r as a deformation parameter. We aim here to recover, in the limit $\eta_r \to 0$, a model with a site of multiplicity two, *i.e.* with a double pole in its twist function. It is then clear from equation (3.2.38) that this is the case if one supposes that the quantities $\ell_{0,r}$ and $\ell_{1,r}$ stay finite when η_r goes to 0. From now on, we will thus define the undeformed limit as taking $\eta_r \to 0$ while keeping $\ell_{0,r}$ and $\ell_{1,r}$ finite (let us note that the levels $\ell_{r,\pm}$ of the sites $(r,\pm)$ then diverge, as one can see from equation (3.2.35)). In this limit, the twist function becomes

$$\varphi(z) \xrightarrow{\eta_r \to 0} \frac{\ell_{1,r}}{(z - z_r)^2} + \frac{\ell_{0,r}}{z - z_r} + \widetilde{\varphi}(z).$$

Following the terminology of chapter 2, this corresponds to the twist function of an AGM with a site (r) of multiplicity two, with position z_r and levels $\ell_{0,r}$ and $\ell_{1,r}$ (and with the other sites, contained in $\widetilde{\varphi}(z)$, as in the deformed model).

A similar argument applies to the Gaudin Lax matrix of the model. Let us suppose that the Kac-Moody currents $\mathcal{J}_{r,\pm}$ are such that the limits

$$\mathcal{J}_{r,[0]} = \lim_{\eta_r \to 0} \left(\mathcal{J}_{r,+} + \mathcal{J}_{r,-}\right) \qquad \text{and} \qquad \mathcal{J}_{r,[1]} = \lim_{\eta_r \to 0} c_r\eta_r \left(\mathcal{J}_{r,+} - \mathcal{J}_{r,-}\right) \tag{3.2.39}$$

are finite. Then the Gaudin Lax matrix becomes in the undeformed limit:

$$\Gamma(z) \xrightarrow{\eta_r \to 0} \frac{\mathcal{J}_{r,[1]}}{(z - z_r)^2} + \frac{\mathcal{J}_{r,[0]}}{z - z_r} + \widetilde{\Gamma}(z).$$

Thus, $\mathcal{J}_{r,[0]}$ and $\mathcal{J}_{r,[1]}$ are the Takiff currents attached to the site (r) of the undeformed model[7]. Let us now discuss this undeformed limit for the Yang-Baxter realisation and the λ-realisation.

[7] Starting from the Kac-Moody Poisson brackets (3.2.21) of the currents $\mathcal{J}_{r,\pm}$, one can indeed show that in the undeformed limit, the currents $\mathcal{J}_{r,[0]}$ and $\mathcal{J}_{r,[1]}$ satisfy the brackets of Takiff currents with levels $\ell_{0,r}$ and $\ell_{1,r}$.

From the Yang-Baxter to the PCM realisation. Let us suppose that the sites $(r,\pm)$ are associated with a Yang-Baxter realisation with Wess-Zumino term, as described in section 3.2.3. Let us first note that for this realisation, the Wess-Zumino coefficient is given by $\Bbbk_r = -\ell_{0,r}/2$. In particular, the undeformed limit defined in the previous paragraph can then be seen as taking η_r to 0 while keeping $\Bbbk_r$ and $\ell_{1,r}$ finite. Let us denote by R_r the R-matrix associated with this Yang-Baxter limit and introduce $R^{(r)} = \text{Ad}_{g_r}^{-1} \circ R_r \circ \text{Ad}_{g_r}$ and $\Pi^{(r)} = \text{Id} - R^{(r)\,2}/c_r^2$. The Kac-Moody currents of the realisation are then given by

$$\mathcal{J}_{r,\pm} = \frac{1}{2}\left(\text{Id} \mp \frac{R^{(r)}}{c_r} \mp \delta_r \Pi^{(r)}\right) Y_r + \left(\left(\pm\frac{\ell_{1,r}}{2c_r\eta_r} - \frac{\Bbbk_r}{2}\right)\text{Id} \pm \frac{\Bbbk_r}{2c_r}R^{(r)} \pm \frac{\Bbbk_r\delta_r}{2}\Pi^{(r)}\right) j_r, \quad (3.2.40)$$

with

$$Y_r = X_r - \Bbbk_r W_r \qquad \text{and} \qquad \delta_r = \ell_{1,r}\frac{1-\sqrt{1-(2c_r\eta_r\Bbbk_r/\ell_{1,r})^2}}{2c_r\eta_r\Bbbk_r}. \quad (3.2.41)$$

Let us now consider the undeformed limit, *i.e.* taking η_r to 0 while keeping $\Bbbk_r$ and $\ell_{1,r}$ finite. One first observes that in this limit, the coefficient δ_r tends to 0. Using this, one finds that the limits $\mathcal{J}_{r,[0]}$ and $\mathcal{J}_{r,[1]}$ defined in equation (3.2.39) are indeed finite and simply read

$$\mathcal{J}_{r,[0]} = X_r - \Bbbk_r W_r - \Bbbk_r j_r \qquad \text{and} \qquad \mathcal{J}_{r,[1]} = \ell_{1,r} j_r. \quad (3.2.42)$$

Thus, the undeformed limit described in the previous paragraph is well defined. Moreover, one recognises in the above equation the Takiff currents of the PCM+WZ realisation (with levels $\ell_{0,r} = -2\Bbbk_r$ and $\ell_{1,r}$), as defined in [59, section 3.1.3]. For $\Bbbk_r = 0$ (3.2.42) coincides with the Takiff realisation considered in the examples section of chapter 2 to describe the PCM without Wess-Zumino term and its coupled version.

From the λ-realisation to the non-abelian T-dual realisation. A similar mechanism to the one described above for the Yang-Baxter realisation provides the undeformed limit of the λ-realisation, yielding the so-called non-abelian T-dual realisation, as defined in [59, section 4.3.1]. This limit requires however a more subtle treatment. Indeed, if one were to consider the currents $\mathcal{J}_{r,\pm}$ of the λ-realisation in terms of the fields g_r and X_r and take the limits (3.2.39) "naively", one would encounter divergent expressions, making the undeformed limit procedure ill-defined. In order to obtain a well defined limit, one has to consider the fields g_r and X_r as depending on the deformation parameter η_r and suppose that they obey a well-chosen asymptotic expansion when η_r goes to 0. In particular, one of the consequences of this more subtle limit is that it changes the phase space of the realisation: from the space of canonical fields on T^*G (generated by g_r and X_r), one goes in the limit to the space of canonical fields on $T^*\mathfrak{g}$, which is the phase space of the non-abelian T-dual realisation. For brevity, we will not re-explain this procedure in this thesis and refer to [59, section 4.4.3] for details.

Undeformed limits of the coupled models. Let us consider the model defined in the previous section by coupling together N_1 copies of the Yang-Baxter model and N_2 copies of the λ-model. For each pair of sites $(r,\pm)$, one can consider the corresponding undeformed limit $\eta_r \to 0$. One would then obtain a model where the r-th copy reduces to either an undeformed PCM with Wess-Zumino term or a non-abelian T-dual of the PCM (depending on whether we started with a Yang-Baxter realisation or a λ-realisation at the sites $(r,\pm)$), still interacting non-trivially with the other $N-1$ copies. One can then consider different combinations of these undeformed limits on any number of copies, yielding various limits of the model. All these limits can be seen as deformations of a completely undeformed model, obtained by taking the limit where all the deformation parameters $\eta_1, \cdots, \eta_N$ are sent to 0. This undeformed model is the coupling of N_1 copies of the PCM with Wess-Zumino terms and N_2 copies of the non-abelian T-dual of the PCM. In particular, if one considers $N_2 = 0$, one obtains the model coupling together N copies of the PCM with Wess-Zumino term: this is the integrable coupled sigma model first introduced in [58] and whose detailed construction was presented in [59, section 3.3]. As discussed above, in the case without Wess-Zumino term one recovers the model described in section 2.2.6 of chapter 2.

Although it is defined in a different way, let us note also that the undeformed model with $N_2 \neq 0$ copies of the non-abelian T-dual of the PCM is in fact canonically equivalent to the model with $N = N_1 + N_2$ copies of the PCM, where N_2 of these copies have no Wess-Zumino term. This is because the non-abelian T-dual realisation is related to the PCM realisation without Wess-Zumino term by a canonical transformation [84]. Thus, the general model with N_1 Yang-Baxter realisations and N_2 λ-realisations can be seen as a deformation of the model coupling N_1 PCM with Wess-Zumino term and N_2 PCM without Wess-Zumino term (which is a particular case of the model introduced in [58]) after having first T-dualised the N_2 copies without Wess-Zumino term.

Homogeneous Yang-Baxter limit. For completeness, let us end this section by mentioning briefly another possible limit of the models considered here, which corresponds to going from an inhomogeneous Yang-Baxter realisation to a homogeneous Yang-Baxter realisation[8]. Let us consider an inhomogeneous Yang-Baxter realisation without Wess-Zumino term and with R-matrix R, which satisfies the mCYBE (3.2.14). So far, we considered the coefficient c appearing in the mCYBE as being either 1 or i, depending on the type of reality conditions imposed on the realisation. However, one easily checks that the construction of the Yang-Baxter realisation as recalled in section 3.2.3 holds without changes for any $c \neq 0$ (the realisation is then equivalent to the one with $c = 1$ or $c = i$ by rescaling the matrix R). The homogeneous limit consists in taking the limit $c \to 0$ of this realisation while also making the corresponding simple poles in the twist function collide (see for example [85]). Similarly to what happens for the undeformed limit described in this section, this yields a model with a site of multiplicity two, to which is attached the so-called homogeneous Yang-Baxter realisation, as defined in [59, section 4.1.1]. This realisation corresponds to a deformation of the PCM realisation without Wess-Zumino term by a homogeneous R-matrix, *i.e.* a solution of the (non-modified) CYBE:

$$[RX, RY] - R([RX, Y] + [X, RY]) = 0, \qquad \forall\, X, Y \in \mathfrak{g},$$

which corresponds to the limit $c \to 0$ of the mCYBE.

Summary. Although introduced as limits, the PCM, non-abelian T-dual and homogeneous Yang-Baxter realisations can be constructed independently, as was done for example in [59] (see also section 2.2.6). One can then consider AGM containing these realisations. In general, one can construct a model coupling together any combination of PCMs, non-abelian T-dual models, homogeneous and inhomogeneous Yang-Baxter models and λ-models. Up to taking appropriate limits, the present chapter then covers all these possibilities. In particular, one can obtain a model with $N - 1$ copies of the PCM and one homogeneous Yang-Baxter realisation: one then recovers the model studied in [59, appendix D] as the simplest illustration of the various possible integrable deformations of coupled integrable sigma models.

[8]This idea was first applied in the article [86] in the context of the deformed superstring on $AdS_5 \times S^5$.

3.3 Lagrangian formulation

In this section, our aim will be to describe the models introduced in section 2.2 in the Lagrangian formulation. Recall that in the Hamiltonian formulation, the degrees of freedom of these models are the fields $g_r(x)$ and $X_r(x)$, describing canonical fields valued in N independent copies of the cotangent bundle T^*G. The fields $g_r(x)$ are the "coordinate fields" valued in the space G. The momentum fields conjugate to these coordinates are then encoded in the fields $X_r(x)$ (see for instance [59, section 3.1.1] for details). In order to pass to the Lagrangian formulation, one has to consider the coordinate fields $g_r(x,t)$ as depending explicitly on the time variable $t \in \mathbb{R}$, defined by the Hamiltonian of the model, and express the momentum fields of the theory, encoded in X_r, in terms of these Lagrangian fields $g_r(x,t)$ and their derivatives $\partial_t g_r(x,t)$ and $\partial_x g_r(x,t)$. Finally, one obtains the action of the model as a functional of $g_r(x,t)$ by performing an inverse Legendre transform on their Hamiltonian.

In the present case, we will obtain the Lagrangian expression of the fields X_r in a rather indirect way. Indeed, as we shall see, these fields can be expressed naturally in terms of the Lax connection of the model. For this reason, we will start by determining the Lagrangian expression of the latter.

3.3.1 Lax connection in the Lagrangian formulation

Maurer-Cartan currents in terms of the Lax connection. Let us begin by considering the time evolution of the fields g_r. In the Hamiltonian formulation, this is given by their Poisson bracket with the Hamiltonian. More explicitly, recalling the definition (3.2.27) of the latter, one expresses the temporal Maurer-Cartan current $j_{t,r} = g_r(x)^{-1}\partial_t g_r(x)$ as

$$j_{t,r}(x) = g_r(x)^{-1}\{\mathcal{H}, g_r(x)\} = \sum_{i=1}^{2N} \epsilon_i \, g_r(x)^{-1}\{\mathcal{Q}_i, g_r(x)\}.$$

From the expression (3.2.26) of the charges $\mathcal{Q}_i$, we then have

$$j_{t,r}(x) = \sum_{i=1}^{2N} \frac{\epsilon_i}{\varphi'(\zeta_i)} \int_{\mathbb{D}} dy \; \kappa_{\underline{\mathbf{2}}} \left(g_{r,\underline{\mathbf{1}}}(x)^{-1}\{g_{r,\underline{\mathbf{1}}}(x), \Gamma_{\underline{\mathbf{2}}}(\zeta_i, y)\}, \Gamma_{\underline{\mathbf{2}}}(\zeta_i, y) \right).$$

The Poisson bracket in the integrand is calculated by inserting the definition (3.2.24) of $\Gamma(z,x)$, yielding:

$$\{g_{r,\underline{\mathbf{1}}}(x), \Gamma_{\underline{\mathbf{2}}}(\zeta_i, y)\} = \frac{1}{\zeta_i - z_r^+}\{g_{r,\underline{\mathbf{1}}}(x), \mathcal{J}_{r,+\underline{\mathbf{2}}}(y)\} + \frac{1}{\zeta_i - z_r^-}\{g_{r,\underline{\mathbf{1}}}(x), \mathcal{J}_{r,-\underline{\mathbf{2}}}(y)\},$$

where we have also used the fact that $\mathcal{J}_{s,\pm}$ is in the s^{th}-factor in T^*G^N and thus Poisson commutes with g_r if $r \neq s$. In order to calculate the Poisson brackets on the right hand side we then use the definition (3.2.22) of the currents $\mathcal{J}_{r,\pm}$ in terms of Y_r and j_r. Note that, firstly, the Poisson brackets of g_r with j_r vanish. Moreover, the brackets of g_r with the operators $\mathcal{B}_r^\pm$ and $\mathcal{C}_r^\pm$ also give no contribution as we assumed that these operators depend only on g_r. Thus, we have to take into account only the terms coming from the Poisson bracket of g_r with Y_r, so that

$$g_{r,\underline{\mathbf{1}}}(x)^{-1}\{g_{r,\underline{\mathbf{1}}}(x), \mathcal{J}_{r,\pm\underline{\mathbf{2}}}(y)\} = -\mathcal{B}^\pm_{r\underline{\mathbf{2}}} C_{\underline{\mathbf{12}}}\delta_{xy} = -{}^t\mathcal{B}^\pm_{r\underline{\mathbf{1}}} C_{\underline{\mathbf{12}}}\delta_{xy},$$

where we have used the fact that for any operator $\mathcal{O}$ on $\mathfrak{g}^{\mathbb{C}}$, one has $\mathcal{O}_{\underline{\mathbf{2}}} C_{\underline{\mathbf{12}}} = {}^t\mathcal{O}_{\underline{\mathbf{1}}} C_{\underline{\mathbf{12}}}$. Putting everything together, we conclude that

$$j_{t,r}(x) = \sum_{i=1}^{2N} \frac{\epsilon_i}{\varphi'(\zeta_i)} \left(\frac{{}^t\mathcal{B}_r^+}{z_r^+ - \zeta_i} + \frac{{}^t\mathcal{B}_r^-}{z_r^- - \zeta_i} \right) \Gamma(\zeta_i, x).$$

Using the expression of the temporal component of the Lax connection (3.2.31), this can be re-expressed in the following way:

$$j_{t,r} = {}^t\mathcal{B}_r^+ \mathcal{M}(z_r^+) + {}^t\mathcal{B}_r^- \mathcal{M}(z_r^-).$$

Moreover, by repeating this argument replacing the Hamiltonian by the momentum $\mathcal{P}$, expressed in terms of the charges $\mathcal{Q}_i$ as in (3.2.30), and using the expression (3.2.31) for the spatial component of the Lax connection, one finds a similar relation for the currents j_r:

$$j_r = {}^t\mathcal{B}_r^+ \mathcal{L}(z_r^+) + {}^t\mathcal{B}_r^- \mathcal{L}(z_r^-).$$

Therefore, using light-cone coordinates, we find that the Maurer-Cartan currents

$$j_{\pm,r} = g_r^{-1}\partial_\pm g_r$$

take the following rather simple form in terms of the Lax connection:

$$j_{\pm,r} = {}^t\mathcal{B}_r^+ \mathcal{L}_\pm(z_r^+) + {}^t\mathcal{B}_r^- \mathcal{L}_\pm(z_r^-). \tag{3.3.1}$$

Lagrangian Lax connection from interpolation. Our goal in this section is to find a Lagrangian expression of the Lax connection, *i.e.* an expression of $\mathcal{L}_\pm(z)$ in terms of the Maurer-Cartan currents $j_{\pm,r}$. We note that equation (3.3.1) relates these currents to the evaluations $\mathcal{L}_\pm(z_r^+)$ and $\mathcal{L}_\pm(z_r^-)$ of the Lax connection at the positions z_r^+ and z_r^-. As we shall now explain, this relation is enough to reconstruct the expression of $\mathcal{L}_\pm(z)$ in terms of $j_{\pm,r}$ for all values of the spectral parameter z. Let us define

$$\mathcal{J}_{\pm,r} = \mathcal{L}_\pm(z_r^\pm), \tag{3.3.2}$$

for $r = 1, \cdots, N$. From equation (3.2.33), one sees that $\mathcal{L}_\pm(z)$ is a rational function of z with N simple poles, situated at the zeroes of the twist function ζ_i, for $i \in \mathcal{I}_\pm$ (recall that we have supposed that the subsets $\mathcal{I}_\pm$ are both of size N). It is a standard result that such a function is completely determined by its evaluation at N pairwise distinct points. In particular, $\mathcal{L}_\pm(z)$ can be expressed in terms of its evaluations at the positions $z_r^\pm$, *i.e.* the currents $\mathcal{J}_{\pm,r}$ introduced above. More precisely, one has the following interpolation formula (see also Lemma B.2 of [59])

$$\mathcal{L}_\pm(z) = \sum_{r=1}^N \frac{\varphi_{\pm,r}(z_r^\pm)}{\varphi_{\pm,r}(z)} \mathcal{J}_{\pm,r}, \tag{3.3.3}$$

where

$$\varphi_{\pm,r}(z) = \frac{\prod_{i\in\mathcal{I}_\pm}(z-\zeta_i)}{\prod_{\substack{s=1\\ s\neq r}}^N (z-z_s^\pm)}. \tag{3.3.4}$$

We are now in a position to rewrite the Lax connection in terms of the currents $j_{\pm,r}$. Indeed, the above equation (3.3.1) can now be rewritten as a system of linear equations between the currents $j_{\pm,r}$ and $\mathcal{J}_{\pm,r}$, which (at least formally) can be inverted. More precisely, reinserting (3.3.3) in (3.3.1), we have that

$$j_{\pm,r} = \sum_{s=1}^N \mathcal{U}_{rs}^\pm \mathcal{J}_{\pm,s}, \tag{3.3.5}$$

where we have defined

$$\mathcal{U}_{rs}^\pm = \delta_{rs}\, {}^t\mathcal{B}_r^\pm + \frac{\varphi_{\pm,s}(z_s^\pm)}{\varphi_{\pm,s}(z_r^\mp)}\, {}^t\mathcal{B}_r^\mp. \tag{3.3.6}$$

In the following, we will see the operators $\mathcal{U}^{\pm}_{rs}$ as the entries of some matrix operators $\mathcal{U}_{\pm}$, so that $\mathcal{U}^{\pm}_{rs} = (\mathcal{U}_{\pm})_{rs}$. Note that $\mathcal{U}_{\pm}$ are then $N \times N$ matrices with non-commutative entries. To conclude, we rewrite the currents $j_{\pm,r}$ in terms of the $j_{\pm,r}$'s by means of the inversion

$$j_{\pm,r} = \sum_{s=1}^{N} (\mathcal{U}_{\pm}^{-1})_{rs} J_{\pm,s}, \tag{3.3.7}$$

where $(\mathcal{U}_{\pm}^{-1})_{rs}$ denote the entries of the inverse of the matrix operators $\mathcal{U}_{\pm}$. Reinserting now (3.3.7) in (3.3.3) then gives an expression of $\mathcal{L}_{\pm}(z)$ in terms of the currents $j_{\pm,r}$:

$$\mathcal{L}_{\pm}(z) = \sum_{s=1}^{N} \left(\sum_{r=1}^{N} \frac{\varphi_{\pm,r}(z_r^{\pm})}{\varphi_{\pm,r}(z)} (\mathcal{U}_{\pm}^{-1})_{rs} \right) j_{\pm,s}. \tag{3.3.8}$$

Note that this is a formal relation, as it involves the inverse of the matrix operators $\mathcal{U}_{\pm}$. Performing explicitly this inversion is in general not straightforward because of the non-commutativity of the entries of $\mathcal{U}_{\pm}$ (for example, one cannot use the general expression for the inverse of a matrix in terms of its comatrix). We will explain in section 3.3.2 how this is done explicitly in the case of two copies.

Different interpolations and factorisations of the twist function. We conclude this section by making an important remark about equations (3.3.2) and (3.3.3). In these equations, we decided to express the component $\mathcal{L}_{+}(z)$, resp. $\mathcal{L}_{-}(z)$, of the Lax connection in terms of its evaluations at the positions z_r^{+}, resp. z_r^{-}. Let us stress here that this choice is arbitrary, as one could have chosen for example to interpolate $\mathcal{L}_{+}(z)$ and $\mathcal{L}_{-}(z)$ through their evaluations at the positions z_r^{-} and z_r^{+}, respectively[9]. More generally, one could have considered the evaluations[10]

$$\widetilde{J}_{\pm,r} = \mathcal{L}_{\pm}(z_r^{\pm\sigma_r}),$$

where the σ_r's take values in the set $\{+1, -1\}$ for every r. The interpolation equation (3.3.3) would then become

$$\mathcal{L}_{\pm}(z) = \sum_{r=1}^{N} \frac{\widetilde{\varphi}_{\pm,r}(z_r^{\pm\sigma_r})}{\widetilde{\varphi}_{\pm,r}(z)} \widetilde{J}_{\pm,r}, \qquad \text{where now} \qquad \widetilde{\varphi}_{\pm,r}(z) = \frac{\prod_{i\in\mathcal{I}_{\pm}} (z-\zeta_i)}{\prod_{\substack{s=1\\ s\neq r}}^{N} (z - z_s^{\pm\sigma_s})}. \tag{3.3.9}$$

Following the method developed in the previous paragraph, one would then express the currents $\widetilde{J}_{\pm,r}$ in terms of the Maurer-Cartan currents $j_{\pm,r}$ by a relation similar to equation (3.3.7), with the operators $\mathcal{U}_{\pm}$ replaced by some different operators $\widetilde{\mathcal{U}}_{\pm}$. Re-inserting this expression in equation (3.3.9) would then give $\mathcal{L}_{\pm}(z)$ in terms of $j_{\pm,r}$, similarly to equation (3.3.8). This expression can be shown to coincide with equation (3.3.8) as one should expect, considering that they correspond to two ways of expressing the same object $\mathcal{L}_{\pm}(z)$. Similarly, all the methods and computations developed in the rest of this section can be applied starting from an arbitrary choice of interpolation, *i.e.* from an arbitrary choice of σ_r's: the end results (in particular the expression of the action of the model in terms of the Maurer-Cartan currents that will be obtained in the next section) can then be shown to be independent of this choice. For this reason, and to avoid unnecessary cumbersome notations, we will use in the rest of this chapter a particular choice of σ_r's, namely $\sigma_r = +1$ for every r, corresponding to the choice

[9]Note in particular that the indices $\pm$ of $\mathcal{L}_{\pm}(z)$ are conceptually totally unrelated to the labels $\pm$ of the positions $z_r^{\pm}$. Indeed, the former are space-time indices corresponding to the light-cone directions in $\mathbb{R}\times\mathbb{D}$ while the latter are abstract labels distinguishing the two sites $(r,+)$ and $(r,-)$.

[10]Note that here the superscripts are abstract indices as before and not exponents.

made originally in the previous paragraph.

To conclude this paragraph, let us discuss a reinterpretation of the functions $\widetilde{\varphi}_{\pm,r}(z)$ appearing in the interpolation formula (3.3.9) and of the freedom encoded in the choice of σ_r's in terms of the twist function (3.2.25) of the model. Let us rewrite the latter in the following factorised form:

$$\varphi(z) = -\ell^\infty \widetilde{\varphi}_+(z)\widetilde{\varphi}_-(z), \qquad \text{where} \qquad \widetilde{\varphi}_\pm(z) = \frac{\prod_{i\in\mathcal{I}_\pm}(z-\zeta_i)}{\prod_{s=1}^N (z - z_s^{\pm\sigma_s})}. \tag{3.3.10}$$

The functions $\widetilde{\varphi}_{\pm,r}(z)$ can then be re-expressed as $\widetilde{\varphi}_{\pm,r}(z) = (z - z_r^{\pm\sigma_r})\widetilde{\varphi}_\pm(z)$. Moreover, we observe that the freedom in the choice of the σ_r's gets now reinterpreted as the existence of different ways of factorising the twist function. Indeed, redistributing the pairs of factors $(z - z_r^+)$ and $(z - z_r^-)$ associated to the paired sites $(r, \pm)$ into the definition (3.3.10) of $\widetilde{\varphi}_\pm(z)$ amounts to changing the values of the σ_r's[11]. In the rest of this chapter and in agreement with the notations of the previous paragraph, we will denote by $\varphi_\pm(z)$ the functions $\widetilde{\varphi}_\pm(z)$ corresponding to the choice $\sigma_r = +1$ for every $r \in \{1, \cdots, N\}$.

3.3.2 Inverse Legendre transform and action of the models

Lagrangian expression of the momentum. We are now in a position to perform the first step towards writing down the inverse Legendre transform of the model, *i.e.* re-expressing the fields X_r, which encode the momentum fields of the theory, in terms of Lagrangian fields. Let us first note from equation (3.2.23) that the fields Y_r and X_r are related through the current W_r. As explained in section 3.2.1, this current W_r is expressed in terms of the field g_r and its spatial derivative (and not the momentum fields) and has thus a direct Lagrangian expression. Thus, finding the Lagrangian expression of X_r is equivalent to finding the Lagrangian expression of Y_r. As we shall now see, the latter is easier to find, using the Lagrangian expression of the Lax connection obtained in the previous paragraph. From the definition (2.2.31) of the Lax matrix $\mathcal{L}(z)$, one can prove that (see also [59, equation (2.22)])

$$\mathcal{L}(z_r^\pm) = \frac{\mathcal{J}_{r,\pm}}{\ell_{r,\pm}} = \frac{\mathcal{B}_r^\pm}{\ell_{r,\pm}} Y_r + \frac{\mathcal{C}_r^\pm}{\ell_{r,\pm}} j_r,$$

where to obtain the second equality we have used the definition (3.2.22) of the currents $\mathcal{J}_{r,\pm}$. Then, using the identities (3.2.12) satisfied by the operators $\mathcal{B}_r^\pm$ and $\mathcal{C}_r^\pm$, we find the following expression for Y_r:

$$Y_r = {}^t\mathcal{C}_r^+ \mathcal{L}(z_r^+) + {}^t\mathcal{C}_r^- \mathcal{L}(z_r^-). \tag{3.3.11}$$

Using the light-cone components of the Lax connection, this can be rewritten as

$$Y_r = \frac{{}^t\mathcal{C}_r^+ \mathcal{L}_+(z_r^+) + {}^t\mathcal{C}_r^- \mathcal{L}_+(z_r^-)}{2} - \frac{{}^t\mathcal{C}_r^+ \mathcal{L}_-(z_r^+) + {}^t\mathcal{C}_r^- \mathcal{L}_-(z_r^-)}{2}.$$

From the Lagrangian expression (3.3.3) of $\mathcal{L}_\pm(z)$, one then finds that

$$Y_r = \sum_{s=1}^N \left[\mathcal{V}_{rs}^+ J_{+,s} + \mathcal{V}_{rs}^- J_{-,s} \right], \tag{3.3.12}$$

[11]Note that contrarily to the poles $z_r^\pm$, the zeroes ζ_i of the twist function cannot be redistributed differently between the functions $\widetilde{\varphi}_+(z)$ and $\widetilde{\varphi}_-(z)$, as they are naturally associated with one or the other depending on whether the index i belongs to the set $\mathcal{I}_+$ or $\mathcal{I}_-$.

where we have defined

$$\mathcal{V}^{\pm}_{rs} = \pm\frac{1}{2}\left(\delta_{rs}\,{}^t\mathcal{C}^{\pm}_r + \frac{\varphi_{\pm,s}(z^{\pm}_s)}{\varphi_{\pm,s}(z^{\mp}_r)}\,{}^t\mathcal{C}^{\mp}_r\right). \tag{3.3.13}$$

Similarly to the operators $\mathcal{U}^{\pm}_{rs}$ in the previous section, we will see the operators $\mathcal{V}^{\pm}_{rs}$ as the entries of some $N \times N$ matrix of operators $\mathcal{V}_{\pm}$, so that $\mathcal{V}^{\pm}_{rs} = (\mathcal{V}_{\pm})_{rs}$.

Action in terms of $j_{\pm,r}$'s and $j_{\pm,r}$'s. The action of the models is obtained as the following inverse Legendre transform of the Hamiltonian (see for instance [59]):

$$S[g_1, \cdots, g_N] = \sum_{r=1}^{N} \iint \mathrm{d}t\,\mathrm{d}x\;\kappa\left(X_r, j_{t,r}\right) - \int \mathrm{d}t\;\mathcal{H},$$

where both X_r and $\mathcal{H}$ should be replaced by their expressions in terms of Lagrangian fields. Recalling the definitions (3.2.23) and (3.2.4), one can rewrite the action in terms of the fields Y_r making the Wess-Zumino terms of g_r appear:

$$S[g_1, \cdots, g_N] = \sum_{r=1}^{N} \iint \mathrm{d}t\,\mathrm{d}x\;\kappa\left(Y_r, j_{t,r}\right) - \int \mathrm{d}t\;\mathcal{H} + \sum_{r=1}^{N} \hbar_r\, I_{\mathrm{WZ}}\big[g_r\big].$$

From here, reinserting the expression (3.3.12) of Y_r in terms of the currents $j_{\pm,r}$, we find:

$$\begin{aligned} S[g_1, \cdots, g_N] = \frac{1}{2}\sum_{r,s}^{N} \iint \mathrm{d}t\,\mathrm{d}x\;\left[\kappa\left(\mathcal{V}^{+}_{sr}J_{+,r}, J_{-,s}\right) + \kappa\left(J_{+,r}, \mathcal{V}^{-}_{rs}J_{-,s}\right)\right] - \int \mathrm{d}t\;\mathcal{H} + \sum_{r=1}^{N} \hbar_r\, I_{\mathrm{WZ}}\big[g_r\big] \\ + \frac{1}{2}\sum_{r,s}^{N} \iint \mathrm{d}t\,\mathrm{d}x\;\left[\kappa\left(\mathcal{V}^{+}_{sr}J_{+,r}, J_{+,s}\right) + \kappa\left(J_{-,r}, \mathcal{V}^{-}_{rs}J_{-,s}\right)\right]. \end{aligned} \tag{3.3.14}$$

We note that the terms in the second line are not Lorentz invariant. However, one shows that these are cancelled by the term containing the Hamiltonian (for brevity, we give the proof of this result in appendix 3.B), so that we eventually get

$$S[g_1, \cdots, g_N] = \frac{1}{2}\sum_{r,s}^{N} \iint \mathrm{d}t\,\mathrm{d}x\;\left[\kappa\left(\mathcal{V}^{+}_{sr}J_{+,r}, J_{-,s}\right) + \kappa\left(J_{+,r}, \mathcal{V}^{-}_{rs}J_{-,s}\right)\right] + \sum_{r=1}^{N} \hbar_r\, I_{\mathrm{WZ}}\big[g_r\big]. \tag{3.3.15}$$

Action in terms of Maurer-Cartan currents. To conclude this section, we proceed to compute the expression of the action in terms of the $j_{\pm,r}$'s only. This is done through the formal inversion relation (3.3.7). As a final result we obtain

$$S[g_1, \cdots, g_N] = \iint \mathrm{d}t\,\mathrm{d}x\;\sum_{r,s=1}^{N} \kappa\left(J_{+,r}, \mathcal{O}_{rs}J_{-,s}\right) + \sum_{r=1}^{N} \hbar_r\, I_{\mathrm{WZ}}\big[g_r\big], \tag{3.3.16}$$

where we have defined $\mathcal{O}_{rs}$ as the entries of the following matrix operator:

$$\mathcal{O} = \frac{1}{2}\left({}^t\mathcal{U}_{+}^{-1}\,{}^t\mathcal{V}_{+} + \mathcal{V}_{-}\mathcal{U}_{-}^{-1}\right). \tag{3.3.17}$$

Finally, using the identities (3.2.9), one proves that the second term in this definition is equal to the first one, so that we get:

$$\mathcal{O} = {}^t\mathcal{U}_{+}^{-1}\,{}^t\mathcal{V}_{+} = \mathcal{V}_{-}\mathcal{U}_{-}^{-1}. \tag{3.3.18}$$

Model with two copies. In this paragraph, we give an explicit expression for the inversion of the operator matrices $\mathcal{U}_\pm$ and consequently for the coupling operator $\mathcal{O}$ in the case of a model with two copies only, i.e. with $N=2$. In order to do so, one has to make a further assumption about the operators $\mathcal{B}_r^\pm$ appearing in the ansatz (3.2.22) made for the Kac-Moody currents $\mathcal{J}_{r,\pm}$. More precisely, we will suppose that they satisfy the following commutation relation

$$\left[\mathcal{B}_r^+, \mathcal{B}_r^-\right] = 0, \qquad \forall\, r \in \{1,2\}. \tag{3.3.19}$$

Let us note that, crucially, this additional condition is satisfied by the Yang-Baxter realisation (with or without Wess-Zumino term) and the λ-realisation, as can be checked easily from equations (3.2.17), (3.2.18) and (3.2.19)[12].

As we have noted in section 3.3.1, the fact that it is not straightforward to invert the operator matrices $\mathcal{U}_\pm$ is due to the non-commutativity of their entries. However, using the additional assumption (3.3.19) made on the operators $\mathcal{B}_r^\pm$, one shows that:

$$\left[\mathcal{U}_{rs}^\pm, \mathcal{U}_{rt}^\pm\right] = 0, \qquad \forall\, r,s,t \in \{1,2\}. \tag{3.3.20}$$

Thus, even if the entries of $\mathcal{U}_\pm$ are not all commutative, this shows that the ones on a same line commute with one another. This fact will allow us to find an explicit expression of the inverse of $\mathcal{U}_\pm$.

Let us introduce the operators

$$\Delta_1^\pm = (\mathcal{U}_{11}^\pm\, \mathcal{U}_{22}^\pm - \mathcal{U}_{12}^\pm\, \mathcal{U}_{21}^\pm)^{-1} \qquad \text{and} \qquad \Delta_2^\pm = (\mathcal{U}_{22}^\pm\, \mathcal{U}_{11}^\pm - \mathcal{U}_{21}^\pm\, \mathcal{U}_{12}^\pm)^{-1}. \tag{3.3.21}$$

If the entries $\mathcal{U}_{rs}^\pm$ of $\mathcal{U}_\pm$ were commutative, the objects $\Delta_1^\pm$ and $\Delta_2^\pm$ would be equal and would correspond to the inverse of the determinant of the 2×2 matrix $\mathcal{U}_\pm$. In the present case, these operators $\Delta_r^\pm$ are the inverse of non-commutative versions of the determinant. In terms of these, the inverse of the operator $\mathcal{U}_\pm$ is then given by

$$\mathcal{U}_\pm^{-1} = \begin{pmatrix} \mathcal{U}_{22}^\pm\, \Delta_1^\pm & -\mathcal{U}_{12}^\pm\, \Delta_2^\pm \\ -\mathcal{U}_{21}^\pm\, \Delta_1^\pm & \mathcal{U}_{11}^\pm\, \Delta_2^\pm \end{pmatrix}. \tag{3.3.22}$$

Indeed, one checks explicitly that

$$\begin{pmatrix} \mathcal{U}_{11}^\pm & \mathcal{U}_{12}^\pm \\ \mathcal{U}_{21}^\pm & \mathcal{U}_{22}^\pm \end{pmatrix} \begin{pmatrix} \mathcal{U}_{22}^\pm\, \Delta_1^\pm & -\mathcal{U}_{12}^\pm\, \Delta_2^\pm \\ -\mathcal{U}_{21}^\pm\, \Delta_1^\pm & \mathcal{U}_{11}^\pm\, \Delta_2^\pm \end{pmatrix} = \begin{pmatrix} (\mathcal{U}_{11}^\pm\, \mathcal{U}_{22}^\pm - \mathcal{U}_{12}^\pm\, \mathcal{U}_{21}^\pm)\Delta_1^\pm & (\mathcal{U}_{12}^\pm\, \mathcal{U}_{11}^\pm - \mathcal{U}_{11}^\pm\, \mathcal{U}_{12}^\pm)\Delta_2^\pm \\ (\mathcal{U}_{21}^\pm\, \mathcal{U}_{22}^\pm - \mathcal{U}_{22}^\pm\, \mathcal{U}_{21}^\pm)\Delta_1^\pm & (\mathcal{U}_{22}^\pm\, \mathcal{U}_{11}^\pm - \mathcal{U}_{21}^\pm\, \mathcal{U}_{12}^\pm)\Delta_2^\pm \end{pmatrix}.$$

The property (3.3.20) then ensures that the off-diagonal terms vanish, while the definition (3.3.21) of the operators $\Delta_r^\pm$ is such that the diagonal terms are the identity operator, thus proving that the matrix (3.3.22) is the inverse of $\mathcal{U}_\pm$[13]. The expression (3.3.22) is a non-commutative generalisation of the standard comatrix formula for the inverse of a 2×2 matrix, where in particular one takes into account the non-commutativity of the entries by considering different "inverse determinants" $\Delta_r^\pm$ in the different columns.

To give a more compact expression of the entries of $\mathcal{U}_\pm^{-1}$, let us introduce the notation $\bar{r}$, defined for every $r \in \{1,2\}$ by $\bar{r} \in \{1,2\} \setminus r$ (*i.e.* $\bar{1}=2$ and $\bar{2}=1$). Then, one has

$$(\mathcal{U}_\pm^{-1})_{rs} = (-1)^{r+s}\, \mathcal{U}_{\bar{s}\bar{r}}^\pm\, \Delta_s^\pm.$$

[12] It is not obvious whether this condition is an accidental property of these particular realisations or if it can be derived more generally as a consequence of the fact that $\mathcal{J}_{r,\pm}$ are Kac-moody currents, as was for example the case for the identities (3.2.9) (see appendix 3.A).

[13] More precisely, this proves that it is the right inverse of $\mathcal{U}_\pm$. However, recalling that the entries of $\mathcal{U}_\pm$ are operators on $\mathfrak{g}^{\mathbb{C}}$, one can see $\mathcal{U}_\pm$ as a $2\dim\mathfrak{g} \times 2\dim\mathfrak{g}$ matrix, for which the left and right inverses coincide.

Reinserting the above results into the expression (3.3.18) of the operator $\mathcal{O}$, one can compute its entries $\mathcal{O}_{rs}$, which appear in the action (3.3.16) of the model, yielding

$$\mathcal{O}_{rs} = {}^t\Delta_r^+ \left({}^t\mathcal{U}^+_{\bar{r}\bar{r}}\, {}^t\mathcal{V}^+_{sr} - {}^t\mathcal{U}^+_{\bar{r}r}\, {}^t\mathcal{V}^+_{s\bar{r}} \right) = \left(\mathcal{V}^-_{rs}\, \mathcal{U}^-_{\bar{s}\bar{s}} - \mathcal{V}^-_{r\bar{s}}\, \mathcal{U}^-_{\bar{s}s} \right) \Delta^-_s. \tag{3.3.23}$$

3.3.3 Parameters of the models

In section 3.2.7, we have discussed what are the defining parameters of the models, from their construction as realisations of affine Gaudin models. Let us briefly give some additional comments on the subject in the light of the Lagrangian formulation of the models.

Functions $\boldsymbol{\varphi_\pm(z)}$. Recall the functions $\varphi_\pm(z)$ and $\varphi_{\pm,r}(z) = (z - z_r^\pm)\varphi_\pm(z)$ introduced in section 3.3.1. It is clear from the results of this section that these functions play an important role in describing the Lagrangian formulation of the models. For example, they are used to obtain the Lagrangian expression (3.3.3) of the Lax connection. Similarly, they enter the definitions (3.3.6) and (3.3.13) of the operators $\mathcal{U}_\pm$ and $\mathcal{V}_\pm$, which are then used to express the operator $\mathcal{O}$ appearing in the action (3.3.16) of the model. Note that the definition of the operators $\mathcal{U}_\pm$ and $\mathcal{V}_\pm$ also involves the operators $\mathcal{B}_r^\pm$ and $\mathcal{C}_r^\pm$, which characterise the choice of Kac-Moody realisations of the model. In particular, these realisations depend on the levels $\ell_{r,\pm}$. For completeness, let us thus note that the latter can also be expressed quite easily using the functions $\varphi_\pm(z)$ and $\varphi_{\pm,r}(z)$:

$$\ell_{r,\pm} = -\ell^\infty \varphi_{\pm,r}(z_r^\pm)\varphi_\mp(z_r^\pm) = \mp \frac{\ell^\infty}{z_r^+ - z_r^-} \varphi_{+,r}(z_r^\pm)\varphi_{-,r}(z_r^\pm). \tag{3.3.24}$$

Finally, let us note that these levels also determine the coefficients $\Bbbk_r$ of the Wess-Zumino terms in the action (3.3.16) of the model. Thus, the datum of the functions $\varphi_\pm(z)$ is enough to describe completely the model in its Lagrangian formulation.

Parameters $\boldsymbol{(z_r^\pm, \nu_r^\pm, \ell^\infty)}$. Recall that in section 3.2.7, we discussed two possible sets of parameters for the model: the "Gaudin parameters" $(z_r^\pm, \ell_{r,\pm}, \ell^\infty)$ and the parameters $(z_r^\pm, \zeta_i, \ell^\infty)$, where the datum of the levels $\ell_{r,\pm}$ has been replaced by the datum of the zeroes ζ_i of the twist function. In particular, recall that the second parametrisation is in general more convenient as the zeroes play an important role in the description of the model and as they cannot be expressed explicitly in terms of the levels, whereas the levels can be expressed rationally in terms of the zeroes. Recall also that choosing the parametrisation using the zeroes is however less convenient to describe models with λ-realisations and/or Yang-Baxter realisations without Wess-Zumino term. Indeed, these realisations require that the levels $\ell_{r,\pm}$ satisfy the additional constraint (3.2.37), which translates into a complicated algebraic condition on the parameters $(z_r^\pm, \zeta_i)$.

These observations motivate the introduction of a third possible set of parameters $(z_r^\pm, \nu_r^\pm, \ell^\infty)$, which is in some sense intermediate between the two sets described above and which circumvents the various issues related to solving algebraic equations. In this parametrisation, the datum of the levels $\ell_{r,\pm}$ or of the zeroes ζ_i is replaced by the datum of the coefficients

$$\nu_r^\pm = \varphi_{\pm,r}(z_r^\pm).$$

Note that these coefficients characterise the partial fraction decomposition of the functions $\varphi_\pm(z)$:

$$\varphi_\pm(z) = 1 + \sum_{r=1}^{N} \frac{\nu_r^\pm}{z - z_r^\pm}.$$

In particular, the levels $\ell_{r,\pm}$ can then be expressed in terms of these parameters as

$$\ell_{r,\pm} = -\ell^{\infty}\nu_r^{\pm}\left(1+\sum_{s=1}^{N}\frac{\nu_s^{\mp}}{z_r^{\pm}-z_s^{\mp}}\right).$$

Thus, the condition (3.2.37), which the levels $\ell_{r,\pm}$ should satisfy in order to attach a λ-realisation or a Yang-Baxter realisation without Wess-Zumino term to the sites $(r,\pm)$, becomes

$$\nu_r^{+}\left(1+\sum_{s=1}^{N}\frac{\nu_s^{-}}{z_r^{+}-z_s^{-}}\right)+\nu_r^{-}\left(1+\sum_{s=1}^{N}\frac{\nu_s^{+}}{z_r^{-}-z_s^{+}}\right)=0. \tag{3.3.25}$$

If one considers a model with N_2 λ-realisations, one has to impose N_2 relations as the one above, which form a linear system on the corresponding set of coefficients ν_r^+ (or equivalently on the corresponding ν_r^-). This is the advantage of this parametrisation, as one then has to solve linear constraints on the parameters instead of algebraic ones when using the zeroes. In particular, the solutions of these constraints are rational expressions of the remaining free parameters (however potentially quite complicated). This will be useful later in section 3.4.3 when we will study the model with N coupled λ-models.

Coefficients $\boldsymbol{\rho_{rs}^{\pm}}$. Let us end this section by introducing some coefficients which will be useful to study the undeformed limit of the models in section 3.3.4 and specific examples of models in section 3.4. We define

$$\rho_{rr}^{\pm} = \mp\frac{\ell^{\infty}}{2}\varphi_{\pm,r}(z_r^{\pm})\frac{\varphi_{\mp,r}(z_r^{\mp})-\varphi_{\mp,r}(z_r^{\pm})}{z_r^{\mp}-z_r^{\pm}}, \tag{3.3.26a}$$

$$\rho_{rs}^{\pm} = \mp\frac{\ell^{\infty}}{2}\frac{\varphi_{\mp,r}(z_r^{\mp})\varphi_{\pm,s}(z_s^{\pm})}{z_r^{\mp}-z_s^{\pm}}, \quad \text{for } r\neq s. \tag{3.3.26b}$$

Using the expression (3.3.24) of the levels $\ell_{r,\pm}$, one shows that these coefficients can be rewritten as

$$\rho_{rs}^{\pm} = \pm\frac{1}{2}\left(\ell_{r,\pm}\,\delta_{rs}+\ell_{r,\mp}\frac{\varphi_{\pm,s}(z_s^{\pm})}{\varphi_{\pm,s}(z_r^{\mp})}\right). \tag{3.3.27}$$

Using this expression, the operators $\mathcal{U}_{rs}^{\pm}$ and $\mathcal{V}_{rs}^{\pm}$ introduced in (3.3.6) and (3.3.13) can be re-expressed as

$$\mathcal{U}_{rs}^{\pm} = \left({}^t\mathcal{B}_r^{\pm}-\frac{\ell_{r,\pm}}{\ell_{r,\mp}}\,{}^t\mathcal{B}_r^{\mp}\right)\delta_{rs}\pm\frac{2\rho_{rs}^{\pm}}{\ell_{r,\mp}}\,{}^t\mathcal{B}_r^{\mp}, \qquad \mathcal{V}_{rs}^{\pm} = \pm\frac{1}{2}\left({}^t\mathcal{C}_r^{\pm}-\frac{\ell_{r,\pm}}{\ell_{r,\mp}}\,{}^t\mathcal{C}_r^{\mp}\right)\delta_{rs}+\frac{\rho_{rs}^{\pm}}{\ell_{r,\mp}}\,{}^t\mathcal{C}_r^{\mp}. \tag{3.3.28}$$

3.3.4 Undeformed limit

As explained in section 3.2.8, one can see the model constructed from N_1 inhomogeneous Yang-Baxter realisations and N_2 λ-realisations as a deformation of a simpler model, coupling together N_1 copies of the PCM with Wess-Zumino term and N_2 copies of the non-abelian T-dual of the PCM. This was understood by means of the so-called undeformed limit, in which the positions z_r^+ and z_r^- collide for every pair of sites $(r,\pm)$, or equivalently by letting the parameters η_r go to 0, while keeping $\ell_{0,r}$ and $\ell_{1,r}$ finite (see section 3.2.8 for details). The goal of this section will be to complete this discussion by studying this limit in the Lagrangian formulation of the model, focusing mostly on Yang-Baxter realisations (as explained in section 3.2.8, the undeformed limit of λ-realisations requires a more subtle treatment which we will not detail here for conciseness).

This will allow us to compare the methods and results presented in the previous sections for deformed models to the ones presented in section 2.2.6 and the articles [58, 59] for undeformed ones. In particular, we will see how the action (2.2.52) discussed in section 2.2.6 is recovered in this case.

Interpolation formula. Let us focus for the moment on a pair of sites $(r,\pm)$, which we suppose to be associated with a Yang-Baxter realisation with Wess-Zumino term. The corresponding operators $\mathcal{B}_r^{\pm}$ are then given by

$$\mathcal{B}_r^{\pm}=\frac{1}{2}\left(\mathrm{Id}\mp\frac{1}{c_r}R^{(r)}\mp\delta_r\Pi^{(r)}\right),$$

where δ_r is defined in equation (3.2.41), $R^{(r)}=\mathrm{Ad}_{g_r}^{-1}\circ R_r\circ\mathrm{Ad}_{g_r}$ and $\Pi^{(r)}=\mathrm{Id}-R^{(r)\,2}/c_r^2$. Recall that in section 3.3.1, we found the Lagrangian expression of the Lax connection by interpolation methods, using the fact that one can express the Maurer-Cartan currents $j_{\pm,r}$ in terms of the evaluation of the Lax connection at the positions $z_r^{\pm}$ by equation (3.3.1). In the present case, this equation can be rewritten as

$$j_{\pm,r}=\frac{\mathcal{L}_\pm(z_r^+)+\mathcal{L}_\pm(z_r^-)}{2}+\left(R^{(r)}-c_r\delta_r\Pi^{(r)}\right)\frac{\mathcal{L}_\pm(z_r^+)-\mathcal{L}_\pm(z_r^-)}{2c_r}. \tag{3.3.29}$$

As recalled above, the undeformed limit corresponds to making the positions $z_r^{\pm}$ collide to the same point z_r. It is then clear that in the undeformed limit, the above formula simply becomes

$$j_{\pm,r}=\mathcal{L}_\pm(z_r). \tag{3.3.30}$$

This is precisely the interpolation formula obtained in [59, equation (3.33)] for the model coupling N PCM with Wess-Zumino terms. In this reference, this formula plays a key role in obtaining the Lagrangian expression of the Lax connection of this model. The method developed in section 3.3.1 of this chapter is thus a generalisation of the one of [59] to include deformed realisations.

Recall from equation (3.3.2) that in the deformed model, the currents $\mathscr{j}_{\pm,r}$ are defined as the evaluations $\mathcal{L}_\pm(z_r^{\pm})$. It is then clear from the above equation that in the undeformed limit, these currents $\mathscr{j}_{\pm,r}$ coincide with the Maurer-Cartan currents $j_{\pm,r}$. The expression (3.3.3) of the Lax connection is thus a natural deformation of the one (3.34) of [59] for the undeformed model. Moreover, this implies that the operator $\mathcal{U}_\pm$, which relates the currents $j_{\pm,r}$ and $\mathscr{j}_{\pm,r}$ (see equation (3.3.5)), becomes the identity in the undeformed limit, or equivalently, in components:

$$\mathcal{U}_{rs}^{\pm}\xrightarrow{\eta_1,\cdots,\eta_N\to 0}\delta_{rs}\mathrm{Id}. \tag{3.3.31}$$

For completeness, let us comment briefly on the homogeneous Yang-Baxter limit considered at the end of section 3.2.8 (note that we considered the homogeneous limit only for realisations without Wess-Zumino term, in which case $\hbar_r=\delta_r=0$). Recall that this limit corresponds to taking the coefficient c_r to 0. Recall also that the positions $z_r^{\pm}$ are given by $z_r\pm c_r\eta_r$. Thus, in the limit $c_r\to 0$, the equation (3.3.29) becomes

$$j_{\pm,r}=\mathcal{L}_\pm(z_r)+\eta_r R^{(r)}\mathcal{L}'_\pm(z_r), \tag{3.3.32}$$

where $\mathcal{L}'_\pm(z)$ denotes the derivative of $\mathcal{L}_\pm(z)$ with respect to the spectral parameter z. This is the equivalent of the equation (D.7) of [59], which was obtained when studying a model with $N-1$ PCM realisations and one homogeneous Yang-Baxter realisation. It is interesting to compare the equations (3.3.29), (3.3.30) and (3.3.32): the undeformed interpolation formula (3.3.30) is corrected by a derivative term $\mathcal{L}'_\pm(z_r)$ for an homogeneous Yang-Baxter deformation and by a finite difference term $\big(\mathcal{L}_\pm(z_r+c_r\eta_r)-\mathcal{L}_\pm(z_r-c_r\eta_r)\big)/2c_r$ for an inhomogeneous Yang-Baxter realisation.

Lagrangian expression of Y_r. Recall from section 3.3.2 that after the derivation of the Lagrangian expression of the Lax connection, the next step for performing the inverse Legendre transform of the model is to find the Lagrangian expression of the field Y_r, which encodes the momentum fields of the model. This was done using equation (3.3.11), which expresses Y_r in terms of the Lax connection, through the operators $\mathcal{C}_r^{\pm}$. For a Yang-Baxter realisation, it can be re-written, after a few manipulations, as

$$Y_r = \left(\ell_{1,r}\mathrm{Id} - \eta_r \cancel{k}_r R^{(r)} + c_r\eta_r\cancel{k}_r\delta_r\Pi^{(r)}\right)\frac{\mathcal{L}(z_r^+) - \mathcal{L}(z_r^-)}{2c_r\eta_r} - \cancel{k}_r\frac{\mathcal{L}(z_r^+) + \mathcal{L}(z_r^-)}{2}.$$

The undeformed limit corresponds to taking η_r to 0 while keeping $\ell_{1,r}$ and $\cancel{k}_r = -\ell_{0,r}/2$ finite. Recalling that $z_r^{\pm} = z_r \pm c_r\eta_r$, the above equation then becomes in this limit

$$Y_r = \ell_{1,r}\mathcal{L}'(z_r) - \cancel{k}_r\mathcal{L}(z_r).$$

This then coincides with the equation (3.36) of [59].

Recall from section 3.3.2 that equation (3.3.11) allows us to rewrite Y_r in terms of the currents $j_{\pm,r}$ and the operators $\mathcal{V}_{rs}^{\pm}$, in equation (3.3.12). In the undeformed limit, the currents $j_{\pm,r}$ are identified with the Maurer-Cartan currents $j_{\pm,r}$. Moreover, one can study the behaviour of the undeformed limit of the operators $\mathcal{V}_{rs}^{\pm}$ using their expression (3.3.28). In particular, the coefficients $\rho_{rs}^{\pm}$ in this expression, defined by equation (3.3.26), can be shown to converge in the undeformed limit to:

$$\rho_{rs}^{+} \xrightarrow{\eta_1,\cdots,\eta_N\to 0} \rho_{sr} - \frac{\cancel{k}_r}{2}\delta_{rs} \qquad \text{and} \qquad \rho_{rs}^{-} \xrightarrow{\eta_1,\cdots,\eta_N\to 0} \rho_{rs} + \frac{\cancel{k}_r}{2}\delta_{rs}, \tag{3.3.33}$$

with the coefficients ρ_{rs} as defined in [59, equation (3.40)]. Note that in this limit, the expression of the coefficient $\cancel{k}_r$ also coincides with its expression in [59, equation (3.38)]. Using the above limit of the coefficients $\rho_{rs}^{\pm}$, as well as the expression (3.2.35) of the levels $\ell_{r,\pm}$ in terms of the coefficients $\ell_{0,r} = -2\cancel{k}_r$ and $\ell_{1,r}$ which stay finite in the undeformed limit, one can compute the limit of the operators $\mathcal{V}_{rs}^{\pm}$ starting from their expression (3.3.28):

$$\mathcal{V}_{rs}^{+} \xrightarrow{\eta_1,\cdots,\eta_N\to 0} \rho_{sr}\,\mathrm{Id} \qquad \text{and} \qquad \mathcal{V}_{rs}^{-} \xrightarrow{\eta_1,\cdots,\eta_N\to 0} \rho_{rs}\,\mathrm{Id}. \tag{3.3.34}$$

In particular, equation (3.3.12) agrees with [59, equation (3.39)] in the undeformed limit:

$$Y_r \xrightarrow{\eta_1,\cdots,\eta_N\to 0} \sum_{s=1}^{N}\left(\rho_{sr}J_{+,s} + \rho_{rs}J_{-,s}\right).$$

Action. Finally, we are now in a position to calculate the undeformed limit of the action of the model with N copies of the Yang-Baxter realisation. By reinserting the limits (3.3.31) and (3.3.34) in the expression (3.3.17) for the operator $\mathcal{O}$, we find:

$$\mathcal{O}_{rs} \xrightarrow{\eta_1,\cdots,\eta_N\to 0} \rho_{rs}\,\mathrm{Id}.$$

Comparing to equation (3.49) of [59], one sees that the action (3.3.16) then reduces to the one of N coupled copies of the PCM with Wess-Zumino term:

$$S\big[g_1,\cdots,g_N\big] = \iint \mathrm{d}t\,\mathrm{d}x\,\sum_{r,s=1}^{N}\rho_{rs}\,\kappa\left(J_{+,r},J_{-,s}\right) + \sum_{r=1}^{N}\cancel{k}_r I_{\mathrm{WZ}}\big[g_r\big]. \tag{3.3.35}$$

We recognise in the equation above the action (2.2.52) discussed in section 2.2.6 for the case $\cancel{k}_r = 0$.

Undeformed and q-deformed symmetries. The undeformed model (3.3.35) possesses N global symmetries acting by left translation on the fields g_r:

$$g_1(t,x) \longmapsto h_1 g_1(t,x), \quad \cdots \quad , \quad g_N(t,x) \longmapsto h_N g_N(t,x), \tag{3.3.36}$$

where $h_1, \cdots, h_N$ are constant elements of G. Indeed, these transformations leave the Maurer-Cartan currents $j_{\pm,r} = g_r^{-1}\partial_\pm g_r$ and the Wess-Zumino terms $I_{\text{WZ}}\big[g_r\big]$ invariant.

These global symmetries are broken by the introduction of deformations. Indeed, let us consider the model with N copies of the Yang-Baxter model studied in this section. The entries of the operators $\mathcal{U}_\pm$ and $\mathcal{V}_\pm$ are expressed in terms of the operators $R^{(r)} = \text{Ad}_{g_r}^{-1} \circ R_r \circ \text{Ad}_{g_r}$ and $\Pi^{(r)} = \text{Ad}_{g_r}^{-1} \circ \Pi_r \circ \text{Ad}_{g_r}$. Because of their dependence on the fields g_r, these operators are not invariant under the left translations (3.3.36), making the operator $\mathcal{O}$ appearing in the deformed action (3.3.16) not invariant. Thus, the transformations (3.3.36) are not symmetries of the action (3.3.16).

It is a well-known result [47] (see also [87–89]) that in the Yang-Baxter model (with one copy and without Wess-Zumino term) the global symmetry of the undeformed model is in fact replaced by a q-deformed Poisson-Lie symmetry. The latter is a symmetry with respect to a Lie group which is equipped with a Poisson structure in such a way that the multiplication map in this group is a Poisson mapping. In this setting, the moment map that generates the symmetry in the canonical case is replaced by a so-called non-abelian moment map. The fact that the Yang-Baxter model has this symmetry descends from this non-abelian moment map commuting with the Hamiltonian of the model. Moreover, this symmetry is q-deformed in the sense that the Poisson brackets satisfied by the conserved charges contained in this non-abelian moment map are deformed to the ones of a Poisson-Hopf algebra $\mathcal{U}_q(\mathfrak{g})$ [88, 90], where q is a function of the deformation parameter. This algebra can be seen as a semi-classical limit of the quantum group $U_{\hat{q}}(\mathfrak{g})$ with $\hat{q} = q^\hbar$. Based on this result, it was explained in [59] that this is in general the case for every affine Gaudin model with a Yang-Baxter realisation (without Wess-Zumino term). In particular, the model coupling N copies of the Yang-Baxter models without Wess-Zumino term then possesses N q-deformed symmetries, which replace the translation symmetries (3.3.36). Their action on the fields of the model can be computed using the results of [91]: in particular, let us note that this action is non-local.

As the bilinear form κ is invariant under conjugacy, the undeformed model also possesses a global symmetry acting by simultaneous right translation on all the fields g_r:

$$g_1(t,x) \longmapsto g_1(t,x)\, h, \quad \cdots \quad , \quad g_1(t,x) \longmapsto g_N(t,x)\, h, \tag{3.3.37}$$

with h a constant element of G. As explained in [59], it corresponds to the diagonal symmetry of the underlying affine Gaudin model. As such, it is not broken by applying Yang-Baxter deformations to the various copies of the model. Indeed, one checks that under the transformation (3.3.37), the operators $\mathcal{O}_{rs}$ entering the action of the model with N Yang-Baxter realisations become $\text{Ad}_h^{-1} \circ \mathcal{O}_{rs} \circ \text{Ad}_h$. Since the Maurer-Cartan currents $j_{\pm,r}$ become $\text{Ad}_h^{-1} j_{\pm,r}$ and the Wess-Zumino terms $I_{\text{WZ}}\big[g_r\big]$ are invariant under this transformation, it is thus a symmetry of the deformed action (3.3.16). Note that a similar result holds for models involving λ-realisations: in this case, the corresponding fields g_r should not transform by right multiplication but by conjugacy $g_r \mapsto h^{-1} g_r h$, while the fields corresponding to Yang-Baxter realisations still transform by right multiplication by h.

3.4 Yang-Baxter and λ-deformed coupled models

The action (3.3.16) presented in the previous section was obtained using the general ansatz introduced in section 3.2.2 for the form of the Kac-Moody realisations defining the model. In this section, we specialise these results to the model constructed from N_1 copies of the Yang-Baxter realisation and N_2 copies of the λ-realisation. As we shall see, the particular form of these realisations will allow us to rewrite the action of this model in a simpler form. In particular, we will show that the integrable sigma model introduced in [63] corresponds to a particular limit of the model constructed from N copies of the λ-realisation. We will then focus on models with two copies and will rewrite their action in a more explicit form, using the expressions (3.3.22) and (3.3.23) of the inverse of $\mathcal{U}_\pm$ and of the operators $\mathcal{O}_{rs}$ obtained in this case.

3.4.1 Deformed model with N_1 Yang-Baxter realisations and N_2 λ-realisations

Let us consider a model made up of N_1 copies of the Yang-Baxter realisation with Wess-Zumino term and N_2 copies of the λ-realisation. Let us now associate the former to the first N_1 pairs of sites $(r,\pm)$ and the latter to the last N_2 pairs. Then, from (3.2.18) and (3.2.19), one obtains, for the operators $\mathcal{B}_r^\pm$ and $\mathcal{C}_r^\pm$, the following expression

$$\mathcal{B}_r^\pm = \frac{1}{2}\text{Id} \mp \frac{1}{2c_r}R^{(r)} \mp \frac{\delta_r}{2}\Pi^{(r)}, \quad \mathcal{C}_r^\pm = \left(\ell_{r,\pm} + \frac{\Bbbk_r}{2}\right)\text{Id} \mp \frac{\Bbbk_r}{2c_r}R^{(r)} \mp \frac{\Bbbk_r\delta_r}{2}\Pi^{(r)}, \qquad 1 \leq r \leq N_1,$$
$$\mathcal{B}_r^+ = \text{Id}, \quad \mathcal{B}_r^- = -\text{Ad}_{g_r}, \quad \mathcal{C}_r^+ = -\Bbbk_r\,\text{Id}, \quad \mathcal{C}_r^- = -\Bbbk_r\,\text{Ad}_{g_r}, \qquad N_1 < r \leq N_2,$$

where $R^{(r)} = \text{Ad}_{g_r}^{-1} \circ R_r \circ \text{Ad}_{g_r}$ and $\Pi^{(r)} = \text{Ad}_{g_r}^{-1} \circ \Pi_r \circ \text{Ad}_{g_r}$. We observe that the relations $\mathcal{C}_r^\pm = \ell_{r,\pm} + \Bbbk_r\mathcal{B}_r^\pm$ and $\mathcal{C}_r^\pm = \mp\Bbbk_r\mathcal{B}_r^\pm$ respectively hold in the first and in the second case. Thus, from (3.3.6) and (3.3.13) and after a few manipulations, we obtain for the entries of the operator $\mathcal{V}_\pm$:

$$\mathcal{V}_{rs}^\pm = \rho_{rs}^\pm\text{Id} \pm \frac{\Bbbk_r}{2}\mathcal{U}_{rs}^\pm, \qquad 1 \leq r \leq N_1, \tag{3.4.1a}$$
$$\mathcal{V}_{rs}^\pm = -\Bbbk_r\delta_{rs}{}^t\mathcal{B}_r^\pm + \frac{\Bbbk_r}{2}\mathcal{U}_{rs}^\pm, \qquad N_1 < r \leq N_2. \tag{3.4.1b}$$

where the coefficients $\rho_{rs}^\pm$ have been defined in (3.3.26).

From the expressions (3.3.17) and (3.3.18) of the operator $\mathcal{O}$ found in the previous section, we are now in a position to write the action of the model. We choose to express the entries $\mathcal{O}_{rs}$ of this operator as in (3.3.17) for $1 \leq r \leq N_1$ and as in the second equality in (3.3.18) for $N_1 < r \leq N_2$[14]. Reinserting (3.4.1) in the form of the action (3.3.16), we obtain

$$\begin{aligned} S[g_1,\cdots,g_N] = \frac{1}{2}\iint \text{d}t\,\text{d}x \, \sum_{r=1}^{N_1}\sum_{s,t=1}^{N} \kappa\left(J_{+,r}, \left(\alpha_{st}^{+}{}^t(\mathcal{U}_+^{-1})_{tr} + \alpha_{rt}^-(\mathcal{U}_-^{-1})_{ts}\right)J_{-,s}\right) + \sum_{r=1}^{N_1}\Bbbk_r\,I_{\text{WZ}}[g_r] \\ + \iint \text{d}t\,\text{d}x \sum_{r=N_1+1}^{N}\sum_{s=1}^{N}\Bbbk_r\,\kappa\left(\text{Ad}_{g_r}J_{+,r}, (\mathcal{U}_-^{-1})_{rs}J_{-,s}\right) + \sum_{r=N_1+1}^{N} S_{\text{WZW},\Bbbk_r}[g_r], \end{aligned} \tag{3.4.2}$$

with

$$\begin{aligned} \alpha_{rs}^\pm &= \rho_{rs}^\pm, \qquad && 1 \leq r \leq N_1, \\ \alpha_{rs}^+ &= -\Bbbk_r\delta_{rs}, && N_1 < r \leq N \end{aligned}$$

[14]This choice makes the discussion of the cases $(N_1 = N, N_2 = 0)$ and $(N_1 = 0, N_2 = N)$ in the next sections simpler. However, we note that due to the relation (3.3.18), different choices are possible in general (see for example section 3.4.4).

and where $S_{\text{WZW},\,\Bbbk_r}[g_r]$ denotes the Wess-Zumino-Witten action of g_r with level $\Bbbk_r$:

$$S_{\text{WZW},\,\Bbbk_r}[g_r] = \frac{\Bbbk_r}{2} \iint \mathrm{d}t\, \mathrm{d}x\; \kappa\big(j_{+,r}, j_{-,r}\big) + \Bbbk_r\, I_{\text{WZ}}\big[g_r\big].$$

3.4.2 Model with N Yang-Baxter realisations

Let us now briefly discuss the model with copies of the Yang-Baxter realisation only. In this case, the action (3.4.2) gets simplified to

$$S\big[\{g_r\}\big] = \frac{1}{2} \iint \mathrm{d}t\, \mathrm{d}x \sum_{r,s=1}^{N} \kappa\Big(g_r^{-1}\partial_+ g_r,\; \big({}^t\mathcal{U}_+^{-1}\, {}^t\varrho_+ + \varrho_-\, \mathcal{U}_-^{-1}\big)_{rs}\; g_s^{-1}\partial_- g_s\Big) + \sum_{r=1}^{N} \Bbbk_r\, I_{\text{WZ}}\big[g_r\big], \tag{3.4.3}$$

where $\varrho_\pm$ are operators on $\mathfrak{g}^N$ which can be seen as $N\times N$ matrices with entries $(\varrho_\pm)_{rs} = \rho^\pm_{rs}\,\text{Id}$.

Let us describe more explicitly the operators $\mathcal{U}_\pm$ appearing in the action (3.4.3). From the expressions of the operators $\mathcal{B}^\pm_r$ and $\mathcal{C}^\pm_r$ for a Yang-Baxter realisation, one finds that

$$\mathcal{U}_\pm = \text{Id} \pm \widetilde{\mathcal{R}}^\pm \vartheta^\pm, \tag{3.4.4}$$

where we defined

$$\widetilde{\mathcal{R}}^\pm_{rs} = \big(R^{(r)} \mp c_r \text{Id} - c_r\delta_r \Pi^{(r)}\big)\, \delta_{rs},$$

with $R^{(r)} = \text{Ad}_{g_r}^{-1} \circ R_r \circ \text{Ad}_{g_r}$ and $\Pi^{(r)} = \text{Id} - R^{(r)\,2}/c_r^2$, and

$$\vartheta^\pm_{rs} = \theta^\pm_{rs}\, \text{Id}, \qquad\qquad \theta^\pm_{rs} = \frac{\mp\rho^\pm_{rs} - \Bbbk_r\delta_{rs}}{c_r \ell_{r,\mp}}. \tag{3.4.5}$$

Let us end this section by presenting an alternative form of the action of the model. Let us introduce the operator c, with entries $c_{rs} = c_r\delta_{rs}\,\text{Id}$. Then, one can further rewrite the operator $\mathcal{U}_\pm$ as

$$\mathcal{U}_\pm = \big(\text{Id} \pm \widetilde{\mathcal{R}}\, \widetilde{\vartheta}^\pm\big)\big(\text{Id} - c\vartheta^\pm\big),$$

where

$$\widetilde{\mathcal{R}}_{rs} = \big(R^{(r)} - c_r\delta_r \Pi^{(r)}\big)\, \delta_{rs} \qquad \text{and} \qquad \widetilde{\vartheta}^\pm = \frac{\vartheta^\pm}{\text{Id} - c\vartheta^\pm}.$$

Finally, introducing $\widetilde{\varrho}_\pm = \varrho_\pm(\text{Id} - c\vartheta^\pm)^{-1}$, one can rewrite the action of the model in the form

$$S[\{g_r\}] = \frac{1}{2} \iint \mathrm{d}t\, \mathrm{d}x \sum_{r,s=1}^{N} \kappa\left(g_r^{-1}\partial_+ g_r, \left(\frac{1}{\text{Id} + {}^t\widetilde{\vartheta}^+\, {}^t\widetilde{\mathcal{R}}}\, {}^t\widetilde{\varrho}_+ + \widetilde{\varrho}_-\, \frac{1}{\text{Id} - \widetilde{\mathcal{R}}\, \widetilde{\vartheta}^-}\right)_{rs} g_s^{-1}\partial_- g_s\right)$$
$$+ \sum_{r=1}^{N} \Bbbk_r\, I_{\text{WZ}}\big[g_r\big].$$

This way of writing the action of the model is quite similar to the way the action of the Yang-Baxter model with one copy is expressed and thus seems rather natural. Let us note however that it has some downsides compared to the expression (3.4.3). Indeed, the entries $\widetilde{\rho}^\pm_{rs}$ and $\widetilde{\theta}^\pm_{rs}$ of the operators $\widetilde{\varrho}_\pm$ and $\widetilde{\vartheta}^\pm$ appearing in the expression above are not straightforwardly expressed in terms of the parameters of the models (contrarily to the coefficients $\rho^\pm_{rs}$ and $\theta^\pm_{rs}$ which were used in the previous formulation) as their definition involves the inversion of the operator $\text{Id} - c\vartheta^\pm$.

From the expression of the action above, one can simply check that its undeformed limit yields the action of N coupled PCMs with Wess-Zumino terms presented in [59] (see also section 2.2.6). Indeed, in this limit, the parameters $\theta^\pm_{rs}$ and thus also the operators $\widetilde{\theta}^\pm$, go to zero. In particular, the coefficients $\widetilde{\rho}^\pm_{rs}$ and $\rho^\pm_{rs}$ have the same limit. From equation (3.3.33), we then see that in this limit, $\widetilde{\rho}^+_{sr} + \widetilde{\rho}^-_{rs} \to 2\,\rho_{rs}$, with ρ_{rs} as defined in [59].

3.4.3 Model with N λ-realisations

Action. Let us now discuss the case where we take λ-realisations only. For this model, the action reads

$$S[\{g_r\}] = \sum_{r=1}^{N} S_{\text{WZW},\,\Bbbk_r}[g_r] + \iint \mathrm{d}t\,\mathrm{d}x \sum_{r,s=1}^{N} \Bbbk_r\,\kappa\left(\partial_+ g_r g_r^{-1}, \left(\mathcal{U}_-^{-1}\right)_{rs} g_s^{-1}\partial_- g_s\right). \tag{3.4.6}$$

From the expression of the operators $\mathcal{B}_r^{\pm}$ of the λ-realisation, one can rewrite the operator $\mathcal{U}_-$ as

$$\mathcal{U}_- = \mathcal{M} - \mathcal{D}^{-1}, \qquad \text{with} \qquad \mathcal{M}_{rs} = \mu_{rs}\,\mathrm{Id} \qquad \text{and} \qquad \mathcal{D}_{rs} = \mathrm{Ad}_{g_r}\delta_{rs}, \tag{3.4.7}$$

where the coefficients μ_{rs} are defined as

$$\mu_{rs} = \frac{\varphi_{-,s}(z_s^-)}{\varphi_{-,s}(z_r^+)}. \tag{3.4.8}$$

The action of the model then takes the simple form

$$S[\{g_r\}] = \sum_{r=1}^{N} S_{\text{WZW},\,\Bbbk_r}[g_r] + \iint \mathrm{d}t\,\mathrm{d}x \sum_{r,s=1}^{N} \Bbbk_r\,\kappa\left(\partial_+ g_r g_r^{-1}, \left(\frac{1}{\mathcal{M}-\mathcal{D}^{-1}}\right)_{rs} g_s^{-1}\partial_- g_s\right). \tag{3.4.9}$$

Parameters. Let us discuss what are the defining parameters of the model. We will use the parameterisation $(z_r^{\pm}, \nu_r^{\pm}, \ell^{\infty})$ introduced in section 3.3.3. As explained in this section, these parameters are convenient to take into account the fact that the levels $\ell_{r,\pm}$ of the models should satisfy the constraints $\ell_{r,+} + \ell_{r,-} = 0$, that one has to impose to consider λ-realisations. Indeed, these constraints translate into the conditions (3.3.25) on the parameters $z_r^{\pm}$ and $\nu_r^{\pm}$. One can solve this condition by expressing the parameters ν_r^+ in terms of $z_r^{\pm}$ and ν_r^-:

$$\nu_r^+ = \sum_{s=1}^{N} (\beta^{-1})_{rs}\nu_s^-, \qquad \text{where} \qquad \beta = \left(\left(1 + \sum_{t=1}^{N}\frac{\nu_t^-}{z_r^+ - z_t^-}\right)\delta_{rs} + \frac{\nu_r^-}{z_r^- - z_s^+}\right)_{r,s=1,\cdots,n}. \tag{3.4.10}$$

The remaining $3N+1$ parameters $(z_r^{\pm}, \nu_r^-, \ell^{\infty})$ are unconstrained: taking into account the translation and dilation redundancy among these parameters (see section 3.2.7), the model is thus defined by $3N-1$ free parameters (for concreteness, one can for example fix this redundancy by fixing the values of ℓ^{∞} and of one of the positions $z_r^{\pm}$). The coefficients μ_{rs} defined in equation (3.4.8) can be expressed in terms of this parametrisation as

$$\mu_{rs} = \frac{\nu_s^-}{z_r^+ - z_s^-}\left(1 + \sum_{t=1}^{N}\frac{\nu_t^-}{z_r^+ - z_t^-}\right)^{-1}. \tag{3.4.11}$$

Similarly, the coefficient $\Bbbk_r$ appearing in the action (3.4.9) is given by

$$\Bbbk_r = \frac{1}{2}\ell^{\infty}\nu_r^+\left(1 + \sum_{s=1}^{N}\frac{\nu_s^-}{z_r^+ - z_s^-}\right),$$

where ν_r^+ is replaced by its expression (3.4.10).

Comparison with [63]. Actions of the form (3.4.9) have been considered in [63] (and in [60–62] for the case $N=2$, see section 3.4.4). More precisely, the action (3.4.9) is identical to the action (2.13) of [63], with the matrix λ^{-1} in this reference identified in the present language with the matrix whose components are $\lambda^{-1}_{rs}=\sqrt{\hbar_r/\hbar_s}\,\mu_{rs}$.

It was shown in [63] that the model defined by taking all entries of λ^{-1} to be zero except for $\lambda^{-1}_{11},\cdots,\lambda^{-1}_{(N-1)1}$ and $\lambda^{-1}_{N2},\cdots,\lambda^{-1}_{NN}$ is integrable. Let us now explain how this model can be obtained as a limit of the one constructed above by coupling together N λ-realisations. We introduce the following reparametrisation of the positions $z_r^\pm$ of the model:

$$z_r^+=y_r\ \text{ for }\ r\in\{1,\cdots,N-1\},\qquad z_N^+=\frac{1}{\gamma},\qquad z_1^-=0,\qquad z_r^-=\widehat{y}_r+\frac{1}{\gamma}\ \text{ for }\ r\in\{2,\cdots,N\},\tag{3.4.12}$$

in terms of new parameters $y_1,\cdots,y_{N-1},\widehat{y}_2,\cdots,\widehat{y}_N$ and γ. We used here the translation redundancy on the parameters $z_r^\pm$ to fix the value of z_1^- to 0. Recall that one can also use the dilation redundancy to fix the value of ℓ^∞: for future convenience, we choose here to fix it to

$$\ell^\infty=2\left(1+\sum_{r=1}^N\frac{\nu_r^-}{z_N^+-z_r^-}\right)^{-1}=\frac{2}{\varphi_-\big(z_N^+\big)}.$$

Using this parametrisation, the model is then described by the $3N-1$ free parameters $y_1,\cdots,y_{N-1}$, $\widehat{y}_2,\cdots,\widehat{y}_N$, $\nu_1^-,\cdots,\nu_N^-$ and γ. The limit we shall consider in this paragraph is $\gamma\to 0$, while keeping the remaining parameters fixed.

Using the expression (3.4.11) of the coefficients μ_{rs}, one checks that in the limit $\gamma\to 0$, these coefficients all vanish except $\mu_{11},\cdots,\mu_{(N-1)1}$ and $\mu_{N2},\cdots,\mu_{NN}$. The matrix λ^{-1}, which has components $\lambda^{-1}_{rs}=\sqrt{\hbar_r/\hbar_s}\,\mu_{rs}$, then takes the same form as in the integrable truncation considered in [63]. As one considered the limit $\gamma\to 0$, the model is now described by the $3N-2$ parameters $y_1,\cdots,y_{N-1}$, $\widehat{y}_2,\cdots,\widehat{y}_N$ and $\nu_1^-,\cdots,\nu_N^-$. This coincides with the number of free parameters considered for the integrable model of [63]. More precisely, the parameters used in this reference are the Wess-Zumino levels $\hbar_1,\cdots,\hbar_N$ and the coefficients $\mu_{11},\cdots,\mu_{(N-1)1}$ and $\mu_{N2},\cdots,\mu_{NN}$ (or equivalently the corresponding coefficients λ^{-1}_{rs}). Using the expressions of these coefficients obtained in the previous paragraph and considering the limit $\gamma\to 0$, one can relate explicitly the parametrisation used here with the parametrisation used in [63]. More precisely, one finds after several computational steps:

$$y_r=\left(\frac{1}{\mu_{r1}}-1\right)\frac{\hbar_1-a}{b},\qquad \widehat{y}_s=\frac{\hbar_s}{\mu_{Ns}}-\hbar_N,\qquad \nu_1^-=\frac{\hbar_1-a}{b}\qquad\text{and}\qquad \nu_s^-=\frac{\hbar_s-\hbar_N\mu_{Ns}}{b},$$

for $r\in\{1,\cdots,N-1\}$ and $s\in\{2,\cdots,N\}$, where we define $a=\sum_{r=1}^{N-1}\hbar_r\mu_{r1}$ and $b=\sum_{s=2}^N\mu_{Ns}-1$.

Let us comment on the limit $\gamma\to 0$ considered above. This limit consists in singling out two sets of positions $\mathcal{Z}_1=\{z_1^-,z_1^+,\cdots,z_{N-1}^+\}$ and $\mathcal{Z}_2=\{z_2^-,\cdots,z_N^-,z_N^+\}$ and sending the distance between these two sets to infinity. It is thus quite similar to the decoupling procedure considered in [59, section 2.3.3][15]. According to this procedure, the sites $(1,-),(1,+),\cdots,(N-1,+)$

[15]The main difference with this procedure comes from the realisations attached to the sites. Indeed, in [59], the two sets $\mathcal{Z}_1$ and $\mathcal{Z}_2$ are associated with independent realisations, in the sense that the phase space of the model takes the form $\mathcal{P}_1\times\mathcal{P}_2$ and the positions in $\mathcal{Z}_1$ are associated with Kac-Moody (or Takiff) currents in the first factor $\mathcal{P}_1$ and the positions in $\mathcal{Z}_2$ are associated with currents in $\mathcal{P}_2$. In the decoupling limit, where the sites $\mathcal{Z}_1$ cease to interact with the sites $\mathcal{Z}_2$, one then obtains two independent models on $\mathcal{P}_1$ and $\mathcal{P}_2$, respectively. In the present case, the currents associated with the sets $\mathcal{Z}_1$ and $\mathcal{Z}_2$ do not belong to independent parts of the phase space.

corresponding to the positions $\mathcal{Z}_1$ thus cease to interact with the sites $(2,-),\cdots,(N,-),(N,+)$ corresponding to the positions $\mathcal{Z}_2$ in the limit $\gamma \to 0$. This explains the structure of the model considered in [63], where the fields $g^{(2)},\cdots,g^{(N-1)}$ have no interactions one with another. The theory before taking the limit $\gamma \to 0$ then defines a non-trivial integrable generalisation of this model: indeed, although it corresponds to adding only one parameter, this introduces non-trivial interactions between all the different fields g_r, as the coefficients μ_{rs} then become generically all non-zero.

Following the decoupling procedure of [59], one describes the integrability of the model in the limit $\gamma \to 0$ using two independent Lax connections, which are obtained as two different limits of the initial Lax connection $\mathcal{L}_\pm(z)$. More precisely, let us consider:

$$\mathcal{L}^{(1)}_\pm(z) = \lim_{\gamma\to 0} \mathcal{L}_\pm(z) \qquad \text{and} \qquad \mathcal{L}^{(2)}_\pm(z) = \lim_{\gamma\to 0} \mathcal{L}_\pm\left(z+\frac{1}{\gamma}\right). \tag{3.4.13}$$

It is clear that, before taking the limit $\gamma \to 0$, both $\mathcal{L}_\pm(z)$ and $\mathcal{L}_\pm(z+\gamma^{-1})$ satisfy a zero curvature equation (as $\mathcal{L}_\pm(z)$ does) and thus still do after taking the limit. The reason behind the necessity of considering these two Lax connections is that, loosely speaking, the Lax connection $\mathcal{L}_\pm(z)$ loses the information about the positions $\mathcal{Z}_2$ in the limit $\gamma \to 0$: the Lax connection $\mathcal{L}^{(1)}_\pm(z)$ then only "corresponds to" the positions $\mathcal{Z}_1$ (see [59] for a more precise treatment). Considering the shift of the spectral parameter by γ^{-1}, as done in the definition of $\mathcal{L}^{(2)}_\pm(z)$, exchanges the roles of the sets $\mathcal{Z}_1$ and $\mathcal{Z}_2$, so that the second Lax connection $\mathcal{L}^{(2)}_\pm(z)$ contains the information about the positions $\mathcal{Z}_2$. This is coherent with [63], where the integrable truncation was described using two Lax connections.

The Hamiltonian analysis of the corresponding Lax matrices was performed recently in [92], where it was shown that their Poisson brackets are described by twist functions. In the language of affine Gaudin models used above, these twist functions are obtained from the twist function $\varphi(z)$ of the original model by a limit similar to the one of equation (3.4.13) (see [59]):

$$\varphi^{(1)}(z) = \lim_{\gamma\to 0} \varphi(z) \qquad \text{and} \qquad \varphi^{(2)}(z) = \lim_{\gamma\to 0} \varphi\left(z+\frac{1}{\gamma}\right).$$

One then finds that the twist function $\varphi^{(1)}$ has poles at the points $\{y_1,\cdots,y_{N-1},0\}$ while the twist function $\varphi^{(2)}(z)$ has poles at the points $\{0,\widehat{y}_2,\cdots,\widehat{y}_N\}$. Up to dilation and translation, these poles coincide with the ones obtained in [92].

3.4.4 Deformed models with two copies

Recall from section 3.3.2 that in the case of a model with two copies, one can rewrite the operators $\mathcal{U}_\pm^{-1}$ and $\mathcal{O}_{rs}$ more explicitly as in equations (3.3.22) and (3.3.23). Using these results, we study in this section the models with two Yang-Baxter realisations and two λ-realisations.

Model with two Yang-Baxter realisations. Let us consider first the model with two Yang-Baxter realisations. In this case, we will use the first expression of the operators $\mathcal{O}_{rs}$ in equation (3.3.23). The entries of the operator $\mathcal{U}_+$ can be read from (3.4.4) while the entries of $\mathcal{V}_+$ are related to the ones of $\mathcal{U}_+$ by equation (3.4.1a). Using the notation $\bar{r}$ introduced in section 3.3.2, one then obtains the following expression for the operators $\mathcal{O}_{rs}$:

$$\mathcal{O}_{rs} = \frac{1}{1+\theta^+_{rr}\widehat{R}^{(r)}+\theta^+_{\bar{r}\bar{r}}\widehat{R}^{(\bar{r})}+\det(\theta^+)\widehat{R}^{(\bar{r})}\widehat{R}^{(r)}}\left(\rho^+_{sr}+\left(\rho^+_{sr}\theta^+_{\bar{r}\bar{r}}-\rho^+_{s\bar{r}}\theta^+_{\bar{r}r}\right)\widehat{R}^{(\bar{r})}\right)+\delta_{rs}\frac{\hslash_r}{2},$$

where $\widehat{R}^{(r)} = c_r\mathrm{Id}+R^{(r)}+c_r\delta_r\Pi^{(r)}$, $\det(\theta^+)=\theta^+_{11}\theta^+_{22}-\theta^+_{12}\theta^+_{21}$ and θ^+_{rs} is given by equation (3.4.5).

Model with two $\boldsymbol{\lambda}$-realisations. Let us now consider the model with two λ-realisations. Its action is given by equation (3.4.6) with $N = 2$. Reinserting the explicit form (3.4.7) of the operator $\mathcal{U}_-$ and calculating its inverse through (3.3.22), we find that in this case the operator $\mathcal{U}_-^{-1}$ appearing in the action is explicitly given by

$$(\mathcal{U}_-^{-1})_{rs} = (-1)^{r+s} \left(\mu_{\bar{s}\bar{r}} - \delta_{rs}\mathrm{Ad}_{g_{\bar{s}}}^{-1}\right) \frac{1}{\det(\mu) - \mu_{\bar{s}\bar{s}}\mathrm{Ad}_{g_s}^{-1} - \mu_{ss}\mathrm{Ad}_{g_{\bar{s}}}^{-1} + \mathrm{Ad}_{g_s}^{-1}\mathrm{Ad}_{g_{\bar{s}}}^{-1}},$$

with $\det(\mu) = \mu_{11}\mu_{22} - \mu_{12}\mu_{21}$ and μ_{rs} given by equation (3.4.8).

Let us end this section by comparing this result with the ones of [60–62]. Indeed, the integrable sigma models introduced in these references can be obtained from the model above by taking limits similar to the one considered in section 3.4.3 (which allowed us to compare the model with N copies of the λ-realisation with the integrable model of [63]).

Let us first consider the following reparametrisation of the positions $z_r^\pm$: $z_1^+ = y$, $z_2^+ = \gamma^{-1}$, $z_1^- = \widehat{y} + \gamma^{-1}$ and $z_2^- = 0$, similar to the parametrisation (3.4.12) used in the model with N copies. We then take the limit $\gamma \to 0$. One checks that in this limit, $\mu_{11} = \mu_{22} = 0$. The model is then identical to the model (2.12) of [61] (see also [60] for the case with equal Wess-Zumino levels $\Bbbk_1 = \Bbbk_2$), where the remaining coefficients μ_{12} and μ_{21} are identified with $\mu_{12} = \lambda_0^{-1}\lambda_2^{-1}$ and $\mu_{21} = \lambda_0\lambda_1^{-1}$, in terms of the parameters λ_i of [61].

Let us now consider another reparametrisation $z_1^+ = 0$, $z_2^+ = y_2$, $z_1^- = y_1$ and $z_2^- = \gamma^{-1}$ and then take the limit $\gamma \to 0$. In this case, μ_{12} and μ_{22} vanish. The model is then identical to the model (3.1) of [62], where the remaining coefficients μ_{11} and μ_{21} are identified with $\mu_{11} = \lambda_0^{-1}\lambda_4^{-1}$ and $\mu_{21} = \lambda_0\lambda_1^{-1}$, in terms of the parameters λ_i of [62].

3.5 Relation with 4d semi-holomorphic Chern-Simons theory

In this section, we explain how the models considered in this chapter can be obtained using the approach proposed recently by Costello and Yamazaki to generate integrable 2d field theories from 4d semi-holomorphic Chern-Simons theory [70] (see [64–69,71,93] for additional references on this variant of Chern-Simons theory and its relation to integrable systems). Note that, in the terminology of [70], we restrict our attention here to 4d Chern-Simons theory with disorder defects. It was shown in [70] that the PCM with Wess-Zumino term and the integrable sigma model coupling N of its copies can be obtained from this approach. It was subsequently shown in [71] that the integrable 2d field theories obtained from 4d Chern-Simons theory with disorder defects are realisations of AGM. Moreover, it was explained in [93] how the Yang-Baxter model and the λ-model can also be derived following this approach. It is thus natural to search for a generalisation of these results for the AGM coupling together N_1 copies of the Yang-Baxter model and N_2 copies of the λ-model, which is the integrable field theory constructed in the present chapter.

3.5.1 4d semi-holomorphic Chern-Simons theory and integrable field theories

In this section, we will briefly sketch the method developed in [70] to generate integrable 2d field theories from 4d semi-holomorphic Chern-Simons theory. We will not explain this method in details here and mainly focus on the aspects that will be concretely relevant for the purpose of this chapter (we then refer to [70,93] for details). We will follow here the conventions of [93], which are in agreement with the ones used in the rest of this chapter.

4d Chern-Simons theory. The semi-holomorphic Chern-Simons theory is defined on the 4d manifold $\mathbb{R}\times\mathbb{D}\times\mathbb{P}^1$: the $\mathbb{R}\times\mathbb{D}$ part of this manifold corresponds to the 2d space-time with coordinates (t,x) of the resulting integrable field theory (here the spatial manifold $\mathbb{D}$ can be either the real line $\mathbb{R}$ or the circle S^1, as in the rest of this chapter), while the Riemann sphere $\mathbb{P}^1$ gives rise to the spectral parameter z of this integrable model. The 4d Chern-Simons theory is partly characterised by the choice of a meromorphic 1-form $\omega=\varphi(z)\mathrm{d}z$ on $\mathbb{P}^1$: as shown in [71], the corresponding rational function $\varphi(z)$ is the twist function of the resulting integrable model. The dynamical fields of the four-dimensional theory are the components A_+, A_- and $A_{\bar z}$ of a $\mathfrak{g}^{\mathbb{C}}$-valued gauge field A along the light-cone directions $x^{\pm}$ of $\mathbb{R}\times\mathbb{D}$ and the anti-holomorphic direction $\bar z$ of $\mathbb{P}^1$ (note that the component of A in the z-direction decouples from the theory and is not a physical degree of freedom). In addition to the choice of ω made above, the theory is then fully determined by specifying appropriate boundary conditions on the field A at the poles $\mathcal{Z}\subset\mathbb{P}^1$ of ω, *i.e.* at the poles of the twist function (see [70,93] and the next section for details). The action of the semi-holomorphic Chern-Simons theory is defined as [64]

$$S_{\mathrm{CS}}[A]=\frac{i}{4\pi}\int_{\mathbb{R}\times\mathbb{D}\times\mathbb{P}^1}\omega\wedge\mathrm{CS}(A), \tag{3.5.1}$$

where $\mathrm{CS}(A)$ is the standard Chern-Simons 3-form of A.

Parametrisation of the gauge field. In order to relate the 4d Chern-Simons theory to an integrable 2d model, one parametrises the gauge field components in the following form

$$A_{\bar z}=\widehat{g}\,\partial_{\bar z}\widehat{g}^{\,-1},\qquad A_{\pm}=\widehat{g}\,\partial_{\pm}\widehat{g}^{\,-1}+\widehat{g}\,\mathcal{L}_{\pm}\,\widehat{g}^{\,-1}, \tag{3.5.2}$$

where $\widehat{g}$ and $\mathcal{L}_{\pm}$ are fields respectively valued in the group $G^{\mathbb{C}}$ and the Lie algebra $\mathfrak{g}^{\mathbb{C}}$. The equation of motion obtained by varying the action (3.5.1) with respect to $A_{\bar z}$ then ensures that the fields $\mathcal{L}_{\pm}$ depend meromorphicaly on z. Moreover, the equations of motion obtained by varying $A_{\pm}$ show that they also satisfy a zero curvature equation $\partial_+\mathcal{L}_- - \partial_-\mathcal{L}_+ + \big[\mathcal{L}_+,\mathcal{L}_-\big]=0$ on $\mathbb{R}\times\mathbb{D}$. These two properties make the field $\mathcal{L}_{\pm}$ a good candidate for being the Lax connection of a 2d integrable model on $\mathbb{R}\times\mathbb{D}$.

The fields of the 2d theory. Let us now explain how this integrable 2d field theory is constructed. For z in the Riemann sphere $\mathbb{P}^1$ and a field ϕ on $\mathbb{R}\times\mathbb{D}\times\mathbb{P}^1$, we will denote by $\phi|_z$ the field on $\mathbb{R}\times\mathbb{D}$ obtained by evaluating ϕ at the point z on the Riemann sphere. It is explained in [70,93] that for a point $z\in\mathbb{P}^1\setminus\mathcal{Z}$ which is not a pole of ω, the 2d field $\widehat{g}|_z$ can be set to a constant field equal to the identity of G by an appropriate gauge transformation on the gauge field A. The fact that we restrict here to points z which are not poles of ω is due to the fact that this gauge transformation on A should preserve the boundary conditions imposed on A at these poles and mentioned above (see [70,93] for details). Thus, the 2d fields $\widehat{g}|_z$, $z\in\mathbb{P}^1\setminus\mathcal{Z}$, are not physical degrees of freedom of the model. The dynamical fields of the 2d model we aim to construct are then defined to be the remaining degrees of freedom contained in $\widehat{g}$, *i.e.* its evaluations $\{\widehat{g}|_{z_0}\}_{z_0\in\mathcal{Z}}$ at the poles of ω. Let us mention that in general, one should also consider the fields $\partial_z^p\,\widehat{g}|_{z_0}$ obtained by evaluating derivatives of $\widehat{g}$ at the points $z_0\in\mathcal{Z}$: however, as explained in [70,93], for the boundary conditions considered in these references and that we shall consider in this chapter, these degrees of freedom can also be eliminated by gauge transformations.

So far, we have considered only the degrees of freedom contained in the field $\widehat{g}$, which, as we see from equation (3.5.2), encodes the component $A_{\bar{z}}$ of the gauge field. Let us now consider the component $A_\pm$ and thus the field $\mathcal{L}_\pm$. As explained above, the equation of motion of $A_{\bar{z}}$ ensures that $\mathcal{L}_\pm$ is meromorphic in z. In fact, it also implies that $\mathcal{L}_\pm$ can have poles in $\mathbb{P}^1$ only at the zeroes of ω. This constrains quite strongly the dependence of $\mathcal{L}_\pm$ in terms of the variable $z \in \mathbb{P}^1$. Let us be more precise. As ω will ultimately be given by the twist function of the resulting 2d theory, let us denote its zeroes $\{\zeta_i\}_{i\in\{1,\cdots,M\}}$, in agreement with what was done in the rest of this chapter. These zeroes can be separated into two sets $\{\zeta_i\}_{i\in\mathcal{I}_\pm}$, labelled by subsets $\mathcal{I}_+$ and $\mathcal{I}_-$ of $\{1,\cdots,M\}$, depending on which of the component $\mathcal{L}_+$ or $\mathcal{L}_-$ has a pole at ζ_i (see [93] for details). This fixes the z-dependence of the fields $\mathcal{L}_\pm$: more precisely, they are of the form

$$\mathcal{L}_\pm(z) = \sum_{i\in\mathcal{I}_\pm} \frac{U_i}{z-\zeta_i} + U_\pm^\infty, \tag{3.5.3}$$

for some 2d $\mathfrak{g}^{\mathbb{C}}$-valued fields U_i, U_+^∞ and U_-^∞ on $\mathbb{R}\times\mathbb{D}$. In this equation, we have written the Lax connection as $\mathcal{L}_\pm(z)$ to stress its dependence on the spectral parameter z: note however that it also depends on the coordinates $(t,x)\in\mathbb{R}\times\mathbb{D}$, through the 2d fields U_i and $U_\pm^\infty$.

Recall that the gauge field A obeys some specific boundary conditions at the poles $z_0\in\mathcal{Z}$ of ω, which translate into conditions on the evaluations $\{\mathcal{L}_\pm|_{z_0}\}_{z_0\in\mathcal{Z}}$ and $\{\widehat{g}|_{z_0}\}_{z_0\in\mathcal{Z}}$. As observed in [70, 93] and as we shall see in this chapter, these boundary conditions, combined with the z-dependence (3.5.3) of $\mathcal{L}_\pm$, specify completely $\mathcal{L}_\pm$ in terms of the 2d fields $\{\widehat{g}|_{z_0}\}_{z_0\in\mathcal{Z}}$. The field $\mathcal{L}_\pm$ then does not contain any additional degrees of freedom and is interpreted as the Lax connection of the resulting 2d field theory on $\{\widehat{g}|_{z_0}\}_{z_0\in\mathcal{Z}}$ (indeed, recall also from the previous paragraph that, on-shell, it satisfies a zero curvature equation on $\mathbb{R}\times\mathbb{D}$).

Let us end this paragraph by the following remark. As argued above, the fields $\{\widehat{g}|_{z_0}\}_{z_0\in\mathcal{Z}}$ describe all the degrees of freedom of the resulting 2d model. However, in general, these degrees of freedom are not all physical: there are some residual gauge symmetries acting on these fields, which depend on the type of boundary conditions considered. Moreover, there always exists an additional redundancy on these fields which consists on multiplying all of them on the right by an arbitrary $G^{\mathbb{C}}$-valued field h on $\mathbb{R}\times\mathbb{D}$ (see [70, 93]). This redundancy can be used to fix one of the fields $\{\widehat{g}|_{z_0}\}_{z_0\in\mathcal{Z}}$ to the identity.

The effective 2d action. To complete the description of the 2d field theory obtained through this method, one has to describe its action. This is done by performing the integration over $\mathbb{P}^1$ in the 4d action (3.5.1), resulting on an effective 2d action on $\mathbb{R}\times\mathbb{D}$ depending on the 2d fields $\{\widehat{g}|_{z_0}\}_{z_0\in\mathcal{Z}}$. However, we will not need the details of this procedure in the following and thus refer to [70, 93] for details. In particular, it was shown in [93] that, for the type of boundary conditions that we shall consider in this chapter, this 2d action simply reads:

$$\begin{aligned} S\big[\{\widehat{g}|_{z_0}\}_{z_0\in\mathcal{Z}}\big] \;=\; & \frac{1}{4}\sum_{z_0\in\mathcal{Z}} \iint_{\mathbb{R}\times\mathbb{D}} \mathrm{d}t\,\mathrm{d}x\, \left(\kappa\Big(\operatorname*{res}_{z=z_0}\varphi(z)\mathcal{L}_+(z)\mathrm{d}z, j_-^{\{z_0\}}\Big) - \kappa\Big(j_+^{\{z_0\}}, \operatorname*{res}_{z=z_0}\varphi(z)\mathcal{L}_-(z)\mathrm{d}z\Big)\right) \\ & -\frac{1}{2}\sum_{z_0\in\mathcal{Z}}\left(\operatorname*{res}_{z=z_0}\varphi(z)\mathrm{d}z\right) I_{\rm WZ}\big[\widehat{g}|_{z_0}\big], \end{aligned} \tag{3.5.4}$$

where $I_{\rm WZ}\big[\widehat{g}|_{z_0}\big]$ is the Wess-Zumino term of $\widehat{g}|_{z_0}$ and $j_\pm^{\{z_0\}}$ is defined as the Maurer-Cartan current

$$j_\pm^{\{z_0\}} = \widehat{g}|_{z_0}^{-1}\partial_\pm \widehat{g}|_{z_0}.$$

3.5.2 The models

Our aim in this section is to show explicitly that a certain class of 2d integrable field theories obtained using the Chern-Simons approach described in the previous section can be identified with the affine Gaudin models coupling together an arbitrary number of copies of inhomogeneous Yang-Baxter realisations and λ-realisations, as considered in the rest of this chapter. Let us then start by defining the particular class of 4d Chern-Simons theories that we shall consider here.

As explained in [70,93] and recalled in the previous section, the 4d semi-holomorphic Chern-Simons theory is characterised by the choice of the meromorphic 1-form ω and of the boundary conditions on A at the poles $\mathcal{Z}$ of ω. Let us then define the 1-form and boundary conditions that we shall consider here.

3.5.3 1-form ω

Following [71] (see also the summary in the previous section), the meromorphic 1-form ω characterising the models obtained from the 4d Chern-Simons approach should coincide with $\varphi(z)\mathrm{d}z$, where $\varphi(z)$ is the twist function of these models when seen as realisations of AGM. As we aim to recover the models constructed in this chapter, we will then choose ω to be given by the twist function (3.2.25) considered in the previous sections, *i.e.*

$$\omega = -\ell^\infty \frac{\prod_{i=1}^{2N}(z-\zeta_i)}{\prod_{s=1}^{N}(z-z_r^+)(z-z_r^-)}\,\mathrm{d}z. \tag{3.5.5}$$

This 1-form has $2N$ simple poles at the points $z_r^\pm$ and a double pole at ∞. In the language of the previous section, one then has $\mathcal{Z} = \{z_1^+, z_1^-, \cdots, z_N^+, z_N^-, \infty\}$. Following the notations of this chapter, let us define $\ell_{r,\pm}$ as the residues of ω at the poles $z_r^\pm$, which coincide with the levels of the model when seen as a realisation of AGM.

3.5.4 Boundary conditions

Boundary condition at the double pole at infinity. Let us consider the double pole at infinity of ω. Following [70] (see also [93]), we will impose at this pole the following simple boundary condition on the Chern-Simons gauge field A:

$$A_\pm|_\infty = 0. \tag{3.5.6}$$

Boundary conditions at the simple poles $z_r^\pm$. Let us now consider a pair of simple poles $z_r^\pm$ and the corresponding evaluations $A_\pm|_{z_r^+}$ and $A_\pm|_{z_r^-}$ of the gauge field at these points. A systematic study of the consistent boundary conditions that can be imposed on these evaluations has been presented in [93] (see also [67,70]). We will consider here two of them.

Yang-Baxter boundary condition. The first one, that we shall call Yang-Baxter boundary condition, is characterised by the choice of a skew-symmetric R-matrix R_r satisfying the mCYBE (3.2.14), with $c_r = 1$ if the poles $z_r^\pm$ are real and $c_r = i$ if they are complex conjugate, and satisfying $R_r^3 = c_r^2 R_r$. Consider the residues $\ell_{r,\pm}$ of ω at $z_r^\pm$, as defined above. Let us define from them the following parameters:

$$\Bbbk_r = -\frac{\ell_{r,+}+\ell_{r,-}}{2}, \qquad \gamma_r = \frac{1}{c_r(\ell_{r,+}-\ell_{r,-})} \qquad \text{and} \qquad \delta_r = \frac{1-\sqrt{1-4c_r^2\Bbbk_r^2\gamma_r^2}}{2c_r\Bbbk_r\gamma_r}.$$

These coincide with the coefficients $\Bbbk_r$, γ_r and δ_r considered in the rest of this chapter for an inhomogeneous Yang-Baxter realisation with Wess-Zumino term (see section 3.2.3). Let us note that the coefficient δ_r satisfy $(\delta_r+1)^2\ell_{r,+}+(\delta_r-1)^2\ell_{r,-}=0$ so that $-c_r\delta_r$ coincides with the parameter θ considered in [93, section 5.6]. The Yang-Baxter boundary condition corresponds to requiring that the evaluations $A_\pm|_{z_r^+}$ and $A_\pm|_{z_r^-}$ satisfy (see [93]):

$$(R_r - c_r\delta_r\Pi_r + c_r)A_\pm|_{z_r^+} = (R_r - c_r\delta_r\Pi_r - c_r)A_\pm|_{z_r^-}, \tag{3.5.7}$$

with $\Pi_r = \mathrm{Id} - R_r^2/c_r^2$ as in section 3.2.3.

λ-boundary condition. Let us describe the second type of boundary condition at the pair of simple poles $z_r^\pm$ that we shall consider, which we call the λ-boundary condition. It can be imposed only if the poles $z_r^\pm$ and the residues $\ell_{r,\pm}$ are real and satisfy the additional condition $\ell_{r,+}+\ell_{r,-}=0$ (note that this is identical to the condition (3.2.37) that one should impose to consider a λ-realisation in an affine Gaudin model). The λ-boundary condition is then simply given by

$$A_\pm|_{z_r^+} = A_\pm|_{z_r^-}. \tag{3.5.8}$$

For a λ-boundary condition, we define the parameter $\Bbbk_r = \ell_{r,-}/2 = -\ell_{r,+}/2$, which is equal to the Wess-Zumino coefficent $\Bbbk_r$ defined for a λ-realisation (see section 3.2.3).

3.5.5 Fields of the model

Let us consider a 4d Chern-Simons theory with ω as in equation (3.5.5) and with N_1 Yang-Baxter boundary conditions and N_2 λ-boundary conditions. Let us describe what are the dynamical fields of this model. As recalled in the previous section, these fields are given by the evaluations $\widehat{g}|_{z_0}$ of the field $\widehat{g}$ at the poles $z_0 \in \mathcal{Z}$ of ω and thus by the $2N+1$ fields $\widehat{g}|_\infty$, $\widehat{g}|_{z_r^+}$ and $\widehat{g}|_{z_r^-}$.

However, as mentionned in the previous section and explained in [70, 93], we can eliminate some of these degrees of freedom. In particular, recall from the previous section that we can fix one of the fields $\widehat{g}|_{z_0}$ to the identity: here, we will choose to fix the field at infinity $\widehat{g}|_\infty$. Moreover, as explained in [93], if one considers a Yang-Baxter boundary condition or a λ-boundary condition at the pair of poles $z_r^\pm$, there exists a residual gauge symmetry on the fields $\widehat{g}|_{z_r^+}$ and $\widehat{g}|_{z_r^-}$. In the case of a Yang-Baxter boundary condition, this gauge symmetry can be fixed by imposing $\widehat{g}|_{z_r^+} = \widehat{g}|_{z_r^-}$: we then define g_r as their common value. For the λ-boundary condition, it can be fixed instead by imposing $\widehat{g}|_{z_r^-} = \mathrm{Id}$: we then define $g_r = \widehat{g}|_{z_r^+}$. To summarise, the fields of the model are the N group-valued fields $g_1, \cdots, g_N$ and we have

$$\widehat{g}|_\infty = \mathrm{Id}, \qquad \text{YB-BC:} \quad \widehat{g}|_{z_r^+} = \widehat{g}|_{z_r^-} = g_r, \qquad \lambda\text{-BC:} \quad \widehat{g}|_{z_r^+} = g_r, \;\; \widehat{g}|_{z_r^-} = \mathrm{Id}.$$

3.5.6 Identification of the two approaches

Let us consider the 2d integrable field theory defined in the previous section with N_1 Yang-Baxter boundary conditions and N_2 λ-boundary conditions. We will prove in this section that it can be identified with the AGM with N_1 Yang-Baxter realisations and N_2 λ-realisations studied in the previous sections. In order to do so, we shall show that the two approaches lead to the same Lax connection as well as the same action.

3.5.7 Identification of the Lax connections

Let us consider the Lax connection of the model coming from 4d Chern-Simons theory as given by equation (3.5.3). Let us now express it in terms of the fields g_r of the model, using the boundary conditions that are imposed on the gauge field A at the poles $z_0 \in \mathcal{Z}$ of ω.

Pole at infinity. Let us start with the pole at $z_0 = \infty$, for which the boundary condition is simply defined by equation (3.5.6). From the fact that $\widehat{g}|_\infty = \mathrm{Id}$ (see section 3.5.5) and the expression (3.5.3) of the Lax connection $\mathcal{L}_\pm(z)$, it is clear that the evaluation of the gauge field (3.5.2) at $z = \infty$ gives

$$A_\pm|_\infty = \mathcal{L}_\pm(\infty) = U_\pm^\infty.$$

Combining this with the boundary condition (3.5.6), we then get that $U_\pm^\infty = 0$.

Pair of poles with Yang-Baxter boundary condition. Let us know consider a pair of simple poles $z_r^\pm$ and let us suppose that we imposed on this pair a Yang-Baxter boundary condition (3.5.7). As explained in section 3.5.5, in this case, we have $\widehat{g}|_{z_r^+} = \widehat{g}|_{z_r^-} = g_r$. Thus, the evaluation of the gauge field (3.5.2) at z_r^ε, for $\varepsilon \in \{+1, -1\}$, is given by

$$A_\pm|_{z_r^\varepsilon} = g_r \partial_\pm g_r^{-1} + g_r \mathcal{L}_\pm(z_r^\varepsilon) g_r^{-1} = \mathrm{Ad}_{g_r}\Big(\mathcal{L}_\pm(z_r^\varepsilon) - j_{\pm,r}\Big),$$

where $j_{\pm,r} = g_r^{-1}\partial_\pm g_r$ are the Maurer-Cartan currents of the field g_r. After a few manipulations, the Yang-Baxter boundary condition (3.5.7) then becomes

$$j_{\pm,r} = \frac{1}{2}\mathrm{Ad}_{g_r}^{-1}\left(\mathrm{Id} + \frac{R_r}{c_r} - \delta_r \Pi_r\right)\mathrm{Ad}_{g_r}\,\mathcal{L}_\pm(z_r^+) + \frac{1}{2}\mathrm{Ad}_{g_r}^{-1}\left(\mathrm{Id} - \frac{R_r}{c_r} + \delta_r \Pi_r\right)\mathrm{Ad}_{g_r}\,\mathcal{L}_\pm(z_r^-).$$

Noting that R_r is skew-symmetric and Π_r is symmetric, this can be rewritten as

$$j_{\pm,r} = {}^t\mathcal{B}_r^+ \mathcal{L}_\pm(z_r^+) + {}^t\mathcal{B}_r^- \mathcal{L}_\pm(z_r^-), \qquad \text{with} \qquad \mathcal{B}_r^\pm = \frac{1}{2}\left(\mathrm{Id} \mp \frac{R^{(r)}}{c_r} \mp \delta_r \Pi^{(r)}\right),$$

where $R^{(r)} = \mathrm{Ad}_{g_r}^{-1} \circ R_r \circ \mathrm{Ad}_{g_r}$ and $\Pi^{(r)} = \mathrm{Ad}_{g_r}^{-1} \circ \Pi_r \circ \mathrm{Ad}_{g_r}$. The operators $\mathcal{B}_r^\pm$ found here coincide exactly with the operators, denoted in the same way in the rest of this chapter, coming from a Yang-Baxter realisation (see section 3.2.3). The above equation is then equivalent to the equation (3.3.1) obtained in the context of affine Gaudin models.

Pair of poles with $\boldsymbol{\lambda}$-boundary condition. Let us now consider a pair of simple poles $z_r^\pm$ with the λ-boundary condition (3.5.8). In this case, we have $\widehat{g}|_{z_r^+} = g_r$ and $\widehat{g}|_{z_r^-} = \mathrm{Id}$ (see section 3.5.5). Thus, the evaluations of the gauge field (3.5.2) at z_r^+ and z_r^- read

$$A_\pm|_{z_r^+} = \mathrm{Ad}_{g_r}\Big(\mathcal{L}_\pm(z_r^+) - j_{\pm,r}\Big) \qquad \text{and} \qquad A_\pm|_{z_r^-} = \mathcal{L}_\pm(z_r^-).$$

Similarly to what was done in the previous paragraph for the Yang-Baxter boundary condition, the λ-boundary condition (3.5.8) can then be rewritten

$$j_{\pm,r} = {}^t\mathcal{B}_r^+ \mathcal{L}_\pm(z_r^+) + {}^t\mathcal{B}_r^- \mathcal{L}_\pm(z_r^-), \qquad \text{with} \qquad \mathcal{B}_r^+ = \mathrm{Id} \quad \text{and} \quad \mathcal{B}_r^- = \mathrm{Ad}_{g_r}.$$

The operators $\mathcal{B}_r^\pm$ coincide with the ones introduced in the previous sections for a λ-realisation (see section 3.2.3). As for the Yang-Baxter boundary condition, we then recover the equation (3.3.1) obtained through the affine Gaudin model approach.

Summary. Let us summarise the results of this section. We have proved from the boundary condition at $z = \infty$ that the fields $U_\pm^\infty$ vanish. The component $\mathcal{L}_\pm(z)$ of the Lax connection (3.5.3) has then no constant term and has simple poles at the zeroes $\{\zeta_i\}_{i\in\mathcal{I}_\pm}$. Thus, it has the same meromorphic z-dependence as the Lax connection (3.2.33) of the corresponding affine Gaudin model. Moreover, we showed that the boundary conditions imposed at the pairs of simple poles $z_r^\pm$ in the Chern-Simons approach coincide exactly with the equation (3.3.1) obtained in the affine Gaudin model approach. Recall from section 3.3.1 that this equation, combined with the meromorphic z-dependence mentioned above, allowed us to express the Lax connection $\mathcal{L}_\pm(z)$ in terms of the Maurer-Cartan currents $j_{\pm,r}$ by means of interpolation techniques. This ensures that the Lax connections obtained from the two approaches can be identified.

3.5.8 Identification of the actions

Let us end this section by showing that the action obtained by the Chern-Simons approach for the model with N_1 Yang-Baxter and N_2 λ-boundary conditions coincides with the one of the AGM with N_1 Yang-Baxter and N_2 λ-realisations, computed in section 3.3. As recalled in section 3.5.1, the former is given by equation (3.5.4). Since we proved in the previous section that the Lax connection $\mathcal{L}_\pm(z)$ of the two models coincide, one can then re-insert in this equation the expression (3.3.3) of $\mathcal{L}_\pm(z)$ obtained in the AGM approach using interpolation techniques. As the twist function has simple poles at $z_r^\pm$ with residues $\ell_{r,\pm}$, we then get

$$\operatorname*{res}_{z=z_r^\pm} \varphi(z)\mathcal{L}_\pm(z)\,\mathrm{d}z = \ell_{r,\pm} j_{\pm,r} \qquad \text{and} \qquad \operatorname*{res}_{z=z_r^\mp} \varphi(z)\mathcal{L}_\pm(z)\,\mathrm{d}z = \ell_{r,\mp}\sum_{s=1}^{N} \frac{\varphi_{\pm,s}(z_s^\pm)}{\varphi_{\pm,s}(z_r^\mp)} j_{\pm,s}.$$

Moreover, recall that the field $\widehat{g}|_\infty$ has been set to the identity. The action (3.5.4) then becomes

$$S = \sum_{r=1}^{N} \iint \mathrm{d}t\,\mathrm{d}x\; \Upsilon_r \;-\sum_{r=1}^{N}\left(\frac{\ell_{r,+}}{2} I_{\mathrm{WZ}}\big[\widehat{g}|_{z_r^+}\big] + \frac{\ell_{r,-}}{2} I_{\mathrm{WZ}}\big[\widehat{g}|_{z_r^-}\big]\right), \tag{3.5.9}$$

where

$$\begin{aligned}\Upsilon_r = {}& \frac{\ell_{r,+}}{4}\kappa\big(j_{+,r}, j_-^{\{z_r^+\}}\big) - \frac{\ell_{r,-}}{4}\kappa\big(j_+^{\{z_r^-\}}, j_{-,r}\big) \\ &+ \sum_{s=1}^{N}\left(\frac{\ell_{r,-}}{4}\frac{\varphi_{+,s}(z_s^+)}{\varphi_{+,s}(z_r^-)}\kappa\big(j_{+,s}, j_-^{\{z_r^-\}}\big) - \frac{\ell_{r,+}}{4}\frac{\varphi_{-,s}(z_s^-)}{\varphi_{-,s}(z_r^+)}\kappa\big(j_+^{\{z_r^+\}}, j_{-,s}\big)\right).\end{aligned}$$

Recall from section 3.5.5 that the fields $\widehat{g}|_{z_r^\pm}$ are related to the fundamental fields g_r of the model, in a way which depends on the type of boundary conditions considered at the poles $z_r^\pm$. Equation (3.5.9) then expresses the action of the model in terms of the Maurer-Cartan currents $j_{\pm,r}$, the currents $j_{\pm,r}$ and the Wess-Zumino terms of the fields g_r. In the AGM approach, we obtained a similar expression for the action in equation (3.3.15). In the rest of this section, we shall show that these two expressions coincide, thus proving that the models obtained from the 4d Chern-Simons and the AGM approaches are identical. For that, we will prove that for every $r \in \{1, \cdots, N\}$, we have

$$\frac{\ell_{r,+}}{2} I_{\mathrm{WZ}}\big[\widehat{g}|_{z_r^+}\big] + \frac{\ell_{r,-}}{2} I_{\mathrm{WZ}}\big[\widehat{g}|_{z_r^-}\big] = -\cancel{k}_r I_{\mathrm{WZ}}\big[g_r\big] \tag{3.5.10}$$

and

$$\Upsilon_r = \frac{1}{2}\sum_{s=1}^{N}\left(\kappa\left(\mathcal{V}_{rs}^+ J_{+,s}, J_{-,r}\right) + \kappa\left(J_{+,r}, \mathcal{V}_{rs}^- J_{-,s}\right)\right). \tag{3.5.11}$$

In order to show these identities, one needs to distinguish the cases where the pair of poles $z_r^\pm$ is associated with a Yang-Baxter boundary condition or a λ-boundary condition in the Chern-Simons approach and, correspondingly, with a Yang-Baxter realisation or a λ-realisation in the AGM approach.

Yang-Baxter boundary condition. Let us start with the Yang-Baxter boundary condition. In this case, recall that $\widehat{g}|_{z_r^+} = \widehat{g}|_{z_r^-} = g_r$ and that we defined the Wess-Zumino coefficient to be $\hbar_r = -(\ell_{r,+} + \ell_{r,-})/2$. The Wess-Zumino terms in equation (3.5.9) corresponding to these poles thus satisfy equation (3.5.10). Let us now focus on the term Υ_r. Note first that $j_\pm^{\{z_r^+\}} = j_\pm^{\{z_r^-\}} = j_{\pm,r}$. One can then rewrite Υ_r as

$$\Upsilon_r = \frac{1}{2}\sum_{s=1}^{N}\left(\rho_{rs}^{+}\,\kappa\left(J_{+,s}, J_{-,r}\right) + \rho_{rs}^{-}\,\kappa\left(J_{+,r}, J_{-,s}\right)\right),$$

with $\rho_{rs}^\pm$ given by equation (3.3.27). For a Yang-Baxter realisation, the operators $\mathcal{B}_r^\pm$ and $\mathcal{C}_r^\pm$ are related by $\mathcal{C}_r^\pm = \ell_{r,\pm}\,\mathrm{Id} + \mathcal{B}_r^\pm$. This implies that the operators $\mathcal{U}_{rs}^\pm$ and $\mathcal{V}_{rs}^\pm$ defined in equations (3.3.6) and (3.3.13) satisfy

$$\mathcal{V}_{rs}^\pm = \rho_{rs}^\pm\,\mathrm{Id} \pm \frac{\hbar_r}{2}\mathcal{U}_{rs}^\pm.$$

Using the expression (3.3.7) of $j_{\pm,r}$, we then get

$$\sum_{s=1}^{N}\rho_{rs}^\pm J_{\pm,s} = \sum_{s=1}^{N}\mathcal{V}_{rs}^\pm J_{\pm,s} \mp \frac{\hbar_r}{2}\sum_{s=1}^{N}\mathcal{U}_{rs}^\pm j_{\pm,s} = \sum_{s=1}^{N}\mathcal{V}_{rs}^\pm J_{\pm,s} \mp \frac{\hbar_r}{2} j_{\pm,r}.$$

Re-inserting this identity in the above expression for Υ_r then shows that it satisfies equation (3.5.11), as required.

$\boldsymbol{\lambda}$-boundary condition. Let us consider now a pair of poles $z_r^\pm$ associated with a λ-boundary condition. One then has $\widehat{g}|_{z_r^+} = g_r$ and $\widehat{g}|_{z_r^-} = \mathrm{Id}$ (see section 3.5.5). Recall moreover from section 3.5.4 that the Wess-Zumino coefficient is defined for λ-boundary conditions as $\hbar_r = -\ell_{r,+}/2$. Thus, the Wess-Zumino terms corresponding to the poles $z_r^\pm$ in the action (3.5.9) are given by equation (3.5.10). Let us now compute Υ_r. For a λ-boundary condition, one has $j_\pm^{\{z_r^+\}} = j_{\pm,r}$ and $j_\pm^{\{z_r^-\}} = 0$. Thus, Υ_r is given by:

$$\Upsilon_r = -\frac{\hbar_r}{2}\kappa\big(j_{+,r}, J_{-,r}\big) + \frac{\hbar_r}{2}\sum_{s=1}^{N}\frac{\varphi_{-,s}(z_s^-)}{\varphi_{-,s}(z_r^+)}\kappa\big(J_{+,r}, j_{-,s}\big). \tag{3.5.12}$$

Let us note that for a λ-realisation, one has $\mathcal{C}_r^+ = -\hbar_r\mathcal{B}_r^+$ and $\mathcal{C}_r^- = \hbar_r\mathcal{B}_r^-$ (see section 3.2.3). In terms of the operators $\mathcal{U}_{rs}^+$ and $\mathcal{V}_{rs}^+$ defined in equations (3.3.6) and (3.3.13), this implies

$$\mathcal{V}_{rs}^+ = \frac{\hbar_r}{2}\mathcal{U}_{rs}^+ - \hbar_r\delta_{rs}\mathrm{Id}.$$

Using the expression (3.3.7) of the currents $j_{\pm,s}$, we then obtain

$$-\hbar_r j_{+,r} = \sum_{s=1}^{N}\mathcal{V}_{rs}^+ j_{+,s} - \frac{\hbar_r}{2}\sum_{s=1}^{N}\mathcal{U}_{rs}^+ j_{+,s} = \sum_{s=1}^{N}\mathcal{V}_{rs}^+ j_{+,s} - \frac{\hbar_r}{2} j_{+,r}. \tag{3.5.13}$$

Moreover, using $\mathcal{C}_r^- = \hbar_r \mathcal{B}_r^-$, $\mathcal{B}_r^+ = \mathrm{Id}$ and $\mathcal{C}_r^+ = -\hbar_r \mathrm{Id}$ (cf. section 3.2.3), one gets

$$\hbar_r \sum_{s=1}^{N} \frac{\varphi_{-,s}(z_s^-)}{\varphi_{-,s}(z_r^+)} J_{-,s} = \sum_{s=1}^{N} \mathcal{V}_{rs}^- J_{-,s} + \frac{\hbar_r}{2} \sum_{s=1}^{N} \mathcal{U}_{rs}^- J_{-,s} = \sum_{s=1}^{N} \mathcal{V}_{rs}^- J_{-,s} + \frac{\hbar_r}{2} J_{-,r}. \tag{3.5.14}$$

Reinserting equations (3.5.13) and (3.5.14) in the expression (3.5.12) of Υ_r, one sees that Υ_r satisfies equation (3.5.11), as required.

3.6 Conclusions

Let us conclude by discussing some perspectives of the work presented in this chapter. As explained in section 3.3.4, the models we constructed involving a Yang-Baxter realisation without Wess-Zumino term possess a q-deformed Poisson-Lie symmetry, which replaces the left translation symmetry of the undeformed model. It is well known that the Yang-Baxter model (with one copy and without Wess-Zumino term) in fact possesses a larger (infinite) symmetry algebra, satisfying the relations of an affine q-deformed Poisson algebra [94] (see also [88,89,95]), which replaces the Yangian symmetry of the undeformed Principal Chiral Model [96]. It would be interesting to understand whether such infinite extensions of the q-deformed symmetries also exist for the deformed coupled models and what would be their underlying algebraic structure.

The integrable deformed models constructed in this chapter still possess an undeformed symmetry, corresponding to the diagonal symmetry of the underlying affine Gaudin model, which acts on the fields $g^{(r)}$ by right multiplication ($g^{(r)} \mapsto g^{(r)}h$) or conjugacy ($g^{(r)} \mapsto h^{-1}g^{(r)}h$), depending on whether the realisations at sites $(r, \pm)$ are Yang-Baxter realisations or λ-realisations. It was explained in [85] that for a general realisation of affine Gaudin model of the type considered in [59], one can construct an integrable Yang-Baxter deformation which breaks its diagonal symmetry. Thus, one could introduce a further integrable deformation of the deformed coupled sigma models constructed in this chapter. As explained in [85], this deformation procedure involves gauging the model and thus requires treating Hamiltonian first-class constraints. For brevity, we chose not to treat these deformations here: however, we expect that they can be studied using similar methods to the ones developed in the present chapter and further in chapter 4. For the case with one copy only, it was conjectured in [85] that these further deformed theories coincide with already known models, namely the bi-Yang-Baxter model (see [21,50,97] for the case without Wess-Zumino term and [98] for the case with Wess-Zumino term) and the generalised λ-model [99].

It is known that the Yang-Baxter and λ-models are Poisson-Lie T-dual [100–102] to one another [49, 99, 103, 104], while the Yang-Baxter model with Wess-Zumino term is Poisson-Lie T-dual to itself with different parameters [105]. It would be interesting to investigate the various possible dualities between the coupled models constructed in this article and how they would manifest themselves in the underlying geometry of their target space G^N.

The results of section 3.5 illustrate once again the deep relation existing between the approaches to two-dimensional integrable field theories from affine Gaudin models [57] and from four-dimensional semi-holomorphic Chern-Simons theory [70], first established in [71] and further supported in [93]. In particular, the analysis conducted in this section strengthens the apparent correspondence between the choice of realisations in the first approach and the choice of boundary conditions in the second one. It would be interesting to understand in more details this correspondence.

A natural question is to explore the quantum properties of these classically integrable deformed sigma models. The one-loop renormalisability of the class of models constructed here was recently proved in [106]. It would be interesting to study these renormalisation properties further. For example, one could look into the existence of conformal fixed points in the space of models. The results obtained in [61–63] about the renormalisation of the coupled λ-models introduced in these same references (which are limits of the models considered here) already show the existence of non-trivial fixed points. A further possible direction would be to investigate the higher-loops renormalisability of these models. In [106], it was shown that they are not renormalisable at 2-loop without quantum corrections of their underlying geometry. It would be interesting to find the explicit form of the quantum corrections that would ensure their higher-loop renormalisability, as it was recently studied in [107–109] for non-coupled models.

3.A Proof of the identities for the operators $\mathcal{B}_\pm$ and $\mathcal{C}_\pm$

In this appendix we will present the calculation of the non-ultralocal terms (*i.e.* terms containing derivatives of the delta distribution) in the bracket (3.2.7), using the ansatz (3.2.8) for the currents $\mathcal{J}_\pm$ in terms of the operators $\mathcal{B}_\pm$ and $\mathcal{C}_\pm$. In particular, we will show that this computation implies that these operators satisfy the identities (3.2.9). Let us start by noting that in order to perform this computation, we need the Poisson brackets between the following objects: $\mathcal{B}_\pm$, Y, $\mathcal{C}_\pm$ and j. However, let us recall that we have assumed the operators $\mathcal{B}_\pm$ and $\mathcal{C}_\pm$ to depend only on the field g (and not on its derivative $\partial_x g$). Thus, the non-ultralocal terms in the brackets of $\mathcal{J}_\pm$ can only come from the brackets between the fields Y and j. More precisely, for $\epsilon, \sigma \in \{\pm\}$, we have

$$\begin{aligned}\{\mathcal{J}_{\epsilon\underline{\mathbf{1}}}(x), \mathcal{J}_{\sigma\underline{\mathbf{2}}}(y)\} &= \{\mathcal{B}_\epsilon Y_{\underline{\mathbf{1}}}(x), \mathcal{C}_\sigma j_{\underline{\mathbf{2}}}(y)\} + \{\mathcal{C}_\epsilon j_{\underline{\mathbf{1}}}(x), \mathcal{B}_\sigma Y_{\underline{\mathbf{2}}}(y)\} + [\text{ultralocal terms}]\\ &= \mathcal{B}_{\epsilon\underline{\mathbf{1}}}(x)\mathcal{C}_{\sigma\underline{\mathbf{2}}}(y)\{Y_{\underline{\mathbf{1}}}(x), j_{\underline{\mathbf{2}}}(y)\} + \mathcal{C}_{\epsilon\underline{\mathbf{1}}}(x)\mathcal{B}_{\sigma\underline{\mathbf{2}}}(y)\{j_{\underline{\mathbf{1}}}(x), Y_{\underline{\mathbf{2}}}(y)\} + [\text{u.l.}].\end{aligned}$$

From here, using the form of the Poisson bracket between Y and j, which can be simply found from (3.2.2) and (3.2.5), we find that

$$\begin{aligned}\{\mathcal{J}_{\epsilon\underline{\mathbf{1}}}(x), \mathcal{J}_{\sigma\underline{\mathbf{2}}}(y)\} &= -\mathcal{B}_{\epsilon\underline{\mathbf{1}}}(x)\mathcal{C}_{\sigma\underline{\mathbf{2}}}(y)C_{\underline{\mathbf{12}}}\delta'_{xy} - \mathcal{C}_{\epsilon\underline{\mathbf{1}}}(x)\mathcal{B}_{\sigma\underline{\mathbf{2}}}(y)C_{\underline{\mathbf{12}}}\delta'_{xy} + [\text{u.l.}]\\ &= -(\mathcal{B}_{\epsilon\underline{\mathbf{1}}}(x)\mathcal{C}_{\sigma\underline{\mathbf{2}}}(x) + \mathcal{C}_{\epsilon\underline{\mathbf{1}}}(x)\mathcal{B}_{\sigma\underline{\mathbf{2}}}(x))C_{\underline{\mathbf{12}}}\delta'_{xy} + [\text{u.l.}],\end{aligned}$$

where we have used the fact that for any function f, $f(y)\delta'_{xy} = f(x)\delta'_{xy} + f'(x)\delta_{xy} = f(x)\delta'_{xy} +$ [u.l.]. Finally, using the fact that for an operator $\mathcal{O}$ on the Lie algebra $\mathfrak{g}$, $\mathcal{O}_{\underline{\mathbf{1}}}C_{\underline{\mathbf{12}}} = {}^t\mathcal{O}_{\underline{\mathbf{2}}}C_{\underline{\mathbf{12}}}$, we get

$$\{\mathcal{J}_{\epsilon\underline{\mathbf{1}}}(x), \mathcal{J}_{\sigma\underline{\mathbf{2}}}(y)\} = -(\mathcal{B}_\epsilon{}^t\mathcal{C}_\sigma + \mathcal{C}_\epsilon{}^t\mathcal{B}_\sigma)_{\underline{\mathbf{1}}}(x)C_{\underline{\mathbf{12}}}\delta'_{xy} + [\text{u.l.}]. \tag{3.A.1}$$

As we want $\mathcal{J}_\pm$ to be Poisson commuting Kac-Moody currents of levels $\ell_\pm$, one should have

$$\begin{aligned}\{\mathcal{J}_{\pm\underline{\mathbf{1}}}(x), \mathcal{J}_{\pm\underline{\mathbf{2}}}(y)\} &= -\ell_\pm C_{\underline{\mathbf{12}}}\delta'_{xy} + [\text{u.l.}],\\ \{\mathcal{J}_{\pm\underline{\mathbf{1}}}(x), \mathcal{J}_{\mp\underline{\mathbf{2}}}(y)\} &= 0.\end{aligned}$$

Comparing with equation (3.A.1), we then see that the operators $\mathcal{B}_\pm$ and $\mathcal{C}_\pm$ should satisfy the identities (3.2.9).

3.B Simplification of the action of the model with N copies

In this appendix, we show that the non-Lorentz invariant terms appearing in the second line of the action (3.3.14) cancel with the term in the first line containing the Hamiltonian. For that, let us start by computing the expression of the Hamiltonian in terms of Lagrangian fields.

Hamiltonian in terms of Lagrangian fields. We will proceed here in a similar fashion to what has been done in [59]. Let us start by noting that, combining the equations (3.2.26) and (4.2.16), the Hamiltonian can be rewritten as

$$\mathcal{H}=\int_{\mathbb{D}}\mathrm{d}x\,\left(\sum_{i\in\mathcal{I}_-}\frac{1}{2\varphi'(\zeta_i)}\kappa(\Gamma(\zeta_i),\Gamma(\zeta_i))-\sum_{i\in\mathcal{I}_+}\frac{1}{2\varphi'(\zeta_i)}\kappa(\Gamma(\zeta_i),\Gamma(\zeta_i))\right), \tag{3.B.1}$$

where we have used the fact that $\epsilon_i=\pm1$ for $i\in\mathcal{I}_\pm$. We then need to look for the Lagrangian expression of the quantities $\Gamma(\zeta_i)$. This is done by relating them to residues of the Lax connection. More precisely, let us fix $i\in\mathcal{I}_\pm$: from (3.2.33), we have

$$\Gamma(\zeta_i)=\pm\frac{1}{2}\varphi'(\zeta_i)\operatorname*{res}_{z=\zeta_i}\mathcal{L}_\pm(z).$$

Using the Lagrangian expression (3.3.3) of $\mathcal{L}_\pm(z)$ and the fact that $\varphi'(\zeta_i)=\varphi'_\pm(\zeta_i)\varphi_\mp(\zeta_i)$, we then get

$$\Gamma(\zeta_i)=\pm\frac{\ell^\infty}{2}\sum_{r=1}^{N}\frac{\varphi_{\pm,r}(z_r^\pm)\varphi_\mp(\zeta_i)}{z_r^\pm-\zeta_i}j_{\pm,r}.$$

Substituting back into equation (3.B.1), we arrive at the following expression for $\mathcal{H}$:

$$\mathcal{H}=\int_{\mathbb{D}}\mathrm{d}x\,\left(\sum_{r,s=1}^{N}c^+_{rs}\,\kappa\big(J_{+,r},J_{+,s}\big)+\sum_{r,s=1}^{N}c^-_{rs}\,\kappa\big(J_{-,r},J_{-,s}\big)\right), \tag{3.B.2}$$

with

$$c^\pm_{rs}=\pm\frac{\ell^\infty}{8}\varphi_{\pm,r}(z_r^\pm)\varphi_{\pm,s}(z_s^\pm)\sum_{i\in\mathcal{I}_\pm}\frac{1}{z_r^\pm-\zeta_i}\frac{1}{z_s^\pm-\zeta_i}\frac{\varphi_\mp(\zeta_i)}{\varphi'_\pm(\zeta_i)}.$$

Before proceeding, we note that one can prove the above coefficients to be equal to

$$c^\pm_{rs}=\pm\delta_{rs}\frac{\ell_{r,\pm}}{8}\pm\frac{1}{2}\sum_{k=1}^{N}\ell_k^\mp\frac{\varphi_{\pm,r}(z_r^\pm)}{\varphi_{\pm,r}(z_k^\mp)}\frac{\varphi_{\pm,s}(z_s^\pm)}{\varphi_{\pm,s}(z_k^\mp)}. \tag{3.B.3}$$

Simplification of non Lorentz invariant terms in the action. Let us consider the terms in the second line of (3.3.14). Using the expression (3.3.5) of $j_{\pm,r}$, they can be rewritten in the following way:

$$\begin{aligned}
&\frac{1}{2}\sum_{r,s}^{N}\iint\mathrm{d}t\,\mathrm{d}x\,\left[\kappa\left(\mathcal{V}^+_{rs}J_{+,s},J_{+,r}\right)+\kappa\left(\mathcal{V}^-_{rs}J_{-,s},J_{-,r}\right)\right]\\
&=\frac{1}{2}\sum_{r,s,t}^{N}\iint\mathrm{d}t\,\mathrm{d}x\,\left[\kappa\left(J_{+,s},{}^t\mathcal{V}^+_{rs}\,\mathcal{U}^+_{rt}J^{(t)}_+\right)+\kappa\left(J_{-,s},{}^t\mathcal{V}^-_{rs}\,\mathcal{U}^-_{rt}J^{(t)}_-\right)\right]\\
&=\frac{1}{4}\sum_{r,s,t}^{N}\iint\mathrm{d}t\,\mathrm{d}x\,\left[\kappa\left(J_{+,s},({}^t\mathcal{V}^+_{rs}\,\mathcal{U}^+_{rt}+{}^t\mathcal{U}^+_{rs}\,\mathcal{V}^+_{rt})J^{(t)}_+\right)+\kappa\left(J_{-,s},({}^t\mathcal{V}^-_{rs}\,\mathcal{U}^-_{rt}+{}^t\mathcal{U}^-_{rs}\,\mathcal{V}^-_{rt})J^{(t)}_-\right)\right]
\end{aligned}$$

We want to prove that these terms are cancelled by the term in (3.3.14) containing the Hamiltonian. From the expression (3.B.2) of the Hamiltonian, one sees that this is the case upon using the following identity:

$$\frac{1}{4}\sum_{r=1}^{N}\left({}^t\mathcal{V}^\pm_{rs}\,\mathcal{U}^\pm_{rt}+{}^t\mathcal{U}^\pm_{rs}\,\mathcal{V}^\pm_{rt}\right)=c^\pm_{st}\,\mathrm{Id},$$

which can be proved using the identities (3.2.9) and the form (3.B.3) of the coefficients $c^\pm_{st}$.

Chapter 4

Integrable multi-parametric coset sigma models

4.1 Introduction

Let G be a connected semi-simple real Lie group, σ an involutive automorphism of G and $G^{(0)}$ the subgroup of fixed-points of σ. Another prototypical example of two-dimensional integrable field theories is the standard sigma model on the symmetric space $G/G^{(0)}$. For our purposes, the most adapted (Lagrangian) formulation of this model is given by a unique G-valued field $g(x,t)$, together with a gauge symmetry with respect to the transformation $g(x,t) \mapsto g(x,t)h(x,t)$ for $h(x,t) \in G^{(0)}$. Since G possesses an involutive automorphism, its Lie algebra $\mathfrak{g}$ admits a $\mathbb{Z}_2$-grading, *i.e.* $\mathfrak{g} = \mathfrak{g}^{(0)} \oplus \mathfrak{g}^{(1)}$, where $\mathfrak{g}^{(0)}$ and $\mathfrak{g}^{(1)}$ are the eigenspaces of eigenvalues 1 and -1 of the automorphism induced by σ on $\mathfrak{g}$. In terms of the grading (1) part of the currents $j_\pm = g^{-1}\partial_\pm g$, the action of the symmetric space sigma model reads:

$$S[g] = \frac{K}{2} \iint_{\mathbb{D}\times\mathbb{R}} \mathrm{d}x \, \mathrm{d}t \; \kappa\big(j_+^{(1)}, j_-^{(1)}\big).$$

One easily checks that this action is invariant under the gauge transformation $g(x,t) \mapsto g(x,t)h(x,t)$. Hence, the physical degrees of freedom of this model are fields in the quotient $G/G^{(0)}$ as anticipated. In the Hamiltonian formulation, this is obtained by imposing a first-class constraint on the phase space of fields of the model.

Our aim in this chapter is to make a further step towards the exploration of the panorama of integrable coset sigma models. Namely, we will explain how to construct integrable sigma models on a coset of the direct product of N copies of G over the diagonal subgroup $G^{(0)}$, generalising the standard symmetric space construction corresponding to the $N = 1$ case. The existence of these models was conjectured in references [58, 59]. Their construction was carried out in the article [110], which I completed during my PhD in collaboration with G. Arutyunov and S. Lacroix and on which this chapter is based.

The formalism that we will use to construct these integrable field theories is the one of dihedral affine Gaudin models, introduced in [57], which is naturally defined in the Hamiltonian formulation of classical field theories. We will not review here the complete construction of these models. This is similar to the one described in chapter 2, generalising the latter to the case in which the Lie algebra $\mathfrak{g}$ admits a $\mathbb{Z}_T$-grading ($T \in \mathbb{Z}_{\geq 1}$). Instead, we will focus in this chapter on the main definitions that will allow us to construct the models we are interested in, restricting in particular to the case $T = 2$. In this context, we will explain how the phase space of the models is constructed as the phase space of canonical fields on the cotangent bundle T^*G^N, together with a first-class constraint encoding the $G^{(0)}_{\text{diag}}$ gauge symmetry.

Let us sketch briefly the results of the construction in this introduction. Let us start by considering the model with two copies, *i.e.* defined on the homogeneous space $G \times G/G^{(0)}_{\text{diag}}$. Its action is described in terms of two G-valued fields g_1 and g_2. In terms of the currents $j_{\pm,r} = g_r^{-1}\partial_\pm g_r$, $r = 1, 2$, we computed this action to be:

$$S[g_1, g_2] = \sum_{r,s=1}^{2} \iint \mathrm{d}x\,\mathrm{d}t \left(\rho^{(0)}_{rs}\,\kappa\left(j^{(0)}_{+,r}, j^{(0)}_{-,s}\right) + \rho^{(1)}_{rs}\,\kappa\left(j^{(1)}_{+,r}, j^{(1)}_{-,s}\right)\right) + \Bbbk\, I_{\text{WZ}}\big[g_1\big] - \Bbbk\, I_{\text{WZ}}\big[g_2\big], \tag{4.1.1}$$

where the superscripts of the currents $j_{\pm,r}$ indicates their grading components as explained above and the $\rho^{(k)}_{rs}$ and $\Bbbk$ are scalar coefficients depending on a total of 4 free parameters. One can check that the model is invariant under the gauge transformation $g_r(x,t) \mapsto g_r(x,t)h(x,t)$ for $h(x,t) \in G^{(0)}$, acting simultaneously on all the fields g_r.

The action (4.1.1) admits a remarkably simple reformulation in terms of the $\mathbb{Z}_2$-graded $\mathcal{R}$-matrix of the theory which underlies the integrable structure of the corresponding dihedral affine Gaudin model [57]. This form allows to conjecture the action for models with arbitrary N, as well as arbitrary order T of the grading of $\mathfrak{g}$, namely $\mathfrak{g} = \bigoplus_{k=0}^{T-1} \mathfrak{g}^{(k)}$. The expression of this action depends on $3N - 2$ free parameters and possesses the following expression:

$$S = \sum_{r=1}^{N} S_{\text{WZW},\,\Bbbk_r}[g_r] - \frac{KT^3}{2} \iint \mathrm{d}x\,\mathrm{d}t \sum_{r,s=1}^{N} \operatorname*{res}_{w=z_s} \operatorname*{res}_{z=z_r} \kappa_{\underline{\mathbf{12}}}\Big(\mathcal{R}^0_{\underline{\mathbf{12}}}(w,z)\varphi_+(z)\varphi_-(w), j_{+,r\underline{\mathbf{1}}}\, j_{-,s\underline{\mathbf{2}}}\Big),$$

where $\Bbbk_r$ are scalar coefficients, φ_+ and φ_- are complex functions depending on the free parameters of the models. Moreover, $\mathcal{R}^0$ denotes the $\mathbb{Z}_T$-graded $\mathcal{R}$-matrix, which was given in (2.1.29).

Having obtained these general results, it is interesting to consider some limits and to focus on some particular models. First, in the $N = 2$ case we define a scaling limit in which one of the four parameters decouples leaving behind a three-parameter $(\lambda, \lambda_1, \lambda_2)$ family of integrable models. We then observe that at the particular point $\lambda_1 = \lambda_2 = \lambda$ the corresponding action coincides with the one of the Guadagnini-Martellini-Mintchev model [111] on the homogeneous space $G \times G/G^{(0)}_{\text{diag}}$, which is an example of a two-dimensional conformal field theory.

Finally, specifying $G = SU(2)$ and $G^{(0)} = U(1)$ in the above $N = 2$ three-parameter model we obtain a gauged sigma model on the coset $SU(2)\times SU(2)/U(1)$. Fixing the gauge by putting one of the Euler angles to zero, we obtain the gauge-fixed action in terms of the five remaining angles $(\theta_1, \theta_2, \phi_1, \phi_2, \psi)$ from which we read off the sigma-model metric and the B-field. The metric turns out to coincide with the three-parameter family of metrics on $T^{1,1}$ manifolds:

$$ds^2 = \lambda_1^2(\mathrm{d}\theta_1^2 + \sin^2\theta_1\,\mathrm{d}\phi_1^2) + \lambda_2^2(\mathrm{d}\theta_2^2 + \sin^2\theta_2\,\mathrm{d}\phi_2^2) + \lambda^2(\mathrm{d}\psi + \cos\theta_1\,\mathrm{d}\phi_1 + \cos\theta_2\,\mathrm{d}\phi_2)^2\,.$$

What follows from our consideration is that the sigma model on a generic three-parameter $T^{1,1}$ is integrable if the following B-field

$$B = \lambda^2(\cos\theta_1\,\mathrm{d}\phi_1 + \mathrm{d}\psi) \wedge (\cos\theta_2\,\mathrm{d}\phi_2 + \mathrm{d}\psi)\,.$$

is present. In particular, changing the overall coefficient λ^2 to any other value destroys integrability. To support this claim, we consider an isometry-preserving setting where the B-field is allowed with an arbitrary coefficient. In order to probe (non-)integrable properties of this generalised model, we reduce the sigma-model equations to a mechanical system by plugging in them the so-called spinning string ansatz, in the spirit of [112–116] where spinning (or wrapped) strings on $T^{1,1}$ were studied. At the end we obtain a coupled system of differential equations for the two angle coordinates θ_1 and θ_2. We then observe that only when the coefficient of the B-field is λ^2, the equations for θ_1 and θ_2 decouple (separate) and can be integrated by quadrature. In any other case there is no decoupling and hence we argue that the corresponding dynamical system exhibits a chaotic behaviour, which is in agreement with what has been found in [114–116] and which has been the subject of later study in [117].

The chapter is organised as follows. In the next section we construct the coset models in the Hamiltonian formulation. In section 4.2 we derive the action of the coset sigma model for $N = 2$, rewrite this action in the form involving the classical $\mathcal{R}$-matrix and discuss its generalisation for arbitrary N. We also consider the limiting case where one of the parameters is scaled away and at a special point in the parameter space we recover the conformal model of Guadagnini, Martellini and Mintchev. Section 4.3 is devoted to integrable sigma models on $T^{1,1}$ manifolds. We relegate some technical details to three appendices.

4.2 Construction of the models in the Hamiltonian formulation

In this section, we apply the formalism of dihedral affine Gaudin models introduced in [57] to construct integrable sigma models on coset spaces $G^N/G^{(0)}_{\rm diag}$, as discussed in the previous section. We will start by defining the structure of the models as dihedral affine Gaudin models in section 4.2.1. In section 4.2.2, we will define the Hamiltonian of these field theories as well as the constraint corresponding to their $G^{(0)}_{\rm diag}$ gauge symmetry. Section 4.2.3 will concern space-time symmetries of the models and in particular the determination of a simple condition ensuring their relativistic invariance. In section 4.2.4 we will prove that these models are integrable. Finally, in section 4.2.5 we describe the panorama of models obtained through this construction and in particular discuss their defining parameters.

4.2.1 Definition of the models as realisations of affine Gaudin models

In this section, we define the models that we will consider in this chapter as realisations of dihedral affine Gaudin models (AGM), following [57]. The adjective dihedral refers to certain equivariance properties under an action of the dihedral group D_{2T} which are satisfied by the twist function and the Gaudin Lax matrix of the models. These properties have to do with the reality conditions described in section 2.2.2 (see equation (2.2.10)) and with the choice of a $\mathbb{Z}_T$-grading of the Lie algebra $\mathfrak{g}$. For the models that we are considering in this chapter, we fix $T = 2$. The corresponding choice of $\mathbb{Z}_2$-grading $\mathfrak{g}^{(0)} \oplus \mathfrak{g}^{(1)}$ of $\mathfrak{g}$ is then given by the choice of an involutive automorphism σ, as we shall recall now in more detail. We will come back to the equivariance properties encoding the dihedrality at the end of this section.

Conventions and notation. As in the previous section let σ be an involutive automorphism of G. It induces an involutive automorphism of the Lie algebra $\mathfrak{g}$ of G, which we also call σ by a slight abuse of notation. As σ is of order two, it has eigenvalues $+1$ and -1. We define the corresponding eigenspaces

$$\mathfrak{g}^{(0)} = \{x \in \mathfrak{g} : \sigma(x) = x\}, \qquad \text{and} \qquad \mathfrak{g}^{(1)} = \{x \in \mathfrak{g} : \sigma(x) = -x\}.$$

These eigenspaces form a $\mathbb{Z}_2$-gradation of $\mathfrak{g}$ ($\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$ is the cyclic group of order two): $\mathfrak{g} = \mathfrak{g}^{(0)} \oplus \mathfrak{g}^{(1)}$, with

$$[\mathfrak{g}^{(0)}, \mathfrak{g}^{(0)}] \subset \mathfrak{g}^{(0)}, \qquad [\mathfrak{g}^{(0)}, \mathfrak{g}^{(1)}] \subset \mathfrak{g}^{(1)} \qquad \text{and} \qquad [\mathfrak{g}^{(1)}, \mathfrak{g}^{(1)}] \subset \mathfrak{g}^{(0)}.$$

The opposite is also true, *i.e.* given a $\mathbb{Z}_2$-gradation of $\mathfrak{g}$, there is a unique automorphism σ which leaves $\mathfrak{g}^{(0)}$ invariant and acts on $\mathfrak{g}^{(1)}$ as multiplication by -1. In particular, $\mathfrak{g}^{(0)}$ is a subalgebra of $\mathfrak{g}$, which is the Lie subalgebra corresponding to the subgroup $G^{(0)}$ in G.

In the following we will use the notation $X^{(i)}$ to indicate the component of an element $X \in \mathfrak{g}$ in $\mathfrak{g}^{(i)}$, $i \in \{0,1\}$. More precisely, if we call $\pi^{(0)} = (\mathrm{Id}+\sigma)/2$ and $\pi^{(1)} = (\mathrm{Id}-\sigma)/2$ the projectors on $\mathfrak{g}^{(0)}$ and $\mathfrak{g}^{(1)}$ respectively, we then have $X^{(i)} = \pi^{(i)}X$, for $X = X^{(0)} + X^{(1)}$ a generic element of $\mathfrak{g}$.

As in previous chapters, we denote by κ the opposite of the Killing form of $\mathfrak{g}$. It is a standard result that the automorphism σ preserves the bilinear form κ. Hence, $\mathfrak{g}^{(0)}$ and $\mathfrak{g}^{(1)}$ are orthogonal with respect to the bilinear form κ, or, in other words, $\kappa\big(\mathfrak{g}^{(0)}, \mathfrak{g}^{(1)}\big) = 0$.

Let us recall the definition (2.1.20) of the split quadratic Casimir of $\mathfrak{g}$. For $i \in \{0,1\}$, we define its projections $C^{(ii)}_{\underline{\mathbf{12}}} = \pi^{(i)}_{\underline{\mathbf{1}}}\pi^{(i)}_{\underline{\mathbf{2}}} C_{\underline{\mathbf{12}}}$ on $\mathfrak{g}^{(i)} \otimes \mathfrak{g}^{(i)}$. Let us note that the orthogonality of $\mathfrak{g}^{(0)}$ and $\mathfrak{g}^{(1)}$ implies that $\pi^{(i)}_{\underline{\mathbf{1}}}\pi^{(j)}_{\underline{\mathbf{2}}} C_{\underline{\mathbf{12}}} = \delta_{ij} C^{(ii)}_{\underline{\mathbf{12}}}$, for $i,j \in \{0,1\}$. Moreover, we have

$$\kappa_{\underline{\mathbf{2}}}\big(C^{(ii)}_{\underline{\mathbf{12}}}, X_{\underline{\mathbf{2}}}\big) = X^{(i)}, \qquad \forall\, X \in \mathfrak{g}. \tag{4.2.1}$$

Sites, levels and twist function. Following [57], let us consider a dihedral AGM with $N \in \mathbb{Z}_{\geq 1}$ real sites of multiplicity two, whose positions will be denoted by z_r with $r \in \{1,\dots,N\}$ and will be supposed to be non zero ($z_r \in \mathbb{R}^*$). Since each site z_r is of multiplicity two, it is associated with two constant numbers $\ell_{r,0} \in \mathbb{R}$ and $\ell_{r,1} \in \mathbb{R}^*$, called the levels. Altogether this data specifies the twist function $\varphi(z)$ of the model, which takes the following form [57]:

$$\varphi(z) = \frac{1}{2}\sum_{r=1}^{N}\sum_{p=0}^{1}\sum_{k=0}^{1} \frac{(-1)^k \ell_{r,p}}{((-1)^k z - z_r)^{p+1}}. \tag{4.2.2}$$

This is the generalisation for the case of multiplicity two of the twist function (2.2.6) seen in the non-dihedral case. The main difference with that expression is the presence of the sum over $k \in \{0,1\}$ and of the factors $(-1)^k$ which encode the $T=2$ dihedrality of the model[1].

In the rest of this chapter, we will suppose that the levels $\ell_{r,0}$ satisfy the following additional hypothesis, which for reasons to be explained later we call the first-class condition:

$$\sum_{r=1}^{N} \ell_{r,0} = 0. \tag{4.2.3}$$

As we shall see in section 4.2.2, this condition will be necessary to ensure that the models that we construct possess a gauge symmetry.

Takiff currents and phase space. As in the non-dihedral case discussed in chapter 2, to each site z_r of the model are attached two $\mathfrak{g}$-valued Takiff currents $\mathcal{J}_{r,[0]}(x)$ and $\mathcal{J}_{r,[1]}(x)$. They satisfy the following Poisson bracket determined by the choice of levels $\ell_{r,p}$:

$$\{\mathcal{J}_{r,[0]\underline{\mathbf{1}}}(x), \mathcal{J}_{s,[0]\underline{\mathbf{2}}}(y)\} = \delta_{rs}\left([C_{\underline{\mathbf{12}}}, \mathcal{J}_{r,[0]\underline{\mathbf{1}}}(x)]\delta_{xy} - \ell_{r,0} C_{\underline{\mathbf{12}}}\delta'_{xy}\right), \tag{4.2.4a}$$

$$\{\mathcal{J}_{r,[0]\underline{\mathbf{1}}}(x), \mathcal{J}_{s,[1]\underline{\mathbf{2}}}(y)\} = \delta_{rs}\left([C_{\underline{\mathbf{12}}}, \mathcal{J}_{r,[1]\underline{\mathbf{1}}}(x)]\delta_{xy} - \ell_{r,1} C_{\underline{\mathbf{12}}}\delta'_{xy}\right), \tag{4.2.4b}$$

$$\{\mathcal{J}_{r,[1]\underline{\mathbf{1}}}(x), \mathcal{J}_{s,[1]\underline{\mathbf{2}}}(y)\} = 0. \tag{4.2.4c}$$

The phase space of the AGM underlying the present construction consists then of configurations of the Takiff currents $\mathcal{J}_{r,[p]}(x)$ ($r \in \{1,\cdots,N\}$ and $p \in \{0,1\}$), equipped with the Poisson

[1] Note that in the dihedral case one also sets ℓ^∞ to zero. This allows to obtain a model with gauge symmetry, as it will become clear later.

bracket (4.2.4). In the present case, we consider the particular realisation of this AGM in the phase space of canonical fields on the cotangent bundle T^*G^N. We described this phase space in section 2.2.6. In particular, canonical fields on T^*G^N can be encoded into N G-valued fields $g_1(x), \cdots, g_N(x)$ and N $\mathfrak{g}$-valued fields $X_1(x), \cdots, X_N(x)$. They are the equivalents of the fields $g(x)$ and $X(x)$ introduced in section 2.2.6 for one copy of T^*G and satisfy then N independent copies of the Poisson bracket (2.2.41):

$$\{g_{r\underline{\mathbf{1}}}(x), g_{s\underline{\mathbf{2}}}(y)\} = 0, \tag{4.2.5a}$$

$$\{X_{r\underline{\mathbf{1}}}(x), g_{s\underline{\mathbf{2}}}(y)\} = \delta_{rs}\, g_{r\underline{\mathbf{2}}}(x) C_{\underline{\mathbf{12}}} \delta_{xy}, \tag{4.2.5b}$$

$$\{X_{r\underline{\mathbf{1}}}(x), X_{s\underline{\mathbf{2}}}(y)\} = \delta_{rs} [C_{\underline{\mathbf{12}}}, X_{r\underline{\mathbf{1}}}(x)] \delta_{xy}. \tag{4.2.5c}$$

Similarly to the construction of the models of chapter 3, we also introduce currents $j_r(x) = g_r(x)^{-1}\partial_x g_r(x)$ and $W_r(x)$ related to the Wess-Zumino term of g_r, which are the equivalents of the currents $j(x)$ and $W(x)$ discussed for one copy of T^*G in section 3.2.1. Let us then define

$$\mathcal{J}_{r,[0]}(x) = X_r(x) + \frac{\ell_{r,0}}{2} j_r(x) + \frac{\ell_{r,0}}{2} W_r(x), \tag{4.2.6a}$$

$$\mathcal{J}_{r,[1]}(x) = \ell_{r,1}\, j_r(x). \tag{4.2.6b}$$

From the Poisson brackets (2.2.41), (3.2.2) and (3.2.5), one can check that the currents above satisfy the Takiff brackets (4.2.4).

Gaudin Lax matrix. Let $\mathfrak{g}^{\mathbb{C}}$ denote the complexification of $\mathfrak{g}$. As in the non-dihedral case, the other basic building block for the construction of the model is the Gaudin Lax matrix. In this context, it is defined as the following $\mathfrak{g}^{\mathbb{C}}$-valued field [57]:

$$\Gamma(z,x) = \frac{1}{2} \sum_{r=1}^{N} \sum_{p=0}^{1} \sum_{k=0}^{1} \frac{(-1)^k \sigma^k \mathcal{J}_{r,[p]}(x)}{((-1)^k z - z_r)^{p+1}}. \tag{4.2.7}$$

In comparison to equation (2.2.5) of chapter 2, the $T = 2$ dihedrality of the model is now taken into account by the sum over $k \in \{0, 1\}$ and by the presence of the involutive automorphism σ. This is how the choice of σ and thus the choice of the subgroup $G^{(0)}$ enters the definition of the model as an AGM.

From (4.2.4), one can show that the Gaudin Lax matrix satisfies a Poisson bracket of the same form (2.2.7) as the AGMs discussed in chapter 2, namely:

$$\begin{aligned}\{\Gamma_{\underline{\mathbf{1}}}(z,x), \Gamma_{\underline{\mathbf{2}}}(w,y)\} &= [\mathcal{R}^0_{\underline{\mathbf{12}}}(z,w), \Gamma_{\underline{\mathbf{1}}}(z,x)]\delta_{xy} - [\mathcal{R}^0_{\underline{\mathbf{21}}}(w,z), \Gamma_{\underline{\mathbf{2}}}(w,x)]\delta_{xy} \\ &\quad - \left(\mathcal{R}^0_{\underline{\mathbf{12}}}(z,w)\varphi(z) + \mathcal{R}^0_{\underline{\mathbf{21}}}(w,z)\varphi(w)\right)\delta'_{xy}.\end{aligned} \tag{4.2.8}$$

For the case of a dihedral AGM, the $\mathcal{R}$-matrix appearing in this bracket is given by the standard $\mathcal{R}$-matrix twisted by the automorphism σ:

$$\mathcal{R}^0_{\underline{\mathbf{12}}}(z,w) = \frac{1}{2} \sum_{k=0}^{1} \frac{\sigma^k_{\underline{\mathbf{1}}} C_{\underline{\mathbf{12}}}}{w - (-1)^k z}. \tag{4.2.9}$$

Dihedrality. As mentioned at the beginning of this section, the AGM that we are considering in this chapter possesses certain equivariance properties under the dihedral group D_4. Let us now discuss these properties.

The general dihedral group D_{2T} contains the cyclic group $\mathbb{Z}_T$ as a subgroup. Recall that for the models considered in this chapter, we have $T = 2$: the corresponding cyclic group $\mathbb{Z}_2$

acts on the complex plane by multiplication by -1 and on the Lie algebra $\mathfrak{g}$ by the involutive automorphism σ, which we extend to the complexification $\mathfrak{g}^{\mathbb{C}}$ by $\mathbb{C}$-linearity. One checks from their expressions (4.2.2) and (4.2.7) that the twist function and the Gaudin Lax matrix are equivariant 1-forms with respect to these actions, *i.e.* that

$$\sigma(\Gamma(z,x)) = -\,\Gamma(-z,x) \qquad \text{and} \qquad \varphi(z) = -\varphi(-z). \tag{4.2.10}$$

Let us note that the sums over $k \in \{0,1\}$ and the presence of the factors $(-1)^k$ and σ^k in equations (4.2.2) and (4.2.7) are crucial for the above conditions to hold.

In addition to the cyclic group $\mathbb{Z}_T$, the dihedral group D_{2T} contains an order two cyclic group $\mathbb{Z}_2$ (which is not to be confused with the $\mathbb{Z}_2$ group discussed above, which arises since we have $T=2$ in the case considered in this chapter). The equivariance properties corresponding to this $\mathbb{Z}_2$ subgroup encode the reality conditions of the model. It acts on the complex plane by conjugation $z \mapsto \bar{z}$ and on the complexified Lie algebra $\mathfrak{g}^{\mathbb{C}}$ by the antilinear involutive automorphism τ, defined such that the real form $\mathfrak{g}$ is the subalgebra of fixed points of τ. One checks that the automorphisms σ and τ of $\mathfrak{g}^{\mathbb{C}}$ satisfy the dihedrality condition $\sigma\circ\tau=\tau\circ\sigma$: the group generated by σ and τ is thus isomorphic to the direct product $\mathbb{Z}_2\times\mathbb{Z}_2$, which is the dihedral group[2] D_4. Using this dihedrality condition and the facts that the Takiff currents $\mathcal{J}_{r,[p]}$ are valued in the real form $\mathfrak{g}$ and the positions z_r and levels $\ell_{r,p}$ are real numbers, one checks that the twist function (4.2.2) and the Gaudin Lax matrix (4.2.7) satisfy the reality conditions

$$\tau(\Gamma(z,x)) = \Gamma(\bar{z},x) \qquad \text{and} \qquad \overline{\varphi(z)} = \varphi(\bar{z}),$$

which are the same as the equivariance conditions that were found in section 2.2.2 of chapter 2. Combining these with the conditions (4.2.10), we then get that $\Gamma(z,x)$ and $\varphi(z)$ are equivariant under the action of the full dihedral group D_4, as expected from the general construction of dihedral AGM in [57].

4.2.2 Hamiltonian, constraint and gauge symmetry

Zeroes of the twist function. Let us begin by studying the zeroes of the twist function (4.2.2). Firstly, we note that $z=0$ is always a zero of $\varphi(z)$. We will suppose that this zero is simple, *i.e.* that $\varphi'(0)\neq 0$. Moreover, the behaviour of $\varphi(z)$ at $z=\infty$ is described by the following asymptotic expansion:

$$\varphi\left(\frac{1}{u}\right) = 2Ku^3 + O(u^5), \qquad \text{where} \qquad K = \frac{1}{2}\sum_{r=1}^{N} z_r\left(z_r\,\ell_{r,0} + 2\,\ell_{r,1}\right). \tag{4.2.11}$$

Let us make a few comments on this expansion. From the equivariance property (4.2.10) of $\varphi(z)$, it is clear that only odd powers of u can appear in the expansion of $\varphi(u^{-1})$ around $u=0$. Moreover, in general, the function $\varphi(z)$ as defined in equation (4.2.2) also possesses a term of order $O(u)$ in its expansion at infinity, which is proportional to the sum $\sum_{r=1}^N \ell_{r,0}$: as we supposed that this sum vanishes (see the first-class condition (4.2.3)), the first term in the expansion is then of order u^3. Let us now consider the 1-form $\varphi(z)\mathrm{d}z$. To study its behaviour at infinity, let us consider the change of coordinate $z=u^{-1}$. We then have

$$\varphi(z)\mathrm{d}z = \chi(u)\mathrm{d}u, \qquad \text{with} \qquad \chi(u) = -\frac{1}{u^2}\varphi\left(\frac{1}{u}\right). \tag{4.2.12}$$

[2] For a general T (*i.e.* when we have σ of order T), the dihedrality condition reads $\sigma\circ\tau=\tau\circ\sigma^{-1}$ and the dihedral group D_{2T} has the structure of a semi-direct product $\mathbb{Z}_T\rtimes\mathbb{Z}_2$ instead of a direct product. For $T=2$, we have $\sigma^{-1}=\sigma$, so that the dihedrality condition becomes the commutation of σ and τ.

According to the asymptotic expansion (4.2.11), the 1-form $\varphi(z)\mathrm{d}z$ thus has a zero at infinity. Moreover, the derivative of this 1-form at $z=\infty$ is given by $\chi'(0)=-2K$. We will suppose that this zero at infinity is simple, *i.e.* that $K\neq 0$.

As $\varphi(z)\mathrm{d}z$ possesses $4N$ poles (counted with multiplicities), it possesses $4N-2$ zeroes in the Riemann sphere: in addition to the one at the origin $z=0$ and the one at infinity $z=\infty$, it thus possesses $4(N-1)$ zeroes in $\mathbb{C}\setminus\{0\}$. From the equivariance property (4.2.10), one sees that these zeroes come as pairs ζ_i and $\zeta_{-i}=-\zeta_i$, with $i\in\{1,\cdots,2N-2\}$. We will suppose that the ζ_i's are pair-wise distinct and are thus simple zeroes of $\varphi(z)$, hence $\varphi'(\zeta_i)\neq 0$. In terms of the z_r's and the ζ_i's, the twist function can then be rewritten as

$$\varphi(z)=2K\frac{z\prod_{i=1}^{2N-2}(z^2-\zeta_i^2)}{\prod_{r=1}^{N}(z^2-z_r^2)^2}. \tag{4.2.13}$$

Hamiltonian. Let us recall the definition of the charge $\mathcal{Q}(z)$ given in (2.2.12):

$$\mathcal{Q}(z)=-\frac{1}{2\varphi(z)}\int_{\mathbb{D}}\mathrm{d}x\ \kappa(\Gamma(z,x),\Gamma(z,x)). \tag{4.2.14}$$

As in section 2.2.3, the quantities

$$\mathcal{Q}_{\pm i}=\operatorname*{res}_{z=\pm\zeta_i}\mathcal{Q}(z)\mathrm{d}z,\qquad i=1,\dots,2N-2, \tag{4.2.15a}$$

$$\mathcal{Q}_0=\operatorname*{res}_{z=0}\mathcal{Q}(z)\mathrm{d}z\qquad\text{and}\qquad\mathcal{Q}_\infty=\operatorname*{res}_{z=\infty}\mathcal{Q}(z)\mathrm{d}z. \tag{4.2.15b}$$

are local quadratic charges in the currents $\mathcal{J}_{r,[p]}$. It is straightforward to show that $\mathcal{Q}_i=\mathcal{Q}_{-i}$, from the equivariance property (4.2.10) of the Gaudin Lax matrix and twist function. From the Poisson bracket (4.2.8) of the Gaudin Lax matrix, they are also in involution *i.e.* they mutually Poisson commute. Given a collection of real numbers $\{\epsilon_0,\epsilon_i,\epsilon_\infty\}$, $i=1,\dots,2N-2$, we define the *naive Hamiltonian* of the model (the term naive will be explained later in this section) as the following sum over the charges introduced above:

$$\mathcal{H}=\epsilon_0\mathcal{Q}_0+2\sum_{i=1}^{2N-2}\epsilon_i\mathcal{Q}_i+\epsilon_\infty\mathcal{Q}_\infty, \tag{4.2.16}$$

where we introduced a factor of 2 for future convenience. Due to the reality conditions introduced in the previous section, $\mathcal{H}$ can be shown to be real [59].

Constraint. In this paragraph, we introduce a constraint on the phase space of canonical fields on T^*G^N and show its consistency with the choice of Hamiltonian made in the previous paragraph. We will use Dirac's theory of constraints in Hamiltonian systems: we refer to [118, 119] for reviews of this formalism. Following the general construction of [57], we define the constraint as

$$\mathcal{C}(x)=-\operatorname*{res}_{z=\infty}\Gamma(z,x)\mathrm{d}z=\lim_{u\to 0}\frac{1}{u}\Gamma\left(\frac{1}{u},x\right). \tag{4.2.17}$$

Using the expression (4.2.7) of the Gaudin Lax matrix $\Gamma(z,x)$ and the fact that $\frac{1}{2}(\mathrm{Id}+\sigma)$ is the projector on the grading $\mathfrak{g}^{(0)}$ of $\mathfrak{g}$, one checks that the constraint explicitly reads

$$\mathcal{C}(x)=\sum_{r=1}^{N}\mathcal{J}^{(0)}_{r,[0]}(x). \tag{4.2.18}$$

In particular, it is a $\mathfrak{g}^{(0)}$-valued field. The models we are interested in are then defined on a reduced phase space, obtained from canonical fields on T^*G^N by imposing

$$\mathcal{C}(x) \approx 0.$$

In this equation, and in the rest of this chapter, we use the notation $\approx$ to denote *weak equalities*, *i.e.* equalities that are true when the constraint is imposed. The standard equality sign $=$ will then indicate *strong equalities*, which are true even without imposing the constraint.

Poisson bracket of the constraint with the naive Hamiltonian. From the Poisson bracket (4.2.8) of the Gaudin Lax matrix with itself, one checks that the local charge $\mathcal{Q}(z)$, defined as in equation (4.2.14), satisfies the following Poisson bracket with the constraint:

$$\{\mathcal{Q}(z), \mathcal{C}(x)\} = -\partial_x \Gamma(z,x)^{(0)}.$$

In particular, as $\Gamma(z,x)$ is regular at $z=0$ and $z=\zeta_i$ for $i=1,\cdots,2N-2$, one has

$$\{\mathcal{Q}_i, \mathcal{C}(x)\} = 0, \qquad \forall\, i \in \{0,\cdots,2N-2\}, \tag{4.2.19}$$

for the charges $\mathcal{Q}_i$ introduced in equation (4.2.15). Moreover, the residue of $\Gamma(z,x)^{(0)}\mathrm{d}z$ at $z=\infty$ is equal to $-\mathcal{C}(x)$. Thus we also have

$$\{\mathcal{Q}_\infty, \mathcal{C}(x)\} = \partial_x \mathcal{C}(x). \tag{4.2.20}$$

Recall that the naive Hamiltonian of the model $\mathcal{H}$ is defined in terms of the charges $\mathcal{Q}_i$, $i \in \{0,\cdots,2N-2,\infty\}$, by equation (4.2.16). Thus, we get

$$\{\mathcal{H}, \mathcal{C}(x)\} = \epsilon_\infty \partial_x \mathcal{C}(x).$$

In particular, we see that the naive Hamiltonian weakly Poisson commutes with $\mathcal{C}(x)$:

$$\{\mathcal{H}, \mathcal{C}(x)\} \approx 0. \tag{4.2.21}$$

This ensures that the Hamiltonian flow of $\mathcal{H}$ preserves the constraint $\mathcal{C}(x) \approx 0$.

First-class property. The Poisson bracket of the constraint with itself can be obtained from its definition (4.2.17) and the Poisson bracket (4.2.8) of the Gaudin Lax matrix (or equivalently from its expression (4.2.18) and the Poisson bracket (4.2.4) of the currents $\mathcal{J}_{r,[0]}$). It reads

$$\left\{\mathcal{C}_{\underline{\mathbf{1}}}(x), \mathcal{C}_{\underline{\mathbf{2}}}(y)\right\} = \left[C^{(00)}_{\underline{\mathbf{12}}}, \mathcal{C}_{\underline{\mathbf{1}}}(x)\right]\delta_{xy}, \tag{4.2.22}$$

where $C^{(00)}_{\underline{\mathbf{12}}} \in \mathfrak{g}^{(0)} \otimes \mathfrak{g}^{(0)}$ is the split Casimir of $\mathfrak{g}^{(0)}$. In fact, this bracket also contains in general a non-ultralocal term $-\left(\sum_{r=1}^N \ell_{r,0}\right) C^{(00)}_{\underline{\mathbf{12}}}\delta'_{xy}$: as we supposed in equation (4.2.3) that the levels $\ell_{r,0}$ sum to zero, this term vanishes. In particular, this shows that the Poisson bracket of the constraint with itself weakly vanishes:

$$\left\{\mathcal{C}_{\underline{\mathbf{1}}}(x), \mathcal{C}_{\underline{\mathbf{2}}}(y)\right\} \approx 0. \tag{4.2.23}$$

Thus, the constraint $\mathcal{C}(x) \approx 0$ is *first-class* (see for instance [118,119]). This justifies a posteriori the name of first-class condition for the assumption (4.2.3) that we made: indeed, without this assumption, the bracket of the constraint would contain a non-ultralocal term which would not vanish weakly and the constraint would then not be first-class.

Total Hamiltonian and Lagrange multiplier. At the beginning of this section, we defined the naive Hamiltonian $\mathcal{H}$ through equation (4.2.16). As we are considering models subject to the constraint $\mathcal{C}(x) \approx 0$, we have to define the *total Hamiltonian* of the system as the sum of the naive Hamiltonian and a generic term proportional to the constraint, so that it coincides weakly with the naive Hamiltonian. It thus takes the form

$$\mathcal{H}_T = \mathcal{H} + \int_{\mathbb{D}} dx \; \kappa\big(\mu(x), \mathcal{C}(x)\big), \tag{4.2.24}$$

where μ is a $\mathfrak{g}^{(0)}$-valued field, called the *Lagrange multiplier*. It is a new dynamical field, independent of the canonical fields on T^*G^N. As we shall see in the next paragraph, the existence of this Lagrange multiplier reflects the presence of a gauge symmetry in the model.

The dynamic of the model is defined by the Hamiltonian flow of $\mathcal{H}_T$, *i.e.* the time evolution of any observable $\mathcal{O}$ is given by

$$\partial_t \mathcal{O} \approx \{\mathcal{H}_T, \mathcal{O}\} \approx \{\mathcal{H}, \mathcal{O}\} + \int_{\mathbb{D}} dx \; \kappa\big(\mu(x), \{\mathcal{C}(x), \mathcal{O}\}\big). \tag{4.2.25}$$

The facts that the naive Hamiltonian Poisson commutes with the constraint (see equation (4.2.21)) and that the constraint is first-class (see equation (4.2.23)) ensure that the constraint $\mathcal{C}(x) \approx 0$ is conserved under time evolution:

$$\partial_t \mathcal{C}(x) \approx 0. \tag{4.2.26}$$

Gauge symmetry. It is a standard result that the presence of first-class constraints in Hamiltonian systems implies the existence of gauge (local) symmetries (see for instance [118, 119]). The infinitesimal action of this gauge symmetry on the observables of the model is given by the Hamiltonian flow generated by the constraint. In the case at hand, the constraint satisfies the bracket (4.2.22), which is a copy of the Kirillov-Kostant bracket of the Lie algebra $\mathfrak{g}^{(0)}$ for every point $x \in \mathbb{D}$. Thus, the gauge symmetry takes the form of a local action of the group $G^{(0)}$. The corresponding infinitesimal transformation of an observable $\mathcal{O}$, with gauge parameter $\epsilon(x,t) \in \mathfrak{g}^{(0)}$, is given by

$$\delta_\epsilon^\infty \mathcal{O} \approx \left\{ \int_{\mathbb{D}} dx \; \kappa\big(\epsilon(x,t), \mathcal{C}(x)\big), \mathcal{O} \right\} \approx \int_{\mathbb{D}} dx \; \kappa\big(\epsilon(x,t), \{\mathcal{C}(x), \mathcal{O}\}\big). \tag{4.2.27}$$

One can then observe that the terms involving the Lagrange multiplier μ in the total Hamiltonian (4.2.24) and the dynamic (4.2.25) of the model correspond to a gauge transformation and thus account for the freedom of performing such a transformation in the time evolution of the system.

Let us study in more details the action of the gauge symmetry on the canonical fields on T^*G^N. For that, recall the expression (4.2.18) of the constraint $\mathcal{C}(x)$ in terms of the Kac-Moody current $\mathscr{J}_{r,[0]}$. It is clear from the definition (4.2.6a) of the latter and the Poisson brackets (2.2.41), (3.2.2) and (3.2.5) that

$$\big\{\mathcal{C}_{\underline{\mathbf{1}}}(x), g_{r\underline{\mathbf{2}}}(y)\big\} = g_{r\underline{\mathbf{2}}}(x) C^{(00)}_{\underline{\mathbf{12}}} \delta_{xy}. \tag{4.2.28}$$

Thus, using equation (4.2.27), one finds that the infinitesimal gauge transformation of the field $g_r(x)$ is given by:

$$\delta_\epsilon g_r(x) = g_r(x) \epsilon(x,t). \tag{4.2.29}$$

Similarly, one can determine the gauge transformation of the fields X_r. It is in fact more convenient to consider the gauge transformation of the field $Y_r = X_r + \ell_{r,0} W_r/2$, which reads

$$\delta_\epsilon Y_r(x) = [Y_r(x), \epsilon(x,t)] + \frac{\ell_{r,0}}{2} \partial_x \epsilon(x,t). \tag{4.2.30}$$

The transformations (4.2.29) and (4.2.30) are infinitesimal actions with local parameter $\epsilon(x,t)$ valued in $\mathfrak{g}^{(0)}$. They can be lifted to an action of the group $G^{(0)}$, depending on a local parameter $h(x,t)$ in $G^{(0)}$, which takes the form:

$$g_r \longmapsto g_r h \qquad \text{and} \qquad Y_r \longmapsto h^{-1} Y_r h + \frac{\ell_{r,0}}{2} h^{-1} \partial_x h. \tag{4.2.31}$$

In particular, we see that the gauge symmetry acts on the set of fields $(g_1, \cdots, g_N) \in G^N$ by right translation of the diagonal subgroup

$$G^{(0)}_{\text{diag}} = \bigl\{ (h, \cdots, h),\ h \in G^{(0)} \bigr\}.$$

Let us summarise what are the physical degrees of freedom of the model. By construction, we start from the phase space of canonical fields on $T^* G^N$. One then needs to restrict to the field configurations such that the constraint $\mathcal{C}(x) \approx 0$ is satisfied. Furthermore, one needs to quotient out by the action of the gauge symmetry (4.2.31) (the fact that this gauge symmetry preserves the constraint $\mathcal{C}(x) \approx 0$ is a direct consequence of the first-class property (4.2.23) of $\mathcal{C}(x)$). As explained above, this gauge symmetry acts on the coordinate fields $(g_1, \cdots, g_N) \in G^N$ by right translation of the subgroup $G^{(0)}_{\text{diag}}$: one can then see the "physical" coordinate fields of the model as fields on the quotient $G^N / G^{(0)}_{\text{diag}}$, by gauging away the coordinate fields in $G^{(0)}_{\text{diag}}$. The constraint $\mathcal{C}(x) \approx 0$ can then be seen as eliminating the corresponding superfluous conjugate momentum fields. The physical phase space of the model can thus be identified with canonical fields on $T^*(G^N / G^{(0)}_{\text{diag}})$: in particular, the Lagrangian formulation of the model will then describe a field theory on $G^N / G^{(0)}_{\text{diag}}$. In this chapter, we will however keep working with the unreduced phase space $T^* G^N$, together with the constraint and the gauge symmetry, to avoid having to consider the quotient.

Gauge transformation of the Gaudin Lax matrix. From the expression (4.2.18) of the constraint and the Poisson bracket (4.2.4) of the Takiff currents $\mathcal{J}_{r,[p]}(x)$, one checks that the gauge transformation of the latter is given by

$$\delta^\infty_\epsilon \mathcal{J}_{r,[p]}(x) = [\mathcal{J}_{r,[p]}(x), \epsilon(x,t)] + \ell_{r,p}\, \partial_x \epsilon(x,t). \tag{4.2.32}$$

The corresponding lifted action of the group $G^{(0)}$, with local parameter $h(x,t) \in G^{(0)}$, reads

$$\mathcal{J}_{r,[p]} \longmapsto h^{-1} \mathcal{J}_{r,[p]} h + \ell_{r,p}\, h^{-1} \partial_x h. \tag{4.2.33}$$

Recall that σ is an automorphism of $\mathfrak{g}$ whose fixed-points form the subalgebra $\mathfrak{g}^{(0)}$. Thus, its lift to G leaves the elements of the subgroup $G^{(0)}$ invariant. As $h \in G^{(0)}$ and $h^{-1}\partial_x h \in \mathfrak{g}^{(0)}$, the gauge transformation of $\sigma\bigl(\mathcal{J}_{r,[p]}\bigr)$ is given by

$$\sigma\bigl(\mathcal{J}_{r,[p]}\bigr) \longmapsto h^{-1} \sigma\bigl(\mathcal{J}_{r,[p]}\bigr) h + \ell_{r,p}\, h^{-1} \partial_x h.$$

From equations (4.2.2) and (4.2.7), we then see that the gauge symmetry acts on the Gaudin Lax matrix $\Gamma(z)$ as

$$\Gamma(z) \longmapsto h^{-1} \Gamma(z) h + \varphi(z)\, h^{-1} \partial_x h. \tag{4.2.34}$$

Gauge transformation of the Lagrange multiplier. It is a standard result that the equations of motion of the model are invariant under gauge symmetries, as one should expect, if one also transforms the Lagrange multiplier appropriately [118, 119]. In the present case, the transformation rule of the Lagrange multiplier is

$$\delta_\epsilon \mu(x) = \big[\mu(x), \epsilon(x,t)\big] + \partial_t \epsilon(x,t) - \epsilon_\infty \partial_x \epsilon(x,t). \tag{4.2.35}$$

This infinitesimal transformation can be lifted to the following action of the gauge group $G^{(0)}$, with local parameter $h(x,t) \in G^{(0)}$:

$$\mu \longmapsto h^{-1}\mu h + h^{-1}\partial_t h - \epsilon_\infty \, h^{-1}\partial_x h. \tag{4.2.36}$$

4.2.3 Space-time symmetries

In this section, we discuss the space-time symmetries of the models constructed in this chapter. In particular, we find a simple condition for their relativistic invariance similar to the one discussed in section 2.2.4 for realisations of non-dihedral AGM.

Momentum. Recall that the momentum of the phase space consisting of canonical fields on T^*G is given by equation (3.2.3). The model constructed in the previous sections is defined on N copies of this phase space and thus has the following momentum:

$$\mathcal{P} = \sum_{r=1}^{N} \mathcal{P}_r = \sum_{r=1}^{N} \int \mathrm{d}x \; \kappa\big(X_r(x), j_r(x)\big). \tag{4.2.37}$$

Using the facts that

$$\Gamma(z) = \frac{\ell_{r,1} j_r}{(z-z_r)^2} + \frac{X_r + \ell_{r,0} W_r/2 + \ell_{r,0} j_r/2}{z - z_r} + O\big((z-z_r)^0\big) \tag{4.2.38}$$

and

$$\frac{1}{\varphi(z)} = (z-z_r)^2 \left(\frac{1}{\ell_{r,1}} - \frac{\ell_{r,0}}{\ell_{r,1}^2}(z-z_r) + O\big((z-z_r)^2\big) \right), \tag{4.2.39}$$

together with the definition (4.2.14) of $\mathcal{Q}(z)$ and the orthogonality relation (3.2.6), one checks that

$$\mathcal{P}_r = - \operatorname*{res}_{z=z_r} \mathcal{Q}(z)\mathrm{d}z, \tag{4.2.40}$$

so that $\mathcal{P} = -\sum_{r=1}^N \operatorname{res}_{z=z_r} \mathcal{Q}(z)\mathrm{d}z$. Thus, we see explicitly that the assumption made in section 2.2.3 is verified. From the results in that section and from the fact that the residues of $Q(z)\mathrm{d}z$ at ζ_i and $-\zeta_i$ are equal for $i \in \{1, \cdots, 2N-2\}$, we then have that

$$\mathcal{P} = \mathcal{Q}_0 + 2\sum_{i=1}^{2N-2} \mathcal{Q}_i + \mathcal{Q}_\infty. \tag{4.2.41}$$

Let us note as in section 2.2.3 the similarity of this expression with the one (4.2.16) of the naive Hamiltonian $\mathcal{H}$: one sees that the momentum would correspond to the choice of all coefficients ϵ_i equal to 1 in equation (4.2.16). This then allows a similar treatment to section 2.2.4 of the energy-momentum tensor of the theory, which can be written in terms of the densities $q_0(x)$, $q_i(x)$ and $q_\infty(x)$ of the charges $\mathcal{Q}_0$, $\mathcal{Q}_i$ and $\mathcal{Q}_\infty$ defined as in (2.2.22), as we shall now briefly describe.

Energy-momentum tensor. Following the discussion in section 2.2.4, the components $T^0{}_0$ and $T^0{}_1$ of the energy-momentum tensor are given by:

$$T^0{}_0 = \epsilon_0\, q_0 + 2\sum_{i=1}^{2N-2} \epsilon_i\, q_i + \epsilon_\infty\, q_\infty \qquad \text{and} \qquad T^0{}_1 = q_0 + 2\sum_{i=1}^{2N-2} q_i + q_\infty. \tag{4.2.42}$$

The other two components $T^1{}_0$ and $T^1{}_1$ of the energy-momentum tensor are defined through the local conservation law $\partial_\mu T^\mu{}_\nu = 0$ by computing the time evolution of the densities $q_i(x)$, as discussed in section 2.2.4. This time evolution is the same as in equation (2.2.26). However, its computation is slightly different than in section 2.2.4, as we now explain. Let us consider the Poisson bracket $\{q_i(x), q_j(y)\}$. A direct computation from the bracket (4.2.8) shows that the densities q_i satisfy

$$\big\{q_i(x), q_j(y)\big\} \approx -\delta_{ij}\lambda_i\big(\partial_x q_i(x)\delta_{xy} + 2q_i(x)\delta'_{xy}\big), \qquad \text{where} \qquad \lambda_i = \begin{cases} 1 & \text{if } i = 0,\infty, \\ 1/2 & \text{if } i = 1,\cdots,2N-2. \end{cases}$$

From this equation, one easily deduces the evolution of $q_i(x)$ under the Hamiltonian flow of $\mathcal{Q}_j$, namely $\{\mathcal{Q}_j, q_i(x)\} \approx \delta_{ij}\lambda_i\,\partial_x q_i(x)$. To obtain the time evolution of $q_i(x)$, one needs to take into account the Lagrange multiplier term in the dynamics (4.2.25). One shows that this term in fact does not contribute, as the densities $q_i(x)$ are first-class and more precisely satisfy $\{\mathcal{C}(y), q_i(x)\} = \delta_{i\infty}\mathcal{C}(x)\delta'_{xy} \approx 0$. Thus, the time evolution of $q_i(x)$ is given by (2.2.26) (note that the factor 2 in front of the charges $\mathcal{Q}_i$ in the Hamiltonian for $i \in \{1,\cdots,2N-2\}$ cancels with the factor λ_i).

Using the expressions (4.2.42) of $T^0{}_0$ and $T^0{}_1$, we find

$$\begin{aligned} \partial_t T^0{}_0 &= \epsilon_0^2\,\partial_x q_0 + 2\sum_{i=1}^{2N-2} \epsilon_i^2\,\partial_x q_i + \epsilon_\infty^2\,\partial_x q_\infty \\ \partial_t T^0{}_1 &= \epsilon_0\,\partial_x q_0 + 2\sum_{i=1}^{2N-2} \epsilon_i\,\partial_x q_i + \epsilon_\infty\,\partial_x q_\infty. \end{aligned}$$

Finally, from the conservation equation $\partial_\mu T^\mu{}_\nu = 0$ we get the following components $T^1{}_0$ and $T^1{}_1$ of the energy momentum tensor:

$$T^1{}_0 = -\epsilon_0^2\, q_0 - 2\sum_{i=1}^{2N-2} \epsilon_i^2\, q_i - \epsilon_\infty^2\, q_\infty \quad \text{and} \quad T^1{}_1 = -\epsilon_0\, q_0 - 2\sum_{i=1}^{2N-2} \epsilon_i\, q_i - \epsilon_\infty q_\infty. \tag{4.2.43}$$

Classical scale invariance. From equations (4.2.42) and (4.2.43), we note that $T^\mu{}_\mu = T^0{}_0 + T^1{}_1 = 0$. This implies the classical scale invariance of the model. We shall see in section 4.3.4 that some particular limit of the model that we are constructing will also maintain this scale invariance at the quantum level and define a conformal field theory.

Relativistic invariance. Let us consider the energy-momentum tensor with both indices down obtained from equations (4.2.42) and (4.2.43) as in section 2.2.4. One then arrives at the following simple condition for the relativistic invariance of the model:

$$\epsilon_i = \pm 1, \qquad \forall\, i \in \{0,\cdots,2N-2,\infty\}, \tag{4.2.44}$$

which is the same condition found in (2.2.30) for the non-dihedral model.

4.2.4 Integrability

Lax matrix. The Lax matrix for a dihedral affine Gaudin model is defined as in equation (2.2.31) of chapter 2, *i.e.* as the ratio of $\Gamma(z)$ by $\varphi(z)$ [57]. To give an explicit description of this Lax matrix, let us determine its partial fraction decomposition. As $\Gamma(z)$ and $\varphi(z)$ have the same poles (at the points z_i and $-z_i$), of the same order, $\mathcal{L}(z)$ has poles at the zeroes of the twist function $\varphi(z)$, *i.e.* at $z=0$, $z=\pm\zeta_i$ for $i\in\{1,\cdots,2N-1\}$ and $z=\infty$. One easily checks that the residues of $\mathcal{L}(z)$ at $z=0$ and ζ_i are respectively equal to $\Gamma(0)/\varphi'(0)$ and $\Gamma(\zeta_i)/\varphi'(\zeta_i)$. Moreover, using the equivariance properties (4.2.10), one finds that the residue of $\mathcal{L}(z)$ at $z=-\zeta_i$ is equal to $\Gamma(-\zeta_i)/\varphi'(-\zeta_i)=-\sigma\big(\Gamma(\zeta_i)\big)/\varphi'(\zeta_i)$. This fixes the non-polynomial part of the partial fraction decomposition of $\mathcal{L}(z)$. To determine the polynomial part, let us study the behaviour of $\mathcal{L}(z)$ around $z=\infty$. The asymptotic expansion of the Gaudin Lax matrix $\Gamma(z,x)$ around infinity reads

$$\Gamma\left(\frac{1}{u},x\right)=u\,\mathcal{C}(x)-u^2\mathcal{B}(x)-u^3\mathcal{B}_1(x)+O(u^4)\approx -u^2\mathcal{B}(x)-u^3\mathcal{B}_1(x)+O(u^4), \tag{4.2.45}$$

where $\mathcal{B}(x)$ and $\mathcal{B}_1(x)$ are the following $\mathfrak{g}$-valued currents:

$$\mathcal{B}(x)=-\sum_{r=1}^{N}\left(z_r\mathcal{J}^{(1)}_{r,[0]}+\mathcal{J}^{(1)}_{r,[1]}\right), \tag{4.2.46a}$$

$$\mathcal{B}_1(x)=-\sum_{r=1}^{N}z_r\left(z_r\mathcal{J}^{(0)}_{r,[0]}+2\mathcal{J}^{(0)}_{r,[1]}\right). \tag{4.2.46b}$$

Moreover, using the expression (4.2.13) of the twist function, we get

$$\frac{1}{\varphi(1/u)}=\frac{1}{u^3}\left(\frac{1}{2K}+O(u^2)\right). \tag{4.2.47}$$

Using the asymptotic expansions (4.2.45) and (4.2.47), one can then express the $O(u^{-1})$ and $O(u^0)$-terms in the expansion of $\mathcal{L}(1/u)$ around $u=0$, which correspond to the linear and constant terms in the polynomial part of the partial fraction decomposition of $\mathcal{L}(z)$. In the end, we then get

$$\mathcal{L}(z,x)\approx\frac{1}{\varphi'(0)}\frac{\Gamma(0,x)}{z}+\sum_{i=1}^{2N-2}\sum_{k=0}^{1}\frac{1}{\varphi'(\zeta_i)}\frac{(-1)^k\sigma^k\big(\Gamma(\zeta_i,x)\big)}{z-(-1)^k\zeta_i}-\frac{\mathcal{B}_1(x)}{2K}-\frac{\mathcal{B}(x)}{2K}z. \tag{4.2.48}$$

Lax connection and integrability. Together with a $\mathfrak{g}^{\mathbb{C}}$-valued field $\mathcal{M}(z,x)$, $\mathcal{L}(z,x)$ forms the Lax connection of the model which satisfies the zero curvature equation (2.1.1). This was proven for general dihedral affine Gaudin models in [57]. Let us find the expression of $\mathcal{M}(z,x)$ by computing the dynamics of $\mathcal{L}(z,x)$, *i.e.* its bracket with the Hamiltonian (4.2.24). In order to do this, one has to calculate the Poisson bracket of $\mathcal{L}(z,x)$ with the charges $\mathcal{Q}_i$ and with the Lagrange multiplier. The first part of this computation is performed as in section 2.2.5. In particular, let us consider the Poisson bracket of $\mathcal{Q}(w)$ and $\mathcal{L}(z,x)$. Taking residues at $w=\zeta_i$, $i\in\{0,\ldots,2N-2,\infty\}$ gives equation (2.2.35), which we recall here for convenience:

$$\{\mathcal{Q}_i,\mathcal{L}(z,x)\}-\partial_x\mathcal{M}_i(z,x)+\big[\mathcal{M}_i(z,x),\mathcal{L}(z,x)\big]=0. \tag{4.2.49}$$

For $i\in\{0,\cdots,2N-2\}$, the fields $\mathcal{M}_i(z,x)$ are given by the following expression:

$$\mathcal{M}_i(z,x)=\frac{1}{2\varphi'(\zeta_i)}\sum_{k=0}^{1}\frac{(-1)^k\sigma^k\big(\Gamma(\zeta_i,x)\big)}{z-(-1)^k\zeta_i},$$

where in comparison to expression (2.2.5) of section 2.2.5 the $T = 2$ dihedrality of the model is now taken into account by the sum over $k \in \{0, 1\}$ and by the presence of the involutive automorphism σ. From the equivariance property (4.2.10), one finds that $\sigma\big(\Gamma(0, x)\big) = -\Gamma(0, x)$. Thus, we get in particular that $\mathcal{M}_0(z, x) = \Gamma(0)/z\varphi'(0)$. To compute $\mathcal{M}_\infty(z, x)$, we use the asymptotic expansions (4.2.45) and (4.2.47), as well as

$$\mathcal{R}^0_{\underline{\mathbf{12}}}\left(z, \frac{1}{u}\right) = u\, C^{(00)}_{\underline{\mathbf{12}}} + u^2 z\, C^{(11)}_{\underline{\mathbf{12}}} + O(u^3). \tag{4.2.50}$$

After a short computation, we get:

$$\mathcal{M}_\infty(z, x) \approx -\frac{\mathcal{B}_1(x) + z\,\mathcal{B}(x)}{2K}. \tag{4.2.51}$$

To complete the derivation of the temporal part $\mathcal{M}(z, x)$ of the Lax connection, we finally need to compute the contribution of the Lagrange multiplier μ to the dynamics of $\mathcal{L}(z, x)$. From the Poisson bracket (2.2.7), the definition (4.2.17) of the constraint and the expansion (4.2.50) we get

$$\big\{\mathcal{C}_{\underline{\mathbf{2}}}(y), \mathcal{L}_{\underline{\mathbf{1}}}(z, x)\big\} = -\big[C^{(00)}_{\underline{\mathbf{12}}}, \mathcal{L}_{\underline{\mathbf{1}}}(z, x)\big]\delta_{xy} + C^{(00)}_{\underline{\mathbf{12}}}\,\delta'_{xy}. \tag{4.2.52}$$

Thus,

$$\int_{\mathbb{D}} dy\; \kappa_{\underline{\mathbf{2}}}\big(\mu_{\underline{\mathbf{2}}}(y), \{\mathcal{C}_{\underline{\mathbf{2}}}(y), \mathcal{L}_{\underline{\mathbf{1}}}(z, x)\}\big) = -\big[\mu(x), \mathcal{L}(z, x)\big] + \partial_x \mu(x).$$

Combining all the results above, we find the following expression for $\mathcal{M}(z, x)$:

$$\mathcal{M}(z, x) \approx \frac{\epsilon_0}{\varphi'(0)}\frac{\Gamma(0, x)}{z} + \sum_{i=1}^{2N-2}\sum_{k=0}^{1}\frac{\epsilon_i}{\varphi'(\zeta_i)}\frac{(-1)^k\sigma^k\big(\Gamma(\zeta_i, x)\big)}{z - (-1)^k\zeta_i} - \epsilon_\infty\frac{\mathcal{B}_1(x)}{2K} - \epsilon_\infty\frac{\mathcal{B}(x)}{2K}z + \mu(x). \tag{4.2.53}$$

Finally, one shows from (4.2.8) that the Lax matrix $\mathcal{L}(z, x)$ satisfies the Maillet bracket, hence establishing the integrability of the models. The $\mathcal{R}$-matrix appearing in this bracket is given by the standard twisted $\mathcal{R}$-matrix (4.2.9) multiplied by the inverse of the twist function $\varphi(w)$.

Integrable local hierarchies. Let us consider the charges $\mathcal{Q}_i$, $i \in \{0, \cdots, 2N-2, \infty\}$ defined in equation (4.2.15). For $i \neq \infty$, we have:

$$\mathcal{Q}_i = -\frac{1}{2\varphi'(\zeta_i)}\int_{\mathbb{D}} \mathrm{d}x\; \kappa\big(\Gamma(\zeta_i, x), \Gamma(\zeta_i, x)\big). \tag{4.2.54}$$

Similarly, one shows that the charge $\mathcal{Q}_\infty$ admits the following weak expression:

$$\mathcal{Q}_\infty \approx -\frac{1}{2\chi'(0)}\int_{\mathbb{D}} \mathrm{d}x\; \kappa\big(\mathcal{B}(x), \mathcal{B}(x)\big). \tag{4.2.55}$$

The function $\chi(u)$ was introduced in equation (4.2.12) to describe the 1-form $\varphi(z)\mathrm{d}z$ around infinity, while the field $\mathcal{B}(x)$ can be seen as the evaluation of the 1-form $\Gamma(z, x)\mathrm{d}z$ at $z = \infty$. The above expression is then a natural generalisation for $i = \infty$ of equation (4.2.54).

As for the affine Gaudin models described in chapter 2, the models that we are describing admit integrable hierarchies of local conserved charges generalising the construction of the quadratic charges $\mathcal{Q}_i$. They consist of charges of increasing degrees whose densities are well-chosen invariant polynomials[3] of $\Gamma(\zeta_i, x)$ for $i \neq \infty$ and of $\mathcal{B}(x)$ for $i = \infty$ (only weakly in this case). We refer to [74] for details on the construction of these charges (for completeness, let us note that these local charges were first constructed in [120] for the symmetric coset sigma model, which corresponds to the model considered here for $N = 1$).

Lax connection in light-cone coordinates. In the following we will often refer to the Lax connection written in light-cone components $\mathcal{L}_\pm(z,x) = \mathcal{M}(z,x) \pm \mathcal{L}(z,x)$. Let us briefly comment on the pole structure of these two quantities. From the expressions (4.2.48) and (4.2.53) of $\mathcal{L}(z)$ and $\mathcal{M}(z)$ respectively, we observe that they contain the same terms. For the case of $\mathcal{M}(z)$, these terms are multiplied by one of the coefficients ϵ_i, $i \in \{0, \cdots, 2N-2, \infty\}$, which we recall can be either $+1$ or -1 to ensure the relativistic invariance of the model. Hence, depending on the values of these numbers, these terms will be present only in one of the two light-cone components of the Lax connection. More precisely, they will appear in the expression of $\mathcal{L}_+(z)$ if the corresponding ϵ_i is equal to $+1$ and in the expression of $\mathcal{L}_-(z)$ if ϵ_i is equal to -1.

Gauge symmetry and integrable structure. Let us now discuss how the integrable structure of the model behaves under the $G^{(0)}_{\text{diag}}$ gauge symmetry introduced in section 4.2.2 and in particular determine how the Lax connection transforms under this symmetry. From its definition (2.2.31) and the transformation (4.2.34) of the Gaudin Lax matrix, one simply finds that $\mathcal{L}(z)$ transforms as

$$\mathcal{L}(z) \longmapsto h^{-1}\mathcal{L}(z)h + h^{-1}\partial_x h. \tag{4.2.56}$$

Let us now focus on $\mathcal{M}(z)$. From (4.2.34), we obtain that the evaluations of the Gaudin Lax matrix at finite zeros of the twist function vary covariantly as $\Gamma(0) \mapsto h^{-1}\Gamma(0)h$ and $\Gamma(\zeta_i) \mapsto h^{-1}\Gamma(\zeta_i)h$. Moreover, inserting the asymptotic expansions (4.2.45) and (4.2.11) in equation (4.2.34), we get

$$\mathcal{B} \longmapsto h^{-1}\mathcal{B}h \qquad \text{and} \qquad \mathcal{B}_1 \longmapsto h^{-1}\mathcal{B}_1 h - 2K\, h^{-1}\partial_x h.$$

Combining the above results with the transformation (4.2.36) of the Lagrange multiplier μ and the expression (4.2.53) of $\mathcal{M}(z)$, one then finds that $\mathcal{M}(z)$ transforms as

$$\mathcal{M}(z) \longmapsto h^{-1}\mathcal{M}(z)h + h^{-1}\partial_t h. \tag{4.2.57}$$

Re-expressing the equations (4.2.56) and (4.2.57) in light-cone components, we finally arrive at

$$\mathcal{L}_\pm(z) \longmapsto h^{-1}\mathcal{L}_\pm(z)h + h^{-1}\partial_\pm h. \tag{4.2.58}$$

Let us make a few comments. Firstly, we note that the transformation (4.2.58) takes the form of a formal gauge transformation $\mathcal{L}_\pm(z) \mapsto \mathcal{L}^h_\pm(z)$ of the Lax connection. Such formal gauge transformations are a general feature of integrable field theories, regardless of whether they possess a gauge symmetry or not. They can be performed for any $h = h(z,x,t)$ in the group G and leave the zero curvature equation invariant: they therefore encode the non-uniqueness of the Lax connection in the integrable field theory under consideration. In the present case, this

[3]The degrees of these polynomials follow a specific pattern which depends on the underlying Lie algebra $\mathfrak{g}$ and the zero which is considered. For zeroes ζ_i, $i \in \{1, \cdots, 2N-2\}$, which are not fixed under the $\mathbb{Z}_2$-transformation $z \mapsto -z$, these degrees are given by one plus the exponents of the untwisted affine algebra of $\mathfrak{g}$. For the zeroes 0 and ∞, which are fixed under $z \mapsto -z$, only a subset of the exponents appears, which depends on the choice of automorphism σ (see [74] for more details).

field theory also possesses a $G^{(0)}_{\text{diag}}$ gauge symmetry, which encodes the presence of unphysical degrees of freedom in the model. The above computation thus shows that the action of this gauge symmetry, with local parameter $h(x,t) \in G^{(0)}$, on the Lax connection coincides with a formal gauge transformation with parameter h. Since such a transformation preserves the zero curvature equation, which is a reformulation of the equations of motion of the model, this provides an alternative check of the invariance of these equations of motion under the $G^{(0)}_{\text{diag}}$ gauge symmetry.

Moreover, it is a standard result that the conserved charges extracted from the monodromy of the Lax matrix are invariant under formal gauge transformations. As a consequence, the above results show that these charges are also invariant with respect to the $G^{(0)}_{\text{diag}}$ gauge symmetry of the model.

Recall that in addition to these charges extracted from the monodromy matrix, the model also admits an infinite number of local conserved charges in involution (see above). The latter are also gauge invariant, as was proven in general in [74]. This fact can also be checked directly using the results derived above. Indeed, we have shown that the currents $\Gamma(0,x)$, $\Gamma(\zeta_i,x)$ and $\mathcal{B}(x)$ are covariant under gauge transformations. As mentioned earlier in this section, the densities of the local conserved charges are obtained by taking conjugacy invariant polynomials of these currents, which are then gauge invariant.

4.2.5 The panorama of the models

Let us end this section by briefly discussing the panorama of integrable models constructed above. A model in this class first depends on the number of sites N of the underlying AGM, which fixes its target space $G^N/G^{(0)}_{\text{diag}}$. Moreover, following the different steps of the construction of the model, one sees that it is characterised by the following parameters:

- the positions $z_1, \cdots, z_N$ of the sites ;
- the levels $\ell_{1,0}, \cdots, \ell_{N,0}$ and $\ell_{1,1}, \cdots, \ell_{N,1}$;
- the coefficients $\epsilon_0, \cdots, \epsilon_{2N-2}, \epsilon_\infty$ entering the definition of the Hamiltonian (4.2.16).

In particular, the parameters in the first two bullets are encoded in the twist function (4.2.2). As explained in section 4.2.3, the coefficients ϵ_i cannot take arbitrary values as they are required to be either $+1$ or -1 to ensure the relativistic invariance of the model. Recall also that the levels $\ell_{r,0}$ are subject to the first-class condition (4.2.3), which imposes one relation between them. Moreover, one shows that the model obtained by considering a dilation of the spectral parameter $z \mapsto az$ is equivalent to the inital model: this induces a redundancy among the parameters of the model, which can be fixed for instance by setting one of the position z_r to a fixed value, say $z_1 = 1$. Thus, the model depends in the end on $3N-2$ continuous free parameters.

Recall from section 4.2.2 that the definition of the Hamiltonian of the model involves the zeroes $\{0, \infty, \zeta_1, \cdots, \zeta_{2N-2}\}$ of the twist function. In general, expressing these zeroes in terms of the positions z_r and the levels $\ell_{r,p}$ is a complicated, if not impossible, task, as it requires solving a polynomial equation of degree $2N-2$. To circumvent this difficulty, one can choose another set of parameters of the model, given by:

- the positions $z_2, \cdots, z_N$ of the sites (fixing $z_1 = 1$) ;
- the constant term K in the twist function (4.2.13) ;

- the zeroes $\zeta_1, \cdots, \zeta_{2N-2}$ of the twist function and the corresponding coefficients $\epsilon_i \in \{+1,-1\}$;
- the coefficients ϵ_0 and ϵ_∞ in $\{+1,-1\}$.

This set of parameters is encoded in the choice of the twist function in its factorised form (4.2.13) (except for the discrete parameters $\epsilon_i = \pm 1$). In particular, if these are chosen as the defining parameters of the model, the levels $\ell_{r,p}$ are defined in terms of this expression of the twist function as the residues

$$\ell_{r,0} = 2 \operatorname*{res}_{z=z_r} \varphi(z) \mathrm{d}z \qquad \text{and} \qquad \ell_{r,1} = 2 \operatorname*{res}_{z=z_r} (z-z_r)\varphi(z)\mathrm{d}z.$$

Note that in this parametrisation, the first-class condition (4.2.3) is automatically satisfied, as the factorised form (4.2.13) of the twist function ensures that $\varphi(z)\mathrm{d}z$ is regular at $z=\infty$. The $3N-2$ continuous parameters listed above are thus unconstrained.

Let us end this section by discussing briefly the simplest example in this panorama of models, the model with one site, *i.e.* $N=1$. This model was first considered in [57], where it was shown that it coincides with the standard sigma model on the symmetric space $G/G^{(0)}$. In the parametrisation discussed above, this model possesses one site with fixed position $z_1 = 1$ and no zeroes ζ_i (0 and ∞ are the only zeroes of the twist function). The only continuous free parameter of the model is then the constant term K. The twist function simply reads

$$\varphi(z) = \frac{2Kz}{(z^2-1)^2}. \tag{4.2.59}$$

We fix the coefficients ϵ_i to[4] $\epsilon_\infty = +1$ and $\epsilon_0 = -1$. The phase space of the model consists of canonical fields on a single copy of T^*G, described by the two fields $g(x)$ and $X(x)$ (as $N=1$, we drop the indices r). A direct computation shows that the naive Hamiltonian of the model (4.2.16) is given in this case by

$$\mathcal{H}_{N=1} = \frac{1}{2}\int_{\mathbb{D}} \mathrm{d}x \, \left(\frac{1}{K}\kappa\big(X^{(1)},X^{(1)}\big) + K\kappa\big(j^{(1)},j^{(1)}\big) + 2\kappa\big(X^{(0)},j^{(0)}\big) \right), \tag{4.2.60}$$

where $j = g^{-1}\partial_x g$ as above. As expected, this coincides with the Hamiltonian of the symmetric space sigma model on $G/G^{(0)}$, formulated as a model on G with a $G^{(0)}$ gauge symmetry. In the present case, the constraint associated with this gauge symmetry simply reads $X^{(0)} \approx 0$.

4.3 Lagrangian formulation of the models with two copies

The Lagrangian formulation of the models we are concerned with in this chapter consists of field theories with fundamental fields $g_r(x,t)$, $r \in \{1,\cdots,N\}$, taking values in G. We will obtain these Lagrangian theories by performing an inverse Legendre transform of the models constructed in section 4.2 in the Hamiltonian formulation. In order to make the computation of the inverse Legendre transform more explicit, we will restrict to the case of two copies, *i.e.* we will fix $N=2$.

[4] The choice $\epsilon_\infty = -1$ and $\epsilon_0 = +1$ would simply lead to the opposite Hamiltonian, while the choices $\epsilon_\infty = \epsilon_0 = \pm 1$ would lead to the Hamiltonian coinciding with (plus or minus) the momentum of the theory, as one can see from equation (4.2.41).

Before that, let us briefly describe, as a simple illustration, the model with only one copy. The model is described in its Lagrangian formulation by a unique G-valued field $g(x,t)$. Performing the inverse Legendre transform of the Hamiltonian (4.2.60), one finds that its action takes the form:

$$S_{N=1}[g] = \frac{K}{2} \iint_{\mathbb{D}\times\mathbb{R}} \mathrm{d}x\,\mathrm{d}t\; \kappa\big(j_+^{(1)}, j_-^{(1)}\big),$$

where $j_\pm = g^{-1}\partial_\pm g$. As expected, this is the action of the standard symmetric space sigma model on $G/G^{(0)}$ in its gauged formulation. One easily checks that this action is invariant under the gauge transformation $g(x,t) \mapsto g(x,t)h(x,t)$ for $h(x,t) \in G^{(0)}$.

Let us return to the models with $N = 2$. Before proceeding to the computation of the inverse Legendre transform, let us describe the parameters of these models. From the discussion in section 4.2.5, they depend on four continuous parameters: the position z_2 of the second site (having fixed the position of the first site to $z_1 = 1$), the global factor in the twist function K, and the zeroes ζ_1 and ζ_2. In the following we will rename $z_2 = x$ to avoid unnecessary indices, although we will sometimes use the notation z_1 and z_2 so that some formulae assume a more compact form. In addition to these continuous parameters, the models are characterised by the choice of four discrete coefficients $(\epsilon_0, \epsilon_1, \epsilon_2, \epsilon_\infty)$ in $\{-1, +1\}$. We will fix these coefficients to the values[5] $\epsilon_0 = -1, \epsilon_1 = -1, \epsilon_2 = +1$ and $\epsilon_\infty = +1$. Motivated by this choice and for future convenience, we will rename ζ_1 as ζ_- and ζ_2 as ζ_+.

4.3.1 Lagrangian expression of the momentum fields

In order to perform the inverse Legendre transform of the models, we first need to express their momentum fields, encoded in the fields X_r introduced in the previous section, in terms of the time derivatives of the coordinate fields g_r, encoded in the temporal Maurer-Cartan current $j_{t,r} = g_r^{-1}\partial_t g_r$.

For that, let us calculate the dynamics of the fields g_r, given by the Poisson bracket of g_r with the total Hamiltonian introduced in section 4.2.2. We start by seeking a more explicit expression of the naive Hamiltonian (4.2.16) in terms of the fields j_r and

$$Y_r = X_r + \frac{\ell_{r,0}}{2} W_r,$$

which we introduce for future convenience. After a few manipulations, one rewrites it in the form

$$\mathcal{H} = \sum_{r,s=1}^{2} \sum_{k=0}^{1} a_{rs}^{(k)} \int_{\mathbb{D}} \mathrm{d}x\; \kappa\left(j_r^{(k)}, j_s^{(k)}\right) + b_{rs}^{(k)} \int_{\mathbb{D}} \mathrm{d}x\; \kappa\left(Y_r^{(k)}, j_s^{(k)}\right) + c_{rs}^{(k)} \int_{\mathbb{D}} \mathrm{d}x\; \kappa\left(Y_r^{(k)}, Y_s^{(k)}\right), \quad (4.3.1)$$

where the coefficients $a_{rs}^{(k)}, b_{rs}^{(k)}$ have slightly long expressions and are hence written in appendix 4.A, while the coefficients $c_{rs}^{(k)}$ are given by:

$$c_{rs}^{(0)} = \frac{\zeta_-^2}{2K} \frac{z_{\bar r}^2 z_{\bar s}^2/\zeta_-^2 - z_{\bar r}^2 - z_{\bar s}^2 + \zeta_-^2}{\zeta_-^2 - \zeta_+^2} \qquad \text{and} \qquad c_{rs}^{(1)} = \frac{z_r z_s}{2K} \frac{z_{\bar r}^2 z_{\bar s}^2/\zeta_+^2 - z_{\bar r}^2 - z_{\bar s}^2 + \zeta_-^2}{\zeta_-^2 - \zeta_+^2}, \quad (4.3.2)$$

where we introduced the notation $\bar r = 3 - r$, $r = 1, 2$.

[5] Other choices would give either equivalent models, up to a redefinition of the parameters, or models for which the inverse Legendre transform is singular and thus which do not possess a Lagrangian formulation.

The form (4.3.1) of the Hamiltonian allows us to calculate easily what $j_{t,r}$ reads in terms of the Hamiltonian fields j_r and Y_r. From the Poisson brackets (4.2.5), (3.2.2) and (3.2.5), as well as the identity (4.2.1), one shows that

$$g_r^{-1}\{\mathcal{H}, g_r\} = \sum_{s=1}^{2}\sum_{k=0}^{1} b_{rs}^{(k)} j_s^{(k)} + 2c_{rs}^{(k)} Y_s^{(k)}.$$

Hence, taking into account the form (4.2.24) of the total Hamiltonian and the Poisson bracket (4.2.28), we have:

$$j_{t,r} \approx g_r^{-1}\{\mathcal{H}_T, g_r\} \approx \sum_{s=1}^{2}\sum_{k=0}^{1} b_{rs}^{(k)} j_s^{(k)} + 2c_{rs}^{(k)} Y_s^{(k)} + \mu. \tag{4.3.3}$$

The above equation is a linear system that can be projected into the gradings and solved to express the fields $Y_r^{(k)}$ in terms of the currents $j_{t,r}$. However, we take a different path to eliminate the Lagrange multiplier μ. For the grading zero, subtracting the equations for $r = 1$ and $r = 2$, we arrive at:

$$2\sum_{s=1}^{2}\left(c_{1s}^{(0)} - c_{2s}^{(0)}\right) Y_s^{(0)} \approx j_{t,1}^{(0)} - j_{t,2}^{(0)} - \sum_{s=1}^{2}\left(b_{1s}^{(0)} - b_{2s}^{(0)}\right) j_s^{(0)}.$$

In order to obtain a second equation independent of the Lagrange multiplier μ, we make use of the constraint (4.2.18), rewritten in the form:

$$Y_1^{(0)} + Y_2^{(0)} \approx -\frac{\ell_{1,0}}{2} j_1^{(0)} - \frac{\ell_{2,0}}{2} j_2^{(0)}.$$

Altogether, the solution for the grading zero is given by:

$$Y_r^{(0)} \approx \frac{1}{2\sum_{s=1}^{2}\left(c_{ss}^{(0)} - c_{s\bar{s}}^{(0)}\right)}\left(j_{t,r}^{(0)} - j_{t,\bar{r}}^{(0)} - \sum_{s=1}^{2}\left(b_{rs}^{(0)} - b_{\bar{r}s}^{(0)} - \ell_{s,0}\left(c_{r\bar{r}}^{(0)} - c_{\bar{r}\bar{r}}^{(0)}\right)\right) j_s^{(0)}\right). \tag{4.3.4}$$

For the grading one, one has the following equations:

$$j_{t,r}^{(1)} \approx \sum_{s=1}^{2} b_{rs}^{(1)} j_s^{(1)} + 2c_{rs}^{(1)} Y_s^{(1)}.$$

If we rename the components of the inverse matrix of $(c^{(1)})_{rs} = c_{rs}^{(1)}$ as $\bar{c}_{rs}^{(1)} = (c^{(1)})_{rs}^{-1}$, the solution then reads:

$$Y_r^{(1)} \approx \frac{1}{2}\sum_{s=1}^{2} \bar{c}_{rs}^{(1)}\left(j_{t,s}^{(1)} - \sum_{t=1}^{2} b_{st}^{(1)} j_t^{(1)}\right). \tag{4.3.5}$$

4.3.2 Action of the model

Inverse Legendre transform. Using the definition of X_r in terms of the canonical fields (see for instance [59] for more details), the action of the model is given by the following inverse Legendre transform[6]:

$$S[g_1, g_2] = \sum_{r=1}^{2}\iint \mathrm{d}x\,\mathrm{d}t\;\kappa\left(X_r, j_{t,r}\right) - \int \mathrm{d}t\;\mathcal{H}.$$

[6] As we are now working in the Lagrangian formulation, in which the constraint always holds, we drop the distinction between weak and strong equalities. In particular, one can use the naive Hamiltonian (and not the total one) to compute the inverse Legendre transform.

In terms of the fields Y_r introduced in the previous section, we can rewrite the above equation as

$$S[g_1, g_2] = \sum_{r=1}^{2} \iint \mathrm{d}x\, \mathrm{d}t\ \kappa\left(Y_r, j_{t,r}\right) - \int \mathrm{d}t\ \mathcal{H} - \sum_{r=1}^{2} \frac{\ell_{r,0}}{2}\, I_{\mathrm{WZ}}\left[g_r\right],$$

where the Wess-Zumino terms of g_r have now appeared, using the definition (3.2.4). To obtain the explicit expression of the action, we now have to replace the Hamiltonian fields Y_r by their Lagrangian expression, given by equations (4.3.4) and (4.3.5), including in the Hamiltonian $\mathcal{H}$, using its expression (4.3.1). Let us introduce the light-cone components of the Maurer-Cartan currents $j_{\pm,r} = g_r^{-1}\partial_\pm g_r = j_{t,r} \pm j_r$. After some manipulations, one finds

$$S[g_1, g_2] = \sum_{r,s=1}^{2} \iint \mathrm{d}x\, \mathrm{d}t \left(\rho^{(0)}_{rs}\, \kappa\left(j^{(0)}_{+,r}, j^{(0)}_{-,s}\right) + \rho^{(1)}_{rs}\, \kappa\left(j^{(1)}_{+,r}, j^{(1)}_{-,s}\right) \right) + \AE\, I_{\mathrm{WZ}}\left[g_1\right] - \AE\, I_{\mathrm{WZ}}\left[g_2\right]. \tag{4.3.6}$$

In terms of the defining parameters of the model K, x, ζ_+ and ζ_-, the coefficients corresponding to the grading zero in this action are given by

$$\rho^{(0)}_{11} = \rho^{(0)}_{22} = \frac{K}{2} \frac{\zeta_-^2 - \zeta_+^2}{(1-x^2)^2}, \qquad \rho^{(0)}_{12} = K \frac{\left(1-\zeta_+^2\right)\left(x^2-\zeta_-^2\right)}{\left(1-x^2\right)^3}, \qquad \rho^{(0)}_{21} = -K \frac{\left(1-\zeta_-^2\right)\left(x^2-\zeta_+^2\right)}{\left(1-x^2\right)^3}, \tag{4.3.7a}$$

while the ones corresponding to the grading one are

$$\begin{aligned} \rho^{(1)}_{11} &= \frac{K}{2} \frac{\left(1-2\zeta_+^2+\zeta_-^2\zeta_+^2\right)}{\left(1-x^2\right)^2}, \qquad \rho^{(1)}_{12} = K \frac{x\left(1-\zeta_+^2\right)\left(x^2-\zeta_-^2\right)}{\left(1-x^2\right)^3}, \\ \rho^{(1)}_{21} &= -K \frac{\left(1-\zeta_-^2\right)\left(x^2-\zeta_+^2\right)}{x\left(1-x^2\right)^3}, \qquad \rho^{(1)}_{22} = \frac{K}{2} \frac{\left(x^4-2\zeta_+^2x^2+\zeta_-^2\zeta_+^2\right)}{x^2\left(1-x^2\right)^2}. \end{aligned} \tag{4.3.7b}$$

Finally, the Wess-Zumino coefficient $\AE$ is defined as $\AE = -\ell_{1,0}/2 = \ell_{2,0}/2$ and explicitly reads

$$\AE = K \frac{2x^2 + 2\zeta_-^2\zeta_+^2 - (1+x^2)(\zeta_-^2+\zeta_+^2)}{(1-x^2)^3}. \tag{4.3.7c}$$

Gauge symmetry. Let us check explicitly that the action (4.3.6) is invariant under the gauge transformation $g_r(x,t) \mapsto g_r(x,t)h(x,t)$ with $h(x,t) \in G^{(0)}$, as expected from the Hamiltonian construction. Under this transformation, the Wess-Zumino terms change according to the Polyakov-Wiegmann formula [121]:

$$I_{\mathrm{WZ}}\left[g_r h\right] = I_{\mathrm{WZ}}\left[g_r\right] + I_{\mathrm{WZ}}\left[h\right] - \frac{1}{2} \iint \mathrm{d}x\, \mathrm{d}t \left[\kappa\left(j^{(0)}_{+,r}, (\partial_- h)h^{-1}\right) - \kappa\left(j^{(0)}_{-,r}, (\partial_+ h)h^{-1}\right) \right].$$

Moreover, the light-cone components of the Maurer-Cartan currents transform as:

$$j^{(0)}_{\pm,r} \longmapsto h^{-1}\big(j^{(0)}_{\pm,r} + (\partial_\pm h)h^{-1}\big)h \qquad \text{and} \qquad j^{(1)}_{\pm,r} \longmapsto h^{-1} j^{(1)}_{\pm,r} h\,.$$

It is then clear that in an action of the form (4.3.6) with general coefficients $\rho^{(k)}_{rs}$ the terms of grading one are invariant under this gauge transformation. The variation of the action thus only contains terms in the grading zero, coming from the variation of the factors $\kappa\big(j^{(0)}_{+,r}, j^{(0)}_{-,s}\big)$ and of the Wess-Zumino terms. Computing explicitly this variation, one finds that gauge invariance is verified if and only if the following conditions are satisfied:

$$\rho^{(0)}_{11} + \rho^{(0)}_{12} - \frac{\AE}{2} = \rho^{(0)}_{12} + \rho^{(0)}_{22} - \frac{\AE}{2} = \rho^{(0)}_{21} + \rho^{(0)}_{22} + \frac{\AE}{2} = \rho^{(0)}_{11} + \rho^{(0)}_{21} + \frac{\AE}{2} = 0\,. \tag{4.3.8}$$

The above relations are indeed all identically satisfied for the choice of coefficients (4.3.7).

Note that one can also rewrite the action (4.3.6) in a manifestly gauge invariant way. Using the Polyakov-Wiegmann identity [121] to make the Wess-Zumino term $I_{\text{WZ}}\big[g_1 g_2^{-1}\big]$ appear, as well as the relations (4.3.8), one finds

$$\begin{aligned} S[g_1,g_2] \;&=\; \rho^{(0)}_{11} \iint \mathrm{d}x\,\mathrm{d}t\;\kappa\left(j^{(0)}_{+,1}-j^{(0)}_{+,2},\, j^{(0)}_{-,1}-j^{(0)}_{-,2}\right) + \hbar\, I_{\text{WZ}}\big[g_1 g_2^{-1}\big] \\ &\quad + \sum_{r,s=1}^{2}\left(\rho^{(1)}_{rs}-\frac{\hbar}{2}\epsilon_{rs}\right)\iint \mathrm{d}x\;\mathrm{d}t\,\kappa\left(j^{(1)}_{+,r},j^{(1)}_{-,s}\right), \end{aligned}$$

where $\epsilon_{12}=-\epsilon_{21}=1$ and $\epsilon_{11}=\epsilon_{22}=0$. As announced, this form of the action is manifestly invariant under a gauge transformation $g_r(x,t)\mapsto g_r(x,t)h(x,t)$ with $h(x,t)\in G^{(0)}$. Indeed, the field $g_1 g_2^{-1}$ is itself invariant and the currents $j^{(0)}_{\pm,1}-j^{(0)}_{\pm,2}$ and $j^{(1)}_{\pm,r}$ are covariant, *i.e.* they transform as

$$j^{(0)}_{\pm,1}-j^{(0)}_{\pm,2}\longmapsto h^{-1}\big(j^{(0)}_{\pm,1}-j^{(0)}_{\pm,2}\big)h \qquad \text{and} \qquad j^{(1)}_{\pm,r}\longmapsto h^{-1}j^{(1)}_{\pm,r}h\,.$$

Global symmetries. Let us briefly discuss the global symmetries of the model (4.3.6), which are given by the left $(G\times G)$-translations on g_1 and g_2:

$$(g_1,g_2)\longmapsto(f_1g_1,f_2g_2), \qquad (f_1,f_2)\in G\times G. \tag{4.3.9}$$

Indeed, these translations leave the Maurer-Cartan currents $j_{\pm,r}=g_r^{-1}\partial_\pm g_r$ invariant and also preserve the Wess-Zumino terms $I_{\text{WZ}}\big[g_r\big]$. Thus, they define global symmetries of the action (4.3.6). Making use of equation (4.3.8), the conserved Noether currents associated to these symmetries read

$$\begin{aligned} \mathcal{K}_{+,r} &= \sum_{s=1}^{2} g_r\left(\rho^{(0)}_{sr}\big(1-\delta_{sr}\big)j^{(0)}_{+,s}+\big(\rho^{(1)}_{sr}-\rho^{(0)}_{sr}\delta_{sr}\big)j^{(1)}_{+,s}\right)g_r^{-1}, \\ \mathcal{K}_{-,r} &= \sum_{s=1}^{2} g_r\left(\rho^{(0)}_{rs}\big(1-\delta_{rs}\big)j^{(0)}_{-,s}+\big(\rho^{(1)}_{rs}-\rho^{(0)}_{rs}\delta_{rs}\big)j^{(1)}_{-,s}\right)g_r^{-1}. \end{aligned}$$

These currents satisfy the conservation equation $\partial_+\mathcal{K}_{-,r}+\partial_-\mathcal{K}_{+,r}=0$. Let us also note that they are gauge-invariant under the $G^{(0)}_{\text{diag}}$ gauge symmetry $g_r(x,t)\mapsto g_r(x,t)h(x,t)$ of the model.

Reformulation of the action. As detailed in appendix 4.B, the coefficients $\rho^{(k)}_{rs}$ and $\hbar_r$ defined in equation (4.3.7) can be re-expressed as residues of well-chosen functions (for the non-dihedral sigma models on G^N defined in [58, 59], a similar result was pointed out in [85]). This allows us to reformulate the action (4.3.6) in the following remarkably simple way:

$$S=\sum_{r=1}^{2} S_{\text{WZW},\hbar_r}[g_r]-4K\iint \mathrm{d}x\,\mathrm{d}t\sum_{r,s=1}^{2}\mathop{\text{res}}_{w=z_s}\mathop{\text{res}}_{z=z_r}\kappa_{\underline{\mathbf{12}}}\Big(\mathcal{R}^0_{\underline{\mathbf{12}}}(w,z)\varphi_+(z)\varphi_-(w),\,j_{+,r\underline{\mathbf{1}}}\,j_{-,s\underline{\mathbf{2}}}\Big), \tag{4.3.11}$$

where $\mathcal{R}^0_{\underline{\mathbf{12}}}$ is the $\mathcal{R}$-matrix (4.2.9) underlying the integrable structure of the model, $S_{\text{WZW},\hbar}[g]$ is the Wess-Zumino-Witten action

$$S_{\text{WZW},\hbar}[g]=\frac{\hbar}{2}\iint \mathrm{d}x\,\mathrm{d}t\;\kappa\left(g^{-1}\partial_+g,g^{-1}\partial_-g\right)+\hbar\, I_{\text{WZ}}\big[g\big] \tag{4.3.12}$$

and $\varphi_\pm(z)$ are functions defined as

$$\varphi_+(z) = \frac{z^2-\zeta_+^2}{(z^2-z_1^2)(z^2-z_2^2)} \qquad \text{and} \qquad \varphi_-(z) = \frac{z(z^2-\zeta_-^2)}{(z^2-z_1^2)(z^2-z_2^2)}. \tag{4.3.13}$$

In particular, note that the reformulation (4.3.11) of the action does not involve an explicit sum over the grading index $k=0,1$ as in the original expression (4.3.6). As explained in the appendix 4.B, this graded structure, and thus the choice of automorphism σ, is accounted for in the $\mathcal{R}$-matrix $\mathcal{R}^0_{\underline{\mathbf{12}}}$.

Conjectured generalisations. Having derived equation (4.3.11), it is natural to formulate conjectures about generalisations of the models considered here. For instance, we expect a similar expression to hold for the models on $G^N/G^{(0)}_{\rm diag}$ with arbitrary N constructed in the Hamiltonian formalism in section 4.2. More generally, we conjecture that it also holds for models on $G^N/G^{(0)}_{\rm diag}$ with arbitrary N and where the subalgebra $\mathfrak{g}^{(0)}$ is the grading zero subspace of a $\mathbb{Z}_T$-gradation with arbitrary T, generalising the case $T=2$ considered here.

Let us be more precise about this conjecture. For $N=1$, the model on the $\mathbb{Z}_T$-coset $G/G^{(0)}$ for arbitrary T was constructed in [122] and was identified with a realisation of D_{2T}-dihedral affine Gaudin model in [57], based on the Hamiltonian analysis carried out in [123]. Although the generalisations of this sigma model on cosets $G^N/G^{(0)}_{\rm diag}$ with arbitrary N have not been considered before in the literature, we expect the procedure of section 4.2 to readily generalise to the construction of such models, using a D_{2T}-dihedral affine Gaudin model [57] instead of a D_4-dihedral model. In this case, the twist function of the model would read[7]

$$\varphi(z) = KT\frac{z^{T-1}\prod_{i=1}^{2N-2}(z^T-\zeta_i^T)}{\prod_{r=1}^{N}(z^T-z_r^T)^2}, \tag{4.3.14}$$

in terms of its zeroes $\zeta_1,\cdots,\zeta_{2N-2}$ and poles $z_1,\cdots,z_N$. One can then factorise this twist function[8] as $\varphi(z)=TK\varphi_+(z)\varphi_-(z)$, similarly to equation (4.B.1) for $T=2$, with

$$\varphi_+(z) = \frac{\prod_{i=N}^{2N-2}\left(z^T-\zeta_i^T\right)}{\prod_{r=1}^{N}\left(z^T-z_r^T\right)} \qquad \text{and} \qquad \varphi_-(z) = \frac{z^{T-1}\prod_{i=1}^{N-1}\left(z^T-\zeta_i^T\right)}{\prod_{r=1}^{N}\left(z^T-z_r^T\right)}.$$

We then conjecture that the action of the model is given by

$$S = \sum_{r=1}^N S_{\text{WZW},\,\Bbbk_r}[g_r] - \frac{KT^3}{2}\iint \mathrm{d}x\,\mathrm{d}t \sum_{r,s=1}^N \mathop{\rm res}_{w=z_s}\mathop{\rm res}_{z=z_r} \kappa_{\underline{\mathbf{12}}}\Big(\mathcal{R}^0_{\underline{\mathbf{12}}}(w,z)\varphi_+(z)\varphi_-(w), j_{+,r\underline{\mathbf{1}}}\, j_{-,s\underline{\mathbf{2}}}\Big), \tag{4.3.15}$$

where $\Bbbk_r = -\frac{T}{2}\mathop{\rm res}_{z=z_r}\varphi(z)\mathrm{d}z$ and $\mathcal{R}^0$ now denotes the $\mathbb{Z}_T$-graded $\mathcal{R}$-matrix which underlies the integrable structure of D_{2T}-dihedral affine Gaudin models [57], namely

$$\mathcal{R}^0_{\underline{\mathbf{12}}}(w,z) = \sum_{k=0}^{T-1}\frac{w^k z^{T-1-k}}{z^T-w^T}\pi^{(k)}_{\underline{\mathbf{1}}}C_{\underline{\mathbf{12}}},$$

with $\pi^{(k)}$, $k\in\{0,\cdots,T-1\}$, the projections along the grading $\mathfrak{g}=\bigoplus_{k=0}^{T-1}\mathfrak{g}^{(k)}$.

[7] The equivariance condition (4.2.10) is then replaced by $\varphi(\omega z)=\omega^{-1}\varphi(z)$, where $\omega=\exp(2i\pi/T)$.

[8] As for the case $T=2$ treated in section 4.2, we expect such a separation of the zeroes of $\varphi(z)$ in two sets $\{0,\zeta_1,\cdots,\zeta_{N-1}\}$ and $\{\zeta_N,\cdots,\zeta_{2N-2},\infty\}$ to come naturally from the relativistic invariance of the model, which requires the coefficients ϵ_i, $i\in\{0,1,\cdots,2N-2,\infty\}$, in the Hamiltonian of the model to be equal to either -1 or $+1$.

As mentioned above, for $N = 1$ and arbitrary T, the corresponding integrable model on the $\mathbb{Z}_T$-coset $G/G^{(0)}$ has been constructed in [122]: we have checked that the action of this model can indeed be reformulated as in (4.3.15). Moreover, for the case of arbitrary N and $T = 1$, the results of [85] show that the action of the model is also given by (4.3.15), with $\mathcal{R}^0_{\underline{\mathbf{12}}}(z, w)$ the standard non-twisted $\mathcal{R}$-matrix $C_{\underline{\mathbf{12}}}/(w - z)$. Finally, we have checked this conjecture by direct computation for all cases with $N \leq 3$ and $T \leq 3$.

4.3.3 Lax connection in the Lagrangian formulation

From the equations (4.2.48) and (4.2.53), the Lax connection can be written in terms of the fields j_r, Y_r and μ. Moreover, from equation (4.3.3), we have:

$$\mu \approx j^{(0)}_{t,r} - \sum_{s=1}^{2} b^{(0)}_{rs} j^{(0)}_s + 2c^{(0)}_{rs} Y^{(0)}_s.$$

We can then express the Lax connection solely in terms of the fields j_r and Y_r. Inserting equations (4.3.4) and (4.3.5), we finally get the Lagrangian expression of the Lax connection. In terms of the light-cone currents $j_{\pm,r}$, it reads:

$$\mathcal{L}_\pm(z) = \sum_{r=1}^{2} \sum_{k=0}^{1} \eta^{(k)}_{\pm,r}(z) j^{(k)}_{\pm,r}, \tag{4.3.16}$$

where

$$\eta^{(0)}_{\pm,1}(z) = \frac{(z^2 - x^2)\left(1 - \zeta_\pm^2\right)}{(z^2 - \zeta_\pm^2)\,(1 - x^2)}, \qquad \eta^{(1)}_{\pm,1}(z) = z^{\pm 1}\, \eta^{(0)}_{\pm,1}(z), \tag{4.3.17}$$

$$\eta^{(0)}_{\pm,2}(z) = \frac{(z^2 - 1)\left(x^2 - \zeta_\pm^2\right)}{(z^2 - \zeta_\pm^2)\,(x^2 - 1)}, \qquad \eta^{(1)}_{\pm,2}(z) = \left(\frac{z}{x}\right)^{\pm 1} \eta^{(0)}_{\pm,2}(z). \tag{4.3.18}$$

In particular, we note as an observation that $\eta^{(k)}_{\pm,s}(z_r) = \delta_{rs}$ (where we recall that $z_1 = 1$ and $z_2 = x$) and therefore

$$\mathcal{L}_\pm(z_r) = j_{\pm,r}. \tag{4.3.19}$$

4.3.4 A limit of the model

Definition of the limit. Let us recall that the model with two copies introduced above depends on the four continuous real parameters x, K, ζ_+ and ζ_-. In this section, we will describe the simple form that this model assumes after taking a particular limit of these parameters. In particular, this limit will be our starting point in section 4.4. We start by considering the following reparametrisation of x, K, ζ_+ and ζ_- in terms of four new parameters α, λ_1, λ_2 and λ:

$$x = \frac{1}{\alpha}, \qquad K = \frac{\lambda_2^2}{\alpha^2}, \qquad \zeta_+ = \frac{\lambda_1}{\lambda}, \qquad \zeta_- = \frac{\lambda}{\lambda_2 \alpha}. \tag{4.3.20}$$

We then define the limit we will be interested in by taking $\alpha \to 0$ while keeping the other parameters λ_1, λ_2 and λ fixed.

Action. Let us look at how the action of the model simplifies in this limit. From their expression (4.3.7), we obtain that the coefficients $\rho_{rs}^{(k)}$ and $\hbar$ simply become:

$$\rho_{11}^{(0)} = \rho_{22}^{(0)} = \frac{\lambda^2}{2}, \qquad \rho_{12}^{(0)} = \rho_{12}^{(1)} = \rho_{21}^{(1)} = 0, \qquad \rho_{21}^{(0)} = -\hbar = -\lambda^2, \qquad \rho_{11}^{(1)} = \frac{\lambda_1^2}{2}, \qquad \rho_{22}^{(1)} = \frac{\lambda_2^2}{2}.$$

Writing the action explicitly, we thus have

$$S[g_1,g_2] = \iint \mathrm{d}x\,\mathrm{d}t \sum_{r=1}^{2} \left(\frac{\lambda^2}{2}\,\kappa\left(j_{+,r}^{(0)}, j_{-,r}^{(0)}\right) + \frac{\lambda_r^2}{2}\,\kappa\left(j_{+,r}^{(1)}, j_{-,r}^{(1)}\right)\right) - \lambda^2\,\kappa\left(j_{+,2}^{(0)}, j_{-,1}^{(0)}\right) \qquad (4.3.21)$$
$$+\,\lambda^2\, I_{\mathrm{WZ}}\big[g_1\big] - \lambda^2\, I_{\mathrm{WZ}}\big[g_2\big].$$

Lax connection. Let us now turn to the Lax connection. Taking the limit on the coefficients $\eta_\pm(z)$ defined in (4.3.17) and reinserting in the expression (4.3.16) of the Lax connection, we get:

$$\mathcal{L}_+(z) = \frac{1}{\lambda^2 z^2 - \lambda_1^2}\left(\left(\lambda^2 - \lambda_1^2\right)\left(j_{+,1}^{(0)} + z\, j_{+,1}^{(1)}\right) + \lambda^2\left(z^2-1\right) j_{+,2}^{(0)}\right), \qquad \mathcal{L}_-(z) = j_{-,1}^{(0)} + \frac{j_{-,1}^{(1)}}{z}. \qquad (4.3.22)$$

One can check that the zero curvature equation for this Lax connection actually does not encode all the equations of motion of the model. To circumvent this difficulty, let us also consider the limit of $\mathcal{L}_\pm(z/\alpha)$, which we will denote as $\widetilde{\mathcal{L}}_\pm(z)$ (by construction, $\widetilde{\mathcal{L}}_\pm(z)$ also satisfies a zero curvature equation). A direct computation shows that

$$\widetilde{\mathcal{L}}_+(z) = j_{+,2}^{(0)} + z\, j_{+,2}^{(1)}, \qquad \widetilde{\mathcal{L}}_-(z) = \frac{1}{z^2\lambda_2^2 - \lambda^2}\left(\lambda^2\left(z^2-1\right) j_{-,1}^{(0)} + \left(\lambda_2^2 - \lambda^2\right)\left(z^2 j_{-,2}^{(0)} + z\, j_{-,2}^{(1)}\right)\right). \qquad (4.3.23)$$

The combined zero curvature equations of $\mathcal{L}_\pm(z)$ and $\widetilde{\mathcal{L}}_\pm(z)$ are equivalent to all the equations of motion of the model.

Additional symmetry. For this paragraph, we will suppose that the pair $\big(G, G^{(0)}\big)$ characterising the model is such that $G^{(0)}$ possesses a center Z. There are many examples of such pairs, which include for instance $\big(SU(p+q), S\big(U(p)\times U(q)\big)\big)$, $\big(SL(p+q), S\big(GL(p)\times GL(q)\big)\big)$ and $\big(SO(2n), U(N)\big)$. As we will now show, in this case, the model (4.3.21) then possesses an additional global Z-symmetry, which acts on the fields $g_1, g_2 \in G$ as

$$(g_1, g_2) \longmapsto (g_1 k, g_2), \qquad k \in Z. \qquad (4.3.24)$$

Note that we could also have considered the action $(g_1,g_2) \mapsto (g_1, g_2 k)$, which is equivalent to the one above *via* the $G^{(0)}_{\text{diag}}$ gauge symmetry. Under the action (4.3.24), the graded components $j_{\pm,r}^{(k)}$ of the Maurer-Cartan currents transform as

$$j_{\pm,1}^{(0)} \longmapsto j_{\pm,1}^{(0)}, \qquad j_{\pm,1}^{(1)} \longmapsto k^{-1} j_{\pm,1}^{(1)} k, \qquad j_{\pm,2}^{(0)} \longmapsto j_{\pm,2}^{(0)} \qquad \text{and} \qquad j_{\pm,2}^{(1)} \longmapsto j_{\pm,2}^{(1)}, \qquad (4.3.25)$$

where we have used the fact that k is central in $G^{(0)}$ and thus that $k^{-1} j_{\pm,1}^{(0)} k = j_{\pm,1}^{(0)}$. Noting also that the Wess-Zumino term of g_1 is invariant under the transformation (4.3.24), *i.e.* $I_{\mathrm{WZ}}\big[g_1 k\big] = I_{\mathrm{WZ}}\big[g_1\big]$, it is direct to check that this transformation defines a symmetry of the action (4.3.21), as claimed.

Guadagnini-Martellini-Mintchev model. Let us now define $U = g_1$ and $V = g_2^{-1}$. We recall that the Wess-Zumino term satisfies the following relation:

$$I_{\text{WZ}}\big[g^{-1}\big] = -I_{\text{WZ}}\big[g\big].$$

Then, in the case in which $\lambda_1 = \lambda_2 = \lambda$, the action (4.3.21) can be rewritten as

$$S[U,V] = S_{\text{WZW},\,\lambda^2}[U] + S_{\text{WZW},\,\lambda^2}[V] + \lambda^2 \iint \mathrm{d}x\,\mathrm{d}t\;\kappa\Big(\big(\partial_+ V V^{-1}\big)^{(0)}, \big(U^{-1}\partial_- U\big)^{(0)}\Big), \quad (4.3.26)$$

where $S_{\text{WZW},\,\Bbbk}$ denotes the Wess-Zumino-Witten action with level $\Bbbk$ as defined in (4.3.12). The action (4.3.26) coincides with the one of the Guadagnini-Martellini-Mintchev model introduced in [111] as a theory on $(G \times G')/H$, when considered in the special case $G' = G$ and $H = G^{(0)}$. This model was shown to preserve scale invariance at the quantum level at one loop in [111] and at two loops in [124]. This thus shows that the integrable sigma model considered in this section is a two-dimensional conformal field theory for the specific choice $\lambda_1 = \lambda_2 = \lambda$ of its defining parameters. The Kac-Moody current algebras of this conformal model have been studied in [125].

Let us finally note that in the case under consideration, the Lax connections $\mathcal{L}_\pm(z)$ and $\tilde{\mathcal{L}}_\pm(z)$, given in (4.3.22) and (4.3.23) respectively, assume the following simple form:

$$\mathcal{L}_+(z) = j^{(0)}_{+,2}, \qquad \mathcal{L}_-(z) = j^{(0)}_{-,1} + \frac{j^{(1)}_{-,1}}{z},$$
$$\tilde{\mathcal{L}}_+(z) = j^{(0)}_{+,2} + z\, j^{(1)}_{+,2}, \qquad \tilde{\mathcal{L}}_-(z) = j^{(0)}_{-,1}.$$

The existence of a Lax connection for this model is consistent with the results of [79], where its integrability was first established.[9]

4.4 Integrable σ-models on $T^{1,1}$ manifolds

4.4.1 The models

Action. Let us consider the model with two copies described in the previous section for the choice $G = SU(2)$, with Lie algebra $\mathfrak{g} = \mathfrak{su}(2)$ generated by $I_a = i\sigma_a/2$, where σ_a is the a-th Pauli matrix. We take σ to be the $\mathbb{Z}_2$-automorphism of $\mathfrak{su}(2)$ defined by the following action on the generators: $\sigma(I_1) = -I_1$, $\sigma(I_2) = -I_2$ and $\sigma(I_3) = I_3$, so that $\mathfrak{g}^{(0)} = \mathfrak{u}(1) = \text{span}\{I_3\}$ and correspondingly $G^{(0)} = U(1) = \exp(\mathbb{R}I_3)$. Let us finally pick the following parametrisation for the fields $(g_1, g_2) \in SU(2) \times SU(2)$ of the model:

$$g_1 = \exp(\phi_1 I_3)\exp(\theta_1 I_2)\exp(\psi I_3), \quad (4.4.1\text{a})$$
$$g_2 = \exp(-\phi_2 I_3)\exp(-\theta_2 I_2)\exp(-\tilde{\psi} I_3). \quad (4.4.1\text{b})$$

Inserting this parametrisation in the action (4.3.21), one finds:

$$\begin{aligned} S = \frac{1}{4}\iint \mathrm{d}x\,\mathrm{d}t\Big(&\big(\lambda^2 + \lambda_1^2 + (\lambda^2 - \lambda_1^2)\cos(2\theta_1)\big)\partial_-\phi_1\partial_+\phi_1 + 2\lambda_1^2\,\partial_-\theta_1\partial_+\theta_1 + 2\lambda^2\,\partial_-\psi\partial_+\psi + 4\lambda^2\,\partial_-\phi_1\partial_+\psi\cos\theta_1 \\ &+ \big(\lambda^2 + \lambda_2^2 + (\lambda^2 - \lambda_2^2)\cos(2\theta_2)\big)\partial_-\phi_2\partial_+\phi_2 + 2\lambda_2^2\,\partial_-\theta_2\partial_+\theta_2 + 2\lambda^2\,\partial_-\tilde{\psi}\partial_+\tilde{\psi} + 4\lambda^2\,\partial_-\tilde{\psi}\partial_+\phi_2\cos\theta_2 \\ &+ 4\lambda^2\big(\cos\theta_1\,\partial_-\phi_1 + \partial_-\psi\big)\big(\cos\theta_2\,\partial_+\phi_2 + \partial_+\tilde{\psi}\big)\Big). \end{aligned} \quad (4.4.2)$$

[9]The integrability of a class of models that includes (4.3.26) was also studied in [60].

Gauge fixing and background. Recall that the model we are considering is invariant under the gauge transformation $g_r \mapsto g_r h$, $h \in U(1)$. In the parametrisation (4.4.1) used above, this gauge symmetry simply becomes the translation $(\psi, \tilde{\psi}) \mapsto (\psi + \eta, \tilde{\psi} - \eta)$ with local parameter $\eta \in \mathbb{R}$. We now use this freedom to set $\tilde{\psi} = 0$. Having fixed the gauge, we can then rewrite the action (4.4.2) as a sigma model on the coset $SU(2) \times SU(2)/U(1)$, with coordinate fields $y = (\theta_1, \theta_2, \phi_1, \phi_2, \psi)$. This defines the background metric G_{ij} and background B-field B_{ij}, in terms of which the action reads

$$S = \frac{1}{2} \iint \mathrm{d}x\, \mathrm{d}t\, \big(G_{ij} + B_{ij}\big) \partial_- y^i \partial_+ y^j. \tag{4.4.3}$$

Setting $\tilde{\psi} = 0$ in (4.4.2), we read for the metric:

$$ds^2 = G_{ij} \mathrm{d}y^i \mathrm{d}y^j = \lambda_1^2(\mathrm{d}\theta_1^2 + \sin^2\theta_1\, \mathrm{d}\phi_1^2) + \lambda_2^2(\mathrm{d}\theta_2^2 + \sin^2\theta_2\, \mathrm{d}\phi_2^2) + \lambda^2(\mathrm{d}\psi + \cos\theta_1\, \mathrm{d}\phi_1 + \cos\theta_2\, \mathrm{d}\phi_2)^2, \tag{4.4.4}$$

while the B-field is given by

$$B = \frac{1}{2} B_{ij}\, \mathrm{d}y^i \wedge \mathrm{d}y^j = \lambda^2 (\cos\theta_1\, \mathrm{d}\phi_1 + \mathrm{d}\psi) \wedge (\cos\theta_2\, \mathrm{d}\phi_2 + \mathrm{d}\psi). \tag{4.4.5}$$

We recognise (4.4.4) as the metric of the so-called $T^{1,1}$ manifolds [126–128]. More precisely, it defines a family of metrics, which depend on the three parameters λ_1, λ_2 and λ. Let us note that certain members of this family possess additional interesting geometrical properties. For instance, the choice $\lambda_1^2 = \lambda_2^2 = 3\lambda^2/2$ yields an Einstein metric, which has well-known applications in supergravity. As explained for a general group G in the paragraph 4.3.4, the case $\lambda_1 = \lambda_2 = \lambda$ yields the conformal model of [111], which for the group $SU(2)$ considered here has been studied in [129], where it has been used to construct a pure NS-NS supergravity solution[10].

By construction, the model considered in this section is integrable for any metric in this family, *i.e.* for all values of the parameters λ_1, λ_2 and λ. However, let us stress that this integrability also requires the presence of a B-field in the model, namely the B-field (4.4.5) whose global prefactor λ^2 is then fixed by the choice of the metric (for other choices of this prefactor, the model is non-integrable, see sections 4.4.2 and 4.4.3).

Lax connection. As proven in section 4.3.4, the model under consideration possesses two independent Lax connections $\mathcal{L}_\pm$ and $\widetilde{\mathcal{L}}_\pm$, which characterise its integrability. Let us discuss their explicit expressions in terms of the coordinate fields $(\theta_1, \theta_2, \phi_1, \phi_2, \psi)$. As it turns out, instead of $\mathcal{L}_\pm(z)$, it will be simpler to describe its gauge transformation $\widehat{\mathcal{L}}_\pm(z) = h^{-1}\mathcal{L}_\pm(z) h + h^{-1}\partial_\pm h$ with $h = \exp(-\psi I_3)$. Let us then write these Lax connections in terms of their components in the decompositions $\widehat{\mathcal{L}}_\pm = \widehat{\mathcal{L}}_\pm^a I_a$ and $\widetilde{\mathcal{L}}_\pm = \widetilde{\mathcal{L}}_\pm^a I_a$ along the basis $I_a = i\sigma_a/2$ of $\mathfrak{su}(2)$. From (4.3.22), using the parametrisation (4.4.1), we get for $\widehat{\mathcal{L}}_\pm$:

$$\widehat{\mathcal{L}}_+^1 = \frac{\big(\lambda^2 - \lambda_1^2\big) z}{\lambda^2 z^2 - \lambda_1^2} \sin\theta_1\, \partial_+\phi_1, \qquad \widehat{\mathcal{L}}_+^2 = \frac{\big(\lambda^2 - \lambda_1^2\big) z}{\lambda^2 z^2 - \lambda_1^2}\, \partial_+\theta_1,$$

$$\widehat{\mathcal{L}}_+^3 = \frac{1}{\lambda^2 z^2 - \lambda_1^2} \Big(\big(\lambda^2 - \lambda_1^2\big) \cos\theta_1\, \partial_+\phi_1 - \lambda^2 (z^2 - 1)(\cos\theta_2\, \partial_+\phi_2 + \partial_+\psi) \Big),$$

[10] A parafermionic integrable deformation of this conformal sigma model on $T^{1,1}$ has been considered in [107], by specifying to $SU(2)$ a class of models studied in [60]. It would be interesting to investigate whether this model can be obtained from a construction similar to the one presented in this article.

together with

$$\widehat{\mathcal{L}}^1_- = \frac{\sin\theta_1\,\partial_-\phi_1}{z}, \qquad \widehat{\mathcal{L}}^2_- = \frac{\partial_-\theta_1}{z}, \qquad \widehat{\mathcal{L}}^3_- = \cos\theta_1\,\partial_-\phi_1.$$

Similarly, for $\widetilde{\mathcal{L}}_\pm$ we get from (4.3.23):

$$\widetilde{\mathcal{L}}^1_+ = z\sin\theta_2\,\partial_+\phi_2, \qquad \widetilde{\mathcal{L}}^2_+ = -z\,\partial_+\theta_2, \qquad \widetilde{\mathcal{L}}^3_+ = -\cos\theta_2\,\partial_+\phi_2,$$

as well as

$$\widetilde{\mathcal{L}}^1_- = -\frac{\big(\lambda^2-\lambda_2^2\big)z}{\lambda_2^2z^2-\lambda^2}\sin\theta_2\,\partial_-\phi_2, \qquad \widetilde{\mathcal{L}}^2_- = \frac{\big(\lambda^2-\lambda_2^2\big)z}{\lambda_2^2z^2-\lambda^2}\,\partial_-\theta_2,$$

$$\widetilde{\mathcal{L}}^3_- = \frac{1}{\lambda_2^2z^2-\lambda^2}\Big(\big(\lambda^2-\lambda_2^2\big)z^2\cos\theta_2\,\partial_-\phi_2 + \lambda^2(z^2-1)(\cos\theta_1\,\partial_-\phi_1+\partial_-\psi)\Big).$$

4.4.2 Modification of the background, isometries and equations of motion

Isometries-preserving modification of the model. Let us now consider a modification of the model described in the previous section (this will allow us to pinpoint the requirements for the integrability of the model and to make connections with other works in the next section). More precisely, let us take again an action of the form (4.4.3), with $y = (\theta_1,\theta_2,\phi_1,\phi_2,\psi)$ and metric given by (4.4.4), but with the following B-field ($k\in\mathbb{R}$):

$$B = k\,(\cos\theta_1\,\mathrm{d}\phi_1+\mathrm{d}\psi)\wedge(\cos\theta_2\,\mathrm{d}\phi_2+\mathrm{d}\psi), \tag{4.4.8}$$

obtained from (4.4.5) by substituting the overall multiplication parameter λ^2 with k. For arbitrary values of k, this modification will break the integrability of the theory, while retaining the same isometries as the original model. In particular, as one can see from equations (4.4.4), (4.4.5) and (4.4.8), the coordinate fields ϕ_1, ϕ_2 and ψ do not appear in the metric and the B-field of both the original and modified model and therefore the shifts $\phi_1\to\phi_1+\epsilon_1$, $\phi_2\to\phi_2+\epsilon_2$ and $\psi\to\psi+\epsilon$ are isometries of both backgrounds.

For the original model, this is to be expected from the general results of section 4.3. Indeed, as explained in section 4.3.2, the model is invariant under the left translations $g_1\mapsto f_1g_1$ and $g_2\mapsto f_2g_2$, for $f_1,f_2\in SU(2)$. In the parametrisation (4.4.1), the corresponding actions of the Cartan subgroup $\exp(\mathbb{R}I_3)$ of $SU(2)$ simply become shifts of the coordinates ϕ_1 and ϕ_2. Similarly, the shift of ψ corresponds to the symmetry discussed in section 4.3.4. Consistently, ϕ_1, ϕ_2 and ψ appear in the action (4.4.2) only through their derivatives.

One can calculate the Noether currents associated with these isometries for both models starting from the modified one. Following the conventions of appendix 4.C, we define the components of these currents as

$$\Pi^\mu_{\phi_1} = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_1)}, \qquad \Pi^\mu_{\phi_2} = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_2)} \qquad\text{and}\qquad \Pi^\mu_{\psi} = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}, \tag{4.4.9}$$

where μ are 2-dimensional space-time indices and $\mathcal{L}$ is the Lagrangian density of the action (4.4.3). In light-cone indices, one finds, using (4.4.4) and (4.4.8):

$$2\Pi^\pm_{\phi_1} = \big(\lambda_1^2-(\lambda_1^2-\lambda^2)\cos^2\theta_1\big)\,\partial_\mp\phi_1 + (\lambda^2\mp k)\cos\theta_1\big(\cos\theta_2\,\partial_\mp\phi_2+\partial_\mp\psi\big), \tag{4.4.10a}$$

$$2\Pi^\pm_{\phi_2} = \big(\lambda_2^2-(\lambda_2^2-\lambda^2)\cos^2\theta_2\big)\,\partial_\mp\phi_2 + (\lambda^2\pm k)\cos\theta_2\big(\cos\theta_1\,\partial_\mp\phi_1+\partial_\mp\psi\big), \tag{4.4.10b}$$

$$2\Pi^\pm_{\psi} = (\lambda^2\pm k)\cos\theta_1\,\partial_\mp\phi_1 + (\lambda^2\mp k)\cos\theta_2\,\partial_\mp\phi_2 + \lambda^2\partial_\mp\psi. \tag{4.4.10c}$$

These Noether currents satisfy the conservation equations:

$$\partial_\mu\Pi^\mu_i = \partial_+\Pi^+_i + \partial_-\Pi^-_i = 0, \qquad \text{for}\ \ i=\phi_1,\phi_2,\psi. \tag{4.4.11}$$

Equations of motion. Let us describe the equations of motion for the modified model. From the action (4.4.3), one obtains the following standard form:

$$\partial_-\partial_+ y^i + \hat{\Gamma}^i_{jk}\, \partial_- y^j \partial_+ y^k = 0,$$

where $\hat{\Gamma}^i_{jk}$ are the components of the Christoffel symbol for the metric G_{ij} modified by the torsion T_{ijk} of the B-field B_{ij}, *i.e.*

$$\hat{\Gamma}^i_{jk} = \Gamma^i_{jk} - T^i_{jk} = \frac{1}{2} G^{im}\big(\partial_j G_{mk} + \partial_k G_{jm} - \partial_m G_{jk}\big) - \frac{1}{2} G^{im}\big(\partial_j B_{mk} + \partial_k B_{jm} + \partial_m B_{kj}\big).$$

From (4.4.4) and (4.4.8), we then find the following equations of motion for θ_1 and θ_2:

$$\frac{\partial_-\partial_+\theta_1}{\sin\theta_1} = \partial_-\phi_1 \left(\left(1 - \frac{\lambda^2}{\lambda_1^2}\right) \cos\theta_1\, \partial_+\phi_1 - \frac{k+\lambda^2}{2\lambda_1^2} \big(\cos\theta_2\, \partial_+\phi_2 + \partial_+\psi\big) \right) - \frac{k-\lambda^2}{2\lambda_1^2} \big(\cos\theta_2\, \partial_-\phi_2 + \partial_-\psi\big) \partial_+\phi_1, \tag{4.4.12a}$$

$$\frac{\partial_-\partial_+\theta_2}{\sin\theta_2} = \partial_-\phi_2 \left(\left(1 - \frac{\lambda^2}{\lambda_2^2}\right) \cos\theta_2\, \partial_+\phi_2 + \frac{k-\lambda^2}{2\lambda_2^2} \big(\cos\theta_1\, \partial_+\phi_1 + \partial_+\psi\big) \right) + \frac{k+\lambda^2}{2\lambda_2^2} \big(\cos\theta_1\, \partial_-\phi_1 + \partial_-\psi\big) \partial_+\phi_2. \tag{4.4.12b}$$

For simplicity and as we will not need them, we have omitted the equations for the isometric coordinates ϕ_1, ϕ_2 and ψ. However, one checks that they can be expressed as particular combinations of the conservation equations (4.4.11) for the currents (4.4.10) and the above equations of motion for θ_1 and θ_2.

4.4.3 Spinning string solutions

In this subsection, we describe a certain class of solutions of the equations of motion of the model with modified B-field (4.4.8), obtained by a spinning string ansatz [130, 131]. Note that spinning strings in $T^{1,1}$ manifolds (or closely related wrapped strings) have already been studied in [112–116] in specific cases. In particular, the non-integrability of these solutions have been discussed in [114–116]: we will compare our results with the ones of [114–116] at the end of this subsection.

Spinning string ansatz. We follow the procedure described in appendix 4.C, where we discuss the spinning string ansatz for a general sigma model with B-field. Since the model we are considering possesses three commuting isometries, in the coordinates ϕ_1, ϕ_2 and ψ, one can then search for spinning string solutions of the form:

$$\theta_i = \theta_i(x), \qquad \phi_i = \omega_i t + \widetilde{\phi}_i(x), \qquad \psi = \psi(x), \tag{4.4.13}$$

with $i \in \{1, 2\}$ and ω_1 and ω_2 constant parameters (more generally, one could also add a term ωt in the expression of ψ as it is also an isometric coordinate: for simplicity, we will not consider this more general case here). The functions $\widetilde{\phi}_1(x)$ and $\widetilde{\phi}_2(x)$ and $\psi(x)$ are the equivalent of the functions $\chi^j(x)$ in appendix 4.C.2. As explained in this appendix, these functions are necessary to ensure the consistency of the ansatz. As we shall now see, they (or more precisely their derivatives) can be determined explicitly, which in the end will allow us to obtain ordinary differential equations governing the functions $\theta_1(x)$ and $\theta_2(x)$.

Equations of motion for the isometric coordinates. As explained in appendix 4.C.2, in the spinning string ansatz (4.4.13), the spatial and temporal components of the Noether currents (4.4.10) do not depend on time, and therefore their conservation equations simply become

$$\partial_x \Pi^x_{\phi_1} = \partial_x \Pi^x_{\phi_2} = \partial_x \Pi^x_{\psi} = 0,$$

where $\Pi^x_i = \Pi^+_i - \Pi^-_i$. The above equations have solutions $\Pi^x_{\phi_1} = \Pi^x_{\phi_2} = \Pi^x_{\psi} = 0$ if we choose the integration constant to be zero for simplicity. Using this, the derivatives of the functions $\widetilde{\phi}_1(x)$, $\widetilde{\phi}_2(x)$ and $\psi(x)$, which we denote with a dot as in appendix 4.C, can be solved for in terms of the functions $\theta_1(x)$ and $\theta_2(x)$. More precisely, applying the equation (4.C.7) in the present case, we get

$$\dot{\widetilde{\phi}}_1 = -\frac{k}{\lambda_1^2}\,\omega_1 \cot^2\theta_1, \qquad \dot{\widetilde{\phi}}_2 = +\frac{k}{\lambda_2^2}\,\omega_2 \cot^2\theta_2, \tag{4.4.14a}$$

$$\dot{\psi} = +\frac{k}{\lambda^2}\,\omega_1 \cos\theta_1\left(1+\frac{\lambda^2}{\lambda_1^2}\cot^2\theta_1\right) - \frac{k}{\lambda^2}\,\omega_2\cos\theta_2\left(1+\frac{\lambda^2}{\lambda_2^2}\cot^2\theta_2\right). \tag{4.4.14b}$$

Equations of motion for the non-isometric coordinates and integrability. Inserting the spinning string ansatz (4.4.13) and the expressions (4.4.14) in the equations of motion (4.4.12a) and (4.4.12b) for the non-isometric coordinates, we get the following:

$$\ddot{\theta}_1 = \omega_1 \sin\theta_1\left(\omega_1\left(\left(\frac{\lambda^2}{\lambda_1^2}-1\right)+\frac{k^2}{\lambda^2\lambda_1^2}\left(\left(1-\frac{\lambda^2}{\lambda_1^2}\right)+\frac{\lambda_1^2}{\sin^4\theta_1}\right)\right)\cos\theta_1 - \omega_2\,\frac{k^2-\lambda^4}{\lambda^2\lambda_1^2}\cos\theta_2\right), \tag{4.4.15a}$$

$$\ddot{\theta}_2 = \omega_2 \sin\theta_2\left(\omega_2\left(\left(\frac{\lambda^2}{\lambda_2^2}-1\right)+\frac{k^2}{\lambda^2\lambda_2^2}\left(\left(1-\frac{\lambda^2}{\lambda_2^2}\right)+\frac{\lambda_2^2}{\sin^4\theta_2}\right)\right)\cos\theta_2 - \omega_1\,\frac{k^2-\lambda^4}{\lambda^2\lambda_2^2}\cos\theta_1\right). \tag{4.4.15b}$$

As justified for a general sigma model in appendix 4.C.2, these are ordinary differential equations which involve only the functions $\theta_1(x)$ and $\theta_2(x)$ corresponding to the non-isometric directions of the background. For generic values of the parameters, these equations are coupled and we expect them to be non-integrable. This is consistent with the analysis carried out in [114,115], where the authors consider wrapped strings solutions in the case $k=0$ (*i.e.* no B-field) and rule out integrability by proving that their motion is chaotic [114] or by using the theory of non-analytic integrability [115][11]. Yet, the general results of appendix 4.C.3 show that starting from an integrable sigma model, for which the equations of motion can be recast as a zero curvature equation, and applying the spinning string ansatz to the latter, one will find (under certain assumptions) a Lax equation for the mechanical system describing the dynamical variables of the spinning string ansatz. In our case, we thus expect the equations (4.4.15) to be integrable if the sigma model we start with is integrable. As explained in the previous subsections, this requires the addition of a B-field with the right coefficient, namely $k=\lambda^2$. This has the effect of cancelling the coupling terms in (4.4.15), hence leaving us with equations of motion of two decoupled 1d systems, which are then trivially integrable.

[11] More precisely, the works [114,115] deal with a string model on $T^{1,1}\times \mathrm{AdS}_5$, described by a Polyakov action. In this case, the equations of motion of the fields are supplemented with the Virasoro constraints coming from the worldsheet diffeomorphism invariance. The wrapped strings solutions considered in [114,115] contain non-trivial dynamical degrees of freedom only in the $T^{1,1}$ part of the target space and more precisely in the coordinates θ_1 and θ_2. The equations obeyed by these coordinates are then the same as the ones obtained here for the sigma model on $T^{1,1}$ alone, *i.e.* equations (4.4.15) with $k=0$. Similar spinning strings solutions have also been studied in [112]. Moreover, the analysis of [114,115] was extended in [116] to the more general class of $L^{a,b,c}$ manifolds, which includes $T^{1,1}$.

4.5 Conclusions

Let us conclude this chapter by stating a number of interesting questions arising from the work presented in this chapter which deserve further study. First of all, it would be desirable to prove that the Lagrangian of models with generic N and T fits our conjectural form (4.3.15) given in terms of the classical $\mathcal{R}$-matrix. We checked the validity of this conjecture up to $(N = 3, T = 3)$, and also for $N = 1$ and T arbitrary [122], but further evidence is welcome. Also, it would be nice to find an independent field-theoretic derivation of (4.3.15) which bypasses performing the Legendre transform.

In the recent work [132] it was proven that the integrable models with $N = 2$ are renormalisable at one-loop and that the one on $T^{1,1}$ is renormalisable also at 2-loop. In the same reference, it was constructed an integrable sigma model on the space $G \times G/H$ with H an abelian subgroup which can be embedded non diagonally in $G\times G$. In particular, this theory possesses two unequal levels $\Bbbk_1$ and $\Bbbk_2$ appearing in front of the Wess-Zumino terms in the action. Moreover, when specified to the $G = SU(2)$, $H = U(1)$ case, one finds a class of integrable sigma models on target spaces of the family $T^{1,q}$, with q related to the ratio of $\Bbbk_1$ and $\Bbbk_2$, generalising the model on $T^{1,1}$ presented here. These models can be interpreted as integrable deformations of the GMM model with unequal levels [111]. Since the latter admits a generalisation to the case $G_1 \times G_2/H$ with $G_1 \neq G_2$ and H not necessarily abelian, it is natural to ask wether there exists an integrable deformation of this model. More generally, one could look for a class of integrable sigma models defined on coset spaces of the form $G_1 \times \cdots \times G_N/H$ with H a gauge group admitting an embedding in each of the $G_1, \ldots, G_N$ (see [133] for conformal models on these type of targets spaces). It would be interesting to construct these models from dihedral affine Gaudin models, generalising the construction presented in this chapter.

Since our approach is applicable for both compact and non-compact groups, one can try to construct in a similar fashion an integrable sigma model on Lorentzian spaces $W_{4,2} = SL(2,\mathbb{R})\times SL(2,\mathbb{R})/U(1)$, that can be viewed as non-compact analogues of $T^{1,1}$. The combined sigma model on the 10-d homogeneous space $W_{4,2} \times T^{1,1}$ should then have a special conformal point in the parameter space which would correspond to a critical NS-NS superstring background [129]. Deviations from this point would be then regarded as integrable deformations of the corresponding conformal field theory.

Finally, it would be very interesting to generalise the present approach to construct integrable coset sigma models based on supergroups. For $N = 2$ one obvious candidate to take for G is the supergroup $PSU(1,1|2)$, that has $SL(2,\mathbb{R}) \times SU(2)$ as its bosonic subgroup. One might speculate that the corresponding integrable sigma model could have a special point in the parameter space corresponding to a critical string background, this time with both NS-NS and R-R fluxes.

4.A Coefficients of the Hamiltonian

In this appendix, we give explicit expressions for the coefficients $a^{(k)}_{rs}$ and $b^{(k)}_{rs}$, where $r, s = 1, 2$ and $k = 0, 1$ appearing in equation (4.3.1). For the coefficients $b^{(k)}_{rs}$, we have:

$$b^{(0)}_{rs} = c^{(0)}_{\bar{r}\bar{s}} \frac{2K\left(2z_{\bar{r}}^4 + \zeta_+^2\left(z_r^2 - 3z_{\bar{r}}^2\right) + \zeta_-^2\left(2\zeta_+^2 - z_1^2 - z_2^2\right)\right)}{\left(z_{\bar{s}}^2 - z_s^2\right)^3},$$

$$b^{(1)}_{rs} = c^{(1)}_{r\bar{s}} \frac{2K\left(z_1^2 z_2^2\left(z_1^2 + z_2^2\right) - \zeta_+^2\left(z_1^4 + z_2^4\right) + \zeta_-^2\left(\zeta_+^2\left(z_1^2 + z_2^2\right) - 2z_1^2 z_2^2\right)\right)}{z_1 z_2\left(z_{\bar{s}}^2 - z_s^2\right)^3},$$

where we introduced the notation $\bar{r} = 3-r$, $r = 1,2$ and where the coefficients $c_{rs}^{(k)}$ with $k = 0,1$ are defined in (4.3.2). For the coefficients $a_{rs}^{(k)}$, we have:

$$a_{rs}^{(0)} = b_{\bar{r}\bar{s}}^{(0)} \frac{K\left(2z_s^4 + \zeta_+^2\left(z_{\bar{s}}^2 - 3z_s^2\right) + \zeta_-^2\left(2\zeta_+^2 - z_1^2 - z_2^2\right)\right)}{2\left(z_r^2 - z_{\bar{r}}^2\right)^3},$$

$$a_{rs}^{(1)} = \frac{(-1)^{r+s} c_{\bar{r}\bar{s}}^{(1)}}{z_1^2 z_2^2 \left(z_1^2 - z_2^2\right)^6} K^2 \Big(z_1^2 z_2^2\left(2\zeta_+^2 - z_1^2 - z_2^2\right)\left(2\zeta_+^2\left(z_1^4 - z_2^2 z_1^2 + z_2^4\right) - z_1^2 z_2^2\left(z_1^2 + z_2^2\right)\right)$$
$$- \zeta_-^2\left(2\zeta_+^2 - z_1^2 - z_2^2\right)\left(\zeta_+^2\left(z_1^2 + z_2^2\right)\left(z_1^4 + z_2^4\right) - 4z_1^4 z_2^4\right) + \zeta_-^4\left(\zeta_+^2\left(z_1^2 + z_2^2\right) - 2z_1^2 z_2^2\right)^2\Big).$$

4.B Reformulation of the action

In this appendix, we give an expression of the coefficients $\rho_{rs}^{(k)}$ and $\hbar_r$ defined in (4.3.7) in terms of residues of well-chosen functions. This will allow us to reformulate the action (4.3.6) in a compact way.

We start with the definition (4.3.13) of the functions $\varphi_\pm(z)$, which we restate here for the reader's convenience:

$$\varphi_+(z) = \frac{z^2 - \zeta_+^2}{(z^2 - z_1^2)(z^2 - z_2^2)} \qquad \text{and} \qquad \varphi_-(z) = \frac{z(z^2 - \zeta_-^2)}{(z^2 - z_1^2)(z^2 - z_2^2)}.$$

We recall that in section 4.3, we have made the choice $z_1 = 1$ and $z_2 = x$ for the parameters z_1 and z_2. Note that in terms of the functions $\varphi_\pm(z)$, the twist function (4.2.13) of the model takes the factorised form

$$\varphi(z) = 2K\varphi_+(z)\varphi_-(z). \tag{4.B.1}$$

Let us also define the functions

$$\alpha_0(z,w) = \frac{z}{z^2 - w^2} \qquad \text{and} \qquad \alpha_1(z,w) = \frac{w}{z^2 - w^2}.$$

Using the expression (4.3.17) of the coefficients $\rho_{rs}^{(k)}$ and $\hbar_r$, one checks that they satisfy

$$\rho_{rs}^{(k)} - \frac{\delta_{rs}}{2}\hbar_r = -4K \operatorname*{res}_{w=z_s} \operatorname*{res}_{z=z_r} \alpha_k(z,w)\varphi_+(z)\varphi_-(w). \tag{4.B.2}$$

Note that the order in which we take the residues in the above equation is important. Indeed, for the opposite order, we have

$$\rho_{rs}^{(k)} + \frac{\delta_{rs}}{2}\hbar_r = -4K \operatorname*{res}_{z=z_r} \operatorname*{res}_{w=z_s} \alpha_k(z,w)\varphi_+(z)\varphi_-(w).$$

Let us relate these expressions to the $\mathcal{R}$-matrix (4.2.9). The latter can be re-expressed in terms of the projections $C_{\underline{12}}^{(kk)}$ of the Casimirs on the gradations $\mathfrak{g}^{(k)}$ (see paragraph 4.2.1) as

$$\mathcal{R}_{\underline{12}}^0(z,w) = \sum_{k=0}^{1} \alpha_k(w,z) C_{\underline{12}}^{(kk)}.$$

This shows that for any elements X, Y in the Lie algebra $\mathfrak{g}$, we have

$$\kappa_{\underline{12}}\Big(\mathcal{R}_{\underline{12}}^0(w,z), X_{\underline{1}} Y_{\underline{2}}\Big) = \sum_{k=0}^{1} \alpha_k(z,w)\kappa\big(X^{(k)}, Y^{(k)}\big).$$

Using this result, and reinserting the equation (4.B.2) in the action (4.3.6), we can rewrite the latter as

$$S = \sum_{r=1}^{2} S_{\text{WZW},\,\&_r}[g_r] - 4K \iint \mathrm{d}x\,\mathrm{d}t \sum_{r,s=1}^{2} \mathop{\text{res}}_{z=z_r} \mathop{\text{res}}_{w=z_s} \kappa_{\underline{\mathbf{12}}}\Big(\mathcal{R}^0_{\underline{\mathbf{12}}}(w,z)\varphi_+(z)\varphi_-(w), j_{+,r\underline{\mathbf{1}}}\, j_{-,s\underline{\mathbf{2}}}\Big),$$

which is the equation (4.3.11) announced in the main text. Note also that, using the property (4.3.19) of the Lax connection $\mathcal{L}_\pm(z)$, this expression can be further rewritten as

$$S = \sum_{r=1}^{2} S_{\text{WZW},\,\&_r}[g_r] - 4K \iint \mathrm{d}x\,\mathrm{d}t \sum_{r,s=1}^{2} \mathop{\text{res}}_{z=z_r} \mathop{\text{res}}_{w=z_s} \kappa_{\underline{\mathbf{12}}}\Big(\mathcal{R}^0_{\underline{\mathbf{12}}}(w,z)\varphi_+(z)\varphi_-(w), \mathcal{L}_+(z)_{\underline{\mathbf{1}}}\, \mathcal{L}_-(w)_{\underline{\mathbf{2}}}\Big).$$

4.C Spinning string ansatz for a sigma model with B-field

4.C.1 Generalities

$\boldsymbol{\sigma}$-models with $\boldsymbol{B}$-field. Let us consider a sigma model with coordinate fields $y^1(x,t),\cdots,y^N(x,t)$, metric $G_{ij}=G_{ji}$ and B-field $B_{ij}=-B_{ij}$, whose action is then

$$S[y^1,\cdots,y^N] = \frac{1}{2}\iint \mathrm{d}x\,\mathrm{d}t\,(G_{ij}+B_{ij})\partial_- y^i\,\partial_+ y^j. \tag{4.C.1}$$

We denote by $\mathcal{L}=\frac{1}{2}(G_{ij}+B_{ij})\partial_- y^i\,\partial_+ y^j$ the corresponding Lagrangian density. Let us define:

$$\Pi^\mu_i = \frac{\partial\mathcal{L}}{\partial(\partial_\mu y^i)},$$

so that

$$\Pi^\pm_i = \frac{1}{2}(G_{ij}\mp B_{ij})\partial_\mp y^j.$$

In space-time coordinates (t,x), this becomes

$$\Pi^t_i = \Pi^+_i + \Pi^-_i = G_{ij}\,\partial_t y^j + B_{ij}\,\partial_x y^j, \tag{4.C.2a}$$
$$\Pi^x_i = \Pi^+_i - \Pi^-_i = -G_{ij}\,\partial_x y^j - B_{ij}\,\partial_t y^j. \tag{4.C.2b}$$

The Euler-Lagrange equations of the action (4.C.1) can then be written as

$$\partial_\mu \Pi^\mu_i = \frac{\partial\mathcal{L}}{\partial y^i}, \tag{4.C.3}$$

for all $i\in\{1,\cdots,N\}$.

Isometries. Let us now suppose that the sigma model possesses an isometry along the coordinate y^i, *i.e.* that the metric G_{ij} and B-field B_{ij} do not depend explicitly on y^i. In this case, the derivative of $\mathcal{L}$ with respect to y^i vanishes and the equation of motion (4.C.3) of y^i becomes the conservation equation

$$\partial_\mu\Pi^\mu_i = \partial_t\Pi^t_i + \partial_x\Pi^x_i = 0. \tag{4.C.4}$$

In particular, the quantities Π^t_i and Π^x_i are identified as the components of the Noether current associated with the global symmetry $y^i\mapsto y^i+\epsilon$ of the model and the Noether charge

$$\mathcal{Q}_i = \int \mathrm{d}x\;\Pi^t_i$$

is conserved under time evolution.

4.C.2 Spinning string ansatz

The ansatz. Let us consider the above sigma model with coordinates $y^1, \cdots, y^N$ and M an integer number smaller than N. We will suppose that the model possesses $N-M$ commuting isometries along its coordinates $y^{M+1}, \cdots, y^N$. Our goal in this section will be to search for particular classical solutions of the equations of motion (4.C.3) of this sigma model, by introducing the following ansatz for the fields $y^1, \cdots, y^N$:

$$\begin{aligned} y^i &= y^i(x), && \text{for } 1 \leq i \leq M, \\ y^i &= \omega_i\, t + \chi^i(x), && \text{for } M+1 \leq i \leq N \end{aligned} \tag{4.C.5}$$

where ω_i, $i \in \{M+1, \cdots, N\}$, are constant numbers and $y^1(x), \cdots, y^M(x), \chi^{M+1}(x), \cdots, \chi^N(x)$ are functions of the worldsheet space coordinate x only. As we shall see, the t-dependence of this ansatz will completely drop out of the equations of motion, yielding a coherent set of equations on the functions $y^i(x)$ and $\chi^i(x)$, in the coordinate x.

The usual spinning string ansatz, see *e.g.* [130, 131], corresponds to the case where the functions $\chi^{M+1}(x)$, $\cdots, \chi^N(x)$ vanish. As we will see, because of the presence of the B-field B_{ij}, these functions will be necessary to obtain a coherent ansatz. Moreover, we will also show that the equations of motion of these functions $\chi^i(x)$ can be explicitly solved in terms of the remaining functions $y^1(x), \cdots, y^M(x)$, yielding in the end a coherent set of coupled ordinary differential equations on the latter (under a certain assumption on the metric). Such a generalisation of the spinning string ansatz was considered in [134].

As a general remark, let us start by recalling that the equations of motion (4.C.3) are expressed in terms of the quantities Π_i^μ defined in the previous section. Inserting the ansatz (4.C.5) in the expression (4.C.2) of Π_i^t and Π_i^x, we get

$$\Pi_i^t = +\sum_{j=1}^{M} B_{ij}\, \dot{y}^j(x) + \sum_{j=M+1}^{N} \big(G_{ij}\, \omega_j + B_{ij}\, \dot{\chi}^j(x) \big), \tag{4.C.6a}$$

$$\Pi_i^x = -\sum_{j=1}^{M} G_{ij}\, \dot{y}^j(x) - \sum_{j=M+1}^{N} \big(B_{ij}\, \omega_j + G_{ij}\, \dot{\chi}^j(x) \big), \tag{4.C.6b}$$

where the dot denotes the derivative with respect to x.

Let us recall that the only dependences of the spinning string ansatz (4.C.5) on the worldsheet time coordinate t are in the coordinates $y^{M+1}, \cdots, y^N$, corresponding to isometries of the model. Because of these isometries, the metric G_{ij} and B-field B_{ij} do not depend explicitly on the coordinates $y^{M+1}, \cdots, y^N$, and thus on the time t under the ansatz (4.C.5). In particular, this shows that the quantities Π_i^t and Π_i^x obtained in equation (4.C.6) do not depend on t.

Equations of motion for the isometric coordinates $y^{M+1}, \cdots, y^N$. Let us first focus on the coordinates $y^{M+1}, \cdots, y^N$. Since they correspond to the isometries of the model, their equations of motion take the form of conservation equations (see equation (4.C.4)) $\partial_t \Pi_i^t + \partial_x \Pi_i^x = 0$, for all $i \in \{M+1, \cdots, N\}$. Then, as Π_i^t does not depend on t in the spinning string ansatz (see previous paragraph), these conservation equations simply become $\partial_x \Pi_i^x = 0$. These are trivially solved by

$$\Pi_i^x = C_i, \qquad \text{for all } i \in \{M+1, \cdots, N\},$$

where $C_{M+1}, \cdots, \cdots, C_N$ are integration constants. From the expression (4.C.6b) of Π_i^x, the above equation can be rewritten as

$$\sum_{j=M+1}^{N} G_{ij}\, \dot{\chi}^j(x) = -\sum_{j=M+1}^{N} B_{ij}\, \omega_j - \sum_{j=1}^{M} G_{ij}\, \dot{y}^j(x) - C_i,$$

for all $i \in \{M+1, \cdots, N\}$. To be able to proceed further, and in the rest of this appendix, we shall make the following assumption:

Assumption: We suppose that the $(N-M) \times (N-M)$ matrix $(G_{ij})_{M+1 \leq i,j \leq N}$ is invertible.

We will then denote by $(H^{ij})_{M+1 \leq i,j \leq N}$ its inverse. Let us briefly comment on this. In other words, this assumption means that we suppose the restriction of the metric to the isometric directions to be invertible. Although the full metric $(G_{ij})_{1 \leq i,j \leq N}$ is of course an invertible matrix, it is possible for its submatrix $(G_{ij})_{M+1 \leq i,j \leq N}$ to be non-invertible. However, in the examples consider in this chapter, this assumption will be satisfied. Using the inverse matrix H, we then solve the above equation for $\dot{\chi}^i(x)$:

$$\dot{\chi}^i(x) = -\sum_{j=M+1}^{N} H^{ij} \left(\sum_{k=M+1}^{N} B_{jk}\, \omega_k + \sum_{k=1}^{M} G_{jk}\, \dot{y}^k(x) + C_j \right), \qquad \text{for all } i \in \{M+1, \cdots, N\}. \tag{4.C.7}$$

In particular, this gives the solution of the equations of motion of $y^{M+1}, \cdots, y^N$ in terms of explicit integrals (indeed, the right hand-side of equation (4.C.7) and in particular the matrix H^{ij} do not depend on the $\chi^j(x)$'s, as the corresponding coordinates y^j are isometries of the model).

Let us briefly comment on the relation of the present results with the standard spinning string ansatz for a model without B-field. As explained in the previous paragraph, this standard ansatz corresponds to taking $\chi^i(x) = 0$ for $i \in \{M+1, \cdots, N\}$. In this case, one has to make another assumption on the metric for the ansatz to be consistent, which is to suppose that its components G_{ij} vanish for $i \in \{M+1, \cdots, N\}$ and $j \in \{1, \cdots, M\}$, *i.e.* that there are no metric terms mixing the isometric coordinates $y^{M+1}, \cdots, y^N$ with the non-isometric coordinates $y^1, \cdots, y^M$. Under this assumption and supposing that there is no B-field (or at least no B-field mixing together the isometric coordinates $y^{M+1}, \cdots, y^N$), the quantities Π_i^x, for $i \in \{M+1, \cdots, N\}$, vanish (see equation (4.C.6b)). The equations of motion $\partial_x \Pi_i^x = 0$ are then trivially satisfied, ensuring the consistency of the standard spinning string ansatz. It is clear that the presence of a B-field in the isometric directions $y^{M+1}, \cdots, y^N$ introduces non-vanishing terms in the expression (4.C.6b) of Π_i^x: in this case, consistency of the equations of motion $\partial_x \Pi_i^x = 0$ then requires choosing non-zero $\chi^j(x)$'s, which is why we introduced these functions in the more general ansatz (4.C.5).

Let us finally note that in the notation of this paragraph, the standard spinning ansatz corresponds to taking the integration constants C_i to be zero, as it gives $\Pi_i^x = 0$. Is is also possible to choose these constants to be non-zero and thus introduce new parameters in the final spinning string equations of motion. However, the consistency of the ansatz then requires to also introduce non-zero functions $\chi^j(x)$, even in the absence of a B-field.

Equations of motion for the non-isometric coordinates $y^1, \cdots, y^M$. Let us now study the equations of motion of the coordinates $y^1, \cdots, y^M$. For that, we will use the following standard form of the field equations of a sigma model:

$$\partial_- \partial_+ y^i + \widehat{\Gamma}^i{}_{jk}\, \partial_- y^j\, \partial_+ y^k = 0, \tag{4.C.8}$$

where $\widehat{\Gamma}^i{}_{jk}$ are the Christoffel symbols of the metric G_{ij} modified by the torsion of the B-field B_{ij}:

$$\widehat{\Gamma}^i{}_{jk} = \Gamma^i{}_{jk} - T^i{}_{jk} = \frac{1}{2} G^{im} \Big(\partial_j G_{mk} + \partial_k G_{jm} - \partial_m G_{jk} \Big) - \frac{1}{2} G^{im} \Big(\partial_j B_{mk} + \partial_k B_{jm} + \partial_m B_{kj} \Big).$$

Considering $i \in \{1, \cdots, M\}$ and inserting the ansatz (4.C.5) in the equation of motion (4.C.8), we get:

$$\ddot{y}^{\,i}(x) + \sum_{j=1}^{M} \sum_{k=1}^{M} \widehat{\Gamma}^{i}{}_{jk}\, \dot{y}^{\,j}(x)\, \dot{y}^{\,k}(x) + \sum_{j=M+1}^{N} \sum_{k=M+1}^{N} \widehat{\Gamma}^{i}{}_{jk}\, (\dot{\chi}^{\,j}(x) - \omega_j)\, (\dot{\chi}^{\,k}(x) + \omega_k)$$
$$+ \sum_{j=1}^{M} \sum_{k=M+1}^{N} \widehat{\Gamma}^{i}{}_{jk}\, \dot{y}^{\,j}(x)\, (\dot{\chi}^{\,k}(x) + \omega_k) + \sum_{j=M+1}^{N} \sum_{k=1}^{M} \widehat{\Gamma}^{i}{}_{jk}\, (\dot{\chi}^{\,j}(x) - \omega_j)\, \dot{y}^{\,k}(x) = 0. \quad (4.C.9)$$

The quantities $\widehat{\Gamma}^{i}{}_{jk}$ are defined in terms of the metric G_{ij} and B-field B_{ij}. As the latter do not depend explicitly on the isometric coordinates $y^{M+1}, \cdots, y^N$, so does $\widehat{\Gamma}^{i}{}_{jk}$. In particular, under the ansatz (4.C.5), the quantities $\widehat{\Gamma}^{i}{}_{jk}$ do not depend on the time coordinate t. The equation (4.C.9) is thus a differential equation only in the variable x. Moreover, let us note that the functions $\dot{\chi}^{\,j}(x)$ appearing in this equation are expressed explicitly in terms of $y^1(x), \cdots, y^M(x)$ and their derivatives through equation (4.C.7). Finally, reinserting this expression in the above equation, one gets Ordinary Differential Equations (ODEs) of the form:

$$\ddot{y}^{\,i}(x) + F^i\big(y^{\,j}(x), \dot{y}^{\,j}(x)\big) = 0, \qquad \forall\, i \in \{1, \cdots, N\}, \quad (4.C.10)$$

for some explicit functions $F^i\big(y^{\,j}, \dot{y}^{\,j}\big)$. We thus get a coherent one-dimensional dynamical system on $y^1(x), \cdots, y^M(x)$.

Let us make a brief comment on the method. We used equation (4.C.7) to eliminate the functions $\chi^j(x)$ of the system. Equation (4.C.7) only allows to express $\chi^j(x)$ as integrals over x, which are thus "non-local" quantities in terms of the functions $y^1(x), \cdots, y^M(x)$. However, it is important to notice that in the above analysis, the functions $\chi^j(x)$ appeared in the system only through their derivatives $\dot{\chi}^j(x)$ (because $y^{M+1}, \cdots, y^N$ are isometric coordinates), which ensures that this replacement does not introduce any non-local terms in $y^1(x), \cdots, y^M(x)$. Thus, in the end, one really obtains an ODE of the form (4.C.10), and not a non-local integro-differential equation.

4.C.3 Integrability

If the sigma model we start from is integrable, a natural question is whether the induced 1d dynamical system (4.C.10) obtained from the spinning string ansatz is itself integrable. We investigate this question in this section. The integrability of the sigma model relies on the zero curvature equation

$$\partial_x \mathcal{M}(z) - \partial_t \mathcal{L}(z) + \big[\mathcal{L}(z), \mathcal{M}(z)\big] = 0, \quad (4.C.11)$$

of a Lax connection $\big(\mathcal{M}(z), \mathcal{L}(z)\big)$, depending on the spectral parameter $z \in \mathbb{C}$. In this section, we will make the following assumption on the Lax connection:

> **Assumption:** The Lax connection $\big(\mathcal{M}(z), \mathcal{L}(z)\big)$ depends on the isometric coordinates $y^{M+1}, \cdots, y^N$ only through their derivatives $\partial_-^k \partial_+^l y^i$ $(k + l > 0)$.

Let us comment briefly on this assumption. The zero curvature equation (4.C.11) on $\big(\mathcal{M}(z), \mathcal{L}(z)\big)$ should be equivalent to the equations of motion of the sigma model (4.C.1). The coordinates $y^{M+1}, \cdots, y^N$ only enter these equations of motion through their derivatives $\partial_- y^i$, $\partial_+ y^i$ and $\partial_- \partial_+ y^i$, as they correspond to isometries of the model. Thus, the zero curvature equation (4.C.11) involves only these derivatives. It is thus rather natural to expect that the Lax connection $\big(\mathcal{M}(z), \mathcal{L}(z)\big)$ itself also only depends on these derivatives. A subtlety in this reasoning is that the zero curvature equation (4.C.11) is invariant under gauge transformations

$\mathcal{M}(z) \mapsto h(z)^{-1}\mathcal{M}(z)h(z) + h(z)^{-1}\partial_t h(z)$ and $\mathcal{L}(z) \mapsto h(z)^{-1}\mathcal{L}(z)h(z) + h(z)^{-1}\partial_x h(z)$. In general, it is thus natural to expect that the Lax connection depends solely on the derivatives $\partial_+ y^i$, $\partial_- y^i$ and $\partial_+\partial_- y^i$ only up to gauge transformations. If this is the case, one would then have to perform a gauge transformation to get to a Lax connection satisfying the above assumption.

We will now suppose that this assumption is verified and study the behaviour of the Lax connection under the spinning string ansatz (4.C.5). For $i \in \{M+1, \cdots, N\}$, the derivatives $\partial_-^k \partial_+^l y^i$ take the form

$$\partial_-^k \partial_+^l y^i = (\delta_{k0}\delta_{l1} + \delta_{k1}\delta_{l0})\omega_i + (-1)^k \frac{\mathrm{d}^{k+l}}{\mathrm{d}x^{k+l}} \chi^i(x).$$

In particular, they do not depend on the worldsheet time coordinate t. As the non-isometric coordinates $y^1, \cdots, y^M$ do not depend on t in the ansatz (4.C.5), we thus conclude that the Lax connection $\bigl(\mathcal{M}(z), \mathcal{L}(z)\bigr)$ does not depend on t. In particular, the zero curvature equation (4.C.11) then takes the form of the Lax equation of a mechanical system:

$$\frac{\mathrm{d}}{\mathrm{d}x}\mathcal{M}(z) = \bigl[\mathcal{M}(z), \mathcal{L}(z)\bigr]. \tag{4.C.12}$$

This is not yet a Lax representation of the dynamical system (4.C.10). Indeed, the matrices $\mathcal{M}(z)$ and $\mathcal{L}(z)$ still depend on the functions $\chi^i(x)$ and not only on the functions $y^i(x)$. However, because of the main assumption made in this section, they depend on these functions $\chi^i(x)$ only through their derivatives $\frac{\mathrm{d}^k}{\mathrm{d}x^k}\chi^i(x)$ ($k > 0$, see above). These derivatives can be expressed in terms of the functions $y^i(x)$ through equation (4.C.7). In then end, we then obtain an expression of the Lax connection $(\mathcal{M}(z), \mathcal{L}(z))$ in terms of the functions $y^1(x), \cdots, y^M(x)$ and their derivatives.

This is a good indication of the integrability of the spinning string system. Let us note however that in general, this does not ensure that the Lax representation (4.C.12) produces a sufficient number of conserved quantities, nor that these conserved quantities are in involution one with another (even if the field theory Lax connection one starts with satisfies a Maillet bracket). It seems difficult to address these questions in full generality. They would thus require a case by case analysis.

Chapter 5

Concluding remarks

In this thesis we focused on the study of integrable sigma models with twist function. In particular, we showed the power of affine Gaudin models for generating new examples of these theories in a systematic way. Using this framework, we constructed in chapters 3 and 4 two new classes of models pertaining to this panorama of theories. In chapter 3 we focused on integrable deformations, constructing a theory coupling together an arbitrary number of Yang-Baxter and λ-deformations of the principal chiral model on the same Lie group. In chapter 4, we presented a new class of integrable coset sigma models defined on the direct product of N copies of a Lie group modulo the action of a certain diagonal gauge subgroup. We then specified the construction of chapter 4 to the case of two copies of the group $SU(2)$, obtaining a new integrable sigma model on the manifold $T^{1,1}$.

Numerous possible directions appear promising for future exploration of this panorama of integrable sigma models. Firstly, the question that arises naturally from the work contained in this thesis is if it is possible to combine the results of chapters 3 and 4 to construct integrable deformations of the models introduced in chapter 4. From the general formalism of dihedral affine Gaudin models we know that these theories should exist. One should then proceed to compute the explicit form of the action obtained from this construction. For the case of $T^{1,1}$ models it would be interesting to compare this action with the models of [135–138], where non-integrable deformations of sigma models on $T^{1,1}$ spaces were studied in the context of the so-called gravity/CYBE correspondence.

Another direction would be to study further the connections of affine Gaudin models with the four-dimensional semi-holomorphic Chern-Simons theory introduced in [64]. In this regard, it would be interesting to show explicitly how the models constructed in chapter 4 can be recovered from the framework of the 4d Chern-Simons theory, similarly to what we did for the models presented in chapter 3. Furthermore, it was shown recently in [139] how the action of very general realisations of affine Gaudin models can be obtained from the 4d Chern-Simons theory, including ones with twist functions with higher order poles which were not studied in this thesis. An interesting possibility would be to apply this construction to obtain explicit examples of new integrable sigma models and explore the properties of these theories.

It would also be very interesting to generalise the approach presented in this thesis to construct new integrable sigma models based on supergroups or cosets thereof. For that, one would have to formulate affine Gaudin models associated with Lie superalgebras. Among the theories resulting from this construction, it would be very interesting to identify integrable superstring models. A first step in this direction would be to understand how to implement diffeomorphism invariance and possibly local forms of supersymmetry such as κ-symmetry in the formalism of affine Gaudin models. As an exercise one could for instance reinterpret the

Hamiltonian integrable structure of the $AdS_5 \times S^5$ superstring in terms of affine Gaudin models[1], starting from the known results about its Maillet bracket [141, 142] (see also previous results in [143–148]). As the following step in the construction one would have to understand which of the resulting new models correspond to critical string backgrounds.

The quantisation of integrable sigma models in the panorama described above is also an appealing problem. For example, a natural question would be to determine the S-matrix and spectrum of these theories. Another important direction for the exploration of the quantum properties of these models is the study of their renormalisability. As mentioned respectively in chapters 3 and 4, the one-loop renormalisability of the two classes of models introduced in this thesis was proved in [106] and [132] (for the latter, in the case of models on $G \times G/H$ only). These works opened interesting future perspectives. For instance, it was found in the same references that both these classes of models are in general not stable under the 2-loop RG flow as they stand, hence requiring quantum corrections to their geometry. It would thus be interesting to determine the form of these corrections. Moreover, it was shown in [149] that for the coupled principal chiral models, the RG flow admits a remarkably simple characterisation at one-loop in terms of a renormalisation group equation for their twist function. This was also shown to be the case for the models constructed in chapter 3 in the reference [106]. Since the twist function packages all the continuous parameters of the underlying affine Gaudin model, this leads to the expectation that a similar renormalisation group equation holds in general for all integrable sigma models characterised by a twist function. In particular, it would be interesting to check this for the models of chapter 4 and see how this result extends to higher loops.

[1]Note that the $AdS_5 \times S^5$ superstring was reinterpreted recently in the context of 4d Chern-Simons theory in [140].

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Eidesstattliche Versicherung / Declaration on oath

Hiermit versichere ich an Eides statt, die vorliegende Dissertationsschrift selbst verfasst und keine anderen als die angegebenen Hilfsmittel und Quellen benutzt zu haben.

Hamburg, den 02 Mai 2021 Unterschrift