Homomorphism thresholds and embeddings of spanning subgraphs in dense graphs

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1. Introduction

In this thesis we will consider graph theoretical problems from extremal and probabilistic combinatorics. Before going into detail of the backgrounds of the specific topics in Section 1.2 and Section 1.3, we give an overview, pointing out the common traits of both sections as well as the vicinity of the results in their respective fields.

1.1 Overview

In this thesis, we will consider simple, undirected, finite graphs. Usually we will denote them with $G = (V, E)$, and the number of vertices $n = |V|$ can be regarded as large or growing towards infinity.

Generally we use standard notation and refer the readers to one of the standard books [5, 7, 13]. We would like to point out, that by a path / walk of length ℓ, we are referring to a path / walk containing exactly ℓ edges, rather than one containing ℓ vertices. By the distance of two vertices we are referring to the length of the shortest path between them.

Although it is sometimes remarked that graph theoretical concepts appear as early as 1736 in a paper of Euler [20], the field has broadened into a large active field of research in the 20th century, with many of its defining theorems being formulated and proved in the latter half of it. Lately it has seen a sharp rise in applications in other mathematical fields as well as in computer science. With the study of larger and more complex graphs, the need to describe the rough structure of such graphs arises. In a sense we will focus on certain structural properties, which we can guarantee for a large class of graphs each in Chapter 2 and Chapter 3. A crucial role in our structural analysis will be played by (graph-)homomorphisms.

1.2 Homomorphism thresholds

Often in extremal graph theory we want to force global behaviour with local conditions. Examples for such local conditions are degree conditions and forbidding the appearance of small subgraphs, where small usually translates to constant size, i.e. independent of n. The global behaviour we want to force may be some kind of structural property (see, e.g., [3, 4, 55] for examples), and one such structural property is the chromatic number of a graph.

Chromatic number

We recall the definition of the *chromatic number* of a graph. First we define a *proper r-colouring* c of a graph $G = (V, E)$ as a map $c\colon V \to [r]$ such that $c(v) \neq c(u)$ whenever we have $vu \in E$. The *chromatic number* of G, or $\chi(G)$, then is the smallest number r, such that there is a proper r-colouring of G.

An obvious sufficient condition for a graph G to have large chromatic number is when G contains a large clique as a subgraph. The inverse is not true, as the following celebrated result of Erdős [16] shows.

Theorem 1.2.1 (Erdős 1959)**.** *Let g, r be arbitrary numbers. There exists a graph G with girth at least g and chromatic number at least r.*

This theorem does not only show that the chromatic number of a graph may be large, even if it does not contain any clique of more than 2 vertices, but that there are graphs with large chromatic number that locally look like trees. Since the chromatic number of trees is 2, the large chromatic number must be a global phenomenon rather than a local one. A natural generalisation of the chromatic number are graph homomorphisms into fixed graphs H, since the chromatic number $\chi(G)$ of a graph G is just the smallest number r, such that there is a graph homomorphism $\varphi\colon V(G) \longrightarrow K_r$.

Here, once again the complete graphs seem to be connected to the chromatic number, but in a not so obvious way. Indeed, it turns out that determining the chromatic number of a graph is computationally complex, in the sense that it is NP-complete, and even determining if a graph has a proper r-colouring for a fixed $r \geqslant 3$ is already NP-complete.

According to [6] "one of the deepest unsolved problems in graph theory" is Hadwigers conjecture [24] that states that every graph with chromatic number at least r contains a K_r as a *minor*. It is proven for $r \leqslant 6$, but already the proofs for $r = 5$ and $r = 6$ used the probably most famous theorem of graph theory, the 4-colouring theorem, that was evading attempts to being proved for quite some time itself. Furthermore, determining if G contains a graph H as a minor turns out to be another NP-complete problem, so even if Hadwiger's conjecture was true for general r, it would still be hard to determine the chromatic number of a given graph, or bound it from above using Hadwiger's conjecture.

Trying to bound the chromatic number of a graph G from below by something else than directly looking for the largest clique contained as a subgraph in G, we might turn to local conditions that force the appearance of a clique inside a graph, and therefore force a large chromatic number indirectly.

The following famous theorem of Turán [54] states that a very large average degree of a graph G forces the appearance of a large clique in G

Theorem 1.2.2 (Turán 1941)**.** *Let $G = (V, E)$ be an K_{r+1} free n-vertex graph, then*

$$|E| \leqslant \frac{r-1}{r}\frac{n^2}{2}.$$

Such a threshold ($\frac{r-1}{r}$ in case of a K_{r+1}) for the share of all possible edges that can be present without forcing the appearance of a certain subgraph H is known as the *Turán density* $\pi(H)$ for any fixed graph H. This notation can naturally be extended to families of graphs simply by replacing the graph H with a family $\mathscr{F}$ of graphs.

A rather unusual way to interpret this theorem is to remark that a very large average degree in G forces G to have a large chromatic number. Since the average degree is a condition that is sometimes difficult to use in proofs, one may consider the question, which minimum degree $\delta(G)$ forces a graph G to have a chromatic number of at least c? An obvious answer would be to use Turán's theorem directly to determine some value for $\delta(G)$, however Theorem 1.2.1 tells us that there are graphs that locally look tree-like and therefore naturally have an upper bound on the maximum degree, while still having large chromatic number. Combining these two theorems, it seems interesting to study graphs with large girth and large minimum degree. We start our discussion with triangle-free graphs for simplicity.

There, Erdős, Simonovits, and Hajnal [19, page 325] proved that for every $\varepsilon > 0$ there exists a sequence of triangle-free graphs $(G_n)_{n\in\mathbb{N}}$ with minimum degree at least $(\frac{1}{3} - \varepsilon)$ with unbounded chromatic number, i.e., $\chi(G_n) \longrightarrow \infty$ as $n \longrightarrow \infty$. Maybe surprisingly, they did not find such sequences where the minimum degree was at least $(\frac{1}{3} + \varepsilon)$, implying that the combination of these two parameters may bound $\chi(G)$ from above instead of bounding it from below. Indeed they conjectured that such a sequence does not exist with minimum degree at least $(\frac{1}{3} + \varepsilon)$. This conjecture was later proved by Thomassen [52], establishing the first known chromatic threshold.

Chromatic threshold

To describe all graphs of interest to us precisely and yet in a general way, we will introduce the following shorthand notation. Let $\mathscr{F}$ be some family of graphs, and let α be an arbitrary number in $[0, 1]$, for the class of $\mathscr{F}$-free graphs G with minimum degree at least $\alpha|V(G)|$ we will simply write $\mathscr{G}_{\mathscr{F}}(\alpha)$, i.e.,

$$\mathscr{G}_{\mathscr{F}}(\alpha) = \big\{G\colon \delta(G) \geqslant \alpha|V(G)| \text{ and } F \nsubseteq G \text{ for all } F \in \mathscr{F}\big\},$$

where for convenience we will drop the brackets for one element sets $\mathscr{F} = \{F\}$, so we write $\mathscr{G}_F(\alpha)$ instead of $\mathscr{G}_{\{F\}}(\alpha)$.

For $\alpha \approx 1$, obviously $\mathscr{G}_{\mathscr{F}}(\alpha)$ does contain just finitely many graphs of bounded size, namely at most $min_{F\in\mathscr{F}}|V(F)|-1$, since otherwise the graph F_{min} realising this minimum would appear in a graph from $\mathscr{G}_{\mathscr{F}}(\alpha)$ with size $|F_{min}|$. This number might increase as α gets smaller, however the limit on the size of the graphs does not change, as long as α is bigger than the *Turán density* $\pi(\mathscr{F})$. When α gets even smaller, in most cases we will see a sharp increase of the size of $\mathscr{G}_{\mathscr{F}}(\alpha)$, and we are interested in structural properties of members of $G \in \mathscr{G}_{\mathscr{F}}(\alpha)$ as α moves from $\pi(\mathscr{F})$ to 0, where structural properties are captured by the existence of (graph) homomorphism $G \xrightarrow{\text{hom}} H$ for some "small" graph H, i.e. the size of H is dependent on α and $\mathscr{F}$, but not on G.

With no further restrictions on H, for a fixed size of H we get the most homomorphisms from other graphs into H by taking $H = K_{|H|}$. Considering the second interpretation of the definition of the chromatic number given in Section 1.2 it is easy to see, why this approach was studied under the name *chromatic threshold*, and the following definition will specify this threshold.

Definition 1.2.3. For a family of graphs $\mathscr{F}$ we define its *chromatic threshold*

$$\delta_\chi(\mathscr{F}) = \inf\big\{\alpha \in [0,1] \colon \text{there is } K = K(\mathscr{F},\alpha) \text{ such that } \chi(G) \leqslant K \text{ for every } G \in \mathscr{G}_{\mathscr{F}}(\alpha)\big\}$$

If $\mathscr{F} = \{F\}$ consists of a single graph only, then we again simply write $\delta_\chi(F)$.

For the smallest reasonable graph to analyse in terms of the number of vertices we get $\mathscr{F} = \{K_3\}$, and rephrasing the above mentioned result of Erdős, Simonovits, and Hajnal [19, page 325], they proved that $\delta_\chi(K_3) \geqslant \frac{1}{3}$ by showing that for every $\varepsilon > 0$ there exists a sequence of graphs $(G_n)_{n\in\mathbb{N}}$ with members from $\mathscr{G}_{K_3}(\frac{1}{3}-\varepsilon)$ with unbounded chromatic number, i.e., $\chi(G_n) \longrightarrow \infty$ as $n \longrightarrow \infty$. They conjectured, that such a sequence does not exist with members from $\mathscr{G}_{K_3}(\frac{1}{3}+\varepsilon)$, which would imply $\delta_\chi(K_3) \leqslant \frac{1}{3}$ and therefore

$$\delta_\chi(K_3) = \frac{1}{3}. \tag{1.2.1}$$

As was noted above, this conjecture was later proved by Thomassen [52], establishing the first known chromatic threshold.

In their paper, Erdős and Simonovits [19] moreover asked for the chromatic threshold for C_5. In another paper Thomassen [53] answered this question by establishing

$$\delta_\chi(C_{2k-1}) = 0, \tag{1.2.2}$$

for $k \geqslant 3$.

Interpreting K_3 as a clique rather than a cycle, (1.2.1) generalises to

$$\delta_\chi(K_k) = \frac{2k-5}{2k-3}, \tag{1.2.3}$$

for $k \geqslant 3$ (see [22, 36]).

Some more progress concerning the understanding of the chromatic threshold was made by Łuczak and Thomassé [38], and Lyle [39], until finally Allen, Böttcher, Griffiths, Kohayakawa, and Morris [1] determined the chromatic threshold $\delta_\chi(\mathscr{F})$ for every finite family of graphs $\mathscr{F}$.

Homomorphism threshold

Recalling the definition of the chromatic threshold, we had no restrictions on H except for its size, which might lead to homomorphic images H of our graphs G that are somehow undesirable. Restricting H further might lead to a different threshold behaviour, and a natural restriction on H seems to be to require H to be $\mathscr{F}$-free, ensuring that $\mathscr{F}$-free graphs stay that way after using the homomorphisms into a smaller graph. This leads to the following definition.

Definition 1.2.4. For a family of graphs $\mathscr{F}$ we define its *homomorphism threshold*

$$\begin{aligned}\delta_{\text{hom}}(\mathscr{F}) = \inf\big\{\alpha \in [0,1]\colon &\text{ there is an } \mathscr{F}\text{-free graph } H = H(\mathscr{F},\alpha) \\ &\text{such that } G \xrightarrow{\text{hom}} H \text{ for every } G \in \mathscr{G}_{\mathscr{F}}(\alpha)\big\}.\end{aligned}$$

If $\mathscr{F} = \{F\}$ consists of a single graph only, then we again simply write $\delta_{\text{hom}}(F)$.

It follows directly from the definition that

$$\pi(\mathscr{F}) \geqslant \delta_{\text{hom}}(\mathscr{F}) \geqslant \delta_\chi(\mathscr{F})$$

and that $\delta_{\text{hom}}(\mathscr{F}) = 0$ for all families $\mathscr{F}$ containing a bipartite graph, because $\pi(\mathscr{F}) = 0$ in this case. The first one to study the homomorphism threshold was Łuczak [35], who showed that

$$\delta_{\text{hom}}(K_3) = \delta_\chi(K_3) = 1/3,$$

proving the same threshold as in (1.2.1). For larger cliques, similar to (1.2.3) we have

$$\delta_{\text{hom}}(K_k) = \delta_\chi(K_k) = \frac{2k-5}{2k-3}, \tag{1.2.4}$$

for $k \geqslant 3$, which was proved by Goddard and Lyle [22] and Nikiforov [36] (see also [42]).

By interpreting K_3 as an odd cycle, rather than a clique, one might extend the result of Łuczak by looking at (families of) odd cycles for $\mathscr{F}$. In that direction, Letzter and Snyder [34] recently showed that

$$\delta_{\hom}(C_5) \leqslant \frac{1}{5} \qquad \text{and} \qquad \delta_{\hom}(\{C_3, C_5\}) = \frac{1}{5}.$$

Further generalising this results to families of odd cycles of arbitrary girth we present the following result.

Theorem 1.2.5. *For every integer $k \geqslant 3$ we have*

(i) $\delta_{\hom}(C_{2k-1}) \leqslant \frac{1}{2k-1}$ *and*

(ii) $\delta_{\hom}(\mathscr{C}_{2k-1}) = \frac{1}{2k-1}$*, where the family $\mathscr{C}_{2k-1} = \{C_3, C_5, \ldots, C_{2k-1}\}$ consists of all odd cycles of length at most $2k-1$.*

This is our first main result of Chapter 2, and will be addressed in the first 3 sections of this chapter. In some of the proofs we might insist on G being maximally $\mathscr{C}_{2k-1}$-free. This however does not restrict our choices of G in these kind of theorems, since every $\mathscr{C}_{2k-1}$-free graph can be made maximally $\mathscr{C}_{2k-1}$-free without lowering the minimum degree by adding edges, and a homomorphism of a graph is also a homomorphism of all its subgraphs. Determining the homomorphisms for maximally $\mathscr{C}_{2k-1}$-free graphs therefore indeed suffices to determine the homomorphisms of all $\mathscr{C}_{2k-1}$-free graphs.

Note that for $k = 2$ part (ii) of Theorem 1.2.5 would include part (i) and this is Łuczak's theorem [35]. For $k = 3$ Theorem 1.2.5 was obtained by Letzter and Snyder [34]. We remark that our approach substantially differs from the work of Łuczak and of Letzter and Snyder. For example, Łuczak's proof relied on Szemerédi's Regularity Lemma, which is not required here.

In fact, in the paper of Letzter and Snyder they proved an even stronger statement, namely that all $\mathscr{C}_5$-free graphs G with $\delta(G) > \alpha_{\delta_{\hom}(\mathscr{C}_5)}|V(G)| = \frac{1}{5}|V(G)|$ are homomorphic to an Andrásfai graph $A_{k,r}$, which is introduced in Section 2.1, while the graphs H from our proof from Section 2.3 are computationally much more complex. It would be nice to have a firmer grasp on the graphs H, just like in the proof of Letzter and Snyder for general k, however for $k = 2$ the analogous statement (that all $\mathscr{C}_3$-free graphs G with $\delta(G) > \alpha_{\delta_{\hom}(\mathscr{C}_3)}|V(G)| = \frac{1}{3}|V(G)|$ are homomorphic to an Andrásfai graph) turns out to be not true. Since all (generalised) Andrásfai graphs have chromatic number 3, but as Häggkvist pointed out [25], the Grötzsch graph (see Figure 1.2.1) is a triangle free graph with chromatic number 4 where suitable unbalanced blow-ups keep these properties while having a minimum degree of $\frac{10}{29} > \frac{1}{3}$.

And for $k \geqslant 4$ the analogous statement turns out to be not true again, as we show in Section 2.6, specifically by Lemma 2.6.2, Observation 2.6.8, and Observation 2.6.9, using some graph from $\mathscr{T}_k$ defined there.

Taking these counterexamples into account, we present the following theorem, which is our second main result from Chapter 2.

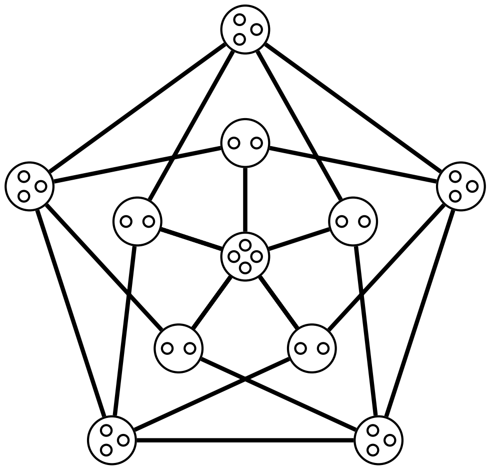


Figure 1.2.1: A blow-up of the Grötzsch graph which is triangle-free, 10-regular on 29 vertices and has chromatic number 4.

Theorem 1.2.6. *Let $k = 3$ and let G be a maximal $\mathscr{C}_{2k-1}$-free graph on n vertices with $\delta(G) > \frac{1}{2k-1}n$.*

If G does not contain a graph of $\mathscr{T}_k$ as a subgraph, then G is homomorphic to $A_{k,r}$ for some r.

While this theorem is slightly weaker than the one proven by Letzter and Snyder in [34], our proofs are mainly done for general k and point out a clear path how to generalise Theorem 1.2.6 for arbitrary large k and we will come back to this in Chapter 4.

1.3 Spanning subgraphs

In Section 1.2 we have seen Turán's theorem, that a large average degree $d(G)$ forces the appearance of a small subgraph in G. An interesting question is, if something similar is possible with large, or even spanning graphs that should appear as a subgraph, without the condition that G should essentially be a complete graph. Since a large average degree can always be achieved having a few vertices with really small degree, which might obstruct the appearance of a spanning subgraph very locally, one should rather consider $\delta(G)$ to be large for these kind of studies.

Graphs with large minimum degree

The most prominent theorem of this type, concerning Hamiltonian cycles, surely is the following theorem of Dirac [14].

Theorem 1.3.1 (Dirac 1952)**.** *Let $G = (V, E)$ be an n-vertex graph with $n \geqslant 3$. If $\delta(G) \geqslant n/2$, then G contains a Hamiltonian cycle.*

This gives an optimal bound to maybe the simplest version of sufficient conditions for the existence of Hamiltonian cycles in a graph and has sparked a wide field of research.

Although Diracs theorem seems hard to improve upon, since the bound is tight while offering a - in a sense maximal - local condition to get a global structure, there have indeed been plenty of different proofs to get related results with a similar approach. From weakening the assumption in the theorem, making it slightly less local in the process [43] to extending the definition of Hamiltonian cycles to hypergraphs [46, 56] or infinite graphs [27] there are plentiful results.

However, in this thesis we want to concentrate on another direction of research, extending the definition of Hamiltonian cycles in simple finite graphs to something called powers of Hamiltonian cycles (see Figure 1.3.1). For this we first need some definitions. Let $k \in \mathbb{N}$ be a natural number, then the *k-th power* of a graph G, or shorthand G^k is a graph on the same vertex set as G, where two vertices are neighbours if they have distance at most k in G. For example for $k = 2$, for every 3 vertices a, b, and c such that $ab, bc \in E(G)$, we would have $ac \in E(G^2)$ as well.

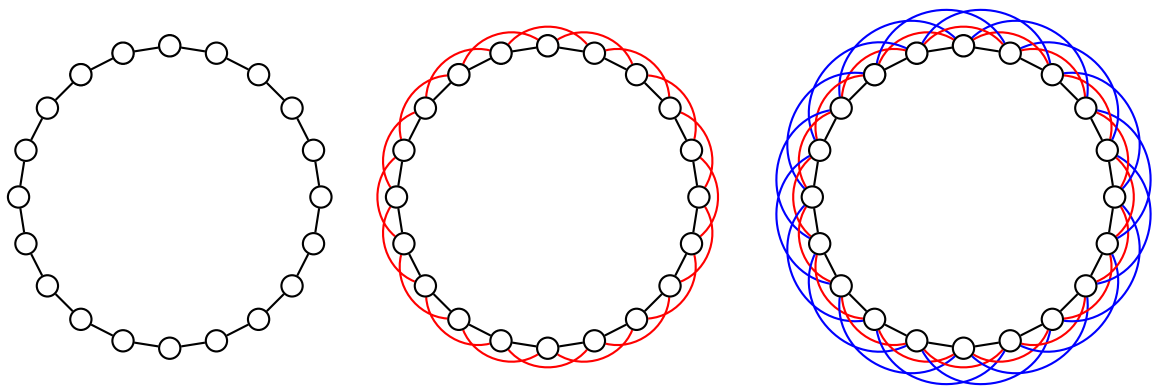


Figure 1.3.1: The graphs C_{20}, $(C_{20})^2$, and $(C_{20})^3$.

For simplicity, we refer to a k-th power of a path with at least k vertices as a *k-path*. Moreover, we refer to the ordered k-tuples of the first and last k vertices of a k-path as *ends* of the k-path and an $(\vec{x}, \vec{y}; k)$-path is a k-path with ends $\vec{x}$ and $\vec{y}$. Note that every $k + 1$ consecutive vertices of a k-path span a clique and if a graph $G = (V, E)$ contains the k-th power of a Hamiltonian cycle, it also contains $\lfloor \frac{|V|}{k+1} \rfloor$ pairwise vertex disjoint copies of K_{k+1} and G contains a K_{k+1}-factor if $|V|$ is divisible by $k + 1$.

Dirac's well known theorem [14] guarantees the existence of a Hamiltonian cycle, just by having a sufficiently big minimum degree. Two disjoint cliques of size $n/2$ show that this minimum degree condition is best possible as well. Forcing the existence of spanning subgraphs and finding the optimal minimum degree condition to do so became a rich field of research in extremal graph theory (see, e.g., [9] and the references therein).

In 1963 Corrádi and Hajnal solved the minimum degree problem for K_{k+1}-factors for $k = 2$ [12], and in 1970 Hajnal and Szemerédi [26] solved this problem for every $k \geqslant 3$. Pósa (see [17]) and Seymour [49] asked for a common generalisation of those results on factors and Dirac's theorem. They conjectured that the best possible minimum degree conditions for K_{k+1}-factors and k-th powers of Hamiltonian cycles are the same. (At least when the number of vertices is divisible by $k + 1$.) In 1998 Komlós, Sárközy, and Szemerédi [31] proved this conjecture for large graphs by giving the following theorem.

Theorem 1.3.2 (Komlós, Sarközy, Szemerédi 1998)**.** *For every positive integer k there exists n_0 such that if G is a graph on $n \geqslant n_0$ vertices with minimum degree $\delta(G) \geqslant \frac{k}{k+1}n$, then G contains the k-th power of a Hamiltonian cycle.*

The complete and nearly balanced $(k + 1)$-partite graphs show that this bound on $\delta(G)$ is optimal.

Uniformly dense and inseparable graphs

Whenever a counterexample seems to be the only one of its kind, in extremal graph theory one may ask if the desired theorem still holds true for relaxed assumptions when excluding this particular example. We have seen similar behaviour in Section 1.2 already, this time however, we want to achieve excluding the appearance of certain graphs indirectly. The following robust restriction that imposes a uniformly positive edge density for subgraphs induced on linear sized subsets of vertices will rule out the appearance of the counterexamples. It also has the benefit of being true for random graphs, giving this a wide range of applications, as well as being relatively easy to use in proofs.

Definition 1.3.3. We say that a graph $G = (V, E)$ is (ϱ, d)-*dense* for $\varrho > 0$ and $d \in [0, 1]$ if

$$e(U) \geqslant d\frac{|U|^2}{2} - \varrho|V|^2$$

for every subset $U \subseteq V$, where $e(U)$ denotes the number of edges of G contained in U.

Staden and Treglown [50] indeed showed the following theorem using (ϱ, d)-denseness.

Theorem 1.3.4. *For every positive integer k, and $d, \mu > 0$ there exists $\varrho > 0$, and an integer n_0 such that if G is a (ϱ, d)-dense graph on $n \geqslant n_0$ vertices with minimum degree $\delta(G) \geqslant (\frac{1}{2} + \mu)n$, then G contains the k-th power of a Hamiltonian cycle.*

(For K_{k+1}-factors see also [45].) Maybe surprisingly the minimum degree condition becomes independent of k. Moreover, this bound on $\delta(G)$ is optimal in this setting, as once again, with smaller $\delta(G)$, counterexamples are arising. For $\delta(G) < n/2$ consider the graph consisting of two disjoint cliques on close to $n/2$ vertices. This graph is not connected and therefore not Hamiltonian at all, furthermore it is (ϱ, d)-dense (for any $d < 1/4$) and has a big minimum degree of almost $n/2$.

Once again trying to rule out this specific example by requiring G to have an additional property Glock and Joos (see [50, Concluding Remarks]) considered the following bipartite version of Definition 1.3.3.

$$e(X,Y) = \big|\{(x,y) \in X \times Y : xy \in E(G)\}\big| \geqslant d|X||Y| - \varrho|V|^2 \tag{1.3.1}$$

for all subsets X, $Y \subseteq V$.

They proceed to prove that requiring G to be uniformly dense and satisfy (1.3.1), as well as having a minimum degree of at least μn for arbitrary $\mu > 0$ guarantees G to contain the k-th power of a Hamiltonian cycle. Once again we note that the only variables depending on k are ϱ and n_0, where we must have that $n = |V(G)| \geqslant n_0$.

We show that the following definition, which is slightly weaker than the one used by Glock and Joos is sufficient to be paired with uniform density of graphs to force a k-th power of a Hamiltonian cycle.

Definition 1.3.5. We say that a graph $G = (V, E)$ is *μ-inseparable* for some $\mu > 0$ if

$$e(X, V \smallsetminus X) \geqslant \mu|X||V \smallsetminus X|$$

for every subset $X \subseteq V$.

Note that there is no error term in our definition, forcing μ-inseparable graphs to have a large minimum degree by invoking this assumption to subsets X consisting of one vertex. We will show that μ-inseparable graphs are "well connected" (see, e.g., Lemma 3.2.2).

Our first main result of Chapter 3 is then the following theorem, combining Definitions 1.3.3 and 1.3.5 to force the appearance of k-th powers of Hamiltonian cycles for every fixed integer $k \geqslant 1$.

Theorem 1.3.6. *For every d, $\mu \in (0,1]$, and $k \in \mathbb{N}$ there exist $\varrho > 0$ and n_0 such that every (ϱ, d)-dense and μ-inseparable graph G on $n \geqslant n_0$ vertices contains the k-th power of a Hamiltonian cycle.*

Since it is rather easy to see that every graph G with minimum degree $\delta(G) \geqslant (1/2 + \mu)|V(G)|$ is μ-inseparable, Theorem 1.3.6 is a strengthening of the result of Staden and Treglown for powers of Hamiltonian cycles [50].

Furthermore it is also a strengthening of the result by Glock and Joos [50, Concluding Remarks]. To see this, consider the graph G consisting of two cliques of size $(1/2 + \mu/2)n$ which intersect in μn vertices. It has n vertices in total and satisfies Definition 1.3.3 as well as Definition 1.3.5, therefore by Theorem 1.3.6 it contains the k-th power of a

Hamiltonian cycle. But since it fails to satisfy property (1.3.1) for arbitrary subsets X and Y, the result of Glock and Joos is not applicable to this graph.

Staden and Treglown and also Glock and Joos did not just consider the embeddings of powers of Hamiltonian cycles, but of a more general class of graphs. We recall that the *bandwidth* $\mathrm{bw}(H)$ of an n-vertex graph H is the maximum distance of two adjacent vertices minimised over all possible orderings of the vertex set of H, i.e.,

$$\mathrm{bw}(H) = \min_{\sigma} \max_{xy \in E(H)} \big|\sigma(x) - \sigma(y)\big|,$$

where the minimum is taken over all possible bijections $\sigma\colon V(H) \longrightarrow [n]$. We may refer to an ordering σ of $V(H)$ achieving this minimum $\mathrm{bw}(H)$ as a *bandwidth ordering* of H.

Staden and Treglown and also Glock and Joos proved that with the requirements they had on their graphs, it is not only possible to embed powers of Hamiltonian cycles, but more generally graphs with small bandwidth which satisfy a few additional conditions.

Using our Theorem 1.3.6, we will establish the following version of the bandwidth theorem from [9] for inseparable and uniformly dense graphs, which is our second main result from Chapter 3.

Theorem 1.3.7. *For every d, $\mu \in (0,1]$, and $\Delta \in \mathbb{N}$ there exist ϱ, $\beta > 0$ and n_0 such that every (ϱ, d)-dense and μ-inseparable graph G on $n \geqslant n_0$ vertices contains every n-vertex graph H satisfying $\Delta(H) \leqslant \Delta$ and $\mathrm{bw}(H) \leqslant \beta n$.*

Finally, another direction which one could investigate further is the question if there is a common generalisation of Theorem 1.3.2 and Theorem 1.3.6. Indeed it is rather easy to see that such a large minimum degree condition forces a graph to be inseparable, so the question is, if it also forces a graph to be uniformly dense. While this turns out not to be true, there is a slightly weaker version of Definition 1.3.3, which is forced by graphs with very large minimum degree. Maesaka and Schacht [40] proved that this weaker version of Definition 1.3.3 for $k = 1$ suffices to guarantee Hamilton cycles to appear in a graph. We will come back to this in Section 3.5, together with some further research in this direction.

2. Homomorphism Thresholds of odd cycles

To prepare for the proof of Theorem 1.2.5, first we will introduce Andrásfai graphs in Section 2.1, which will be used for the lower bound of part (ii) of Theorem 1.2.5. To be able to give the proof of the upper bound for both parts of Theorem 1.2.5 in Section 2.3, we will make sure that some subgraphs may not appear in the graphs we consider in Theorem 1.2.5, these statements and proofs will be collected in Section 2.2.

To prepare for the proof of Theorem 1.2.6, we will start off with introducing a property of subgraphs, namely being well-behaved, and prove that some subgraphs we will consider have this property in Section 2.4. This property will be of great help in making other statements and proofs relatively compact. Analogously to the preparation for the proof of Theorem 1.2.5, we will then prove that some (induced) subgraphs may not appear in the graphs we consider in Theorem 1.2.6 and collect these statements and proofs in Section 2.5. After introducing odd tetrahedra in Section 2.6, which motivate our formulation of Theorem 1.2.6 in the first place, we will then give the proof of Theorem 1.2.6 in Section 2.7.

2.1 Generalised Andrásfai graphs

In this section we establish the lower bound of part (ii) of Theorem 1.2.5, which will be given by a sequence of so-called *Andrásfai graphs.* For $k = 2$ those graphs already appeared in the work of Erdős [15] and were also considered by Andrásfai [2,3].

Definition 2.1.1. For every integer $k \geqslant 2$ we define the class $\mathscr{A}_k$ of *Andrásfai graphs* consisting of all graphs $G = (V, E)$, where V is a finite subset of the unit circle $\mathbb{R}/\mathbb{Z}$ and two vertices are adjacent if and only if their distance in $\mathbb{R}/\mathbb{Z}$ is bigger than $\frac{k-1}{2k-1}$, i.e., the neighbourhood of any vertex $v \in V \subseteq \mathbb{R}/\mathbb{Z}$ is given by the set $V \cap \big(v + \big(\frac{k-1}{2k-1}, \frac{k}{2k-1}\big)\big)$, where

$$v + \left(\tfrac{k-1}{2k-1}, \tfrac{k}{2k-1}\right) = \left\{v + x \colon x \in \left(\tfrac{k-1}{2k-1}, \tfrac{k}{2k-1}\right)\right\} \subseteq \mathbb{R}/\mathbb{Z}.$$

Moreover, for integers $k \geqslant 2$ and $r \geqslant 1$ the *Andrásfai graph* $A_{k,r}$ is isomorphic to a graph from $\mathscr{A}_k$ having the corners of a regular $((2k-1)(r-1)+2)$-gon as its vertices (see Figur 2.1.1 for an illustration).

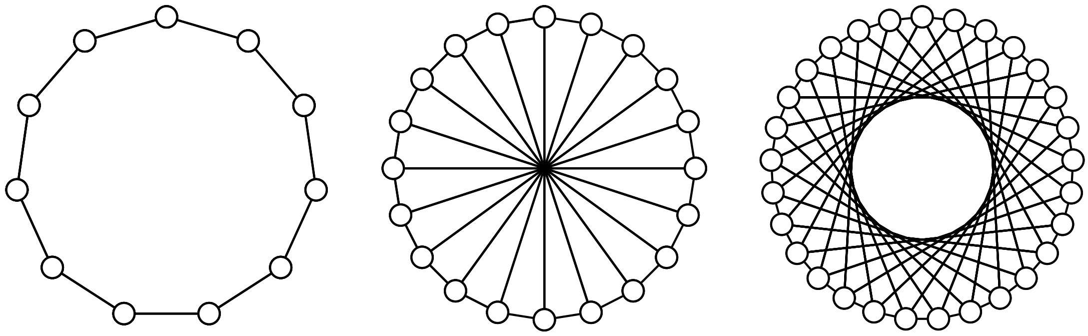


Figure 2.1.1: The Andrásfai graphs $A_{5,2}$, $A_{5,3}$ and $A_{5,4}$ in the Hamiltonian ordering from Equation (2.1.2).

We remark that one can show that every graph $G \in \mathscr{A}_k$ is homomorphic to $A_{k,r}$ for some sufficiently large r. The following properties of Andrásfai graphs are well-known and we include the proof for completeness.

Proposition 2.1.2. *For all integers $k \geqslant 2$ and $r \geqslant 1$ the following properties hold*

(*a*) *$A_{k,r}$ is r-regular,*

(*b*) *$A_{k,r}$ is $\mathscr{C}_{2k-1}$-free,*

(*c*) *if $r \geqslant 2$ then any two vertices of $A_{k,r}$ are contained in a cycle of length $2k+1$, and*

(*d*) *if $A_{k,r} \xrightarrow{\text{hom}} H$ for some graph H with $|V(H)| < |V(A_{k,r})|$, then H contains an odd cycle of length at most $2k-1$.*

In particular, it follows from (a), $|V(A_{k,r})| = (2k-1)(r-1)+2$, (b), and (d) that $\delta_{\text{hom}}(\mathscr{C}_{2k-1}) \geqslant \frac{1}{2k-1}$, as r can be chosen arbitrarily big.

Proof. For given integers $k \geqslant 2$ and $r \geqslant 1$ set

$$n = |V(A_{k,r})| = (2k-1)(r-1)+2$$

and let $v_0, \ldots, v_{n-1}$ be the vertices of $A_{k,r}$ in cyclic order, i.e., we assume $v_i \equiv i/n \in \mathbb{R}/\mathbb{Z}$ for every $i = 0, \ldots, n-1$. By definition of $A_{k,r}$ the neighbourhood of v_0 is contained in the open interval $\left(\frac{k-1}{2k-1}, \frac{k}{2k-1}\right) \subseteq \mathbb{R}/\mathbb{Z}$. Consequently,

$$N(v_0) = \{v_i \colon i = (k-1)(r-1)+1, \ldots, k(r-1)+1\} \tag{2.1.1}$$

and part (*a*) follows by symmetry.

For part (b) we observe that for any closed walk $u_1 \dots u_\ell u_1$ of length ℓ in $A_{k,r}$ we have $(u_\ell - u_1) + \sum_{i=1}^{\ell-1}(u_i - u_{i+1}) = 0$ and owing to the definition of $A_{k,r}$ each term of that sum lies in $\left(\frac{k-1}{2k-1}, \frac{k}{2k-1}\right) \subseteq \mathbb{R}/\mathbb{Z}$. However, for every integer $j = 2, \dots, k$ we have

$$(j-1) \leqslant (2j-1)\frac{k-1}{2k-1} < (2j-1)\frac{k}{2k-1} \leqslant j.$$

Consequently, $(u_\ell - u_1) + \sum_{i=1}^{\ell-1}(u_i - u_{i+1}) \in (j-1, j)$. Since $0 \notin (j-1, j)$, no walk in $A_{k,r}$ of length $2j-1$ for $j \leqslant k$ can be closed and part (b) follows.

For part (c) we show below that starting in $u_0 = v_0$ and always choosing the closest clockwise neighbour in $A_{k,r}$, i.e., setting

$$u_j \equiv u_{j-1} + \frac{(k-1)(r-1)+1}{n} \equiv j\frac{(k-1)(r-1)+1}{n} \in \mathbb{R}/\mathbb{Z}, \tag{2.1.2}$$

defines a Hamiltonian cycle $C = u_0 \dots u_{n-1}u_0$ with the property that

$$u_1, \quad u_{(2k-1)+1}, \quad u_{2(2k-1)+1}, \quad \dots, \quad u_{(r-1)(2k-1)+1} = u_{n-1}$$

are the r neighbours of $u_0 = v_0$ in $A_{k,r}$. In other words, every $(2k-1)$-st vertex on the subpath $u_1 \dots u_{n-1}$ of the Hamiltonian cycle C is a neighbour of u_0. Considering the C_{2k+1}'s created by the chords between u_0 and its neighbours $u_{(2k-1)+1}, \dots, u_{(r-2)(2k-1)+1}$ shows that $u_0 = v_0$ lies on a cycle of length $2k+1$ with every other vertex of $A_{k,\ell}$, which by symmetry verifies part (c).

It is left to show that the C defined above, has the desired properties, i.e. is Hamiltonian with the stated distribution of $N(v_0)$. It follows from the definition of C that $u_{n-1}u_0$ and u_iu_{i+1} are edges of $A_{k,r}$ for every $i = 0, \dots, n-2$ and, hence, C is a closed walk of length n. However, since

$$n = (2k-1)(r-1) + 2 = 2\big((k-1)(r-1)+1\big) + (r-1)$$

and $(k-1)(r-1)+1$ are relatively prime, it follows that C is indeed a Hamiltonian cycle. Moreover, we observe for $s = 0, \dots, r-1$ that

$$\begin{aligned}
u_{s(2k-1)+1} &\overset{(2.1.2)}{\equiv} (s(2k-1)+1)\frac{(k-1)(r-1)+1}{n} \\
&\equiv (s(2k-1)+1)\frac{(k-1)(r-1)+1}{(2k-1)(r-1)+2} \\
&\equiv \frac{(k-1)(r-1)+1+s}{(2k-1)(r-1)+2} + s(k-1) \\
&\equiv \frac{(k-1)(r-1)+1+s}{n} \equiv v_{(k-1)(r-1)+1+s} \overset{(2.1.1)}{\in} N(v_0),
\end{aligned}$$

which shows the stated distribution of $N(v_0)$ on C.

Finally, assertion (d) is a direct consequence of part (c). Suppose $\varphi\colon A_{k,r} \longrightarrow H$ is a graph homomorphism with $|V(H)| < n$. Then there are two vertices $x, y \in V(A_{k,r})$ such that $\varphi(x) = \varphi(y)$. In particular $xy \notin E(A_{k,r})$ and in view of (c) the vertex $\varphi(x) = \varphi(y)$ must be contained in a closed odd walk of length at most $2k-1$ in H and, consequently, H contains an odd cycle of length at most $2k-1$. □

2.2 Forbidden subgraphs in C_{2k-1}-free graphs

In this section we collect a few observations on local properties of graphs with high minimum degree and without an odd cycle of given length.

The main result of this section is the proof of Proposition 2.2.5, which gives some structural information on such graphs by excluding long odd cycles and pairs of odd cycles connected by a path of length 4.

We remark that in the following lemmas and in Proposition 2.2.5 the additional εn term in the minimum degree condition could be replaced by some polynomial in k. However, since we do not strive for the optimal condition in these auxiliary results, we chose to state them with the same assumption as in Theorem 1.2.5. Recall that by the *length* of a path or more generally the length of a walk, we refer to the number of edges, where each edge is counted with its multiplicity. In particular, we denote by P_r the path on $r+1$ vertices.

Lemma 2.2.1. *Let $k \geqslant 2$, $\varepsilon > 0$, and let $G = (V, E)$ be a C_{2k-1}-free graph satisfying $|V| = n \geqslant 4k/\varepsilon$ and $\delta(G) \geqslant \big(\frac{1}{2k-1} + \varepsilon\big)n$.*

(i) For every vertex $v \in V$ we have $d(M) := 2e(M)/|M| < 2k$ for all $M \subseteq N(v)$.

(ii) For every two vertices $v, u \in V$, if there is an odd v-u-path of length at most $2k-3$ in G, then u and v have less than $5k^2$ common neighbours in G.

In the proof of Lemma 2.2.1 we will use the following consequence of the Erdős-Gallai theorem on paths [18], also stated Theorem 1 of [23].

Theorem 2.2.2. (Erdős & Gallai 1959)

(i) Let G be an n-vertex graph. If $e(G) \geqslant \frac{1}{2}kn$, then G contains a path with k vertices.

(ii) Let $G = (A, B, E)$ be a bipartite graph with $|A| \geqslant |B| \geqslant k$. If $e(A, B) > (|A| + |B|)k$, then G contains an even path of length k.

Proof of Lemma 2.2.1. Assertion (i) is a direct consequence of Theorem 2.2.2 (i). Indeed, it implies that $d(M) \geqslant 2k$ yields a copy of P_{2k-3} in $M \subseteq N(v)$, which together with v would form a cycle C_{2k-1} in G.

For the proof of (ii) assume for a contradiction that $|N(v) \cap N(u)| \geqslant 5k^2$, and there is an odd v-u-path P of length at most $2k-3$. Let $A' = (N(v) \cap N(u)) \setminus V(P)$, clearly, $|A'| \geqslant 4k^2$ so let $A \subseteq A'$ be a subset of A' with exactly $4k^2$ vertices and $B = N(A) \setminus (A \cup V(P))$. Since every vertex in A has at most $2k-2 < 2k$ neighbours in P we have

$$e(A,B) \geqslant |A| \cdot \delta(G) - 2e(A) - |A| \cdot 2k \overset{(i)}{>} |A|\left(\frac{1}{2k-1}n + \varepsilon n - 4k\right) \geqslant \frac{4k^2}{2k-1}n > 2k \cdot n.$$

Consequently, $|B| > 2k$ and Theorem 2.2.2 (ii) yields a P_{2k-2} in $G[A,B]$ and, hence, for every $\ell \in [k-2]$ there exists a $P_{2\ell}$ in $G[A,B]$ with end vertices in A. Together with the path P this yields a cycle C_{2k-1} in G, which is a contradiction to the assumption that G is C_{2k-1}-free. □

Lemma 2.2.1 yields the following corollary, which asserts that the first and the second neighbourhoods of a short odd cycle cover the "right" proportion of vertices.

Lemma 2.2.3. *Let $k \geqslant 2$, $\varepsilon > 0$, and let $G = (V,E)$ be a C_{2k-1}-free graph satisfying $|V| = n \geqslant 20k^3/\varepsilon$ and $\delta(G) \geqslant \left(\frac{1}{2k-1} + \varepsilon\right)n$. If $C = c_1 \dots c_\ell c_1$ is an odd cycle of length $\ell < 2k-1$ in G, then for every $i \in [\ell]$ there are subsets $M_i \subseteq N(c_i) \setminus V(C)$, vertices $m_i \in M_i$, and subsets $L_i \subseteq N(m_i) \setminus V(C)$ such that the sets $M_1, \dots, M_\ell, L_1, \dots, L_\ell$ are mutually disjoint and each of those sets contains at least $\frac{1}{2k-1}n$ vertices.*

Proof. Let $C = c_1 \dots c_\ell c_1$ be an odd cycle of length ℓ in $G = (V,E)$, where $\ell < 2k-1$. Since there is a path of odd length at most $\ell - 2 < 2k-3$ between any two vertices of C, Lemma 2.2.1 (ii) tells us, that $|N(c_i) \cap N(c_j)| < 5k^2$ for all distinct $i,\, j \in [\ell]$. Consequently, we may discard up to at most $(\ell-1) \cdot 5k^2 + \ell < 10k^3$ vertices from the neighbourhoods $N(c_i)$ and obtain mutually disjoint sets $M_i \subseteq N(c_i) \setminus V(C)$ of size at least

$$\delta(G) - 10k^3 \geqslant \frac{1}{2k-1}n + \varepsilon n - 10k^3 > \frac{1}{2k-1}n.$$

For every $i \in [\ell]$ fix an arbitrary vertex $m_i \in M_i$. Since there is a path of odd length at most $\ell - 2 < 2k-3$ between any two vertices of C, there is a path of odd length at most $(\ell-2)+2 = \ell \leqslant 2k-3$ between any two vertices m_i and m_j. Again we infer from Lemma 2.2.1 (ii) that $|N(m_i) \cap N(m_j)| < 5k^2$ for all distinct $i,\, j \in [\ell]$ and in the same way as before, we obtain mutually disjoint sets $L_i' \subseteq N(m_i) \setminus V(C)$ of size at least $\delta(G) - 10k^3$.

Furthermore, since there also is a path of even length at most $\ell - 1 < 2k-3$ between any two (not necessarily distinct) vertices of C, there is a path of odd length at most $(\ell-1)+1 = \ell \leqslant 2k-3$ between any pair of vertices c_i and m_j. Again Lemma 2.2.1 (ii) implies that $|N(c_i) \cap N(m_j)| < 5k^2$ for all $i,\, j \in [\ell]$ and discarding at most $\ell \cdot 5k^2 < 10k^3$ vertices from each L_i' yields sets $L_i \subseteq N(m_i)$ such that $M_1, \dots, M_\ell, L_1, \dots, L_\ell$ are mutually disjoint and disjoint from $V(C)$. Moreover, the assumption $n \geqslant 20k^3/\varepsilon$ implies

$$|L_i| \geqslant |L_i'| - 10k^3 \geqslant \delta(G) - 20k^3 \geqslant \frac{1}{2k-1}n + \varepsilon n - 20k^3 \geqslant \frac{1}{2k-1}n,$$

which concludes the proof of the lemma. □

In the proof of part (i) of Theorem 1.2.5 it will be useful to exclude the graphs described in Definition 2.2.4 as subgraphs of a C_{2k-1}-free graph of sufficiently high minimum degree.

Definition 2.2.4. We denote by D_ℓ the graph on $2\ell + 3$ vertices that consist of two disjoint cycles of length ℓ and a path of length 4 joining these two cycles, which is internally disjoint to both cycles.

The following proposition excludes the appearance of some short odd cycles and D_ℓ's in the graphs G considered in Theorem 1.2.5.

Proposition 2.2.5. *Let $k \geqslant 2$, $\varepsilon > 0$, and $G = (V, E)$ be a C_{2k-1}-free graph satisfying $|V| = n \geqslant 20k^3/\varepsilon$ and $\delta(G) \geqslant \big(\frac{1}{2k-1} + \varepsilon\big)n$. Then*

(i) G is C_ℓ-free for every odd ℓ with $k \leqslant \ell \leqslant 2k-1$.

(ii) G is D_ℓ-free for every odd ℓ with $\max\{3, 2k-7\} \leqslant \ell \leqslant 2k-1$.

Proof. Assertion (i) is a direct consequence of Lemma 2.2.3, as the mutually disjoint sets $M_1, \ldots, M_\ell, L_1, \ldots, L_\ell$ would not fit into $V(G)$.

For the proof of assertion (ii) we assume for a contradiction that $G = (V, E)$ contains a subgraph D_ℓ for some odd ℓ with $\max\{3, 2k-7\} \leqslant \ell \leqslant 2k-1$. Since the graph D_ℓ contain a cycle of length ℓ, we immediately infer from part (i), that we may assume $\ell < k$. Consequently, $k > \ell \geqslant 2k-7$ implies $k \leqslant 6$ and owing to $k > \ell \geqslant \max\{3, 2k-7\}$ we see that the only remaining cases we have to consider are $(k, \ell) \in \{(4, 3), (5, 3), (6, 5)\}$. We discuss each of the cases below.

Case: $k = 6$ and $\ell = 5$.

Let $C = c_1 \ldots c_5 c_1$ and $C' = c_1' \ldots c_5' c_1'$ be the two cycles of length 5 appearing in $D_5 \subseteq G$ and suppose the path P of length 4 connects c_1 and c_1'. We observe that c_5' is connected to every vertex of C by an odd path of length at most 9, as seen in Figure 2.2.1. In fact, $Q = c_5' P$ connects c_5' and c_1 by a path of length 5 and every other vertex of C can be reached by an even path of length at most 4 from c_1.

Furthermore, c_5' is connected to every vertex in $N(C)$ by an odd path of length at most 9. For the vertices in $N(C) \smallsetminus N(c_1)$ we again follow the path Q and since c_2, c_3, c_4, and c_5 can be reached by an odd path of length at most 3 from c_1, as seen in

Figure 2.2.1, every vertex in $N(C) \smallsetminus N(c_1)$ can be reached by an odd path of length at most $5+3+1=9$. For the vertices in $N(c_1)$ we utilise the path of length 4 from c_5' to c_1' in C'. Continuing then along P to c_1 shows that there are paths of length 9 connecting c_5' with every vertex in $N(c_1)$.

As $9 = 2k-3$, we infer from Lemma 2.2.1 (ii) that c_5' has at most $10 \cdot (5k^2 + |Q|) < 10k^3$ neighbours in the sets $M_1, \ldots, M_5, L_1, \ldots, L_5$ given by Lemma 2.2.3 applied to C. However, since

$$\left|M_1 \cup \ldots \cup M_5 \cup L_1 \cup \ldots \cup L_5\right| \geqslant \frac{10}{11}n$$

this implies $\deg(c_5') \leqslant \frac{n}{11} + 10k^3 < \frac{n}{11} + \varepsilon n$ by the assumption that $n > 20k^3/\varepsilon$, which contradicts the minimum degree assumption on G in this case.

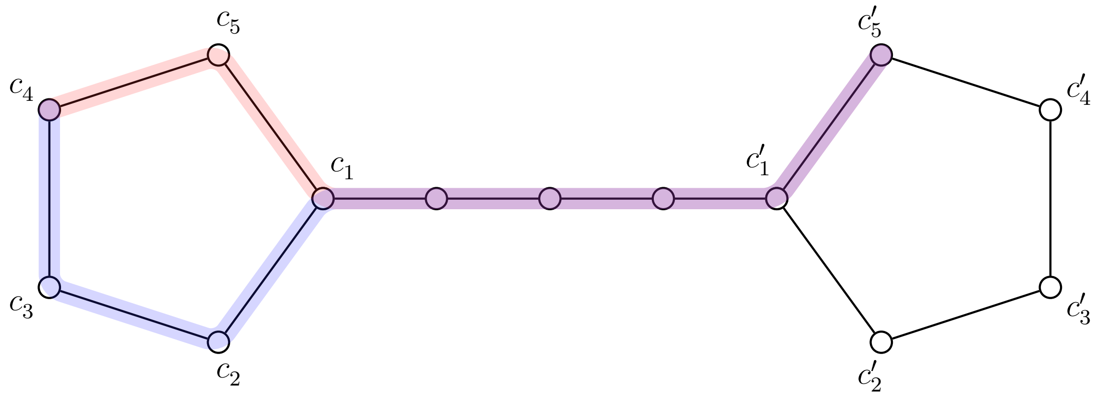


Figure 2.2.1: An odd path of length 7 from c_5' to c_4 in red and an even path of length 8 from c_5' to c_4 in blue as used in the proof of case $k = 6$ and $\ell = 5$.

Case: $k = 5$ and $\ell = 3$.

Let $C = c_1c_2c_3c_1$ and $C' = c_1'c_2'c_3'c_1'$ be the two triangles of $D_3 \subseteq G$ and suppose the path of length 4 connects c_1 and c_1'. Moreover, Lemma 2.2.3 applied with C yields vertices m_1, m_2, m_3 and vertex sets M_1, M_2, M_3 and L_1, L_2, L_3. It is easy to check that c_2' and c_3' can reach each c_i and m_i for every $i \in [3]$ by an odd path of length at most $7 = 2k-3$, as seen in Figure 2.2.2 on the left. In view of Lemma 2.2.1 (ii), and since $|N(c_2')|, |N(c_3')| \geqslant \delta(G) \geqslant n/9$ it follows that

$$\left|M_1 \cup M_2 \cup M_3 \cup L_1 \cup L_2 \cup L_3 \cup N(c_2') \cup N(c_3')\right| \geqslant \frac{8}{9}n.$$

Consequently, we infer from $|N(c_1')| \geqslant \delta(G) \geqslant n/9 + \varepsilon n > n/9 + 40k^2$ that the vertex c_1' must have at least $5k^2$ common neighbours with one of the eight vertices c_1, c_2, c_3, m_1, m_2, m_3, c_2', c_3'. Since c_1' can be connected by an odd path of length at most 7 to all of these eight vertices but c_1, we infer that c_1 and c_1' have $5k^2$ common neighbours and we can fix such a neighbour disjoint from m_1, m_2, m_3, C and C'. In other words, we found a graph D_3' consisting of C, C', and a path of length 2 between c_1 and c_1'.

Consequently, c'_2 and c'_3 are connected to each c_i and each m_i for every $i \in [3]$ by an odd path of length at most 5. Hence, we can fix a neighbour m'_2 of c'_2, which can be connected to each c_i and each m_i for $i \in [3]$ and to c'_2 and c'_3 by an odd path of length at most 7, as seen in Figure 2.2.2 on the right. In other words, any two of the 9 vertices from $c_1, c_2, c_3, m_1, m_2, m_3, c'_2, c'_3$ and m'_2 are connected by an odd path of length at most 7 and thus have fewer than $5k^2$ common neighbours by Lemma 2.2.1 (*ii*). However, since $\varepsilon n > 40k^2$ the minimum degree assumption implies that at least one of the former 8 vertices must have at least $5k^2$ common neighbours with m'_2.

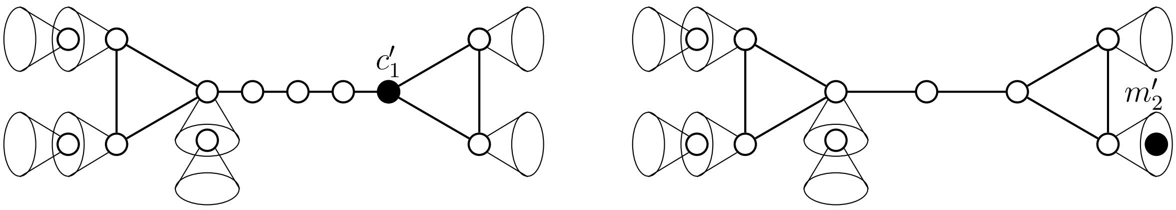


Figure 2.2.2: On the left the graph D_3 of the case $k = 5$ and $\ell = 3$ where the vertex c'_1 does not have enough neighbours, and on the right the graph D'_3 where the vertex m'_2 does not have enough neighbours.

Case: $k = 4$ **and** $\ell = 3$.

Again we consider the two triangles $C = c_1c_2c_3c_1$ and $C' = c'_1c'_2c'_3c'_1$ of $D_3 \subseteq G$ and assume c_1 and c'_1 are connected by a path $c_1p_1p_2p_3c'_1$ of length 4. We consider the vertices m_1, m_2, m_3 and sets M_1, M_2, M_3, L_1, L_2, L_3 and M'_1, M'_2, M'_3 given by Lemma 2.2.3 applied with C and with C'.

Note that there can only be one edge between a vertex of C and a vertex of C', namely $c_1c'_1$, otherwise there is a C_7 in D_3. Therefore, if there are vertices c_i and c'_j with $i, j \in [3]$ such that they have at least two common neighbours, G contains a graph D'_3 consisting of C, C' and a path of length 2 between c_i and c'_j. By symmetry, we may assume $i = j = 1$. However, in this case we see that c'_2 is connected to c_1, c_2, c_3 and m_1, m_2, m_3 by an odd path of length at most 5, as seen in Figure 2.2.3 on the right. Since

$$\left|M_1 \cup M_2 \cup M_3 \cup L_1 \cup L_2 \cup L_3\right| \geqslant \frac{6}{7}n,$$

the minimum degree assumption yields at least $(\varepsilon n - 4)/6 \geqslant 5k^2$ common neighbours of c'_2 and one of the vertices of $\{c_1, c_2, c_3, m_1, m_2, m_3\}$, which is a contradiction to Lemma 2.2.1 (*ii*).

Assuming that no two vertices of C and C' have more than one common neighbour, we notice that p_1 can be connected to all three vertices of C and to all three vertices of C' by an odd path of length at most $5 = 2k - 3$, as seen in Figure 2.2.3 on the left.

This implies that

$$\left|M_1 \cup M_2 \cup M_3 \cup M_1' \cup M_2' \cup M_3'\right| \geqslant \frac{6}{7}n - 9.$$

Consequently, the minimum degree assumption yields at least $(\varepsilon n - 9 - 9)/6 \geqslant 5k^2$ common neighbours of p_1 and one of the vertices of C_1 or C_1', which is a contradiction to Lemma 2.2.1 (*ii*).

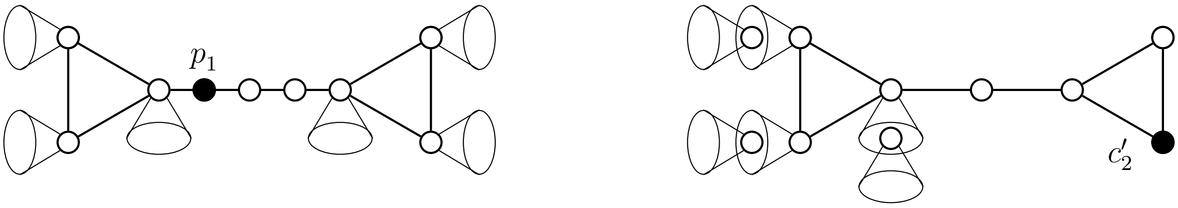


Figure 2.2.3: On the left the graph D_3 of the case $k = 4$ and $\ell = 3$ where the vertex p_1 does not have enough neighbours, and on the right the graph D_3' where the vertex c_2' does not have enough neighbours.

□

2.3 Proof of the main theorem

Proof of Theorem 1.2.5. We first prove assertion (i) of Theorem 1.2.5. Given a sufficiently large C_{2k-1}-free n-vertex graph $G = (V, E)$ with $\delta(G) \geqslant (\frac{1}{2k-1} + \varepsilon)n$ for $k \geqslant 3$ and $\varepsilon > 0$, it suffices to show that there exists a C_{2k-1}-free graph H with $|V(H)| \leqslant K = K(k, \varepsilon)$ and $G \xrightarrow{\text{hom}} H$. The required graph $H(C_{2k-1}, \alpha)$ for Definition 1.2.4 can then be taken to be the disjoint union of all non-isomorphic C_{2k-1}-free graphs on K vertices.

In particular, the constant K must be independent of n. Without loss of generality we may assume that $2/\varepsilon$ is an integer. In order to define K, consider the function $f\colon \mathbb{R} \longrightarrow \mathbb{R}$ with $x \longmapsto x2^x$ and set

$$m = \max\left\{\left\lceil \frac{2\ln(3/\varepsilon)}{\varepsilon^2}\right\rceil, 8k^2\right\} \qquad \text{and} \qquad K = \underbrace{f \circ f \circ \cdots \circ f}_{2k\text{-times}}\left((2/\varepsilon + 1)^{\binom{m}{4k}}\right), \quad (2.3.1)$$

i.e., K is given by a $2(k+1)$-times iterated exponential function in $\mathrm{poly}(1/\varepsilon, k)$.

Considering a random m-element subsets $X \subseteq V$, it follows from the concentration of the hypergeometric distribution (see e.g. [28, inequality (2.6) and Theorem 2.10]) for any fixed vertex $v \in V$

$$\mathbb{P}\left(|N(v) \cap X| \leqslant (\tfrac{1}{2k-1} + \varepsilon)m - t\right) \leqslant \exp\left(-\tfrac{t^2}{2m}\right),$$

for every $t > 0$. Since our choice of m in (2.3.1) yields $m/2k > 4k$ it follows with

$t = \varepsilon m$, that there exists a set X of size m, such that all but at most $\varepsilon n/3$ vertices of G have at least $4k$ neighbours in X. We fix such a set $X = \{x_1, \dots, x_m\}$ and set

$$Y = \big\{v \in V \colon |N(v) \cap X| \geqslant 4k\big\}.$$

For every $y \in Y$ fix a set $X(y)$ of exactly $4k$ neighbours of y in X in an arbitrary way. We partition Y into $\binom{m}{4k}$ sets, where two vertices y, $y' \in Y$ belong to the same partition class if $X(y) = X(y')$. Removing all the classes with fewer than $8k/\varepsilon$ vertices from this partition yields a partition $\mathcal{Q}$ of a subset of Y of size

$$\Big|\bigcup \mathcal{Q}\Big| \geqslant |Y| - \binom{m}{4k}\frac{8k}{\varepsilon} \geqslant \Big(n - \frac{\varepsilon}{3}n\Big) - \binom{m}{4k}\frac{8k}{\varepsilon} > n - \frac{\varepsilon}{2}n, \tag{2.3.2}$$

where the last inequality holds for sufficiently large n. For convenience we may index the partition classes of $\mathcal{Q}$ by a suitable set $I = [M]$ with $M \leqslant \binom{m}{4k}$, i.e., $\mathcal{Q} = (Q_i)_{i \in I}$.

Next we define a partition $\mathcal{R}$ of the whole vertex set V, based on the neighbourhoods with respect to the partition classes of $\mathcal{Q}$. More precisely we assign to each vertex $v \in V$ a vector $\mu(v) = (\mu_i(v))_{i \in I}$, where $\mu_i(v)$ equals the proportion of vertices in Q_i that are neighbours of v "rounded down" to the next integer multiple of $\varepsilon/2$, i.e.

$$\mu_i(v) = \left\lfloor \frac{|N(v) \cap Q_i|}{|Q_i|} \cdot \frac{2}{\varepsilon} \right\rfloor \cdot \frac{\varepsilon}{2}. \tag{2.3.3}$$

In particular, since every class from $\mathcal{Q}$ has at least $8k/\varepsilon$ vertices, we have

$$\big|N(v) \cap Q_i\big| \geqslant 4k \tag{2.3.4}$$

for every $v \in V$ with $\mu_i(v) > 0$.

We now define the partition $\mathcal{R}$. The classes of $\mathcal{R}$ are given by the equivalence classes of the relation $\mu_i(v) = \mu_i(v')$ for every $i \in I$. Owing to the discretisation of $\mu_i(v)$ the partition $\mathcal{R}$ has at most

$$(2/\varepsilon + 1)^{|I|} \leqslant (2/\varepsilon + 1)^{\binom{m}{4k}}$$

parts. Furthermore, we note

$$\begin{aligned} \sum_{i \in I} \mu_i(v)|Q_i| \;&\geqslant\; d(v) - \Big|V \smallsetminus \bigcup \mathcal{Q}\Big| - \sum_{i \in I} \frac{\varepsilon}{2}|Q_i| \\ &\overset{(2.3.2)}{>} \Big(\frac{1}{2k-1} + \varepsilon\Big) n - \frac{\varepsilon}{2}n - \frac{\varepsilon}{2}n \\ &\geqslant \Big(\frac{1}{2k-1}\Big) n \end{aligned} \tag{2.3.5}$$

for every $v \in V$. For later reference we make the following observation.

Claim 2.3.1. *For every $i \in I$ no two distinct vertices $v, v' \in V$ with $\mu_i(v), \mu_i(v') > 0$ are joined by an odd v-v'-path of length at most $2k-5$ in G.*

Proof. Suppose for a contradiction, that for some $i \in I$ and $v \neq v'$ we have $\mu_i(v), \mu_i(v') > 0$ and there is an odd v-v'-path P of length at most $2k-5$ in G. Let q_i be a neighbour of v in Q_i and let q_i' be a neighbour of v' in Q_i, such that $q_i \neq q_i'$ and both not contained in P (see (2.3.4)). Consequently, there is a q_i-q_i'-path $P' \subseteq G$ of odd length $2k-1-2\ell$ for some $\ell \in [k-2]$.

Since all vertices of Q_i have $4k$ common neighbours in X, there is a set X' consisting of ℓ of these neighbours from $X \setminus V(P')$. Similarly, there is a set $Q_i' \subseteq Q_i$ of $\ell - 1$ vertices in $Q_i \setminus (V(P') \cup X')$. Clearly, $X' \cup Q_i' \cup \{q_i, q_i'\}$ spans a q_i-q_i'-path P'' of length 2ℓ, which together with P' yields a copy of C_{2k-1} in G. This, however, contradicts the assumption that G is C_{2k-1}-free. □

Starting with the partition $\mathcal{R}^0 = \mathcal{R}$ we inductively refine this partition $2k$ times and obtain partitions $\mathcal{R}^0 \geqslant \mathcal{R}^1 \geqslant \cdots \geqslant \mathcal{R}^{2k}$. Given $\mathcal{R}^i$ we define $\mathcal{R}^{i+1}$ by subdividing every partition class such that vertices remain in the same class if and only if they have neighbours in the same classes of R^i. More precisely, two vertices v, v' from some partition class of $\mathcal{R}^i$ stay in the same class in $\mathcal{R}^{i+1}$ if and only if for every class R_j^i from $\mathcal{R}^i$ we have

$$N(v) \cap R_j^i \neq \varnothing \quad \Longleftrightarrow \quad N(v') \cap R_j^i \neq \varnothing.$$

Owing to this inductive process and our choice of K in (2.3.1) the partition $\mathcal{R}^{2k}$ consists of at most K classes. Since $k \geqslant 3$, Claim 2.3.1 implies that the classes of $\mathcal{R}^0$ are independent sets in G and, therefore, also the classes of $\mathcal{R}^{2k}$ are independent. Hence, we may define the reduced graph H of $\mathcal{R}^{2k}$, where each class $\mathcal{R}^{2k}$ is a vertex of H and two vertices are adjacent, if the corresponding partition classes induce at least one crossing edge in G. Obviously, we have

$$G \xrightarrow{\text{hom}} H \qquad \text{and} \qquad |V(H)| \leqslant K \tag{2.3.6}$$

and it is left to show that H is also C_{2k-1}-free (see Claim 2.3.4). For the proof of this property we first collect a few observations concerning the interplay of odd paths in H and walks in G (see Claims 2.3.2 and 2.3.3).

Denote by $\mathcal{R}^i(v)$ the unique class of the partition $\mathcal{R}^i$ which contains the vertex $v \in V$. Similarly, for $j \geqslant i$ let $\mathcal{R}^i(R)$ be the unique class of the partition $\mathcal{R}^i$ which is a superset of $R \in \mathcal{R}^j$.

Claim 2.3.2. *If there is a walk $W_H = h_1 h_2 \dots h_s$ in H for some integer $s \leqslant 2k$, then there are vertices $w_i \in \mathcal{R}^{2k-i+1}(h_i) \subseteq \mathcal{R}^0(h_i)$ for every $i \in [s]$ such that $W = w_1 w_2 \dots w_s$ is a walk in G. Moreover, w_1 can be chosen arbitrarily in $h_1 = \mathcal{R}^{2k}(h_1)$.*

Proof. We shall locate the walk W in an inductive manner and note that for $s = 1$ it is trivial.

For $s \geqslant 2$ let a walk $W' = w_1 w_2 \dots w_{s-1}$ satisfying $w_i \in \mathcal{R}^{2k-i+1}(h_i)$ for every $i \in [s-1]$ be given. The walk W_H in H guarantees an edge between $\mathcal{R}^{2k}(h_{s-1})$ and $\mathcal{R}^{2k}(h_s)$ and, hence, there is an edge between $\mathcal{R}^{2k-(s-1)+1}(h_{s-1})$ and $\mathcal{R}^{2k-(s-1)+1}(h_s)$. Consequently, the construction of the refinements shows that $w_{s-1} \in \mathcal{R}^{2k-(s-1)+1}(h_{s-1})$ must have a neighbour $w_s \in \mathcal{R}^{2k-s+1}(h_s)$ and the walk $W = W'w_s = w_1 \dots w_{s-1}w_s$ has the desired properties. $\square$

Even if we assume in Claim 2.3.2 that W_H is a path in H and, in particular, $h_i \neq h_j$ for all distinct $i, j \in [s]$, it may happen that $\mathcal{R}^0(h_i) = \mathcal{R}^0(h_j)$ and, hence, we cannot guarantee $w_i \neq w_j$. In other words, even if we apply Claim 2.3.2 to a path in H, the promised walk W might not be a path. However, combined with Proposition 2.2.5 we can get the following improvement.

Claim 2.3.3. *If there is an odd path $P_H = h_1 \dots h_{s+1}$ of length $s \leqslant 2k-1$ in H, then there are vertices $v_1 \in \mathcal{R}^0(h_1)$ and $v_{s+1} \in \mathcal{R}^0(h_{s+1})$ such that there is an odd path of length at most s between them.*

Proof. Consider a walk $W = w_1 w_2 \dots w_{s+1}$ in G with $w_i \in \mathcal{R}^0(h_i)$ given by Claim 2.3.2. If this walk does not contain an odd w_1-w_{s+1}-path already, then W must contain an odd cycle. Below we shall show that this leads to a contradiction and, hence, W contains an odd w_1-w_{s+1}-path.

Consider an odd cycle $C = c_1 \dots c_\ell c_1$ contained in $W \subseteq G$, such that

$$c_1 = w_{i_1}, c_2 = w_{i_2}, \dots, c_\ell = w_{i_\ell}, \qquad \text{and} \qquad c_1 = w_{i_{\ell+1}} = w_{i_1}$$

for some set of indices satisfying $1 \leqslant i_1 < i_2 < \dots < i_\ell < i_{\ell+1} \leqslant s+1$. To find such a cycle, consider the walk W, delete an even w_1-w_{s+1}-path. Now the remaining edges of W are a family of closed walks, take an odd closed walk $W' = w'_1 w'_2 \dots w'_{s'+1} = w_{j_1} w_{j_2} \dots w_{j_{s'+1}}$, where $w'_1 = w'_{s'+1}$. If W' is not a cycle, there is a smaller closed walk $W'' \subset W'$, such that the edges of W' without the edges of W'' are also a closed walk, both retaining the order of the vertices from W' and therefore from W. One of $\{W', W''\}$ needs to be odd. Iterating this process eventually gives rise to the odd cycle C.

In view of Proposition 2.2.5 (i) we must have $3 \leqslant \ell < k$. Consequently, $\ell \leqslant 2k-5$ and since ℓ is odd, it follows from Claim 2.3.1 that there is no path of length ℓ between any two vertices from $\mathcal{R}^0(c_1) = \mathcal{R}^0(h_{i_1}) = \mathcal{R}^0(h_{i_{\ell+1}})$. Moreover, Claim 2.3.1 tells us that the ℓ classes $\mathcal{R}^0(c_1) = \mathcal{R}^0(h_{i_1}) = \mathcal{R}^0(h_{i_{\ell+1}}), \dots, \mathcal{R}^0(c_\ell) = \mathcal{R}^0(h_{i_\ell})$ from $\mathcal{R}^0$ are distinct, since otherwise the cycle C would contain an odd path of length at most $2k-7$ between two vertices of some class in $\mathcal{R}^0$.

Since P_H is a path in H, we have $h_{i_1} \neq h_{i_{\ell+1}}$ and the cycle C avoids at least one of the sets h_{i_1} or $h_{i_{\ell+1}}$. Without loss of generality we may assume C avoids h_{i_1} and we fix an arbitrary vertex $c'_1 \in h_{i_1}$.

We are going to locate a second cycle of length ℓ in G that starts and ends in c'_1. By construction this cycle is going to visit the same partition classes of $\mathcal{R}^0$ as C. For that we shall repeat the argument from Claim 2.3.2 starting with $h_{i_1} \dots h_{i_\ell} h_{i_{\ell+1}}$ even though this is not necessarily a subpath of P_H. However, since $h_{i_1} \dots h_{i_\ell} h_{i_{\ell+1}}$ appear in that order in P_H, we can repeat the reasoning of Claim 2.3.2 starting with the vertex $c'_1 \in h_{i_1}$. Continuing in an inductive manner, for $j \in [\ell]$ we have to consider the two cases $i_{j+1} = i_j + 1$ and $i_{j+1} > i_j + 1$.

In the first case, we can indeed proceed as in the proof of Claim 2.3.2, since this means that $h_{i_j} h_{i_{j+1}}$ is an edge of P_H. The second case, by construction of C, only occurs, when $w_{i_j} = w_{i_{j+1}-1}$ and

$$\mathcal{R}^{2k-i_j+1}(h_{i_j}) \subseteq \mathcal{R}^{2k-(i_{j+1}-1)+1}(h_{i_{j+1}-1}).$$

Owing to the fact that $w_{i_{j+1}-1} w_{i_{j+1}}$ is an edge of W and that $w_{i_{j+1}-1} \in \mathcal{R}^{2k-i_j+1}(h_{i_j})$ and $w_{i_{j+1}-1} \in \mathcal{R}^{2k-(i_{j+1}-1)+1}(h_{i_{j+1}-1})$, we infer from the construction of the refinements that $w_{i_j} = w_{i_{j+1}-1}$ also has a neighbour in $\mathcal{R}^{2k-i_{j+1}+1}(h_{i_{j+1}})$, which concludes the induction step.

Therefore, we obtain another walk $C' = c'_1 \dots c'_\ell c'_{\ell+1}$ where $c'_j \in \mathcal{R}^0(h_{i_j}) = R^0(c_j)$. Recalling that the ℓ classes $\mathcal{R}^0(h_{i_1}), \dots, \mathcal{R}^0(h_{i_\ell})$ are pairwise distinct, this implies that C' is either a path or a cycle of odd length $\ell \leqslant 2k-5$. Moreover, since $\mathcal{R}^0(h_{i_1}) = \mathcal{R}^0(h_{i_{\ell+1}})$ we infer from Claim 2.3.1 that C' cannot be a path and, hence, it must be an odd cycle of length $\ell \leqslant 2k-5$. By construction c'_1 avoids C, and hence C' and C are disjoint, as otherwise we would have an odd path of length ℓ connecting c_1 and c'_1 in $\mathcal{R}^0(c_1)$, which would contradict Claim 2.3.1 again.

Consequently, C and C' form a copy of D_ℓ since c_1 and c'_1 are connected by a path of length 4 whose three internal vertices avoid C and C' (and the middle vertex is from X). Owing to Proposition 2.2.5 (ii) we have $\ell \leqslant 2k-9$, but in D_ℓ there exists an odd path of length $\ell + 4 \leqslant 2k-5$ between c_i and c'_i for every $i = 2, \dots, \ell$, which again contradicts Claim 2.3.1. □

After these preparations we are now ready to conclude the proof of part (i) of Theorem 1.2.5.

Claim 2.3.4. *The graph H is C_{2k-1}-free.*

Proof. Assume for a contradiction that there is a cycle $C_H = h_1 \dots h_{2k-1} h_1$ of length $2k-1$ in H. We recall that the vertices of H are partition classes of $\mathcal{R}^{2k}$ and for a simpler notation we set for any vertex h_x of C_H

$$\mu_i(h_x) := \mu_i(v),$$

where v is an arbitrary vertex from $\mathcal{R}^0(h_x)$ and the definition of $\mathcal{R} = \mathcal{R}^0$ shows that the definition of $\mu_i(h_x)$ is indeed independent of the choice of $v \in \mathcal{R}^0(h_x)$.

By (2.3.5) we have

$$\sum_{x=1}^{2k-1} \sum_{i \in I} \mu_i(h_x)|Q_i| > n \geqslant \sum_{i \in I} |Q_i|$$

and, hence, there is some $i \in I$ such that

$$\sum_{x=1}^{2k-1} \mu_i(h_x) > 1. \tag{2.3.7}$$

In particular, there are at least two distinct vertices h_x and h_y of C_H such that $\mu_i(h_x) > 0$ and $\mu_i(h_y) > 0$. On the other hand, among three vertices of C_H two are connected by an odd path of length at most $2k - 5$ in C_H, since the negation is only true for vertices with distance 2 on C_H. Therefore it follows from Claim 2.3.3 and Claim 2.3.1, that no other vertex h_z with $z \in [2k-1] \smallsetminus \{x, y\}$ satisfies $\mu_i(h_z) > 0$. Consequently, we have $\mu_i(h_x) + \mu_i(h_y) > 1$, which means that any two vertices $v \in \mathcal{R}^0(h_x)$ and $u \in \mathcal{R}^0(h_y)$ have a common neighbour in Q_i. In fact, since $2/\varepsilon$ is assumed to be an integer, v and u have at least $2|Q_i|/\varepsilon > 4k$ joint neighbours. Moreover, again Claim 2.3.3 and Claim 2.3.1 imply that h_x and h_y are connected by a path of length $2k - 3$ in C_H and that there is a path P of length $2k - 3$ in G connecting some $v \in \mathcal{R}^0(h_x)$ and $u \in \mathcal{R}^0(h_y)$. Using one of the joint neighbours in Q_i outside P yields a copy of C_{2k-1} in G. This contradicts the C_{2k-1}-freeness of G and concludes the proof of Claim 2.3.4. $\square$

Claim 2.3.4 together with (2.3.6) establishes the proof of part (i) of Theorem 1.2.5 and it remains to consider part (ii), when G is assumed to be $\mathscr{C}_{2k-1}$-free.

In view of Proposition 2.1.2 it suffices to verify the upper bound of assertion (ii) of Theorem 1.2.5. Compared to the proof of part (i) of Theorem 1.2.5, we have the additional assumption that G is not only C_{2k-1}-free, but also contains no cycle C_ℓ for any odd $\ell < 2k - 1$. Consequently, the graph H defined in the paragraph before (2.3.6) in the proof of part (i) satisfies (2.3.6) in this case as well and owing to Claim 2.3.4 it is C_{2k-1}-free. Hence, we only have to show that the C_ℓ-freeness of G for every odd $\ell \leqslant 2k - 3$ can be carried over to H in this situation, which is rendered by the following claim.

Claim 2.3.5. *If G is $\mathscr{C}_{2k-1}$-free, then H is also $\mathscr{C}_{2k-1}$-free.*

Proof. Recall, that we assume $k \geqslant 3$. Suppose for a contradiction that H contains a cycle $C_H = h_1 \dots h_\ell h_1$ for some odd integer ℓ with $3 \leqslant \ell \leqslant 2k - 1$. In fact, it follows from Claim 2.3.4 that $\ell \leqslant 2k-3$. Moreover, applying Claim 2.3.2 to C_H yields a walk W of length ℓ in G which starts and ends in $\mathcal{R}^0(h_1)$. Since G contains no odd cycle of length at most ℓ, the walk W contains an odd path of length at most ℓ connecting two vertices in $\mathcal{R}^0(h_1)$. Therefore, Claim 2.3.1 implies that $\ell = 2k-3$ and by symmetry we infer that for every $x \in [2k-3]$ there exists an odd path of length $2k-3$ between two vertices v_x, $u_x \in \mathcal{R}^0(h_x)$.

As in the proof of Claim 2.3.4 we infer from (2.3.5) that

$$\sum_{x=1}^{2k-3} \sum_{i \in I} \mu_i(h_x)|Q_i| > \frac{2k-3}{2k-1} n > \frac{1}{2} \sum_{i \in I} |Q_i|,$$

where we used $k \geqslant 3$ for the last inequality. Consequently, there is some index $i \in I$ such that $\sum_{x=1}^{2k-3} \mu_i(h_x) > 1/2$. Since for every distinct $x, y \in [2k-3]$ there exists an odd path of length at most $2k-5$ connecting a vertex from $\mathcal{R}^0(h_x)$ with a vertex from $\mathcal{R}^0(h_y)$ there is only one vertex of C_H such that $\mu_i(h_x) > 0$ and, hence, for that $x \in [2k-3]$ we have $\mu_i(h_x) > 1/2$. In particular, every two distinct vertices $v, u \in \mathcal{R}^0(h_x)$ have a common neighbour in Q_i and, since $2/\varepsilon$ is assumed to be an integer, v and u have at least $2|Q_i|/\varepsilon > 4k$ joint neighbours. Applying this observation to v_x and u_x leads to an odd cycle of length $2k-1$ in G, which is a contradiction and concludes the proof of Claim 2.3.5. $\square$

This concludes the proof of Theorem 1.2.5. $\square$

2.4 Well behaved graphs

Following Letzter and Snyder [34], we will give the following definition. Recall that $N_{G,U}(v)$ for a vertex $v \in G$ and a vertex set U is defined as $N_G(v) \cap U$, and if G is clear from the context we might simply write $N_U(v)$ or $N_H(v)$ for a subgraph H of G with $V(H) = U$ instead. If H is a graph that is not necessarily a subgraph of G, instead of $N_{G,V(H)}(v)$ we will write $N^H(v)$ to emphasise H. Of course this notation does just make sense if $V(G) \cap V(H) \neq \varnothing$, but for the remaining sections of this chapter we will have $V(H) \subseteq V(G)$ whenever we use this notation, so it should not be that confusing.

Definition 2.4.1. A subgraph H of a graph G is called *well-behaved* (in G) if for every vertex v in G, there is a vertex u in H, such that $N_H(v) \subseteq N_H(u)$.

This definition will help us to briefly state the occurrence of graphs where we know exactly how neighbourhoods of vertices look like, which will often be useful in an in-depth analysis. Sometimes we will use the notion of well-behaved vertices or vertices that act well-behaved, which is used as shorthand term for a vertex which with its neighbourhood in a graph H does not contradict H being well-behaved. We might also drop G or H if it is clear from the context. Moreover, we will also have a slightly weaker definition, which we will use analogously for not quite well-behaved graphs.

Definition 2.4.2. Let $H = (V_H, E_H)$ be a subgraph of $G = (V, E)$ and $H' = (V_H, E_{H'})$ be a supergraph of H on the same vertex set, not necessarily a subgraph of G. We call H *semi well-behaved* (in G) with respect to H', if for every vertex v in G, there is a vertex u in H', such that $N_H(v) \subseteq N^{H'}(u)$.

Note, that any well-behaved graph H is also semi well-behaved with respect to itself.

We will now collect and prove some statements concerning (semi) well-behaved graphs, which we will primarily use throughout Section 2.5.

We start with the Andrásfai graphs, defined in Section 2.1.

Lemma 2.4.3. *Let $k \geqslant 2$ and let G be a $\mathscr{C}_{2k-1}$-free graph, furthermore let the Andrásfai graph $A_{k,r}$ be a subgraph of G, then it is well-behaved in G for any $r \geqslant 1$.*

Proof. For $r = 1$, the graph $A_{k,r}$ consists of a single edge, so the lemma is trivial.

For $r = 2$, the graph $A_{k,r}$ is just a C_{2k+1}. Assuming a vertex v has two neighbours on any odd cycle gives rise to two new cycles containing v, one of which is even and one of which is odd. If the even cycle contains more than 4 edges, the odd cycle will be shorter than $2k + 1$, contradicting our assumption on G. Therefore the even cycle has length exactly four, so a C_{2k+1} is indeed well-behaved.

For $r > 2$, consider the Hamiltonian cycle C used in the proof of Proposition 2.1.2 (*c*). By symmetry let v be a neighbour of u_0, owing to the structure of C and the fact that C_{2k+1} are well-behaved, the possible neighbours of v in C are a subset of

$$N = \{u_{1+i(2k-1)+h} \colon 0 \leqslant i \leqslant r-1, h \in \{-1, 1\}\} \qquad \text{(see Figure 2.4.1.)}$$

Note, that $1 + 0(2k-1) - 1 = 0 \equiv 1 + (r-1)(2k-1) + 1$.

It is easy to see that v is either well-behaved, or contradicts some C_{2k+1} of C being well-behaved, if there is an i, such that both $u_{1+i(2k-1)+1}$ and $u_{1+i(2k-1)-1}$ are in $N(v)$. It is equally easy to see that v is well-behaved, if $N_{C \smallsetminus \{u_0\}}(v) \subseteq \{u_{1+i(2k-1)+1} \colon 0 \leqslant i \leqslant r-1\}$, or $N_{C \smallsetminus \{u_0\}}(v) \subseteq \{u_{1+i(2k-1)-1} \colon 0 \leqslant i \leqslant r-1\}$.

So assuming v is not well-behaved, we have $\{u_{1+i(2k-1)-1}, u_{1+j(2k-1)+1}\} \subseteq N(v)$.

If $j < i$, the 4-path $u_{1+j(2k-1)+1} u_{1+i(2k-1)+2} u_{1+i(2k-1)+1} u_{1+i(2k-1)} u_{1+(2k-1)-1}$ is part of a C_{2k+1} in C, so v would be contradicting C_{2k+1} being well-behaved.

If $j > i$, take a maximal i with this property. If v does not contradict some $C_{2k+1} \subseteq C$ being well-behaved, there will only be one swap from $h = -1$ to $h = 1$, so there is an i such that $u_{1+i(2k-1)-1} \in N(v)$, but no $u_{1+i'(2k-1)-1} \in N(v)$, for $i < i' < r-1$. It is easy to see that either $N_C(v) \subseteq N_C(1 + i(2k-1))$, or v contradicts some $C_{2k+1} \subseteq C$ being well-behaved. Therefore any v not contradicting the case $r = 2$ is well-behaved in

C, and since v was an arbitrary vertex having neighbours in C, C itself (which is $A_{k,r}$) is well-behaved. □

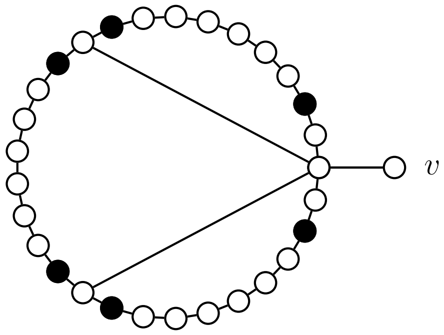


Figure 2.4.1: The possible additional neighbours of v in the Andrásfai graph, according to the case $r = 4$ are marked black. To highlight the important edges, the other edges not lying on the outer circle have been left out of the picture.

In the next lemma we are looking at an $A_{k,3}$ where some edges are missing, stating that it still is well-behaved with respect to the original $A_{k,3}$. For the purpose of a more streamlined proof, we will introduce some notation beforehand. By a *diagonal* in an even cycle $C_{2\ell}$, we are referring to an edge joining two vertices with distance ℓ on the cycle. The distance of two diagonals on an even cycle is the shortest distance between any two end vertices of the diagonals, just using edges of the underlying cycle. A Möbius ladder $M_{4k} = m_0m_1 \dots m_{4k-1}$ is a C_{4k} with all diagonals added, or equivalently any $A_{k,3}$ where $u_i = m_i$ for the ordering from the proof of Proposition 2.1.2 (c).

Lemma 2.4.4. *Let $k \geqslant 3$ and let G be a $\mathscr{C}_{2k-1}$-free n-vertex graph with $\delta(G) > \frac{1}{2k-1}n$, then any $C_{4k} = c_0c_1 \dots c_{4k-1}c_0$ in G with two added diagonals with distance at least 2 is semi well-behaved with respect to an M_{4k} where $c_i \longrightarrow m_i$.*

Proof. Throughout this proof, we will make extensive use of the fact that a C_{2k+1} is well-behaved, so sometimes we will omit this fact and simply state things as being obvious. Let C be the C_{4k}, all indices along it are regarded to be in cyclic order, so $c_{4k} = c_0$ etc.

Let $d_0 = c_0c_{2k}$ be the first diagonal, and let $d_i = c_ic_{i+2k}$ be the second. By symmetry let $2 \leqslant i \leqslant 2k-2$.

First we are analysing the possible neighbourhoods of vertices which have at least 3 neighbours in C.

Let v be such a vertex with $c_0 \in N(v)$. Obviously $N_C(v) \subseteq \{c_{-2}, c_0, c_2, c_{2k-1}, c_{2k+1}\}$, and $c_{\pm 2} \in N(v) \Rightarrow c_{2k\mp 1} \notin N(v)$. Observe that since $2 \leqslant i \leqslant 2k-2$, we also have $c_2 \in N(v) \Rightarrow c_{-2} \notin N(v)$. But than $N_C(v) \in \{\{c_{-2}, c_{2k-1}, c_0\}, \{c_{2k-1}, c_0, c_{2k+1}\}, \{c_0, c_{2k+1}, c_2\}\}$ concluding the case that v has an end vertex of one of the diagonals as a neighbour.

Let v be a vertex with at least 3 neighbours in C, such that $c_{-1}, c_1 \in N(v)$. Obviously $N_C(v) \subseteq \{c_{-3}, c_{-1}, c_{2k}, c_1, c_3\}$, and $c_{2k} \in N(v) \Rightarrow c_{-3}, c_3 \notin N(V)$ (see Figure 2.4.2 on the left).

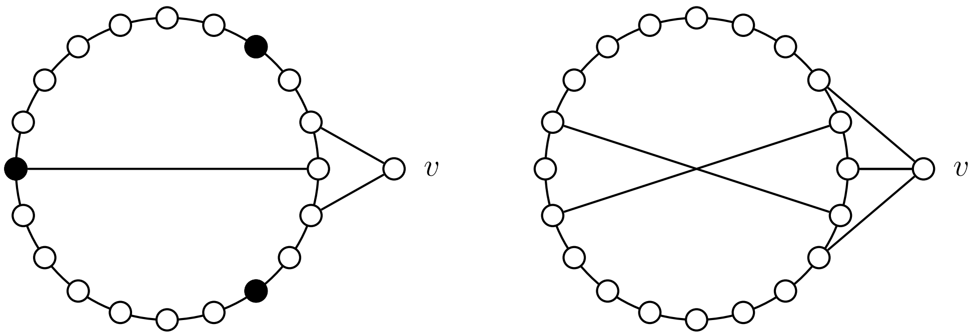


Figure 2.4.2: On the left, the possible additional neighbours of v in C are marked black. On the right, a special case is depicted.

Assuming $c_3 \in N(v)$ implies $i = 2$ and therefore $c_{-3} \notin N(v)$ (see Figure 2.4.2 on the right). This case however, does not comply with C being semi well-behaved with respect to M_{4k}, and it is here precisely we will make use of the minimal degree condition of G to assure that this case does not appear. For this part of the proof, we will need the left over cases of vertices having at least 3 neighbours in C, which we will therefore proof right now, and we will come back to this case later.

Let v be a vertex with at least 3 neighbours in C, such that none of these neighbours is part of d_0 or d_i. Since C together with d_0 consists of two C_{2k+1}, v must have two of its neighbours in one of these C_{2k+1}, so let these neighbours be c_{j-1} and c_{j+1}, such that $2 \leqslant j \leqslant 2k - 2$. In the light of the above open case, we might assume $i \notin \{j-1, j, j+1\}$, and by symmetry we might assume $i > j + 1$.

The diagonals d_0 and d_i are dividing C into 4 segments, and since c_{j-1} and c_{j+1} lie in one of them, it is easy to see that the third neighbour c_h of v in C must lie on the segment, that is not adjacent to the one where c_{j-1} and c_{j+1} lie (see Figure 2.4.3).

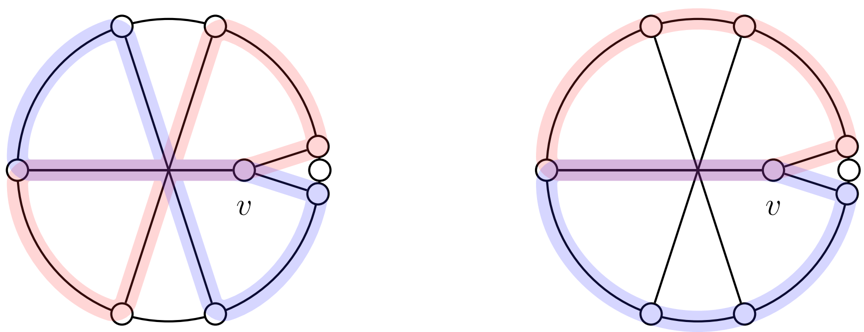


Figure 2.4.3: On the left, the two cycles must be even. On the right, the two cycles are either even or both odd, but the same colour cycles on the left and right have different parity since their symmetric difference is an odd cycle.

Consider the two cycles $C_1 = vc_{j+1}Cc_ic_{i+2k}Cc_hv$ and $C_2 = vc_{j-1}Cc_0c_{2k}Cc_hv$ (see Figure 2.4.3 on the left). Since the sum of their edges is $4k-2(2k-i)+2-2+4 < 4k+2$ and even, they both have the same parity, and since G is $\mathscr{C}_{2k-1}$-free they both must be even.

Now consider the two cycles $C_3 = vc_{j+1}Cc_hv$ and $C_4 = vc_{j-1}Cc_hv$ (see Figure 2.4.3 on the right). Since the sum of their edges is $4k - 2 + 4 = 4k + 2$ and therefore even, they both have the same parity, and since G is $\mathscr{C}_{2k-1}$-free they both must be even, or both be a C_{2k+1}.

Since the symmetric difference between C_1 and C_3 is a C_{2k+1} and therefore odd, and C_1 is an even cycle, C_3 must be odd and therefore an C_{2k+1}. A simple calculation yields $h = j + 2k$, so v behaves semi well-behaved with respect to M_{4k}.

Coming back to the Case $i = 2$ and $N_C(v) = \{c_{-1}, c_1, c_3\}$, there might be other vertices v' with 3 neighbours in C, but they either have the neighbourhood $N_C(v') = \{c_{2k-1}, c_{2k+1}, c_{2k+3}\}$, or behave semi well-behaved like some m_i. If there really is a vertex v', with the above mentioned neighbourhood in C however, there can only be other vertices with 3 neighbours in C who behave like one of the vertices from the set $A' = \{v, v', m_0, m_1, m_2, m_{2k}, m_{2k+1}, m_{2k+2}\}$, since otherwise either c_{-1}, c_1 and c_3, or c_{2k-1}, c_{2k+1} and c_{2k+3} would lie on a common C_{2k+1}, contradicting Lemma 2.4.3 (see Figure 2.4.4 on the left). But since all the vertices in A' are a neighbour of either c_0 or c_{2k}, the set $V(C) \setminus \{c_0, c_{2k}\}$ consisting of $4k - 2 = 2(2k - 1)$ vertices has the property that no vertex has more than 2 neighbours inside, contradicting $\delta(G) > \frac{1}{2k-1}n$ by a simple double counting argument.

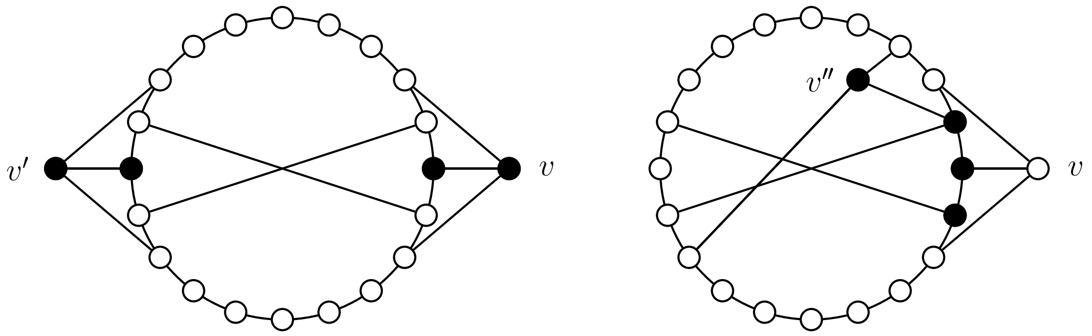


Figure 2.4.4: On the left, if there is a vertex v', the white vertices are a set of $4k - 2$ vertices, such that no vertex has more than 2 neighbours in this set, contradicting the minimum degree of G. On the right, another case of an additional vertex v' is depicted. This case is brought to a contradiction analogously.

Assuming there is no such vertex v', there might be a vertex v'' behaving like m_3 or m_{-1} having 3 neighbours in C in addition to the vertices behaving like these from $A' \setminus \{v'\}$. If there is no such vertex v'', we easily get the same contradiction as in the case that the vertex v' was present, so assume by symmetry that there is indeed a vertex v'' behaving like m_3 (see Figure 2.4.4 on the right).

In this case it is relatively easy to see, that just vertices behaving like one of the vertices from $A'' = (A' \setminus \{v'\}) \cup \{m_3, m_{-1}\}$ can have 3 neighbours in C. Notice, that all of them have at least one neighbour in $\{c_0, c_1, c_2\}$.

We would like to get the same contradiction as above, but since this time we would have to remove 3 vertices from C, the numbers do not add up. To solve this problem, we will add v to $C \setminus \{c_0, c_1, c_2\}$, and claim that this vertex set, consisting of $4k-2$ vertices also has the property that no vertex has more than 2 neighbours inside, leading to the same contradiction with $\delta(G)$ and therefore finishing the proof of this case.

It is left to verify that no vertex has more than 2 neighbours in the vertex set $B = (C \setminus \{c_0, c_1, c_2\}) \cup \{v\}$. Indeed, we did verify this claim beforehand for all vertices which are not neighbours of v, so consider a neighbour u of v. Since replacing c_0 with v gives rise to an C_{2k+1} in C with d_i and replacing c_i with v gives rise to an C_{2k+1} in C with d_0, the possible neighbours of u are merely $\{v, c_{-2}, c_0, c_2, c_4, c_{2k+1}\}$. Because $c_0, c_2 \notin B$, we just have to make sure, that u does not have more than 2 neighbours in $\{v, c_{-2}, c_4, c_{2k+1}\}$. Obviously u cannot be a neighbour of c_{2k+1} and either c_{-2} or c_4 at the same time, therefore the only possible case left is that $\{v, c_{-2}, c_4\} \subseteq N(u)$. However, considering the C_{2k+1} formed by $vc_3c_4v''c_{2k+3}Cc_{-1}v$, this case leads to a contradiction as well, concluding the case of vertices with at least 3 neighbours in C.

Since all vertices with at most one neighbour in C naturally act well-behaved, the only case left for consideration are the vertices with exactly 2 neighbours in C. Let v be such a vertex.

It is easy to see, that v acts well-behaved, if $N(v) \cap d_0 \neq \varnothing$ or $N(v) \cap d_i \neq \varnothing$, and it follows directly from the fact that C_{2k+1} is well-behaved, that v acts well-behaved, if both neighbours of v lie on a common C_{2k+1} in $C \cup d_0 \cup d_i$.

Therefore we might assume that both neighbours of v lie in non-adjacent segments of C, once again considering the 4-split produced by d_0 and d_i, let these neighbours be c_j and c_h (see Figure 2.4.5).

Consider the two cycles $C_1 = vc_jCc_ic_{i+2k}Cc_hv$ and $C_2 = vc_jCc_0c_{2k}Cc_hv$ (see Figure 2.4.5 on the left). Since the sum of their edges is $4k - 2(2k-i) + 2 + 4 \leqslant 4k + 2$ and even, they both have the same parity, and since G is $\mathscr{C}_{2k-1}$-free in the case that both are odd, they must both be a C_{2k+1}, implying that the distance between d_0 and d_i is exactly two.

This however implies, that there is no additional diagonal present in C except for maybe $d_1 = c_1c_{2k+1}$. Furthermore, there is no vertex having at least 3 neighbours in C apart from these acting like one of the end vertices of the diagonals d_0, d_1 or d_2 or like m_j or m_h. Excluding the latter two cases, we get a contradiction with $\delta(G) > \frac{1}{2k-1}n$ like in the above case with the vertices v and v'. Therefore, by symmetry we might assume there is a vertex acting like m_j.

Let m be that vertex. Switching c_j with m gives rise to a new diagonal in a C_{4k},

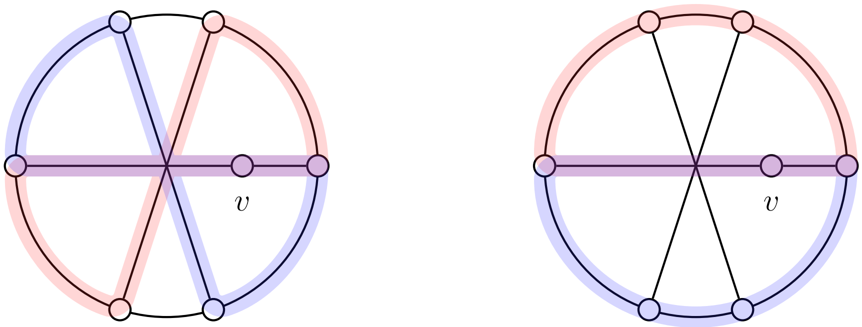


Figure 2.4.5: On the left, the two cycles must be even, or each is a C_{2k+1}, in this case however, we may interfere that the distance between the diagonals is small. On the right, the two cycles are either even or both odd, in which case one of them is a C_{2k+1}, concluding this case. Since the same colour cycles on the left and right have different parity since their symmetric difference is an odd cycle, at least one pair of odd cycles occurs.

closing two C_{2k+1}. One of them will contain c_h, let C' be that C_{2k+1}, and by symmetry let $c_{j+1} \in C'$. Now consider the cycle $C'' = mc_{2k+j}Cc_hvc_jc_{j-1}m$. Since $j - h \geqslant 4$ and even, C'' is odd with at most $2k+1$ edges. To avoid a contradiction with G being $\mathscr{C}_{2k-1}$-free, the only case left is $j - h = 4$, implying $c_j = c_{-1}$ and $c_h = c_3$ or vice versa. In this case however, m is behaving as v'' from the case with 3 neighbours which do not act well-behaved, and v does behave in the same way as the v of that case was, leading to the exact same contradiction with $\delta(G) > \frac{1}{2k-1}n$.

Now consider the two different cycles $C_3 = vc_jCc_hv$ and $C_4 = vc_jCc_hv$ (see Figure 2.4.5 on the right). Since the sum of their edges is $4k + 4 = 4k + 4$ and therefore even, they both have the same parity, and since G is $\mathscr{C}_{2k-1}$-free they both must be even, or one of them is a C_{2k+1}.

Since the symmetric difference between C_1 and C_3 (or C_1 and C_4) is a C_{2k+1} and therefore odd, and C_1 is an even cycle by the case outlined above, C_3 and C_4 must be odd and therefore one of them must be an C_{2k+1}. A simple calculation yields $h = j + 2k \pm 1$, so v behaves semi well-behaved with respect to M_{4k}.

This finally concludes the proof of Lemma 2.4.4. $\square$

This next lemma is a strengthening of Lemma 2.4.4, but since Lemma 2.4.4 is often strong enough we split up the proof, which is essentially some more special case analysis, to give the not that interested reader a good point to continue reading just if this stronger statement is really needed.

Lemma 2.4.5. *Let $k \geqslant 3$ and let G be a $\mathscr{C}_{2k-1}$-free n-vertex graph with $\delta(G) > \frac{1}{2k-1}n$, then any $C_{4k} = c_0c_1 \dots c_{4k-1}v_0$ with two added diagonals in G is semi well-behaved with respect to one of the following:*

1. *An M_{4k} where $c_i \longrightarrow m_i$, or*

2. *an M_{4k} where $c_i \longrightarrow m_i$ for $2k \leqslant i \leqslant 4k-1$, and $c_i \longrightarrow m_{2k-1-i}$ for $0 \leqslant i \leqslant 2k-1$.*

In the latter case the two diagonals must be c_0c_{2k} and $c_{4k-1}c_{2k-1}$.

Proof. Assume there is a third added diagonal in the C_{4k}, resulting in two of them having distance at least 2, or the two added diagonals having distance of at least 2, then Lemma 2.4.5 part 1 follows directly from Lemma 2.4.4.

So it is left to verify this lemma for the case that the two added diagonals have distance 1. Let C be the C_{4k} and by symmetry let $d_0 = c_0c_{2k}$ be the first and $d_{-1} = c_{4k-1}c_{2k-1}$ be the second diagonal.

Note, that the graph $c_{2k}c_{2k+1} \dots c_{4k-1}c_{2k-1}c_{2k-2} \dots c_0c_{2k}$ is a C_{4k}, and c_0c_{4k-1} and $c_{2k}c_{2k-1}$ are diagonals with distance 1 in this C_{4k} (see Figure 2.4.6).

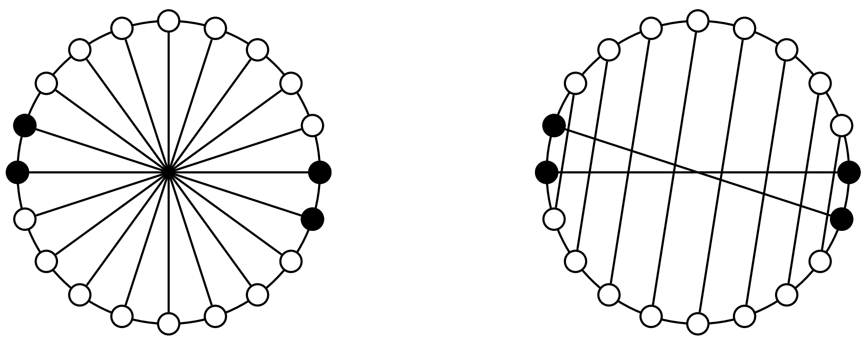


Figure 2.4.6: The two possible M_{4k} a C_{4k} with two diagonals can be semi well-behaved to.

Let v be a vertex with at least 3 neighbours in C, such that none of these neighbours is part of d_0 or d_{-1}. Since C together with d_0 consists of two C_{2k+1}, v must have two of its neighbours in one of these C_{2k+1}, so let these neighbours be c_{j-1} and c_{j+1}, such that $2 \leqslant j \leqslant 2k-3$.

The diagonals d_0 and d_{-1} are dividing C into 4 segments, and since c_{j-1} and c_{j+1} lie in one of them, it is easy to see that the third neighbour c_h of v in C must lie on the segment, that is not adjacent to the one where c_{j-1} and c_{j+1} lie (see Figure 2.4.3).

Consider the two cycles $C_1 = vc_{j+1}Cc_{2k-1}c_{4k-1}Cc_hv$ and $C_2 = vc_{j-1}Cc_0c_{2k}Cc_hv$ (see Figure 2.4.3 on the left). Since the sum of their edges is $4k-2+2-2+4 = 4k+2$ and therefore even, they both have the same parity, and since G is $\mathscr{C}_{2k-1}$-free they both must be even, or they both must be a C_{2k+1}. In the latter case, a simple calculation shows that v is well-behaved with respect to the M_{4k} from part 2 of this lemma.

Now consider the two cycles $C_3 = vc_{j+1}Cc_hv$ and $C_4 = vc_{j-1}Cc_hv$ (see Figure 2.4.3 on the right). Since the sum of their edges is $4k - 2 + 4 = 4k + 2$ and therefore even, they both have the same parity, and since G is $\mathscr{C}_{2k-1}$-free they both must be even, or both be a C_{2k+1}. In the latter case, a simple calculation shows that v is well-behaved with respect to the M_{4k} from part 1 of this lemma.

Since the symmetric difference between C_1 and C_3 is a C_{2k+1} and therefore odd, exactly one of the two cases holds.

Let v be a vertex with at least 3 neighbours in C, such that $c_0 \in N(v)$.

Since C together with d_0 consists of two C_{2k+1}, we can easily interfere that $N_C(v) \subseteq \{c_0, c_2, c_{-2}, c_{2k-1}, c_{2k+1}\}$ (see Figure 2.4.7 on the left).

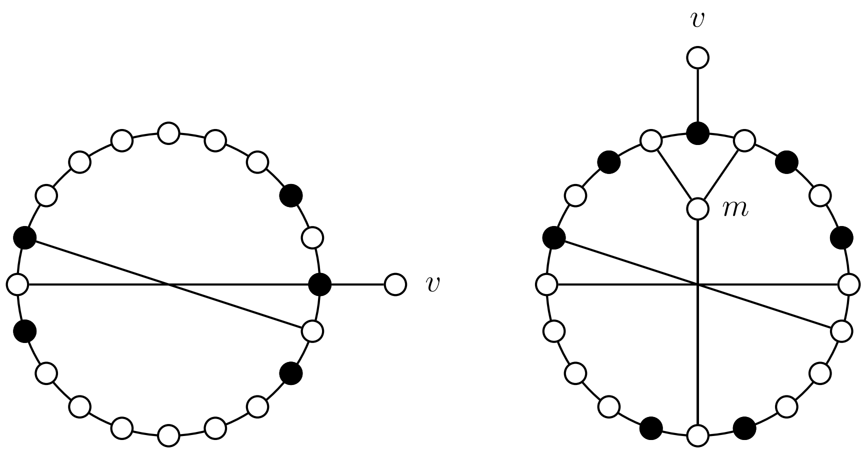


Figure 2.4.7: On the left are the possible neighbours of a vertex v with $c_0 \in N(v)$. On the right are the possible neighbours of a vertex v which has a neighbour c_j, such that there is a vertex m acting like a m_j.

Furthermore, using that a C_{2k+1} is well-behaved, we see that indeed

$$N_C(v) \in \{\{c_{2k-1}, c_0, c_{2k+1}\}, \{c_{-2}, c_0, c_{2k-1}\}, \{c_{2k+1}, c_0, c_2\}, \{c_{-2}, c_0, c_2\}\}.$$

In the former 3 cases, v is well-behaved with respect to the M_{4k} from part 1 of this lemma, and in the latter case, v is well-behaved with respect to the M_{4k} from part 2 of this lemma.

Summarising the proof of this lemma so far, every vertex with at least 3 neighbours in C has exactly 3 neighbours in C which act well-behaved either according to part 1 or part 2 of the lemma.

Assuming there is no vertex with 3 neighbours in C where not at least one of the neighbours is in $\{c_0, c_{2k}\}$ quickly gives rise to a contradiction with $\delta(G) > \frac{1}{2k-1}n$. Therefore and by symmetry (of part 1 or part 2) we might assume there is a vertex m acting like a m_j from part 1 of this lemma, where $1 \leqslant j \leqslant 2k - 2$.

Replacing c_j in C with m gives rise to a new graph C', which is well-behaved with respect to a M_{4k} and the obvious mapping of vertices.

But since C' is, with the exception of one vertex, exactly C we can interfere that C is well-behaved with respect to the M_{4k} from part 1, with the exception of vertices which are neighbours of c_j.

Assume there exists such a vertex v. Because v is part of a C_{2k+3} with m, c_{j+2k} and c_{j+1}, as well as part of a C_{2k+3} with m, c_{j+2k} and c_{j-1}, we know that

$$N_C(v) \subseteq \{c_{j-4}, c_{j-2}, c_j, c_{j+2}, c_{j+4}, c_{j+2k-1}, c_{j+2k+1}\}.$$

(See Figure 2.4.7 on the right.)

However, because v is well-behaved in C', it is easy to see that

$$N_C(v) \in \{\{c_j, c_{j-2}, c_{j-4}\}, \{c_j, c_{j+2}, c_{j+4}\}\}$$

if v is not well-behaved in C.

Assuming by symmetry that $N_C(v) = \{c_j, c_{j-2}, c_{j-4}\}$ now implies $j \in \{1, 2\}$, since otherwise there would be a not well-behaved C_{2k+1}. Relabelling m as v'' and we can now once again derive a contradiction to $\delta(G) > \frac{1}{2k-1}n$ by the same analysis as done in the proof of Lemma 2.4.4 where v has 3 not well-behaved neighbours in C and a vertex v'' emerges. $\square$

Before we state the last lemma in this section, we will need some additional terminology, which will be introduced here.

By a *2-diagonal* in an even cycle $C_{2\ell}$, we are referring to a P_2 joining two vertices with distance ℓ on the cycle. The distance of 2-diagonals is defined analogously to the distance of diagonals. Let L_{4k} be the graph on $4k$ vertices, consisting of a C_{4k-2} together with two adjacent 2-diagonals in this C_{4k-2}. For a L_{4k} consisting of the $C_{4k-2} = c_0c_1 \dots c_{4k-3}c_0$ and the two 2-diagonals $D_0 = c_0d_0c_{2k-1}$ and $D_{-1} = c_{4k-3}d_{-1}c_{2k-2}$ let L^*_{4k} be the supergraph, arising from this L_{4k} by adding all the edges of the form c_ic_{4k-3-i} for $1 \leqslant i \leqslant 2k-3$ as well as the edge d_0d_{-1} (see Figure 2.4.8).

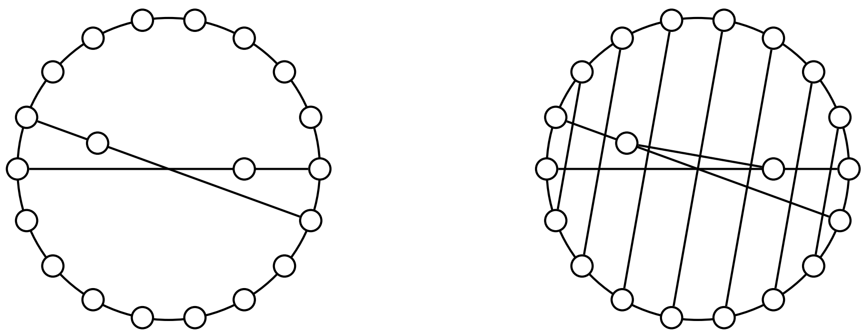


Figure 2.4.8: The graphs L_{4k} and L^*_{4k} for $k = 5$.

Lemma 2.4.6. *Let $k \geqslant 3$ and let G be a $\mathscr{C}_{2k-1}$-free n-vertex graph with $\delta(G) > \frac{1}{2k-1}n$, then any L_{4k} is semi well-behaved with respect to the L^*_{4k} arising from it.*

Proof. Let L be an L_{4k} consisting of the $C_{4k-2} = c_0c_1 \dots c_{4k-3}c_0$ and the two 2-diagonals $D_0 = c_0d_0c_{2k-1}$ and $D_{-1} = c_{4k-3}d_{-1}c_{2k-2}$.

Consider the cycle $C = d_0c_0c_1 \dots c_{2k-2}d_{-1}c_{4k-3}c_{4k-4} \dots c_{2k-1}d_0$ of length $4k$ with two diagonals c_0c_{4k-3} and $c_{2k-1}c_{2k-2}$ with distance 2.

Applying Lemma 2.4.4 to C yields Lemma 2.4.6. $\square$

For the following two corollaries we will just give a sketch of the proof for each of them. In the proof no surprises happen and it is just a tedious case analysis. Nonetheless they are used a few times such that stating them explicitly seems reasonable.

Corollary 2.4.7. *Let $k \geqslant 2$, let G be a $\mathscr{C}_{2k-1}$-free graph, and let C be a C_{4k} with exactly one diagonal. Then, either there is a vertex v with 3 neighbours in C such that exchanging a vertex from C with v yields a C_{4k} with two diagonals or C is semi well-behaved with respect to a blow up of a C_{2k+1}.*

Proof (Sketch). As usually it is rather easy to prove the claim for vertices witch have a neighbour in one of the end vertices of the diagonal. Assuming a vertex v would contradict the claimed corollary, it has to have a neighbour on both "sides" of the diagonal. Fixing a neighbour on one of the sides and considering the two sets of cycles containing v with and without using the diagonal there are only five possible distances (3 by using symmetries) the second neighbour can have from the first. Using that a C_{2k+1} is well-behaved in G and by a standard case analysis it follows straight forwardly that v either behaves like a diagonal, or acts well-behaved with respect to a blow-up of a C_{2k+1} as claimed. $\square$

Corollary 2.4.8. *Let $k \geqslant 2$, let G be a $\mathscr{C}_{2k-1}$-free graph, and let C be a C_{4k-2} with one 2-diagonal. Then, either there is a second 2-diagonal in C, or C is semi well-behaved with respect to a blow up of a C_{2k+1}.*

Proof (Sketch). This proof is analogous to the one from Corollary 2.4.7. $\square$

2.5 Forbidden subgraphs in $\mathscr{C}_{2k-1}$-free graphs

In this section we are going to use the collected lemmas from Section 2.4 as well as some other tools we will introduce throughout the section to prove that some small graphs may not appear as (induced) subgraphs in any sufficiently dense $\mathscr{C}_{2k-1}$-free graph.

Our main goal is to proof Lemma 2.5.9, which we will rely on in Section 2.7, furthermore Lemma 2.5.2 will occasionally be used in Section 2.6.

We will start with a tool that will drastically decrease the cases to be considered in each of the following lemmas, but before we state this tool we need a small lemma.

Lemma 2.5.1. *Let $G = (V_G, E_G)$ be a (maximal) $\mathscr{C}_{2k-1}$ free graph on $n > 1$ vertices with $\delta(G) = \alpha n$, let $H = (V_H, E_H)$ be a balanced blow-up of G on $f \cdot n$ vertices, then the following hold.*

(i) *H is (maximal) $\mathscr{C}_{2k-1}$-free, and*

(ii) *$\delta(H) = \alpha(f \cdot n) = f \cdot \delta(G)$.*

Proof. Proving part *(ii)* is rather simple. Let v be a vertex that realizes the minimum degree in G, since each of its neighbours will be mapped to a set of f independent vertices, its degree will also multiply by a factor of f. Since every other vertex in G has at least as much neighbours as v, their degree will also be at least as big in H, therefore one of the image vertices for v ("still") realizes the new minimum degree.

It is left to prove part *(i)*. Assume there is an odd cycle C of length $\ell \leqslant 2k-1$ in H. Since every vertex from G was mapped to an independent set of vertices, we will get a walk W without any loops by mapping C to G such that every edge between the image of two vertices v and u gets mapped to the edge vu. Since C was an odd cycle, W is a closed odd walk. It is not hard to see that any closed odd walk contains an odd cycle, indeed if W is not already an odd cycle it will contain a vertex twice, splitting the edges from the first to the second occurrence of this vertex from W to create W_1 and defining $W_2 = W \setminus W_1$ creates two new closed walks, exactly one of them is odd. Repeating this process, induction eventually will ensure the occurrence of an odd cycle. But than, G contains an short odd cycle, contradicting G being $\mathscr{C}_{2k-1}$-free, therefore H must be $\mathscr{C}_{2k-1}$-free as well.

Now assume that G is maximal $\mathscr{C}_{2k-1}$-free. Consider an edge $e = vu \notin E_H$. Since $n > 1$ and G is maximal $\mathscr{C}_{2k-1}$-free, there is no isolated vertex in G, therefore if v and u are copies of the same vertex in H, we can not add e to H because it would close a C_3. However, considering an edge that joins two vertices in H that are not copies of the same vertices in G will also create a short odd cycle, namely "the same" as the addition of the edge joining the pre images of v and u in G would close. Since we can not add an edge to H without closing a short odd cycle, H is maximal $\mathscr{C}_{2k-1}$-free as well. □

We will use the above lemma as so called *Schnitzers principle* [48]: When proving a statement of the form "Let G be a (maximal) $\mathscr{C}_{2k-1}$-free graph on n vertices with $\delta(G) \geqslant \alpha n$, then G does not contain a (n induced) subgraph H of the form ... " and in the proof we infer the existence of some vertices $M \subseteq V(G)$ with some properties leading to a contradiction, we might assume all the vertices of M to be disjoint from each other and from all the vertices of H to reduce the cases to be considered.

To validate Schnitzers Principle, assume that G does indeed contain a (n induced) subgraph H of a certain form, and there are indeed some vertices $M \subseteq V(G)$ with certain properties, but they are either not disjoint from H or from each other, so the contradiction constructed in the proof does not apply. Furthermore assume that $|V(H)| = h$ and we inferred the existence of m vertices with certain properties. Now by Lemma 2.5.1 we know, that there is a $(h+m)$-balanced blow-up G' of G, that is (maximal) $\mathscr{C}_{2k-1}$-free as well and has the same relative minimum degree. We can now locate a copy H' of H in G' and follow our proof in G' furthermore each time we infer the existence of a vertex with some properties which might not be disjoint from H' or the other vertices we inferred so far, we might select a different copy of the vertices it is potentially identical to. Since there are $h+m$ copies of each vertex, we never run out of vertex copies during our proof, enabling us to construct a contradiction with either G' being (maximal) $\mathscr{C}_{2k-1}$-free or $\delta(G') = (h+m)\delta(G)$. But now Lemma 2.5.1 tells us, that the only reason this could happen is either that G is not (maximal) $\mathscr{C}_{2k-1}$-free, $\delta(G) \neq \alpha n$ or G does not contain a (n induced) subgraph H of a certain form. Since the first two are false by our assumptions on G, the third must hold, proving our desired statement.

In the following lemma, we would like to strengthen Lemma 2.4.6 analogously to how Lemma 2.4.5 gets strengthened to Lemma 2.4.4 by a bigger distance between the two diagonals. As is turns out however, for our given δ of $\frac{1}{2k-1}n$, this strengthening implies that the graphs do not appear in G at all.

Lemma 2.5.2. *Let $k \geqslant 3$ and let G be a $\mathscr{C}_{2k-1}$-free n-vertex graph with $\delta(G) > \frac{1}{2k-1}n$, then G does not contain a C_{4k-2} with 2 different 2-diagonals with distance at least 2.*

Proof. Assume that $G = (V, E)$ contains a subgraph L consisting of a cycle $C = c_0c_1 \dots c_{4k-3}c_0$ and two 2-diagonals $D_0 = c_0d_0c_{2k-1}$ and $D_j = c_jd_jc_{j+2k-1}$ with $2 \leqslant j \leqslant 2k-3$. As usually, we will consider indices of c_i to be cyclic in C.

We will first consider the case that $3 \leqslant j \leqslant 2k-4$. This additional assumption will allow us to make the following claim, leading to an easy contradiction with $\delta(G) > \frac{1}{2k-1}n$.

Claim 2.5.3. *No vertex $v \in V$ has more than 2 neighbours in C.*

Proof. Assume v is a neighbour of one of the vertices from $\{c_0, c_{2k-1}, c_j, c_{j+2k-1}\}$, by symmetry assume $c_0 \in N(v)$, then since C_{2k+1} is well-behaved in G by Lemma 2.4.3 $N_C(v) \subseteq \{c_0, c_{2k-1}, c_2, c_{4k-4}\}$. But since c_2 and c_{4k-4} both lie in one C_{2k+1} induced by C and D_j, it is easy to see v can have at most one neighbour in $\{c_{2k-1}, c_2, c_{4k-4}\}$.

Assume v has no neighbour in $\{c_0, c_{2k-1}, c_j, c_{j+2k-1}\}$, but at least 3 on C. By symmetry two of the neighbours must be c_{h-1}, c_{h+1} with $2 \leqslant h \leqslant 2k-3$ and one neighbour must be c_ℓ with $2k \leqslant \ell \leqslant 4k-3$.

If $j = h$ we have $N_C(v) \subseteq \{c_{j-1}, c_{j+1}, c_{j-3}, c_{j+3}\}$. But since $\ell \notin \{j-3, j+3\}$ because of $3 \leqslant j \leqslant 2k-4$, v can only have two neighbours on C.

If $j \neq h$, ℓ must lie in the segment of C that is not adjacent to the one where c_{h-1} and c_{h+1} lie. By symmetry let $0 < h-1, h+1, < j$. Consider the cycles

$C_1 = vc_{h-1}Cc_0d_0c_{2k-1}Cc_\ell v$ and $C_2 = vc_{h+1}Cc_jd_jc_{j+2k-1}Cc_\ell v$. Since the sum of their edges is $4k-2-2(2k-1-j)-2+4+4 \leqslant 4k < 4k+2$ and even, they both must have the same parity and therefore be even.

Now consider the cycles $C_3 = vc_{h-1}Cc_\ell v$ and $C_4 = vc_{h+1}Cc_\ell v$. Since the sum of their edges is $4k-2-2+4 = 4k < 4k+2$ and even, they both must have the same parity and therefore be even.

But the symmetric difference of C_1 and C_3 is a C_{2k+1}, contradicting the fact that both are even. Therefore v can have at most 2 neighbours in C. $\square$

Since no vertex has more than 2 neighbours in C, but C consist of $4k-2$ vertices, a simple double counting argument yields a contradiction to $\delta(G) > \frac{1}{2k-1}n$.

Now we will consider the case $j \in \{2, 2k-3\}$, and by symmetry we assume $j = 2$.

Since Claim 2.5.3 leads to a contradiction and the proof of Claim 2.5.3 also works for $j = 2$ with the exception of the case $j = h$, we might assume there is a vertex v with $c_1, c_3 \in N(v)$. It is not hard to see that the only other neighbour, v can have in C without having 3 neighbours in a C_{2k+1} is c_{4k-3}, and indeed any vertex u with 3 neighbours in C has either $N_C(u) = \{c_{4k-3}, c_1, c_3\}$ or $N_C(u) = \{c_{2k-2}, c_{2k}, c_{2k+2}\}$.

Note, that there cannot be a vertex that gives rise to a 2-diagonal in C apart from the already present 2-diagonals or one joining c_1 and c_{2k}, if there is a vertex with 3 neighbours in C, as this diagonal and this vertex would give rise to a C_{2k-1}.

Therefore any vertex with exactly one neighbour in $\{d_0, d_j\}$ has at most 2 neighbours in C, at least one of which is in $\{c_1, c_{2k}\}$. Additionally, no vertex that is a neighbour of both, d_0 and d_j has any neighbours in $C \setminus \{c_1, c_{2k}\}$.

Now consider the subset $V(C) \setminus \{c_1, c_{2k}\} \cup \{d_0, d_j\}$ of vertices of G. It consists of $4k-2$ vertices, and no vertex can have more than 2 neighbours in it, contradicting $\delta(G) > \frac{1}{2k-1}n$. $\square$

The following lemma appears in [34] already for $k = 3$ and although our proof here is streamlined via the use of Lemma 2.5.2 the basic idea stays the same.

Lemma 2.5.4. *Let $k \geqslant 3$ and let G be a maximal $\mathscr{C}_{2k-1}$-free graph on n vertices with $\delta(G) > \frac{1}{2k-1}n$, then G does not contain an induced copy of C_6.*

Proof. Assume for a contradiction, that G contains a C_6, namely $C = c_0c_1\dots c_5c_0$ as an induced subgraph.

Since G is maximal $\mathscr{C}_{2k-1}$-free, the absence of the edges c_0c_3 lets us infer the presence of an even path P_x between c_0 and c_3, containing at most $2k-2$ edges. Considering the closed odd walk $c_0P_xc_3c_2c_1c_0$ consisting of $|P_x|+3$ edges, P_x must contain exactly $2k-2$ edges. Analogously there are paths P_y and P_z, joining c_2 and c_5 as well as c_4 and c_1 respectively, which also consist of exactly $2k-2$ edges. By Schnitzers Principle, we can assume these P_i to be disjoint from C as well as from each other.

Let L be the graph, induced by $V(C) \cup V(P_x) \cup V(P_y) \cup V(P_z)$ (see Figure 2.5.1 on the left). We will analyse, how vertices with neighbours in L behave in detail and thereby prove the following claim.

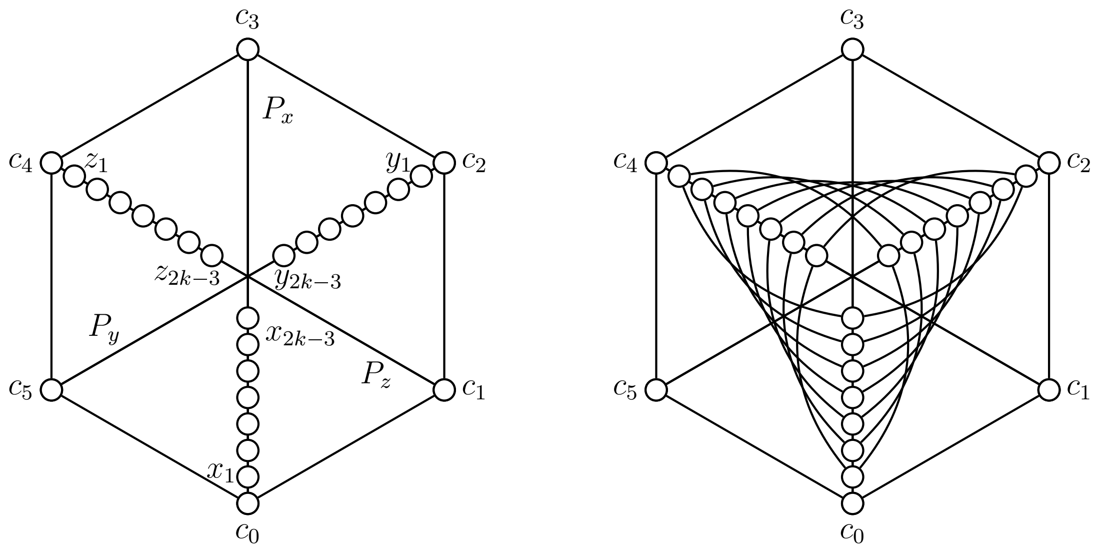


Figure 2.5.1: On the left, the graph L for $k = 5$ with the P_i. On the right, the graph L^*. Each vertex with neighbours in L either acts well-behaved with respect to L^*, or has 3 neighbours in the C_6.

Claim 2.5.5. *No vertex has more than* 3 *neighbours in* L.

Proof. First, for easier notation, let the vertices on P_x be named x_i such that $P_x = c_0x_1x_2 \dots x_{2k-3}c_3$. Similarly let $P_y = c_2y_1y_2 \dots y_{2k-3}c_5$ and let $P_z = c_4z_1z_2 \dots z_{2k-3}c_1$.

Obviously, there can be no vertex with more than 3 neighbours in C.

Let v be a vertex with exactly 3 neighbours in C. These neighbours must be $\{c_0, c_2, c_4\}$ or $\{c_1, c_3, c_5\}$. In both cases, v cannot have any neighbour in any P_i, since this would be a third neighbour in an C_{2k+1}.

Let v be a vertex with exactly 2 neighbours in C. By symmetry let $c_0, c_2 \in N(v)$, then v can not have any additional neighbour on P_x or P_y. Also, on P_z v can only have c_4 and y_{2k-3} as additional neighbours, but $c_4 \in N(v)$ would contradict that v has exactly 2 neighbours on C. In total, v cannot have more than 3 neighbours in L.

Let v be a vertex with exactly 1 neighbour in C. By symmetry let this neighbour be c_0. Then $N_L(v) \subseteq \{c_0, c_2, c_4, x_2, y_{2k-3}, z_{2k-3}\}$, but since v has exactly 1 neighbour on C we have $N_L(v) \subseteq \{c_0, x_2, y_{2k-3}, z_{2k-3}\}$.

Consider the cycle $C_x = c_2P_xc_5c_4P_zc_1c_2$ consisting of $4 + 2(2k - 3) = 4k - 2$ vertices. The vertex c_0 with the edges c_0c_1 and c_0c_5 is a 2-diagonal in C_x, and the vertex c_3 analogously defines another 2-diagonal in C_x. Assuming there is a vertex v with $N_L(v) \supseteq \{c_0, x_2, y_{2k-3}, z_{2k-3}\}$, this vertex would form a third 2-diagonal in C_x, two of which therefore would have distance at least 2, contradicting Lemma 2.5.2. Because of this, no vertex v with exactly 1 neighbour in C can have more than 3 neighbours in L.

Finally, let v be a vertex with no neighbours in C. If v has more than 3 neighbours in L, two of them must lie at one P_i, by symmetry let it be P_x. Since P_x is part of a C_{2k+1}, these two neighbours must be of the form x_{i-1} and x_{i+1}. Now consider the cycle $C_z = c_0 P_x c_3 c_2 P_y c_5 c_0$ of length $4k-2$ with the two 2-diagonals spawned by c_4 and c_1, as well as the cycle $C_y = c_0 P_x c_3 c_4 P_z c_1 c_0$ of length $4k-2$ with the two 2-diagonals spawned by c_2 and c_5, since by Lemma 2.4.6 both of these cycles are semi well-behaved, we have that $N(v) \subseteq \{x_{i-1}, x_{i+1}, y_{2k-2-i}, z_{2k-2-i}\}$.

Indeed, adding all the edges from the "reference graph" L^*_{4k} to our graph L for each of the cycles C_x, C_y and C_z, as shown in Figure 2.5.1 on the rights, yields a graph L^*, such that L is semi well-behaved with respect to L^* except for vertices which have 3 neighbours in C.

But considering C_x again, a vertex v with $N(v) \supseteq \{x_{i-1}, x_{i+1}, y_{2k-2-i}, z_{2k-2-i}\}$ would yield a third 2-diagonal in C_x, once again contradicting Lemma 2.5.2. Since every vertex with 4 neighbours in $L \setminus C$ does construct such an additional 2-diagonal for one of the three cycles C_x, C_y or C_z, there cannot be such a vertex. $\square$

Observing that L contains exactly $3 \cdot (2k-1)$ vertices, together with Claim 2.5.5 once again leads to a contradiction with $\delta(G) > \frac{1}{2k-1} n$ by a standard double counting argument. $\square$

The following lemma is the only one we are not yet able to proof for general k, and although some ideas about handling cycles of length relatively close to $4k$ are there, the small cycles for large k seem to be a bit more difficult to exclude with a general proof. In a sense it is just a technical lemma breaking the seemingly circular dependence from Lemma 2.5.8 and Lemma 2.5.9, so if one would be able to proof these two lemmas for general k without using it, this would also be fine, even though this seems unlikely due to the structural resemblance of Lemma 2.5.6 and Lemma 2.5.9.

Lemma 2.5.6. *Let $k = 3$ and let G be a maximal $\mathscr{C}_{2k-1}$-free graph on n vertices with $\delta(G) > \frac{1}{2k-1} n$, then G does not contain a well-behaved $C_{2\ell}$ with $2\ell \geqslant 6$ as an induced subgraph.*

For the sake of readability and compactness we will introduce some notation that will be used in the proofs of the following lemmas from this chapter. Let H be a subgraph of G, $V'' \subseteq V' \subseteq V(H)$, where V'' usually will be very small, and $h \in V(H)$, we will then express $h \in N(v) \Rightarrow N_{V'}(v) \subseteq V'' \cup \{h\}$ as

$$\begin{array}{c||c} \cdot & V' \\ \hline\hline h & V''. \end{array}$$

Furthermore we will extend this notation naturally for sets of vertices instead of a single vertex h and more than one vertex set V'.

Proof. First we remark that an induced well-behaved $C_{2\ell}$ with $2\ell \geqslant 4k-2$ in G would lead to a contradiction with $\delta(G)$ by a standard double counting argument. Therefore we only have to concern ourselves with the proof of the cases where $6 \leqslant 2\ell \leqslant 4k-4$.

For $k=3$ we have $4k-4=8$, and so considering Lemma 2.5.4 it suffices to show that G does not contain a well-behaved C_8 as an induced subgraph.

Assume this is not the case and let $C = c_0c_1 \dots c_7$ be an induced and well-behaved C_8 in G. As usually we will consider the indices on the c_i to be circular.

Claim 2.5.7. *For any 4 vertices c_i, c_{i+1}, c_{i+4}, and c_{i+5} there is a vertex v, such that exchanging one of these 4 vertices with v gives rise to an induced and well-behaved C_8 where there is a P_3 between c_i and c_{i+4} or between c_{i+1} and c_{i+5}, if such a path does not already exist in C.*

Proof. By symmetry, let $c_0, c_1, c_4, c_5 \in C$ be vertices that contradict the claim. In particular there is no P_3 between c_0 and c_4 or between c_1 and c_5. Since G is maximal $\mathscr{C}_{2k-1}$-free, there is a path P_x of length 4 between c_0 and c_5 and a path P_y of length 4 between c_1 and c_4. Let $P_x = c_0x_1x_2x_3c_5$ and $P_y = c_1y_1y_2y_3c_4$ (see Figure 2.5.2 on the left for an illustration.).

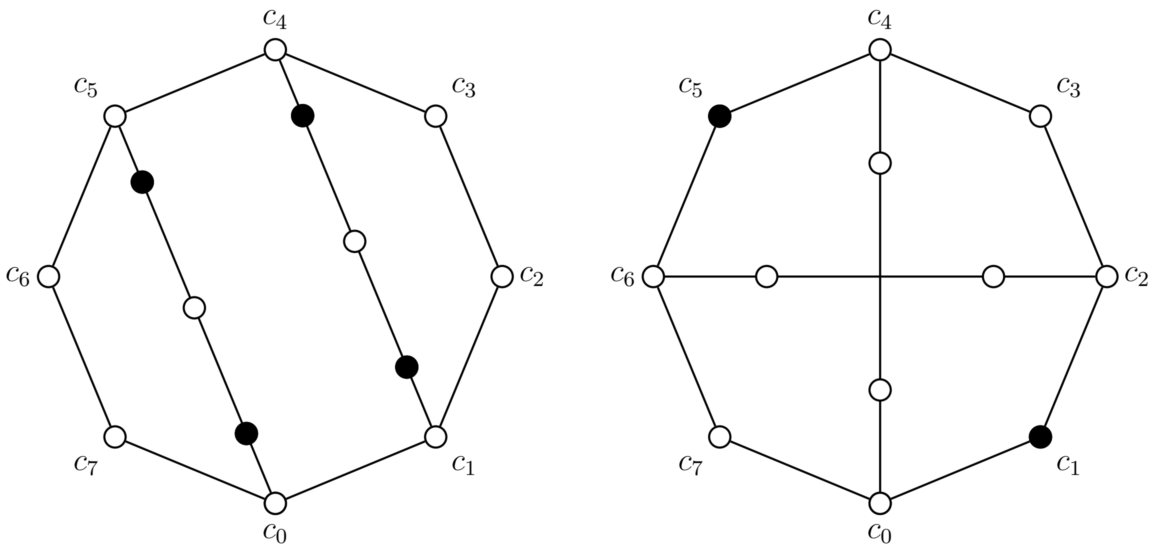


Figure 2.5.2: On the left a C_8 with two adjacent P_4. On the right a C_8 with two non adjacent P_3.

Now consider the set $U = V(C) \cup \{x_2, y_2\}$ of size $8+2 = 2(2k-1)$, we claim that no vertex can have 3 neighbours inside, leading to a contradiction with $\delta(G)$ by double counting.

First we note that the vertices of U are all disjoint, since every unification of x_2 or y_2 with another vertex of U would lead to a short odd cycle.

Secondly we note that we have the following statements.

$\cdot$	C	$U \setminus C$
x_2	c_0, c_2, c_3, c_5	y_2
y_2	c_1, c_4, c_6, c_7	x_2

And since C is well-behaved and induced in G, any vertex in G with at least 3 neighbours in U has to be a neighbour of either x_2 or y_2. But a vertex that has both x_2 and y_2 as neighbours can have no other neighbours on C, and a vertex with exactly one of x_2 and y_2 as a neighbour, as well as 2 well-behaved neighbours on C leads to Claim 2.5.7 being true. $\square$

By using Claim 2.5.7 up to 3 times on C and the resulting cycles, after relabelling the new vertices with the names of the vertices they replaced, we can ensure a P_3 between c_i and c_{i+4} as well as one between c_{i+2} and c_{i+6} for some i. Note that in each replacement step we get another induced and well-behaved C_8 again. By symmetry let these paths be P_s between c_0 and c_4 as well as P_t between c_2 and c_6 (see Figure 2.5.2 on the right for an illustration).

Now however $c_0 P_s c_4 c_3 c_2 P_t c_6 c_7 c_0$ is a C_{4k-2} and c_1 and c_5 give rise to two 2-diagonals that are not adjacent, contradicting Lemma 2.5.2. $\square$

The following lemma, where the proof is quite lengthy is probably provable without using Lemma 2.5.6 or Lemma 2.5.9, however the length of it will probably explode, and since it is just a tool to prove Lemma 2.5.8 anyway there seems no motivation to do so. It will allow us to handle the appearance of a not well-behaved vertex in an induced even cycle creating no shorter induced even cycle of length > 4 in the proof of Lemma 2.5.9.

Lemma 2.5.8. *Let $k = 3$ and let G be a maximal $\mathscr{C}_{2k-1}$-free graph on n vertices with $\delta(G) > \frac{1}{2k-1}n$, then G does not contain a C_8 with exactly 2 chords that meet in a vertex as an induced subgraph.*

Proof. Note that the upper bound restriction on k originates from Lemma 2.5.6 and does not appear elsewhere in the proof.

Assume that G contains a C_8 with exactly two chords that meet in a vertex as an induced subgraph. For convenience in notation we will refer to the C_8 as $C = c_0 c_1 \dots c_7 c_0$ and let the chords be $c_6 c_1$ and $c_6 c_3$.

From all these graphs G contains as an induced subgraph, let H be the one that maximizes ℓ in the sense that there is a P_ℓ from c_2 to a vertex t_ℓ, a $P_{2k-2-\ell}$ from t_ℓ to c_5 as well as a $P_{2k-2-\ell}$ from t_ℓ to c_7. These three paths form a P_{2k-2} from c_2 to c_5 as well as a P_{2k-2} from c_2 to c_7. Let the former path be $P_y = c_2 y_1 y_2 \dots y_{2k-3} c_5$ and the latter path be $P'_y = c_2 y'_1 y'_2 \dots y'_{2k-3} c_7$ and whenever $y_i = y'_i$, so $i \leqslant \ell$, we will call this vertex t_i instead of y_i or y'_i (see Figure 2.5.3 for an illustration).

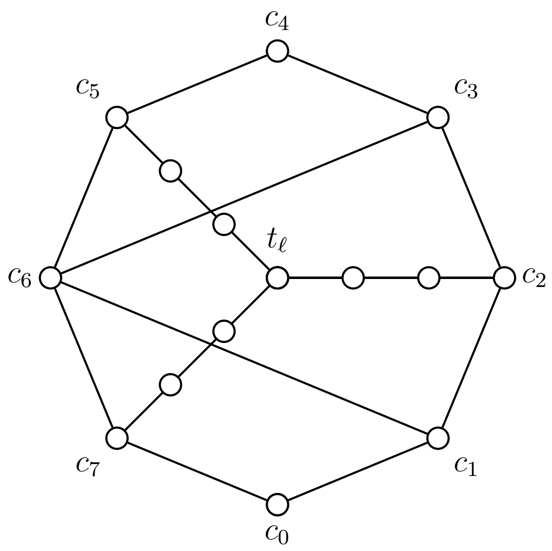


Figure 2.5.3: The graph C together with the auxiliary paths P_y and P'_y for $k = 4$ and $\ell = 3$.

Note that G is maximal $\mathscr{C}_{2k-1}$-free and there are no edges between c_2 and c_5 or c_2 and c_7. Therefore ℓ might be 0, implying that $t_\ell = c_2$, but the two paths P_y and P'_y do exists.

We will now contradict $0 \leqslant \ell \leqslant 2k-3$, implying that G does not containing a C_8 with exactly 2 chords that meet in a vertex as an induced subgraph, which is the statement of this lemma. Depending on the value of ℓ we will arrive at different contradictions, we will therefore split this proof in 3 cases and we start with the broadest case.

Case: $1 \leqslant \ell \leqslant 2k-4$

Let $1 \leqslant \ell \leqslant 2k-4$ be given. First we will ensure, that $l \leqslant 2k-5$.

Because of the cycles $c_2 P_y c_5 c_6 c_1 c_2$ and $c_2 P'_y c_7 c_6 c_1 c_2$ of length C_{2k+1} the vertex t_ℓ can not be a neighbour of c_6. Additionally, neither the edge $y_{2k-3}c_7$ nor the edge $y'_{2k-3}c_5$ can be present due to the maximality of ℓ. But if $\ell = 2k-4$ would be the case, then the cycle $t_\ell y_{2k-3} c_5 c_6 c_7 y'_{2k-3} t_\ell$ would contradict Lemma 2.5.4.

Therefore $\ell \leqslant 2k-5$, and since $\ell > 0$, there can not be a 2-diagonal joining other vertices than c_2 and c_6 in the C_{4k-2} formed by P_y, P'_y, and c_6.

Consider the even cycle $C' = t_\ell P_y c_5 c_6 c_7 P'_y t_\ell$. Since it is a subset of the vertices of the above mentioned C_{4k-2}, Corollary 2.4.8 ensures that just certain neighbourhoods in C' can occur. Apart from neighbours of t_ℓ or c_6, factoring in the maximality of ℓ, a vertex v which does not act well-behaved in C' must have a neighbourhood of the form $N_{C'}(v) = \{y_i, y'_i\}$.

A neighbour of t_ℓ, because of the two C_{2k+1} containing t_ℓ, can only not act well-behaved, if it is also a neighbour to both $y_{\ell+2}$ and $y'_{\ell+2}$, contradicting the maximality of ℓ.

Now consider the smallest index i such that there is a vertex v with $\{y_i, y'_i\} \subseteq N_{C'}(v)$ that does not act well-behaved in C'.

Obviously $i \geqslant \ell + 2$, so consider the even induced cycle $C'' = t_\ell P_y y_i v y'_i P'_y t_\ell$. Note that by the choice of v, neighbours of v have no other neighbours in C'' except y_{i-2} and y'_{i-2}, and if they are a neighbour to both of them, then $i = \ell + 2$ and C'' is a C_6 contradicting Lemma 2.5.4. Otherwise C'' is well-behaved by construction and therefore contradicts Lemma 2.5.6

Now if there is no vertex v that does not act well-behaved in C', C' itself is induced and well-behaved, contradicting Lemma 2.5.6 again.

Case: $\ell = 0$

Assume $\ell = 0$.

Recall that $t_0 = c_2$ and consider the Cycle C' from the last case, by construction it is an induced even cycle. Since it contains $2(2k-3)+4 = 4k-2$ vertices however, there must be a vertex with 3 neighbours in C', otherwise we get a contradiction t $\delta(G)$ by a double counting argument.

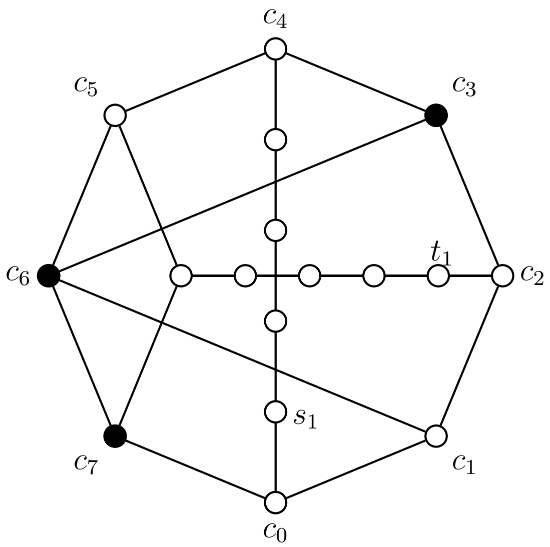
By construction (maximality of ℓ) and Corollary 2.4.8, no vertex can have two neighbours in either P_y or P'_y and also at least one neighbour in the other one. Therefore, there must be a vertex v with 3 neighbours in C', with at most one neighbour in each P_y and P'_y. Since $V(C') = V(P_y) \cup V(P'_y) \cup \{c_6\}$, this vertex is a neighbour of c_6 and has exactly one neighbour in each P_y and P'_y. By using Corollary 2.4.8 once again, we know that $N_{C'}(v) = \{c_6, y_{2k-3}, y'_{2k-3}\}$.

This however implies that the cycle $C'' = v y_{2k-3} P_y y_1 c_2 y'_1 P'_y y'_{2k-3} v$ is even, induced, and in regards to being well-behaved acts exactly like the cycle C' from the previous case. For the remainder of the proof an analogous procedure as in the case above will lead to a similar contradiction, concluding this case as well.

Case: $\ell = 2k-3$

For this final case we will refer to $P_y = c_2 t_1 t_2 \dots t_{2k-3} c_5$ as P_t, and also introduce some more auxiliary paths. The proof will deviate from the ones for the previous cases, indeed it will look more like the proof of Lemma 2.5.4.

First, assume that there is a P_{2k-3} joining c_0 and c_4. Let this path be $P_s = c_0 s_1 s_2 \dots s_{2k-4} c_4$, see Figure 2.5.4 on the left. Then the cycle $c_0 P_s c_4 c_5 t_{2k-3} P_t c_2 c_1 c_0$ contains $(2k-4)+(2k-3)+5 = 4k-2$ vertices, therefore the three 2-diagonals $c_4c_3c_2$, $c_5c_6c_1$, and $t_{2k-3}c_7c_0$ contradict Lemma 2.5.2.



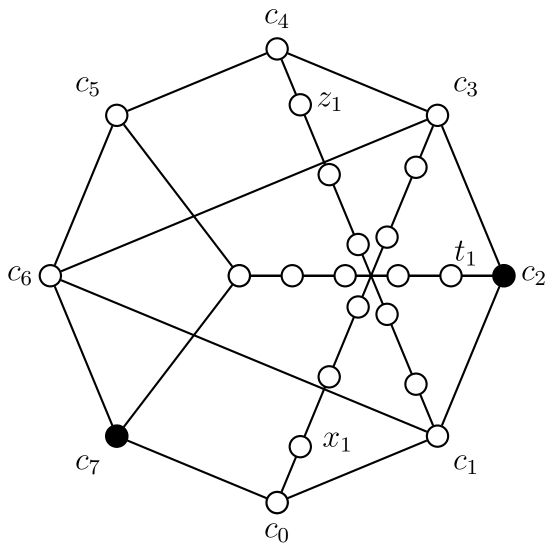


Figure 2.5.4: The graph C together with the auxiliary path P_t and the edge $t_{2k-3}c_7$ for $k = 4$ and $l = 3$. On the left with the additional auxiliary path P_s. On the right with the additional auxiliary paths P_x and P_z.

Since G is maximal $\mathscr{C}_{2k-1}$-free, and there is an induced P_3 between both c_0 and c_3 as well as c_1 and c_4, there are paths of length $2k-2$ between these pairs of vertices. Let $P_x = c_0 x_1 x_2 \ldots x_{2k-3} c_3$ and $P_z = c_4 z_1 z_2 \ldots z_{2k-3}$ be these paths. Note that P_z "starts" at c_4 for having a more symmetric notation.

Now let $H = G[V(C) \cup V(P_t) \cup V(P_x) \cup V(P_z)]$ (see Figure 2.5.4 on the right for an illustration).

Note that the statements from Lemma 2.5.2 as well as Corollary 2.4.8 hold, even if some of the vertices from the statements are unified, since "false" neighbourhoods would create short odd cycles in this case.

To get the statements from Table 2.1, in addition to using that G is $\mathscr{C}_{2k-1}$-free and considering the statements of Lemma 2.5.2 as well as the statement from Corollary 2.4.8, we will also use the maximality of ℓ and finally the fact that there is no P_s from c_0 to c_4 as described above.

Note, that each pair of P_{2k-2} paths together with some vertices from C forms a C_{4k-2} with one or two 2-diagonals, and that a common neighbour of x_i and z_{2k-5-i} or x_{2k-5-i} and z_i would give rise to a P_s.

Now with simple set intersections we get Table 2.2 from Table 2.1.

Analysing the statements from Table 2.2 we notice, that there are very few possibilities for a vertex to have at least 2 neighbours in C or the inner vertices of one of the P_{2k-2} and have more than 3 neighbours in H.

First we note that no vertex can be a neighbour to x_i and t_i or a neighbour to z_i and t_i at the same time due to Table 2.1. Furthermore from Lemma 2.5.2 we infer that no vertex can be a neighbour to x_i and z_i at the same time if $i \neq 2k-3$. Together

$\cdot$	C	$P_t \setminus C$	$P_x \setminus C$	$P_z \setminus C$
c_0	c_2, c_4, c_6	t_{2k-3}	x_2	z_{2k-3}, z_{2k-5}
c_1	c_3, c_5, c_7	t_1	x_1	z_{2k-4}
c_2	c_0, c_4, c_6	t_2	x_{2k-3}	z_{2k-3}
c_3	c_1, c_5, c_7	t_1	x_{2k-4}	z_1
c_4	c_0, c_2, c_6	t_{2k-3}	x_{2k-3}, x_{2k-5}	z_2
c_5	c_1, c_3, c_7	t_{2k-4}	x_1, x_{2k-4}	z_1
c_6	c_0, c_2, c_4	t_{2k-3}	x_{2k-3}	z_{2k-3}
c_7	c_1, c_3, c_5	t_{2k-4}	x_1	z_1, z_{2k-4}
t_i	C	t_{i-2}, t_{i+2}	x_{2k-1-i}, x_{2k-3-i}	z_{2k-1-i}, z_{2k-3-i}
x_i	C	t_{2k-1-i}, t_{2k-3-i}	x_{i-2}, x_{i+2}	$z_{2k-1-i}, z_{2k-3-i}, z_i$
z_i	C	t_{2k-1-i}, t_{2k-3-i}	$x_{2k-1-i}, x_{2k-3-i}, x_i$	z_{i-2}, z_{i+2}

Table 2.1: Here are all the possible neighbours a neighbour of one of the vertices of H can have in H.

with $i \geqslant 1$ this eliminates all the possibilities for a vertex with 4 neighbours in H that has at least 2 neighbours in the inner vertices of one of the P_{2k-2}. This also ensures that there is in fact no vertex with more than 4 neighbours in H that has at least 2 neighbours in either C or in the inner vertices of one of the P_{2k-2}.

Concerning the vertices with 4 neighbours in H that have at least 2 neighbours in C, we claim that each of them has at least one neighbour in $\{c_2, c_7\}$.

The only counterexample to this claim in Table 2.2 is a vertex v with $N_H(v) = \{c_3, c_5, x_{2k-4}, z_1\}$. If such a vertex would exist however, exchanging c_4 for v would give rise to a path P_s and therefore the already constructed contradiction to Lemma 2.5.2.

If there would be no vertex with at least 4 neighbours in $V(H) \setminus \{c_2, c_7\}$, we can use Schnitzers Principle to may assume all the inner vertices form P_t, P_x, and P_z to be disjoint from C and from each other and then get a contradiction to $\delta(G)$ by a double counting argument, since $|V(H) \setminus \{c_2, c_7\}| = 8 + 3(2k-3) - 2 = 3(2k-1)$.

A vertex with 4 neighbours in $V(H) \setminus \{c_2, c_7\}$, factoring in our reasoning after Table 2.2, must have one neighbour in $C \setminus \{c_2, c_7\}$, and one neighbour in the inner vertices of each P_{2k-2} each. Table 2.3 shows that such a vertex can not exists, concluding the proof of this case.

□

The following lemma is our main goal for this section. It gives a firm grasp on the feeling that G should somehow be (the blow-up of) a union of odd cycles such that basically all vertices have neighbours in all C_{2k+1}. In this case, even cycles would just appear by traversing images of a single vertex in a blow-up multiple times or by the symmetric difference of some C_{2k+1}, which both would not be induced. Even if this described feeling turns out not to be true as we shall see in Section 2.6, if Conjecture 4.2.2 does hold, it is true for graphs not containing the specific counter

$\cdot$	$P_t \setminus C$	$P_x \setminus C$	$P_z \setminus C$
c_0, c_2	$\varnothing$	$\varnothing$	z_{2k-3}
c_1, c_3	t_1	$\varnothing$	$\varnothing$
c_2, c_4	$\varnothing$	x_{2k-3}	$\varnothing$
c_3, c_5	$\varnothing$	x_{2k-4}	z_1
c_4, c_6	t_{2k-3}	x_{2k-3}	$\varnothing$
c_5, c_7	t_{2k-4}	x_1	z_1
c_6, c_0	t_{2k-3}	$\varnothing$	z_{2k-3}
c_7, c_1	$\varnothing$	x_1	z_{2k-4}
c_2, c_6	$\varnothing$	x_{2k-3}	z_{2k-3}
c_0, c_2, c_6	$\varnothing$	$\varnothing$	z_{2k-3}
c_2, c_4, c_6	$\varnothing$	x_{2k-3}	$\varnothing$
c_1, c_3, c_5	$\varnothing$	$\varnothing$	$\varnothing$
c_1, c_3, c_7	$\varnothing$	$\varnothing$	$\varnothing$
c_0, c_2, c_4, c_6	$\varnothing$	$\varnothing$	$\varnothing$
c_1, c_3, c_5, c_7	$\varnothing$	$\varnothing$	$\varnothing$
t_{i-1}, t_{i+1}	$\varnothing$	x_{2k-2-i}	z_{2k-2-i}
x_{i-1}, x_{i+1}	t_{2k-2-i}	$\varnothing$	z_{2k-2-i}
z_{i-1}, z_{i+1}	t_{2k-2-i}	x_{2k-2-i}	$\varnothing$

Table 2.2: Here are all the possible neighbours a neighbour of at least two of the vertices of C or a neighbour of at least two inner vertices of one of the P_{2k-2} can have in H. Note that no vertex v can have $N_C(v) = \{c_0, c_2\}$ or $N_C(v) = \{c_2, c_4\}$ (as highlighted in red), even if one would expect so, since this would give rise to an induced C_6. Also, note that a vertex v can have $N_C(v) = \{c_2, c_6\}$ (as highlighted in blue) without giving rise to an induced C_6.

examples from Section 2.6.

Lemma 2.5.9. *Let $k = 3$ and let G be a maximal $\mathscr{C}_{2k-1}$-free graph on n vertices with $\delta(G) > \frac{1}{2k-1}n$, then G does not contain a $C_{2\ell}$ with $6 \leqslant 2\ell \leqslant 4k-2$ as an induced subgraph.*

Proof. This lemma states almost the same as Lemma 2.5.6, but we drop the requirement of $C_{2\ell}$ to be well-behaved. Note that the upper bound restriction on k originates from Lemma 2.5.6 and does not appear elsewhere in the proof.

We will prove this lemma by induction. For $2\ell = 6$, this lemma is Lemma 2.5.4.

For $8 \leqslant 2\ell \leqslant 4k-4$ assume that $C = c_0 c_1 \dots c_{2\ell-1}$ is a $C_{2\ell}$. If it is well-behaved, we have a contradiction to Lemma 2.5.6, therefore there is a vertex v which does not act well-behaved in C. However, since $|C| \leqslant 4k-4$ is even, the graph $G[V(C) \cup \{v\}]$ may not contain any odd cycle. If v has less than ℓ neighbours in C, we will find two neighbours c_i and c_j of v with distance at least 4 on C, such that v has no other neighbours on one of the two paths on C from c_i to c_j. But then v together with this path forms an induced cycle of length at least 6 and at most $2\ell - 2$, contradiction our induction assumption.

$\cdot$	$P_t \setminus C$	$P_x \setminus C$	$P_z \setminus C$
c_0	t_{2k-3}	x_2	z_{2k-3}, z_{2k-5}
c_1	t_1	x_1	z_{2k-4}
c_3	t_1	x_{2k-4}	z_1
c_4	t_{2k-3}	x_{2k-3}, x_{2k-5}	z_2
c_5	t_{2k-4}	x_1, x_{2k-4}	z_1
c_6	t_{2k-3}	x_{2k-3}	z_{2k-3}

$\cdot$	$P_x \setminus C$	$P_z \setminus C$
c_0, t_{2k-3}	x_2	$\varnothing$
c_1, t_1	$\varnothing$	z_{2k-4}
c_3, t_1	x_{2k-4}	$\varnothing$
c_4, t_{2k-3}	$\varnothing$	z_2
c_5, t_{2k-4}	x_1	z_1
c_6, t_{2k-3}	$\varnothing$	$\varnothing$

Table 2.3: There is no vertex with 1 neighbour in $C \setminus \{c_2, c_7\}$ and one neighbour in the inner vertices of each of the P_{2k-2} each, since x_1 and z_1 do not have a common neighbour.

If v does have exactly ℓ neighbours on C, say $c_0, c_2, \ldots, c_{2\ell-2}$ then the cycle $vc_0c_1 \ldots c_6v$ contradicts Lemma 2.5.8.

For $2\ell = 4k-2$ assume that C is a $C_{2\ell}$. We know by the minimum degree assumption on G that there must be a vertex v with at least 3 neighbours in C. Using this vertex as a choice for a not well-behaved vertex in the case $8 \leqslant 2\ell \leqslant 4k-4$ leads to the same contradiction. $\square$

2.6 Odd tetrahedra

We begin this section by a definition that appeared slightly different in a paper of Messuti and Schacht [41], and before that in a paper by Gerards [21] and describes a subdivision of K_4, such that the faces stay odd cycles. Note that our definition differs slightly from the one Messuti and Schacht used since the additional graphs that were considered to be odd tetrahedra there were indeed homomorphic to an Andrásfai graph, making the statement of our lemmas in this section more complicated without providing any benefit. Our central lemma in this section is Lemma 2.6.5 that states that any graph G with the usual prerequisites that contains an odd tetrahedra does also contain a well-behaved one with some additional properties. We will then use this specific odd tetrahedra to prove Corollary 2.6.7.

Definition 2.6.1 ($(2k+1)$-tetrahedra)**.** Given $k \geqslant 2$ we denote by $\mathscr{T}_k$ the set of graphs T consisting of

(i) one cycle C_T with three branch vertices a_T, b_T, and $c_T \in V(C_T)$,

(ii) a center vertex z_T, and

(iii) internally vertex disjoint (from C_T as well as each other) paths (called spokes) P_{az}, P_{bz}, P_{cz} connecting the branch vertices with the center vertex.

Furthermore, we require that each cycle in T containing z_T and exactly two of the branch vertices must have length $2k+1$, and the spokes have length at least 2. See

Figure 2.6.1 for an illustration.

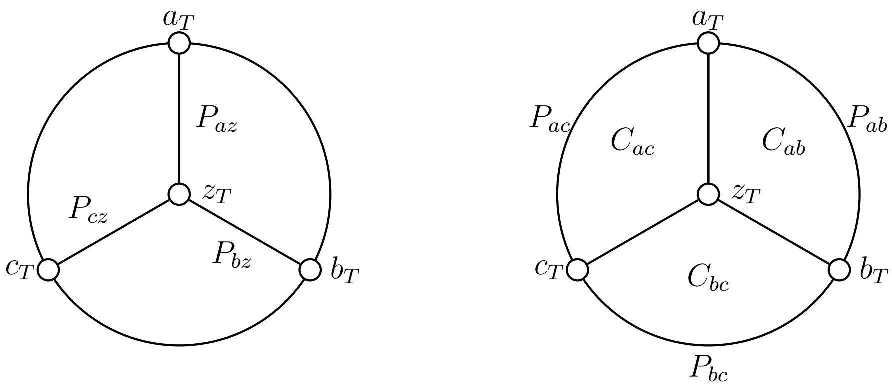


Figure 2.6.1: The $(2k+1)$-tetrahedra as defined in Definition 2.6.1. On the right some more terms we will be using.

In this first lemma concerning odd tetrahedra we motivate that there might be graphs with suitable odd girth and minimum degree where not all vertices have a neighbour in all C_{2k-1} and are therefore not homomorphic to Andrásfai graphs. With Observation 2.6.8 and Observation 2.6.9 we will then proof that these graphs can actually realise a sufficiently large minimum degree.

Lemma 2.6.2. *For all integers $k \geqslant 2$ and $r \geqslant 1$ there is no $(2k+1)$-tetrahedra $T \in \mathscr{T}_k$ that is homomorphic to the Andrásfai graph $A_{k,r}$.*

Proof. Let $T \in \mathscr{T}_k$ be given. Suppose for a contradiction that $T \xrightarrow{\text{hom}} A_{k,r}$ and let φ be such a homomorphism. Since T contains an odd cycle we have $r \geqslant 2$ and let $C_A = u_0 \dots u_{(2k-1)(r-1)+1}u_0$ be the Hamiltonian cycle of $A_{k,r}$ such that $N(u_0) = \{u_{i(2k-1)+1} \colon i = 0, \dots, r-1\}$ (c.f. proof of Proposition 2.1.2 (*c*)).

Claim 2.6.3. *Let v, v' be two vertices of a $2k+1$ cycle C in T with distance $d \geqslant 2$ in C. If $\varphi(v) = u_0$, than $\varphi(v') \in \{u_{i(2k-1)+d}, u_{i(2k-1)+(2k+1-d)}\}$ for some integer $0 \leqslant i \leqslant r-2$.*

Proof. In C there are two paths between v and v'. Let d and d' be their lengths. There cannot be a path of length $d-2s$ or $d'-2s$ with $s \geqslant 1$ between $\varphi(v)$ and $\varphi(v')$, since this path together with the embedding of the v-v'-path of other parity from C would form a closed odd walk of length less than $2k+1$, contradicting Proposition 2.1.2 (*b*). Similarly, $\varphi(v')$ is not in the neighbourhood of $\varphi(v) = u_0$ in $A_{k,r}$, since $2 \leqslant d \leqslant k < d' \leqslant 2k-1$.

Consequently, $\varphi(v')$ will lie on a segment S between $u_{i(2k-1)+1}$ and $u_{(i+1)(2k-1)+1}$ on the Hamiltonian cycle C_A for some integer $0 \leqslant i \leqslant r-2$. The segment S, together with $u_0 = \varphi(v)$ forms a C_{2k+1}, and since there are only two vertices with distance d from $u_0 = \varphi(v)$ on this C_{2k+1}, an embedding of v' onto any other vertex gives rise to a v-v'-path of length $d-2s$ or $d'-2s$ with $s \geqslant 1$. Therefore, $\varphi(v') \in \{u_{i(2k-1)+d}, u_{i(2k-1)+(2k+1-d)}\}$ as claimed. $\square$

Claim 2.6.4. *Let v, v', v'' be distinct vertices of a $2k+1$ cycle C in T. Let P' be the path from v to v' avoiding v'' on C and let P'' be the path from v to v'' avoiding v' on C. Suppose $d', d'' \geqslant 2$ are the lengths of P' and P''. If $\varphi(v) = u_0$, then $\varphi(v') = u_{i(2k-1)+d'}$ and $\varphi(v'') = u_{j(2k-1)+(2k+1-d'')}$, or $\varphi(v') = u_{i(2k-1)+(2k+1-d')}$ and $\varphi(v'') = u_{j(2k-1)+d''}$, for some integers $0 \leqslant i, j \leqslant r-1$.*

Proof. By Claim 2.6.3 it suffices to show, that $\varphi(v') = u_{i(2k-1)+d'}$ implies $\varphi(v'') \neq u_{j(2k-1)+d''}$ and $\varphi(v') = u_{i(2k-1)+(2k+1-d')}$ implies $\varphi(v'') \neq u_{j(2k-1)+(2k+1-d'')}$, for all $0 \leqslant i, j \leqslant r-1$.

In the first case, we may assume that $i \leqslant j$. Since $u_{j(2k-1)+2}$ is a neighbour of $u_{i(2k-1)+1}$, we may consider the path P starting with the path in C_A from $u_{i(2k-1)+d'}$ to $u_{i(2k-1)+1}$ together with the edge from $u_{i(2k-1)+1}u_{j(2k-1)+2}$ and then following C_A to $u_{j(2k-1)+d''}$. The path P consists of $(d'-1)+1+(d''-2) = d'+d''-2$ edges. Together with the embedding of the path between v' and v'' from C avoiding v, this yields a closed odd walk of length at most $2k-1$ in $A_{k,r}$, contradicting Proposition 2.1.2 (b). A similar argument for the second case concludes the proof of the claim. □

Note that $i(2k-1)+d \neq i(2k-1)+(2k+1-d)$ for all integers $d, i \geqslant 0$. Since z_T lies in three C_{2k+1}, each also containing two of the vertices a_T, b_T, c_T, if $\varphi(z_T) = u_0$, then it follows from Claim 2.6.4, that not all three branch vertices can be embedded onto $A_{k,r}$. Consequently, there is no homomorphism from T to $A_{k,r}$ and Lemma 2.6.2 is proved. □

For the following lemma and its proof, we introduce more notation on $(2k+1)$-tetrahedra.

We will call the three C_{2k+1} containing z_T and exactly two of the branch vertices by these two branch vertices, i.e. C_{ab}, C_{ac}, and C_{bc}. Furthermore we will call the subpath on C_T connecting two branch vertices and not containing the third by these two branch vertices as well, i.e. P_{ab}, P_{ac}, and P_{bc}, see Figure 2.6.1 for an illustration.

Note that since C_T is the symmetric difference of C_{ab}, C_{ac}, and C_{bc}, it is an odd cycle. Assuming our host graph G to be $\mathscr{C}_{2k-1}$-free, C_T has length $2k+1+2j$. An easy calculation yields that the sum of the edges contained in the spokes is $2k+1-j$, and therefore $|P_{ab}| = |P_{cz}| + j$, $|P_{ac}| = |P_{bz}| + j$, and $|P_{bc}| = |P_{az}| + j$. For easier notation, we will refer to the length of the spokes with ℓ_a, ℓ_b, and ℓ_c.

Furthermore we will call the interior vertices of the spokes by their respective branch vertex, so $P_{az} = z_T a_1 a_2 \dots a_{\ell_a - 1} a_T$ and for easier notation $z_T = a_0$ and $a_T = a_{\ell_a}$. Analogous definitions hold for P_{bz} and P_{cz}. Starting from $a_T = u_0$ we will label the vertices of C_T clockwise (such that $b_T < c_T$ in this ordering), and here we will use the indices modulo $|C_T|$ as usually.

The following lemma will give us a rather detailed look on the vertex minimal graph of $\mathscr{T}_k$ that is a subgraph of G, allowing us a very detailed analysis of graphs that contains a graph of $\mathscr{T}_k$ as a subgraph in return.

Lemma 2.6.5. *Let $k \geqslant 3$ and let G be a maximally $\mathscr{C}_{2k-1}$-free graph on n vertices with $\delta(G) > \frac{1}{2k-1}n$, furthermore let T be the vertex minimal graph of $\mathscr{T}_k$ that is a subgraph of G, then $|V(T)| \in \{4k, 4k+1\}$, all spokes have length at least 3 and T is well-behaved.*

Proof. In this proof we will once again extensively use the fact that C_{2k+1} is well-behaved in G and therefore not explicitly mention Lemma 2.4.3 each time we use this fact.

Let T be a vertex minimal $(2k+1)$-tetrahedra in G. We begin with a detailed analysis of the possible neighbourhoods a vertex can have inside T, which we summarise in the following claim.

Claim 2.6.6. *Every vertex v that does not act well-behaved in T has $|N_T(v)| - 2$ neighbours in $\{a_{\ell_a-1}, b_{\ell_b-1}, c_{\ell_c-1}\}$.*

Proof. Assume for a contradiction that v is a vertex which does not act well-behaved in T.

Case: v is a neighbour to one of the 4 special vertices

If v is a neighbour of z_T, then $N_T(v) \subseteq \{z_T, a_2, b_2, c_2\}$. But since $\ell_a, \ell_b, \ell_c \geqslant 2$, v can have at most one neighbour in $\{a_2, b_2, c_2\}$, therefore v acts well-behaved.

If v is a neighbour of one of the branch vertices, by symmetry let $a_T \in N(v)$, then $N_T(v) \subseteq \{a_T, a_{\ell_a-2}, u_{-2}, u_2, u_{i-1}, u_{i+1}\}$ for $\ell_c + j + 2 \leqslant i \leqslant \ell_c + \ell_a + 2j - 2$.

If $a_T, u_i \in N(v)$ for $\ell_c + j + 1 \leqslant i \leqslant \ell_c + \ell_a + 2j - 1$, either the cycle $C_1 = va_T P_{ab} b_T P_{bc} u_i v$ or the cycle $C_2 = va_T P_{ac} c_T P_{bc} u_i v$ is odd, since their symmetric difference is C_T. Let C_1 be odd, than $C_1' = va_T P_{az} z_T P_{cz} c_T P_{bc} u_i v$ is odd as well, and since they both must contain at least $2k+1$ edges, we have $|P_{bz}| = 2$, and v together with C_{ab} and C_{bc} forms a C_{4k-2} with two 2-diagonals with distance at least 2, contradicting Lemma 2.5.2. Therefore $N_T(v) \subseteq \{a_T, a_{\ell-2}, u_{-2}, u_2\}$.

Obviously $a_{\ell-2} \in N(v)$ implies $u_{-2}, u_2 \notin N(v)$ making v act well-behaved, leaving $N_T(v) \subseteq \{a_T, u_{-2}, u_2\}$ to be considered.

If $N_T(v) = \{a_T, u_{-2}\}$ or $N_T(v) = \{a_T, u_2\}$, v does act well-behaved, so assume $N_T(v) = \{a_T, u_{-2}, u_2\}$. Now however, replacing a_T with v, the cycle C_{ab} with the cycle $z_T P_{az} a_T v u_2 P_{ab} b_T P_{bz} z_T$, and the cycle C_{ac} with the cycle $z_T P_{az} a_T v u_{-2} P_{ac} c_T P_{cz} z_T$ yields another $(2k+1)$-tetrahedra with the additional vertex v but without the vertices u_1 and u_{-1}, contradicting the minimality of T. Therefore, v acts well-behaved.

Case: v has two neighbours in a spoke

If v has two neighbours inside one of the spokes, say P_{az}, it can only have additional neighbours on the opposite segment of C_T. So let $\{a_{i-1}, a_{i+1}, u_h\} \subseteq N(v)$ with $2 \leqslant i \leqslant \ell_a - 2$ and $\ell_c + j + 1 \leqslant h \leqslant \ell_c + \ell_a + 2j - 1$. But than, replacing a_i by v yields a diagonal cutting either the symmetrical difference of C_{ab} and C_{bc} or the symmetrical difference of C_{ac} and C_{bc} into two odd cycles, where at least one of the odd cycles in shorter than $2k + 1$, because $\ell_a, \ell_b, \ell_c \geqslant 2$. Therefore $N(v) = \{a_{i-1}, a_{i+1}\}$ and v acts well-behaved.

Case: v has two neighbours in one of the segments of C_T

If v has two neighbours inside one of the segments of C_T, say P_{bc}, it can have additional neighbours in P_{ac}, P_{ab}, or P_{az}. However, if $\{u_{i-1}, u_{i+1}, a_h\} \subseteq N(v)$ with $1 \leqslant h \leqslant \ell_a - 1$ and $\ell_c + j + 2 \leqslant i \leqslant \ell_c + \ell_a + 2j - 2$, replacing u_i with v leads to the same contradiction as in the last case.

If $\{u_{i-1}, u_{i+1}, u_h\} \subseteq N(v)$ with $1 \leqslant h \leqslant \ell_a + j - 1$ and $\ell_c + j + 2 \leqslant i \leqslant \ell_c + \ell_a + 2j - 2$, replacing u_i with v yields a chord in C_T from $u_i' = v$ to u_h. Since this edge is segmenting the symmetric difference of C_{ab} and C_{bc} into two even cycles (otherwise there is a short odd cycle), we have $|b_T P_{bc} u_i'| \neq |b_T P_{ab} u_h|$, by symmetry we may assume $|b_T P_{bc} u_i'| > |b_T P_{ab} u_h|$.

Now however, replacing b_T with u_h, and C_{bc} with $z_T P_{cz} c_T P_{bc} u_i' u_h P_{ab} b_T P_{bz} z_T$ yields another $(2k+1)$-tetrahedra without the vertices between u_i' and b_T, which exist since $|b_T P_{bc} u_i'| > |b_T P_{ab} u_h| \geqslant 1$, contradicting the minimality of T.

The case that v has a neighbour in P_{ac} works analogous to the case where v has a neighbour in P_{ab}. Therefore, v is well-behaved.

Case: v has neighbours in multiple spokes

If v has a neighbour in 2 spokes, say P_{az} and P_{bz}, they lie in a common C_{2k+1}, and their distance using P_{ab} is bigger than 2, therefore these neighbours are a_1 and b_1. If v has a neighbour in P_{cz} as well, this neighbours must be c_1 and v acts well-behaved. If v has a neighbour in $P_{ac} \setminus \{a_T\}$, this neighbour must be u_{-1}, and $\ell_a = 2$, therefore $a_1 = a_{\ell_a - 1} \in N(v)$. If v has a neighbour in $P_{bc} \setminus \{b_T\}$, this neighbour must be u_{ℓ_c+j+1}, and $\ell_b = 2$, therefore $b_1 = b_{\ell_b - 1} \in N(v)$.

Case: v has a neighbour in exactly one spoke

If v has a neighbour in exactly one spoke, say a_i in P_{az}, then $N_T(v) \subseteq \{a_i, u_1, u_{-1}, u_h\}$ with $\ell_c + j + 1 \leqslant h \leqslant \ell_c + \ell_a + 2j - 1$. If $u_h \notin N(v)$, then either $|N(v)| = 1$ and v acts well-behaved, or $N_T(v) \subseteq \{a_{\ell_a - 1}, u_1, u_{-1}\}$ and v acts well-behaved. So assume $u_h \in N(v)$, then either $C_1 = v a_i P_{az} z_T P_{cz} c_T P_{bc} u_h v$ or $C_2 = v a_i P_{az} z_T P_{bz} b_T P_{bc} u_h v$ is odd. Assume by symmetry that C_1 is odd, then the symmetric difference of C_1 and $C_1' = va_iP_{az}a_TP_{ab}b_TP_{bc}u_hv$ is equal to the symmetric difference of C_{ab} and C_{bc}, therefore both are odd and since they must have length at least $2k+1$ all 4 of these cycles are a C_{2k+1} and $|P_{bz}| = 2$. Since now v and its neighbours are a second 2-diagonal in a C_{4k-2}, and to avoid a contradiction with Lemma 2.5.2, we must have $a_i = a_1$ and $u_h = u_{\ell_c+\ell_a+2j-1}$. Because $u_1 \in N(v)$ would be v's third neighbour in a C_{2k+1}, the only possibility for v to have 3 neighbours in T is if $N_T(v) \subseteq \{a_1, u_{-1}, u_{\ell_c+\ell_a+2j-1}\}$, but for a_1 and u_{-1} to have a common neighbour, $|P_{az}| = 2$ must hold, and therefore $a_1 = a_{\ell_a-1} \in N(v)$.

Case: v has multiple neighbours in C_T but none in $T \setminus C_T$

As we have discussed the case of v having two neighbours inside a segment of C_T already, assume that v has a neighbour in at least 2 different segments of T_C, say $u_i \in P_{ab} \setminus \{a_T, b_T\}$ and $u_h \in P_{bc} \setminus \{b_T, c_T\}$. If the cycle $C_1 = vu_iP_{ab}b_TP_{bc}u_hv$ is odd, the only option in which v is not giving rise to a short odd cycle is if v forms a 2-diagonal in the symmetric difference of C_{ab} and C_{bc} and $|P_{bz}| = 2$, but this contradicts Lemma 2.5.2 again.

Therefore, C_1 is even. Let $x = |u_iP_{ab}b_T|$ and $y = |u_hP_{bc}b_T|$, since C_1 is even, x and y have the same parity. If they are not equal, say by symmetry $y \geqslant x+2$, we can replace the longer segment to find a T with fewer vertices similar to the case where v has 2 neighbours in one segment of C_T. If $x = y > 1+j$, the symmetrical difference of C_T and C_1 contains less than $2k+1+2j-2(1+j)+2 = 2k+1$ edges, contradicting G being $\mathscr{C}_{2k-1}$-free. Therefore $x = y \leqslant 1+j$.

Since having 3 neighbours on C_T would require to segment C_T into 3 even segments, v can have at most 2 neighbours on C_T. $\square$

In the light of the above claim, it is easy to see that we must have $j \leqslant 1$, indeed, assuming $j > 1$ the set $V(T) \setminus \{a_1, a_{\ell_a-1}, b_{\ell_b-1}, c_{\ell_c-1}\}$ would consist of at least $4k-2$ vertices, but no vertex can have more than 2 neighbours in this set, contradicting $\delta(G) > \frac{1}{2k-1}n$ with a standard double counting argument.

Furthermore, if $j = 1$, and one of the spokes, say P_{az}, has length 2 the set $V(T) \setminus \{a_1, a_{\ell_a-1}, b_{\ell_b-1}, c_{\ell_c-1}\} = V(T) \setminus \{a_1, b_{\ell_b-1}, c_{\ell_c-1}\}$ leads to the same contradiction as above. Note, that any not well-behaved vertex with neighbours in at least one spoke forces at least one spoke to be of length 2, so they can not appear for $j = 1$, also a vertex with neighbours in multiple segments of C_T would give rise to an induced C_6, contradicting Lemma 2.5.4. Therefore T is well-behaved, if $j = 1$.

Finally for $j = 0$, the vertices with neighbours in multiple segments of C_T act well-behaved. If any spoke, say P_{az} has length 2, the opposite segment of C_T has the same length, so the set $S = V(T) \setminus \{a_1, u_{\ell_c+1}\}$ would once again consist of $4k-2$

vertices, such that every well-behaved vertex has just two neighbours in it. It remains to check, that no vertex that is not well-behaved has more than 2 neighbours in S to arrive at the usual contradicting.

Note, that for $j = 0$ at most 2 of the spokes may have length 2, since otherwise we would have $C_{2k+1} = C_6$. Since all vertices act well-behaved if all spokes have at least length 3, we might assume that P_{az} has length 2 and define S as above.

Considering vertices with exactly one neighbour in a spoke, if they have 3 neighbours in T, there are at least 2 spokes of length 2, such that they have both vertices in the opposite segments of C_T from the short spokes in their neighbourhood, therefore they do not have more than 2 neighbours in S.

Considering vertices with neighbours in exactly two spokes, if they have 4 neighbours in T, both the spokes they have neighbours on must have length 2, and the other two neighbours are the one on the opposite segment of C_T from the short spokes, therefore they do not have more than 2 neighbours in S. If they have 3 neighbours in T, one of the spokes they have a neighbour on must have length 2, if this is P_{az}, we are done, so assume $|P_{bz}| = 2$ as well and let v be a vertex with $N_T(v) = \{b_1, c_1, u_{\ell_c - 1}\}$. Note that $u_{-1} \in N(v)$ would imply that $|P_{cz}| = 2$ as well which we already excluded.

Now however, v does form a 2-diagonal in the symmetric difference of C_{ab} and C_{bc} with distance one from the 2-diagonal P_{bz}. By Lemma 2.4.6 there can only be very specific vertices with 3 neighbours in this C_{4k-2}, but because of $\delta(G) > \frac{1}{2k-1}n$ there must be such vertices, and by the case analysis done in the proof of Claim 2.6.6 there are only 2 cases left to consider, namely the ones where the 2 neighbours in the C_{4k-2} with distance 2 are neither both in C_T nor both outside of it.

If there is a vertex u with 3 neighbours in this C_{4k-2} such that $\{a_1, u_1\} \subseteq N(u)$, than since $k \geqslant 3$ and by Lemma 2.4.6, the third neighbour must lie on $P_{cz} \setminus \{z_T, c_1\}$, leading to u having 2 neighbours with distance more than 2 in C_{ac} and therefore a contradiction.

If there is a vertex u with 3 neighbours in this C_{4k-2} such that $\{c_{\ell_c - 1}, u_{\ell_c + \ell_a - 1}\} \subseteq N(u)$, than since $k \geqslant 3$ and by Lemma 2.4.6, the third neighbour must lie on P_{ab}, leading to u having a neighbour on P_{cz}, the opposite segment of C_T and a third neighbour, which implies that $|P_{cz}| = 2$ and therefore a contradiction. $\square$

The following corollary of Lemma 2.6.5, gives a rather precise upper bound as to how large δ can be for an odd tetrahedra to appear in G.

Corollary 2.6.7. *Let $k \geqslant 3$ and let G be a maximally $\mathscr{C}_{2k-1}$-free graph on n vertices with $\delta(G) > \frac{1}{2k-2}n$, than G does not contain any $T \in \mathscr{T}_k$.*

Proof. Let $k \geqslant 3$ and let G be a maximally $\mathscr{C}_{2k-1}$-free graph on n vertices with $\delta(G) > \frac{1}{2k-2}n$, furthermore let $T \in \mathscr{T}_k$ be a subgraph of G. Since G contains a $(2k+1)$-

tetrahedra, it must also contain one on a minimal amount of vertices. Let T^* be such a $(2k+1)$-tetrahedra.

From Lemma 2.6.5 we know, that T^* is well-behaved, and following the notation from the proof of Lemma 2.6.5 we know that no vertex has more than 2 neighbours in $V(T^*) \setminus \{a_1, a_{\ell_a-1}, b_{\ell_b-1}, c_{\ell_c-1}\}$, observing that $|V(T^*) \setminus \{a_1, a_{\ell_a-1}, b_{\ell_b-1}, c_{\ell_c-1}\}| \geqslant 4k-4 = 2(2k-2)$ together with a standard double counting argument however would contradict $\delta(G) > \frac{1}{2k-2}n$. □

In the following observation, we proof that for even k, there are indeed odd tetrahedra that may appear in $\mathscr{C}_{2k-1}$-free graphs G with $\delta(G) > \frac{1}{2k-1}n$, however we do not quite reach the upper bound on the appearance of odd tetrahedra of $\frac{1}{2k-2}n$ from Corollary 2.6.7. We do currently not know, which should be the exact threshold here.

Observation 2.6.8. *For $k \geqslant 4$ even there are $\mathscr{C}_{2k-1}$-free graphs on n vertices with minimum degree $\delta(G) = \frac{3}{6k-4}n > \frac{1}{2k-1}n$, that contain a $T \in \mathscr{T}_k$ for arbitrary large n.*

Proof. Let $k \geqslant 4$ and even be given. Consider the $(2k+1)$-tetrahedra constructed by replacing 2 not adjacent edges in a K_4 with a path of length $2k-5$ and all the other edges of the K_4 by a P_3. Now let T_{even} be the graph obtained from this $(2k+1)$-tetrahedra by blowing up with a factor of $2 \cdot f$ (for arbitrarily large f) all the vertices with degree 3 as well as all the vertices with distance $3 + 4 \cdot \ell$ or $4 + 4 \cdot \ell$ for some $\ell \geqslant 0$ to the closest vertex with degree 3, and blowing up all the other vertices with a factor of $1 \cdot f$ (see Figure 2.6.2).

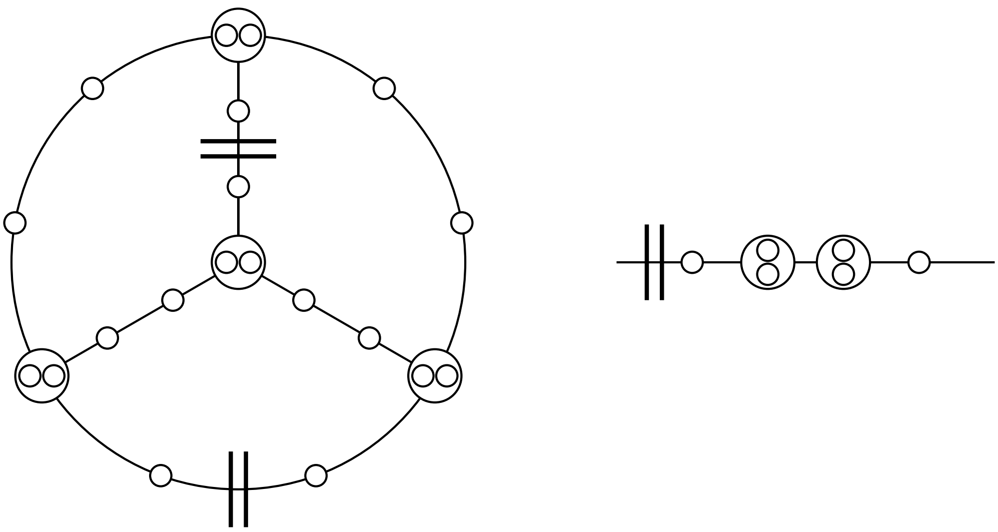


Figure 2.6.2: Depicted on the left is T_{even} for $k = 4$. By replacing the edges indicated by $||$ in a T_{even} with the path on the right for some k, one obtains the T_{even} for the next even k.

It is easy to see that T_{even} is $\mathscr{C}_{2k-1}$ free, 3-regular, and contains $6k-4$ vertices (for the last one inductive thinking is helpful, see Figure 2.6.2).

By Lemma 2.5.1, balanced blow-ups of T_{even} therefore prove this observation. □

In the following observation there is no gap between the bound proven and the one provided by Corollary 2.6.7, making this bound optimal for odd k at least. Other than that it is quite similar to Observation 2.6.8 with a slightly different blow-up.

Observation 2.6.9. *For $k \geqslant 5$ odd there is a sequence of $\mathscr{C}_{2k-1}$-free graphs on n vertices where the minimum degree converges to $\frac{1}{2k-2}n > \frac{1}{2k-1}n$ as n tends to infinity that each contain a $T \in \mathscr{T}_k$.*

Proof. Let $k \geqslant 5$ and odd be given. Let T_{odd} be the $(2k+1)$-tetrahedra constructed by replacing 2 not adjacent edges in a K_4 with a path of length $2k-5$ and all the other edges of the K_4 by a P_3 (see Figure 2.6.3).

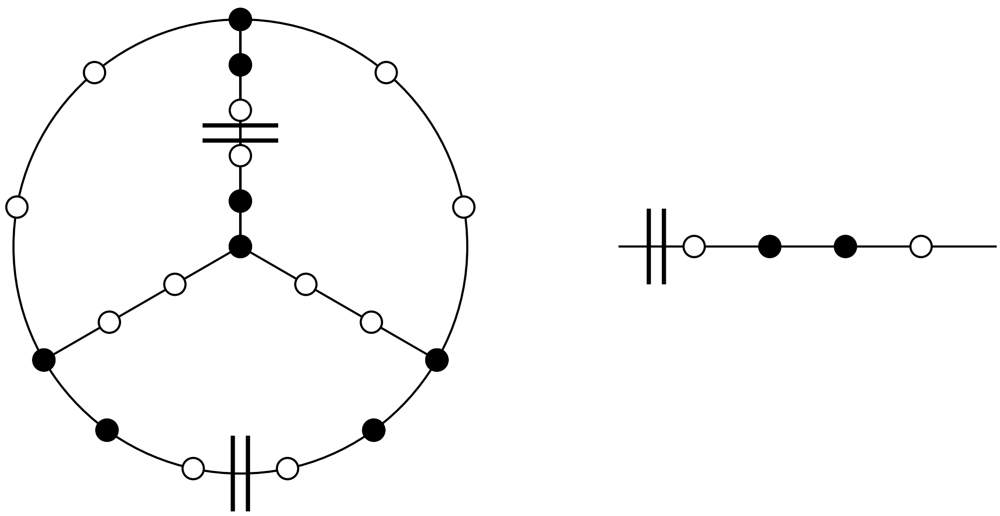


Figure 2.6.3: Depicted on the left is T_{odd} for $k = 5$. By replacing the edges indicated by $||$ in a T_{odd} with the path on the right for some k, one obtains the T_{odd} for the next odd k. The black vertices are the ones which will be blown up indefinetely in the proof of Observation 2.6.9.

Let S be the set of vertices, containing all the vertices with degree 3 as well as all the vertices with distance $3 + 4 \cdot \ell$ or $4 + 4 \cdot \ell$ for some $\ell \geqslant 0$ to the closest vertex with degree 3 in T_{odd} (see the black vertices in Figure 2.6.3).

It is easy to see that T_{odd} is $\mathscr{C}_{2k-1}$ free, every vertex has exactly one neighbour in S, and S contains $2k-2$ vertices (for the last one inductive thinking is helpful, see Figure 2.6.3).

Blowing up the vertices in S with a blow-up factor of f and the other vertices with a factor of 1, the minimum degree of these blow-ups will converge to f, and the size of the whole graph will converge to $|S| = f(2k-2)$ as f tends to infinity. By construction, these blow-ups will also stay $\mathscr{C}_{2k-1}$-free, and therefore prove this observation. □

2.7 Proof of the more detailed version of the main theorem for $k = 3$

In this final section of the chapter, we will proof Theorem 1.2.6. Both lemmas and the theorem in this chapter closely resemble the ones from [34], however there are some differences which we will point out. Also, note that while our statements are all made for $k \leqslant 3$ in this chapter, this is solely due to Lemma 2.5.6 and if Conjecture 4.2.2 (which states that Lemma 2.5.6 holds for arbitrarily large k) turns out to be true, this upper bound on k might be lifted on this chapter without any other change.

For the first lemma of this section, our proof is quite differnt from the one provided in [34], as the cases for larger k would grow indefinitely, which makes them too many to handle manually. It will be rather helpful in showing that any "incomplete" Andrásfai graph showing up in G, after closer consideration must be complete after all.

Lemma 2.7.1. *Let $3 \leqslant k \leqslant 3$ and let G be a maximal $\mathscr{C}_{2k-1}$-free graph on n vertices with $\delta(G) > \frac{1}{2k-1}n$.*

Furthermore let $C = c_0c_1 \ldots c_{4k-1}c_0$ be a C_{4k} with at least two added diagonals, then $G[V(C)]$ is isomorphic to one of the following two.

1. *An M_{4k} where $c_i \longrightarrow m_i$, or*
2. *an M_{4k} where $c_i \longrightarrow m_i$ for $2k \leqslant i \leqslant 4k-1$, and $c_i \longrightarrow m_{2k-1-i}$ for $0 \leqslant i \leqslant 2k-1$.*

Where in the latter case the two diagonals must be c_0c_{2k} and $c_{4k-1}c_{2k-1}$.

Proof. Note that the upper bound restriction on k originates from Lemma 2.5.6 and does not appear elsewhere in the proof.

Assume that $G[V(C)]$ does contain two diagonals with distance at least 2, then by Lemma 2.4.4 C is semi well-behaved with respect to an M_{4k} as in item 1 of the lemma. If there is at least one diagonal missing, we pick a vertex of C without the corresponding diagonal and move along C on both directions until we hit a vertex with a diagonal on both ends, using these diagonals and afterwards changing direction on C these walks will meet at the vertex with distance $2k$ from the starting vertex. This walk will then define an induced even cycle of length at most $4k-2$, contradicting Lemma 2.5.9.

Assume that $G[V(C)]$ does contain exactly two diagonals which are adjacent, by symmetry let these diagonals be c_0c_{2k} and $c_{4k-1}c_{2k-1}$.

Since $U = V(C) \setminus \{c_0, c_{2k}\}$ is a set of $4k-2$ vertices, there must be a vertex v with at least 3 neighbours inside U. By Lemma 2.4.5 we know that $N_C(v)$ is a superset of $N_C(c_i)$ for some vertex v_i with $2 \leqslant i \leqslant 4k-2, i \neq 2k-1, 2k, 2k+1$. Exchanging this c_i for v to create a new C_{4k} with 3 diagonals contradicts the first part of this proof since

$k \geqslant 3$ and therefore $4k/2 \geqslant 6$ which implies that there is a diagonal missing in this new C_{4k}. □

The following lemma, which appeared in [34] without the restriction on odd tetrahedra is the reason the original conjecture of Letzter and Snyder, that all $\mathscr{C}_{2k-1}$-free graphs G with $\delta(G) > \alpha_{\delta_{\hom}(\mathscr{C}_{2k-1})}|V(G)| = \frac{1}{2k-1}|V(G)|$ are homomorphic to an Andrásfai graph $A_{k,r}$, does not hold for general k. Our proof centres around the appearance of such an odd tetrahedra, and is therefore quite different from the one provided by Letzter and Snyder.

Lemma 2.7.2. *Let $3 \leqslant k \leqslant 3$ and let G be a maximal $\mathscr{C}_{2k-1}$-free graph on n vertices with $\delta(G) > \frac{1}{2k-1}n$.*

If G does not contain an odd Tetrahedra from $\mathscr{T}_k$ as a subgraph, then every vertex in G has a neighbour in every C_{2k+1} in G.

Proof. Note that the upper bound restriction on k originates from Lemma 2.5.6 and does not appear elsewhere in the proof.

Assume there is a vertex $v' \in V$ and a cycle $C = c_0c_1 \dots c_{2k}c_0$ in G, such that v' does not have any neighbours on C.

Since G is maximal $\mathscr{C}_{2k-1}$-free, G must be connected, otherwise an edge that is a bridge could be added without creating an additional cycle. Therefore, there is a shortest path from v' to $V(C)$, and on this path there is a first vertex v that does not have any neighbour on C. By construction v has a common neighbour u with a vertex of C, and by symmetry we may assume this vertex of C to be c_0.

As v does not have any neighbours on C, it does not have any neighbours in $\{c_1, c_3, c_{-1}, c_{-3}\}$ either. Owing to the maximality of G there must be even paths P_1, P_3, P_{-1}, and P_{-3} connecting these vertices with v (see Figure 2.7.1 on the left for an illustration).

We will construct an odd tetrahedra from these paths along C and $\{v, u\}$. In the first step we will prove the existence of the necessary edges and in the second step we will prove that they are disjoint where they need to.

Assume there is a path P from v to c_3 (to c_{-3}) of length $2k-4$, then the walk $C^+ = vuc_0c_1c_2c_3Pv$ (the walk $C^- = vuc_0c_{-1}c_{-2}c_{-3}Pv$) has length $2k+1$, contains vuc_0, and intersects with C in at least two incident edges. We will then use it as a building block of our odd tetrahedra.

If there is no such path then P_3 must have length $2k-2$ since the above constructed odd walk, by replacing P with P_3, would otherwise have a length of at most $2k-1$ leading to the existence of a short odd cycle in G.

Apparently both P_1 and P_{-1} must have length $2k-2$ exactly. Furthermore P_3 and P_1 must be induced paths, since otherwise we would find a short odd cycle or a path of length $2k-4$ from v to c_3. Considering the closed even walk $vP_1c_1c_2c_3P_3v$ of length

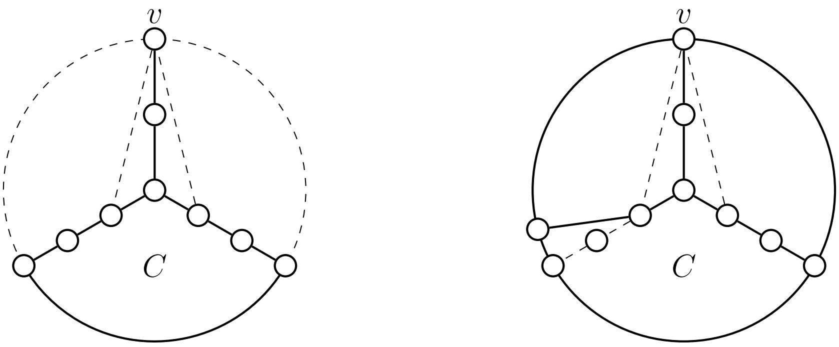


Figure 2.7.1: On the left the non-edges from v to a C_{2k+1} that yield relatively short even paths. On the right the from v and C constructed odd tetrahedra.

$4k-2$ there must be either less than $4k-2$ vertices or more induced edges, otherwise it would contradict Lemma 2.5.9. Since subtracting a closed odd walk from a closed even walk yields another closed odd walk and $4k-2 < 2(2k+1)$ all closed subwalks must be even.

For a more precise notation, let $P_1 = vx_1x_2 \dots x_{2k-3}c_1$ and let $P_3 = vz_1z_2 \dots z_{2k-3}c_1$. We will find a vertex c_2' that has both c_1 and c_3 as a neighbour and an odd path P of length $2k-3$ to v.

Consider the walk $W = x_{2k-4}x_{2k-3}c_1c_2c_3z_{2k-3}z_{2k-4}$ of length 6. If it is an induced path consisting of 7 vertices, it will be part of an induced even cycle of length at least 6 and at most $4k-2$, contradicting Lemma 2.5.9, since no vertex from $P_1 \setminus W$ can have neighbours in $W \setminus P_3$ and the other way around without creating a short odd cycle.

The only way for W to contain less than 7 vertices is by either identifying x_{2k-3} and z_{2k-3}, or identifying x_{2k-4} and z_{2k-4}. In the former case, this new vertex would be a neighbour of both c_1 and c_3 and have an odd path of length $2k-3$ to v, so we found a vertex c_2'. In the latter case, W would be a C_6 and should therefore not be induced, leading to additional edges being present in W.

If there are additional edges in W, the only ones that can be present without creating a short odd cycle are the ones from $\{x_{2k-4}c_2, z_{2k-4}c_2, x_{2k-3}c_3, z_{2k-3}c_1\}$. In the former two cases c_2 satisfies all the properties we want c_2' to satisfy. In the latter two cases x_{2k-3} or z_{2k-3} satisfies all the properties we want c_2' to satisfy.

After we ensured the existence of a vertex c_2', the walk $C' = c_2'c_3c_4 \dots c_{2k}c_0c_1c_2'$ has length $2k+1$. Furthermore, the walk $C^+ = vPc_2'c_1c_0uv$ has length $2k+1$, contains vuc_0, and intersects with C' in at least two incident edges.

By symmetry, if C^- was not defined beforehand, we find a vertex c_{-2}' and possibly after modifying C' to C'' we can define a walk $C^- = vPc_{-2}'c_{-1}c_0uv$ that has length $2k+1$, contains vuc_0, and intersects with C'' in at least two incident edges (see Figure 2.7.1 on the right for an illustration).

If the 3 walks C'', C^+ and C^- indeed intersect in these 2 or 3 consecutive edges with each other and nowhere else, they obviously form an odd tetrahedra with the spokes being the intersections.

First we notice, that these closed odd walks of length $2k+1$ must be cycles, since otherwise they would contain a shorter odd cycle, contradicting the odd girth of G. Next we ensure, that these cycles do indeed form an odd tetrahedra together.

Claim 2.7.3. *If two C_{2k+1} are intersecting in at least* 1 *edge, they are identical, or there is a subset of their vertices, such that this subset induces two C_{2k+1} and their intersection is a path.*

Proof. Let $A = a_0a_1 \dots a_{2k}a_0$ and $B = b_0b_1 \dots b_{2k}b_0$ be the given two C_{2k+1} and assume that $a_0 = b_0$ as well as $a_1 = b_1$ is the intersecting edge.

Assume $a_i = b_j$ for indices i and j such that at least one of them is neither 0 nor 1. Obviously $i, j > 1$ and if $i \neq j$ either the closed walk $W = a_1a_2 \dots a_ib_{j-1}b_{j-2} \dots a_1$ is odd, in which case either W or $(A \cup B) \setminus W$ would contain a short odd cycle, contradicting the odd girth of G. Or W is even, but then the paths from a_1 to a_i would have the same parity but different length since $i \neq j$, however assuming that $|a_1a_2 \dots a_i| > |b_1b_2 \dots b_j|$ leads to the short odd cycle $a_0b_1b_2 \dots b_ja_{i+1}a_{i+2} \dots a_{2k}a_0$, and $|a_1a_2 \dots a_i| < |b_1b_2 \dots b_j|$ to an analogous contradiction.

Therefore $i = j$, which implies $|a_1a_2 \dots a_i| = |b_1b_2 \dots b_j|$ and we may discard $\{b_2, b_3, \dots, b_{j-1}\}$. Iterating this process for new intersections leads to the cycles being identical or fulfilling the claimed property.

If the starting cycles were not identical but by greedily discarding vertices the resulting cycles are, with a more careful discarding order it is easy to ensure that the resulting cycles are not identical as well, just go back to the last step where vertices were discarded and add them back into the picture. $\square$

With this claim we know that the constructed cycles indeed intersect in paths, and by construction these paths have a length of at least 2 and all 3 intersect in c_0, which will be our central vertex. The only reason we might not use the intersection paths as spokes and the end vertices of these paths as branch vertices, and $C'' \cup C^+ \cup C^-$ without c_0 and the inner vertices of the spokes as cycle C_T, is that the 3 paths may be intersecting in more than just c_0.

But since by construction $u \notin C$ and apparently $c_1 \notin C^-$ as well as $c_{-1} \notin C^+$, using the proof of Claim 2.7.3 we will expand the given intersections of the cycles "away from c_0" such that we will have 3 distinct cycles which each intersect in a path afterwards.

Assuming the third cycle would intersect with the intersecting path of the other two, by symmetry let C'' intersect in $C^+ \cap C^-$, by construction it would need to intersect with the whole of $C^+ \setminus C^-$ as well as the whole of $C^- \setminus C^+$. But since C'' is a cycle and therefore does not contain a vertex of degree greater than 2, $|C'' \cap C^+ \cap C^-| = 2$ and C'' would be the union of C^+ and C^- without the inner vertices of their intersecting path. This however would imply that C'' is an even cycle, contradicting C'' being a C_{2k+1}. $\square$

Just like Letzter and Snyder did, we will not prove Theorem 1.2.6 directly, but instead prove the following theorem which clearly implies Theorem 1.2.6.

This proof is basically the same as the one appearing in [34], just extended to general k, but there is no surprise, and everything works out just as in the $k = 3$ case. We merely include this proof for completeness and the nice pictures at the very end. Note that the k used in [34] is called r in this proof.

Theorem 2.7.4. *Let $3 \leqslant k \leqslant 3$ and let G be a maximal $\mathscr{C}_{2k-1}$-free graph on n vertices with $\delta(G) > \frac{1}{2k-1}n$.*

If G does neither contain an odd Tetrahedra from $\mathscr{T}_k$ nor an $A_{k,r+1}$ as a subgraph, then G is homomorphic to $A_{k,r}$.

Proof. Note that the upper bound restriction on k originates from Lemma 2.5.6 and does not appear elsewhere in the proof.

We will use induction to prove this theorem. For $r = 1$, G not containing a $A_{k,r+1}$ implies that G is bipartite and therefore homomorphic to $A_{k,1} = K_2$. To see this, assume that G does not contain a $A_{k,r+1} = C_{2k+1}$ as a subgraph, but is not bipartite. Obviously, G must contain an odd cycle, let C be the shortest odd cycle in G. Since G is $\mathscr{C}_{2k+1}$-free this cycle C has length at least $2k + 3$. Consider two vertices v and v' with distance 3 on C, they cannot be neighbours, since otherwise there would be a shorter odd cycle contained in G, but owing to G being maximal $\mathscr{C}_{2k+1}$-free, there must be an even path P of length at most $2k - 2$ between these to vertices for them not to be neighbours. Now however the odd cycle $vPv'Cv$ has length at most $2k + 1$ contradicting C being the shortest odd cycle in G.

Recall that any Andrásfai graph $A_{k,r}$ is well-behaved in G by Lemma 2.4.3 and r-regular by Proposition 2.1.2 (a). For the latter cases the following claim will be useful.

Claim 2.7.5. *Every vertex of G has either r or $r - 1$ neighbours in any $A_{k,r}$ that is a subgraph of G. Furthermore for every vertex v that has $r - 1$ neighbours in an $A_{k,r}$ there is a vertex $u_i \in A_{k,r}$ such that $N_{A_{k,r}}(v) = N_{A_{k,r}}(u_i) \setminus u_{i-1}$, where the u_j are ordered like in the proof of Proposition 2.1.2 (c).*

Proof. Let $A = u_0u_1 \dots u_{(2k-1)(r-1)+2}u_0$ be an $A_{k,r}$ with the vertex order from the proof of Proposition 2.1.2 (c).

For $r \in \{1, 2\}$ the claim is trivial since every $A_{k,r}$ is well-behaved in G by Lemma 2.4.3 and Lemma 2.7.2 holds.

For $r \geqslant 3$ let v be a vertex of G. Because of Lemma 2.4.3 if v has fewer than $r - 1$ neighbours in A, there will be $2k + 1$ vertices with consecutive indices such that v is a neighbour to none of them. But from the proof of Proposition 2.1.2 (c) we know that these vertices of A induce a C_{2k+1}, contradicting Lemma 2.7.2.

If v has exactly $r-1$ neighbours in A we will once again find $2k+1$ consecutive vertices such that v is a neighbour to none of them unless $N_{A_{k,r}}(v) = N_{A_{k,r}}(u_i) \setminus u_{i-1}$ or $N_{A_{k,r}}(v) = N_{A_{k,r}}(u_i) \setminus u_{i+1}$. However in the latter case, we might interpret the neighbours of v to be of the form $N_{A_{k,r}}(u_{i+(2k-1)}) \setminus u_{(i+2k-1)-1}$ by symmetry.

Because of Lemma 2.4.3 no vertex of G can have more than r neighbours in A, and this finishes this proof. $\square$

We proceed with our induction on r, so let $r \geqslant 2$ be given. First we ensure that G does contain a copy of $A_{k,r}$. Assume that $A_{k,r}$ is not a subgraph of G, then, by induction hypothesis, G is homomorphic to $A_{k,r-1}$, but since $A_{k,r-1}$ is a subgraph of $A_{k,r}$ this implies that G is homomorphic to $A_{k,r}$ as well and we are done.

Let A^* be a vertex maximal blow-up of $A_{k,r}$ that is a subgraph of G. For easier notation, let $A^* \supseteq A = u_0u_1 \dots u_{(2k-1)(r-1)+1}u_0$ be an $A_{k,r}$ with the vertex order from the proof of Proposition 2.1.2 (c), such that $u_i \in U_i$, where $U_i, 0 \leqslant i \leqslant (2k-1)(r-1)+1$ are the independent blow-up sets of A^*.

If $V(A^*) = V(G)$ we are done, so assume there is a vertex $v \in V(G) \setminus V(A^*)$. Claim 2.7.5 now ensures, that v has neighbours in r or $r-1$ classes U_i in A^*, since otherwise by a suitable choice of vertices from each class there would be an $A_{k,r}$ containing less than $r-1$ or more than r neighbours of v.

We will first consider the case that v has neighbours in r classes U_i, and make the following claim.

Claim 2.7.6. *If v has neighbours in r classes U_i, it is joined to all the vertices of all these classes.*

Proof. By symmetry we may assume that v has r neighbours in A and these r neighbours coincide with the neighbours of u_0 in A. If there is a non-neighbour in any of the classes apart from U_{-1} or U_1 this non-neighbour gives rise to an $A_{k,r}$ contradicting Claim 2.7.5 together with v. By symmetry assume there is a vertex $u_1' \in U_1$ that is a non-neighbour of v (see Figure 2.7.2 on the left for an illustration).

Since G is maximal $\mathscr{C}_{2k-1}$-free, the absence of the edge $u_1'v$ implies the existence of an even path of length at most $2k-2$ between v and u_1', but considering the path $vu_1u_0u_1'$ this path $P = u_1'p_1p_2 \dots p_{2k-3}v$ must have length exactly $2k-2$.

The cycle $C = u_0u_1'Pvu_1u_2 \dots u_{(2k-1)(r-1)+1}u_0$ consists of exactly $(2k-1)(r-1)+2+(2k-1)$ vertices, and we will show that it is isomorphic to an $A_{k,r+1}$, contradicting the assumption that G does not contain a copy of $A_{k,r+1}$.

It is easy to see that all the edges that should be present in an $A_{k,r+1}$ on C that are not incident to one of the vertices from P are inherited from A being an $A_{k,r}$. Indeed, recalling the proof of Proposition 2.1.2 (c) and traversing C "backwards" from u_0 all

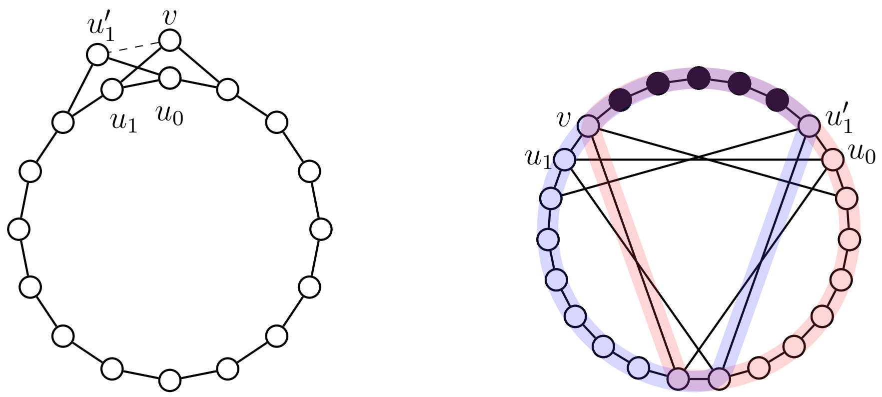


Figure 2.7.2: On the left the graph $A_{4,3}$ with additional vertices v and u_1' and a dashed non-edge. On the right the constructed subgraph of $A_{4,4}$ with the added vertices marked in black. In both pictures most of the chords were not drawn for clarity. The coloured coatings mark possible $A_{4,3}$ subgraphs including the new vertices.

the distances of neighbours are still the same until the path P is reached. If during the consideration of neighbours along C one crosses P, the fact that it consists of $2k-1$ vertices implies that there was simply one neighbour left out that should have been present.

Considering the cycle $C_0 = u_1'Pvu_1Cu_{2k+1}u_1'$ (blue cycle on the right of Figure 2.7.2) notice that is has $4k$ vertices and the 3 diagonals $u_1'u_2$, vu_{2k}, and u_1u_{2k+1} are present. By Lemma 2.7.1 all the other diagonals must be present as well. To ensure that these additional edges are the ones that should be present in an $A_{k,r+1}$, consider adding these diagonals one at a time, starting with the one adjacent to the diagonal $D = u_1'u_2$. Since D is part of the $A_{k,r+1}$, its end vertices have distance $2k + j(2k-1)$ for some $j < r-1$ on C, namely $j = 0$, and by shifting one vertex along C on both ends, this distance stays the same. therefore, all the diagonals are between vertices with the right distance on C to each other and must be part of an $A_{k,r+1}$.

Let $1 \leqslant i \leqslant r-2$. Now considering the cycle $C_i = u_1'Pvu_{1+i(2k-1)}Cu_{2k+1+i(2k-1)}u_1'$, recall that u_0, u_1, v, and u_1' each have r neighbours in A, the 3 diagonals $u_1'u_{2+i(2k-1)}$, $vu_{2k+i(2k-1)}$, and $u_{1+i(2k-1)}u_{2k+1+i(2k-1)}$ are present. Once again, Lemma 2.7.1 ensures the existence of all the other diagonals, and by adding them one at a time it is clear, that they belong in an $A_{k,r+1}$ (see the red cycle in Figure 2.7.2 for an illustration of the case $i = r-2$).

Since in each of the $r-1$ steps we added $2k-3$ diagonals, one for each vertex of the $2k-3$ inner vertices of P, each of these is now incident to $2 + (r-1) = r+1$ distinct edges that all are part of an $A_{k,r+1}$, so there is no more missing edge, and C is indeed isomorphic to an $A_{k,r+1}$. This however contradicts the assumption of G not containing a copy of an $A_{k,r+1}$. $\square$

Following Claim 2.7.6 however, we could add the vertex v to A^* enlarging the size of a vertex maximal blow-up, a contradiction. Therefore, any vertex $v \in V(G) \setminus V(A^*)$ must have neighbours in exactly $r-1$ classes U_i of A^*. Owing to Claim 2.7.5 we might assume that there is a set W_j of vertices that have neighbours in the same classes U_i as u_j with the exception of U_{j-1} and that these sets W_j for $0 \leqslant j \leqslant (2k-1)(r-1)+1$ partition $V(G) \setminus V(A^*)$. We will now formulate our last Claim of this chapter.

Claim 2.7.7. *If $u_s u_t$ is a non-edge, $W_s \cup W_t$ is an independent set.*

Proof. Assume there is an edge contradicting this claim. By symmetry we might assume one of the end vertices to be $w_0 \in W_0$. Let the other end vertex be $w_z \in W_z$. First we note, that $z \neq 0$, since every vertex in U_j is a neighbour of u_{j+1} and therefore U_j is an independent set for any $0 \leqslant j \leqslant (2k-1)(r-1)+1$.

For a simpler notation let $C = u_0 u_1 \dots u_{(2k-1)(r-1)+1} u_0$ be the Hamiltonian cycle of A from the proof of Proposition 2.1.2 (c).

For $z = x + y(2k-1)$, with $2 \leqslant x \leqslant 2k-3$ and $0 \leqslant y \leqslant r-2$ consider the cycle $C^* = w_0 u_1 u_{2+y(2k-1)} C u_{z+1} w_z w_0$. It is induced and has length at least $1+1+1+1+1 = 5$ and at most $1+1+(x-1)+1+1 = x+3 \leqslant 2k$ (see red cycles in Figure 2.7.3 on the left and on the right), contradicting either G being $\mathscr{C}_{2k-1}$-free or Lemma 2.5.9 since $2k < 4k-2$ for $k \geqslant 3$.

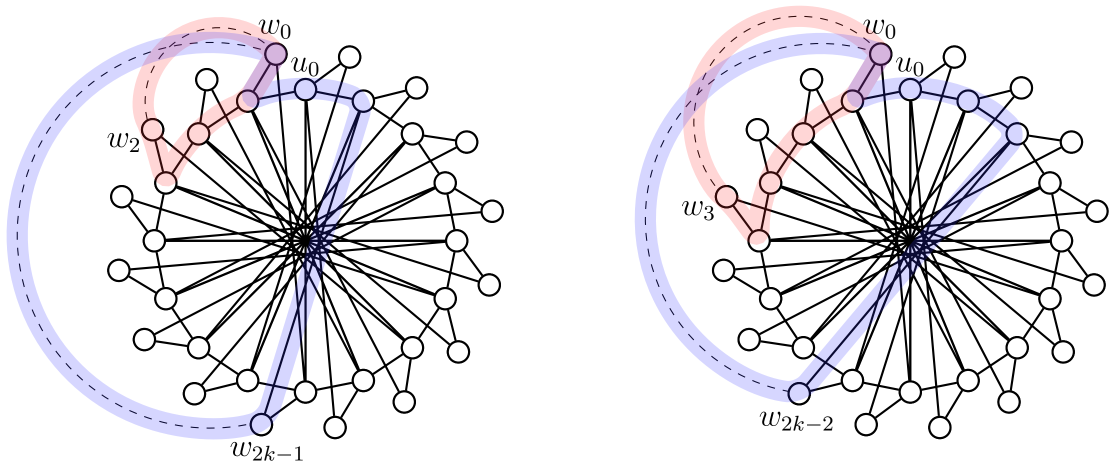


Figure 2.7.3: The graph A with representatives from each set W_j. On the left an odd induced cycle for small (red) x and large (blue) x. On the right an even induced cycle for small (red) x and large (blue) x.

For $z = x + y(2k-1)$, with $2k-2 \leqslant x \leqslant 2k-1$ and $0 \leqslant y \leqslant r-2$ consider the cycle $C^* = w_0 u_1 C u_{x-2k} w_z w_0$. It is induced and has length at least $1+2+1+1 = 5$ and at most $1+3+1+1 = 6$ (see blue cycles in Figure 2.7.3 on the left and on the right), contradicting either G being $\mathscr{C}_{2k-1}$-free or Lemma 2.5.9 since $6 < 4k-2$ for $k \geqslant 3$.

Therefore z must be of the form $1 + y(2k-1)$ with $0 \leqslant y \leqslant r-1$, but then $u_0 u_z$ is an edge and $w_0 w_z$ can not contradict this claim. $\square$

Let $U_i' = U_i \cup W_i$ for $0 \leqslant i \leqslant (2k-1)(r-1)+1$. Note that these sets are independent by construction and partition $V(G)$ due to Claim 2.7.5, Claim 2.7.6, and the definition of A^*. Claim 2.7.7 ensures that there is no edge between U_i' and U_j' whenever u_iu_j is a non-edge, the maximality of G then implies that all the edges between U_i' and U_j' are present whenever u_iu_j is an edge, implying that G is a blow-up of $A_{k,r}$. This obviously makes G homomorphic to $A_{k,r}$ and therefore finishes the proof of Theorem 2.7.4. $\square$

3. Spanning subgraphs in uniformly dense and inseparable graphs

The proof of Theorem 1.3.6 utilises the absorption method of Rödl, Ruciński, and Szemerédi [47]. We discuss this approach and give the details of the proof in Section 3.3. For the proof we use some observations on uniformly dense and inseparable graphs, which we collect in Section 3.1 and Section 3.2.

Theorem 1.3.7 follows from Theorem 1.3.6 combined with Szemerédi's Regularity Lemma [51] and the accompanying Blow-up Lemma [30]. Similar reductions appeared in the proofs of the bandwidth theorems in [9, 50], and we give the proof in Section 3.4.

In Section 3.5 we will then conclude this chapter with some research regarding the common generalisation of Theorem 1.3.2 and Theorem 1.3.6.

3.1 Properties of uniformly dense graphs

In this section we shall explore some properties of uniformly dense graphs that are crucial for the proof of Theorem 1.3.6. In the latter half of this section, we will then state some properties of uniformly dense graphs that will come into play in Section 3.4.

We start with the following well known fact that uniformly dense graphs contain many cliques of given size.

Lemma 3.1.1. *For every $k \in \mathbb{N}$, $d \in [0,1]$, and $\varrho > 0$, every (ϱ, d)-dense n-vertex graph contains at least $\big(d^{\binom{k}{2}} - (k-1)k\varrho\big)n^k$ ordered copies of K_k.*

Proof. Let $G = (V, E)$ be a (ϱ, d)-dense graph and $|V| = n$. For $k = 1$, the assertion is trivial. For $k = 2$, we are counting the number of edges twice. Since G is (ϱ, d)-dense, we have $2|E| \geqslant 2(d/2 - \varrho)n^2$ and the lemma follows.

We continue by induction. Let $k \geqslant 2$ and assume that for every ϱ', d', it is true that every (ϱ', d')-dense graph H contains at least $\big(d'^{\binom{k}{2}} - (k-1)k\varrho'\big)|V(H)|^k$ ordered copies of K_k. For counting the ordered copies of K_{k+1} in G, consider the subset $V^* \subseteq V$ of the vertices $v \in V$ with $|N(v)| \geqslant 1$. Let $\hom(K_{k+1}, G)$ denote the number of ordered copies of K_{k+1} in G. Consequently, we have that

$$\hom(K_{k+1}, G) = \sum_{v \in V^*} \hom(K_k, G[N(v)]).$$

Since G is (ϱ, d)-dense, for every $v \in V^*$ and $X \subseteq N(v)$ we have

$$e(X) \geqslant \frac{d}{2}|X|^2 - \varrho n^2 = \frac{d}{2}|X|^2 - \varrho_v |N(v)|^2,$$

for $\varrho_v = \varrho n^2/|N(v)|^2$. Thus $G[N(v)]$ is (ϱ_v, d)-dense and we can apply the induction hypothesis to get

$$\begin{aligned}
\hom(K_{k+1}, G) &\geqslant \sum_{v \in V^*} \left(d^{\binom{k}{2}} - (k-1)k\varrho_v\right)|N(v)|^k \\
&= d^{\binom{k}{2}} \sum_{v \in V^*} |N(v)|^k - (k-1)k \sum_{v \in V^*} \varrho_v |N(v)|^k \\
&\geqslant d^{\binom{k}{2}} |V^*| \left(\frac{(d-2\varrho)n^2}{|V^*|}\right)^k - (k-1)k \sum_{v \in V^*} \frac{\varrho n^2}{|N(v)|^2}|N(v)|^k,
\end{aligned}$$

where the last estimate employed Jensen's inequality and $\sum_{v \in V^*} N(v) = 2|E| \geqslant (d - 2\varrho)n^2$. Hence, from $k \geqslant 2$ we derive

$$\begin{aligned}
\hom(K_{k+1}, G) &\geqslant d^{\binom{k}{2}}(d-2\varrho)^k n^{k+1} - (k-1)k\varrho n^{k+1} \\
&\geqslant d^{\binom{k}{2}}(d^k - 2k\varrho)n^{k+1} - (k-1)k\varrho n^{k+1} \\
&\geqslant \left(d^{\binom{k+1}{2}} - k(k+1)\varrho\right)n^{k+1},
\end{aligned}$$

which concludes the proof of the lemma. □

As a corollary, we obtain the following result, which ensures the existence of fairly long k-paths in uniformly dense graphs. These k-paths will be the building blocks for an almost perfect k-path cover in the proof of Theorem 1.3.6. In that proof, we will connect these k-paths to an almost spanning k-path. For the connection, it will be convenient to insist that the ends of the k-paths are contained in many K_{k+1}'s. For that we say a clique K_k is *ζ-connectable* in G if it is contained in at least $\zeta|V(G)|$ cliques of order $k+1$.

Corollary 3.1.2 (Path Lemma)**.** *For every $d \in (0, 1]$ and positive integer k, there exist $\varrho, \zeta > 0$, and n_0 such that if G is a (ϱ, d)-dense graph on $n \geqslant n_0$ vertices, then G contains a k-path P with ζn vertices, where every consecutive K_k in P is ζ-connectable.*

Proof. Given $d \in (0, 1]$ and a positive integer k we define the constants

$$\varrho = \frac{d^{\binom{k+1}{2}}}{2k(k+1)} \qquad \text{and} \qquad \zeta = \frac{d^{\binom{k+1}{2}}}{3(k+1)}. \tag{3.1.1}$$

Let $G = (V, E)$ be a (ϱ, d)-dense graph with $|V| = n$ sufficiently large. Applying Lemma 3.1.1 and considering the choice of constants in (3.1.1) show that the number of ordered copies of K_{k+1} in G is at least

$$\left(d^{\binom{k+1}{2}} - k(k+1)\varrho\right)n^{k+1} = \frac{d^{\binom{k+1}{2}}}{2}n^{k+1}. \tag{3.1.2}$$

Define the auxiliary $(k+1)$-uniform hypergraph $\mathcal{H}_0$ with $V(\mathcal{H}_0) = V$ and

$$E(\mathcal{H}_0) = \left\{e \in V^{(k+1)} \colon e \text{ spans a } K_{k+1} \text{ in } G\right\}.$$

Successively remove the hyperedges of $\mathcal{H}_0$ which contain a k-tuple that is in at most ζn hyperedges and let $\mathcal{H}$ be the resulting subhypergraph. Note that the number of erased edges is at most

$$\binom{n}{k} \cdot \zeta n \overset{(3.1.1)}{<} \frac{d^{\binom{k+1}{2}}}{2(k+1)!}n^{k+1} \overset{(3.1.2)}{\leqslant} |E(\mathcal{H}_0)|.$$

Every k-tuple of vertices of G which is contained in some edge of $\mathcal{H}$ is now ζ-connectable in G.

Consider *tight paths* in $\mathcal{H}$, which are subhypergraphs P with $V(P) = \{x_1, \ldots, x_\ell\}$ and $e \in E(P)$ if and only if $e = \{x_i, x_{i+1}, \ldots, x_{i+k}\}$ for every $i = 1, \ldots, \ell - k$. In particular, consecutive hyperedges in such a path intersect in k vertices. Observe that any tight path in $\mathcal{H}$ induces a k-path in G with every consecutive K_k being ζ-connectable.

Take the longest tight path P_0 in $\mathcal{H}$. Let K be the set of the last k vertices in P_0. If $e \in E(\mathcal{H})$ is of the form $e = K \cup \{u\}$ for some $u \in V$, then u is already contained in P_0, otherwise the tight path could be enlarged. Since every k-tuple contained in some hyperedge of $\mathcal{H}$ is in at least ζn hyperedges, we know that P_0 has at least ζn vertices. □

Here we start the part of the section that is needed for Section 3.4, and the readers that are mainly interested in the proof of Theorem 1.3.6 might proceed to Section 3.2 and come back here later.

First we state that a balanced blow-up of a graph inherits its property to be (ϱ, d)-dense with slightly worse d that is, however, independent of the blow-up factor.

Lemma 3.1.3. *Let $G = (V_G, E_G)$ be a (ϱ, d)-dense n-vertex graph, and $H = (V_H, E_H)$ be a balanced blow-up of G. Then there is an integer n_0 such that H is $(\varrho, d/2)$-dense, if $n \geqslant n_0$.*

Proof. Let f be the blow-up factor of H. Suppose there is a set $X_H \subseteq V_H$ contradicting the $(\varrho, d/2)$-denseness of H.

Since X_H may contain up to f copies of each vertex of G, we will assign to each vertex $v_i \in V_G$ an integer f_i representing, how many copies of v_i are present in X_H. Obviously $0 \leqslant f_i \leqslant f$ for all $i \in [n]$.

Now we are going to display a set $X_G \subseteq V_G$ which contradicts the (ϱ, d)-denseness of G. Let $X \subseteq V_G$ be a randomly chosen set of vertices, where $\mathbb{P}(v_i \in X) = f_i/f$.

Clearly, we have

$$\mathbb{E}(|X|) = \sum_i \frac{f_i}{f} = \frac{|X_H|}{f}.$$

And since X_H contradicts the $(\varrho, d/2)$-denseness of H, we also have

$$\begin{aligned}
\mathbb{E}(|E(X)|) &= \sum_{1 \leqslant i < j \leqslant n} \frac{f_i}{f}\frac{f_j}{f} \mathbb{1}_{v_i v_j \in E_G} \\
&= \frac{1}{f^2}|E(X_H)| \\
&< \frac{1}{f^2}\left(\frac{d}{4}|X_H|^2 - \varrho|V_H|^2\right) \\
&= \frac{d}{4}\left(\frac{|X_H|}{f}\right)^2 - \varrho n^2.
\end{aligned}$$

Now Markov's inequality tell us, that

$$\mathbb{P}\left(|E(X)| \geqslant \frac{3}{2}\mathbb{E}(|E(X)|)\right) \leqslant \frac{2}{3}.$$

And Chernoff's inequality tells us, that

$$\mathbb{P}(|X| \leqslant (1-\delta)\mathbb{E}(|X|)) \leqslant exp\left(\frac{-\delta^2 \mathbb{E}(|X|)}{2}\right) < \frac{1}{3}.$$

Where in the last step we used that n and therefore $\mathbb{E}(|X|)$ is large. In particular, with positive probability there is a set X which contradicts the (ϱ, d)-denseness of G.

Note that if n is large but $\mathbb{E}(|X|)$ is small, X_H is small relative to n and f and therefore the error term $\varrho(f \cdot n)^2$ would directly imply that there are enough edges in $E(X_H)$, contradiction our assumption on X_H. $\square$

We will use the Regularity Lemma (Lemma 3.4.1), which we will explicitly state in Section 3.4, to formulate the following lemma, that basically guarantees that the reduced graph of a regular partition of a uniformly dense graph is still uniformly dense, but with other constants.

Lemma 3.1.4. *Let $G = (V_G, E_G)$ be a (ϱ, d)-dense graph on n vertices. Furthermore let $V = V_0 \cup V_1 \cup \ldots \cup V_t$ be a ε-regular partition and $R = R(\varepsilon, \delta)$ be the reduced graph of G from the Regularity Lemma (Lemma 3.4.1) with $\delta \leqslant d/13$ and $\varrho, 1/t_0 \ll \varepsilon, d$.*

Then R is $(\varepsilon, d/2)$-dense.

The symbol $\ll$ here should state that $\varrho, 1/t$ are smaller than any *reasonable* function f of ε or d In the case of this lemma, $f(x) = x/156$ would suffice.

Proof. Assume, there is a set $U = \{v_{i_1}, \ldots, v_{i_s}\} \subseteq V_R$ that spans less than $(d/2)|U|^2/2 - \varepsilon t^2$ edges in R. Note that $t \geqslant |U| \geqslant \sqrt{(\varepsilon/d)}2t$, and therefore $1/|U| \leqslant \sqrt{d/(4\varepsilon t^2)} \ll d$. Consider the set $U_G = V_{i_1} \cup \ldots \cup V_{i_s}$ of vertices in G. Since G is (ϱ, d)-dense, it will span at least $d|U_G|^2/2 - \varrho n^2$ edges. For a simper notation, let $n' = n - |V_0|$.

By the definition of R, and since U spans so little edges, we know, that there are at most

$$
\begin{aligned}
&\Big(\frac{d}{2}\frac{|U|^2}{2} - \varepsilon t^2\Big)\Big(\frac{n'}{t}\Big)^2 + |U|\Big(\frac{n'}{t}\Big)^2 + \varepsilon t^2\Big(\frac{n'}{t}\Big)^2 + \Big(|U|^2 - \Big(\frac{d}{2}\frac{|U|^2}{2} - \varepsilon t^2\Big) - \varepsilon t^2\Big)\Big(\frac{n'}{t}\Big)^2\delta \\
&= \Big(\frac{d}{2}\frac{|U|^2}{2}\Big)\Big(\frac{n'}{t}\Big)^2 + |U|\Big(\frac{n'}{t}\Big)^2 + \Big(|U|^2 - \frac{d}{2}\frac{|U|^2}{2}\Big)\Big(\frac{n'}{t}\Big)^2\delta \\
&< \Big(\frac{d}{2}\frac{|U|^2}{2} + |U| + \delta|U|^2\Big)\Big(\frac{n'}{t}\Big)^2 \\
&= \Big(\frac{d}{4} + \frac{1}{|U|} + \delta\Big)|U|^2\Big(\frac{n'}{t}\Big)^2 \\
&= \Big(\frac{d}{4} + \frac{1}{|U|} + \delta\Big)|U_G|^2 \\
&\leqslant \frac{d}{3}|U_G|^2 \\
&\leqslant d\frac{|U_G|^2}{2} - \varrho n^2
\end{aligned}
$$

edges in $G[U_G]$, contradicting the assumption that G is (ϱ, d)-dense. In the penultimate equation we used that $\delta \leqslant d/13$ and $1/|U| \ll d$, and in the last equation we used that $\varrho \ll \varepsilon$ and $|U_G|^2 \geqslant (\varepsilon/d)n^2$. $\square$

3.2 Properties of inseparable graphs

In this section we shall explore some properties of inseparable graphs that are crucial for the proof of Theorem 1.3.6. In the latter half of this section, we will then, analogously to Section 3.1 state some properties of inseparable graphs that will come into play in Section 3.4.

First we note that removing a small set of vertices has only little effect on the inseparability.

Lemma 3.2.1. *For every $\mu \in (0,1]$ and $\beta \in [0,1/2)$, the following holds. If $G = (V,E)$ is μ-inseparable and $U \subseteq V$ with $|U| \leqslant \beta\mu n$, then $G[V \setminus U]$ is $(1-2\beta)\mu$-inseparable.*

Proof. Suppose by contradiction that $G[V \setminus U] = (V', E')$ is not $(1-2\beta)\mu$-inseparable. Thus, there exists $X \subseteq V'$ with $|X| \leqslant n/2$ such that $e(X, V' \setminus X) < (1-2\beta)\mu|X||V' \setminus X|$. Consider the partition of V into the sets X and $(V' \setminus X) \cup U = V \setminus X$. We have that

$$\begin{aligned} e(X, V \setminus X) &< (1-2\beta)\mu|X||V' \setminus X| + |U||X| \\ &= (1-2\beta)\mu|X|(|V| - |U| - |X|) + |U||X| \\ &= \mu|X||V \setminus X| - 2\beta\mu|X||V \setminus X| + (1-(1-2\beta)\mu)|U||X|. \end{aligned} \tag{3.2.1}$$

Since $|V \setminus X| \geqslant n/2$, $\beta\mu n \geqslant |U|$, and $\beta < 1/2$ we have

$$2\beta\mu|X||V \setminus X| \geqslant \beta\mu n|X| \geqslant |U||X| \geqslant (1-(1-2\beta)\mu)|U||X|.$$

Together with (3.2.1), we derive that $e(X, V \setminus X) < \mu|X||V \setminus X|$, which contradicts the assumption that G is μ-inseparable. □

The key property of inseparable graphs is that between any pair of vertices there exist many paths of bounded length.

Lemma 3.2.2. *For every $\mu \in (0,1]$, there exist $c > 0$ and integers L, n_0 such that every μ-inseparable graph $G = (V,E)$ on $|V| = n \geqslant n_0$ vertices satisfies the following.*

For every two distinct vertices x, $y \in V$, there is some integer ℓ with $0 \leqslant \ell \leqslant L$ such that the number of (x,y)-walks with ℓ inner vertices in G is at least cn^ℓ.

Proof. Given μ we define

$$L = \left\lfloor \frac{8}{\mu} \right\rfloor, \qquad \delta_i = \left(\frac{\mu^2}{3}\right)^i \left(\frac{1}{2}\right)^{\binom{i+1}{2}}, \qquad \text{and} \qquad c = \frac{\mu^2}{48}\delta^2_{\lceil 4/\mu \rceil}. \tag{3.2.2}$$

Let G be a sufficiently large μ-inseparable graph on n vertices and x, y be two distinct vertices of G. Consider for each $i \geqslant 0$ the set of vertices v that can be reached from x by "many" walks in G with i inner vertices. For that we define

$$X_i = \{v \in V \colon \text{there are } \delta_i n^i \ (x,v)\text{-walks with } i \text{ inner vertices}\} \quad \text{and} \quad X^i = \bigcup_{0 \leqslant j \leqslant i} X_j.$$

Analogously, consider the vertices v that can be reached from y by $\delta_i n^i$ walks in G with i inner vertices and define the sets Y_i and Y^i in the same way.

Observe that $X_0 = X^0 = N(x)$ and since G is μ-inseparable, $|N(x)| \geqslant \mu(n-1)$. Moreover, $X^i \subseteq X^{i+1}$ and we shall show that as long as $|X^i|$ is not too large, then $|X^{i+1}|$ is substantially larger than $|X^i|$. More precisely, we show for every $i \geqslant 0$ that

$$|X^i| \leqslant \frac{2}{3}n \quad \Longrightarrow \quad |X^{i+1} \smallsetminus X^i| \geqslant \frac{\mu}{6}n. \tag{3.2.3}$$

Before verifying (3.2.3), we conclude the proof of Lemma 3.2.2. In fact, (3.2.3) implies that there is some $i_0 < \lfloor 4/\mu \rfloor$ such that $|X^{i_0}| > 2n/3$. Applying the same argument for Y^i, we get some $j_0 < \lfloor 4/\mu \rfloor$ such that $|Y^{j_0}| > 2n/3$ and, hence, $|X^{i_0} \cap Y^{j_0}| \geqslant n/3$.

Each vertex $v \in X^{i_0} \cap Y^{j_0}$ can be used to create many (y,x)-walks with possibly different number of inner vertices. However, by the Pigeonhole Principle there are integers a, b with $0 \leqslant a \leqslant i_0$ and $0 \leqslant b \leqslant j_0$ such that

$$|X_a \cap Y_b| \geqslant \frac{|X^{i_0} \cap Y^{j_0}|}{(i_0+1)(j_0+1)} \geqslant \frac{\mu^2 n}{48}. \tag{3.2.4}$$

For each $v \in X_a \cap Y_b$ there exist $\delta_a n^a$ (x,v)-walks and $\delta_b n^b$ (v,y)-walks with a and b inner vertices, respectively. Concatenating these walks leads to at least

$$\delta_a \delta_b n^{a+b} \cdot |X_a \cap Y_b|$$

(x,y)-walks, with $\ell = a+b+1$ inner vertices. The choice of constants in (3.2.2) and (3.2.4) conclude the proof.

It is left to verify (3.2.3). Suppose $|X^i| \leqslant 2n/3$ and consider the complement $Z = V \smallsetminus X^i$. Owing to the μ-inseparability of G we have

$$e(X^i, Z) \geqslant \mu |X^i||Z|. \tag{3.2.5}$$

Note that each vertex v with at least $\delta_{j+1}n/\delta_j$ neighbours in X_j belongs to X_{j+1}. Since Z is disjoint from X^i, we have

$$e(X^{i-1}, Z) < |Z| \cdot \sum_{j=0}^{i-1} \frac{\delta_{j+1}}{\delta_j} n. \tag{3.2.6}$$

Moreover, supposing by contradiction that (3.2.3) fails, we also have

$$e(X_i, Z) < |Z| \cdot \frac{\delta_{i+1}}{\delta_i} n + \frac{\mu}{6}|X_i| n. \tag{3.2.7}$$

Combining (3.2.6) and (3.2.7) we arrive at

$$e(X^i, Z) < |Z| \cdot \sum_{j=0}^{i-1} \frac{\delta_{j+1}}{\delta_j} n + |Z| \cdot \frac{\delta_{i+1}}{\delta_i} n + \frac{\mu}{6} n|X_i| = |Z| \cdot \sum_{j=0}^{i} \frac{\delta_{j+1}}{\delta_j} n + \frac{\mu}{6} n|X_i|. \tag{3.2.8}$$

Owing to the choice of δ_j in (3.2.2) we have

$$\sum_{j=0}^{i} \frac{\delta_{j+1}}{\delta_j} = \frac{\mu^2}{3} \sum_{j=0}^{i} \left(\frac{1}{2}\right)^{j+1} \leqslant \frac{\mu^2}{3}.$$

Furthermore, since $|X^i| \geqslant |X^0| = |N(x)| \geqslant \mu(n-1)$ and $|Z| = |V \smallsetminus X^i| \geqslant n/3$, we derive for sufficiently large n from (3.2.8) that

$$e(X^i, Z) < \frac{\mu^2}{3}|Z|n + \frac{\mu}{6}|X_i|n \leqslant \frac{\mu}{2}|Z||X^i| + \frac{\mu}{2}|X_i||Z| \leqslant \mu|X^i||Z|,$$

which contradicts (3.2.5). □

Here we start the part of the section that is needed for Section 3.4, and the readers that are mainly interested in the proof of Theorem 1.3.6 might proceed to Section 3.3 and come back here later.

First we state that a balanced blow-up of a graph inherits its property to be μ-inseparable with slightly worse μ that is, however, independent of the blow-up factor.

Lemma 3.2.3. *Let $G = (V_G, E_G)$ be a μ-inseparable n-vertex graph, and $H = (V_H, E_H)$ be a balanced blow-up of G. Then there is an integer n_0 such that H is $(\mu/2)$-inseparable, if $n \geqslant n_0$.*

Proof. Let f be the blow-up factor of H. Suppose there is a set $X_H \subseteq V_H$, such that the separation $(X_H, V_H \smallsetminus X_H)$ is contradicting the $(\mu/2)$-inseparability of H. We will proceed analogously to the proof of Lemma 3.1.3.

Since X_H may contain up to f copies of each vertex of G, we will assign to each vertex $v_i \in V_G$ an integer f_i representing, how many copies of v_i are present in X_H. Obviously $0 \leqslant f_i \leqslant f$ for all $i \in [n]$.

Now we are going to display a set $X_G \subseteq V_G$ which contradicts the μ-inseparability of G. Let $X \subseteq V_G$ be a randomly chosen set of vertices, where $\mathbb{P}(v_i \in X) = f_i/f$.

Clearly, we have

$$\mathbb{E}(|X|) = \sum_i \frac{f_i}{f} = \frac{|X_H|}{f}.$$

And since X_H contradicts the $(\mu/2)$-inseparability of H, we also have

$$\begin{aligned}
\mathbb{E}(|E(X, V_G \smallsetminus X)|) &= \sum_{1 \leqslant i < j \leqslant n} \frac{f_i}{f}\left(1 - \frac{f_j}{f}\right)\mathbb{1}_{v_i v_j \in E_G} \\
&= \frac{1}{f^2}|E(X_H, V_H \smallsetminus X_H)| \\
&< \frac{1}{f^2}\left(\frac{\mu}{2}|X_H||V_H \smallsetminus X_H|\right) \\
&= \frac{\mu}{2}\frac{|X_H|}{f}\frac{|V_H \smallsetminus X_H|}{f} \\
&= \frac{\mu}{2}\frac{|X_H|}{f}\left(|V_G| - \frac{|X_H|}{f}\right).
\end{aligned}$$

Now Markov's inequality tell us, that

$$\mathbb{P}\left(|E(X, V_G \smallsetminus X)| \geqslant \frac{3}{2}\mathbb{E}(|E(X, V_G \smallsetminus X)|)\right) \leqslant \frac{2}{3}.$$

And Chernoff's inequality tells us, that

$$\mathbb{P}(|X| \leqslant (1-\delta)\mathbb{E}(|X|)) \leqslant exp\left(\frac{-\delta^2\mathbb{E}(|X|)}{2}\right) < \frac{1}{6},$$

and

$$\mathbb{P}(|X| \geqslant (1+\delta)\mathbb{E}(|X|)) \leqslant exp\left(\frac{-\delta^2\mathbb{E}(|X|)}{3}\right) < \frac{1}{6},$$

Where in the last step of the latter two inequalities we used that n and therefore $\mathbb{E}(|X|)$ is large. So in particular, with positive probability there is a set X which contradicts the μ-inseparability of G.

Note that if n is large but $\mathbb{E}(|X|)$ is small, X_H is small relative to n and f and therefore the minimum degree of $f\mu(n-1)$ in H would directly imply that there are enough edges in $E(X_H, V_H \smallsetminus X_H)$, contradiction our assumption on X_H. $\square$

Once again, we will use the Regularity Lemma (Lemma 3.4.1) which we will explicitly state in Section 3.4 to formulate the following lemma, that basically guarantees that the reduced graph of a regular partition of an inseparable graph is still inseparable, with slightly worse constants.

Lemma 3.2.4. *Let $G = (V_G, E_G)$ be a μ-inseparable graph on n vertices. Furthermore let $V = V_0 \cup V_1 \cup \ldots \cup V_t$ be a ε-regular partition and $R = R(\varepsilon, \delta)$ be the reduced graph of G from the Regularity Lemma (Lemma 3.4.1) with $\delta \leqslant \mu/4$ and $\varepsilon \leqslant \mu^2/1000$.*

Then there is a set $S \subseteq [t]$ of size at most $\sqrt{\varepsilon} t$, such that $R' = R[\{v_i | i \in ([t] \setminus S)\}]$ is $(\mu/2)$-inseparable.

Proof. First we will move all the partition classes V_i to V_0 which are part of more than $\sqrt{\varepsilon} t$ irregular pairs. At the start we have $|V_0| \leqslant \varepsilon n \leqslant \sqrt{\varepsilon} n$. Since there are at most εt^2 irregular pairs, we can only have $\sqrt{\varepsilon} t$ classes, which are in $\sqrt{\varepsilon} t$ irregular pairs each, enlarging V_0 by at most $\sqrt{\varepsilon} t \cdot n/t = \sqrt{\varepsilon} n$. Therefore the enlarged V_0, say V_0', will have size at most $2\sqrt{\varepsilon} n$. Let S be the set of indices of V_i that were moved to V_0.

Assume, there is a set $A = \{v_{i_1}, \ldots, v_{i_x}\} \subseteq V_R$ such that there are less than $\mu/2|A||V_R \setminus A|$ crossing edges between A and $V_R \setminus A$ in R. By symmetry we may assume $|A| \leqslant t - |A|$.

Consider the set $A_G = V_{i_1} \cup \ldots \cup V_{i_x}$ of vertices in G. For a simper notation, let $n' = n - |V_0|$.

By the definition of R, and since there are so little crossing edges in $(A, V_R \setminus A)$, we know that there are at most

$$\begin{aligned}
&\frac{\mu}{2}|A|(t-|A|)(\frac{n'}{t})^2 + |A|\sqrt{\varepsilon} t(\frac{n'}{t})^2 + |A|\frac{n'}{t}|V_0|' + |A|(t-|A|)\delta(\frac{n'}{t})^2 \\
\leqslant\ & [\frac{\mu}{2} + \sqrt{\varepsilon} + \delta]|A|(t-|A|)(\frac{n'}{t})^2 + |A|\sqrt{\varepsilon}|A|(\frac{n'}{t})^2 + |A|\frac{n'}{t} 2\sqrt{\varepsilon} n \\
=\ & [\frac{\mu}{2} + \sqrt{\varepsilon} + \delta]|A_G|(n' - |A_G|) + |A_G|^2\sqrt{\varepsilon} + |A_G| 2\sqrt{\varepsilon} n \\
\leqslant\ & [\frac{\mu}{2} + \sqrt{\varepsilon} + \delta + \sqrt{\varepsilon} + 4\sqrt{\varepsilon}]|A_G|(n - |A_G|) \\
<\ & \mu|A_G|(n - |A_G|)
\end{aligned}$$

edges crossing $(A_G, V \setminus A_G)$, contradicting the μ-inseparability of G. In the penultimate equation we used that $|A| \leqslant t - |A|$ and $n - |A_G| \geqslant n/2$, and in the last equation we used that $\delta \leqslant \mu/4$ and $\varepsilon \leqslant \mu^2/1000$. □

3.3 Embedding powers of Hamiltonian cycles

The proof of Theorem 1.3.6 is based on the absorption method and follows the strategy from [47]. Roughly speaking, this method splits the problem of finding a k-th power of a Hamiltonian cycle into the following three parts:

1. finding an almost perfect cover with only "few" k-paths,

2. ensuring the abundant existence of so-called *absorbers*, and

3. connecting those absorbers and paths to an almost spanning k-th power of a cycle.

The first part is achieved by Corollary 3.1.2 and only makes use of the (ϱ, d)-denseness of G. For the second part of the absorption method again the (ϱ, d)-denseness of G suffices. However, for the connection of these absorbers the μ-inseparability is required. The appropriate *Connecting Lemma*, which is also utilised for connecting the paths of the almost perfect cover from the first part, is given in Section 3.3. In Section 3.3 we establish the *Absorbing Path Lemma* and in Section 3.3 we combine these results and deduce Theorem 1.3.6.

Connecting Lemma

The Connecting Lemma asserts that any two connectable K_k's in a uniformly dense and inseparable graph G are connected by "many" k-paths of bounded length. As shown in Lemma 3.2.2, for $k = 1$ this is a direct consequence of the inseparability. For $k \geqslant 2$ we combine Lemma 3.2.2 with Lemma 3.1.1 by a standard supersaturation argument to obtain the desired k-paths.

Lemma 3.3.1 (Connecting Lemma). *For every d, $\mu \in (0, 1]$, $\zeta > 0$, and every integer $k \geqslant 1$, there exist ϱ, $\xi > 0$ and integers M, $n_0 \in \mathbb{N}$ such that every (ϱ, d)-dense and μ-inseparable graph $G = (V, E)$ on $|V| = n \geqslant n_0$ vertices satisfies the following.*

For every pair $\vec{x}$, $\vec{y} \in V^k$ of disjoint ζ-connectable K_k in G, there is some integer $m \leqslant M$ such that the number of $(\vec{x}, \vec{y}; k)$-paths with m inner vertices in G is least ξn^m.

Proof. Given d, μ, ζ and k, we shall fix constants ϱ, ξ, M and n_0. For that we first apply Lemma 3.2.2 for μ and obtain L and c. Next we define auxiliary constants ξ_i for integers $i \geqslant 0$ inductively through

$$\xi_0 = \frac{\zeta^2 c}{L+1} \qquad \text{and} \qquad \xi_{i+1} = \frac{d^{\binom{k}{2}}}{2k!} \left(\frac{\xi_i}{2}\right)^{k+1}. \tag{3.3.1}$$

Finally, we set

$$\xi = \frac{\xi_{L+2}}{2}, \qquad \varrho = \frac{d^{\binom{k}{2}} \xi^2}{8k^2}, \qquad \text{and} \qquad M = (L+2)k, \tag{3.3.2}$$

and let n be sufficiently large.

Let G be a (ϱ, d)-dense and μ-inseparable graph on n vertices and let $\vec{x} = (x_1, \ldots, x_k)$ and $\vec{y} = (y_1, \ldots, y_k)$ be two disjoint ζ-connectable K_k in G.

We consider the following type of graphs that will be useful to obtain the desired $(\vec{x}, \vec{y}; k)$-paths. For integers $k \geqslant 1$, $\ell \geqslant 0$ and $0 \leqslant a \leqslant \ell$, a graph R is a (k, ℓ, a)-*rope* (see Figure 3.3.1 for an illustration) if it can be obtained from a path on $\ell + 2$ vertices by blowing up the first, the last, and the first a inner vertices into K_k. More precisely, the vertex set of R is

$$V(R) = Z_0 \cup \cdots \cup Z_{\ell+1}$$

such that

$$|Z_0| = |Z_{\ell+1}| = k = |Z_1| = \cdots = |Z_a| \quad \text{and} \quad |Z_{a+1}| = \cdots = |Z_\ell| = 1.$$

The edges of R are such that $Z_0, Z_1, \ldots, Z_a$, and $Z_{\ell+1}$ each induce a K_k and between any consecutive pair (Z_i, Z_{i+1}), for $i = 0, \ldots, \ell$, all $|Z_i||Z_{i+1}|$ edges are present. Note that we do not insist that the sets Z_i are pairwise disjoint. If the vertices in Z_0 are those of $\vec{x}$ and the vertices in $Z_{\ell+1}$ are those of $\vec{y}$, then the rope is said to be a $(\vec{x}, \vec{y}; k, \ell, a)$-rope and the sets $(Z_1, \ldots, Z_a)$ are called the *inner parts* of the rope.

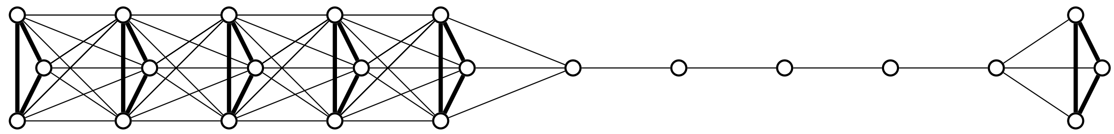


Figure 3.3.1: A $(3, 9, 4)$-rope.

We shall prove the following assertion for fixed cliques $\vec{x}$ and $\vec{y}$.

Claim 3.3.2. *There exists $\ell \leqslant L+2$ such that for every $a = 0, \ldots, \ell$ there are $\xi_a n^{ak+(\ell-a)}$ $(\vec{x}, \vec{y}; k, \ell, a)$-ropes in G.*

Note that, for $a = 0$, Claim 3.3.2 ensures many walks between $N(\vec{x})$ and $N(\vec{y})$, which indeed are provided by Lemma 3.2.2. For $a = \ell$, it is easy to see that a $(\vec{x}, \vec{y}; k, \ell, \ell)$-rope (without any vertex repetition) contains a $(\vec{x}, \vec{y}; k)$-path with $m = \ell k$ inner vertices. The number of $(\vec{x}, \vec{y}; k, \ell, \ell)$-ropes with vertex repetitions is bounded by $m^2 n^{m-1}$. Excluding these ropes from those obtained by Claim 3.3.2 for $a = \ell$ yields at least $\xi_\ell n^m - m^2 n^{m-1} \geqslant \xi_\ell n^m / 2 \geqslant \xi n^m$ many $(\vec{x}, \vec{y}; k)$-paths, for sufficiently large n. Thus, for $a = \ell$, the claim leads to the conclusion of Lemma 3.3.1 and it is left to verify the claim. $\square$

Proof of Claim 3.3.2. First we fix the integer ℓ. Consider the neighbourhoods $N(\vec{x})$ and $N(\vec{y})$. Since $\vec{x}$ and $\vec{y}$ are ζ-connectable, we have $|N(\vec{x})|, |N(\vec{y})| \geqslant \zeta n$. Since G is μ-inseparable, for each pair of distinct vertices $(x, y) \in N(\vec{x}) \times N(\vec{y})$, by Lemma 3.2.2, there are at least $cn^{\ell(x,y)}$ many (x, y)-walks with $\ell(x, y)$ inner vertices and $0 \leqslant \ell(x, y) \leqslant L$. Hence, by the Pigeonhole Principle, there is some ℓ with $2 \leqslant \ell \leqslant L + 2$ such that there are at least $\zeta^2 cn^\ell / (L + 1) = \xi_0 n^\ell$ (x_k, y_1)-walks with exactly ℓ inner vertices and such that the first and last inner vertex is from $N(\vec{x})$ and $N(\vec{y})$, respectively. This yields the claim for $a = 0$.

We proceed in an inductive manner. Let $a \geqslant 0$ and assume by induction that G contains at least $\xi_a n^{ak+(\ell-a)}$ many $(\vec{x}, \vec{y}; k, \ell, a)$-ropes. For many of such ropes we now blow up the 1-element part Z_{a+1}. Consider collections $\mathcal{Z} = (Z_1, \ldots, Z_a, Z_{a+2}, \ldots, Z_\ell)$ and sets $U_{\mathcal{Z}}$ such that $u \in U_{\mathcal{Z}}$ if and only if

$$(Z_1, \ldots, Z_a, \{u\}, Z_{a+2}, \ldots, Z_\ell)$$

are the inner parts of a $(\vec{x}, \vec{y}; k, \ell, a)$-rope. By a standard averaging argument, it is easy to see that there exist $\xi_a n^{ak+(\ell-a)-1}/2$ collections $\mathcal{Z}$ such that the set $U_{\mathcal{Z}}$ has size at least $\xi_a n/2$.

Note that each K_k contained in $U_{\mathcal{Z}}$ yields a $(\vec{x}, \vec{y}; k, \ell, a+1)$-rope. Lemma 3.1.1 applied to every set $U_{\mathcal{Z}}$ of size at least $\xi_a n/2$ leads to

$$\frac{1}{k!}\left(d^{\binom{k}{2}} - (k-1)k\frac{\varrho n^2}{|U_{\mathcal{Z}}|^2}\right)|U_{\mathcal{Z}}|^k \overset{(3.3.2)}{\geqslant} \frac{d^{\binom{k}{2}}}{2k!}\left(\frac{\xi_a}{2}\right)^k n^k$$

unordered copies of K_k in a given $U_{\mathcal{Z}}$ of size at least $\xi_a n/2$. Hence, there are at least

$$\frac{\xi_a}{2} n^{ak+(\ell-a)-1} \cdot \frac{d^{\binom{k}{2}}}{2k!}\left(\frac{\xi_a}{2}\right)^k n^k \overset{(3.3.1)}{=} \xi_{a+1} n^{(a+1)k+(\ell-(a+1))}$$

$(\vec{x}, \vec{y}; k, \ell, a+1)$-ropes in G, which concludes the proof of Claim 3.3.2. □

Absorbing Path Lemma

For a given vertex v a clique K_{2k} contained in $N(v)$ can be used as an absorber for v, in the sense that obviously the K_{2k} induces a k-path on $2k$ vertices. Moreover, placing v in the middle of this K_{2k} yields a k-path containing v that starts and ends with the same K_k's. Since inseparable and uniformly dense graphs $G = (V, E)$ have a minimum degree linear in the number of vertices, by Lemma 3.1.1 the uniform density yields many cliques K_{2k} in the neighbourhood of any given vertex v. Moreover, for many of these K_{2k} all the K_k's contained in it are connectable and, hence, together with the Connecting Lemma we can build an *absorbing path* defined below.

Definition 3.3.3. For a graph $G = (V, E)$ on n vertices, an integer $k \geqslant 1$, and $\alpha \geqslant 0$, we say that a $(\vec{x}, \vec{y}; k)$-path P in G is α-absorbing if for every set $X \subseteq V \smallsetminus V(P)$ of size $|X| \leqslant \alpha n$, there is a $(\vec{x}, \vec{y}; k)$-path Q in G with vertex set $V(Q) = V(P) \cup X$.

Lemma 3.3.4 (Absorbing Path Lemma). *For every d, $\mu \in (0,1]$, and integer $k \geqslant 1$, there exist ϱ, ζ, $\alpha > 0$, with $\zeta \leqslant \mu/2$, and n_0 such that every (ϱ, d)-dense and μ-inseparable graph G on $n \geqslant n_0$ vertices contains an α-absorbing $(\vec{x}, \vec{y}; k)$-path P_A of size $|V(P_A)| \leqslant \zeta n/2$, and $\vec{x}$, $\vec{y}$ being ζ-connectable.*

Proof. Given d, μ, and k, we set

$$\zeta = \frac{d^{\binom{2k+1}{2}} \mu^{2k+1}}{2^{2k+3}}. \tag{3.3.3}$$

Applying Lemma 3.1.1 for d, $\mu/2$, and $\zeta/2$ yields constants ϱ', ξ, M, and n_0' and we fix

$$\alpha = \frac{\zeta^2}{24(10k^2 + M)} \qquad \text{and} \qquad \varrho = \min\left\{ \frac{\varrho'}{4}, \frac{d^{\binom{2k+1}{2}} \mu^2}{8(2k+1)^2} \right\}, \tag{3.3.4}$$

and let a sufficiently large (ϱ, d)-dense and μ-inseparable graph $G = (V, E)$ on n vertices be given.

For every vertex $v \in V$, call an ordered K_{2k} contained in $N(v)$ with ordered vertex set $(x_1, \ldots, x_{2k})$ an v-*absorber*, if both $\vec{x}_v = (x_1, \ldots, x_k)$ and $\vec{y}_v = (x_{k+1}, \ldots, x_{2k})$ are ζ-connectable (ordered) K_k's in G.

Note that $(x_1, \ldots, x_{2k})$ and $(x_1, \ldots, x_k, v, x_{k+1}, \ldots, x_{2k})$ are both $(\vec{x}_v, \vec{y}_v; k)$-paths with ζ-connectable ends. We denote by $\mathcal{A}_v \subseteq V^{2k}$ the set of all v-absorbers and we let

$$\mathcal{A} = \bigcup_{v \in V} \mathcal{A}_v$$

be the set of all absorbers in G.

The α-absorbing $(\vec{x}, \vec{y}; k)$-path P_A is constructed by considering a collection $A \subseteq \mathcal{A}$ independently at random. We shall show that a.a.s. for every vertex v the collection A will contain "many" v-absorbers from $\mathcal{A}_v$. After erasing intersecting absorbers, we shall connect the remaining ones to a path P_A by repeated applications of Lemma 3.1.1.

First we prove the existence of "many" v-absorbers for every $v \in V$ in G. Since G is μ-inseparable, we have $|N(v)| \geqslant \mu(n-1) \geqslant \mu n/2$ for sufficiently large n. Consequently, the induced subgraph $G[N(v)]$ is $(4\varrho/\mu^2, d)$-dense and Lemma 3.1.1 shows that the number of ordered K_{2k+1} in $G[N(v)]$ is at least

$$\left(d^{\binom{2k+1}{2}} - (2k)(2k+1)\frac{4\varrho}{\mu^2}\right)|N(v)|^{2k+1} \overset{(3.3.4)}{\geqslant} \frac{d^{\binom{2k+1}{2}}}{2} \cdot \left(\frac{\mu}{2}\right)^{2k+1} \cdot n^{2k+1} \overset{(3.3.3)}{=} 2\zeta n^{2k+1}.$$

Since there are at most n^{2k} different K_{2k} and each K_{2k} is contained in at most n different K_{2k+1}, by a simple averaging argument, there exist at least ζn^{2k} different K_{2k} each contained in ζn different K_{2k+1} in $G[N(v)]$. Consequently, $|\mathcal{A}_v| \geqslant \zeta n^{2k}$ for every $v \in V$.

Set

$$p = \frac{\zeta}{6(10k^2 + M)n^{2k-1}}. \tag{3.3.5}$$

and consider a random collection $A \subseteq \mathcal{A}$, in which every ordered $K_{2k} \in \mathcal{A}$ is included independently with probability p. Let X_v be the random variable $|A \cap \mathcal{A}_v|$. Then,

$$\mathbb{E}X_v \geqslant \zeta n^{2k} p \overset{(3.3.5)}{=} \frac{\zeta^2 n}{6(10k^2 + M)}.$$

Since X_v is binomially distributed, by the union bound and Chernoff's inequality (see, e.g., [28, Theorem 2.1]), we have

$$\begin{aligned}\mathbb{P}\Big(\exists v\colon X_v \leqslant \frac{\zeta^2 n}{12(10k^2 + M)}\Big) &\leqslant n \cdot \max_{v \in V} \mathbb{P}\Big(X_v \leqslant \frac{\zeta^2 n}{12(10k^2 + M)}\Big) \\ &\leqslant n \cdot \exp\Big(-\frac{\zeta^2 n}{48(10k^2 + M)}\Big) < \frac{1}{3},\end{aligned} \tag{3.3.6}$$

for sufficiently large n.

Consider now the pairs of absorbers in A that share some vertex. Let Y be the random variable that counts the number of such intersecting pairs. There are at most $5k^2 n^{4k-1}$ possible intersecting pairs in $\mathcal{A}$ and, hence, $\mathbb{E}Y < 5k^2 n^{4k-1} p^2$. Markov's inequality yields

$$\mathbb{P}\Big(Y \geqslant \frac{\zeta^2 n}{24(10k^2 + M)}\Big) \overset{(3.3.5)}{\leqslant} \mathbb{P}(Y \geqslant 15k^2 n^{4k-1} p^2) \leqslant \frac{\mathbb{E}Y}{15k^2 n^{4k-1} p^2} < \frac{1}{3}. \tag{3.3.7}$$

For the size of A we note that $\mathbb{E}|A| < n^{2k} p$ and another application of Markov's inequality shows

$$\mathbb{P}\Big(|A| \geqslant \frac{\zeta n}{2(10k^2 + M)}\Big) \overset{(3.3.5)}{=} \mathbb{P}(|A| \geqslant 3n^{2k} p) \leqslant \frac{\mathbb{E}|A|}{3n^{2k} p} < \frac{1}{3}. \tag{3.3.8}$$

Thus, by (3.3.6), (3.3.7), and (3.3.8), there exists a collection $A_0 \subseteq \mathcal{A}$ satisfying

(i) $|A_0 \cap \mathcal{A}_v| \geqslant \zeta^2 n/(12(10k^2 + M)$ for every $v \in V$,

(ii) there are at most $\zeta^2 n/(24(10k^2 + M))$ pairs of intersecting absorbers in A_0,

(iii) the size $|A_0|$ is at most $\zeta n/(2(10k^2 + M))$.

From each pair of intersecting absorbers, delete one of them in an arbitrary way and let $A_1 = \{\hat{K}_1, \ldots, \hat{K}_m\} \subseteq A_0$ be the set of absorbers in A_0 obtained this way. It follows from properties (i) and (ii) that

$$|A_1 \cap \mathcal{A}_v| \geqslant \frac{\zeta^2 n}{24(10k^2 + M)} = \alpha n$$

for every $v \in V$, i.e., A_1 contains at least αn many v-absorber for every vertex $v \in V$.

In the final step we connect the absorbers in A_1 to a k-path P_A. We construct P_A inductively by repeated applications of Lemma 3.3.1. Suppose we already obtained a path P^i that contains $\hat{K}_1, \ldots, \hat{K}_i$ and

$$|V(P^i)| \leqslant i \cdot 2k + (i-1)M,$$

below we establish the existence of P^{i+1} that in addition contains $\hat{K}_{i+1}$ and satisfies

$$|V(P^{i+1})| \leqslant (i+1) \cdot 2k + i \cdot M.$$

For that, set

$$Z^i = V(P^i) \cup \bigcup_{j=i+1}^{m} V(\hat{K}_j)$$

and observe that for every $i \leqslant m$,

$$\begin{aligned} |Z^i| &\leqslant i \cdot 2k + (i-1)M + (m-i) \cdot 2k < m \cdot (2k+M) \\ &\overset{(iii)}{\leqslant} \frac{\zeta n}{2(10k^2+M)}(2k+M) \leqslant \frac{\zeta n}{2} \overset{(3.3.3)}{\leqslant} \frac{\mu n}{4}. \end{aligned} \tag{3.3.9}$$

Let $\vec{x}$ be the last k vertices of $\hat{K}_i$ and let $\vec{y}$ be the first k vertices of $\hat{K}_{i+1}$. In view of the choice of constants in (3.3.4), the observation (3.3.9), and Lemma 3.2.1, the induced subgraph $G' = G[(V \smallsetminus Z^i) \cup V(\vec{x}) \cup V(\vec{y})]$ is (ϱ', d)-dense and $(\mu/2)$-inseparable, and $\vec{x}$ and $\vec{y}$ are $\zeta/2$-connectable in G'. Consequently, there is an $(\vec{x}, \vec{y}; k)$-path in G' with at most M inner vertices outside Z^i. Together with P^i this yields P^{i+1} with the desired properties.

Since P^m contains at least αn distinct v-absorbers for every $v \in V$, it is an α-absorbing path. Moreover, the first k vertices in P^m are from $\hat{K}_1$ and the last k vertices are from $\hat{K}_m$, which by definition are ζ-connectable cliques in G, which shows that $P_A = P^m$ has the desired properties. □

Proof of the main theorem

Having established the Connecting Lemma (Lemma 3.3.1), the Absorbing Path Lemma (Lemma 3.3.4), and the Path Lemma (Corollary 3.1.2), we are ready to deduce Theorem 1.3.6.

Proof of Theorem 1.3.6. The proof of Theorem 1.3.6 is based on the absorption method and we start by fixing all involved constants. Given d, μ, and k, applying Lemma 3.3.4 (Absorbing Path Lemma) yields constants ϱ_{A}, ζ_{A}, α_{A}, and an application of Corollary 3.1.2 (Path Lemma) yields ϱ_{P} and ζ_{P}. For an application of Lemma 3.3.1 (Connecting Lemma), set

$$\zeta_{\mathrm{C}} = \min\left\{\frac{\zeta_{\mathrm{A}}}{2}, \frac{\alpha_{\mathrm{A}}\zeta_{\mathrm{P}}}{2}\right\}, \tag{3.3.10}$$

and apply the Connecting Lemma with $\mu/2$, d, ζ_{C}, and k to attain constants ϱ_{C}, ξ_{C}, and M_{C}. Finally, set

$$\varrho = \min\left\{\varrho_{\mathrm{A}}, \frac{\alpha_{\mathrm{A}}^2\varrho_{\mathrm{P}}}{4}, \frac{\varrho_{\mathrm{C}}}{4}\right\}, \tag{3.3.11}$$

and let n be sufficiently large. In particular, we may assume that

$$\frac{2}{\alpha_{\mathrm{A}}\zeta_{\mathrm{P}}} M_{\mathrm{C}}^2 < \frac{\xi_{\mathrm{C}}}{4}\left(\frac{\alpha_{\mathrm{A}}}{8}\right)^{M_{\mathrm{C}}} n. \tag{3.3.12}$$

Let $G = (V, E)$ be a (ϱ, d)-dense and μ-inseparable graph on n vertices. By the absorbing Path Lemma, there is an α_{A}-absorbing $(\vec{x}_A, \vec{y}_A; k)$-path P_A contained in G with $|V(P_A)| \leqslant \zeta_{\mathrm{A}} n/2$ and $\vec{x}_A$, $\vec{y}_A$ being ζ_{A}-connectable in G. This path will be set aside and with it, another special set of vertices which we call the reservoir. On the remaining graph, we shall find an almost perfect covering of its vertices by i_0 disjoint $(\vec{x}_i, \vec{y}_i; k)$-paths with $\vec{x}_i$, $\vec{y}_i$ being ζ_{C}-connectable.

The reservoir $R \subseteq V$ will be used to connect the absorbing path and the paths in the almost perfect covering to attain an almost spanning cycle. For that it is convenient to choose the set R in such a way that for any connectable $\vec{x}$ and $\vec{y}$ there are still "many" $(\vec{x}, \vec{y}; k)$-paths having all their inner vertices in R. In order to define the reservoir, consider the induced subgraph

$$G' = G[(V \smallsetminus V(P_A)) \cup V(\vec{x}_A) \cup V(\vec{y}_A)].$$

Since the number of deleted vertices is $|V(P_A)| - 2k \leqslant \zeta_{\mathrm{A}} n/2 \leqslant \mu n/4$, Lemma 3.2.1 shows that G' is $\mu/2$-inseparable and, since $\mu n/4 < n/2$, it follows from the (ϱ, d)-denseness of G that G' is also $(4\varrho, d)$-dense. Moreover, if $\vec{x}$ is a clique in G' that is ζ_{A}-connectable in G, then it is $\zeta_{\mathrm{A}}/2$-connectable in G'. By our choice of constants in (3.3.10) and (3.3.11), it follows from the Connecting Lemma that if $\vec{x}$ and $\vec{y}$ are two disjoint ζ_{C}-connectable cliques in G', then there are at least $\xi_{\mathrm{C}}(n/2)^m$ distinct $(\vec{x}, \vec{y}; k)$-paths with m inner vertices in G', for some $m = m(\vec{x}, \vec{y}) \leqslant M_{\mathrm{C}}$.

We wish the reservoir to be disjoint from P_A, thus consider the induced subgraph $G - V(P_A) = G' - (V(\vec{x}_A) \cup V(\vec{y}_A))$. The number of $(\vec{x}, \vec{y}; k)$-paths in G' intersecting $\vec{x}_A$ or $\vec{y}_A$ is at most $2k \cdot m \cdot (n/2)^{m-1}$, implying that for n sufficiently large, there are at least

$$\xi_{\mathrm{C}}\left(\frac{n}{2}\right)^m - 2k \cdot m \cdot \left(\frac{n}{2}\right)^{m-1} \geqslant \frac{\xi_{\mathrm{C}}}{2}\left(\frac{n}{2}\right)^{m(\vec{x},\vec{y})}, \tag{3.3.13}$$

$(\vec{x}, \vec{y}; k)$-paths with inner vertices in $G - V(P_A)$. Note that this is true for any two disjoint ζ_{C}-connectable cliques in G', which in particular allows $\vec{x}_A$ or $\vec{y}_A$ to be one of them.

For the reservoir R we choose vertices from $V \smallsetminus V(P_A)$ independently with probability

$$p = \frac{\alpha_{\mathrm{A}}}{4}. \tag{3.3.14}$$

The desired properties for the reservoir are

(i) $|R| \leqslant \alpha_{\mathrm{A}} n/2$ and

(ii) for any two disjoint ζ_{C}-connectable cliques $\vec{x}$ and $\vec{y}$ in G', there are at least

$$\frac{\xi_{\mathrm{C}}}{4}\Big(\frac{\alpha_{\mathrm{A}}}{4}\Big)^m\Big(\frac{n}{2}\Big)^m,$$

$(\vec{x}, \vec{y}; k)$-paths with all their $m = m(\vec{x}, \vec{y})$ inner vertices in R.

For property (i) we observe that Markov's inequality implies

$$\mathbb{P}\Big(|R| \geqslant \frac{\alpha_{\mathrm{A}} n}{2}\Big) \leqslant \frac{2\mathbb{E}|R|}{\alpha_{\mathrm{A}} n} = \frac{n - |V(P_A)|}{2n} < \frac{1}{2}.$$

For property (ii), let $\vec{x}$ and $\vec{y}$ be two disjoint ζ_{C}-connectable cliques in G' and define X to be the random variable that counts the number of $(\vec{x}, \vec{y}; k)$-paths with all their $m = m(\vec{x}, \vec{y})$ inner vertices in R. Note that the inclusion or exclusion of a vertex in R affects X by at most $m \cdot n^{m-1}$ and that

$$\mathbb{E}X \overset{(3.3.13)}{\geqslant} \frac{\xi_{\mathrm{C}}}{2}\Big(\frac{n}{2}\Big)^m\Big(\frac{\alpha_{\mathrm{A}}}{4}\Big)^m.$$

Consequently, the Azuma-Hoeffding inequality (see, e.g., [28, Corollary 2.27]) asserts that

$$\begin{aligned}\mathbb{P}\Big(X \leqslant \frac{\xi_{\mathrm{C}}}{4}\Big(\frac{\alpha_{\mathrm{A}}}{4}\Big)^m\Big(\frac{n}{2}\Big)^m\Big) &\leqslant \exp\Big(-\frac{(\xi_{\mathrm{C}}/4)^2(\alpha_{\mathrm{A}}/4)^{2m}(n/2)^{2m}}{2n \cdot m^2 n^{2m-2}}\Big)\\ &\leqslant \exp\Big(-\frac{\xi_{\mathrm{C}}^2}{32} \cdot \frac{\alpha_{\mathrm{A}}^{2M_{\mathrm{C}}}}{8^{2M_{\mathrm{C}}} M_{\mathrm{C}}^2} \cdot n\Big).\end{aligned} \tag{3.3.15}$$

Since there are at most n^{2k} pairs $(\vec{x}, \vec{y})$, the union bound and (3.3.15) show that the probability of R not having property (ii) is less than $1/2$ for sufficiently large n. Hence, there is some set R satisfying both (i) and (ii).

Set aside P_A and R and cover the vertices of $G - (V(P_A) \cup V(R))$ by disjoint k-paths until at most $\alpha_{\mathrm{A}} n/2$ vertices are left uncovered. We obtain these k-paths by repeated applications of the Path Lemma. Assume we already have $\{P_1, \dots, P_i\}$, where for $j = 1, \dots, i$ the k-path P_j is a $(\vec{x}_j, \vec{y}_j; k)$-path with $\vec{x}_j$, $\vec{y}_j$ being ζ_{C}-connectable in G' and

$$|V(P_j)| \geqslant \frac{\alpha_{\mathrm{A}} \zeta_{\mathrm{P}}}{2} n. \tag{3.3.16}$$

Let $L = V - V(P_A) - V(R) - \bigcup_{j=1}^{i} V(P_j)$ be the set of vertices not yet covered and suppose that $|L| \geqslant \alpha_{\mathrm{A}} n/2$. Hence, the subgraph $G[L]$ is $(4\varrho/\alpha_{\mathrm{A}}^2, d)$-dense. By the choice in (3.3.11) and the Path Lemma, there is a $(\vec{x}_{i+1}, \vec{y}_{i+1}; k)$-path P_{i+1} in $G[L]$ with

$$|V(P_{i+1})| \geqslant \zeta_{\mathrm{P}}|L| \geqslant \frac{\alpha_{\mathrm{A}}\zeta_{\mathrm{P}}}{2} n,$$

and $\vec{x}_{i+1}$, $\vec{y}_{i+1}$ being ζ_{P}-connectable in $G[L]$. By the choice in (3.3.10), we have

$$\zeta_{\mathrm{P}}|L| \geqslant \zeta_{\mathrm{C}} n \geqslant \zeta_{\mathrm{C}}|V(G')|,$$

and hence, $\vec{x}_{i+1}$, $\vec{y}_{i+1}$ are ζ_{C}-connectable in G'. Therefore, we may enlarge the partial k-path covering until the set of leftover vertices has size at most

$$|L| < \frac{\alpha_{\mathrm{A}}}{2} n. \tag{3.3.17}$$

Let $\{P_1, \dots, P_{i_0}\}$ be such a family of k-paths. Note that (3.3.16) yields

$$i_0 < \frac{n}{\alpha_{\mathrm{A}}\zeta_{\mathrm{P}}/2 \cdot n} = \frac{2}{\alpha_{\mathrm{A}}\zeta_{\mathrm{P}}}. \tag{3.3.18}$$

The next step is to connect the k-paths in $\{P_A, P_0, P_1, \dots, P_{i_0}\}$ using the reserved vertices in R to obtain the k-th power of an almost spanning cycle. Assume we already obtained for some $0 \leqslant j \leqslant i_0$ a $(\vec{x}_A, \vec{y}_j; k)$-path Q^j containing the paths $P_A, P_1, \dots, P_j$ and such that

$$|V(Q^j) \smallsetminus (V(P_A) \cup \dots \cup V(P_j))| = |V(Q^j) \cap R| \leqslant j \cdot M_{\mathrm{C}}. \tag{3.3.19}$$

If $j < i_0$, we will connect Q^j to P_{j+1} and obtain the $(\vec{x}_A, \vec{y}_{j+1}; k)$-path Q^{j+1} that in addition contains P_{j+1} and satisfies $|V(Q^{j+1}) \cap R| \leqslant (j+1) \cdot M_{\mathrm{C}}$. For $j = i_0$, we have that Q^{i_0} is a $(\vec{x}_A, \vec{y}_{i_0}; k)$-path including all $\{P_A, P_1, \dots, P_{i_0}\}$ and we will connect $\vec{y}_{i_0}$ to $\vec{x}_{i_0+1} = \vec{x}_A$, obtaining the k-th power of a cycle.

By property (ii) of the reservoir, since $\vec{y}_j$ and $\vec{x}_{j+1}$ are ζ_{C}-connectable cliques in G', there are at least

$$\frac{\xi_{\mathrm{C}}}{4}\left(\frac{\alpha_{\mathrm{A}}}{4}\right)^m \left(\frac{n}{2}\right)^m,$$

$(\vec{y}_j, \vec{x}_{j+1}; k)$-paths with all its $m = m(\vec{y}_j, \vec{x}_{j+1})$ inner vertices in R. We need to ensure that one of these k-paths is disjoint from Q^j. By (3.3.19), the number of k-paths with m inner vertices that intersect Q^j is at most

$$j \cdot M_{\mathrm{C}} \cdot m \cdot n^{m-1} \overset{(3.3.18)}{<} \frac{2}{\alpha_{\mathrm{A}}\zeta_{\mathrm{P}}} \cdot M_{\mathrm{C}}^2 \cdot n^{m-1} \overset{(3.3.12)}{\leqslant} \frac{\xi_{\mathrm{C}}}{4}\left(\frac{\alpha_{\mathrm{A}}}{8}\right)^{M_{\mathrm{C}}} n^m.$$

Hence, there is a $(\vec{y}_j, \vec{x}_{j+1}; k)$-path with all its $m \leqslant M_{\mathrm{C}}$ inner vertices in R that is disjoint from Q^i. If $j < i_0$, this $(\vec{y}_j, \vec{x}_{j+1}; k)$-path can be used to build Q^{j+1}, which satisfies that

$$|V(Q^{j+1}) \cap R| \leqslant j \cdot M_{\mathrm{C}} + M_{\mathrm{C}} = (j+1) \cdot M_{\mathrm{C}}.$$

If $j = i_0$, use this $(\vec{y}_{i_0}, \vec{x}_{i_0+1}; k)$-path to close the k-th power of an almost spanning cycle H'.

The vertices from G which are not in H' are those from L, which were not covered by the almost perfect k-path covering, plus the vertices in R that were not used to connect the paths in $\{P_A, P_1, \ldots, P_{i_0}\}$. Hence,

$$|V \smallsetminus V(H')| \leqslant |L| + |R| \overset{(3.3.17)}{\leqslant} \frac{\alpha_{\mathrm{A}}}{2} n + \frac{\alpha_{\mathrm{A}}}{2} n.$$

Since P_A is a segment of H' and P_A is α_{A}-absorbing, we may replace P_A by a $(\vec{x}_A, \vec{y}_A; k)$-path with vertex set $V(P_A) \cup (V \smallsetminus V(H'))$ and obtain the desired k-th power of a Hamiltonian cycle in G. $\square$

3.4 Embedding graphs with small bandwidth

In this section we give the proof of Theorem 1.3.7, which is based on the regularity method for graphs and on Theorem 1.3.6.

We recall that a bipartite graph $G = (A \cup B, E)$ is called (ε, d)-regular, if

$$\big||E(A', B')| - d|A'||B'|\big| \leqslant \varepsilon |A||B|$$

for all subsets $A' \subseteq A$, and $B' \subseteq B$. Whenever we have that $d = |E(A, B)|/(|A||B|)$ is the edge density of (A, B), we simply say that (A, B) is ε-regular.

There are quite a few examples of results obtained by the regularity method, which are based on reductions to simpler or seemingly weaker results that are applied to the reduced graph of the regular partition obtained by an application of Szemerédi's Regularity Lemma [51] (see, e.g., [32, Sections 2, 4-6] and [29]). In particular, the proofs of the bandwidth theorems in [9, 50] were based on reductions to the corresponding theorems for powers of Hamiltonian cycles. We will follow the same route and start the discussion by recalling this approach. For the discussion below we assume the reader to be familiar with the regularity method for graphs and the *Blow-up Lemma* from [30].

Furthermore, since there are many versions, we refer to the following version of Szemerédi's Regularity Lemma [51].

Lemma 3.4.1. *For every $\varepsilon > 0$ and $t_0 \in \mathbb{N}$ there is some $T_0 = T_0(\varepsilon, t_0)$ such that every graph $G = (V, E)$ with $|V| = n \geqslant T_0$ admits a vertex partition $V_0 \,\dot\cup\, V_1 \,\dot\cup\, \dots \,\dot\cup\, V_t = V$ satisfying the following properties:*

1. *$|V_0| \leqslant \varepsilon n$ and $|V_1| = \dots = |V_t|$,*
2. *$t_0 \leqslant t \leqslant T_0$, and*
3. *all but at most εt^2 pairs (V_i, V_j) with $1 \leqslant i < j \leqslant t$ are ε-regular.*

Let $R = R(\varepsilon, \delta) = (V_R, E_R)$ be the auxiliary graph defined by $V_R = \{v_i : i \in [t]\}$ and $E_R = \{v_iv_j : (V_i, V_j)$ is ε-regular with density at least $\delta\}$. We call this graph R the reduced graph of G.

Preparations for the proof

For the proof of Theorem 1.3.7 we will need some tools that we will collect beforehand. We start with a slightly altered version of Theorem 1.3.6.

Theorem 3.4.2. *For every d, $\mu \in (0, 1]$, and $k \in \mathbb{N}$ there exist ϱ, α, $\gamma \in (0, 1)$ and $n_0 \in \mathbb{N}$ such that the following holds. Suppose $G = (V, E)$ is a (ϱ, d)-dense and μ-inseparable graph on $n \geqslant n_0$ vertices and subsets $U_1, \dots, U_m \subseteq V$ each of size at least $\mu n/2$ for some $m \leqslant 2^{\gamma n}$ are given.*

Then G contains the k-th power of a Hamiltonian cycle C with the additional property that for every $i \in [m]$ there are at least αn cliques K_{2k} contained in $C[U_i]$.

Proof. There is only one difference between Theorem 3.4.2 and Theorem 1.3.6. It concerns the additionally given vertex subsets $U_1, \dots, U_m \subseteq V = V(G)$ of size $\mu|V|/2$ for which we require that the guaranteed k-th power of a Hamiltonian cycle shares $\alpha|V|$ many K_{2k} with each of these sets. This additional restriction can be achieved by adjusting the proof of the Absorbing Path Lemma (Lemma 3.3.4), and we will give these adjustments here. Recall the proof of Lemma 3.3.4.

Given d, μ, and k, we set

$$\zeta = \frac{d^{\binom{2k+1}{2}} \mu^{2k+1}}{2^{2k+3}}. \tag{3.4.1}$$

Applying Lemma 3.1.1 for d, $\mu/2$, and $\zeta/2$ yields constants ϱ', ξ, M, and n_0' and we fix

$$\alpha = \frac{\zeta^2}{48(10k^2 + M)}, \qquad \gamma = \alpha/2 \qquad \text{and} \qquad \varrho = \min\left\{\frac{\varrho'}{4}, \frac{d^{\binom{2k+1}{2}}\mu^2}{8(2k+1)^2}\right\}, \tag{3.4.2}$$

and let a sufficiently large (ϱ, d)-dense and μ-inseparable graph $G = (V, E)$ on n vertices be given. Note that the only things we changed from the original version of Lemma 3.3.4

are α, which decreased by a factor of $1/2$ and will also be the α from Theorem 3.4.2, and we included $\gamma = \alpha/2$.

After defining the constants, we showed that the number of ordered K_{2k+1} in $G[N(v)]$ is at least $2\zeta n^{2k+1}$, resulting in at least ζn^{2k} different ζ-connectible K_{2k} inside $G[N(v)]$ for every $v \in V$. For that we used that G is (ϱ, d)-dense, and that $G[N(v)] \geqslant \mu n/2$ for sufficiently large n. Since we have $|U_i| \geqslant \mu n/2$ for all $i \in [m]$ by definition, the same argument yields that there are at least ζn^{2k} different ζ-connectible K_{2k} inside every U_i.

Later we defined $X_v = |A \cap \mathcal{A}_v|$, and in (3.3.6) we showed that

$$\begin{aligned}\mathbb{P}\Big(\exists v \colon X_v \leqslant \frac{\zeta^2 n}{12(10k^2 + M)}\Big) &\leqslant n \cdot \max_{v \in V} \mathbb{P}\Big(X_v \leqslant \frac{\zeta^2 n}{12(10k^2 + M)}\Big) \\ &\leqslant n \cdot \exp\Big(- \frac{\zeta^2 n}{48(10k^2 + M)}\Big) < \frac{1}{3},\end{aligned}$$

for sufficiently large n. But the $1/3$ was very generously chosen, and the equation stays true for $1/6$ instead.

Defining $Y_i = |A \cap U_i|$, we might do an analogous calculation to get that

$$\begin{aligned}\mathbb{P}\Big(\exists i \colon Y_i \leqslant \frac{\zeta^2 n}{12(10k^2 + M)}\Big) &\leqslant m \cdot \max_{i \in [m]} \mathbb{P}\Big(Y_i \leqslant \frac{\zeta^2 n}{12(10k^2 + M)}\Big) \\ &\leqslant m \cdot \exp\Big(- \frac{\zeta^2 n}{48(10k^2 + M)}\Big) < \frac{1}{6},\end{aligned}$$

for sufficiently large n, where in the last equation we used that

$$m \leqslant 2^{\gamma n} \overset{(3.4.2)}{=} 2^{\frac{\zeta^2 n}{96(10k^2+M)}}.$$

Therefore, our collection A_0 will ensure that

(i) $|A_0 \cap \mathcal{A}_v| \geqslant \zeta^2 n/(12(10k^2 + M)$ for every $v \in V$,

(ii) $|A_0 \cap U_i| \geqslant \zeta^2 n/(12(10k^2 + M)$ for every $i \in [m]$,

(iii) there are at most $\zeta^2 n/(24(10k^2 + M))$ pairs of intersecting absorbers in A_0,

(iv) the size $|A_0|$ is at most $\zeta n/(2(10k^2 + M))$.

And after switching to A_1 we will still have

$$|A_1 \cap \mathcal{A}_v| \geqslant \frac{\zeta^2 n}{24(10k^2 + M)} = 2\alpha n$$

for every $v \in V$, as well as

$$|A_1 \cap U_i| \geqslant \frac{\zeta^2 n}{24(10k^2 + M)} = 2\alpha n$$

for every $i \in [m]$.

After designating this path as α-absorbing path and using it in the main proof to get a k-th power of a Hamiltonian cycle, at least αn disjoint absorbers in each U_i will still be unchanged. By definition each of these absorbers contains a K_{2k}, concluding the proof of Theorem 3.4.2. $\square$

For convenience in the proof of Theorem 1.3.7 we will be using the following strengthening of the Regularity Lemma (Lemma 3.4.1).

Lemma 3.4.3. *For all constants d, μ, δ, $\varepsilon \in (0,1]$, and $t_0, \Delta \in \mathbb{N}$ such that $\delta \leqslant min(\mu/4, d/13)$, and $\varepsilon \leqslant \mu^2/1000 \ll 1/\Delta$ there are constants T_0, ϱ, and n_0, such that for all $n \geqslant n_0$ all n-vertex graphs $G = (V, E)$ which are (ϱ, d)-dense and μ-inseparable admit a partition $V = V_0 \cup V_1 \cup \ldots \cup V_t$ of the vertices of G such that the following holds.*

(*i*) $|V_0| \leqslant 2\sqrt{\varepsilon}n$, *and* $|V_1| = |V_2| = \cdots = |V_t|$.

(*ii*) $t_0 \leqslant t \leqslant T_0$, *and* $2(\Delta + 1)|t$.

(*iii*) *All but at most εt^2 pairs (V_i, V_j) for $1 \leqslant i < j \leqslant t$ are ε-regular.*

Furthermore, the reduced graph $R = R(\varepsilon, \delta)$ of this partition satisfies the following.

(I) *R is $(2\varepsilon, d/2)$-dense.*

(II) *R is $(\mu/2)$-inseparable.*

Proof. To set up the constants, we first let

$$s_0 \geqslant 2(t_0 + 2(\Delta + 1))$$

be big enough to comply with $1/s_0 \ll \varepsilon, d$ from Lemma 3.1.4 and let $T_0 = T_0(\varepsilon, s_0)$ be given by the Regularity Lemma (Lemma 3.4.1). We also let ϱ be small enough to comply with $\varrho \ll \varepsilon, d$ from Lemma 3.1.4, and finally let n_0 be large enough.

Now let $G = (V, E)$ be a given (ϱ, d)-dense and μ-inseparable graph. By Lemma 3.4.1 we get a ε-regular partition $V = U_0 \cup U_1 \cup \ldots \cup U_s$ with $s_0 \leqslant s \leqslant T_0$ of G. By Lemma 3.1.4 the reduced graph $Q = Q(\varepsilon, \delta)$ of this partition is $(\varepsilon, d/2)$-dense.

Now, following the proof of Lemma 3.2.4 we will find a set $S \subset [s]$ of size at most $\sqrt{\varepsilon}s$ such that moving all sets U_i with $i \in S$, to U_0 will leave us with a partition $V = V_0 \cup V_1 \cup \ldots \cup V_{t'}$ where the reduced graph is $(\mu/2)$-inseparable. Now moving up to $2(\Delta + 1)$ suitable of these partition classes to V_0, we get another $(\mu/2)$-inseparable partition $V = V_0 \cup V_1 \cup \ldots \cup V_t$ such that $2(\Delta + 1)|t$, let $R = R(\varepsilon, \delta)$ be the reduced graph of this partition.

The additionally moved partition classes will again be classes that are part of many non regular pairs of partition classes, so the proof of Lemma 3.2.4 will still work without them. Furthermore, obviously we have $t \leqslant T_0$ because we started with T_0 partition classes, but we can also guarantee $t_0 \leqslant t$, since by removing up to $\sqrt{\varepsilon} s$ of the original partition, by our assumptions on the constants, we still had at least half of them left, so $t' \geqslant t_0 + 2(\Delta+1)$. Indeed by our choice on s_0 we can ensure that $t \geqslant s/2$, since Δ and therefore $2(\Delta+1)$ is constant, and we will use this fact later.

Now having moved up to $\sqrt{\varepsilon} s + 2(\Delta+1)$ classes to U_0 to get V_0, the size of V_0 is at most

$$|U_0| + \sqrt{\varepsilon} n + 2(\Delta+1) \leqslant \varepsilon n + \sqrt{\varepsilon} n + 2(\Delta+1) \leqslant 2\sqrt{\varepsilon} n.$$

Since in the proof of Lemma 3.2.4 we moved the classes which were part of many irregular pairs to U_0, and for our suitable additional classes we moved to U_0 afterwards we choose ones that are part of many irregular pairs again, we will have at most εt^2 pairs V_iV_j for $1 \leqslant i < j \leqslant t$ that are not ε-regular, even if $t < s$.

Obviously our reduced graph R is $(\mu/2)$-inseparable by Lemma 3.2.4, but it remains to check, if R is indeed $(2\varepsilon, d/2)$-dense as claimed. By Lemma 3.1.4 we know that our first reduced graph Q is $(\varepsilon, d/2)$-dense, and since all the sets we consider in $V(R)$ are also sets in $V(Q)$, we only need to be concerned about the error term $\varepsilon s^2 > \varepsilon t^2$, but since $t^2 > s^2/2$, R is still $(2\varepsilon, d/2)$-dense. □

In the proof of Theorem 1.3.7 the following lemma will prove to be quite useful.

Lemma 3.4.4. *Let (A, B) be an (ε, d)-regular pair of density $d(A, B)$ at least d such that $|A| = |B|$.*

For any given f, let $A_1 \cupdot \ldots \cupdot A_f = A$ and $B_1 \cupdot \ldots \cupdot B_f = B$ be partitions of A and B such that

$$\frac{|A|}{2f} \leqslant |A_i|, |B_i| \leqslant \frac{2|A|}{f}$$

for every $i \in [f]$, then the pair (A_i, B_j) is $(4f^2\varepsilon, d)$-regular for every $i, j \in [f]$. In particular the density $d(A_i, B_j)$ of any pair of the partition classes is at least $d - 4f^2\varepsilon$.

Proof. Let $A' \subseteq A_i$ and $B' \subseteq B_j$ be given. By the definition of (ε, d)-regularity we have

$$\big||E(A', B')| - d|A'||B'|\big| \leqslant \varepsilon|A||B| \leqslant \varepsilon(2f|A_i|)(2f|B_j|) = (4f^2\varepsilon)|A_i||B_j|,$$

proving the first assertion.

The second assertion is another direct consequence of the definition of (ε, d)-regularity. Observing that $\varepsilon|A||B| \leqslant (4f^2\varepsilon)|A_i||B_j|$ again and dividing the defining equation for $A' = A_i$ and $B' = B_j$ by $|A_i||B_j|$ yields

$$\big|d(A_i, B_j) - d(A, B)\big| \leqslant 4f^2\varepsilon,$$

ensuing that the second assertion holds as well. $\square$

The last tool we need for the proof of Theorem 1.3.7 is the following theorem, which is a direct consequence of [10, Theorem 14]. This theorem uses the notion of super-regular partitions. A bipartite graph $G = (A \cup B, E)$ is called (ε, δ)-*super-regular*, if it is (ε, δ)-regular, and we have $|N(v, B)| \geqslant \delta|B|$ for all $v \in A$ and $|N(v, A)| \geqslant \delta|A|$ for all $v \in B$. We then extend this notion to partition classes $(V_i)_{i\in[t]}$ and a graph $R = ([t], E)$ by saying $(V_i)_{i\in[t]}$ is (ε, δ)-(super-)regular on R, if when $ij \in E(R)$, the pair (V_i, V_j) is (ε, δ)-(super-)regular.

Theorem 3.4.5. *For all $\Delta \in \mathbb{N}$, and $\delta \in (0, 1]$ there exist $\varepsilon \in (0, 1]$ such that for every s there is n_0 such that the following is true for every $n_1, \ldots, n_s$ with $n_0 \leqslant n = \sum n_i$ and $n_i \leqslant 2n_j$ for all $i, j \in [s]$. Assume we are given a graph R with $V(R) = [s]$, $\Delta(R) \leqslant 8\Delta$, and graphs G, H on $V(G) = V_1 \cup \ldots \cup V_s$, $V(H) = W_1 \cup \ldots \cup W_s$ and $\Delta(H) \leqslant \Delta$ with*

1. *$|V_i| = |W_i| = n_i$ for every $i \in [s]$,*
2. *$(V_i)_{i\in[s]}$ is (ε, δ)-super-regular on R,*
3. *for every edge $uv \in E(H)$, where $u \in W_i$ and $v \in W_j$ we have $ij \in E(R)$.*

Then $H \subseteq G$.

The proof of Theorem 1.3.7

The proof of Theorem 1.3.7 is based on the Regularity Lemma and the Blow-up Lemma and we start by fixing all involved constants.

Proof of Theorem 1.3.7. Given d, μ, and Δ, set

$$\delta = min(\mu/32, d/13).$$

Applying Theorem 3.4.5 with Δ and $\delta/4$ yields ε_B and $n_{0,B}$. For an application of Theorem 3.4.2 set

$$k = 2(\Delta + 1), \tag{3.4.3}$$

and apply Theorem 3.4.2 with $d/4$, $\mu/4$, and k to attain constants ϱ_H, α_H, γ_H, and $n_{0,H}$. For an application of Lemma 3.4.3 set

$$f = \left\lceil \frac{1}{\gamma_H} \right\rceil \qquad \text{and} \qquad \xi = min\Big(\frac{\varepsilon_B^2}{1440}, \frac{\delta}{160}\Big), \tag{3.4.4}$$

$$\text{as well as} \qquad \varepsilon_R = min\Big(\frac{\varepsilon_B}{16f^2}, \frac{\varrho_H}{2}, \frac{\delta^2}{(24f(\Delta + 1))^2}, \frac{\xi^2\alpha_H^2}{12(\Delta + 1)}\Big), \tag{3.4.5}$$

and apply Lemma 3.4.3 with $d, \mu, 2\delta, \varepsilon_R, \Delta$, and $t_0 = 0$ to attain constants $T_{0,R}$, ϱ_R, and $n_{0,R}$. Finally, set

$$\varrho = min\Big(\varrho_R, \frac{d}{8f^2T_0^2}\Big), \qquad \beta = \frac{\Delta}{fT_0}, \tag{3.4.6}$$

and let n be sufficiently large.

Finally, let a (ϱ, d)-dense and μ-inseparable graph $G = (V_G, E_G)$ on $n \geqslant n_0$ vertices, as well as an n-vertex graph $H = (V_H, E_H)$ satisfying $\Delta(H) \leqslant \Delta$ and $\mathrm{bw}(H) \leqslant \beta n$ be given.

Let $V(H) = \{h_1, h_2, \ldots, h_n\}$ be the vertices of H in a bandwidth ordering. We try to find an embedding of H into G by first defining an order v_i of the vertices of G such that their order will closely resemble the order of the vertices h_i in H.

Apply Lemma 3.4.3 with $d, \mu, 2\delta, \varepsilon_R, \Delta$ to G to get a partition $V = V_0 \mathbin{\dot\cup} V_1 \mathbin{\dot\cup} \ldots \mathbin{\dot\cup} V_t$ of the vertices of G such that the following holds.

(i) $|V_0| \leqslant 2\sqrt{\varepsilon_R}n$, and $|V_1| = |V_2| = \cdots = |V_t|$.

(ii) $t_0 \leqslant t \leqslant T_0$, and $(\Delta + 1)|t$.

(iii) All but at most $\varepsilon_R t^2$ pairs (V_i, V_j) for $1 \leqslant i < j \leqslant t$ are ε_R-regular.

Furthermore, the reduced graph $R = R(\varepsilon_R, 2\delta)$ of this partition satisfies the following.

(I) R is $(2\varepsilon_R, d/2)$-dense.

(II) R is $(\mu/2)$-inseparable.

Preparing for an application of Theorem 3.4.2 on R, we will define sets

$$U'_v = \big\{i \in t\colon |N_G(v) \cap V_i| \geqslant 2\delta|V_i|\big\}$$

and remark that by construction we have at most 2^t different such sets.

It follows by a standard averaging argument, that $|U'_v| \geqslant \mu t/8$ for all $v \in V$, since G is μ-inseparable and $|V_0| \leqslant \mu n/2$.

However, since this would still possibly yield too many different sets U_v for Theorem 3.4.2 to handle, we will refine each class V_i for $i \in [t]$ of our partition into sets $V_{i,1} \mathbin{\dot\cup} \ldots \mathbin{\dot\cup} V_{i,f}$. The following claim will ensure that this refined partition keeps all our desired properties. In particular, the number of different sets U_v stays 2^t, while the number of clusters goes up by a factor of f.

Claim 3.4.6. *There is a partition $V_i = V_{i,1} \cupdot \ldots \cupdot V_{i,f}$ of each V_i into f subsets, such that the following holds.*

1. *$(1-\xi)|V_i|/f \leqslant |V_{i,\ell}| \leqslant (1+\xi)|V_i|/f$ for each $\ell \in [f]$.*
2. *The reduced graph $S = S(4f^2\varepsilon_R, \delta)$ of the refined partition of $V \smallsetminus V_0$ contains an f-blow-up of $R = R(\varepsilon_R, 2\delta)$ as a subgraph.*
3. *There are sets*

$$U_v \subseteq \big\{(i,\ell) \in [t] \times [f] \colon |N_G(v) \cap V_{i,\ell}| \geqslant \delta |V_{i,\ell}|\big\},$$

of size at least $\mu t f/8$ each. Such that for every two vertices v, and w we have $U_v = U_w$ if $U'_v = U'_w$.

Proof. For a more convenient notation let $n' = n - |V_0| \geqslant n/2$. For a given set V_i we will randomly define the partition $V_{i,1} \cupdot \ldots \cupdot V_{i,f}$ of V_i such that for any vertex $v \in V_i$ we have $\mathbb{P}(v \in V_{i,\ell}) = 1/f$ for all $\ell \in [f]$.

Apparently for all $i \in [t]$ and $\ell \in [f]$ we have

$$\mathbb{E}(|V_{i,\ell}|) = \frac{1}{f}|V_i| = \frac{n'}{f \cdot t}.$$

Chernoff's inequality then tells us, that

$$\mathbb{P}\Big(|V_{i,\ell}| \leqslant (1-\xi)\frac{|V_i|}{f}\Big) \leqslant exp\Big(-\frac{\xi^2 \frac{n'}{f \cdot t}}{2}\Big) \quad \text{and} \quad \mathbb{P}\Big(|V_{i,\ell}| \geqslant (1+\xi)\frac{|V_i|}{f}\Big) \leqslant exp\Big(-\frac{\xi^2 \frac{n'}{f \cdot t}}{3}\Big).$$

Since we only have $f \cdot t \ll n'$ classes $V_{i,\ell}$, this implies that property 1 holds with probability more than $\frac{1}{2}$ by the union bound for sufficiently large n.

Let $X_{v,i,\ell}$ be the random variable counting the number of neighbours of v in $V_{i,\ell}$. Given that $\mathbb{E}(X_{v,i,\ell}) \geqslant 2\delta n'/(ft)$ for every $i \in U'_v$ and $\ell \in [t]$, another application of Chernoff's inequality yields

$$\mathbb{P}\Big(X_{v,i,\ell} \leqslant (1-\xi)\frac{2\delta n'}{ft}\Big) \leqslant exp\Big(-\frac{2\xi^2\delta \frac{n'}{f \cdot t}}{2}\Big),$$

for all $v \in V$, $i \in U'_v$, and $\ell \in [f]$. Since we have at most $f \cdot t \cdot n$ random variables $X_{v,i,\ell}$, by the union bound, the probability of our randomly defined refinement of the V_i giving rise to a "to small" $X_{v,i,\ell}$ with $i \in U'_v$ is less than $\frac{1}{2}$ for sufficiently large n.

Note that $\xi \leqslant 1/4$. Consequently by defining $U_v = \{(i,\ell) \colon i \in U'_v \text{ and } \ell \in [t]\}$ we get that there is refinement of the V_i such that item 1 and item 3 hold.

Item 2 then easily follows from Lemma 3.4.4, since $\xi \leqslant 1/2$ and $2\delta - 4f^2\varepsilon_R \geqslant \delta$. $\square$

Let $s = |V(S)| = t \cdot f$ and $\varepsilon'_R = f^2 \varepsilon_R$ for a more convenient notation. Recall that $|U_v| \geqslant \mu t f/8 = \mu s/8$ for all $v \in V$.

By Lemma 3.1.4 and Lemma 3.2.4 the reduced graph $S = S(4\varepsilon'_R, \delta)$ is $(2\varepsilon_R, d/4)$-dense and $(\mu/4)$-inseparable, and by our choice of constants in (3.4.3), in (3.4.4), in (3.4.5), and in (3.4.6) Theorem 3.4.2 gives us a graph $C \subseteq S$ that is a k-th power of a Hamiltonian cycle, with the additional requirement that there are at least αs cliques $K_{4(\Delta+1)}$ in each $C[U_v]$. In particular, since $(\Delta + 1)|s$, C contains a $(\Delta + 1, \frac{s}{\Delta+1} - 2, \frac{s}{\Delta+1} - 2)$-rope (see Figure 3.3.1 for an illustration) on s vertices as a spanning subgraph. Let $R_{\Delta+1}$ be this spanning subgraph of C.

Claim 3.4.7. *There is a partition $V = V_1^* \cup \ldots \cup V_s^*$ of the vertex set of G such that the following holds.*

1. $|V_i^*| \leqslant 2|V_j^*|$ *for all* $i, j \in [s]$.
2. $|V_i^*| \geqslant |V_{i+1}^*| \geqslant \cdots \geqslant |V_{i+\Delta}^*| \geqslant |V_i^*| - 1$ *for* $i = 1 + (j-1)(\Delta + 1)$ *and* $j \in [\frac{s}{\Delta+1}]$.
3. $(V_i^*)_{i\in[s]}$ *is* $(\varepsilon_B, \delta/4)$*-super-regular on* $R_{\Delta+1}$.

Proof. Consider the sets $\{V_{i,\ell} : i \in [t], \ell \in [f]\}$ that partition $V \setminus V_0$, and are all roughly of the same size. We will distribute the vertices of V_0 to these sets and slightly manipulate these sets to get our desired partition of V.

Let $(i_1, \ell_1), (i_2, \ell_2), \ldots, (i_s, \ell_s)$ be the indices of the $V_{i,\ell}$ in the order of C, starting at a $V_{i,\ell}$ of the "start $K_{\Delta+1}$" of $R_{\Delta+1}$ and traversing C alongside $R_{\Delta+1}$. Relabel each V_{i_j,ℓ_j} as V'_j. Apparently $(V'_i)_{i\in[s]}$ is $(4\varepsilon'_R, \delta)$-regular on $R_{\Delta+1}$.

Since $\Delta(R_{\Delta+1}) \leqslant 3(\Delta + 1)$, according to a proposition of Böttcher, Schacht, and Taraz [9, Proposition 13] (the proof can be found in [33, Proposition 8]) there are subsets $V''_i \subseteq V'_i$ of size at least $(1 - 4\varepsilon'_R(3(\Delta+1)))|V'_i|$, such that $(V''_i)_{i\in[s]}$ is $(8\varepsilon'_R, \delta/2)$-super-regular on $R_{\Delta+1}$. Removing these excessive vertices from V'_i for $i \in [s]$ enlarges V_0 to up to $3\sqrt{\varepsilon_R}n$, because $\varepsilon'_R \leqslant f^2 \frac{1}{(24f(\Delta+1))^2)}$.

Redistributing the vertices of V_0 to the V''_i, we have to distribute up to $3\sqrt{\varepsilon'_R}n$ vertices. Since by construction for each $v \in V_0$ we have a set U_v of indices of classes V'_i where these vertices have at least $\delta|V'_i|$ neighbours, and since we reduced each V'_i by a factor of at most $(1 - 4\varepsilon'_R(3(\Delta + 1)))$, these vertices still have at least $\delta/2|V'_i|$ neighbours in these V'_i.

For $j \in [\frac{s}{\Delta+1}]$ let $K^j = \{V'_i : i = 1 + (j-1)(\Delta + 1), \ldots, 1 + \Delta + (j-1)(\Delta + 1)\}$ be the distinguished $K_{\Delta+1}$ that form a $\Delta + 1$-factor in $R_{\Delta+1}$ alongside its vertex ordering. By the construction of $R_{\Delta+1}$, each $\{V'_i : i \in U_v\}$ meets $R_{\Delta+1}$ in at least $\alpha_H s$ distinct segments of $4(\Delta+1)$ vertices, consequently each set $\{V'_i : i \in U_v\}$ meets $R_{\Delta+1}$ in at least $\alpha_H s$ sets K^j, together with its successor K^{j+1} and predecessor K^{j-1} (if they exist). Embedding v into any V''_i in the middle $K_{\Delta+1}$, creating V'''_i, will therefore not meddle with the minimum degree that is required of the super-regular pairs by much.

Recall that we had

$$(1-\xi)\frac{n}{s} \leqslant |V_i'| \leqslant (1+\xi)\frac{n}{s} \text{ for each } i \in [s],$$

and after moving some vertices to V_0 we have that

$$(1-2\xi)\frac{n}{s} \leqslant (1-\xi)(1-4\varepsilon_R'(3(\Delta+1)))\frac{n}{s} \leqslant |V_i''| \leqslant (1+\xi)\frac{n}{s} \text{ for each } i \in [s],$$

and $(V_i'')_{i\in[s]}$ is $(8\varepsilon_R', \delta/2)$-super-regular on $R_{\Delta+1}$.

Creating $(V_i''')_{i\in[s]}$ by redistributing the vertices of V_0 in a balanced way, since we have $\alpha_H s$ choices to redistribute every vertex, we do not add more than $|V_0|/(\alpha_H s)$ vertices to any $K_{\Delta+1}$, and consequently to any V_i''. Therefore we have

$$(1-2\xi)\frac{n}{s} \leqslant |V_i'''| \leqslant (1+\xi)\frac{n}{s} + \frac{3\sqrt{\varepsilon_R'}n}{\alpha_H s} \leqslant (1+2\xi)\frac{n}{s} \text{ for each } i \in [s].$$

To get to $(V_i'''')_{i\in[s]}$ we will move some vertices, that we will specify in the penultimate paragraph of this proof, inside each K^j to balance the included partition sets in such a way that they differ in size by at most 1. Since for each $i \in [s]$ the symmetric difference of V_i'' and V_i'''' is at most

$$\left[(1+2\xi)\frac{n}{s} - (1-2\xi)\frac{n}{s}\right] + \frac{3\sqrt{\varepsilon_R'}n}{\alpha_H s} \leqslant 5\xi\frac{n}{s} \leqslant 10\xi|V_i''|$$

Quoting another proposition of Böttcher, Schacht, and Taraz [9, Proposition 14], and since $(V_i'')_{i\in[s]}$ is $(8\varepsilon_R', \delta/2)$-super-regular on $R_{\Delta+1}$, we have that $(V_i'''')_{i\in[s]}$ is $(\varepsilon_B, \delta/4)$-super-regular on $R_{\Delta+1}$, as long as the minimum degree condition of $\delta/4$ is uphold by every vertex.

Considering that $|V_i'' \cap V_i''''| \geqslant |V_i''|/2$ for every $i \in [s]$ and that we are only $\delta/4$-dense on $R_{\Delta+1}$ in $(V_i'''')_{i\in[s]}$, the minimum degree condition is still uphold for every vertex that started in V_i'' and ended up in V_i''''. While redistributing the vertices from V_0, we did it in such a way, that they had enough neighbours in all the necessary V_j'' and by dropping in density requirement they still have enough neighbours in V_j''''.

While moving vertices in the step from $(V_i''')_{i\in[s]}$ to $(V_i'''')_{i\in[s]}$ they are required to have a large neighbourhood in the same vertex classes as before with one exception. Some vertices that were moved might not have enough neighbours in the V_i''' they came from, but this can be avoided by choosing suitable vertices to move. Owing to G being (ϱ, d)-dense, and by a standard averaging argument, there are more than $10\xi|V_i'''|$ vertices inside each V_i''' having at least $\delta/2|V_i'''|$ neighbours inside V_i'''.

Obviously $(V_i'''')_{i\in[s]}$ does satisfy the properties the claim guarantees for a partition of V, so this concludes the proof of Claim 3.4.7. □

The partition of G provided by Claim 3.4.7 is the one we want to use in our application of Theorem 3.4.5. Since the sizes of the partition classes do not differ by too much, $\Delta(R_{\Delta+1}) \leqslant 3(\Delta+1) \leqslant 8\Delta$ and $\Delta(H) \leqslant \Delta$ are given, therefore we just need to find a partition $W_1^* \cup \ldots \cup W_s^* = V(H)$ of H in same size blocks as the partition of G such that for every edge $uv \in E(H)$, where $u \in W_i^*$ and $v \in W_j^*$ we have $ij \in E(R)$.

Finding this partition of H however is not that hard. Recall that $V(H) = \{h_1, h_2, \ldots, h_n\}$ are the vertices of H in the bandwidth ordering of H, and let $r_j = \sum_{i \in I_j} |V_i^*|$ with $I_j = 1 + (j-1)(\Delta+1), \ldots, 1 + \Delta + (j-1)(\Delta+1)\}$ be the size of the j-th "$(\Delta+1)$-superset of V_i^*" for $j = 1, \ldots, s/(\Delta+1)$. Now inductively define W_j for $j = 1, \ldots, s/(\Delta+1)$ by letting W_j be the first r_j vertices of H in the bandwidth ordering of H that are not already part of some W_i with $i < j$.

Using that $\Delta(H) \leqslant \Delta$ and therefore also $\Delta(H[V(W_i)]) \leqslant \Delta$, the Hajnal-Szemerédi theorem [26] yields for every W_i an equitable $(\Delta+1)$-colouring, and we will be choosing these colour classes $W^*_{1+(i-1)(\Delta+1)} \cup W^*_{2+(i-1)(\Delta+1)} \cup \ldots \cup W^*_{\Delta+1+(i-1)(\Delta+1)} = W_i$ as our refinement of the partition $(W_i)_{i \in [s/(\Delta+1)]}$. By labelling the colour classes in such a way that we avoid matching a W_i^* to a V_j^* where the sizes differ by 1 is sufficient to get the partition classes $(W_i^*)_{i \in [s]}$ such that $|W_i^*| = |V_i^*|$ for every $i \in [s]$ because of Claim 3.4.7 part 2 and our choice on the size of the W_i.

The partition classes $(W_i^*)_{i \in [s]}$ moreover satisfy Theorem 3.4.5 3, since $\mathrm{bw}(H) \leqslant \beta n$, and

$$\beta n \leqslant (\Delta+1)\frac{n}{2s}$$

implying that a vertex $v \in W_j$ might only have neighbours in W_i for $i \in \{j-1, j, j+1\}$ but not in its own sub partition class, since these are independent. In $R_{\Delta+1}$ however, all of the sub partition classes of W_j span a complete graph together with all of the sub partition classes of W_{j-1}, and similarly all of the sub partition classes of W_j span a complete graph together with all of the sub partition classes of W_{j+1}.

Therefore Theorem 3.4.5 now tells us, that $H \subseteq G$, which concludes the proof of Theorem 1.3.7. $\square$

3.5 Robust Hamiltonian graphs

In this section we are making some first steps in the direction of finding a common generalisation of the approximate version of Theorem 1.3.2 and Theorem 1.3.6. We will restrict ourself to $k = 1$ for this purpose. We note that, as mentioned in Section 1.3, uniform density cannot be forced by a reasonable minimum degree condition. In particular, for $k = 1$, an approximate version of Theorem 1.3.1 would require a considered graph G to have $\delta(G) \geqslant (\frac{1}{2} + \varepsilon)|V(G)|$, which would not force the graph to be uniformly dense at all. One can show however, that it does force G to abide the following definition.

Definition 3.5.1. For $d, \varrho, \eta \in (0,1]$ we say that a graph $G = (V, E)$ on $|V| = n$ vertices is *(ϱ, d, η)-robust matchable*, if for all $U \subseteq V$ we have one of the following.

$$e(U) \geqslant d\frac{|U|^2}{2} - \varrho n^2,$$

or

$$|U| \leqslant \frac{n}{2} + \eta n \quad \text{and} \quad \left|\left\{v \in V \smallsetminus U \colon |N(v) \cap U| \geqslant d|U|\right\}\right| \geqslant |U| - 2\eta n.$$

And as it turns out, this is sufficient to guarantee the appearance of Hamiltonian cycles, as the following theorem by Maesaka and Schacht [40] ensures.

Theorem 3.5.2. *For every $d, \mu, \eta \in (0,1]$ there exist $\varrho > 0$ and $n_0 \in \mathbb{N}$ such that every graph that is μ-inseparable and (ϱ, d, η)-robust matchable on $n \geqslant n_0$ vertices is Hamiltonian.*

As graphs which are uniformly dense are uniformly robust matchable as well, this theorem indeed is a common generalisation of Theorem 1.3.2 and Theorem 1.3.6 for $k = 1$.

In both, Theorem 1.3.6 and Theorem 3.5.2 the Hamiltonian cycles seem to appear in a "robust" way, in a sense that changing small parts of G does not destroy its Hamiltonicity. In an attempt to capture this robustness in a formal way, we give the following definition.

Definition 3.5.3. For $\delta > 0$ we say a graph $G = (V, E)$ on $|V| = n$ vertices is *δ-robust Hamiltonian*, if every subgraph $H = (V, E')$, which satisfies

(*i*) $e_H(X, V \smallsetminus X) \geqslant e_G(X, V \smallsetminus X) - \delta n^2$

(*ii*) $d_H(v) \geqslant d_G(v)/2 - \delta n$

for every subset $X \subseteq V$ and vertex $v \in V$ is Hamiltonian.

The factor 1/2 in Definition 3.5.3 (*ii*) stems from the graph given in Figure 3.5.1, that would without this factor be considered to be robust Hamiltonian as well. The Hamiltonicity of this graphs depends on just 2 vertices however, which does not seem to be very robust, and more generally, we want to exclude few vertices of very high degree to force a graph to contain a Hamilton cycle even if the rest of the graph does not support this cycle on its own. For the definition to capture some reasonable considered robust Hamilton graphs then, we restrict the amount of edges that may be deleted globally in Definition 3.5.3 (*i*).

As it turns out, this notion of being robust Hamiltonian, in a sense, even is equivalent to being inseparable and uniformly robust matchable. With the help of Theorem 3.5.2 we are able to present the following theorem.

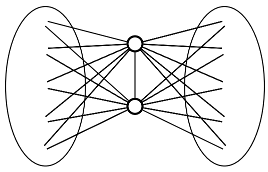


Figure 3.5.1: The shown $2n$-vertex graph, consisting of two K_{n-1} together with two vertices with complete neighbourhood is Hamiltonian, but not δ-robust Hamiltonian for any $\delta > 0$.

Theorem 3.5.4.

(*a*) *For every $d, \mu, \eta \in (0, 1]$ there exist $\varrho, \delta > 0$ and $n_0 \in \mathbb{N}$ such that every graph that is μ-inseparable and (ϱ, d, η)-robust matchable on $n \geqslant n_0$ vertices is δ-robust Hamiltonian.*

(*b*) *For every $\delta \in (0, 1/2]$ there exist $d, \mu, \eta, \varrho > 0$ and $n_0 \in \mathbb{N}$ such that every δ-robust Hamiltonian graph on $n \geqslant n_0$ vertices is μ-inseparable and (ϱ, d, η)-robust matchable.*

Proof of Theorem 3.5.4 (a). For this direction we will use the following claim, stating that we can move from a μ-inseparable and (ϱ, d, η)-robust matchable graph G to a subgraph H as in Definition 3.5.3 without losing any of the properties, maybe with slightly worse constants.

Claim 3.5.5. *For every $d, \mu, \eta \in (0, 1]$ there exist $d', \mu', \eta', \varrho', \varrho, \delta > 0$ and $n_0 \in \mathbb{N}$ such that every subgraph H that complies with Definition 3.5.3 of a graph G that is μ-inseparable and (ϱ, d, η)-robust matchable on $n \geqslant n_0$ vertices is μ'-inseparable and (ϱ', d', η')-robust matchable.*

Let $d, \mu, \eta \in (0, 1]$ be given. We will choose the constants from Claim 3.5.5 such that the subgraphs $H \subseteq G$ from Definition 3.5.3 are still μ-inseparable and (ϱ, d, η)-robust matchable (for slightly worse constants). Applying Theorem 3.5.2 to these subgraphs then implies this direction of Theorem 3.5.2.

It is left to verify Claim 3.5.5.

Proof of Claim 3.5.5. Let $d, \mu, \eta \in (0, 1]$ be given. We define the constants

$$\mu' = \frac{\mu}{5} \qquad \text{as well as} \qquad d' = \frac{d}{2} \quad \text{and} \quad \eta' = 2\eta.$$

We will also need the constants

$$\varrho > 0 \quad \text{and} \quad \delta = min\left\{\frac{\mu}{25}, \frac{\mu^2}{16}, \frac{\varrho}{5}, 2\eta\sqrt{d\varrho}\right\} \qquad \text{as well as} \qquad \varrho' = 2\varrho.$$

Now let G be a sufficiently large μ-inseparable and (ϱ, d, η)-robust matchable graph.

First we are going to ensure that any graph H we might consider is μ'-inseparable. Consider a set X and the partition $(X, V \setminus X)$, by symmetry we may assume $|X| \leqslant n/2$, furthermore we will assume $|X| \geqslant \mu n/4$. Apparently

$$e_G(X, V \setminus X) \geqslant \mu(\mu\frac{n}{4})(n - \mu\frac{n}{4}) \geqslant \frac{3\mu^2 n^2}{16}$$

and by deleting at most $\delta n^2 \leqslant \mu^2 n^2/16$ crossing edges we will have a lot more than a fifth of the original crossing edges remaining, so at least for sets X with $|X| \geqslant \mu n/4$ our potential graphs H appear to be μ'-inseparable.

Now assume $|X| < \mu n/4$. Since the minimum degree of G is at least $\mu(n-1) \geqslant 99n/100$ for sufficiently large n, after removing half of its neighbours and subtracting another δn while accounting for possible neighbours inside X, every vertex of X still has at least

$$\frac{99}{200}\mu n - \delta n - |X| > \frac{49}{100}\mu n - \frac{4}{100}\mu n - \frac{25}{100}\mu n = \frac{1}{5}\mu n$$

neighbours in $V \setminus X$. Therefore all potential graphs H are μ'-inseparable as claimed.

Next we are going to ensure that any graph H we might consider is (ϱ', d', η')-robust matchable.

Assume that $U \subseteq V$ is a set of vertices satisfying

$$e_G(U) \geqslant d\frac{|U|^2}{2} - \varrho n^2,$$

but not

$$e_H(U) \geqslant d'\frac{|U|^2}{2} - \varrho' n^2 = d\frac{|U|^2}{4} - 2\varrho n^2.$$

This implies that $d|U|^2/4 - 2\varrho n^2 > 0$ and therefore that $e_G(U) > \varrho n^2$. Since we have $d' = d/2$ and even allow for a larger error term with $\varrho' = 2\varrho$, we must have discarded at least half the edges in U to get from G to H. It is a well known fact [44] that in a graph with m edges there is a partition with at least $m/2$ crossing edges, using this fact we find a partition in U where we have discarded at least $\varrho n^2/2/2$ crossing edges to get from G to H. This however contradicts $\delta \leqslant \varrho/5$, since this partition can be extended to a partition of V where we have deleted at least the same amount of edges to get from G to H.

Assume that $U \subseteq V$ is a set of vertices satisfying

$$\left|\left\{v \in V \setminus U \colon |N_G(v) \cap U| \geqslant d|U|\right\}\right| \geqslant |U| - 2\eta n,$$

but not

$$\left|\left\{v \in V \setminus U : |N_H(v) \cap U| \geqslant d'|U|\right\}\right| \geqslant |U| - 2\eta' n.$$

Let $U' \subseteq (V \setminus U)$ be the set of vertices with at least $d|U|$ neighbours in U from G. Since $\eta' = 2\eta$ and $d' = d/2$, at least $2\eta n$ vertices from U' must not have enough neighbours in U in H and therefore lost at least $|U|d/2$ edges each. In total this are at least $d\eta n|U|$ edges that were deleted from the partition $(U, V \setminus U)$.

Observer that when $|U| < \sqrt{2\varrho'/d'}n$, the set U complies with Definition 3.5.3 trivially, therefore we may assume $|U| \geqslant \sqrt{2\varrho'/d'}n$. But then we deleted at least $d\eta n(\sqrt{2\varrho'/d'}n) = \sqrt{8}\eta\sqrt{d\varrho}n^2$ crossing edges from a partition, contradicting $\delta \leqslant 2\eta\sqrt{d\varrho}$. □

This concludes the proof of this direction from Theorem 3.5.4 (*a*). □

Proof of Theorem 3.5.4 (*b*). Let $G = (V, E)$ be a δ-robust Hamiltonian graph. We define the constants

$$\mu = \delta \qquad \text{as well as} \qquad d = \delta^2 \quad \text{and} \quad \eta = \delta \quad \text{and} \quad \varrho > 0$$

First, we are going to show that G is μ-inseparable.

Assume G is not μ-inseparable. Hence, there must be a partition $V = X \cup Y$ of the vertex set, such that $e(X, Y) = \hat{\mu}|X||Y| < \mu|X||Y|$.

Assume by symmetry, that $|X| \geqslant |Y|$. Additionally we might assume there is no partition $V = A \cup B$ such that $\frac{e(A,B)}{|A||B|} < \frac{e(X,Y)}{|X||Y|}$.

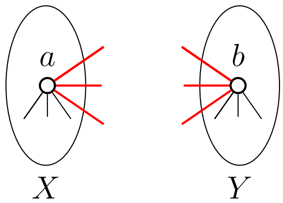


Figure 3.5.2: The partition (X, Y) of G with minimal density. Deleting the crossing edges will separate a from b, making G non-Hamiltonian.

Let $a \in X$ and $b \in Y$ be vertices. We would like to define H as G without the crossing edges from the partition (X, Y). This will separate a and b in H (see Figure 3.5.2) therefore, H cannot be Hamiltonian. On the other hand, we deleted at most $\hat{\mu}|X||Y| < \delta n^2$ edges in total, therefore, we did not delete more than δn^2 edges from any cut in G. Additionally, the following Claim 3.5.6 guarantees that we comply with Definition 3.5.3 (*ii*).

Claim 3.5.6.

- $d_X(b) \leqslant d_Y(b) + \hat{\mu}n$
- $d_Y(a) \leqslant d_X(a) + \hat{\mu}n$

Therefore, this H is a legitimate choice for a Hamiltonian subgraph of G by Definition 3.5.3, but not Hamiltonian. A contradiction. It is left to verify Claim 3.5.6.

Proof. Assume, $d_X(b) > d_Y(b) + \hat{\mu}n$. With the following minor calculations, one can see, that the partition $V = (X \cup \{b\}) \mathbin{\dot\cup} (Y \setminus \{b\})$ would contradict our assumption on the partition (X, Y) to have the smallest density of crossing edges.

$$
\begin{aligned}
\frac{e(X,Y)}{|X||Y|} &\leqslant \frac{e(X \cup \{b\}, Y \setminus \{b\})}{(|X|+1)(|Y|-1)} \\
\Leftrightarrow e(X,Y)\big[|X||Y| - |X| + |Y| - 1\big] &\leqslant \big[e(X,Y) - d_X(b) + d_Y(b)\big]|X||Y| \\
\Leftrightarrow \big[d_X(b) - d_Y(b)\big]|X||Y| &\leqslant e(X,Y)\big[|X| - |Y| + 1\big] \\
&= \hat{\mu}|X||Y|\big[|X| - |Y| + 1\big] \\
\Leftrightarrow d_X(b) &\leqslant d_Y(b) + \hat{\mu}\big[|X| - |Y| + 1\big] \\
&\leqslant d_Y(b) + \hat{\mu}n
\end{aligned}
$$

In the case of $|X| = |Y|$ we can prove $d_Y(a) \leqslant d_X(a) + \hat{\mu}n$ with the same calculation as above. Otherwise we have $|X| > |Y|$, and assuming $d_Y(a) > d_X(a)$ the partition $V = (X \setminus \{a\}) \mathbin{\dot\cup} (Y \cup \{a\})$ would contradict our assumption on the partition (X, Y) to have the smallest density of crossing edges, since it would have no smaller volume but fewer crossing edges. □

In the second step, we are going to show, that G is (ϱ, d, η)-robust matchable as well. Consider a subset $U \subseteq V$ of the vertices of G. The following claim will help us in the proof.

Claim 3.5.7. *If $e(U) < d\frac{|U|^2}{2} - \varrho n^2$, then $\big|\{u \in U \colon |N(u) \cap U| \geqslant \delta n\}\big| \leqslant \eta n$.*

Proof. Since $\eta = \delta$ and $d = \delta^2$, assuming $\big|\{u \in U \colon |N(u) \cap U| \geqslant \delta n\}\big| > \eta n$ would imply that there are at least

$$(\eta n)(\delta n)\frac{1}{2} = \delta^2 \frac{n^2}{2} = d\frac{n^2}{2} \geqslant d\frac{|U|^2}{2}$$

edges inside of U, contradicting our assumption on $e(U)$. □

To proceed in the proof, we note that the minimum degree of G is at least $2\delta n$. Otherwise G would not satisfy Definition 3.5.3 (*ii*).

Consider a subset $U \subseteq V$ such that $|U| > \frac{n}{2} + \eta n$.

If U does not comply with Definition 3.5.1, Claim 3.5.7 ensures that there are at most ηn vertices in U which have a degree of at least δn in U. Lets call this subset U'' and define $U' = U \setminus U''$ (see Figure 3.5.3).

Deleting all the edges inside U' leaves an independent set of more than $n/2$ vertices, therefore this graph cannot be Hamiltonian. Furthermore, this graph is indeed a

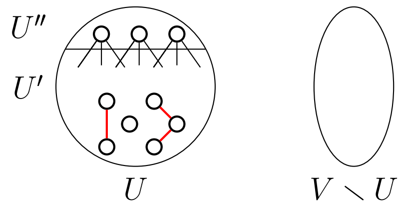


Figure 3.5.3: A big subset $U \subseteq V$ of the vertices, where high degree vertices are in U'' and low degree vertices in U'. Since $|U'| > n/2$, deleting the edges inside U' gives rise to a matching problem, making G non-Hamiltonian.

legitimate choice for the graph H in Definition 3.5.3, since we deleted no more than δn edges from a single vertex, ensuring Definition 3.5.3 (i) trivially and Definition 3.5.3 (ii) since the minimum degree of G is at least $2\delta n$.

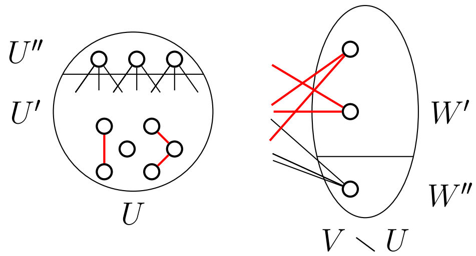


Figure 3.5.4: A small subset $U \subseteq V$ of the vertices, where high degree vertices are in U'' and low degree vertices in U'. Since the set W'' of vertices with high degree inside U is small, deleting the edges inside U' and between U' and W' gives rise to a matching problem, making G non-Hamiltonian.

Consider a subset $U \subseteq V$ such that $|U| \leqslant \frac{n}{2} + \eta n$.

We may assume, that $\left|\left\{v \in V \smallsetminus U \colon |N(v) \cap U| \geqslant d|U|\right\}\right| < |U| - 2\eta n$, (and therefore $|U| > 2\eta n$) since otherwise we would comply with Definition 3.5.1 anyway. Lets call this subset of $V \smallsetminus U$ by the name of W'' and define $W' = (V \smallsetminus U) \smallsetminus W''$ (see Figure 3.5.4). Similarly we many assume $e(U) < d\frac{|U|^2}{2} - \varrho n^2$. Claim 3.5.7 ensures that there are at most ηn vertices in U witch have a degree of at least δn in U. Lets call this subset U'' and define $U' = U \smallsetminus U''$ (see Figure 3.5.4).

Once again deleting all the edges inside U', as well as all the edges between U' and W' leaves us with a big independent set U', which has all its neighbours in $U'' \cup W''$. Since $|U'' \cup W''| = |U''| + |W''| < \eta n + |U| - 2\eta n = |U| - \eta n \leqslant |U'|$, this graph cannot contain a Hamilton cycle. Furthermore, this graph is indeed a legitimate choice for the graph H in Definition 3.5.3, since we deleted no more than $\delta n > d|U|$ edges from a single vertex, ensuring Definition 3.5.3 (i) trivially and Definition 3.5.3 (ii) since the minimum degree of G is at least $2\delta n$. $\square$

4. Concluding remarks

We discuss some outlook and further research regarding our three main theorems.

Homomorphism thresholds for graphs

While we provide an upper bound for $\delta_{\hom}(C_{2k-1})$, at this point it is not clear if it is best possible. Proving a matching lower bound or just showing $\delta_{\hom}(C_{2k-1}) > 0$, would require to establish the existence of a sequence of graphs $(G_n)_{n\in\mathbb{N}}$ with members from $\mathscr{G}_{C_{2k-1}}(\alpha)$ for some $\alpha > 0$ having no homomorphic C_{2k-1}-free image H of bounded size. However, without imposing H to be C_{2k-1}-free itself, no such sequence exists for $k \geqslant 3$, as was shown by Thomassen [53], as the chromatic threshold of odd cycles other than the triangle is 0, which makes the problem somewhat delicate and for the first open case we raise the following question.

Question 4.1.1. Is it true that $\delta_{\hom}(C_5) > 0$?

The affirmative answer to Question 4.1.1 would, in particular, show that there is a graph F with $\delta_{\hom}(F) > \delta_{\chi}(F)$. To our knowledge such a strict inequality is only known for families of graphs $\mathscr{F}$, like for $\mathscr{F} = \mathscr{C}_{2k-1}$ for $k \geqslant 3$.

The lack of lower bounds for families consisting of a single graph, may suggest the following natural variation of the homomorphic threshold

$$\begin{aligned}\delta'_{\hom}(F) = \inf\big\{\alpha \in [0,1] \colon & \text{there is an } \mathscr{F}\text{-free graph } H = H(\mathscr{F},\alpha) \\ & \text{such that } G \xrightarrow{\hom} H \text{ for every } G \in \mathscr{G}_F(\alpha)\big\},\end{aligned}$$

where $\mathscr{F}$ consists of all surjective homomorphic images of F. For odd cycles we have $\delta'_{\hom}(C_{2k-1}) = \delta_{\hom}(\mathscr{C}_{2k-1})$ and in view of Theorem 1.2.5 it seems possible that $\delta'_{\hom}(F)$ is easier to determine.

In Proposition 2.2.5 (i) we observed that C_{2k-1}-free graphs G of high minimum degree are in addition also C_{2j-1}-free for some sufficiently large $j < k$ depending on the imposed minimum degree. A more careful analysis of the argument may yield the correct dependency between j and the minimum degree of G and, moreover, yield a stability version of such a result. However, for a shorter presentation we used the same minimum degree assumption as given by Theorem 1.2.5, which sufficed for our purposes. It would also be interesting to see, if the excluded cycles of shorter odd length can be also excluded for the homomorphic image H in the proof of Theorem 1.2.5.

Structure of small homomorphic images

Recall that for $k = 2$ the Grötzsch graph proved that not all $\mathscr{C}_{2k-1}$-free graphs with minmum degree larger than 1/3 are homomorphic to an Andrásfai graph.

Letzter and Snyder then conjectured that any $\mathscr{C}_{2k-1}$-free graph G with $\delta(G) > \frac{1}{2k-1}|V(G)|$ for $k \geqslant 3$ is homomorphic to some Andrásfai graph, which, as it turns out, is not true as well. Indeed our Lemma 2.6.2, Observation 2.6.8, and Observation 2.6.9 from Section 2.6 show that odd tetrahedra are counterexamples to this conjecture for every $k \geqslant 4$. This makes the case of $k = 3$ somewhat special, but for $k = 2$ a slightly weaker result (see [11]) basically states that triangle free graphs with a high minimum degree are homomorphic to an Andrásfai graph or a generalised form of the Grötzsch graph (see [37, Section 4] for a nice visualisation).

Hoping for a similar theorem to be true for $k \geqslant 4$, and in particular, our counterexamples from Section 2.6 to be essentially the only counterexamples, we raise the following conjecture.

Conjecture 4.2.1. Let $k \geqslant 4$ and let G be a $(\mathscr{C}_{2k-1} \cup \mathscr{T}_k)$-free graph on n vertices with $\delta(G) > \frac{1}{2k-1}n$. Then G is homomorphic to $A_{k,r}$ for some r.

If this conjecture would be true, similar considerations as in the paper of Letzter and Snyder would lead to natural levels of homomorphism images depending on r as shown in Table 4.1.

Note that Theorem 1.2.6 is just Conjecture 4.2.1 for $k = 3$. Furthermore it is easy to see that for $k = 3$ maximal $\mathscr{C}_{2k-1}$-free graph on n vertices with $\delta(G) > \frac{1}{2k-1}n$ do not contain odd tetrahedra as subgraphs by Lemma 2.6.5, implying the Theorem of Letzter and Snyder from Theorem 1.2.6.

So, while we essentially give an alternative proof for the theorem of Letzter and Snyder, our approach had Conjecture 4.2.1 in mind, and in fact, the only step of the proof, where an upper bond on k is needed, is Lemma 2.5.6. Taking this into account, we may formulate another conjecture, that, if true, does imply Conjecture 4.2.1 by our proof of Theorem 1.2.6.

Conjecture 4.2.2. Let $k \geqslant 3$ and let G be a maximal $\mathscr{C}_{2k-1}$-free graph on n vertices with $\delta(G) > \frac{1}{2k-1}n$, then G does not contain a well-behaved $C_{2\ell}$ with $2\ell \geqslant 6$ as an induced subgraph.

We remark, that it is not necessary to prove the statement of Conjecture 4.2.2 for $2\ell \geqslant 4k - 2$ by hand as it is rather trivial for these values of ℓ. Considering this, for

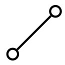
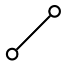
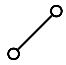
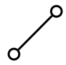
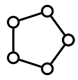
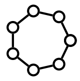
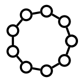
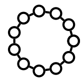
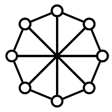
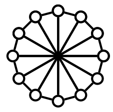
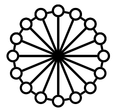
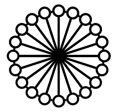
	$\mathscr{C}_3$-free	$\mathscr{C}_5$-free	$\mathscr{C}_7$-free	...	$\mathscr{C}_{2k-1}$-free	
$\delta(G) > \frac{2}{2k+1}n$				...		K_2
$\delta(G) > \frac{3}{4k}n$				...		C_{2k-1}
$\delta(G) > \frac{4}{6k-1}n$				...		M_{4k}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
$\delta(G) > \frac{1+r}{2+r(2k-1)}n$	$A_{2,r}$, $\mathscr{V}$	$A_{3,r}$	$A_{4,r}$	...	...	$A_{k,r}$
$\delta(G) > \frac{1}{2k-1}n$	$\mathscr{A}_2$, $\mathscr{V}$	$\mathscr{A}_3$	$\mathscr{A}_4$	...	...	$\mathscr{A}_k$

Table 4.1: The homomorphic images of $(\mathscr{C}_{2k-1} \cup \mathscr{T}_k)$-free graphs if Conjecture 4.2.1 is true. The rows indicate the assumed minimum degree, the columns indicate the size of k. In the cells than are the graphs that may be used as homomorphic image H for the homomorphism threshold. The $\mathscr{V}$ in the second (or first) column stands for Vega graphs, and the entire columns stems from [11].

small cases of k Conjecture 4.2.2 might be provable by a computer aided proof, and we might come back to this in the future.

For bigger cases of k we seem to be able to exclude certain big even cycles, but the small ones are still troublesome.

We remark that with Corollary 2.6.7, Conjecture 4.2.1 would imply that Letzter and Snyder's conjecture holds for slightly bigger minimum degree.

Finally, it would be nice if a similar statement as for $k = 2$ would be true for $k > 3$ as well, namely that every graph containing an odd tetrahedra is homomorphic to a combination of an odd tetrahedra blow-up and an Andrásfai graph blow-up. Currently however, we do not have sufficient evidence to make this a conjecture.

Enforcing spanning subgraphs

In Section 3.1 and Section 3.2 we collected some properties of uniformly dense and inseparable graphs. In the lemmas concerning blow-ups, namely Lemma 3.1.3 and Lemma 3.2.3, we lost a factor of 2 in our constants due to the use of probabilistic arguments. While one can certainly reduce the loss in the constants by more elaborate computations, it would be interesting to know, if such a loss in the constants can be prevented altogether.

It would furthermore be interesting to know how resilient the property of being robust matchable is. A starting point may be to see how resilient it is under blow-ups or regularizing, like we have seen for uniformly denseness in Section 3.1, and then see how far this concept would support further theorems.

Going back to uniformly dense and inseparabe graphs, as we are using the Blow-up

Lemma [30] as a central part in our proof of Theorem 1.3.7, it seems plausible to use the versions of the Blow-up Lemma developed in [8] to extend Theorem 1.3.7 in the spirit of [10, Theorem 3] to so called a-arrangeable graphs (see [10, Definition 2] for example).

Another interesting, and to our knowledge open, question would be the following.

Question 4.3.1. Is there a common generalisation of Theorem 1.3.2 and Theorem 1.3.6, for arbitrary k?

For such a theorem, the new version of Definition 1.3.3, namely Definition 3.5.1, seems applicable, but Definition 1.3.5 must probably be adjusted in a generalised way as well, since it is not enough to guarantee the existence of $(\vec{x}, \vec{y}; k)$-walks for arbitrary k.

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Appendix

English summary

In this thesis we investigate structures in dense graphs. In the first part we analyse certain aspects of the interplay of minimum degree conditions and structural properties of large graphs with forbidden subgraphs, which is a central topic in extremal graph theory. More precisely, for a given family of graphs $\mathscr{F}$ we define the *chromatic threshold* as the infimum over all $\alpha \in [0, 1]$ such that every n-vertex $\mathscr{F}$-free graph G with minimum degree at least αn has a homomorphic image H of bounded order, in other words, we insist that G has bounded chromatic number. In 2013 the chromatic threshold was determined by Allen et al. [1] for all finite families of graphs.

Insisting that the homomorphic image H of G is $\mathscr{F}$-free as well, we get the definition of the *homomorphism threshold*, that is less well understood, and Theorem 1.2.5 solves this question for the family of odd cycles up to a given length $2k - 1$. Moreover in Theorem 1.2.6 we analyse the graph H in more detail to get a deeper understanding for the structure of G.

In the second part we consider sufficient conditions for the existence of k-th powers of Hamiltonian cycles. Confirming a conjectures of Pósa (see [17]) and Seymour [49], more then 20 years ago Komlós, Sarközy, and Szemerédi [30] obtained optimal minimum degree conditions for this problem by showing that every n-vertex graph with minimum degree at least μn for $\mu = \frac{k}{k+1}$ contains the k-th power of a Hamiltonian cycle. For smaller values of μ the given graph G must satisfy additional assumptions, and in this direction Staden and Treglown [50] showed that $\mu = \frac{1}{2} + \varepsilon$ is sufficient when we insist that every linear size induced subgraph has density $d > 0$. This bound on μ is optimal under the given circumstances again. In Theorem 1.3.6 we show that μ can be chosen arbitrarily small as long as $\mu > 0$ if, in addition to linear size induced subgraph having density $d > 0$ we also insist that every cut has density at least μ.

In fact Staden and Treglown [50] showed that the graphs they consider do not just contain k-th powers of Hamiltonian cycles, but a broader class of graphs, namely graphs with bounded degree and sublinear bandwidth. The bandwidth of a graph is defined as the maximum distance of two vertices in a linear ordering of the vertices of a graph, where we take the minimum over all possible vertex orderings of the graph. In Theorem 1.3.7 we show that we can keep this property for our relaxed assumptions.

German summary (Deutsche Zusammenfassung)

In dieser Doktorarbeit befassen wir uns mit Strukturen in dichten Graphen. Im ersten Teil analysieren wir Zusammenhänge zwischen Minimalgradbedingungen und strukturellen Eigenschaften großer Graphen, die gewisse Teilgraphen nicht enthalten. Dies ist eines der zentralen Forschungsgebiete der extremalen Graphentheorie. Wir gehen näher auf den sogenannten *chromatic threshold* ein, dieser ist wie folgt definiert. Für eine Familie $\mathscr{F}$ von Graphen sei der *chromatic threshold* das Infimum aller $\alpha \in [0,1]$, sodass jeder $\mathscr{F}$-freie graph G auf n Ecken mit Minimalgrad mindestens αn ein homomorphes Bild H begrenzter Größe hat. Anders ausgedrückt wollen wir die chromatische Zahl von G durch eine Konstante $K(\mathscr{F}, \alpha)$ beschränken. 2013 wurde der genaue *chromatic threshold* für alle endlichen Familien $\mathscr{F}$ von Allen et al. [1] bestimmt.

Bestehen wir jedoch darauf, dass auch das homomorphe Bild H von G selbst wieder $\mathscr{F}$-frei sein soll, so führt dies zur Definition des *homomorphic threshold*. Dieser ist bisher noch kaum verstanden, und Theorem 1.2.5 bestimmt diesen für die Familie $\mathscr{C}_{2k-1}$ der ungeraden Kreise bis zu einer länge von $2k-1$. In Theorem 1.2.6 gehen wir noch genauer auf das homomorphe Bild H ein um ein ausdifferenzierteres Bild von der Struktur von G zu erhalten.

Im zweiten Teil untersuchen wir hinreichende Bedingungen für die Existenz k-ter Potenzen von Hamiltonkreisen in Graphen. Vor über 20 Jahren bestätigten Komlós, Sarközy und Szemerédi [30] eine Vermutung von Pósa [17] und Seymour [49] indem sie zeigten, dass jeder Graph auf n Ecken mit einem Minimalgrad von mindestens μn für $\mu = \frac{k}{k+1}$ die k-te Potenz eines Hamiltonkreises enthält. Diese Schranke für μ ist optimal. Staden und Treglown [50] konnten jedoch zeigen, dass man μ auf $\frac{1}{2} + \varepsilon$ reduzieren kann, wenn man zusätzlich fordert, dass G in induzierten Teilgraphen linearer Größe eine Dichte von wenigstens $d > 0$ aufweißt. Diese Schranke für μ ist optimal für das erzwingen k-ter Potenzen von Hamiltonkreisen unter der gegebenen Nebenbedingung. In Theorem 1.3.6 zeigen wir, dass mit dem einführen einer zweiten Nebenbedingung, nämlich dass jeder Schnitt mindestens die Dichte μ aufweist, μ auf einen beliebig kleinen positiven Wert abgesenkt werden kann.

Tatsächlich zeigten Staden und Treglown [50] in ihrem Artikel eine stärkere Aussage, und zwar das G nicht nur die k-te Potenz eines Hamiltonkreises enthält, sondern jeden Graphen H mit beschränktem Maximalgrad und sublinearer Bandweite. Die Bandweite eines Graphen ist dabei definiert als der maximale Abstand der Endecken einer Kante in einer linearen Ordnung der Ecken, wobei wir dies über alle linearen Ordnungen des Graphen minimieren. In Theorem 1.3.7 zeigen wir, dass wir diese Eigenschaft für G auch mit unseren schwächeren Nebenbedingungen erzwingen können.

Publications related to this dissertation

Articles

[A1] O. Ebsen, G. S. Maesaka, C. Reiher, M. Schacht, and B. Schülke, *Embedding spanning subgraphs in uniformly dense and inseparable graphs*, Random Structures and Algorithms (2019), available at arXiv:1909.13071. accepted. ↑A.4

[A2] O. Ebsen and M. Schacht, *Homomorphism Thresholds for Odd Cycles*, Combinatorica **40** (2020), no. 1, 39–62, DOI 10.1007/s00493-019-3920-8. MR4078811 ↑A.4

Extended abstracts

[E1] O. Ebsen, G. S. Maesaka, C. Reiher, M. Schacht, and B. Schülke, *Powers of Hamiltonian cycles in μ-inseparable graphs*, Acta Math. Univ. Comenian. (N.S.) **88** (2019), no. 3, 637–641. MR4014133 ↑A.4

Declaration of contributions

In this thesis, that is mostly based on [A1] and [A2], my co-authors and I share an equal amount of work.

The first half of Chapter 2 (Sections 2.1-2.3) is based on the paper *Homomorphism thresholds for odd cycles* [A2] which is joint work together with my advisor Mathias Schacht. This research project is a continuation of my master thesis. We discussed this problem in several approaches and eventually came up with the idea for Section 2.3. After I wrote up the first draft of the paper, Mathias Schacht proofread everything and we finalised the manuscript together.

The second half of Chapter 2 (Secions 2.4-2.7) is based on independent research by myself.

Chapter 3 is based on the paper *Embedding spanning subgraphs in uniformly dense and inseparable graphs* [A1], which is joint work with Giulia Maesaka, Christian Reiher, Mathias Schacht, and Bjarne Schülke and was also presented by Giulia Maesaka at Eurocomb 2019 [E1]. This extension of the theorem in [50] was first suggested by Christian Reiher and Mathias Schacht. Together with Bjarne Schülke and Giulia Maesaka we worked out the details of the proof and drafted an early version of the paper. After proofreading and streamlining the proof in varying subsets of 2 to 3 people, eventually this article was finalised, and is the basis for the Sections 3.1-3.3.

The extension of the work from [A1] to the bandwidth version (Theorem 1.3.7) was mainly discussed by Mathias Schacht and myself, and we included an outline of the proof in [A1]. The details of this proof were worked out by myself and appear in Section 3.4 (and for that we also added the last two lemmas in Section 3.1 and Section 3.2). A similar process as the one described above also applies to Section 3.5.

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Eidesstattliche Versicherung

Hiermit versichere ich an Eides statt, die vorliegende Dissertation selbst verfasst und keine anderen als die angegebenen Hilfsmittel benutzt zu haben. Darüber hinaus versichere ich, dass diese Dissertation nicht in einem früheren Promotionsverfahren eingereicht wurde.