Harmonic Analysis in Conformal and Superconformal Field Theory

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Abstract

Conformal partial waves are fundamental objects in conformal field theory and their knowledge is a necessary prerequisite for the bootstrap programme. The study of partial waves is naturally a part of harmonic analysis on the conformal group. We take this observation as the starting point and use it to derive various results about partial waves. These include relations of four/higher-point conformal blocks to wavefunctions of Calogero-Moser-Sutherland/Gaudin integrable models and new explicit expressions for superconformal and defect blocks in terms of special functions.

Zussamenfassung

Konforme Partialwellen sind fundamentale Objekte in konformen Feldtheorien und sie zu verstehen ist eine notwendige Voraussetzung für das Bootstrap-Progamm. Die Untersuchung der Partialwellen ist ein natürlicher Bestandteil bei der harmonischen Analyse der konformen Gruppe. Diese Beobachtung ist unser Ausgangspunkt von dem aus wir verschiedene weitere Resultate zu Partialwellen herleiten. Diese Resultate sind beispielswiese Relationen zwischen vier-/vielpunkt konformen Blöcken und Wellenfunktionen von Calogero-Moser-Sutherland/Gaudin integrablen Modellen sowie neue explizite Ausdrücke für superkonforme und defect Blöcke gegeben durch spezielle Functionen.

Dani i Majčetu

Declaration on oath

I hereby affirm, in lieu of an oath, that I have written this dissertation myself and that I have not used any aids or sources other than those allowed.
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Hamburg, 2021

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This thesis is based on the following articles

1. *Superconformal Blocks: General Theory*, JHEP 01 (2020) 159, arxiv:1904.04852, with Volker Schomerus and Evgeny Sobko

2. *Conformal Group Theory of Tensor Structures*, JHEP 10 (2020) 004, arxiv:1910.08099, with Mikhail Isachenkov and Volker Schomerus

3. *The Superconformal X-ing Equation*, JHEP 10 (2020) 147, arxiv:2005.13547, with Volker Schomerus and Evgeny Sobko

4. *Harmonic Analysis in d-dimensional Superconformal Field Theory*, arxiv:2009.00393, SIGMA 17 (2021) 007

From Gaudin Integrable Models to Multipoint Conformal Blocks, arxiv:2009.11882, with Sylvain Lacroix, Jeremy Mann, Lorenzo Quintavalle and Volker Schomerus, *Phys. Rev. Lett.* 126 (2021) 2, 021602

6. *Crossing Symmetry for Long Multiplets in 4D $\mathcal{N} = 1$ SCFTs*, with Volker Schomerus and Evgeny Sobko, arxiv:2011.14116

7. *Defect Conformal Blocks from Appell functions*, with Volker Schomerus, arxiv:2012.12489

Contents

Chapter 1

Introduction

1.1 The importance of conformal field theory

Conformal field theories describe quantum systems that are invariant under angle-preserving transformations of spacetime. In particular, such systems do not have a length scale, which may at first suggest to discard them as physically irrelevant. Yet, it turns out that conformal invariance plays an important role in many areas of theoretical physics, two prominent examples of which are statistical mechanics and quantum gravity.

Let us explain why such relations are to be expected. Consider a statistical physics system with a large number N of particles. It is well known that in the thermodynamic limit, i.e. when $N \to \infty$, the system becomes formally equivalent to a Euclidean quantum field theory. Indeed, in the continuum, possible states of the system can be approximated by fields. In this process, the probability distribution over states becomes a distribution over the space of field configurations, i.e. a path-integral measure. In general, the field theory defined by this measure will be massive. However, at a phase transition, when the correlation length of the statistical system becomes infinite, the resulting field theory enjoys scale invariance and often the full conformal invariance. Thus, conformal field theories model continuum limits of *critical* statistical physics systems.

A typical example of systems that we have in mind is the Ising model. This model was introduced to describe magnets and is defined as a system of spins $\sigma_k \in \{\pm 1\}$ on a lattice Λ. Each spin interacts with its nearest neighbours and they all lie in an external magnetic field. Thus, the Hamiltonian

$$H(\{\sigma_k\}) = -J \sum_{\langle ij \rangle} \sigma_i \sigma_j - h \sum_i \sigma_i, \tag{1.1}$$

involves two parameters, the interaction strength J and the magnetic field strength h. (The notation $\langle ij \rangle$ means that the first sum runs over pairs of neighbouring sites on the lattice.) If $J > 0$, the model describes a ferromagnet. There are various versions and generalisations of the Ising model - one can study it on different lattices Λ, allow the interaction strength to depend on the sites (i, j) etc. We will discuss the simplest case of a hypercubic lattice in $\mathbb{R}^d$ with vanishing magnetic field. Already in this case, the exact computation of the partition function

$$Z = \sum_{\{\sigma_k\}} e^{-\beta H(\{\sigma_k\})}, \tag{1.2}$$

is notoriously difficult for $d \geq 3$. In one dimension, the model is not very interesting and does not exhibit a phase transition. In two, it has been analytically solved by Onsager in 1944.

Some interesting properties of system's behaviour are encoded in the average magnetisation m (if the number of sites is N, this is $\langle \frac{1}{N} \sum \sigma_i \rangle$). At zero temperature, the system rests in its lowest energy state where all spins point in the same direction and $|m| = 1$. In particular, if we know the direction of any one spin, every other one is fixed by it. On the other extreme of very high temperatures, the spins behave independently of one another and m tends to zero. Somewhere in between, there is a *critical temperature* T_c (for the Ising model, $T_c = 2dJ$) at which the propagation of local interactions and effects of randomness balance each other so that the spins are distributed in roughly the same way on all length scales. At the critical temperature, the magnetisation becomes zero in a non-differentiable manner, to remain zero for all $T > T_c$. We say that the system (or rather, the family of systems parametrised by T) exhibits a phase transition at T_c.

A familiar, but highly nontrivial, fact from statistical mechanics is that phase transitions are in some sense universal. For example, as a liquid changes into a gas, the heat capacity diverges as a power law of the temperature, $c \sim |T - T_c|^{-\gamma}$. The exponent γ turns out to be the same for all gases. However, the universality goes way beyond this and the liquid - gas phase transition can be shown to be equivalent to certain critical systems that describe magnets and are similar to the Ising model.

For the same reasons as in statistical mechanics, CFTs occupy a significant place in the space of all quantum field theories, viewed in the Wilsonian paradigm. In this heuristic picture, there is a (renormalisation group) flow on the space of all theories, provided by the integration of field modes. Fixed points of the flow are, by definition, scale invariant theories. When combined with other properties of a QFT, scale invariance "in most cases" enhances to the full conformal invariance. Thus, conformal theories may be used to organise universality classes of QFTs. Quantum theories of a class are all the ones that flow into a given fixed point. If one could solve a certain CFT, then one could hope to access the neighbouring quantum field theories through perturbation theory.

There are also important applications of CFTs to systems that do not exhibit conformal symmetry. The most famous such example is the conjectured AdS/CFT correspondence, according to which quantum theories of gravity with asymptotically AdS_{d+1} metric are equivalent to d-dimensional conformal field theories. The conformal group $SO(d,2)$ is in the original gravitational theory interpreted as the group of isometries of AdS_{d+1}. Here, the CFT is an auxiliary object whose correlation functions compute the scattering amplitudes of the original theory. Indeed, the idea that properties of CFTs secretly teach us about gravity has been one of the driving sources for their study.

We mentioned a number of reasons why one may wish to study conformal field theories. Of course, there would be hardly any sense in using conformal theories to classify more general QFTs or describe gravity, if they themselves were intractable. And indeed, there exist powerful techniques for the study of CFTs. It was realised in the early works [11, 12, 13, 14] that in a CFT one can define a certain algebra of local operators and that all properties of the theory are encoded in this algebra (denoted $\mathcal{A}$). Compared to ordinary QFTs, which have also been studied through their algebras of local operators, expansions of operator products in CFTs are convergent rather than asymptotic. The construction of conformal theories reduces to that of their associated algebras $\mathcal{A}$. In this language, the most constraining consistency condition that has to be satisfied in any theory is associativity of $\mathcal{A}$. It can also be formulated as the set of *crossing symmetry equations* satisfied by four-point correlation functions.

We will go into more details in later chapters, but for now let us note that the problem cast in the new language is still a very difficult one. Local operators are in 1-1 correspondence with the irreducible representations of the conformal group that make up the Hilbert space. One can easily show that there are necessarily infinitely many irreducible components. Therefore, there are infinitely many crossing equations. In two dimensions, it is sometimes possible to organise the irreducible representations into a finite number of modules of (two copies of) the Virasoro algebra. Theories with this property are known as minimal models. Their construction and subsequent identification with two-dimensional statistical systems was the major achievement of the field in the 1980s, [15] .

Nowadays, conformal field theories constitute a substantial part of research in mathematical physics. They lie at the intersection of several non-perturbative approaches to QFTs. Many of these apply to theories with supersymmetry and include localisation, superconformal index calculations and techniques coming from chiral algebras. It is believed that there are continuous families of consistent SCFTs (parametrising so-called moduli spaces). The maximally supersymmetric Yang-Mills theory with the gauge group $SU(N)$ is superconformal and has been the subject of numerous investigations. For some time it was thought that the exact computations in this theory were possible mainly due to supersymmetry, but there exist deformations of SYM that break all the supersymmetries but preserve the conformal symmetry and are still solvable. (These *fishnet* CFTs are, however, not unitary).

Besides the analysis of concrete models, the approach of constraining general CFTs using crossing symmetry, known as the conformal bootstrap programme, has seen a revival since 2008 when it was realised in [16] that small subsets of the infinite system of crossing equations can be studied numerically in their own right to produce interesting conclusions. The crowning achievement of the bootstrap to date is a precise determination of critical exponents in the 3d Ising model. Since it is the one that we will follow, let us briefly discuss the basic ideas of this approach.

1.2 Invitation to the conformal bootstrap

We will illustrate the workings of the bootstrap programme on a simple example that was originally considered in [16]. Assume that we have a four-dimensional conformal field theory that is weakly interacting. The free field in four dimensions has conformal weight $\Delta = 1$. Thus, we assume the existence of a field φ that has dimension $\Delta_\varphi \approx 1$. Two- and three-point functions of scalar fields are fixed by conformal symmetry - this follows from the fact that any triple of points in general position can be brought by conformal transformations to some predetermined configuration. The correlators read

$$\langle \varphi_1(x_1)\varphi_2(x_2)\rangle = \frac{\delta_{12}}{x_{12}^{2\Delta_\varphi}}, \quad \langle \varphi_1(x_1)\varphi_2(x_2)\varphi_3(x_3)\rangle = \frac{\lambda_{\varphi_1\varphi_2\varphi_3}}{|x_{12}|^{\Delta_1+\Delta_2-\Delta_3}|x_{23}|^{\Delta_2+\Delta_3-\Delta_1}|x_{31}|^{\Delta_3+\Delta_1-\Delta_2}} \, .$$

Here, we consider conformal fields on a flat Euclidean space $M = \mathbb{R}^d \cup \{\infty\}$ and use the notation $x_{ij} = x_i - x_j$. The four-point function of φ is no longer fixed by symmetry, which only constrains it to take the general form

$$\langle\varphi(x_1)...\varphi(x_4)\rangle = \frac{1}{x_{12}^{2\Delta_\varphi} x_{34}^{2\Delta_\varphi}} g(u,v), \qquad u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}, \quad v = \frac{x_{14}^2 x_{23}^2}{x_{12}^2 x_{34}^2}, \tag{1.3}$$

for some arbitrary function g. The number of *conformal invariants* on which g depends is two because, starting from four points x_i in general position, one can use conformal transformations to map them to

$$x_1 \mapsto 0, \quad x_2 \mapsto \frac{z_1+z_2}{2} e_1 + \frac{z_1-z_2}{2i} e_2, \quad x_3 \mapsto e_1, \quad x_4 \mapsto \infty, \tag{1.4}$$

where $\{e_i\}$ is a standard orthonormal basis of $\mathbb{R}^d$. The coordinates (z_1, z_2) of the point x_2 are then related to the cross ratios u, v through

$$z_1 z_2 = u, \qquad (1-z_1)(1-z_2) = v \ . \tag{1.5}$$

As we had said already, the Hilbert space of physical states carries a representation of the conformal group. This representation decomposes as a direct sum (or an integral) of irreducible components V_I. Let P_I be the projectors to spaces V_I. Then, by inserting the identity $1 = \sum P_I$ into (1.3) we get

$$\langle\varphi(x_1)...\varphi(x_4)\rangle = \sum_I \langle\varphi(x_1)\varphi(x_2) P_I \varphi(x_3)\varphi(x_4)\rangle = \frac{1}{x_{12}^{2\Delta_\varphi} x_{34}^{2\Delta_\varphi}} \sum_I g_I(u,v) \ . \tag{1.6}$$

Another crucial property of conformal field theories is the state-operator correspondence. A way to formulate this property is by saying that the projectors P_I are of the form

$$P_I = \sum_i \mathcal{O}_{Ii} |0\rangle\langle 0| \tilde{\mathcal{O}}_{Ii} \ . \tag{1.7}$$

Here, $\mathcal{O}_I$ and $\tilde{\mathcal{O}}_I$ are a local primary operator and its *shadow*, i runs over all their descendants and $|0\rangle$ is the vacuum of the theory. The state-operator correspondence allows to rewrite the four-point function as a sum of products of three-point functions. Since the three-point functions that involve descendants of a primary field $\mathcal{O}$ are fixed in terms of three-point functions with $\mathcal{O}$ itself, one can perform the summation over i

$$\langle\varphi(x_1)...\varphi(x_4)\rangle = \frac{1}{x_{12}^{2\Delta_\varphi} x_{34}^{2\Delta_\varphi}} \sum_{\Delta,l} p_{\Delta,l} g_{\Delta,l}(u,v), \quad p_{\Delta,l} = \lambda^2_{\varphi\varphi\mathcal{O}_{\Delta,l}} \ . \tag{1.8}$$

The last equation defines *conformal partial waves*, or *conformal blocks* $g_{\Delta,l}$. By definition, the blocks are given as sums over i of three-point functions, with a fixed normalisation. They are therefore some fixed functions. However, explicit expressions for blocks in terms of special functions were only obtained in the work of Dolan and Osborn [30] almost 30 years after the integral expressions that capture the above summation process were written down. In four dimensions, partial waves are

$$g^{(4d)}_{\Delta,l} = \frac{z_1 z_2}{z_1 - z_2} \Big(k_{\Delta+l}(z_1) k_{\Delta-l-2}(z_2) - k_{\Delta-l-2}(z_1) k_{\Delta+l}(z_2) \Big), \tag{1.9}$$

where k is given in terms of the hypergeometric function by

$$k_{2\rho}(x) = x^{\rho}\ {}_2F_1(\rho,\rho;2\rho;x)\ . \tag{1.10}$$

The final property of correlators needed for formulation of the bootstrap equations is their invariance under permutations of the arguments x_i. This is one of the axioms of Euclidean quantum field theory. When combined with the partial wave decomposition, this property leads to non-trivial functional equations for g

$$g(u,v) = \left(\frac{u}{v}\right)^{\Delta_\varphi} g(v,u), \quad g(u,v) = g(u/v, 1/v)\ . \tag{1.11}$$

The idea of conformal bootstrap is to substitute the conformal block decomposition for g on both sides of these equations and try to find solutions with positive coefficients $p_{\Delta,l}$. The second of these equations is actually satisfied by each block of even spin, while for odd spins the two sides differ by a sign. Therefore, we learn that only operators of even spin appear in the decomposition. The first equation, however, leads to a much less trivial condition on the $p_{\Delta,l}$

$$v^{\Delta_\varphi} \sum_{\Delta,l} p_{\Delta,l} g_{\Delta,l}(u,v) = u^{\Delta_\varphi} \sum_{\Delta,l} p_{\Delta,l} g_{\Delta,l}(v,u)\ . \tag{1.12}$$

It is common to separate the contribution of the identity operator on both sides. This leads to

$$v^{\Delta_\varphi}\left(1 + \sum_{\Delta,l} p_{\Delta,l} g_{\Delta,l}(u,v)\right) = u^{\Delta_\varphi}\left(1 + \sum_{\Delta,l} p_{\Delta,l} g_{\Delta,l}(v,u)\right), \tag{1.13}$$

which we can further rewrite as

$$1 = \sum_{\Delta,l} p_{\Delta,l} F_{\Delta_\varphi,\Delta,l}, \quad F_{\Delta_\varphi,\Delta,l}(u,v) = \frac{v^{\Delta_\varphi} g_{\Delta,l}(u,v) - u^{\Delta_\varphi} g_{\Delta,l}(v,u)}{u^{\Delta_\varphi} - v^{\Delta_\varphi}}\ . \tag{1.14}$$

This is an infinite set of equations for $p_{\Delta,l}$ since u and v can assume infinitely many values. The implications of these equations can only be studied once the properties of conformal blocks are taken into account. In Euclidean kinematics, the coordinates z_i are complex conjugates of one another. In Lorentzian kinematics that one really wishes to study, z_i are real and $0 < z_i < 1$.

Let us put $\Delta_\varphi = 1$ and plot $F_{\Delta_\varphi,\Delta,l}$. To simplify matters, we set $z_1 = z_2$. This subregion of the unit square will already put restrictions on operator dimensions. For $l \geq 2$ and any value Δ above the unitarity bound, $\Delta \geq l + 2$, we get a similar plot

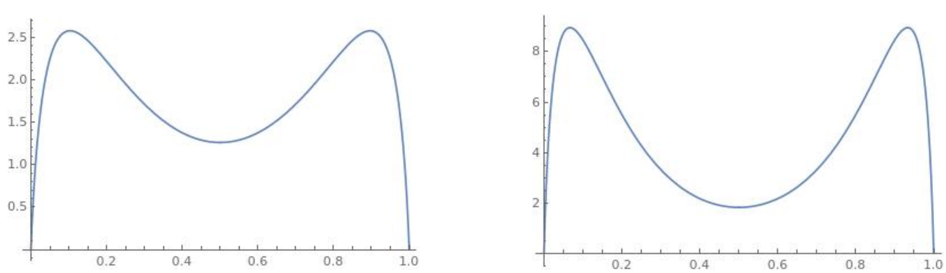

Figure 1.1: $F_{1,\Delta,l}(z,z)$ for $(\Delta,l) = (5,2)$ and $(\Delta,l) = (6,4)$, respectively

For any $l \geq 2$ the function on the right hand side of (1.14) is convex at $z = 1/2$. Since the left hand side is a constant, there need to be terms with $l = 0$ in the decomposition as well. But these terms are also of the above shape as long as $\Delta > \Delta_c \approx 3.62$ (see Figure 2). We conclude that there exists an operator in the decomposition with conformal dimension less than Δ_c. This may not be a very strong bound, but we used a very small amount of information contained in the crossing equation (1.14).

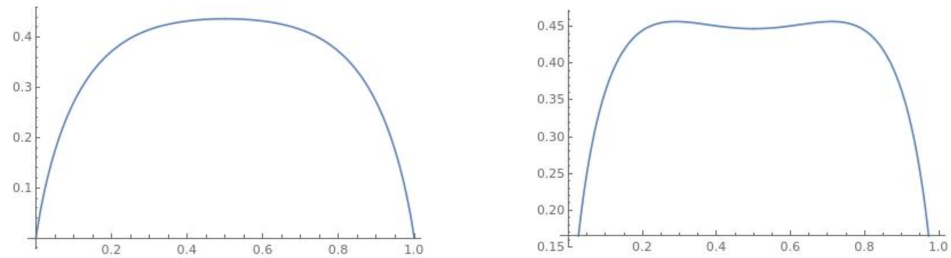

Figure 1.2: $F_{1,\Delta,0}(z,z)$ for $\Delta = 3.4$ and $\Delta = 3.8$, respectively

To get a stronger bound, we may observe that the above argument amounts to applying the linear functional $(\partial_{z_1} + \partial_{z_2})^2|_{z_1=z_2=1/2}$ to both sides of (1.14) and noticing that for any $F_{1,\Delta,l}$ with $l > 0$ it produces a positive number (convexity at the midpoints in Figure 1). To obtain stronger bounds, one looks for other functionals with certain positivity properties. Let us write

$$\Lambda^{(2m,2n)} = (\partial_{z_1} + \partial_{z_2})^{2m} (\partial_{z_1} + \partial_{z_2})^{2n}|_{z_1=z_2=1/2} \ . \tag{1.15}$$

It is then possible to show that $\Lambda^{(2,0)} - \Lambda^{(0,2)}$ is positive when acting on $F_{1,\Delta,l}$ for arbitrary (Δ, l) that satisfy the unitarity bound. We show a few values of this functional as a function of Δ in Figure 3, for spins 0, 2 and 4.

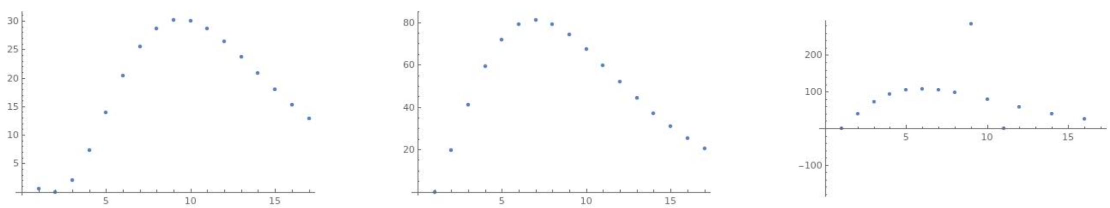

Figure 1.3: $(\Lambda^{(2,0)} - \Lambda^{(0,2)})(F_{1,\Delta,l})$ as a function of Δ for spin $l = 0, 2, 4$, respectively

It can be further shown that the functional $\Lambda^{(2,0)} - \Lambda^{(0,2)}$ becomes zero only for $(\Delta, l) = (2, 0)$ and on the unitarity bound $(\Delta = l + 2, l)$ for non-zero spins l. This proves, in particular, that in the OPE of a dimension one scalar φ with itself only the operators of twist (the difference of dimension and spin) two appear, as is the case in the free theory. In passing, we observe that since all functions $F_{\Delta_\varphi,\Delta,l}(z,z)$ vanish at the boundaries $z \in \{0, 1\}$, the crossing equation immediately requires the presence of infinitely many operators in the OPE.

Bootstrap arguments have become more refined, but they all follow the above logic. After writing down a crossing symmetry equation, one tries to understand what kind of operators can/have to appear in the OPE and bound the positive coefficients $p_{\Delta,l}$. Properties of conformal partial waves that allow for any kind of bound to exist, like convexity of functions $F_{1,\Delta,l}$ at

$z_1 = z_2 = 1/2$, are detected by acting with judiciously chosen linear functionals. For some representative works in the numerical bootstrap programme, the reader is referred to [17, 18, 19, 20, 21]. The review article [22] contains many more details, examples and references.

1.3 Outline of the thesis

Let us describe the structure of the rest of this thesis. Chapters 2 and 3 give an introduction to conformal field theory in more than two dimensions. In particular, chapter 2 introduces basic notions of CFTs and describes some of their simplest properties. For the most part, we focus on properties themselves rather than specific computational methods of deriving them (e.g. no mention of embedding space is made). The exception to this rule is made in the discussion of conformal blocks, for which we also review their derivation as solutions to Casimir equations. Chapter 3 gives several generalisations of the basic setup considered in chapter 2. This includes theories with symmetry enhanced to the superconformal group, or broken to a subgroup by the presence of a defect. The structure introduced in chapter 2 carries over to these cases with only minor modifications (at least conceptually). We hope that this first part of the thesis may be useful for someone who wishes to enter the field. Its purpose is also to fix the terminology and notations used in the rest of the text.

The next three chapters are devoted to mathematical underpinnings of our work. Mathematical objects that describe symmetries of quantum systems are groups and their linear representations. In chapter 4, we study properties of the conformal group in some detail. This includes various group decompositions and a large class of irreducible representations. Representations that are relevant for us are infinite-dimensional. They are realised on spaces of functions and constructed using parabolic induction. Chapter 5 focuses on a particular branch of representation theory - its relation to special functions and integrable systems. In particular, we will explain how certain irreducible matrix elements of Lie groups, called spherical functions, arise as wavefunctions of integrable Schrödinger problems (Calogero-Sutherland models). Since the matrix elements are eigenfunctions of the group Laplacian, their study belongs to the field of harmonic analysis. The discussion of chapters 4 and 5 is in part about general Lie groups and in part restricted to the conformal group.

Since superconformal theories occupy a significant portion of our work, there is a chapter about supermathematics, i.e. mathematics of $\mathbb{Z}_2$-graded objects. In its first part, we discuss super linear algebra and notions such as supermatrices and the Berezinian. This part ends with the discussion of finite-dimensional Lie superalgebras. Then we move to the general notion of a supermanifold and explain how they can be reconstructed from their "algebras of functions". Super analogues of standard concepts from differential geometry, such as vector fields, differential forms and integration are also introduced. The algebraic and geometric aspects of supersymmetry come together in the notion of a Lie supergroup. The last part of the chapter discusses supergroups and their actions on cosets.

In chapters 7-12, we present the results of our research. The main result of chapter 7 is a representation of four-point functions of a superconformal field theory in terms of K-spherical functions on the superconformal group. In order to arrive at this representation, we lift the fields of the theory to functions on the supergroup and then construct an isomorphism between the set of invariant vectors in the four-fold tensor product of principal series representations of the superconformal group and the space of K-spherical functions. The end result is a simple expression for $G_4(x_i)$ in terms of a K-spherical function F, which we refer to as the lifting formula, (7.40). The all important property of the map (7.40) is that it takes conformal partial waves to harmonic functions. Due to relations of the latter with Calogero-Sutherland models, we will sometimes say that F is the expression for $G_4(x_i)$ in the Calogero-Sutherland gauge.

The lifting formula is the starting point for discussions of almost all later chapters. Chapter 8 is devoted to the study of crossing equations in the Calogero-Sutherland gauge. Its main result is the formula for the so-called crossing factor (2.39) that measures the ratio of four-point tensor structures in two different OPE channels. The crossing factor for theories without supersymmetry takes a very simple form that we give explicitly. Chapter 9 uses (7.40) to derive some previously "experimentally observed" relations between conformal blocks and Calogero-Sutherland wavefunctions.

Some of the central ideas of the thesis are contained in chapter 10. In it, we present a systematic derivation of superconformal blocks for four-point functions in theories that have type I supersymmetry. The idea is to reduce the problem to a simpler question about ordinary bosonic blocks. In mathematical terms, we will compute the K-spherical harmonics on a supergroup by staring with K-spherical harmonics on its underlying Lie group. The method is illustrated on examples in one and four dimensions, where it produces simple formulas for superconformal partial waves as finite sums of bosonic ones.

In chapter 11 we turn to conformal field theories with defects. It turns out that the technology of chapter 7 goes a long way towards embedding the theory of defect conformal blocks into harmonic analysis. For this, one more ingredient is needed, namely the lift of bulk fields to functions on the defect conformal group $G_{d,p}$. This lift is constructed in the first part of the chapter using the Iwasawa decomposition of $G_{d,p}$. It allows us to write formulas similar to (7.40) and represent correlators of various numbers of bulk and defect fields as functions on $G_{d,p}$. As a concrete application, we will compute the partial waves for three-point functions of one defect and two bulk fields and express them in terms of Appell's functions.

Finally, chapter 12 is concerned with correlation functions with more than four field insertions. It is shown that for any number of points, there is a natural action of the Gaudin algebra on the space of solutions to the Ward identities. Furthermore, upon specialisation of parameters of this algebra, conformal partial waves may be characterised as simultaneous eigenfunctions of Gaudin Hamiltonians. This is a promising starting point for investigations that we plan to take in the future.

We conclude in chapter 13 with a summary of results and the discussion of natural or interesting directions in which the presented studies might be taken.

Each chapter begins with an introduction that tries to summarise its main points on a non-technical level.

This thesis is based on articles [1, 2, 3, 4, 5, 6, 7]. Some of the ideas on which it builds were introduced in [8, 9, 10].

Chapter 2

Elements of Conformal Field Theory

2.1 Defining properties

Conformal field theories admit a mathematical description, or *axiomatisation*, as do quantum field theories in general. Here, we do not wish to go into the details of these axioms, but rather formulate a list of properties that we want a conformal field theory to satisfy. The properties will be sufficiently clear to provide a working definition of a CFT for all our purposes. They will also point to the correct axiomatisation, should one wish to obtain one.

Any conformal field theory is, first of all, a quantum field theory. That is, it has a Hilbert space of physical states $\mathcal{H}$ and quantum fields which are distributions valued in $\mathrm{End}(\mathcal{H})$ (operator-valued distributions). The spacetime M, over which these distributions are defined, will always be assumed to be either the Euclidean space $\mathbb{R}^d$ or the Minkowski space $\mathbb{R}^{1,d-1}$, conformally compactified at infinity.

The conformal group of M is the group of its angle preserving diffeomorphisms. It is isomorphic to $O(d+1,1)$ or $O(d,2)$ in the Euclidean or Lorentzian signature, respectively. The symmetry group of the theory, denoted by G, is locally isomorphic to the conformal group, e.g. in the Euclidean case it may be $O(d+1,1)$, but also just its identity component $SO^+(d+1,1)$ or the connected and simply connected group $\mathrm{Spin}(d+1,1)$ etc. The precise group depends on the physical situation. For instance, we may or may not wish parity to be a symmetry of the physical system under consideration. We leave the discussion of these various choices for the following chapters and assume that some choice has been made. The statement of symmetry is formulated as

Property 1 The Hilbert space of physical states $\mathcal{H}$ carries a representation of G.

This representation will be denoted by Π. In the Lorentzian signature, the representation is required to be unitary. In the Euclidean signature, the representation should be an analytic continuation of a unitary Lorentzian representation, a property called reflection positivity.

So, what are concretely the conformal transformations of $\mathbb{R}^d$? Any isometry is a conformal transformation, so translations and rotations are examples. Also, rescalings (also called dilations)

$$x \mapsto \lambda x, \quad x \in \mathbb{R}^d,\ \lambda > 0, \tag{2.1}$$

are obviously conformal. Finally, it is not difficult to see that the inversion

$$I : \mathbb{R}^d \cup \{\infty\} \to \mathbb{R}^d \cup \{\infty\}, \quad I(x^\mu) = \frac{x^\mu}{x^2}, \tag{2.2}$$

is also a conformal map. These four types of transformations generate the group $O(d+1,1)$. We will often work with the Lie algebra $\mathfrak{g} = \mathfrak{so}(d+1,1)$. It has a basis $\{P_\mu, M_{\mu\nu}, D, K_\mu\}$ with non-vanishing brackets

$$[M_{\mu\nu}, P_\rho] = \delta_{\nu\rho}P_\mu - \delta_{\mu\rho}P_\nu, \quad [M_{\mu\nu}, K_\rho] = \delta_{\nu\rho}K_\mu - \delta_{\mu\rho}K_\nu, \tag{2.3}$$
$$[M_{\mu\nu}, M_{\rho\sigma}] = \delta_{\nu\rho}M_{\mu\sigma} - \delta_{\mu\rho}M_{\nu\sigma} + \delta_{\nu\sigma}M_{\rho\mu} - \delta_{\mu\sigma}M_{\rho\nu}, \tag{2.4}$$
$$[D, P_\mu] = P_\mu, \quad [D, K_\mu] = -K_\mu, \quad [K_\mu, P_\nu] = 2(M_{\mu\nu} - \delta_{\mu\nu}D)\ . \tag{2.5}$$

Here P_μ and $M_{\mu\nu}$ are the standard generators of translations and rotations, D generates dilations and K_μ, related to P_μ by $K_\mu = IP_\mu I$, are called the special conformal generators. Obviously, indices μ, ν run from 1 to d. The conformal algebra $\mathfrak{g}$ can also be viewed as the Lorentz Lie algebra in $d+2$ dimensions. Indeed, the generators

$$L_{01} = D, \quad L_{0\mu} = \frac{1}{2}(P_\mu + K_\mu), \quad L_{1\mu} = \frac{1}{2}(P_\mu - K_\mu), \quad L_{\mu\nu} = M_{\mu\nu}, \tag{2.6}$$

satisfy the standard relations

$$[L_{\alpha\beta}, L_{\gamma\delta}] = \eta_{\beta\gamma}L_{\alpha\delta} - \eta_{\alpha\gamma}L_{\beta\delta} + \eta_{\beta\delta}L_{\gamma\alpha} - \eta_{\alpha\delta}L_{\gamma\beta}, \tag{2.7}$$

with $\mu, \nu = 2, ..., d+1$, $\alpha, \beta... = 0, 1, ..., d+1$ and η being the mostly-positive Minkowski metric.

Having described the symmetry group, let us turn to the representation that the Hilbert space carries. The space decomposes into a direct sum (or an integral) of irreducible components V_i. Further, each V_i is a parabolic Verma module of the form

$$V_i = \mathrm{Ind}_{\mathfrak{p}}^{\mathfrak{g}} W_i = U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} W_i\ . \tag{2.8}$$

Here, we are simply saying that one starts with a finite-dimensional representation W_i of the Lie algebra of rotations and dilations

$$\mathfrak{k} = \mathfrak{so}(1,1) \oplus \mathfrak{so}(d)\ . \tag{2.9}$$

Such a representation can be characterised by a conformal weight Δ and an $\mathfrak{so}(d)$ highest weight λ (spin). Next, we impose that special conformal generators should act trivially on W_i. This defines a representation of the subalgebra of $\mathfrak{g}$ generated by $\{K_\mu, M_{\mu\nu}, D\}$, that we denote by $\mathfrak{p}$. Finally, we postulate that acting with P_μ on vectors in W_i creates new states. This gives an infinite-dimensional vector space V_i and the action of $\mathfrak{p}$ can be uniquely extended from W_i to it. The notion of induced representations (2.8) that we just described will be given a more precise treatment in later chapters.

Note that P_μ and K_μ act as raising and lowering operators for the dilation generator D, while rotations commute with it. It follows that in any representation V_i, and thus the full Hilbert space as well, the spectrum of D is bounded from below. Vectors that are annihilated by all special conformal generators are called primaries. Any vector obtained from a primary by an application of P_μ-s is called a descendant.

Let us pause for a moment and discuss the Lorentzian case. Now, the conformal algebra is $\mathfrak{g} = \mathfrak{so}(d,2)$. It is spanned by generators $\{L_{\alpha\beta}\}$ that obey the brackets (2.7), only with the metric $\eta = \mathrm{diag}(-1,1,...,1,-1)$. The generators $\{D, P_\mu, K_\mu, M_{\mu\nu}\}$, still defined by (2.6) obey the non-zero brackets

$$[M_{\mu\nu}, P_\rho] = \eta_{\nu\rho}P_\mu - \eta_{\mu\rho}P_\nu, \quad [M_{\mu\nu}, K_\rho] = \eta_{\nu\rho}K_\mu - \eta_{\mu\rho}K_\nu, \tag{2.10}$$

$$[M_{\mu\nu}, M_{\rho\sigma}] = \eta_{\nu\rho}M_{\mu\sigma} - \eta_{\mu\rho}M_{\nu\sigma} + \eta_{\nu\sigma}M_{\rho\mu} - \eta_{\mu\sigma}M_{\rho\nu}, \tag{2.11}$$

$$[D, P_\mu] = P_\mu, \quad [D, K_\mu] = -K_\mu, \quad [K_\mu, P_\nu] = 2(M_{\mu\nu} - \eta_{\mu\nu}D), \tag{2.12}$$

where $\alpha, \beta, ... = 0, ..., d+1$ and $\mu, \nu, ... = 2, ..., d+1$. The identification of conformal generators with Lorentz generators is of course not unique. Notice, however, that in both signatures the dilation generator has to be picked as a Lorentz generator along directions one of which is spacelike, while the other is timelike.

We move to the second property of conformal field theories, which goes under the name of the operator-state correspondence. Fields in the theory are local operators $\mathcal{O}(x)$ and as such are acted on by G via the adjoint representation of $\mathcal{H}$

$$g \cdot \mathcal{O}(x) = \Pi(g)\mathcal{O}(x)\Pi(g)^{-1} \ . \tag{2.13}$$

By definition we have

$$\mathcal{O}(x) = e^{x\cdot P} \cdot \mathcal{O}(0) = \Pi(e^{x\cdot P})\mathcal{O}(0)\Pi(e^{-x\cdot P}) \ . \tag{2.14}$$

Furthermore, the space $\mathcal{O}(0)$ carries a representation ρ of the Lie algebra $\mathfrak{p}$. We suppressed the indices that $\mathcal{O}$ carries. The field is said to be primary if K_μ act trivially on $\mathcal{O}(0)$. In this case, we can write $\rho = (\Delta, \lambda)$ as above, that is, label the representation by a conformal weight and a spin. Given any generator of $\mathfrak{g}$, we can compute its action on $\mathcal{O}(x)$ using the Leibniz rule and the Baker-Campbell-Hausdorff formula. For example, in the case of rotations

$$\begin{aligned} M_{\mu\nu} \cdot \mathcal{O}(x) &= [\Pi(M_{\mu\nu}), \mathcal{O}(x)] = [\Pi(M_{\mu\nu}), \Pi(e^{x\cdot P})\mathcal{O}(0)\Pi(e^{-x\cdot P})] = \\ &= \Pi(e^{x\cdot P})\left(\Pi(e^{-x\cdot P}M_{\mu\nu}e^{x\cdot P})\mathcal{O}(0) - \mathcal{O}(0)\Pi(e^{-x\cdot P}M_{\mu\nu}e^{x\cdot P})\right)\Pi(e^{-x\cdot P}) = \\ &= \Pi(e^{x\cdot P})[\Pi(M_{\mu\nu} - x_\mu P_\nu + x_\nu P_\mu), \mathcal{O}(0)]\Pi(e^{-x\cdot P}) = (x_\nu\partial_\mu - x_\mu\partial_\nu + \lambda(M_{\mu\nu}))\mathcal{O}(x) \ . \end{aligned}$$

Let us denote the generators $M_{\mu\nu}$ in the representation λ by $\Sigma_{\mu\nu}$. If the last calculation is repeated for other generators we arrive at a representation of the conformal algebra by differential operators

$$p_\mu = \partial_\mu, \tag{2.15}$$

$$m_{\mu\nu} = x_\nu\partial_\mu - x_\mu\partial_\nu + \Sigma_{\mu\nu}, \tag{2.16}$$

$$d = x^\mu\partial_\mu + \Delta, \tag{2.17}$$

$$k_\mu = x^2\partial_\mu - 2x_\mu(x^\nu\partial_\nu + \Delta) + 2x^\nu\Sigma_{\mu\nu} \ . \tag{2.18}$$

These operators act on classical fields, i.e. functions on M valued in the carrier space of the representation ρ. Indeed, field configurations form an infinite-dimensional representation of $\mathfrak{g}$, which is generically (for general values of Δ) irreducible and non-unitary. As we shall see later, these representations belong to the principal series of $\mathfrak{g}$. The corresponding representation π of the conformal group is given by

$$(\pi_g \varphi)(gx) = \rho(dg_x)\varphi(x) \ . \tag{2.19}$$

Here, g is an arbitrary element of G and dg is its differential when g is considered as a diffeomorphism of M (the fact that the diffeomorphism g is a conformal map means that the differential dg_x belongs to the group $K = SO(1,1) \times SO(d)$, so evaluating ρ at it makes sense).

As in any quantum field theory, the Hilbert space $\mathcal{H}$ should contain a distinguished vacuum vector, which is invariant under the symmetry group. Given a local primary field $\mathcal{O}(x)$, one can produce a state in $\mathcal{H}$ by acting with $\mathcal{O}(0)$ on the vacuum

$$|\psi\rangle = \mathcal{O}(0)|0\rangle \ . \tag{2.20}$$

The state is also primary, by conformal invariance of the vacuum

$$\Pi(K_\mu)|\psi\rangle = \Pi(K_\mu)\mathcal{O}(0)|0\rangle = [\Pi(K_\mu), \mathcal{O}(0)]|0\rangle = 0 \ . \tag{2.21}$$

Now we can state the operator-state correspondence

Property 2 Any primary vector in $\mathcal{H}$ is obtained by acting with a primary field on the vacuum, (2.20).

Descendant vectors are obtained by acting with derivatives of a primary field, e.g

$$\Pi(P_\mu)|\psi\rangle = \Pi(P_\mu)\mathcal{O}(0)|0\rangle = [\Pi(P_\mu), \mathcal{O}(0)]|0\rangle = \mathcal{O}'(0)|0\rangle \ . \tag{2.22}$$

A useful corollary of the operator-state correspondence is the so-called operator product expansion (OPE). This is a way to substitute a product of two fields at different points by an infinite sum of fields at just one point. In order for the expansion to be valid, the operators must be applied to the vacuum. Indeed, let $\mathcal{O}_1$ and $\mathcal{O}_2$ be two primary fields. Then

$$\begin{aligned}
\mathcal{O}_1(0)\mathcal{O}_2(x)|0\rangle = |\varphi\rangle = \sum_i |\varphi_i\rangle &= \sum_i \Pi(F_i^x(P_\mu))|\psi_i\rangle \\
&= \sum_i \Pi(F_i^x(P_\mu))\mathcal{O}_i(0)|0\rangle = \sum_i F_i^x(\partial_\mu)\mathcal{O}_i(0)|0\rangle \ .
\end{aligned}$$

We first decomposed the resulting vector $|\varphi\rangle$ as a sum of its projections $|\varphi_i\rangle$ to irreducible components V_i. Each of $|\varphi_i\rangle$-s was then written in terms of a corresponding primary vector $|\psi_i\rangle$ through an application of translation generators. Functions F_i^x are completely fixed up to overall scaling by the compatibility of the above equalities with conformal transformations. We therefore write them as $F_i^x(\partial_\mu) = c_{12i}C(x, \partial_\mu)$. Then the OPE is schematically written as

$$\mathcal{O}_1(0)\mathcal{O}_2(x) \sim \sum_i c_{12i}C(x, \partial_\mu)\mathcal{O}_i(0) \ . \tag{2.23}$$

The constants c_{12i} are called OPE coefficients. Morally speaking, they are the structure constants of the operator product algebra. The set of representations $\{V_i\}$ and the OPE coefficients

$\{c_{ijk}\}$ define what is called a set of CFT data. If one applies the operator product expansion inside a correlation function of n fields, it reduces to an infinite sum of $(n-1)$-point functions. Therefore, by induction, the knowledge of the CFT data allows for a computation of any correlation function of local fields, and thus the construction of the theory. However, since the product of operators is associative, the structure constants c_{ijk} satisfy a set of equations, or consistency conditions. Analysis of these conditions is the object of study of the conformal bootstrap programme.

2.2 Correlation functions

Quantities of physical interest in a conformal field theory are correlation functions. These are vacuum expectation values of products of operators. We will be looking at correlation functions of n local primary fields

$$G_n(x_1, ..., x_n) = \langle 0|\mathcal{O}_1(x_1)...\mathcal{O}_n(x_n)|0\rangle \ . \tag{2.24}$$

The function G_n is a map from n copies of the spacetime to the vector space $W = W_1 \otimes ... \otimes W_n$. In a Lorentzian theory, the operators inside the correlator should be time ordered. In the Euclidean case, the order does not play a role. The correlation function is invariant under permutations of its arguments - this is an axiom of a local quantum field theory that expresses causal independence of events that are spacelike separated.

Correlation functions are constrained by conformal symmetry through Ward identities. For, let $X \in \mathfrak{g}$ be any conformal generator and denote by X_i the representation on a field $\mathcal{O}_i$ that was introduced above. We have

$$\sum_{i=1}^{n} X_i G_n = \sum_{i=1}^{n} \langle 0|\mathcal{O}_1(x_1)...[\Pi(X), \mathcal{O}_i(x_i)]...\mathcal{O}_n(x_n)|0\rangle = \langle 0|[\Pi(X), \mathcal{O}_1(x_1)...\mathcal{O}_n(x_n)]|0\rangle = 0 \ . \tag{2.25}$$

In the last step we have used the invariance of the vacuum and in the second to last the Leibniz rule. We have written the local version of the Ward identities corresponding to the action of the conformal algebra on the fields. Their global counterparts that correspond to the action of the group read

$$G_n(gx_i) = (\rho_1(dg_{x_1}) \otimes \cdots \otimes \rho_n(dg_{x_4}))\, G_n(x_i) \ . \tag{2.26}$$

Either way, the identities state that G_n is an invariant vector in the tensor product of n principal series representations of G. Indeed, the space of function $M^n \to W$ is naturally the tensor product of n principal series and Ward identities simply express the G-invariance. We will return to this point of view in later chapters and focus for the moment on the simplest constraints that conformal invariance imposes on correlation functions.

We start with one-point functions. Translation invariance, $\partial_\mu G_1(x) = 0$, implies that G_1 is a constant function. Next, using the Ward identity for dilations

$$DG_1(x) = (x^\mu \partial_\mu + \Delta) G_1(x) = \Delta G_1(x) = 0 \ .$$

Therefore, a one-point function can be non-zero only for fields of vanishing conformal weight. The identity operator is the only such field. Moving on to two-point functions of scalars we find

$$\langle \varphi_1(x_1)\varphi_2(x_2)\rangle = \frac{c\delta_{\Delta_1,\Delta_2}}{x_{12}^{2\Delta_1}},$$

for some constant c. The constant can be absorbed in the normalisation of fields, leaving us with the standardised form of two-point functions

$$\langle \varphi_i(x_1)\varphi_j(x_2)\rangle = \frac{\delta_{ij}}{x_{12}^{2\Delta_\varphi}} \ . \tag{2.27}$$

The fact that the two-point function is of fixed form follows from the possibility to move any two points in general position to some predetermined configuration, e.g. $(0,\infty)$. That is, the action of conformal transformations on M is (generically) two-point transitive. The same is true for three points, and the three-point function of scalars reads

$$\langle \varphi_i(x_1)\varphi_j(x_2)\varphi_k(x_3)\rangle = \frac{c_{ijk}}{|x_{12}|^{\Delta_j+\Delta_j-\Delta_k}|x_{23}|^{\Delta_j+\Delta_k-\Delta_i}|x_{31}|^{\Delta_k+\Delta_i-\Delta_j}} \ . \tag{2.28}$$

The constants c_{ijk} are precisely the OPE coefficients introduced earlier. To see this, one should expand the product of the first two fields inside the three-point function and use (2.27).

Two- and three-point functions of spinning fields are still fixed by symmetry up to a finite number of constants. The fixed kinematical expressions that these constants multiply are known as invariant tensor structures. For example, a two-point function of traceless symmetric tensors that have the same conformal weight Δ reads, in canonical normalisation

$$\langle \mathcal{O}_{\mu_1\dots\mu_l}(x_1)\mathcal{O}_{\nu_1\dots\nu_l}(x_2)\rangle = \frac{1}{x_{12}^{2\Delta}}\left(\frac{1}{l!}\sum_{\sigma\in S_l} J_{\mu_1\sigma(\nu_1)}(x_{12})\dots J_{\mu_l\sigma(\nu_l)}(x_{12}) - \text{traces}\right), \tag{2.29}$$

where $J^\mu{}_\nu(x) = s_x$ is the reflection in the hyperplane orthogonal to x

$$J^\mu{}_\nu(x) = \delta^\mu_\nu - 2\frac{x^\mu x_\nu}{x^2} \ . \tag{2.30}$$

Due to the fact that it appears in the differential of the conformal inversion, $dI_x = x^{-2}J(x)$, $J(x)$ is also sometimes called the inversion tensor.

Starting from four-point functions, correlators are no longer fixed by symmetry. Instead, the Ward identities constrain them to depend on an arbitrary function of conformal invariants $\{u_a\}$

$$G^\alpha_n(x_i) = \Omega^\alpha{}_I(x_i)F^I(u_a) \ . \tag{2.31}$$

The index α runs over a basis of the space of field polarisations W, while I runs over a basis of a generally lower dimensional space of n-point tensor structures. Conformal invariants can be constructed from cross ratios

$$u_{ijkl} = \frac{x_{ij}^2 x_{kl}^2}{x_{ik}^2 x_{jl}^2} \ . \tag{2.32}$$

Indeed, we see that such cross ratios are invariant under conformal transformations. In order to construct them, we need at least four points, in which case there are two independent invariants

$$u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}, \quad v = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2} \ . \tag{2.33}$$

In general, not all cross ratios that one can build out of n points are functionally independent. The number of independent ones for an n-point function in d dimensions is

$$\# = \frac{1}{2}m(m-3) + d(n-m), \quad m = \min(n, d+2) \ . \tag{2.34}$$

Let us spend a moment to explain the nature of indices I. For concreteness, we focus on four-point functions, the analysis for higher-point functions being entirely analogous. Notice that conformal transformations can be used to put four points in general position to the configuration

$$x_1 \mapsto 0, \quad x_2 \mapsto \frac{z+\bar{z}}{2}e_1 + \frac{z-\bar{z}}{2i}e_2, \quad x_3 \mapsto e_1, \quad x_4 \mapsto \infty \ . \tag{2.35}$$

The configuration space M^4 of four points is foliated into orbits of G under the diagonal action and the four-point function is completely specified by giving its values on one point of each orbit. Let us denote the space of orbits by $X = M^4/G$. The structure of this space might be complicated, but there is an open dense subset of X which is a smooth manifold with local coordinates (u_a). Since the action of G on M^4 is not free, not every function $X \to W$ gives a well-defined correlation function. To see this, let $x_1, ..., x_4$ be four points in general position. The stabiliser of $(x_1, ..., x_4)$ in G under the diagonal action is locally isomorphic to $SO(d-2)$. Indeed, one can notice that this is the stabiliser when points are chosen as in (2.35) - it consists of rotations of the space spanned by vectors $e_3, ..., e_d$. For other choices of the four points, the stabiliser subgroup is related to this one by conjugation. Let us denote the points from (2.35) by x_i^0 and their stabiliser by B. For any $b \in B$, the Ward identities imply

$$G_4(x_i^0) = \left(\rho_1(db_{x_1^0}) \otimes ... \otimes \rho_4(db_{x_4^0})\right) G_4(x_i^0) = \left(\rho_1(b) \otimes ... \otimes \rho_4(b)\right) G_4(x_i^0), \tag{2.36}$$

where in the last equality we used that all elements of B act on M as linear transformations. In conclusion, $G_4(x_i^0)$ belongs to the space of invariants W^B. As a vector space, this is the direct sum of trivial representations of B that appear in the decomposition

$$\mathrm{Res}^K_B(\rho_1 \otimes ... \otimes \rho_4) \ .$$

A generic orbit in M^4 contains a point of the form $\chi = (0, \infty, x_3, x_4)$ and corresponding spaces $W^{\mathrm{Stab}_G(\chi)}$ all have the same dimension. This allows to write the correlation function as in (2.31) (with $n = 4$), where I labels a basis of W^B. For works on tensor structures, see [23, 24, 25, 26, 27, 28].

The simplest correlator that is not fixed by symmetry is the four-point function of scalars, which takes the form

$$G_4(x_i) = \frac{1}{(x_{12}^2)^{\frac{\Delta_1+\Delta_2}{2}}(x_{34}^2)^{\frac{\Delta_3+\Delta_4}{2}}} \left(\frac{x_{14}^2}{x_{24}^2}\right)^a \left(\frac{x_{14}^2}{x_{13}^2}\right)^b F(u,v), \tag{2.37}$$

for some function F, with $2a = \Delta_2 - \Delta_1$ and $2b = \Delta_3 - \Delta_4$. Much of the bootstrap research has been devoted to the study of this correlation function for reasons that we shall now explain.

2.3 Conformal partial waves and crossing symmetry equations

We have said that the construction of a conformal field theory amounts to solving the constrains on the CFT data $\{\rho_i, c_{ijk}\}$ that come from associativity of the operator product algebra. To formulate these constraints concretely as a set of equations, we consider a four-point function

$$G_4^\alpha(x_i) = \langle \mathcal{O}_1(x_1)...\mathcal{O}_4(x_4)\rangle = \Omega^\alpha{}_I(x_i)F^I(u,v) \ . \tag{2.38}$$

If we permute points 2 and 4 and compare the resulting equation with the one above, we get

$$F^I(u,v) = \mathcal{M}^I{}_J(u,v)F^J(v,u) \ . \tag{2.39}$$

The matrix $\mathcal{M}^I{}_J$ is termed the crossing factor and it depends on the representations that characterise fields $\mathcal{O}_i$. A distinguishing feature of the crossing factor is its conformal invariance. Therefore, whereas the tensor factors $\Omega(x_i)$ depend non-trivially on coordinates of all insertion points, the crossing factor is a function of cross ratios only. For scalars, the equation becomes

$$F(u,v) = \frac{u^{\frac{\Delta_3+\Delta_4}{2}}}{v^{\frac{\Delta_2+\Delta_3}{2}}}F(v,u) \ . \tag{2.40}$$

Let P_i be the projection operator to the space $V_i \subset \mathcal{H}$. Using $1 = \sum P_i$, the correlation function decomposes as

$$G_4(x_i) = \sum_i \langle \varphi_1(x_1)\varphi_2(x_2)P_i\varphi_3(x_3)\varphi_4(x_4)\rangle \ . \tag{2.41}$$

This decomposition defines conformal partial waves (conformal blocks) as

$$G_4(x_i) = \frac{1}{(x_{12}^2)^{\frac{\Delta_1+\Delta_2}{2}}(x_{34}^2)^{\frac{\Delta_3+\Delta_4}{2}}}\left(\frac{x_{14}^2}{x_{24}^2}\right)^a\left(\frac{x_{14}^2}{x_{13}^2}\right)^b\sum_i c_{12i}c_{i34}g_i(u,v) \ . \tag{2.42}$$

An alternative expression for the blocks can be obtained using the operator product expansion. We expand the products of first two and second two fields

$$\begin{aligned}G_4(x_i) &= \langle\sum_{\mathcal{O}} c_{12\mathcal{O}}C(x_{12},\partial_{x_2})^{\mu_1...\mu_l}\mathcal{O}_{\mu_1...\mu_l}(x_2)\sum_{\mathcal{O}'} c_{\mathcal{O}'34}C(x_{34},\partial_{x_4})^{\nu_1...\nu_k}\mathcal{O}'_{\nu_1...\nu_k}(x_4)\rangle\\ &= \sum_{\mathcal{O}} c_{12\mathcal{O}}c_{\mathcal{O}34}C(x_{12},\partial_{x_2})^{\mu_1...\mu_l}C(x_{34},\partial_{x_4})^{\nu_1...\nu_l}\langle\mathcal{O}_{\mu_1...\mu_l}(x_2)\mathcal{O}_{\nu_1...\nu_l}(x_4)\rangle \ .\end{aligned}$$

Using expressions for $C(x,\partial)$ and the two-point function of symmetric traceless tensors, it is possible to compute the partial waves from above. However, a much simpler derivation of these functions was found by Dolan and Osborn, [30], who characterised the conformal blocks as solutions of two differential equations. Their idea, that we will now review, has an immense bearing on the content of this thesis.

Let $C_2 = \kappa_{ab}X^aX^b$ be the quadratic Casimir element of the conformal algebra. We fix an irreducible representation V_i and denote the corresponding contribution to the sum (2.41) by

G_4^i. Then

$$\begin{aligned}
&\langle\varphi_1(x_1)\varphi_2(x_2)\Pi(C_2)P_i\varphi_3(x_3)\varphi_4(x_4)\rangle = \langle\varphi_1(x_1)\varphi_2(x_2)\kappa_{ab}\Pi(X^a)\Pi(X^b)P_i\varphi_3(x_3)\varphi_4(x_4)\rangle \\
&= \kappa_{ab}\Big(\langle[\Pi(X^a),\varphi_1(x_1)]\varphi_2(x_2)\Pi(X^b)P_i\varphi_3(x_3)\varphi_4(x_4)\rangle \\
&\qquad\qquad\qquad + \langle\varphi_1(x_1)[\Pi(X^b),\varphi_2(x_2)]\Pi(X^b)P_i\varphi_3(x_3)\varphi_4(x_4)\rangle\Big) \\
&= \kappa_{ab}(X_1^a+X_2^a)\langle\varphi_1(x_1)\varphi_2(x_2)\Pi(X^b)P_i\varphi_3(x_3)\varphi_4(x_4)\rangle = \kappa_{ab}(X_1^a+X_2^a)(X_1^b+X_2^b)G_4^i\ .
\end{aligned}$$

We have simply commuted the operators $\Pi(X^a)$ and $\Pi(X^b)$ past the fields $\varphi_{1,2}$ and used that they annihilate the vacuum. However, if we look on the right, the presence of the projector P_i implies that the expression above is also equal to $C_2(V_i)G_4^i$, where $C_2(V_i)$ is the value of the quadratic Casimir in the irreducible representation V_i. The eigenvalue problem

$$\kappa_{ab}(X_1^a+X_2^a)(X_1^b+X_2^b)G_4^i(x_j) = C_2(V_i)G_4^i(x_j), \tag{2.43}$$

is a differential equation in $4d$ variables x_j^μ. However, the operator on the left by construction preserves the functional form of $G_4(x_j)$, which allows one to reduce the equation to one in the cross ratios. Another equation satisfied by partial waves is obtained in the same way from the fourth order Casimir.

So, what do conformal blocks look like? It turns out that Casimir equations take a simple form in coordinates $(z,\bar z)$, where they reduce to the eigenvalue problem of the operator

$$\Delta_2 = D_z + D_{\bar z} + (d-2)\frac{z\bar z}{z-\bar z}((1-z)\partial_z - (1-\bar z)\partial_{\bar z})\ . \tag{2.44}$$

Here, the operator $D_x = D_x^{(a,b;c)}$ reads

$$D_x^{(a,b)} = x^2(1-x)\partial_x^2 - ((a+b+1)x^2 - cx)\partial_x - abx, \tag{2.45}$$

and parameters a and b are related to conformal dimensions of fields in the correlator as before. The third parameter is actually $c=0$, but we have introduced it in order to emphasise the similarity of D_x to the hypergeometric differential operator $H_x = H(a,b,c,x,\partial_x)$. Namely, $D_x = xH_x$. Consequently, eigenfunctions of D_x can be expressed in terms of hypergeometric functions. We see that in two dimensions the term in Δ_2 that couples D_z and $D_{\bar z}$ vanishes. With some additional work, it is possible to decouple the equations in any even dimension. Conformal blocks $g_{\Delta,l}$ are eigenfunctions of Δ_2 with eigenvalues $2\Delta(\Delta-d)+2l(l+d-2)$. We have written the four-dimensional blocks already in (1.9). Here, we give two-dimensional ones

$$g^{(2d)}_{\Delta,l}(z,\bar z) = k_{\Delta+l}(z)k_{\Delta-l}(\bar z) + k_{\Delta-l}(z)k_{\Delta+l}(\bar z)\ . \tag{2.46}$$

In this formula, it is assumed that the fields have equal conformal weights.

Since [30], conformal partial waves have been derived in a number of circumstances. In odd dimensions, no closed form expression for blocks is known and in even dimensions blocks for correlators of spinning fields are given by increasingly complicated expressions in terms of hypergeometric functions. However, for any four-point function of arbitrary spinning fields, there is an efficient algorithm that produces the corresponding conformal blocks. We refer the reader to [31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43] for details.

2.4 Energy-momentum tensor and the central charge

A conformal field theory is said to be local if the symmetry transformations are generated by an energy-momentum tensor. This is a symmetric traceless primary field $T_{\mu\nu}$ of spin two and dimension d. Unlike other fields, the normalisation of the energy-momentum tensor is not fixed by its two-point function, but rather by the requirement of Ward identities

$$\partial_\mu \langle T_{\mu\nu}(x)\mathcal{O}_1(x_1)...\mathcal{O}_n(x_n)\rangle = -\sum_{i=1}^{n} \delta(x-x_i)\langle \mathcal{O}_1(x_1)...\partial_\nu \mathcal{O}_i(x_i)...\mathcal{O}_n(x_n)\rangle \ . \tag{2.47}$$

On the other hand, the two-point function of $T_{\mu\nu}$ defines the *central charge* C_T of the theory by

$$\langle T_{\mu\nu}(x_1)T_{\rho\sigma}(x_2)\rangle = \frac{C_T}{S_d^2}\frac{1}{x_{12}^{2d}}\left(\frac{1}{2}(J_{\mu\rho}(x_{12})J_{\nu\sigma}(x_{12}) + J_{\mu\sigma}(x_{12})J_{\nu\rho}(x_{12})) - \frac{1}{d}\delta_{\mu\nu}\delta_{\rho\sigma}\right) \ . \tag{2.48}$$

Here, S_d is the volume of the unit d-sphere. We will denote the rescaled energy momentum tensor, whose two-point function is normalised as for other fields, by $\tilde{T} = TS_d/\sqrt{C_T}$. In a somewhat vague sense, the central charge measures the number of degrees of freedom in the theory. This intuition comes from the fact that the free four-dimensional theory of N_φ scalars, N_ψ Dirac fermions and N_A vectors has central charge

$$C_T = \frac{4}{3}N_\varphi + 8N_\psi + 16N_A \ . \tag{2.49}$$

It is worth mentioning that one of the first applications of the numerical conformal bootstrap was to bound the central charge of any four-dimensional CFT in terms of the dimension of its lowest lying operator. We recall the argument. To see why such a bound should be possible, let us write schematically the OPE of φ with itself as

$$\varphi \times \varphi = 1 + S_\Delta + T_{\mu\nu} + ... \ . \tag{2.50}$$

On the right hand side are written the operators that appear in the expansion, with S_Δ being the lowest appearing scalar operator except for the identity field. In the ... are other operators of spin 0,2,... . With the normalisations from above, the three-point function $\langle\varphi\varphi\tilde{T}\rangle$ reads

$$\langle\varphi(x_1)\varphi(x_2)\tilde{T}_{\mu\nu}(0)\rangle = -\frac{d\Delta_\varphi}{(d-1)\sqrt{C_T}}\frac{1}{(x_{12}^2)^{\Delta_\varphi-1}x_1^2x_2^2}\left(Z_\mu Z_\nu - \frac{1}{d}\delta_{\mu\nu}Z^2\right), \quad Z = Ix_1 - Ix_2 \ . \tag{2.51}$$

By comparing with the standard form of three-point functions that will be reviewed in the next chapter, we see that the central charge is related to the $c_{\varphi\varphi T}$ OPE coefficient by

$$c_{\varphi\varphi T} = -\frac{d\Delta_\varphi}{(d-1)\sqrt{C_T}} \ . \tag{2.52}$$

Now, similarly as in the example from the introduction, any linear functional Λ that satisfies $\Lambda[F_{\Delta_\varphi,\Delta,l}] > 0$ for all Δ and l leads to an upper bound on $c_{\varphi\varphi T}^2$ and consequently, a lower bound on the central charge

$$c_{\varphi\varphi T}^2 \leq \frac{\Lambda[1]}{\Lambda[F_{\Delta_\varphi,4,2}]}, \quad C_T \geq \frac{d^2\Delta_\varphi^2}{(d-1)^2}\frac{\Lambda[F_{\Delta_\varphi,4,2}]}{\Lambda[1]} \ . \tag{2.53}$$

This simply follows by applying Λ to both sides of (1.14). By ranging over functionals Λ of the form (1.15) one obtains a bound $C_T \geq f(\Delta_\varphi)$ for some numerical function f, [17]. For Δ_φ sufficiently close to one, f is above the free-field theory value 4/3. Thus we learn that interacting theories have more degrees of freedom than the free one.

2.5 Some open problems of conformal field theory

The ultimate goal to be achieved in the conformal bootstrap is the complete classification of conformal field theories. As we have seen in this chapter, the classification problem is a well-defined question in mathematics. However, it is believed to be far beyond the present status of the field and one looks for more modest temporary goals.

One natural direction of research is to try and construct explicit examples of CFTs. The starting point in this endeavour are the free theories. In any spacetime dimension, there is the free scalar theory with the Lagrangian

$$S[\varphi] = -\int d^d x \ \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi \ , \quad \Delta_\varphi = \frac{1}{2}(d-2) \ . \tag{2.54}$$

Similarly, free fermions give an example of a CFT

$$S[\psi,\bar{\psi}] = -\int d^d x \ \bar{\psi}\bar{\gamma}^\mu\partial_\mu\psi \ , \quad \Delta_\psi = \Delta_{\bar{\psi}} = \frac{1}{2}(d-1) \ . \tag{2.55}$$

Here γ^μ are d-dimensional gamma matrices. Further, in even dimensions, $d=2n$, we have the free theory of $(n-1)$-forms $A_{\mu_1\ldots\mu_{n-1}}$

$$S[A] = -\int d^{2n} x \frac{1}{2n!} F^{\mu_1\ldots\mu_n} F_{\mu_1\ldots\mu_n} \ , \quad \Delta_A = n-1 \ . \tag{2.56}$$

Conformal weights of fundamental fields φ, ψ, $\bar{\psi}$ and A ensure that the above examples are classically conformal. Since the theories are free, they are also conformal on the quantum level.

Free theories can be slightly generalised to *mean field theories*. In the scalar case, the mean field theory (MFT) is defined by the set of primary operators $\{\mathcal{O}_{n,l} \mid n,l \in \mathbb{N}_0\}$ of scaling weights $\Delta_{n,l} = 2\Delta_\varphi + 2n + l$ and the rule that any n-point function is equal to the sum of all possible Wick contractions. Compared to the free theory given by the Lagrangian (2.54), the MFT is more general in that it allows for the lowest scaling dimension Δ_φ to be arbitrary, rather than fixed by the spacetime dimension d. One can verify that the result is a consistent conformal field theory. The four-point function of φ in an MFT reads

$$\langle\varphi(x_1)...\varphi(x_4)\rangle = \frac{1}{(x_{12}x_{34})^{2\Delta_\varphi}} + \frac{1}{(x_{13}x_{24})^{2\Delta_\varphi}} + \frac{1}{(x_{14}x_{23})^{2\Delta_\varphi}} = \frac{1}{(x_{13}x_{24})^{2\Delta_\varphi}}(1 + u^{-\Delta_\varphi} + v^{-\Delta_\varphi}) \ . \tag{2.57}$$

It is manifestly crossing symmetric. This four-point function has been decomposed in conformal blocks in any dimension, i.e. the values of the squared OPE coefficients $p_{\Delta_{n,l},l}$ such that

$$1 + u^{-\Delta_\varphi} + v^{-\Delta_\varphi} = v^{-\Delta_\varphi}\Big(1 + \sum_{n,l} p_{\Delta_{n,l},l} \ g_{\Delta_{n,l},l}\Big),$$

are known. They can be found, for instance, in [52]. At the moment, mean field theories exhaust the full set of rigorously constructed CFTs in dimensions higher than two (in two dimensions, minimal models are a prominent class of further examples).

Perhaps more in line with the philosophy and the methods of conformal bootstrap are questions about generic properties of CFTs. For example, what is the average shape of the spectrum? Or, what is the asymptotic behaviour of OPE coefficients?

Some questions of this kind have been given definite answers. We show here an example from one-dimensional theories. In this case, one can prove that the asymptotic behaviour of a four-point function determines the integrated asymptotic density of states. More precisely, consider the correlation function

$$\langle \varphi(0)\varphi(z)\varphi(1)\varphi(\infty)\rangle = z^{-2\Delta_\varphi} G(z), \tag{2.58}$$

and expand $G(z)$ in one-dimensional conformal blocks (1.10)

$$G(z) = \int_{\mathbb{R}} d\Delta\ p(\Delta) k_{2\Delta}(z)\ . \tag{2.59}$$

Here we allowed for a continuous spectrum. The squared OPE coefficients define a positive spectral density $p(\Delta)$, which is supported on $\Delta \geq 0$. The conformal block decomposition converges for $z \in (0,1)$. By looking at the OPE in the t-channel, and assuming that φ is the lowest primary above the identity operator, one shows that as $z \to 1$ the correlator diverges according to a power-law

$$G(1-x) \sim x^{-2\Delta_\varphi} \quad \text{as} \quad x \to 0\ . \tag{2.60}$$

The $\sim$ sign means that the ratio of two sides approaches one in the limit. We wish to understand what constraints on the spectral density does (2.60) put. In [44] it was shown that the asymptotic integrated density of states has to behave as

$$\int_0^Y d\Delta\ 4^\Delta \sqrt{\Delta} p(\Delta) \sim \frac{4\sqrt{\pi}}{\Gamma(2\Delta_\varphi)^2} Y^{2\Delta_\varphi} \quad \text{as} \quad Y \to \infty, \tag{2.61}$$

in order to achieve (2.60). (This result is kind of a tauberian theorem - asymptotic behaviour of a function is determined from that of its integral transform. The integral transform at hand that has conformal blocks as the kernel goes under names of Jacobi, Wilson or the α-space transform, [45].)

In any higher dimensional CFT, one can consider the crossing equations restricted to the line $z = \bar{z}$ and "forget" the spin labels of intermediate fields that propagate in the OPE. Therefore, the one-dimensional crossing appears as a part of any conformal field theory. Partly for this reason, and partly for simplicity, the crossing equations for the simplest correlator (2.58) received a lot of attention, [46, 47, 48]. These investigations have shown that any solution to crossing behaves *on average* as a mean field theory. For example, for any $n \in \mathbb{N}$, there is an operator of dimension between the values of the n-th primary in the spectrum of a generalised free boson and a generalised free fermion

$$2\Delta_\varphi + 2n = \Delta_n^b < \Delta < \Delta_n^f = 2\Delta_\varphi + 2n + 1\ . \tag{2.62}$$

This and some other results can be proven by acting on the crossing equations by a set of judiciously chosen *extremal functionals*. The extension of extremal functional methods to the full-fledged crossing in higher dimensions is a subject of intensive research, [49, 50].

Fundamentally, since crossing expresses a duality between different OPE channels, to formulate a good bootstrap question, one looks for a correlator and a kinematical regime such that: 1) in one channel the behaviour of $G_4(x_i)$ is under good control, e.g. it is dominated by a finite number of operators, 2) to reproduce the same behaviour in the other channel, the spectrum has to be of some particular form.

Above we saw an example of this idea: the asymptotic behaviour (2.60) follows from the presence of the identity operator in the t-channel. In the s-channel, this information gives a bound on operators of high scaling dimensions. This idea, that goes under the name of the *lightcone bootstrap* has led to most of the analytic results obtained over the last ten years, [51, 52, 53, 54]. More recently, an analysis similar in spirit was done for the "stress tensor sector" of the correlator that involves two light and two heavy operators, [55, 56].

Often further progress can be made with theories that satisfy additional special assumptions. A prominent example here are supersymmetric CFTs. They can be studied by a variety of non-perturbative techniques and contain classes of protected operators over which one has uncommonly good control. (This fact has the origin in representation theory of Lie superalgebras. While generic representations of a Lie superalgebra are labelled by certain parameters that can be varied continuously, the so-called short representations imply relations between quantum numbers and cannot be deformed. As a consequence, the appropriate operators in an SCFT cannot have anomalous dimensions.) Another example are conformal theories with defects. We may ask for instance, "what kind of defects does a free theory admit?", [57, 58].

In time, some of these directions will prove to be more useful than others, but each of them seems to steam for sound physical questions, as well as being tractable by current methods.

Chapter 3

Extensions: Spinning Fields, Defects and Supersymmetry

In the previous chapter, we discussed the structure of a conformal field theory in the most basic setup and often focused on correlators of scalar fields. A similar analysis can be carried out in various slightly different contexts. These include correlation functions of fields with spin or certain modifications of CFTs such as superconformal theories or theories with defects. In all these setups, in order to study a correlator, one first determines its most general form as allowed by symmetry and then expands the remaining kinematically unconstrained function in a basis of partial waves. These waves are in principle determined by symmetry and our approach will be to always characterise them as solutions to appropriate differential equations. Finally, the coefficients in the partial wave expansion, that carry the dynamical information about the theory, are constrained from crossing equations.

Even though the general logic in all setups is the same, each of them comes with, often significant, technical difficulties. The aim of this chapter is to introduce some standard techniques to treat spinning, supersymmetric and defect correlators. We will be rather brief and mention only those methods that have a direct bearing on later chapters.

3.1 Spinning fields and tensor structures

As we have seen, correlation functions of spinning fields depend on a finite number of invariant tensor structures. The aim of this section is to introduce *polarisation vectors*, [59], which provide a way to convert vector-valued fields into scalars that depend on additional auxiliary variables. Kinematical constraints that spinning correlators satisfy, are often written in a very simple way using the polarisation vectors.

Before coming to this, however, let us introduce the so-called *conformal vectors*. Given three points x_i we denote

$$X_{i;jk} = Ix_{ij} - Ix_{ik} \ . \tag{3.1}$$

From the definition, one finds that the norm of the conformal vector is $X^2_{i;jk} = x^2_{jk}/(x^2_{ij}x^2_{ik})$. Another nice property of $X_{i;jk}$ is related to the inversion tensor that we introduced above

$$J(X_{i;jk})^\mu{}_\nu = J(x_{ij})^\mu{}_\rho J(x_{jk})^\rho{}_\sigma J(x_{ki})^\sigma{}_\nu \ . \tag{3.2}$$

With the help of conformal vectors, the three-point function of two scalar fields and a symmetric traceless tensor can be written as

$$\langle\varphi_1(x_1)\varphi_2(x_2)\mathcal{O}_3^{\mu_1...\mu_l}(x_3)\rangle = \frac{(-1)^l c_{123}(X_{3;12}^{\mu_1}...X_{3;12}^{\mu_l} - \text{traces})}{(x_{12}^2)^{\frac{1}{2}(\Delta_{12;3}+l)}(x_{23}^2)^{\frac{1}{2}(\Delta_{23;1}-l)}(x_{31}^2)^{\frac{1}{2}(\Delta_{31;2}-l)}}, \tag{3.3}$$

with $\Delta_{ij;k} = \Delta_i + \Delta_j - \Delta_k$. Notice that the form of the the correlator is still completely fixed by symmetry up to overall scaling. Indeed, by the same argument that we have given for four-point functions in the previous chapter, one can count the number of invariant tensor structures for three fields that transform in representations ρ_i as

$$N_3(\rho_1,\rho_2,\rho_3) = \dim\left(W_1\otimes W_2\otimes W_3\right)^{SO(d-1)} \ . \tag{3.4}$$

The group $SO(d-1)$ appears as the stabiliser of three points in general position. So, if W_1 and W_2 carry the trivial representation of the rotation group, the number of three-point tensor structures equals the number of $SO(d-1)$-invariants in the irreducible representation W_3. By standard group theory, this number is one, in accordance with the general form of the correlator (3.3).

We have not written the correlator (3.3) fully explicitly yet, as there remains to do the subtraction of traces. This seems like a tedious process, but it is captured quite elegantly by polarisation vectors. Let z be some auxiliary null vector and write

$$\mathcal{O}(x,z) = \mathcal{O}^{\mu_1...\mu_l}(x)z_{\mu_1}...z_{\mu_l} \ . \tag{3.5}$$

In order to be null, the vector z must be complex, $z\in\mathbb{C}^d$. The three-point function (3.3) can now be rewritten as

$$\langle\varphi_1(x_1)\varphi_2(x_2)\mathcal{O}_3(x_3,z)\rangle = (-1)^l c_{123}\frac{(X_{3;12}\cdot z)^l}{(x_{12}^2)^{\frac{1}{2}(\Delta_{12;3}+l)}(x_{23}^2)^{\frac{1}{2}(\Delta_{23;1}-l)}(x_{31}^2)^{\frac{1}{2}(\Delta_{31;2}-l)}} \ . \tag{3.6}$$

Similarly, the two-point function (2.29) from the last chapter is given by the simple expression

$$\langle\mathcal{O}(x_1,z_1)\mathcal{O}(x_2,z_2)\rangle = \frac{1}{x_{12}^{2\Delta}}\left(z_{1\mu}J^{\mu\nu}(x_{12})z_{2\nu}\right)^l \ . \tag{3.7}$$

Clearly, the function $\mathcal{O}(x,z)$ is uniquely determined by $\mathcal{O}^{\mu_1...\mu_l}(x)$. The converse is also true - $\mathcal{O}^{\mu_1...\mu_l}(x)$ can be constructed from $\mathcal{O}(x,z)$ by repeated differentiation with respect to z. The contraction of tensor indices can be written as the integral

$$\mathcal{O}^{\mu_1...\mu_l}(x)\mathcal{O}'_{\mu_1...\mu_l}(x') = \int_{\mathbb{C}^d} d^{2d}z\ \delta(z^2)\rho(\bar{z}\cdot z)\mathcal{O}(x,\bar{z})\mathcal{O}(x',z), \tag{3.8}$$

where the function ρ appearing in the kernel is expressed in terms of the Bessel function of the second kind

$$\rho(t) = \left(\frac{2}{\pi}\right)^{d-1}\frac{(16t)^{1-d/4}}{\Gamma(d/2-1)}K_{(d/2-2)}(2\sqrt{t}) \ . \tag{3.9}$$

Another nice feature of polarisation vectors emerges when we consider correlators with more than one invariant tensor structure. The simplest such example is a three-point function involving one scalar and two symmetric traceless tensors. It takes the form

$$\langle \mathcal{O}_1(x_1,z_1)\varphi_3(x_3)\mathcal{O}_2(x_2,z_2)\rangle = \frac{(X_{1;32}\cdot z_1)^{l_1}(X_{2;13}\cdot z_2)^{l_2}}{(X^2_{3;21})^{-\frac{\Delta_3}{2}}(X^2_{2;13})^{\frac{l_2-\Delta_2}{2}}(X^2_{1;32})^{\frac{l_1-\Delta_1}{2}}}t(X)\ . \tag{3.10}$$

Here, the variable X is constructed from the insertion points as

$$X = \frac{1}{x_{12}^2}\frac{z_{1\mu}J^{\mu\nu}(x_{12})z_{2\nu}}{(X_{1;32}\cdot z_1)(X_{2;13}\cdot z_2)}\ . \tag{3.11}$$

The nature of the function $t(X)$, as dictated by conformal symmetry, depends on the spacetime dimension. In $d>3$, it is a polynomial of degree at most $\min(l_1,l_2)$. In three dimensions, $t(X)$ can also be of the form $\sqrt{X(1-X)}P(X)$, where $P(X)$ is a polynomial of degree less than of equal to $\min(l_1,l_2)-1$. The corresponding class of tensor structures is of odd $O(d)$ parity and thus not allowed in parity-invariant theories. For future reference, we will introduce the notation

$$n_{12} = \begin{cases} 2min(l_1,l_2)+1, & d=3, \\ min(l_1,l_2)+1, & d>3, \end{cases} \tag{3.12}$$

for the total number of invariant tensor structures. What we have achieved is to trade a finite-dimensional vector space (of three-point tensor structures) for a space of functions in one variable X. Since the latter space has to be finite-dimensional as well, it consists of (essentially) polynomials of bounded degree. Nevertheless, for many manipulations that one may want to perform, the precise space of functions is not important. Examples are addition or composition of differential operators. The idea to replace the carrier space of a representation given by an explicit basis $\{e_i\}$, by a certain function space is a common one. In particular, when working with infinite-dimensional representations, the function space description is almost always the simpler one to use.

One can study more complicated correlators along the same lines. We shall not do this presently, but will return to this idea in a later chapter.

3.2 Defect conformal field theories

One setup in which CFT techniques can be applied with mild modifications are the so-called defect conformal field theories. By a defect, we will mean a (conformally compactified) subspace $\mathbb{R}^p$ of the spacetime $\mathbb{R}^d$. Typically, it represents an impurity in a critical system or the boundary of an experimental setup. Alternatively, one can think of defect theories as ordinary CFTs, in which we are no longer interested only in correlation functions of local fields. Defects are then non-local operators, such as Wilson or 't Hooft lines etc.

The presence of a defect reduces the symmetry group of the system to consist of those conformal transformations of $\mathbb{R}^d$ that preserve the p-dimensional subspace $\mathbb{R}^p$ along which the defect is localised. These form the subgroup $G_{d,p} = SO(p+1,1)\times SO(d-p)$ of the conformal group $G_d = SO(d+1,1)$. We will adopt the notation

$$M = S^d = \mathbb{R}^d \cup \{\infty\},\quad N = S^p = \mathbb{R}^p\cup\{\infty\}, \tag{3.13}$$

and often refer to the spacetime M as the *bulk space*. Indices $a, b... = 1, ..., p$ will be used to label a basis of the defect subspace $\{e_a\}$ and indices $i, j... = p+1, ..., d$ for its orthogonal complement in $\mathbb{R}^d$. Then, the $SO(p+1,1)$ factor of $G_{d,p}$ is generated by dilations D, translations and special conformal transformations along the defect P_a, K_a, and rotations in the defect plane M_{ab}, while the $SO(d-p)$ factor is generated by transverse rotations M_{ij}, $i, j = p+1, ..., d$. We write $q = d - p$ and for any element g of the defect conformal group, g_p and g_q will stand for its unique factors in $SO(p+1,1)$ and $SO(q)$.

Correlation functions that we will be considering admit two kinds of field insertions, those of bulk and defect fields. Similarly as in an ordinary conformal field theory, the correlators satisfy a set of consistency conditions. However, there are two types of conditions. Firstly, defect fields close with respect to operator products and in fact define a p-dimensional CFT. The group $SO(q)$ of transverse rotations acts as an internal symmetry of this p-dimensional theory. We will write a generic spacetime point as x and a point on the defect as $\hat{x}$. Similarly, defect fields will carry a hat. Then, the defect OPE can be written as

$$\hat{\mathcal{O}}_1(0)\hat{\mathcal{O}}_2(\hat{x}) \sim \sum_i c_{12i} C(\hat{x}, \partial_a)\hat{\mathcal{O}}_i(0) \ . \tag{3.14}$$

In addition to the p-dimensional theory operator product coefficients $c_{\hat{i}\hat{j}\hat{k}}$, there is another set of new data a defect brings in: the bulk-defect operator product coefficients $b_{i\hat{j}}$. The latter appear when a local bulk field is moved close to the defect and expanded in terms of defect fields

$$\mathcal{O}(x) \sim \sum_i b_{\mathcal{O}\hat{\mathcal{O}}_i} B(\hat{x}, \partial_a)\hat{\mathcal{O}}_i(\hat{x}) \ . \tag{3.15}$$

Consistency constraints arise from considering mixed correlators of bulk and defect fields

$$G_{m,n}(x_1, ..., x_m, \hat{x}_1, ..., \hat{x}_n) = \langle \varphi_1(x_1)...\varphi_m(x_m)\hat{\varphi}_1(\hat{x}_1)...\hat{\varphi}_n(\hat{x}_n)\rangle \ . \tag{3.16}$$

Let $\rho_1, ..., \rho_m, \hat{\rho}_1, ..., \hat{\rho}_n$ be representations that label the fields entering the correlation function. Bulk fields are labelled by representations of the group $K_d = SO(1,1) \times SO(d)$, while the defect fields are labelled by representations of $K_p \times SO(q)$. The Ward identities read

$$G_{m,n}(gx_1, ..., gx_m, h\hat{x}_1, ..., g\hat{x}_n) = \Big(\rho_1(dg_{x_1}) \otimes ... \otimes \hat{\rho}_n(dg_{\hat{x}_n})\Big) G_{m,n}(x_1, ..., \hat{x}_n) \ . \tag{3.17}$$

Similarly as in the non-defect case, symmetry fixes the form of correlation functions with a small number of field insertions. However, there are some important differences between the two setups. Firstly, one-point functions of bulk fields do not necessarily vanish. The general form allowed by symmetry of the scalar one-point function is

$$\langle \varphi(x)\rangle = \frac{a_\varphi}{|x_\perp|^{\Delta_\varphi}}, \tag{3.18}$$

for some constant a_φ. Here, $x_\perp$ is the component of x orthogonal to the defect subspace.

Also fixed by symmetry is the two-point function of one bulk and one defect field. For scalars, it takes the form

$$\langle \varphi(x_1)\hat{\varphi}(\hat{x}_2)\rangle = b_{\varphi\hat{\varphi}} |x_{1\perp}|^{\hat{\Delta}-\Delta} (\hat{x}_{12}^2 + x_{1\perp}^2)^{-\hat{\Delta}} \ . \tag{3.19}$$

Of course, two- and three-point functions that involve only defect fields are fixed as in an ordinary CFT. The smallest correlator that is no longer fixed by symmetry is the two-point function of bulk fields. It admits two invariants

$$u_1 = \frac{x_1^\perp \cdot x_2^\perp}{x_{12}^2}, \quad u_2 = \frac{|x_1^\perp||x_2^\perp|}{x_{12}^2}, \tag{3.20}$$

and takes the general form

$$\langle \varphi_1(x_1)\varphi_2(x_2)\rangle = |x_1^\perp|^{-\Delta_1}|x_2^\perp|^{-\Delta_2}F(u_1,u_2), \tag{3.21}$$

for an arbitrary function F. Again, we have assumed the fields to be scalar. If the codimension of the defect is $q=1$, the two invariants u_i become equal to each other.

In some sense, in the defect setup, the correlators (3.19) and (3.21) play the role of three- and four-point functions of an ordinary conformal field theory. Knowledge of the two-point functions (3.19) is equivalent to that of the coefficients $b_{\mathcal{O}\hat{\mathcal{O}}}$. Higher correlators are used to constrain the OPE data through crossing equations and the simplest of these higher-point functions is (3.21). Due to its non-trivial dependence on cross ratios, this two-point function can be expanded in conformal partial waves. These were first obtained by McAvity and Osborn in 1995, [60], and later more generally in [61, 62]. Such bulk-bulk conformal blocks are similar to those of one-dimensional CFTs and can be written as products of hypergeometric functions and Gegenbauer polynomials.

Foundational results about defect CFTs that we described above were obtained in [63, 64, 60, 61, 62]. A more complete list of references will be given in a later chapter, when we come to analyse defects using group theory.

3.3 Superconformal theories

While general conformal field theories have been significantly constrained by consistency, the bootstrap programme is still very far from classifying them. In principle self-sufficient, its methods greatly benefit from our knowledge or intuition that come from other sources, such as statistical mechanics.

Another class of theories that we know a lot about are those that posses supersymmetry. While the belief in its phenomenological relevance has diminished over the last decades, supersymmetry is a rare tool that allows for exact computations in quantum field theory. When combined with conformal symmetry, the usual supersymmetry algebra is extended to the superconformal algebra. In four dimensions, the superconformal algebra is spanned by supertranslations $\{Q_{\dot\alpha}{}^J, Q_I{}^\beta\}$, special superconformal transformations[1] $\{S_\alpha{}^J, S_I{}^{\dot\beta}\}$ and internal symmetries $\{R, R_I{}^J\}$ in addition to the usual conformal algebra. Here, the indices $I,J=1,...,\mathcal{N}$ are that of the fundamental representation of $\mathfrak{su}(\mathcal{N})$. Indices $\alpha,\dot\alpha=1,2$ are that of the fundamental and the anti-fundamental representation of the rotation Lie algebra $\mathfrak{so}(4)$. They are raised and

[1] We will use the terminology "special superconformal transformations" rather than "super special conformal transformations", although the latter may be logically more appropriate, as ordinary special conformal transformations are also superconformal maps.

lowered using the Levi-Civita symbol

$$\psi_\alpha = \varepsilon_{\alpha\beta}\psi^\beta, \quad \varepsilon_{\alpha\beta} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} . \tag{3.22}$$

The same rule holds for dotted indices. To convert a vector index into a pair of a fundamental and an anti-fundamental index, we make use of the matrices

$$(\gamma_\mu)^{\dot\alpha}{}_\alpha = (-\sigma_3, -iI_2, \sigma_1, -\sigma_2) \quad \text{i.e.} \quad x^{\dot\alpha}{}_\alpha = \begin{pmatrix} -x_1 - ix_2 & x_3 + ix_4 \\ x_3 - ix_4 & x_1 - ix_2 \end{pmatrix} . \tag{3.23}$$

Using the spinor notation, conformal group generators may be written as $\{D, P_{\dot\alpha}{}^\beta, K_\alpha{}^{\dot\beta}, M_\alpha{}^\beta, M_{\dot\alpha}{}^{\dot\beta}\}$ and their bracket relations read

$$[D, P_{\dot\alpha}{}^\beta] = P_{\dot\alpha}{}^\beta \ , \ [D, K_\alpha{}^{\dot\beta}] = -K_\alpha{}^{\dot\beta}, \tag{3.24}$$

$$[M_\alpha{}^\beta, P_{\dot\gamma}{}^\delta] = \frac{1}{2}\delta_\alpha{}^\beta P_{\dot\gamma}{}^\delta - \delta_\alpha{}^\delta P_{\dot\gamma}{}^\beta, \ [M_{\dot\alpha}{}^{\dot\beta}, P_{\dot\gamma}{}^\delta] = -\frac{1}{2}\delta_{\dot\alpha}{}^{\dot\beta} P_{\dot\gamma}{}^\delta + \delta_{\dot\gamma}{}^{\dot\beta} P_{\dot\alpha}{}^\delta, \tag{3.25}$$

$$[M_\alpha{}^\beta, K_\gamma{}^{\dot\delta}] = -\frac{1}{2}\delta_\alpha{}^\beta K_\gamma{}^{\dot\delta} + \delta_\gamma{}^\beta K_\alpha{}^{\dot\delta}, \ [M_{\dot\alpha}{}^{\dot\beta}, K_\gamma{}^{\dot\delta}] = \frac{1}{2}\delta_{\dot\alpha}{}^{\dot\beta} K_\gamma{}^{\dot\delta} - \delta_{\dot\alpha}{}^{\dot\delta} K_\gamma{}^{\dot\beta}, \tag{3.26}$$

$$[M_{\dot\alpha}{}^{\dot\beta}, M_{\dot\gamma}{}^{\dot\delta}] = \delta_{\dot\gamma}{}^{\dot\beta} M_{\dot\alpha}{}^{\dot\delta} - \delta_{\dot\alpha}{}^{\dot\delta} M_{\dot\gamma}{}^{\dot\beta}, \ [M_\alpha{}^\beta, M_\gamma{}^\delta] = \delta_\gamma{}^\beta M_\alpha{}^\delta - \delta_\alpha{}^\delta M_\gamma{}^\beta, \tag{3.27}$$

$$[K_\alpha{}^{\dot\beta}, P_{\dot\gamma}{}^\delta] = \delta_{\dot\gamma}{}^{\dot\beta} M_\alpha{}^\delta - \delta_\alpha{}^\delta M_{\dot\gamma}{}^{\dot\beta} - 2\delta_{\dot\gamma}{}^{\dot\beta}\delta_\alpha{}^\delta D \ . \tag{3.28}$$

Conformal symmetry commutes with internal $U(1)\times SU(\mathcal{N})$ symmetry. On the other hand, the commutation relations involving one odd and one even generator are indicated by the indices that Q-s and S-s carry. They are in either fundamental or anti-fundamental representations of both Spin(4) and $SU(\mathcal{N})$. The Q-s have dilation weight $1/2$ and S-s have $-1/2$. They both split in two sets according to the R-charge. Explicitly

$$[R, Q_{\dot\alpha}{}^J] = Q_{\dot\alpha}{}^J, \ [R, Q_I{}^\beta] = -Q_I{}^\beta, \ [R, S_\alpha{}^J] = S_\alpha{}^J, \ [R, S_I{}^{\dot\beta}] = -S_I{}^{\dot\beta}, \tag{3.29}$$

$$[D, Q_{\dot\alpha}{}^J] = \frac{1}{2}Q_{\dot\alpha}{}^J, \ [D, Q_I{}^\beta] = \frac{1}{2}Q_I{}^\beta, \ [D, S_\alpha{}^J] = -\frac{1}{2}S_\alpha{}^J, \ [D, S_I{}^{\dot\beta}] = -\frac{1}{2}S_I{}^{\dot\beta}, \tag{3.30}$$

$$[M_\alpha{}^\beta, Q_K^\delta] = \frac{1}{2}\delta_\alpha{}^\beta Q_K^\delta - \delta_\alpha{}^\delta Q_K^\beta, \ [M_{\dot\alpha}{}^{\dot\beta}, Q_{\dot\gamma}{}^L] = -\frac{1}{2}\delta_{\dot\alpha}{}^{\dot\beta} Q_{\dot\gamma}{}^L + \delta_{\dot\gamma}{}^{\dot\beta} Q_{\dot\alpha}{}^L, \tag{3.31}$$

$$[M_\alpha{}^\beta, S_\gamma{}^L] = -\frac{1}{2}\delta_\alpha{}^\beta S_\gamma{}^L + \delta_\gamma{}^\beta S_\alpha{}^L, \ [M_{\dot\alpha}{}^{\dot\beta}, S_K^{\dot\delta}] = \frac{1}{2}\delta_{\dot\alpha}{}^{\dot\beta} S_K^{\dot\delta} - \delta_{\dot\alpha}{}^{\dot\delta} S_K^{\dot\beta}, \tag{3.32}$$

$$[P_{\dot\alpha}{}^\beta, S_\gamma{}^L] = \delta_\gamma{}^\beta Q_{\dot\alpha}{}^L, \ [P_{\dot\alpha}{}^\beta, S_K^{\dot\alpha}] = -\delta_{\dot\alpha}{}^{\dot\delta} Q_K^\beta, \ [K_\alpha{}^{\dot\beta}, Q_{\dot\gamma}{}^L] = \delta_{\dot\gamma}{}^{\dot\beta} S_\alpha{}^L, \ [K_\alpha{}^{\dot\beta}, Q_K^\delta] = -\delta_\alpha{}^\delta S_K^{\dot\beta} \ . \tag{3.33}$$

Finally, we give the brackets between odd generators

$$\{Q_{\dot\alpha}{}^J, Q_I{}^\beta\} = \delta_I{}^J P_{\dot\alpha}{}^\beta, \ \{S_\alpha{}^J, S_I{}^{\dot\beta}\} = \delta_I{}^J K_\alpha{}^{\dot\beta}, \tag{3.34}$$

$$\{Q_{\dot\alpha}{}^J, S_I{}^{\dot\beta}\} = \delta_I{}^J M_{\dot\alpha}{}^{\dot\beta} + \delta_{\dot\alpha}{}^{\dot\beta} R_I{}^J + \delta_I{}^J \delta_{\dot\alpha}{}^{\dot\beta}(aD + bR), \tag{3.35}$$

$$\{Q_I{}^\beta, S_\alpha{}^J\} = \delta_I{}^J M_\alpha{}^\beta + \delta_\alpha{}^\beta R_I{}^J + \delta_I{}^J \delta_\alpha{}^\beta (cD + dR) \ . \tag{3.36}$$

Together, these generators form the Lie superalgebra $\mathfrak{sl}(4|\mathcal{N})$. If we make $\alpha, \dot{\alpha}$ run over a one-element set instead, they form the superconformal algebra in one dimension, $\mathfrak{sl}(2|\mathcal{N})$. The full classification of superconformal algebras will be reviewed in a later chapter.

Fields in a superconformal theory are organised in superconformal multiplets. The way to formulate this that is most useful for our purposes is to say that classical fields (superfields) are functions on the superspace. They belong to the algebra generated by coordinates $\{x^{\dot{\alpha}}{}_{\alpha}, \theta^{\dot{\alpha}}{}_{I}, \bar{\theta}^{I}{}_{\alpha}\}$. Here x-s are commuting and θ-s are Grassmann variables. Each superfield can be expanded in component fields multiplied by products of Grassmann variables, e.g. for $\mathcal{N}=1$

$$\Phi(x,\theta) = \phi(x) + \psi_{\dot{\alpha}}(x)\theta^{\dot{\alpha}} + \psi^{\alpha}(x)\bar{\theta}_{\alpha} + F_{\dot{\alpha}}{}^{\alpha}(x)\theta^{\dot{\alpha}}\bar{\theta}_{\alpha} + \dots \; . \tag{3.37}$$

The superspace is acted on by the supersymmetry algebra $\langle P_{\dot{\alpha}}{}^{\alpha}, Q_{\dot{\alpha}}{}^{I}, Q_{I}{}^{\alpha}\rangle$ through differential operators

$$p_{\dot{\alpha}}{}^{\alpha} = \partial_{\dot{\alpha}}{}^{\alpha}, \quad q_{\dot{\alpha}}{}^{I} = \partial_{\dot{\alpha}}{}^{I} - \frac{1}{2}\bar{\theta}^{I}{}_{\beta}\partial_{\dot{\alpha}}{}^{\beta}, \quad q_{I}{}^{\alpha} = q_{I}{}^{\alpha} - \frac{1}{2}\theta^{\dot{\beta}}{}_{I}\partial_{\dot{\beta}}{}^{\alpha}, \tag{3.38}$$

and this action extends to a representation of the superconformal algebra on field configurations. Such representations ρ are characterised by their conformal weights, spins and internal symmetry quantum numbers.

Similarly to the bosonic case, the Hilbert space of physical states decomposes as a sum of parabolic Verma modules. These representations are constructed by taking the finite-dimensional representation ρ, extending it trivially to special conformal transformations and their super-cousins and then acting freely with P-s and Q-s

$$V = \mathrm{Ind}_{\mathfrak{p}}^{\mathfrak{g}}\rho, \quad \mathfrak{p} = \mathrm{span}\{D, M_{\alpha}{}^{\beta}, M_{\dot{\alpha}}{}^{\dot{\beta}}, R, R_{I}{}^{J}, S_{\alpha}{}^{J}, S_{I}{}^{\dot{\beta}}, K_{\alpha}{}^{\dot{\beta}}\} \; . \tag{3.39}$$

More precisely, irreducible components of $\mathcal{H}$ are quotients of these modules. A notable difference in the representation theory of Lie superalgebras compared to Lie algebras is the existence of so-called atypical representations. These are representations that are not irreducible and cannot be written as a sum of irreducible components. Irreducible quotients of atypical modules are called short or BPS multiplets. Two important kinds of short representations are chiral and anti-chiral multiplets. A chiral multiplet is obtained by imposing $Q_{\alpha}{}^{I}v = 0$ on the lowest weight vector $v \in V$ (i.e. quotienting out from V the subrepresentation generated from vectors $\{Q_{\alpha}{}^{I}v\}$). In the dual context of representations on fields, short multiplets arise as submodules of the full field representation. The representation spaces for them consist of those field configurations that satisfy differential equations $\bar{D}\Phi = 0$ ($D\bar{\Phi} = 0$ for anti-chiral) where

$$D_{\dot{\alpha}}{}^{I} = \partial_{\dot{\alpha}}{}^{I} + \frac{1}{2}\bar{\theta}^{I}{}_{\beta}\partial_{\dot{\alpha}}{}^{\beta}, \quad \bar{D}_{I}{}^{\alpha} = -\partial_{I}{}^{\alpha} - \frac{1}{2}\theta^{\dot{\beta}}{}_{I}\partial_{\dot{\beta}}{}^{\alpha} \; . \tag{3.40}$$

The operators $D, \bar{D}$ are called covariant derivatives and they realise the right-regular action of the supersymmetry algebra on M. On the other hand, the operators (3.38) realise the left-regular action. Clearly, the property of being chiral is preserved by operator products. Therefore, all chiral operators form a subalgebra of the full operator product algebra, called the chiral ring.

The importance of short multiples for quantum field theory lies in the fact that they exist only when certain relations between the quantum numbers of the representation are satisfied.

They are, therefore, protected against continuous deformations of the theory, at least if such deformations preserve some of the quantum numbers (typically this can be inferred for R-charges).

We could now repeat large parts of the previous chapter and define superconformal Ward identities, superconformal blocks and crossing equations. These notions will be regarded as evident. Generalising basic results about the partial waves and crossing factors, rather than merely their definitions, is more involved. The guiding principle for deriving superconformal blocks is that they can be written as sums of ordinary bosonic blocks. This idea has been used to compute the blocks by making an ansatz and fixing undetermined coefficients either by Ward identities [65, 66, 67, 68, 69, 70, 71, 72] or appropriate Casimir differential equations [73, 74, 75, 76, 77] (sometimes, the integrals that define superconformal partial waves can also be evaluated directly, [78, 79]). What is common to these works is that they either focus on correlators of short operators, or otherwise restrict to the superprimary component of blocks by setting all Grassmann variables to zero. It is desirable to extend the analysis and derive all components of superconformal partial waves (termed *long blocks*) because they lead to a larger set of crossing equations for the same OPE data. First steps in this direction have been performed in [80, 81, 82, 83].[2] The question of long superconformal blocks will occupy a significant place in this thesis.

[2]There is another, very general but less explicit, approach to superconformal partial waves proposed in [84].

Chapter 4

Conformal Symmetry and Group Theory

In this chapter, we will analyse the properties of the conformal group and its representations in some detail. Since the conformal group is locally isomorphic to $SO(d+1,1)$, we are led to the field of representation theory of non-compact semisimple Lie groups.

The need to extend representation theory from the case of compact groups, solved so successfully by the Peter-Weyl theorem, to non-compact ones arose in quantum theory, through the work of Dirac and Wigner. The basic reason for this interest lies in the fact that the Hilbert space of a quantum system carries a (projective) unitary representation of its symmetry group. An immediate observation is that a unitary representation of a non-compact group has to be infinite-dimensional. For, a faithful finite-dimensional unitary representation $\pi : G \to \mathrm{Aut}(\mathbb{C}^n)$ would realise G as a closed bounded subset of $\mathbb{C}^{n^2}$. However, all such subsets are compact by the Heine-Borel theorem.

Unitary irreducible representations (UIRs) of the physically most important groups were constructed by Stone and von Neumann (Heisenberg group 1930, 1931), Wigner (Poincare group 1939), Bargmann ($SL(2,\mathbb{R})$ 1947) and Gelfand and Naimark (Lorentz group 1947). In fact, as was shown by Gelfand and his collaborators, any locally compact group possess UIRs in abundance.

The starting point for the construction of representations is the group algebra $L^1(G, dg)$. This is the space of L^1-functions on the group under the convolution product

$$(f_1 \star f_2)(h) = \int dg\ f_1(g) f_2(g^{-1}h)\ . \tag{4.1}$$

For a finite group, this is simply the algebra spanned by vectors $\{e_g\}$ where $g \in G$, with the multiplication $e_{g_1} e_{g_2} = e_{g_1 g_2}$. For infinite groups, we can consider various classes of functions, such as continuous functions $C(G)$ or functions with finite support $\mathbb{C}[G]$, that all give rise to different versions of the group algebra. Given a function $f \in L^1(G, dg)$ and a representation π of G on a Hilbert space $\mathcal{H}$, we define the operator $\pi(f)$ by

$$\pi(f) = \int_G dg\ f(g)\pi(g)\ . \tag{4.2}$$

In this way, we obtain a representation of the group algebra due to

$$\pi(f_1)\pi(f_2) = \int dg_1 dg_2 \; f_1(g_1) f_2(g_2) \pi(g_1 g_2) = \pi(f_1 \star f_2) \; .$$

So, there is a close connection between representations of G and those of $L^1(G, dg)$. The correct version of the group algebra to use turns out to be the so-called *enveloping C^*-algebra of G*, denoted $C^*(G)$. It is defined as the completion of $L^1(G, dg)$ with respect to the norm

$$||f|| := \sup\{||\pi(f)||, \; \pi \in \hat{G}\} \; .$$

Here, π runs over the dual group of G (the set of its unitary irreducible representations). With such a definition, there is an isomorphism between categories of unitary representations of G and non-singular representations of $C^*(G)$.

Therefore, representation theory for non-compact groups can be studied using C^*-algebra techniques. The latter were developed by Gelfand, Naimark and Segal using the notion of a *state*, which is simply a positive linear functional on a C^*-algebra A. It is easy to see that all states of a C^*-algebra form a convex set. Boundary points of this set are called pure states. There are certain relations between states on A, representations of A and ideals in A, that can be summarised as surjective maps

$$\{\text{states}\} \to \{\text{representations}\} \to \{\text{ideals}\} \; .$$

These maps specialise to another set of surjections

$$\{\text{pure states}\} \to \{\text{irreducible representations}\} \to \{\text{primitive ideals}\} \; .$$

Given a representation π of A on a Hilbert space $\mathcal{H}$ and a unit cyclic vector ξ in $\mathcal{H}$ one constructs a state as

$$\omega : A \to \mathbb{C}, \quad \omega(a) = \langle \pi(a)\xi, \xi \rangle \; .$$

If two pairs (π, ξ) and (π', ξ') produce the same state, there is a unitary equivalence between their Hilbert spaces $U : \mathcal{H} \to \mathcal{H}'$ that intertwines between π and π' and such that $U(\xi) = \xi'$. In the case of π being irreducible, Schur's lemma implies that U is unique.

The converse process, in which one starts form a state ω and obtains from it a pair (π, ξ) goes under the name of the Gelfand-Naimark-Segal construction. One uses the state ω to construct a positive semi-definite bilinear form on A via $\langle x, y \rangle = \omega(y^* x)$. The Hilbert space of the representation is constructed as (the completion of) the quotient of A by the set of zero-norm elements. The algebra A acts on this space simply by reducing the left-regular representation. One shows that π is irreducible if and only if the state from which one started was pure. Therefore, pure states correspond to rays in irreducible representations of A.

It follows that the first surjection in the above diagram becomes a bijection precisely when each irreducible representation of A contains only one ray. In other words, irreducible representations of A are one-dimensional. This only happens when A is commutative. For a group algebra $C^*(G)$, it means in turn that G is abelian. For non-abelian groups, there are many more states than representations.

The question of when the second arrow in the diagram becomes a bijection is considerably more complicated. Primitive ideals are defined as kernels of irreducible representations. Therefore, we are asking whether irreducible representations of A are completely determined by their kernels. This is not true for all C^*-algebras (those for which it is are said to be of class I), but by a theorem of Harish-Chandra it is true for group algebras of semisimple Lie groups.

The relation between unitary representations and states enables one to prove the existence of many such representations. Further, the decomposition into irreducible components corresponds to writing some arbitrary element of the convex set of states as a linear combination of its boundary points. Standard results of convex geometry (Choquet's theorem) tell us that such a decomposition is always possible. Therefore, unitary representations are completely reducible.

Theorem (Gelfand-Rajkov) Every unitary representation π of a locally compact group G on a Hilbert space can be decomposed into a direct integral of irreducible representations.

It turns out that for semisimple Lie groups, the dual group $\hat{G}$ admits a much more concrete description. All UIRs come in series of three types - principal, discrete and complementary. They are constructed using *parabolic induction* (in fact, parabolic induction is used to construct non-unitary irreducible representations as well). Given a parabolic subgroup $P \subset G$, the representation space consists of vector-valued functions on G that are covariant with respect to the (right) regular action of P. On such a function, group elements act simply by (left) multiplication of the argument. Equivalently, one can realise the representation on vector-valued functions on the coset space G/P.

The infinitesimal action of the Lie algebra in a parabolically induced representation of G is by first order differential operators. When these operators are made to act on an appropriate space of polynomials of bounded degree, one arrives at finite-dimensional representations. We remark that there is a notion of induction for the Lie algebra $\mathfrak{g}$ itself which is not equivalent to taking the infinitesimal action of the induced G-module. Rather, the two are related by duality.

The chapter is organised as follows. We begin by discussing topology of several locally isomorphic groups that can all be in different circumstances called the conformal group. Next, we describe three important decompositions of this group. The first one is the Bruhat (or Gauss) decomposition and it is closely related to transformation properties of fields in a conformal theory, thus being very relevant for our considerations. The second one is the Iwasawa decomposition, which is of central importance in representation theory. It will also find applications when we consider defect CFTs later. Finally, the Cartan decomposition is used to define spherical functions on the group and we will discuss it further in the next chapter to make contact with integrable systems. After this part, we will introduce an important class of parabolically induced representations, called the non-unitary principal series, and explain how all UIRs of the conformal group are constructed from them. The final part discusses the Lorentzian conformal group.

In treatment of general topics from group and representation theory, we mostly follow [85, 86]. Their specialisation to the conformal group is made after [87].

4.1 Topology

The Euclidean conformal group is locally isomorphic to $SO(d+1,1)$. There are several Lie groups with the Lie algebra $\mathfrak{so}(d+1,1)$ that we will now describe.

We start with the group $O(d+1,1)$ of pseudo-orthogonal matrices. This group has four connected components. Two of these are contained in $SO(d+1,1)$ and the identity component is denoted by $G = SO^+(d+1,1)$. The fundamental group of $SO^+(d+1,1)$ depends on the value of d - we have

$$\pi_1(SO^+(1,1)) = 1, \quad \pi_1(SO^+(2,1)) = \mathbb{Z}, \quad \pi_1(SO^+(d+1,1)) = \mathbb{Z}_2,\ d \geq 2\ . \tag{4.3}$$

These results follow from the fact that π_1 of any Lie group coincides with that of its maximal compact subgroup, which is in the case of $SO^+(d+1,1)$ the group $SO(d+1)$. We will mostly be interested in $d \geq 3$ for which the fundamental group is $\mathbb{Z}_2$. The simply-connected double cover of $SO^+(d+1,1)$ will be denoted by $\tilde{G} = \mathrm{Spin}(d+1,1)$. We have

$$Z(\tilde{G}) \cong \pi_1(G)\ . \tag{4.4}$$

The canonical projection from $\tilde{G}$ to G will be denoted by Π. Since any representation π of G gives a representation of $\tilde{G}$ by precomposing with the projection, $\Pi \circ \pi$, the simply connected group $\tilde{G}$ is often the best candidate for the symmetry group of the field theory.

The conformal inversion is an element of $O(d+1,1)$ not connected to the identity and thus may not be part of the symmetry. However, by composing it with a reflection in any hyperplane, we get an element of $SO^+(d+1,1)$. We will pick the hyperplane orthogonal to the the d-th unit vector e_d and denote the reflection by s_{e_d}. As the action of $SO^+(d+1,1)$ on the compactified Euclidean space is faithful, $I \circ s_{e_d}$ is a unique element of G. In $\mathrm{Spin}(d+1,1)$ there are two elements that project to it, one of which is

$$w = e^{\pi \frac{K_d - P_d}{2}}\ . \tag{4.5}$$

The element $w \in \tilde{G}$ is called the Weyl inversion. One can check that its square is the non-trivial element of the centre $Z(\tilde{G})$, i.e. that $w^2 = -1$. By a slight abuse of notation, we will use the same letter w for the element (4.5) of $SO^+(d+1,1)$.

4.2 Decompositions of the conformal group

Let G be a real semisimple connected linear non-compact Lie group and K_I its maximal compact subgroup. We denote their Lie algebras by $\mathfrak{g}$ and $\mathfrak{k}_I$. It can be shown that there exists an involutive automorphism θ of $\mathfrak{g}$, called a Cartan involution, for which $\mathfrak{k}_I$ is the stationary subspace. Since $\theta^2 = 1$, its eigenvalues are ± 1 and we denote the -1 eigenspace by $\mathfrak{p}$. Then

$$\mathfrak{g} = \mathfrak{k}_I \oplus \mathfrak{p}\ . \tag{4.6}$$

Notice that the Killing form is positive-definite on $\mathfrak{p}$ and negative-definite on $\mathfrak{k}_I$. Let $\mathfrak{a}_I$ be a subalgebra of $\mathfrak{g}$ entirely contained in $\mathfrak{p}$. Since θ is an automorphism we have

$$[\mathfrak{k}_I, \mathfrak{k}_I] \subset \mathfrak{k}_I, \quad [\mathfrak{k}_I, \mathfrak{p}] \subset \mathfrak{p}, \quad [\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}_I\ . \tag{4.7}$$

Therefore $[\mathfrak{a}_I, \mathfrak{a}_I]$ is contained both in $\mathfrak{p}$ and in $\mathfrak{k}_I$, so $\mathfrak{a}_I$ must be abelian. From now, we will assume that $\mathfrak{a}_I$ is of maximal possible dimension (any two such subalgebras are conjugate to one another) and denote the corresponding abelian subgroup of G by A_I. The dimension of $\mathfrak{a}_I$ is called the real rank of G.

The Lie algebra $\mathfrak{g}$ carries the representation of $\mathfrak{a}_I$ under the adjoint action and we denote the space spanned by positive weight vectors (restricted roots) by $\mathfrak{n}$. Then one can show that $\mathfrak{n}$ is a maximal nilpotent subalgebra of $\mathfrak{g}$ and that the latter decomposes as

$$\mathfrak{g} = \mathfrak{k}_I \oplus \mathfrak{a}_I \oplus \mathfrak{n} \ . \tag{4.8}$$

This is the Iwasawa decomposition of $\mathfrak{g}$. Of fundamental importance in the representation theory of G is the corresponding group decomposition

$$G = K_I A_I N \ . \tag{4.9}$$

Let us introduce some more notation. The centraliser of A_I in K_I will be denoted M_I. The group $P = M_I A_I N$ is called the minimal parabolic subgroup of G. Other parabolic subgroups P' are defined by $P \subset P' \subset G$. Since we shall have no occasion to consider these other groups, we will often drop the adjective "minimal" when referring to P.

We wish to describe two more standard decompositions of G. For the first one, let $\mathfrak{m}$ be the space spanned by negative restricted roots of $\mathfrak{a}_I$. Then the following factorisation holds

$$G = M M_I A_I N \ . \tag{4.10}$$

This is usually referred to as the Gauss decomposition. Finally, the Cartan decomposition reads

$$G = K_I A_I K_I \ . \tag{4.11}$$

Iwasawa and Cartan decompositions are global, while the Gauss decomposition is not. However, the set of Gauss-decomposable elements is dense in G, which will suffice for applications that we have in mind. More important for our considerations is the question of uniqueness of these factorisations. It is possible to show that factorisations (4.9) and (4.10) are unique. The Cartan decomposition is not unique as soon as the stabiliser of A_I in K_I is non-trivial, which is often the case.

For the conformal group $G = SO^+(d+1, 1)$, the maximal compact subgroup is $K_I = SO(d+1)$, generated by rotations and differences $P_\mu - K_\mu$ of translation and special conformal generators. The abelian group $A_I = SO(1, 1)$ is that of dilations and $M_I = SO(d)$ is the group of rotations. Finally, M and N are the groups of translations and special conformal transformations, respectively. Note that these two groups are actually abelian, as opposed to just being nilpotent.

Apart from the factorisation (4.9) we will sometimes use a closely analogous decomposition $G = M A_I K_I$ and refer to both of them as Iwasawa decompositions. The order of factors in these decompositions is a matter of convention.

In the context of the conformal group, the Gauss decomposition is also known as the Bruhat decomposition. It is the unique factorisation of conformal transformations into translations, rotations, dilations and special conformal transformations. Validity of the Bruhat decomposition is clear near the identity $e \in G$ by the corresponding decomposition of the conformal Lie algebra

$$\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{n} \oplus \mathfrak{k} \ . \tag{4.12}$$

Recall that we denote by $\mathfrak{k}$ the Lie algebra of dilations and rotations. For any decomposable group element, we will write

$$g = m(g)\ n(g)\ k(g)\ . \tag{4.13}$$

The Lie algebra $\mathfrak{p}$ of the parabolic subgroup P consists of all elements in $\mathfrak{g}$ that have a non-positive dilation weight, $\mathfrak{p} = \mathfrak{k} \oplus \mathfrak{n}$. This is consistent with the notation used in previous chapters. The conformally compactified space is diffeomorphic to the quotient G/P. We will write

$$m(x) = e^{x^\mu P_\mu}, \quad n(x) = w^{-1} m(x) w = e^{(s_{e_d} x)^\mu K_\mu}\ . \tag{4.14}$$

If the space is realised as a right quotient of the conformal group as above, the action of conformal transformations is the one that descends from the left-regular action of G on itself. Therefore, we can write

$$g m(x) = m(y(x,g))\ n(z(x,g))\ k(x,g), \tag{4.15}$$

with $y(x,g) = gx$ and some functions $z(x,g)$ and $k(x,g)$. It is not difficult to determine these functions for all different types of conformal transformations. Obviously

$$z(x, m(x')) = 0, \quad k(x, m(x')) = 1, \quad z(x,k') = 0, \quad k(x,k') = k'\ . \tag{4.16}$$

This follows from the grading with respect to the dilation weight and the Baker-Campbell-Hausdorff formula. Finally, for $g = w$ we have

$$z(x,w) = -x, \quad k(x,w) = |x|^{-2D} s_{e_d} s_x\ . \tag{4.17}$$

It is important to understand how the Iwasawa and Bruhat decompositions are related to each other. For the conformal group, we can spell out the relation explicitly. It will be known as soon as we find the Iwasawa factors of $g = e^{x^\mu K_\mu}$. Elements of G which are Bruhat factors of other types, that is, translations, rotations and dilations, are by themselves Iwasawa factors as well. We have

$$e^{x^\mu K_\mu} = e^{\frac{x^\mu}{1+x^2} P_\mu} (1+x^2)^{-D} k_I(x)\ . \tag{4.18}$$

Here, the last factor $k_I(x)$ reads in the $(d+2)$-dimensional representation of G

$$k_I(x) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1-x^2}{1+x^2} & \frac{-2x^\mu}{1+x^2} \\ 0 & \frac{2x^\mu}{1+x^2} & \delta_{\mu\nu} - \frac{2x^\mu x^\nu}{1+x^2} \end{pmatrix}\ . \tag{4.19}$$

We have written the matrix on the right hand side in the block form, as indicated by indices carried by matrix elements. The usefulness of the formula (4.18) will be seen in the following chapters. One can conjugate both sides by w to obtain another variant of it, that we will use frequently

$$e^{x^\mu P_\mu} = e^{\frac{x^\mu}{1+x^2} K_\mu} (1+x^2)^{D} k_I(-x)\ . \tag{4.20}$$

It is possible to read equations (4.18) and (4.20) as Bruhat decompositions of $k_I(x)$ as well

$$k_I(x) = e^{-x^\mu P_\mu} e^{\frac{x^\mu}{1+x^2} K_\mu} (1+x^2)^D = (1+x^2)^{-D} e^{\frac{x^\mu}{1+x^2} K_\mu} e^{-x^\mu P_\mu}\ . \tag{4.21}$$

4.2.1 Cartan decomposition

Apart from the factorisation (4.11) that will play some role in our considerations, we will be also interested in its "non-compact" cousin, $G = KAK$. Here, A is the abelian subgroup of G generated by $\{P_1 + K_1, P_2 - K_2\}$. The KAK decomposition is similar to the Cartan decomposition of the Lorentzian conformal group, in the sense that the factors of the two have the same complexified Lie algebras. We will parametrise A by complex coordinates (u_1, u_2) as

$$a(u_1, u_2) = e^{\frac{u_1+u_2}{4}(P_1+K_1)-i\frac{u_1-u_2}{4}(P_2-K_2)} \equiv e^{\frac{u_1+u_2}{2}A_+-i\frac{u_1-u_2}{2}A_-} \quad . \tag{4.22}$$

Almost all elements of the conformal group can be factorised as

$$g = k_l\, a(u_1, u_2)\, k_r, \tag{4.23}$$

with $k_l, k_r \in K$. The factorisation is far from unique since elements of A commute with the group $B \sim SO(d-2)$ of rotations of the space spanned by $\{e_3, ..., e_d\}$. Consequently, the group element g is invariant under the action $(k_l, k_r) \mapsto (k_l b, b^{-1} k_r)$ with $b \in B$. Any factorisation of the form (4.23) will be referred to as a Cartan decomposition of g.

4.2.2 Decompositions in the vector representation

It will be useful for us to have, besides the abstract notation from above, concrete expressions for various group elements in the $(d+2)$-dimensional vector representation of $SO(d+1, 1)$. In this representation, the Lorenz-like generators (2.6) of the conformal Lie algebra are

$$L_{\alpha\beta} = \eta_{\alpha\gamma}E_{\gamma\beta} - \eta_{\beta\gamma}E_{\gamma\alpha}, \tag{4.24}$$

where $(E_{\alpha\beta})_{ij} = \delta_{\alpha i}\delta_{\beta j}$. Thus in particular

$$P_\mu = E_{1\mu} - E_{\mu 1} - E_{0\mu} - E_{\mu 0}, \quad K_\mu = -E_{1\mu} + E_{\mu 1} - E_{0\mu} - E_{\mu 0} \ . \tag{4.25}$$

We will write matrices in the vector representation in block form. For example

$$x^\mu P_\mu = \begin{pmatrix} 0 & 0 & -x^T \\ 0 & 0 & x^T \\ -x & -x & 0 \end{pmatrix}, \quad x^\mu K_\mu = \begin{pmatrix} 0 & 0 & -x^T \\ 0 & 0 & -x^T \\ -x & x & 0 \end{pmatrix} \ . \tag{4.26}$$

The matrices representing translations and special conformal transformation are easily found using nilpotency of P_μ and K_μ. Namely $(x^\mu P_\mu)^3 = (x^\mu K_\mu)^3 = 0$ and

$$x^\mu x^\nu P_\mu P_\nu = x^\mu x_\mu (E_{00} - E_{11} + E_{01} - E_{10}), \quad x^\mu x^\nu K_\mu K_\nu = x^\mu x_\mu (E_{00} - E_{11} - E_{01} + E_{10}) \ . \tag{4.27}$$

Therefore

$$e^{x^\mu P_\mu} = \begin{pmatrix} 1+\frac{1}{2}x^2 & \frac{1}{2}x^2 & -x^T \\ -\frac{1}{2}x^2 & 1-\frac{1}{2}x^2 & x^T \\ -x & -x & 1 \end{pmatrix}, \quad e^{x^\mu K_\mu} = \begin{pmatrix} 1+\frac{1}{2}x^2 & -\frac{1}{2}x^2 & -x^T \\ \frac{1}{2}x^2 & 1-\frac{1}{2}x^2 & -x^T \\ -x & x & 1 \end{pmatrix} \ . \tag{4.28}$$

The dilations are represented as

$$D = -E_{01} - E_{10}, \quad e^{\lambda D} = \begin{pmatrix} \cosh\lambda & -\sinh\lambda & 0 \\ -\sinh\lambda & \cosh\lambda & 0 \\ 0 & 0 & 1 \end{pmatrix} . \tag{4.29}$$

The relation between Iwasawa and Bruhat decompositions written above now follows from the matrix identity

$$\begin{pmatrix} 1+\frac{1}{2}x^2 & -\frac{1}{2}x^2 & -x^T \\ \frac{1}{2}x^2 & 1-\frac{1}{2}x^2 & -x^T \\ -x & x & 1 \end{pmatrix} =$$
$$= \begin{pmatrix} 1+\frac{1}{2}y^2 & \frac{1}{2}y^2 & -y^T \\ -\frac{1}{2}y^2 & 1-\frac{1}{2}y^2 & y^T \\ -y & -y & 1 \end{pmatrix} \begin{pmatrix} \cosh\lambda & -\sinh\lambda & 0 \\ -\sinh\lambda & \cosh\lambda & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1-x^2}{1+x^2} & \frac{-2x^\mu}{1+x^2} \\ 0 & \frac{2x^\mu}{1+x^2} & \delta_{\mu\nu} - \frac{2x^\mu x^\nu}{1+x^2} \end{pmatrix},$$

where

$$e^\lambda = \frac{1}{1+x^2}, \quad y^\mu = \frac{x^\mu}{1+x^2} . \tag{4.30}$$

Let us also spell out the Weyl inversion in the vector representation

$$w = e^{\pi \frac{K_d - P_d}{2}} = \mathrm{diag}(1, -1, 1, 1, ..., 1, -1) . \tag{4.31}$$

This concludes our discussion of decompositions of the conformal group.

4.3 Unitary irreducible representations

4.3.1 Induced and coinduced representations

A standard method to construct irreducible representations of a Lie group is by induction from its various subgroups. We will now describe this process in a more general context of algebras, which will allow us to treat induction for Lie groups and Lie algebras in a unified manner. It turns out that the induction of representations in the context of groups corresponds to the dual notion of coindunction (or *production*) of the Lie algebra representations, as was first shown by Blattner in [88].

Given any algebra $\mathcal{A}$, a subalgebra $\mathcal{B}$ and a representation $\rho : \mathcal{B} \to \mathrm{End}(W)$ of $\mathcal{B}$, we can define two representations of $\mathcal{A}$ on the following spaces

$$\mathrm{Ind}_{\mathcal{B}}^{\mathcal{A}}\rho = \mathcal{A} \otimes_{\mathcal{B}} W, \quad \mathrm{Coind}_{\mathcal{B}}^{\mathcal{A}}\rho = \mathrm{Hom}_{\mathcal{B}}(\mathcal{A}, W), \tag{4.32}$$

Elements of the first space are linear combinations of vectors $a \otimes w$, under identifications

$$ab \otimes w \sim a \otimes bw, \quad a \in \mathcal{A}, b \in \mathcal{B}, w \in W,$$

and the action of $\mathcal{A}$ is the left regular one. In the second space, elements are $\mathcal{B}$-equivariant maps

$$\varphi : \mathcal{A} \to W, \quad \varphi(ba) = b\varphi(a),$$

and the action now is $(a\varphi)(a') = \varphi(a'a)$. The two modules introduced are called induced and coinduced modules, respectively. We defined them as left $\mathcal{A}$-modules and there is an obvious analogue for right modules. For an arbitrary algebra, induced and coinduced modules are formally related by duality. We shall now explain this relation in the context of representations of Lie groups and Lie algebras.

When studying representations of groups and Lie algebras, one can replace these algebraic objects by associative algebras that have the same representation theory. For groups, this is the group algebra and for Lie algebras, it is the universal enveloping algebra. Thus, the above constructions give definitions of induction and coinduction for groups and Lie algebras. For example, if G is any group, $H \subset G$ a subgroup and ρ a representation of H on the space W, we put $\mathcal{A} = L^1(G)$ and $\mathcal{B} = L^1(H)$. Thus, the induced module of $L^1(G)$ (and thereby G) is

$$\mathrm{Ind}_H^G W = L^1(G) \otimes_{L^1(H)} W,$$

with the left-regular action. In turn, we can view this module as the space of covariant vector-valued functions on G

$$\Gamma = \{f : G \to W \mid f(gh^{-1}) = \rho(h)f(g)\}, \tag{4.33}$$

under the action $(g \cdot f)(x) = f(g^{-1}x)$.

There is a close relation between induced representations of Lie groups and coinduced representations of their Lie algebras. Let G be a Lie group, $H \subset G$ a Lie subgroup and $\mathfrak{g} = Lie(G)$, $\mathfrak{h} = Lie(H)$. Let W be a finite-dimensional representation of H and use the same letter for the derivative representation of $\mathfrak{h}$. Then

$$d(\mathrm{Ind}_H^G W) \cong \mathrm{Coind}_{\mathfrak{h}}^{\mathfrak{g}} W = \mathrm{Hom}_{U(\mathfrak{h})}(U(\mathfrak{g}), W)\ . \tag{4.34}$$

To see how this comes about, recall that the representation space on the right hand side consists of linear maps $U(\mathfrak{g}) \to W$ which commute with the action of $U(\mathfrak{h})$ on $U(\mathfrak{g})$ (by left multiplication) and W. The action of $x \in \mathfrak{g}$ on such a map is given by

$$(x\psi)(A) = \psi(Ax),\ \ A \in U(\mathfrak{g})\ .$$

Let us now consider an analytic function $f : G \to W$. This function defines a linear map on the universal enveloping algebra through its Taylor coefficients

$$\psi : U(\mathfrak{g}) \to W, \quad \psi(A) = \mathcal{R}_A f(e)\ . \tag{4.35}$$

Here $\mathcal{R}_A$ is a differential operator corresponding to the element A of the universal enveloping algebra, constructed out of right-invariant vector fields on G. Conversely, the knowledge of all Taylor coefficients can be used to recover f. If f is has covariance properties as in (4.33) the resulting ψ is also covariant and belongs to $\mathrm{Coind}_{\mathfrak{h}}^{\mathfrak{g}} W$.

We mentioned that there is a formal relation of duality between induced and coinduced representation of arbitrary algebras. For Lie algebras, the duality takes a concrete form

$$\mathrm{Coind}_{\mathfrak{h}}^{\mathfrak{g}}(W^*) \cong \left(\mathrm{Ind}_{\mathfrak{h}}^{\mathfrak{g}} W\right)^*\ . \tag{4.36}$$

To see that this is true, let $V = \mathrm{Ind}_{\mathfrak{h}}^{\mathfrak{g}} W$. Given $f \in V^*$ and $A \in U(\mathfrak{g})$ define the function

$$\psi = \hat{f} : U(\mathfrak{g}) \to W^*, \quad \psi(A)(w) = f(\sigma(A) \otimes w), \tag{4.37}$$

where σ is the antipode in $U(\mathfrak{g})$. It is clear that $\psi(A)$ is an element of W^* and that ψ is a linear map. It also belongs to the coinduced module $\pi = \mathrm{Coind}_{\mathfrak{p}}^{\mathfrak{g}} W^*$. This follows from the computation

$$\psi(BA)(w) = f(\sigma(A)\sigma(B) \otimes w) = \psi(A)(\sigma(B)w) = \big(B(\psi(A))\big)(w) \ .$$

Here, B is an element of $U(\mathfrak{h})$. The last step uses the definition of the dual representation for the algebra $U(\mathfrak{h})$. The map $f \mapsto \psi$ is clearly linear. It also commutes with the action of $U(\mathfrak{g})$. To see this, let $C \in U(\mathfrak{g})$. Then

$$(\hat{Cf})(A)(w) = (Cf)(\sigma(A)w) = f(\sigma(C)\sigma(A)w) = f(\sigma(AC)w) = \psi(AC)(w) = (C\psi)(A)(w) \ .$$

It is a simple matter to show that $f \mapsto \psi$ is a bijection. Therefore, the map establishes an isomorphism between the coinduced representation from W^* and the dual of the induced representation from W.

In the context of conformal field theory, the states of the Hilbert space belong to representations induced from a parabolic subalgebra of the conformal Lie algebra $\mathfrak{g}$. These representations are known as the parabolic Verma modules. Their dual modules form the algebraic principal series of $\mathfrak{g}$. By above, algebraic principal series are naturally realised as coinduced representations. Their name steams from the connection with the principal series representation of the conformal group G that we shall now define. The space of smooth vectors in a principal series representation of G forms the algebraic principal series representation of the Lie algebra $\mathfrak{g}$.

4.3.2 Non-unitary principal series representations

Consider a Lie group G with an Iwasawa decomposition $G = K_I A_I N$ and the corresponding minimal parabolic subgroup $P = M_I A_I N$. A *non-unitary principal series representation* of G is a representation π induced from a finite-dimensional irreducible representation ρ of P that is trivial on N. Such a representation is also called *elementary*. If the inducing representation ρ is trivial on M_I as well, then it is simply given by a character of A_I. The corresponding elementary representation is said to be spherical.

As their name suggest, elementary representations are typically non-unitary. However for a particular choice of ρ they can become so. Moreover, elementary representations can be used to construct all UIRs, at least for many important groups. In particular, this is true for the conformal group, [87]

Fact Every unitary irreducible representation of $SO^+(d+1,1)$ is equivalent to a subquotient of an elementary representation.

Recall that in the conformal case, the representation $\rho = (\Delta, \lambda)$ is specified by the conformal dimension and spin. An elementary representation is said to be *of type I* if $\lambda = (0, ..., 0, l)$ is a symmetric traceless tensor. Elementary representations are generically irreducible. Further, they are multiplicity free - they contain each irreducible representation of K_I at most once. To see this, notice that the Frobenius reciprocity tells us that the multiplicity of a representation μ of $K_I \sim SO(d+1)$ in $\pi = \mathrm{Ind}\rho$ is equal to the multiplicity of λ in μ. Now, the claim follows from the well-known fact that the restriction of irreducibles from $SO(d+1)$ to $SO(d)$ is multiplicity-free.

From the Bruhat decomposition, we see that if a vector-valued function f on the group belongs to an elementary representation, it is completely determined by the values it assumes on M. This means that we can realise an elementary representation on the space of vector-valued functions on M. This realisation is nothing else but the representation on the space of fields in a CFT that was defined in a previous chapter. We will elaborate on this point later on.

Now, we turn to the questions of which elementary representations are unitary. This leads to the classification of unitary irreducible representations of G. Such representations come in following series

Principal series

These are elementary representations with $\Delta \in d/2 + i\mathbb{R}$ and arbitrary λ. One can define an invariant inner product on the representation space V as

$$(f_1, f_2) = \int_N dx\ \langle \bar{f}_1, f_2 \rangle, \tag{4.38}$$

where integration is over a submanifold of G that intersects each P-orbit once and $\langle f_1, f_2 \rangle$ is the standard inner product on W. The condition $\Delta + \bar{\Delta} = d$ ensures that (4.38) is independent of how we choose the section of P-orbits. It leads to the constraint on Δ written above.

Type I complementary series

For $\Delta \neq d/2 + ic$, the inner product (4.38) is not well-defined. However, in some cases, there exist other invariant scalar products which make the elementary representations unitary. These constitute the complementary series. They are elementary representations of type I with the following constraints on Δ

$$l = 0:\ 0 < \Delta < d, \quad l > 0:\ 1 < \Delta < d - 1, \tag{4.39}$$

Complementary series representations can be obtained by analytic continuation of the holomorphic discrete series of $\tilde{SO}(d, 2)$. Unlike the principal and the discrete series representations, they do not appear in the decomposition of the regular representation $L^2(G)$, so one can say that these representations lie outside the scope of harmonic analysis on G.

Discrete series

Discrete series representations are defined by the condition that their matrix coefficients are square-integrable functions on the group. For such a representation π on a Hilbert space $\mathcal{H}$, there exist a positive real number $d(\pi)$, called its formal dimension, such that

$$\int_G dg\ (\phi, \pi(g)\psi)^2 = \frac{1}{d(\pi)} ||\phi||^2 ||\psi||^2, \quad \forall \phi, \psi \in \mathcal{H}\ . \tag{4.40}$$

Discrete series representations are not elementary. Rather, they are subquotients of elementary representations. As indicated by their name, discrete series have $\Delta = d/2 + n,\ n \in \mathbb{N}$. In order for this series to exist, the group G and its maximal compact subgroup have to be of the same rank. Therefore, only conformal groups with odd d admit discrete series representations.

Example The conformal group in one dimension is isomorphic to $SL(2, \mathbb{R})$. It has two series of discrete representations commonly denoted as T_l^- and T_l^+. Here, l is a half-integer and in the first case $l \leq -1$, while in the latter $l \geq 1$. We describe the series T_l^-, the other one being similar. The representation space is that of analytic functions in the upper half plane H^+, square-integrable with respect to the scalar product

$$(F_1, F_2) = \frac{1}{2\Gamma(-2l-1)} \int_{-\infty}^{\infty} dx \int_0^{\infty} dy\, y^{-2l-2} F_1(z)\overline{F_2(z)} \; . \tag{4.41}$$

The action of the $SL(2,\mathbb{R})$ matrix $g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ on a function F is defined as

$$\big(T_l^-(g)F\big)(z) = (\beta z + \delta)^{2l}\, F\Big(\frac{\alpha z + \gamma}{\beta z + \delta}\Big) \; . \tag{4.42}$$

The T_l^- are lowest-weight representations. Note however, that it is not the case that there is some eigenvector of D that is annihilated by K, from which one would construct the module by acting with P. Indeed, this would imply that D has a discrete spectrum and only spacelike vectors (with respect to the Killing form) in $\mathfrak{sl}(2,\mathbb{R})$ have discrete spectra in T_l^-. To exhibit the lowest-weight structure, one needs to consider an $SO(2)$ subgroup generator H and its raising and lowering operators $E_\pm$, which are complex linear combinations of D, P and K. In the above realisation, the lowest-weight vector is the function $(z+i)^{2l}$. For more details, the reader is referred to the original paper [89], or the books [86, 90].

Contrary to the discrete series, representations from the unitary principal series have no lowest (or highest) weight vectors.

4.4 Euclidean and Lorentzian signature

Let us now make a few comments about the Lorentzian conformal group and its representations. This group in four dimensions was analysed by Mack and shown to posses some rather different properties compared to its Euclidean counterpart, [91].

Theorem (Mack) Let π be a unitary, irreducible representation of $\tilde{G}$ of positive energy. Then π possesses a unique lowest-weight vector. Any two such representations with the same lowest weight are unitarily equivalent.

Let us clarify the terminology. Similarly to the Euclidean case, the group $O(d,2)$ has four connected components and the identity component is denoted by $G = SO^+(d,2)$. The maximal compact subgroup of G is $K_L = SO(d) \times SO(2)$. Assuming that $d \geq 3$, we conclude that the fundamental group of G is

$$\pi_1(SO^+(d,2)) = \mathbb{Z}_2 \times \mathbb{Z} \; . \tag{4.43}$$

The universal covering group is denoted by $\tilde{G} = \tilde{SO}^+(d,2)$ and is an infinitely-sheeted covering. The relation (4.4) is still valid and we denote the centre of $\tilde{G}$ by $\Gamma = \langle \gamma_1, \gamma_2 \rangle$.

Let us now specialise to four spacetime dimensions. The Iwasawa decomposition of the Lorentzian conformal group has the form

$$G = K_L A_L N_L,$$

where now A_L is a two-dimensional abelian group of dilations and Lorentz boosts in the z-direction and N_L is the six-dimensional group that consists of special conformal transformations and the two-dimensional abelian group contained in the Wigner's little group of a vector pointing in the z-direction. Both A_L and N_L are simply-connected and the Iwasawa decomposition of the universal cover $\tilde{G}$ is

$$\tilde{G} = \tilde{K_L} A_L N_L, \tag{4.44}$$

where $\tilde{K_L} \cong \mathbb{R} \times SU(2) \times SU(2)$ is the infinitely sheeted universal covering group of K_L. It is worth noting that the centre $Z(\tilde{G})$ is contained in $\tilde{K_L}$.

A representation $(\pi, \mathcal{H})$ of $\tilde{G}$ is said to have positive energy if the operator P^0 is positive on its domain of definition in $\mathcal{H}$. Positivity of P^0 implies that of the *conformal Hamiltonian*

$$H_0 = \frac{1}{2}(P^0 + K^0) \ . \tag{4.45}$$

Indeed, denote by $D_\pi(P^0)$ the domain of definition of $\pi(P^0)$. It is an invariant subset of the representation space. Given $\psi \in D_\pi(P^0)$, let $\psi' = \pi(w^{-1})\psi$ (recall that w is the Weyl inversion). We have

$$\langle \psi, \pi(H_0)\psi \rangle = \frac{1}{2}\langle \psi, \pi(P^0)\psi \rangle + \frac{1}{2}\langle \psi, \pi(K^0)\psi \rangle = \frac{1}{2}\langle \psi, \pi(P^0)\psi \rangle + \frac{1}{2}\langle \psi', \pi(P^0)\psi' \rangle \geq 0,$$

using $wP^0w^{-1} = K^0$. It is important to note that $\gamma_2 = we^{i\pi H_0}$. Therefore, for any natural number n

$$\gamma_2^{2n} = (we^{i\pi H_0})^{2n} = e^{2\pi i n H_0} \ .$$

The element γ_2^{2n} is central, so in the representation $(\pi, \mathcal{H})$ is acts as a number. It follows from the spectral theorem for self-adjoint operators that $\pi(H_0)$ has an integer-spaced spectrum of the form

$$S(H_0) = \{h_0 + m \mid m \in \mathbb{N}_0\} \ . \tag{4.46}$$

The restriction that $m \geq 0$ comes from the positive-energy condition.

Consider now the restriction of $\mathcal{H}$ to the group $\tilde{K_L}$. The non-compact factor $\mathbb{R}$ of $\tilde{K_L}$ is generated by H_0 so the restriction decomposes over lowest-weight irreducible representations labelled by a discrete set

$$\mathcal{H} = \bigoplus m(\mu) V_\mu, \tag{4.47}$$

just as if the group $\tilde{K_L}$ was compact. It is a standard result from representation theory of semisimple Lie groups of finite centre that when a UIR is restricted to the maximal compact subgroup K, all irreducibles of K appear with finite multiplicities. For the restriction to $\tilde{K_L}$ from above, Lüscher was able to show that the multiplicities $m(\mu)$ are also finite. Furthermore, let us denote by V the vector space (4.47) with the Hilbert space sum being replaced by the algebraic direct sum. That is, only finite linear combinations of vectors from different summands are allowed. Elements of V are called $\tilde{K_L}$-finite vectors. The space V is the common domain of definition for operators $X \in \mathfrak{g}$, on which they are essentially self-adjoint. Conversely, any representation of $\mathfrak{g}$ by skew-hermitian operators integrates to a representation of $\tilde{G}$. Equivalence of representations of $\mathfrak{g}$ implies that of representation of the group $\tilde{G}$.

We can now show that π is a lowest-weight module. For, let h_0 be the lowest eigenvalue of $\pi(H_0)$. Then there has to be a weight $\mu = (h_0, j_1, j_2)$ in the above decomposition (4.47). Denote its lowest-weight vector by v

$$\pi(H_0)v = h_0 v, \quad \pi(H_i)v = -j_i v \ .$$

Here $H_i = H_{1,2}$ are the Cartan generators of the two $SU(2)$-s in $\tilde{K}_L$. By the lowest-weight condition, v has to be annihilated by lowering operators of $\{H_0, H_1, H_2\}$ and is therefore a lowest-weight vector for the whole $\mathfrak{g}$-module $\mathcal{H}$ (here it is manifestly important that $\mathfrak{g}$ and $\mathfrak{k}_L$ are of the same rank). From here, Mack's result follows by standard Lie-algebra-theoretic arguments. The last part of the claim is an instance of the general fact that a lowest-weight $\mathfrak{g}$-module admits at most one invariant inner product, up to a normalisation.

Despite having lowest-weight vectors, representations π can be constructed in a similar manner to principal series ones. This is similar to the example of $SL(2, \mathbb{R})$, where the discrete and principal series have very different properties, but are realised by the same kind of differential operators that act on appropriate classes of functions. For the function space realisation of positive energy UIRs of $\tilde{G}$, the reader is referred to [91].

Chapter 5

Harmonic Analysis and Quantum Integrable Systems

Since the 18^{th} century, mathematical physics made use of various functions such as Legendre polynomials, Bessel functions, Jacobi polynomials etc, that were more complicated than trigonometric and exponential functions, and yet enjoyed many remarkable properties and relations. One refers to these kind of functions by a the vague term *special functions*. Initially, most of the special functions arose as solutions of the Laplacian eigenvalue problem in separated variables. In quantum mechanics, other functions such as Hermite and Laguerre polynomials appeared as solutions to Schrödinger problems.

There have been several attempts to unify the theory of special functions, that seemed to consist of a very large number of curious identities without any apparent order. An early unification was obtained by Chebyshev in the 19^{th} century, who constructed a general theory of orthogonal polynomials. The notion of self-adjoint operators reveals some basic properties of these polynomials and other families of special functions parametrised by the spectral value. Another unification was achieved by classifying differential equations that special functions satisfy. In this direction, a very important role is played by Gauss' hypergeometric function

$$ {}_2F_1(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!} \ . \tag{5.1} $$

In fact, most one-variable special functions that one meets in applications are either particular cases of ${}_2F_1$ or limits thereof.

However, a truly unified view on special functions came from another branch of mathematics - the representation theory of Lie groups. In 1947, Bargmann found that unitary irreducible matrix elements of $SL(2,\mathbb{R})$ could be expressed in terms of the hypergeometric function. Moreover, matrix elements of discrete series representations could be written using Jacobi polynomials. In a similar way, Bessel functions are related to irreducible matrix elements of the group $ISO(2)$ of isometries of the Euclidean plane. Fundamentally, the Lie groups appear as symmetry groups of the operator whose eigenfunctions one is considering (Laplacian, d'Alembertian...).

Early observations of Bargmann were vastly extended by many others and notably the Gelfand school. Curious properties of special functions were derived from the group multiplication law, Clebsch-Gordan decompositions and other basic constructions of representation theory.

Matrix elements of unitary irreducible representations of a general Lie group G are eigenfunctions of the Laplace-Beltrami operator on G (at least for the principal and discrete series). They are functions of $\dim G$ variables and trying to write them in terms of a few special functions like ${}_2F_1$ is presumably a hopeless task. Thus, for higher-dimensional Lie groups, only very special irreducible matrix elements are expected to have a nice theory, and indeed such a theory exists for the so-called spherical functions and some of their generalisations. To give a sense of the scope of these generalisations, we say that a Lie group G and a Lie subgroup K form a *Gelfand pair* if the convolution algebra of left-right K-invariant functions on G is commutative. Elements of this algebra are called K-spherical functions and satisfy many interesting properties.

The group Laplacian Δ commutes with left and right regular actions and can therefore be restricted to any space of covariant functions. On this space, one can regard Δ as an operator in a smaller number of variables. One often observes that the reduced operator coincides with the Hamiltonian of some integrable quantum problem. In fact, integrability of the problem can be *explained* from its group-theoretic origin as higher order Casimirs (which act as differential operators on $C^\infty(G)$ and reduce to the space of covariant functions) provide integrals of motion. Famously, trigonometric and hyperbolic Calogero-Moser-Sutherland models are of this kind.

The chapter is organised as follows. We will start by reviewing some elementary results from the representation theory of the simplest noncommutative Lie groups, $SU(2)$ and $SL(2,\mathbb{R})$. Next, we will give a general definition of K-spherical functions and discuss some examples of them. Then we change the topic and introduce the Calogero-Sutherland quantum mechanics problem (and its special case, the Pöschl-Teller Hamiltonian). The two topics are tied together by showing how the BC_2 Calogero-Sutherland Hamiltonian arises from a reduction of the Laplace-Beltrami operator on the conformal group. The last two sections treat somewhat different problems, that still belong to the same broader theme. The first one is about Appell's hypergeometric functions and the second is about Gaudin integrable systems. All results will have a direct relevance for conformal field theories, to be studied in later chapters.

5.1 Simplest groups: $SU(2)$ and $SL(2,\mathbb{R})$

We begin the discussion by considering the simplest noncommutative Lie groups, $SU(2)$ and $SL(2,\mathbb{R})$. Representation theory of these groups is very well-known.

Let us denote the generators of the Lie algebra $\mathfrak{su}(2)$ by M_{ij} and write them as 2×2 matrices $M_{jk} = -\frac{i}{2}\varepsilon_{ijk}\sigma_i$. Elements of the group $SU(2)$ can be parametrised by Euler angles (ϕ,θ,ψ) defined as

$$g(\phi,\theta,\psi) = e^{-\phi M_{12}}e^{-\theta M_{23}}e^{-\psi M_{12}} = \begin{pmatrix} \cos\frac{\theta}{2}e^{i\frac{\phi+\psi}{2}} & i\sin\frac{\theta}{2}e^{i\frac{\phi-\psi}{2}} \\ i\sin\frac{\theta}{2}e^{-i\frac{\phi-\psi}{2}} & \cos\frac{\theta}{2}e^{-i\frac{\phi+\psi}{2}} \end{pmatrix} \equiv \pi_{1/2}(\phi,\theta,\psi)\ . \tag{5.2}$$

The spin-l representation of $SU(2)$ is spanned by the basis vectors $\{|l,m\rangle, m=-l,-l+1,...,l\}$. These are eigenvectors of M_{12} and obey $M_{12}|l,m\rangle = -im|l,m\rangle$. Matrix elements in the spin-l representation are given by

$$t^l_{mn}(\phi,\theta,\psi) = \langle l,m|g(\phi,\theta,\psi)|l,n\rangle = e^{-i(m\phi+n\psi)}d^l_{mn}(\theta)\ . \tag{5.3}$$

Here, the function d^l_{mn} is the the Wigner d-function and it can be expressed in terms of Jacobi polynomials as

$$d^l_{mn}(\theta) = i^{m-n}\sqrt{\frac{(l+m)!(l-m)!}{(l+n)!(l-n)!}}\Big(\sin\frac{\theta}{2}\Big)^{m-n}\Big(\cos\frac{\theta}{2}\Big)^{m+n} P^{(m-n,m+n)}_{l-m}(\cos\theta)\ . \tag{5.4}$$

Therefore, matrix elements of all unitary irreducible representations of $SU(2)$ are very simple special functions.

The group $SL(2,\mathbb{R})$ admits two kinds of Cartan decompositions. If we regard it as at the conformal group in one dimension, then $K = SO(1,1)$ is the non-compact subgroup of dilations. We parametrise the group elements as

$$g(\phi,u,\psi) = e^{\phi D}e^{u\frac{P+K}{2}}e^{\psi D} = \begin{pmatrix} \cosh\frac{u}{2}e^{\frac{\phi+\psi}{2}} & \sinh\frac{u}{2}e^{\frac{\phi-\psi}{2}} \\ \sinh\frac{u}{2}e^{-\frac{\phi-\psi}{2}} & \cosh\frac{u}{2}e^{-\frac{\phi+\psi}{2}} \end{pmatrix}\ . \tag{5.5}$$

This is the KAK decomposition, in the terminology introduced in the last chapter. To write the $K_IA_IK_I$ decomposition, it is useful to regard $SL(2,\mathbb{R})$ as the group $SU(1,1)$. The Lie algebra of the latter is spanned by M_{12}, iM_{23} and iM_{31} and the Euler angles read

$$g(\phi,t,\psi) = e^{-\phi M_{12}}e^{-tiM_{23}}e^{-\psi M_{12}} = \begin{pmatrix} \cosh\frac{t}{2}e^{i\frac{\phi+\psi}{2}} & \sinh\frac{t}{2}e^{i\frac{\phi-\psi}{2}} \\ \sinh\frac{t}{2}e^{-i\frac{\phi-\psi}{2}} & \cosh\frac{t}{2}e^{-i\frac{\phi+\psi}{2}} \end{pmatrix}\ . \tag{5.6}$$

For applications that we have in mind, the decomposition (5.5) is more useful because of the prominent role played by the dilation generator. The second decomposition (5.6) is somewhat more natural in mathematics because the group K_I is compact and thus has a discrete set of unitary representations. By general results, any UIR π of $SL(2,\mathbb{R})$ decomposes over K_I with finite multiplicities, so the restriction to it provides a discrete basis for π.

5.2 Spherical functions

In the previous chapter, we have defined the Cartan decomposition $G = K_IA_IK_I$ for a rather general non-compact semisimple Lie group and another decomposition $G = KAK$ of the Euclidean conformal group. With either of these decompositions, there is an associated space of so-called spherical functions. We will define this space for the KAK decomposition, the $K_IA_IK_I$ case being entirely analogous.

Let ρ_l and ρ_r be two finite-dimensional representations of K with carrier spaces V_l and V_r. The space of K-spherical functions is that of vector-valued functions on G that are covariant with respect to both left and right regular actions of K

$$\Gamma = \Gamma_{\rho_l,\rho_r} = \{f: G\to V_l\otimes V_r \mid f(k_lgk_r) = (\rho_l(k_l)\otimes\rho_r(k_r^{-1}))f(g)\},\quad k_{l,r}\in K,\ g\in G\ . \tag{5.7}$$

It is not difficult to define many K-spherical functions using representation theory. For, let π be an irreducible representation of G on a vector space V. Pick an orthonormal basis $\{e_i\}$ of V. The matrix elements of $\pi_{ij}(g) = \langle e_i|\pi(g)|e_j\rangle$ are functions on the group and satisfy

$$\pi_{ij}(k_lgk_r) = \langle e_i|\pi(k_l)\pi(g)\pi(k_r)|e_j\rangle = \pi_{ik}(k_l)\pi_{kl}(g)\pi_{lj}(k_r), \tag{5.8}$$

where we have simply inserted the identity $1 = |e_k\rangle\langle e_k|$ twice. The representation π restricts to a direct sum of irreducible representations of the subgroup K. Assume that V_l and V_r^* are among the irreducible components and denote their bases by $\{e_a\}$ and $\{e_\alpha\}$, respectively. Then (5.8) tells us that the collection $\{\pi_{a\alpha}\}$ is a K-spherical function on G. If either of the representations $\rho_{l,r}$ is trivial, $(\pi_{a\alpha})$ is called an *associated* spherical function. If both $\rho_{l,r}$ are trivial, it is a *zonal* spherical function, [86].

Example Obviously, any K-spherical function is determined by values it takes on A, so it can be regarded as a function of $\dim A$ variables. In particular, K_I-spherical functions on $SO^+(p+1,1)$ depend on one variable. Let $\pi^{\hat{\Delta}}$ be the non-unitary principal series representation with labels $(\hat{\Delta}, 0)$ (zero spin). Zonal spherical functions on $SO^+(p+1,1)$ are expressible in terms of the hypergeometric function

$$\psi_{p,\hat{\Delta}}(\lambda) = \pi^{\hat{\Delta}}_{00}(g(\lambda)) = (\cosh\lambda)^{-\hat{\Delta}}\ {}_2F_1\left(\frac{\hat{\Delta}+1}{2}, \frac{\hat{\Delta}}{2}; \frac{p+1}{2}; \tanh^2\lambda\right)\ . \tag{5.9}$$

The coordinate λ is the same as in the discussion of the vector representation of $SO^+(p+1,1)$ in the previous chapter. The function $\psi_{p,\hat{\Delta}}$ can be expressed in terms of a Legendre function using a hypergeometric identity. By analytic continuation, one obtains zonal $SO(q-1)$-spherical functions on $SO(q)$

$$\psi_{q,s}(\kappa) = \frac{s!(q-3)!}{(s+q-3)!} C_s^{(q-2)/2}(\cos\kappa)\ . \tag{5.10}$$

Here $C_s^{(q-2)/2}$ is the Gegenbauer polynomial. Associated spherical functions in these two cases are also expressible in terms of ${}_2F_1$. The reader is referred to [86] for details.

5.3 Calogero-Moser-Sutherland models

5.3.1 Pöschl-Teller Hamiltonian

One of the few exactly solvable one-dimensional Schrödinger problems that one learns about in the first course on quantum mechanics was discovered by Pöschl and Teller, [92], and has the Hamiltonian

$$H^{(a,b)}_{PT} = -\partial_u^2 + V^{PT}_{(a,b)}(u) = -\partial_u^2 - \frac{ab}{\sinh^2\frac{u}{2}} + \frac{(a+b)^2 - \frac{1}{4}}{\sinh^2 u}\ . \tag{5.11}$$

The potential depends on two arbitrary parameters a and b. It diverges at the origin, so one often considers the problem on a half-line, e.g. $\{u > 0\}$.

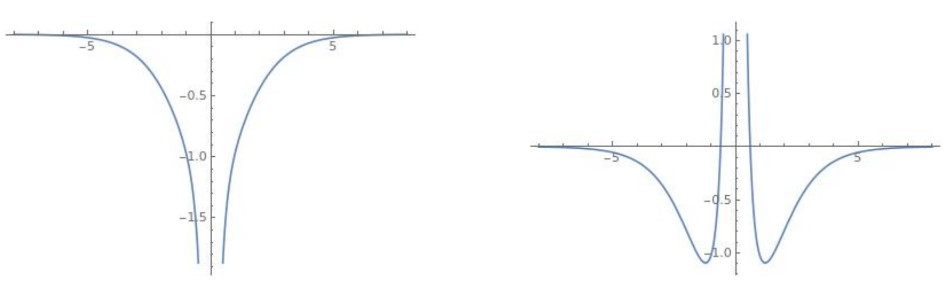

Figure 5.1: Pöschl-Teller Hamiltonians for $(a, b) = (1, 1)$ and $(a, b) = (2, 1)$, respectively

The problem is exactly solvable in the sense that the wavefunctions can be written in terms of standard special functions. Namely, they are given by

$$\Psi_\lambda^{(a,b)} = 4^\lambda c(\lambda,a,b)\left(\tanh\frac{u}{2}\right)^{a-b+\frac{1}{2}}\left(\cosh\frac{u}{2}\right)^{2\lambda}{}_2F_1\Big(\frac{1}{2}+a-\lambda,\frac{1}{2}-b-\lambda,1-2\lambda,\frac{1}{\cosh^2\frac{u}{2}}\Big), \quad (5.12)$$

with the normalisation constant

$$c(\lambda,a,b) = 4^{-\lambda+a+\frac{1}{2}}\frac{\Gamma(a-b+1)\Gamma(2\lambda)}{\Gamma(\frac{1}{2}+\lambda+a)\Gamma(\frac{1}{2}+\lambda-b)}\ .$$

We have labelled the wavefunctions by a parameter λ which is related to the energy as $E=-\lambda^2$. Scattering states have positive energies and $\lambda\in i\mathbb{R}$. They behave as incoming and outgoing plane waves at $u=\infty$, where the potential vanishes. Indeed, eigenfunctions $\Psi_\lambda^{(a,b)}$ are easily seen to have such behaviour. These functions are ill-defined at $u=0$ and there is a unique linear combination of them that is regular at zero. With our normalisations, it is simply their sum $\psi_\lambda^{(a,b)}=\Psi_\lambda^{(a,b)}+\Psi_{-\lambda}^{(a,b)}$, which can be written as

$$\psi_\lambda^{(a,b)} = \left(2\cosh\frac{u}{2}\right)^{2a+1}\left(\tanh\frac{u}{2}\right)^{a-b+\frac{1}{2}}{}_2F_1\Big(\frac{1}{2}+a+\lambda,\frac{1}{2}+a-\lambda,1+a-b,-\sinh^2\frac{u}{2}\Big)\ .$$

Of course, the spectrum of $H_{PT}^{(a,b)}$ is continuous.

It is common to complexify the coordinate u and consider the Pöschl-Teller problem in other regions of the complex plane. By substituting $u=i\mu$ and requiring $\mu\in[0,\pi]$ we arrive at the trigonometric version

$$H_{PT}^{(a,b)} = -\partial_\mu^2-\frac{ab}{\sin^2\frac{\mu}{2}}+\frac{(a+b)^2-\frac{1}{4}}{\sin^2\mu}\ . \quad (5.13)$$

In the trigonometric case, the spectrum is discrete and the hypergeometric functions degenerate to Jacobi polynomials $P_n^{(\alpha,\beta)}$. The wavefunctions now read

$$\psi_n^{a,b}(\mu) = c_n^{(a,b)}\sin^{a-b+\frac{1}{2}}\frac{\mu}{2}\cos^{a+b+\frac{1}{2}}\frac{\mu}{2}\ P_n^{(a-b,a+b)}(\cos\mu),$$

where the normalisation constants are

$$c_n^{(a,b)} = \Big(\frac{2(2n+2a+1)n!\Gamma(n+2a+1)}{\Gamma(n+a-b+1)\Gamma(n+a+b+1)}\Big)^{\frac{1}{2}}\ .$$

5.3.2 BC_N Calogero-Sutherland system

The Pöschl-Teller Hamiltonian admits integrable generalisations to Schrödinger problems that involve an arbitrary number of interacting particles moving on a line. For any root system Φ, it is possible to construct one such generalisation. They are called Calogero(-Moser)-Sutherland (CS) models, [93, 94, 95]. We will describe these models for (non-reduced) root systems BC_N as they are the ones that appear in connection with the pseudo-orthogonal groups, and thereby conformal field theory.

Positive roots of the BC_N root system come in three types

$$\Phi^+ = \{e_i, e_i\pm e_j, 2e_i \mid 1\le i<j\le N\}, \quad (5.14)$$

of lengths 1, $\sqrt{2}$ and 2. Here $\{e_i\}$ denotes the standard orthonormal basis of $\mathbb{R}^N$. If the shortest roots are thrown away, one ends up with the set of positive roots for the Lie algebra $C_N = \mathfrak{sp}(2N)$. The Calogero-Sutherland Hamiltonian associated to Φ reads

$$H^{CS} = -\sum_{i=1}^{N} \partial_{u_i}^2 + V^{CS}(u_i) = -\sum_{i=1}^{N} \partial_{u_i}^2 + \sum_{\alpha\in\Phi^+} \frac{k_\alpha(k_\alpha + 2k_{2\alpha} - 1)\langle\alpha,\alpha\rangle}{4\sinh^2\frac{\langle\alpha,u\rangle}{2}} \ . \tag{5.15}$$

Here u_i are coordinates of the particles and $u = u_i e_i$. Numbers k_α are parameters of the model and are usually referred to as multiplicities. It may seem that there are many of these parameters, but in fact one requires them to depend only on the orbit of the root α under the Weyl group W. Since Φ_+ decomposes into three orbits according to different possible lengths of vectors α, there are in total three parameters. We will express these in terms of another set of parameters $\{a, b, \epsilon\}$ as

$$k_{e_i} = k_1 = -2b, \quad k_{2e_i} = k_2 = a + b + \frac{1}{2}, \quad k_{e_i\pm e_j} = k_3 = \frac{\epsilon}{2} \ . \tag{5.16}$$

For $N = 1$, the Calogero-Sutherland Hamiltonian reduces to the Pöschl-Teller Hamiltonian. For $N = 2$ the potential is

$$V^{CS}_{(a,b,\epsilon)}(u_i) = V^{PT}_{(a,b)}(u_1) + V^{PT}_{(a,b)}(u_2) + \frac{\epsilon(\epsilon-2)}{8\sinh^2\frac{u_1-u_2}{2}} + \frac{\epsilon(\epsilon-2)}{8\sinh^2\frac{u_1+u_2}{2}} \ . \tag{5.17}$$

Integrability of CS models can be studied from various points of view. We will be mostly interested in the group-theoretic origin of Hamiltonians, but mention here another common way to construct them. It uses the so-called Dunkl operators, [96], which are first order "differential operators" that include reflections in the roots. Let us describe them in more details for $N = 2$, the discussion for higher N being similar.

The set of positive roots for the BC_2 system is $\Phi^+ = \{e_1, e_2, e_1+e_2, e_1-e_2, 2e_1, 2e_2\}$. Its Weyl group is isomorphic to the dihedral group, $W \cong D_2$, and can be presented as

$$W = \langle w_1, w_2 \mid w_1^2 = w_2^2 = 1,\ w_1w_2w_1w_2 = w_2w_1w_2w_1\rangle \ .$$

The generators $w_{1,2}$ are the Weyl reflections $w_1 = w_{e_1-e_2}$ and $w_2 = w_{e_2}$. The Weyl group has five irreducible representations, four one-dimensional and one two-dimensional ($1^2+1^2+1^2+1^2+2^2 = 8$). For the two-dimensional representation, there is a corresponding representation on the space of functions of two variables, $(wf)(u) = f(wu)$.

Let $x_i = e^{u_i}$ and $\partial_i = \partial_{u_i}$. Then $x_i^{\pm1}, \partial_i$ and elements of the group algebra $\mathbb{C}[W]$ all act in the appropriate space of functions of (u_1, u_2). They form an algebra and it is a simple matter to verify the relations

$$w_1x_iw_1 = x_{i+1}, \quad w_1x_i^{-1}w_1 = x_{i+1}^{-1}, \quad w_2x_1^{\pm1}w_2 = x_1^{\pm1}, \quad w_2x_2^{\pm1}w_2 = x_2^{\mp1}, \tag{5.18}$$

$$w_1\partial_1w_1 = \partial_2, \quad w_1\partial_2w_1 = \partial_1, \quad w_2\partial_1w_2 = \partial_1, \quad w_2\partial_2w_2 = -\partial_2 \ . \tag{5.19}$$

Together with the obvious relations between coordinates, their inverses and the derivatives, these define the so-called degenerate double affine Hecke algebra, [97]. Among various elements of this algebra, especially interesting are the Dunkl operators

$$y_i = \partial_i - \sum_{\alpha\in\Phi_+} \frac{k_\alpha\langle\alpha, e_i\rangle}{1 - e^{-\langle\alpha,u\rangle}} w_\alpha \ . \tag{5.20}$$

The fundamental property of Dunkl operators is their commutativity, $[y_i, y_j] = 0$. Hamiltonians of the Calogero-Sutherland model are constructed as sums of powers of y_i-s. In the BC_2 case, there are two Hamiltonians

$$H_2 = \sum_{i=1}^{2} y_i^2, \quad H_4 = \sum_{i=1}^{2} y_i^4 \ . \tag{5.21}$$

We have defined the Dunkl operators and CS Hamiltonians as differential operators valued in the group algebra $\mathbb{C}[W]$. Explicitly

$$\begin{aligned}
y_1 &= \partial_1 - \frac{k_1}{1-x_1^{-1}} w_1 w_2 w_1 - \frac{k_3}{1-x_1^{-1}x_2^{-1}} w_2 w_1 w_2 - \frac{k_3}{1-x_1^{-1}x_2} w_1 - \frac{2k_2}{1-x_1^{-2}} w_1 w_2 w_1, \\
y_2 &= \partial_2 - \frac{k_1}{1-x_2^{-1}} w_2 - \frac{k_3}{1-x_1^{-1}x_2^{-1}} w_2 w_1 w_2 + \frac{k_3}{1-x_1^{-1}x_2} w_1 - \frac{2k_2}{1-x_2^{-2}} w_2,
\end{aligned}$$

and

$$\begin{aligned}
H_2 = &\sum_{i=1}^{2} \partial_i^2 - \Big(\frac{1}{4}k_1^2 + \frac{1}{2}k_1 k_2\Big) \sum_{i=1}^{2} \frac{1}{\sinh^2 \frac{u_i}{2}} - k_2^2 \sum_{i=1}^{2} \frac{1}{\sinh^2 u_i} \\
&- \frac{1}{2}k_3^2 \Big(\frac{1}{\sinh^2 \frac{u_1+u_2}{2}} + \frac{1}{\sinh^2 \frac{u_1-u_2}{2}}\Big) + \frac{1}{4}k_1 \Big(\frac{1}{\sinh^2 \frac{u_1}{2}} w_1 w_2 w_1 + \frac{1}{\sinh^2 \frac{u_2}{2}} w_2\Big) \\
&+ k_2 \Big(\frac{1}{\sinh^2 u_1} w_1 w_2 w_1 + \frac{1}{\sinh^2 u_2} w_2\Big) + \frac{1}{2}k_3 \Big(\frac{1}{\sinh^2 \frac{u_1+u_2}{2}} w_2 w_1 w_2 + \frac{1}{\sinh^2 \frac{u_1-u_2}{2}} w_1\Big) \ .
\end{aligned}$$

To get ordinary differential operators, we need to evaluate the elements of the Weyl group in a one-dimensional representation. Doing this for the choice $w_i = 1$ turns H_2 into (minus) the Calogero-Sutherland Hamiltonian (5.15), or explicitly

$$H_{cs}^{(a,b,\epsilon)} = H_{PT}^{(a,b)}(u_1) + H_{PT}^{(a,b)}(u_2) + \frac{\epsilon(\epsilon-2)}{8} \left(\frac{1}{\sinh^2 \frac{u_1-u_2}{2}} + \frac{1}{\sinh^2 \frac{u_1+u_2}{2}}\right) \ . \tag{5.22}$$

Other choices $w_i = \pm 1$ also give the CS Hamiltonian from above with different values of the parameters.

5.3.3 Calogero-Sutherland Hamiltonian from the group Laplacian

We now come to the very important topic of constructing CS Hamiltonians in harmonic analysis. Irreducible (admissible) matrix elements of a Lie group are eigenfunctions of the quadratic and higher order Casimirs constructed out of invariant vector fields. Since they commute with left and right regular actions, these Casimir operators reduce to the space of K-spherical functions and can be thus regarded as differential operators in $\dim A$ coordinates.

For example, the Laplace-Beltrami operator (quadratic Casimir) on $SL(2,\mathbb{R})$ in the Cartan coordinates (ϕ, u, ψ) takes the form

$$\Delta = \partial_u^2 + \coth u \ \partial_u - \frac{1}{\sinh^2 u} \left(\partial_\phi^2 - 2\cosh u \ \partial_\phi \partial_\psi + \partial_\psi^2\right) \ . \tag{5.23}$$

Acting on a K-spherical function f that obeys $\partial_\phi f = af$ and $\partial_\psi f = bf$, it reduces to the operator

$$\Delta_{red} = \partial_u^2 - \coth u\,\partial_u - \frac{1}{\sinh^2 u}\left(a^2 - 2ab\cosh u + b^2\right)\ . \tag{5.24}$$

There are still first order terms in Δ_{red} and they can be eliminated by conjugation with a unique (up to normalisation) function of u. Once this is done, we get the Pöschl-Teller Hamiltonian, namely

$$\sqrt{\sinh u}\ \Delta_{red}\ \frac{1}{\sqrt{\sinh u}} = -H_{PT}^{(a,b)} - \frac{1}{4}\ . \tag{5.25}$$

Calogero-Sutherland models arise in a similar way when the dimension of A exceeds one. For our purposes, the most relevant case is $\dim A = 2$, that occurs for the KAK decomposition of the conformal group.

Let ρ_l and ρ_r be finite-dimensional representations of K. Due to the Cartan decomposition, any K-spherical function (5.7) on the conformal group is uniquely determined by the values it takes on the two-dimensional abelian group A. The reduced Laplacian Δ_{red}, defined by

$$(\Delta f)(k_l a(u_1,u_2) k_r) = \left(\rho_l(k_l)\otimes\rho_r(k_r^{-1})\right)(\Delta_{red} f)(a(u_1,u_2)),$$

again contains first order derivatives with respect to the two coordinates u_1 and u_2. One can eliminate these terms through a transformation of the form

$$f(u_1,u_2) = \omega^{-1/2}(u_1,u_2)\psi(u_1,u_2), \tag{5.26}$$

with the prefactor

$$\omega(u_1,u_2) = 4(-1)^{2-d}(\sinh\frac{u_1}{2}\sinh\frac{u_2}{2})^{2d-2}\coth\frac{u_1}{2}\coth\frac{u_2}{2}|\sinh^{-2}\frac{u_1}{2} - \sinh^{-2}\frac{u_2}{2}|^{d-2}\ . \tag{5.27}$$

Let us assume that the representations $\rho_{l,r}$ are one-dimensional. Then, they are simply two characters of the dilation group. Denote these characters by $2a = \rho_l(D)$ and $2b = \rho_r(D)$. In this case, the conjugated reduced Laplacian is the BC_2 Calogero-Sutherland Hamiltonian

$$\omega^{1/2}\ \Delta_{red}\ \omega^{-1/2} = \frac{1}{2}H_{cs}^{(a,b,d-2)} + \frac{1}{8}(d-1)^2\ . \tag{5.28}$$

Notice that the result holds for any dimension $d \geq 2$. The dependence of the CS model on d is through its parameter $\epsilon = d-2$. At this point, the reader may want to compare these observations to the Dolan-Osborn Casimir equations satisfied by conformal partial waves. They are also second order differential equations in two variables, characterised by three parameters, $2a = \Delta_2 - \Delta_1$, $2b = \Delta_3 - \Delta_4$ and $\epsilon = d-2$. This similarity is of course not accidental and the mapping between Casimir equations and the BC_2 Calogero-Sutherland problem will be given in a later chapter.

5.4 Appell's hypergeometric functions

We have seen how Calogero-Sutherland models provide a sophisticated generalisation of the hypergeometric equation. A much more basic generalisation of ${}_2F_1$ to two variables was found by Appell in the 19^{th} century. Appell's functions are defined by convergent powers series expansions in variables x and y. There are four functions

$$F_1(a,b_1,b_2;c;x,y) = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}(b_1)_m(b_2)_n}{(c)_{m+n}m!n!}x^m y^n, \tag{5.29}$$

$$F_2(a,b_1,b_2;c_1,c_2;x,y) = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}(b_1)_m(b_2)_n}{(c_1)_m(c_2)_n m!n!}x^m y^n, \tag{5.30}$$

$$F_3(a_1,a_2,b_1,b_2;c;x,y) = \sum_{m,n=0}^{\infty} \frac{(a_1)_m(a_2)_n(b_1)_m(b_2)_n}{(c)_{m+n}m!n!}x^m y^n, \tag{5.31}$$

$$F_4(a,b;c_1,c_2;x,y) = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}(b)_{m+n}}{(c_1)_m(c_2)_n m!n!}x^m y^n \ . \tag{5.32}$$

More generally, Horn defined double power series $A_{mn}x^m y^n$ of hypergeometric type by the requiring the two ratios $A_{m+1,n}/A_{mn}$ and $A_{m,n+1}/A_{mn}$ to be rational functions of m and n. The highest degree of the four polynomials that appear in these rational expressions is called the order of the series. It was shown that there are 34 distinct series of order two. These make up the so-called *Horn's list*, the first four entries of which are Appell's functions $F_1, ..., F_4$, [98].

Each function from Horn's list may be characterised as a solution to a pair of second order partial differential equations in x and y. We will now explain this on the example of F_4. To write the equations, we start with the one-variable hypergeometric differential operator. It depends on three parameters a, b, c and reads

$$H(a,b,c,x,\partial_x) = x(1-x)\partial_x^2 + (c-(a+b+1)x)\partial_x - ab = (x\partial_x + c)\partial_x - (x\partial_x + a)(x\partial_x + b) \ . \tag{5.33}$$

The hypergeometric differential equation, $H(a,b,c,x,\partial_x)f = 0$, has two independent solutions near the origin of the complex x-plane

$$f_1 = \ {}_2F_1(a,b,c,x), \quad f_2 = \ x^{1-c}\ {}_2F_1(1+a-c, 1+b-c, 2-c, x) \ . \tag{5.34}$$

Appell's, or more generally Horn's differential equations are obtained by promoting the parameters a, b, c to commuting operators in a variable y, [99]. Concretely, for the Appell's function F_4 we set

$$H_1 = H(a + y\partial_y, b + y\partial_y, c_1, x, \partial_x), \quad H_2 = H(a + x\partial_x, b + x\partial_x, c_2, y, \partial_y) \ . \tag{5.35}$$

The associated system of equations reads $H_1 f(x,y) = H_2 f(x,y) = 0$. There are four independent solutions around the origin. We write only one of them

$$\begin{aligned} F_4(a,b,c_1,c_2,x,y) &= \ {}_2F_1(a + y\partial_y, b + y\partial_y, c_1, x)\ {}_2F_1(a,b,c_2,y) \\ &= \ {}_2F_1(a + x\partial_x, b + x\partial_x, c_2, y)\ {}_2F_1(a,b,c_1,x) \ . \end{aligned}$$

It is clear from the first representation of F_4 that it solves the equation $H_1 F_4 = 0$ and similarly from the second that it solves $H_2 F_4 = 0$. What is non-trivial is that the two representations give the same function. This can be directly verified from the series expansion for F_4.

To solve the hypergeometric eigenvalue problem $H(a,b,c,x,\partial_x)f = \lambda f$ with $\lambda \neq 0$, one observes that it takes again the form of a hypergeometric equation with parameters a', b', c such that $a'b' = ab + \lambda$ and $a' + b' = a + b$. As a side remark, note that in the analysis of conformal blocks for four-point functions, a significant role is played by the differential operator $D_x^{a,b,c} = xH(a,b,c,x,\partial_x)$, that was extensively studied by Dolan and Osborn, [30, 31]. The eigenvalue problem $D_x^{a,b,c} f = \lambda(\lambda + c - 1)f$ has independent solutions

$$f_1 = x^\lambda\, {}_2F_1(a{+}\lambda, b{+}\lambda, c{+}2\lambda, x), \quad f_2 = x^{1-c-\lambda}\, {}_2F_1(1{+}a{-}c{-}\lambda, 1{+}b{-}c{-}\lambda, 2{-}c{-}2\lambda, x)\ . \tag{5.36}$$

This follows from the identity

$$x^{-\lambda} H(a,b,c,x,\partial_x) x^\lambda = H(a+\lambda, b+\lambda, c+2\lambda, x, \partial_x) + \frac{\lambda(\lambda+c-1)}{x}\ . \tag{5.37}$$

Thus indeed

$$D_x^{a,b,c} x^\lambda\, {}_2F_1(a+\lambda, b+\lambda, c+2\lambda, x) = x^{\lambda+1}\left(H(a+\lambda, b+\lambda, c+2\lambda, x, \partial_x) + \frac{\lambda(\lambda+c-1)}{x}\right)$$
$${}_2F_1(a+\lambda, b+\lambda, c+2\lambda, x) = \lambda(\lambda+c-1)x^\lambda\, {}_2F_1(a+\lambda, b+\lambda, c+2\lambda, x)\ .$$

Our definitions of operators H_1 and H_2 make it easy to derive similar relations in the two-variable case. Clearly

$$x^{-\lambda} H_1 x^\lambda = H(a+\lambda+y\partial_y, b+\lambda+y\partial_y, c_1+2\lambda, x) + \frac{\lambda(\lambda+c_1-1)}{x}\ . \tag{5.38}$$

Further, one can readily verify

$$y^{-\mu} H_1 y^\mu = H(a+\mu+y\partial_y, b+\mu+y\partial_y, c_1, x)\ . \tag{5.39}$$

When combined, these two equations lead to

$$y^{-\mu} x^{-\lambda} H_1 x^\lambda y^\mu = H(a+\lambda+\mu+y\partial_y, b+\lambda+\mu+y\partial_y, c_1+2\lambda, x) + \frac{\lambda(\lambda+c_1-1)}{x}\ . \tag{5.40}$$

Analogous statements hold for the operator H_2. These formulas will play a significant role in later chapters.

5.5 Gaudin models

Gaudin models were originally defined in the context of quantum spin chains based on the Lie algebra $\mathfrak{sl}(2)$, [100]. It was later realised that these models can be naturally defined for any semisimple Lie algebra and have relations to deep and beautiful mathematics, [101, 102]. Here we will only give the definition of the model and its most basic properties. This will be sufficient to establish the relevance of Gaudin models for the theory of n-point conformal partial waves in the later parts of this work.

Let $\mathfrak{g}$ be a finite-dimensional simple Lie algebra. We denote a basis for $\mathfrak{g}$ by $\{X_a\}$ and write the corresponding structure constants as f_{ab}^c. Indices on generators are raised and lowered using the Killing form κ.

Fix n numbers $z_1, ..., z_n \in \mathbb{C}P^1$ and imagine attaching a copy of $\mathfrak{g}$ to each point z_i. More precisely, we consider the tensor product of n commuting copies of the universal enveloping algebra of $\mathfrak{g}$, $\mathcal{A} = U(\mathfrak{g})^{\otimes n}$. The Gaudin algebra is defined as a certain commutative subalgebra of $\mathcal{A}$. It is constructed with the help of the Lax matrix with a spectral parameter z

$$\mathcal{L}_a(z) = \sum_{i=1}^{n} \frac{X_a^{(i)}}{z - z_i} \ . \tag{5.41}$$

Here, generators of the Lie algebra attached to the site z_i carry the superscript (i). If we take the generators X^a in some matrix representation and denote $\mathcal{L}(z) = \mathcal{L}_a(z)X^a$, then the object $\mathcal{L}(z)$ is indeed a matrix with values in $\mathcal{A}$, justifying its name. Of fundamental importance in the analysis of the Gaudin model are the commutation relations

$$\begin{aligned}
[\mathcal{L}_a(z), \mathcal{L}_b(w)] = \sum_{i,j=1}^{n} \frac{[X_a^{(i)}, X_b^{(j)}]}{(z - z_i)(w - z_j)} &= \sum_{i=1}^{n} \frac{f_{ab}^c X_c^{(i)}}{(z - z_i)(w - z_i)} \\
&= f_{ab}^c \sum_{i=1}^{n} \frac{X_c^{(i)}}{w - z} \left(\frac{1}{z - z_i} - \frac{1}{w - z_i} \right) = f_{ab}^c \frac{\mathcal{L}_c(z) - \mathcal{L}_c(w)}{w - z} \ .
\end{aligned}$$

Gaudin Hamiltonians are constructed using the Lax matrix and invariant tensors of $\mathfrak{g}$ (i.e. Casimir elements). The quadratic Hamiltonians are extracted from the object

$$\mathcal{H}_2(z) = \frac{1}{2} \kappa^{ab} \mathcal{L}_a(z) \mathcal{L}_b(z) \ . \tag{5.42}$$

Let us look at the commutator of the Lax matrix and $\mathcal{H}_2$

$$\begin{aligned}
[\mathcal{L}_a(w), \mathcal{H}_2(z)] &= \frac{1}{2} \kappa^{bc} [\mathcal{L}_a(w), \mathcal{L}_b(z) \mathcal{L}_c(z)] \\
&= \frac{\kappa^{bc}}{2(w - z)} \left(f_{ac}^d \mathcal{L}_b(z)(\mathcal{L}_d(w) - \mathcal{L}_d(z)) + f_{ab}^d (\mathcal{L}_d(w) - \mathcal{L}_d(z)) \mathcal{L}_c(z) \right) \\
&= \frac{\kappa^{bc} f_{ac}^d}{2(w - z)} \left(\mathcal{L}_b(z) \mathcal{L}_d(w) + \mathcal{L}_d(w) \mathcal{L}_b(z) \right) \ .
\end{aligned}$$

To get to the last line we used the ad-invariance of the Killing form. Starting from this relation, a slightly more involved computation shows that the operators $\mathcal{H}_2$ commute for any values of spectral parameters

$$[\mathcal{H}_2(z), \mathcal{H}_2(w)] = 0 \ . \tag{5.43}$$

The presence of the spectral parameter allows one to extract from $\mathcal{H}_2(z)$ many elements of the algebra $\mathcal{A}$, which will by (5.43) all commute among themselves. The standard way of extracting operators is by the partial fraction decomposition

$$\mathcal{H}_2(z) = \frac{1}{2} \sum_{i,j=1}^{n} \frac{\kappa^{ab}}{(z - z_i)(z - z_j)} = \frac{1}{2} \sum_{i=1}^{n} \frac{\kappa^{ab} X_a^{(i)} X_b^{(i)}}{(z - z_i)^2} + \frac{1}{2} \sum_{i \neq j} \frac{\kappa^{ab} X_a^{(i)} X_b^{(j)}}{z_i - z_j} \left(\frac{1}{z - z_i} - \frac{1}{z - z_j} \right) \ .$$

If we denote the quadratic Casimir at site z_i by $C_2^{(i)}$, the last expression can be written as

$$\mathcal{H}_2(z) = \sum_{i=1}^{n} \frac{C_2^{(i)}}{(z - z_i)^2} + \sum_{i=1}^{n} \frac{\mathcal{H}_2^{(i)}}{z - z_i}, \tag{5.44}$$

where the operators $\mathcal{H}_2^{(i)}$ are given by

$$\mathcal{H}_2^{(i)} = \sum_{j\neq i} \frac{\kappa^{ab} X_a^{(i)} X_b^{(j)}}{z_i - z_j} \ . \tag{5.45}$$

These are known as the (quadratic) Gaudin Hamiltonians. Besides commuting among themselves, they also commute with the *diagonal* generators $X_a^{\text{diag}} = X_a^{(1)} + ... + X_a^{(n)}$. Indeed

$$\begin{aligned}
[X_a^{\text{diag}}, \mathcal{H}_2^{(1)}] &= \kappa^{bc} \left[\sum_{i=1}^{n} X_a^{(i)}, \sum_{j=2}^{n} \frac{X_b^{(1)} X_c^{(j)}}{z_1 - z_j} \right] = \kappa^{bc} \sum_{j=2}^{n} \frac{f_{ab}^{d} X_d^{(1)} X_c^{(j)} + f_{ac}^{d} X_b^{(1)} X_d^{(j)}}{z_1 - z_j} \\
&= \left(\sum_{j=2}^{n} \frac{\kappa^{bc} f_{ab}^{d} + \kappa^{db} f_{ab}^{c}}{z_1 - z_j} \right) X_d^{(1)} X_c^{(j)} = 0,
\end{aligned}$$

where in the last step we have used the ad-invariance of κ. The $\mathcal{H}_2^{(i)}$ commute with diagonal generators by a similar calculation and the same is clearly true for the Casimirs $C_2^{(i)}$. For the Lie algebra $\mathfrak{sl}(2)$, where the model was initially defined, the elements $\{C_2^{(i)}, \mathcal{H}_2^{(i)}\}$ generate the full Gaudin algebra. For Lie algebras of higher rank, any invariant tensor $\tau^{a_1 \ldots a_p}$ gives rise to the object

$$\mathcal{H}_p(z) = \frac{1}{p} \tau^{a_1 \ldots a_p} \mathcal{L}_{a_1}(z) ... \mathcal{L}_{a_p}(z) + ... \ . \tag{5.46}$$

The dots represent a *correction* term that can be written as a polynomial of degree strictly less than p in the generators. In the classical version of the model, the correction terms are absent. The elements $\mathcal{H}_p$ obey

$$[\mathcal{H}_p(z), \mathcal{H}_q(w)] = 0, \quad \forall z, w \ . \tag{5.47}$$

Similarly as above one can perform partial fractioning or some other method to extract from them commuting higher order Hamiltonians. All these Hamiltonians commute with the diagonal action, which is therefore a symmetry of the model

$$[X_a^{\text{diag}}, \mathcal{H}_p(z)] = 0 \ . \tag{5.48}$$

From the partial fraction decomposition

$$\mathcal{H}_p(z) = \sum_{i=1}^{n} \sum_{k=1}^{p} \frac{\mathcal{H}_{p,k}^{(i)}}{(z - z_i)^k}, \tag{5.49}$$

it may seem that the Gaudin algebra has $n \sum p$ generators $\mathcal{H}_{p,k}^{(i)}$. However, not all of these elements are independent. For example, it is easy to see that $\mathcal{H}_2^{(1)} + ... + \mathcal{H}_2^{(n)} = 0$. Subtracting these and similar relations gives an upper bound

$$M = (n-1) \frac{\text{dim}\mathfrak{g} - \text{rk}\mathfrak{g}}{2}, \tag{5.50}$$

on the number of abelian generators of the Gaudin algebra (we say an upper bound, because there might be additional *accidental* relations). In this counting we also did not include Casimir elements at individual sites, which may be regarded as somewhat trivial elements of the algebra. If one quotients $\mathcal{A}$ by the ideal generated by elements X_a^{diag}, the number of non-trivial Hamiltonians reduces to

$$M_{red} = (n-2)\frac{\mathrm{dim}\mathfrak{g} - \mathrm{rk}\mathfrak{g}}{2} - \mathrm{rk}\mathfrak{g} \ . \tag{5.51}$$

A typical question in the theory of Gaudin integrable systems arises when the copies of the universal enveloping algebra are replaced by representations $\pi_{(i)}$ of $\mathfrak{g}$. Then the Gaudin Hamiltonians become commuting operators on the carrier space of $\pi_{(1)} \otimes ... \otimes \pi_{(n)}$ and one seeks for solutions of their simultaneous eigenvalue problem. If representations are of highest weight, the most common way of approaching the diagonalisation is through Bethe-ansatz techniques.

Chapter 6

Superconformal symmetry

The aim of this chapter is to introduce the mathematical underpinnings of supersymmetry. Supersymmetry emerged as an important symmetry principle in quantum field theory in the 1970s through the work of Golfand, Likhtman, Wess, Zumino and others, [104, 105]. It relates bosonic and fermionic excitations of quantum fields and leads to remarkable cancellations of divergences between Feynman diagrams. One early reason to study supersymmetry was that the super-Poincare algebra provided a possible non-trivial extension of the symmetry algebra of an S-matrix for a local relativistic quantum field theory. No such non-trivial extensions exist if one restricts to ordinary Lie algebras, by the Coleman-Mandula theorem.

Partially motivated by the interest of physicists, the study of $\mathbb{Z}_2$-graded, or super, objects became an integral part of algebra and geometry as well. On the one hand, Lie superalgebras were studied by Nahm, Rittenberg and Scheunert and independently by Kac, [106, 107]. Kac obtained a complete classification of finite-dimensional complex simple Lie superalgebras not so dissimilar to that of ordinary Lie algebras. He also developed the theory of their finite-dimensional representations. The main difference compared to the ordinary theory is that Lie superalgebras, or their enveloping algebras, contain nilpotent elements. Recall that in representation theory of finite groups, the identity $g^{|G|} = 1$ implies that in any finite-dimensional representation π, matrices $\pi(g)$ are diagonalisable. This is a crucial property from which one proves complete reducibility of finite-dimensional representations etc. Similarly, in a unitary representation of a Lie group, elements of the Lie algebra are anti-hermitian and therefore diagonalisable. For Lie superalgebras no such statements hold as nilpotent elements can never be represented by non-zero diagonalisable operators. Indeed, Lie superalgebras possess so-called atypical representations which have proper invariant subspaces and yet are not direct sums of irreducible modules. It turns out that precisely these representations, or their irreducible quotients, are of special interest in physics.

Parallel to the algebraic developments, supergeometry was introduced by Berezin and Leites and by Kostant, [108, 109]. Both of these approaches have the spirit of algebraic geometry and define supermanifolds from their structure algebras (to be thought of as algebras of functions). The necessity of such a view comes from the fact that while it makes sense to evaluate a real or complex variable at a point, there is no sense of evaluating a Grassmann variable. Indeed, in differential geometry one can recover a manifold M (as a set) from its commutative algebra of functions $C^\infty(M)$ by taking all one-dimensional representations of $C^\infty(M)$. In the super-context, $C^\infty(M)$ is replaced by a super-commutative algebra $\mathcal{A}$ generated by real and

Grassmann variables and one-dimensional representations of $\mathcal{A}$ are in bijective correspondence with points of the ordinary topological space underlying the supermanifold under consideration.

Despite these obvious differences, one can develop much of the differential geometry for supermanifolds as in the ordinary commutative case. Starting from the structure algebra $\mathcal{A}$, vector fields are defined as its graded derivations and form an $\mathcal{A}$-bimodule (i.e. the product of a vector field and a function is a vector field). One-forms are linear maps $\mathrm{Der}(\mathcal{A}) \to \mathcal{A}$. From vector fields and one-forms one obtains by means of tensor products arbitrary tensors of type (p, q). The set of all forms $\Omega(\mathcal{A})$ constitutes a differential graded algebra whose zero-degree subalgebra is $\mathcal{A}$. However, one notes that differentials of Grassmann variables commute, $d\theta_i d\theta_j = d\theta_j d\theta_i$, so there are no forms of highest degree. Related to this is also the fact that differential forms do not provide the theory of integration for supermanifolds. However, there is a very natural such theory provided by the notion of the Berezin integral.

General aspects of supergeometry and the theory of Lie superalgebras come together in the study of Lie supergroups (or just supergroups). A natural and concrete description of these objects comes from Hopf algebras. Recall that both the universal enveloping algebra of a Lie algebra, $U(\mathfrak{g})$, and the convolution algebra of a group, $L^1(G)$, are cocommutative Hopf algebras. Conversely, any cocommutative Hopf algebra may be constructed from these two types by means of *smash products*. Similarly, the universal enveloping algebra of a Lie superalgebra is a super-cocommutative Hopf algebra. We will say that a supermanifold is a supergroup if a coalgebra $\mathcal{A}^*$ related to its structure algebra $\mathcal{A}$ (distributions with finite support) is a (by definition super-cocommutative) Hopf algebra. There is a correspondence between supergroups and Lie superalgebras analogous to the one between Lie groups and Lie algebras. Having defined supergroups, the meaning of them acting on supermanifolds will be obvious.

Let us note that supergeometry lies somewhere in between ordinary and general noncommutative geometry. Indeed, while the structure algebra $\mathcal{A}$ of a supermanifold is noncommutative, it is so in a very mild way. The approach we follow here is well adopted to further, truly noncommutative, generalisations, but these will not enjoy all the nice properties from above. For example, vector fields can be defined for any algebra as the set of derivations, but generally $\mathrm{Der}(\mathcal{A})$ is not a module over $\mathcal{A}$. Any noncommutative algebra $\mathcal{A}$ can be embedded in a differential graded algebra $\Omega(\mathcal{A})$, but $\Omega(\mathcal{A})$ is by no means unique. All differential calculi over $\mathcal{A}$ are quotients of the canonical *universal calculus* $\Omega_u(\mathcal{A})$. Also, the question of the integration becomes much more subtle. Famously, Hopf algebras and quantum groups as their special cases, are the correct framework to study symmetries of quantum spaces. Supergeometry itself may be studied by another approach, pioneered by de Witt and Rogers, [110, 111], that is more in line with classical differential geometry. It was shown that this alternative formulation is equivalent to that of Berezin-Leites-Kostant and since it lacks the manifest generality of principles present in the latter, we will not discuss it here.

The chapter is organised as follows. We will start by giving elements of superalgebra and supergeometry according to the introduction above. The reader mainly interested in applications may not care for this part, as formal manipulations with Grassmann variables more often than not lead to correct results. In the later parts, we will tailor the discussion to applications that we have in mind and consider actions of supergroups on supercosets (including some explicit methods for computations) and the classification of superconformal algebras.

6.1 Super linear algebra

Let us recall some basic notions of super ($\mathbb{Z}_2$-graded) linear algebra. A super vector space is a direct sum of two vector spaces

$$V = V_0 \oplus V_1 \ . \tag{6.1}$$

The elements of V_0 are called even and the elements of V_1 odd. A vector which is either even or odd is said to be homogeneous. A homogeneous subspace of V is a subspace spanned by homogeneous elements. Not every linear subspace of a super vector space is homogeneous. For example, if $V = \mathbb{C}^{1|1} = \mathrm{span}\{v_0, v_1\}$, the one-dimensional subspace spanned by $v_0 + v_1$ is not homogeneous.

For any super vector space, we denote by ΠV the same space with reversed parity, i.e.

$$(\Pi V)_0 = V_1, \quad (\Pi V)_1 = V_0 \ . \tag{6.2}$$

Any linear map between two super vector spaces V and W can be written uniquely as a sum of a grade-preserving one and a grade-reversing one. We shall call the grade-preserving linear maps *morphism* of super vector spaces and denote the set of all such maps by $\mathrm{Hom}(V, W)$. On the other hand, the set of all linear maps has the structure of a super vector space as well. We write

$$\mathbf{Hom}(V, W) = \mathrm{Hom}(V, W) \oplus \mathrm{Hom}(V, \Pi W) \ .$$

Dual spaces, direct sums and tensor products of super vector spaces are again super vector spaces defined as

$$\begin{aligned} &V^* = \mathbf{Hom}(V, \mathbb{F}^{1|0}), \quad (V \oplus W)_0 = V_0 \oplus W_0, \quad (V \oplus W)_1 = V_1 \oplus W_1, \\ &(V \otimes W)_0 = (V_0 \otimes W_0) \oplus (V_1 \otimes W_1), \quad (V \otimes W)_1 = (V_0 \otimes W_1) \oplus (V_1 \otimes W_0), \end{aligned}$$

where $\mathbb{F}$ denotes the underlying field. In the language of category theory, super vector spaces over the field $\mathbb{F}$ with the above morphisms and the super tensor product form a monoidal category. The unit object is the field itself, considered as a purely even super vector space, $\mathbb{F}^{1|0}$. The braiding map

$$\tau_{V,W} : V \otimes W \to W \otimes V, \quad \tau_{V,W}(x \otimes y) = (-1)^{|x||y|} y \otimes x,$$

makes the category of super vector spaces into a symmetric monoidal category. Here, and always, $|x|$ stands for the degree of a homogeneous element x. The braiding map expresses the familiar rule that swapping two odd elements comes with a sign.

A *superalgebra* is an algebra R, which is also a super vector space $R = R_0 \oplus R_1$ such that

$$R_i R_j \subset R_{i+j} \ . \tag{6.3}$$

The summation of indices is mod 2. The *supercommutator* of two homogeneous elements x, y of R is

$$[x, y]_s = xy - (-1)^{|x||y|} yx \ . \tag{6.4}$$

The supercommutator with any fixed element $x \in R$ is a graded derivation

$$[x, yz]_s = [x, y]_s z + (-1)^{|x||y|} y [x, z]_s \ . \tag{6.5}$$

We say that a superalgebra is supercommutative if the supercommutator of any two homogeneous elements is zero.

The tensor product of two superalgebras A and B is naturally a superalgebra with the multiplication

$$(a_1 \otimes b_1)(a_2 \otimes b_2) = (-1)^{|a_2||b_1|} a_1 a_2 \otimes b_1 b_2, \quad a_i \in A,\ b_i \in B \ . \tag{6.6}$$

Example The Grassmann algebra in n variables, $\Lambda(n)$, is the complex algebra given by the generators and relations

$$\Lambda(n) = \langle \theta_1, ..., \theta_n \mid \theta_i \theta_j = -\theta_j \theta_i \rangle \ .$$

The generators θ_i are defined to be odd and this gives the unique usual grading on $\Lambda(n)$. This is a supercommutative superalgebra. We clearly have $\Lambda(m+n) \cong \Lambda(m) \otimes \Lambda(n)$.

Let us now fix a supercommutative superalgebra R. Typically, we are interested in the case where R is a Grassmann algebra. A *left supermodule* over R is a module E which can be written as a direct sum of abelian groups $E = E_0 \oplus E_1$ such that

$$R_i E_j \subset E_{i+j} \ . \tag{6.7}$$

Right supermodules and super bimodules are defined similarly. Since R is supercommutative, every left supermodule is naturally a super bimodule. The right action is defined by

$$rx = (-1)^{|r||x|} xr, \quad r \in R,\ x \in E \ . \tag{6.8}$$

Then one can check that indeed $r_1(xr_2) = (r_1 x) r_2$. From now on, we shall only speak of super bimodules, assuming the above identifications.

A superalgebra A over R is a super bimodule over R, together with an R-bilinear map $A \times A \to A$ which respects the grading. R-bilinearity means that for all homogeneous $r \in R$ and $x, y \in A$

$$r(xy) = (rx)y = (-1)^{|r||x|} x(ry) \ .$$

Supermodules over R form a category similar to the one of super vector spaces introduced above. A morphism of supermodules is a module morphism which respects the grading. The set of all morphisms between two R-modules E and F is denoted $\mathrm{Hom}(E, F)$. On the other hand, the set of all module morphisms, including those that do not respect the grading, is denoted by $\mathbf{Hom}(E, F)$. They are naturally graded and satisfy

$$\varphi(xr) = \varphi(x) r, \quad \varphi(rx) = (-1)^{|\varphi||r|} r \varphi(x), \quad \varphi \in \mathbf{Hom}(E, F),\ r \in R \ .$$

The set $\mathbf{Hom}(E, F)$ is itself a bimodule over R, by

$$(r \cdot \varphi)(x) = r\varphi(x), \quad (\varphi \cdot r)(x) = \varphi(rx) \ .$$

Superalgebras over R are the monoids in the monoidal category of R-supermodules. The unit object in the category is R.

Example An important class of superalgebras that we shall consider are algebras of supermatrices. Let R be a Grassmann algebra. We consider the space of matrices with entries in R, of the block form

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \ . \tag{6.9}$$

If A is of size $r \times p$ and D of size $s \times q$, we sat that M has size $(r|s) \times (p|q)$. The supermatrix M is said to be square if both A and D are square. It is even if the entries of A, D are even and of B, C odd. If the elements of B, C are even an of A, D odd, then M is odd. Two supermatrices M_1 and M_2 are allowed to be multiplied together if and only if $(p_1|q_1) = (r_2|s_2)$ and then the product is defined as the ordinary product of matrices. However, the scalar multiplication differes from the usual one

$$rM = \begin{pmatrix} rA & rB \\ (-1)^{|r|}rC & (-1)^{|r|}rD \end{pmatrix}, \quad Mr = \begin{pmatrix} Ar & B(-1)^{|r|}r \\ Cr & D(-1)^{|r|}r \end{pmatrix}, \quad r \in R \ . \tag{6.10}$$

With these definitions, one has

$$rM = (-1)^{|r||M|} Mr, \tag{6.11}$$

and the set of all $(r|s) \times (p|q)$ supermatrices becomes a bimodule over R. The set of $(p|q) \times (p|q)$ supermatrices, on which the multiplication is well-defined, is a superalgebra over R.

The supertrace of a homogeneus square supermatrix M is given by

$$\mathrm{str}(M) = \mathrm{tr}A - (-1)^{|M|}\mathrm{tr}D \ . \tag{6.12}$$

On the other hand, the role of the determinant is played by the Berezinian. Let M be an even square supermatrix. Its Berezinian is

$$\mathrm{Ber}(M) = \det(A - BD^{-1}C)\det(D)^{-1} \ . \tag{6.13}$$

The supertrace and the Berezinian satisfy graded analogues of the familiar relations

$$\mathrm{str}(M_1M_2) = (-1)^{|M_1||M_2|}\mathrm{str}(M_2M_1), \quad \mathrm{Ber}(M_1M_2) = \mathrm{Ber}(M_1)\mathrm{Ber}(M_2), \quad \mathrm{Ber}(e^M) = e^{\mathrm{str}M} \ .$$

6.2 Lie superalgebras

A *Lie superalgebra* is a super vector space $\mathfrak{g} = \mathfrak{g}_{(0)} \oplus \mathfrak{g}_{(1)}$ together with a bilinear map $[,] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, called the Lie bracket, which is graded anti-symmetric

$$[x, y] = (-1)^{|x||y|+1}[y, x], \tag{6.14}$$

and satisfies the graded Jacobi identity

$$[x, [y, z]] + (-1)^{|x|(|y|+|z|)}[y, [z, x]] + (-1)^{|z|(|x|+|y|)}[z, [x, y]] = 0 \ . \tag{6.15}$$

We will mostly use physicists' notation and write the bracket of two odd elements as $\{x, y\}$. Two immediate corollaries of the definition are that $\mathfrak{g}_{(0)}$ is a Lie algebra and $\mathfrak{g}_{(1)}$ carries a representation of $\mathfrak{g}_{(0)}$ under the adjoint action. Further, the bracket between odd elements defines a homomorphism of $\mathfrak{g}_{(0)}$-modules $\varphi : S^2\mathfrak{g}_{(1)} \to \mathfrak{g}_{(0)}$.

Unless specified otherwise, we will be considering complex Lie superalgebras. Given any complex superalgebra A, the supercommutator satisfies defining properties of a Lie bracket and thus turns A into a Lie superalgebra. In this way, the algebra of $(m|n)^{\times 2}$ complex supermatrices gives rise to the Lie superalgebra $\mathfrak{gl}(m|n)$. It is possible to impose the super-tracelessness condition on these matrices, which is easily seen to be preserved by the bracket. In this way we obtain the Lie superalgebra $\mathfrak{sl}(m|n)$.

Example The Lie superalgebra $\mathfrak{g} = \mathfrak{sl}(2|1)$ is spanned by even matrices

$$D = \begin{pmatrix} 1/2 & 0 & 0 \\ 0 & -1/2 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ P = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ K = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ R = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{pmatrix},$$

and odd matrices

$$Q_- = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \ Q_+ = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ S_- = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \ S_+ = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

The even subalgebra $\mathfrak{g}_{(0)}$ isomorphic to $\mathfrak{sl}(2) \oplus \mathfrak{u}(1)$. Finite-dimensional irreducible representations $[j, r]$ of this Lie algebra are labelled by a spin j and an R-charge r. We see that the odd subspace $\mathfrak{g}_{(1)}$ decomposes into a sum of two irreducible representations,

$$\mathfrak{g}_{(1)} = \mathfrak{g}_+ \oplus \mathfrak{g}_- = [1/2, 1] \oplus [1/2, -1] .$$

The theory of finite-dimensional Lie superalgebras was constructed by Kac. It has many similarities with the theory of ordinary Lie algebras, but also some important differences.

A Lie superalgebra is said to be solvable if its derived series terminates in the zero subalgebra. It is (semi)simple if it has no (solvable) non-trivial ideals. It is useful to also introduce the notions of classical and basic Lie superalgebras. The latter is related to properties of invariant bilinear forms. An *inner product* on $\mathfrak{g}$ is a bilinear form $\kappa : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ which satisfies

$$\kappa(\mathfrak{g}_0, \mathfrak{g}_1) = 0, \quad \kappa(x, y) = (-1)^{|x||y|}\kappa(y, x), \quad \kappa([x, y], z) = \kappa(x, [y, z]) . \tag{6.16}$$

The three conditions are called consistency, symmetry and invariance. One can verify that $\kappa(x, y) = \mathrm{str}(xy)$ is an inner product on $\mathfrak{gl}(m|n)$. Moreover, given any representation π of $\mathfrak{g}$, $\kappa(x, y) = \mathrm{str}(\pi(x)\pi(y))$ is an inner product. If we take π to be the adjoint representation, κ is called the Killing form. Clearly, the kernel of an invariant form is an ideal in $\mathfrak{g}$. Therefore, for a simple Lie superalgebra, any invariant form is either non-degenerate or identically zero. Unlike for Lie algebras, both possibilities can occur. Lie superalgebras for which a non-degenerate inner product exists are said to be basic. Otherwise, they are called strange.

Finally, a Lie superalgebra is said to be classical if it is simple and the representation of $\mathfrak{g}_{(0)}$ on $\mathfrak{g}_{(1)}$ is completely reducible. Kac showed that in such a case either $\mathfrak{g}_{(1)}$ is irreducible or it decomposes as a sum of two irreducible modules, dual to one another. According to these two cases, Lie superalgebras $\mathfrak{g}$ are classified in types II and I, respectively.

With all this terminology, we can summarise Kac's classification of finite-dimensional simple classical basic Lie superalgebras

Type I: $A(m-1, n-1) = \mathfrak{sl}(m|n)$, $A(n-1, n-1) = \mathfrak{sl}(n|n)/\mathfrak{u}(1)$, $C(n+1) = \mathfrak{osp}(2|2n)$, $G(3)$, $F(4)$

Type II: $B(m, n) = \mathfrak{osp}(2m+1|2n)$, $D(m, n) = \mathfrak{osp}(2m|2n)$, $D(n+1, n) = \mathfrak{osp}(2n+2|2n)$, $D(2, 1; \alpha)$

As the notation suggests, Lie superalgebras $\mathfrak{osp}(m|n)$ consist of $(m+n)\times(m+n)$ complex supermatrices that satisfy certain linear conditions. Out of the above algebras, $A(n-1,n-1)$, $D(n+1,n)$ and $D(2,1;\alpha)$ have vanishing Killing forms. There are also two families of strange Lie superalgebras, denoted $P(n)$ and $Q(n)$. Kac furthermore discovered non-classical finite-dimensional Lie superalgebras of Cartan type, which are in a certain sense similar to infinite-dimensional Lie algebras. For many more details, see [106, 107, 112].

6.2.1 Universal enveloping algebra

Let $\mathfrak{g}$ be a finite-dimensional Lie superalgebra with a homogeneous basis $\{z_a\}$. Its *universal enveloping algebra* is the associative algebra with generators and relations

$$U(\mathfrak{g})=\langle z_a \mid [z_a,z_b]=z_az_b-(-1)^{|z_a||z_b|+1}z_bz_a\rangle\ . \tag{6.17}$$

Similarly as for Lie algebras, there is a bijective correspondence between representations of $\mathfrak{g}$ and those of $U(\mathfrak{g})$. If $\mathfrak{g}_{(0)}$ is nontrivial, the algebra $U(\mathfrak{g})$ is infinite-dimensional. Its linear basis is described by the super-version of the Poincare-Birkhoff-Witt theorem:

Theorem Let $\{x_1,...,x_m\}$ and $\{y_1,...,y_n\}$ be bases for even and odd subspaces of $\mathfrak{g}$. Then the set $\{x_1^{k_1}...x_m^{k_m}y_1^{\epsilon_1}...y_n^{\epsilon_n} \mid k_i\in\mathbb{N}_0,\ \epsilon_i\in\{0,1\}\}$ is a linear basis for $U(\mathfrak{g})$.

The universal enveloping algebra is a super-cocommutative Hopf algebra. Its coproduct is the homomorphism $\Delta: U(\mathfrak{g})\to U(\mathfrak{g})\otimes U(\mathfrak{g})$ of superalgebras defined on the generators $x\in\mathfrak{g}\subset U(\mathfrak{g})$ by

$$\Delta(x)=x\otimes 1+1\otimes x\ . \tag{6.18}$$

From here one can extend Δ uniquely to the entire universal enveloping algebra as a homomorphism of superalgebras. Of course, the tensor product $U(\mathfrak{g})\otimes U(\mathfrak{g})$ is to be understood in the graded sense. The coproduct is the algebraic structure that allows one to build tensor products of representations of the Lie superalgebra $\mathfrak{g}$. Namely, given two representations $\pi_i: U(\mathfrak{g})\to \mathrm{End}(V_i)$, their tensor product is defined as

$$\pi=(\pi_1\otimes\pi_2)\circ\Delta: U(\mathfrak{g})\to \mathrm{End}(V_1\otimes V_2)\ . \tag{6.19}$$

Properties of comultiplication ensure that the tensor product of representations is associative. The antipode on $U(\mathfrak{g})$ is defined by

$$\sigma: U(\mathfrak{g})\to U(\mathfrak{g}),\quad \sigma(x)=-x,\quad \sigma(AB)=(-1)^{|A||B|}\sigma(B)\sigma(A),\quad x\in\mathfrak{g},\ A,B\in U(\mathfrak{g}), \tag{6.20}$$

and the counit $\eta: U(\mathfrak{g})\to\mathbb{C}$ by $\eta(x)=0$. As for any Hopf algebra, the existence of the counit ensures that there is a trivial representation and that of the antipode that any module has a dual.

6.3 Elements of supergeometry

We now move to the geometric part of the theory, where the basic notion is that of a supermanifold.

A supermanifold M is a topological space X together with a sheaf A of superalgebras, such that around any point $x \in X$ there is an open neighbourhood U with $A(U) \cong C^\infty(U) \otimes \Lambda(n)$, where $\Lambda(n)$ is the Grassmann algebra on n generators. The number n is called the odd dimension of M. Recall that for any open set $U \subset X$, the sheaf associates to it a supercommutative algebra $A(U)$, in a way compatible with restrictions to subsets.

In the first place, one may ask whether a supermanifold is different from a vector bundle of rank 2^n over X. The difference between the two is in the notion of morphisms: while a morphism $\varphi : M \to N$ of ordinary smooth manifolds automatically determines the pull-back $\varphi^* : C^\infty(N) \to C^\infty(M)$, for supermanifolds (X, A) and (Y, B) one independently specifies a continuous map of underlying spaces $\varphi : X \to Y$ and homomorphisms of superalgebras $\varphi^*_V : B(V) \to A(\varphi^{-1}(V))$ for all open sets $V \subset Y$, and only requires compatibility of these homomorphisms with restrictions. So, we can say that supermanifolds admit more morphisms than ordinary ones.

6.3.1 The reconstruction theorem

The all important fact about supermanifolds is that they can be completely recovered from their structure algebras, defined as $A(X)$. We wish to explain this in more detail. For simplicity, we will restrict our attention to *superdomains*, where the main new features compared to commutative differential geometry are already present. A superdomain is a supermanifold whose underlying topological space is a domain U in $\mathbb{R}^m$ and $A(U) \cong C^\infty(U) \otimes \Lambda(n)$. We will denote a superdomain as $\mathcal{U} = (U, A)$. One has the following theorem of Leites [109]

Theorem (Leites) Let $\mathcal{U} = (U, A)$ and $\mathcal{V} = (V, B)$ be two superdomains. For any homomorphism $\psi^* : B(V) \to A(U)$ of superalgebras there exists a precisely one morphism $\varphi : \mathcal{U} \to \mathcal{V}$ of superdomains for which $\varphi^* = \psi^*$.

Here, φ^* denotes the map $B(V) \to A(U)$ that is a part of the definition of the morphism φ. We will refer to the above result as the *reconstruction theorem* and the goal of this section is to give its proof.

One can describe superdomains using coordinates $x = (u_i, \theta_j)$, where u_i are coordinate functions on the underlying domain U and θ_j are the Grassmann generators. Elements of the algebra $A(U)$ take the form

$$f = f_0(u_i) + f_j(u_i)\theta_j + ... + f_{1...n}(u_i)\theta_1...\theta_n \; . \tag{6.21}$$

Clearly, the map $f \mapsto f_0$ gives rise to a unique morphism of supermanifolds $U \to \mathcal{U}$, called the *canonical embedding* (U is considered as a supermanifold of zero odd dimension). We shall write $f_0 = \tilde{f}$.

Lemma Canonical embeddings commute with morphisms of superdomains.

Proof: Consider a morphism of superdomains $\varphi : \mathcal{U} \to \mathcal{V}$. Let $f \in B(V)$ and $f^* = \varphi^*(f)$. Further denote by φ_0 the continuous map $U \to V$ from the definition of φ. Let $u \in U$ and $v = \varphi_0(u)$.

Consider the maps $\tilde{f} \in C^\infty(V)$ and $\tilde{f}^* \in C^\infty(U)$. Assume that $\tilde{f}^*(u) \neq \tilde{f}(v)$. Without loss of generality, assume $\tilde{f}^*(u) = 0$ (this can be done, because we can add constant functions - by the homomorphism conditions they are mapped to the same constant functions under morphisms). Now, $\tilde{f}$ is invertible in a neighbourhood V' of v, so f is an invertible element of the algebra

$B(V')$. Therefore f^* is an invertible element of $\varphi^*(B(V'))$. So, we conclude that $\tilde{f}^*$ is non-zero on $\varphi_0^{-1}(V')$. But this contradicts the fact that $\tilde{f}(u) = 0$. All in all, we have proved

$$\tilde{f}^*(u) = \tilde{f}(v) \ .$$

But this precisely says that $\tilde{f}^* = \tilde{f} \circ \varphi_0$ which is what we wanted to show. $\square$

In particular, since for any smooth f, f^* is also smooth, we conclude that φ_0 must be smooth as well. The essential ingredient of the proof was the observation that an element of a Grassmann algebra is invertible iff its part with no Grassmann generators is non-zero. Therefore, this part of the element, sometimes called its body, admits an algebraic description.

Let us move on and describe points of supermanifolds. It is not immediately obvious what the right definition of a point should be, but we can recall the familiar idea from the theory of C^*-algebras that points of a topological space can be described as characters of its abelian algebra of functions. If one copies this definition, points of a supermanifold turn out to be just those of its underlying topological space.

Lemma All homomorphisms $s : A(U) \to \mathbb{R}$ are of the form $s(f) = \tilde{f}(u) \equiv s_u(f)$ for some $u \in U$.

Proof: Let s be any homomorphism $A(U) \to \mathbb{R}$ and $x = (u_i, \theta_j)$ a coordinate system on $\mathcal{U}$. We set $u = (s(u_1), ..., s(u_m))$. It will be shown that $s = s_u$. We first need to establish that $u \in U$. If this was not the case, the function $h = f_1^2 + ... + f_m^2$, where $f_i = u_i - s(u_i)$, would be non-zero on U, and thus invertible. However

$$s(h) = \sum_{i=1}^{m} s\big((u_i - s(u_i))^2\big) = \sum_{i=1}^{m} \big(s(u_i) - s(u_i)s(1)\big)^2 = 0,$$

which contradicts the fact that s is a homomorphism. So $u \in U$. Assume now that there exists a function $f \in A(U)$ such that $s(f) \neq s_u(f)$ and consider

$$g = f_1^2 + ... + f_m^2 + (f - s(f))^2 \ .$$

We argue that g is invertible. If $\tilde{g}(u') = 0$ for some u' then $\tilde{f}_i(u') = 0$ for all i, so $u' = u$. But $\tilde{f}(u) = s_u(f) \neq s(f)$, so $\tilde{g}(u) \neq 0$. Therefore g is invertible. However, similarly as above $s(g) = 0$, which is a contradiction. This completes the proof of the lemma. $\square$

For obvious reasons, homomorphisms s_u are called *evaluations*. A third important technical result about superdomains is a generalisation of the classical Hadamard lemma

Lemma (Hadamard) Let U be an open subset of $\mathbb{R}^m$, star-shaped around a and $f : U \to \mathbb{R}$ a smooth function. Then there exist smooth $g_1, ..., g_m : U \to \mathbb{R}$ such that for any $x \in U$ we have $f(x) = f(a) + \sum_i (x_i - a_i) g_i(x)$.

Hadamard's result has a one line proof. Define $h(t) = f(tx + (1-t)a)$. Then

$$f(x) - f(a) = h(1) - h(0) = \int_0^1 h'(t)dt = \sum_i (x_i - a_i) \int_0^1 \partial_i f dt,$$

so we can define $g_i(x) = \int_0^1 \partial_i f(tx + (1-t)a)dt$. It is useful to rephrase the conclusion of the lemma in algebraic terms. It reads that the smooth function $f(x) - f(a)$ lies in the ideal

$\langle x_1 - a_1, ..., x_n - a_n \rangle$ of $C^\infty(U)$. A very similar reasoning can be used to establish the superversion

Lemma Let $\mathcal{U}$ be a superdomain with coordinates $x = (u_i, \theta_j)$ and u a point of the underlying set U. Denote by I_u the ideal in $A(U)$ generated by $\{u_1 - s_u(u_1), ..., u_n - s_u(u_m), \theta_1, ..., \theta_n\}$. Then for any $f \in A(U)$ and any $k \in \mathbb{N}_0$ there exists a polynomial P_k in (u_i, θ_j) of degree $\leq k$ such that $f - P_k \in I_{u_0}^{k+1}$.

The reader is referred to [109] for the proof. Note that as a corollary we have $I_u = \ker(s_u)$. For clearly, s_u evaluates to zero on the ideal I_u and the above result implies that $A(U) = \mathbb{R} + I_u$. Another simple consequence is that if $f \in I_u^{n+1}$ then $f(u) = 0$. Indeed, a product of $n+1$ Grassmann variables is zero, so to get a non-zero element in $f \in I_u^{n+1}$ we need to include a factor $u_i - s_u(u_i)$ in it. But then $f(u) = 0$. This gives us an algebraic way to compare two elements $f_1, f_2 \in A(U)$. Namely, if $f_1 - f_2 \in I_u^{n+1}$ for every $u \in U$ then we necessarily have $f_1 = f_2$.

We can now prove Leites' theorem. Let $x = (u_i, \theta_j)$ and $y = (v_i, \eta_j)$ be coordinates on superdomains $\mathcal{U}$ and $\mathcal{V}$. Given any point $u \in U$, the map $s_u \circ \psi^*$ is a homomorphism $B(V) \to \mathbb{R}$, so it is of the form s_v for some $v \in V$. Denote the coordinates of the point v by (v_i^0). By definition, ideals I_u and I_v satisfy $\psi^*(I_v) \subset I_u$, so we conclude

$$\psi^*(v_i - v_i^0) = v_i^* - v_i^0 \in I_u, \quad \text{with} \quad v_i^* = \psi^*(v_i) \ . \tag{6.22}$$

Consequently $\tilde{v}_i^*(u) = v_i^0$. In particular $\tilde{v}_i^*(u) \in V$.

Next, we want to show that any homomorphism $\psi^* : B(V) \to A(U)$ is uniquely determined by images of coordinate functions $\psi^*(y)$. For this, let $z = (w_1, ..., w_m, \zeta_1^, ..., \zeta_n)$ be arbitrary functions in $A(U)$ such that w_i are even, ζ_j are odd and for any $u \in U$ we have $(\tilde{w}_i(u)) \in V$. We show that there is a unique superalgebra homomorphism $\psi^* : B(V) \to A(U)$ such that $\psi^*(y) = z$.

Uniqueness: Assume that we have found ψ^* such that $\psi^*(y) = z$ and let $u \in U$. Similarly as above, there is a unique $v \in V$ such that $\psi^*(I_v) \subset I_u$. Given $f \in B(V)$, pick a polynomial $P(y)$ such that $f - P(y) \in I_v^{n+1}$. Then $\psi^*(f) - P(y^*) \in I_u^{n+1}$. Therefore, the image of ψ^* in every $A(U)/I_u^{n+1}$ is fixed. It follows that ψ^* is unique.

Existence: Assume for simplicity that $\mathcal{V} = V$. The argument in more general cases is very similar. We write $z_i = z_i' + z_i''$, where $z_i' \in C^\infty(U)$ and z_i'' is nilpotent. Functions $\tilde{z}_i$ give us a smooth map $\tilde{\varphi} : U \to V$ such that $\tilde{\varphi}^*(v_i) = z_i'$.

For any $f \in B(V)$ let $\hat{f}$ be the formal power series expansion of $f(y_1 + t_1, ..., y_m + t_m)$ in t. Coefficients of this expansion elements $f_\alpha \in B(V)$. These series satisfy

$$\widehat{f+g} = \hat{f} + \hat{g}, \quad \widehat{fg} = \hat{f}\hat{g} \ .$$

Notice that, computing each term in the product requires only finitely many terms in f and g, so the operations are well-defined. Define $\psi^*(f)$ by replacing f_α by $\varphi^*(f_\alpha)$ and t_i by z_i'' in $\hat{f}$. Since z_i'' are nilpotent, the series truncates to give a well-defined element of $A(U)$ and ψ^* is clearly a homomorphism. Finally

$$\psi^*(y_i) = \varphi^*(y_i) + z_i'' = z_i' + z_i'' = z_i \ . \tag{6.23}$$

Thus, we have constructed the required homomorphism. This establishes Leites' theorem. $\square$

The way supermanifolds are built by patching up superdomains is entirely analogous to one in ordinary differential geometry. As we have mentioned, the superalgebra $A(X)$ that the sheaf assigns to the whole underlying space is called the structure algebra of (X, A). Some constructions regarding supermanifolds are more easily formulated in terms of a certain coalgebra $A(X)^*$ rather than $A(X)$ itself, [108]. The $A(X)^*$ is defined as the space of all elements in the full dual $A(X)'$ which vanish on some ideal of finite codimension in $A(X)$. Elements of $A(X)^*$ are referred to as *distributions with finite support*. One observes that $A(X)^*$ is a supercocommutative coalgebra. Namely, let i and Δ be the natural injection and the diagonal map

$$\begin{aligned} &i : A(X)' \otimes A(X)' \to (A(X) \otimes A(X))', \quad i(v \otimes w)(f \otimes g) = (-1)^{|w||f|} v(f)w(g), \\ &\Delta : A(X)' \to (A(X) \otimes A(X))', \quad (\Delta v)(f \otimes g) = v(fg), \quad v, w \in A(X)',\ f, g \in A(X)\ . \end{aligned}$$

Then one can show $\Delta(A(X)^*) \subset A(X)^* \otimes A(X)^*$, so the diagonal map makes $A(X)^*$ into a coalgebra. One again has that $A(X)^*$ determines the sheaf A. For example, X as a set can be recovered as the set of all group-like elements in $A(X)^*$. The coalgebra $A(X)^*$ also plays a prominent role in the theory of Lie supergroups and their actions on supermanifolds.

6.3.2 Vector fields and differential forms

Having dealt with the basic theory in some detail, we will go through other notions of differential geometry more quickly. We continue to use the notation for superdomains introduced above.

A *vector field* on $\mathcal{U}$ is a graded derivation of $\mathcal{A} = A(U)$. That is, it is a linear map $D : \mathcal{A} \to \mathcal{A}$, which satisfies the super Leibniz rule

$$D(fg) = D(f)g + (-1)^{|D||f|} f D(g)\ . \tag{6.24}$$

The set of vector fields $\mathrm{Der}(\mathcal{A})$ has the structure of a Lie superalgebra. As in the ordinary differential geometry, vector fields form a module over $\mathcal{A}$. This is a somewhat fortunate fact that is no longer true in more general noncommutative geometries. It can be shown that $\mathrm{Der}(\mathcal{A})$ is spanned over $\mathcal{A}$ by coordinate derivatives ∂_{x_i}, defined in the obvious way, and that it is a free $\mathcal{A}$-module of rank $m + n$.

One can define the tensor algebra $T(U)$ starting from vector fields as in ordinary differential geometry. Similarly, differential forms $\Omega(U)$ are defined as graded antisymmetric multilinear maps on $\mathrm{Der}(\mathcal{A})$ with values in $\mathcal{A}$. Important is to observe that $T(U)$ and $\Omega(U)$ are bi-graded. One degree a is the same as in differential geometry, and the other one b comes form the grading on $\mathcal{A}$. The commutation rule for forms with respect to the bi-grading reads

$$\omega_1\omega_2 = (-1)^{a_1a_2 + b_1b_2} \omega_2\omega_1\ . \tag{6.25}$$

In particular, $d\theta d\theta$ is not zero. Thus, there are no top forms on a supermanifold. One shows that on a superdomain, dx_i span the space of one-forms $\Omega^1(\mathcal{A})$ (if only one degree of a form is indicated, it is the first one, a). Higher forms are generated by $\Omega^1(\mathcal{A})$ over $\mathcal{A}$.

There is a unique derivation $d : \Omega(U) \to \Omega(U)$ of bi-degree $(1, 0)$ that on one-forms evaluates to $df(D) = D(f)$ and satisfies $d^2 = 0$. It is called the differential. For future reference, we give here some properties of d

$$df = dx_i \frac{\partial f}{\partial x_i}, \quad ((df)h)(D) = (df(D))h = D(f)h, \quad f, h \in \mathcal{A},\ D \in \mathrm{Der}(\mathcal{A}) \ . \tag{6.26}$$

Our conventions agree with [108, 109, 113].

6.3.3 Berezin integration

Unlike for ordinary manifolds, differential forms do not provide us with the theory of integration on supermanifolds. In particular, we saw that top forms do not even exist. The right notion of the integral was found by Berezin, but before spelling it out, we shall motivate it with an example.

Let S be the space of Schwarz functions $\mathbb{R} \to \mathbb{C}$. For such functions, we have the usual Lebesgue integral

$$I : S \to \mathbb{C}, \quad I(f) = \int_{-\infty}^{+\infty} f(x) dx \ .$$

The Lebesgue integral obeys the obvious properties of linearity and translational invariance

$$I(af + bg) = aI(f) + bI(g), \quad I(T_a f) = I(f), \quad \text{where} \quad (T_a f)(x) = f(x + a) \ .$$

Further, it is a continuous map, when S carries the usual Schwarz topology. It turns out that these three properties determine I uniquely up to a normalisation constant. Let us see how this comes about. First, the integral of any derivative is zero

$$I(f') = I\left(\lim_{a \to 0} \frac{T_a f - f}{a}\right) = \lim_{a \to 0} \left(\frac{I(T_a f) - I(f)}{a}\right) = 0 \ .$$

We have used continuity to move I inside the limit and then the other two properties stated above. Next, let $f \in S$ be a function such that $\int f = 0$. It is a fact that there exists a Schwarz function $F \in S$ with $F' = f$. Thus

$$\int f = 0 \implies I(f) = 0 \ .$$

It follows that I is a scalar multiple of $\int$, as required. One can use the above three properties as an abstract definition of integration which generalises to other spaces of functions. Let us see what they lead to for the space P_d of polynomials $\mathbb{R} \to \mathbb{C}$ of degree less than or equal to d. If P is a polynomial of degree n so is $T_a P$, so translations are well-defined on P_d. Let $I : P_d \to \mathbb{C}$ be linear, translationally invariant and continuous. As above, we have $I(f') = 0$ for any $f \in P_d$. Any polynomial of degree less than d is a derivative of some function in P_d. Thus the only solutions for I are

$$I(a_0 + a_1 x + ... + a_d x^d) = c a_d,$$

for a constant $c = I(x^d)$. The integral only *sees* the top degree coefficient in f. Integrals on P_d and S_d (Schwarz functions $\mathbb{R}^d \to \mathbb{C}$) have some rather different properties. Let us denote $(\delta_\lambda f)(x) = f(x/\lambda)$. Then

$$I_{S_d}(\delta_\lambda f) = \lambda^d I_{S_d}(f), \quad I_{P_d}(\delta_\lambda f) = \lambda^{-d} I_{P_d}(f) \ .$$

Integration on P_d already captures many of the main features of the Berezin integral, that we turn to now. Let $\varphi : \mathcal{U} \to \mathcal{V}$ be a morphism of superdomains and x and y coordinate systems on $\mathcal{U}$ and $\mathcal{V}$. *The matrix of derivatives* of φ with respect to these coordinates is

$$(I_{xy})_{ij} = \partial_{x_i}\varphi^*(y_j) \ . \tag{6.27}$$

In is an even square supermatrix of size $(m|n)$, so it makes sense to define the *Jacobian* of φ as the Berezinian of I_{xy}

$$J(\varphi) = \frac{D(y)}{D(x)} = \mathrm{Ber}(I_{xy}) \ . \tag{6.28}$$

A *volume form* on $\mathcal{U}$ is a correspondence between coordinate systems of $\mathcal{U}$ and functions $f \in A(U)$, with the identification

$$(x, f) \sim \left(y, \frac{D(x)}{D(y)} f\right) \ .$$

We write the volume form as $\rho = [(x, f)]$. Its integral over $\mathcal{U}$ is defined as

$$\int_{(\mathcal{U},x)} \rho := \int_U f_{1\ldots n}(u)du \ . \tag{6.29}$$

It can be shown that the integral depends on (x, f) (up to a sign) only through the equivalence class $[(x, f)]$ (i.e. is well-defined) provided that f has compact support. Therefore, we see that the Berezin integral combines the two types of integrals on S_m and P_n from above.

6.4 Supergroups

Supergroups can be defined as group objects in the the category of supermanifolds. However, we will find it useful for applications to have several other, more concrete descriptions of these objects.

Let $\mathfrak{g}$ be a Lie superalgebra, H a group and $\pi : H \to \mathrm{Aut}(U(\mathfrak{g}))$ a representation of H by Lie superalgebra automorphisms. Further, write $L^1(H)$ for the group algebra of H. The *smash product* $E(H, \mathfrak{g}, \pi)$ is a super-cocommutative Hopf algebra constructed as follows:

1) As a vector space $E = L^1(H) \otimes U(\mathfrak{g})$.
2) The comultiplication Δ, counit η and the antipode σ are defined on $L^1(H)$ and $U(\mathfrak{g})$ as usual.
3) The multiplication in $L^1(H)$ and $U(\mathfrak{g})$ is defined in the usual way and $e_h x e_h^{-1} = \pi(h)x$.

Here $h \in H$ and $x \in \mathfrak{g}$. (The alert reader will have noticed that we glossed over some subtleties here. Namely, e_h are delta-functions on H and it takes care to define them. See [108].) The set of group-like elements of E is precisely H and that of primitive elements is $\mathfrak{g}$. Here $\mathfrak{g}$ is identified with a subspace of $U(\mathfrak{g})$ in the obvious way. Conversely, given a super-cocommutative Hopf algebra E with the group of group-like elements H and the Lie superalgebra of primitive elements $\mathfrak{g}$ one can show that a representation π exists such that $E = E(H, \mathfrak{g}, \pi)$.

Now assume that $\mathfrak{g} = \mathfrak{g}_{(0)} \oplus \mathfrak{g}_{(1)}$ is a Lie superalgebra and $G_{(0)}$ the connected, simply connected Lie group whose Lie algebra is $\mathfrak{g}_{(0)}$. Then there is a unique representation π on $\mathfrak{g}$ by Lie superalgebra automorphisms which reduces to the adjoint representation on $\mathfrak{g}_{(0)}$. The smash product $E(G_{(0)}, \mathfrak{g}, \pi)$ is called the simply-connected Lie-Hopf algebra associated with $\mathfrak{g}$ and it is denoted by $E(\mathfrak{g})$.

We will say that a supermanifold (X, A) is a (Lie) supergroup if the coalgebra $A(X)^*$ is a Hopf algebra. By the above remarks, in this case $A(X)^*$ is a smash product $E(G_{(0)}, \mathfrak{g}, \pi)$ with $X = G_{(0)}$. In fact, if X is simply connected, it can be shown that $A(X)^* = E(\mathfrak{g})$ for some Lie superalgebra, called the Lie superalgebra of (X, A).

Remark In the rest of the text, we will be abusing notation as follows. A Lie supergroup is typically denoted by $G = (G_{(0)}, A)$. Up to now, we have written $A(G_{(0)})$ for the structure algebra of G, whereas $A(G)$ had no meaning. From now on, $A(G)$ should also be understood to mean the structure algebra of G.

6.4.1 Supergroup actions

Assume now that $G = (G_{(0)}, A)$ is a Lie supergroup and $M = (Y, B)$ another supermanifold. We will say that G acts on M if there is a map $A(G_{(0)})^* \otimes B(Y)^* \to B(Y)^*$, $u \otimes w \mapsto u \cdot w$, which satisfies

$$\Delta u = \sum_i u_i' \otimes u_i'',\ \Delta w = \sum_j w_j' \otimes w_j'' \implies \Delta(u \cdot w) = \sum_{i,j} (-1)^{|u_i''||w_j'|} u_i' \cdot w_j' \otimes u_i'' \cdot w_j'' \ . \tag{6.30}$$

In this case, the structure algebra $B(Y)$ is a $A(G_{(0)})^*$-module through

$$\pi : A(G_{(0)})^* \to \mathrm{End}(B(Y)),\ \langle w, \pi(u) f \rangle = (-1)^{|u||w|} \langle \sigma(u) \cdot w, f \rangle \ . \tag{6.31}$$

The latter is called the coaction representation of G. The action of G is fully determined by the corresponding coaction representation. Bearing in mind that $A(G_{(0)})^* = E(\mathfrak{g})$, we see that a Lie supergroup action can be though of as a pair of representations of the underlying group $G_{(0)}$ and of the Lie superalgebra $\mathfrak{g}$ on the vector space $B(Y)$, which satisfy a compatibility condition.

Dually, there is a map $\varphi : B(Y) \to B(Y) \otimes A(G_{(0)})$ that makes $B(Y)$ into a comodule-algebra of $A(G_{(0)})$. This means that φ is a morphism of algebras which is compatible with the Hopf algebra structure of $A(G_{(0)})$. For example, φ satisfies

$$(1 \otimes \Delta) \circ \varphi = (\varphi \otimes 1) \circ \varphi : B(Y) \to B(Y) \otimes A(G_{(0)}) \otimes A(G_{(0)}), \tag{6.32}$$

along with a number of other compatibility conditions, see e.g. [115]. Let p be a point in $G_{(0)}$, regarded as a morphism $p : A(G_{(0)}) \to \mathbb{R}$. Then one can form the map $(1 \otimes p) \circ \varphi : B(Y) \to B(Y)$. For obvious reasons, we refer to such compositions with p as *evaluations*. Running over all points p, we get a representation of the $G_{(0)}$ on $B(Y)$. This agrees with the coaction representation π from above.

6.4.2 Maurer-Cartan form and invariant vector fields

In this section we will explain how one can compute invariant vector fields on a supergroup. We will consider a supergroup G with local coordinates x^A and denote by $\mathfrak{g}$ its Lie superalgebra.

We can use the same indexing set to label a basis for $\mathfrak{g}$, denoted $\{X_A\}$. Coordinates and generators can be chosen such that they have the same parity, i.e. $|x^A| = |X_A|$.

The two algebras we have associated to G, $A(G_{(0)})$ and $U(\mathfrak{g})$, are closely related. In the case of bosonic groups, the generators X of the Lie algebra give rise to (right) invariant vector fields that act on functions as first order differential operators. These differential operators $\mathcal{R}_X$ can be multiplied and added and thereby provide an action of elements A in the universal enveloping algebra $U(\mathfrak{g})$ through differential operators $\mathcal{R}_A$ of higher order. One may combine the application of any such differential operator to a function on the group with the evaluation at the group unit e to obtain a map that assigns a number

$$\mathcal{R}_A(f)(e) = f(A) = \langle f, A\rangle \in \mathbb{C} \tag{6.33}$$

to a pair of an element $A \in U(\mathfrak{g})$ and a (complex valued) function f on the group. In other words, elements of $U(\mathfrak{g})$ give linear functionals of the structure algebra $A(G_{(0)})$ and vice versa. In this form, the statement remains true for Lie superalgebras and is often expressed by saying that there is a duality between $A(G_{(0)})$ and $U(\mathfrak{g})$. See also [114] for a nice discussion of this point.

Let us consider a sort of "supergroup element" $g = \exp(x_A X^A)$ within an appropriate closure of the tensor product $U(\mathfrak{g}) \otimes A(G_{(0)})$. A quick formal calculation gives an interesting property of this element

$$(\Delta \otimes id)g = e^{x_A(X^A\otimes 1 + 1\otimes X^A)} = e^{x_A(X^A\otimes 1)} e^{x_A(1\otimes X^A)} = \overset{1}{g}\,\overset{2}{g} \ . \tag{6.34}$$

Here, the application of the co-product Δ to the first tensor factor produces an element in $U(\mathfrak{g}) \otimes U(\mathfrak{g}) \otimes A(G_{(0)})$. The factors on the right hand side are elements in the same threefold tensor product. More concretely, $\overset{2}{g}$ is the element $1 \otimes g$ with the trivial entry in the first tensor factor. Similarly $\overset{1}{g}$ denotes the element g with the trivial entry in the second tensor factor. In writing the single exponential as a product of exponentials we used the fact that the exponent is an even object so that $x_A(X^A \otimes 1)$ commutes with $x_A(1 \otimes X^A)$. We will call all elements $g \in U(\mathfrak{g}) \otimes A(G_{(0)})$ that satisfy

$$(\Delta \otimes id)g = \overset{1}{g}\,\overset{2}{g}, \tag{6.35}$$

supergroup elements. In physics, it is customary to evaluate g in some representation π of the Lie superalgebra $\mathfrak{g}$. Thereby one obtains a finite-dimensional supermatrix $g^\pi = (\pi \otimes id)g$ with entries from the structure algebra $A(G_{(0)})$. In the following chapters we will often use the symbol g for such a matrix rather than an element of $U(\mathfrak{g}) \otimes A(G_{(0)})$.

Since the structure algebra $A(G_{(0)})$ is contained in the differential graded algebra $\Omega(G)$ (which is generated by x^A and dx^A) we can also regard the supergroup element g as an element of the differential graded algebra $U(\mathfrak{g}) \otimes \Omega(G)$, with the additional rule that $dX^A = 0$, i.e. we regard the generators X^A of the Lie superalgebra as constants. Now it makes sense to consider the Maurer-Cartan form

$$dgg^{-1} \in U(\mathfrak{g}) \otimes \Omega(G) \ . \tag{6.36}$$

If we apply the differential to the equation (6.35) that characterises g we obtain

$$\Delta(dgg^{-1}) = \left(d\overset{1}{g}\,\overset{2}{g} + \overset{1}{g}\,d\overset{2}{g}\right)\ \overset{2}{g}{}^{-1}\ \overset{1}{g}{}^{-1} = d\overset{1}{g}\,\overset{1}{g}{}^{-1} + \ d\overset{2}{g}\,\overset{2}{g}{}^{-1} \ . \tag{6.37}$$

We conclude that the Maurer-Cartan form takes values in the Lie superalgebra $\mathfrak{g} \subset U(\mathfrak{g})$, as is the case for usual bosonic Lie groups. Consequently, it may be expanded as

$$dgg^{-1} = dx_A C_{AB} X^B \quad \text{where} \quad C_{AB} \in A(G_{(0)}) \ . \tag{6.38}$$

The matrix elements C_{AB} possess degree $|A|+|B|$, i.e. they are even elements of the structure algebra if $|A| = |B|$ and odd otherwise. We also stress that coefficients C_{AB} depend on the choice of the supergroup element g. One of the main uses of the matrix elements C_{AB} is to construct the right-invariant vector fields. These vector fields are given by

$$\mathcal{R}_{X^A} = \mathcal{R}_A := \mathcal{C}^G_{AB}\partial_B,$$

where $\mathcal{C} = C^{-1}$ denotes the inverse of C and ∂_B is the (graded) derivative with respect to the coordinate x_B. Since we have assumed that the differential d acts trivially on the generators X^A of the universal enveloping algebra, we conclude that $\partial_\beta X^A = 0$, i.e. the generators X^A are constant objects on the supergroup satisfying

$$\partial_B X^A = (-1)^{|A||B|} X^A \partial_B \ . \tag{6.39}$$

6.4.3 Actions on supercosets

We will now extend the previous discussion and explain how to obtain the infinitesimal action of a supergroup G on a supercoset. The crucial example that we have in mind is the action of the superconformal group on the superspace. In general, we are interested in a case when the Lie superalgebra $\mathfrak{g}$ can be written as a direct sum of two subalgebras

$$\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{p} \ . \tag{6.40}$$

The unique simply connected super subgroup of G whose Lie superalgebra is $\mathfrak{p}$ is denoted by P. Then the quotient $M = G/P$ is a supermanifold (we are not concerned with possible singularities of G/P). We are considering here a rather particular case when M itself is a supergroup. Looking ahead, the standard choice in superconformal field theory is to define $\mathfrak{p}$ as the span of all elements in $\mathfrak{g}$ that have non-positive dilation weights. For this choice, $\mathfrak{m}$ consists of generators P of translations and the supercharges Q. We shall briefly comment on other choices below. We also choose a basis X^A of elements in $\mathfrak{g}$ that is compatible with the decomposition (6.40). Elements X^A that lie in the subspace $\mathfrak{m}$ will be labelled by lower case Latin indices while those that lie in the complement $\mathfrak{p}$ carry Greek indices.

The decomposition of the Lie superalgebra $\mathfrak{g}$ into $\mathfrak{m}$ and $\mathfrak{p}$ determines a decomposition of the corresponding universal enveloping algebra $U(\mathfrak{g}) = U(\mathfrak{m}) \otimes U(\mathfrak{p})$ as well as of the structure algebra $A(G_{(0)}) = A(M_{(0)}) \otimes A(P_{(0)})$. The structure algebras $A(M_{(0)})$ and $A(P_0)$ are generated by the coordinates x_a and x_α, respectively.

Let us now construct the infinitesimal action of G on the coset space M. We shall follow the logic of the previous section and introduce supergroup elements $m = m(x_a)$ and $p = p(x_\alpha)$. In case of m we work with the following standard choice

$$m(x_a) = e^{x_a X^a} \ . \tag{6.41}$$

The infinitesimal action of the Lie superalgebra on the coordinates x_a of the coset space M descends from the left-regular action of $\mathfrak{g}$ and thus can be computed from the Maurer-Cartan form,

$$dgg^{-1} = dx_A C^G_{AB} X^B \ . \tag{6.42}$$

To compute the Maurer-Cartan form for G we express it terms of Maurer-Cartan forms of M and P

$$dmm^{-1} = dx_a C^M_{ab} X^b, \quad dpp^{-1} = dx_\alpha C^P_{\alpha\beta} X^\beta \ .$$

With our choice $g = mp$ of the supergroup element g as a product of the two elements m and p it follows that

$$\begin{aligned} dgg^{-1} &= dx_A \partial_A (mp)(mp)^{-1} = dx_a(\partial_a m)m^{-1} + dx_\alpha m(\partial_\alpha p)p^{-1}m^{-1} = \\ &= dx_a C^M_{ab} X^b + dx_\alpha m C_{\alpha\beta} X^\beta m^{-1} = dx_a C^M_{ab} X^b + dx_\alpha C^P_{\alpha\beta}\Big((M_1)_{\beta a} X^a + (M_2)_{\beta\gamma} X^\gamma \Big) \ . \end{aligned} \tag{6.43}$$

The last equality defines the two matrices $M_{1,2}$,

$$mX^\beta m^{-1} = (M_1)_{\beta a} X^a + (M_2)_{\beta\gamma} X^\gamma \ . \tag{6.44}$$

From the equation (6.43) we can read off the coefficients C^G_{AB} of the Maurer-Cartan form for $\mathfrak{g}$. The inverse $\mathcal{C}^G$ of this matrix is easily seen to take the form

$$\mathcal{C}^G = \begin{pmatrix} \mathcal{C}^M & 0 \\ -M_2^{-1} M_1 \mathcal{C}^M & M_2^{-1}\mathcal{C}^P \end{pmatrix} \ ,$$

where the first row/column corresponds to directions in $\mathfrak{m}$ while the second row/column collects all the directions in $\mathfrak{p}$. As stated before, the matrix $\mathcal{C}^G$ provides us with the right-invariant vector fields (6.39) on the supergroup G. To project these operators to the superspace one simply sets $\partial_\alpha = 0$,

$$\mathcal{R}^{(M)} = \begin{pmatrix} \mathcal{C}^M & 0 \\ -M_2^{-1} M_1 \mathcal{C}^M & M_2^{-1}\mathcal{C}^P \end{pmatrix} \begin{pmatrix} \partial \\ 0 \end{pmatrix} = \begin{pmatrix} \mathcal{C}^M_{ab}\partial_b \\ -(M_2^{-1} M_1 \mathcal{C}^M)_{\alpha b}\partial_b \end{pmatrix} \ . \tag{6.45}$$

This is the main result of this subsection. As advertised above, differential operators $\mathcal{R}^{(M)}$ depend on the choice of coordinates on M, but not on that on P. In practical computations, it is way more economic to use (6.45) directly, rather than first derive the vector fields in all supergroup coordinates and then reduce them to the superspace.

Remark The formula (6.45) applies to all decompositions of $\mathfrak{g}$ into two Lie subalgebras $\mathfrak{m}$ and $\mathfrak{p}$. As we pointed out above, the standard choice is to take $\mathfrak{p}$ to contain generators that have non-positive conformal weight. In that case, the structure algebra $\mathcal{M} = A(M_{(0)})$ is called the standard superspace. If the superconformal algebra $\mathfrak{g}$ is of type I, however, there exist other natural choices to which the constructions of this subsection apply. Such a Lie superalgebra contains a $\mathfrak{u}(1)$ subalgebra (in the R-symmetry part) that commutes with $\mathfrak{g}_{(0)}$ and such that the two irreducible modules that make up $\mathfrak{g}_{(1)}$ have charges ± 1. It turns out that half of supertranslations Q have charge 1 and half -1, $\mathfrak{q} = \mathfrak{q}_+ \oplus \mathfrak{q}_-$. With this in mind we can introduce two new decompositions $\mathfrak{g} = \mathfrak{m}_\pm \oplus \mathfrak{p}_\pm$ of the superconformal algebra where

$$\mathfrak{p}_\pm = \mathfrak{g}_{\leq 0} \oplus \mathfrak{q}_\pm \ , \quad \mathfrak{m}_\pm = \mathfrak{g}_1 \oplus \mathfrak{q}_\mp = \mathfrak{g}/\mathfrak{p}_\pm \ .$$

From the properties of type I Lie superalgebras, one may easily show that both $\mathfrak{p}_\pm$ and $\mathfrak{m}_\pm$ are subalgebras of $\mathfrak{g}$. The associated superspaces $M_\pm$ are called the chiral and anti-chiral superspace, respectively.

Example: As an example, let us illustrate the construction of superspace and the differential operators in the case of the one-dimensional $\mathcal{N}=2$ superconformal algebra $\mathfrak{g}=\mathfrak{sl}(2|1)$. The smallest faithful representation of $\mathfrak{g}$ is three-dimensional. Here we shall consider the decomposition $\mathfrak{g}=\mathfrak{m}\oplus\mathfrak{p}$ with the Lie superalgebra $\mathfrak{m}$ spanned by P, Q_+ and Q_-. The corresponding superspace M is generated by one bosonic variable u along with two Grassmann variables θ and $\bar\theta$. The supergroup element m we introduced above takes the following matrix form

$$m(x) = e^{uP+\theta Q_+ + \bar\theta Q_-} = \begin{pmatrix} 1 & X & \theta \\ 0 & 1 & 0 \\ 0 & -\bar\theta & 1 \end{pmatrix} , \tag{6.46}$$

where $X = u - \frac{1}{2}\theta\bar\theta$ and $x=(u,\theta,\bar\theta)$ represents the three generators of the structure algebra $A(M)$.

The construction we described above provides us with an action of the superconformal algebra $\mathfrak{g}$ on this superspace with differential operators $\mathcal{R}_X$ of the form

$$p = \partial_u , \qquad k = -u^2\partial_u - u\theta\partial_\theta - u\bar\theta\partial_{\bar\theta} , \tag{6.47}$$

$$d = u\partial_u + \frac{1}{2}\theta\partial_\theta + \frac{1}{2}\bar\theta\partial_{\bar\theta} , \qquad r = \theta\partial_\theta - \bar\theta\partial_{\bar\theta} , \tag{6.48}$$

$$q_+ = \partial_\theta - \frac{1}{2}\bar\theta\partial_u , \qquad q_- = \partial_{\bar\theta} - \frac{1}{2}\theta\partial_u , \tag{6.49}$$

$$s_+ = -(u + \frac{1}{2}\theta\bar\theta)q_+ , \qquad s_- = (u - \frac{1}{2}\theta\bar\theta)q_- . \tag{6.50}$$

As we pointed out in our discussion above, the choice of p is not relevant for the final result. We encourage the reader to derive these explicit expressions from our general formula (6.45).

6.5 Superconformal algebras

In this section we will give the defining properties of a superconformal algebra and list all such algebras. To obtain the classification, one simply has to go through Kac's list and select from it the Lie superalgebras that satisfy the additional properties. Indeed, this was the way in which Nahm arrived at the result in [116]. Famously, no superconformal algebras exist in dimensions above six.

Let $\mathfrak{g}=\mathfrak{g}_{(0)}\oplus\mathfrak{g}_{(1)}$ be a finite-dimensional Lie superalgebra. We say that $\mathfrak{g}$ is a superconformal algebra if its even part $\mathfrak{g}_{(0)}$ contains the conformal Lie algebra $\mathfrak{so}(d+1,1)$ as a direct summand and the odd part $\mathfrak{g}_{(1)}$ decomposes as a direct sum of spinor representations of $\mathfrak{so}(d)\subset\mathfrak{so}(d+1,1)$ under the adjoint action.

If this is the case, we denote the dilation generator of the bosonic conformal Lie algebra by D. Eigenvalues with respect to ad_D give a decomposition of $\mathfrak{g}$ into the sum of eigenspaces

$$\mathfrak{g} = \mathfrak{g}_{-1}\oplus\mathfrak{g}_{-1/2}\oplus\mathfrak{g}_0\oplus\mathfrak{g}_{1/2}\oplus\mathfrak{g}_1 = \mathfrak{g}_{-1}\oplus\mathfrak{s}\oplus\mathfrak{k}\oplus\mathfrak{q}\oplus\mathfrak{g}_1 . \tag{6.51}$$

The even part of $\mathfrak{g}$ is composed of $\mathfrak{g}_{\pm 1}$ and $\mathfrak{k}$, where $\mathfrak{g}_{-1} = \mathfrak{n}$ contains the generators K_μ of special conformal transformations, while $\mathfrak{g}_1 = \mathfrak{m}$ is spanned by translations P_μ. Dilations, rotations and internal symmetries make up

$$\mathfrak{k} = \mathfrak{so}(1,1) \oplus \mathfrak{so}(d) \oplus \mathfrak{u} \ .$$

Generators of $\mathfrak{g}_{\pm 1/2}$, are supertranslations Q_α and special superconformal transformations S_α. We shall also denote these summands as $\mathfrak{s} = \mathfrak{g}_{-1/2}$ and $\mathfrak{q} = \mathfrak{g}_{1/2}$. All elements of non-positive degree make up a subalgebra $\mathfrak{p}$ of $\mathfrak{g}$ that will be referred to as the parabolic subalgebra

$$\mathfrak{p} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_{-1/2} \oplus \mathfrak{g}_0 \ . \tag{6.52}$$

Let G be the superconformal group, i.e. some supergroup with $\mathfrak{g} = Lie(G)$. There is a unique (connected) corresponding subgroup $P \subset G$ such that $\mathfrak{p} = Lie(P)$. The superspace can be identified with the supergroup of translations and supertranslations. It is defined as the homogeneous space $M = G/P$.

The above structure is present in any superconformal algebra. We shall often focus on Lie superalgebras that are in addition of type I and denote the two irreducibles representations of $\mathfrak{g}_{(0)}$ that make up $\mathfrak{g}_{(1)}$ as $\mathfrak{g}_\pm$

$$\mathfrak{g}_{(1)} = \mathfrak{g}_+ \oplus \mathfrak{g}_- \ . \tag{6.53}$$

The two modules $\mathfrak{g}_\pm$ are then necessarily dual to each other and further satisfy

$$\{\mathfrak{g}_\pm, \mathfrak{g}_\pm\} = 0 \ . \tag{6.54}$$

In addition, the bosonic algebra assumes the form

$$\mathfrak{g}_{(0)} = [\mathfrak{g}_{(0)}, \mathfrak{g}_{(0)}] \oplus \mathfrak{u}(1) \ . \tag{6.55}$$

The $\mathfrak{u}(1)$ summand is a part of the internal symmetry algebra. Its generator will be denoted by R. All elements in $\mathfrak{g}_+$ possess the same R-charge. The same is true for the elements of $\mathfrak{g}_-$, but the R-charge of these elements has the opposite value. Elements in the even subalgebra $\mathfrak{g}_{(0)}$, on the other hand, commute with R.

Let us denote the intersections of the subspaces $\mathfrak{q}$ and $\mathfrak{s}$ with $\mathfrak{g}_\pm$ by

$$\mathfrak{q}_\pm = \mathfrak{q} \cap \mathfrak{g}_\pm \quad , \quad \mathfrak{s}_\pm = \mathfrak{s} \cap \mathfrak{g}_\pm \ . \tag{6.56}$$

The subspaces $\mathfrak{q}_\pm$ and $\mathfrak{s}_\pm$ do not carry a representation of $\mathfrak{g}_{(0)}$, but they do carry a representation of $\mathfrak{k}$. This also means that in type I superconformal algebras, the action of $\mathfrak{k}$ on supertranslations decomposes into two or more irreducible representations. It turns out that

$$\dim(\mathfrak{q}_\pm) = \dim(\mathfrak{s}_\pm) = \frac{1}{4}\dim(\mathfrak{g}_{(1)}) \ . \tag{6.57}$$

Kac's classification leads to the following list of complexified type I superconformal algebras

$$\mathfrak{sl}(2|\mathcal{N}), \quad \mathfrak{sl}(2|\mathcal{N}_1) \oplus \mathfrak{sl}(2|\mathcal{N}_2), \quad \mathfrak{psl}(2|2), \quad \mathfrak{sl}(2|\mathcal{N}) \oplus \mathfrak{psl}(2|2), \quad \mathfrak{osp}(2|4), \quad \mathfrak{sl}(4|\mathcal{N}), \quad \mathfrak{psl}(4|4) \ .$$

In dimensions higher than two, complexified superconformal algebras of type II are

$$\mathfrak{osp}(\mathcal{N}|4), \quad F(4), \quad \mathfrak{osp}(8|2\mathcal{N}) \ .$$

In addition, each of $\mathfrak{osp}(\mathcal{N}|2)$, $\mathfrak{osp}(4|2\mathcal{N})$, $F(4)$, $G(3)$ and $D(2,1;\alpha)$ may be regarded as a one-dimensional superconformal algebra, [117]. Their direct sums give rise to possible algebras in two dimensions.

In physics, one is interested in real Lie superalgebras and for different spacetime signatures one considers various real forms of the above. For the classification of real forms, the reader is referred to the work of Monique Parker, [118].

Chapter 7

Correlators as covariant functions on the superconformal group

Starting from the present chapter, we turn to the new results obtained in our work (while we do not know if some formulas from previous chapters exist in the literature, they are certainly not new in any essential way). Therefore, it may be an opportune moment to state our aims and describe the general strategy that we will follow over the next chapters to achieve them.

The central object of study in the conformal bootstrap programme, the crossing symmetry equations, consist of two ingredients. These are the conformal partial waves and tensor structures. It is perhaps fair to say that out of these two, the partial waves are more difficult to come by, since even in the simplest description they are defined as solutions to second order PDEs. Conformal blocks for four-point functions of arbitrary spinning fields are fairly well understood, but no so well in other cases such as superconformal theories, defect CFTs or n-point functions with $n \geq 5$. This is the main issue that we want to address.

One may ask if a well-behaved theory of conformal blocks should exist at all. There are several reasons to believe that it actually should, the first one being compact expressions for certain blocks found by Dolan and Osborn. Secondly, from their definition, partial waves come with a group theoretical interpretation. It is therefore desirable to identify them in the vast literature on representation theory. Once this is done, it could open the way to establish relations of blocks with integrable systems and special functions, derive integral representations for them etc.

With the motive and the means in place, all that we need is an opportunity. This brings us to the topic of the present chapter. The main result that we will show is that there is a bijective correspondence between solutions of four-point conformal Ward identities and K-spherical functions on the conformal group. The map that gives a solution $G_4(x_i)$ from a K-spherical function F is written in (7.40). In fact, this formula applies equally well to bosonic and supersymmetric setups. However, to define its meaning for a SCFT, we will need to introduce the analogue of the Bruhat decomposition for the superconformal group. This is the subject of the first section. In the second section we introduce the non-unitary principal series representations of the superconformal group by mimicking the bosonic construction, only using our new Bruhat decomposition. Next, we explain the fact that was alluded to a few times already, that principal series representations are naturally associated with fields in a (S)CFT.

This will be achieved by lifting an arbitrary vector-valued function defined on a superspace, to a covariant function on the superconformal group. If all fields of the theory are lifted in this way, n-point correlation functions become functions on a number of copies of the supergroup, namely G^n. However, covariance laws of these functions allow us to map them to functions on a smaller number of copies of G. For example, a function on $G \times G$ that is covariant with respect to $P \times P$ can be mapped to a function on G covariant with respect to K. In particular, this observation allows for a nice reinterpretation of four-point functions

$$A\left(G/P \times G/P \times G/P \times G/P\right)^G \cong A(G/K \times G/K)^G \cong A(K\backslash G/K) \ . \tag{7.1}$$

The is a heuristic explanation of how spherical functions appear and its precise version is presented in the fourth section. Here $A(X)$ denotes the structure algebra of the supermanifold X and $A(X)^G$ stands for the space of G-invariants when X carries an action of G (using the theory of the last chapter, we could also write $A(X/G)$). The leftmost space above models solutions to Ward identities. The last two sections go into more details of the superconformal Bruhat decomposition. The first one discusses some functorial properties of Bruhat factors (Assume we have a factorisation of some group element g and apply some specific transformation to g. We ask how does this change each individual factor of g.) The second one computes the Bruhat decomposition of various group elements for a class of supergroups using a representation by supermatrices.

Having found the representation (7.40) of four-point functions, we may ask what it is useful for. This is the subject of the following chapters. The main reason was already hinted at: under (7.40) Casimir equations map to the eigenvalue problem for the Laplacian. Four-point correlators are particularly convenient as we end up with K-spherical functions that have a well-behaved theory. However, the methods of this chapter can be adopted with very slight modifications to obtain similar representations of other types of correlators. In particular, this will be done for two- and three-point functions of defect CFTs later on. Further extensions are also possible and often straightforward. In this sense, the next pages contain some of the main ideas of this thesis. They are largely based on [2, 3].

Unless specified otherwise, G will stand for the superconformal group, P for its parabolic subgroup and the corresponding superspace will be denoted by $M = G/P$. The structure algebra of M is written as $\mathcal{M}$ and the Lie superalgebra of M as $\mathfrak{m}$.

7.1 Bruhat decomposition of the superconformal group

We shall now introduce a supersymmetric generalisation of the Bruhat decomposition that was described in a previous chapter. Consider a number of commuting copies of some superspace M. We label these by an index i and write their coordinates as x_{ia}. The corresponding supergroup elements $m(x_i)$ are defined by

$$m(x_i) = e^{x_{ia} X^a} \ . \tag{7.2}$$

Here, $\{X^a\}$ is a basis for $\mathfrak{m}$ and the summation over a is understood. Given any pair of labels i, j we define the variables $x_{ij} = (x_{ija}) \in \mathcal{M}_i \otimes \mathcal{M}_j$ through

$$m(x_{ij}) = m(x_j)^{-1} m(x_i) \ . \tag{7.3}$$

The possibility to define these variables follows from the fact that M itself is a supergroup. Recall that we have defined the Weyl inversion as the element (4.5) of the conformal group. We use the same formula as the definition of the Weyl inversion of the superconformal group G. It is an element of the underlying even subgroup $G_{(0)}$, trivial in the internal symmetry factor U. Using that conjugation of the generator D of dilations with w is given by $\mathrm{Ad}_w(D) = -D$, we obtain

$$\frac{1}{2}\mathrm{Ad}_w(Q) = \mathrm{Ad}_w([D,Q]) = [\mathrm{Ad}_w(D), \mathrm{Ad}_w(Q)] = -[D, \mathrm{Ad}_w(Q)] ,$$

i.e. when a supercharge Q is acted upon by the Weyl inversion it is sent to a generator whose conformal weight is $-1/2$. Consequently, the Weyl inversion interchanges generators of supertranslations and special superconformal transformations. For superconformal algebras of type I, one can similarly use that $\mathrm{Ad}_w(R) = R$ to deduce

$$\mathrm{Ad}_w(\mathfrak{q}_\pm) \subset \mathfrak{s}_\pm . \tag{7.4}$$

Remark Recall that in the superconformal context, $G_{(0)}$ is required to be simply connected. We defined the bosonic Weyl inversion for the simply connected conformal group, so w is well-defined for any superconformal group. This is to be contrasted with the fact that a supersymmetric analogue of the ordinary conformal inversion may actually not exist. Assuming that one could choose the superconformal group such that the inversion I belonged to its even subgroup, then the arguments leading to (7.4) with w replaced by I would remain valid. On the other hand, as the example $\mathfrak{g} = \mathfrak{sl}(4|1)$ shows, the fact that I commutes with rotations is inconsistent with (7.4), bearing in mind that $\mathfrak{q}_+$ and $\mathfrak{s}_+$ are non-isomorphic modules of the rotation group.

With the help of the Weyl inversion, we define a new family of supergroup elements $n(x)$ through

$$n(x) = w^{-1}m(x)w . \tag{7.5}$$

Since m involves only generators of the superconformal algebra of positive conformal weight, the element n is built using generators of negative weight. This means that n involves special conformal generators K and the fermionic generators S.

The Bruhat decomposition of the superconformal algebra is that into subspaces of positive, negative and zero conformal weights

$$\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{n} \oplus \mathfrak{k} . \tag{7.6}$$

The corresponding decomposition of the superconformal group will also be called the Bruhat decomposition, or the Bruhat factorisation. It makes use of elements $m(x)$ and $n(y)$, together with $k = k(t)$ that are build by exponentiating the generators $X \in \mathfrak{g}$ that commute with D, i.e. dilations, rotations and R-symmetry transformations.

Using the Bruhat decomposition we can, similarly as in the bosonic case, define functions $y(x,h)$, $z(x,h)$ and $t(x,h)$ through the factorisation

$$hm(x) = m(y(x,h))\, n(z(x,h))\, k(t(x,h)) . \tag{7.7}$$

Here h is an arbitrary element of G. In the case $h = w$ we simply write $y(x) = y(x,w)$ etc, i.e.

$$wm(x) = m(y(x))\, n(z(x))\, k(t(x)) . \tag{7.8}$$

Explicit expressions for $y(x)$, $z(x)$ and $t(x)$ in a number of cases will be worked out later. The first factor $y(x,h)$ gives by definition the action of h on the superspace, i.e. $y(x,h) = hx$ (It is easy to see from (7.7) that $y(x,h)$ is indeed an action. This will be shown later on.) In particular $y(x) = wx$. Finally, we will write $y_{ij} = y(x_{ij})$ etc. That is

$$wm(x_{ij}) = m(y_{ij})\, n(z_{ij})\, k(t_{ij}) \; . \tag{7.9}$$

The components of x_{ij}, y_{ij}, z_{ij} and t_{ij} are elements in the tensor product $\mathcal{M}^{\otimes 2}$ of superspace structure algebras $\mathcal{M}$, one copy for each insertion point.

Example Let us briefly discuss superconformal transformations and in particular the Weyl inversion for the Lie superalgebra $\mathfrak{sl}(2|1)$. As we said in the last chapter, this Lie superalgebra admits a three-dimensional representation. All generators have been spelled out in this representations above. Within the three-dimensional representation, the supergroup element $m(x)$ takes the form (6.46). The subgroup K is generated by dilations and $U(1)$ R-symmetry transformations, $\mathfrak{k} = \mathrm{span}\{D, R\}$. Under the action of elements $k = \exp(\lambda D + \vartheta R)$ the superspace coordinates $x = (u, \theta, \bar{\theta})$ transform as

$$y(x,k) = (e^{\lambda} u, e^{\frac{1}{2}\lambda + \vartheta}\theta, e^{\frac{1}{2}\lambda - \vartheta}\bar{\theta}) \; . \tag{7.10}$$

Here we can either regard λ and ϑ as elements of the structure algebra $A(G)$ or some real or complex parameters. Supertranslations with an element $m(c) = m(v, \eta, \bar{\eta})$ act as $m(c)m(x) = m(y(x,c))$ with

$$y(x,c) = (u + v + \frac{1}{2}\theta\bar{\eta} + \frac{1}{2}\bar{\theta}\eta, \theta + \eta, \bar{\theta} + \bar{\eta}) \; . \tag{7.11}$$

This action admits only the first interpretation from above and v, η, $\bar{\eta}$ have to be regarded as elements of $A(G)$. It remains to discuss the Weyl inversion. Within the three-dimensional representation it is straightforward to find w from (4.5),

$$w = e^{\pi \frac{K-P}{2}} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \; . \tag{7.12}$$

Note that $w^2 = \mathrm{diag}(-1,-1,1)$, i.e. it squares to -1 within the bosonic conformal group and is trivially extended to the R-symmetry group.

Going further, the elements $n(x)$ read

$$n(x) = w^{-1} m(x) w = \begin{pmatrix} 1 & 0 & 0 \\ -X & 1 & -\theta \\ -\bar{\theta} & 0 & 1 \end{pmatrix} \; . \tag{7.13}$$

The Bruhat decomposition of $wm(x)$ now takes the from of the matrix identity

$$\begin{pmatrix} 0 & -1 & 0 \\ 1 & X & \theta \\ 0 & -\bar{\theta} & 1 \end{pmatrix} = \begin{pmatrix} 1 & -\frac{1}{u}\left(1 + \frac{\theta\bar{\theta}}{2u}\right) & \theta/u \\ 0 & 1 & 0 \\ 0 & -\bar{\theta}/u & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ u + \frac{1}{2}\theta\bar{\theta} & 1 & \theta \\ \bar{\theta} & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{u}\left(1 - \frac{\theta\bar{\theta}}{2u}\right) & 0 & 0 \\ 0 & u\left(1 - \frac{\theta\bar{\theta}}{2u}\right) & 0 \\ 0 & 0 & 1 - \frac{\theta\bar{\theta}}{u} \end{pmatrix}, \tag{7.14}$$

from which we can read off the functions $y(x)$, $z(x)$ and $k(t(x))$. Comparing the first of the three factors with the expression (6.46) for $m(y)$ we deduce

$$y(x) = w(u, \theta, \bar{\theta}) = \left(\frac{-1}{u}, \frac{\theta}{u}, \frac{\bar{\theta}}{u} \right) . \tag{7.15}$$

Note that the action of w on the bosonic coordinate u is the same as in bosonic conformal field theory. This had to be the case, since in the chosen coordinate system on the superspace M the action of the conformal algebra generators on u is the same as in the bosonic theory. The second and third factor are similarly computed by comparing matrices in (7.14) with definitions of $n(x)$ and k to give

$$z(x) = (-u, -\theta, -\bar{\theta}), \quad k(t(x)) = e^{-\log u^2 D + \frac{\theta\bar{\theta}}{2u} R} . \tag{7.16}$$

Putting everything together, the matrix equation (7.14) leads to the following Bruhat factorisation for supergroup elements

$$wm(x) = e^{w(x)\cdot X} e^{-x\cdot X^w} e^{-\log u^2 D + \frac{\theta\bar{\theta}}{2u} R}, \tag{7.17}$$

where $X^w = w^{-1}(P, Q_+, Q_-)w = (-K, -S_+, S_-)$. Notice that the last identity does not make any reference to a particular representation of G.

7.2 Non-unitary principal series representations

The Bruhat decomposition of the ordinary conformal group gives rise to the non-unitary principal series of representations. We can use the analogous construction which starts from the supersymmetric Bruhat decomposition to define the non-unitary principal series of the superconformal group. In fact, we shall define these representations as coinduced modules of the superconformal algebra $\mathfrak{g}$, because the construction is simplest and cleanest in this algebraic setup. However, later we will freely move between the superalgebra and supergroup representations according to whichever is better suited to the problem at hand.

Let ρ be a finite-dimensional representation of the parabolic subalgebra $\mathfrak{p} = \mathfrak{k} \oplus \mathfrak{n}$ of $\mathfrak{g}$ on the space W. The coinduced module $\pi = \mathrm{Coind}_{\mathfrak{p}}^{\mathfrak{g}} W$ is called an algebraic principal series representation if ρ is trivial on $\mathfrak{n}$. The carrier space of π is that of left-covariant linear maps

$$V = \mathrm{Hom}_{U(\mathfrak{p})}(U(\mathfrak{g}), W) . \tag{7.18}$$

Explicitly, the maps $\varphi \in V$ satisfy

$$\varphi(xA) = (-1)^{|x||\varphi|} \pi(x) \varphi(A) \quad \text{for} \quad x \in \mathfrak{p}, \tag{7.19}$$

where, as always, $|.|$ denotes the parity of a homogeneous element in a super vector space. The equation above *defines* the parity of the covariant map φ. For later use we also note that our definition (7.18) implies that any element φ of V satisfies

$$\varphi(yU(\mathfrak{g})) = 0 \quad \text{for } y \in \mathfrak{n}, \tag{7.20}$$

as a consequence of the fact that special (super)conformal transformations act trivially on the space W. The action of $x \in \mathfrak{g}$ on maps $\varphi \in V$ is given by

$$(x\varphi)(A) = (-1)^{|x|(|\varphi|+|A|)}\varphi(Ax), \quad A \in U(\mathfrak{g}) \ . \tag{7.21}$$

Therefore, the coinduction for Lie superalgebra representations is analogous to its bosonic counterpart. One only needs to take care of additional minus signs in (7.19) and (7.21).

7.2.1 Tensor products of principal series representations

We now turn to tensor products of principal series representations. They will also be realised as coinduced representations, however from the smaller subalgebra $\mathfrak{k}$. Such a tensor product is typically highly reducible, but we are not interested here in its decomposition into irreducible components. Our result is a natural generalisation of the theorem 9.2 from [87] to the supersymmetric setting. (See also [119] for similar results about ordinary Lie groups.)

The possibility to realise the tensor product of two principal series representations π_i as a module coinduced from $\mathfrak{k}$ should not be very surprising. Indeed, vectors in a principal series representation may be regarded as functions on the superspace M. Therefore, vectors in the tensor product of two principal series representations should be realisable as functions on two copies of the superspace. Recall that the Lie superalgebra $\mathfrak{m}$ of M has another isomorphic copy $\mathfrak{n}$ in the full superconformal algebra. The two subalgebras are conjugate to each other under the Weyl inversion. With the help of w, one can replace the second of the two copies of the superspace by the supergroup generated by $\mathfrak{n}$. Since $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{n} \oplus \mathfrak{k}$, the resulting functions can be regarded as those on G/K.

To turn this heuristics into a theorem, let us denote by $s = \mathrm{Ad}_w$ the automorphism of $\mathfrak{g}$ corresponding to w. (To see that Ad_w is uniquely defined as an automorphism of $\mathfrak{g}$, recall our definition of a Lie-Hopf algebra). Given any representation π of $\mathfrak{g}$, we denote by π^s the representation obtained by precomposing it with s, $\pi^s = \pi \circ s$. Clearly, for the coinduced module π_W the carrier space of π^s_W takes the form

$$\bar{V}' := \mathrm{Hom}_{U(s(\mathfrak{p}))}\left(U(\mathfrak{g}), W'\right) \ , \tag{7.22}$$

where the representation W' of $s(\mathfrak{p})$ is obtained from the representation W of $\mathfrak{k}$ by composition with the Weyl inversion and the trivial extension to $s(\mathfrak{p})$. As we recalled above, the inversion flips the sign of D and acts trivially on the generators of $\mathfrak{u}$. We will denote the representation of $\mathfrak{g}$ on the space $\bar{V}'$ by $\bar{\pi}_{W'}$. The bar is supposed to remind us that we perform our coinduction from $s(\mathfrak{p})$ rather than $\mathfrak{p}$ itself. Since s is an inner automorphism, we have $\pi_W \cong \bar{\pi}_{W'}$. Now we are prepared to state our theorem.

Theorem Let V_i be two principal series representations (7.18). Their tensor product is isomorphic to

$$V_1 \otimes V_2 \cong \mathrm{Hom}_{U(\mathfrak{k})}\left(U(\mathfrak{g}), W_1 \otimes W_2'\right) \ . \tag{7.23}$$

Proof: We will show that following map F is an isomorphism of modules on the left and the right hand sides,

$$F : V_1 \otimes \bar{V}_2' \to \mathrm{Hom}_{U(\mathfrak{k})}(U(\mathfrak{g}), W_1 \otimes W_2'),$$

$$\varphi_1 \otimes \varphi_2 \mapsto \psi = (\varphi_1 \otimes \varphi_2) \circ \Delta \ .$$

The symbol Δ denotes the coproduct in $U(\mathfrak{g})$. Using standard properties of the coproduct, it is easy to see that F is well-defined and a homomorphism of $\mathfrak{g}$-modules, so we will only show that it is invertible. Let ψ be an arbitrary element of the module on the right hand side. We shall reconstruct from it a function

$$\varphi : U(\mathfrak{g}) \otimes U(\mathfrak{g}) \to W_1 \otimes W_2',$$

which is its preimage in the representation space on the left. Due to covariance properties, such a function is completely specified by the values

$$\varphi(P_1^{n_1}...P_d^{n_d}Q_1^{\varepsilon_1}...Q_k^{\varepsilon_k} \otimes 1), \quad \varphi(1 \otimes S_1^{\eta_1}...S_k^{\eta_k}K_1^{m_1}...K_d^{m_d}) \ . \tag{7.24}$$

Further, by the equation (7.20)

$$\varphi(S_1^{\eta_1}...S_k^{\eta_k}K_1^{m_1}...K_d^{m_d} \otimes 1) = 0 = \varphi(1 \otimes P_1^{n_1}...P_d^{n_d}Q_1^{\varepsilon_1}...Q_k^{\varepsilon_k}).$$

Using that $\Delta(X) = X \otimes 1 + 1 \otimes X$ for all elements $X \in \mathfrak{g}$ we conclude that

$$\psi(P_1^{n_1}...P_d^{n_d}Q_1^{\varepsilon_1}...Q_k^{\varepsilon_k}) = \varphi(P_1^{n_1}...P_d^{n_d}Q_1^{\varepsilon_1}...Q_k^{\varepsilon_k} \otimes 1), \tag{7.25}$$

and similarly for the second type of elements. Hence, we are able to recover φ from ψ. This completes the proof of the theorem. □

7.2.2 Shortening conditions

We shall now extend the above discussion, so far restricted to typical representations of the superconformal algebra $\mathfrak{g}$, and consider multiplet shortening. In fact, our comments will also apply to bosonic theories, where short representations occur e.g. by "imposing equations of motion" $\partial^2\varphi = 0$.

To set the stage, let $\mathfrak{g}$ be some superconformal algebra and $V = \mathrm{Ind}_{\mathfrak{p}}^{\mathfrak{g}} W$ a parabolic Verma module with a highest-weight vector v_0. Pick some finite number of elements A_α of the universal enveloping algebra $U(\mathfrak{g})$ and set

$$u_\alpha = A_\alpha v, \quad X = \mathrm{span}\{u_\alpha\}, \quad U = U(\mathfrak{g})X \ .$$

By construction, U is a subrepresentation of V. Therefore, we can form the quotient representation $M = V/U$. As vector spaces $V = M \oplus U$ and there is a natural projection

$$p : V \to M, \quad p(v) = v + U \ .$$

The module M is clearly a highest-weight representation with a highest-weight vector $v_0 + U$. Moreover

$$A_\alpha(v_0 + U) = u_\alpha + U = 0 \quad \text{in} \quad M \ .$$

For this reason, the representation M will be called *a short multiplet obtained from V by setting $A_\alpha v_0 = 0$*.

Let M^* be the dual module to M. Since M is a quotient of V, M^* is a subrepresentation of $\pi = V^*$. It is the subspace of functions $F : V \to \mathbb{C}$ which are well-defined on M. This in turn means that Fs vanish on U. Therefore, we have

$$F(U(\mathfrak{g})A_\alpha v_0) = 0 \ .$$

As mentioned before, π can be regarded as the coinduced module $\mathrm{Coind}_{\mathfrak{p}}^{\mathfrak{g}}W^*$. In terms of ψ introduced in (4.37), the conditions on F translate to

$$\psi(\sigma(A_\alpha)U(\mathfrak{g}))(w_0) = 0, \tag{7.26}$$

where w_0 is the highest weight vector of W. Obviously, the space of functions ψ that satisfy (7.26) is $\mathfrak{g}$ invariant

$$(x \cdot \psi)(\sigma(A_\alpha)B)(w_0) = \pm\psi(\sigma(A_\alpha)Bx)(w_0) = 0, \quad x \in X,\ B \in U(\mathfrak{g}) \ .$$

If ψ is interpreted as the Taylor expansion of a function on the supergroup $f : G \to W$, the condition (7.26) can be rewritten as the set of differential equations

$$(\mathcal{R}_{\sigma(A_\alpha)}f)(w_0) = 0, \tag{7.27}$$

where, recall, $\mathcal{R}_A$ is the right-invariant differential operator corresponding to $A \in U(\mathfrak{g})$. If W is one-dimensional, we simply have $\mathcal{R}_{\sigma(A_\alpha)}f = 0$, which is the often seen form of shortening. Another case that we wish to describe is when the elements $\sigma(A_\alpha)$ form a representation of $\mathfrak{k}$. Let $k \in \mathfrak{k}$ and $B \in U(\mathfrak{g})$ be arbitrary and $[k, \sigma(A_\alpha)] = \sigma(A_\beta)$. Then we have the sequence of equalities

$$\psi(\sigma(A_\alpha)B)(kw_0) = \Big(k\cdot\psi(\sigma(A_\alpha)B)\Big)(w_0) = \psi(k\sigma(A_\alpha)B)(w_0) = \psi(\sigma(A_\alpha)kB+\sigma(A_\beta)B)(w_0) = 0 \ .$$

In the first step we used duality of representations W and W^* and in the second covariance property of the function ψ. Finally, we applied the commutation relation above and used the shortening conditions. Since the representation W is irreducible, we can get any vector in it by applying an appropriate k to w_0. Therefore, the condition (7.26) again reduces to $\mathcal{R}_{\sigma(A_\alpha)}f = 0$.
In the context of superconformal algebras, the shortening is usually performed by a set of supercharges Q_α. That is, we restrict to vectors in the principal series representation (7.18) that satisfy

$$\varphi(Q_\alpha U(\mathfrak{g}))(w_0) = 0 \ . \tag{7.28}$$

Let us now consider two, possibly short, representations that are submodules of principal series $\pi_i = \mathrm{Coind}_{\mathfrak{p}}^{\mathfrak{g}}W_i$. According to our discussion above, the tensor product $\pi_1 \otimes \pi_2$ is isomorphic to $\mathrm{Coind}_{\mathfrak{k}}^{\mathfrak{g}}(W_1 \otimes W_2')$. By substituting the explicit form of the isomorphism, $\psi = (\varphi_1 \otimes \varphi_2) \circ \Delta$, we see that shortening conditions imply that elements ψ satisfy

$$\psi(Q_\alpha U(\mathfrak{g}))(w_1 \otimes w_2') = \psi(s(Q_\beta)U(\mathfrak{g}))(w_1 \otimes w_2') = 0, \tag{7.29}$$

for all $\alpha \in I_1$ and $\beta \in I_2$. Note that $s(Q_\beta) = S_\beta$ are generators of special superconformal transformations. If the representation W_i are scalar, there is only one vector in the dual of $W_1 \otimes W_2'$ and conditions consequently simplify to

$$\psi(Q_\alpha U(\mathfrak{g})) = \psi(s(Q_\beta)U(\mathfrak{g})) = 0 \ . \tag{7.30}$$

In summary, we have defined principal series representations of $\mathfrak{g}$ on the space of covariant linear maps on $U(\mathfrak{g})$ that take values in some vector space. We can think of these as covariant vector-valued functions on the superconformal group. Tensor products of two such representations admits a similar description in terms of functions on G that are covariant with respect to K. If some of the representations are short, functions in these spaces are required to satisfy further differential equations.

7.3 From quantum fields to covariant functions

The space of fields $\varphi : M \to W$ in a superconformal theory carries a representation of G. This representation, denoted π, was written for bosonic theories in (2.19) and the same formula holds in the supersymmetric case. Thanks to the identity $dg_x = k(t(x,g))$, we can rewrite (2.19) as

$$(\pi_g \varphi)(gx) = \rho(k(t(x,g)))\varphi(x), \tag{7.31}$$

where the element $k(t(x,g))$ is defined by means of the Bruhat decomposition. The statement that $dg_x = k(t(x,g))$ is a familiar one. To explain it a bit more, we focus on a bosonic CFT. If g is a conformal transformation, by definition its differential at any point is of the form $dg_x = \Omega(x)R^{\mu}{}_{\nu}(x)$, with some scale factor $\Omega(x)$ and some rotation matrix $R^{\mu}{}_{\nu}(x)$. These are precisely the dilation and rotation Bruhat factors of $gm(x)$. E.g. if g is a translation, then $\Omega(x) = 1$ and $R(x) = 1$. For a rotation $g = r$, one has $\Omega(x) = 1$ and $R(x) = r$, while dilations $g = e^{\lambda D}$ have $\Omega(x) = e^{\lambda}$ and $R(x) = 1$. Finally, the differential of the Weyl inversion decomposes as

$$dw_x = \frac{1}{x^2} s_{e_d} s_x \qquad \Longrightarrow \qquad \Omega(x) = \frac{1}{x^2}, \quad R(x) = s_{e_d} s_x \ . \tag{7.32}$$

These results should be compared with the Bruhat factors given in (4.16) and (4.17). This leads to the conclusion $dg_x = k(t(x,g))$.

A fundamental observation that we mentioned already a few times is that the field representation (7.31) belongs to the non-unitary principal series. Indeed, let us promote φ to a vector-valued function $f : G \to W$ by

$$f(m(x)) = \varphi(x), \quad f(gp) = \rho(p)^{-1} f(g), \quad g \in G, \ p \in P \ . \tag{7.33}$$

Clearly, the function f is uniquely determined almost everywhere on G by these properties. The space of functions with covariance properties as above is nothing else but a non-unitary principal series representation of the superconformal group under (the restriction of) the left regular-representation (denoted L). Moreover, we have

$$\begin{aligned} &L_{h^{-1}} f_\varphi(g) = f_\varphi(hg) = f_\varphi(hm(x)p) = f_\varphi\left(m(y(x,h))n(z(x,h))k(t(x,h))p\right) \\ &= \rho(p^{-1})\rho(k(t(x,h))^{-1})\varphi(y(x,h)) = \rho(p^{-1})\pi_{h^{-1}}\varphi(x) = \rho(p^{-1}) f_{\pi_{h^{-1}}\varphi}(m(x)) = f_{\pi_{h^{-1}}\varphi}(g) \ . \end{aligned}$$

In this calculation we used that $y(x,h) = hx$, which is true by definition. The result shows that the lift (7.33) is an intertwiner between a field representation and a principal series representation. Since bijectivity of the lift is clear, the two representations are isomorphic. The argument is valid for bosonic and super conformal groups alike.

Given n fields, the Ward identities (2.26) say precisely that the correlation function $G_n(x_i)$ is an invariant vector in the tensor product of representations π_i, $G_n(x_i) \in (\pi_1 \otimes ... \otimes \pi_n)^G$. We can lift each field to a function on the superconformal group as above, so to obtain from the correlator a function $F_n : G^n \to W$ (W is the tensor product of spaces of polarisations of the fields) which satisfies

$$F_n(m(x_i)) = G_n(x_i), \quad F_n(g_1 p_1, ..., g_n p_n) = \left(\rho_1(p_1^{-1}) \otimes ... \otimes \rho_n(p_n^{-1})\right) F_n(g_1, ..., g_n) \ . \tag{7.34}$$

These two properties ensure that F_n is defined almost everywhere on G^n. Now, the invariance of G_n and the intertwining property of lifts imply that F_n is invariant under the diagonal left regular action of G

$$F_n(hg_i) = F_n(g_i) \ . \tag{7.35}$$

The function F_n is the representation of the correlator that will be the starting point for several other representations to be constructed below.

Remark In the mathematics literature, the two different realisations of principal series representations as sometimes referred to as the *non-compact picture* (the field representation) and the *induced picture* (the representation on covariant functions on G), [120].

7.4 Lifting formula: a new representation of four-point functions

Now we come to one of the main results of this chapter. Starting from the representation of a conformal four-point function by a covariant function F_4 introduced above, we will derive a new representation of it as a K-spherical function. This will allow us later to embed the theory of superconformal partial waves into a well-established branch of representation theory and make contact with integrable models. Our result reads

Theorem There is a bijective correspondence between solutions of Ward identities (2.26) for four-point functions and K-spherical functions on the superconformal group G. The latter are elements of the space $A(G) \otimes W$ which satisfy, for all $k_l, k_r \in K$

$$F(k_l g k_r) = \Big(\rho_1(k_l) \otimes \rho_2(wk_lw^{-1}) \otimes \rho_3(k_r^{-1}) \otimes \rho_4(wk_r^{-1}w^{-1})\Big) F(g) \ . \tag{7.36}$$

Proof: We would first like to show how a solution of Ward identities can be used to produce a K-spherical function. Given any solution $G_4(x_i)$, let $F_4 \in A(G)^{\otimes 4} \otimes W$ be the covariant function as constructed in the last section. Using F_4 and the Weyl inversion w we can construct a new object $F \in \ A(G) \otimes W$ by

$$F(g) := F_4(e, w^{-1}, g, gw^{-1}) \ . \tag{7.37}$$

While the motivation for such a map might not be clear, it is readily verified that F is a K-spherical function. Indeed, from the definition of F, and covariance properties of F_4 we obtain

$$\begin{aligned} F(k_l g k_r) &= F_4(e, w^{-1}, k_l g k_r, k_l g k_r w^{-1}) = F_4(k_l^{-1}, w^{-1}wk_l^{-1}w^{-1}, gk_r, gw^{-1}wk_rw^{-1}) \\ &= \Big(\rho_1(k_l) \otimes \rho_2(wk_lw^{-1}) \otimes \rho_3(k_r^{-1}) \otimes \rho_4(wk_r^{-1}w^{-1})\Big) F(g) \ . \end{aligned}$$

We shall now go in the other direction and show how to recover G_4 from F. Suppressing the last two arguments and their corresponding prefactors for simplicity, we have

$$\begin{aligned} F_4(m(x_1), m(x_2)) &= \left(1 \otimes \rho_2(k(t_{21})^{-1})\right) F_4\left(m(x_1)n(y_{21}), m(x_2)k(t_{21})^{-1}n(z_{21})^{-1}\right) \\ &= \left(1 \otimes \rho_2(k(t_{21})^{-1})\right) F_4\left(m(x_1)n(y_{21}), m(x_1)m(x_{21})k(t_{21})^{-1}n(z_{21})^{-1}\right) \\ &= \left(1 \otimes \rho_2(k(t_{21})^{-1})\right) F_4\left(m(x_1)n(y_{21}), m(x_1)w^{-1}m(y_{21})\right) \\ &= \left(1 \otimes \rho_2(k(t_{21})^{-1})\right) F_4\left(m(x_1)n(y_{21}), m(x_1)n(y_{21})w^{-1}\right) . \end{aligned}$$

In the first step we used the right covariance covariance property of F_4 and the fact that the compensating prefactors are trivial on elements of the form $n(x)$. Next, we inserted $m(x_{21})$ using its definition (7.3) and applied the formula

$$m(x_{21}) = w^{-1} m(y_{21}) n(z_{21}) k(t_{21}),$$

which is essentially the definition of y_{21}, z_{21} and t_{21}. Finally we commuted the element w^{-1} past $m(y_{21})$ by an application of (7.5). The same steps can be repeated for the second two arguments of F_4 to arrive at

$$F_4(m(x_i)) = \left(1 \otimes \rho_2(k(t_{21})^{-1}) \otimes 1 \otimes \rho_4(k(t_{43})^{-1})\right) F_4\left(g_{12}(x_i), g_{12}(x_i)w^{-1}, g_{34}(x_i), g_{34}(x_i)w^{-1}\right),$$

where we introduced the elements

$$g_{ij} = m(x_i) n(y_{ji}) \ . \tag{7.38}$$

To complete the derivation, we use the left invariance property of F_4, with $h = g_{12}^{-1}$

$$F_4(m(x_i)) = \left(1 \otimes \rho_2(k(t_{21})^{-1}) \otimes 1 \otimes \rho_4(k(t_{43})^{-1})\right) F_4\left(e, w^{-1}, g(x_i), g(x_i)w^{-1}\right),$$

where the element $g(x_i)$ is defined as

$$g(x_i) = g_{12}^{-1} g_{34} = n(y_{21})^{-1} m(x_{31}) n(y_{43}) \ . \tag{7.39}$$

Putting everything together, the correlation function G_4 is recovered from the corresponding K-spherical function F as

$$G_4(x_i) = \Big(1 \otimes \rho_2(k(t_{21}))^{-1} \otimes 1 \otimes \rho_4(k(t_{43}))^{-1}\Big) F(g(x_i)) \ . \tag{7.40}$$

This establishes the theorem. The last relation will be referred to as the *lifting formula* and it is the main result of this section. $\square$

Example Let us write the lifting formula explicitly for $\mathfrak{g} = \mathfrak{sl}(2|1)$. By matrix multiplication, the variables $x_{ij} = (u_{ij}, \theta_{ij}, \bar\theta_{ij})$ defined in (7.3) are

$$u_{ij} = u_i - u_j - \frac{1}{2}\theta_i\bar\theta_j - \frac{1}{2}\bar\theta_i\theta_j \ , \quad \theta_{ij} = \theta_i - \theta_j \ , \quad \bar\theta_{ij} = \bar\theta_i - \bar\theta_j \ . \tag{7.41}$$

Let $G_4(x_i)$ be a four-point function of primary fields with conformal weights Δ_i and R-charges r_i for $i = 1, \dots, 4$. Given Δ and r, the corresponding representation ρ of the group $K = SO(1,1) \times U(1)$ reads

$$\rho_{\Delta,r}(e^{\lambda D + \kappa R}) = e^{-\Delta\lambda + r\kappa} \ . \tag{7.42}$$

Since the group K is abelian, the spaces of polarisations W_i are one-dimensional and so is their tensor product $W = W_1 \otimes \cdots \otimes W_4$. According to (7.40), there exists a unique function F with the covariance properties

$$F(e^{\lambda_l D + \kappa_l R} g e^{\lambda_r D + \kappa_r R}) = e^{(\Delta_2 - \Delta_1)\lambda_l + (r_1 + r_2)\kappa_l} e^{(\Delta_3 - \Delta_4)\lambda_r - (r_3 + r_4)\kappa_r} F(g) \ , \tag{7.43}$$

such that

$$G_4(x_i) = \frac{e^{r_2 \frac{\theta_{12}\bar\theta_{12}}{2u_{12}} + r_4 \frac{\theta_{34}\bar\theta_{34}}{2u_{34}}}}{u_{12}^{2\Delta_2} u_{34}^{2\Delta_4}} F(e^{-w(x_{21})\cdot X^w} e^{x_{31}\cdot X} e^{w(x_{43})\cdot X^w}) \ . \tag{7.44}$$

7.5 Transformation properties of Bruhat factors

Before we go on to applications of the lifting formula, we will pause to investigate how the various factors introduced in previous sections transform under $x \mapsto hx$. Keeping track of these transformation properties will become especially important in the analysis of crossing symmetry equations.

Proposition Under a superconformal transformation h, elements g_{ij} and $k(t_{ji})$ transform as

$$g_{ij}(x^h) = hg_{ij}(x)k(t(x_i,h))^{-1}, \quad k(t^h_{ji}) = wk(t(x_i,h))w^{-1}k(t_{ji})k(t(x_j,h))^{-1} \ . \tag{7.45}$$

Let us make a comment on the notation. Various objects in this chapter depend on the insertion points x_i. However, to avoid having long expressions we have not kept explicitly this dependence in the notation, e.g. we have written $y_{ij} = y(x_{ij})$ etc. We will adopt the rule that if the insertion points are transformed by a group element h, the corresponding objects will carry an upper index h, e.g. y^h_{ij}, t^h_{ij} etc. In particular, we alternatively write x^h or hx (or very rarely $y(x,h)$).

Proof: Consider the system of equations

$$m(x_i)n(y_{ji}) = g_{ij}(x), \quad m(x_j)k(t_{ji})^{-1}n(z_{ji})^{-1} = g_{ij}(x)w^{-1} \ . \tag{7.46}$$

The first equation is the definition of $g_{ij}(x)$ and the second one was proved in the previous section. Let us apply a transformation h to all x_i-s and use

$$m(x^h) = hm(x)k(t(x,h))^{-1}n(z(x,h))^{-1} \ . \tag{7.47}$$

This relation follows at once from definitions of $k(t(x,h))$ and $n(z(x,h))$. Doing these two steps, we get another system of equations

$$hm(x_i)k(t(x_i,h))^{-1}n(z(x_i,h))^{-1}n(y^h_{ji}) = g_{ij}(x^h), \tag{7.48}$$

$$hm(x_j)k(t(x_j,h))^{-1}n(z(x_j,h))^{-1}k(t^h_{ji})^{-1}n(z^h_{ji})^{-1} = g_{ij}(x^h)w^{-1}. \tag{7.49}$$

We can compare this system to (7.46). Elements $g_{ij}(x)$ and $h^{-1}g_{ij}(x^h)$ have the same m Bruhat factor and similarly $g_{ij}(x)w^{-1}$ and $h^{-1}g_{ij}(x^h)w^{-1}$. It follows that they are related by

$$h^{-1}g_{ij}(x^h) = g_{ij}(x)k_{ij}n_{ij}, \quad h^{-1}g_{ij}(x^h)w^{-1} = g_{ij}(x)w^{-1}k'_{ij}n'_{ij}, \tag{7.50}$$

for some $k_{ij}, k'_{ij}, n_{ij}, n'_{ij}$. Putting these two equations together, we have

$$k_{ij}n_{ij} = (w^{-1}k'_{ij}w)(w^{-1}n'_{ij}w) \ . \tag{7.51}$$

We now make the key observation - the grading with respect to the dilation weight requires $n_{ij} = n'_{ij} = 1$. Also, by looking at elements of conformal weight zero in the first equation of (7.46) and (7.48), we see that $k_{ij} = k(t(x_i,h))^{-1}$. Having established these facts, the proposition follows from (7.50). To get the first claim, one simply substitutes the expressions for k_{ij} and n_{ij} into the first equation. The second claim requires a few more steps. Let us begin by substituting $n'_{ij} = 1$ and $k'_{ij} = wk_{ij}w^{-1}$ into the second equation in (7.50). After cancelling w^{-1} factors on the right

$$h^{-1}g_{ij}(x^h) = g_{ij}(x)k(t(x_i,h))^{-1} \ . \tag{7.52}$$

Next, we use (7.49) and the second equation of (7.46) the to expand $g_{ij}(x^h)$ and $g_{ij}(x)$ on the two sides and cancel the $m(x_j)$ factors

$$k(t(x_j,h))^{-1}n(z(x_j,h))^{-1}k(t^h_{ji})^{-1}n(z^h_{ji})^{-1}w = k(t_{ji})^{-1}n(z_{ji})^{-1}wk(t(x_i,h))^{-1} \ . \tag{7.53}$$

The grading on $\mathfrak{g}$ allows to equate the k-factors from the two sides

$$k(t(x_j,h))^{-1}k(t^h_{ji})^{-1} = k(t_{ji})^{-1}wk(t(x_i,h))^{-1}w^{-1} \ . \tag{7.54}$$

Rearranging terms now gives the second claim and completes the proof of the proposition. □

For bosonic conformal groups, one could have proven the proposition by a cumbersome but explicit calculation. In the supersymmetric setting, such an approach is not at all efficient because there is no uniform description of all superconformal groups and different groups would require separate computations. As the above argument shows however, for results to be true, one only needs to use the grading on $\mathfrak{g}$ with respect to the dilation weight and the properties of the Weyl inversion. Indeed, these are among very few structures shared by all superconformal groups.

7.6 Bruhat decomposition for $\mathfrak{sl}(2m|\mathcal{N})$

We end this chapter by determining explicitly the Bruhat decomposition of $wm(x)$ for two infinite families of superconformal groups. This is an important step required in order to apply the abstract results of above sections to field theory.

Let $\mathfrak{g}$ be a simple complex superconformal algebra of type I. These include a few isolated Lie superalgebras, namely $\mathfrak{osp}(2|4)$, $\mathfrak{psl}(2|2)$ and $\mathfrak{psl}(4|4)$, as well as two infinite families $\mathfrak{sl}(2|\mathcal{N})$ and $\mathfrak{sl}(4|\mathcal{N})$. We shall focus on these infinite families and set $\mathfrak{g} = \mathfrak{sl}(2m|\mathcal{N})$. For this choice of $\mathfrak{g}$ one finds

$$\mathfrak{g}_{(0)} = \mathfrak{sl}(2m) \oplus \mathfrak{sl}(\mathcal{N}) \oplus \mathfrak{u}(1), \quad \mathfrak{g}_+ = (2m, \overline{\mathcal{N}}, 1), \quad \mathfrak{g}_- = (\overline{2m}, \mathcal{N}, -1) \ . \tag{7.55}$$

Generators $\{D, P_{\dot\alpha}{}^{\beta}, K_\alpha{}^{\dot\beta}, M_\alpha{}^\beta, M_{\dot\alpha}{}^{\dot\beta}\}$, $\{R, R_I{}^J\}$ and $\{Q_{\dot\alpha}{}^J, Q_I{}^\beta, S_\alpha{}^J, S_I{}^{\dot\beta}\}$ have already been introduced. The four irreducible $\mathfrak{k}$-modules that make up the odd subspace and whose existence is guaranteed by the type I condition are

$$\mathfrak{q}_+ = \text{span}\{Q_{\dot\alpha}{}^J\}, \quad \mathfrak{q}_- = \text{span}\{Q_I{}^\beta\}, \quad \mathfrak{s}_+ = \text{span}\{S_\alpha{}^J\}, \quad \mathfrak{s}_- = \text{span}\{S_I{}^{\dot\beta}\} \ . \tag{7.56}$$

The representations of $\mathfrak{k}$ which these carry are indicated by the type of indices of their generators. The dual basis of $\mathfrak{g}^*_{(1)}$ to the one above will be denoted by $\{q^{\dot\alpha}{}_J, q^I{}_\beta, s^\alpha{}_J, s^I{}_{\dot\beta}\}$. Spaces $\mathfrak{g}_\pm$ carry representations of $\mathfrak{g}_{(0)}$ which are dual to each other. Explicitly, the dual bases are

$$(S_\alpha{}^I)^* = Q_I{}^\alpha, \quad (Q_{\dot\alpha}{}^I)^* = S_I{}^{\dot\alpha} \ . \tag{7.57}$$

The Lie superalgebra $\mathfrak{g}$ has a fundamental $(2m+\mathcal{N})$-dimensional representation. We will denote by $E_i{}^j$ the matrix with 1 at position (i,j) and zeros elsewhere, $i,j = 1,...,2m+\mathcal{N}$. Such indices are split in three pieces $\dot\alpha, \alpha, I$, that is, we write

$$A = A^i{}_j E_i{}^j = \begin{pmatrix} A^{\dot\alpha}{}_{\dot\beta} & A^{\dot\alpha}{}_\beta & A^{\dot\alpha}{}_J \\ A^\alpha{}_{\dot\beta} & A^\alpha{}_\beta & A^\alpha{}_J \\ (-1)^{|A^I{}_{\dot\beta}|}A^I{}_{\dot\beta} & (-1)^{|A^I{}_\beta|}A^I{}_\beta & (-1)^{|A^I{}_J|}A^I{}_J \end{pmatrix} \ . \tag{7.58}$$

We can choose the generators so that the $\mathfrak{sl}(2m)$ and $\mathfrak{sl}(\mathcal{N})$ algebras sit in the top left and bottom right corners, respectively, while the subspaces $\mathfrak{g}_\pm$ occupy the top right and bottom left corners. Schematically

$$\begin{pmatrix} \mathfrak{k}\cap\mathfrak{sl}(2m) & \mathfrak{g}_1 & \mathfrak{q}_+ \\ \mathfrak{g}_{-1} & \mathfrak{k}\cap\mathfrak{sl}(2m) & \mathfrak{s}_+ \\ \mathfrak{s}_- & \mathfrak{q}_- & \mathfrak{sl}(\mathcal{N}) \end{pmatrix} . \tag{7.59}$$

For the precise definition of the fundamental representation, see subsection (7.6.1).[1]

In the remainder of this section, we will derive expression for $y(x)$, $z(x)$ and $t(x)$ appearing in the decomposition (7.8). In order to do this, we spell out the supermatrices representing various factors in this equation. The Weyl inversion and its inverse take the form

$$w = \begin{pmatrix} 0 & -w^{\dot\alpha}{}_{\beta} & 0 \\ -w^{\alpha}{}_{\dot\beta} & 0 & 0 \\ 0 & 0 & \delta^I{}_J \end{pmatrix} , \quad w^{-1} = \begin{pmatrix} 0 & w^{\dot\alpha}{}_{\beta} & 0 \\ w^{\alpha}{}_{\dot\beta} & 0 & 0 \\ 0 & 0 & \delta^I{}_J \end{pmatrix} , \tag{7.60}$$

where $-w^{\dot\alpha}{}_{\beta} = w^{\alpha}{}_{\dot\beta} = \sigma_2$ for $m=2$ and $w^{\dot\alpha}{}_{\beta} = -w^{\alpha}{}_{\dot\beta} = 1$ for $m=1$.

The superspace structure algebra $\mathcal{M}$ is generated by variables $x^{\dot\alpha}{}_{\beta}, \theta^{\dot\alpha}{}_J, \bar\theta^I{}_\beta$, obeying the usual (anti)commutation relations. We see that

$$m(x) = e^{x^{\dot\alpha}{}_\beta P_{\dot\alpha}{}^\beta + \theta^{\dot\alpha}{}_J Q_{\dot\alpha}{}^J + \bar\theta^I{}_\beta Q_I{}^\beta} = \begin{pmatrix} \delta^{\dot\alpha}{}_{\dot\beta} & X^{\dot\alpha}{}_\beta & \theta^{\dot\alpha}{}_J \\ 0 & \delta^\alpha{}_\beta & 0 \\ 0 & -\bar\theta^I{}_\beta & \delta^I{}_J \end{pmatrix} , \quad \text{with} \quad X^{\dot\alpha}{}_\beta = x^{\dot\alpha}{}_\beta - \frac{1}{2}\theta^{\dot\alpha}{}_K \bar\theta^K{}_\beta . \tag{7.61}$$

Using $w^{\dot\alpha}{}_\delta w^\delta{}_{\dot\beta} = -\delta^{\dot\alpha}{}_{\dot\beta}$ and $w^\alpha{}_{\dot\delta} w^{\dot\delta}{}_\beta = -\delta^\alpha{}_\beta$ we get for elements $n(x)$

$$n(x) = w^{-1} m(x) w = \begin{pmatrix} \delta^{\dot\alpha}{}_{\dot\beta} & 0 & 0 \\ -w^\alpha{}_{\dot\gamma} X^{\dot\gamma}{}_\delta w^\delta{}_{\dot\beta} & \delta^\alpha{}_\beta & w^\alpha{}_{\dot\gamma}\theta^{\dot\gamma}{}_J \\ \bar\theta^I{}_\delta w^\delta{}_{\dot\beta} & 0 & \delta^I{}_J \end{pmatrix} . \tag{7.62}$$

Finally, elements of the subgroup K assume the form

$$k(t) = \begin{pmatrix} e^{\frac{\mathcal{N}\kappa}{\mathcal{N}-2m}+\frac{1}{2}\lambda}(r_1)^{\dot\alpha}{}_{\dot\beta} & 0 & 0 \\ 0 & e^{\frac{\mathcal{N}\kappa}{\mathcal{N}-2m}-\frac{1}{2}\lambda}(r_2)^\alpha{}_\beta & 0 \\ 0 & 0 & e^{\frac{2m\kappa}{\mathcal{N}-2m}} U^I{}_J \end{pmatrix} \equiv \mathrm{diag}(k_1,k_2,k_3) . \tag{7.63}$$

Matrices $r_{1,2}$ are purely rotational. That is, they belong to $SL(2,\mathbb{C})$ for $m=2$ and are equal to 1 if $m=1$.

In the following we will suppress indices where no confusion can arise. They can be put back at any point by looking at what type of indices a certain object carries and contracting over the appropriate number and type of dummy indices. We shall agree to write $J = w^\alpha{}_{\dot\beta}$, then

[1] The early works [121, 122, 123] use the same representation both of the Lie superalgebra and the supergroup and have inspired some of our calculations.

$-J^{-1} = w^{\dot\alpha}{}_\beta$. With these conventions, the above expressions can be rewritten as

$$w = \begin{pmatrix} 0 & J^{-1} & 0 \\ -J & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad w^{-1} = \begin{pmatrix} 0 & -J^{-1} & 0 \\ J & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{7.64}$$

$$m(x) = \begin{pmatrix} 1 & X & \theta \\ 0 & 1 & 0 \\ 0 & -\bar\theta & 1 \end{pmatrix}, \quad n(x) = \begin{pmatrix} 1 & 0 & 0 \\ -JXJ & 1 & J\theta \\ \bar\theta J & 0 & 1 \end{pmatrix} . \tag{7.65}$$

Therefore, the equation $wm(x) = m(y)n(z)k(t)$ reads

$$\begin{pmatrix} 0 & J^{-1} & 0 \\ -J & -JX & -J\theta \\ 0 & -\bar\theta & 1 \end{pmatrix} = \begin{pmatrix} (1 - YJZJ + \eta\bar\zeta J)k_1 & Yk_2 & (YJ\zeta + \eta)k_3 \\ -JZJk_1 & k_2 & J\zeta k_3 \\ (\bar\eta JZJ + \bar\zeta J)k_1 & -\bar\eta k_2 & (1 - \bar\eta J\zeta)k_3 \end{pmatrix} . \tag{7.66}$$

Here, the notation is $y = (y^{\dot\alpha}{}_\beta, \eta^{\dot\alpha}{}_J, \bar\eta^I{}_\beta)$, $z = (z^{\dot\alpha}{}_\beta, \zeta^{\dot\alpha}{}_J, \bar\zeta^I{}_\beta)$ and Y, Z are introduced analogously to X. To write down the solution for y, z and t we introduce

$$T = 1 + \bar\theta X^{-1}\theta, \quad \Lambda = 1 + X^{-1}\theta\bar\theta . \tag{7.67}$$

Then one observes that $\Lambda^{-1} = 1 - X^{-1}\theta T^{-1}\bar\theta$. Using this, the solution to the above system is found

$$(Y, \eta, \bar\eta) = (-(JXJ)^{-1}, -(XJ)^{-1}\theta T^{-1}, -\bar\theta(JX)^{-1}), \tag{7.68}$$
$$(Z, \zeta, \bar\zeta) = (-X - \theta\bar\theta, -\theta T^{-1}, -\bar\theta\Lambda), \tag{7.69}$$
$$(k_1, k_2, k_3) = (-((X + \theta\bar\theta)J)^{-1}, -JX, T) . \tag{7.70}$$

In particular $z^{\dot\alpha}{}_\beta = -x^{\dot\alpha}{}_\beta$, as in the bosonic theory. This completes our analysis of the equation (7.8) for superconformal algebras $\mathfrak{sl}(2m|\mathcal{N})$ and with it the discussion of the present chapter.

7.6.1 Fundamental representation of $\mathfrak{sl}(2m|\mathcal{N})$

$$D = \frac{1}{2}\mathrm{diag}(I_m, -I_m, 0), \quad R = \frac{1}{\mathcal{N} - 2m}\mathrm{diag}(\mathcal{N} I_m, \mathcal{N} I_m, 2m I_\mathcal{N}) . \tag{7.71}$$

$$M_{\dot1}{}^{\dot2} = E_{\dot1}{}^{\dot2}, \quad M_{\dot2}{}^{\dot1} = E_{\dot2}{}^{\dot1}, \quad M_{\dot1}{}^{\dot1} = -M_{\dot2}{}^{\dot2} = \frac{1}{2}(E_{\dot1}{}^{\dot1} - E_{\dot2}{}^{\dot2}), \tag{7.72}$$

$$M_1{}^2 = E_1{}^2, \quad M_2{}^1 = E_2{}^1, \quad M_1{}^1 = -M_2{}^2 = \frac{1}{2}(E_1{}^1 - E_2{}^2) . \tag{7.73}$$

$$P_{\dot\alpha}{}^\beta = E_{\dot\alpha}{}^\beta, \quad Q_{\dot\alpha}{}^J = E_{\dot\alpha}{}^J, \quad Q_I{}^\beta = E_I{}^\beta, \quad K_\alpha{}^{\dot\beta} = E_\alpha{}^{\dot\beta}, \quad S_\alpha{}^J = E_\alpha{}^J, \quad S_I{}^{\dot\beta} = E_I{}^{\dot\beta} . \tag{7.74}$$

$$R_I{}^J = E_I{}^J, \quad R_I{}^I = \frac{1}{2}(E_I{}^I - E_{I+1}^{I+1}) . \tag{7.75}$$

Chapter 8

Crossing factors from the Cartan decomposition

In the last chapter, we have seen how to interpret four-point functions of an SCFT as K-spherical functions on the superconformal group. The next task, that will keep us occupied in the present and the following chapter, is to derive crossing symmetry equations within this new representation.

The starting point for analysis of K-spherical functions is the Cartan KAK decomposition. Covariance properties of these functions allow one to regard them as functions on A, possibly satisfying additional constraints (which come from non-uniqueness of the KAK factorisation). However, there is no standard definition of the Cartan decomposition for supergroups. In principle, one could write several sensible definitions that would apply to a certain class of supergroups and then study their properties. Any such definition should give the usual Cartan decomposition when applied to the bosonic conformal group.

To elaborate on possible choices, let us assume that G is some superconformal group and that we have already decomposed its underlying Lie group as

$$G_{(0)} = KAK \ . \tag{8.1}$$

Notice that the internal symmetry group U is contained in K. Therefore, A from above is the usual two-dimensional abelian factor of the bosonic conformal group. To obtain a coordinate system on G, we still need to include exponentials of fermionic generators. There is an even number of these generators and we can distribute them democratically on both sides of A by including an exponential $e^{\theta Q}$ on the left and $e^{\bar{\theta} S}$ on the right. Here, we are being schematic and use θ, $\bar{\theta}$ to denote collections of fermionic variables.

While legitimate, it seems that this choice of coordinates does not lead to a particularly nice expression for the Laplacian. We can improve the situation considerably by requiring the supergroup to be of type I. Then there is a natural split of fermionic generators into two sets that span two irreducible modules of $\mathfrak{g}_{(0)}$. When exponentials built out of these sets of generators are placed on either side of A as above, we arrive at what will be called the (supersymmetric) Cartan decomposition.

Having defined the coordinate system on G, we will turn to questions of conformal field theory. First, the coordinates on which K-spherical functions essentially depend are naturally related to cross ratios. For the bosonic group, these are the coordinates (u_1, u_2) on $A_{(0)}$, that we will explicitly relate to (z_1, z_2) using the lifting formula (7.40). It is well-known that superconformal four-point functions depend on additional nilpotent invariants. These are related to the fermionic variables in the decomposition. In fact, we will see how to give a simple count of nilpotent invariants using the Cartan coordinates.

Next, we will turn to the computation of the crossing factor $\mathcal{M}^I{}_J$. Recall that this matrix, introduced in (2.39), is roughly speaking the ratio of factors $\Omega(x_i)$ that capture the kinematical dependence of $G_4(x_i)$, in two different channels. The $\Omega(x_i)$-s are not uniquely fixed, because multiplying them by any function of cross ratios gives a correlator of the correct kinematical form. Within our approach, there is a relatively natural way of fixing these factors.

Notice that $\Omega(x_i)$ is a relatively complicated object. In a bosonic CFT, it is a (matrix-valued) function $4d$ of coordinates x_i^μ and depends on four representations of $SO(1,1) \times SO(d)$. In contrast, $\mathcal{M}^I{}_J$ is a function of just two cross ratios. Furthermore, its matrix elements can be computed using representation theory of $SO(1,1) \times SO(2)$ only. A quick explanation of this can be given as follows. Since $\mathcal{M}^I{}_J$ is conformally invariant, we can without loss of generality assume that all points x_i lie in the e_1-e_2 plane of $\mathbb{R}^d$. Then all the group elements of G that appear in (7.40) belong to the $SO(3,1)$ subgroup G_Π of conformal transformations of this plane (the Weyl inversion w drops out because it always appears together with its inverse). Crucially, the group A is contained in G_Π, so the Cartan decomposition can be consistently restricted to this group. Therefore, the computation of $\mathcal{M}^I{}_J$ uses only a conformal group in two dimensions, for which $K = SO(1,1) \times SO(2)$.

The chapter contains a number of explicit examples and calculations for both bosonic and supersymmetric conformal theories. It is mostly based on [2, 3].

8.1 Cartan decomposition of the superconformal group

Let G be a superconformal group which is also a supergroup of type I. There is a basis $\{X^A\} = \{X^a, X^\mu, X_\mu\}$ of the Lie superalgebra $\mathfrak{g} = Lie(G)$ such that

$$\mathfrak{g}_{(0)} = \text{span}\{X^a\}, \quad \mathfrak{g}_+ = \text{span}\{X^\mu\}, \quad \mathfrak{g}_- = \text{span}\{X_\mu\} \ . \tag{8.2}$$

Since $\mathfrak{g}_\pm$ are modules of $\mathfrak{g}_{(0)}$ dual to each other, we can choose $\{X^\mu\}$ and $\{X_\mu\}$ as their dual bases. We will denote the representation of $\mathfrak{g}_{(0)}$ on the space $\mathfrak{g}_+$ by π. Then

$$g_{(0)} X^\mu g_{(0)}^{-1} = \pi(g_{(0)})^\mu{}_\nu X^\nu, \quad g_{(0)} X_\nu g_{(0)}^{-1} = \pi(g_{(0)}^{-1})^\mu{}_\nu X_\mu, \quad g_{(0)} \in G_{(0)} \ . \tag{8.3}$$

Supergroup elements of G may be factorised as, [124],

$$g = e^{x^\mu X_\mu} g_{(0)} e^{x_\nu X^\nu} \ . \tag{8.4}$$

Here, the middle factor $g_{(0)}$ is an element of the underlying Lie group $G_{(0)} = G_{bos} \times U$. To parametrise these elements, we use the Cartan coordinates on G_{bos} and some arbitrary set of coordinates on U. As usual, the full coordinate system on G is denoted by (x_A) and we have $|x_A| = |X^A|$. Since the internal transformations are contained in K, the subgroup A is still two-dimensional and we parametrise it by (u_1, u_2) as in (4.22). We shall also write

$$g = \eta_l \, k_l \, a \, k_r \, \eta_r, \quad k_l, k_r \in K \ . \tag{8.5}$$

In order to have a simple formulation of K-covariance laws in terms of coordinates, it is convenient to move the factors $k_{l,r}$ past $\eta_{l,r}$ to the furthest left and right positions. This can be done using relations (8.3) and the Baker-Campbell-Hausdorff formula to find

$$g = k_l \, e^{\pi(k_l)^\mu{}_\rho x^\rho X_\mu} \, a \, e^{\pi(k_r)^\sigma{}_\nu x_\sigma X^\nu} \, k_r \equiv k_l \, \eta_l' \, a \, \eta_r' \, k_r \ . \tag{8.6}$$

We will refer to (8.5) and (8.6) as unprimed and primed Cartan coordinates, respectively.

Let us note that the Cartan factorisations of supergroup elements g are not unique. Focusing for concreteness on the primed version, we see that, given any one factorisation, we can produce another by the transformation

$$(k_l, \ \eta_l', \ k_r, \ \eta_r') \mapsto \left(k_l b, \ b^{-1}\eta_l' b, \ b^{-1}k_r, \ b^{-1}\eta_r' b\right) , \tag{8.7}$$

where b belongs to the stabiliser of A in K. The stabiliser group will be denoted by B and one observes that $B \sim SO(d-2) \times U$. Let P be the element of the group algebra $L^1(B)$ given by

$$P = \frac{1}{\text{Vol } B} \int_B db \ b \ . \tag{8.8}$$

In any representation of B, the element P acts as a projection operator to invariant vectors. Indeed, it satisfies

$$bP = P, \quad P^2 = P \ . \tag{8.9}$$

The first property follows from left invariance of the Haar measure db. Once the first equation is integrated over B, it gives the second one.

Let us consider a K-spherical function F whose transformation properties under K are determined by representations ρ_l and ρ_r. For any $b \in B$, we have

$$\left(\rho_l(k_l) \otimes \rho_r(k_r^{-1})\right) F(\eta_l' a \eta_r') = \left(\rho_l(k_l b) \otimes \rho_r(k_r^{-1} b)\right) F(b^{-1}\eta_l' b \ a \ b^{-1}\eta_r' b) \ . \tag{8.10}$$

If we denote the fermionic primed Cartan coordinates by $\{y^\mu, y_\mu\}$, it follows that

$$F(e^{y^\mu X_\mu} a e^{y_\nu X^\nu}) = (\rho_l(b) \otimes \rho_r(b)) \, F(e^{\pi(b)^\mu{}_\rho y^\rho X_\mu} a e^{\pi(b^{-1})^\sigma{}_\nu y_\sigma X^\nu}) \ . \tag{8.11}$$

Let us define the action of the projector P on functions $f(u_i, y^\mu, y_\nu)$ that take values in the representation space W of B by

$$\mathcal{P}[f(u_i, y^\mu, y_\nu)] = \frac{1}{\text{Vol } B} \int_B db \ \ (\rho_l(b) \otimes \rho_r(b)) \, f(u_i, \pi(b)^\mu{}_\rho y^\rho, \pi(b^{-1})^\sigma{}_\nu y_\sigma) \ . \tag{8.12}$$

Then, if f arises as a restriction of a K-spherical function F, clearly $\mathcal{P}[f] = f$. Conversely, any f that is preserved by $\mathcal{P}$ can be extended to a K-spherical function on G.

Let us note that in practical computations it is convenient to make some specific choices for the Cartan factors that remove the gauge freedom (8.7). Such gauge fixing conditions are arbitrary and at the end of every calculation one has to check that the result does not depend on them.

8.1.1 Cross ratios in bosonic theories

We have seen that K-spherical functions on the conformal group can be regarded as functions of two variables (u_1, u_2) that parametrise the abelian group A. This is a reformulation of the familiar fact that four-point functions in a CFT depend on two cross ratios (u, v). Our first task in establishing the theory of conformal correlators in the gauge (7.40) is two relate these two sets of variables. In order to do this, we decompose $g(x_i) = k_l a(u_1, u_2) k_r$ and determine the factor a. Let us work in the $(d+2)$-dimensional representation of G and focus on the upper-left 2×2 blocks of the two sides. For an element $g = r_l e^{\lambda_l D} a(u_1, u_2) e^{\lambda_r D} r_r$, with $r_{l,r}$ rotation matrices, this block has entries

$$\begin{aligned}
g_{11} &= \cosh u^+ \cosh\lambda_l \cosh\lambda_r + \cosh u^- \sinh\lambda_l \sinh\lambda_r ,\\
g_{12} &= -\cosh u^+ \cosh\lambda_l \sinh\lambda_r - \cosh u^- \sinh\lambda_l \cosh\lambda_r ,\\
g_{12} &= -\cosh u^+ \sinh\lambda_l \cosh\lambda_r - \cosh u^- \cosh\lambda_l \sinh\lambda_r ,\\
g_{22} &= \cosh u^+ \sinh\lambda_l \sinh\lambda_r + \cosh u^- \cosh\lambda_l \cosh\lambda_r \ .
\end{aligned}$$

Here $u^\pm = (u_1 \pm u_2)/2$. On the other hand, the upper-left corner of $g(x_i)$ reads

$$\frac{1}{2}\begin{pmatrix} x_{13}^2 + \frac{x_{23}^2}{x_{12}^2} + \frac{x_{14}^2}{x_{34}^2} + \frac{x_{24}^2}{x_{12}^2 x_{34}^2} & x_{13}^2 + \frac{x_{23}^2}{x_{12}^2} - \frac{x_{14}^2}{x_{34}^2} - \frac{x_{24}^2}{x_{12}^2 x_{34}^2} \\ -x_{13}^2 + \frac{x_{23}^2}{x_{12}^2} - \frac{x_{14}^2}{x_{34}^2} + \frac{x_{24}^2}{x_{12}^2 x_{34}^2} & -x_{13}^2 + \frac{x_{23}^2}{x_{12}^2} + \frac{x_{14}^2}{x_{34}^2} - \frac{x_{24}^2}{x_{12}^2 x_{34}^2} \end{pmatrix},$$

By comparing the entries of the two matrices we conclude

$$e^{u_i} = 1 - \frac{2}{z_i}\left(1 + \sqrt{1 - z_i}\right), \quad i = 1, 2 \ . \tag{8.13}$$

As a side result, we also write the dilation coordinates

$$e^{2\lambda_l} = \frac{x_{12}^2 x_{14}^2}{x_{24}^2} \frac{1}{\sqrt{(1-z_1)(1-z_2)}}, \quad e^{2\lambda_r} = \frac{x_{14}^2}{x_{13}^2 x_{34}^2} \frac{1}{\sqrt{(1-z_1)(1-z_2)}} \ . \tag{8.14}$$

Coordinates u_i appeared in the literature before. Their exponentials are equal to the so-called radial coordinates of [34]. The relation (8.13) has already been observed on a case-by-case basis in [8, 9, 10]. Our analysis here provides a proper derivation for all four-point functions, of both scalar and spinning fields.

8.2 Tensor and crossing factors

The Cartan decomposition (8.6) and the lifting formula (7.40) are everything that we need in order to construct the four-point tensor structures in the Calogero-Sutherland gauge. Note that (7.40) treats each of the four insertion points differently and hence breaks the permutation symmetry of Euclidean field theory correlators. Different permutations σ of the four points are associated with different channels. We will refer to the channel that corresponds to the identity permutation $\sigma_s = 1$ as the s-channel. The t-channel has $\sigma_t = (24)$. Given any choice of the channel σ, we can extend the lifting formula to become

$$G_4(x_i) = \rho_{\sigma(2)}(k(t_{\sigma(2)\sigma(1)})^{-1})\rho_{\sigma(4)}(k(t_{\sigma(4)\sigma(3)})^{-1}) F_\sigma(g(x_{\sigma(i)})) \ . \tag{8.15}$$

Here, the matrix $\rho_{\sigma(i)}$ acts on the $\sigma(i)^{th}$ tensor factor in the space of polarisations and it acts trivially on all other tensor factors. The last equation defines the function F_σ, which is K-spherical by the proof of (7.40). Let us now apply the Cartan factorisation to the argument $g(x_{\sigma(i)})$

$$g(x_{\sigma(i)}) = k_{\sigma,l}(x_i)\eta'_{\sigma,l}(x_i)a_\sigma(x_i)\eta'_{\sigma,r}(x_i)k_{\sigma,r}(x_i)\ . \tag{8.16}$$

Covariance properties of F_σ give us

$$\begin{aligned} G_4(x_i) &= \rho_{\sigma(2)}(k(t_{\sigma(2)\sigma(1)})^{-1})\rho_{\sigma(4)}(k(t_{\sigma(4)\sigma(3)})^{-1})F_\sigma(g(x_{\sigma(i)})) \\ &= \rho_{\sigma(2)}\left(k(t_{\sigma(2)\sigma(1)})\right)^{-1}\rho_{\sigma(4)}\left(k(t_{\sigma(4)\sigma(3)})\right)^{-1}F_\sigma(k_{\sigma,l}\eta'_{\sigma,l}a_\sigma\eta'_{\sigma,r}k_{\sigma,r}) \\ &= \rho_{\sigma(1)}(k_{\sigma,l})\rho_{\sigma(2)}\left(k(t_{\sigma(2)\sigma(1)})^{-1}k^w_{\sigma,l}\right)\rho_{\sigma(3)}(k^{-1}_{\sigma,r})\rho_{\sigma(4)}\left(k(t_{\sigma(4)\sigma(3)})^{-1}(k^{-1}_{\sigma,r})^w\right)F_\sigma(\eta'_{\sigma,l}a_\sigma\eta'_{\sigma,r})\ . \end{aligned} \tag{8.17}$$

For simplicity of notation, we dropped the dependence of Cartan factors on the insertion points, i.e. for example $k_{\sigma,l} = k_{\sigma,l}(x_i) = k_l(x_{\sigma(i)})$, and further have written $k^w = wkw^{-1}$. Let us spell out the previous formula for the s- and t-channels. In the s-channel one obtains

$$G_4(x_i) = \rho_1(k_{s,l})\rho_2(k(t_{21})^{-1}k^w_{s,l})\rho_3(k^{-1}_{s,r})\rho_4(k(t_{43})^{-1}(k^w_{s,r})^{-1})\mathcal{P}_sF_s(\eta'_{s,l}a_s\eta'_{s,r}), \tag{8.18}$$

while the t-channel gives

$$G_4(x_i) = \rho_1(k_{t,l})\rho_4(k(t_{41})^{-1}k^w_{t,l})\rho_3(k^{-1}_{t,r})\rho_2(k(t_{23})^{-1}(k^w_{t,r})^{-1})\mathcal{P}_tF_t(\eta'_{t,l}a_t\eta'_{t,r})\ . \tag{8.19}$$

Here we introduced projectors $\mathcal{P}_s$ and $\mathcal{P}_t$ explicitly to stress that $F_{s,t}(\eta'_l a\eta'_r)$ are invariant under their action. The prefactors that multiply F_s and F_t are the s- and t-channel tensor structures. The ratio of these tensor structures is referred to as the supercrossing factor and we denote it by $\mathcal{M}$. More precisely, $\mathcal{M}_{st}$ is defined as

$$\mathcal{M}_{st}(x_i) = \mathcal{P}_t\bigotimes_{i=1}^4\rho_i(\kappa_i)\,\mathcal{P}_s, \tag{8.20}$$

where the four elements κ_i are given by

$$\kappa_1 = k^{-1}_{t,l}k_{s,l},\quad \kappa_2 = k^w_{t,r}k(t_{23})k(t_{21})^{-1}k^w_{s,l}, \tag{8.21}$$

$$\kappa_3 = k_{t,r}k^{-1}_{s,r},\quad \kappa_4 = (k^w_{t,l})^{-1}k(t_{41})k(t_{43})^{-1}(k^w_{s,r})^{-1}\ . \tag{8.22}$$

Obviously, there is an analogous definition of $\mathcal{M}_{\sigma_1\sigma_2}$ for any two channels $\sigma_{1,2}$. It is important to stress the two projectors in (8.20) make the supercrossing factor independent of any gauge fixing conditions for our gauge symmetry (8.7). Indeed, using (8.9) one can easily check that any gauge transformation with some element b is absorbed by the projectors.

The matrix $\mathcal{M}_{st}$ that we wish to compute depends on the insertion points x_i through the Bruhat factors $k(t_{ij}) = k(t(x_{ij}))$, as well as the factors $k_{l,r}$ in the Cartan decomposition (8.16) of the supergroup elements $g_{s,t}(x_i)$. Our strategy will be to first show that $\mathcal{M}_{st}$ is invariant under superconformal transformations, i.e. $\mathcal{M}_{st}(x_i^h) = \mathcal{M}_{st}(x_i)$, and then evaluate it after moving the insertion points into special positions.

To see that $\mathcal{M}_{st}$ is superconformally invariant we shall study the dependence of the four tensor components one after another. In this endeavour, the principal role is played by the proposition (7.45). Notice that as its direct consequence we have

$$g_\sigma(x_i^h) = k(t(x_{\sigma(1)},h))\, g_\sigma(x_i)\, k(t(x_{\sigma(3)},h))^{-1} \ . \tag{8.23}$$

Because of the gauge freedom of the Cartan decomposition which we described in (8.7), knowing the behaviour of $g_\sigma(x_i)$ under superconformal transformations does not allow us to uniquely determine the transformation law of the factors, but we can conclude that

$$k_{\sigma,l}(x_i^h) = k(t(x_{\sigma(1)},h))k_{\sigma,l}(x_i)b_\sigma(x_i,h), \quad k_{\sigma,r}(x_i^h) = b_\sigma^{-1}(x_i,h)k_{\sigma,r}(x_i)k(t(x_{\sigma(3)},h))^{-1}, \tag{8.24}$$

for some element $b \in B$ that may depend on the channel, the superspace insertion points x_i and the superconformal transformation h, but must be the same for the left and right factors k_l and k_r. For the case of s- and t-channels, the transformation laws become

$$k_{s/t,l}(x_i^h) = k(t(x_1,h))k_{s/t,l}b_{s/t}(x_i,h), \quad k_{s/t,r}(x_i^h) = b_{s/t}^{-1}(x_i,h)k_{s/t,r}k(t(x_3,h))^{-1} \ . \tag{8.25}$$

It is now easy to verify that all four tensor components κ_i of the super-crossing factor $\mathcal{M}$ are invariant under superconformal transformations, up to gauge transformations, i.e.

$$\kappa_i(x_k^h) = b_t^{-1}(x_k,h)\,\kappa_i(x_k)\,b_s(x_k,h), \quad \kappa_j(x_k^h) = wb_t^{-1}(x_kh)w^{-1}\,\kappa_j(x_k)\,wb_s(x_k,h)w^{-1} \ , \tag{8.26}$$

where $i = 1,3$ and $j = 2,4$. To get the last two relations one, employs the formula for $k(t_{ji}^h)$ written in (7.45). Using the properties (8.9) of P, we see that the factors by which $\kappa_{i,j}$ get multiplied are absorbed by projectors $\mathcal{P}_s$ and $\mathcal{P}_t$. Therefore, $\mathcal{M}_{st}(x_i)$ is indeed invariant under superconformal transformations.

A similar argument can be used to show that $\mathcal{M}_{\sigma_1\sigma_2}$ is invariant for any two permutations $\sigma_{1,2}$.

8.2.1 Crossing factor in bosonic theories

The analysis we have performed in this section holds for conformal and superconformal symmetry alike. In the bosonic setup, it leads to a very simple expression for $\mathcal{M}_{st}$ for arbitrary spinning fields. As we shall now show, in this case it is possible to reduce the problem to one on the two-dimensional conformal group. For the following argument, we will write $G = SO^+(d+1,1)$ and assume $d > 2$.

Since the crossing factor is conformally invariant, in computing $\mathcal{M}(u,v)$ we may assume that x_i are any points that give the correct cross ratios u and v. In particular, all points can be assumed to lie in the two-dimensional plane Π that is spanned by the first two unit vectors e_1, e_2 of the d-dimensional space $\mathbb{R}^d$. In this case, the element $g_\sigma(x_i)$ is seen to belong to the conformal group of the plane, i.e. $g_\sigma(x_i) \in G_\Pi = SO^+(3,1) \subset G$. Within this group, $g_\sigma(x_i)$ admits a unique Cartan decomposition. Since A is a subgroup of G_Π, the latter serves as a valid Cartan decomposition of $g(x_i)$ in G as well. Put in another way, the Cartan decomposition of G_Π defines a particular gauge fixing for Cartan factors of $g(x_i)$. Note that all relevant rotations are generated by M_{12}, which commutes with the Weyl inversion w when $d > 2$. Hence, we conclude that the factors κ_i that arise in the transition from s- to t-channels must be of the form

$$\kappa_i = e^{\gamma_i D} e^{\varphi_i M_{12}} \ , \tag{8.27}$$

for some functions γ_i and φ_i that depend on the insertion points x_i of the four fields through their two cross ratios. Having determined the general from of κ_i, we can find the undetermined coefficients by a direct calculation. Since we can perform the calculation in any conformal frame, let us set for convenience

$$x_1 = \frac{\cosh^2\frac{u_1}{2} + \cosh^2\frac{u_2}{2}}{2\cosh^2\frac{u_1}{2}\cosh^2\frac{u_2}{2}}e_1 - i\frac{\cosh^2\frac{u_1}{2} - \cosh^2\frac{u_2}{2}}{2\cosh^2\frac{u_1}{2}\cosh^2\frac{u_2}{2}}e_2\ ,\ x_2 = 0\ ,\ x_3 = e_1\ ,\ x_4 = \infty e_1\ . \quad (8.28)$$

Then it follows

$$\kappa_1 = \kappa_3 = e^{\gamma D + \alpha M_{12}}\ , \quad \kappa_2 = \kappa_4 = e^{\gamma D - \alpha M_{12}} \quad \text{where} \quad e^{4\gamma} = \frac{x_{12}^2 x_{34}^2}{x_{14}^2 x_{23}^2}\ , \quad e^{2i\alpha} = \frac{\cosh\frac{u_1}{2}}{\cosh\frac{u_2}{2}}\ . \quad (8.29)$$

Let us note that $\mathcal{M}$ was originally defined using representations of $K = SO(1,1) \times SO(d)$, but is computed using only representation theory of $SO(1,1) \times SO(2)$.

Example To make the last point manifest, we give some more details for conformal theories in three dimensions. Let us decompose the factors $k_l = d_l r_l$ and $k_r = d_r r_r$ into dilations $d_{l/r}$ and rotations $r_{l/r}$. We parametrise the elements r of the three-dimensional rotation group through Euler angles (5.2). With this choice of coordinates, the elements κ_i have $\phi = \pm\alpha$ and $\theta = \psi = 0$. For $\theta = 0$, the only non-zero matrix elements are those with $m = n$. Furthermore

$$t^j_{nn}(\pm\alpha, 0, 0) = e^{\mp in\alpha} P^{(0,2n)}_{j-n}(1) = e^{\mp in\alpha} = \left(\frac{\cosh\frac{u_1}{2}}{\cosh\frac{u_2}{2}}\right)^{\mp\frac{n}{2}}\ . \quad (8.30)$$

Since the stabiliser group $B \sim SO(d-2)$ for a bosonic conformal field theory in three dimensions is trivial, so it the projector P. Putting all this together we conclude that the crossing factor reads

$$(\mathcal{M}_{st})^{ijkl}_{pqrs} = \left(\frac{u}{v}\right)^{-\frac{1}{4}\sum\Delta_i} \left(\frac{\cosh\frac{u_1}{2}}{\cosh\frac{u_2}{2}}\right)^{\frac{1}{2}(i+k-j-l)} \delta^i_p \delta^j_q \delta^k_r \delta^l_s\ , \quad (8.31)$$

where u, v are the usual s-channel cross ratios. The first term in this result is the crossing factor for correlators of scalar fields. The corrections it receives for spinning correlators are diagonal in the space of polarisations but depend on the eigenvalues of the generator of rotations around one particular direction.

8.2.2 Expansion in fermionic variables

In contrast to the bosonic case, where the previous subsection provides a complete solution for $\mathcal{M}$, in the supersymmetric context, we need to make a further expansion in the fermionic variables on G. Let $f(u_i, y^\mu, y_\nu)$ be a function that arises by the restriction of a K-spherical function on G. If W_i are the polarisation spaces of the four fields in the correlation function, then upon expansion f takes values in

$$W_l = W_1 \otimes W_2' \otimes \Lambda\mathfrak{g}_-, \quad W_r = W_3' \otimes W_4 \otimes \Lambda\mathfrak{g}_+\ . \quad (8.32)$$

Here, W' denotes the module of K that is obtained from W by conjugation with the Weyl inversion. The two spaces (8.32) carry representations of B by restriction and we have shown that f takes values in $T = (W_l \otimes W_r)^B$. Assume for a moment that W_i are trivial. Then

$$\dim(W_l \otimes W_r)^B - 1 = \dim(\Lambda\mathfrak{g}_{(1)})^B - 1, \quad (8.33)$$

computes the number of nilpotent four-point invariants. In the present framework, this statement is obvious and the number itself is easily computed. In conventional approaches to SCFTs however, already this number is not trivial to come by.

Let us fix some basis v of T. Through the Cartan decomposition of $g(x_i)$, the Grassmann coordinates y^μ, y_ν and thereby v becomes a function of insertion points x_i. Computation of $v(x_i)$ is what is needed in addition to $\mathcal{M}$ in order to write the superconformal crossing equations. Examples that illustrate how this works will be given presently.

8.3 Illustration for a one-dimensional superconformal algebra

We will now use the above theory to compute the crossing factor between s- and t-channels for the $\mathcal{N}=2$ superconformal algebra in one dimension. For all computations, we will use the three-dimensional representation introduced in previous chapters.

Our first task is to write the elements $g_s(x_i)$ and $g_t(x_i)$ in Cartan coordinates. The ingredients of which these elements are built, $m(x_{ij})$ and $n(x_{ij})$, were determined previously and this in principle gives expressions for $g_s(x_i)$ as 3×3 whose entries are complicated functions of x_i-s. However, we can now use superconformal invariance of the crossing factor and fix the coordinates of the four insertion points. The following choice turns out to be convenient

$$x_1 = (x,\theta_1,\bar\theta_1), \quad x_2 = (0,0,0), \quad x_3 = (1,\theta_3,\bar\theta_3), \quad x_4 = (\infty,0,0)\ . \tag{8.34}$$

With this gauge choice, the entries of the matrices $g_s(x_i)$ and $g_t(x_i)$ depend on the bosonic coordinate x and the four Grassmann variables $\theta_{1,3}$, $\bar\theta_{1,3}$ only.

The primed Cartan coordinates on $SL(2|1)$ are introduced via

$$g = e^{\kappa R} e^{\lambda_l D} e^{\bar q' Q_- + \bar s' S_-} e^{\frac{u}{2}(P+K)} e^{q' Q_+ + s' S_+} e^{\lambda_r D}\ . \tag{8.35}$$

This agrees with the general prescription (8.6), except that the abelian group A is parametrised by a single variable u in this case. Through simple manipulations of supermatrices one finds expressions for the Cartan coordinates of $g_s(x_i)$ and $g_t(x_i)$ in the gauge (8.34). For the bosonic Cartan coordinates in the s-channel one has

$$\cosh^2\frac{u_s}{2} = \frac{1}{x}\Big(1 - \frac{1}{2}\theta_3\bar\theta_3 - \frac{\theta_1\bar\theta_1}{2x} + \frac{\theta_1\bar\theta_3}{x} + \frac{\theta_1\bar\theta_1\theta_3\bar\theta_3}{4x}\Big)\ , \quad e^{-2\kappa_s} = 1 + \frac{\theta_1}{x}(\bar\theta_1 - \bar\theta_3)\ , \tag{8.36}$$

$$e^{\lambda_{s,l}-\lambda_{s,r}} = \Big(1 - x - \frac{1}{2}\theta_1\bar\theta_1 - \frac{1}{2}\theta_3\bar\theta_3 + \theta_1\bar\theta_3\Big)\Big(x - \frac{1}{2}\theta_1\bar\theta_1\Big)\ , \tag{8.37}$$

$$e^{\lambda_{s,l}+\lambda_{s,r}} = \Big(1 + \frac{1}{2}\theta_3\bar\theta_3\Big)\Big(x - \frac{1}{2}\theta_1\bar\theta_1\Big)\ , \tag{8.38}$$

while in the t-channel these coordinates read

$$\cosh^2\frac{u_t}{2} = x\Big(1+\frac{1}{2}\theta_3\bar\theta_3+\frac{\theta_1\bar\theta_1}{2x}-\theta_1\bar\theta_3+\frac{\theta_1\bar\theta_1\theta_3\bar\theta_3}{4x}\Big)\ , \quad e^{-2\kappa_t} = 1+\bar\theta_3(\theta_3-\theta_1)\ , \tag{8.39}$$

$$e^{\lambda_{t,l}-\lambda_{t,r}} = -\Big(1-x-\frac{1}{2}\theta_1\bar\theta_1-\frac{1}{2}\theta_3\bar\theta_3+\theta_1\bar\theta_3\Big)\Big(1+\frac{1}{2}\theta_3\bar\theta_3\Big)\ , \tag{8.40}$$

$$e^{\lambda_{t,l}+\lambda_{t,r}} = \Big(1-\frac{1}{2}\theta_3\bar\theta_3\Big)\Big(x+\frac{1}{2}\theta_1\bar\theta_1\Big)\ . \tag{8.41}$$

The fermionic Cartan coordinates, on the other hand, are given by the following expressions

$$q'_s = e^{\frac{1}{2}\lambda_{s,r}}\Big(\theta_3-\frac{\theta_1}{x}\Big(1-\frac{1}{2}\theta_3\bar\theta_3\Big)\Big)\ , \quad s'_s = e^{-\frac{1}{2}\lambda_{s,r}}\frac{\theta_1}{x}\ , \tag{8.42}$$

$$\bar q'_s = e^{-\frac{1}{2}\lambda_{s,l}}(\bar\theta_3-\bar\theta_1)\ , \quad \bar s_s = -e^{\frac{1}{2}\lambda_{s,l}}\frac{\bar\theta_3}{x}\ , \tag{8.43}$$

$$q'_t = e^{\frac{1}{2}\lambda_{t,r}}(\theta_3-\theta_1)\ , \quad s'_t = -e^{-\frac{1}{2}\lambda_{t,r}}\theta_1\Big(1-\frac{1}{2}\theta_3\bar\theta_3\Big)\ , \tag{8.44}$$

$$\bar q'_t = -e^{-\frac{1}{2}\lambda_{t,l}}\Big(\bar\theta_1-\bar\theta_3\Big(x+\frac{1}{2}\theta_3\bar\theta_1\Big)\Big)\ , \quad \bar s'_t = e^{\frac{1}{2}\lambda_{t,l}}\bar\theta_3\ . \tag{8.45}$$

As the next step, we want to compute the crossing factor $\mathcal{M}_{st}$. Since all irreducible representations of K are one-dimensional, $\mathcal{M}_{st}$ is a single function in the variables x, $\theta_{1,3}$ and $\bar\theta_{1,3}$. It depends, of course, on the choice of representations for the external superfields. We shall pick four such representations (Δ_i, r_i). Note that in our gauge (8.34) the factors $k(t_{41})$ and $k(t_{43})$ are trivial. Therefore, we have

$$\kappa_1 = e^{(\lambda_{s,l}-\lambda_{t,l})D+(\kappa_s-\kappa_t)R}\ , \quad \kappa_4 = e^{(\lambda_{t,l}+\lambda_{s,r})D-\kappa_t R}, \tag{8.46}$$

$$\kappa_3 = e^{(\lambda_{t,r}-\lambda_{s,r})D}\ , \quad \kappa_2 = e^{-(\lambda_{t,r}+\lambda_{s,l}-\log x^2)D+(\kappa_s-\frac{1}{2}\theta_3\bar\theta_3+\frac{\theta_1\bar\theta_1}{2x})R}. \tag{8.47}$$

The function $\mathcal{M}_{st}$ is written in terms of superspace coordinates by inserting in κ_i the formulas (8.36)-(8.41) for the Cartan coordinates in the s- and t-channels. This gives

$$\mathcal{M}_{st} = e^{\frac{i\pi}{2}(\Delta_2+\Delta_4-\Delta_1-\Delta_3)}x^{-2\Delta_1}\alpha^{\frac{3}{2}\Delta_1-\frac{1}{2}\Delta_2-\frac{1}{2}\Delta_3-\frac{1}{2}\Delta_4}\times$$
$$\times\,\beta^{\frac{1}{2}\Delta_1+\frac{1}{2}\Delta_2-\frac{3}{2}\Delta_3+\frac{1}{2}\Delta_4}e^{r_1(\kappa_s-\kappa_t)+r_2(\kappa_s-\frac{1}{2}\theta_3\bar\theta_3+\frac{\theta_1\bar\theta_1}{2x})-r_4\kappa_t}, \tag{8.48}$$

where α and β denote the following superspace elements

$$\alpha = x+\frac{1}{2}\theta_1\bar\theta_1, \quad \beta = 1-\frac{1}{2}\theta_3\bar\theta_3\ . \tag{8.49}$$

Finally, we want to expand the functions F_s and F_t in fermionic variables and write the crossing factor as a matrix of functions that depend on x. The space of $B=R$-invariants in $\Lambda\mathfrak{g}^*$ has a basis

$$v = (1, q'\bar q', q'\bar s', s'\bar q', s'\bar s', q's'\bar q'\bar s')\ . \tag{8.50}$$

Once we insert the expressions (8.36)-(8.45) for Cartan coordinates in the two channels we obtain

$$v_s = \left(1, -\frac{(\bar\theta_1-\bar\theta_3)(\theta_1-x\theta_3)}{x^{3/2}\sqrt{1-x}}, \frac{(\theta_1-x\theta_3)\bar\theta_3+\frac{1}{4}\Omega}{x^{3/2}}, \frac{(\bar\theta_1-\bar\theta_3)\theta_1+\frac{1}{4}\Omega}{x^{3/2}}, \frac{-\theta_1\bar\theta_3\sqrt{1-x}}{x^{3/2}}, \frac{\Omega}{x^2}\right) , \tag{8.51}$$

$$v_t = \left(1, i\frac{(\theta_1-\theta_3)(\bar\theta_1-x\bar\theta_3)}{\sqrt{1-x}}, \frac{x\bar\theta_3(\theta_1-\theta_3)+\frac{1}{4}\Omega}{\sqrt{x}}, \frac{\theta_1(\bar\theta_1-x\bar\theta_3)+\frac{1}{4}\Omega}{\sqrt{x}}, i\theta_1\bar\theta_3\sqrt{1-x}\ , \Omega\right), \tag{8.52}$$

where $\Omega = \theta_1\bar\theta_1\theta_3\bar\theta_3$. All factors that enter our expression for $\mathcal{M}_{st}$ belong to the algebra $\mathbb{C}[x,x^{-1}]\otimes\mathcal{A}$, where $\mathcal{A}$ is the six-dimensional algebra that is spanned by the elements

$$e_1 = 1\ ,\ e_2 = \theta_1\bar\theta_1\ ,\ e_3 = \theta_1\bar\theta_3\ ,\ e_4 = \theta_3\bar\theta_1\ ,\ e_2 = \theta_3\bar\theta_3\ ,\ e_6 = \Omega\ . \tag{8.53}$$

If we represent the e_i by column vectors, the row vectors $v_{s/t}$ become 6×6 matrices whose entries are functions of x. Similarly we can also turn the factor $\sqrt{\frac{\sinh u_t}{\sinh u_s}}\mathcal{M}_{st}$ into a 6×6 matrix if we replace the elements e_i by their representatives in the left-regular representation of $\mathcal{A}$. Multiplying all these matrices gives M_{st} as a 6×6 matrix of functions in x

$$M_{st} = v_t^{-1}\sqrt{\frac{\sinh u_t}{\sinh u_s}}\mathcal{M}_{st}v_s\ . \tag{8.54}$$

Having computed the crossing factor between s- and t-channels, there is one final step left, namely to relate the s- and t-channel cross ratios. Since the arguments of the functions f in the two channels are related by a change of variables that involves Grassmann coordinates, we need to perform a fermionic Taylor expansion in order to write the crossing equation in terms of functions of the bosonic cross ratio x only. For example, in the t-channel this expansion of $f_t(\cosh^2\frac{u_t}{2})$ takes the following form

$$f_t = \left(1 + x\Big(\frac{1}{2}\theta_3\bar\theta_3 + \frac{\theta_1\bar\theta_1}{2x} - \theta_1\bar\theta_3 + \frac{\theta_1\bar\theta_1\theta_3\bar\theta_3}{4x}\Big)\partial + \frac{1}{4}x\Omega\partial^2\right) f_t(x)\ . \tag{8.55}$$

Upon substitution, the crossing factor is a 6×6 matrix of second order differential operators in x. This concludes our construction of the crossing symmetry equations for long multiplets of $\mathcal{N}=2$ superconformal field theories in one dimension.

8.3.1 Multiplet shortening

The formulas from above simplify drastically when some of the operators in the four-point function are short. To see this, let us set either $\theta_1 = \bar\theta_3 = 0$ or $\bar\theta_1 = \theta_3 = 0$ in (8.34). Then the crossing factor $\mathcal{M}_{st}$ is simply

$$\mathcal{M}_{st}^{red} = e^{\frac{i\pi}{2}(\Delta_2+\Delta_4-\Delta_1-\Delta_3)}x^{-\frac{1}{2}\sum\Delta_i}\ . \tag{8.56}$$

In the first case, $\theta_1 = \bar\theta_3 = 0$, the fermionic coordinates $s'_{s,t}$ and $\bar s'_{s,t}$ are seen to vanish, while

$$q'_s = e^{\frac{1}{2}\lambda_{s,r}}\theta_3,\quad \bar q'_s = -e^{-\frac{1}{2}\lambda_{s,l}}\bar\theta_1,\quad q'_t = e^{\frac{1}{2}\lambda_{t,r}}\theta_3,\quad \bar q'_t = -e^{-\frac{1}{2}\lambda_{t,l}}\bar\theta_1\ .$$

The R-symmetry coordinate is also equal to zero in both channels, $\kappa_s = \kappa_t = 0$. Other bosonic coordinates do not depend on remaining Grassmann variables and are the same as for the bosonic conformal group

$$\cosh^2 \frac{u_s}{2} = \frac{1}{x}, \quad \cosh^2 \frac{u_t}{2} = x,$$

and

$$e^{\lambda_{s,l}-\lambda_{s,r}} = (1-x)x, \quad e^{\lambda_{s,l}+\lambda_{s,r}} = x, \quad e^{\lambda_{t,l}-\lambda_{t,r}} = x-1, \quad e^{\lambda_{t,l}+\lambda_{t,r}} = x \; .$$

Furthermore, all components of v_s and v_t except the first two vanish. We write

$$v_s^{red} = \Big(1, \frac{\bar\theta_1\theta_3}{\sqrt{x(1-x)}}\Big) \equiv \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{\sqrt{x(1-x)}} \end{pmatrix}, \quad v_t^{red} = \Big(1, i\frac{\bar\theta_1\theta_3}{\sqrt{1-x}}\Big) \equiv \begin{pmatrix} 1 & 0 \\ 0 & \frac{i}{\sqrt{1-x}} \end{pmatrix} \; .$$

We have turned v_s and v_t into matrices as explained above. Now, they are 2×2 matrices, rather than 6×6. The expanded crossing factor is

$$M_{st}^{red} = (v_t^{red})^{-1}\sqrt{\frac{\sinh u_t}{\sinh u_s}}\mathcal{M}_{st}^{red} v_s^{red} = e^{\frac{i\pi}{2}(\Delta_2+\Delta_4-\Delta_1-\Delta_3)} x^{-\frac{1}{2}\sum\Delta_i}(-x^3)^{\frac{1}{4}} \begin{pmatrix} 1 & 0 \\ 0 & i\sqrt{x} \end{pmatrix} \; . \tag{8.57}$$

Its entries no longer involve derivatives with respect to x because the relation between cross ratios in two channels does not depend on Grassmann variables. The features that we exhibited here are generic and happen for shortening in a large class of superconformal algebras. This will be proven elsewhere. Simplifications in the other case, $\bar\theta_1 = \theta_3 = 0$, are not as dramatic, but still considerable.

8.4 Cartan decomposition and the crossing factor for $SL(2m|\mathcal{N})$

The second supersymmetric example that we will treat is the family of superconformal algebras $\mathfrak{sl}(2m|\mathcal{N})$. The elements $m(x)$, $n(x)$, $k(t)$ and Bruhat decompositions of the corresponding supergroups $SL(2m|\mathcal{N})$ were studied in the last chapter. Now we will determine the crossing factor $\mathcal{M}_{st}$ for this two-parameter family in the case of scalar external fields. Let us use superconformal transformations to set

$$x_1 = (ae_1 + be_2, \theta_1, \bar\theta_1), \quad x_2 = (0,0,0), \quad x_3 = (e_1, \theta_3, \bar\theta_3), \quad x_4 = (\infty e_1, 0, 0) \; . \tag{8.58}$$

To write the crossing symmetry equations, one should consider the primed Cartan decomposition of G. We start from its unprimed cousin

$$g = e^{q^I{}_\beta Q_I{}^\beta + s^I{}_{\dot\beta} S_I{}^{\dot\beta}}\, k_l a(u_1,u_2) k_r\, e^{q^{\dot\alpha}{}_J Q_{\dot\alpha}{}^J + s^\alpha{}_J S_\alpha{}^J} \; . \tag{8.59}$$

In the fundamental representation this reads

$$g = \begin{pmatrix} \delta^{\dot\alpha}_{\dot\gamma} & 0 & 0 \\ 0 & \delta^\alpha{}_\gamma & 0 \\ -s^I{}_{\dot\gamma} & -q^I{}_\gamma & \delta^I{}_K \end{pmatrix} \begin{pmatrix} e^{\frac{\mathcal{N}\kappa}{\mathcal{N}-2m}}(g_b)^{\dot\gamma}{}_{\dot\delta} & e^{\frac{\mathcal{N}\kappa}{\mathcal{N}-2m}}(g_b)^{\dot\gamma}{}_\delta & 0 \\ e^{\frac{\mathcal{N}\kappa}{\mathcal{N}-2m}}(g_b)^\gamma{}_{\dot\delta} & e^{\frac{\mathcal{N}\kappa}{\mathcal{N}-2m}}(g_b)^\gamma{}_\delta & 0 \\ 0 & 0 & e^{\frac{2m\kappa}{\mathcal{N}-2m}} U^K_L \end{pmatrix} \begin{pmatrix} \delta^{\dot\delta}_{\dot\beta} & 0 & q^{\dot\delta}{}_J \\ 0 & \delta^\delta{}_\beta & s^\delta{}_J \\ 0 & 0 & \delta^L{}_J \end{pmatrix}, \tag{8.60}$$

where $g_b = k_l^b a k_r^b$ is an element of the bosonic conformal group. We see that the top left $2m \times 2m$ corner is simply a scalar multiple of g_b. When written without indices, generators of $\mathfrak{g}_-^*$ will carry a bar, to be distinguished from generators of $\mathfrak{g}_+^*$. With this convention, the above Cartan decomposition reads

$$g = e^{\frac{\mathcal{N}\kappa}{\mathcal{N}-2m}} \begin{pmatrix} A & B & Aq + Bs \\ C & D & Cq + Ds \\ -\bar{s}A - \bar{q}C & -\bar{s}B - \bar{q}D & e^{-\kappa}U - (\bar{s}A + \bar{q}C)q - (\bar{s}B + \bar{q}D)s \end{pmatrix} . \tag{8.61}$$

Here A, B, C, D are $m \times m$ blocks of g_b and how to extract Cartan coordinates from them will be explained in the next chapter. The elements that we want to decompose are

$$g_s(x_i) = n(y_{21})^{-1} m(x_{31}) n(y_{43}), \quad g_t(x_i) = n(y_{41})^{-1} m(x_{31}) n(y_{23}) . \tag{8.62}$$

As can be seen from the solutions of (7.8) found in the last chapter, when x is sent to $(\infty e_1, \theta, \bar{\theta})$ then $y(x) = 0$ and consequently $n(y(x)) = 1$. Therefore, in the special configuration that we chose, one has

$$g_s(x_i) = n(y_{21})^{-1} m(x_{31}), \quad g_t(x_i) = m(x_{31}) n(y_{23}) . \tag{8.63}$$

Thus, we are led to consider the decomposition of elements that take the general form $n(y)m(x)$ and $m(x)n(y')$. We treat these in turn. In the notation of the previous chapter

$$n(y)m(x) = \begin{pmatrix} 1 & X & \theta \\ -JYJ & 1 - JYJX - J\eta\bar{\theta} & J\eta - JYJ\theta \\ \bar{\eta}J & \bar{\eta}JX - \bar{\theta} & 1 + \bar{\eta}J\theta \end{pmatrix} .$$

One immediately finds

$$s_s = (1 - J\eta\bar{\theta})^{-1} J\eta, \; q_s = \theta - X s_s, \; \bar{q}_s = \bar{\theta}(1 - J\eta\bar{\theta})^{-1}, \; \bar{s}_s = (\bar{q}_s JY - \bar{\eta})J, \; e^{\frac{2m\kappa_s}{\mathcal{N}-2m}} U_s = 1 + \bar{\theta} s_s . \tag{8.64}$$

The last expression can be simplified by substituting for s_s and performing the following manipulation

$$1 + \bar{\theta} s_s = 1 + \bar{\theta}(1 - J\eta\bar{\theta})^{-1} J\eta = 1 + \bar{\theta}\Big(\sum_{n=0}^{\infty} (J\eta\bar{\theta})^n\Big) J\eta = \sum_{n=0}^{\infty} (\bar{\theta} J\eta)^n = (1 - \bar{\theta}J\eta)^{-1} .$$

Therefore, taking the determinant of the last equation in (8.64) gives

$$e^{\frac{2m\mathcal{N}\kappa_s}{\mathcal{N}-2m}} = \det(1 - \bar{\theta}J\eta)^{-1} . \tag{8.65}$$

Next, by looking at determinants of top left four $m \times m$ blocks, we obtain the coordinates associated with dilations

$$e^{2(\lambda_{s,l} + \lambda_{s,r})} = -\det(J^{-1} - YJX - \eta\bar{\theta})^{-1}, \quad e^{2(\lambda_{s,l} - \lambda_{s,r})} = \det X \det Y^{-1}, \tag{8.66}$$

as well as the coordinates (u_1, u_2) of the abelian torus

$$\sinh^2 \frac{u_1^s}{2} \sinh^2 \frac{u_2^s}{2} = \det X \det Y \det(1 - \bar{\theta}J\eta), \tag{8.67}$$

$$\cosh^2 \frac{u_1^s}{2} \cosh^2 \frac{u_2^s}{2} = -\det(J^{-1} - YJX - \eta\bar{\theta}) \det(1 - \bar{\theta}J\eta) . \tag{8.68}$$

We have already put a label s on the coordinates, as they are indeed the s-channel coordinates for appropriate choices of x and y as in (8.63). For the other channel, we decompose

$$m(x)n(y') = \begin{pmatrix} 1 - XJY'J + \theta\bar{\eta}'J & X & XJ\eta' + \theta \\ -JY'J & 1 & J\eta' \\ \bar{\theta}JY'J + \bar{\eta}'J & -\bar{\theta} & 1 - \bar{\theta}J\eta' \end{pmatrix} .$$

Following similar steps as above, we find

$$q_t = (1+\theta\bar{\eta}'J)^{-1}\theta,\ s_t = J(\eta'+Y'Jq_t),\ \bar{s}_t = -\bar{\eta}'J(1+\theta\bar{\eta}'J)^{-1},\ \bar{q}_t = \bar{\theta}-\bar{s}_tX,\ e^{\frac{2m\kappa_t}{\mathcal{N}-2m}}U_t = 1-\bar{\eta}Jq_t, \tag{8.69}$$

and therefore

$$e^{\frac{2m\kappa_t}{\mathcal{N}-2m}} = \det(1+\bar{\eta}'J\theta)^{-1} . \tag{8.70}$$

Dilation coordinates are now

$$e^{2(\lambda_{t,l}+\lambda_{t,r})} = -\det(J^{-1} - XJY' + \theta\bar{\eta}'), \quad e^{2(\lambda_{t,l}-\lambda_{t,r})} = \det X \det Y'^{-1} . \tag{8.71}$$

Finally the coordinates on the abelian torus read

$$\sinh^2\frac{u_1^t}{2}\sinh^2\frac{u_2^t}{2} = \det X\det Y'\det(1+\bar{\eta}'J\theta), \tag{8.72}$$

$$\cosh^2\frac{u_1^t}{2}\cosh^2\frac{u_2^t}{2} = -\det(J^{-1} - XJY' + \theta\bar{\eta}')\det(1+\bar{\eta}'J\theta) . \tag{8.73}$$

Expressions written so far are sufficient to determine the crossing factor for fields which transform trivially under rotations and $SU(\mathcal{N})$ internal symmetries. For applications that we have in mind in this work, these conditions are satisfied. A field that transforms trivially both under spatial rotations and $SU(\mathcal{N})$ internal symmetries is associated with a one-dimensional representation $\rho_{\Delta,r}$ of K. Here Δ is the conformal weight and r the $U(1)_R$-charge of the field. Our parametrisation of K is such that

$$\rho_{\Delta,r}(e^{\lambda D+\kappa R}e^{r^\alpha{}_\beta M_\alpha{}^\beta + r^{\dot\alpha}{}_{\dot\beta}M_{\dot\alpha}{}^{\dot\beta} + u^I{}_J R_I{}^J}) = e^{-\Delta\lambda+r\kappa}. \tag{8.74}$$

Therefore, the tensor factors appearing in $\mathcal{M}_{st}$ are

$$\rho_3(\kappa_3) = e^{\Delta_3(\lambda_{s,r}-\lambda_{t,r})},\quad \rho_4(\kappa_4) = e^{-\Delta_4(\lambda_{t,l}+\lambda_{s,r})-r_4\kappa_t}, \tag{8.75}$$

$$\rho_1(\kappa_1) = e^{\Delta_1(\lambda_{t,l}-\lambda_{s,l})+r_1(\kappa_s-\kappa_t)},\quad \rho_2(\kappa_2) = e^{\Delta_2(\lambda_{t,r}+\lambda_{s,l})+r_2\kappa_s}\rho_2(k(t_{23})k(t_{21})^{-1}) . \tag{8.76}$$

In the last expression we have used that the middle two factors in κ_4 trivialise in our gauge. All the coordinates appearing on right hand sides of previous equations have been spelled out and one simply substitutes for them to find the product.

8.5 Four-dimensional $\mathcal{N} = 1$ SCFTs

Let us apply the results from previous sections to the complexified $\mathcal{N} = 1$ superconformal algebra in four dimensions, $\mathfrak{g} = \mathfrak{sl}(4|1)$. We use the same notation as above, only $\mathfrak{sl}(\mathcal{N})$ indices become redundant as this summand disappears for $\mathcal{N} = 1$.

The correlation function we want to consider is that of two long multiplets $\mathcal{O}$, along with one anti-chiral field $\bar{\varphi}_1$ and one chiral φ_3

$$G_4(x_i) = \langle \bar{\varphi}(x_1)\mathcal{O}(x_2)\varphi(x_3)\mathcal{O}(x_4)\rangle \ . \tag{8.77}$$

The fields have conformal weights Δ_i and R-charges r_i, and we assume that $\sum r_i = 0$. Therefore, we can write $r = r_1 + r_2 = -r_3 - r_4$. Chirality conditions further imply

$$\Delta_1 = -\frac{3}{2}r_1, \quad \Delta_3 = \frac{3}{2}r_3 \ . \tag{8.78}$$

The general solution for $y(x)$ specialises in the case $m=2$, $\mathcal{N}=1$ to

$$y = -\frac{1+\frac{\Omega}{4\mathrm{det}x}}{\mathrm{det}x}x^t,\ \eta = \frac{-i}{\mathrm{det}x}\begin{pmatrix} x^{\dot{2}}{}_1\theta^{\dot{1}} - x^{\dot{1}}{}_1\theta^{\dot{2}} + \frac{1}{2}\bar{\theta}_1\theta^{\dot{1}}\theta^{\dot{2}} \\ x^{\dot{2}}{}_2\theta^{\dot{1}} - x^{\dot{1}}{}_2\theta^{\dot{2}} + \frac{1}{2}\bar{\theta}_2\theta^{\dot{1}}\theta^{\dot{2}} \end{pmatrix},\ \bar{\eta}^T = \frac{-i}{\mathrm{det}x}\begin{pmatrix} x^{\dot{1}}{}_1\bar{\theta}_2 - x^{\dot{1}}{}_2\bar{\theta}_1 - \frac{1}{2}\theta^{\dot{1}}\bar{\theta}_1\bar{\theta}_2 \\ x^{\dot{2}}{}_1\bar{\theta}_2 - x^{\dot{2}}{}_2\bar{\theta}_1 - \frac{1}{2}\theta^{\dot{2}}\bar{\theta}_1\bar{\theta}_2 \end{pmatrix} \ .$$

In these formulas, x and y denote 2×2 matrices of *bosonic* coordinates of super-points x and y. This is a slight abuse of notation, but in any equation the meaning of symbols x, y is clear from the context. By Ω we denote the element $\theta^{\dot{1}}\theta^{\dot{2}}\bar{\theta}_1\bar{\theta}_2$. The covariant derivatives, realising the right-regular action of $\mathfrak{m}$, read in our coordinates

$$D_{\dot{\alpha}}{}^I = \partial_{\dot{\alpha}}{}^I + \frac{1}{2}\bar{\theta}^I{}_\beta\partial_{\dot{\alpha}}{}^\beta, \quad \bar{D}_I{}^\alpha = -\partial_I{}^\alpha - \frac{1}{2}\theta^{\dot{\beta}}{}_I\partial_{\dot{\beta}}{}^\alpha \ . \tag{8.79}$$

One can verify that they anti-commute with the right-invariant vector fields written in the subsection (8.5.1). Let us introduce the corresponding chiral and anti-chiral coordinates

$$x'^{\dot{\alpha}}{}_\beta = x^{\dot{\alpha}}{}_\beta + \frac{1}{2}\theta^{\dot{\alpha}}{}_I\bar{\theta}^I{}_\beta, \quad x''^{\dot{\alpha}}{}_\beta = x^{\dot{\alpha}}{}_\beta - \frac{1}{2}\theta^{\dot{\alpha}}{}_I\bar{\theta}^I{}_\beta \ . \tag{8.80}$$

We further set $\theta' = \theta'' = \theta$ and $\bar{\theta}' = \bar{\theta}'' = \bar{\theta}$. Then the following equalities hold

$$Dx'' = D\bar{\theta}'' = 0, \quad \bar{D}x' = \bar{D}\theta' = 0 \ . \tag{8.81}$$

The chirality conditions satisfied by the fields allow us to set θ_1 and $\bar{\theta}_3$ to zero. Let us write $\alpha = a+ib$, $\alpha^* = a - ib$ and fix the insertion points to positions as explained in the previous subsection. Further, we write $y = -y_{21}$ and $y' = y_{23}$. Then, a computation gives

$$y = \left(\begin{pmatrix} -1/\alpha^* & 0 \\ 0 & 1/\alpha \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}, i\begin{pmatrix} (\bar{\theta}_1)_2/\alpha^* \\ (\bar{\theta}_1)_1/\alpha \end{pmatrix}^t\right),\ y' = \left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, i\begin{pmatrix} (\theta_3)^{\dot{2}} \\ (\theta_3)^{\dot{1}} \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}^t\right) \ . \tag{8.82}$$

Next, the factor $m(x_{31})$ is found

$$m(x_{31}) = \begin{pmatrix} 1 & X & \theta_3 \\ 0 & 1 & 0 \\ 0 & \bar{\theta}_1 & 1 \end{pmatrix}, \text{ with } X = X_3 - X_1 = \begin{pmatrix} -1+\alpha & 0 \\ 0 & 1-\alpha^* \end{pmatrix} \ . \tag{8.83}$$

We are now ready to consider the Cartan decomposition of $n(y)m(x)$ and $m(x)n(y')$. Using the formulas of the previous section, the fermionic coordinates and dilation coordinates are

$$q_s = q_t = \theta_3, \quad \bar{q}_s = \bar{q}_t = -\bar{\theta}_1, \quad s_s = \bar{s}_s = s_t = \bar{s}_t = 0, \tag{8.84}$$

$$e^{4\lambda_{s,l}} = \alpha^2(\alpha^*)^2(1-\alpha)(1-\alpha^*), \quad e^{4\lambda_{s,r}} = \frac{1}{(1-\alpha)(1-\alpha^*)}, \tag{8.85}$$

$$e^{4\lambda_{t,l}} = \alpha\alpha^*(1-\alpha)(1-\alpha^*), \quad e^{4\lambda_{t,r}} = \frac{\alpha\alpha^*}{(1-\alpha)(1-\alpha^*)} \ . \tag{8.86}$$

The other factors that appear in $k_{s/t,l/r}$, which are products of rotations and R-symmetry transformations, assume the following diagonal form

$$r_{s/t,l} = \begin{pmatrix} L_{s/t} & 0 & 0 \\ 0 & L_{s/t}^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \; r_{s/t,r} = \begin{pmatrix} R_{s/t} & 0 & 0 \\ 0 & R_{s/t}^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \text{ with } L_{s,t} = \begin{pmatrix} l_{s,t} & 0 \\ 0 & l_{s,t}^{-1} \end{pmatrix}, \; R_{s,t} = \begin{pmatrix} r_{s,t} & 0 \\ 0 & r_{s,t}^{-1} \end{pmatrix}, \tag{8.87}$$

and $l_{s,t}$, $r_{s,t}$ are in turn given by

$$l_s = \Big(\frac{\alpha^2(1-\alpha)}{(\alpha^*)^2(1-\alpha^*)}\Big)^{1/8}, \; r_s = \Big(\frac{1-\alpha^*}{1-\alpha}\Big)^{1/8}, \; l_t = \sqrt{-i}\Big(\frac{\alpha(1-\alpha)}{\alpha^*(1-\alpha^*)}\Big)^{1/8}, \; r_t = \frac{1}{\sqrt{-i}}\Big(\frac{\alpha(1-\alpha^*)}{\alpha^*(1-\alpha)}\Big)^{1/8} .$$

Finally, the coordinates on the torus are

$$\cosh^2\frac{u_1^s}{2} = \frac{1}{\alpha}, \quad \cosh^2\frac{u_2^s}{2} = \frac{1}{\alpha^*}, \quad \cosh^2\frac{u_1^t}{2} = \alpha, \quad \cosh^2\frac{u_2^t}{2} = \alpha^* . \tag{8.88}$$

This completes the determination of Cartan coordinates of the elements g_s and g_t. To find the matrix $\mathcal{M}_{st}$, it is still required to determine $k(t_{21})$ and $k(t_{23})$. These are

$$k(t_{21}) = (\alpha\alpha^*)^{-D}, \quad k(t_{23}) = 1 . \tag{8.89}$$

This allows for the computation of factors appearing in $\mathcal{M}_{st}$. The computation gives

$$\rho_i(\kappa_i) = (\alpha\alpha^*)^{-\frac{\Delta_i}{4}} . \tag{8.90}$$

To derive the crossing equations, there is one remaining step, namely to perform the expansion in nilpotent invariants in both channels. In order to do this, we need to switch to the primed Cartan coordinates by moving the exponentials containing fermionic variables past the elements of the left and right K-subgroups. We have in both channels that $s' = \bar{s}' = 0$ and

$$\bar{q}_s' = -\bar{\theta}_1 L_s^{-1} e^{-\frac{1}{2}\lambda_{s,l}}, \quad q_s' = R_s e^{\frac{1}{2}\lambda_{s,r}}\theta_3, \quad \bar{q}_t' = -\bar{\theta}_1 L_t^{-1} e^{-\frac{1}{2}\lambda_{t,l}}, \quad q_t' = R_t e^{\frac{1}{2}\lambda_{t,r}}\theta_3 . \tag{8.91}$$

Recall that B is the commutant in $G_{(0)}$ of the two-dimensional abelian group A. In the case at hand, $B \cong SO(2) \times SO(2)$ and Lie algebras of A and B are

$$\mathfrak{a} = \text{span}\{P_1 + K_1,\; P_2 - K_2\}, \quad \mathfrak{b} = \text{span}\{R, M_1{}^1 + M_{\dot{1}}{}^{\dot{1}}\} . \tag{8.92}$$

Irreducible finite-dimensional representations of $\mathfrak{k}$ are labelled by two spins, a conformal weight and an R-charge, $(j_1, j_2)_r^{\Delta}$. In such notation, the four modules $\mathfrak{q}_\pm, \mathfrak{s}_\pm$ are

$$\mathfrak{q}_+ = (0,1/2)_1^{1/2}, \quad \mathfrak{q}_- = (1/2,0)_{-1}^{1/2}, \quad \mathfrak{s}_+ = (1/2,0)_1^{-1/2}, \quad \mathfrak{s}_- = (0,1/2)_{-1}^{-1/2} . \tag{8.93}$$

According to our general theory, to the correlator (8.77) is associated a function on A with values in the space

$$(W_1 \otimes W_2' \otimes \Lambda\mathfrak{q}_- \otimes W_3 \otimes W_4' \otimes \Lambda\mathfrak{q}_+)^{\mathfrak{b}} = (\Lambda\mathfrak{q})^{\mathfrak{b}} . \tag{8.94}$$

Under the action of $\mathfrak{k}$ the 16-dimensional exterior algebra inside the brackets transforms as

$$\Lambda\mathfrak{q} \cong \mathbf{1}_0^0 \oplus \mathbf{1}_2^1 \oplus \mathbf{1}_{-2}^1 \oplus \mathbf{1}_0^2 \oplus (1/2,0)_{-1}^{1/2} \oplus (0,1/2)_1^{1/2} \oplus (1/2,1/2)_0^1 \oplus (1/2,0)_1^{3/2} \oplus (0,1/2)_{-1}^{3/2} .$$

We have written $\mathbf{1}$ for the trivial representation of $SU(2)\times SU(2)$. Two of the singlets are $\mathfrak{b}$-invariant and the four-dimensional representation $(1/2,1/2)$ contains a two-dimensional invariant subspace. Hence, the space of invariants is four-dimensional and spanned by

$$W_{\bar{\varphi}\mathcal{O}\varphi\mathcal{O}} = (\Lambda\mathfrak{q})^B = \text{span}\{1, Q_{\dot{1}}Q^1, Q_{\dot{2}}Q^2, Q_{\dot{1}}Q^1Q_{\dot{2}}Q^2\} \ . \tag{8.95}$$

Indeed, from the bracket relations one checks that these combinations of generators commute, in the universal enveloping algebra $U(\mathfrak{g})$, with $M_1{}^1 + M_{\dot{1}}{}^{\dot{1}}$ and R. In the two channels, the invariant combinations read

$$(\bar{q}'_t)_1(q'_t)^{\dot{1}} = -i(1-\alpha)^{-1/2}(\bar{\theta}_1)_1(\theta_3)^{\dot{1}},\ (\bar{q}'_t)_2(q'_s)^{\dot{2}} = i(1-\alpha^*)^{-1/2}(\bar{\theta}_1)_2(\theta_3)^{\dot{2}}, \tag{8.96}$$

$$(\bar{q}'_s)_1(q'_s)^{\dot{1}} = -\alpha^{-1/2}(1-\alpha)^{-1/2}(\bar{\theta}_1)_1(\theta_3)^{\dot{1}},\ (\bar{q}'_s)_2(q'_s)^{\dot{2}} = -(\alpha^*)^{-1/2}(1-\alpha^*)^{-1/2}(\bar{\theta}_1)_2(\theta_3)^{\dot{2}} \ . \tag{8.97}$$

Putting everything together, the crossing factor between s- and t-channels reads

$$M_{st} = (\alpha\alpha^*)^{\frac{7}{4}-\frac{1}{4}\sum\Delta_i} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & i\sqrt{\alpha} & 0 & 0 \\ 0 & 0 & -i\sqrt{\alpha^*} & 0 \\ 0 & 0 & 0 & \sqrt{\alpha\alpha^*} \end{pmatrix}^{-1} \ . \tag{8.98}$$

The factor $(\alpha\alpha^*)^{7/4}$ is the ratio of Haar measure densities in the two channels. We may observe that variables α, α^* are related to the usual variables $z, \bar{z}$ by

$$\alpha = \frac{z}{z-1}, \quad \alpha^* = \frac{\bar{z}}{\bar{z}-1} \ . \tag{8.99}$$

Thus the top left entry of the crossing matrix is the one that we would get in the bosonic theory, as expected.

8.5.1 Conventions for $\mathfrak{sl}(4|1)$

Here we write the relation between primed and unprimed Cartan coordinates. The primed bosonic coordinates are equal to the unprimed, so we only need to consider the fermionic ones. The relation between two sets of coordinates reads

$$\begin{pmatrix} q'_1 \\ q'_2 \end{pmatrix} = e^{\kappa-\frac{1}{2}\lambda_l}\pi_{1/2}(-\psi_1^l, -\theta_2^l, \varphi_2^l)\begin{pmatrix} q_1 \\ q_2 \end{pmatrix}, \quad \begin{pmatrix} q'^{\dot{1}} \\ q'^{\dot{2}} \end{pmatrix} = e^{\frac{1}{2}\lambda_r}\pi_{1/2}(\varphi_1^r, \theta_1^r, \psi_1^r)\begin{pmatrix} q^{\dot{1}} \\ q^{\dot{2}} \end{pmatrix}, \tag{8.100}$$

$$\begin{pmatrix} s'_{\dot{1}} \\ s'_{\dot{2}} \end{pmatrix} = e^{\kappa+\frac{1}{2}\lambda_l}\pi_{1/2}(\psi_1^l, \theta_1^l, \varphi_1^l)\begin{pmatrix} s_{\dot{1}} \\ s_{\dot{2}} \end{pmatrix}, \quad \begin{pmatrix} s'^1 \\ s'^2 \end{pmatrix} = e^{-\frac{1}{2}\lambda_r}\pi_{1/2}(\varphi_2^r, -\theta_2^r, \psi_2^r)\begin{pmatrix} s^1 \\ s^2 \end{pmatrix} \ . \tag{8.101}$$

From these equations, one gets relations between partial derivatives in the two systems. Among others, we have

$$\partial_{\kappa'} = \partial_\kappa - q_\alpha\partial_{q_\alpha}, \quad \partial_{\lambda'_l} = \partial_{\lambda_l} + \frac{1}{2}q_\alpha\partial_{q_\alpha}, \quad \partial_{\lambda'_r} = \partial_{\lambda_r} - \frac{1}{2}q^{\dot{\alpha}}\partial_{q^{\dot{\alpha}}}, \tag{8.102}$$

Chapter 9

Explicit tensor structures

In the previous chapter we have analysed four-point tensor structures in order to derive the crossing factor for an arbitrary four-point function in the Calogero-Sutherland gauge. Indeed, whereas the invariance of $G_4(x_i)$ under permutations of insertion points is easy to impose in coordinate space, it becomes less obvious in our group-theoretic coordinates. In principle, one of our aims is to be able to forget the coordinate space altogether and implement all aspects of the bootstrap programme in the group-theoretic gauge. The previous chapter was one major step in this direction and the treatment of conformal partial waves, to be done in the next, is another one. However, it is certainly useful to keep track of the relation between objects in the CS gauge and their counterparts in the coordinate space. By doing this, one can make consistency checks of both approaches. More importantly, it may turn out that the two gauges complement each other, in the sense that some difficult problems in one simplify in the other. It seems that the complexity of the map (7.40) is sufficient to allow for such phenomena to occur.

Our goal in this chapter is to provide explicit relations between certain four-point functions in the above two coordinate systems. In particular, this will be done for four-point functions of scalars in any spacetime dimension, arbitrary spinning correlators in three dimensions and seed-1/2 correlation functions in four dimensions. The main technical tool used for computations of this chapter is the four-dimensional representation of the four-dimensional conformal group, i.e. the accidental isomorphism of Lie algebras $\mathfrak{so}(6) \cong \mathfrak{su}(4)$. Of course, the three-dimensional conformal group is contained in the four-dimensional one, a fact related to another accidental isomorphism $\mathfrak{so}(5) \cong \mathfrak{sp}(4)$. The existence of four-dimensional representations will allow us to treat the two groups simultaneously, at least in parts of our discussion.

Our starting point will be the relation between the four-point function and the restriction F of the corresponding K-spherical function to A, which for bosonic conformal groups in the s-channel reads

$$G_4(x_i) = \Big(\rho_2(k(t_{21})^{-1})\rho_4(k(t_{43})^{-1})\Big)\Big(\rho_1(k_l)\rho_2(k_l^w)\rho_3(k_r^{-1})\rho_4((k_r^{-1})^w)\Big)F(u_1,u_2)\ . \tag{9.1}$$

The two factors that multiply $F(u_1, u_2)$ will be denoted for short by $\Theta_1(x_i)$ and $\Theta_2(x_i)$.

9.1 Scalar fields

The first case that we want to discuss is that of a four-point function of scalars in a bosonic CFT. Let us denote by f the restriction of the K-spherical function F to the two-dimensional group A. Then our expression for the correlator in the s-channel reads

$$
\begin{aligned}
G_4(x_i) &= \frac{1}{x_{12}^{\Delta_1+\Delta_2} x_{34}^{\Delta_3+\Delta_4}} \left(\frac{x_{14}}{x_{24}}\right)^{2a} \left(\frac{x_{14}}{x_{13}}\right)^{2b} [(1-z_1)(1-z_2)]^{-\frac{a}{2}-\frac{b}{2}} f(u_1,u_2) \qquad (9.2)\\
&= \frac{1}{x_{12}^{\Delta_1+\Delta_2} x_{34}^{\Delta_3+\Delta_4}} \left(\frac{x_{14}}{x_{24}}\right)^{2a} \left(\frac{x_{14}}{x_{13}}\right)^{2b} \cdot \frac{(-1)^{\frac{d-2}{2}}}{2} \frac{(z_1 z_2)^{\frac{d-1}{2}} |z_1 - z_2|^{-\frac{d-2}{2}}}{[(1-z_1)(1-z_2)]^{\frac{a}{2}+\frac{b}{2}+\frac{1}{4}}} \cdot \psi(u_1,u_2) \ .
\end{aligned}
$$

In the second line we expressed f in terms of a functions ψ as $f = \omega^{-1/2}\psi$, with the factor ω from (5.27). This conventional factor is introduced in order for the Laplacian on G to reduce to an operator of Schrödinger type, as we have explained before. The final result is written as a product of two terms that multiply ψ. The first term coincides with the function $\Omega(x_i)$ that was used in the works of Dolan and Osborn. The second one is, therefore, a function of cross ratios only. It can be viewed as a gauge transformation that takes the Dolan-Osborn Casimir equations to the BC_2 Calogero-Sutherland problem. Throughout this chapter, the scalar factor from above will be denoted by $\tilde{\Omega}(x_i)$, i.e. $G_4(x_i) = \tilde{\Omega}(x_i)\psi(u_1,u_2)$.

9.2 Spinor representation of the conformal group

Let us move to correlators of spinning fields. What we wish to achieve is to explicitly compute the Cartan factors of $g(x_i)$, thereby relating the four-point function to f, or alternatively ψ. We will consider examples in three and four dimensions. In these two cases, the smallest faithful representation of the complexified conformal algebra is four-dimensional. We will use this representation to do computations and refer to it as the spinor representation. The three-dimensional conformal group Spin(4, 1) is a subgroup of Spin(5, 1) and hence we can pass from the latter to the former by restricting the range of indices μ, ν.

In the four-dimensional representation, we work in a basis in which the generators of dilations, translations and special conformal transformations read

$$
D = \frac{1}{2}\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad P_\mu = \begin{pmatrix} 0 & \sigma_\mu \\ 0 & 0 \end{pmatrix}, \quad K_\mu = \begin{pmatrix} 0 & 0 \\ \bar{\sigma}_\mu & 0 \end{pmatrix}, \qquad (9.3)
$$

while the rotation generators $M_{\mu\nu}$ are

$$
M_{23} = -\frac{i}{2}\begin{pmatrix} \sigma_1 & 0 \\ 0 & -\sigma_1 \end{pmatrix}, \quad M_{13} = -\frac{i}{2}\begin{pmatrix} \sigma_2 & 0 \\ 0 & \sigma_2 \end{pmatrix}, \quad M_{12} = -\frac{i}{2}\begin{pmatrix} \sigma_3 & 0 \\ 0 & -\sigma_3 \end{pmatrix}, \qquad (9.4)
$$

$$
M_{14} = -\frac{i}{2}\begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_1 \end{pmatrix}, \quad M_{24} = -\frac{i}{2}\begin{pmatrix} -\sigma_2 & 0 \\ 0 & \sigma_2 \end{pmatrix}, \quad M_{34} = -\frac{i}{2}\begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix} . \qquad (9.5)
$$

All these matrices are written in terms of 2×2 blocks, with I denoting the 2×2 identity matrix and σ_μ and $\bar{\sigma}_\mu$ given by[1]

$$\sigma_\mu = (-\sigma_3, -iI, \sigma_1, -\sigma_2), \quad \bar{\sigma}_\mu = (-\sigma_3, iI, \sigma_1, -\sigma_2)\ .$$

In particular $\det(x_\mu \sigma^\mu) = -x_\mu x^\mu$. With these conventions, the elements $m(x)$ take the form

$$m(x) = e^{x^\mu P_\mu} = \begin{pmatrix} 1 & x^\mu \sigma_\mu \\ 0 & 1 \end{pmatrix} \equiv \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\ . \tag{9.6}$$

In the last expression, we have written the 2×2 matrix x without any indices. We will adhere to such simplified notation, as it should not cause any confusion.

By definition, the two groups have different Weyl inversions that we will denote by w_3 and w_4. They read

$$w_3 = e^{\pi \frac{K_3 - P_3}{2}} = \begin{pmatrix} 0 & -\sigma_1 \\ \sigma_1 & 0 \end{pmatrix}, \quad w_4 = e^{\pi \frac{K_4 - P_4}{2}} = \begin{pmatrix} 0 & \sigma_2 \\ -\sigma_2 & 0 \end{pmatrix}\ . \tag{9.7}$$

One can readily verify that both these matrices square to the negative identity matrix. The Weyl inversion gives rise to elements $n(x)$ which take the form

$$n(x) = \begin{pmatrix} 1 & 0 \\ (s_{e_d} x)^\mu \bar{\sigma}_\mu & 1 \end{pmatrix}\ . \tag{9.8}$$

We already said that the functions $y(x)$, $z(x)$ and $k(x)$ defined by the Bruhat decomposition of $wm(x)$ are given by $y(x) = wx$, $z(x) = -x$ and $k(x) = |x|^{-2D} s_{e_d} s_x$ in any dimension d. In the spinor representation, the matrices $k(x)$ read explicitly

$$k_3(x) = \begin{pmatrix} -y_3 & y_1 + iy_2 & 0 & 0 \\ -y_1 + iy_2 & -y_3 & 0 & 0 \\ 0 & 0 & x_3 & x_1 - ix_2 \\ 0 & 0 & -x_1 - ix_2 & x_3 \end{pmatrix},$$

$$k_4(x) = \begin{pmatrix} iy_3 - y_4 & y_1 - iy_2 & 0 & 0 \\ iy_1 + y_2 & -iy_3 - y_4 & 0 & 0 \\ 0 & 0 & ix_3 + x_4 & ix_1 + x_2 \\ 0 & 0 & ix_1 - x_2 & -ix_3 + x_4 \end{pmatrix}.$$

Above formulas make it easy to work out the group elements $g(x_i)$. This is the matrix we want to decompose into a product $k_l a k_r$. If we split the elements $k \in K$ explicitly into dilations and rotations, the Cartan decomposition in the spinor representation and with our notation using 2×2 matrices, assumes the form

$$g(x_i) = \begin{pmatrix} e^{\frac{1}{2}\lambda_l} & 0 \\ 0 & e^{-\frac{1}{2}\lambda_l} \end{pmatrix} r_l(x_i) \begin{pmatrix} C & S \\ S & C \end{pmatrix} r_r(x_i) \begin{pmatrix} e^{\frac{1}{2}\lambda_r} & 0 \\ 0 & e^{-\frac{1}{2}\lambda_r} \end{pmatrix}, \tag{9.9}$$

where $\exp(\lambda_l/2)$ should be read as a scalar multiple of the 2×2 identity matrix etc. The matrix in the middle is the representative of the factor $a(x_i)$. Its blocks are given by

$$C = \begin{pmatrix} \cosh \frac{u_1}{2} & 0 \\ 0 & \cosh \frac{u_2}{2} \end{pmatrix}, \quad S = \begin{pmatrix} -\sinh \frac{u_1}{2} & 0 \\ 0 & \sinh \frac{u_2}{2} \end{pmatrix}\ . \tag{9.10}$$

[1] There is an obvious abuse of notation here: if we put $\mu = 1$ in σ_μ, the resulting matrix is not the first Pauli matrix σ_1. However, there should be no confusion because μ usually appears as an abstract index.

The rotation matrices r_l and r_r are block diagonal. They read

$$r_l(x_i) = \begin{pmatrix} L(x_i) & 0 \\ 0 & L'(x_i) \end{pmatrix}, \; r_r(x_i) = \begin{pmatrix} R(x_i) & 0 \\ 0 & R'(x_i) \end{pmatrix} .$$

where L, L', R, R' are 2×2 rotation matrices with unit determinant. The coordinates λ_i and u_i can be extracted from g by considering determinants of its 2×2 blocks. Computing the four possible 2×2 subdeterminants of $g(x_i)$ yields the following relations

$$e^{4\lambda_l} = \frac{x_{12}^4 x_{13}^2 x_{14}^2}{x_{23}^2 x_{24}^2}, \qquad e^{4\lambda_r} = \frac{x_{14}^2 x_{24}^2}{x_{13}^2 x_{23}^2 x_{34}^4}, \tag{9.11}$$

and

$$\cosh^2\frac{u_1}{2}\cosh^2\frac{u_2}{2} = \frac{x_{14}^2 x_{23}^2}{x_{12}^2 x_{34}^2}, \quad \sinh^2\frac{u_1}{2}\sinh^2\frac{u_2}{2} = \frac{x_{13}^2 x_{24}^2}{x_{12}^2 x_{34}^2} . \tag{9.12}$$

These equations are equivalent to (8.13) and (8.14) that we derived in general dimension previously. Our main goal now, however, is to determine r_l and r_r. This analysis is case dependent and will be carried out separately for $d=3$ and $d=4$.

9.3 Three-dimensional spinning correlators

Elements of the three-dimensional rotation group are parametrised by three Euler angles (5.2) and since the stabiliser subgroup B is trivial in this case, we need to determine six angles in terms of the insertion points x_i by decomposing $g(x_i)$. In the four-dimensional representation

$$r_{l,r} = \begin{pmatrix} \pi_{1/2}(\phi_{l,r}, \theta_{l,r}, \psi_{l,r}) & 0 \\ 0 & \pi_{1/2}(-\phi_{l,r}, -\theta_{l,r}, -\psi_{l,r}) \end{pmatrix} . \tag{9.13}$$

When these matrices are inserted into (9.9), we obtain eight equations that allow to determine the six unknown angles in terms of x_i. Once these angles are known, we can compute the tensor factor for fields of arbitrary spin.

This may sound tedious, but it can be carried out quite efficiently. We will first see how it is done for seed correlators and then use elementary $SU(2)$ representation theory to write (9.1) for correlators of fields that have arbitrary spin.

9.3.1 Seed four-point functions

The seed correlation function that we want to look at

$$G^{\alpha}{}_{\beta}(x_i) = \langle \psi_1^{\alpha}(x_1)\phi_2(x_2)\phi_3(x_3)\psi_{4\beta}(x_4)\rangle, \tag{9.14}$$

involves two scalar fields at x_2 and x_3 of weights Δ_2 and Δ_3, respectively, along with two fermionic fields of spin 1/2 and weights Δ_1 and Δ_4. In this case, the spaces W_2 and W_3 are one-dimensional while W_1 and W_4 are two-dimensional (vectors in the spin-1/2 representation carry a Greek index α, β...). Hence, the space W of polarisations has four basis elements, as does the space $W^B = W$ of tensor structures.

We compute the two factors $\Theta_{1,2}(x_i)$ in turn. The first one is easy to determine. Since ρ_2 is trivial and ρ_4 is the spin 1/2 representation, we find

$$\Theta_1(x_i) = x_{21}^{-2\Delta_2} x_{43}^{-2\Delta_4} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \frac{1}{|x_{34}|} \begin{pmatrix} x_{34}^3 & -x_{34}^1 - i x_{34}^2 \\ x_{34}^1 - i x_{34}^2 & x_{34}^3 \end{pmatrix} . \tag{9.15}$$

Calculating $\Theta_2(x_i)$ is a bit more of a challenge. From its definition, one quickly gets

$$\Theta_2(x_i) = e^{2a\lambda_l} e^{2b\lambda_r} \pi_{1/2}(r_l) \otimes \pi_{1/2}((r_r^{-1})^w) . \tag{9.16}$$

Through an explicit computation one may show that the product of the last two factors can be expressed in terms of g itself in the four-dimensional representation as

$$\pi_{1/2}(r_l) \otimes \pi_{1/2}((r_r^{-1})^w) = p(g) D_1^{-1} p(a)^{-1}. \tag{9.17}$$

Here, p is the following linear map on the space of 4×4 matrices M,

$$p(M) = \begin{pmatrix} -M_{13} & M_{12} & -M_{43} & -M_{42} \\ M_{14} & M_{11} & M_{44} & -M_{41} \\ -M_{23} & M_{22} & M_{33} & M_{32} \\ M_{24} & M_{21} & -M_{34} & M_{31} \end{pmatrix} . \tag{9.18}$$

Finally, the matrix D_1 reads

$$D_1 = \mathrm{diag}\Big(e^{\frac{\lambda_1 - \lambda_2}{2}}, e^{\frac{\lambda_1 + \lambda_2}{2}}, e^{\frac{-\lambda_1 - \lambda_2}{2}}, e^{\frac{-\lambda_1 + \lambda_2}{2}} \Big) . \tag{9.19}$$

The simplicity of the relation (9.16) follows from the fact that in the four-dimensional representation of the conformal group, rotations are represented essentially as in the spin-1/2 representation of $SU(2)$ in terms of Pauli matrices. The identity holds true for any element g that admits a Cartan decomposition. Once it is applied to special elements $g(x_i)$, we obtain $\Theta_2(x_i)$. More precisely, one also needs to insert expressions for u_i and λ_i in terms of insertion points into (9.16) and the formula for D_1. This completes the construction of $\Theta_2(x_i)$ and hence, along with the expression (9.15) for $\Theta_1(x_i)$, of the tensor structures for seed four-point functions in three dimensions.

9.3.2 Tensor structures for arbitrary spins

The tensor structures for seed correlators we just constructed and in particular the factor $\Theta_2(x_i)$ were so simple because $\Theta_2(x_i)$ contained the same combinations of left and right Euler angles as the matrix elements of $g(x_i)$ in the four-dimensional representation. When dealing with more general spinning fields, we need to construct the left and right Euler angles separately in terms of the insertion points. This is possible, but the formulas are a bit more cumbersome than ones above. Starting from the 4×4 matrix defined in (9.16), we can reconstruct the individual tensor factors as

$$\pi_{1/2}(r_l) = \begin{pmatrix} \sqrt{\frac{\varpi_{33}}{\varpi_{11}}(\varpi_{11}\varpi_{44} - \varpi_{12}\varpi_{43})} & i\sqrt{\frac{\varpi_{13}}{\varpi_{31}}(1 - \varpi_{11}\varpi_{44} + \varpi_{12}\varpi_{43})} \\ i\sqrt{\frac{\varpi_{31}}{\varpi_{13}}(1 - \varpi_{11}\varpi_{44} + \varpi_{12}\varpi_{43})} & \sqrt{\frac{\varpi_{11}}{\varpi_{33}}(\varpi_{11}\varpi_{44} - \varpi_{12}\varpi_{43})} \end{pmatrix} , \tag{9.20}$$

and

$$\pi_{1/2}((r_r^{-1})^w) = \begin{pmatrix} \sqrt{\frac{\varpi_{22}}{\varpi_{11}}(\varpi_{22}\varpi_{33} - \varpi_{13}\varpi_{42})} & i\sqrt{\frac{\varpi_{12}}{\varpi_{21}}(1 - \varpi_{22}\varpi_{33} + \varpi_{13}\varpi_{42})} \\ i\sqrt{\frac{\varpi_{21}}{\varpi_{12}}(1 - \varpi_{22}\varpi_{33} + \varpi_{13}\varpi_{42})} & \sqrt{\frac{\varpi_{11}}{\varpi_{22}}(\varpi_{22}\varpi_{33} - \varpi_{13}\varpi_{42})} \end{pmatrix} . \tag{9.21}$$

Here $\varpi(x_i)$ is the inverse of the matrix (9.16). Note that the matrix elements of ϖ and hence the matrix entries of $\pi_{1/2}(r_l)$ and $\pi_{1/2}((r_r^{-1})^w)$ are functions of the insertion points x_i.

Having constructed the factors $\Theta_2(x_i)$ for fields in the fundamental two-dimensional representation of $SU(2)$, we can now obtain these factors for all other representations with the help of some standard group theory. From the matrix elements of the fundamental representation, we can obtain matrix elements of any other irreducible representation as

$$t^l_{mn}(g) = (-1)^{m-n}\sqrt{\frac{(l+m)!(l-m)!}{(l+n)!(l-n)!}}\tau_{22}^{m+n}\tau_{21}^{m-n}P_{l-m}^{(m-n,m+n)}(\tau_{11}\tau_{22}+\tau_{12}\tau_{21}) . \tag{9.22}$$

Suppose now that we want to find the tensor structures for correlators in which we insert a field of spin l at x_1, while keeping the field at x_2 to be scalar. Then the corresponding matrix factor $\rho_1(r_l)\rho_2(r_l)^w$ can be obtained by inserting the matrix elements of the matrix defined in (9.20) into the previous formula. Other spin assignments may be dealt with similarly and hence, our equations (9.20), (9.21) and (9.22) completely solve the problem of constructing tensor structures for all three-dimensional spinning correlators in the Calogero-Sutherland gauge.

Remark While the above prescription for the evaluation of tensor structures is entirely explicit, the resulting formulas are still relatively complicated. We believe that there is an inherent complexity in tensor structures which no method of calculation can circumvent. For some purposes, one actually does not need the tensor structures - this was the topic of the last chapter.

9.3.3 Comparison with the literature

Let us compare our findings with those of [39]. The constructions in that work are performed in the Minkowski space with the metric

$$g_{\mu\nu} = \mathrm{diag}(-1,1,1) .$$

Greek indices μ,ν... are raised and lowered with this metric. Greek indices α,β... from the beginning of the alphabet are raised and lowered with the Levi-Civita symbol according to

$$\psi_\alpha = \varepsilon_{\alpha\beta}\psi^\beta, \quad \varepsilon_{12} = 1 .$$

A vector x^μ is turned into a 2×2 matrix with the help of three-dimensional gamma matrices

$$x^\alpha{}_\beta = x_\mu(\gamma^\mu)^\alpha{}_\beta, \quad (\gamma^\mu)^\alpha{}_\beta = (i\sigma_2,\sigma_1,\sigma_3) .$$

The correlation function of two scalar and two spinor fields then assumes the form

$$\langle\psi_1^\alpha(x_1)\varphi_2(x_2)\varphi_3(x_3)\psi_4^\beta(x_4)\rangle = \Omega(x_i)\sum_{I=1}^{4} t_I^{\alpha\beta}g^I(z,\bar z),$$

where the index $I = 1, \dots, 4$ runs over the following four four-point tensor structures

$$t_1^{\alpha\beta} = i\frac{(x_{14} i\sigma_2)^{\alpha\beta}}{|x_{14}|}, \quad t_2^{\alpha\beta} = -i\frac{(x_{12}x_{23}x_{34} i\sigma_2)^{\alpha\beta}}{|x_{12}||x_{23}||x_{34}|}, \quad t_3^{\alpha\beta} = i\frac{(x_{12}x_{24} i\sigma_2)^{\alpha\beta}}{|x_{12}||x_{24}|}, \quad t_4^{\alpha\beta} = i\frac{(x_{13}x_{34} i\sigma_2)^{\alpha\beta}}{|x_{13}||x_{34}|} .$$

The matrix $T = (t_I^{\alpha\beta})$ that collects these tensor structures bears some resemblance to the matrix $p(g(x_i))$ defined above. More precisely one can see that

$$\Omega(x_i)\, T\, \chi = \Theta_1(x_i)\, e^{2a\lambda_l} e^{2b\lambda_r}\, p(g(x_i))\, D_1^{-1}\, p(a(x_i))^{-1} = \Theta_1(x_i)\Theta_2(x_i),$$

where the matrix χ takes the form

$$\chi = \begin{pmatrix} 0 & A(\chi_1-\chi_2) & A(-\chi_1-\chi_2) & 0 \\ 0 & A(-\chi_1-\chi_2) & A(\chi_1-\chi_2) & 0 \\ B(\chi_3-\chi_4) & 0 & 0 & B(\chi_3+\chi_4) \\ B(-\chi_3-\chi_4) & 0 & 0 & B(-\chi_3+\chi_4) \end{pmatrix} . \tag{9.23}$$

Here, $A = -i/2$ and $B = -1/2$. The functions $\chi_i(u_1, u_2)$ are defined in (A.10) of [9], with $x = u_1$ and $y = u_2$. In order to check (9.23), it is useful to first establish

$$\frac{\Omega(x_i)}{\tilde{\Omega}(x_i)}\, \chi\, p(a(x_i)) = \begin{pmatrix} 0 & v^{\frac{1}{4}} & 0 & 0 \\ 0 & 0 & v^{\frac{1}{4}} & 0 \\ 0 & 0 & 0 & u^{-\frac{1}{4}} \\ u^{-\frac{1}{4}} & 0 & 0 & 0 \end{pmatrix} .$$

The matrix χ thus relates conformal blocks in coordinate space and the CS gauge. This fact was observed already in [9] by comparing associated sets of Casimir equations. [2] With this remark, we conclude the discussion of tensor structures for three-dimensional four-point functions.

9.4 Four-dimensional spinning seed correlators

Our goal in this section is to construct the tensor structures for the simplest non-trivial seed correlators in four dimensions. In order to do so, we follow the steps described previously. In the four-dimensional theory, we need to determine 11 Euler angles in total. The rotation group Spin(4) itself is six-dimensional, so left and right rotations together are parametrised by 12 angles. However, one of these angles is redundant because of the non-trivial stabiliser subgroup $B \cong \mathrm{Spin}(2)$. To be specific, we parametrise the left and right rotations r_l and r_r as

$$r_l = e^{\varphi_1^l X_1} e^{\theta_1^l Z_1} e^{\psi_1^l X_1} e^{\varphi_2^l X_2} e^{\theta_2^l Z_2} e^{\psi_2^l X_2} , \quad r_r = e^{\varphi_1^r X_1} e^{\theta_1^r Z_1} e^{\psi_1^r X_1} e^{\varphi_2^r X_2} e^{\theta_2^r Z_2} e^{\psi_2^r X_2} . \tag{9.24}$$

In order to reduce down to 11 angles, we impose the additional condition $\psi_2^l = -\psi_1^l$. The symbols X_i and Z_i with $i = 1, 2$ denote the following linear combinations of the rotation matrices M_{ij}

$$X_1 = -\frac{1}{2}(M_{12}+M_{34}),\ X_2 = \frac{1}{2}(M_{12}-M_{34}),\ Z_1 = -\frac{1}{2}(M_{14}+M_{23}),\ Z_2 = \frac{1}{2}(M_{14}-M_{23}) .$$

[2] The comparison of Casimir equations that was used in [9] to determine χ does not determine the numerical factors A and B. The map that was spelled out in that work corresponds to $A = 1/\sqrt{2} = B$.

We shall study correlation functions that involve the non-trivial four-dimensional seed blocks of [37]. These blocks appear in the decomposition of

$$G_4(x_i)^b{}_{\dot a} = \langle \Phi_{0,0}(x_1)\Phi_{s,0}(x_2)\Phi_{0,0}(x_3)\Phi_{0,s}(x_4)\rangle, \tag{9.25}$$

where $s \in (0, 1/2, 1, \dots)$. Labels $(s_1, s_2) = (j_1, j_2)$ that we attached to the operators Φ_{s_1,s_2} refer to the representation of the rotation group. We consider the case with $s = 1/2$. The corresponding representations W_2 and W_4 are then both two-dimensional. Their vectors are written with undotted and dotted Latin indices, respectively. The space of polarisations W has dimension four and the space of B-invariants is two-dimensional.

We turn to the construction of factors $\Theta_{1,2}(x_i)$. The first one is easily found

$$\Theta_1(x_i) = \frac{x_{21}^{-2\Delta_2}x_{43}^{-2\Delta_4}}{|x_{12}||x_{34}|}\begin{pmatrix} x_{21}^4 - ix_{21}^3 & x_{21}^2 - ix_{21}^1 \\ -x_{21}^2 - ix_{21}^1 & x_{21}^4 + ix_{21}^3 \end{pmatrix} \otimes \begin{pmatrix} x_{34}^4 - ix_{34}^3 & x_{34}^2 + ix_{34}^1 \\ -x_{34}^2 + ix_{34}^1 & x_{34}^4 + ix_{34}^3 \end{pmatrix} . \tag{9.26}$$

This is obtained by evaluating the rotation $s_x s_{e_d}$ in the two-dimensional representations $(1/2, 0)$ and $(0, 1/2)$ of the rotation group. Calculating $\Theta_2(x_i)$ is a bit more involved. Similarly to the three-dimensional case

$$\Theta_2(x_i) = e^{2a\lambda_l}e^{2b\lambda_r}\pi_{(1/2,0)}(r_l^w) \otimes \pi_{(0,1/2)}((r_r^{-1})^w) . \tag{9.27}$$

With the Cartan coordinates as introduced above, the last two factors give

$$\pi_{(1/2,0)}(r_l^w) \otimes \pi_{(0,1/2)}((r_r^{-1})^w) = \pi_{1/2}(-\phi_2^l, \theta_2^l, -\psi_2^l) \otimes \pi_{1/2}(\psi_1^r, -\theta_1^r, \phi_1^r) \equiv \hat\pi(r_l, r_r),$$

where $\psi_2^l = -\psi_1^l$. We need to calculate this tensor product as a function of the insertion points x_i to obtain the main building block for the desired tensor structure. As in the previous section, the resulting expression is quite simple. It requires to introduce the following linear map $q : M_{2\times2} \times M_{2\times2} \to M_{2\times4}$ that sends a pair of 2×2 matrices M, N to a rectangular 2×4 matrix of the form

$$q(M,N) = \begin{pmatrix} r(M) \\ r(N) \end{pmatrix}, \quad r(M) = \begin{pmatrix} M_{12} & M_{11} & M_{22} & M_{21} \end{pmatrix} .$$

From the four 2×2 matrix blocks of the group element g in the four-dimensional representation we can construct a pair of 2×2 matrices $M = M_2$ and $N = M_1$ as

$$M_i = \sinh^2\frac{u_i}{2}DB^{-1}A - \cosh^2\frac{u_i}{2}C . \tag{9.28}$$

In terms of these two matrices one can now compute $\hat\pi$ as

$$\hat\pi(r_l, r_r)^{-1} := \mathcal{P}(\pi_{(1/2,0)}(r_l^w)\otimes\pi_{(0,1/2)}((r_r^{-1})^w))^{-1} = \frac{2e^{\frac{\lambda_1-\lambda_2}{2}}}{\cosh u_1 - \cosh u_2}\begin{pmatrix} \sinh\frac{u_1}{2} & 0 \\ 0 & \sinh\frac{u_2}{2} \end{pmatrix} q(M_2, M_1) . \tag{9.29}$$

Here, $\mathcal{P}$ is the projector to the space of B-invariants

$$\mathcal{P} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} .$$

Note that the matrices $\hat{\pi}$ and $\hat{\pi}^{-1}$ are not square. Rather, they are pseudo-inverses of one another in the sense that

$$\hat{\pi}(r_l, r_r)^{-1}\hat{\pi}(r_l, r_r) = I_2 \ .$$

The identity (9.29) holds for any element of the conformal group that has a Cartan decomposition, as can be checked by calculating both sides. Tensor structures arise upon substituting the expression for $g(x_i)$ into (9.29).

9.4.1 Comparison with the literature

The four-point function (9.25) was analysed in [37, 26] and we will quickly review results of these works. They are done in Lorentzian signature with the metric

$$g_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1) \ .$$

Greek indices from the second half of the alphabet are raised and lowered with this metric g. Indices α, β... from the beginning of the Greek alphabet label a basis of the two-dimensional representation $(1/2, 0)$ of $SO(1,3)$. Similarly, the dotted indices $\dot{\alpha}, \dot{\beta}$... enumerate a basis in the representation $(0, 1/2)$. These are raised and lowered with the Levi-Civita symbol according to

$$\psi_\alpha = \varepsilon_{\alpha\beta}\psi^\beta, \quad \varepsilon_{12} = -1 \ .$$

The same formulas hold for the dotted indices. The vector representation is equivalent to the tensor product $(1/2, 0) \otimes (0, 1/2)$. This equivalence can be realised explicitly with the help of σ-matrices

$$x_{\alpha\dot{\beta}} = x_\mu \sigma^\mu_{\alpha\dot{\beta}}, \quad \sigma^\mu_{\alpha\dot{\beta}} = (-I, \sigma^i) \ .$$

Further, we write $(\bar{\sigma}^\mu)^{\dot{\alpha}\beta} = (-I, -\sigma^i)$ and define the corresponding $\bar{x}^{\dot{\alpha}\beta}$ in the obvious way.

With the notation set up, let us now look at the correlation function (9.25). It decomposes over the tensor structures introduced above as[3]

$$G_4^{\dot{\alpha}\beta}(x_i) = \mathcal{K}_4(x_i) T_I^{\dot{\alpha}\beta}(x_i)\, g^I(u,v) = \mathcal{K}_4(x_i) \sum_{e=0}^{2s} g_e^{(2s)}(u,v) I_{42}^e J_{42,31}^{p-e},$$

where the scalar prefactor $\mathcal{K}_4$ is defined by

$$\mathcal{K}_4(x_i) = \Omega(x_i)\sqrt{\frac{x_{13}}{x_{12}x_{24}x_{34}}},$$

and the tensors I and J on the right hand side take the form

$$I_{ij}^{\dot{\alpha}\beta} = x_{ij}^\mu \bar{\sigma}_\mu^{\dot{\alpha}\beta}, \quad J_{ij,kl}^{\dot{\alpha}\beta} = \frac{2}{x_{kl}^2} \bar{x}_{ik}^{\dot{\alpha}\gamma} (x_{kl})_{\gamma\dot{\delta}} \bar{x}_{lj}^{\dot{\delta}\beta} \ .$$

The case we analyse corresponds to $2s = p = 1$ and hence the correlation function has the form

$$G_4^{\dot{\alpha}\beta}(x_i) = \mathcal{K}_4(x_i)\Big(g_0^{(1)}(u,v) J_{42,31}^{\dot{\alpha}\beta} + g_1^{(1)}(u,v) I_{42}^{\dot{\alpha}\beta}\Big) \ .$$

[3] Following the conventions in [37] we label seed blocks by an integer $p = 2s$ rather than the spin s itself.

There are two tensor structures, in agreement with the dimension of the space W^B in our analysis.

In order to compare the tensor structures from [37] with our tensor structures given in (9.26) and (9.27), we note that the two discussions of the seed correlators use a different basis in the space of polarisations. Comparing the conventions in [26] and [10] one can see that the basis transformation is mediated by the following matrix

$$M^{b\ \beta}_{\ \dot{a}\ \dot{\alpha}} = \frac{i}{2}\begin{pmatrix} 1 & 1 & -1 & -1 \\ -1 & -1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ -1 & 1 & -1 & 1 \end{pmatrix} .$$

Taking this necessary change of basis into account we have to prove that

$$\frac{1}{2\sqrt{2}} M\Omega(x_i)\, T(x_i)\, S = \Theta_1(x_i)\Theta_2(x_i) \ . \tag{9.30}$$

It is not difficult to check that this equation is indeed satisfied with S given by

$$S(u_1, u_2) = \frac{2\sqrt{2}\left(\tanh\frac{u_1}{2}\tanh\frac{u_2}{2}\right)^{\frac{1}{2}+a+b}}{(\cosh u_1 - \cosh u_2)^2}\begin{pmatrix} \sinh\frac{u_1}{2} & \sinh\frac{u_2}{2} \\ \frac{2}{\sinh\frac{u_1}{2}} & \frac{2}{\sinh\frac{u_2}{2}} \end{pmatrix} . \tag{9.31}$$

This coincides with the map between Calogero-Sutherland eigenfunctions and conformal blocks that was found in [10] based on the comparison of the Casimir differential equations.

Chapter 10

Superconformal partial waves

Casimir equations of Dolan and Osborn are probably the best known characterisation of conformal partial waves. From their works [29, 30] one can take at least two important messages. Firstly, there *is* a differential equation that characterises the partial waves and can be derived in a relatively straightforward way, and secondly, the equation simplifies considerably once one changes coordinates from (u, v) to $(z, \bar{z})$. The second of these facts could be *explained* by noticing that $(z, \bar{z})$ are coordinates of x_2 once x_1, x_3 and x_4 are sent to 0, e_1 and ∞ using conformal transformations. Therefore, $(z, \bar{z})$ have, in some sense, a geometric origin. We want to take the idea that not all coordinate systems are good coordinate systems seriously.

The Casimir differential operator of Dolan and Osborn is constructed as the quadratic Casimir built out of sums of operators that represent the action of $\mathfrak{g}$ on fields 1 and 2 in the correlator. When we map the four-point function $G_4(x_i)$ to a K-spherical function F using (7.40), the Casimir is carried to the Laplace-Beltrami operator on the (super)conformal group. This is not difficult to see and will be shown in a later chapter. Thus, we are led to study the reduction of the Laplacian to the space of K-spherical functions. For bosonic conformal groups and covariance properties characterised by one-dimensional representations of K, the resulting operator was seen to be conjugate to the BC_2 Calogero-Sutherland Hamiltonian.

The situation becomes more complicated if we allow for higher-dimensional representations of K. Via (7.40), this corresponds to considering correlation functions of spinning fields. Any pair of K-modules (ρ_l, ρ_r) leads to a Schrödinger problem for functions in two variables with $\dim(\rho_l \otimes \rho_r)^B$ components. As far as we know, there is no general theory of all such Schrödinger problems. An idea that has proved fruitful in the CFT community is to construct a set of covariant differential operators that produce spinning conformal blocks by acting on scalar ones. In the CS gauge, some of these *weight-shifting operators* are obtained from invariant vector fields on G. They can be used to construct eigenfunctions of infinitely many matrix Calogero-Sutherland Hamiltonians, although not all of them.

A lot less seems to be known about superconformal partial waves. From the physics side, they have been computed in a number of cases, which all however contain considerable simplifying assumptions. One usually either considers correlation functions where all field multiplets are short, or allows for long multiplets but only seeks for the superprimary component of the blocks (i.e. sets all Grassmann variables to zero). Finally, the full blocks with all representations being long have been computed in some cases, but all these cases are essentially one-dimensional. In mathematics, spherical functions on supergroups received much less attention than those on Lie groups.

In view of these comments, it makes sense to try and reduce the computation of superconformal blocks to that of spinning bosonic blocks. Or, in other words, to construct K-spherical harmonics on a supergroup from those on its underlying Lie group. This is the problem addressed in the present chapter.

Our result is a method that gives K-spherical harmonics on G as finite sums of (a small number of) K-spherical harmonics on $G_{(0)}$. Terms that appear in the sum are easy to determine from representation theory of $G_{(0)}$. The coefficients that multiply these terms are also easy to *define* as certain Clebsch-Gordan coefficients, but it may still be difficult to compute these in practice. From the point of view of Schrödinger problems, the potential of the super Calogero-Sutherland Hamiltonian equals that of a matrix CS Hamiltonian with two corrections. The first correction is a set of constants along the diagonal and the second is an upper triangular matrix of functions. Notice that if the bosonic Hamiltonian H_0 was hermitian with respect to the standard inner product on vector-valued wavefunctions, the supersymmetric one H will not be so (one can make the latter operator hermitian by changing the inner product in a way dictated by the Berezin integral on G. However, we shall not do this.). Regardless of this fact, it is possible to apply the standard quantum mechanical perturbation theory and construct eigenfunctions of H by starting from those of H_0. Since the perturbation is by a nilpotent operator, the procedure gives exact results at a finite order.

The structure of the supersymmetric Hamiltonian H follows from the form of the Laplacian on G. The required split into a bosonic piece and two simple correction terms is a property satisfied by Laplacians on supergroups of type I (and is exhibited in unprimed Cartan coordinates). Thus, our results in this chapter only apply to such supergroups.

The chapter is organised as follows. We begin by giving two examples of bosonic matrix CS Hamiltonians. They appear in relation with three- and four-dimensional seed correlators that were studied in the last chapter. Next, some properties of scalar and seed blocks, such as their symmetries, asymptotic behaviour and identities that follow from their description as spherical functions, are discussed. In the second section we derive the expression for the Laplacian on a supergroup of type I. The third one reviews quantum mechanical perturbation theory, specialising to the case of a nilpotent perturbation. The rest of the chapter treats two examples. We compute partial waves for four-point functions of long multiples in one-dimensional $\mathcal{N}=2$ theories and certain multiplets involving long and short operators in four-dimensional $\mathcal{N}=1$ theories. The chapter is mostly based on the articles [1, 6].

10.1 Matrix Calogero-Sutherland models

Let us consider a four-point function of some arbitrary spinning fields in a bosonic CFT. As usual, we denote the spaces of field polarisations by $W_1, ..., W_4$. The spherical function F that (7.40) associates to $G_4(x_i)$ is left K-covariant according to $W_l = W_1 \otimes W_2'$ and right K-covariant according to $W_r = W_3 \otimes W_4'$. Recall that W' denotes the representation of K obtained from W by conjugation with the Weyl inversion. The function F is uniquely determined by its restriction to A and moreover the restriction takes values in the space of invariants $T = (W_l \otimes W_r)^B$. Conversely, any function $f : A \to T$ extends to a K-spherical function.

It is in principle an easy matter to compute how the group Laplacian acts on f for any given pair of modules (W_l, W_r) (at least in three and four dimensions. The complexity of the calculation grows for larger groups.). Upon conjugation with $\omega^{1/2}$ this gives rise to the corresponding matrix Calogero-Sutherland Hamiltonian. Let us state the results from the literature for the two examples that we considered when studying four-point tensor structures in the last chapter.

10.1.1 Two matrix Hamiltonians

The Hamiltonian for the three-dimensional seed correlator (9.14) has the block-diagonal[1] form $H = \text{diag}\{H_1^{3d}, H_2^{3d}\}$ with

$$H_1^{3d} = \begin{pmatrix} \frac{1}{2}H_{PT}^{(a+\frac{1}{4},b-\frac{1}{4})}(u_1) + \frac{1}{2}H_{PT}^{(a-\frac{1}{4},b+\frac{1}{4})}(u_2) & -\frac{1}{2}V \\ -\frac{1}{2}V & \frac{1}{2}H_{PT}^{(a-\frac{1}{4},b+\frac{1}{4})}(u_1) + \frac{1}{2}H_{PT}^{(a+\frac{1}{4},b-\frac{1}{4})}(u_2) \end{pmatrix} + \frac{U}{4} + \frac{5}{8}, \tag{10.1}$$

and

$$H_2^{3d} = \begin{pmatrix} \frac{1}{2}H_{PT}^{(a+\frac{1}{4},b+\frac{1}{4})}(u_1) + \frac{1}{2}H_{PT}^{(a-\frac{1}{4},b-\frac{1}{4})}(u_2) & -\frac{1}{2}V' \\ -\frac{1}{2}V' & \frac{1}{2}H_{PT}^{(a-\frac{1}{4},b-\frac{1}{4})}(u_1) + \frac{1}{2}H_{PT}^{(a+\frac{1}{4},b+\frac{1}{4})}(u_2) \end{pmatrix} + \frac{U}{4} + \frac{5}{8}\ . \tag{10.2}$$

Here, the functions V and V' read

$$V = \frac{(2 + \cosh u_1 + \cosh u_2)\sinh\frac{u_1}{2}\sinh\frac{u_2}{2}}{4\sinh^2\frac{u_1+u_2}{2}\sinh^2\frac{u_1-u_2}{2}}, \quad V' = \frac{(-2 + \cosh u_1 + \cosh u_2)\cosh\frac{u_1}{2}\cosh\frac{u_2}{2}}{4\sinh^2\frac{u_1+u_2}{2}\sinh^2\frac{u_1-u_2}{2}},$$

and U is given by

$$U = \frac{1}{4\sinh^2\frac{u_1+u_2}{2}} + \frac{1}{4\sinh^2\frac{u_1-u_2}{2}}\ . \tag{10.3}$$

The second Hamiltonian that we spell out arises in relation with the four-dimensional correlator (9.25) with $s = 1/2$. It is similar to operators H_i^{3d} from above and can be compactly written in terms of the same functions

$$H_{\frac{1}{2}}^{a,b} = \begin{pmatrix} \frac{1}{2}H_{PT}^{(a+\frac{1}{4},b-\frac{1}{4})}(u_1) + \frac{1}{2}H_{PT}^{(a-\frac{1}{4},b+\frac{1}{4})}(u_2) & V \\ V & \frac{1}{2}H_{PT}^{(a-\frac{1}{4},b+\frac{1}{4})}(u_1) + \frac{1}{2}H_{PT}^{(a+\frac{1}{4},b-\frac{1}{4})}(u_2) \end{pmatrix} + U + \frac{19}{16}\ . \tag{10.4}$$

10.1.2 Calogero-Sutherland wavefunctions

This section is dedicated to various properties of Calogero-Sutherland eigenfunctions. In particular, we will study their asymptotic behaviour as $u_i \to \infty$, symmetries with respect to their arguments and parameters and also some special identities that are derived using Clebsch-Gordan decompositions for $G_{(0)}$. All these properties will be important for the analysis of superconformal blocks.

[1] In the basis we use throughout this work, the first block matrix H_1 actually appears in the second and third row/column while H_2 acts on the subspace spanned by the first and fourth basis vector

Our main focus will be on partial waves in four dimensions. So, let us begin by reviewing some of the known facts about them. To write the blocks for non-identical scalars, we introduce a generalisation of the function $k_{2\rho}$ given in (1.10) by

$$k_{2\rho}^{(a,b;c)}(x) = x^\rho\ {}_2F_1(a+\rho, b+\rho; c+2\rho; x)\ . \tag{10.5}$$

The partial waves are expressed in terms of these functions as

$$g_{\Delta,l}^{(4d)}(z_1,z_2) = \frac{z_1 z_2}{z_1 - z_2}\left(k_{\Delta+l}^{(a,b;0)}(z_1)k_{\Delta-l-2}^{(a,b;0)}(z_2) - (z_1 \leftrightarrow z_2)\right)\ . \tag{10.6}$$

Conformal blocks for arbitrary spinning fields can be constructed in a systematic way from the so-called *seed conformal blocks*. These are labelled by a positive integer p and have been computed in a large number of cases in [37]. Seed conformal blocks $G_e^{(p)}$ appear in the decomposition of four-point functions (9.25) that involve two scalar and two spinning fields. They take the general form

$$G_e^{(p)}(z_1,z_2) = \left(\frac{z_1 z_2}{z_1 - z_2}\right)^{2p+1} \sum_{m,n} c_{mn}^e \mathcal{F}_{\rho_1+m,\rho_2+n}^{-,(a_e,b_e;c_e)}(z_1,z_2)\ . \tag{10.7}$$

Here, $e = 0,1,...,p$ are components of the block $G^{(p)}$, the sum runs over a finite set of points (m,n) and functions $\mathcal{F}^\pm$ are defined as

$$\mathcal{F}_{\rho_1,\rho_2}^{\pm,(a,b;c)} = \left(k_{2\rho_1}^{(a,b;c)}(z_1)k_{2\rho_2}^{(a,b;c)}(z_2) \pm (z_1 \leftrightarrow z_2)\right)\ . \tag{10.8}$$

For the parameters of these functions that appear in the sum (10.7) and coefficients c_{mn}^e, the reader is referred to [37]. The seed block is associated to the propagation of a field with spins $(2j_1, 2j_2) = (l, l+p)$ in the OPE. The contribution of the field with spins $(2j_1,2j_2) = (l+p,l)$ is captured by the conjugate seed block $\bar{G}^{(p)}$ that has an expansion similar to (10.7). In what follows, we will denote the lowest seed blocks with $p=1$ by $G_+ = (G_e^{(1)})$ and their conjugates by $G_- = (\bar{G}_e^{(1)})$.

In order to characterise any conformal block, the Casimir equations that it solves have to be supplemented by boundary conditions. What the appropriate conditions are can be determined in the OPE limit, $z_i \to 0$. In this limit, the scalar blocks behave as

$$g_{\Delta,l}^{(4d)}(z_1,z_2) \sim z_1^{\frac{\Delta-l}{2}} z_2^{\frac{\Delta+l}{2}} \quad \text{as} \quad z_i \to 0\ . \tag{10.9}$$

We write $\sim$ to indicate that the ratio of two sides is finite in the limit, and not necessarily equal to one. The asymptotics of seed blocks were derived in [37]. For $p=1$ and assuming $l \geq 1$, we have

$$G_e^{(1)} \sim z_1^{\frac{\Delta-l}{2}+\frac{1}{4}} z_2^{\frac{\Delta+l}{2}-\frac{3}{4}+e},\quad \bar{G}_e^{(1)} \sim z_1^{\frac{\Delta-l}{2}-\frac{1}{4}+e} z_2^{\frac{\Delta+l}{2}-\frac{1}{4}} \quad \text{as} \quad z_i \to 0\ . \tag{10.10}$$

The case of zero spin has to be treated separately, see [37].

Let us now map the scalar and seed blocks to Calogero-Sutherland eigenfunctions. For the scalar ones, it is convenient to use a hypergeometric identity and write $\mathcal{F}^-$ as

$$\mathcal{F}_{\rho_1,\rho_2}^{\pm,(a,b;c)}(z_1,z_2) = \left(\coth\frac{u_1}{2}\right)^{-2a}\left(-\cosh\frac{u_1}{2}\right)^{-2\rho_1}\ {}_2F_1\left(a+\rho_1, c-b+\rho_1; c+2\rho_1; \frac{1}{\cosh^2\frac{u_1}{2}}\right)$$
$$\left(\coth\frac{u_2}{2}\right)^{-2a}\left(-\cosh\frac{u_1}{2}\right)^{-2\rho_2}\ {}_2F_1\left(a+\rho_2, c-b+\rho_2; c+2\rho_2; \frac{1}{\cosh^2\frac{u_2}{2}}\right) \pm (u_1 \leftrightarrow u_2)\ .$$

The Calogero-Sutherland wavefunctions are obtained by applying (9.2) to the Dolan-Osborn blocks (10.6). This gives us the functions

$$\phi^{a,b}_{\Delta,l}(u_1,u_2) = \left(\coth\frac{u_1}{2}\coth\frac{u_2}{2}\right)^{a+b+\frac{1}{2}} \sinh\frac{u_1}{2}\sinh\frac{u_2}{2}\ \mathcal{F}^{-(a,b;0)}_{\frac{\Delta+l}{2},\frac{\Delta-l-2}{2}} \tag{10.11}$$

$$= C(a,b,\Delta,l)\left(\Psi^{(a,b)}_{\frac{1-\Delta-l}{2}}(u_1)\Psi^{(a,b)}_{\frac{3-\Delta+l}{2}}(u_2) - \Psi^{(a,b)}_{\frac{1-\Delta-l}{2}}(u_2)\Psi^{(a,b)}_{\frac{3-\Delta+l}{2}}(u_1)\right), \tag{10.12}$$

where

$$C(a,b,\Delta,l) = (-1)^{-4\Delta}4^{-2a-1}\frac{\Gamma\left(a+\frac{3-\Delta-l}{2}\right)\Gamma\left(a+\frac{5-\Delta+l}{2}\right)\Gamma\left(-b+\frac{3-\Delta-l}{2}\right)\Gamma\left(-b+\frac{5-\Delta+l}{2}\right)}{\Gamma(1-\Delta-l)\Gamma(3-\Delta+l)\Gamma(a-b+1)^2}\ . \tag{10.13}$$

According to previous chapters, the $\phi^{a,b}_{\Delta,l}$ should be eigenfunctions of the scalar CS Hamiltonian $H^{a,b}_{sc} = \frac{1}{2}H^{(a,b,2)}_{cs} + \frac{5}{4}$. Indeed, this operator is a sum of two Pöschl-Teller Hamiltonians in u_1 and u_2, so the above are manifestly its eigenfunctions. We read off the eigenvalues

$$H^{a,b}_{sc}\,\phi^{a,b}_{\Delta,l} = C^{sc}_{\Delta,l}\,\phi^{a,b}_{\Delta,l}, \qquad C^{sc}_{\Delta,l} = -\frac{1}{4}\Delta(\Delta-4) - \frac{1}{4}l(l+2)\ . \tag{10.14}$$

The CS wavefunctions related to seed blocks $G_\pm$ are obtained by applying the map S from (9.31)

$$\Psi^{a,b}_{\pm,\Delta,l}(u_i) = S^{-1}(u_i)G_\pm(u_i)\ . \tag{10.15}$$

These $\Psi^{a,b}_{\pm,\Delta,l}$ are eigenfunctions of the seed CS Hamiltonian of the last section. They have the same eigenvalue

$$H^{a,b}_{\frac{1}{2}}\,\Psi^{a,b}_{\pm,\Delta,l} = C^{seed}_{\Delta,l}\,\Psi^{a,b}_{\pm,\Delta,l}, \qquad C^{seed}_{\Delta,l} = -\frac{1}{4}\Delta(\Delta-4) - \frac{1}{4}l(l+3) - \frac{3}{8}\ . \tag{10.16}$$

Wavefunctions $\Psi^{(a,b)}_\lambda$, $\phi^{a,b}_{\Delta,l}$ and $\Psi^{a,b}_{\pm,\Delta,l}$ have many satisfactory properties. We start the discussion of them with some elementary symmetries under transformations of parameters of the Calogero-Sutherland model. Notice first that Hamiltonians $H^{a,b}_{sc}$ and $H^{a,b}_{\frac{1}{2}}$ admit no automorphism of the form

$$H \mapsto U^{-1}HU\ . \tag{10.17}$$

Here U is an invertible function in the scalar case and a 2×2 matrix of functions in the seed case. The only exception occurs when $a=b$. Then the seed Hamiltonian is invariant under conjugation by $U=\sigma_1$.

The Pöschl-Teller potential is invariant under transformations $(a,b)\mapsto(b,a)$ and $(a,b)\mapsto(-a,-b)$. Eigenfunctions $\Psi^{(a,b)}_\lambda$ are not invariant under these, but are invariant under their product, i.e.

$$\Psi^{(a,b)}_\lambda(u) = \Psi^{(-b,-a)}_\lambda(u)\ . \tag{10.18}$$

The scalar Hamiltonian $H^{a,b}_{sc}$ is the sum of two Pöschl-Teller Hamiltonians in independent variables u_1 and u_2. Therefore, it enjoys all the symmetries of $H^{(a,b)}_{PT}$, together with the additional one, $(u_1,u_2)\mapsto(u_2,u_1)$. Under this transformation the wavefunctions are antisymmetric. They enjoy

$$\phi^{a,b}_{\Delta,l}(u_1,u_2) = \phi^{b,a}_{\Delta,l}(u_1,u_2), \quad \phi^{a,b}_{\Delta,l}(u_1,u_2) = -\phi^{a,b}_{\Delta,l}(u_2,u_1)\ . \tag{10.19}$$

For vector-valued functions, we can combine transformations of parameters with a change of basis of the target vector space. Let us denote by $\{e_1, e_2\}$ the basis for the target space of seed blocks. Then the seed Hamiltonian is invariant under transformations

$$(a, b, u_1, u_2) \mapsto (b, a, u_2, u_1), \quad (a, b, u_1, u_2) \mapsto (-a, -b, u_2, u_1), \quad (u_1, u_2, e_1, e_2) \mapsto (u_2, u_1, e_2, e_1) .$$

We will denote the permutation of components of a two-vector by a tilde. Then

$$\Psi^{a,b}_{\pm,\Delta,l}(u_1, u_2) = \Psi^{-b,-a}_{\pm,\Delta,l}(u_1, u_2), \quad \Psi^{a,b}_{\pm,\Delta,l}(u_1, u_2) = \tilde{\Psi}^{a,b}_{\pm,\Delta,l}(u_2, u_1) . \tag{10.20}$$

Finally, the two solutions $\Psi_\pm$ can be obtained from one another by swapping the parameters a and b

$$\Psi^{a,b}_{+,\Delta,l}(u_1, u_2) = -\frac{1}{2}\tilde{\Psi}^{b,a}_{-,\Delta,l}(u_1, u_2) = -\frac{1}{2}\tilde{\Psi}^{-a,-b}_{-,\Delta,l}(u_1, u_2) . \tag{10.21}$$

Certain identities for Calogero-Sutherland eigenfunctions may be derived from Clebsch-Gordan decompositions of irreducible representations of the conformal group. Recall that K-spherical functions on G are provided by matrix elements of vectors that transform irreducibly under K. Furthermore, products of matrix elements of two representations π_1, π_2 of G are matrix elements of the tensor product $\pi_1 \otimes \pi_2$. By decomposing the latter into irreducible components, one can write the product of two matrix elements as a sum of some other matrix elements determined by the Clebsch-Gordan decomposition. In one dimension, this method leads to the identities

$$\cosh\frac{u}{2}\Psi^{(a,b)}_\lambda = \gamma^{++}_+\Psi^{(a+\frac{1}{2},b+\frac{1}{2})}_{\lambda+\frac{1}{2}} + \gamma^{++}_-\Psi^{(a+\frac{1}{2},b+\frac{1}{2})}_{\lambda-\frac{1}{2}} = \gamma^{--}_+\Psi^{(a-\frac{1}{2},b-\frac{1}{2})}_{\lambda+\frac{1}{2}} + \gamma^{--}_-\Psi^{(a-\frac{1}{2},b-\frac{1}{2})}_{\lambda-\frac{1}{2}}, \tag{10.22}$$

$$\sinh\frac{u}{2}\Psi^{(a,b)}_\lambda = \gamma^{+-}_+\Psi^{(a+\frac{1}{2},b-\frac{1}{2})}_{\lambda+\frac{1}{2}} + \gamma^{+-}_-\Psi^{(a+\frac{1}{2},b-\frac{1}{2})}_{\lambda-\frac{1}{2}} = \gamma^{-+}_+\Psi^{(a-\frac{1}{2},b+\frac{1}{2})}_{\lambda+\frac{1}{2}} + \gamma^{-+}_-\Psi^{(a-\frac{1}{2},b+\frac{1}{2})}_{\lambda-\frac{1}{2}}, \tag{10.23}$$

with the coefficients

$$\gamma^{++}_+ = \frac{\frac{1}{2}+a+\lambda}{4\lambda}, \quad \gamma^{++}_- = -\frac{\frac{1}{2}+a-\lambda}{4\lambda}, \quad \gamma^{--}_+ = \frac{\frac{1}{2}-b+\lambda}{\lambda}, \quad \gamma^{--}_- = -\frac{\frac{1}{2}-b-\lambda}{\lambda}, \tag{10.24}$$

$$\gamma^{+-}_+ = \frac{\left(\frac{1}{2}+a+\lambda\right)\left(\frac{1}{2}-b+\lambda\right)}{4\lambda(a-b+1)}, \quad \gamma^{+-}_- = -\frac{\left(\frac{1}{2}+a-\lambda\right)\left(\frac{1}{2}-b-\lambda\right)}{4\lambda(a-b+1)}, \quad \gamma^{-+}_+ = -\gamma^{-+}_- = \frac{a-b}{\lambda} . \tag{10.25}$$

Here, γ^{++}_+ etc. are regarded as functions of a, b and λ and we have only suppressed this dependence for simplicity. These relatively simple relations can be also directly derived using Gauss' contiguous relations for ${}_2F_1$.

In four dimensions, the same method can be used to establish the following identities between scalar and seed blocks, which are to the best of our knowledge, new

$$2\left(\sinh\tfrac{u_1}{2}\ ,\quad -\sinh\tfrac{u_2}{2}\right)\tilde{\Psi}^{a-\frac{1}{4},b+\frac{1}{4}}_{-,\Delta+1,l} = \gamma^{-1}_{\Delta,l}\,\phi^{a,b}_{\Delta+\frac{1}{2},l} + \gamma^{-2}_{\Delta,l}\,\phi^{a,b}_{\Delta+\frac{3}{2},l+1}, \tag{10.26}$$

$$2\left(\sinh\tfrac{u_1}{2}\ ,\quad -\sinh\tfrac{u_2}{2}\right)\tilde{\Psi}^{a-\frac{1}{4},b+\frac{1}{4}}_{+,\Delta+1,l} = \gamma^{+1}_{\Delta,l}\,\phi^{a,b}_{\Delta+\frac{1}{2},l+1} + \gamma^{+2}_{\Delta,l}\,\phi^{a,b}_{\Delta+\frac{3}{2},l}, \tag{10.27}$$

$$2\begin{pmatrix}\sinh\frac{u_2}{2}\\ -\sinh\frac{u_1}{2}\end{pmatrix}\phi^{a-\frac{1}{2},b+\frac{1}{2}}_{\Delta+1,l} = \gamma^{1-}_{\Delta,l}\,\tilde{\Psi}^{a-\frac{1}{4},b+\frac{1}{4}}_{-,\Delta+\frac{1}{2},l} + \gamma^{2-}_{\Delta,l}\,\tilde{\Psi}^{a-\frac{1}{4},b+\frac{1}{4}}_{-,\Delta+\frac{3}{2},l-1} + \gamma^{1+}_{\Delta,l}\,\tilde{\Psi}^{a-\frac{1}{4},b+\frac{1}{4}}_{+,\Delta+\frac{1}{2},l-1} + \gamma^{2+}_{\Delta,l}\,\tilde{\Psi}^{a-\frac{1}{4},b+\frac{1}{4}}_{+,\Delta+\frac{3}{2},l} . \tag{10.28}$$

The coefficients in this sum are certain $SO(6)$ Clebsch-Gordan coefficients. This characterisation does not give a simple way to compute them, but the coefficients can be derived by expanding both sides in z_i and comparing the first few terms. In any case

$$\gamma^{-1}_{\Delta,l} = \frac{2\sqrt{2}}{i(-1)^{a+b}}\frac{l+2}{l+1}, \quad \gamma^{-2}_{\Delta,l} = \frac{i(-1)^{-a-b}}{\sqrt{2}}\frac{(4b-2l-2\Delta-1)(2\Delta-3)(4a+2l+2\Delta+1)}{(2\Delta-1)(2\Delta+2l+1)(2\Delta+2l+3)}, \tag{10.29}$$

$$\gamma^{+1}_{\Delta,l} = \frac{\sqrt{2}}{i(-1)^{a+b}}, \quad \gamma^{+2}_{\Delta,l} = \frac{(-1)^{-a-b}}{2\sqrt{2}i}\frac{(l+2)(4b+2l-2\Delta+5)(2\Delta-3)(-4a+2l-2\Delta+5)}{(l+1)(2l-2\Delta+3)(2l-2\Delta+5)(2\Delta-1)}, \tag{10.30}$$

$$\gamma^{1-}_{\Delta,l} = \frac{\sqrt{2}}{i}(-1)^{a+b}, \quad \gamma^{2-}_{\Delta,l} = \frac{(-1)^{a+b}}{2\sqrt{2}i}\frac{\Delta l(2b+l-\Delta+2)(-2a+l-\Delta+2)}{(\Delta-1)(l+1)(l-\Delta+1)(l-\Delta+2)}, \tag{10.31}$$

$$\gamma^{1+}_{\Delta,l} = \frac{2\sqrt{2}}{i}(-1)^{a+b}\frac{l}{l+1}, \; \gamma^{2+}_{\Delta,l} = \frac{i(-1)^{a+b}}{\sqrt{2}}\frac{\Delta(2b-l-\Delta)(2a+l+\Delta)}{(\Delta-1)(l+\Delta)(l+\Delta+1)} \; . \tag{10.32}$$

10.2 Laplacian on supergroups of type I

The superconformal algebra is represented on the structure algebra of the superconformal group by left and right invariant vector fields. As in the bosonic case, quadratic Casimirs constructed out of left and right invariant fields coincide and are both equal to the Riemannian Laplace-Beltrami operator associated with the bi-invariant metric.

We continue to use the same notation for a general Lie superalgebra of type I that was introduced in the discussion of Cartan coordinates. The quadratic Casimir element for such an algebra takes the form

$$C_2 = K_{ab}X^a X^b - X^\mu X_\mu + X_\mu X^\mu, \tag{10.33}$$

where K_{ab} is the Killing form of the even subalgebra. Indeed, the bases $\{X^\mu\}$ and $\{X_\mu\}$ of modules $\mathfrak{g}_\pm$ are dual to each other and so the Killing form reads

$$K^{ab} = \langle X^a, X^b\rangle = \langle X^b, X^a\rangle, \quad \langle X^\mu, X_\nu\rangle = -\langle X_\nu, X^\mu\rangle = \delta^\mu_\nu, \tag{10.34}$$

other inner products being zero. The non-vanishing brackets between odd generators read

$$\{X^\mu, X_\nu\} = K_{ab}\pi(X^a)^\mu{}_\nu X^b \; . \tag{10.35}$$

These brackets imply that the Killing form is ad-invariant, $\langle X^A, [X^B, X^C\}\rangle = \langle [X^A, X^B\}, X^C\rangle$. Indeed the only non-trivial cases to check are

$$\langle X^a, \{X^\mu, X_\nu\}\rangle = \langle [X^a, X^\mu], X_\nu\rangle = \pi(X^a)^\mu{}_\nu, \quad \langle X^\mu, [X^a, X_\nu]\rangle = \langle [X^\mu, X^a], X_\nu\rangle = -\pi(X^a)^\mu{}_\nu \; .$$

From the bracket relations written above, one can directly verify that C_2 commutes with all the generators X^A in the universal enveloping algebra $U(\mathfrak{g})$.

Let us turn to (right-)invariant vector fields on the supergroup G. The Maurer-Cartan form reads

$$dgg^{-1} = dx^\mu X_\mu + dx^a e^{x^\mu X_\mu}(\partial_{x^a} g_{(0)})g^{-1}_{(0)}e^{-x^\mu X_\mu} + dx_\nu e^{x^\mu X_\mu} g_{(0)} X^\nu g^{-1}_{(0)} e^{-x^\mu X_\mu} \; . \tag{10.36}$$

Let us denote the coefficients of the bosonic Maurer-Cartan form by $C^{(0)}_{ab}$, that is

$$(\partial_{x^a} g_{(0)}) g^{-1}_{(0)} = C^{(0)}_{ab} X^b \ . \tag{10.37}$$

By using the relations

$$e^{x^\rho X_\rho} X^a e^{-x^\sigma X_\sigma} = X^a + \pi(X^a)^\mu{}_\nu x^\nu X_\mu, \tag{10.38}$$

$$e^{x^\rho X_\rho} X^\mu e^{-x^\sigma X_\sigma} = X^\mu + K_{ab}\pi(X^a)^\mu{}_\nu x^\nu X^b + \frac{1}{2} K_{ab}\pi(X^a)^\mu{}_\nu \pi(X^b)^\rho{}_\sigma x^\nu x^\sigma X_\rho, \tag{10.39}$$

the Maurer-Cartan form can be further evaluated to

$$dgg^{-1} = dx^\mu X_\mu + dx^a C^{(0)}_{ab} \left(X^b + \pi(X^b)^\mu{}_\nu x^\nu X_\mu \right) \tag{10.40}$$

$$+ dx_\nu \pi(g_{(0)})^\nu{}_\mu \left(X^\mu + K_{ab}\pi(X^a)^\mu{}_\rho x^\rho X^b + \frac{1}{2} K_{ab}\pi(X^a)^\mu{}_\lambda \pi(X^b)^\rho{}_\sigma x^\lambda x^\sigma X_\rho \right) \ . \tag{10.41}$$

The right-invariant vector fields therefore read

$$\mathcal{R}_{X_\mu} = \partial_{x^\mu}, \quad \mathcal{R}_{X^b} = \mathcal{R}^{(0)}_{X^b} - \pi(X^b)^\mu{}_\nu x^\nu \partial_{x^\mu}, \tag{10.42}$$

$$\mathcal{R}_{X^\mu} = \pi(g^{-1}_{(0)})^\mu{}_\nu \partial_{x_\nu} - K_{ab}\pi(X^a)^\mu{}_\nu x^\nu \left(\mathcal{R}^{(0)}_{X^b} - \frac{1}{2}\pi(X^b)^\rho{}_\sigma x^\sigma \partial_{x^\rho} \right) \ . \tag{10.43}$$

Here $\mathcal{R}^{(0)}_{X^a}$ are the right-invariant vector fields of the underlying Lie group $G_{(0)}$. From the expressions (10.42) – (10.43), we compute the Laplacian

$$\mathcal{R}_{C_2} = \mathcal{R}^{(0)}_{C_2} - 2\pi(g^{-1}_{(0)})^\mu{}_\nu \partial_{x_\nu}\partial_{x^\mu} + \frac{1}{2} K_{ab} \left(\pi(X^a)^\mu{}_\nu \pi(X^b)^\rho{}_\mu + \pi(X^a)^\mu{}_\mu \pi(X^b)^\rho{}_\nu \right) x^\nu \partial_{x^\rho} - K_{ab}\pi(X^a)^\mu{}_\mu \mathcal{R}^{(0)}_{X^b} \ .$$

We can simplify the result using the Jacobi identity

$$[X_\mu, \{X_\nu, X^\rho\}] + [X_\nu, \{X^\rho, X_\mu\}] + [X^\rho, \{X_\mu, X_\nu\}] = 0, \tag{10.44}$$

which implies

$$K_{ab} \left(\pi(X^a)^\rho{}_\nu \pi(X^b)^\sigma{}_\mu + \pi(X^a)^\rho{}_\mu \pi(X^b)^\sigma{}_\nu \right) = 0 \ . \tag{10.45}$$

Upon substitution in the above formula, we are left with

$$\mathcal{R}_{C_2} = \mathcal{R}^{(0)}_{C_2} - 2\pi(g^{-1}_{(0)})^\mu{}_\nu \partial_{x_\nu}\partial_{x^\mu} - K_{ab}\pi(X^a)^\mu{}_\mu \mathcal{R}^{(0)}_{X^b} \ . \tag{10.46}$$

This is the main result of the present section. It expresses the Laplacian on a type I supergroup G in terms of the Laplace operator of its underlying Lie group and the representation π that is a part of the defining data of the supergroup. As can be seen, $\mathcal{R}_{C_2}$ differs from $\mathcal{R}^{(0)}_{C_2}$ by two terms. The reduction of the Laplacian to K-spherical functions and the role played in it by these terms will be discussed presently.

10.3 Reduction to the bosonic case

Consider a K-spherical function F the superconformal group. Assume for the moment that F is a scalar function. Upon expansion in fermionic coordinates, F becomes a vector-valued function on the underlying Lie group, $F : G_{(0)} \to \Lambda\mathfrak{g}_1$. Depending on which fermionic coordinates are used to perform the expansion, the resulting function can satisfy different covariance properties. In the primed Cartan coordinates, all components of F obey the same covariance laws. However, in the unprimed coordinates, this is no longer the case because the generators of $\mathfrak{g}_\pm$ transform non-trivially under K. As the effect, the components F_i mix under the left and right multiplication so to make $\{F_i\}$ a $(\Lambda\mathfrak{g}_-, \Lambda\mathfrak{g}_+)$-spherical function. More generally, a vector-valued function F on G with covariance laws dictated by representations W_l and W_r, gives upon the expansion a K-spherical function on $G_{(0)}$ with the representations

$$V_l = W_l \otimes \Lambda\mathfrak{g}_-, \quad V_r = W_r \otimes \Lambda\mathfrak{g}_+ \; . \tag{10.47}$$

Therefore, we have turned the space of K-spherical functions on G into what we have been denoting Γ_{V_l,V_r}. Since the internal symmetry subgroup of $G_{(0)}$ is a part of K, we deduce that functions in Γ_{V_l,V_r} are still determined by their values on the abelian group A. Moreover, their restrictions have to take values in $(V_l \otimes V_r)^B$ where, recall, the stabiliser group is $B \sim SO(d-2) \times U$.

With this description of the space of functions and the expression for the Laplacian (10.46), the use of unprimed Cartan coordinates allows us to construct the CS Casimir equations. By performing the ordinary bosonic reduction, we end up with the super Calogero-Sutherland Hamiltonian of the form

$$H = H_0 + A \; . \tag{10.48}$$

Here H_0 comes form the first and the third (trace) term in the Laplacian. It is the matrix CS Hamiltonian determined by the pair of representations (V_l, V_r), with added diagonal terms. These additional terms come from two sources. Firstly, the quadratic Casimir of $G_{(0)}$ contains and internal symmetry part. Since the internal symmetries are included in K, this part acts as a set of constants on K-spherical functions. The second contribution comes from the trace term in (10.46) and is by the same token equal to a constant diagonal matrix. Finally, let us look at the operator A, defined as the reduction of the second term in the Laplacian. Since the coefficients that multiply the fermionic derivatives in this term are purely bosonic, A is a nilpotent operator. Therefore, it is a triangular matrix of functions. Its entries involve matrix coefficients of the abelian group A and therefore are extremely simple.[2]

Unprimed Cartan coordinates are also convenient for the implementation of shortening conditions. In particular, if we have half-BPS multiplets with $Q_-\mathcal{O}_{1,2} = Q_+\mathcal{O}_{3,4} = 0$, the second term in (10.46) vanishes and hence superconformal partial waves coincide with bosonic ones. This applies to the case in which we insert chiral fields in positions $1, 2$ and anti-chiral fields in positions $3, 4$. If, on the other hand, we insert pairs of chiral and anti-chiral fields in positions $1, 2$ and $3, 4$, the shortening conditions are not quite as simple. We shall study these in more detail on concrete examples in the following section sections.

[2]The reader should not confuse the two-dimensional group A with the matrix A that appears in (10.48).

10.4 Nilpotent perturbation theory

Having seen that the Casimir operator for a type I superconformal symmetry can be regarded as a nilpotent perturbation of the Casimir operator for a set of spinning bosonic conformal partial waves, our strategy is to construct supersymmetric partial waves as a perturbation of spinning bosonic ones. Since the perturbing term A is nilpotent, we can obtain exact formulas at some finite order N, which depends on the precise setup, and is trivially bounded as $N \leq \dim\mathfrak{g}_{(1)}$. The general methods to solve for eigenfunctions of a Hamiltonian $H = H_0 + A$ in terms of those of H_0 are certainly well established. In our exposition we shall follow [125], and assume for simplicity that H and H_0 have discrete spectra and finite-dimensional eigenspaces. By a limiting process, the construction can be extended to more general spectra.

The Hilbert space on which the operators act is denoted by $\mathcal{H}$ and H_0 is assumed to be hermitian. We will attach an index [0] and write $H_{[0]}$ to mean either H_0 or H and use the similar notation for other objects as well. Eigenspaces of $H_{[0]}$ are written as $V_n^{[0]}$ and have eigenvalues $\varepsilon_n^{[0]}$. Projectors to these eigenspaces are denoted by $P_n^{[0]}$. Recall the definition of the resolvent

$$G_{[0]} : \mathbb{C} \to L(\mathcal{H}) \ , \quad G_{[0]}(z) = (z - H_{[0]})^{-1} \ . \tag{10.49}$$

The resolvent of an operator can be expanded in the projectors to eigenspaces with simple poles at the eigenvalues. Conversely, projectors are residues of the resolvent

$$G_{[0]}(z) = \sum_n \frac{1}{z - \varepsilon_n^{[0]}} P_n^{[0]}, \quad P_n^{[0]} = \frac{1}{2\pi i} \oint_{\Gamma_n} G_{[0]}(z) dz \ . \tag{10.50}$$

Here, Γ_n is a small contour encircling $\varepsilon_n^{[0]}$ and none of the other eigenvalues. If we insert the relation $H = H_0 + A$ into the definition of the resolvent G and perform an expansion in A we get

$$G = G_0 \sum_{n=0}^{\infty} (AG_0)^n = G_0 \sum_{n=0}^{N} (AG_0)^n \ . \tag{10.51}$$

We have used that $A^{N+1} = 0$ for some N to truncate the sum, and also the fact that $A^k = 0$ implies $(AG_0)^k = 0$, which is clearly true for the kind of operators that we wish to consider. In particular, from (10.51) it follows that G has the same singularities ε_i^0 as G_0. Computing residues of the above expansion for G at ε_i^0 we obtain a finite expansion for the projector P_i

$$P_i = P_i^0 + \sum_{n=1}^{N} \mathrm{Res}(G_0(AG_0)^n, \varepsilon_i^0) \equiv P_i^0 + P_i^{(1)} + ... + P_i^{(N)} \ . \tag{10.52}$$

All terms in the above sum are expressed trough projectors P_i^0, the perturbation A and the operator

$$S_i = \sum_{j \neq i} \frac{P_j^0}{\varepsilon_i^0 - \varepsilon_j^0} \ . \tag{10.53}$$

For example, the first two terms read

$$P_i^{(1)} = P_i^0 A S_i + S_i A P_i^0, \tag{10.54}$$

$$P_i^{(2)} = P_i^0 A S_i A S_i + S_i A P_i^0 A S_i + S_i A S_i A P_i^0 - P_i^0 A P_i^0 A S_i^2 - P_i^0 A S_i^2 A P_i^0 - S_i^2 A P_i^0 A P_i^0, \tag{10.55}$$

It is a simple matter to write the higher order terms as well. Thus, what we have achieved is to write P_i in a finite manner in terms of known operators. For any vector v in the Hilbert space, $P_i v$ is an eigenvector of H. Therefore, one can obtain all eigenvectors of H as soon as one is able to evaluate $P_i v$ on sufficiently many vectors v. In practical applications, we will apply P_i to the eigenbasis for the unperturbed operator H_0. Under assumption that $P_i : V_i^0 \to V_i$ are vector space isomorphisms, this produces an eigenbasis for H.

10.5 Superconformal blocks for one-dimensional $\mathcal{N} = 2$ SCFTs

The goal of this section is to illustrate the general theory we have developed at the example of $\mathcal{N} = 2$ supersymmetry in one dimension, i.e. for the superconformal algebra $\mathfrak{sl}(2|1)$.

10.5.1 Calogero-Sutherland Casimir equations

The superconformal group $SL(2|1)$ was introduced in previous chapters. To study the Laplacian, we start by defining the unprimed Cartan coordinates

$$g = e^{\bar{q}Q_- + \bar{s}S_-} e^{\kappa R + \lambda_l D} e^{\frac{u}{2}(P+K)} e^{\lambda_r D} e^{qQ_+ + sS_+} \ . \tag{10.56}$$

They are related to primed Cartan coordinates (8.35) by

$$\bar{q}' = e^{\kappa - \frac{1}{2}\lambda_l}\bar{q}, \quad \bar{s}' = e^{\kappa + \frac{1}{2}\lambda_l}\bar{s}, \quad q' = e^{\frac{1}{2}\lambda_r} q, \quad s' = e^{-\frac{1}{2}\lambda_r} s \ . \tag{10.57}$$

The quadratic Casimir is given in terms of generators by

$$C_2 = 2D^2 + \{P, K\} - \frac{1}{2}R^2 - [Q_+, S_-] + [Q_-, S_+] \ . \tag{10.58}$$

There is one more algebraically independent Casimir element, of third order

$$C_3 = \left(D^2 - \frac{1}{4}R^2 + PK\right) R - Q_+S_- \left(D + \frac{3}{2}R\right) - Q_-S_+ \left(D - \frac{3}{2}R\right) - KQ_+Q_- + PS_-S_+ - D - \frac{1}{2}R \ .$$

Typical representations of $\mathfrak{sl}(2|1)$ can be distinguished by the values of these two Casimir elements. This is no longer the case for short multiplets, in which both Casimirs vanish, [106]. We will assume that the R-charges of fields that enter into the correlation function add up to zero, $\sum r_i = 0$. The restriction of a K-spherical function $F : G_{(0)} \to \Lambda\mathfrak{g}_1$ is denoted $\omega^{1/2}G(u)$, with

$$G(u) = G^{(1)}(u) + G^{(2)}(u)\bar{q}q + G^{(3)}(u)\bar{q}s + G^{(4)}(u)\bar{s}q + G^{(5)}(u)\bar{s}s + G^{(6)}(u)\bar{q}\bar{s}qs \ . \tag{10.59}$$

Recall that in one dimension $\omega = \sinh^{-1} u$. Other components of G vanish due to B-invariance. In primed Cartan coordinates any component of F is a covariant function that obeys

$$F_i(e^{\kappa R + \lambda_l D} g_{(0)} e^{\lambda_r D}) = e^{r\kappa + a\lambda_l + b\lambda_r} F_i(g_{(0)}), \tag{10.60}$$

where $a=\Delta_2-\Delta_1$, $b=\Delta_3-\Delta_4$ and $r=r_1+r_2$. Therefore, in the unprimed coordinates, F satisfies

$$\omega^{-1/2}F = e^{r\kappa+a\lambda_l+b\lambda_r}G^{(1)}(u) + e^{(r+1)\kappa+(a-\frac{1}{2})\lambda_l+(b+\frac{1}{2})\lambda_r}G^{(2)}(u)\bar{q}q + e^{(r+1)\kappa+(a-\frac{1}{2})\lambda_l+(b-\frac{1}{2})\lambda_r}G^{(3)}(u)\bar{q}s$$
$$+ e^{(r+1)\kappa+(a+\frac{1}{2})\lambda_l+(b+\frac{1}{2})\lambda_r}G^{(4)}(u)\bar{s}q + e^{(r+1)\kappa+(a+\frac{1}{2})\lambda_l+(b-\frac{1}{2})\lambda_r}G^{(5)}(u)\bar{s}s + e^{(r+2)\kappa+a\lambda_l+b\lambda_r}G^{(6)}(u)\bar{q}\bar{s}qs\ .$$

The reduced Laplacian can be written down directly using the general theory from above. Its two pieces H_0 and A are

$$H_0 = -2\mathrm{diag}\Big(H_{PT}^{(a,b)} + \frac{(r-1)^2}{4},\ H_{PT}^{(a-\frac{1}{2},b+\frac{1}{2})} + \frac{r^2}{4},\ H_{PT}^{(a-\frac{1}{2},b-\frac{1}{2})} + \frac{r^2}{4}, \tag{10.61}$$
$$H_{PT}^{(a+\frac{1}{2},b+\frac{1}{2})} + \frac{r^2}{4},\ H_{PT}^{(a+\frac{1}{2},b-\frac{1}{2})} + \frac{r^2}{4},\ H_{PT}^{(a,b)} + \frac{(r+1)^2}{4}\Big), \tag{10.62}$$

and

$$A = -2\begin{pmatrix} 0 & {a^2}_1 & {a^2}_2 & {a^1}_1 & {a^1}_2 & 0\\ 0&0&0&0&0& -{a^1}_2\\ 0&0&0&0&0& {a^1}_1\\ 0&0&0&0&0& {a^2}_2\\ 0&0&0&0&0& -{a^2}_1\\ 0&0&0&0&0&0\end{pmatrix} = -2\begin{pmatrix} 0 & -\sinh\frac{u}{2} & \cosh\frac{u}{2} & \cosh\frac{u}{2} & -\sinh\frac{u}{2} & 0\\ 0&0&0&0&0& \sinh\frac{u}{2}\\ 0&0&0&0&0& \cosh\frac{u}{2}\\ 0&0&0&0&0& \cosh\frac{u}{2}\\ 0&0&0&0&0& \sinh\frac{u}{2}\\ 0&0&0&0&0&0\end{pmatrix}. \tag{10.63}$$

To derive these formulas, note that the $SL(2)$-part of the Laplacian in the above conventions (see (5.25)), reduces upon conjugation with $\omega^{-1/2}$ to $-2H_{PT}^{(a,b)}-\frac{1}{2}$. This "basic operator" receives two corrections, from $-R^2/2$ term in the quadratic Casimir and the trace terms in (10.46) to give the top left entry of H_0. In other terms along the diagonal of H_0, the parameters a, b and r receive shifts according to covariance laws in unprimed Cartan coordinates.

Two obtain the nilpotent term A, we need to compute the matrix $a(u)^{-1}$ in the representation of $\mathfrak{g}_0$ on $\mathfrak{g}_+$. To notations for fermionic generators that we have used are related by

$$X^1 = Q_+,\quad X^2 = S_+,\quad X_1 = S_-,\quad X_2 = S_+\ .$$

Therefore, the representation matrix is

$$\pi(a(u)^{-1})^\mu{}_\nu = \begin{pmatrix} \cosh\frac{u}{2} & -\sinh\frac{u}{2}\\ -\sinh\frac{u}{2} & \cosh\frac{u}{2}\end{pmatrix} \equiv ({a^\mu}_\nu)\ . \tag{10.64}$$

We will introduce an additional factor for convenience and write the super CS Hamiltonian as $H_s=-\frac{1}{2}(H_0+A)$. Notice that the above derivation did not require us to do any calculations. The supergroup $SL(2|1)$ is sufficiently small and one can actually obtain H_s by computing the Laplacian in all supergroup coordinates and then imposing covariance conditions with respect to K. We have done this as a check of our method and obtained the same operator from above.

10.5.2 Superconformal partial waves

Having derived the supersymmetric Calogero-Sutherland Hamiltonian, we turn its eigenfunctions. The form of the wavefunctions is tightly constrained by representation theory. Technically the easiest way to obtain them is to write the most general ansatz as allowed by representation theory and then fix the undetermined coefficients using identities (10.22)-(10.23).

For each value of λ, there are six solutions of the form

$$
\begin{aligned}
G^{(1)}_\lambda &= \Psi^{(a,b)}_\lambda e_1, \\
G^{(2)}_\lambda &= \Psi^{(a-\frac{1}{2},b+\frac{1}{2})}_\lambda e_2 + \left(\alpha_{2,+}\Psi^{(a,b)}_{\lambda+\frac{1}{2}} + \alpha_{2,-}\Psi^{(a,b)}_{\lambda-\frac{1}{2}}\right) e_1, && (10.65)\\
G^{(3)}_\lambda &= \Psi^{(a-\frac{1}{2},b-\frac{1}{2})}_\lambda e_3 + \left(\alpha_{3,+}\Psi^{(a,b)}_{\lambda+\frac{1}{2}} + \alpha_{3,-}\Psi^{(a,b)}_{\lambda-\frac{1}{2}}\right) e_1, && (10.66)\\
G^{(4)}_\lambda &= \Psi^{(a+\frac{1}{2},b+\frac{1}{2})}_\lambda e_4 + \left(\alpha_{4,+}\Psi^{(a,b)}_{\lambda+\frac{1}{2}} + \alpha_{4,-}\Psi^{(a,b)}_{\lambda-\frac{1}{2}}\right) e_1, && (10.67)\\
G^{(5)}_\lambda &= \Psi^{(a+\frac{1}{2},b-\frac{1}{2})}_\lambda e_5 + \left(\alpha_{5,+}\Psi^{(a,b)}_{\lambda+\frac{1}{2}} + \alpha_{5,-}\Psi^{(a,b)}_{\lambda-\frac{1}{2}}\right) e_1 && (10.68)\\
G^{(6)}_\lambda &= \Psi^{(a,b)}_\lambda e_6 + \left(\beta_{2,+}\Psi^{(a-\frac{1}{2},b+\frac{1}{2})}_{\lambda+\frac{1}{2}} + \beta_{2,-}\Psi^{(a-\frac{1}{2},b+\frac{1}{2})}_{\lambda-\frac{1}{2}}\right) e_2 && (10.69)\\
&+ \left(\beta_{3,+}\Psi^{(a-\frac{1}{2},b-\frac{1}{2})}_{\lambda+\frac{1}{2}} + \beta_{3,-}\Psi^{(a-\frac{1}{2},b-\frac{1}{2})}_{\lambda-\frac{1}{2}}\right) e_3 + \left(\beta_{4,+}\Psi^{(a+\frac{1}{2},b+\frac{1}{2})}_{\lambda+\frac{1}{2}} + \beta_{4,-}\Psi^{(a+\frac{1}{2},b+\frac{1}{2})}_{\lambda-\frac{1}{2}}\right) e_4 && (10.70)\\
&+ \left(\beta_{5,+}\Psi^{(a+\frac{1}{2},b-\frac{1}{2})}_{\lambda+\frac{1}{2}} + \beta_{5,-}\Psi^{(a+\frac{1}{2},b-\frac{1}{2})}_{\lambda-\frac{1}{2}}\right) e_5 + \left(\beta_{1,+}\Psi^{(a,b)}_{\lambda+1} + \beta_{1,0}\Psi^{(a,b)}_{\lambda} + \beta_{1,-}\Psi^{(a,b)}_{\lambda-1}\right) e_1 \,. && (10.71)
\end{aligned}
$$

They have the same eigenvalues as eigenfunctions of the unperturbed Hamiltonian. Explicitly

$$
E^{(1)}_\lambda = \frac{1}{4}(r-1)^2 - \lambda^2, \quad E^{(2)}_\lambda = ... = E^{(5)}_\lambda = \frac{1}{4}r^2 - \lambda^2, \quad E^{(6)}_\lambda = \frac{1}{4}(r+1)^2 - \lambda^2 \,. \tag{10.72}
$$

Coefficients $\alpha_{i,\pm}$, $\beta_{j,\pm}$ and $\beta_{1,0}$ are, upon substitution of $(10.22) - (10.23)$, obtained by solving a linear algebraic system. This gives

$$
\begin{aligned}
\alpha_{2,\pm} &= \mp\frac{\gamma^{+-}_\pm(a-\frac{1}{2},b+\frac{1}{2},\lambda)}{\lambda\pm\frac{1}{2}r}, \quad \beta_{2,\pm} = \frac{\gamma^{-+}_\pm(a,b,\lambda)}{\pm\lambda+\frac{1}{2}(r+1)}, && (10.73)\\
\alpha_{3,\pm} &= \pm\frac{\gamma^{++}_\pm(a-\frac{1}{2},b-\frac{1}{2},\lambda)}{\lambda\pm\frac{1}{2}r}, \quad \beta_{3,\pm} = \frac{\gamma^{--}_\pm(a,b,\lambda)}{\pm\lambda+\frac{1}{2}(r+1)}, && (10.74)\\
\alpha_{4,\pm} &= \pm\frac{\gamma^{--}_\pm(a+\frac{1}{2},b+\frac{1}{2},\lambda)}{\lambda\pm\frac{1}{2}r}, \quad \beta_{4,\pm} = \frac{\gamma^{++}_\pm(a,b,\lambda)}{\pm\lambda+\frac{1}{2}(r+1)}, && (10.75)\\
\alpha_{5,\pm} &= \mp\frac{\gamma^{-+}_\pm(a+\frac{1}{2},b-\frac{1}{2},\lambda)}{\lambda\pm\frac{1}{2}r}, \quad \beta_{5,\pm} = \frac{\gamma^{+-}_\pm(a,b,\lambda)}{\pm\lambda+\frac{1}{2}(r+1)} \,. && (10.76)
\end{aligned}
$$

and

$$\beta_{1,\pm 1} = \frac{-1}{\pm 2\lambda + r + 1}\Big(\beta_{2,\pm}\gamma_{\pm}^{+-}(a - \frac{1}{2}, b + \frac{1}{2}, \lambda \pm \frac{1}{2}) - \beta_{3,\pm}\gamma_{\pm}^{++}(a - \frac{1}{2}, b - \frac{1}{2}, \lambda \pm \frac{1}{2}) - \beta_{4,\pm}\gamma_{\pm}^{--}(a + \frac{1}{2}, b + \frac{1}{2}, \lambda \pm \frac{1}{2}) + \beta_{5,\pm}\gamma_{\pm}^{-+}(a + \frac{1}{2}, b - \frac{1}{2}, \lambda \pm \frac{1}{2})\Big),$$

$$\beta_{1,0} = \frac{-1}{r}\sum_{\pm}\Big(\beta_{2,\pm}\gamma_{\mp}^{+-}(a - \frac{1}{2}, b + \frac{1}{2}, \lambda \pm \frac{1}{2}) - \beta_{3,\pm}\gamma_{\mp}^{++}(a - \frac{1}{2}, b - \frac{1}{2}, \lambda \pm \frac{1}{2}) - \beta_{4,\pm}\gamma_{\mp}^{--}(a + \frac{1}{2}, b + \frac{1}{2}, \lambda \pm \frac{1}{2}) + \beta_{5,\pm}\gamma_{\mp}^{-+}(a + \frac{1}{2}, b - \frac{1}{2}, \lambda \pm \frac{1}{2})\Big)\ .$$

Notice that, with the preparation from previous sections, solving the eigenvalue problem required hardly any computation. Essentially, equations (10.46), (10.22) and (10.23) allow to more or less directly write down the wavefunctions.

10.5.3 Multiplet shortening

Superconformal blocks that we just computed are the most complicated ones in one-dimensional $\mathcal{N} = 2$ theories and they can be used to derive the partial waves for cases in which some of the fields in the correlator are short. However, it would be desirable if these simpler blocks could be obtained directly, without the use of the above results. For some types of correlators this can indeed be done. Assume that a K-spherical function F either does not depend on variables $(s, \bar{s})$ or on the other pair $(q, \bar{q})$. We can impose these conditions on the expansion (10.59) by keeping only the terms $(G^{(1)}, G^{(2)})$ and $(G^{(1)}, G^{(5)})$, respectively. The resulting Hamiltonians read

$$H_{\pm} = -2\begin{pmatrix} H_{PT}^{(a,b)} + \frac{(r-1)^2}{4} & -\sinh\frac{u}{2} \\ 0 & H_{PT}^{(a\mp\frac{1}{2}, b\pm\frac{1}{2})} + \frac{r^2}{4}\end{pmatrix}\ . \tag{10.77}$$

Obviously, eigenfunctions of H_+ are $G_\lambda^{(2)}$ and of H_- are $G_\lambda^{(5)}$. Indeed, out of all eigenfunctions of the full super CS Hamiltonian, these were the ones that did not depend on $(s, \bar{s})$ and $(q, \bar{q})$.

10.6 Superconformal blocks for four-dimensional $\mathcal{N} = 1$ SCFTs

The goal of this section is to apply the general theory we have developed to the example of $\mathcal{N} = 1$ supersymmetry in four dimensions, i.e. for the superconformal algebra $\mathfrak{sl}(4|1)$.

10.6.1 Casimir equations for four-dimensional $\mathcal{N} = 1$ SCFTs

Even with the methods of previous sections, the computation of completely general scalar K-spherical harmonics on the supergroup $SL(4|1)$ requires significant additional efforts. To appreciate this fact, notice that if the R-charges of modules W_l and W_r add up to zero, K-spherical functions depend on 35 nilpotent invariants. Therefore, we arrive at the eigenvalue problem for a 36×36 matrix Hamiltonian, albeit of a relatively simple structure. We will leave the analysis of this full Hamiltonian for another occasion and impose here further conditions on K-spherical functions that correspond through (7.40) to multiplet shortening in the field theory.

We will make use of both primed and unprimed Cartan coordinates on the supergroup $SL(4|1)$. The unprimed ones read

$$g = e^{q_\alpha Q^\alpha + s_{\dot\alpha} S^{\dot\alpha}} e^{\kappa R} e^{\lambda_l D} r_l e^{\frac{u_1+u_2}{4}(P_1+K_1) - i\frac{u_1-u_2}{4}(P_2-K_2)} r_r e^{\lambda_r D} e^{q^{\dot\alpha} Q_{\dot\alpha} + s^\alpha S_\alpha}, \tag{10.78}$$

with r_l, r_r as defined in (9.24). On the other hand, the primed Cartan coordinates are

$$g = e^{\kappa R} e^{\lambda_l D} r_l e^{q'_\alpha Q^\alpha + s'_{\dot\alpha} S^{\dot\alpha}} e^{\frac{u_1+u_2}{4}(P_1+K_1) - i\frac{u_1-u_2}{4}(P_2-K_2)} e^{q'^{\dot\alpha} Q_{\dot\alpha} + s'^\alpha S_\alpha} r_r e^{\lambda_r D} . \tag{10.79}$$

The explicit relation between coordinate systems (10.78) and (10.79) is written in section (8.5.1).

Let ρ_l and ρ_r be two characters of K given by

$$\rho_l(D) = 2a, \quad \rho_l(R) = r, \quad \rho_r(D) = -2b, \quad \rho_r(R) = -r . \tag{10.80}$$

In particular, $\rho_{l,r}$ are trivial representations of the rotation group Spin(4). The pair (ρ_l, ρ_r) defines the space of K-spherical functions $\Gamma_{\rho_l,\rho_r} = \Gamma^r_{a,b}$ whose elements are functions $f \in A(SL(4|1))$ with covariance properties

$$(\partial_{\lambda'_l} - 2a) f = (\partial_{\lambda'_r} - 2b) f = (\partial_{\kappa'} - r) f = \partial_{\varphi'^l_1} f = ... = \partial_{\psi'^r_2} f = 0 . \tag{10.81}$$

As we explained before, $\lambda'_l = \lambda_l$ and similarly for all other variables that appear in (10.81), but the partial derivatives of course depend on the full system of coordinates and $\partial_{\lambda'_l} \neq \partial_{\lambda_l}$ etc. Inside Γ_{ρ_l,ρ_r} we consider two subspaces of functions that satisfy further conditions

$$\Gamma_1 = \{f \in \Gamma_{\rho_l,\rho_r} \mid \partial_{s_{\dot\alpha}} f = \partial_{s^\alpha} f = 0\}, \quad \Gamma_2 = \{f \in \Gamma_{\rho_l,\rho_r} \mid \partial_{q^{\dot\alpha}} f = \partial_{q_\alpha} f = 0\} . \tag{10.82}$$

The rest of this section is devoted to solving the eigenvalue problem for the Laplacian on spaces Γ_i.

The quadratic Casimir on $SL(4|1)$ takes the general form (10.33). The Killing form can, up to normalisation, be computed from the supertrace in any faithful representation of $\mathfrak{sl}(4|1)$. In the five-dimensional representation that we have been using, the correct normalisation is

$$K^{ab} = \mathrm{str}(X^a X^b) . \tag{10.83}$$

Then, if we set $(X^\mu) = (Q_{\dot 1}, Q_{\dot 2}, S_1, S_2)$ and $(X_\mu) = (S^{\dot 1}, S^{\dot 2}, Q^1, Q^2)$, one can verify that the anticommutation relations (10.35) hold. We will rescale the Casimir (10.33) by a factor 1/4, thereby obtaining

$$C_2 = -\frac{3}{16} R^2 + \frac{1}{4} D^2 + ... \tag{10.84}$$

With such a normalisation, the coefficients of second derivatives $\partial^2_{u_i}$ in the super CS Hamiltonian will be equal to 1/2.

On the two-dimensional abelian group A, a function $f \in \Gamma_1$ restricts to $\omega^{1/2} G_1$ with

$$G_1 = G_1^{(1)}(u_i) + G_1^{(2)}(u_i) q_1 q^{\dot 1} + G_1^{(3)}(u_i) q_2 q^{\dot 2} + G_1^{(4)}(u_i) q_1 q^{\dot 1} q_2 q^{\dot 2} . \tag{10.85}$$

Other components of G_1 vanish due to requirements of B-invariance. The unperturbed part of the Hamiltonian reads

$$H_{01} = -\begin{pmatrix} H_{sc}^{a,b} + \frac{3}{16}r^2 - \frac{3}{4}r & 0 & 0 \\ 0 & \tilde{H}_{\frac{1}{2}}^{a-\frac{1}{4},b+\frac{1}{4}} + \frac{3}{16}(r+1)^2 - \frac{3}{4}(r+1) & 0 \\ 0 & 0 & H_{sc}^{a-\frac{1}{2},b+\frac{1}{2}} + \frac{3}{16}(r+2)^2 - \frac{3}{4}(r+2) \end{pmatrix} . \tag{10.86}$$

Here, the operators on the diagonal are the scalar and seed Calogero-Sutherland Hamiltonians. Indeed, covariance conditions (10.81) and the above normalisations imply that the $SO(5,1)$-part of the Casimir gives the scalar CS Hamiltonian with an overall minus sign. This operator receives $-3r^2/16$ correction from the R-symmetry part in the Casimir (10.84) and a further correction $3r/4$ from the trace term in (10.46). Hence, we end up with the top left entry of H_0.

To obtain the other elements on the diagonal, notice that in primed Cartan coordinates all components of f (which is expanded in Grassmann variables) have covariance laws

$$f_i'(e^{\kappa R+\lambda_l D} g_{(0)} e^{\lambda_r D}) = e^{r\kappa+2a\lambda_l+2b\lambda_r} f_i'(g_{(0)}) \ . \tag{10.87}$$

In the unprimed coordinates, these laws receive shifts. For instance

$$f^{\alpha}_{\ \dot{\alpha}}(x_a) q_\alpha q^{\dot{\alpha}} = f'^{\beta}_{\ \dot{\beta}}(x_a) q'_\beta q'^{\dot{\beta}} = e^{\kappa-\frac{1}{2}\lambda_l+\frac{1}{2}\lambda_r} f'^{\beta}_{\ \dot{\beta}}(x_a) \mathcal{L}_\alpha^{\ \beta} \mathcal{R}^{\dot{\alpha}}_{\ \dot{\beta}} q_\alpha q^{\dot{\alpha}} \tag{10.88}$$

The $SU(2)$ matrices $\mathcal{L}$ and $\mathcal{R}$ can read off from equations of the section (8.5.1). Therefore, components $f^{\alpha}_{\ \dot{\alpha}}$ are covariant functions with parameters a and b shifted by $\mp 1/4$ and r shifted by 1. By considering rotations, we similarly conclude that $f^{\alpha}_{\dot{\alpha}}$ is a $(\rho_l = (1/2,0), \rho_r = (0,1/2))$ spherical function with respect to Spin(4) (while $f'^{\alpha}_{\ \dot{\alpha}}$ are invariant under rotations). This leads to the seed Hamiltonian with parameters written above. Finally, the bottom right entry of H_0 is deduced by the same kind of arguments.

Having described the unperturbed Hamiltonian, let us turn to the perturbation A. It reads

$$A = -2\begin{pmatrix} 0 & a^3_{\ 1} & a^4_{\ 2} & 0 \\ 0 & 0 & 0 & a^4_{\ 2} \\ 0 & 0 & 0 & a^3_{\ 1} \\ 0 & 0 & 0 & 0 \end{pmatrix} = -2\begin{pmatrix} 0 & \sinh\frac{u_1}{2} & -\sinh\frac{u_2}{2} & 0 \\ 0 & 0 & 0 & -\sinh\frac{u_2}{2} \\ 0 & 0 & 0 & \sinh\frac{u_1}{2} \\ 0 & 0 & 0 & 0 \end{pmatrix} . \tag{10.89}$$

To derive this, notice that in the representation $\mathfrak{g}_+$, with the basis $\{X^\mu\}$ as defined above, the element $a(u_1,u_2)$ is given by

$$\pi\left(a(u_1,u_2)^{-1}\right)^\mu_{\ \nu} = \begin{pmatrix} \cosh\frac{u_1}{2} & 0 & \sinh\frac{u_1}{2} & 0 \\ 0 & \cosh\frac{u_2}{2} & 0 & -\sinh\frac{u_2}{2} \\ \sinh\frac{u_1}{2} & 0 & \cosh\frac{u_1}{2} & 0 \\ 0 & -\sinh\frac{u_2}{2} & 0 & \cosh\frac{u_2}{2} \end{pmatrix} \equiv (a^\mu_{\ \nu}) \ . \tag{10.90}$$

Now, an inspection of the second term in (10.46) leads to (10.89).

The reduction of the Laplacian to the space Γ_2 is very similar. Now, the functions restricted to A assume the form

$$G_2 = G_2^{(1)}(u_i) + G_2^{(2)}(u_i) s_{\dot{1}} s^1 + G_2^{(3)}(u_i) s_{\dot{2}} s^2 + G_2^{(4)}(u_i) s_{\dot{1}} s^1 s_{\dot{2}} s^2 \ . \tag{10.91}$$

The unperturbed part of the Hamiltonian is given by

$$H_{02} = -\begin{pmatrix} H_{sc}^{a,b} + \frac{3}{16}r^2 - \frac{3}{4}r & 0 & 0 \\ 0 & H_{\frac{1}{2}}^{a+\frac{1}{4},b-\frac{1}{4}} + \frac{3}{16}(r+1)^2 - \frac{3}{4}(r+1) & 0 \\ 0 & 0 & H_{sc}^{a+\frac{1}{2},b-\frac{1}{2}} + \frac{3}{16}(r+2)^2 - \frac{3}{4}(r+2) \end{pmatrix}. \tag{10.92}$$

Compared to the previous case, parameters a and b get shifted in the opposite direction, while the shift in r stays the same. Due to the fact that $(a^\mu{}_\nu)$ is a symmetric matrix, the nilpotent term is the same in two cases.

$$A = -2\begin{pmatrix} 0 & a^1{}_3 & a^2{}_4 & 0 \\ 0 & 0 & 0 & a^2{}_4 \\ 0 & 0 & 0 & a^1{}_3 \\ 0 & 0 & 0 & 0 \end{pmatrix} = -2\begin{pmatrix} 0 & \sinh\frac{u_1}{2} & -\sinh\frac{u_2}{2} & 0 \\ 0 & 0 & 0 & -\sinh\frac{u_2}{2} \\ 0 & 0 & 0 & \sinh\frac{u_1}{2} \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{10.93}$$

To summarise, the Laplacian on $SL(4|1)$ reduces on the spaces Γ_i of K-spherical functions to operators $H_i = H_{0i} + A$, with H_{0i} written in (10.86) and (10.92) and A given in (10.89). We now turn to solutions of the eigenvalue equations for H_i.

10.6.2 Construction of superconformal blocks

Eigenfunctions of Hamiltonians H_i are found similarly as in the one-dimensional example. We will derive them for the operator H_1 and just state the result for H_2. The eigenfunctions of the unperturbed Hamiltonian H_{01} read

$$G_1^0 = \phi_{\Delta+1,l}^{a,b} e_1, \quad G_2^0 = \tilde{\Psi}_{-,\Delta+1,l}^{a-\frac{1}{4},b+\frac{1}{4}}, \quad G_3^0 = \tilde{\Psi}_{+,\Delta+1,l}^{a-\frac{1}{4},b+\frac{1}{4}}, \quad G_4^0 = \phi_{\Delta+1,l}^{a-\frac{1}{2},b+\frac{1}{2}} e_4 \ . \tag{10.94}$$

Here, $\{e_1, ..., e_4\}$ denotes the standard basis of $\mathbb{C}^4$ and it is understood that the two non-zero components of G_2 and G_3 are in the space spanned by e_2 and e_3. The corresponding eigenvalues are

$$C_2 = C_3 = \frac{1}{4}\Delta(\Delta-2) + \frac{1}{4}l(l+3) - \frac{3}{16}(r-1)^2 + \frac{3}{8}, \tag{10.95}$$

$$C_1 = \frac{1}{4}\Delta(\Delta-2) + \frac{1}{4}l(l+2) - \frac{3}{16}(r-2)^2, \ C_4 = \frac{1}{4}\Delta(\Delta-2) + \frac{1}{4}l(l+2) - \frac{3}{16}r^2 \ . \tag{10.96}$$

Now, the most general form of eigenfunctions of H as allowed by representation theory is[3]

$$G_1 = \phi^{a,b}_{\Delta+1,l} e_1, \tag{10.98}$$

$$G_2 = \tilde{\Psi}^{a-\frac{1}{4},b+\frac{1}{4}}_{-,\Delta+1,l} + \left(c^{-1}_{\Delta,l}\, \phi^{a,b}_{\Delta+\frac{1}{2},l} + c^{-2}_{\Delta,l}\, \phi^{a,b}_{\Delta+\frac{3}{2},l+1}\right) e_1, \tag{10.99}$$

$$G_3 = \tilde{\Psi}^{a-\frac{1}{4},b+\frac{1}{4}}_{+,\Delta+1,l} + \left(c^{+1}_{\Delta,l}\, \phi^{a,b}_{\Delta+\frac{1}{2},l+1} + c^{+2}_{\Delta,l}\, \phi^{a,b}_{\Delta+\frac{3}{2},l}\right) e_1, \tag{10.100}$$

$$G_4 = \phi^{a-\frac{1}{2},b+\frac{1}{2}}_{\Delta+1,l} e_4 + c^{1-}_{\Delta,l}\, \tilde{\Psi}^{a-\frac{1}{4},b+\frac{1}{4}}_{-,\Delta+\frac{1}{2},l} + c^{2-}_{\Delta,l}\, \tilde{\Psi}^{a-\frac{1}{4},b+\frac{1}{4}}_{-,\Delta+\frac{3}{2},l-1} + c^{1+}_{\Delta,l}\, \tilde{\Psi}^{a-\frac{1}{4},b+\frac{1}{4}}_{+,\Delta+\frac{1}{2},l-1} + c^{2+}_{\Delta,l}\, \tilde{\Psi}^{a-\frac{1}{4},b+\frac{1}{4}}_{+,\Delta+\frac{3}{2},l} +$$
$$+ \left(k^{00}_{\Delta,l}\, \phi^{a,b}_{\Delta,l} + k^{01}_{\Delta,l}\, \phi^{a,b}_{\Delta+1,l+1} + k^{10}_{\Delta,l}\, \phi^{a,b}_{\Delta+1,l-1} + k^{11}_{\Delta,l}\, \phi^{a,b}_{\Delta+2,l}\right) e_1 \,. \tag{10.101}$$

Each G_i is obtained as a perturbation of $G_i^{(0)}$ and they have the same eigenvalue. The construction of these four solutions requires increasing orders of perturbation theory. The solution G_1 is obviously obtained at the zeroth order and equal to the scalar bosonic block G_1^0. The second and the third solution G_2 and G_3 are obtained at the first order while the last solution G_4 required to go to the second order. However, as before, once the Clebsch-Gordan identities $(10.26)-(10.28)$ are used, the above ansatz turns the eigenvalue problem into a system of linear algebraic equations for the coefficients $c^{\pm i}$, $c^{i\pm}$ and k^{ij}. The solutions read

$$c^{\pm i} = \frac{-\gamma^{\pm i}}{\frac{3}{16}(3-2r) + C^{sc}_{\pm i} - C^{seed}_{\Delta+1,l}} \quad , \qquad c^{i\pm} = \frac{\gamma^{i\pm}}{\frac{3}{16}(1-2r) + C^{seed}_{i\pm} - C^{sc}_{\Delta+1,l}} \tag{10.102}$$

and

$$k^{00}_{\Delta,l} = -\frac{c^{1-}_{\Delta,l}\gamma^{-1}_{\Delta-\frac{1}{2},l} + c^{1+}_{\Delta,l}\gamma^{+1}_{\Delta-\frac{1}{2},l-1}}{\frac{3}{4}(1-r) - C_{\Delta+1,l} + C_{\Delta,l}} \,, \qquad k^{01}_{\Delta,l} = -\frac{c^{1-}_{\Delta,l}\gamma^{-2}_{\Delta-\frac{1}{2},l} + c^{2+}_{\Delta,l}\gamma^{+1}_{\Delta+\frac{1}{2},l}}{\frac{3}{4}(1-r) - C_{\Delta+1,l} + C_{\Delta+1,l+1}}, \tag{10.103}$$

$$k^{10}_{\Delta,l} = -\frac{c^{2-}_{\Delta,l}\gamma^{-1}_{\Delta+\frac{1}{2},l-1} + c^{1+}_{\Delta,l}\gamma^{+2}_{\Delta-\frac{1}{2},l-1}}{\frac{3}{4}(1-r) - C_{\Delta+1,l} + C_{\Delta+1,l-1}} \,, \qquad k^{11}_{\Delta,l} = -\frac{c^{2-}_{\Delta,l}\gamma^{-2}_{\Delta+\frac{1}{2},l-1} + c^{2+}_{\Delta,l}\gamma^{+2}_{\Delta+\frac{1}{2},l}}{\frac{3}{4}(1-r) - C_{\Delta+1,l} + C_{\Delta+2,l}} \,. \tag{10.104}$$

Here, we have introduced the notation

$$\begin{pmatrix} C^\sigma_{-1} & C^\sigma_{-2} \\ C^\sigma_{+1} & C^\sigma_{+2} \end{pmatrix} = \begin{pmatrix} C^\sigma_{\Delta+\frac{1}{2},l} & C^\sigma_{\Delta+\frac{3}{2},l+1} \\ C^\sigma_{\Delta+\frac{1}{2},l+1} & C^\sigma_{\Delta+\frac{3}{2},l} \end{pmatrix} \,, \quad \begin{pmatrix} C^\sigma_{1-} & C^\sigma_{1+} \\ C^\sigma_{2-} & C^\sigma_{2+} \end{pmatrix} = \begin{pmatrix} C^\sigma_{\Delta+\frac{1}{2},l} & C^\sigma_{\Delta+\frac{1}{2},l-1} \\ C^\sigma_{\Delta+\frac{3}{2},l-1} & C^\sigma_{\Delta+\frac{3}{2},l} \end{pmatrix} \,, \tag{10.105}$$

[3] The tensor product of any given finite dimensional $SO(d+1,1)$ representation T_ν labelled by a Young tableau ν and the induced representation $\pi_{\Delta,\mu}$ can be decomposed into a finite sum of induced representations as

$$T_\nu \otimes \pi_{\Delta,\mu} = \bigoplus_{i=-j}^{j} \bigoplus_{\lambda\in\nu^i\otimes\mu} \pi_{\Delta+i,\lambda} \tag{10.97}$$

where indices (i,ν^i) are defined through the decomposition of the $SO(d+2)$ representation ν with respect to its $SO(2)\times SO(d)$ subgroup and enumerate a (semi) integer $SO(2)$ conformal weight i along with an $SO(d)$ Young tableau ν^i.Bosonic conformal blocks are particular matrix elements of some representation $\pi_{\Delta,\mu}$ of $G_{(0)}$, [9]. In the course of perturbation theory they are multiplied by matrix elements of the fundamental representation $\pi=\pi_f$ of $G_{(0)}$. Therefore, the bosonic blocks that appear in the n-th order of the perturbation theory are matrix elements of $\pi_{\Delta,\mu}\otimes\pi_f^{\otimes n}$. This fixes the functional form of our solutions.

for $\sigma = sc, seed$. This completes our discussion of the solution to the eigenvalue problem of H_1 and thereby the construction of harmonic functions in the space Γ_1. Harmonics in Γ_2 are very similar. This is in particular due to symmetry properties of seed blocks (10.20) – (10.21). In fact, these relations can be used to find eigenfunctions of H_2 based on the solutions for H_1. They are

$$G_1 = \phi^{a,b}_{\Delta+1,l} e_1, \tag{10.106}$$

$$G_2 = \Psi^{a+\frac{1}{4},b-\frac{1}{4}}_{+,\Delta+1,l} + \left(d^{+1}_{\Delta,l}\, \phi^{a,b}_{\Delta+\frac{1}{2},l} + d^{+2}_{\Delta,l}\, \phi^{a,b}_{\Delta+\frac{3}{2},l+1} \right) e_1, \tag{10.107}$$

$$G_3 = \Psi^{a+\frac{1}{4},b-\frac{1}{4}}_{-,\Delta+1,l} + \left(d^{-1}_{\Delta,l}\, \phi^{a,b}_{\Delta+\frac{1}{2},l+1} + d^{-2}_{\Delta,l}\, \phi^{a,b}_{\Delta+\frac{3}{2},l} \right) e_1, \tag{10.108}$$

$$\begin{aligned} G_4 = {}& \phi^{a+\frac{1}{2},b-\frac{1}{2}}_{\Delta+1,l} e_4 + d^{1+}_{\Delta,l}\, \Psi^{a+\frac{1}{4},b-\frac{1}{4}}_{+,\Delta+\frac{1}{2},l} + d^{2+}_{\Delta,l}\, \Psi^{a+\frac{1}{4},b-\frac{1}{4}}_{+,\Delta+\frac{3}{2},l-1} + d^{1-}_{\Delta,l}\, \Psi^{a+\frac{1}{4},b-\frac{1}{4}}_{-,\Delta+\frac{1}{2},l-1} + d^{2-}_{\Delta,l}\, \Psi^{a+\frac{1}{4},b-\frac{1}{4}}_{-,\Delta+\frac{3}{2},l} + \\ & + \left(\kappa^{00}_{\Delta,l}\, \phi^{a,b}_{\Delta,l} + \kappa^{01}_{\Delta,l}\, \phi^{a,b}_{\Delta+1,l+1} + \kappa^{10}_{\Delta,l}\, \phi^{a,b}_{\Delta+1,l-1} + \kappa^{11}_{\Delta,l}\, \phi^{a,b}_{\Delta+2,l} \right) e_1, \end{aligned} \tag{10.109}$$

with the coefficients

$$d^{+i} = -\frac{1}{2}(-1)^{-2a-2b} c^{-i}(-a,-b), \quad d^{-i} = -2(-1)^{-2a-2b} c^{+i}(-a,-b), \tag{10.110}$$

$$d^{i-} = -\frac{1}{2}(-1)^{2a+2b} c^{i+}(-a,-b), \quad d^{i+} = -2(-1)^{2a+2b} c^{i-}(-a,-b), \tag{10.111}$$

and

$$\kappa_{ij} = k_{ij}(-a,-b)\ . \tag{10.112}$$

This concludes our discussion of four-dimensional superconformal blocks.

Chapter 11

Defect conformal correlators and conformal blocks

In the introductory chapters we have mentioned that techniques of conformal bootstrap apply to theories in which conformal symmetry is broken to a subgroup. The subgroup in question consists of all those conformal transformation that preserve some submanifold of the full space, conventionally referred to as the defect. Defects typically model an impurity in a statistical system or a boundary of the experimental setup. In order to be able to apply the bootstrap methods, the broken symmetry group has to be sufficiently large and therefore, the defect is usually assumed to be a p-dimensional subspace of $\mathbb{R}^d$. Then, in a way not very different from that in ordinary CFTs, one can expand correlators in partial waves and formulate the appropriate crossing symmetry relations.

Since the symmetry group is reduced, correlation functions are less constrained compared to those of CFTs. The functional form of correlators, as allowed by symmetry, was determined in [63, 64, 126] and the possible tensor structures that appear in them were subsequently studied in [127, 128, 129]. A simple analysis shows that the smallest correlator not fixed by Ward identities is the two-point function of bulk fields. Corresponding conformal blocks were first computed in [60] and slightly more generally in [61, 62] (see also [130, 131] for related work). While these kinematical aspects are similar in spirit to those of ordinary conformal theories, the bootstrap analysis proceeds very differently. Namely, coefficients that appear in the crossing equations are no longer positive, which makes it difficult to use numerical techniques (see however [132, 133]). However, the equations may be amenable to analytic treatments, [134, 135]. In this regard, especially popular are defects of codimension one, i.e. boundaries. In the presence of a boundary, two-point functions of bulk operators depend on a single cross ratio and can be analysed using methods of one-dimensional CFTs, [136, 137, 138, 139].

In this chapter, we will not be concerned with these interesting questions about dynamics, but instead try to construct a unified theory for the kinematical aspects of defect CFTs. It turns out that harmonic analysis provides a very satisfactory (in our opinion) such a theory, based on the notion of a lift of a conformal field. In ordinary conformal field theory, functions on the Euclidean space can be lifted to covariant functions on the conformal group G_d in a way that carries the action on fields to the left-regular action of G_d. In other words, the lifting is an intertwiner between two different realisations of a non-unitary principal series representation of G_d. The correspondence between functions on M and covariant functions on G_d becomes particularly clear when the conformal group is decomposed into Bruhat factors. Then, one of the factors is diffeomorphic to the spacetime and the remaining ones make up the parabolic subgroup P, with respect to which covariance laws are imposed.

It turns out that a similar realisation of fields as functions on the symmetry group is possible also in defect CFTs. Fields on the defect can be lifted to the group in the obvious way, but it is a priori not clear how to lift the bulk fields, since the spacetime M can no longer be naturally regarded as a subset of $G_{d,p}$. However, we will see that the Iwasawa decomposition of G_p provides us with the correct group to replace P with. This will allow us to explicitly lift bulk fields to functions on $G_{d,p}$ so that the CFT action on them is carried to the left-regular one.

This initial step places us in a favourable situation where correlation functions of m bulk and n defect fields are represented as certain covariant functions on $G_{d,p}^{m+n}$. By making some very natural transformations from the group theory point of view, such functions are reduced to ones on a smaller number of copies of $G_{d,p}$. Notice that this is precisely what we have done in ordinary CFTs to relate four-point functions to K-spherical functions through (7.40). We will obtain formulas similar to (7.40) for various two- and three-point functions. In each case, the space of solutions to Ward identities is put in bijective correspondence with that of certain covariant functions on $G_{d,p}$.

Conformal partial waves in defect theories can, similarly as in ordinary CFTs, be characterised as solutions to appropriate Casimir equations. However, for bulk-bulk-defect three-point functions or higher, there is more than one quadratic Casimir operator (here, we are not merely talking about the fact that quadratic Casimirs of two simple factors of $G_{d,p}$ commute). Under the above correspondence, one of these Casimirs is mapped to the Laplacian on $G_{d,p}$. For bulk-bulk two-point functions and bulk-defect-defect three-point functions, there is essentially one non-trivial cross ratio and one quadratic Casimir. It is mapped to the Laplacian and the eigenfunctions are constructed in terms of one-variable hypergeometric functions. We say "essentially", because all defect correlators depend on one "transverse" cross ratio κ. Casimir equations factorise in the transverse and longitudinal variables and any conformal block is a function of the longitudinal cross ratios multiplied by a Gegenbauer polynomial in $\cos\kappa$.

In this sense, the first truly non-trivial correlator to consider is the bulk-bulk-defect three-point function. It depends on three cross ratios, κ and two longitudinal variables $v_{1,2}$. Correspondingly, there are two quadratic Casimir operators that arise physically by bringing each of the bulk fields to the defect and performing the bulk-defect OPE. Either of these Casimirs can be brought to the Laplacian by an appropriate version of our map, but obviously not both simultaneously. Nevertheless, one can pick one of these maps and compute the two second order operators in v_i. It turns out that the operators form the Appell's system of hypergeometric equations. This will allow us to construct the partial waves in terms of Appell's function F_4 (and Gegenbauer polynomials in $\cos\kappa$). Our result is the first of its kind - all previous studies focused on blocks that can be built by solving one-variable problems. Indeed, blocks for the three-point function of one bulk and two defect fields were constructed only recently in [58], where the authors also considered the more difficult case of two bulk and one defect field. The latter blocks were determined for special configurations of points that are parametrised by two cross ratios. Upon restriction to a two-dimensional submanifold of configurations, our results reproduce those of [58]. Before computing bulk-bulk-defect partial waves, we will treat two-point functions and bulk-defect-defect three-point functions. Our findings agree with the existing literature wherever a comparison is possible.

Constructions just described lead to some natural questions. Can we go on and compute partial waves for higher-point functions? And should we care? To start with the second question, the study of multipoint functions in (ordinary or defect) conformal field theories is in its early stages, but holds a promise of being very rewarding. We will give the basic reasoning for such an expectation in the next chapter. That said, the four-point function of two bulk and two defect fields seems to be within reach of our current methods. It still admits a representation as a function on a single copy of $G_{d,p}$ and depends on five variables. In fact, the representation takes $G_{2,2}(x_i)$ to a spherical function (which is e.g. not the case for $G_{2,1}(x_i)$ or $G_{1,2}(x_i)$), so despite a large number of cross ratios, there is a hope of developing a well-behaved theory for the blocks. Analysis of this correlator is certainly an interesting problem for the future.

The chapter is organised as follows. In the first section, we will introduce the notion of a lift and discuss under which circumstances is such a lift compatible with the action of the symmetry group. In the second section, the lifts of scalar bulk and defect conformal fields to functions on the defect conformal group are constructed. Later parts of the chapter use these results to provide new representations of various two- and three-point functions and compute associated partial waves. In particular, our main new results are given in the last section. The chapter is almost entirely based on the article [7].

11.1 Lifting conformal primary fields

In earlier chapters we have on many occasions used the fact that fields in a CFT carry a principal series representation of the conformal group. The mathematical origin of this fact was that the spacetime $\mathbb{R}^d \cup \{\infty\}$ could be identified with the nilpotent factor of the Bruhat (Gauss) decomposition of G. Thus, we could extend any function on M to the whole group using some specified covariance law with respect to the parabolic subgroup P. The resulting function was a vector in a (non-unitary) principal series representation of G. Finally, the extension was shown to carry the usual action on spacetime fields to the (restriction of the) left-regular action of G.

We shall now generalise this construction and realise the field representations of a defect conformal theory as induced representations of the defect conformal group. Let $\varphi : M \to W$ be a bulk primary field valued in some vector space W. The space of fields carries a representation of the conformal group G_d, and thus by restriction, of the defect group $G_{d,p}$ as well. The representation, denoted π, is given by

$$(\pi_h\varphi)(hx) = \rho(dh_x)\varphi(x) = \rho(k(x,h))\varphi(x) \ . \tag{11.1}$$

Here, ρ is the representation of the group $K_d = SO(1,1) \times SO(d)$ that characterises transformation properties of φ. Further, h is an arbitrary element of $G_{d,p}$ and dh is its differential when h is regarded as a smooth map $M \to M$.

In order to extend the field φ to a vector valued function $f : G_{d,p} \to W$ on the defect conformal group, we will specify four pieces of data. First, we need an embedding of the bulk space M into the defect conformal group,

$$g_d : M \to G_{d,p} \ . \tag{11.2}$$

This replaces the identification of the spacetime with the group of translations in G_d. Since the group of translations $\mathbb{R}^d$ is not contained in $G_{d,p}$ there is no obviously natural way to regard M as a submanifold of $G_{d,p}$, and therefore we keep g_d as part of the lift data. On the manifold $g_d(M)$ we require that f agrees with φ, possibly up to some specified prefactor $\Phi(x)$

$$f(g_d(x)) = \Phi(x)\varphi(x) \ . \tag{11.3}$$

Next, we pick a subgroup $S_{d,p} \subset G_{d,p}$ such that almost all of its orbits in $G_{d,p}$ (under the right-regular action) intersect $g_d(M)$ exactly once. We postulate that f is right-covariant with respect to $S_{d,p}$

$$f(gs) = \mu(s)^{-1}f(g), \quad g \in G_{d,p},\ s \in S_{d,p} \ . \tag{11.4}$$

Here, $\mu : S_{d,p} \to \mathrm{Aut}(W)$ is some finite-dimensional representation of $S_{d,p}$. Clearly, the function f is uniquely determined almost everywhere on $G_{d,p}$ by the properties (11.3) and (11.4). We shall call the quadruple $(g_d, S_{d,p}, \mu, \Phi)$ a lift of the bulk field. By definition, the function f is an element of the induced module $\mathrm{Ind}_{S_{d,p}}^{G_{d,p}}\mu$ and there is a bijective correspondence between $\mathrm{Ind}_{S_{d,p}}^{G_{d,p}}\mu$ and the space of fields φ. We wish to determine under what conditions is this correspondence an isomorphism of representations.

As the first step in this direction, note that the data just introduced allows to factorise almost all elements of $G_{d,p}$ uniquely as $g = g_d(x)s$. Moreover, it defines an action of $G_{d,p}$ on the space M as follows. Given any $h \in G_{d,p}$, the factorisation

$$hg_d(x) = g_d(y(x,h))s_d(x,h), \tag{11.5}$$

defines functions $y(x,h)$ and $s_d(x,h)$. In particular, the function $y(x,h)$ is an action of the group $G_{d,p}$ on the bulk space. Indeed, we can write $h_1h_2g_d(x)$ in two ways

$$\begin{aligned} &h_1h_2g_d(x) = g_d(y(x,h_1h_2))s_d(x,h_1h_2) \\ &= h_1g_d(y(x,h_2))s_d(x,h_2) = g_d(y(y(x,h_2),h_1))s_d(y(x,h_2),h_1)s_d(x,h_1) \ . \end{aligned}$$

The product of last two terms in the second line is again an element of $S_{d,p}$, so we conclude

$$y(x,h_1h_2) = y(y(x,h_2),h_1) \ .$$

This precisely says that $y(x,h)$ is an action of $G_{d,p}$ on M. In principle, this action may or may not coincide with the geometric action of the defect conformal group on the bulk.[1] This depends both on the map g_d and on the subgroup $S_{d,p}$, but not on Φ or μ. If the action $y(x,h)$ is the geometric action on M we shall say that the lift is geometric.

Having a geometric lift goes a long way towards a construction of an intertwiner. For, let f_φ be the lift of a function φ and denote by L the left regular representation of $G_{d,p}$. Then we have

$$\begin{aligned} &L_{h^{-1}}f_\varphi(g) = f_\varphi(hg) = f_\varphi(hg_d(x)s) = f_\varphi\left(g_d(y(x,h))s_d(x,h)s\right) = \mu(s^{-1})\mu(s_d(x,h)^{-1}) \\ &\Phi(y(x,h)))\varphi(y(x,h)) = \mu(s^{-1})\Phi(x)\pi_{h^{-1}}\varphi(x) = \mu(s^{-1})f_{\pi_{h^{-1}}\varphi}(g_d(x)) = f_{\pi_{h^{-1}}\varphi}(g) \ . \end{aligned}$$

In order to get to the second line we have used the fact that the lift is geometric and the identity

$$\mu(s_d(x,h)^{-1})\Phi(hx)\varphi(hx) = \Phi(x)\pi_{h^{-1}}\varphi(x) \ .$$

[1] The geometric action of a transformation h on the point x is always written as hx.

Looking back at (11.1) , a sufficient condition for this identity to hold is

$$\Phi(hx)^{-1}\mu(s_d(x,h))\Phi(x) = \rho(k(x,h)) \ . \tag{11.6}$$

This is an equation for both Φ and μ that were left completely arbitrary by the requirement of geometricity of the lift. If they are satisfied, the lift is an intertwiner between the representation on fields φ and the left regular representation of $G_{d,p}$ restricted to the space of right $S_{d,p}$-covariant functions. We observe that, if ρ is trivial, then a geometric lift with $\Phi = 1$ and $\mu = 1$ is an intertwiner.

11.2 Construction of the lift

Let us now explicitly construct a geometric lift of bulk fields to the defect conformal group. The group with respect to which the associated function f is required to be covariant is

$$S_{d,p} = SO(p+1) \times SO(q-1) \ . \tag{11.7}$$

It is generated by rotations in the defect plane, transverse rotations that preserve one particular direction, say e_d, together with elements of the form $P_a - K_a$. Next, we embed the bulk space into $G_{d,p}$ by the map

$$g_d : \mathbb{R}^d \to G_{d,p}, \quad g_d(x) = e^{x^a P_a}|x_\perp|^D e^{\varphi^i M_{id}} \ . \tag{11.8}$$

Here $\varphi^{p+1}, ..., \varphi^{d-1}$ are the angles of a spherical coordinate system on $\mathbb{R}^q$. To be precise, these coordinates are defined in such a way that $e^{\varphi^i M_{id}}$ maps the vector e_d to $x_\perp/|x_\perp|$ in $\mathbb{R}^q$. There is a unique element with this property of the above form. To show that the pair $(S_{d,p}, g_d)$ defines a geometric lift, we determine $y(x,h)$ and $s_d(x,h)$

$$\begin{aligned}
m(\hat{x}')e^{x^a P_a}|x_\perp|^D e^{\varphi^i M_{id}} &= e^{(\hat{x}'+x)^a P_a}|x_\perp|^D e^{\varphi^i M_{id}} &&\Longrightarrow \quad y(x, m(\hat{x}')) = m(\hat{x}')x, \quad s_d(x, m(\hat{x}')) = 1,\\
e^{\lambda D}e^{x^a P_a}|x_\perp|^D e^{\varphi^i M_{id}} &= e^{(e^\lambda x)^a P_a}(e^\lambda|x_\perp|)^D e^{\varphi^i M_{id}} &&\Longrightarrow \quad y(x, e^{\lambda D}) = e^{\lambda D}x, \quad s_d(x, e^{\lambda D}) = 1,\\
r_p e^{x^a P_a}|x_\perp|^D e^{\varphi^i M_{id}} &= e^{(r_p x)^a P_a}|x_\perp|^D e^{\varphi^i M_{id}} r_p &&\Longrightarrow \quad y(x, r_p) = r_p x, \quad s_d(x, r_p) = r_p \ .
\end{aligned}$$

Here $r_p \in SO(p)$. Let us now consider transverse rotations $r_q \in SO(q)$. In order for the decomposition

$$r_q e^{x^a P_a}|x_\perp|^D e^{\varphi^i M_{id}} = e^{x^a P_a}|x_\perp|^D r_q e^{\varphi^i M_{id}} = e^{x^a P_a}|x_\perp|^D e^{\psi^i M_{id}} r'_{q-1}, \tag{11.9}$$

to hold, we must have the equality

$$r_q e^{\varphi^i M_{id}} = e^{\psi^i M_{id}} r'_{q-1} \ . \tag{11.10}$$

Here the factor r'_{q-1} belongs to the group $SO(q-1)$ that stabilises the vector e_d. If one acts with both sides on e_d, one learns

$$\frac{1}{|x_\perp|} r_q(x_\perp) = r_q\left(\frac{x_\perp}{|x_\perp|}\right) = e^{\psi^i M_{id}}(e_d) = \frac{y_\perp(x, r_q)}{|y_\perp(x, r_q)|} \ . \tag{11.11}$$

But we already know from the dilation factor in the decomposition (11.9) that $|y_\perp(x,r_q)| = |x_\perp|$, so we conclude

$$y(x,r_q) = r_q x, \quad s_d(r_q,x) = r'_{q-1}\ . \tag{11.12}$$

The precise form of r'_{q-1} is not important for us at the moment, but we observe that $r'_{q-1} = r_q$ whenever $r_q \in SO(q-1)$. This follows from the fact that the space spanned by $\{M_{id}\}$ is closed under conjugation by elements in $SO(q-1)$ (it carries the vector representation under the adjoint action). Finally, the action of the Weyl inversion w_p is found with the help of the Iwasawa decomposition

$$\begin{aligned}
w_p g_d(x) &= e^{\left(\frac{s_p x_\parallel}{|x_\parallel|^2}\right)^a P_a} e^{-s_p x_\parallel^a K_a} |x_\parallel|^{-2D} s_{e_p} s_{x_\parallel} |x_\perp|^D e^{\varphi^i M_{id}}\\
&= e^{\left(\frac{s_p x_\parallel}{|x_\parallel|^2}\right)^a P_a} \left(\frac{|x_\perp|}{|x_\parallel|^2}\right)^D e^{-\frac{|x_\perp|}{|x_\parallel|^2} s_p x_\parallel^a K_a} s_{e_p} s_{x_\parallel} e^{\varphi^i M_{id}}\\
&= e^{\left(\frac{s_p x_\parallel}{|x_\parallel|^2}\right)^a P_a} \left(\frac{|x_\perp|}{|x_\parallel|^2}\right)^D e^{\left(-\frac{|x_\perp| s_p x_\parallel/|x_\parallel|^2}{1+|x_\perp|^2/|x_\parallel|^2}\right)^a P_a} \left(1+\frac{|x_\perp|^2}{|x_\parallel|^2}\right)^{-D} k_I\left(\frac{-|x_\perp| s_p x_\parallel}{|x_\parallel|^2}\right) s_{e_p} s_{x_\parallel} e^{\varphi^i M_{id}}\\
&= e^{\left(\frac{s_p x_\parallel}{|x|^2}\right)^a P_a} \left(\frac{|x_\perp|}{|x|^2}\right)^D e^{\varphi^i M_{id}} k_I\left(\frac{-|x_\perp| s_p x_\parallel}{|x_\parallel|^2}\right) s_{e_p} s_{x_\parallel}\ .
\end{aligned}$$

The decomposition was used to get to the second to last line by an application of (4.18). We also applied the G_p-Bruhat decomposition in the first step. Other manipulations in the derivation above, such as moving dilations past rotations and special conformal transformations, are evident. We read off

$$y(x,w_p) = w_p x, \quad s_d(x,w_p) = k_I\left(\frac{-|x_\perp| s_p x_\parallel}{|x_\parallel|^2}\right) s_{e_p} s_{x_\parallel}\ . \tag{11.13}$$

Elements of the form $m(\hat{x}'), e^{\lambda D}, r_p, r_q$ together with the Weyl inversion w_p generate the whole defect conformal group. Therefore, we have the following important corollary

$$y(x,h) = hx, \quad h \in G_{d,p}\ . \tag{11.14}$$

The action $y(x,h)$ defined in the manner explained above through the choice of the group (11.7) and the embedding (11.8) is precisely the action of the defect conformal group on the bulk space. That is, $(g_d, S_{d,p})$ gives rise to a geometric lift. The equation (11.14) is the most important result of this chapter and all subsequent applications will rely on it in an essential way.

Having found the group $S_{d,p}$ and the map g_d, we still need to solve equations (11.6) in order to turn a lift into a morphism of representations. For various types of group elements, the equations read

$$\Phi(m(\hat{x}')x)^{-1}\Phi(x) = 1, \quad \Phi(e^{\lambda D}x)^{-1}\Phi(x) = e^{-\Delta\lambda}, \quad \Phi(r_p x)^{-1}\mu(r_p)\Phi(x) = \rho(r_p), \tag{11.15}$$

$$\Phi(r_q x)^{-1}\mu(r'_{q-1})\Phi(x) = \rho(r_q), \quad \Phi(w_p x)^{-1}\mu\left(k_I\left(\frac{-|x_\perp| s_p x_\parallel}{|x_\parallel|^2}\right) s_{e_p} s_{x_\parallel}\right)\Phi(x) = \rho(k(x,w_p))\ . \tag{11.16}$$

We learn that Φ is a homogeneous function of $x_\perp$, that is

$$\Phi = \Phi(x_\perp), \quad \Phi(\lambda x_\perp) = \lambda^\Delta \Phi(x_\perp) \ . \tag{11.17}$$

Under these conditions, the remaining equations simplify

$$\Phi(x_\perp)\rho(r_p) = \mu(r_p)\Phi(x_\perp), \quad \Phi(r_q x_\perp)\rho(r_q) = \mu(r'_{q-1})\Phi(x_\perp), \tag{11.18}$$

$$\Phi(x_\perp)^{-1}\mu\left(k_I\left(\frac{-|x_\perp|s_p x_\parallel}{|x_\parallel|^2}\right)s_{e_p}s_{x_\parallel}\right)\Phi(x_\perp) = \rho(s_{e_p}s_x) \ . \tag{11.19}$$

Let us consider the case of a scalar bulk field and try to put μ to be the trivial representation. Then the equations (11.18) and (11.19) give only one non-trivial condition

$$\Phi(r_q x_\perp) = \Phi(x_\perp) \ . \tag{11.20}$$

Combining it with (11.17) we arrive at the unique solution

$$\Phi(x) = |x_\perp|^\Delta \ . \tag{11.21}$$

Thus, we have constructed an isomorphism between scalar fields in the bulk and a class of covariant functions on $G_{d,p}$. As this is the only setup that we will consider in applications of later sections, we will not discuss extensions to the case of spinning bulk fields at present.

Example Let us illustrate parts of the above discussion on the simplest non-trivial example, that of a line defect in a two-dimensional conformal field theory. The conformal group of the Euclidean plane is $G_d \sim SO(3,1)$ and the defect group is $G_{d,p} \sim SO(2,1)$. Let $\mathfrak{g}_d = \mathfrak{so}(3,1)$ be the complexified Lie algebra of G_d. We choose its basis

$$\mathfrak{g}_d = \text{span}\{P_\mu, K_\mu, D, M\}, \quad \mu = 1,2 \ . \tag{11.22}$$

The notation here is $M = M_{12} = -M_{21}$. The representation on fields of conformal weight Δ and spin l by differential operators reads

$$p_\mu = \partial_\mu, \quad m_{\mu\nu} = x_\nu\partial_\mu - x_\mu\partial_\nu - l\epsilon_{\mu\nu}, \quad d = x^\mu\partial_\mu + \Delta, \quad k_\mu = x^2\partial_\mu - 2x_\mu d - 2lx^\nu\epsilon_{\mu\nu} \ . \tag{11.23}$$

The Levi-Civita symbol has $\epsilon_{12} = 1$. Differential operators satisfy the opposite brackets compared to the generators (11.22). The defect algebra is spanned by $\{P_1, K_1, D\}$ and is isomorphic to the Lie algebra of the conformal group in one dimension. We write the differential operators explicitly

$$p_1 = \partial_1, \quad d = x_1\partial_1 + x_2\partial_2 + \Delta, \quad k_1 = (x_2^2 - x_1^2)\partial_1 - 2x_1x_2\partial_2 - 2\Delta x_1 - 2lx_2 \ . \tag{11.24}$$

These operators should be compared with the right-invariant vector fields on $SL(2,\mathbb{R})$ in the coordinates specified by

$$g = g_d(x)e^{\mu(P_1-K_1)} = e^{x_1P_1}e^{\log x_2 D}e^{\mu(P_1-K_1)} \ . \tag{11.25}$$

The vector fields are computed form the Maurer-Cartan form to give

$$\tilde{p} = \partial_1, \quad \tilde{d} = x_1\partial_1 + x_2\partial_2, \quad \tilde{k} = (x_2^2 - x_1^2)\partial_1 - 2x_1x_2\partial_2 - x_2\partial_\mu \ . \tag{11.26}$$

If we look at scalar fields, the construction above instructs us to set $\partial_\mu = 0$ and conjugate the operators (11.26) by $|x_2|^{-\Delta}$ in order to obtain (11.24) with $l = 0$. A simple calculation verifies that this is indeed the case.

11.2.1 Lifts of defect fields

Lifts of defect fields $\hat{\varphi} : N \to \hat{W}$ are defined completely analogously to those of fields in the bulk. These can be constructed in the same way as for ordinary CFTs. The representation of $G_{d,p}$ on $\hat{\varphi}$ reads

$$(\hat{\pi}_h \hat{\varphi})(h\hat{x}) = \hat{\rho}(dh_{\hat{x}})\hat{\varphi}(\hat{x}) \; . \tag{11.27}$$

In order to lift $\hat{\varphi}$ from N to the defect conformal group, we must fix an embedding $g_p : N \to G_{d,p}$ that intersects almost all orbits of the right regular action of $\hat{S}_{d,p}$ on $G_{d,p}$ exactly once. The embedding we will be using is given by

$$g_p(\hat{x}) = e^{\hat{x}^a P_a} \; . \tag{11.28}$$

Clearly, there is a decomposition $G_{d,p} = g_p(N)\hat{S}_{d,p}$ with the group

$$\hat{S}_{d,p} = (SO(1,1) \times SO(p)) \ltimes \mathbb{R}^p \times SO(q), \tag{11.29}$$

generated by dilations, special conformal transformations and (parallel and transverse) rotations. The action of the defect conformal group on the points $\hat{x}$ is transitive and the stabiliser of any point is isomorphic to $\hat{S}_{d,p}$ (stabilisers of different points are related by conjugation). As the lift of the defect primary $\hat{\varphi}$ we define the function $\hat{f} : G_{d,p} \to \hat{W}$ which agrees with $\hat{\varphi}$ on $g_p(N)$ and transforms covariantly under the right multiplication by $\hat{s} \in \hat{S}_{d,p}$,

$$\hat{f}(g_p(\hat{x})) = \hat{\varphi}(\hat{x}), \quad \hat{f}(g\hat{s}) = \hat{\mu}(\hat{s})^{-1}\hat{f}(g), \quad g \in G_{d,p}, \; \hat{s} \in \hat{S}_{d,p} \; . \tag{11.30}$$

Here, $\hat{\mu}$ is the finite-dimensional representation of $\hat{S}_{d,p}$ obtained from the representation $\hat{\rho}$ by trivial extension to the abelian factor $\mathbb{R}^p$. Clearly, conditions (11.30) define f uniquely almost everywhere on $G_{d,p}$ and the lift defined in this way is an intertwiner.

For future reference, we introduce the notation for the factorisation

$$hg_p(\hat{x}) = g_p(\hat{y}(\hat{x},h))\hat{s}_p(\hat{x},h) \quad \text{for} \;\; h \in G_{d,p} \; . \tag{11.31}$$

These factors are essentially the same as in an ordinary conformal field theory, since transverse rotations commute with the image of g_p.

11.3 Lifting correlation functions

The lifts of bulk and defect fields that we just constructed can be used to write down a new representation of correlation functions as functions on a number of copies of the defect conformal group. A correlator of m bulk and n defect fields will be written in terms of a covariant function on $G_{d,p}^{m+n}$. Our goal is to eventually end up with functions on just one copy of $G_{d,p}$ which can be done if the number of insertion points is sufficiently small. As a first step in this direction, we will show how one can lift pairs of bulk and defect fields, a trick that will be useful when we come to analyse three-point functions later.

Ward identities satisfied by a correlation function of m bulk and n defect fields can be compactly written using the field representations π_i and $\hat{\pi}_j$ as

$$G_{m,n} = (\pi_1(h) \otimes ... \otimes \hat{\pi}_n(h))G_{m,n} \; . \tag{11.32}$$

That is, $G_{m,n}$ is an invariant vector in the tensor product $\pi_1 \otimes ... \otimes \hat{\pi}_n$. Let us now assume that we are given two sets of intertwiners $(g_d, S_{d,p}, \mu_i, \Phi_i)$ and $(g_p, \hat{S}_{d,p}, \hat{\mu}_j, \hat{\Phi}_j)$ as described in the last section (in particular $\hat{\Phi}_j = 1$). They allow to lift any solution to the Ward identities $G_{m,n}$ to a function $F_{m,n} : G_{d,p}^{m+n} \to W$ (W is the tensor product of spaces of polarisations of the fields) which satisfies

$$F_{m,n}(g_d(x_1), ..., g_p(\hat{x}_n)) = \Phi_1(x_1)...\hat{\Phi}_n(\hat{x}_n)G_{m,n}(x_1, ..., \hat{x}_n), \tag{11.33}$$

and is right covariant

$$F_{m,n}(g_1 s_1, ..., g_m s_m, g_{m+1}\hat{s}_1, ..., g_{m+n}\hat{s}_n) = \tag{11.34}$$
$$= \Big(\mu_1(s_1^{-1}) \otimes ... \otimes \mu_m(s_m^{-1}) \otimes \hat{\mu}_1(\hat{s}_1^{-1}) \otimes ... \otimes \hat{\mu}_n(\hat{s}_n^{-1})\Big) F_{m,n}(g_1, ..., g_{m+n}) \ . \tag{11.35}$$

These two properties ensure that $F_{m,n}$ is defined almost everywhere on $G_{d,p}^{m+n}$. Now, the invariance of $G_{m,n}$, (11.32), and the intertwining property of lifts imply that $F_{m,n}$ is invariant under the diagonal left regular action of $G_{d,p}$

$$F_{m,n}(hg_i) = F_{m,n}(g_i) \ . \tag{11.36}$$

The function $F_{m,n}$ is our new representation of the correlator and the starting point for several other representations that will be constructed below.

11.3.1 Pairing up bulk and defect fields

A tool that efficiently reduces the number of copies needed to encode the function $F_{m,n}$ is "pairing up bulk and defect points". We can think of it as lifting pairs of fields, rather than individual ones. We now explain this process in more detail.

As mentioned before, functions $f, \hat{f}$ belong to induced representations of the defect conformal group

$$f \in \pi = \mathrm{Ind}_{S_{d,p}}^{G_{d,p}} W, \quad \hat{f} \in \hat{\pi} = \mathrm{Ind}_{\hat{S}_{d,p}}^{G_{d,p}} \hat{W} \ . \tag{11.37}$$

We have used the same notation $\pi, \hat{\pi}$ as for the representations on fields because our analysis indeed showed that these representations are isomorphic to one another. The tensor product $\pi \otimes \hat{\pi}$ is naturally realised in the space of functions

$$F : G_{d,p}^2 \to W \otimes \hat{W}, \quad F(g_1 s, g_2 \hat{s}) = \big(\mu(s)^{-1} \otimes \hat{\mu}(\hat{s})^{-1}\big) F(g_1, g_2), \tag{11.38}$$

under the diagonal left-regular action. We will be interested in another way of realising this representation:

Proposition Let $K = SO(p) \times SO(q-1)$ be the stabiliser of a pair of one bulk and one defect point in $G_{d,p}$. The following is an isomorphism of $G_{d,p}$-modules

$$Q : \pi \otimes \hat{\pi} \to \chi = \mathrm{Ind}_K^{G_{d,p}}(\mu \otimes \hat{\mu}), \quad (QF)(g) = F(g, g) \ . \tag{11.39}$$

Thus, Q is essentially composing a function F with the coproduct map on the group algebra $L^1(G)$. It is the properties of the coproduct that ensure Q respects the $G_{d,p}$-action.

Proof: First, observe that the representation χ is well-defined. Indeed, both μ and $\hat{\mu}$ are representations of K by restriction. Let us show that $Q(F) \in \chi$, that is, that is has the required covariant properties

$$Q(F)(gk) = F(gk, gk) = \left(\mu(k)^{-1} \otimes \hat{\mu}(k)^{-1}\right) F(g, g) = (\mu \otimes \hat{\mu})(k^{-1})Q(F)(g) \ . \tag{11.40}$$

Thus, Q is well-defined. It is clearly a $G_{d,p}$-module homomorphism

$$Q(g \cdot F)(g') = (g \cdot F)(g', g') = F(gg', gg') = Q(F)(gg') = (g \cdot Q(F))(g') \ . \tag{11.41}$$

It remains to prove that Q is a bijection. To this end, notice that almost any element $g \in G_{d,p}$ can be written as $g = \hat{s}s$ with $\hat{s} \in \hat{S}_{d,p}$ and $s \in S_{d,p}$. This is true by the following argument

$$g = g_p g_q = n_I(g_p)a_I(g_p)k_I(g_p)g_q = n_I(g_p)a_I(g_p)g_q \ k_I(g_p) \ . \tag{11.42}$$

In the first step, we have written g as a product of elements in $SO(p+1,1)$ and $SO(q)$. Then we have factorised the first term according to the Iwasawa decomposition and moved g_q past $k_I(g_p)$. The last expression is of the correct form $\hat{s}s$.

We can now reconstruct F from $Q(F)$. Given two elements $g_1, g_2 \in G_{d,p}$, let $s_1, \hat{s}_2$, be the above solutions to the decomposition $g_2^{-1}g_1 = \hat{s}_2 s_1^{-1}$. Then we have

$$\begin{aligned} F(g_1, g_2) &= (\mu(s_1) \otimes \hat{\mu}(\hat{s}_2))\, F(g_1 s_1, g_2 \hat{s}_2) \\ &= (\mu(s_1) \otimes \hat{\mu}(\hat{s}_2))\, F(g_1 s_1, g_1 s_1) = (\mu(s_1) \otimes \hat{\mu}_2(\hat{s}_2))\, Q(F)(g_1 s_1) \ . \end{aligned}$$

This completes the proof of the proposition. □

As a consequence, we can lift a pair of primary fields, one bulk and one defect, by composing the individual lifts with the isomorphism Q. The constructions allows us to uplift correlation functions of m bulk and n defect fields to the product group with $max(m,n) - 1$ factors $G_{d,p}$. In particular, correlation functions of two bulk and two defect fields can be lifted to functions on a single copy of the defect conformal group.

11.3.2 An example: bulk-defect two-point function

As a simple example of the above ideas, let us determine the form of a two-point function of one bulk and one defect field, $G_{1,1}(x_i)$. The correlator $G_{1,1}(x_i)$ lifts to a function $F_{1,1} : G_{d,p}^2 \to V$ which satisfies

$$F_{1,1}(g_1 s_1, g_2 \hat{s}_2) = \left(\mu_1(s_1^{-1}) \otimes \hat{\mu}_2(\hat{s}_2^{-1})\right) F_{1,1}(g_1, g_2), \quad F_{1,1}(hg_i) = F_{1,1}(g_i) \ . \tag{11.43}$$

Let us put $F = Q(F_{1,1})$, that is $F(g) = F_{1,1}(g, g)$. Then F is a constant function

$$F(hg) = F_{1,1}(hg, hg) = F_{1,1}(g, g) = F(g) \ . \tag{11.44}$$

To write the two-point function in terms of F we need to Iwasawa-decompose

$$g_{12}(x_i) = g_p(\hat{x}_2)^{-1} g_d(x_1) = e^{(x_1^a - \hat{x}_2^a)P_a} |x_{1\perp}|^D e^{\varphi^i M_{id}} \equiv \hat{s}_{12} s_{12}^{-1} \ . \tag{11.45}$$

As explained in the previous section, s_{12} and $\hat{s}_{12}$ are essentially the Iwasawa factors of g_{12}. We have

$$e^{\hat{x}_{12}^a P_a}|x_{1\perp}|^D = |x_{1\perp}|^D e^{\frac{\hat{x}_{12}^a}{|x_{1\perp}|}P_a} = |x_{1\perp}|^D e^{\frac{|x_{1\perp}|\hat{x}_{12}^a}{x_{1\perp}^2+\hat{x}_{12}^2}K_a}\left(\frac{\hat{x}_{12}^2+x_{1\perp}^2}{x_{1\perp}^2}\right)^D k_I\left(-\frac{\hat{x}_{12}}{|x_{1\perp}|}\right)$$
$$= e^{\frac{\hat{x}_{12}^a}{x_{1\perp}^2+\hat{x}_{12}^2}K_a}\left(\frac{\hat{x}_{12}^2+x_{1\perp}^2}{|x_{1\perp}|}\right)^D k_I\left(\frac{\hat{x}_{21}}{|x_{1\perp}|}\right) .$$

Therefore, the factors are

$$\hat{s}_{12} = e^{\frac{\hat{x}_{12}^a}{x_{1\perp}^2+\hat{x}_{12}^2}K_a}\left(\frac{\hat{x}_{12}^2+x_{1\perp}^2}{|x_{1\perp}|}\right)^D e^{\varphi_1^i M_{id}}, \quad s_{12} = k_I\left(\frac{\hat{x}_{21}}{|x_{1\perp}|}\right)^{-1} . \tag{11.46}$$

In terms of F, the two point function reads

$$G_{1,1}(x_i) = \frac{1}{\Phi_1(x_1)}F_{1,1}(g_d(x_1),g_p(\hat{x}_2)) = \frac{1}{\Phi_1(x_1)}\left(\mu_1(s_{12})\otimes\hat{\rho}_2(\hat{s}_{12})\right)F\left(g_d(x_1)s_{12}\right) . \tag{11.47}$$

Let us evaluate this expression further in the case where the bulk field is a scalar. Then we should put $\mu_1 = 1$, hence

$$G_{1,1}(x_i) = c\frac{(x_{1\perp}^2+\hat{x}_{12}^2)^{-\Delta_{\hat{2}}}}{|x_{1\perp}|^{\Delta_{1\hat{2}}}}\hat{\rho}_2(e^{\varphi_1^i M_{id}}), \tag{11.48}$$

for some constant c (such that $F \equiv c$). We have written the conformal dimensions of two fields as Δ_1 and $\Delta_{\hat{2}}$ and used the shorthand notation $\Delta_{1\hat{2}} = \Delta_1 - \Delta_{\hat{2}}$. If one assumes the transverse (internal) spin of the second field to be trivial, i.e. $\hat{\rho}_2(e^{\varphi_1^i M_{id}}) = 1$, one recognises the usual expression for the two-point function.

11.4 Bulk-bulk two-point function

Let us move to two-point functions of bulk fields. Their kinematical form is no longer completely fixed by symmetry and there are two invariants on which they can depend. According to the general theory, the correlator $G_{2,0}(x_i)$ lifts to a function $F_{2,0} : G_{d,p}^2 \to W$ which satisfies

$$F_{2,0}(g_1s_1,g_2s_2) = \left(\mu_1(s_1^{-1})\otimes\mu_2(s_2^{-1})\right)F_{2,0}(g_1,g_2), \quad F_{2,0}(hg_i) = F_{2,0}(g_i) . \tag{11.49}$$

Let us define a function $F : G_{d,p} \to W$ by $F(g) = F_{2,0}(e,g)$. We can easily recover $F_{2,0}$ from F using the above covariance properties

$$F_{2,0}(g_1,g_2) = F_{2,0}(e,g_1^{-1}g_2) = F(g_1^{-1}g_2) .$$

On the other hand, F is left-right covariant with respect to the subgroup $S_{d,p} \subset G_{d,p}$

$$F(s_1gs_2) = F_{2,0}(e,s_1gs_2) = F_{2,0}(s_1^{-1},gs_2) = \left(\mu_1(s_1)\otimes\mu_2(s_2^{-1})\right)F_{2,0}(e,g) = \left(\mu_1(s_1)\otimes\mu_2(s_2^{-1})\right)F(g) .$$

Therefore F can be regarded as a function on the double quotient $M_{2pt} = S_{d,p}\backslash G_{d,p}/S_{d,p}$. This space is two-dimensional as almost any element of $G_{d,p}$ can be written in the form

$$g = r_l^{p+1}e^{\lambda D}r_r^{p+1}\ r_l^{q-1}e^{\kappa M_{d-1,d}}r_r^{q-1} , \tag{11.50}$$

with $r_{l,r}^{p+1} \in SO(p+1)$ and $r_{l,r}^{q-1} \in SO(q-1)$. We will refer to this factorisation as the Cartan decomposition of g, as it is indeed the Cartan decomposition on each simple factor of $G_{d,p}$. The space M_{2pt} is the direct product of two double quotients of similar forms, $SO(p+1)\backslash SO(p+1,1)/SO(p+1)$ and $SO(q-1)\backslash SO(q)/SO(q-1)$, each of which is one-dimensional. The function F satisfies

$$F(g) = \left(\mu_1(r_l^{p+1}r_l^{q-1}) \otimes \mu_2(r_r^{p+1}r_r^{q-1})^{-1}\right) F(e^{\lambda D+\kappa M_{d-1,d}}) \ . \tag{11.51}$$

The restriction of F to the two-dimensional abelian subgroup generated by D and $M_{d-1,d}$ will be denoted by $\psi(\lambda,\kappa) = F(e^{\lambda D+\kappa M_{d-1,d}})$. We can relate ψ and the two-point function as soon as the Cartan decomposition of $g_d(x_1)^{-1}g_d(x_2)$ is known

$$G_{2,0}(x_i) = \frac{1}{\Phi_1(x_1)\Phi_2(x_2)}F_{2,0}(g_d(x_1),g_d(x_2)) = \frac{1}{\Phi_1(x_1)\Phi_2(x_2)}F\left(g_d(x_1)^{-1}g_d(x_2)\right) \ . \tag{11.52}$$

Let us denote the argument of F by $g_{2pt}(x_i)$. We determine its Cartan factors

$$g_{2pt}(x_i)_p = |x_{1\perp}|^{-D}e^{x_{21}^a P_a}|x_{2\perp}|^D = r_l^{p+1}(x_i)e^{\lambda D}r_r^{p+1}(x_i) \quad \text{with} \quad \cosh\lambda = \frac{x_{1\perp}^2+x_{2\perp}^2+\hat{x}_{12}^2}{2|x_{1\perp}||x_{2\perp}|} \ . \tag{11.53}$$

For a simple proof of (11.53), see the end of this section, (11.4.1). The factors $r_{l,r}^{p+1}$ are computed similarly. Let us now turn to the $SO(q)$-part. Again, in the subsection (11.4.1), it is shown that

$$g_{2pt}(x_i)_q = e^{-\varphi_1^i M_{id}}e^{\varphi_2^j M_{jd}} = r_l^{q-1}(x_i)e^{\kappa M_{d-1,d}}r_r^{q-1}(x_i) \quad \text{with} \quad \cos\kappa = \frac{x_1^i x_2^i}{|x_{1\perp}||x_{2\perp}|} \ . \tag{11.54}$$

Hence, the correlation function becomes

$$G_{2,0}(x_i) = \frac{1}{\Phi_1(x_1)\Phi_2(x_2)}\left(\mu_1(r_l^{p+1}(x_i)r_l^{q-1}(x_i)) \otimes \mu_2(r_r^{p+1}(x_i)r_r^{q-1}(x_i))^{-1}\right)\psi(\lambda,\kappa) \ . \tag{11.55}$$

Let us evaluate this expression further in the case when the fields are scalar. Then we should put $\mu_i = 1$, so

$$G_{2,0}(x_i) = \frac{1}{|x_{1\perp}|^{\Delta_1}|x_{2\perp}|^{\Delta_2}}\psi(\lambda,\kappa) \ . \tag{11.56}$$

The coordinates (λ,κ) are the two independent conformal invariants. They are related to coordinates (ϕ,χ) used in [62] by

$$\kappa = \phi, \quad \cosh\lambda = \frac{1}{2}\chi \ . \tag{11.57}$$

We recognise in (11.56) the usual expression for the two point function.

Conformal blocks are eigenfunctions of the Laplace-Beltrami operator within the space of covariant functions (11.51) . Let us show how this comes about. Partial waves for the two-point function $G_{2,0}(x_i)$ are characterised as eigenfunctions of the quadratic Casimir that is constructed out of the vector fields that represent the action of the defect conformal algebra $\mathfrak{g}_{d,p}$ on a scalar field. We may chose either the first or the second point for these differential operators. Let us choose the second one to be concrete. After the correlation function is lifted to $F_{2,0}$, results of the previous section tell us that the action generated by these vector fields maps to the left-regular action *on the second copy* of $G_{d,p}$. The corresponding geometric representation on $F_{2,0}$ reads

$$(g \cdot F_{2,0})(g_1, g_2) = F_{2,0}(g_1, g^{-1}g_2) \ . \tag{11.58}$$

This is indeed a representation of $G_{d,p}$ on the space $L^1(G_{d,p}^2, V)$, but clearly it does not respect the covariance properties satisfied by $F_{2,0}$. However, the quadratic Casimir does respect the covariance properties and equals the Riemannian Laplace-Beltrami operator Δ on the second copy of $G_{d,p}$, denoted $\Delta^{(2)}$. Furthermore, by the definition of F

$$\Delta F = \Delta^{(2)} F_{2,0}, \tag{11.59}$$

and hence the Casimir operator acting on F coincides with the Laplacian, as claimed. Conformal blocks factorise according to the direct product structure of $G_{d,p}$. It is possible to write the restriction $\psi_{\hat{\Delta},s}(\lambda, \kappa) = \psi_{p,\hat{\Delta}}(\lambda)\psi_{q,s}(\kappa)$ using standard representation theory. For scalar fields, conditions (11.51) tell us that $\psi_{p,\hat{\Delta}}$ and $\psi_{q,s}$ are zonal spherical functions written in (5.9) and (5.10). The function $\psi_{p,\hat{\Delta}}$ can be expressed in terms of a Legendre function using a hypergeometric identity. In fact, the functions $\psi_{p,\hat{\Delta}}$ and $\psi_{q,s}$ are very similar to each other, which is clear from the fact that they come from quotients that are related by analytic continuation. See [86] for more details.

Let us compare our conformal blocks to those of [62]. For the transverse part, we observe that the polynomials $\psi_{q,s}(\kappa)$ readily agree with the functions (4.9) from that paper. As for $\psi_{p,\hat{\Delta}}$, notice that the function

$$\tilde{\psi}_{p,\hat{\Delta}} = (\cosh\lambda)^{\hat{\Delta}} \ {}_2F_1\left(\frac{\hat{\Delta}+1}{2}, \frac{\hat{\Delta}}{2}; 1+\hat{\Delta}-\frac{p}{2}; 1-\tanh^2\lambda\right), \tag{11.60}$$

solves the same hypergeometric equation as $\psi_{p,\hat{\Delta}}$. Indeed, in our discussion above, we did not include the analysis of boundary conditions that supplement the Casimir differential equation. Once this is done, it turns out the $\tilde{\psi}_{p,\hat{\Delta}}$ is the correct eigenfunction to use. Now from (11.57) we can rewrite

$$\tilde{\psi}_{p,\hat{\Delta}} = \left(\frac{\chi}{2}\right)^{-\hat{\Delta}} \ {}_2F_1\left(\frac{\hat{\Delta}+1}{2}, \frac{\hat{\Delta}}{2}, 1+\hat{\Delta}-\frac{p}{2}, \frac{4}{\chi^2}\right), \tag{11.61}$$

in agreement with the equation (4.7) of [62]. This concludes our analysis of two-point correlation functions of scalar bulk fields in the presence of a defect. Of course our results here are not new, but the well-studied setup illustrates nicely how the group theoretic approach works.

11.4.1 Calculation of cross ratios

In this subsection we prove relations (11.53) and (11.54) . The formula (11.53) is obtained by taking the $(1,1)$ matrix element in the vector representation of both sides of

$$|x_{1\perp}|^{-D} e^{x_{21}^a P_a} |x_{2\perp}|^D = r_l^{p+1}(x_i) e^{\lambda D} r_r^{p+1}(x_i) \ . \tag{11.62}$$

For the coordinate defined in (11.54) consider the space $\mathbb{R}^q$ spanned by the vectors $e_{p+1},...,e_d$. A direct calculation shows that

$$e^{\varphi^i M_{id}} e_d = \left(\frac{\varphi^{p+1}\sin|\varphi|}{|\varphi|}, ..., \frac{\varphi^{d-1}\sin|\varphi|}{|\varphi|}, \cos|\varphi| \right)^t, \quad |\varphi| = \left(\sum_{p+1}^{d-1} \varphi^i \varphi^i \right)^{\frac{1}{2}} . \tag{11.63}$$

By definition, this is the vector $x_\perp/|x_\perp|$ if the element $e^{\varphi^i M_{id}}$ is associated to x. Furthermore, the bottom right matrix element of $e^{-\varphi_1^i M_{id}} e^{\varphi_2^j M_{jd}}$ is seen to be

$$\cos|\varphi_1|\cos|\varphi_2| + \frac{\sin|\varphi_1|\sin|\varphi_2|}{|\varphi_1||\varphi_2|} \sum_{j=p+1}^{d-1} \varphi_1^j \varphi_2^j = \frac{x_1^i x_2^i}{|x_{1\perp}||x_{2\perp|}} . \tag{11.64}$$

This is compared with the bottom right element of the matrix $r_l^{q-1} e^{\kappa M_{d-1,d}} r_r^{q-1}$, which is $\cos\kappa$. Thus, (11.54) follows.

11.5 Bulk-defect-defect three-point function

In order to address the less studied example of a three-point function involving one bulk and two defect fields, we start with by now familiar lift of the correlator $G_{1,2}(x_i)$ to a function $F_{1,2} : G^3_{d,p} \to W$ which satisfies

$$F_{1,2}(g_1 s_1, g_2 \hat{s}_2, g_3 \hat{s}_3) = \left(\mu_1(s_1^{-1}) \otimes \hat{\mu}_2(\hat{s}_2^{-1}) \otimes \hat{\mu}_3(\hat{s}_3^{-1})\right) F_{1,2}(g_1, g_2, g_3), \quad F_{1,2}(hg_i) = F_{1,2}(g_i) . \tag{11.65}$$

To simplify the lift, we can pair up the two defect fields in the correlation function and set $F(g) = F_{1,2}(e, g, gw_p^{-1})$. The function $F_{1,2}$ is now reconstructed from F as

$$F_{1,2}(g_1, g_2, g_3) = (\hat{\mu}_2(\hat{s}_2) \otimes \hat{\mu}_3(\hat{s}_3)) F_{1,2}(g_1, g_2\hat{s}_2, g_3\hat{s}_3) = (\hat{\mu}_2(\hat{s}_2) \otimes \hat{\mu}_3(\hat{s}_3)) F(g_1^{-1} g_2 \hat{s}_2), \tag{11.66}$$

where the elements $\hat{s}_2, \hat{s}_3 \in \hat{S}_{d,p}$ solve the equation $g_3\hat{s}_3 = g_2\hat{s}_2 w_p^{-1}$. The function F has left-right covariance properties

$$F(sgl) = F_{1,2}(e, sgl, sglw_p^{-1}) = F_{1,2}(s^{-1}, gl, gw_p^{-1}w_p l w_p^{-1}) = \left(\mu_1(s) \otimes \hat{\mu}_2(l^{-1}) \otimes \hat{\mu}_3(w_p l^{-1} w_p^{-1})\right) F(g),$$

with $l \in L = SO(1,1) \times SO(p) \times SO(q)$. The way we constructed F mimics the pairing up of points in a bulk conformal field theory without defects. Indeed, there one associates to a pair of fields in $\mathbb{R}^d$ a K_d-covariant function on the conformal group. The group K_d naturally appears as the stabiliser of the pair $(0,\infty)$. Here the same construction is performed along the defect, while transverse directions play no role. The function F can be regarded as a function on the coset space $S_{d,p}\backslash G_{d,p}/L$. This space is one-dimensional

$$X = S_{d,p}\backslash G_{d,p}/L = SO(p+1)\backslash SO(p+1,1)/\left(SO(1,1) \times SO(p)\right) . \tag{11.67}$$

We parametrise it by writing group elements as

$$g = k_I e^{yK_1} r^p e^{\lambda D}\ r^q, \tag{11.68}$$

with $k_I \in SO(p+1)$, $r^p \in SO(p)$ and $r^q \in SO(q)$. The function F is determined by its restriction to the group generated by K_1

$$F(g) = \left(\mu_1(k_I) \otimes \hat{\mu}_2(r^p e^{\lambda D} r^q)^{-1} \otimes \hat{\mu}_3(w_p r^p e^{\lambda D} r^q w_p^{-1})^{-1}\right) \varphi(y) \ . \tag{11.69}$$

We have denoted the restriction by $\varphi(y) = F(e^{yK_1})$. The correlation function is related to F by

$$G_{1,2}(x_i) = \frac{1}{\Phi_1(x_1)} F_{1,2}(g_d(x_1), g_p(\hat{x}_2), g_p(\hat{x}_3)) = \frac{1}{\Phi_1(x_1)} \left(\hat{\mu}_2(\hat{s}_2) \otimes \hat{\mu}_3(\hat{s}_3)\right) F(g_d(x_1)^{-1} g_p(\hat{x}_2)\hat{s}_2) \ . \tag{11.70}$$

Group elements $\hat{s}_2$ and $\hat{s}_3$ are determined as in the non-defect theory on $SO(p+1,1)$ and read

$$\hat{s}_2 = w_p^{-1} m(w_p \hat{x}_{32}) w_p, \quad \hat{s}_3 = k_p(t_{32})^{-1} w_p^{-1} m(\hat{x}_{32}) w_p \ . \tag{11.71}$$

Therefore, the argument of F has the $SO(p+1,1)$-part

$$\begin{aligned}
(g_d(x_1)^{-1} g_p(\hat{x}_2)\hat{s}_2)_p &= |x_{1\perp}|^{-D} e^{\hat{x}_{21}^a P_a} e^{\frac{\hat{x}_{32}^a}{\hat{x}_{32}^2} K_a} = e^{\frac{\hat{x}_{21}^a}{|x_{1\perp}|} P_a} e^{\frac{|x_{1\perp}|\hat{x}_{32}^a}{\hat{x}_{32}^2} K_a} |x_{1\perp}|^{-D} \\
&= k_I \left(\frac{\hat{x}_{12}}{|x_{1\perp}|}\right) \left(\frac{\hat{x}_{12}^2 + x_{1\perp}^2}{x_{1\perp}^2}\right)^{-D} e^{\left(\frac{|x_{1\perp}|\hat{x}_{21}^a}{\hat{x}_{12}^2 + x_{1\perp}^2} + \frac{|x_{1\perp}|\hat{x}_{32}^a}{\hat{x}_{32}^2}\right) K_a} |x_{1\perp}|^{-D} \\
&= k_I \left(\frac{\hat{x}_{12}}{|x_{1\perp}|}\right) e^{\frac{1}{|x_{1\perp}|}\left(\hat{x}_{21} + (\hat{x}_{12}^2 + x_{1\perp}^2)\frac{\hat{x}_{32}}{\hat{x}_{32}^2}\right)^a K_a} \left(\frac{\hat{x}_{12}^2 + x_{1\perp}^2}{|x_{1\perp}|}\right)^{-D} \ .
\end{aligned}$$

In the first step we used the result (4.20). The coordinate y may be read off as

$$y = \frac{1}{|x_{1\perp}||\hat{x}_{23}|} \sqrt{(\hat{x}_{12}^2 + x_{1\perp}^2)(\hat{x}_{13}^2 + x_{1\perp}^2) - x_{1\perp}^2 \hat{x}_{23}^2} = \sqrt{u_{23,1}^{-1} - 1} \ , \tag{11.72}$$

where the cross ratio $u_{23,1}$ is defined as in [126][2]

$$u_{23,1} = \frac{x_{1\perp}^2 \hat{x}_{23}^2}{(\hat{x}_{12}^2 + x_{1\perp}^2)(\hat{x}_{13}^2 + x_{1\perp}^2)} \ . \tag{11.73}$$

For $SO(p)$-scalar fields, the correlation function (11.70) is further evaluated

$$G_{1,2}(x_i) = \frac{1}{|x_{1\perp}|^{\Delta_1}} \frac{1}{\hat{x}_{23}^{2\Delta_{\hat{3}}}} \left(\frac{\hat{x}_{12}^2 + x_{1\perp}^2}{|x_{1\perp}|}\right)^{\Delta_{\hat{3}\hat{2}}} (\hat{\rho}_2 \otimes \hat{\rho}_3)(e^{\varphi_1^i M_{id}}) \varphi(y) \ . \tag{11.74}$$

Let us now solve for eigenfunctions of the Laplacian on the space of left-right covariant functions F. From the simple relation

$$\Delta^{(1)} F_{1,2} = \Delta^{(23)} F_{1,2} \mapsto \Delta F, \tag{11.75}$$

under the above mapping, it follows that these eigenfunctions are conformal blocks. Here, $\Delta^{(23)}$ is the quadratic Casimir constructed from the vector fields that generate the diagonal

[2]Cross ratios $u_{i,jk}$, $u_{ij,k}$ used in this work may differ from those of [126] by factors such as $2, -1$ etc. In all formulas, these functions mean the ones explicitly defined in the present paper.

left-regular action on the last two copies of $G_{d,p}$. We can first consider only the left quotient $S_{d,p}\backslash G_{d,p}$ and parametrise it according to

$$g = k_I e^{y^a K_a} e^{\lambda D} \; . \tag{11.76}$$

Since the $SO(q)$-factor will be trivialised by the right quotient we omitted writing it in the above equation. Later, we will trade (y^a) for spherical polar coordinates (y, ϕ) on $\mathbb{R}^p$. We compute the Laplacian from the left-invariant vector fields - ones that generate the right regular action. The action of dilations, special conformal transformations and rotations is simple

$$k_I e^{y^a K_a} e^{\lambda D} e^{\mu D} = k_I e^{y^a K_a} e^{(\lambda+\mu)D}, \quad k_I e^{y^a K_a} e^{\lambda D} e^{z^a K_a} = k_I e^{(y^a + e^{-\lambda} z^a)K_a} e^{\lambda D}, \tag{11.77}$$

$$k_I e^{y^a K_a} e^{\lambda D} r^p = k_I r^p e^{((r^p)^{-1} y)^a K_a} e^{\lambda D} \; . \tag{11.78}$$

Finally, the action of translations is found by the following calculation

$$\begin{aligned}
k_I e^{y^a K_a} e^{\lambda D} e^{z^b P_b} &= k_I w_p e^{(s_p y)\cdot P} w_p e^{e^{\lambda} z\cdot P} e^{\lambda D} \\
&= k_I' e^{s_p(y + z^{-2} e^{-\lambda} z)\cdot P} e^{-e^{-\lambda} s_p z \cdot K} s_p s_z \left(\frac{1}{e^{2\lambda} z^2}\right)^D e^{\lambda D} = k_I'' e^{(s_z y - z^{-2} e^{-\lambda} z)\cdot P} e^{e^{\lambda} z \cdot K} \left(\frac{1}{e^{\lambda} z^2}\right)^D \\
&= k_I''' e^{(s_z y - z^{-2} e^{-\lambda} z + (1 + (s_z y - z^{-2} e^{-\lambda} z)^2) e^{\lambda} z)\cdot K} \left(\frac{1}{(1 + (s_z y - z^{-2} e^{-\lambda} z)^2) e^{\lambda} z^2}\right)^D \; .
\end{aligned}$$

Group elements k_I', k_I'', k_I''' all belong to $SO(p+1)$ and their precise form does not matter for the action on the coset. By linearising the above action, the Lie algebra is found to be represented by differential operators

$$k_a = e^{-\lambda}\partial_{y^a}, \quad d = \partial_\lambda, \quad m_{ab} = y_a \partial_{y^b} - y_b \partial_{y^a}, \quad p_a = e^{\lambda}\left((1+y^2)\partial_{y^a} - 2 y_a \partial_\lambda\right) \; . \tag{11.79}$$

The quadratic Casimir, restricted to functions of (λ, y) is computed

$$C_2 = \partial_\lambda^2 + (1+y^2)\partial_y^2 - 2y\partial_y\partial_\lambda + \left((p+1)y + \frac{p-1}{y}\right)\partial_y - p\partial_\lambda \; . \tag{11.80}$$

To pass to the final quotient, we set $\partial_\lambda \to \Delta_{\hat{2}\hat{3}}$. Therefore, conformal blocks satisfy the eigenvalue equation

$$\left((1+y^2)\partial_y^2 + \left((p+1-2\Delta_{\hat{2}\hat{3}})y + \frac{p-1}{y}\right)\partial_y + \Delta_{\hat{2}\hat{3}}(\Delta_{\hat{2}\hat{3}} - p)\right)\varphi = \hat{\Delta}(\hat{\Delta} - p)\varphi \; . \tag{11.81}$$

This equation is solved by hypergeometric functions

$$\begin{aligned}
\varphi &= A\, {}_2F_1\left(\frac{p - \hat{\Delta} - \Delta_{\hat{2}\hat{3}}}{2}, \frac{\hat{\Delta} - \Delta_{\hat{2}\hat{3}}}{2}; \frac{p}{2}; -y^2\right) \\
&\quad + B y^{2-p}\, {}_2F_1\left(\frac{2 - \hat{\Delta} - \Delta_{\hat{2}\hat{3}}}{2}, \frac{2 - p + \hat{\Delta} - \Delta_{\hat{2}\hat{3}}}{2}; 2 - \frac{p}{2}; -y^2\right) \; .
\end{aligned} \tag{11.82}$$

Let us compare our results with conformal blocks from [58]. The authors there take the limit $\hat{x}_3 \to \infty$, in which their coordinate $\hat{\chi}$ is related to y as $\hat{\chi} = y^2$. They consider the three-point function

$$\hat{x}_3^{2\Delta_{\hat{3}}} \langle \mathcal{O}_1(x_1)\hat{\mathcal{O}}_2(\hat{x}_2)\hat{\mathcal{O}}_3(\hat{x}_3)\rangle \sim \frac{e^{i(s_{\hat{2}}+s_{\hat{3}})\varphi_1}}{|x_{1\perp}|^{\Delta_1+\Delta_{\hat{2}\hat{3}}}} \sum \mathcal{F}^{\hat{\mathcal{O}}_2\hat{\mathcal{O}}_3}_{\mathfrak{p},s}(\hat{\chi}) \ . \tag{11.83}$$

Conformal blocks $\mathcal{F}^{\hat{\mathcal{O}}_2\hat{\mathcal{O}}_3}_{\mathfrak{p},s}$ read

$$\mathcal{F}^{\hat{\mathcal{O}}_2\hat{\mathcal{O}}_3}_{\mathfrak{p},s}(\hat{\chi}) = \hat{\chi}^{-\frac{1}{2}(\hat{\Delta}+\Delta_{\hat{2}\hat{3}})}\ {}_2F_1\left(\frac{\hat{\Delta}+\Delta_{\hat{2}\hat{3}}}{2}, \frac{2-p+\hat{\Delta}+\Delta_{\hat{2}\hat{3}}}{2}, 1-\frac{p}{2}+\hat{\Delta}; -\frac{1}{\hat{\chi}}\right) \ . \tag{11.84}$$

By expanding around zero instead of infinity, we can rewrite these as

$$\mathcal{F}^{\hat{\mathcal{O}}_2\hat{\mathcal{O}}_3}_{\mathfrak{p},s}(\hat{\chi}) = \ {}_2F_1\left(\frac{p-\hat{\Delta}+\Delta_{\hat{2}\hat{3}}}{2}, \frac{\hat{\Delta}+\Delta_{\hat{2}\hat{3}}}{2}, \frac{p}{2}; -\hat{\chi}\right) \ . \tag{11.85}$$

On the other hand, our expression for $G_{1,2}(x_i)$ becomes in the $\hat{x}_3 \to \infty$ limit

$$\hat{x}_3^{2\Delta_{\hat{3}}} G_{1,2}(x_i) = \frac{(\hat{\rho}_2 \otimes \hat{\rho}_3)(e^{\varphi_1^i M_{id}})}{|x_{1\perp}|^{\Delta_1+\Delta_{\hat{2}\hat{3}}}} (1+y^2)^{\Delta_{\hat{3}\hat{2}}} \varphi(y) \ . \tag{11.86}$$

Therefore, we should have the relation $\mathcal{F}^{\hat{\mathcal{O}}_2\hat{\mathcal{O}}_3}_{\mathfrak{p},s} = (1+y^2)^{\Delta_{\hat{3}\hat{2}}}\varphi$. There is a number of ways to verify that this is true. Perhaps the simplest one is to conjugate the operator on the left hand side of eq. (11.81) by $(1+y^2)^{\Delta_{\hat{3}\hat{2}}}$. Then eigenfunctions of this new operator should coincide with $\mathcal{F}^{\hat{\mathcal{O}}_2\hat{\mathcal{O}}_3}_{\mathfrak{p},s}$. Indeed, the eigenfunctions read

$$A\, {}_2F_1\left(\frac{p-\hat{\Delta}+\Delta_{\hat{2}\hat{3}}}{2}, \frac{\hat{\Delta}+\Delta_{\hat{2}\hat{3}}}{2}; \frac{p}{2}; -y^2\right) + B y^{2-p}\, {}_2F_1\left(\frac{2-\hat{\Delta}+\Delta_{\hat{2}\hat{3}}}{2}, \frac{2-p+\hat{\Delta}+\Delta_{\hat{2}\hat{3}}}{2}; 2-\frac{p}{2}; -y^2\right),$$

so putting $A = 1$ and $B = 0$ gives us $\mathcal{F}^{\hat{\mathcal{O}}_2\hat{\mathcal{O}}_3}_{\mathfrak{p},s}$. With this we end the discussion of three-point functions of two defect and one bulk field.

11.6 Bulk-bulk-defect three point function

We move to the three-point function that involves two fields in the bulk and one on the defect. Following the familiar strategy, we start by lifting the three-point function $G_{2,1}(x_i)$ to a function $F_{2,1} : G^3_{d,p} \to W$ which satisfies

$$F_{2,1}(g_1 s_1, g_2 s_2, g_3 \hat{s}_3) = \left(\mu_1(s_1^{-1}) \otimes \mu_2(s_2^{-1}) \otimes \hat{\mu}_3(\hat{s}_3^{-1})\right) F_{2,1}(g_1, g_2, g_3), \quad F_{2,1}(hg_i) = F_{2,1}(g_i) \ . \tag{11.87}$$

Let us pair up the last two fields by setting $F'(g_1, g_2) = F_{2,1}(g_1, g_2, g_2)$. Then F' obeys $F'(hg_i) = F'(g_i)$ and we put $F(g) = F'(e, g) = F_{2,1}(e, g, g)$. In particular, this implies that solutions to $\Delta F = cF$ will correspond to eigenfunctions of the quadratic Casimir acting at the point x_1. One reconstructs $F_{2,1}$ from F by

$$F_{2,1}(g_1, g_2, g_3) = (\mu_2(s_2) \otimes \hat{\mu}_3(\hat{s}_3))\, F_{2,1}(g_1, g_2 s_2, g_2 s_2) = (\mu_2(s_2) \otimes \hat{\mu}_3(\hat{s}_3))\, F\left(g_1^{-1} g_2 s_2\right) , \tag{11.88}$$

where $s_2, \hat{s}_3$ solve the equation $g_2 s_2 = g_3 \hat{s}_3$. The function F is right-covariant with respect to the group K and left-covariant with respect to $S_{d,p}$

$$F(sgk) = F_{2,1}(e, sgk, sgk) = F_{2,1}(s^{-1}, gk, gk) = \big(\mu_1(s) \otimes \mu_2(k^{-1}) \otimes \hat{\mu}_3(k^{-1})\big)\, F(g)\ . \tag{11.89}$$

Therefore, it can be regarded as a function on the double quotient

$$Y = S_{d,p}\backslash G_{d,p}/K = SO(p+1)\backslash SO(p+1,1)/SO(p)\ \times\ SO(q-1)\backslash SO(q)/SO(q-1)\ . \tag{11.90}$$

Both direct factors were already analysed in previous sections. The first one is two-dimensional and the second one one-dimensional. Cartan coordinates on the double coset are introduced by writing elements of $G_{d,p}$ as

$$g = k_I e^{yK_1} e^{\lambda D} r^p\ r_l^{q-1} e^{\kappa M_{d-1,d}} r_r^{q-1}, \tag{11.91}$$

with $r^p \in SO(p)$ and $r_{l,r}^{q-1} \in SO(q-1)$. The function F is determined in terms of its restriction $\psi(\lambda, y, \kappa) = F(e^{yK_1} e^{\lambda D} e^{\kappa M_{d-1,d}})$ by

$$F(g) = \big(\mu_1(k_I r_l^{q-1}) \otimes \mu_2(r^p r_r^{q-1})^{-1} \otimes \hat{\mu}_3(r^p r_r^{q-1})^{-1}\big)\, \psi(\lambda, y, \kappa)\ . \tag{11.92}$$

The correlation function is related to F by

$$G_{2,1}(x_i) = \frac{1}{\Phi_1(x_1)\Phi_2(x_2)} F_{2,1}(g_d(x_1), g_d(x_2), g_p(\hat{x}_3)) = \frac{\mu_2(s_{23}) \otimes \hat{\mu}_3(\hat{s}_{23})}{\Phi_1(x_1)\Phi_2(x_2)} F(g_d(x_1)^{-1} g_d(x_2) s_{23})\ . \tag{11.93}$$

Here s_{23} and $\hat{s}_{23}$ are given analogously to (11.46) . We can evaluate the argument of F similarly as before using the Bruhat and Iwasawa decompositions. For the $SO(p+1)$-part

$$\begin{aligned}
(g_d(x_1)^{-1} g_d(x_2) s_{23})_p &= |x_{1\perp}|^{-D} e^{\hat{x}_{21}^a P_a} |x_{2\perp}|^D k_I \left(\frac{\hat{x}_{23}}{|x_{2\perp}|}\right) \\
&= |x_{1\perp}|^{-D} e^{\hat{x}_{21}^a P_a} |x_{2\perp}|^D e^{-\frac{\hat{x}_{23}^a}{|x_{2\perp}|} P_a} \left(\frac{\hat{x}_{23}^2 + x_{2\perp}^2}{x_{2\perp}^2}\right)^D e^{\frac{\hat{x}_{23}^a}{|x_{2\perp}|} K_a} = e^{\frac{\hat{x}_{31}^a}{|x_{1\perp}|} P_a} \left(\frac{\hat{x}_{23}^2 + x_{1\perp}^2}{|x_{1\perp}||x_{2\perp}|}\right)^D e^{\frac{\hat{x}_{23}^a}{|x_{2\perp}|} K_a} \\
&= k_I \left(\frac{\hat{x}_{13}}{|x_{1\perp}|}\right) \left(\frac{\hat{x}_{13}^2 + x_{1\perp}^2}{x_{1\perp}^2}\right)^D e^{\frac{|x_{1\perp}|\hat{x}_{31}^a}{\hat{x}_{13}^2 + x_{1\perp}^2} K_a} \left(\frac{\hat{x}_{23}^2 + x_{2\perp}^2}{|x_{1\perp}||x_{2\perp}|}\right)^D e^{\frac{\hat{x}_{23}^a}{|x_{1\perp}|} K_a} \\
&= k_I \left(\frac{\hat{x}_{13}}{|x_{1\perp}|}\right) e^{\frac{1}{|x_{1\perp}|}\left(\hat{x}_{31} + \frac{\hat{x}_{13}^2 + x_{1\perp}^2}{\hat{x}_{23}^2 + x_{2\perp}^2}\hat{x}_{23}\right)\cdot K} \left(\frac{|x_{1\perp}|(\hat{x}_{23}^2 + x_{2\perp}^2)}{|x_{2\perp}|(\hat{x}_{13}^2 + x_{1\perp}^2)}\right)^D\ .
\end{aligned}$$

We read off the coordinate y

$$\begin{aligned}
y &= \frac{1}{|x_{1\perp}|(\hat{x}_{23}^2 + x_{2\perp}^2)} \sqrt{\hat{x}_{12}^2(\hat{x}_{13}^2 + x_{1\perp}^2)(\hat{x}_{23}^2 + x_{2\perp}^2) + (\hat{x}_{13}^2 x_{2\perp}^2 - \hat{x}_{23}^2 x_{1\perp}^2)(\hat{x}_{23}^2 + x_{2\perp}^2 - \hat{x}_{13}^2 - x_{1\perp}^2)} \\
&= \frac{1}{|x_{1\perp}|(\hat{x}_{23}^2 + x_{2\perp}^2)} \sqrt{(\hat{x}_{12}^2 + x_{1\perp}^2 + x_{2\perp}^2)(\hat{x}_{13}^2 + x_{1\perp}^2)(\hat{x}_{23}^2 + x_{2\perp}^2) - x_{2\perp}^2(\hat{x}_{13}^2 + x_{1\perp}^2)^2 - x_{1\perp}^2(\hat{x}_{23}^2 + x_{2\perp}^2)^2}\ .
\end{aligned}$$

Therefore, the Cartan coordinates (y, λ) are given by

$$y = \sqrt{u_{12}^{\bullet} u_{3,12} - u_{3,12}^2 - 1}, \quad e^{\lambda} = u_{3,12}^{-1}, \tag{11.94}$$

where $u_{12}^{\bullet}$ and $u_{3,12}$ are cross ratios of [126]

$$u_{12}^{\bullet} = \frac{\hat{x}_{12}^2 + x_{1\perp}^2 + x_{2\perp}^2}{|x_{1\perp}||x_{2\perp}|}, \quad u_{3,12} = \frac{\hat{x}_{13}^2 + x_{1\perp}^2}{\hat{x}_{23}^2 + x_{2\perp}^2}\frac{|x_{2\perp}|}{|x_{1\perp}|} \ . \tag{11.95}$$

The coordinate κ was determined in (11.54). These results give the relation between the correlator $G_{2,1}(x_i)$ and ψ in the case the fields are scalars

$$G_{2,1}(x_i) = \frac{1}{|x_{1\perp}|^{\Delta_1}|x_{2\perp}|^{\Delta_2}}\left(\frac{\hat{x}_{23}^2 + x_{2\perp}^2}{|x_{2\perp}|}\right)^{-\Delta_{\hat{3}}} \psi(\lambda, y, \kappa) \ . \tag{11.96}$$

Eigenfunctions of the Laplacian correspond to eigenfunctions of the quadratic Casimir at the first insertion point x_1. Clearly, it is possible to repeat the whole argument with points x_1 and x_2 interchanged. This would lead to another representation of the correlator

$$G_{2,1}(x_i) = \frac{1}{|x_{1\perp}|^{\Delta_1}|x_{2\perp}|^{\Delta_2}}\left(\frac{\hat{x}_{13}^2 + x_{1\perp}^2}{|x_{1\perp}|}\right)^{-\Delta_{\hat{3}}} \tilde{\psi}(\tilde{\lambda}, \tilde{y}, \kappa), \tag{11.97}$$

with $\tilde{y}, \tilde{\lambda}$ obtained from y, λ by swapping indices 1 and 2. One can parametrise the coset space Y by $(y, \tilde{y})$ and it is not hard to establish that $\tilde{y} = e^{\lambda} y \equiv x$. The Laplacian (11.80) in these coordinates reads

$$C_2^{(2)} = (1+y^2)\partial_y^2 + \frac{x^2}{y^2}\partial_x^2 + \frac{2x}{y}\partial_y\partial_x + \left((p+1)y + \frac{p-1}{y}\right)\partial_y + \frac{(p-1)x}{y^2}\partial_x \ . \tag{11.98}$$

If we performed the construction with x_1 and x_2 exchanged, the Laplacian would be given by the operator $C_2^{(1)}$ that is obtained from $C_2^{(2)}$ by exchanging x and y. We have to remember that the prefactors multiplying ψ and $\tilde{\psi}$ are different. Conformal blocks are therefore simultaneous eigenfunctions of $C_2^{(2)}$ and $(x/y)^{\Delta_{\hat{3}}} C_2^{(1)} (x/y)^{-\Delta_{\hat{3}}}$. These two operators are easily seen to commute. We will consider a more symmetric pair of differential operators

$$L_1 = \frac{1}{4}\left(\frac{x}{y}\right)^{\frac{\Delta_{\hat{3}}}{2}} C_2^{(1)} \left(\frac{x}{y}\right)^{-\frac{\Delta_{\hat{3}}}{2}}, \quad L_2 = \frac{1}{4}\left(\frac{x}{y}\right)^{-\frac{\Delta_{\hat{3}}}{2}} C_2^{(2)} \left(\frac{x}{y}\right)^{\frac{\Delta_{\hat{3}}}{2}}, \tag{11.99}$$

and solve an equivalent eigenvalue problem

$$L_1 f(x,y) = \frac{1}{4}\hat{\Delta}(\hat{\Delta}-p) f(x,y), \quad L_2 f(x,y) = \frac{1}{4}\hat{\Delta}'(\hat{\Delta}'-p) f(x,y) \ . \tag{11.100}$$

To proceed, let us introduce variables $v_1 = -x^{-2}$, $v_2 = -y^{-2}$. Then the two operators can be written as

$$L_1 = v_1 D_{v_1 v_2}\left(0, \frac{2-p}{2}, \frac{\Delta_{\hat{3}} - p + 2}{2}\right) + \frac{\frac{\Delta_{\hat{3}}}{2}\left(\frac{\Delta_{\hat{3}}}{2} - p\right)}{4},$$
$$L_2 = v_2 D_{v_2 v_1}\left(0, \frac{2-p}{2}, \frac{\Delta_{\hat{3}} - p + 2}{2}\right) + \frac{\frac{\Delta_{\hat{3}}}{2}\left(\frac{\Delta_{\hat{3}}}{2} - p\right)}{4},$$

where $D_{xy}(a,b,c)$ is defined as

$$D_{xy}(a,b,c) = x(1-x)\partial_x^2 - 2xy\partial_x\partial_y - y^2\partial_y^2 + (c-(a+b+1)x)\partial_x - (a+b+1)y\partial_y - ab\ . \tag{11.101}$$

The significance of this operator is that it appears in connection with Appell's hypergeometric function F_4. Namely, the system of equations satisfied by F_4 with labels (a,b,c_1,c_2) can be written as

$$D_{xy}(a,b,c_1)F_4(x,y) = D_{yx}(a,b,c_2)F_4(x,y) = 0\ . \tag{11.102}$$

Our equations are not quite in the form of the Appell's system, but they become so once we introduce $f(v_1,v_2) = v_1^{\frac{\hat\Delta}{2}-\frac{\Delta_{\hat 3}}{4}} v_2^{\frac{\hat\Delta'}{2}-\frac{\Delta_{\hat 3}}{4}} F(v_1,v_2)$. Then, using formulas from a previous chapter, the eigenvalue equations (11.99) can be written in terms of F as

$$v_1 D_{v_1 v_2}\left(\frac{\hat\Delta+\hat\Delta'-\Delta_{\hat 3}}{2}, \frac{\hat\Delta+\hat\Delta'-\Delta_{\hat 3}+2-p}{2}, \hat\Delta - \frac{p}{2}+1\right) F = 0, \tag{11.103}$$

$$v_2 D_{v_2 v_1}\left(\frac{\hat\Delta+\hat\Delta'-\Delta_{\hat 3}}{2}, \frac{\hat\Delta+\hat\Delta'-\Delta_{\hat 3}+2-p}{2}, \hat\Delta' - \frac{p}{2}+1\right) F = 0\ . \tag{11.104}$$

Therefore, the Appell function

$$F(v_1,v_2) = F_4\left(\frac{\hat\Delta+\hat\Delta'-\Delta_{\hat 3}}{2}, \frac{\hat\Delta+\hat\Delta'-\Delta_{\hat 3}+2-p}{2}, \hat\Delta - \frac{p}{2}+1, \hat\Delta' - \frac{p}{2}+1; v_1, v_2\right), \tag{11.105}$$

solves the eigenvalue problem. There are three more independent solutions, all expressible in terms of Appell functions, but the one we have written has the correct boundary behaviour and we will see that it reproduces the result of [58] in a special limit. Before doing that, let us give the final formula for Laplacian eigenfunctions that correspond to conformal blocks. They are labelled by three quantum numbers $(\hat\Delta, \hat\Delta', s)$ and read[3]

$$\Psi_{\hat\Delta,\hat\Delta',s}(v_1,v_2,\kappa) = v_1^{\frac{\hat\Delta}{2}-\frac{\Delta_{\hat 3}}{4}} v_2^{\frac{\hat\Delta'}{2}-\frac{\Delta_{\hat 3}}{4}} F(v_1,v_2)\ C_s^{(q-2)/2}(\cos\kappa)\ . \tag{11.106}$$

The authors of [58] consider the three-point function of two bulk and one defect field in the limit $\hat x_3 \to \infty$ and in the special configuration $x_{1\perp} = x_{2\perp}$. In such a configuration, $v_1 = v_2$ and there are two independent cross ratios, $\varphi = \kappa$ and $\hat\chi = -v_1^{-1}$. The correlator in [58] reads

$$\hat x_3^{2\Delta_{\hat 3}} \langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\hat{\mathcal{O}}_3(\hat x_3)\rangle \sim \frac{e^{-is_1\varphi_1}}{|x_{1\perp}|^{\Delta_1+\Delta_2-\Delta_{\hat 3}}} \sum \mathcal{F}^{\hat{\mathcal{O}}_3}_{\hat{\mathcal{O}}\hat{\mathcal{O}}'}(\hat\chi)F(\varphi)\ . \tag{11.107}$$

The conformal blocks factorise in $\hat\chi$ and φ in the usual way and the transverse parts agree with ours by the same calculation as in the previous sections. Let us focus therefore on longitudinal parts, which are given in [58] by

$$\mathcal{F}^{\hat{\mathcal{O}}_3}_{\hat{\mathcal{O}}\hat{\mathcal{O}}'}(\hat\chi) = \hat\chi^{-\frac{1}{2}(\hat\Delta+\hat\Delta'-\Delta_{\hat 3})}\ {}_4F_3\Big(\frac{\hat\Delta+\hat\Delta'-p+1}{2}, \frac{\hat\Delta+\hat\Delta'-p+2}{2}, \tag{11.108}$$

$$\frac{\hat\Delta+\hat\Delta'-\Delta_{\hat 3}}{2}, \frac{\hat\Delta+\hat\Delta'-\Delta_{\hat 3}-p+2}{2}; \hat\Delta - \frac{p}{2}+1, \hat\Delta' - \frac{p}{2}+1, \hat\Delta+\hat\Delta'-p+1; -\frac{4}{\hat\chi}\Big)\ . \tag{11.109}$$

[3]The relation between Ψ and ψ is $\Psi = (v_1/v_2)^{\Delta_{\hat 3}}\psi$. We use Ψ in the final formula as it gives the most symmetric form of blocks.

We can rewrite this using an identity due to Burchnall as

$$\mathcal{F}^{\hat{\mathcal{O}}_3}_{\hat{\mathcal{O}}\hat{\mathcal{O}}'}(\hat{\chi}) = \hat{\chi}^{-\frac{1}{2}(\hat{\Delta}+\hat{\Delta}'-\Delta_{\hat{3}})}\, F_4\Big(\frac{\hat{\Delta}+\hat{\Delta}'-\Delta_{\hat{3}}}{2}, \frac{\hat{\Delta}+\hat{\Delta}'-\Delta_{\hat{3}}-p+2}{2}; \hat{\Delta}-\frac{p}{2}+1, \hat{\Delta}'-\frac{p}{2}+1, ; -\frac{1}{\hat{\chi}}, -\frac{1}{\hat{\chi}}\Big) .$$

The prefactor in the correlation function that multiplies $\mathcal{F}^{\hat{\mathcal{O}}_3}_{\hat{\mathcal{O}}\hat{\mathcal{O}}'}(\hat{\chi})$ is the same as the prefactor of f, so we need to show that $f(v_1, v_1)$ and $\mathcal{F}^{\hat{\mathcal{O}}_3}_{\hat{\mathcal{O}}\hat{\mathcal{O}}'}(\hat{\chi})$ agree, up to a multiplicative constant. But one readily observes that

$$\mathcal{F}^{\hat{\mathcal{O}}_3}_{\hat{\mathcal{O}}\hat{\mathcal{O}}'}(\hat{\chi}) = (-1)^{-\frac{1}{2}(\hat{\Delta}+\hat{\Delta}'-\Delta_{\hat{3}})} f(v_1, v_1) \; . \tag{11.110}$$

Therefore, the blocks from [58] follow from those written in (11.106).

Chapter 12

Multipoint correlation functions and Gaudin models

A promising direction in which the present bootstrap studies may be extended is the analysis of correlation functions that involve more than four field insertions. Since the associativity of the operator product algebra is equivalent to crossing symmetry of all four-point functions in the theory, in principle no additional constraints arise from consistency of higher-point functions and one may jump to a conclusion that these correlators are irrelevant.

However, such a reasoning is very far from the truth. As we have seen in the first few chapters, most conformal bootstrap studies focus on a handful of (or simply one) four-point functions. Considering systems of infinitely many correlators looks like a notoriously difficult task. In the light of these issues, higher-point functions arise as an economical repackaging of bootstrap equations. This comes from the fact that a single five-point function contains information that is available only by looking at an infinite number of four-point functions. Indeed, when one expands the product of the first two fields in the OPE, an infinite number of internal fields that appear in their product act as external operators for the remaining four-point function. Similarly, any higher-point correlator captures information available from infinitely many four-point functions.

This simple heuristic argument suggests that the so called *multipoint bootstrap* holds a great promise. However, the derivation of crossing symmetry equations for higher-point functions meets significant technical difficulties, the main of which is the computation of conformal blocks, [140, 141, 142, 143, 144, 145, 146, 147, 148, 149, 150, 151, 152].

To start appreciating the challenges that appear in developing a theory of multipoint conformal blocks, one can simply observe that these are functions depending on a large number of variables. We have written the number of conformal invariants for an n-point function in d-dimensions in (2.34). This number stabilises in high enough dimensions, starting from the critical one $d_c = n-2$, and equals $n(n-3)/2$. Thus, the simplest interesting case beyond four-point functions, a five-point function in $d=3$, already depends on five cross ratios.

The issues do not end here though, as there remains the question of how to characterise multipoint blocks. For any n-point correlation function, partial waves admit a basic description using the shadow formalism of [11]. The latter provides integral formulas for conformal blocks that may be regarded simply as definitions of these functions. Starting with such integrals, it requires significant effort to find analytical expressions for blocks in terms of special functions or even just efficiently numerically evaluate them. In the four-point case, the decisive step forward was made by Dolan and Osborn who characterised the blocks as solutions of Casimir differential equations. We have seen how their approach leads to wonderfully simple formulas for scalar fields in even number of dimensions, but the scope of applications of the basic idea goes far beyond these cases. Even when no explicit solution in terms of known special functions can be found, differential equations can be used to efficiently obtain series expansions for blocks to an arbitrary order.

As soon as one looks at a five-point function in $d \geq 3$ and tries to apply the same strategy as above and write a complete set of differential equations satisfied by partial waves, one runs into trouble. Let us for definiteness focus on a five-point function of scalar fields. The corresponding OPE diagram contains two intermediate fields, characterised by weights Δ and spins l. Once these intermediate fields are fixed, one remains with a three-point function of a scalar and two symmetric traceless tensors. Such a three-point function depends on a number of tensor structures. Therefore, to specify a conformal block, one needs to fix a vector in the space of three-point tensor structures, in addition to the two intermediate fields.

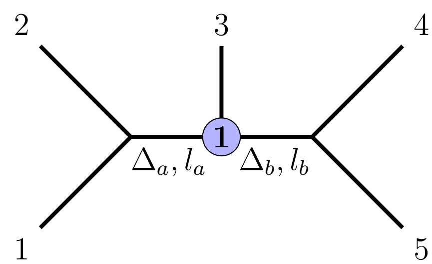

Figure 12.1: OPE diagram for a five-point function

There are four Dolan-Osborn type operators that measure the quantum numbers $\Delta_{a,b}$ and $l_{a,b}$ of the intermediate fields, but it is not obvious what operator could be used to measure a fifth quantum number. It is at this point that Gaudin models come to our rescue. We will show that the set of commuting Hamiltonians for the Gaudin model based on the conformal Lie algebra $\mathfrak{g} = \mathfrak{so}(d+1,1)$, with five sites that carry field representations π_i and some particular values of parameters z_i, contains the four Dolan-Osborn operators and also a fifth operator of order four that we will call the vertex operator. Therefore, the Gaudin model provides us with the desired set of differential equations.

The relation between Gaudin models and correlation functions is by no means accidental and holds for any number of insertion point and in any dimension. To be precise, on the space of solutions to Ward identities, there is an action of the Gaudin algebra. For an appropriate choice of parameters z_i, the algebra contains the Dolan-Osborn type operators that measure quantum numbers of intermediate fields propagating in the OPE diagram. It seems that number of independent operators always matches that of cross ratios, although this is yet to be proved.

The chapter is organised as follows. In the first section, we will show how the Gaudin Hamiltonians act on the space of n-point correlation functions and illustrate this on the example of five-point functions. The second section will focus on the vertex of the OPE diagram (12.1) and the choice of tensor structures for which conformal partial waves diagonalise the fifth Gaudin

Hamiltonian. These tensor structures are through the use of polarisation vectors characterised as eigenfunctions of a particular fourth-order operator H. The operator H is itself a quite interesting mathematical object, but its further study is left for future work. The chapter is mostly based on the article [5].

12.1 Gaudin models for correlation functions

It is not difficult to understand how Gaudin models appear in the context of n-point conformal correlation functions. Let us recall that these functions live in the space $(\pi_1 \otimes ... \otimes \pi_n)^G$ of invariant vectors in the tensor product of n principal series representations of G.

Now consider a Gaudin model with n arbitrary sites, based on the conformal Lie algebra $\mathfrak{so}(d+1,1)$. To each site i we attach the corresponding principal series representation π_i. Gaudin Hamiltonians then act on the carrier space of $\pi_1 \otimes ... \otimes \pi_n$. However, as we have seen, the Hamiltonians also commute with the diagonal action of $\mathfrak{g}$. Therefore, they map invariant vectors to other invariant vectors, thereby preserving the space of solutions to Ward identities, $(\pi_1 \otimes ... \otimes \pi_n)^G$.

This is our basic observation, but it is not quite enough in order to be able to use Gaudin Hamiltonians to characterise conformal blocks. For this to be the case, the Dolan-Osborn-type operators that measure the quantum numbers of intermediate fields have to belong to the Gaudin algebra. For concreteness, let us focus on the scalar five-point function from the introduction to this chapter. In agreement with our previous notation for Gaudin models, a basis of the conformal Lie algebra at the site i is denoted $\{X_a^{(i)}\}$. The four Dolan-Osborn operators are of the form

$$C_p^{(ij)} = \frac{1}{p}\kappa^{a_1 ... a_p}(X_{a_1}^{(i)} + X_{a_1}^{(j)})...(X_{a_p}^{(i)} + X_{a_p}^{(j)}) \ . \tag{12.1}$$

The operators are either quadratic, $p = 2$, or quartic, $p = 4$, and inserted between the pairs of points (12) and (45).

To recover these operators from the Gaudin model, we start by fixing three of the sites to $z_1 = 0$, $z_{n-1} = 1$ and $z_n = \infty$. This can always be done, since two Gaudin models whose parameters are related by an $SL(2,\mathbb{C})$ transformation are equivalent. The remaining two parameters are set to be $z_2 = \varpi^2$ and $z_3 = \varpi$ and we take the limit $\varpi \to 0$[1]

$$\tilde{\mathcal{H}}_p(z) := \lim_{\varpi \to 0} \varpi^p \mathcal{H}_p(\varpi z) \ . \tag{12.2}$$

The limiting process does not spoil commutativity of these Hamiltonians. They are regular in z everywhere except form $z \in \{0, 1, \infty\}$. At this point, $\tilde{\mathcal{H}}_p(z)$ are differential operators in spacetime coordinates x_i. However, since they preserve the functional form of correlators, they reduce to differential operators in cross ratios.

After performing the special limit on the parameters z_i we can now extract the multipoint Casimir operators rather easily. In fact, it is not difficult to check that

$$C_p^{(12)} = \lim_{z \to 0} z^p \tilde{\mathcal{H}}_p(z), \quad C_p^{(45)} = \lim_{z \to \infty} z^p \tilde{\mathcal{H}}_p(z), \tag{12.3}$$

[1] Such limits have also been considered in [153, 154] to study bending flow Hamiltonians and their generalisations [155, 156, 157, 158].

for $p = 2, 4$. But now we come to the upshot of the construction: any additional independent operator we can obtain from $\tilde{\mathcal{H}}_p(z)$ may be used to measure a fifth quantum number. One can show that the two second order Casimir operators $C_2^{(ij)}$ exhaust all the independent operators that can be obtained from $\tilde{\mathcal{H}}_2(z)$. The family $\tilde{\mathcal{H}}_4(z)$, on the other hand, indeed provides one independent operator in addition to the fourth order Casimir operators $C_4^{(ij)}$. We propose to use the operator $\mathcal{V}_4$ defined through

$$\tilde{\mathcal{H}}_4\,(z = 1/2) = 16\,\mathcal{V}_4 + \dots\,, \tag{12.4}$$

where the dots represent quadratic terms coming from the corrections in (5.46). In the particular limit $\varpi \to 0$ that we consider here, these corrections can be re-expressed in terms of the quadratic Casimirs $C_2^{(ij)}$, and can thus be discarded without spoiling commutativity of $\mathcal{V}_4$ with the Casimirs. An explicit computation then shows that $\mathcal{V}_4$ is expressed in terms of the conformal generators $X_a^{(i)}$ as

$$\mathcal{V}_4 = \kappa_4^{a_1\dots a_4}(X_{a_1}^{(1)} + X_{a_1}^{(2)} - X_{a_1}^{(3)}) \cdots (X_{a_4}^{(1)} + X_{a_4}^{(2)} - X_{a_4}^{(3)})\ . \tag{12.5}$$

The explicit form of $\mathcal{V}_4$ as a differential operator acting on functions $\psi(u)$ of five cross ratios will be spelled out in our forthcoming work.

In the end, the vertex operator $\mathcal{V}_4$ is quite simple and obviously commutes with Dolan-Osborn operators, so one cannot help but wander whether we could have found it without the help of Gaudin models. Indeed, we probably could have. Let us therefore briefly sketch how the above exposition extends to some more involved cases. We focus on the the comb channel, [140], of general n-point functions in arbitrary dimension d. In this case, the Lax matrix of the Gaudin model depends on n complex parameters z_i. We can set three of these to the values $z_1 = 0$, $z_{n-1} = 1$ and $z_n = \infty$, before scaling the remaining ones as $z_i = \varpi^{n-i-1}$, $i = 2, \dots, n-2$ in terms of a single complex parameter ϖ that we send to zero. Generalising our construction of the commuting families of operators in (12.3), we now introduce

$$\tilde{\mathcal{H}}_p^{[i]}(z) := \lim_{\varpi\to 0} \varpi^{(n-i-2)p}\mathcal{H}_p(\varpi^{n-i-2}z), \tag{12.6}$$

where $p = 2, 4, \dots$ enumerates the different (Casimir) invariants of the d-dimensional conformal algebra and $z \in \mathbb{C}$ is the spectral parameter. Through the label $i \in \{1, \dots, n-2\}$ we characterise different ways to perform the scaling limit of the original Gaudin Hamiltonians. It is not difficult to show that the resulting family of commuting Hamiltonians includes all the Casimir operators that are needed to measure the weight and spin of intermediate fields, similarly to (12.3). The other Hamiltonians extracted from the families (12.6) then provide additional commuting operators characterising the vertices in the n-point OPE diagram (note that the range of our index i indeed allows us to enumerate these vertices). One thereby expects to complete the full set of Casimir operators into a system of independent commuting operators that suffices to characterise the dependence of n-point comb channel blocks on all conformal cross ratios, for any dimension d and an arbitrary choice of representations for external fields. We have checked this claim for various choices of n and d.

For $d = 3$, an n-point function with scalar external fields involves $3n - 10$ cross ratios. The intermediate fields in the comb channel OPE diagram are characterised by $2n - 6$ Casimir operators, of degree two and four. In addition, each of the $n-4$ internal vertices is associated with an operator $\mathcal{V}_4^{[i]}$, extracted similarly to $\mathcal{V}_4$ in (12.4) as

$$\tilde{\mathcal{H}}_4^{[i]}\,(z=1/2) = 16\,\mathcal{V}_4^{[i]} + \dots\,, \tag{12.7}$$

where $i \in \{2,\dots,n-3\}$ [2]. The spectrum of these $n-4$ operators is independent of i and we will say more about it in the next section. With the additional index $i \in \{2,\dots,n-3\}$ on the left hand side of the vertex eigenvalue equation (12.11), we obtain enough differential equations to characterise three-dimensional n-point blocks in the comb channel.

12.2 Vertex systems

The operator $\mathcal{V}_4$ provided us with the definition the fifth quantum number needed to characterise five-point partial waves, but did not shed much light on the intuition that a choice this quantum number is related to fixing a particular basis in the space of tensor structures at the vertex of OPE diagram. The tensor structures that appear in the three-point function of a scalar and two symmetric traceless tensor fields were written down in (3.10). Let us denote this three-point function by Φ^t_{123}. If of the symmetric traceless tensors is actually a scalar, there is only one tensor structure and we can drop the superscript t.

Having described the vertex, we can now write down the shadow integral representations of partial waves. Let Ψ be the integral

$$\Psi(x_i) = \prod_{s=a,b}\int_{\mathbb{R}^d} d^dx_s \int_{\mathbb{C}^d} d^{2d}z_s\ \delta(z_s^2)\rho(\bar{z}_s\cdot z_s) \tag{12.8}$$

$$\Phi_{12\tilde{a}}(x_1,x_2,x_a;\bar{z}_a)\Phi^t_{a3b}(x_a,x_3,x_b;z_a,z_b)\Phi_{\tilde{b}45}(x_b,x_4,x_5;\bar{z}_b)\ . \tag{12.9}$$

Here, the tilde on the indices of the first and third vertex means that we use (3.10) for two scalar legs but with Δ_a and Δ_b replaced by $d-\Delta_a$ and $d-\Delta_b$, respectively. The integral depends on external weights Δ_i, internal weights and spins $\Delta_{a,b}$, $l_{a,b}$ and the choice of the function $t(X)$. For simplicity, we do not display these dependencies in the notation. After splitting off the factor $\Omega(x_i)$ that accounts for the nontrivial covariance law of the scalar fields under conformal transformations

$$\Psi(x_i) = (X^2_{23;1})^{\frac{\Delta_1}{2}}(X^2_{34;5})^{\frac{\Delta_5}{2}}\prod_{i=2}^{4}(X^2_{i+1,i-1;i})^{\frac{\Delta_i}{2}}\psi(u_a),$$

the shadow integral (12.9) gives rise to a finite conformal integral that defines the conformal block ψ as a function of five conformally invariant cross ratios u_a.[3] The function ψ is labelled by (Δ_a,l_a), (Δ_b,l_b) and the function $t(X)$. By the argument of Dolan and Osborn, ψ satisfies eigenvalue equations

$$\mathcal{D}_p^{(12)}\psi(u) = C_p(\Delta_a,l_a)\psi(u),\quad \mathcal{D}_p^{(45)}\psi(u) = C_p(\Delta_b,l_b)\psi(u), \tag{12.10}$$

[2] Note that for the case of scalar external fields, the extremal vertices of the comb channel diagram are trivial, which is why we restrict i to the range $\{2,\dots,n-3\}$ in this case.

[3] There is no relation between the index a on cross ratios and the index a used for one of the intermediate fields.

where $p = 2, 4$ and $\mathcal{D}_p^{(ij)}$ denotes the operator $C_p^{(ij)}$ reduced to the space of cross ratios. Our addition to these results is the characterisation of the tensor structures $t_n(X)$ such that the fifth equation

$$\mathcal{V}_4\ \psi(u) = \tau_n\,\psi(u), \tag{12.11}$$

holds. Functions $t_n(X)$ are eigenfunctions of a particular fourth order differential operator

$$H^{(d,\Delta_i,l_i)} = h_0(X) + \sum_{q=1}^{4} h_q(X) X^{q-1}(1-X)^{q-1}\partial_X^q\ , \tag{12.12}$$

where $h_q = h_q^{(d,\Delta_i,l_i)}$ are polynomials of order at most three. Except for a constant term in h_0 which depends a bit on the precise choice of the fifth Gaudin Hamiltonian we extract, all coefficients are symmetric w.r.t. exchange of a and b. Hence we will write them as

$$h^{(\Delta_a,l_a;\Delta_c;\Delta_b,l_b)}(X) = \chi^{(\Delta_a,l_a;\Delta_c;\Delta_b,l_b)}(X) +\ a \leftrightarrow b,$$

and display the polynomials $\chi(X)$ instead of $h(X)$

$$\begin{aligned}
&\chi_4 = 8,\\
&\chi_3 = 32X\,(l_a-2) - 4\,(4l_a + 2\Delta_c - d - 8)\,,\\
&\chi_2 = 16X^2\Big(l_a^2 + 2l_al_b - 9l_a + 7\Big)\\
&- 4X\Big(4l_a^2 + 8l_al_b + 2l_a\,(2\Delta_c - d - 18) + 2\Delta_a\Delta_b - 2d\Delta_a - (4+d)\Delta_c + d^2 + 2d + 28\Big)\\
&+ 2\Big((l_a+l_b)^2 + 2l_a\,(2\Delta_c - 2d - 4) - 2\Delta_a^2 + \Delta_c^2 + 2\Delta_a\Delta_b - 2(d+2)\Delta_c + 6d + 4\Big)\\
&\chi_1 = 16X^3\,(l_a-1)\,(l_b-1)\,(l_a+l_b-2)\\
&- 2X^2\Big(24l_a^2(l_b-1) + 2l_al_b(2\Delta_c - 24 - d) + (4l_a-2)(2\Delta_a\Delta_b - d(\Delta_a+\Delta_b+\Delta_c) + 18 + d^2) + 12\Big)\\
&+ 2X\Big(2l_a^2\,(4l_b - d - 2) + 4l_al_b\,(\Delta_c - d - 3) + 2l_a\big(4\Delta_a\Delta_b - 2d(\Delta_a+\Delta_b+\Delta_c-3) + d^2 + 4\big)\\
&+ (d-2)(2\Delta_a^2 - \Delta_c^2) - 4\Delta_a\Delta_b - 2d(d-4)\Delta_a + d^2\Delta_c - 8d\Big)\\
&+ (d-2)\Big((l_a+l_b)^2 + 4l_a(\Delta_c-2) - 2\Delta_a^2 + \Delta_c^2 + 2\Delta_a\Delta_b - 4\Delta_c + 4\Big)\\
&\chi_0 = -8X^2\,l_a\,(l_a-1)\,l_b\,(l_b-1) + 4Xl_al_b\Big(2l_al_b - 4l_a + 2\Delta_a\Delta_b - 2d\Delta_a - (d-2)\Delta_c + d^2 - d + 2\Big)\ .
\end{aligned}$$

The operator H has many interesting properties that we shall discuss on a future occasion. For the present, it is most important to note that H leaves the two subspaces $W_t^{\pm}$ invariant whenever both l_a and l_b are integers. Consequently, it specifies a special basis t_n of functions $t(X)$ in the space of tensor structures,

$$H^{(d,\Delta_i,l_i)} t_n(X) = \tau_n t_n(X)\ , \quad n = 0, \ldots, n_{ab}\ . \tag{12.13}$$

Explicit formulas for the eigenvalues τ_n and the eigenfunctions $t_n(X)$ can be worked out, and it is this basis of three-point tensor structures that we use to write down differential equations for the associated conformal partial waves.

Starting from six points, there exist topologically distinct channels that can include vertices in which all three legs carry spin, such as the so-called snowflake channel for $n = 6$, [146]. These vertices involve functions t of several variables and hence the choice of basis in the space of tensor structures needs to be extended. As we increase the dimensions d, internal edges of the OPE diagram can carry new representations beyond symmetric traceless tensors. Treating these more general representations requires us to consider higher order Casimir operators, but does not seem to bring significant new complications for the construction of multipoint blocks in any d, at least conceptually.

Chapter 13

Concluding remarks

We wish to conclude with a summary and suggest some directions for future developments of the ideas presented over the previous chapters.

The main purpose of the text was to embed the theory of conformal partial waves into harmonic analysis of the conformal group. Actually, we hope to have convinced the reader that the theory of partial waves *is* a part of harmonic analysis by its very definition. We merely tried to thoroughly explore the consequences of this initial observation. The main results that were obtained are explicit expressions for various previously unknown blocks in terms of special functions.

The first two chapters after the introduction reviewed elementary properties of higher-dimensional CFTs, focusing in particular on kinematical aspects. The next two chapters introduced some basic notions from the representation theory of non-compact Lie groups and harmonic analysis. While our discussion was tailored to the applications to CFTs, it did not deviate in a significant way from mathematical treatments of the subject, such as [86]. For example, the three decompositions most relevant in harmonic analysis on general non-compact Lie groups - the Iwasawa, Cartan and Gauss decompositions, all feature prominently in the analysis of conformal correlators. Similarly, conformal primary fields are associated to the main class of representations of the conformal group, the non-unitary principal series.

If the importance of equations in this work is to be measured by the number of times they were referred to throughout the text, then our main result is undoubtedly (7.40). This formula relates conformal blocks of four-point functions to spherical functions on the conformal group. The latter are a particularly well-behaved class of functions of two variables (two cross ratios) that can for even spacetime dimensions be written in terms of Gauss' hypergeometric function. The corresponding properties of conformal blocks were discovered by Dolan and Osborn in [30]. The proof of the relation (7.40) does not rely on the explicit description of the conformal group as an indefinite orthogonal group, but uses only a few of its properties. Namely, these are the grading structure of the conformal Lie algebra with respect to the dilation weight and the existence of an automorphism of $\mathfrak{g}$ that exchanges spaces of positive and negative dilation weights (conformal inversion). As such, the proof applies to superconformal groups as well.

Another prominent equation of this thesis is (10.46), which expresses the Laplace-Beltrami operator on a type I supergroup in terms of the Laplacian of its underlying Lie group. This equation is not new, but it acquires a new significance when restricted to the space of K-

spherical functions. It allows to write the eigenvalue equation for K-spherical harmonics on G as a nilpotent perturbation of a corresponding bosonic equation. Once this equation is derived, any method of solving it produces superconformal blocks via (7.40). In particular cases on groups $SL(2|1)$ and $SL(4|1)$ we solved the equation by reducing it to a linear algebraic one with the help of certain Clebsch-Gordan identities for the bosonic conformal group.

The formula (7.40) is only one instance of a representation of a certain space of conformal correlation functions in terms of covariant functions on the conformal group itself. Ideas that lead to it can be extended to produce several other representations of different kinds of correlators. It seems that the context of defect CFTs is particularly well-suited to such representations. Roughly, the reason for this is that the presence of defects allows for analogues of higher-point functions (in an ordinary CFT, we use the term "higher-point function" if there are at least five field insertions) that are still simple enough for partial waves to be exactly computable. An example is the bulk-bulk-defect three-point function, whose partial waves were expressed in terms of Appell's functions, (11.106).

Once various decompositions of the conformal group are introduced, our new representations of correlators can be written in a compact way, but they in fact involve relatively complicated changes of coordinates and function redefinitions. This is a desirable property, because only a transformation of sufficient complexity can be hoped to map a difficult problem to a simple one. However, the price we pay is that some notions that are obvious in the coordinate representation, such as the permutation invariance of correlation functions, become obscured in the group-theoretic coordinates. To gain control over these aspects of the theory, one should establish covariance (or functorial) properties of factors in the Bruhat and other decompositions of G. Once this is done, one can write the crossing equations in the group-theoretic gauge, without any reference to the coordinate space. An important step in this direction was the computation of the four-point crossing factor (8.20).

The systematic study of partial waves in harmonic analysis provided us with new results, but one should wander about the aims and limitations of this approach. We have said that it is hopeless to try and write closed form expressions for arbitrary irreducible matrix elements of a Lie group such as $SO(d+1,1)$. Probably is equally unlikely to obtain closed form expressions for 2437-point conformal blocks. With every new variable, the solutions to partial differential equations become significantly more difficult to come by.

If the future developments of the bootstrap will rely on higher-point functions, then the observation that the Gaudin algebra acts on these correlators and can be used to characterise partial waves, (12.3), is a useful starting point. A complete solution of the scalar five-point problem would be a great achievement for this, or any other approach.

Interesting variations on the theme of higher-point functions occur when defects are involved. There is a richer class of examples to consider and some of them may turn out to be particularly well-behaved. This is expected of the four-point function that involves two bulk and two defect field insertions. There is a realisation of this correlator as a spherical function on $G_{d,p}$ which depends on four longitudinal and one transverse cross ratio. Moreover, the correlator is particularly interesting because it satisfies a simple crossing equation obtained by switching the two bulk fields. It would be certainly worthwhile to try and compute the corresponding partial waves.

Superconformal blocks for fields that belong to long representations have for some time resisted computation. There was a feeling, however, that the required methods are in place, and that by a skilled application of weight-shifting operators one should be able to compute these partial waves. In some sense, the missing element was the universal and simple form of Casimir equations, such as (10.46). Indeed, once we had (10.46) and the Clebsch-Gordan identities (10.26) − (10.28), which are essentially a type of weight-shifting, it was extremely simple to find the blocks. It would be of great importance to use this method to obtain partial waves of four long operators in four-dimensional $\mathcal{N} = 1$ theories and other four-point functions in $\mathcal{N} \geq 2$ theories. From the mathematical point of view, this would amount to computing various spherical harmonics on supergroups $SL(4|\mathcal{N})$.

Lastly, there has recently been some interest in defects in superconformal field theories. There is a variety of ways in which a super-defect can be embedded in a superspace so that a certain amount of supersymmetry is preserved, [159, 160, 161]. We do not see any obstacle to extending our theory of defect kinematics and lifts (11.7) − (11.8) to the supersymmetric case. In particular, all our new results, such as (11.106) should have their super-partners.

At this point we will stop. Ever since the early works of Schrödinger, Heisenberg and Dirac, quantum theory has inspired truly magnificent mathematics. Some of the most fruitful interactions between the two disciplines came from the study of symmetry. The universal mathematical notion for the description of symmetry is a group. Linearity inherent in quantum mechanics allows one not to study the intractable space of all group actions, but a particularly beautiful subset of them - linear representations.

All known exactly solvable quantum systems, be it the harmonic oscillator, hydrogen atom or minimal models of two-dimensional CFTs, are so because of symmetry. Solubility is closely related to the concept of integrability, if not taken as the definition of the latter. It is, therefore, no surprise that representation theory and integrable systems are so heavily intertwined fields of mathematics. All our investigations have been based on methods of these two fields.

For a period of time, especially through the popularity of string theory, systems with finite-dimensional symmetry groups fell out of fashion. However, rather than searching for ever larger groups, a possible direction of study that one may take is to explore consequences of some symmetry more fully. In the context of conformal field theories, this idea was revived with the realisation that $SO(d + 1, 1)$ symmetry can be combined with numerical techniques to great effects, [16]. Aided by physical intuition in order to ask useful questions, methods of [16] paved way to obtaining rigorous results about CFTs that were previously out of reach. Conformal bootstrap is similar in spirit to the S-matrix approach to quantum field theory and the latter also received renewed attention in the last few years.

In relation to these facts, the mathematics that we used was not exactly modern. Its foundations were laid down in the 1940s and 1950s by Gelfand, Bargmann, Harish-Chandra and their collaborators and developed over the next decades into an immense body of knowledge. Surprisingly few of these results found their way into conformal field theory literature, at least until recently (with notable exceptions such as [87]). However, with a number of relevant physics questions lined up, it is a good moment to return to this beautiful part of the 20^{th} century mathematics. We view our work as a small step in this direction.

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